Finite Simplicial Sets are Algorithms

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1 Introduction

In this thesis we defend the following proposition: simplicial sets are algorithms for constructing topological spaces. Recall that a simplicial set X is a collection of sets $\{X_n\}_{n\geq 0}$ consisting of *n-simplices* X_n for each $n \geq 0$, together with face and degeneracy maps between these sets satisfying certain equations, see [**?**, §I.1.xii], [**?**, §10]. The idea is that simplicial sets are combinatorial models of topological spaces, which can be built up from a set X_0 of vertices, a set X_1 of edges, a set X_2 of triangles, and so on, by gluing these basic spaces together in a particular way.

Here by a vertex we mean the standard *0-simplex* $\Delta^0 = \{0\} \subseteq \mathbb{R}^0$, by an edge we mean the standard *1-simplex* $\Delta^1 = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ and by a triangle we mean the standard *2-simplex* $\Delta^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq x_2 \leq 1\}$, and more generally by a standard *n-simplex* we mean the topological space

$$\Delta^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq ... \leq x_n \leq 1\}$$

The simplicial set $X = \Delta[2]$ (where $\Delta[2]$ is the image of $[2]$ in the simplex category, see Definition 28, under the Yoneda embedding) has three nondegenerate 0-simplices $a, b, c \subseteq X_0$, three nondegenerate 1-simplices $e, f, g \subseteq X_1$ and one nondegenerate 2-simplex $h \subseteq X_2$. This is read as a combinatorial model of a topological space according to the following algorithm: we take three copies $\Delta^0_a, \Delta^0_b, \Delta^0_c$ of the standard 0-simplex, three copies $\Delta^1_e, \Delta^1_f, \Delta^1_g$ of the standard 1-simplex and one copy Δ^2_h of the standard 2-simplex and glue them together according to the aforementioned face and degeneracy maps in the simplicial set. For $\Delta[2]$ this means that we identify 0 in Δ^0_a with 0 in the Δ^1_e and with 1 in Δ^1_f, 0 in Δ_b with 0 in Δ^1_f and with Δ^1_g, 0 in Δ^0_c with 0 in Δ_g and with 1 in Δ^1_a. As well as the subspace $\{(x_1, x_2) \in \Delta^2 \mid x_1 = 0\}$ with Δ^1_e, the subspace $\{(x_1, x_2) \in \Delta^2_f \mid x_2 = 0\}$ with Δ^1_f, and the subspace $\{(x_1, x_2) \in \Delta^2_2 \mid x_1 + x_2 = 1\}$ with Δ^1_g. The result of all this gluing is the topological space Δ^2. In general this process of gluing together standard n-simplices Δ^n according to the combinatorial information in X is called **geometric realisation**. In the case of $X = \Delta[2]$ we can see that some parts of this process are redundant, since the end result is just homeomorphic to one of the starting pieces Δ^2_h, but this does not hold in general.

The question that motivates this thesis is the following: can we give a precise sense in which this construction is algorithmic? That is, can we give a precise sense in which a simplicial set **is** an algorithm for constructing a topological space? To grapple with this question we must first decide what counts as an algorithm. This is no trivial matter, but several satisfactory (and different) Definitions were given by Godel [**?**], Turing [**?**] and Church [**?**]. In each case the notion of algorithm is predicated on certain fundamental operations that are assumed to be allowed in the list of instructions which constitute the algorithm: for example, in Turing's Definition, the semantic interpretation of his machines involves a pre-existing notion of a tape, and the operations of reading from and writing to such a tape. In our context, the basic operations seem to necessarily include (a) making copies of the interval I, (b) any construction involving the standard order $\leq$ on I, the end points $0, 1 \in I$, and (c) any construction permitted by higher order intuitionistic logic.

Realising simplicial sets as algorithms in this sense will be done using the language of classifying topoi [**?**, §VIII] and the link between topoi and higher-order logic given by the Mitchell-Benabou language [**?**], [**?**]. In short, it was proven by Joyal that the category of simplicial sets is the classifying topos of a particular first-order theory, called the theory of **linear orders**, which is roughly speaking a logic in which all we can talk about is an object I which carries a partial order $\leq$ with a bottom element b and a top element t, satisfying some axioms that make $(I, \leq, b, t)$ resemble the unit interval. A topos is a category which admits all finite limits and all finite colimits, as well as all exponentials, and also admits a subobject classifier (see 7), with the natural example being the category of sets or sheaves of sets on a topological space. A useful intuition is a topos is a "generalised universe of sets", see [**?**, §Prologue, p.1]. Then the classifying topos of linear orders is the universal domain of mathematical discourse where it is possible to talk about something like an interval (see 27 for a precise Definition).

The way we can "talk about" this platonic interval is using a type theory called the Mitchell-Benabou language, and this language contains as its primitive elements precisely the **fundamental operations that are necessary in order to realise simplicial sets as algorithms**. We establish this formally by giving a function

$$\text{Objects}(\underline{\text{sSet}}) \to \text{Terms and formulas}$$

which sends a simplicial set X to a corresponding term in this language, where the term contains in explicit form the instructions for constructing a topological space that are implicit in X. For example, the term corresponding to the triangle names several copies of Δ^1, and a copy of Δ^2 (or rather, their logical incarnation I, and the relation $\leq$) and describes how to glue the copies of Δ^1 together along copies of their endpoints (whose logical incarnations are b and t), and also describes how to glue Δ^2 to copies of Δ^1, see Section 6.

Most of the work involved in establishing this picture lies in Joyal's Theorem, but at least from our point of view there is a gap, since the Mitchell-Benabou language does not (in its usual presentation) talk directly about the colimit diagram (gluing) that is necessary for us to view a simplicial set as a single term. So the main contribution of this thesis is to explain how to talk in the Mitchell-Benabou language about colimits, and thus define the map mentioned above, and thus establish the sense in which simplicial sets are algorithms.

2 Topoi

We assume the reader is familiar with category theory, for background see [?] or the categorical preliminaries chapter of [?].

This section covers the topos theory content which will be required for the remainder of this thesis.

Definition 1. *Let $\mathscr{C}$ be a category which admits all products, and let X be an object in $\mathscr{C}$. Then a family of* ***exponentials*** *with* ***exponent*** *X is a collection of objects $\{Z^X\}_{Z\in\mathscr{C}}$ and a collection of morphisms $\{ev_{X,Z} : X \times Z^X \to X\}_{Z\in\mathscr{C}}$ such that for every morphism $f : X \times Y \to Z$, there exists a unique morphism $\ulcorner f \urcorner : Y \to Z^X$ such that the diagram*

$$\begin{tikzcd} X \times Y \arrow[d, "id_Z \times \ulcorner f \urcorner"'] \arrow[dr, "f"] & \\ X \times Z^X \arrow[r, "ev_{X,Z}"'] & Z \end{tikzcd}$$

commutes. The morphism $\ulcorner f \urcorner$ is the ***transpose*** *of f, and f is the* ***inverse transpose*** *of $\ulcorner f \urcorner$.*

Definition 2. *We say a category $\mathscr{C}$* ***admits all exponentials*** *if for every object $X \in \mathscr{C}$, there exists a family of exponentials with exponent X.*

Definition 3. *Let $\mathscr{C}$ be a category which admits a terminal object $\mathbb{1}$. Then a* ***subobject classifier*** *is an object Ω together with a monomorphism $true : \mathbb{1} \rightarrowtail \Omega$, such that for any object $A \in \mathscr{C}$ and any monomorphism $A' \rightarrowtail A$, there exists a unique morphism $\chi_{A'} : A \to \Omega$ such that the following is a pullback diagram,*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail] & \mathbb{1} \arrow[d, tail, "true"] \\ A \arrow[r, "\chi_{A'}"'] & \Omega \end{tikzcd}$$

Definition 4. *Let C be an object of a category $\mathscr{C}$, let $\leq$ be the preorder on the collection of monomorphisms with codomain C, which is such that $f : A \rightarrowtail C \leq g : B \rightarrowtail C$ if there exists a morphism $h : A \to B$ such that the triangle*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, tail, "f"'] & B \arrow[d, tail, "g"] \\ & C \end{tikzcd}$$

commutes. Let $\sim$ be the smallest equivalence relation on this same collection such that $f \sim g$ if and only if $f \leq g$ and $g \leq f$. An equivalence class of monomorphisms is a ***subobject of*** *C, and $Sub(C)$ is the collection of subobjects of C.*

Remark 1. *Any partially ordered set is naturally a category, and in particular $Sub(C)$ can be made into a category, where the objects are given by the subobjects, and given two subobjects $f : A \rightarrowtail C$ and $g : B \rightarrowtail C$, there is a morphism $f \to g$ if and only if $f \leq g$.*

Theorem 1. *Let $\mathscr{C}$ be a category which admits all pullbacks and a subobject classifier Ω. Then there is a natural isomorphism*

$$\eta_X : Sub(X) \cong Hom(X, \Omega)$$

This isomorphism maps a subobject $m : X' \rightarrowtail X$ to the unique morphism $f : X \to \Omega$ such that the following is a pullback diagram,

$$\begin{tikzcd} X' \arrow[r] \arrow[d, tail, "m"'] & \mathbb{1} \arrow[d, tail, "true"] \\ X \arrow[r] & \Omega \end{tikzcd}$$

The inverse of this isomorphism maps a morphism $f : X \to \Omega$ to the monic $m : X' \rightarrowtail X$ such that the following is a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, tail, "m"'] & \mathbb{1} \arrow[d, tail, "true"] \\ X \arrow[r, "f"'] & \Omega \end{tikzcd}$$

Proof. See proposition 1 of [**?**, §I] $\square$

Definition 5. *Given an object A in a category which admits all exponentials and a subobject classifier, let $\in_A \rightarrowtail A \times \Omega^A$ be the subobject such that the following is a pullback diagram,*

$$\begin{tikzcd} \in_A \arrow[r] \arrow[d, tail] & \mathbb{1} \arrow[d, tail, "true"] \\ A \times \Omega^A \arrow[r, "ev_A"'] & \Omega \end{tikzcd}$$

Definition 6. *Let $\mathscr{C}$ be a category which admits all pullbacks. For every morphism $f : A \to B$ of $\mathscr{C}$ there exists a functor $f^{-1} : Sub(B) \to Sub(A)$ which is such that given a subobject $m : B' \rightarrowtail B$, the image under f^{-1} is the subobject $n : A' \rightarrowtail A$ such that the following is a pullback diagram,*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail, "n"'] & B' \arrow[d, tail, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Notice that this is a functor as the partial order $\leq$ is preserved.

Definition 7. *A **topos** (plural: **topoi**) is a category $\mathscr{E}$ which admits*

- *all finite limits and all finite colimits,*
- *all exponentials,*
- *a subobject classifier.*

For a list of examples of topoi, see [**?**, §1.1]. A topos with all colimits is called **cocomplete**.

Many familiar concepts from the topos <u>Sets</u> can be generalised to an arbitrary topos, including the image of a morphism:

Definition 8. *Let $\mathscr{C}$ be a category which admits all pushouts and equalisers. Then the **image** of a subobject $f : A \to B$ is the morphism e such that the following is an equaliser diagram,*

$$\mathrm{Equaliser}(\iota_1, \iota_2) \xrightarrow{e} B \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} \mathrm{Pushout}(f, f)$$

where ι_1, ι_2, and $\mathrm{Pushout}(f, f)$ are such that the following is a pushout diagram,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "\iota_1"] \\ B \arrow[r, "\iota_2"'] & \mathrm{Pushout}(f, f) \end{tikzcd}$$

The domain of e will be notated $\mathrm{Im}(f)$. By the universal properties of $\mathrm{Equaliser}(\iota_1, \iota_2)$ and $\mathrm{Pushout}(f, f)$, there exists a unique morphism $k : A \to \mathrm{Im}(f)$ such that the diagram

$$\begin{tikzcd} A \arrow[d, "k"'] \arrow[dr, "f"] & \\ \mathrm{Im}(f) \arrow[r, "e"'] & B \end{tikzcd}$$

*commutes. Moreover, k an epimorphism (see [**?**, IV 6.1]).*

Later, a type theory will be associated to an arbitrary cocomplete topos. Part of this type theory will be universal and existential quantifiers, $\forall$ and $\exists$, as well as the usual logical connective, $\wedge$, $\vee$, and $\Rightarrow$. We now recall the categorical structures required to interpret these connectives in a topos.

Theorem 2. *Let $\mathscr{E}$ be a topos and $f : A \to B$ a morphism. Then the functor f^{-1} : $Sub(B) \to Sub(A)$ admits both a left and a right adjoint.*

Proof. See Johnstone [**?**, §A 1.4.10]. □

Definition 9. *Let $f : A \to B$ be a morphism in a topos. Then the left adjoint to f^{-1} will be denoted $\exists_f$, and the right adjoint by $\forall_f$.*

The reason why this notation is used, is because in the topos Set, these adjoints are given by the following explicit maps

$$\begin{gathered}\exists_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)\\ A' \mapsto \{b \in B \mid \text{ there exists } a \in A' \text{ such that } f(a) = b\}\end{gathered}$$

and

$$\begin{gathered}\forall_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)\\ A' \mapsto \{b \in B \mid \text{ for all } a \in A \text{ if } f(a) = b \text{ then } a \in A'\}\end{gathered}$$

Proof. See [**?**, §1 9.2]. □

Definition 10. *A* ***Heyting algebra*** *is a set with a partial order, which as a category admits*

- *initial and terminal objects,*
- *binary products, binary coproducts,*
- *all exponentials.*

Many examples are given by the following Theorem.

Theorem 3. *Let $\mathscr{E}$ be a topos, and let E be any object of $\mathscr{E}$. Then the set $Sub(E)$, with partial order given by inclusion, is a Heyting algebra, with*

- *terminal object given by $id_E : E \to E$,*
- *initial object given by the monic $0 \rightarrowtail E$, where 0 is the initial object of the topos $\mathscr{E}$. Note; this morphism $0 \rightarrowtail E$ is necessarily monic as $\mathscr{E}$ is a topos (see [**?**, §A1.4.1]),*

- *the binary product of subobjects $m : A \rightarrowtail E$ and $n : B \rightarrowtail E$ is given by the triple $(A \times_E B, \pi_A, \pi_B)$, which is such that the following is a pullback diagram in $\mathscr{E}$,*

$$\begin{tikzcd} A \times_E B \arrow[r, "\pi_A"] \arrow[d, tail, "\pi_B"'] & A \arrow[d, tail, "m"] \\ B \arrow[r, tail, "n"'] & E \end{tikzcd}$$

- *the binary coproduct of subobjects $m : A \to E$ and $n : B \to E$ given by the triple $(\mathrm{Im}(m \coprod n), k\iota_A, k\iota'_B)$ where $(A \coprod B, \iota_A, \iota_B)$ is a coproduct in $\mathscr{E}$, k is as in Definition 8, and $m \coprod n$ is the unique morphism $A \coprod B \to E$ such that the diagram*

$$\begin{tikzcd} & E & \\ A \arrow[ur, tail, "m"] \arrow[r, tail, "\iota_A"'] & A \coprod B \arrow[u, "m \coprod n"'] & B \arrow[ul, tail, "n"'] \arrow[l, tail, "\iota_B"] \end{tikzcd}$$

commutes in $\mathscr{E}$.

- *the exponentials are more complicated, see [**?**, §A1.4.13].*

Proof. See [**?**, §A1.4], for exponentials specifically see [**?**, §A1.4.13]. □

Definition 11. *We denote the initial and terminal objects of a Heyting algebra by 0 and 1 respectively. The object corresponding to the binary product of objects A and B will be denoted $A \wedge B$, and the object corresponding the coproduct, $A \vee B$. Lastly, the objects B^A will be denoted $A \Rightarrow B$.*

Remark 2. *If $\mathscr{E}$ admits arbitrary coproducts, then for any object $E \in \mathscr{E}$, the category $Sub(E)$ also admits arbitrary coproducts. See [**?**, §A1.4].*

3 Type Theory

Type theories were originally suggested by Russell and Whitehead [**?**] as a foundation for mathematics, for a historical view, see [**?**, p.124]. For an introduction to type theories, see [**?**, §3.3]. The idea of approaching logic from the perspective of category theory is originially due to Lawvere reference. In the modern form, associated to every *topos* $\mathscr{E}$ is a type theory, called the *Mitchell-Benabou language of* $\mathscr{E}$. This type theory can be used to describe constructions of objects and momorphisms in $\mathscr{E}$ (see Section 3.3). First, the Definition of a type theory will be given, then in section 3.1, some helpful Lemmas will be proved.

Definition 12. *A **type theory** consists of*

- *a class of **types**, including special types $\Omega, \mathbb{1}$. Also, for each type τ, there is a countably infinite set of **variables** of type τ,*

- *a class of **function symbols** $f : \tau \to \sigma$, where f is a formal symbol, τ and σ are types,*

- *a class of **relation symbols** $R \subseteq \tau$, where R is a formal symbol, and τ is a type,*

- *a class of **terms**, where to each term t, there is an associated type, τ. $t : \tau$ means "t is of type τ". Also, there is an associated set $FV(t)$ of **free variables**.*

- *a class of **formulas**, where similarly to terms, to each formula there is an associated set of free variables, however unlike terms, there is no associated type,*

- *for every finite sequence $\Delta = (x_1 : \tau_1, ..., x_n : \tau_n)$ of variables, a binary relation, $\vdash_\Delta$, of **entailment** between formulas whose free variables appear in Δ. An expression $p \vdash_\Delta q$, where p and q are formulas, is called a **sequent**.*

Subject to,

1. *if τ and σ are types, then so are $\tau \times \sigma$ and $P\tau$. Identification between types is also allowed.*

   The following axiom describes how the class of terms and formulas along with their associated type (for terms) and free variable set is constructed. First a class of *preterms* will be defined by induction, then appropriate equivalence classes of preterms will constitute the terms, a similar process will be undertaken for formulas.

2. *the class of **preterms** is such that,*

   (a) *there is the class of **atomic preterms**, containing all the variables, as well as the special term $* : \mathbb{1}$. There is also assumed to be **atomic preformulas** which are $\top, \bot$.*

   (b) *the preterms and preformulas are closed under the following **preterm and preformula formation rules**,*

      i. *if $t : \tau$ and $s : \sigma$ are preterms, then $\langle t, s \rangle : \tau \times \sigma$ is a preterm,*
      ii. *if $t : \tau \times \sigma$ is a preterm, then $fst(t) : \tau$ and $snd(t) : \sigma$ are preterms,*
      iii. *if $f : \tau \to \sigma$ is a function symbol, and $t : \tau$ is a preterm, then $ft : \sigma$ is a preterm,*
      iv. *if $R \subseteq \tau$ is a relation symbol, and $t : \tau$ is a term, then $R(t)$ is a preformula,*
      v. *if $t, s : \tau$, then $t = s$ is a preformula,*
      vi. *if p and q are preformulas, then $p \wedge q$, $p \vee q$, and $p \Rightarrow q$, are all preformulas,*

vii. if $\{p_i\}_{i=0}^{\infty}$ *is a countable set of preformulas, then* $\bigvee_{i=0}^{\infty} p_i$ *is a preformula,*

viii. if $x : \tau$ *is a variable, and* p *is a preformula, then* $\forall x : \tau, p$ *and* $\exists x : \tau, p$ *are both preformulas.*

ix. if p *is a preformula, and* $x : \tau$ *is a variable, then* $\{x : \tau \mid p\}$ *is a preterm of type* $P\tau$*,*

x. if $t : \tau$ *and* $T : P\tau$*, then* $t \in T$ *is in a preformula.*

(c) the preterms and preformulas have free variable sets such that

i. $FV(*) = FV(\perp) = FV(\top) = \varnothing$,

ii. $FV(x : \tau) = \{x : \tau\}$*, if* $x : \tau$ *is any variable,*

iii. $FV(\langle t, s\rangle) = FV(t) \cup FV(s)$,

iv. $FV(\mathit{fst}(t)) = FV(\mathit{snd}(t)) = FV(t)$,

v. $FV(ft) = FV(t)$,

vi. $FV(R(t)) = FV(t)$,

vii. $FV(t = s) = FV(t) \cup FV(s)$,

viii. $FV(t \in T) = FV(t) \cup FV(T)$,

ix. $FV(p \wedge q) = FV(p \vee q) = FV(p \Rightarrow q) = FV(p) \cup FV(q)$,

x. $FV(\bigvee_{i=0}^{\infty} p_i) = \bigcup_{i=0}^{\infty} FV(p_i)$

xi. $FV(\{x : \tau \mid p\}) = FV(\forall x : \tau, p) = FV(\exists x : \tau, p) = FV(p) \setminus \{x\}$

An ***occurence*** *of a variable* x *in a term will mean any which is not the "*$x \in \tau$*" part of any term of the form* $\{x : \tau \mid p\}$, $\forall x : \tau, p$*, or* $\exists x : \tau, p$*. Any occurence of a variable which is not part of a term's free variable set is a* ***bound variable****.*

(d) The terms and formulas are α*-equivalence classes of preterms and preformulas respectively, where* α***-equivalence*** *is the smallest relation on the collection of preterms or preformulas respectively, such that*

i. if $t =_\alpha s$*, then* t *and* s *are of the same type,*

ii. if $t =_\alpha t'$ *and* $s =_\alpha s'$*, then* $\langle t, s\rangle =_\alpha \langle t', s'\rangle$*,*

iii. if $t =_\alpha s$*, then* $\mathit{fst}(t) =_\alpha \mathit{fst}(t)$*, and* $\mathit{snd}(t) =_\alpha \mathit{snd}(s)$*,*

iv. if $t =_\alpha t'$*, then* $ft =_\alpha ft'$*,*

v. if $t =_\alpha t'$*, then* $R(t) =_\alpha R(t')$*,*

vi. if $t =_\alpha t'$ *and* $s =_\alpha s'$*, then* $(t = s) =_\alpha (t' = s')$*,*

vii. if $p =_\alpha p'$ *and* $q =_\alpha q'$*, then* $p \wedge q =_\alpha p' \wedge q'$*,* $p \vee q =_\alpha p' \vee q'$*, and* $p \Rightarrow q =_\alpha p' \Rightarrow q'$*,*

viii. if for all i*,* $p_i =_\alpha p_i'$*, then* $\bigvee_{i=0}^{\infty} p_i =_\alpha \bigvee_{i=0}^{\infty} p_i'$*,*

ix. if $p =_\alpha p'$*, then* $\{x : \tau \mid p\} =_\alpha \{x : \tau \mid p'\}$*,* $\forall (x : \tau) p =_\alpha \forall (x : \tau) p'$*, and* $\exists (x : \tau) p =_\alpha \exists (x : \tau) p'$*,*

x. if $t =_\alpha t'$ *and* $T =_\alpha T'$*, then* $t \in T =_\alpha t' \in T'$*,*

xi. $\{x:\tau \mid p\} =_\alpha \{y:\tau \mid p[x:=y]\}$,
$\forall x:\tau, p =_\alpha \forall y:\tau, (p[x:=y])$,
$\exists x:\tau, p =_\alpha \exists y:\tau, (p[x:=y])$, *provided that no free occurrence of x in p is such that y in place of x would be bound. In the above, the notation $p[x:=y]$ means the term p but with every occurrence of x replaced with y.*

(e) *The free variable set of a term $[t]$ is $FV(t)$, where t is any representative of $[t]$, and similarly for formulas. For convenience, equivalence class brackets will be dropped.*

(f) *There is also the following shorthand notation,*

i. $\neg p$ *means* $p \Rightarrow \bot$,

ii. $p \Leftrightarrow q$ *means* $(p \Rightarrow q) \wedge (q \Rightarrow p)$,

iii. $\{x\}$ *means* $\{x':\tau \mid x = x'\}$, *where* $x:\tau$,

iv. $\varnothing_{x:\tau}$ *means* $\{x:\tau \mid \bot\}$

3. *Finally, there are two sets of axioms concerning the entailment relation, these are*

(a) *the* ***structural rules****:*

$$\text{i.} \quad p \vdash_\Delta p$$

$$\text{ii.} \quad \frac{p \vdash_\Delta q \qquad q \vdash_\Delta r}{p \vdash_\Delta r}$$

iii. *For any sequence of variables $(x_1:\tau_1, ..., x_n:\tau_n)$, sequence of terms $(t_1:\tau_1, ..., t_n:\tau_n)$, and context Σ such that each variable which appears in Δ, except for those in $(x_1, ..., x_n)$, also appear in Σ,*

$$\frac{p \vdash_\Delta q}{p[(x_1, ..., x_n) := (t_1, ..., t_n)] \vdash_\Sigma q[(x_1, ..., x_n) := (t_1, ..., t_n)]}$$

where $p[(x_1, ..., x_n) := (t_1, ..., t_n)]$ means the term given by p after simultaneously substituting each x_i for t_i (similarly for q). It is assumed that no free variable in any t_i becomes bound in $p[(x_1, ..., x_n) := (t_1, ..., t_n)]$ nor $q[(x_1, ..., x_n) := (t_1, ..., t_n)]$. This can always be achieved by remaining bound variables. [1]

Note: this axiom also allows introducing superfluous variables to the context Δ, and also for rearrangement of elements.

[1] This final rule is the one to which contexts owe their existence. The essential point is that from $p \vdash_{\Delta, x:X} q$ (where $\Delta, x:\tau$ is the context given by appending $x:\tau$ to the end of Δ), one can infer that $p[x:=t] \vdash_\Delta q[x:=t]$ only if there exists a term $t:X$ such that $\mathrm{FV}(t) \subseteq \Delta$. Lambek and Scott point out [**?**, §II.1 p.131] that to deduce $\forall x:X, p \vdash \exists x:X, p$ from $\forall x:X, p \vdash_{x:X} \exists x:X, p$ without there existing a closed term of type X is undesirable from a logical point of view, as although "for all unicorns x, x has a horn", it is not the case that "there exists a unicorn x, such that x has a horn", because (presumably) there does not exist any unicorns at all!

(b) *the* ***logical rules****, in the following, the notation* $\Delta, x : \tau$ *means the context given by* Δ *with the variable* $x : \tau$ *appended to the end,*

   i. $p \vdash_\Delta \top$,
   ii. $\bot \vdash_\Delta p$,
   iii. $r \vdash_\Delta p \wedge q$ *if and only if* $r \vdash_\Delta p$ *and* $r \vdash_\Delta q$,
   iv. $p \vee q \vdash_\Delta r$ *if and only if* $p \vdash_\Delta r$ *and* $q \vdash_\Delta r$,
   v. $p \vdash_\Delta q \Rightarrow r$ *if and only if* $p \wedge q \vdash_\Delta r$,
   vi. $p \vdash_\Delta \forall x \in \tau, q$ *if and only if* $p \vdash_{\Delta, x:\tau} q$*, for any variable* $x : \tau$*, and*
   vii. $\exists x \in \tau, p \vdash_\Delta q$ *if and only if* $p \vdash_{\Delta, x:\tau} q$*, for any variable* $x : \tau$.
   viii. $\top \vdash_\Delta x = x$*, for any variable* $x : \tau$,
   ix. $(x_1 = y_1) \wedge ... \wedge (x_n = y_n) \wedge p \vdash_\Delta p[(x_1, ..., x_n) := (y_1, ..., y_n)]$*, for any sequence of variables* $(x_1 : \tau_1, ..., x_n : \tau_n)$.

   We write $t \vdash s$ *for* $t \vdash_\varnothing s$ *and* $\vdash_\Delta p$ *for* $\top \vdash_\Delta p$.

Definition 13. *If a term has a single free variable* $a : A$*, it will be denoted* $t(a)$*, and the notation* $t(a')$ *will be used for* $t(a)[a := a']$*. Similarly for formulas.*

Definition 14. *Given a term* t *in a type theory, a* ***suitable context*** *for* t *is a context* Δ *such that every free variable of* t *appears in* Δ*. Similarly for formulas.*

3.1 Some type theory Lemmas

Many reasonable sounding statements concerning the entailment relation do in fact follow from the axioms. This section provides a collection of particularly helpful ones. Many of these will be used in sections 3.3 and 4. In what follows, $\Delta, x : \tau$ will always mean the context given by appending $x : \tau$ to the end of the sequence Δ, we give full proofs to illustrate the basic methods involved in working with type theories.

Lemma 1. 1. $p \wedge q \vdash_\Delta r$ *if and only if* $q \wedge p \vdash_\Delta r$

2. $p \vdash_\Delta \neg(\neg p)$,

3. *if* $p \vdash_\Delta q$*, then* $p \wedge r \vdash_\Delta q$,

4. $\neg q \vdash_\Delta \neg(q \wedge p)$,

5. $p \wedge (q \vee r) \vdash_\Delta (p \wedge q) \vee (p \wedge r)$,

6. $(p \vee q) \wedge \neg q \vdash_\Delta p$*, and*

7. $(\exists x : \tau, p) \wedge q \vdash_\Delta \exists x : \tau, p \wedge q$ *and* $\exists x : \tau, p \wedge q \vdash_\Delta (\exists x : \tau, p) \wedge q$

Proof. 1. There is the following proof tree,

$$
\dfrac{\dfrac{\dfrac{q \wedge p \vdash_\Delta q \wedge p}{q \wedge p \vdash_\Delta p}\,3.b.iii \qquad \dfrac{q \wedge p \vdash_\Delta q \wedge p}{q \wedge p \vdash_\Delta p}\,3.b.iii}{q \wedge p \vdash_\Delta p \wedge q}\,3.b.iii \qquad \begin{array}{c}\vdots\\ p \wedge q \vdash_\Delta r\end{array}}{q \wedge p \vdash_\Delta r}\,3.a.ii
$$

2. Let π denote the following proof tree,

$$
\dfrac{\dfrac{p \wedge (p \Rightarrow \bot) \vdash_\Delta p \wedge (p \Rightarrow \bot)}{p \wedge (p \Rightarrow \bot) \vdash_\Delta p \Rightarrow \bot}\,3.b.iii \qquad \dfrac{p \wedge (p \Rightarrow \bot) \vdash_\Delta p \wedge (p \Rightarrow \bot)}{p \wedge (p \Rightarrow \bot) \vdash_\Delta p}\,3.b.iii}{p \wedge (p \Rightarrow \bot) \vdash_\Delta (p \Rightarrow \bot) \wedge p}\,3.b.iii
$$

Then, there is the following proof tree,

$$
\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ p \wedge (p \Rightarrow \bot) \vdash_\Delta (p \Rightarrow \bot) \wedge p\end{array} \qquad \dfrac{p \Rightarrow \bot \vdash_\Delta p \Rightarrow \bot}{(p \Rightarrow \bot) \wedge p \vdash_\Delta \bot}\,3.b.v}{p \wedge (p \Rightarrow \bot) \vdash_\Delta \bot}\,3.a.ii}{p \vdash_\Delta (p \Rightarrow \bot) \Rightarrow \bot}\,3.b.v
$$

3. Observe the following proof tree,

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ p \vdash_\Delta q\end{array} \qquad \dfrac{q \wedge r \vdash_\Delta q \wedge r}{q \vdash_\Delta r \Rightarrow (q \wedge r)}\,3.b.v}{p \vdash_\Delta r \Rightarrow (q \wedge r)}\,3.a.ii}{p \wedge r \vdash_\Delta q \wedge r}\,3.b.v}{p \wedge r \vdash_\Delta q}\,3.b.iii
$$

4. By axiom 3.$b.v$, it suffices to show $(q \Rightarrow \bot) \wedge (q \wedge p) \vdash_\Delta \bot$, for which it suffices to show $((q \Rightarrow \bot) \wedge q) \wedge p) \vdash_\Delta \bot$. By part 2 of this Lemma, it suffices to show $(q \Rightarrow \bot) \wedge q \vdash \bot$, which follows from axiom 3.$b.v$, as $q \Rightarrow \bot \vdash_\Delta q \Rightarrow \bot$.

5. By 3.$b.v$, it suffices to show $q \vee r \vdash_\Delta p \Rightarrow ((p \wedge q) \vee (p \wedge r))$, for which by 3.$b.iv$, it suffices to show both $q \vdash_\Delta p \Rightarrow ((p \wedge q) \vee (p \wedge r))$, and $r \vdash_\Delta p \Rightarrow ((p \wedge q) \vee (p \wedge r))$. The first sequent can be proved by the following, where the label (1) is referring to the first part of this Lemma,

$$
\dfrac{\dfrac{\dfrac{(p \wedge q) \vee (p \wedge r) \vdash_\Delta (p \wedge q) \vee (p \wedge q)}{p \wedge q \vdash_\Delta (p \wedge q) \vee (p \wedge q)}\,3.b.iv}{q \wedge p \vdash_\Delta (p \wedge q) \vee (p \wedge q)}\,(1)}{q \vdash_\Delta p \Rightarrow ((p \wedge q) \vee (p \wedge r))}\,3.b.v
$$

with a similar proof tree for the remaining sequent.

6. From part 4 of this Lemma, it suffices to show $(p \wedge \neg q) \vee (\neg q \wedge q) \vdash_\Delta p$, for which there is the following proof tree,

$$\dfrac{\dfrac{p \wedge (q \Rightarrow \bot) \vdash_\Delta p \wedge (q \Rightarrow \bot)}{p \wedge (q \Rightarrow \bot) \vdash_\Delta p}\,3.b.iii \qquad \dfrac{\dfrac{(q \Rightarrow \bot) \vdash_\Delta (q \Rightarrow \bot)}{(q \Rightarrow \bot) \wedge q \vdash_\Delta \bot}\,3.b.v \qquad \dfrac{}{\bot \vdash_\Delta p}\,3.b.ii}{(q \Rightarrow \bot) \wedge q \vdash_\Delta p}\,3.a.ii}{(p \wedge (q \Rightarrow \bot)) \vee ((q \Rightarrow \bot) \wedge q) \vdash_\Delta p}\,3.b.iv$$

7. Observe the following proof tree,

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x : \tau, p \wedge q \vdash_\Delta \exists x : \tau, p \wedge q}{p \wedge q \vdash_{\Delta, x:\tau} \exists x : \tau, p \wedge q}\,3.b.vii}{p \vdash_{\Delta, x:\tau} q \Rightarrow (\exists x : \tau, p \wedge q)}\,3.b.v}{\exists x : \tau, p \vdash_\Delta q \Rightarrow (\exists x : \tau, p \wedge q)}\,3.b.vii}{(\exists x : \tau, p) \wedge q \vdash_\Delta \exists x : \tau, p \wedge q}\,3.b.v$$

Reading this same proof tree from bottom to top gives a proof tree for the second sequent. □

Lemma 2. *If $x : \tau$ is a variable, and $t : \tau$ is a term, then*

$$x = t \vdash_{\Delta, x:\tau} s \qquad \text{if and only if} \qquad \vdash_\Delta s[x := t]$$

Proof. The "only if" direction follows from the following proof tree,

$$\dfrac{\dfrac{}{\vdash_\Delta t = t}\,3.b.viii \qquad \dfrac{\begin{matrix}\vdots \\ x = t \vdash_{\Delta, x:\tau} s\end{matrix}}{t = t \vdash_\Delta s[x := t]}\,3.a.iii}{\vdash_\Delta s[x := t]}\,2.a.ii$$

For the other direction, let $x' : \tau$ be such that $x' \notin \mathrm{FV}(s)$, then there is the following proof tree,

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash_{x,\Delta} s[x := t]\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{x = x' \wedge s[x := x'] \vdash_{\Delta, x:\tau, x':\tau} s}\,3.b.xi}{x = t \wedge s[x := t] \vdash_{\Delta, x:\tau} s}\,3.a.iii}{s[x := t] \wedge x = t \vdash_{\Delta, x:\tau} s}\,1\ (1)}{s[x := t] \vdash_{\Delta, x:\tau} (x = t) \Rightarrow s}\,3.b.v}{\vdash_{\Delta, x:\tau} (x = t) \Rightarrow s[x := t]}\,3.a.ii}{x = t \vdash_{\Delta, x:\tau} s[x := t]}\,3.b.v$$

where the label 1 (1) is referring to the first part of Lemma 1. □

Lemma 3. *If $p \vdash_{\Delta, x:\tau} q$, then $\exists x : \tau, p \vdash_\Delta \exists x : \tau, q$.*

Proof. First, let π denote the following proof tree,

$$\dfrac{\dfrac{\forall x:\tau, q \vdash_\Delta \forall x:\tau, q}{\forall x:\tau, q \vdash_{\Delta,x:\tau} q}\,3.b.vi \qquad \dfrac{\exists x:\tau, q \vdash_\Delta \exists x:\tau, q}{q \vdash_{\Delta,x:\tau} \exists x:\tau, q}\,3.b.vii}{\forall x:\tau, q \vdash_{\Delta,x:\tau} \exists x:\tau, q}\,3.a.ii$$

Then there is the following,

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ p \vdash_{\Delta,x:\tau} q\end{array}}{p \vdash_\Delta \forall x:\tau, q}\,3.b.vii}{p \vdash_{\Delta,x:\tau} \forall x:\tau, q}\,3.a.iii \qquad \begin{array}{c}\pi\\ \vdots\\ \forall x:\tau, q \vdash_{\Delta,x:\tau} \exists x:\tau, q\end{array}}{p \vdash_{\Delta,x:\tau} \exists x:\tau, q}\,3.a.ii}{\exists x:\tau, p \vdash_\Delta \exists x:\tau, q}\,3.b.vii$$

□

Lemma 4. *Let $t:\tau$ be a term and p a formula. Then*

$$p[x := t] \vdash_\Delta \exists x:\tau, p$$

The term t can be thought of as a witness of the statement p, so this Lemma states that to entail $\exists x:\tau, p$, it suffices to bear a witness t.

Proof. Observe the following proof tree,

$$\dfrac{\dfrac{\exists x:\tau, p \vdash_\Delta \exists x:\tau, p}{p \vdash_{\Delta,x:\tau} \exists x:\tau, p}\,3.b.vii}{p[x := t] \vdash_\Delta \exists x:\tau, p}\,3.a.iii$$

□

3.2 The Mitchell-Benabou language

As already mentioned, if $\mathscr{E}$ is a cocomplete topos, then there is an associated type theory in the sense of Definition 12, called the *Mitchell-Benabou language of $\mathscr{E}$*.

Definition 15. *Choose a colimit for every cocone, and a limit for every finite cone. Also choose an initial object $\mathbb{1}$, a subobject classifer Ω, along with a representative for every subobject, and lastly a family of exponentials for Ω. Let $\mathscr{E}$ be a topos which admits all colimits. The* ***Mitchell-Benabou language of*** $\mathscr{E}$ *is a type theory in the sense of Definition 12, which for every object $A \in \mathscr{E}$ admits a type with the same name. If C in $\mathscr{E}$ is the chosen product of objects A and B, then identify the type C with the product type $A \times B$. Similarly, if C is the chosen exponent Ω^A for some obejct $A \in \mathscr{E}$, then identify the type PA with the type C. The function symbols are all $f : A \to B$, where f is a morphism in $\mathscr{E}$, the relation symbols are all $R \subseteq A$ where $r : R \rightarrowtail A$ is a monomorphism in $\mathscr{E}$, and terms and formulas consisting only of those which can be constructed from the formation rules as given in Definition 12.*

The Definition of entailment for this type theory is delayed until after Definition 16 below.

Largely following Johnstone [**?**, §D4.1], associated to the Mitchell-Benabou language of a topos $\mathscr{E}$ is an *interpretation* of this type theory in $\mathscr{E}$, ie, an assignment of an object in $\mathscr{E}$ to every type, a morphism with codomain B to every tuple $(\Delta, t : B)$ consisting of a term t along with a suitable context (where, as was given in Definition 14, a context Δ is a suitable context for t if every free variable of t appears in Δ), and a monomorphism to every tuple (Δ, p) where p is a formula and Δ is a suitable context. In more detail, if $\Delta = (x_1 : A_1, ..., x_n : A_n)$ is a context and we have assigned objects $[\![A_i]\!]$ to each type A_i, and $t : B$ is a term for which Δ is a suitable context, then $[\![\Delta \mid t]\!]$ is a morphism

$$[\![A_1]\!] \times ... \times [\![A_n]\!] \xrightarrow{[\![\Delta|t]\!]} [\![B]\!]$$

If p is a formula for which Δ is a suitable context, then $[\![\Delta \mid p]\!]$ is a subobject

$$[\![\Delta \mid p]\!] \rightarrowtail \overline{\Delta} = [\![A_1]\!] \times ... \times [\![A_n]\!]$$

Definition 16. *The* ***interpretation*** $[\![\cdot]\!]$*, of a type, term or formula of the Mitchell-Benabou language of* $\mathscr{E}$ *is defined by induction on the collection of objects and morphisms:*

- ***Types:*** *If A is an object of $\mathscr{E}$ viewed as a type, then $[\![A]\!]$ is this object. If $A \times B$ is a product type, then let $[\![A \times B]\!]$ be the chosen product $[\![A]\!] \times [\![B]\!]$. For terms of the form PA, let $[\![PA]\!]$ be the object Ω^A.*
- ***Terms and formulas:*** *assume in what follows that Δ is always a suitable context. Also, for any context $\Sigma = (x_1 : A_1, ..., x_n : A_n)$, let $\overline{\Sigma}$ be $[\![A_1]\!] \times ... \times [\![A_n]\!]$ in $\mathscr{E}$,*
  - *if $x : A$ is a variable, then*

    $$[\![\Delta \mid x]\!] = \pi_A : \overline{\Delta} \to [\![A]\!]$$

    is the projection onto $[\![A]\!]$,
  - $[\![\Delta \mid *]\!] = \overline{\Delta} \to [\![\mathbb{1}]\!]$*, the unique morphism into the terminal object,*
  - *Recall from Theorem 3 that $Sub(\overline{\Delta})$ is a Heyting algebra. In light of this, let $[\![\Delta \mid \top]\!]$ be the terminal object of this category, and $[\![\Delta \mid \bot]\!]$ the initial object of this category.*
  - *if $t : A$ and $s : B$, then*

    $$[\![\Delta \mid \langle t, s\rangle]\!] = \langle [\![\Delta \mid t]\!], [\![\Delta \mid s]\!]\rangle$$
  - *if $t : A \times B$, then $[\![\Delta \mid fst(t)]\!] = \pi_L[\![\Delta \mid t]\!]$, similarly for $snd(t)$.*

– *if* $f : A \to B$ *is a function symbol, and* $t : A$ *a term, then* $[\![\Delta \mid ft]\!]$ *is the composite*

$$\overline{\Delta} \xrightarrow{[\![\Delta|t]\!]} [\![A]\!] \xrightarrow{f} [\![B]\!]$$

– *if* $R \subseteq A$ *is a relation symbol corresponding to a monomorphism* $r : R \to A$*, and* $t : A$ *a term, then* $[\![\Delta \mid R(t)]\!]$ *is the subobject* $B \rightarrowtail \overline{\Delta}$ *such that the following is a pullback diagram,*

$$\begin{tikzcd} B \arrow[r] \arrow[d, tail] & R \arrow[d, tail, "r"] \\ \overline{\Delta} \arrow[r, "{[\![\Delta|t]\!]}"'] & A \end{tikzcd}$$

– $[\![\Delta \mid t = s]\!]$ *is the subobject*

$$\text{Equaliser}([\![\Delta \mid t]\!], [\![\Delta \mid s]\!]) \rightarrowtail \overline{\Delta}$$

– *In the notation of Definition 11, if* p *and* q *are formulas, then*

$$[\![\Delta \mid p \wedge q]\!] = [\![\Delta \mid p]\!] \wedge [\![\Delta \mid q]\!],$$
$$[\![\Delta \mid p \vee q]\!] = [\![\Delta \mid p]\!] \vee [\![\Delta \mid q]\!],$$
$$[\![\Delta \mid p \Rightarrow q]\!] = [\![\Delta \mid p]\!] \Rightarrow [\![\Delta \mid q]\!].$$

– *similarly, if* $\{p_i\}_{i=0}^{\infty}$ *is a collection of formulas, then*

$$[\![\Delta \mid \bigvee_{i=0}^{\infty} p_i]\!] = \bigvee_{i=1}^{\infty} \left([\![\Delta \mid p_i]\!]\right)$$

where $\bigvee_{i=1}^{\infty}$ *is the coproduct of the category* $Sub(\overline{\Delta})$*,*

– *As was done at the beginning of Section 3.1, let* $\Delta, a : A$ *mean the context* Δ *with* $a : A$ *appended to the end. Also,* $\Delta \setminus a : A$ *will mean* Δ *with* $a : A$ *omitted (assuming* $a : A$ *appears in* Δ*).* $[\![\Delta \mid \exists a : A, p]\!]$ *is the image (see Definition 8) of the composite*

$$[\![\Delta \mid p]\!] \rightarrowtail \overline{\Delta} \xrightarrow{\pi} \overline{\Delta \setminus a : A}$$

where the morphism π *is given by a product of projection morphisms,*

– *Recall from Theorem 2 that the functor* $\pi^* : Sub(\overline{\Delta \setminus a : A}) \to Sub(\overline{\Delta})$ *admits a right adjoint, in accordance with Definition 9, we denote this adjoint by* $\forall_\pi$*. Then*

$$[\![\Delta \mid \forall a : A, p]\!] = \forall_\pi([\![\Delta, a : A \mid p]\!])$$

– *$[\![\Delta \mid \{x : A \mid p\}]\!]$ is the transpose of the morphism $\chi_{[\![\Delta,a:A|p]\!]}$, where, as per the notation defined in 3, $\chi_{[\![\Delta|\{x:A|p\}]\!]}$ is the unique morphism such that the following is a pullback diagram*

$$\begin{tikzcd}
{[\![\Delta, a : A \mid p]\!]} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "true"] \\
\overline{\Delta} \times A \arrow[r, "\chi_{[\![\Delta,a:A|p]\!]}"'] & \Omega
\end{tikzcd}$$

That is, $[\![\Delta \mid \{x : A \mid p\}]\!] : \overline{\Delta} \to \omega^A$ corresponds under adjunction to $\chi_{[\![\Delta,a:A|p]\!]}$.

– *$[\![\Delta \mid t \in \{x : A \mid p\}]\!]$ is the monic $C \rightarrowtail \overline{\Delta}$ such that the following is a pullback diagram*

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & \in_A \arrow[d, tail] \\
\overline{\Delta} \arrow[r] & A \times \Omega^A
\end{tikzcd}$$

where the bottom morphism is $\langle [\![\Delta \mid t]\!], [\![\Delta \mid \{x : A \mid p\}]\!] \rangle$.

Now, Definition 15 can be completed by defining $p \vdash_\Delta q$ to hold only if $[\![\Delta \mid p]\!] \leq [\![\Delta \mid q]\!]$, where $\leq$ is the preorder on the set $\mathrm{Sub}(\overline{\Delta})$.

3.3 Applications of the Mitchell-Benabou Language

We begin with some useful Lemmas (Lemma 5, Lemma 6) which illustrate how the Mitchell-Benabou language may be used to prove statements about the topos. In Section 3.4 we establish some of the most important technical Lemmas of the thesis, which help make the Mitchell-Benabou language useable in practice to deal with morphisms out of subobjects.

Some sequents hold in the Mitchell-Benabou language but not in arbitrary type theories.

Lemma 5. *In the setting of the Mitchell-Benabou language associated to a topos,*

1. $\vdash_{z:\mathbb{1}} z = *$,
2. $\vdash_{X,x:A} x \in \{x' : A \mid p\} \Leftrightarrow p[x' := x]$, *and*
3. $\vdash_{z:A\times B} \exists x : A, \exists y : B, z = \langle x, y \rangle$.

Proof. The proofs are exercises in "compiling" the sequent to a statement about the category and then proving that statement using category theory. For example, to show

(3), observe first that $[\![z : A \times B \mid \exists x : A, \exists y : B, z = \langle x, y\rangle]\!]$ is the image of the composition

$$\mathrm{Equaliser}(\pi_1, \langle \pi_2, \pi_3\rangle) \overset{e}{\rightarrowtail} (A \times B) \times A \times B \xrightarrow{\pi} (A \times B) \times A \xrightarrow{\pi'} A \times B$$

Thus $\vdash_{z:A\times B} \exists x : A, \exists y : B, z = \langle x, y\rangle$ is equivalent to the categorical statement that there exists a morphism $A \times B \to \mathrm{im}(\pi'\pi e)$ such that the triangle

$$\begin{tikzcd} A \times B \arrow[r] \arrow[dr] & \mathrm{Im}(\pi'\pi e) \arrow[d] \\ & A \times B \end{tikzcd}$$

commutes. Then, since $\Delta_{A\times B}$, the diagonal morphism, is such that $\pi_1\Delta_{A\times B} = \langle \pi_2, \pi_3\rangle\Delta_{A\times B}$, by the universal property of $\mathrm{Equaliser}(\pi_1, \langle \pi_2, \pi_3\rangle)$, there exists a morphism $\varphi : A\times B \to \mathrm{Equaliser}(\pi_1, \langle \pi_2, \pi_3\rangle)$ such that the triangle

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[dr, "\Delta_{A\times B}"'] & \mathrm{Equaliser}(\pi_1, \langle \pi_2, \pi_3\rangle) \arrow[d, tail] \\ & (A \times B) \times A \times B \end{tikzcd}$$

commutes. Thus, the composite

$$A \times B \xrightarrow{\varphi} \mathrm{Equaliser}(\pi_1, \langle \pi_2, \pi_3\rangle) \twoheadrightarrow \mathrm{Im}(\pi'\pi e)$$

is a suitable morphism. $\square$

The power of the Mitchell-Benabou language of a topos $\mathscr{E}$ comes from its ability to describe morphisms and objects of $\mathscr{E}$ as though they have "elements" in them.

Lemma 6. *Let $f, g : A \to B$ be morphisms, then $\vdash_{a:A} fa = ga$ if and only if $f = g$ (as morphisms in the topos).*

Proof. This is a matter of unwinding Definitions. The morphism $[\![a : A \mid fa = ga]\!]$ is the monic

$$\begin{tikzcd} \mathrm{Equaliser}([\![a : A \mid fa]\!], [\![a : A \mid ga]\!]) \arrow[r, tail] & A \arrow[r, shift left, "{[\![a:A|fa]\!]}"] \arrow[r, shift right, "{[\![a:A|ga]\!]}"'] & B \end{tikzcd}$$

Moreover, the morphism $[\![a : A \mid fa]\!]$ is

$$A \xrightarrow{[\![a:A|a]\!]} A \xrightarrow{f} B$$

and similar for $[\![a : A \mid ga]\!]$. Then, $[\![a : A \mid a]\!]$ is the identity morphism $\mathrm{id}_A : A \to A$. Thus, if $\vdash_{a:A} fa = ga$, there exists a morphism $A \to \text{equaliser}(f, g)$ such that the diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr, "\mathrm{id}_A"] & \\
\text{Equaliser}(f,g) \arrow[rr, tail] & & A
\end{tikzcd}$$

commutes, and so $f = g$. Conversely, if $f = g$, then such a morphism exists by the universal property of the equaliser, and so $\vdash_a fa = ga$. $\square$

The Mitchell-Benabou language also has the power to prove existence of morphisms in $\mathscr{E}$. An example of this is Lemma 8 below. The proof of this Lemma will require the following one, which in itself is helpful for making arguments about the interpretation of formulas of the form $t(a) \in \{x : A \mid p\}$.

Lemma 7. *Let $t(a)$ be a term of type B with free variable set $FV(t(a)) = \{a : A\}$, and let $p(b)$ be a formula with free variable set $FV(p(b)) = \{b : B\}$. Let $f = [\![a : A \mid t(a)]\!]$, then*

$$[\![a : A \mid p(t(a))]\!] \cong f^{-1}[\![b : B \mid p(b)]\!]$$

where f^{-1} is as defined in Definition 6.

Proof. By Lemma 5, it suffices to show that

$$[\![a : A \mid t(a) \in \{b : B \mid q(b)\}]\!] \cong f^{-1}[\![b : B \mid q(b)]\!]$$

By unwinding Definitions, $[\![a : A \mid t(a) \in \{b : B \mid q(b)\}]\!]$ is the monomorphism with codomain A such that the following is a pullback diagram,

$$\begin{tikzcd}
{[\![a : A \mid t(a) \in \{b : B \mid q(b)\}]\!]} \arrow[r] \arrow[d, tail] & \in_B \arrow[d, tail] \\
A \arrow[r] & B \times \Omega^B
\end{tikzcd} \tag{1}$$

where $\in_B$ is as in Definition 5, and the morphism in the bottom row is

$$\langle [\![a : A \mid t(a)]\!], [\![a : A \mid \{b : B \mid q(b)\}]\!] \rangle$$

Also, let B' be the domain of the monomorphism $[\![b : B \mid q(b)]\!]$, then there is the following commuting diagram,

$$\begin{tikzcd}
A \arrow[r, "{f, \ulcorner \chi_{B'} \urcorner 0_B \rangle}"] \arrow[ddr, "f"'] & B \times \Omega^B \arrow[r, "\mathrm{ev}_B"] & \Omega \\
& B \times \mathbb{1} \arrow[u, "\mathrm{id}_B \times \ulcorner \chi_{B'} \urcorner"] & \\
& B \arrow[u] \arrow[uur, "\chi_{B'}"'] &
\end{tikzcd} \tag{2}$$

By combining diagrams 1 and 2 and using the defining pullback square of $\in_B$, it follows that

$$\begin{tikzcd} \llbracket a : A \mid t(a) \in \{b : B \mid q(b)\} \rrbracket \arrow[rr] \arrow[d] & & \mathbb{1} \arrow[d, "\mathrm{true}"] \\ A \arrow[r, "f"'] & B \arrow[r, "\chi_{B'}"'] & \Omega \end{tikzcd} \tag{3}$$

is a pullback diagram. Then, by unravelling Definitions, the following are also pullback diagrams

$$\begin{tikzcd} B' \arrow[r] \arrow[d, tail] & \mathbb{1} \arrow[d, "\mathrm{true}"] \\ B \arrow[r, "\chi_{B'}"'] & \Omega \end{tikzcd}$$

and

$$\begin{tikzcd} f^{-1}\llbracket b : B \mid q(b) \rrbracket \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Thus $\llbracket a : A \mid t(a) \in \{b : B \mid q(b)\} \rrbracket$ and $f^{-1}\llbracket b : B \mid q(b) \rrbracket$ are isomorphic by essential uniqueness of pullbacks. $\square$

3.4 Dealing with Subobjects

Recall that a term $t(a)$ of type B with free variable set $\mathrm{FV}(t(a)) = \{a : A\}$ has for its interpretation a morphism $\llbracket a : A \mid t(a) \rrbracket : A \to B$. It makes sense that this morphism factors through the subobject $\llbracket b : B \mid p(b) \rrbracket$ of "elements" satisfying a formula $p(b)$ if and only if the sequent $p(a) \vdash_{a:A} t(a) \in \{b : B \mid q(b)\}$:

Lemma 8. *If $t(a)$ is a term of type B with free variable $a : A$, and $\llbracket a : A \mid p(a) \rrbracket \rightarrowtail A$ is a subobject of A, then $p(a) \vdash_{a:A} t(a) \in \{b : B \mid q(b)\}$ if and only if there exists a (necessarily unique) morphism $g : \llbracket a : A \mid p(a) \rrbracket \to \llbracket b : B \mid q(b) \rrbracket$ such that the diagram*

$$\begin{tikzcd} \llbracket a : A \mid p(a) \rrbracket \arrow[r, "g"] \arrow[d, tail] & \llbracket b : B \mid q(b) \rrbracket \arrow[d, tail] \\ A \arrow[r, "\llbracket a:A \mid t(a) \rrbracket"'] & B \end{tikzcd}$$

commutes.

Proof. Let $f = \llbracket a : A \mid t(a) \rrbracket$. Then g exists if and only if $\llbracket a : A \mid p(a) \rrbracket \le f^{-1}\llbracket b : B \mid q(b) \rrbracket$, where $\le$ is the order on the set $\mathrm{Sub}(A)$. By Lemma 8, this is the case if and only if

$$\llbracket a : A \mid t(a) \rrbracket \le \llbracket a : A \mid t(a) \in \{b : B \mid q(b)\} \rrbracket$$

which by the Definition of entailment in the Mitchell-Benabou language is true if and only if

$$p(a) \vdash_{a:A} t(a) \in \{b : B \mid q(b)\}$$

$\square$

The remaining results of this section will be used heavily in the proofs of Theorems 9 and 7 in section 4. The following Lemma gives sufficient conditions on when a particular type of morphism is a monomorphism or an epimorphism. Later, this will be used to imply the existence of a morphism for which it seems difficult to give a direct description.

Lemma 9. *If $f : A \twoheadrightarrow B$ is an epimorphism, $\{a : A \mid p(a)\}$ is a subobject of A such that $FV(\{a : A \mid p(a)\}) = \varnothing$, then*

- *if $a \in \{a : A \mid p(a)\} \wedge a' \in \{a : A \mid p(a)\} \vdash_{a:A,a':A} (fa = fa') \Rightarrow (a = a')$, then $f[\![a : A \mid p(a)]\!]$ is monic, and*
- *if $\vdash_{b:B} \exists a : A,\ a \in \{a : A \mid p(a)\} \wedge fa = b$ then $f[\![a : A \mid p]\!]$ is epic.*

Proof. Let T be the domain of the morphism $[\![a : A \mid p(a)]\!] \to A$. For the first dot point, it suffices to show that for any $g, g' : C \to A$ which factor through T such that $fg = fg'$, that $g = g'$. Let $T := \{a : A \mid p\}$ and say $g, g' : C \to A$ both factor through T, and are such that $fg = fg'$. The equaliser of fg and fg' is isomorphic to C, so by Definition, $\vdash_{c:C} f(gc) = f(g'c)$. Also, since g and g' both factor through T, it follows from Lemma 8 that $\vdash_c gc \in T \wedge g'c \in T$. Thus

$$\vdash_c gc \in T \wedge gc' \in T \wedge f(gc) = f(g'c) \tag{4}$$

Thus, if $a \in T \wedge a' \in T \vdash_{a:A,a':A} (fa = fa') \wedge (a = a)$, ie, $a \in T \wedge a' \in T \wedge fa = fa' \vdash_{a,a':A} a = a$, it follows from axiom $3.a.iii$ of Definition 12 that,

$$gc \in T \wedge g'c \in T \wedge f(gc) = f(g'c) \vdash_{c:C} gc = g'c \tag{5}$$

It then follows by axiom $3.a.ii$ of Definition 12 that sequents 4 and 5 imply that $\vdash_c gc = g'c$, which by Lemma 6, implies that $g = g'$ as morphisms in the topos.

Next, say $g, g' : B \to C$ are such that $gf[\![a : A \mid p]\!] = g'f[\![a : A \mid p(a)]\!]$. This assumption implies that there exists a morphism $h : T \to \text{Equaliser}(gr, g'f)$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \\
T \arrow[u, tail] \arrow[r, "h"'] & \text{Equaliser}(gf, g'f) \arrow[ul, tail] &
\end{tikzcd}$$

commutes. From Lemma 8, it follows that $p(a) \vdash_{a:A} gfa = g'fa$, which by Lemma 5, implies that $a \in T \vdash_a g(fa) = g'(fa)$. So, if $\vdash_{b:B} \exists a : A, a \in T \wedge fa = b$, then

$$\vdash_{b:B} \exists a : A, a \in T \wedge fa = b \wedge g(fa) = g'(fa) \tag{6}$$

Also, by axiom 3.*b.ix* in Definition 12,

$$b' = b \wedge gb' = g'b' \vdash_{a:A,b,b':B} gb = g'b$$

so

$$(b' = b \wedge gb' = g'b')[b' := fa] \vdash_{a:A,b,b':B} (gb = g'b)[b' := fa]$$

ie,

$$fa = b \wedge g(fa) = g'(fa) \vdash_{a:A,b:B} gb = g'b$$

and so

$$fa = b \wedge g(fa) = g'(fa) \wedge a \in T \vdash_{a:A,b:B} gb = g'b$$

which follows from Lemma 1 (2). Lemma 3 implies that

$$\exists a : A, fa = b \wedge g(fa) = g'(fa) \wedge a \in A' \vdash_b \exists a : A, gb = g'b$$

Combining this with Equation 6, it then follows that $\vdash_{a:A,b:B} gb = g'b$, then, using axiom 3.*a.iii* of Definition 12, it follows that $\vdash_{a:A} gb = g'b$. □

A special case of this is the following:

Corollary 1. *Let $f : A \to B$ be a morphism. Then,*

- *if $\vdash_{a,a'} fa = fa' \Rightarrow a = a'$, then f is monic, and*
- *if $\vdash_{b:B} \exists a : A, fa = b$, then f is epic.*

Lemma 10. *Let $f_1^{q(y)} : [\![y : A \times B \mid q(y)]\!] \rightarrowtail A \times B$ be a subobject, and let $t_{q(y)} = \{y : A \times B \mid q(y)\}$. Then if t_{qy} is "provably functional", ie, the sequents*

$$x \in t_{q(y)} \wedge x' \in t_{q(y)} \vdash_{x,x':A \times B} \mathit{fst}(x) = \mathit{fst}(x') \Rightarrow x = x'$$

and

$$\vdash_{z:A} \exists x : A \times B, x \in t_{q(y)} \wedge \mathit{fst}(x) = z$$

hold, then there exists a morphism $f : A \to A \times B$ in the topos such that $\pi_L f = id_A$.

Proof. First, by unwinding Definition, $[\![\Delta \mid \mathrm{fst}(x)]\!] = [\![\Delta \mid \pi_L(x)]\!]$, for any suitable context Δ, so the two hypotheses are equivalent to the sequents given by replacing "fst" with "π_L". It then follows from Lemma 9 that the morphism

$$[\![y : A \times B \mid q(y)]\!] \xrightarrow{f_1^{q(y)}} A \times B \xrightarrow{\pi_L} A$$

is an isomorphism, as any morphism which is both a monomorphism and an epimorphism in a topos is an isomorphism. Let $f_2^{q(y)}$ denote the inverse of this isomorphism. The desired morphism f is then $f_1^{q(y)} f_2^{q(y)}$. □

In the constructions of Section 4 we will need to encode at the level of the Mitchell-Benabou language the following construction: we have a formula $p(z)$ determining a subobject $[\![z : Z \mid p(z)]\!] \rightarrowtail Z$ and a term $t(b) : Z$ determining a morphism $B \to Z$ which factors through the subobject, or what is the same, $\vdash_{b:B} t(b) \in \{z : Z \mid p(z)\}$.

Suppose $h : B \to U$ is a morphism such that $t(b) = t(b') \vdash_{b:B,b':B} hb = hb'$. In the topos Sets, this implies h factors through the image of the morphism $B \to Z$ by the construction "given $z \in Z$ in the image, choose $b \in B$ such that $z = t(b)$ and send z to $h(b)$".

For our purposes, it is also useful to realise this construction as a morphism **out of Z** (rather than the image) which sends any $z \in Z$ not in the image to a special value.

Lemma 11. *Let $[\![z : Z \mid p(z)]\!] \rightarrowtail Z$ be a subobject, $t(b) : Z$ be a term with free variable $b : B$, such that*

$$\vdash_{b:B} t(b) \in \{z : Z \mid p(z)\}$$

Also, let $h : B \to U$ be a morphism such that

$$t(b) = t(b') \vdash_{b,b':B} hb = hb'$$

Then there exists a morphism $f : Z \to Z \times \Omega^U$ such that the diagram

$$\begin{tikzcd}
B \arrow[rr, "h"] \arrow[d, "{[\![b:B|t(b)]\!]}"'] & & U \arrow[dd, "{[\![u:U|\{u\}]\!]}"] \\
{[\![z : Z \mid p(z)]\!]} \arrow[d, tail] & & \\
Z \arrow[r, "f"'] & Z \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U
\end{tikzcd}$$

commutes, and $\pi_L f = id_Z$.

We first give the proof of a stronger result:

Lemma 12. *Let $[\![z : Z \mid p(z)]\!] \rightarrowtail Z$ be a subobject, $t(a), s(b) : Z$ be a terms with free variables $a : A$ and $b : B$ respectively such that*

$$\vdash_{a:A,b:B} t(a) \in \{z : Z \mid p(z)\} \wedge s(b) \in \{z : Z \mid p(z)\}$$

and

$$\vdash_{a:A,b:B} \neg(t(a) = s(b))$$

Also, let $h_1 : A \to U$ and $h_2 : B \to U$ be a morphisms such that

$$t(a) = t(a') \vdash_{a,a':A} h_1 a = h_1 a'$$

and

$$s(b) = s(b') \vdash_{b,b':B} h_2 b = h_2 b'$$

Then there exists a morphism $f : Z \to Z \times \Omega^U$ such that the diagrams

$$\begin{tikzcd}
A \arrow[rr, "h_1"] \arrow[d, "{[\![a:A \mid t(a)]\!]}"'] & & U \arrow[dd, "{[\![u:U \mid \{u\}]\!]}"] \\
{[\![z : Z \mid p(z)]\!]} \arrow[d, tail] & & \\
Z \arrow[r, "f"'] & Z \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[rr, "h_2"] \arrow[d, "{[\![b:B \mid s(b)]\!]}"'] & & U \arrow[dd, "{[\![u:U \mid \{u\}]\!]}"] \\
{[\![z : Z \mid p(z)]\!]} \arrow[d, tail] & & \\
Z \arrow[r, "f"'] & Z \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U
\end{tikzcd}$$

both commute, and $\pi_L f = id_Z$.

The idea of this technical sounding Theorem is that f is designed with the graph of the familiar mapping of sets

$$Z \to P(U)$$
$$z \mapsto \begin{cases} \{h_1 a\}, & \text{where } a \in A \text{ is such that } z = t(a) \\ \{h_2 b\}, & \text{where } b \in B \text{ is such that } z = s(b) \\ \varnothing, & \text{else} \end{cases}$$

in mind. Notice that this map is well defined by the hypotheses on h_1 and h_2. So indeed the terms t and s can be thought of as conditions on "elements" in $[\![z : Z \mid p(z)]\!]$. The reason why the map out of Z and into the power set of U is considered instead of simply mapping $[\![z : Z \mid p(z)]\!]$ into U is because there seems to be no simple way of describing maps out of subobjects using the Mitchell-Benabou language. To work around this, a map out of Z is considered instead, but then the action of f on elements in Z which are not in $[\![z : Z \mid p]\!]$ need to be accounted for. So U is injected into Ω^U by the map $u \mapsto \{u\}$, and these elements can now be mapped to the canonical element $\varnothing$.

Proof. First, a term which generalises the graph of the above mapping will be defined. In alignment with the notation of Theorem 10, let $t_{q(y)} = \{y : Z \times \Omega^U \mid q(y)\}$, where

$$\begin{aligned} q(y) = \; & (\exists a : A, y = \langle t(a), \{h_1 a\}\rangle) \\ & \vee (\exists b : B, y = \langle s(b), \{h_2 b\}\rangle) \\ & \vee (\neg(\exists a : A, \mathrm{fst}(y) = t(a)) \wedge \neg(\exists b : B, \mathrm{fst}(y) = s(b)) \wedge \mathrm{snd}(y) = \varnothing) \end{aligned}$$

The goal now is to prove that the hypotheses of Theorem 10 are satisfied, which will then yield the desired morphism f. So it must be shown that

$$y \in t_{q(y)} \wedge y' \in t_{q(y)} \vdash_{y,y':Z\times\Omega^U} \mathrm{fst}(y) = \mathrm{fst}(y') \Rightarrow y = y' \tag{7}$$

and

$$\vdash_{z:Z} \exists y : Z \times \Omega^U, y \in t_{q(y)} \wedge \mathrm{fst}(y) = x \tag{8}$$

The proofs for each of these are technical exercises in type theory manipulation, only the proof of Equation 7 will be shown here, but the same general idea is used to prove Equation 8. To prove that sequent 7 holds, it suffices to show

$$q(y) \wedge q(y') \wedge \mathrm{fst}(y) = \mathrm{fst}(y') \vdash_{y,y':Z\times\Omega^U} y = y'$$

for which it suffices to show

$$q(y) \wedge q(y') \wedge \mathrm{fst}(y) = \mathrm{fst}(y') \vdash_{y,y':Z\times\Omega^U} \mathrm{snd}(y) = \mathrm{snd}(y')$$

as

$$\mathrm{fst}(y) = \mathrm{fst}(y') \wedge \mathrm{snd}(y) = \mathrm{snd}(y') \vdash_{y,y'} y = y'$$

which follows from Lemma 5. By multiple applications of Lemma 1, $q(y) \wedge q(y') \wedge \mathrm{fst}(y) = \mathrm{fst}(y')$ entails the disjunction of nine terms, corresponding to the nine "cases" of y and y'. $\mathrm{fst}(y) = \mathrm{fst}(y')$ entail the negation of six of these, corresponding to the cases where y and y' are of different forms. Thus by multiple applications of Lemma 1,

$$\begin{aligned}
&q(y) \wedge q(y') \wedge \mathrm{fst}(y) = \mathrm{fst}(y')\\
&\vdash_{y,y'} \big(\exists a : A, y = \langle t(a), \{h_1 a\}\rangle\big) \wedge \big(\exists a' : A, y' = \langle t(a'), \{h_1 a'\}\rangle\big)\\
&\qquad \vee \big(\exists b : B, y = \langle s(b), \{h_2 b\}\rangle \wedge \exists b' : B, y' = \langle s(b'), \{h_2 b'\}\rangle\big)\\
&\qquad \vee \Big((\neg(\exists a : A, \mathrm{fst}(y) = t(a)) \wedge \neg(\exists b : B, \mathrm{fst}(y) = s(b)) \wedge \mathrm{snd}(y) = \varnothing)\\
&\qquad\qquad \wedge (\neg(\exists a' : A, \mathrm{fst}(y') = t(a')) \wedge \neg(\exists b' : B, \mathrm{fst}(y') = s(b')) \wedge \mathrm{snd}(y') = \varnothing)\Big)
\end{aligned}$$

It now needs to be shown that these three disjuncts each entail $y = y'$. The last of these follows similarly to the first two, but in fact is easier, and the arguments for the first two disjuncts are essentially the same as each other, so only the argument for the first one will be shown.

First, by Lemma 1, it suffices to show

$$\begin{aligned}
\exists a : A, \exists a' : A, y = \langle t(a), \{h_1 a\}\rangle \wedge y' = \langle t(a'), \{h_1 a'\}\rangle \wedge \mathrm{fst}(y) = \mathrm{fst}(y')&\\
\vdash_{y,y'} \mathrm{snd}(y) = \mathrm{snd}(y')&
\end{aligned}$$

Since $y = \langle t(a), \{h_1 a\}\rangle \vdash_{y,a} \mathrm{fst}(y) = t(a)$, it follows that

$$\begin{aligned}
&\exists a : A, \exists a' : A, y = \langle t(a), \{fa\}\rangle \wedge y' = \langle t(a'), \{fa'\}\rangle \wedge \mathrm{fst}(y) = \mathrm{fst}(y')\\
&\vdash_{y,y'} \exists a : A. \exists a' : A, y = \langle t(a), \{h_1 a\}\rangle \wedge y' = \langle t(a'), \{h_1 a'\}\rangle \wedge t(a) = t(a')
\end{aligned}$$

So by Lemmas 4 and 2, it suffices to show

$$t(a) = t(a') \vdash_{y,y'} \mathrm{snd}(\langle t(a), \{h_1 a\}\rangle) = \mathrm{snd}(\langle t(a'), \{h_1 a'\}\rangle)$$

which follows from the hypothesis

$$t(a) = t(a') \vdash_{a,a'} h_1 a = h_1 a'$$

This establishes existence of the morphism f, to show that the respective diagrams commute, it suffices to show

$$\vdash_{a:A} \mathrm{snd}(ft(a)) = \{h_1 a\} \qquad \text{and} \qquad \vdash_{b:B} \mathrm{snd}(fs(b)) = \{h_2 b\}$$

The proofs of each of these are almost identical. To see the first one, the fact that $\pi_L f = \mathrm{id}_Z$ is used to show that $\vdash_{a:A} \mathrm{fst}(ft(a)) = t(a)$ which in turn is used to show $\vdash_a ft(a) = \langle t(a), \{h_1 a\}\rangle$, from which the result follows. $\square$

The proof of Lemma 11 follows in almost exactly the same way as the proof of Lemma 12, except rather than considering the term $q(y)$ defined there, instead the term

$$\begin{aligned}\{y : Z \times \Omega^U \mid &(\exists b : B, y = \langle t(b), \{hb\}\rangle) \\ &\vee \big(\neg(\exists b : B, \mathrm{fst}(y) = t(b)) \wedge \mathrm{snd}(y) = \varnothing\big)\end{aligned}$$

is considered.

4 Describing Colimits using the Mitchell-Benabou Language

This section is the main contribution of this thesis. Here, it will be shown how to describe finite colimits of a topos $\mathscr{E}$ using its associated Mitchell-Benabou language. Since all colimits are either an initial object, or a coequaliser of a diagram consisting of coproducts, it suffices to show how to describe an initial object, as well as coproducts, and coequalisers using the Mitchell-Benabou language.

The inclusion maps corresponding to the coproduct will be described by defining two terms, both of type $\Omega^A \times \Omega^B$, one with free variable of type A, and the other with free variable of type B, corresponding respectively to the inclusion map of A and the inclusion map of B. Then it will be shown that these maps factor uniquely through the object of the coproduct, these unique morphisms will be the inclusions. A similar process will be done to define the projection map corresponding to the coequaliser.

First though, a few preliminary results which will be used in both Sections 4.2 and 4.3 will be proved. In the following, Δ_E is the diagonal morphism of E, see Definitions 1 and 3 for Definition of the notation $\ulcorner \chi_{\Delta_E} \urcorner$.

Lemma 13. *The map* $\ulcorner\chi_{\Delta_E}\urcorner$ *is monic, for any object* $E \in \mathscr{E}$.

Proof. Let $b_1, b_2 : B \to E$ be two maps such that $\ulcorner\chi_{\Delta_E}\urcorner b_1 = \ulcorner\chi_{\Delta_E}\urcorner b_2$. Since the diagram

$$\begin{tikzcd} B \arrow[r, "b_i"] \arrow[d, "\langle \mathrm{id}_B, b_i\rangle"'] & E \arrow[d, "\Delta_E"] \\ B \times E \arrow[r, "b_i \times 1_E"'] & E \times E \end{tikzcd}$$

is a pullback square for each i, the following,

$$\begin{tikzcd} B \arrow[r, "b_i"] \arrow[d, "\langle \mathrm{id}_B, b_i\rangle"'] & E \arrow[r] \arrow[d, "\Delta_E"] & \mathbb{1} \arrow[d, "\mathrm{true}"] \\ B \times E \arrow[r, "b_i \times \mathrm{id}_E"'] & E \times E \arrow[r, "\chi_{\Delta_E}"'] & \Omega \end{tikzcd}$$

is a pullback diagram. It follows that $\langle \mathrm{id}_B, b_1\rangle$ and $\langle \mathrm{id}_B, b_2\rangle$ represent the same subobject of $B \times E$, so by the Definition of the equivalence relation on subobjects, there exists an isomorphism $h : B \to B$ such that $\langle \mathrm{id}_B, b\rangle h = \langle \mathrm{id}_B, b'\rangle$. Projecting onto the first component shows that $h = \mathrm{id}_B$, and projecting onto the second shows that $b = b'$. □

Corollary 2. *The morphism* $[\![u : U \mid \{u\}]\!]$ *is monic.*

Proof. By unravelling Definitions, $[\![u : U \mid \{u\}]\!] = \ulcorner\chi_{\Delta_U}\urcorner$. □

Recall from Definition 12, that $\{u\}$ means the term $\{u' : U \mid u' = u\}$.

Definition 17. *Given an object* U*, let Single*(U) *be the domain of the monic* $[\![z : \Omega^U \mid \exists u : U, z = \{u\}]\!] \rightarrowtail \Omega^U$. *Let this monic be denoted* l.

The idea of Single(U) is to be "the set of singleton subsets of U".

Lemma 14. *There exists an isomorphism* $g : U \to$ *Single*(U) *such that the diagram*

$$\begin{tikzcd} U \arrow[d, "g"'] \arrow[dr, "{[\![u|\{u\}]\!]}"] & \\ \mathit{Single}(U) \arrow[r, tail, "l"'] & \Omega^U \end{tikzcd}$$

commutes.

Proof. First, notice that since $\vdash_{u:U} \{u\} \in \left\{y : \Omega^U \mid \exists u' : U, z = \{u'\}\right\}$, g exists at least as a morphism, so it only remains to show that it is an isomorphism. By Lemmas 9 and 8, it suffices to show that

$$g(u) = g(u') \vdash_{u:U, u':U} u = u'$$

and

$$z \in \{z' : \Omega^U \mid \exists u : U, \{u\} = z'\} \vdash_{z:\Omega^U} \exists u : U, gu = z$$

which are both exercises. □

4.1 Initial object

Definition 18. *Define*

$$\mathit{Initial} = \{z : \mathbb{1} \mid \bot\}$$

Let I be such that the following is a pullback diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "true"] \\ \mathbb{1} \arrow[r] & \Omega \end{tikzcd}$$

where the bottom morphism is the interpretation of Initial *with respect to the empty context.*

Theorem 4. *I is the initial object.*

Proof. By unravelling Definitions, the morphism $I \to \mathbb{1}$ is $[\![z : \mathbb{1} \mid \bot]\!]$, which in turn is the unique morphism $0 \to \mathbb{1}$. Thus there is the following diagram

$$\begin{tikzcd} I \arrow[r] \arrow[dr] & U \arrow[d] \\ & \mathbb{1} \end{tikzcd}$$

where all the maps are the unique according to the universal property of either the terminal object or the initial object. Thus I is initial. $\square$

4.2 Finite Coproducts

Recall that in the category Sets the coproduct is the disjoint union. Here we emulate the disjoint union in a roundabout way by viewing A, B as subsets of "marked" singletons in $\Omega^A \times \Omega^B$. Theorem 5 will give a description of finite coproducts using the Mitchell-Benabou language, then the details of the proof will be given for the binary coproduct case.

Recall from Definition 12 that, $\{a\} = \{a' : A \mid a' = a\}$, and $\varnothing_{a:A} = \{a : A \mid \bot\}$.

Definition 19. *Let $\{A_i\}_{i=0}^n$ be a finite set of objects, then $\coprod_{i=0}^n A_i$ is the following term,*

$$\{z : \prod_{i=0}^n \Omega^{A_i} \mid \bigvee_{i=0}^n (\exists a_i : A_i, z = \langle \varnothing_{a_0:A_0}, ..., \varnothing_{a_{i-1}:A_{i-1}}, \{a_i\}, \varnothing_{a_{i+1}:A_{i+1}}, ..., \varnothing_{a_n:A_n}\rangle)\}$$

where $\langle \varnothing_{a_0:A_0}, ..., \varnothing_{a_{i-1}:A}, \{a_i\}, \varnothing_{a_{i+1}:A}, ..., \varnothing_{a_n:A}\rangle$ is the term

$$\langle\langle ... \langle \varnothing_{a_0:A_0}, \varnothing_{a_1:A_1}\rangle, \varnothing_{a_2:A_2}\rangle, ...\rangle, \varnothing_{a_{i-1}:A_{i-1}}\rangle, \{a_i\}\rangle, \varnothing_{a_{i+1}:A_{i+1}}\rangle, ...\rangle, \varnothing_{a_n:A_n}\rangle$$

which will be denoted $\iota_{A_i}(a_i)$.

Lemma 15. *The interpretation of $\coprod_{i=0}^{n} A_i$ is a morphism $\mathbb{1} \to P(\prod_{i=0}^{n} \Omega^{A_i})$ which we identify with its transpose $\prod_{i=0}^{n} \Omega^{A_i} \to \Omega$. Let $j : C \rightarrowtail \prod_{i=0}^{n} \Omega^{A_i}$ be the subobject classified by this morphism. For each i, there exists a unique morphism $A_i \to C$ such that the triangle*

$$\begin{tikzcd} A_i \arrow[r] \arrow[dr, "{[\![a:A \mid \iota_{A_i}(a)]\!]}"'] & C \arrow[d, tail] \\ & \prod_{i=0}^{n} \Omega^{A_i} \end{tikzcd}$$

commutes. We denote these morphisms $A_i \to C$ also by $[\![a_i : A \mid \iota_{A_i}(a_i)]\!]$.

Theorem 5. *$(C, \{[\![\iota_{A_i}(a_i)]\!]\}_{i=0}^{n})$ is the coproduct of $\{A_i\}_{i=0}^{n}$.*

We give the proof in the special case of binary coproducts, as the general case is similar.

Definition 20. *Define the following term,*

$$A \coprod B := \big\{z : \Omega^A \times \Omega^B \mid \big(\exists a : A, z = \langle \{a\}, \varnothing_{b:B}\rangle\big) \vee \big(\exists b : B, z = \langle \varnothing_{a:A}, \{b\}\rangle\big)\big\}$$

Also define

$$\iota_A(a) := \langle \{a\}, \varnothing_{b:B}\rangle \qquad \text{and} \qquad \iota_B := \langle \varnothing_{a:A}, \{b\}\rangle$$

Theorem 6. *The interpretation of $A \coprod B$ is a morphism $\mathbb{1} \to P(\Omega^A \times \Omega^B)$ which we identify with its transpose $\Omega^A \times \Omega^B \to \Omega$. Let $j : C \rightarrowtail \Omega^A \times \Omega^B$ be the subobject classified by this morphism. Then there exists a unique morphism $A \to C$ such that the triangle*

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "{[\![a:A \mid \iota_A(a)]\!]}"'] & C \arrow[d, tail] \\ & \Omega^A \times \Omega^B \end{tikzcd}$$

commutes, and similarly for $B \to C$. We denote these morphisms respectively by $[\![a : A \mid \iota_A(a)]\!]$ and $[\![b : B \mid \iota_B(b)]\!]$.

Proof. The proofs for A and B are almost identical, so only the case corresponding to A is given. Let p be the formula such that $A \coprod B = \{z : \Omega^A \times \Omega^B \mid p\}$. By unwinding Definitions, $[\![z : \Omega^A \times \Omega^B \mid p]\!]$ is equal to the morphism j. Thus, by Lemma 8, it suffices to show that $\vdash_{a:A} \iota_A(a) \in A \coprod B$. By the Definition of $\iota_A(a)$ and Lemma 5, it suffices to show $p[z := \langle \{a\}, \varnothing_{b:B}\rangle]$. In turn, it suffices to show $\vdash_{a:A} \exists a' : A, \langle \{a\}, \varnothing_{b:B}\rangle = \langle \{a'\}, \varnothing_{b:B}\rangle$, which since $\vdash_{a:A} \langle \{a\}, \varnothing_{b:B}\rangle = \langle \{a\}, \varnothing_{b:B}\rangle$, follows from Lemma 4. $\square$

Theorem 7. *$(C, [\![a : A \mid \iota_A(a)]\!], [\![b : B \mid \iota_B(b)]\!])$ is the coproduct of A and B.*

Here is a sketch of the proof: Let $h_1 : A \to U$ and $h_2 : B \to U$ be arbitrary. There seems to be no easy way of describing morphisms out of subobjects in any direct way using the Mitchell-Benabou language. So instead, Theorem 12 will be used to define a morphism $f : \Omega^A \times \Omega^B \to (\Omega^A \times \Omega^B) \times \Omega^U$ which will be precomposed with the monic j and post composed with the epic $\pi_R : (\Omega^A \times \Omega^B) \times \Omega^U$ to yield a morphism $C \to \Omega^U$. Lemma 8 will then be used to show that this induces a morphism $f' : C \to \mathrm{Single}(U)$, where $\mathrm{Single}(U)$ is as defined in Definition 17. The universal map will then be $g^{-1}f'$, where g is as given in Lemma 14. The facts that $g^{-1}f'[\![a : A \mid \iota_A(a)]\!] = h_1$ and $g^{-1}f'[\![b : B \mid \iota_B(b)]\!] = h_2$ will follow easily from the Lemmas used in the construction of $g^{-1}f'$. This can all be summarised in the following commuting diagram,

$$\begin{tikzcd}
A \arrow[rr, "h_1"] \arrow[d, "{[\![a:A|\iota_A(a)]\!]}"'] & & U \arrow[dl, "g"'] \arrow[dd, "{[\![u|\{u\}]\!]}"] \\
C \arrow[r, "f'"] \arrow[d, tail, "j"'] & \mathrm{Single}(U) \arrow[dr, tail, "l"] & \\
\Omega^A \times \Omega^B \arrow[r, "f"'] & (\Omega^A \times \Omega^B) \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U
\end{tikzcd}$$

along with a similar one corresponding to B. That $g^{-1}f'$ is the unique such map will require a bit more work.

Proof. It will first be shown that the hypotheses of Theorem 12 are satisfied, in the notation used there, let $Z = \Omega^A \times \Omega^B$, and let p be such that $A \coprod B = \{z : \Omega^A \times \Omega^B \mid p\}$. The terms $t(a)$ and $s(b)$ will respectively be $\iota_A(a)$ and $\iota_B(b)$. It was shown in Theorem 6 that $\vdash_a \iota_A(a) \in A \coprod B \wedge \iota_B(b) \in A \coprod B$, so it needs to be shown that $\iota_A(a) = \iota_A(a') \vdash_{a:A,a':A} h_1 a = h_1 a'$ and similarly for B, these arguments are essentially the same, and so only the first one will be shown. It suffices to show $\langle \{a\}, \varnothing_{b:B}\rangle = \langle \{a'\}, \varnothing_{b:B}\rangle \vdash_{a,a'} a = a'$, for which it suffices to show that $\{a\} = \{a'\} \vdash_{a,a'} a = a'$, which follows from the facts that $\{a\} = \{a'\} \vdash_{a,a'} a \in \{a'\}$ and $a \in \{a'\} \vdash_{a,a'} a = a'$. Thus there exists the desired morphism f.

Next, the existence of f' is to be validated. By Lemma 8, it suffices to show $z \in A \coprod B \vdash_{z:\Omega^A \times \Omega^B} \mathrm{snd}(fz) \in \mathrm{Single}(U)$. This ammounts to showing

$$\exists a : A, z = \iota_A(a) \vdash_{z:\Omega^A \times \Omega^B} \exists u : U, \mathrm{snd}(fz) = \{u\}$$

and

$$\exists b : B, z = \iota_B(b) \vdash_{b:B,z:\Omega^A \times \Omega^B} \exists u : U, \mathrm{snd}(fz) = \{u\}$$

for which, by Lemma 4, it suffices to show

$$z = \iota_A(a) \vdash_{a:A,z:\Omega^A \times \Omega^B} \mathrm{snd}(f\iota_A(a)) = \{h_1 a\}$$

and

$$z = \iota_B(b) \vdash_{b:B,z:\Omega^A \times \Omega^B} \mathrm{snd}(f\iota_B(b)) = \{h_2 b\}$$

for which, by Lemma 2, reduces to showing

$$\vdash_{a:A} \mathrm{snd}(f\iota_A(a)) = \{h_1 a\} \qquad \text{and} \qquad \vdash_{b:B} \mathrm{snd}(f\iota_B(b)) = \{h_2 b\}$$

By the construction of f, these are also easily verified, as

$$\pi_R f j[\![a : A \mid \iota_A(a)]\!] = [\![u \mid \{u\}]\!]h_1$$

which amounts to saying $\vdash_{a:A} \mathrm{snd}(f\iota_A(a)) = \{h_1 a\}$, and similarly for B.

To show that

$$g^{-1}f'[\![a : A \mid \iota_A(a)]\!] = h_1 \text{ and } g^{-1}f'[\![b : B \mid \iota_B(b)]\!] = h_2$$

it suffices to show that

$$[\![u : U \mid \{u\}]\!]g^{-1}f'[\![a : A \mid \iota_A(a)]\!] = [\![u : U \mid \{u\}]\!]h_1$$

and

$$[\![u : U \mid \{u\}]\!]g^{-1}f'[\![b : B \mid \iota_B(b)]\!] = [\![u : U \mid \{u\}]\!]h_2$$

as by Lemma 13, the morphism $[\![u : U \mid \{u\}]\!]$ is monic. Indeed both of these equations follow from commutativity of the morphisms involved in the construction of $g^{-1}f'$.

To show uniqueness, it is required to recall the construction of the morphism f as given in the proofs of Lemmas 10 and 12. It is recommended that these proofs are read and understood before this one. Say $l_1, l_2 : C \to U$ where morphisms such that $l_1[\![a : A \mid \iota_A(a)]\!] = l_2[\![a : A \mid \iota_A(a)]\!] = h_1$ and $l_1[\![b : B \mid \iota_B(b)]\!] = l_2[\![b : B \mid \iota_B(b)]\!] = h_2$. Then, since $[\![u : U \mid \{u\}]\!]$ is monic, it suffices to show that $[\![u : U \mid \{u\}]\!]l_1 = [\![u : U \mid \{u\}]\!]l_2$. In accordance with the notation of Lemmas 12 and 10, f is the composite

$$\Omega^A \times \Omega^B \xrightarrow{f_1^{q(y)}} [\![y : (\Omega^A \times \Omega^B) \times \Omega^U \mid q(y)]\!] \xrightarrow{f_2^{q(y)}} (\Omega^A \times \Omega^B) \times \Omega^U$$

where

$$\begin{aligned} q(y) = {} & (\exists a : A, y = \langle \iota_A(a), \{h_1 a\}\rangle) \\ & \vee (\exists b : B, y = \langle \iota_B(b), \{h_2 a\}\rangle) \\ & \vee (\neg(\exists a : A, \mathrm{fst}(y) = \iota_A(a)) \wedge \neg(\exists b : B, \mathrm{fst}(y) = \iota_B(b)) \wedge \mathrm{snd}(y) = \varnothing_{u:U}) \end{aligned}$$

The aim is to show that for $i = 1, 2$, the diagram

$$\begin{tikzcd} C \arrow[rrr, "l_i"] \arrow[d, "j"'] & & & U \arrow[d, "{[\![u:U\mid\{u\}]\!]}"'] \\ \Omega^A \times \Omega^B \arrow[r, "f_1^{q(y)}"'] & {[\![y : (\Omega^A \times \Omega^B) \times \Omega^U \mid q(y)]\!]} \arrow[r, tail, "f_2^{q(y)}"'] & (\Omega^A \times \Omega^B) \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U \end{tikzcd}$$

commutes. For each i, the same diagram but with $f_1^{q(y)}, f_2^{q(y)}, q(y)$ replaced respectively by $f_1^{q_i(y)}, f_2^{q_i(y)}, q_i(y)$ commutes, where

$$\begin{aligned} q_i(y) = {} & (\exists a : A, y = \langle j\iota_A(a), \{l_i\iota_A(a)\}\rangle) \\ & \vee (\exists b : B, y = \langle j\iota_B(b), \{l_i\iota_B(b)\}\rangle) \\ & \vee (\neg(\exists a : A, \mathrm{fst}(y) = \langle \{a\}, \varnothing_{b:B}\rangle) \wedge \neg(\exists b : B, \mathrm{fst}(y) = \langle \varnothing_{a:A}, \{b\}\rangle) \wedge \mathrm{snd}(y) = \varnothing) \end{aligned}$$

The final observation to make is that due to the hypotheses on l_1 and l_2, the terms $q(y)$, $q_1(y)$, and $q_2(y)$ are all equal. $\square$

4.3 Coequalisers

In the topos Sets the coequaliser of functions $f, g : A \to B$ is given by ${}^{B}\!/\!_{\sim}$, where $\sim$ is the smallest equivalence relation on B such that $f(a) \sim g(a)$, for all $a \in A$. We emulate this by taking the subobject of Ω^B consisting only of those the "equivalence classes" of $\sim$, where in turn, $\sim$ is emulated by a the subobject of $B \times B$ consisting of elements $\langle b_0, b_1\rangle$ such that either $b_0 = b_1$, or b_0 is related to b_1 by a finite number of applications of transitivity and symmetry.

Definition 21. *Given morphisms $g_0, g_1 : A \to B$ in $\mathscr{E}$, let R_{g_0,g_1} be the term*

$$\begin{aligned}\Big\{z : B \times B \;|\exists b_0, b_1 : B,\ z = \langle b_0, b_1\rangle \wedge \Big(b_0 = b_1 \\ &\vee \bigvee_{m=1}^{n} \bigvee_{\alpha \in \mathscr{A}_n} \big(\exists a_0, ..., a_{n-1}, (b_0 = g_{\alpha_0} a_0) \\ &\wedge (g_{\alpha_0+1} a_0 = g_{\alpha_1} a_1) \wedge ... \wedge (g_{\alpha_{n-2}+1} a_{n-2} = g_{\alpha_{n-1}} a_{n-1}) \\ &\wedge (g_{\alpha_{n-1}+1} a_{n-1} = b_1)\big)\Big)\Big\}\end{aligned}$$

where $\mathbb{Z}_2^n$ is the set of length n sequences of elements of $\mathbb{Z}_2$.

The idea of this term is that it is a generalisation of the smallest equivalence relation $\sim$ such that for all a, $g_0(a) \sim g_1(a)$. It does this by declaring $\langle b_0, b_1\rangle$ to be in the relation if either $b_0 = b_1$, or there exists a sequence $(a_0, ..., a_{n-1})$ which "connect" b_0 and b_1 by the images of these a_i under g_0 and g_1. A diagram representing an example of this is the following,

$$\begin{tikzcd}
 & a_0 \arrow[dl] \arrow[dr] & & a_1 \arrow[dl] \arrow[dr] & & a_2 \arrow[dl] \arrow[dr] & \\
b_0 = g_0(a_0) & & g_1(a_0) = g_0(a_1) & & g_1(a_1) = g_0(a_2) & & g_1(a_2) = b_2
\end{tikzcd}$$

However, it must also be allowed for that g_0 and g_1 do not always appear in this order, due to the symmetry axiom of an equivalence relation, this is why the set $\mathscr{A}_n$ is considered. The next Definition generalises "the set of elements related to b under R_{g_0,g_1}", and "the set of equivalence classes of R_{g_0,g_1}".

Definition 22. *In the setting of Definition 21, for any variable $b : B$, define the term*

$$[b]_{g_0,g_1} := \{b' : B \mid \langle b, b'\rangle \in R_{g_0,g_1}\}$$

and finally,

$$\text{Coequaliser}(g_0, g_1) := \{z : PB \mid \exists b : B, z = [b]_{g_0,g_1}\}$$

The subscripts g_0, g_1 on R_{g_0,g_1} and $[b]_{g_0,g_1}$ will now be dropped to avoid clutter. The interpretation of Coequaliser(g_0, g_1) is a morphism

$$[\![\text{Coequaliser}(g_0, g_1)]\!] : \mathbb{1} \to \Omega^{\Omega^B}$$

which corresponds under adjunction to a morphism $\Omega^B \to \Omega$ we also denote $[\![\text{Coequaliser}(g_0, g_1)]\!]$. Let $j_R : C \to \Omega^B$ be the subobject classified by this morphism.

Theorem 8. *There exists a unique morphism $B \to C$ such that the triangle*

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "{[\![b:B|[b]]\!]}"'] & C \arrow[d, tail, "j_R"] \\ & \Omega^B \end{tikzcd}$$

commutes.

Proof. By unwinding Definitions, $[\![z : \Omega^B \mid \exists b : B, z = [b]]\!]$ is equal to the morphism j. Thus, by Lemma 8, it suffices to show that $\vdash_{b:B} [b] \in \text{Coequaliser}(g_0, g_1)$, ie, that $\vdash_{b:B} \exists b' : B, [b] = [b']$, which is an easy application of Lemma 4. $\square$

From now on, $[\![b : B \mid [b]]\!]$ will always refer to this induced morphism $B \to C$.

Theorem 9. $(C, [\![b : B \mid [b]]\!])$ *is the coequaliser of g_0 and g_1.*

Proof sketch: Following the proof of Theorem 7, given arbitrary $h : B \to U$ such that $hg_0 = hg_1$ Theorem 11 will be used to define a morphism $f : \Omega^B \to \Omega^B \times \Omega^U$ which will be precomposed with the monic j and post composed with the epic $\pi_R : \Omega^B \to \Omega^U$ to yield a morphism $C \to \Omega^U$. Then, Lemma 8 will be used to show there exists a morphism $f' : C \to \text{Single}(U)$, where, as in the proof of Theorem 7, Single(U) is the domain of the "subobject of singleton subsets of U", ie, $k : [\![z : \Omega^U \mid \exists u : U, z = \{u\}]\!] \rightarrowtail \Omega^U$. Next, by Lemma 14, there exists an isomorphism $g : U \to \text{Single}(U)$. The claimed universal map is then $g^{-1}f'$. That $g^{-1}f'[\![b : B \mid [b]]\!] = h$ will follow easily from the Lemmas used in the construction of $g^{-1}f'$. This is all summarised by the following commuting diagram

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[d, "{[\![b:B|[b]]\!]}"'] & & U \arrow[dl, "g"'] \arrow[dd, "{[\![u|\{u\}]\!]}"] \\ C \arrow[r, "f'"] \arrow[d, tail, "j"'] & \text{Single}(U) \arrow[dr, "k"] & \\ \Omega^B \arrow[r, "f"'] & \Omega^B \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U \end{tikzcd}$$

Proof. It will now be shown that the hypotheses of Theorem 11 are satisfied. In the notation used there, let $Z = \Omega^B$, and let $t(b) = [b]$. Clearly, $\vdash_{b:B} [b] \in \text{coeq}(g_0, g_1)$, so it remains to show that $t(b) = t(b') \vdash_{b,b':B} hb = hb'$. This can be done by first noticing that

$$[b] = [b'] \vdash_{b,b'} b' \in [b]$$

and then showing

$$b' \in [b] \vdash_{b,b'} hb = hb'$$

This in turn amounts to showing that any of the infinite amount of terms which appear in the infinite disjunction in the Definition of R entail $hb = hb'$. This can be done using a general argument, although notation becomes burdensome. Instead, an example which highlights the general idea will be presented. To show

$$\exists a_0...a_{n-1} : A, b = g_0a_0 \wedge g_1a_0 = g_0a_1 \wedge ... \wedge g_1a_{n-2} = g_0a_{n-1} \wedge g_1a_n = b \vdash_{b,b'} hb = hb'$$

It follows immediately from Lemma 2 that if $1 \leq i \leq n-2$, then $g_1a_i = g_0a_{i+1} \vdash_{a_i:A} hg_1a_i = hg_0a_{i+1}$. Similarly, $b = g_0a_0 \vdash_{a_0:A,b:B} hb = hg_0a_0$ and $g_1a_n = b \vdash_{a_n:A,b':B} hg_1a_n = hb$. Thus, by multiple applications of Lemma 1 and 3, it follows that notice first that

$$\begin{aligned}&\exists a_0...a_{n-1} : A, b = g_0a_0 \wedge g_1a_0 = g_0a_1 \wedge ... \wedge g_1a_{n-2} = g_0a_{n-1} \wedge g_1a_n = b\\ &\vdash_{b,b':B} \exists a_0...a_{n-1} : A, hb = hg_0a_0 \wedge hg_1a_0 = hg_0a_1 \wedge ... \wedge hg_1a_{n-2} = hg_0a_{n-1}\\ &\wedge hg_1a_n = hb'\end{aligned}$$

Since h as a morphism is assumed to be such that $hg_0 = hg_1$, it then follows that

$$\begin{aligned}&\exists a_0...a_{n-1} : A, hb = hg_0a_0 \wedge hg_1a_0 = hg_0a_1 \wedge ... \wedge hg_1a_{n-2} = hg_0a_{n-1} \wedge hg_1a_n = hb'\\ &\vdash_{b,b':B} \exists a_0...a_{n-1} : A, hb = hg_0a_0 \wedge hg_0a_0 = hg_0a_1 \wedge ... \wedge hg_0a_{n-2} = hg_0a_{n-1}\\ &\wedge hg_0a_n = hb'\end{aligned}$$

(where the difference between the two sides of the sequent is that in the second, all g_i have been set to g_0). It then follows from multiple applications of Lemma 2, that

$$\begin{aligned}&\exists a_0...a_{n-1} : A, hb = hg_0a_0 \wedge hg_0a_0 = hg_0a_1 \wedge ... \wedge hg_0a_{n-2} = hg_0a_{n-1} \wedge hg_0a_n = hb'\\ &\vdash_{b,b':B} \exists a_0...a_{n-1} : A, hb = hb'\end{aligned}$$

It then follows from n applications of Lemma 1 and axiom $3.b.vii$ of Definition 12 that

$$\exists a_0...a_{n-1} : A, hb = hb' \vdash_{b,b':B} hb = hb'$$

Thus the morphism f exists.

To show that the morphism f' exists, it suffices by Lemma 8 to show that $z \in \text{Coequaliser}(g_0, g_1) \vdash_{z:\Omega^B} \text{snd}(fz) \in \text{Single}(U)$, ie, that

$$\exists b : B, z = [b] \vdash_{z:\Omega^B} \exists u : U, \text{snd}(fz) = \{u\}$$

Similarly to the proof of Theorem 7, to show this it suffices to show by Lemma 4 that

$$z = [b] \vdash_{z:\Omega^B, b:B} \mathrm{snd}(f[b]) = \{[b]\}$$

which by Lemma 2, reduces to showing

$$\vdash_{b:B} \mathrm{snd}(f[b]) = \{[b]\}$$

Indeed this holds as this sequent is equivalent to $\pi_R f j [\![b : B \mid [b]]\!] = [\![u : U \mid \{u\}]\!] h$ which holds by the construction of f.

Lastly, to show that

$$g^{-1} f' [\![b : B \mid [b]]\!] = h$$

it suffices to show that

$$[\![u \mid \{u\}]\!] g^{-1} f' [\![b : B \mid [b]]\!] = [\![u : U \mid \{u\} h]\!] l_2$$

as $[\![u : U \mid \{u\}]\!]$ is monic, by Lemma 13. Indeed this equation holds by commutativity of the morphisms inolved in the construction of $g^{-1} f'$.

To show that this is the unique such map, the construction of the morphism f as given in the proofs of Lemmas 10 and 11 will be used, similarly to the uniqueness part of Theorem 7, again, it is recommended that the proofs of these are read and understood before this one.

Say $l_1, l_2 : C \to U$ are morphisms such that $l_1 [\![b : B \mid [b]]\!] = l_2 [\![b : B \mid [b]]\!]$. Then since $[\![u : U \mid \{u\}]\!]$ is monic, it suffices to show that

$$[\![u : U \mid \{u\}]\!] l_1 = [\![u : U \mid \{u\}]\!]$$

In accordance with the notation of Lemmas 10 and 11, f is the composite

$$\Omega^B \xrightarrow{f_1^{q(y)}} [\![y : \Omega^B \times \Omega^U \mid q(y)]\!] \xrightarrow{f_2^{q(y)}} \Omega^B \times \Omega^U$$

where

$$\begin{aligned} q(y) = &(\exists b : B, y = \langle [b], \{hb\} \rangle) \\ &\vee \big(\neg(\exists b : B, \mathrm{fst}(y) = t(b)) \wedge \mathrm{snd}(y) = \varnothing\big) \end{aligned}$$

The proof will be done once it has been shown that for $i = 1, 2$ the diagram

$$\begin{tikzcd} C \arrow[rrr, "l_i"] \arrow[d, "j"'] & & & U \arrow[d, tail, "{[\![u:U \mid \{u\}]\!]}"'] \\ \Omega^B \arrow[r, "f_1^{q(y)}"'] & {[\![y : \Omega^B \times \Omega^U \mid q(y)]\!]} \arrow[r, tail, "f_2^{q(y)}"'] & \Omega^B \times \Omega^U \arrow[r, "\pi_R"'] & \Omega^U \end{tikzcd}$$

commutes. For each i, the same diagram but with $f_1^{q(y)}, f_2^{q(y)}, q(y)$ replaces respectively by $f_1^{q_i(y)}, f_2^{q_i(y)}, q_i(y)$ commutes, where

$$\begin{aligned} q_i(y) = {} & (\exists b : B, y = \langle [b], \{l_i[b]\}\rangle) \\ & \vee \big(\neg(\exists b : B, \mathrm{fst}(y) = t(b)) \wedge \mathrm{snd}(y) = \varnothing\big) \end{aligned}$$

The final observation to make is that due to the hypotheses on l_1 and l_2, the terms $q(y), q_1(y)$, and $q_2(y)$ are all equal. $\square$

5 The Map From Simplicial Sets to Algorithms

The aim of this Section is to outline the general method of realising a simplicial set X as an algorithm, ie, describe X using the Mitchell-Benabou language. One of the key Theorems behind this description is that the category of simplicial sets is the classifying topos for the theory of linear orders. The full weight of this Theorem will not be required, but some of the preliminary Definitions and Lemmas will be.

Definition 23. *A* ***linear order*** *is a set I along with two distinguished elements $b, t \in I$ and a binary relation $\leq$, such that*

- *for all $x \in I$, $x \leq x$,*
- *for all $x, y, z \in I$, if $x \leq y$ and $y \leq z$, then $x \leq z$,*
- *for all $x, y \in I$, if $x \leq y$ and $y \leq x$, then $x = y$,*
- *for all x, $b \leq x$ and $x \leq t$,*
- *$b \neq t$,*
- *for all $x, y \in I$, either $x \leq y$ or $y \leq x$.*

This in fact is the particular case of a more general notion of a linear order in an arbitrary topos $\mathscr{E}$, where $\mathscr{E} = \mathrm{set}$.

Definition 24. *Given a topos $\mathscr{E}$, a* ***linear order element*** *is a tuple (I, r, b, t) consisting of an object I, a monomorphism $r : R \rightarrowtail I \times I$, and two morphisms $b, t : \mathbb{1} \to I$, such that*

- *there exists a morphism $I \to R$ such that the following diagram commutes,*

$$\begin{tikzcd} I \arrow[r] \arrow[dr, "\Delta"'] & R \arrow[d, tail, "r"] \\ & I \times I \end{tikzcd}$$

  where $\Delta : I \to I \times I$ is the diagonal map,

- *there is a morphism* $(I \times R) \times_{I^3} (R \times I) \to R$ *such that the following diagram commutes,*

$$\begin{tikzcd} (I \times R) \times_{I^3} (R \times I) \arrow[r] \arrow[d] & I^3 \arrow[d, "\langle \pi_1, \pi_3 \rangle"] \\ R \arrow[r, tail, "r"] & I^2 \end{tikzcd}$$

where $(I \times R) \times_{I^3} (R \times I)$ *be such that the following is a pullback diagram*

$$\begin{tikzcd} (I \times R) \times_{I^3} (R \times I) \arrow[r] \arrow[d] & R \times I \arrow[d] \\ I \times R \arrow[r] & I^3 \end{tikzcd}$$

- *there is a monomorphism* $R \times_{I^2} R \to I$ *such that the following diagram commutes,*

$$\begin{tikzcd} R \times_{I^2} R \arrow[r] \arrow[dr] & I \arrow[d, "\Delta"] \\ & I^2 \end{tikzcd}$$

where $R \times_{I^2} R$ *is such that the following is a pullback diagram*

$$\begin{tikzcd} R \times_{I^2} R \arrow[r] \arrow[dd] & R \arrow[d, tail, "r"] \\ & I^2 \arrow[d, "\tau"] \\ R \arrow[r, tail, "r"'] & I^2 \end{tikzcd}$$

where $\tau : I^2 \to I^2$ *is the "twist map",* $\langle \pi_2, \pi_1 \rangle$*,*

- *there are morphisms* $\mathbb{1} \times I \to R$ *and* $I \times \mathbb{1} \times I \to R$ *such that the diagrams*

$$\begin{tikzcd} \mathbb{1} \times I \arrow[r] \arrow[dr, "b \times id_I"'] & R \arrow[d, tail, "r"] \\ & I \times I \end{tikzcd} \qquad \begin{tikzcd} I \times \mathbb{1} \arrow[r] \arrow[dr, "id_I \times t"'] & R \arrow[d, tail, "r"] \\ & I \times I \end{tikzcd}$$

both commute,

- *the following is a pullback diagram,*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "t"] \\ \mathbb{1} \arrow[r, "b"'] & I \end{tikzcd}$$

where 0 *is the initial object of* $\mathscr{E}$*,*

- *the morphisms r and τr, where τ is the "twist map", form an epimorphic family.*

See [?, §VII 8] for a discussion on how Definition 24 generalises Definition 23.

Definition 25. *Given two linear order objects of a topos $\mathscr{E}$, (I, r, b, t) and (I', r', b', t'), a pair of morphisms $(f : I \to I', g : R \to R')$ of $\mathscr{E}$ is a morphism between such orders if the following diagrams commute,*

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, tail, "r"'] & R' \arrow[d, tail, "r'"] \\ I \times I \arrow[r, "f \times f"'] & I' \times I' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "b"] \arrow[dr, "b'"'] & I \arrow[d, "f"] & \mathbb{1} \arrow[l, "t"'] \arrow[dl, "t'"] \\ & I' & \end{tikzcd}$$

This defines the category Orders($\mathscr{E}$) of linear orders in the topos $\mathscr{E}$.

Similarly to how classifying *spaces* in cohomology theory are used in Topology, there is a notion of a classifying *topoi* which exist in a more general setting. Here, the classifying topos for the theory of linear orders will be considered, the idea of which is to be a generalised universe of mathematics which contains the universal interval, see [?, §VIII].

Definition 26. *Given topoi $\mathscr{E}$ and $\mathscr{F}$, a* ***geometric morphism*** *$\mathscr{E} \to \mathscr{F}$ is a pair of adjoint functors $F^* : \mathscr{F} \to \mathscr{E}$ and $F_* : \mathscr{E} \to \mathscr{F}$ such that F^* is left adjoint to F_*, and F^* preserves all finite limits. F^* is the* ***inverse image*** *functor, and F_* the* ***direct image****. Given geometric morphisms (F^*, F_*) and (G^*, G_*), a morphism $(F^*, F_*) \to (G^*, G_*)$ is a natural transformation $\eta : F^* \to G^*$. This defines a category Geo($\mathscr{E}, \mathscr{F}$).*

Definition 27. *A* ***classifying topos for the theory of linear orders*** *is a topos $\mathscr{E}$ together with a family of bijections such that for every cocomplete topos $\mathscr{F}$, there is an equivalence of categories*

$$\mathit{Orders}(\mathscr{F}) \cong \mathit{Geo}(\mathscr{F}, \mathscr{E})$$

which is natural in $\mathscr{F}$.

It was first proved by Joyal that the category of *simplicial sets* is a classifying topos for the theory of linear orders.

Definition 28. *The* ***simplex category*** *Δ is the category whose objects are sets of the form $\{0, 1, ..., n\}$ for some n, these will be denoted $[n]$. The morphisms of this category are order preserving functions. For any positive integer k, let $\Delta_{\leq k}$ be the full subcategory of Δ with objects $\{[0], ..., [k]\}$.*

There is a canonical way of factorising morphisms in the simplex category:

Definition 29. *Define*

$$\begin{aligned}\epsilon_n^i : [n-1] &\to [n] \\ j &\mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}\end{aligned}$$

and

$$\begin{aligned}\eta_n^i : [n+1] &\to [n] \\ j &\mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}\end{aligned}$$

Theorem 10. *Any morphism $[n] \to [m]$ in Δ can be written uniquely as*

$$\epsilon_m^{i_1}\epsilon_{m-1}^{i_2}...\epsilon_{m-k+1}^{i_l}\eta_{m-k}^{j_1}\eta_{m-k+1}^{j_2}...\eta_{m-1}^{j_{k-1}}\eta_m^{j_k}$$

with $m \geq i_1 \geq i_2 \geq ... \geq i_l \geq 0$, and $0 \leq j_1 \leq j_2 \leq ... \leq j_k \leq n$.

Proof. See [**?**, §VIII.5.1]. □

Definition 30. *A* ***simplicial set*** *is a functor $\Delta^{op} \to \underline{Set}$, where $\underline{Set}$ is the category of sets. The collection of these, along with the collection of natural transformations between them, forms a category $\underline{sSet}$, the category of simplicial sets.*

Theorem 11. *$\underline{sSet}$ is a classifying topos for the theory of linear orders.*

Proof. See [**?**, §VIII 8] □

Definition 31. *Since $\underline{sSet}$ is the classifying topos for the theory of linear orders, it admits a universal linear order object given by the image of $id_{\underline{sSet}}$ under the correspondence $Geo(\underline{sSet}, \underline{sSet}) \cong Orders(\underline{sSet})$. Let $(I, r : R \rightarrowtail I \times I, b, t)$ denote this order.*

5.1 The General Method

In this section we explain the general method for taking a finite simplicial set X and translating it into a term in the Mitchell-Benabou language. A simplicial set X is called *finite* if it has only finitely many nondegenerate simplices, which means that there is a canonical diagram $J : \mathscr{J} \to \Delta$ with $\mathscr{J}$ finite such that X is the colimit of the diagram

$$\mathscr{J} \to \Delta \to \underline{\text{sSet}}$$

where the second functor is the Yoneda embedding. Since it was already seen, roughly speaking, the process which will be taken to realise a simplicial set as an algorithm will be to begin with a finite simplicial set X and then write it as a finite colimit of an indexing category $\mathscr{J}$ through a functor $J : \mathscr{J} \to \underline{\text{sSet}}$. This can then be written as a coequaliser diagram in a canonical way, which only uses objects and morphism in the image of Δ under the Yoneda embedding $\mathbf{y} : \Delta \rightarrowtail \underline{\text{sSet}}$. Since it was already seen in Section 4 how to write finite coproducts and coequalisers using the Mitchell-Benabou language, it will then only require a way of describing the subcategory Δ of $\underline{\text{sSet}}$ using this language to then have a full description of an object isomorphic to X in $\underline{\text{sSet}}$ using this type theory.

In order to describe the subcategory Δ of $\underline{\text{sSet}}$ using the Mitchell-Benabou language, it suffices to define a functor $F : \Delta \to \underline{\text{sSet}}$ whose image is described by this language, and is naturally isomorphic to the Yoneda embedding $\mathbf{y}$. A description of such a functor is given in [**?**, §VIII.8.2], and proved to be isomorphic to the Yoneda embedding in [**?**, §VIII.8.6], but this functor is insufficient, as although it describes the image of *objects* using the Mitchell-Benabou language, it does **not** give a term level description of the image of *morphisms*. Motivated by what was done there, we define the following construction.

Definition 32. *Let $F : \Delta \to \underline{sSet}$ be the functor which sends an object $[n]$ to the domain of the morphism*

$$[\![x_1 : I, ..., x_n : I \mid b \leq x_1 \leq ... \leq x_n \leq t]\!]$$

where $b \leq x_1 \leq ... \leq x_n \leq t$ means $(b \leq x_1) \wedge (x_1 \leq x_2) \wedge ... \wedge (x_{n-1} \leq x_n) \wedge (x_n \leq t)$, and $x_i \leq x_{i+1}$ means $R(\langle x_i, x_{i+1}\rangle)$.

By Theorem 10, in order to define F_I on morphisms, it suffices to define it on the morphisms ϵ^i_n and η^i_n. In the following, $\langle x_1, ..., x_n\rangle$ means the term $\langle ...\langle x_1, x_2\rangle, x_3\rangle, ...\rangle, x_n\rangle$. Define

$$F(\epsilon^i_n) = \begin{cases} [\![x_1 : I, ..., x_{n-1} : I \mid \langle b*, x_1, x_2, ..., x_{n-1}\rangle]\!] & i = 0 \\ [\![x_1 : I, ..., x_{n-1} : I \mid \langle x_1, ..., x_i, x_i, ..., x_{n-1}\rangle]\!] & 0 < i < n \\ [\![x_1 : I, ..., x_{n-1} : I \mid \langle x_1, ..., x_{n-1}, t*\rangle]\!] & i = n \end{cases}$$

and

$$F(\eta^i_n) = [\![x_1 : I, ..., x_{n+1} : I \mid \langle x_1, ..., x_{i-1}, \hat{x}_i, x_{i+1}, ..., x_{n+1}\rangle]\!]$$

where the notation $\hat{x}_i$ means to leave x_i omitted.

As already mentioned, given a finite simplicial set X, a finite colimit diagram will be chosen for X, and then this colimit will be written as a coequaliser diagram, which will then allow F to be extended to a functor $\bar{F} : \underline{\text{sSet}} \to \underline{\text{sSet}}$, see Lemma 16 below. The process behind the construction of this functor $\bar{F}$ will then make it clear how to write the image of $\bar{F}$ using the language of the Mitchell-Benabou language.

Lemma 16. *Given a functor $F : \mathscr{C} \to \mathscr{D}$ where $\mathscr{C}$ is a small category, and $\mathscr{D}$ admits all colimits, there exists a unique (up to natural isomorphism) colimit preserving functor $\bar{F} : \underline{Set}^{\mathscr{C}^{op}} \to \mathscr{D}$ such that the diagram*

$$\begin{tikzcd} \mathscr{C} \arrow[d, hook, "y"'] \arrow[dr, "F"] & \\ \underline{Set}^{\mathscr{C}^{op}} \arrow[r, "\bar{F}"'] & \mathscr{D} \end{tikzcd}$$

where $y : \mathscr{C} \hookrightarrow \underline{Set}^{\mathscr{C}^{op}}$ is the Yoneda embedding.

The action of this functor on objects is given by the following. Let $P \in \underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ be a presheaf. Then P is a colimit of representable presheaves. Pick a colimit diagram for each such presheaf. Also choose a coproduct for every collection of objects of $\mathscr{D}$, and also a coequaliser for every pair of suitable morphisms. Let $J : \mathscr{J} \to \mathscr{C}$ be a functor such that the colimit of $\mathscr{J}$ through J is isomorphic to P, and construct the following diagram,

$$\begin{tikzcd} \coprod_{f:j\to j'} Jj \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & \coprod_{j\in\mathscr{J}} Jj \arrow[r] & \mathrm{coeq}(g_0, g_1) \cong P \end{tikzcd}$$

where g_0 is such that for all $f : j \to j'$, the following diagram commutes

$$\begin{tikzcd} \coprod_{f:j\to j'} Jj \arrow[r, "g_0"] & \coprod_{j\in\mathscr{J}} Jj \\ Jj \arrow[u, tail] \arrow[r, "\mathrm{id}_{Jj}"] & Jj \arrow[u, tail] \end{tikzcd}$$

and g_1 is such that for all $j \in \mathscr{J}$, the following diagram commutes

$$\begin{tikzcd} \coprod_{f:j\to j'} Jj \arrow[r, "g_1"] & \coprod_{j\in\mathscr{J}} Jj \\ Jj \arrow[u, tail] \arrow[r, "Jf"] & Jj' \arrow[u, tail] \end{tikzcd}$$

Then the image of P under F_* is defined to be the coequaliser,

$$\begin{tikzcd} \coprod_{f:j\to j'} AJj \arrow[r, shift left, "g_0 A"] \arrow[r, shift right, "g_1 A"'] & \coprod_{j\in\mathscr{J}} AJj \arrow[r] & \mathrm{Coequaliser}(g_0 A, g_1 A) \end{tikzcd}$$

where $g_0 A$ and $g_1 A$ are defined similarly to g_0 and g_1 above.

Since the functor $\bar{F}$ is essentially unique, assuming that the functor F is naturally isomorphic to the Yoneda embedding, this then implies that $\bar{F}$ is naturally isomorphic to the identity functor $\mathrm{id}_{\underline{\mathrm{sSet}}}$. This ensures that the description of the image of a simplicial set X under $\bar{F}$ using the Mitchell-Benabou language is indeed (isomorphic to) X.

6 Example

In this section we give the explicit example of the 2-simplex $\Delta[2]$ which is the functor $\underline{\text{sSet}}(-,[2]) : \Delta^{\text{op}} \to \underline{\text{Set}}$. The canonical diagram alluded to at the beginning of Section 5.1 coming from the nondegenerate simplices is: The explicit example of $\Delta[2] = \underline{\text{sSet}}(\_,[2])$ is now considered. In what follows, the objects and morphisms of Δ will be identified with their image in $\underline{\text{Set}}$ under the Yoneda embedding. Let $\overline{T}$ be the partially ordered set $\{a \leq b \leq c\}$, and then define the following sets,

$$\begin{aligned}
T_0 &= \big\{\{a\},\{b\},\{c\}\big\} \\
T_1 &= \big\{\{a,b\},\{a,c\},\{b,c\}\big\} \\
T_2 &= \big\{\{a,b,c\}\big\} \\
T &= T_0 \cup T_1 \cup T_2
\end{aligned}$$

Then T is a simplicial complex whose geometric realisation is homeomorphic to the triangle. Let S_T be the corresponding simplicial set as given in Definition **??**. To apply the functor $\bar{F}$ to T, first a diagram whose colimit is isomorphic to T is needed. The following is defined with figure 1 in mind.

Figure 1: Colimit diagram of the triangle

Lemma 17. *Define the indexing category $\mathscr{J}$ generated by the following diagram, where all object and morphism names are formal symbols,*

$$\begin{tikzcd}
 & & {[0]_1} \arrow[dl, "{}^{(1,1)}\epsilon_1^0"'] \arrow[dr, "{}^{(1,2)}\epsilon_1^1"] & & \\
 & {[1]_1} \arrow[r, "{}^{(1,1)}\epsilon_2^1"'] & {[2]_1} & {[1]_2} \arrow[l, "{}^{(2,1)}\epsilon_2^2"] & \\
{[0]_2} \arrow[ur, "{}^{(2,1)}\epsilon_1^1"] \arrow[rr, "{}^{(2,3)}\epsilon_1^0"'] & & {[1]_3} \arrow[u, "{}^{(3,1)}\epsilon_2^0"] & & {[0]_3} \arrow[ul, "{}^{(3,2)}\epsilon_1^0"'] \arrow[ll, "{}^{(3,3)}\epsilon_1^1"]
\end{tikzcd}$$

Then let $J : \mathscr{J} \to \underline{sSet}$ be the functor which "removes the label", ie, $J([i]_j) = [i]$, and $J({}^{(i,j)}\eta_k^l) = \eta_k^l$. Then the colimit of $\mathscr{J}$ through J is isomorphic to $\Delta[2]$.

This colimit can be written as a coequaliser in a canonical way:

Lemma 18. *Let g_1 and g_0 be the unique morphisms such that the following diagrams commute,*

$$\begin{tikzcd}
\coprod_{{}^{(i,j)}\epsilon_k^l:[k-1]_i \to [k]_j} J[k-1]_i \arrow[r, "g_1"] & \coprod_{[i]_j \in \mathscr{J}} J[i]_j \\
J[k-1]_i \arrow[u, hook] \arrow[r, "{}^{(i,j)}\epsilon_k^l"] & J[k]_j \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\coprod_{(i,j)\epsilon^l_k:[k-1]_i\to[k]_j} J[k-1]_i \arrow[r, "g_0"] & \coprod_{[i]_j\in\mathscr{J}} J[i]_j \\
J[k-1]_i \arrow[u, hook] \arrow[r, "id"] & J[k-1]_j \arrow[u, hook]
\end{tikzcd}$$

then there is the following coequaliser diagram,

$$\coprod_{(i,j)\epsilon^l_k:[k-1]_i\to[k]_j} J[k-1]_i \underset{g_1}{\overset{g_0}{\rightrightarrows}} \coprod_{[i]_j\in\mathscr{J}} J[i]_j \dashrightarrow \text{Coequaliser}(g_0, g_1)$$

which induces

$$\coprod_{(i,j)\epsilon^l_k:[k-1]_i\to[k]_j} F[k-1] \underset{g_1F}{\overset{g_0F}{\rightrightarrows}} \coprod_{[i]_j\in\mathscr{J}} F[i] \dashrightarrow \text{Coequaliser}(g_0F, g_1F)$$

where g_0F and g_1F are defined similarly to g_0 and g_1, and $\text{Coequaliser}(g_0F, g_1F) \cong S_T$*.*

Now, $\text{Coequaliser}(g_0F, g_1F)$ will be written using the Mitchell-Benabou language.

Example 1. *$\coprod_{[i]_j\in\mathscr{J}} F[i]$ is isomorphic to the object given by the domain of*

$$\begin{aligned}
&[\![z : \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I} \times \Omega^{R} \mid \\
&(z = \langle\{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee (z = \langle\varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee (z = \langle\varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing\rangle) \\
&\vee (\exists y : R, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{y\}\rangle)]\!]
\end{aligned}$$

Similarly, $\coprod_{(i,j)\epsilon^l_k:[k-1]_i\to[k]_j} F[k-1]$ is isomorphic to the object given by the domain of

$$\begin{aligned}
&[\![z : \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I} \mid \\
&z = \langle\{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee \ldots \\
&\vee z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing\rangle) \\
&\vee (\exists x : I, z = \langle\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}\rangle)]\!]
\end{aligned}$$

Then the morphism g_0 is induced by the morphism,

$$
\begin{aligned}
&[\![z : (\Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I}) \\
&\times (\Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I} \times \Omega^{R}) \mid \\
&z = \big\langle \langle \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle\ \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\} \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing \rangle \big\rangle \\
&\vee \ \textit{remaining term}\ ]\!]
\end{aligned}
$$

where "remaining term" is the term consisting of the conjunction of ten terms, the first of which is

$$\neg(\mathit{fst}(z) = \langle \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$$

the next five are similar, but with $\{*\}$ appearing respectively in the 2^{nd} through to 6^{th} position, the next three are similar but with $\{*\}$ replaced with $\{x\}$, and with the position of $\{x\}$ shifted to the 3^{rd} last, 2^{nd} last, and last position respectively. The final term is

$$\mathit{snd}(z) = \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle$$

This final term can be thought of as describing the map which sends the remaining cases of the form fst(z) can take to a special element. Indeed by Lemma 12, this induces a *morphism which is the morphism g_0. In a similar way, the morphism g_1 is induced by*

$$
\begin{aligned}
&[\![z : (\Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I}) \\
&\times (\Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{\mathbb{1}} \times \Omega^{I} \times \Omega^{I} \times \Omega^{I} \times \Omega^{R}) \mid \\
&z = \big\langle \langle \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \{t*\}, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \{b*\}, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \{b*\}, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{t*\}, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \{t*\}, \varnothing, \varnothing, \varnothing \rangle \big\rangle \\
&\vee z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{*\}, \varnothing, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{b*\}, \varnothing \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{\langle x, t* \rangle\} \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\}, \varnothing \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{\langle x, x \rangle\} \rangle \big\rangle \\
&\vee \exists x : I, z = \big\langle \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{x\} \rangle, \langle \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{\langle b*, x \rangle\} \rangle \big\rangle \\
&\vee \ \textit{remaining term} \ ]\!]
\end{aligned}
$$