Open Curry-Howard

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1 (Annotated) Intuitionistic Sequent Calculus

1.1 Preproofs

Definition 1.1.1. *Assume there is a countably infinite set of* ***atomic propositions*** *and assume the set* $\Psi_{\Rightarrow}$ *of* ***propositions*** *is built up from the atomic propositions using the connective* $\Rightarrow$*. That is, all atomic propositions are propositions and if* p *and* q *are propositions then so is* $p \Rightarrow q$*. For each proposition* p *let* Y_p *be a countably infinite set of* ***variables*** *associated with* p*. We will write* $x : p$ *for* $x \in Y_p$ *and say that* x *has type* p*. Let* $\mathcal{P}^n$ *be the set of all length* n *sequences of variables (of possibly different type), and* $\mathcal{P} := \cup_{n=1}^{\infty} \mathcal{P}^n$*. A* ***sequent*** *is a pair* $(\boldsymbol{x}, x)$ *where* $\boldsymbol{x} \in \mathcal{P}$ *and* $x \in \Psi_{\Rightarrow}$*.* ***Entailment*** $\vdash$ *is the smallest relation over* $\mathcal{P}$ *and* $\Psi_{\Rightarrow}$ *subject to the following axioms, notationally,* Γ *is an arbitrary element of* $\mathcal{P}$ *and* $\Gamma, x : p$ *means the element of* $\mathcal{P}$ *given by appending* $x : p$ *to the end of* Γ*:*

- *the* ***axiom law****:*

$$\frac{}{x : p \vdash p}\ ax$$

- *the following* ***structural rules****:*
  - ***Contraction***

$$\frac{\Gamma, x : p, x' : p, \Gamma' \vdash q}{\Gamma, x : p, \Gamma' \vdash q}\ (ctr)$$

  - ***Weakening***

$$\frac{\Gamma, \Gamma' \vdash q}{\Gamma, x : p, \Gamma' \vdash q}\ (weak)$$

  - ***Exchange***

$$\frac{\Gamma, x : p, y : q, \Gamma' \vdash r}{\Gamma, y : q, x : p, \Gamma' \vdash r}\ (ex)$$

- *the following* ***logical rules***
  - ***Right Abstraction***

$$\frac{\Gamma, x : p, \Gamma' \vdash q}{\Gamma, \Gamma' \vdash p \Rightarrow q}\ (R \Rightarrow)$$

  - ***Left Abstraction***

$$\frac{\Gamma \vdash p \qquad \Gamma', x : q, \Gamma'' \vdash r}{\Gamma, \Gamma', \Gamma'', y : p \Rightarrow q \vdash r}\ (L \Rightarrow)$$

– ***Cut***

$$\frac{\Gamma \vdash p \qquad \Gamma', x : p, \Gamma'' \vdash q}{\Gamma, \Gamma', \Gamma'' \vdash q}\,(cut)$$

*A **preproof** is a (finite) tree where each node is a sequent and each edge is a valid deduction rule. Let $\mathscr{S}_{pre}$ be the set of all preproofs.*

Proofs will then be defined to be appropriate equivalence classes of preproofs. For the sake of clarity the appropriate equivalence relation is broken up into five smaller ones. The first of these, *commuting equivalence*, relates two proofs which have insignificant rearranging of deduction rules, for instance, we identify a preproof which at some point applies the *exchange* rule immediately after applying the *weakening* rule, and the preproof which differs from this one only by applying this instance of this pair of rules in the opposite order.

Detour equivalence is inspired by *detour elimination*, which is usually observed when considering the *natural deduction* presentation of intuitionistic logic. The point is that since labels of hypotheses have been used in Definition 1.1 which are *eliminated* by the (ctr), $(R \Rightarrow)$, and $(L \Rightarrow)$ rules and are introduced by the (weak) and $(L \Rightarrow)$ rules, it is possible for a preproof to introduce a superfluous variables and then eliminate it, we call this a *detour* and indentify any preproof with a detour with the preproof which is identical but without this detour. Note: The (cut) rule also eliminates a variable, but detours involving this rule will be treated by *cut equivalence* (see below).

There are preproofs which differ only by insignificant relabelling of hypotheses, for example

$$\frac{x : p \vdash p}{\vdash p \Rightarrow p}\,(R \Rightarrow) \qquad \text{and} \qquad \frac{y : p \vdash p}{\vdash p \Rightarrow p}\,(R \Rightarrow)$$

These preproofs are identified by α-equivalence. The name is due to the fact that it aligns with α-equivalence of λ-terms in a way made precise in Section which section??

Something about cut elimination motivates *cut equivalence*.

η-equivalence seems hard to motivate from the perspective of proofs. For λ-terms you can just say that we're imposing equiality of λ-terms which are "functionally equal" (ie, if $tM = sM$ for all M then $t = s$) but imposing that here on the level of proofs seems really contrived. Ideally I would like to remove η-equivalence from the picture (as I did from my head a long time ago) but then we don't have a category. This seems to be a real downside of insisting on category theory for the sake of this equivalence. η-equivalence.

Definition 1.1.2. *An equivalence relation $\sim$ on $\mathscr{S}_{pre}$ is **compatible** if two equivalent proofs remain equivalent after applying some structural or logical rule to both. For example,*

$$\begin{array}{c}\pi_1\\ \vdots\\ Q \vdash p\end{array} \sim \begin{array}{c}\pi_2\\ \vdots\\ Q \vdash p\end{array} \Rightarrow \begin{array}{c}\pi_1\\ \vdots\\ \dfrac{Q \vdash p}{Q, x : q \vdash p}\,(weak)\end{array} \sim \begin{array}{c}\pi_2\\ \vdots\\ \dfrac{Q \vdash p}{Q, x : q \vdash p}\,(weak)\end{array}$$

Notice first that it does not make sense to commute with the *axiom* law as this law must always be placed at a leaf of a preproof.

Definition 1.1.3. ***Commuting Equivalence*** *$\sim_c$ is the smallest, compatible equivalence relation on preproofs satisfying:*

- $$\dfrac{\dfrac{\Gamma, \Gamma', \Gamma'' \vdash q}{\Gamma, x : p, \Gamma', \Gamma'' \vdash q}\,(weak)}{\Gamma, x : p, \Gamma', y : r, \Gamma'' \vdash q}\,(weak) \sim_c \dfrac{\dfrac{\dfrac{\Gamma, \Gamma', \Gamma'' \vdash q}{\Gamma, y : r, \Gamma' \vdash q}\,(weak)}{\Gamma, y : r, \Gamma', x : p, \Gamma'' \vdash q}\,(weak)}{\Gamma, x : p, \Gamma', y : r, \Gamma'' \vdash q}\,(ex)$$

- $$\dfrac{\dfrac{\Gamma, x : p, x' : p \vdash q}{\Gamma, x : p \vdash q}\,(ctr)}{\Gamma, x : p, y : r \vdash q}\,(weak) \sim_c \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x : p, x' : p \vdash q}{\Gamma, x : p, x' : p, y : r \vdash q}\,(weak)}{\Gamma, y : r, x : p, x' : p \vdash q}\,(ex)}{\Gamma, y : r, x : p \vdash q}\,(ctr)}{\Gamma, x : p, y : r \vdash q}\,(ex)$$

- $$\dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma' \vdash r}{\Gamma, y : q, x : p, \Gamma' \vdash r}\,(ex)}{\Gamma, y : q, x : p, \Gamma', z : s \vdash r}\,(weak) \sim_c \dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma' \vdash r}{\Gamma, x : p, y : q, \Gamma', z : s \vdash r}\,(weak)}{\Gamma, y : q, x : p, \Gamma', z : s \vdash r}\,(ex)$$

- $$\dfrac{\dfrac{\Gamma \vdash p \qquad \Gamma, x : p \vdash q}{\Gamma \vdash q}\,(cut)}{\Gamma, y : r \vdash q}\,(weak) \sim_c \dfrac{\dfrac{\Gamma \vdash p}{\Gamma, y : r}\,(weak) \qquad \dfrac{\dfrac{\Gamma, x : p \vdash q}{\Gamma, x : p, y : r \vdash q}\,(weak)}{\Gamma, y : r, x : p \vdash q}\,(ex)}{\Gamma, y : r \vdash q}\,(cut)$$

- $$\dfrac{\dfrac{\Gamma, x : p, x' : p, x'' : p \vdash q}{\Gamma, x : p, x' : p \vdash q}\,(ctr)}{(\Gamma, x : p \vdash q)}\,(ctr) \sim_c \dfrac{\dfrac{\dfrac{\Gamma, x : p, x', p, x'' : p \vdash q}{\Gamma, x, p, x'' : p, x' : p \vdash q}\,(ex)}{\Gamma, x : p, x'' : p \vdash q}\,(ctr)}{\Gamma, x : p \vdash q}\,(ctr)$$

- $$\dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma', z : r, z' : r \vdash s}{\Gamma, y : q, x : p, \Gamma', z : r, z' : r \vdash s}\,(ex)}{\Gamma, y : q, x : p, \Gamma', z : r \vdash s}\,(ctr) \sim_c \dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma', z : r, z' : r \vdash s}{\Gamma, x : p, y : q, \Gamma', z : r \vdash s}\,(ctr)}{\Gamma, y : q, x : p, \Gamma', z : r \vdash s}\,(ex)$$

- $$\dfrac{\dfrac{\Gamma, x : p, x' : p \vdash q \qquad \Gamma, x : p, x' : p, y : q \vdash r}{\Gamma, x : p, x' : p \vdash r}\,(cut)}{\Gamma, x : p \vdash r}\,(ctr)$$

$$\sim_c$$

$$\dfrac{\dfrac{\Gamma, x : p, x' : p \vdash q}{\Gamma, x : p \vdash q}\,(ctr) \qquad \dfrac{\dfrac{\dfrac{\Gamma, x : p, x' : p, y : q \vdash r}{\Gamma, y : q, x : p, x' : p \vdash r}\,(ex)}{\Gamma, y : q, x : p \vdash r}\,(ctr)}{\Gamma, x : p, y : q \vdash r}\,(ex)}{\Gamma, x : p \vdash r}\,(cut)$$

- $$\dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma', z : r, w : s\Gamma'' \vdash l}{\Gamma, y : q, x : p, \Gamma', z : r, w : s, \Gamma'' \vdash l}\,(ex)}{\Gamma, y : q, x : p, \Gamma', w : s, z : r, \Gamma'' \vdash l}\,(ex) \sim_c \dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma', z : r, w : s, \Gamma'' \vdash l}{\Gamma, x : p, y : q, \Gamma', w : s, z : r, \Gamma'' \vdash l}\,(ex)}{\Gamma, y : q, x : p, \Gamma', w : s, z : r, \Gamma'' \vdash l}\,(ex)$$

- $$\dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma' \vdash r \qquad \Gamma, x : p, y : q, \Gamma', z : r \vdash s}{\Gamma, x : p, y : q, \Gamma' \vdash s}\,(cut)}{\Gamma, y : q, x : p, \Gamma' \vdash s}\,(ex)$$

$$\sim_c$$

$$\dfrac{\dfrac{\Gamma, x : p, y : q, \Gamma' \vdash r}{\Gamma, y : q, x : p, \Gamma' \vdash r}\,(ex) \qquad \dfrac{\Gamma, x : p, y : q, \Gamma', z : r \vdash s}{\Gamma, y : q, x : p, \Gamma', z : r \vdash s}\,(ex)}{\Gamma, y : q, x : p, \Gamma' \vdash s}\,(cut)$$

- $(R \Rightarrow) \times (ctr) \sim_c (ctr) \times (R \Rightarrow)$
- $(R \Rightarrow) \times (weak) \sim_c (weak) \times (R \Rightarrow)$
- $(R \Rightarrow) \times (ex) \sim_c (ex) \times (R \Rightarrow)$

- $(R \Rightarrow) \times (L \Rightarrow) \sim_c (L \Rightarrow) \times (R \Rightarrow)$
- $(L \Rightarrow) \times (ctr) \sim_c (ctr) \times (L \Rightarrow)$
- $(L \Rightarrow) \times (weak) \sim_c (weak) \times (L \Rightarrow)$
- $(L \Rightarrow) \times (ex) \sim_c (ex) \times (L \Rightarrow)$
- $(L \Rightarrow) \times (cut) \sim_c (cut) \times (L \Rightarrow)$

Total = 17, I should justify this number.

Definition 1.1.4. ***Detour Equivalence*** $\sim_d$ *is the smallest, compatible equivalence relation on preproofs satisfying:*

$$\dfrac{\dfrac{\Gamma, x : p \vdash q}{\Gamma, x : p, x' : p \vdash q}\,(weak)}{\Gamma, x : p \vdash q}\,(ctr) \quad \sim_d \quad \Gamma, x : p \vdash q$$

Remark 1.1.1. *We do no impose detour equivalence on any other combination as doing so will conflate non-equal λ-terms and so in this sense are "computationally significant differences between preproofs".*

Definition 1.1.5. ***α-equivalence*** $\sim_\alpha$ *is the smallest, compatible equivalence relation on preproofs such that*

$$\dfrac{\Gamma, x : p \vdash q}{\Gamma \vdash p \Rightarrow q}\,(R \Rightarrow) \sim_\alpha \dfrac{\Gamma, y : p \vdash q}{\Gamma \vdash p \Rightarrow q}\,(R \Rightarrow)$$

for any pair of variables with the same type $(x : p, y : p)$.

Definition 1.1.6. ***Single step cut reduction*** $\rightarrow_{cut}$ *is the smallest, compatible relation (not necessarily an equivalence relation) on preproofs which increases the height of an occurrence of a (cut) rule (where it is presumed that an occurrence of a (cut) rule with infinite height does not appear in the tree). Since the (cut) rule branches to two subpreproofs, and there are 7 rules, there are in total 49 cases to consider. These are methodically considered now:*

- *for any deduction rule (r) $M[y := x] \rightarrow M$ (changes the preterm) (I think that we want $x = y$ here)*

  $$\dfrac{\dfrac{}{x : p \vdash p}\,(ax) \quad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma, y : p \vdash q}\,(r)}{\Gamma, x : p \vdash q}\,(cut) \quad \rightarrow_{cut} \quad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma, y : p \vdash q}\,(r)$$

  and $x[x := M] \rightarrow M$

  $$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash p}\,(r) \quad \dfrac{}{x : p \vdash p}\,(ax)}{\Gamma \vdash p}\,(cut) \quad \rightarrow_{cut} \quad \dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma \vdash q}\,(r)$$

  This covers 13 cases (7 by the first rule, and a further 6 in the second as the case when $(r) = (ax)$ in the second is already considered in the first), there are thus a remaining 36 to consider.

- *Let (r_0) be a structual rule and (r) any rule of inference, then $N[y := M] \rightarrow N[y := M]$ if $r_0 = (ctr)$ and $N[y := M[x' := x]] \rightarrow (N[y := M])[x' := x]$ if $r_0 = (ctr)$ so need $x' \notin N$?*

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p}}{\Gamma' \vdash p}\,(r_0) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma'', y : p \vdash s}\,(r)}{\Gamma', \Gamma'' \vdash s}\,(cut) \quad \rightarrow_{cut} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma'', y : p \vdash s}\,(r)}{\Gamma, \Gamma'' \vdash s}\,(cut)}{\Gamma'', \Gamma \vdash s}\,(ex)}{\Gamma'', \Gamma' \vdash s}\,(r_0)}{\Gamma', \Gamma'' \vdash s}\,(ex)$$

which covers a further 18 cases, leaving a remaining 18. 9 of these correspond to a logical rule on the left and a structural rule on the right, and the remaining 9 correspond to a logical rule on the left and a logical rule on the right.

- *for any logical rule* (r_1)*:* $((N[y' := y])[y := M] \rightarrow (N[y' := M])[y := M])$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p}\,(r_1) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y : p, y' : p \vdash s\end{array}}{\Gamma', y : p \vdash s}\,(ctr)}{\Gamma, \Gamma' \vdash s}\,(cut) \quad \rightarrow_{cut} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p}\,(r_1) \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p}\,(r_1) \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y : p, y' : p \vdash s\end{array}}{\Gamma, \Gamma', y : p \vdash s}\,(cut)}{\Gamma, \Gamma' \vdash s}\,(cut)$$

For the next one, we need that $x : p \notin \Gamma'$ $(N[x := M] \rightarrow N)$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash p}\,(r_1) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma' \vdash q\end{array}}{\Gamma', x : p \vdash q}\,(weak)}{\Gamma, \Gamma' \vdash q}\,(cut) \quad \rightarrow_{cut} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma' \vdash q}\,(r_1)$$

- *the remaining 9 cases correspond to having a logical rule on both the left and right hand side of the* (cut) *rule.*
  - $(R \Rightarrow)$ *on the left and* $(R \Rightarrow)$ *on the right:* $((\lambda z.N)[y := \lambda x.M] \rightarrow \lambda z.(N[y := \lambda x.M]))$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x : p \vdash q\end{array}}{\Gamma \vdash p \Rightarrow q}\,(R \Rightarrow) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y : p \Rightarrow q, z : s \vdash l\end{array}}{\Gamma', y : p \Rightarrow q \vdash s \Rightarrow l}\,(R \Rightarrow)}{\Gamma, \Gamma' \vdash s \Rightarrow l}\,(cut)$$

$$\rightarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x : p \vdash q\end{array}}{\Gamma \vdash p \Rightarrow q}\,(R \Rightarrow) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y : p \Rightarrow q, z : s \vdash l\end{array}}{\Gamma, z : s, y : p \Rightarrow q \vdash l}\,(ex)}{\Gamma, \Gamma', z : s \vdash l}\,(cut)}{\Gamma, \Gamma' \vdash s \Rightarrow l}\,(R \Rightarrow)$$

  - $(R \Rightarrow)$ *on the left and* $(L \Rightarrow)$ *on the right:* $((O[x' := (\lambda x.M)N]) = (O[x' := yN])[y := (\lambda x.M)] \rightarrow O[x' := M[x := N]]$ *so this one actually changes the preterm (mod* α*-equivalence) (provided no free occurrence of a variable in* N *becomes bound in* $M[x := N]$*))*

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x:p \vdash q\end{array}}{\Gamma \vdash p \Rightarrow q}\,(R\Rightarrow) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma' \vdash p\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Gamma'', x':q \vdash s\end{array}}{\Gamma', \Gamma'', y: p\Rightarrow q \vdash s}\,(L\Rightarrow)}{\Gamma, \Gamma', \Gamma'' \vdash s}\,(cut)$$

$$\rightarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma' \vdash p\end{array} \qquad \begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x:p \vdash q\end{array}}{\Gamma', \Gamma \vdash q}\,(cut) \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Gamma''.x':q \vdash s\end{array}}{\Gamma', \Gamma, \Gamma'' \vdash s}\,(cut)}{\Gamma, \Gamma', \Gamma'' \vdash s}\,(ex)$$

- $(R\Rightarrow)$ *on the left and* (cut) *on the right (I believe here we need the assumption that $y : p \Rightarrow q \notin \Gamma''$): $((O[z := N])[y := \lambda x.M] \to O[z := N[y := \lambda x.M]]$*

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x:p \vdash q\end{array}}{\Gamma \vdash p \Rightarrow q}\,(R\Rightarrow) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y:p\Rightarrow q \vdash l\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Gamma'', z:l \vdash s\end{array}}{\Gamma', \Gamma'', y: p\Rightarrow q \vdash s}\,(cut)}{\Gamma, \Gamma' \vdash s}\,(cut)$$

$$\rightarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, x:p \vdash q\end{array}}{\Gamma \vdash p \Rightarrow q}\,(R\Rightarrow) \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma', y:p\Rightarrow q \vdash l\end{array}}{\Gamma, \Gamma' \vdash l}\,(cut) \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Gamma'', z:l \vdash s\end{array}}{\Gamma, \Gamma', \Gamma'' \vdash s}\,(cut)$$

- $(L\Rightarrow)$ *on the left and any logical rule* (r_1) *on the right (I believe here we need the assumption that $x : p \notin \Gamma''$): $O[z := (N[x := yM])] \to (O[z := N])[x := yM]$*

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash p\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma', x:q \vdash s\end{array}}{\Gamma, \Gamma', y:p\Rightarrow q \vdash s}\,(L\Rightarrow) \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma'', z:s \vdash l}\,(r_1)}{\Gamma, \Gamma', y:p\Rightarrow q, \Gamma'' \vdash l}\,(cut)$$

$$\rightarrow_{cut}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash p\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma', x:q \vdash s\end{array} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Gamma'', z:s \vdash l}\,(r_1)}{\Gamma', x:q, \Gamma'' \vdash l}\,(cut)}{\Gamma', \Gamma'', x:q \vdash l}\,(ex)}{\Gamma, \Gamma', \Gamma'', y:p\Rightarrow q \vdash l}\,(L\Rightarrow)}{\Gamma, \Gamma', y:p\Rightarrow q, \Gamma'' \vdash l}\,(ex)$$

- (cut) *on the left and any logical rule* (r_1) *on the right (I believe here we need the assumption that $x : p \notin \Gamma''$): $O[y := N[x := M]] \to (O[y := N])[x := M]$*

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma\vdash p\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdots\\ \Gamma',x:p\vdash q\end{matrix}}{\Gamma,\Gamma'\vdash q}\,(cut) \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma'',y:q\vdash s}\,(r_1)}{\Gamma,\Gamma',\Gamma''\vdash s}\,(cut)$$

$$\to_{cut}$$

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma\vdash p\end{matrix}\qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\\ \Gamma',x:p\vdash q\end{matrix}\qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Gamma'',y:q\vdash s}\,(r_1)}{\Gamma',x:p,\Gamma''\vdash s}\,(cut)}{\Gamma',\Gamma'',x:p\vdash s}\,(ex)}{\Gamma,\Gamma',\Gamma''\vdash s}\,(cut)$$

All cases have now been considered.

Definition 1.1.7. ***cut-equivalence*** $\sim_{cut}$ *is the smallest equivalence relation containing the relation* $\to_{cut}$.

Definition 1.1.8. η***-equivalence*** $\sim_\eta$ *is the smallest, compatible equivalence relation such that*

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma\vdash p\Rightarrow q\end{matrix}\qquad \dfrac{x:p\vdash p\qquad y:q\vdash q}{x:p,z:p\Rightarrow q\vdash q}\,(L\Rightarrow)}{x:p\vdash q}\,(cut)}{p\Rightarrow q}\,(R\Rightarrow)\quad\sim_\eta\quad \begin{matrix}\pi\\ \vdots\\ \Gamma\vdash p\Rightarrow q\end{matrix}$$

Definition 1.1.9. ***Proof equivalence*** $\sim_p$ *is the smallest, compatible equivalence relation containing the union of commuting equivalence (Definition 1.1.3), detour equivalence (Definition 1.1.4), α-equivalence (Definition 1.1.5), cut equivalence (Definition 1.1.7), and η-equivalence (Definition 1.1.8). A **proof** is an equivalence class of preproofs under proof equivalence.*

1.2 The Category of Proofs

Definition 1.2.1. *Let $Q\in\mathcal{P}$ be a finite sequence of variables. The category $\mathscr{S}_Q$ has objects*

$$\mathrm{Ob}(\mathscr{S}_Q)=\Psi_\Rightarrow\cup\{\mathbf{1}\}$$

and morphisms given for propositions $p,q\in\Psi_\Rightarrow$ by

$$\begin{aligned}\mathscr{S}_Q(p,q)&=\Sigma^Q_{p\Rightarrow q}/\sim_p\\ \mathscr{S}_Q(\mathbb{1},q)&=\Sigma^Q_q/\sim_p\\ \mathscr{S}_Q(p,\mathbb{1})&=\{*\}\\ \mathscr{S}_Q(\mathbb{1},\mathbb{1})&=\{*\}\end{aligned}$$

where $*$ is a new symbol. To compose a proof π_1 of $Q\vdash p\Rightarrow q$ with a proof π_2 of $Q\vdash q\Rightarrow r$ we wish to utilise the cut *axiom. Since the* cut *axiom requires the formulas p and q to be on the left hand side of their respective turnstiles a bit of manipulation is required. Say π_1 is a proof of $Q\vdash p\Rightarrow q$, then let $\pi(\pi_1,x:p,y:q,z:p\Rightarrow q)$ be represented by:*

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ Q\vdash p\Rightarrow q\end{matrix}\qquad \dfrac{x:p\vdash p\qquad y:q\vdash q}{x:p,z:p\Rightarrow q\vdash q}\,(L\Rightarrow)}{Q,x:p\vdash q}\,(cut)$$

For $p, q, r \in \Psi_{\Rightarrow}$ the composition rule is the function

$$\mathscr{S}_Q(q,r) \times \mathscr{S}_Q(p,q) \to \mathscr{S}_Q(p,r)$$

$$\left(\dfrac{\overset{\pi_1}{\vdots}}{Q \vdash q \Rightarrow r}, \dfrac{\overset{\pi_2}{\vdots}}{Q \vdash p \Rightarrow q} \right) \mapsto \dfrac{\dfrac{\overset{\pi(\pi_2, x:p, y:q, z:p \Rightarrow q)}{\vdots}}{Q, x : p \vdash q} \qquad \dfrac{\overset{\pi(\pi_1, x':q, y':r, z':q \Rightarrow r)}{\vdots}}{Q, x' : q \vdash r}}{\dfrac{Q, x' : q \vdash r}{Q \vdash q \Rightarrow r}\,(R \Rightarrow)}\,(cut)$$

In the remaining special cases, composition is given by the following functions:

$$\mathscr{S}_Q(p,q) \times \mathscr{S}_Q(\mathbf{1},p) \to \mathscr{S}_Q(\mathbf{1},q)$$

$$\left(\dfrac{\overset{\pi_1}{\vdots}}{Q \vdash p \Rightarrow q}, \dfrac{\overset{\pi_2}{\vdots}}{Q \vdash p} \right) \mapsto \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{Q \vdash p} \qquad \dfrac{\overset{\pi(\pi_1, x:p, y:q, z:p \Rightarrow q)}{\vdots}}{Q, x : p \vdash q}}{Q \vdash q}$$

$$\mathscr{S}_Q(\mathbf{1},q) \times \mathscr{S}_Q(p,\mathbf{1}) \to \mathscr{S}_Q(p,q)$$

$$\left(\dfrac{\overset{\pi_1}{\vdots}}{Q \vdash q}, * \right) \mapsto \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{Q \vdash q}}{Q, x : p \vdash q}}{Q \vdash p \Rightarrow q}$$

and lastly,

$$\mathscr{S}_Q(\mathbf{1},p) \times \mathscr{S}_Q(\mathbf{1},\mathbf{1}) \to \mathscr{S}_Q(\mathbf{1},p)$$

is the map which projects onto the first coordinate. All other cases are trivial. Note that these functions which have been described using a choice of representatives from a proof equivalence class, are nonetheless well-defined haven't actually checked this.

1.3 The Curry-Howard Isomorphism

This Section is the main contribution of this paper, which provides two maps (which we give the same name) one maps $\text{Obj}(\mathscr{L}_Q) \to \text{Obj}(\mathscr{S}_Q)$ and the other has domain given by the collection of preproofs (modulo no equivalence relations) and codomain given by the collection of preterms (modulo no equivalence relations). The later of these maps establishes correspondences between equivalence relations sitting on the two sides of the fence; Sequent Calculi and λ-terms, in a way made precise in Lemmas Lemmas. The main result is Theorem Theorem which for each choice of a finite sequence of variables Q, establishes a categorical equivalence $\mathscr{F}_Q : \mathscr{S}_Q \cong \mathscr{L}_Q$. This last result is similar to what is commonly referred to as the *Curry-Howard Isomorphism* in literature need references.

Similar results have been found in literature references, but these results are unsatisfactory as add vitriol here.

Recall from Definition don't have a reference for the definition of λ-calc in this doc that there is assumed to be a countably infinite sequence of atomic types. Similarly in Definition 1.1.1, it is assumed that there is a countably infinite equence of atomic propositions. In what follows we will assume that the set of atomic types is equal to the set of atomic propositions so that $\Phi_{\Rightarrow} = \Psi_{\Rightarrow}$.

Definition 1.3.1. *Let Q be a finite sequence of variables. We define $\mathscr{F}_Q$ to map a proposition p to the type with the same name, and define $\mathscr{F}_Q(\mathbb{1}) = \mathbb{1}$.*

The definition of $\mathscr{F}_Q$ on preproofs will be described by annotating the axiom, structural, *and* logical *rules of Definition 1.1.1 with λ-terms so that a preproof prescribes instructions for building a corresponding preterm. Consider the following preterm annotations of the* axiom, structural, *and* logical *rules of Definition 1.1.1:*

$$x : p \vdash x : p$$

$$\frac{Q, x : p, x' : p \vdash M : q}{Q, x : p \vdash M[x' := x] : q}\,(ctr) \qquad \frac{Q \vdash M : q}{Q, x : p \vdash M : q}\,(weak) \qquad \frac{Q, x : p, y : q \vdash M : r}{Q, y : q, x : p \vdash M : r}\,(ex)$$

$$\frac{Q, x : p \vdash M : q}{Q \vdash \lambda x.M : p \Rightarrow q}\,(R \Rightarrow) \qquad \frac{Q \vdash N : p \qquad Q, x : q \vdash M : r}{Q, y : p \Rightarrow q \vdash M[x := yN] : r}\,(L \Rightarrow)$$

$$\frac{Q \vdash N : p \qquad Q, x : p \vdash M : q}{Q \vdash M[x := N]q}\,(cut)$$

For a proof π of a formula p, the term $\mathscr{F}_Q(p)$ is constructed by taking a preproof representative of π', then annotating this preproof with λ-terms on the right hand side of the turnstile according to the above rules, and then $\mathscr{F}_Q(p)$ is defined to be the $\alpha\beta\eta$ equivalence class represented by the preterm which appears on the right hand side of the turnstile of the root node in the preproof π'.

For example, let π be the preproof

$$\frac{w : p \Rightarrow q \vdash p \Rightarrow q \qquad \dfrac{x : p \vdash p \qquad y : q \vdash q}{x : p, z : p \Rightarrow q \vdash q}\,(L \Rightarrow)}{w, x : p \vdash q}\,(\text{cut})$$

which is annotated to become

$$\frac{w : p \Rightarrow q \vdash w : p \Rightarrow q \qquad \dfrac{x : p \vdash x : p \qquad y : q \vdash y : q}{x : p, z : p \Rightarrow q \vdash zx : q}\,(L \Rightarrow)}{w, x : p \vdash wx : q}\,(\text{cut})$$

then $\mathscr{F}_Q(\pi) = wx : p$.

The next result establishes a correspondence between preproofs modulo commuting equivalence and detour equivalence and preterms (modulo no equivalences): need to check notation

Lemma 1.3.1. *For every choice of finite sequence of variables Q, the map $\mathscr{F}_Q/{\sim_{cd}} : \Sigma^Q/{\sim_{cd}} \to \Lambda'$ is a well defined bijection.*

Proof. Recall that commuting equivalence is the smallest *compatible* equivalence relation on preterms containing a collection of pairs of the form $\pi_1 \sim_c \pi_2$ (see Definition 1.1.3). To show well definedness, by compatibility it suffices to show that $\mathscr{F}_Q(\pi_1) = \mathscr{F}_Q(\pi_2)$ for each of these pairs. These are simple checks □

1.4 Notes to the author

Remark 1.4.1. *It is not the case that $FV(\mathscr{F}_Q(\pi)) = Q$ but instead $FV(\mathscr{F}_Q(\pi)) \subseteq Q$. For example, say $Q = (x : p)$, and let π be*

$$\frac{\dfrac{x : p \vdash x : p}{\vdash \lambda x.x : p \Rightarrow p}}{x : p \vdash \lambda x.x : p \Rightarrow p}$$

is such that $FV(\mathscr{F}_Q(\pi)) = \varnothing \subsetneq Q$.