Linear Logic

Will Troiani

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Contents

1 Multiplicatives

Definition 1.0.1. There is a countably infinite set of atomic formulas $\mathscr{F}_0 = \{p, q, r \ldots\}$ and a set of **preformulas** $\operatorname{Pre}\Psi_{\otimes,\parr}$ (or simply $\operatorname{Pre}\Psi$) which is the smallest set subject to:

1. all atomic formulas are formulas, that is, if $p \in \mathscr{F}_0$ then $p \in \Psi$,
2. if A and B are formulas then so are $A \otimes B$, and $A \parr B$,
3. if A is a formula then $\sim A \in \Psi$,

Let $\mathcal{P}^n$ be the set of all length n sequences of variables with $\mathcal{P}^0 := \{\varnothing\}$, and $\mathcal{P} := \cup_{n=0}^{\infty}\mathcal{P}^n$. A *sequent* is a pair (Γ, A) where $\Gamma \in \mathcal{P}$ and $A \in \Psi$, written $\Gamma \vdash A$. We call Γ the *antecedent* and A the *succedent* of the sequent. Given Γ and a formula A we write Γ, A for the element of $\mathcal{P}$ given by appending A to the end of Γ. We write $\vdash A$ for $\varnothing \vdash A$.

Definition 1.0.2. We define the following map $\gamma : \operatorname{Pre}\Psi \longrightarrow \operatorname{Pre}\Psi$:

$$\sim (A) \longmapsto \sim A \qquad\qquad \sim(\sim A) \longmapsto A$$
$$\sim (A \otimes B) \longmapsto \sim A \parr \sim B \qquad\qquad \sim (A \parr B) \longmapsto \sim A \otimes \sim B$$

Lemma 1.0.3. *For any formula A, there exists $n > 0$ and a formula B such that for all $m \geq n$ we have $\gamma^m(A) = B$.*

Proof. Follows from induction on the number of pairs of brackets in A. $\square$

Definition 1.0.4. In the notation of Lemma 1.0.3, B is the **negation normal form** corresponding to A, we write $\operatorname{NF}(A) = B$.

There is a map $\reflectbox{$\gamma$}$: $\operatorname{Pre}\Psi \longrightarrow \operatorname{Pre}\Psi$ similar to γ but with the arrows of Definition 1.0.2 reversed. This leads to the **negation abnormal form** of a formula A which we denote by $\operatorname{NAB}(A)$.

Definition 1.0.5. We let $\cong$ be the smallest equivalence relation on the set of preformulas $\mathrm{Pre}\,\Psi$ such that $A \cong \gamma(A)$. The set Ψ of **formulas** is the set of equivalence classes of preformulas under $\cong$.

Definition 1.0.6. A **multiplicative, linear logic deduction rule** (or simply **deduction rule**) results from one of the schemata below by a substitution of the following kind: replace p, q, r by arbitrary formulas, x, y, by arbitrary variables, and Γ, Δ, Θ by arbitrary (possibly empty) sequences of formulas separated by commas:

- the **identity group**:
  - **Axiom**

$$\frac{}{\vdash A, \sim A}\ (\mathrm{ax})$$

  - **Cut**:

$$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, \sim A, \Delta'}{\vdash \Gamma, \Gamma', \Delta, \Delta'}\ (\mathrm{cut})$$

- the **multiplicative rules**
  - **Times**:

$$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, B, \Delta'}{\vdash \Gamma, \Gamma', A \otimes B, \Delta, \Delta'}\ \otimes$$

  - **Parr**

$$\frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \parr B, \Gamma'}\ \parr$$

- the **structural rule**:
  - **Exchange**

$$\frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, B, A, \Gamma'}\ (\mathrm{ex})$$

Definition 1.0.7. A **proof in MLL** is a finite rooted planar tree where each edge is labelled by a sequent and each node except for the root is labelled by a valid deduction rule. If the edge connected to the root is labelled by the sequent $\vdash A, \Gamma$ then we call the proof a **proof of** $\vdash A, \Gamma$.

There is a lot of redundancy in Definition 1.0.7 (for instance, $\vdash$) so we introduce another way of writing proofs:

Definition 1.0.8. A **multiplicative proof structure** (or simply **proof-structure**) consists of:

- a finite set of **occurrences of formulas**, ie, tuples (A, i) consisting of a formula A and an integer i,
- a finite set consisting of **links** which may be:
  - **axiom links**: tuples $(\mathrm{ax}, A, i, \sim A, j)$ where $(A, i), (\sim A, j)$ are occurrences of formulas,
  - **cut links**: tuples $(\mathrm{cut}, A, i, \sim A, j)$ where $(A, i), (\sim A, j)$ are occurrences of formulas,
  - **tensor links**: tuples $(A, i, B, j, A \otimes B, k)$ or, $(B, i, A, j, A \otimes B, k)$ where $(A, i), (B, j), (A \otimes B, k)$ are occurrences of formulas,
  - **par links**: $(A, i, B, j, A \parr B, k)$ or $(B, i, A, j, A \parr B, k)$ where $(A, i), (B, j), (A \parr B, k)$ are occurrences of formulas.

subject to:

1. every occurrence of a formula is distinct. Ie, for two occurrences of formulas $(A, i), (B, j)$ in a proof-structure π, we have either $A \neq B$ or $i \neq j$,

2. every occurrence of a formula is a conclusion of one and only one link,

3. every occurrence of a formula is the premise of at most one link.

For tensor and par links, the second last element of the tuple is the **conclusion** of the link, the first two occurrences of formulas in any link are the **premises**. If the conclusion of a tensor link is $A \otimes B$, then A is the **left premise**, and B the **right premise**, similarly for par links.

A proof-structure without any cut links is **cut-free**.

Remark 1.0.9. We have introduced axiom 1 explicitly here whereas in other sources ([2]) it has been left implicit.

We draw proof-structures diagrammatically, there is an obvious map from the set of proofs to the set of proof-structures, this map is non-surjective. The proof structure given by the image of a proof π is the **translation** of π. The image of this translation is the set of **proof-nets**, there is a characterisation of proof-nets involving *long trips* and *switchings*. Loosely speaking, a *trip* of a proof structure π is a sequence of occurrences of formulas in π representing a path on the underlying graph. A *switching* will be instructions which define this path.

Definition 1.0.10. The set of **intuitionistic formulas** $I\Psi$ is defined in the same way as Ψ in Definition 1.0.1 but we omit 3. An **intuitionistic, multiplicative deduction rule** (or simply **deduction rule**) results from one of the schemata below by a substitution of the following kind: replace A, B, C by arbitrary formulas, x, y by arbitrary intuitionistic formulas, and Γ, Δ, Θ by arbitrary (possibly empty) sequences of intuitionistic formulas separated by commas:

- The **identity group**:

Axiom
$$\frac{}{A \vdash A}\,(\mathrm{ax}) \tag{1}$$

Cut
$$\frac{\Gamma \vdash A \qquad \Delta, A, \Theta \vdash B}{\Gamma, \Delta, \Theta \vdash B}\,(\mathrm{cut}) \tag{2}$$

- The **logical rules**:

Left/right times
$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, A \otimes B, \Gamma' \vdash C}\,(\mathrm{L}\otimes) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\mathrm{R}\otimes) \tag{3}$$

Right/left implication
$$\frac{\Gamma, A, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \multimap B}\,(\mathrm{R} \multimap) \qquad \frac{\Gamma \vdash A \qquad \Delta, B, \Delta' \vdash C}{A \multimap B, \Gamma, \Delta \vdash C}\,(\mathrm{L} \multimap) \tag{4}$$

- The **structural rule**

Exchange $$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,(\mathrm{ex}) \tag{5}$$

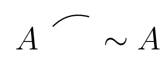
A **proof in IMLL** is defined the same way as in Definition 1.0.7.

Definition 1.0.11. Let Σ denote the set of multiplicative, linear logic proofs and MPS the set of multiplicative proof-structures. We let

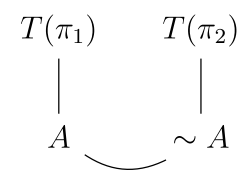
$$T : \Sigma \longrightarrow \mathrm{MPS} \tag{6}$$

denote the function defined inductively by associating to each deduction rule of Definition 1.0.6 a multiplicative proof-structure:

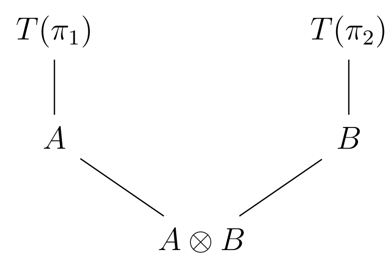
Axiom $$\frac{}{\vdash A, \sim A}\,(\mathrm{ax}) \quad \xrightarrow{T}$$



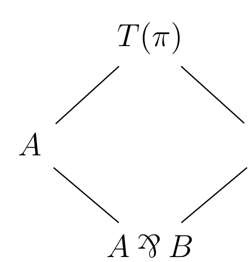
Cut $$\frac{\dfrac{\pi_1 \atop \vdots}{\vdash \Gamma, A, \Gamma'} \qquad \dfrac{\pi_2 \atop \vdots}{\vdash \Delta, \sim A, \Delta'}}{\vdash \Gamma, \Gamma', \Delta, \Delta'}\,(\mathrm{cut}) \quad \xrightarrow{T}$$



Times $$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, B, \Delta'}{\vdash \Gamma, \Gamma', A \otimes B, \Delta, \Delta'}\,\otimes \quad \xrightarrow{T}$$



Par $$\frac{\dfrac{\pi \atop \vdots}{\vdash \Gamma, A, B, \Gamma'}}{\vdash \Gamma, A \parr B, \Gamma'}\,\parr \quad \xrightarrow{T}$$



Exchange $$\frac{\dfrac{\pi \atop \vdots}{\vdash \Gamma, A, B, \Gamma'}}{\vdash \Gamma, B, A, \Gamma'}\,(\mathrm{ex}) \quad \xrightarrow{T} \quad T(\pi)$$

A **multiplicative proof-net** (or simply **proof-net**) is a multiplicative proof-structure which lies in the image of T.

Definition 1.0.12. Let Π denote the set of intuitionistic, multiplicative, linear logic proofs. Then again, there is a translation

$$S : \Pi \longrightarrow \mathrm{MPS} \tag{7}$$

defined inductively:

Axiom

$$\frac{}{A \vdash A}\,(\mathrm{ax}) \quad \xrightarrow{T} \quad \begin{tikzcd} A \arrow[r, dash, bend left] & \sim A \end{tikzcd}$$

Cut

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta, A, \Theta \vdash B}}{\Gamma, \Delta, \Theta \vdash B}\,(\mathrm{cut}) \quad \xrightarrow{T} \quad \begin{tikzcd} T(\pi_1) \arrow[d, dash] & T(\pi_2) \arrow[d, dash] \\ A \arrow[r, dash, bend right] & \sim A \end{tikzcd}$$

Left times

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Gamma, A, B, \Gamma' \vdash C}}{\Gamma, A \otimes B, \Gamma' \vdash C}\,(\mathrm{L}\otimes) \quad \xrightarrow{T} \quad \begin{tikzcd} & T(\pi) \arrow[dl, dash] \arrow[dr, dash] & \\ \sim A \arrow[dr, dash] & & \sim B \arrow[dl, dash] \\ & \sim A \parr \sim B & \end{tikzcd}$$

Right times

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\,(\mathrm{R}\otimes) \quad \xrightarrow{T} \quad \begin{tikzcd} T(\pi_1) \arrow[d, dash] & & T(\pi_2) \arrow[d, dash] \\ A \arrow[dr, dash] & & B \arrow[dl, dash] \\ & A \otimes B & \end{tikzcd}$$

Right implication

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Gamma, A, \Gamma' \vdash B}}{\Gamma, \Gamma' \vdash A \multimap B}\,(\mathrm{R}\multimap) \quad \xrightarrow{T} \quad \begin{tikzcd} & T(\pi) \arrow[dl, dash] \arrow[dr, dash] & \\ \sim A \arrow[dr, dash] & & B \arrow[dl, dash] \\ & \sim A \parr B & \end{tikzcd}$$

Left implication

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta, B, \Delta' \vdash C}}{A \multimap B, \Gamma, \Delta \vdash C}\,(\mathrm{L}\multimap) \quad \xrightarrow{T} \quad \begin{tikzcd} T(\pi_1) \arrow[d, dash] & & T(\pi_2) \arrow[d, dash] \\ A \arrow[dr, dash] & & \sim B \arrow[dl, dash] \\ & A \otimes \sim B & \end{tikzcd}$$

Exchange

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\vdash \Gamma, A, B, \Gamma'}}{\vdash \Gamma, B, A, \Gamma'}\,(\mathrm{ex}) \quad \xrightarrow{T} \quad T(\pi)$$

A **intuitionistic, multiplicative proof-net** (or simply **intuitionistic proof-net**) is an proof-structure which lies in the image of S.

Definition 1.0.13. There is also a map $R : \Pi \longrightarrow \Sigma$ which simply moves formulas to the right of the turnstile.

It is easy to see that the following diagram:

$$\begin{tikzcd} \Pi \arrow[r, "R"] \arrow[dr, "S"'] & \Sigma \arrow[d, "T"] \\ & \mathrm{MPN} \end{tikzcd} \tag{8}$$

commutes.

Lemma 1.0.14. *The map R is injective.*

Proof. There is a map $\operatorname{im} R \longrightarrow \Pi$ which puts all formulas of a proof $\pi \in \operatorname{im} R$ into negation abnormal form which will leave every formula A of every sequent in the form $\sim B$ for some B except for one (as $\pi \in \operatorname{im} R$). We move all formulas except this special one per sequent to the left of the turnstile. $\square$

Lemma 1.0.15. *The map R is not surjective.*

Proof. Define the following function

$$f : \Pi \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{9}$$

which, for a proof $\pi \in \Pi$, computes the following element of $\mathbb{Z}_2 \times \mathbb{Z}_2$: beginning at $(0,0)$, add $(0,1)$ for every occurrence of $(\mathrm{R} \multimap)$ and add $(0,0)$ for every instance of $(\mathrm{L} \otimes)$. Define also the function

$$g : \Sigma \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{10}$$

which, for a proof $\pi \in \Sigma$, computes the following element of $\mathbb{Z}_2 \times \mathbb{Z}_2$: beginning at $(0,0)$, add $(0,1)$ for every $\parr$ rule in π involving a formula A such that $\mathrm{NF}(A) =\sim A'$ for some A' on the left and a formula B on the right such that $\mathrm{NF}(B) \neq\sim B'$ for any B'. Add $(1,0)$ for every $\parr$ rule involving a formula A on the left satisfying $\mathrm{NF}(A) \neq\sim A'$ for any A' and a formula B on the right satisfying $\mathrm{NF}(B) =\sim B'$. Similarly for $(0,0)$ and $(1,1)$. Then the following diagram commutes:

$$\begin{tikzcd} \Pi \arrow[r, "R"] \arrow[dr, "f"'] & \Sigma \arrow[d, "g"] \\ & \mathbb{Z}_2 \times \mathbb{Z}_2 \end{tikzcd} \tag{11}$$

Moreover, g is surjective, the four elements $(0,0), (1,0), (0,1), (1,1)$ are respectively mapped to by

$$\dfrac{\dfrac{\dfrac{}{\vdash A, \sim A}\,(\mathrm{ax}) \qquad \dfrac{}{\vdash A, \sim A}\,(\mathrm{ax})}{\vdash A \otimes A, \sim A, \sim A}\,\otimes}{\vdash A \otimes A, \sim A \parr \sim A}\,\parr \qquad\qquad \dfrac{\dfrac{}{\vdash A, \sim A}\,(\mathrm{ax})}{\vdash A \parr \sim A}\,\parr$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A, \sim A}\,(\mathrm{ax}) \qquad \dfrac{}{\vdash A \sim A}\,(\mathrm{ax})}{\vdash A, A, \sim A \otimes \sim A}\,\otimes}{\vdash A \parr A, \sim A \otimes \sim A}\,\parr \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash A, \sim A}\,(\mathrm{ax})}{\vdash \sim A, A}\,(\mathrm{ex})}{\vdash (\sim A) \parr A}\,\parr$$

and f is clearly not surjective as there are no proofs which map to $(1,0)$ nor $(1,1)$. Thus R is not surjective. $\square$

Remark 1.0.16. As proof-structures suppress exchanges, neither of the maps T, S are injective. By definition of proof-nets, the map T is surjective. The image of S is the set of **multiplicative, intuitionistic proof-nets**. These will not be considered again in these notes, but it would be interesting to find a correctness criterion for intuitionistic proof-nets similar to the *long trip condition* (Section 2).

Remark 1.0.17. The proof of Lemma 1.0.15 would be more interesting if Σ/g was in bijection with Π.

In Section 2 we will see what the image of the map T is.

2 The Sequentialisation Theorem

Definition 2.0.1. Let π be a proof-structure and denote the set of tensor and par links of π by $\mathrm{Link}_{\otimes,\parr}\,\pi$ (or simply $\mathrm{Link}\,\pi$). A **switching** of π is a function

$$S : \mathrm{Link}(\pi) \longrightarrow \{L, R\} \tag{12}$$

A **switching** of a particular link τ is a choice of L, R associated to τ.

Definition 2.0.2. Let π be a proof-structure. Let $\mathcal{O}(\pi)$ denote the set of occurrences of formulas in π. We consider two disjoint copies of this set

$$\mathcal{U}(\pi) := \mathcal{O}(\pi) \coprod \mathcal{O}(\pi) \tag{13}$$

where elements from the first copy are the **up elements**, and elements from the second copy are the **down elements**. We write $\uparrow A$ for the up element corresponding to an occurrence of a formula A in π, and similarly for $A \downarrow$. Given a switching S of π, a **pretrip of π with respect to S** is a finite sequence $(x_1, ..., x_n)$ of elements of $\mathcal{U}(\pi)$ such that:

1. the sequence is a loop, that is, $x_1 = x_n$, and all elements (except the first and the last) are distinct,
2. if $x_j = \uparrow A$ and A is part of an axiom link then $x_{j+1} = \sim A \downarrow$,
3. if $x_j = A \downarrow$ and A is part of a cut link then $x_{j+1} = \uparrow \sim A$,
4. for any tensor link τ with premises A, B such that τ has switching L, we have:
   - if $x_j = A \downarrow$ then $x_{j+1} = (A \otimes B) \downarrow$,
   - if $x_j = \uparrow (A \otimes B)$ then $x_{j+1} = \uparrow B_j$,
   - if $x_j = B \downarrow$ then $x_{j+1} = \uparrow A$.

   and if τ has switching R, we have:
   - if $x_j = A \downarrow$ then $x_{j+1} = \uparrow B$,
   - if $x_j = \uparrow (A \otimes B)$ then $x_{j+1} = \uparrow A$,
   - if $x_j = B \downarrow$ then $x_{j+1} = (A \otimes B) \downarrow$.

   (see Figure 1)
5. for any par link τ with premises A, B such that τ has switching L, we have:
   - if $x_j = \uparrow (A \parr B)$ then $x_{j+1} = \uparrow A$,
   - if $x_j = A \downarrow$ then $x_{j+1} = (A \parr B) \downarrow$,
   - if $x_j = B \downarrow$ then $x_{j+1} = \uparrow B$.

   and if τ evaluates under S to R, we have:

- if $x_j = A\downarrow$ then $x_{j+1} = \uparrow A$,
- if $x_j = \uparrow(A \parr B)$ then $x_{j+1} = \uparrow B$,
- if $x_j = B\downarrow$ then $x_{j+1} = (A \parr B)\downarrow$.

(see Figure 2)

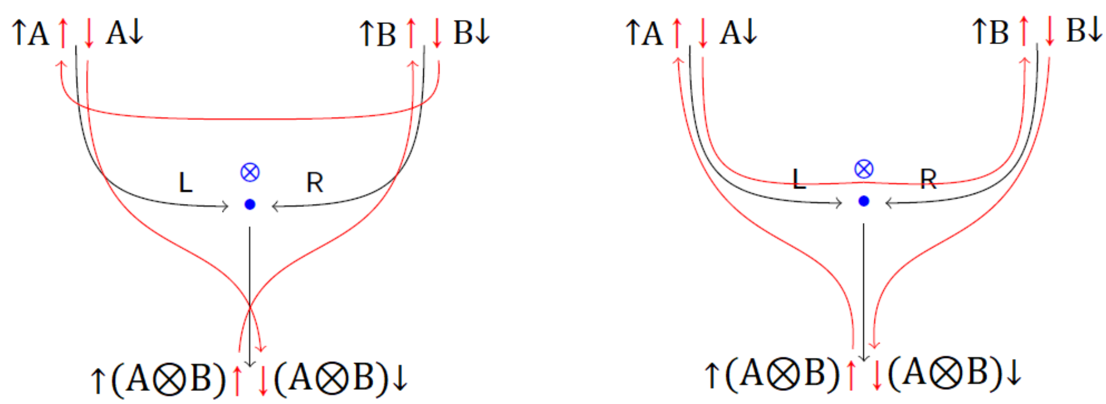


Figure 1: Tensor link, L switching, R switching

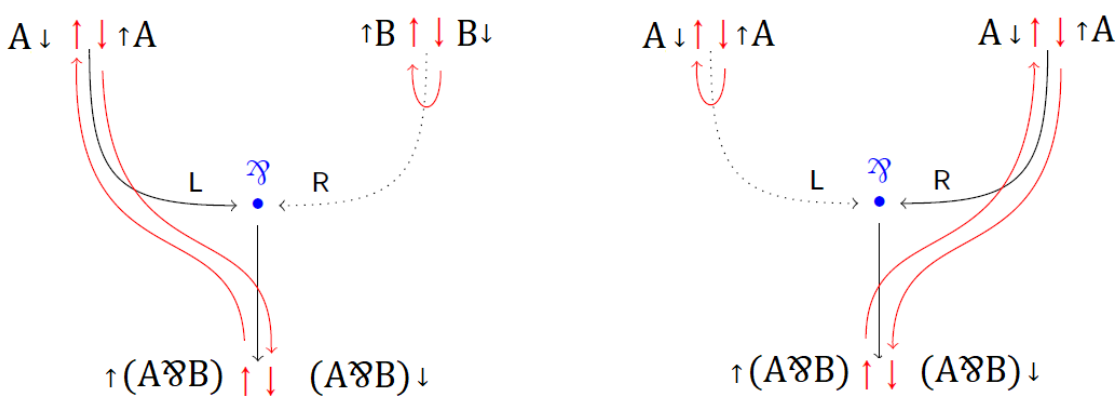


Figure 2: Par link, L switching, R switching.

Definition 2.0.3. Let $\mathrm{Pre}\,\mathcal{T}(\pi, S)$ denote the set of all pretrips of π with respect to S. We define an equivalence relation on this set $\sim$ where two pretrips $(x_1, ..., x_n)$ and $(y_1, ..., y_m)$ are equivalent if $n = m$, and there exists an integer k such that $x_{i+k} = y_i$ (where $i + k$ means $\bmod n$) for all $i = 1, ..., n$.

A **trip** of π with respect to S is an equivalence class of pretrips. We denote the set of all trips by $\mathcal{T}(\pi, S)$. If the set $\mathcal{T}(\pi, S)$ admits more than one element, these elements are called **short trips**, and if it admits only one element, this element is the **long trip**. We refer to the statement "for all switchings S, the set $\mathcal{T}(\pi, S)$ contains exactly one element" as the **long trip condition**.

A **short pretrip** is a choice of representative for a pretrip, and a **long pretrip** is a choice of representatitive of a long trip.

Given a proof-structure π satisfying the long trip condition and a tensor link $\tau := (A, i, B, j, A \otimes B, k)$ of π, let S be a switching of π and $t := (x_1, ..., x_n)$ be the long pretrip of π satisfying $x_1 = A\downarrow$. Since π satisfies the long trip condition, it must be the case that $\uparrow(A \otimes B)$ and $B\downarrow$ occur somewhere in t, can we determine which occurs earlier? Let $m, l > 0$ be such that $x_m = \uparrow(A \otimes B), x_l = B\downarrow$ and assume $l < m$. Say $S(\tau) = L$, then t has the shape

$$(A\downarrow, (A \otimes B)\downarrow, ..., B\downarrow, \uparrow A, ..., \uparrow(A \otimes B), \uparrow B, ..., A\downarrow) \tag{14}$$

Now consider the switching given by

$$\hat{S}(\sigma) = \begin{cases} S(\sigma), & \sigma \neq \tau \\ R, & \sigma = \tau \end{cases}$$

Then (14) becomes:

$$(A \downarrow, \uparrow B, ..., A \downarrow) \tag{15}$$

which is a short pretrip, contradicting the assumption that π satisfies the long trip condition. Thus $m < l$. We have proven (the first half) of:

Lemma 2.0.4. *Let π be a proof-structure satisfying the long trip condition, $\tau := (A, i, B, j, A \otimes B, k)$ be a tensor link of π, S be a switching of π and $(x_1, ..., x_n)$ the long pretrip satisfying $x_1 = A \downarrow$. If $m, l > 0$ are such that $x_m = \uparrow (A \otimes B), x_l = B \downarrow$, then*

- *if $S(\tau) = L$ then $m < l$,*
- *if $S(\tau) = R$ then $l < m$*

The proof of the other half is similar to what has already been written, however since Lemma 2.0.4 contradicts [2, Lemma 2.9.1] we write out the details here:

Proof. Say $m < l$, then t has the shape

$$(A \downarrow, \uparrow B, ..., \uparrow (A \otimes B), \uparrow A, ..., B \downarrow, (A \otimes B) \downarrow, ..., A \downarrow) \tag{16}$$

Now consider the switching given by

$$S'(\sigma) = \begin{cases} S(\sigma), & \sigma \neq \tau \\ L, & \sigma = \tau \end{cases}$$

Then (16) becomes:

$$(A \downarrow, (A \otimes B) \downarrow, ..., A \downarrow) \tag{17}$$

which is a short pretrip. □

Lemma 2.0.5. *Let π be a proof-structure satisfying the long trip condition, $\tau := (A, i, B, i, A \parr B, k)$ be a par link of π, S be a switching of π and $(x_1, ..., x_n)$ be the long pretrip satisfying $x_1 = A \downarrow$. If $m, l > 0$ are such that $x_m = \uparrow (A \parr B), x_l = B \downarrow$, then*

- *if $S(\tau) = L$ then $m < l$,*
- *if $S(\tau) = R$ then $l < m$*

Proof. Exercise. □

Remark 2.0.6. A long pretrip starting at position 1 of Figure 3 necessarily moves to $1'$, granted the switching of the displayed tensor link is L. The long pretrip will necessarily return to this link, and moreover it will do so for the first time after leaving $1'$ either at position 2 or 3 (position 1 will lead to a short trip). Lemma 2.0.4 states that in fact position 2 will be taken next, then position 3 at a later point. A similar story rings true if the switching is R. This gives a nice interpretation of Lemma 2.0.4 that long trips *return to where they left* at each tensor link.

The situation is a bit different for par links; we visit the premises before returning to the conclusion, see Figure 4.

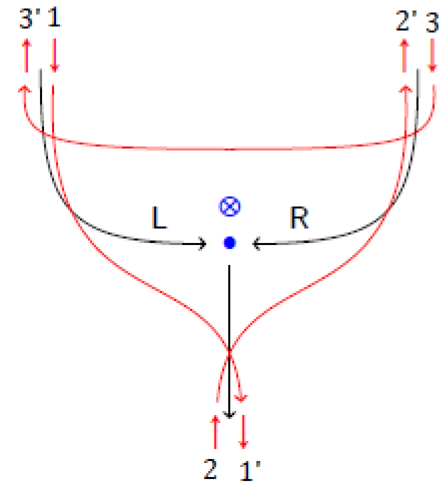


Figure 3: Left switching

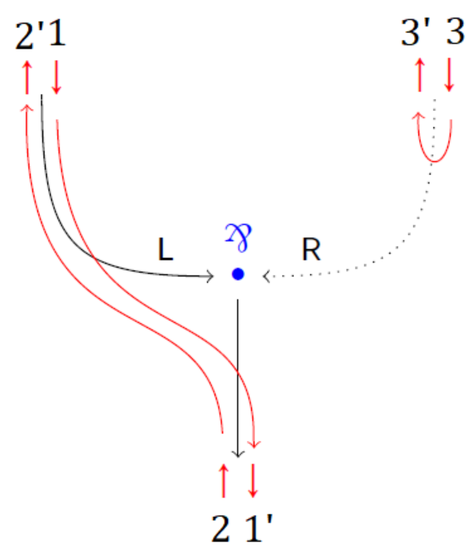


Figure 4: Left switching

Loosely speaking, for a proof-structure π satisfying the long trip condition to "split" at a tensor link $\tau := (A, i, B, j, A \otimes B, k)$ into two distinct proof-structures π_1, π_2 each satisfying the long trip condition, it would necessarily be the case that any pretrip σ of π starting at $\uparrow A$ visits the entirety of $\mathcal{U}(\pi_1)$ before returning to the tensor link τ, lest π_1 admit a short trip. Moreover, it must be the case that σ admits no occurrence of formulas in π_2 lest the result of removing the tensor link τ not result in disjoint proof-structures. Thus, if such a link τ exists, it is *maximal* in the sense that there is no other tensor link $\tau' := (A', i', B', j', A' \otimes B', k')$ where a pretrip starting at A' contains the entirety of any pretrip starting at A. The remainder of this Section will amount to proving the converse, that any such maximal tensor link "splits" π.

Definition 2.0.7. Let π be a proof-structure satisfying the long trip condition, S a switching of π, and A an occurrence of a formula in π. Consider the long pretrip $(x_1, ..., x_n)$ satisfying $x_1 = \uparrow A$. We denote by

$$\mathrm{PTrip}(\pi, S, A, \uparrow) \tag{18}$$

the subsequence $(x_1, ..., x_m)$ of $(x_1, ..., x_n)$ satisfying $x_m = \uparrow A$. We define

$$\mathrm{PTrip}(\pi, S, A, \downarrow) \tag{19}$$

similarly.

Also, for $a \in \{\uparrow, \downarrow\}$ we define the following set

$$\mathrm{Visit}_S(A, a) := \{C \in \mathcal{O}(\pi) \mid \uparrow C, C\downarrow \text{ occur in } \mathrm{PTrip}(\pi, S, A, a)\} \tag{20}$$

The **up empire of** A is the following set:

$$\mathrm{Emp}_\uparrow A := \{C \in \mathcal{O}(\pi) \mid \text{For all switchings } S \text{ we have } \uparrow C, C\downarrow \text{ occur in } \mathrm{PTrip}(\pi, S, A, \uparrow)\} \tag{21}$$

The **down empire of** A is defined symmetrically.

Remark 2.0.8. Notice that by Lemmas 2.0.4, 2.0.5, for any formula A which is a premise to either a tensor or par link, we have:

$$\uparrow C \text{ occurs in } \mathrm{Trip}(\pi, S, A, \uparrow) \qquad \text{if and only if} \qquad C\downarrow \text{ occurs in } \mathrm{PTrip}(\pi, S, A, \uparrow)$$

and similarly for $\mathrm{PTrip}(\pi, S, A, \downarrow)$.

With this new terminology we now have some corollaries of Lemmas 2.0.4 and 2.0.5:

Corollary 2.0.9. *Let π be a proof-structure satisfying the long trip condition, we have:*

1. *for any axiom link with conclusions $A, \sim A$:*

$$\mathrm{Emp}_\uparrow A = \mathrm{Emp}_\downarrow(\sim A) \cup \{A\} \tag{22}$$

2. *for any cut link with premises $A, \sim A$:*

$$\mathrm{Emp}_\downarrow A = \mathrm{Emp}_\uparrow(\sim A) \cup \{A\} \tag{23}$$

3. *for any tensor link with premises A, B:*

$$\mathrm{Emp}_\uparrow A \cap \mathrm{Emp}_\uparrow B = \varnothing \tag{24}$$

4. *for any tensor or par link with premises A, B and conclusion C:*

$$\mathrm{Emp}_\uparrow C = \mathrm{Emp}_\uparrow A \cup \mathrm{Emp}_\uparrow B \cup \{C\} \tag{25}$$

5. *for any tensor link with premises A, B:*

$$\mathrm{Emp}_\downarrow B = \mathrm{Emp}_\uparrow A \cup \mathrm{Emp}_\downarrow(A \otimes B) \cup \{B\} \tag{26}$$

   and similarly,

$$\mathrm{Emp}_\downarrow A = \mathrm{Emp}_\uparrow B \cup \mathrm{Emp}_\downarrow(A \otimes B) \cup \{A\} \tag{27}$$

Remark 2.0.10. Recall Definition 1.0.8 that there are two types of tensor links $(A, i, B, j, A \otimes B, k)$, $(B, j, A, i, A \otimes B, k)$ and two types of par links $(A, i, B, j, A \parr B, k), (B, j, A, i, A \parr B, k)$. Lemmas 2.0.4 and 2.0.5 were only stated for links of the form $(A, i, B, j, A \otimes B, k), (A, i, B, j, A \parr B, k)$ however they hold for *all* tensor and par links. One merely replaces all instances of $A \otimes B$ with $B \otimes A$, and all instances of $A \parr B$ with $B \parr A$ in the proofs.

Definition 2.0.11. Given any link τ we write $B \in \tau$ if B occurs as either a premise or a conclusion of τ.

Let π be a proof-structure satisfying the long trip condition, and $a \in \{\uparrow, \downarrow\}$. The set of **links of** A **with respect to** S is the set

$$\mathrm{Link}_a A := \{\tau \in \mathrm{Link}\, \pi \mid \forall B \in \tau, B \in \mathrm{Emp}_a A\} \tag{28}$$

Definition 2.0.12. Let π be a proof-structure satisfying the long trip condition and let $a \in \{\uparrow,\downarrow\}$. Define the set

$$\mathrm{Link}^0_{\parr,a} A := \{\tau \in \mathrm{Link}_a A \mid \text{Exactly one premise of } \tau \text{ is in } \mathrm{Emp}_a A\} \tag{29}$$

Lemma 2.0.13. *Let π be a proof-structure satisfying the long trip condition, let $a \in \{\uparrow,\downarrow\}$ and A an occurrence of a formula in π. Define the following function:*

$$\begin{aligned} S : \mathrm{Link}^0_{\parr,a} A &\longrightarrow \{L, R\} \\ \tau &\longmapsto \begin{cases} L, & \textit{if the right premise of } \tau \textit{ is in } \mathrm{Emp}_a A \\ R, & \textit{if the left premise of } \tau \textit{ is in } \mathrm{Emp}_a A \end{cases} \end{aligned}$$

and extend this to a switching $\hat{S} : \mathrm{Link}\,\pi \longrightarrow \{L, R\}$ arbitrarily. Then

$$\mathrm{Emp}_a A = \mathrm{Visit}_{\hat{S}}(A, a) \tag{30}$$

Proof. We proceed by induction on the size $|\mathrm{Link}_a(A)|$ of the set $\mathrm{Link}_a(A)$. For the base case, assume $|\mathrm{Link}_a(A)| = 0$. The formula A is connected to a (possibly empty) chain of axiom and/or cut links which do not form a loop of the underlying graph of π. The result follows from Corollary 2.0.9, (1),(2).

Now assume that $|\mathrm{Link}_a A| = n > 0$ and the result holds for any formula B such that $|\mathrm{Link}_a B| < n$. First say $a =\uparrow$, and A is a conclusion of either a tensor or a par link

$$\begin{tikzcd}[column sep=small] A_1 \arrow[dr, dash] & & \sim A_2 \arrow[dl, dash] \\ & A & \end{tikzcd}$$

where $A = A_1 \otimes A_2$ or $A = A_1 \parr A_2$. By (4) we have

$$\begin{aligned} \mathrm{Emp}_\uparrow A &= \mathrm{Emp}_\uparrow A_1 \cup \mathrm{Emp}_\uparrow A_2 \cup \{A\} \\ &= \mathrm{Visit}_S(A_1, \uparrow) \cup \mathrm{Visit}_S(A_2, \uparrow) \cup \{A\} \\ &= \mathrm{Visit}_S(A, \uparrow) \end{aligned}$$

where the second equality follows from the inductive hypothesis.

Assume A is part of an axiom link, we prove the case when $a =\uparrow$. By (1)

$$\mathrm{Emp}_\uparrow A = \mathrm{Emp}_\downarrow(\sim A) \cup \{A\} \tag{31}$$

with

$$|\mathrm{Link}_\uparrow A| = |\mathrm{Link}_\downarrow(\sim A)| \tag{32}$$

Since $|\mathrm{Link}_\downarrow(\sim A)| > 0$ we necessarily have that $\sim A$ is not a conclusion. Thus, A is connected via a chain of axiom and cut links to an occurrence $\sim A$ which is a premise to either a tensor link or a par link. In the case of the former, we have:

$$\begin{tikzcd}[column sep=small] C \arrow[dr, dash] & & \sim A \arrow[dl, dash] \\ & C \otimes \sim A & \end{tikzcd}$$

then by (5):

$$\begin{aligned} \mathrm{Emp}_\downarrow(\sim A) &= \mathrm{Emp}_\uparrow C \cup \mathrm{Emp}_\downarrow(C \otimes \sim A) \cup \{\sim A\} \\ &= \mathrm{Visit}_{\hat{S}}(C, \uparrow) \cup \mathrm{Visit}_{\hat{S}}(C \otimes \sim A, \downarrow) \cup \{\sim A\} \\ &= \mathrm{Visit}_{\hat{S}}(\sim A, \downarrow) \end{aligned}$$

where the second equality follows from the inductive hypothesis.

If $\sim A$ is a premise of a par link

$$\begin{tikzcd}[column sep=small] C \arrow[dr, no head] & & \sim A \arrow[dl, no head] \\ & C \parr \sim A & \end{tikzcd}$$

then by construction of $\hat{S}$, where we use the specific definition of S for the first time,

$$\begin{aligned} \mathrm{Emp}_{\downarrow}(\sim A) &= \{\sim A\} \\ &= \mathrm{Visit}_{\hat{S}}(\sim A, \downarrow) \end{aligned}$$

The case when A is part of a cut link is exactly symmetrical to what has been shown already, and so we omit the proof. $\square$

Definition 2.0.14. A tensor or par link is **terminal** if it is a conclusion.

Corollary 2.0.15. *Let π be a proof-structure satisfying the long trip condition. Let*

$$\tau := \begin{tikzcd}[column sep=small] A \arrow[dr, no head] & & B \arrow[dl, no head] \\ & A \otimes B & \end{tikzcd}$$

be a terminal tensor link of π. Then π admits a par link

$$\sigma := \begin{tikzcd}[column sep=small] C \arrow[dr, no head] & & D \arrow[dl, no head] \\ & C \parr D & \end{tikzcd}$$

such that either $C \in \mathrm{Emp}_{\uparrow} A$ and $D \in \mathrm{Emp}_{\uparrow} B$ or $C \in \mathrm{Emp}_{\uparrow} B$ and $D \in \mathrm{Emp}_{\uparrow} A$ if and only if for any switching S of π we have that either

$$\mathrm{Emp}_{\uparrow} A \subsetneq \mathrm{Visit}_S(A, \uparrow) \qquad \textit{or} \qquad \mathrm{Emp}_{\uparrow} B \subsetneq \mathrm{Visit}_S(B, \uparrow)$$

Proof. Say π admitted σ and $C \in \mathrm{Emp}_{\uparrow} A$ and $D \in \mathrm{Emp}_{\uparrow} B$. If the switching S is such that $S(\tau) = L$ then $C \parr D \in \mathrm{Visit}_S(B) \setminus \mathrm{Emp}_{\uparrow} B$ and if $S(\tau) = R$ then $C \parr D \in \mathrm{Visit}_S(A) \setminus \mathrm{Emp}_{\uparrow} A$. The other case is similar.

Conversely, say π admits no such par link σ, that is, assume

$$\mathrm{Link}^0_{\parr,\uparrow}(A) \cap \mathrm{Link}^0_{\parr,\uparrow}(B) = \varnothing \tag{33}$$

Then there is by Lemma 2.0.13 a well defined function

$$S : \mathrm{Link}^0_{\parr,\uparrow}(A) \cup \mathrm{Link}^0_{\parr,\uparrow}(B) \longrightarrow \{L, R\}$$

which extends to a switching $\hat{S}$ such that

$$\mathrm{Emp}_{\uparrow} A = \mathrm{Visit}_{\hat{S}}(A, \uparrow) \qquad \text{and} \qquad \mathrm{Emp}_{\uparrow} B = \mathrm{Visit}_{\hat{S}}(B, \uparrow) \tag{34}$$

$\square$

Lemma 2.0.16 (Separation Lemma)**.** *A proof-structure π satisfying the long trip condition, with only tensor links amongst its conclusions admits a tensor link*

$$\tau := \begin{tikzcd}[column sep=small] A \arrow[dr, no head] & & B \arrow[dl, no head] \\ & A \otimes B & \end{tikzcd}$$

satisfying

$$\mathcal{O}(\pi) = \mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\uparrow} B \cup \{A \otimes B\} \tag{35}$$

Moreover, removing $A \otimes B$ results in a disconnected graph with each component a proof-structure satisfying the long trip condition.

Proof. Consider the set of tensor links $\mathrm{Link}_{\otimes}(\pi)$ of π. We endow this with the following partial order $\leq$: a pair of links:

$$\sigma := \begin{array}{ccc} A & & B \\ & \diagdown \quad \diagup & \\ & A \otimes B & \end{array} \qquad \rho := \begin{array}{ccc} C & & D \\ & \diagdown \quad \diagup & \\ & C \otimes D & \end{array}$$

are such that $\tau \leq \sigma$ if $\mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\uparrow} B \subseteq \mathrm{Emp}_{\uparrow} C \cup \mathrm{Emp}_{\uparrow} D$. Let τ (with conclusion $A \otimes B$ say) be a tensor link maximal with respect to $\leq$. We show that τ satisfies the required property.

Say $\mathcal{O}(\pi) \neq \mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\uparrow} B \cup \{A \otimes B\}$. Then by Lemma 2.0.15 there exists a par link

$$\sigma := \begin{array}{ccc} C & & D \\ & \diagdown \quad \diagup & \\ & C \parr D & \end{array}$$

such that either $C \in \mathrm{Emp}_{\uparrow} A$ and $D \in \mathrm{Emp}_{\uparrow} B$ or $C \in \mathrm{Emp}_{\uparrow} B$ and $D \in \mathrm{Emp}_{\uparrow} A$. We show the proof in the case of the former. Since π admits no terminal par links, this link is above a tensor link

$$\rho := \begin{array}{ccc} E & & F \\ & \diagdown \quad \diagup & \\ & E \otimes F & \end{array}$$

Notice that if $\rho = \tau$, then either $C \parr D \in \mathrm{Emp}_{\uparrow} A$ or $C \parr D \in \mathrm{Emp}_{\uparrow} B$ which in either case implies $\mathrm{Emp}_{\uparrow} A \cap \mathrm{Emp}_{\uparrow} B \neq \varnothing$, contradicting Corollary 2.0.9, 3, and so $\rho \neq \tau$. Without any loss of generality, assume that σ sits above F. Let S be a switching of π so that $\mathrm{Emp}_{\uparrow} F = \mathrm{Visit}_S(F, \uparrow)$ and so that $S(\sigma) = L$, which exists by Lemma 2.0.15. Let $t = (x_1, ..., x_n)$ be the long pretrip of π with respect to S satisfying $x_1 = F\uparrow$. We have by Lemma 2.0.5 that t takes the following shape:

$$\uparrow F, ..., \uparrow (C \parr D), \uparrow C, ..., D \downarrow, \uparrow D, ..., C \downarrow, (C \parr D) \downarrow, ..., F \downarrow, ... \tag{36}$$

We have that $D \in \mathrm{Emp}_{\uparrow} B$ so for simplicity, rewrite (36) as $t' = (x_{1+k}, ..., x_{n+k})$ for some $k > 0$ (where $i + k$ means $i + k \bmod n$) so that t takes the shape

$$..., \uparrow F, ..., \uparrow (C \parr D), \uparrow C, ..., D \downarrow, \uparrow D, ..., C \downarrow, (C \parr D) \downarrow, ..., F \downarrow, ... \tag{37}$$

with $\uparrow B$ occurring to the left of $D \downarrow$ and $B \downarrow$ occurring to the right of $\uparrow D$. We have that $C \notin \mathrm{Emp}_{\uparrow} B$ and so by Remark 2.0.8:

$$\uparrow B \text{ occurrs in } \uparrow C, ..., D \downarrow \text{ and } B \downarrow \text{ occurrs in } \uparrow D, ..., C \downarrow \tag{38}$$

However, this implies that $B \in \mathrm{Visit}_S(F, \uparrow)$ which by Lemma 2.0.13 implies $B \in \mathrm{Emp}_{\uparrow} F$.

By reversing the switching of σ and interchanging the rolls of C, D in the above argument, we also have that $A \in \mathrm{Emp}_{\uparrow} F$, contradicting the maximality of τ. This proves the first claim.

For the second claim, since $\mathcal{O}(\pi) = \mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\uparrow} B \cup \{A \otimes B\}$ we have by Lemma 2.0.15 that

$$\mathrm{Link}^0_{\parr, \uparrow}(A \otimes B) = \varnothing \tag{39}$$

and we saw in the proof of Lemma 2.0.13 that a switching S which realises $\mathrm{Emp}_{\uparrow} A$ is given by setting all switchings arbitrarily except for those in $\mathrm{Link}^0_{⅋,\uparrow}(A \otimes B)$. This means that for any switching S of π:

$$\mathrm{Visit}_S(A, \uparrow) = \mathrm{Emp}_{\uparrow} A \qquad \text{and} \qquad \mathrm{Visit}_S(B, \uparrow) = \mathrm{Emp}_{\uparrow} B \tag{40}$$

which is to say the two subproof-structures given by removing $A \otimes B$ never admit a short trip, that is, they each satisfy the long trip condition. □

Theorem 2.0.17 (The Sequentialisation Theorem)**.** *A proof-structure π satisfies the long trip condition if and only π is a proof-net.*

Proof. We proceed by induction on the number of links of π. If there are none then π consists of a sequence of axiom and cut links which does not form a loop of the underlying graph of π, the result in this case is clear. For the inductive step, we consider two cases, first say π admits a par link for a conclusion. Then removing this par link clearly results in two subproof-structures satsifying the long trip condition and so the result follows from the inductive hypothesis. If no such terminal par link exists, then by the Separation Lemma there exists some tensor link in the conclusion for which we can remove and apply the inductive hypothesis. □

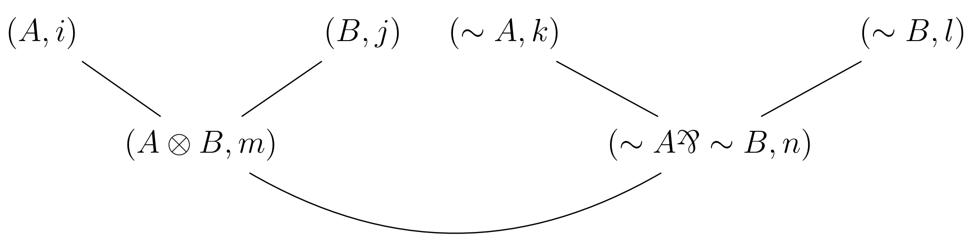
3 Cut

Definition 3.0.1 (Cut-reduction)**.** Cut-reduction $\longrightarrow_{\text{cut}}$ is the smallest, *compatible* equivalence relation on the set of all proof-nets containing the following generators (for clarity, the occurrence labels have been written in explicitly):

1.

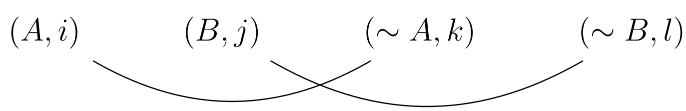
$$(A, i) \quad (\sim A, j) \quad (A, k) \longrightarrow_{\text{cut}} (A, i) \tag{41}$$

2.



(42)

$\longrightarrow_{\text{cut}}$



(43)

The reflexive, symmetric, transitive closure of cut-reduction is **cut-equivalence**, and is denoted $\sim_{\text{cut}}$.

A pair consisting of an axiom link and a cut link in the shape of (41) is a **axiom redex**, and a set consisting of occurrences of formulas, a tensor link, a par link, and a cut link in the shape of (42) is a **tensor-par redex**.

Proposition 3.0.2 (Church-Rosser)**.** *If π_1 is a proof-structure and $\pi_1 \longrightarrow_{\text{cut}} \pi_2, \pi_1 \longrightarrow_{\text{cut}} \pi_3$ then there exists a proof-structure π_4 such that $\pi_2 \longrightarrow_{\text{cut}} \pi_4, \pi_3 \longrightarrow_{\text{cut}} \pi_4$.*

Proof. The key observation is that reducing any redex in a proof does not eliminate any other redex. □

Consider a proof-structure π and S a switching. Let $(x_1, ..., x_n)$ be a representing short pretrip. If $x_i = A \downarrow$ with A a premise to either a tensor or par link, then $x_{i+1} = B \downarrow$ for some formula B. Similarly, if $x_i = \uparrow A$ with A a conclusion of a tensor or par link then $x_{i+1} = \uparrow B$ for some formula B. Thus, tensor and par links preserve the *direction* ($\uparrow$ or $\downarrow$) of the trip. We thus have:

Proposition 3.0.3. *Let π be a proof-structure which admits a short trip for some switching S. Every representative of any short trip $(x_1, ..., x_n)$ contains at least one pair $\uparrow C, C \downarrow$ for some conclusion C of π, and at least one pair $\uparrow D, \sim D \downarrow$ for some atom D part of an axiom link of π.*

Definition 3.0.4. Let π be a proof-net with n axiom links. Assume the occurrences of the axioms of π have been labelled by integers $1, ..., 2n$. For each $1 \leq m \leq 2n$ let $\alpha_\pi(m)$ denote the integer such that the formulas labelled $m, \alpha_\pi(m)$ are connected by an axiom link in π. This defines a permutation (which is a disjoint union of transpositions) which we call the **axiom link permutation associated to** π.

There is another permutation of $\{1, ..., 2n\}$ defined by π. Let S be a switching of π and for each $1 \leq m \leq 2n$ let $\beta_\pi^S(m)$ denote the integer such that the first occurrence of any $\uparrow A_1, ..., \uparrow A_{2n}$ in $\mathrm{PTrip}(\pi, S, A_m, \downarrow)$ (Definition 2.0.7) is $\uparrow A_{\beta_\pi(m)}$.

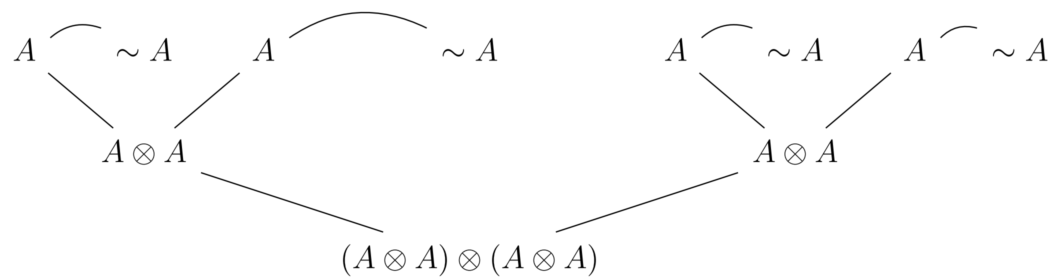
The set of all premutations of the second form is denoted:

$$\Sigma(\pi) := \{\beta_\pi^S \mid S \text{ is a switching of } \pi\} \tag{44}$$

We will often denote elements of $\beta_\pi^S \in \Sigma(\pi)$ simply by β.

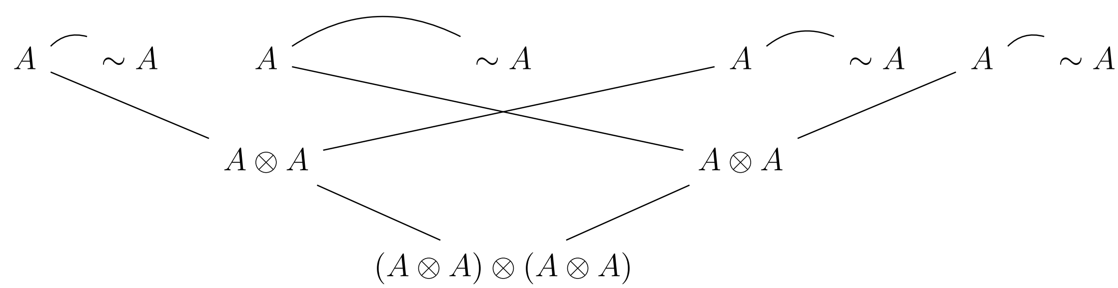
Remark 3.0.5. Strictly speaking, two proof-structures which differ only in the labelling of the occurrences of formulas are considered distinct proof-structures. Thus the following are two distinct proof-structures:

$\pi_1 :=$



(45)

$\pi_2 :=$



(46)

One could consider now proof-nets ζ_1, ζ_2 given by forming par links (in some arbitrary way) connecting the conclusions of π_1, π_2 respectively so that ζ_1, ζ_2 have only one conclusion A. Even if the same par links are taken in the construction of ζ_1, ζ_2 we still (by Proposition 3.0.2) have distinct proof-nets ζ_1, ζ_2. This contradicts a claim made in [3] that "the tree T_A" (here: the underlying trees of ζ_1, ζ_2 given by removing the edges which form axiom links) "is common to all proofs of A". One could consider the underlying tree where the labels of all occurrences of formulas are removed, and then Girard's claim is true, however then his definition of f_S (which for us β_π^S of Definition 3.0.4) is ill-defined.

Definition 3.0.6. Let π be a proof-net possibly containing cut links. A **reduction sequence** is a sequence

$$\pi = \pi_0 \longrightarrow_{\text{cut}} \pi_1 \longrightarrow_{\text{cut}} \ldots \longrightarrow_{\text{cut}} \pi_n \tag{47}$$

with π_n cut-free.

Lemma 3.0.7. *Every proof-net π admits a reduction sequence.*

Proof. Given a cut link $\tau := (A, i, \sim A, j)$ in π, the **complexity of** τ, $c(\tau)$ is the sum of the number of occurrences of $\otimes$ and the number of occurrences of $⅋$ in A. We proceed by induction on the maximum of the complexities of all cut links in π.

Say this maximum is 0. Then all cut-links have the shape of (41) (using the fact that π is a *proof-net*, not merely a proof-structure). We can use (41) finitely many times (in any order) to deduce the result.

Now say the maximum is $n > 0$. We then apply (42) to all cut links of complexity n (in any order) to obtain a new proof-structure ζ. It follows from Lemmas 3, 4 that π satisfying the long trip condition ensures that ζ does, and so we may apply the inductive hypothesis. □

Definition 3.0.8. Let $\mathrm{Red}\,\pi$ denote the set of all reduction sequences of π. The **length** $l(\underline{x})$ of a reduction sequence $\underline{x} \in \mathrm{Red}\,\pi$ is the length of the sequence $\underline{x}$.

Corollary 3.0.9. *The length of a reduction path is independent of the choice of reduction path.*

Proof. The proof is purely geometric. Let

$$\underline{x} := (\pi = \pi_1 \longrightarrow_{\text{cut}} \dots \longrightarrow_{\text{cut}} \pi_n) \tag{48}$$

be the reduction path described by Lemma 3.0.7 and let

$$\underline{y} := (\pi = \zeta_0 \longrightarrow_{\text{cut}} \dots \longrightarrow_{\text{cut}} \zeta_n) \tag{49}$$

be any other reduction sequence. By Lemma 3.0.2 we have $\pi_n = \zeta_n$. Also using 3.0.2, the pair of reduction paths can be completed to some grid defined by a subset of $\mathbb{N} \times \mathbb{N}$. All paths p consisting of only upwards steps or right steps such that p is bound to this grid have the same length and so $l(\underline{x}) = l(\underline{y})$. □

Definition 3.0.10. The proof of Corollary 3.0.9 shows that every reduction path of a proof-net π leads to the same cut-free proof ζ. We call ζ the **normal form** of π.

Corollary 3.0.11. *Multiplicative proof-nets are strongly normalising.*

4 Orthogonality

Proposition 4.0.1. *Let π be a proof-structure, then π is a proof-net if and only if for all $\beta \in \Sigma(\pi)$ the permutation $\alpha_\pi \beta$ is cyclic.*

Proof. By Proposition 3.0.3 we have that every short trip involves a pair $\uparrow D, \sim D \downarrow$ with $D, \sim D$ part of an axiom link. This exists if and only if $\sigma_\pi \tau$ is non-cyclic for some $\tau \in \Sigma(\pi)$. □

Lemma 4.0.2. *Let π be a proof-net with conclusions $A_1, ..., A_n$ and let ζ be a proof-net obtained by beginning with π and in any order forming par links which connect all the conclusions $A_1, ..., A_n$ so that ζ has conclusions $B_1, ..., B_m$ where $m \leq n$ and each B_i is constructed only by $⅋$ and a subset of the formulas $A_1, ..., A_n$. Then $\Sigma(\pi) = \Sigma(\zeta)$.*

Proof. Easy proof by induction on the integer given by the number of par links in ζ minus the number of par links in π. □

Example 4.0.3. In what follows, π_1 is as defined in (45). By consulting Figure 45 we see that

$$\Sigma(\pi_1) = \{(1375), (1357), (1753), (1573)\} \tag{50}$$

Remark 4.0.4. Let π_1, π_2 be as defined in (45),(46). By consulting Figure 6 we see

$$(1537) \in \Sigma(\pi_2) \tag{51}$$

Combining this with Example 4.0.3, we see $\Sigma(\pi_1) \neq \Sigma(\pi_2)$ and by Lemma 4.0.2, if ζ_1, ζ_2 denote the proof-nets given by forming par links which connect the conclusions of π_1, π_2 respectively (in any way) so that ζ_1, ζ_2 have only one conclusion A, then

$$\Sigma(\zeta_1) = \Sigma(\pi_1) \neq \Sigma(\pi_2) = \Sigma(\zeta_2) \tag{52}$$

which contradicts the claim made in [3, §4.1] that these sets are uniquely determined by the conclusion A.

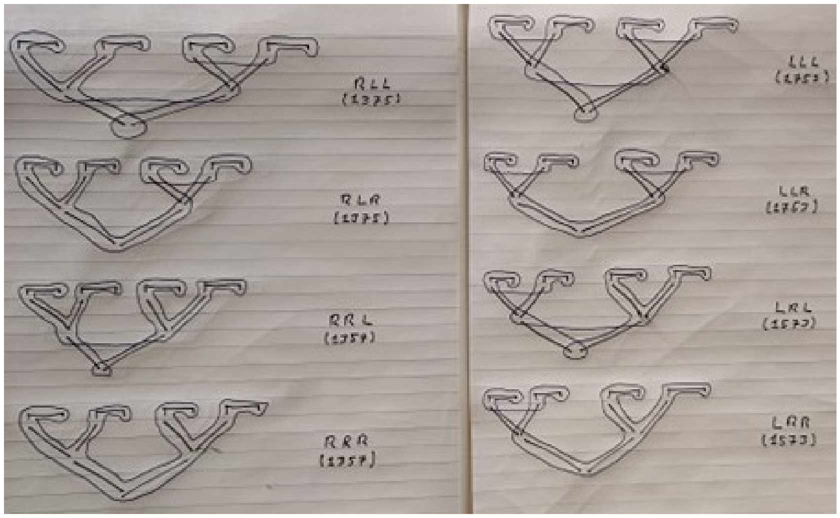


Figure 5: The set $\Sigma(\pi_1)$

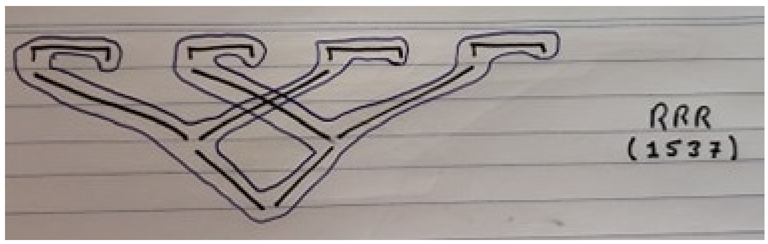


Figure 6: An element of $\Sigma(\pi_2)$

5 Geometry of Interaction

Definition 5.0.1. Let π be a proof-net possibly with cuts and assume the axioms have been labelled $1, ..., 2n$. Perform the cut-reduction step (42) as many times as possible and perform *no* cut-reduction step of the form (41). The result is a proof-net ζ where each axiom link is part of a chain

$$A \frown \sim A \smile A \frown \dots \frown \sim A \smile A \frown \sim A \tag{53}$$

The cut-links of ζ define a permutation (which is a disjoint union of transpositions) γ_π (which is easily shown to be well defined by Lemma 3.0.2) on $\{1, ..., n\}$ where $2n$.

Say the atoms $(A_1, k_1), ..., (A_m, k_m)$ are at the ends of these chains. Recall Definition 3.0.4 the permutation α_π induced by the axiom links of π. For each i let d_i denote the least integer such that

$$(\alpha_\pi \circ \gamma_\pi)^{d_i}(i) \in \{k_1, ..., k_m\} \tag{54}$$

We then define the following permutation:

$$\begin{aligned} \delta_\pi : \{1, ..., m\} &\longrightarrow \{1, ..., m\} \\ i &\longmapsto \text{the integer } j \text{ such that } (\alpha_\pi \circ \gamma_\pi)^{d_i}(k_i) = k_j \end{aligned}$$

Theorem 5.0.2 (Geometry of Interaction)**.** *Let π be a proof-net possibly with cuts and let ζ be the normal form of π (Definition 3.0.10). Then*

$$\delta_\pi = \alpha_\zeta \tag{55}$$

Proof. Let ξ be the proof-net obtained from π given by reducing all tensor-redexes (referred to as ζ in Definition 5.0.1). Then atom (A_i, k_i) is connected to atom $(\sim A, k_{\delta_\pi(i)})$ by definition of α_ζ. $\square$

References

[1] Intuitionistic, Linear Sequent Calculus, W. Troiani.

[2] *Linear Logic*, J.Y. Girard

[3] *Multiplicatives*, J.Y. Girard.

[4] *Geometry of Interaction I*, J.Y. Girard.

[5] *Intuitionistic, linear sequent calculus* W. Troiani.

[6] *Proof-nets*, W Troiani