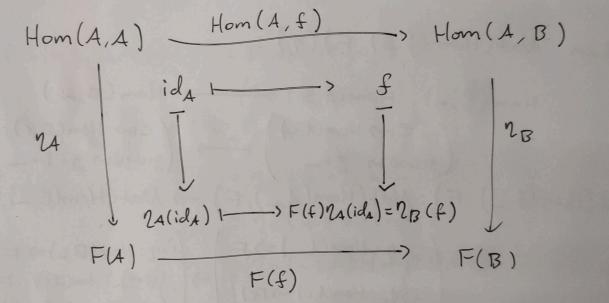


Lemma: Leb F: E-> <u>set</u> be a function. For each object A E e there is a natural bijection:

 $\Phi_A: Nat(Hom(A, -), F) \cong F(A).$ $2 \longrightarrow 2A(Id_A)$

Proof: Since 2 is natural, there is the following commutative diagram for any f: A-> BEE:



thus, if $u \in F(A)$, then defining $\eta_A(id_A) = u$ completely determines the natural transformation η (proving surjectivity and injectivity simultaneously).

To prove naturality, let $f:A \longrightarrow B$ be a morphism in E. Consider the following diagram:

Nat
$$(Hom(A,-),F)$$

Nat $(Hom(B,-),F)$

PB

Leb $\gamma \in Nat(Hom(A,-),F)$

Then $Nat(Hom(A,-),F)$.

Then $Nat(Hom(A,-),F)$.

Hom $(f,-):Hom(A,-)$
 $(g:c>0\mapsto g\circ-)$

Nat $(Hom(B,-),F):Nat(Hom(A,-),F) \to Nat(Hom(B,-),F)$
 $(g:c>0\mapsto g\circ-)$

Nat $(Hom(A,-)\to F):Nat(Hom(A,-)\to F)$
 $(g:c>0\mapsto g\circ-)$

Proof that this is well defined:

Need commutativity of: (for any $h:C\to P\in E$).

Hom (B,C)

Hom (B,C)

Hom (B,C)

F(0)

 $(f\circ\gamma)\circ=\gamma\circ(-\circ f)$

So we need

Since n is natural, we have commutativity of:

So for any j.B-, c we have:

So Nat (Hom (f, _), F) is well defined.

Now we prove commutationity of (x).

Ve need F(f) · \$\Phi_A = \Pi_B · Nat(Hom(f,-), F).

Leb 2: Hom(A, -) => F.

Then
$$(F(f) \circ \overline{\Phi}_A)(\gamma) = F(f)(\gamma_A(Fd_A))$$

= $\gamma_B(f)$

On the other hand, (\$00 Nor(Hom (f, _), F))(7) = \$\Pi_B (for) = (for)B (IdB) = 20(f) -So in particular: Take F = Hom(B, -): Nat (Hom (A, -), Hom (B, -)) = Hom (B, A). Also, if F: E -> Set is a contavariant functor, then $Nat(Hom(-,A),F) \subseteq F(A)$ 2 1-) 2A(IdA). So if F= Hom (-, B): Nat (Hom (-, A), Hom (-, B)) = Hom (A, B). In particular, this means there is a full and faithful embedding; e - set en

C --- Hom (-, c).

The category <u>set</u> en has nice properties which e may not:

Set:

- Admits all products,
- · Admits all limits and colimits (to come).
 - . Is a topos (very significant but outside the scope of this course).