# Groups, categories, and homological algebra.

MAST90068
Assignment 1

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This note is essentially meant to be an introduction to homological algebra with a with a discussion on relevant embedding theorems.

On this contents page, a description of the motivation and goals of each section is given.

Section 1: basic ideas.

In homological algebra, there are many simple statements which sound true (and indeed are) which are very tempting to use without proof. The philosophy of this section is that one must walk before learning to run, so the elementary statements which will be used for the remainder of this assignment are proved here. The main result of this section will be that the first isomorphism theorem holds in an arbitrary abelian category, which, of course, will first require a suitable notion of the "image" of a morphism.

Section 2: homology.

Here, the definition of homology is suitably generalised. The main result will be that the $i^{th}$-homology functor is in fact a functor.

Section 3: Injective objects, resolutions, and basic properties.

The goal of this section will be to get as far through right derived functor theory before an embedding theorem is demanded for. Main results will include, any map $A \to B$ gives rise to a chain map of any pair of injective resolutions for $A$ & $B$, and moreover, that any pair of such induced maps are chain homotopic.

Section 4, proofs of some theorems in the category of abelian groups.

This section lays down the foundation for section 6. It will be shown here that the category of abelian groups has enough injectives. The snake lemma for abelian groups will also be presented.

Section 5, embedding theorems.

A presentation and discussion of two embedding theorems with proofs of some of the easy parts.

Section 6, proofs for abelian categories using the embedding theorem.

This section extends section 3 a little bit further, now with the extra power of the embedding theorem(s).

## Section 1: basic ideas.

A category $\mathcal{A}$ is said to be <u>abelian</u> if

- Every hom set is an abelian group.
- Composition is a bilinear function.
- $\mathcal{A}$ admits a zero object.
- $\mathcal{A}$ admits all finite products.
- Every morphism has a kernel.
- Every morphism has a cokernel.
- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of it kernel.

Note: some authors (Hartshorne included) also require as an axiom that every morphism can be factored as an epimorphism followed by a monomorphism, however, as will be shown, this requirement is superfluous.

There are a few basic ideas from module-theory which will be helpful throughout this note, in fact, several of them will be used to even define cohomology. So these are established first.

Lemma 1.1, a) A morphism $f: A \to B$ in an abelian category is a monomorphism if and only if for all $g: C \to A$, $fg = 0 \Rightarrow g = 0$.
b) A morphism $f: A \to B$ in an abelian category is an epimorphism if and only if for all $g: B \to C$, $gf = 0 \Rightarrow g = 0$.

Proof: a)
Say $f: B \to C$ is monic. Then since $f0 = 0$, it follows that $fg = 0 \Rightarrow fg = f0 \Rightarrow g = 0$.
Conversely, if $fg = 0 \Rightarrow g = 0$, then say $fg = fg'$. Then $f(g - g') = 0 \Rightarrow g - g' = 0$.
b)
Moreover, if $f: B \to C$ is an epi, then since $0f = 0$, it follows that $gf = 0 \Rightarrow gf = 0f \Rightarrow g = 0$.
Conversely, if $gf = 0 \Rightarrow g = 0$, then say $gf = g'f$. Then $(g - g')f = 0 \Rightarrow g - g' = 0$. $\square$.

Lemma 1.2, a) For every morphism $f$ in an abelian category, $f0 = 0$, and $0f = 0$
b) $\ker(f)$ is a monic, $\mathrm{coker}(f)$ is an epi.

Proof:
a) follows easily from the uniqueness of morphisms into and out of the zero object $0$.
b) Let $g_1, g_2: C \to \ker(f)$, and consider the following diagram,

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] & B \\ C \arrow[u, shift left, "g_2"'] \arrow[u, shift right, "g_1"] & & \end{tikzcd}$$

, PTO

If $ig_1 = ig_2$, then $fig_1 = fig_2 = 0$, so by the universal property of ker(f), there exists a unique map $C \longrightarrow \ker(f)$ such that the diagram

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] & B \\ & \ker(f) \arrow[u, "i"'] & \\ C \arrow[uu, "g_1"] \arrow[ur, "g_2"'] & & \end{tikzcd}$$

commutes. $g_1$ & $g_2$ are both such maps, thus $g_1 = g_2$.
The last claim holds by reversing arrows in the previous argument. □

Lemma 1.3, let $f: A \to B$ be a morphism in an abelian category, then
a) The cokernel of a 0 morphism is an isomorphism.
b) The kernel of a 0 morphism is an isomorphism.
c) If $f$ is monic and epic, then $f$ is an isomorphism.

Proof:
a) Consider the diagram

$$\begin{tikzcd} A \arrow[r, "0_{AB}"] & B \arrow[r, "j"] \arrow[d, "id"'] & \mathrm{coker}(0_{AB}) \arrow[dl, dashed, "h"] \\ & B & \end{tikzcd}$$

, where $h$ exists by the UP of coker$(0_{AB})$, and the fact that $id\, 0_{AB} = 0$, this show that $hj = id_B$. To get that $jh = id_{\mathrm{coker}(0_{AB})}$, notice that there is a unique map $\mathrm{coker}(0_{AB}) \longrightarrow \mathrm{coker}(0_{AB})$ such that the diagram

$$\begin{tikzcd} & & \mathrm{coker}(0_{AB}) \arrow[d, dashed] \\ A \arrow[r, "0_{AB}"] & B \arrow[ur, "j"] \arrow[r, "j"] & \mathrm{coker}(0_{AB}) \end{tikzcd}$$

commutes, and both $jh$ and $id_{\mathrm{coker}(0_{AB})}$ satisfy this. □

b) Similar, reverse arrows.
c) Construct the following diagram,

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\pi"'] & B \arrow[r] & \mathrm{coker}(f) \\ & \mathrm{coker}(i) \arrow[ur, dashed, "\varphi"'] & & \end{tikzcd}$$

Since $f$ is epic, and all epimorphisms are cokernels of their kernels, the isomorphism $\varphi$ exists.

Since $f$ is monic, and $fi = 0$, $\Rightarrow i = 0$. So part a) of this theorem show that $\pi$ is an isomorphism.
Thus $f$ is an isomorphism. □.

Note: throughout, the common abuse of terminology of referring to both the inclusion map and the object as the kernel will be used. Similar for cokernel.

The next step is to define a suitable notion of image in an arbitrary abelian category.

Def$^n$
The *image* of a morphism $f: A \longrightarrow B$ of an abelian category is $\mathrm{coker}(\ker(f))$, and is denoted $\mathrm{im}(f)$.

There are two reasons, at this point, why the image is important. The first is because it is the first out of two pieces which will be used to show that every morphism can be factored as an epi followed by a monic, and the second is that it will provide a key component of the definition of homology. The first of these two points is addressed now.

The key to the proof that morphisms factor into epis followed by monics is the first isomorphism theorem, which, when stated in the language of abelian categories, is the following.

Lemma 1.4, For every morphism $f: A \to B$ in an abelian category $\mathcal{A}$, there is an isomorphism,

$$\mathrm{coker}(\ker(f)) \cong \ker(\mathrm{coker}(f)) \quad \big(= \mathrm{im}(f)\big)$$

The proof of this will require 4 lemmas, fortunately, two of which are dual to the other two, so only the work of two proofs is required.

Lemma 1.5.
Let $f: A \to B$ be a morphism in an abelian category.
Let $p: B \to \mathrm{coker}(f)$ be the cokernel, and $i: \ker(p) \to B$ the kernel of $p$.
Then if $m: C \to B$, $g: A \to C$ are two morphisms such that $f = mg$, with $m$ a monic, then there exists a morphism $u: \ker(p) \to C$ such that $i = mu$.

The statement of this lemma is a bit confusing, but can be seen in a clearer light with the aide of the following diagram. With respect to this diagram, the lemma states "in the situation of black, given blue, there exists green".

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "p"] & \mathrm{coker}(f) \\
C \arrow[ur, "m"] & & \\
\mathrm{im}(f) \arrow[u, dashed, "u"] \arrow[uur, "i"'] & &
\end{tikzcd}$$

PTO for proof.

Proof of lemma 1.5:
Consider the solid lines in the diagram on the right.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[rr, "p"] \arrow[dr, "q"] & & \mathrm{coker}(f) \arrow[d, dashed, "\varphi"] \\ & C \arrow[ur, "m"] & & & \mathrm{Coker}(m) \\ & \mathrm{im}(f) \arrow[uur, "i"'] & & & \end{tikzcd}$$

Then $qf = qmg = 0g = 0$. So by the UP of $\mathrm{coker}(f)$, there exists the dashed arrow $\varphi$.
Then, $\varphi pm = qm = 0$, which implies $\varphi p = 0$, as $m$ is monic. This in turn implies that $qi = 0$. Then, since monics are kernels of cokernels, $m = \ker(q)$, and so by the universal property of kernels, there exists a morphism $\mathrm{im}(f) \xrightarrow{u} C$ as required. □

This argument seems wrong.
Can be corrected though: $\varphi$ exists as claimed.
(does not use UP of $\mathrm{Coker}(m)$)
Then $\varphi p i = 0 \Rightarrow qi = 0$
So since $m$ is a kernel of $q$,
$\exists\, \mathrm{im}(f) \to C$.

Lemma 1.6. Let $f: A \to B$ be a morphism, let $p$ be the cokernel, $i$ the kernel of $p$, and $e$ the map induced by the UP of $\ker(p)$, which fit together into the following diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dashed, "e"'] & B \arrow[r, "p"] & \mathrm{coker}(f) \\ \mathrm{im}(f) \arrow[ur, "i"'] & & \end{tikzcd}$$

Then $e$ is epic.
Proof: This will be done by showing that if $g: \mathrm{im}(f) \longrightarrow C$ is a morphism such that $ge = 0$, then $g = 0$.
So assume this is the case, and consider the kernel of $g$, which fits into the following diagram

This is a cool argument. I should rephrase it in terms of "$k$ admits a section" though.

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "e"] \arrow[ddl, dashed, "\omega"'] & B \arrow[r, "p"] & \mathrm{coker}(f) \\ & \mathrm{im}(f) \arrow[ur, "i"'] \arrow[dr, "g"] \arrow[dl, dashed, "u"] & & \\ \ker(g) \arrow[ur, "k", bend left] & & C & \end{tikzcd}$$

The universal property of $\ker(g)$, along with the fact that $ge = 0$ implies the existence of the dashed arrow $\omega$. Then, since $i$ and $k$ are monic, being kernel arrows, it follows that $ik$ is monic. This fact combined with the fact that $ik\omega = f$, means all hypotheses of lemma 1.5 are satisfied, so there exists the dashed arrow $u$.

Then $iku = i \Rightarrow ku = \mathrm{id}$, as $i$ is monic.
Then $g = gku = 0u = 0$. □

A dual argument to lemma 1.5 and 1.6 give
Lemma 1.7. Let $f: A \to B$ be a morphism, let $i$ be the kernel, $j$ the cokernel of $i$, and $g$ the map induced by the UP of $\mathrm{coker}(i)$, which fit together into the following diagram.

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "j"'] & B \\ & \mathrm{coker}(i) \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Then $g$ is monic. □

As mentioned, this allows for a proof of the first isomorphism theorem.

Proof of lemma 1.4:
Consider the solid lines in the following diagram.

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[dl, "\pi"'] \arrow[d, "\varphi"] & B \arrow[r, "j"] & \operatorname{coker}(f) \\ \operatorname{coker}(i) \arrow[r, dashed, "\psi"'] & \operatorname{im}(f) \arrow[ur, "\iota"'] & & \end{tikzcd}$$

where $\varphi$ exists by the UP of $\ker(j)$, and the fact that $jf=0$, and is epic by lemma 1.6.

$\iota$ is monic, and $\iota\varphi i = fi = 0$, so $\varphi i = 0$, so by the UP of $\operatorname{coker}(i)$, the dashed arrow $\psi$ exists.

$\psi\pi = \varphi$, and $\varphi$ is epic, by lemma 1.6, so since $\pi$ is epic, it follows that $\psi$ is epic.
Furthermore, lemma 1.7 applied to the diagram

$$\begin{tikzcd} \ker(f) \arrow[r] & A \arrow[dl] \arrow[d, "\varphi"] \\ \operatorname{coker}(i) \arrow[r, dashed, "\psi"'] & \operatorname{im}(f) \end{tikzcd}$$

implies that $\psi$ is monic.

Since $\psi$ is monic and epic, it is thus an isomorphism. $\square$

Lemma 1.8, Every morphism can be factored as an epi followed by a monic.
Proof: Let $f: A \longrightarrow B$ be a morphism. Then consider the diagram

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, dashed, "\varphi"] & B \arrow[r, "j"] & \operatorname{coker}(f) \\ & \operatorname{coker}(i) \arrow[r, "\sim", "\psi"'] & \operatorname{im}(f) \arrow[u, "\iota"'] & \end{tikzcd}$$

Where $\pi, \iota, \varphi$, and $\psi$ are all as in the proof of lemma 1.4. $\square$.

This concludes section 1. //

# Section 2, homology.

**Def$^{n}$**

Let $\mathcal{A}$ be an abelian category. A <u>cochain complex</u> in $\mathcal{A}$ consists of a

- Collection of objects $\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}$, $A_i \in \mathcal{A}$, and
- a collection of morphisms $\{d_i : A_i \to A_{i+1}\}_{i \in \mathbb{Z}_{\geq 0}}$

such that

- for all $i$, $d_{i+1} d_i = 0$.

There is an evident notion of a cochain complex morphism.

The next goal is to define the $i^{th}$ cohomology of a cochain complex (also called a cocomplex).

Let $A_\cdot$ be a cocomplex in an abelian category $\mathcal{A}$. Then there is the following diagram.

$$\begin{tikzcd}
 & & & \mathrm{coker}(d_{i-1}) \arrow[dr, dashed, "p"] & & \\
\cdots \arrow[r] & A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, dashed, "\pi"'] & A_i \arrow[ur, "j"] \arrow[rr, "d_i"] & & A_{i+1} \arrow[r] & \cdots \\
 & \mathrm{im}(d_{i-1}) \arrow[ur, "\iota"] \arrow[rr, dashed, "k_i"] & & \mathrm{ker}(d_i) \arrow[ul, "\gamma"'] & &
\end{tikzcd}$$

Where

- $d_i d_{i-1} = 0$, so the arrow $p$ exists by the UP of $\mathrm{Coker}(d_i)$,
- $j\, d_{i-1} = 0$, so the arrow $\pi$ exists by the UP of $\ker(j)$, moreover, by lemma 1.6, $\pi$ is epic.
- $d_i d_{i-1} = 0 \Rightarrow d_i\, \iota\, \pi = 0 \Rightarrow d_i\, \iota = 0$, as $\pi$ is an epi, and so $k_i$ exists by the universal property of $\ker(d_i)$.

Moreover, the arrow $k$ is monic, to see this, assume $kg_1 = kg_2$, and consider the following commuting diagram,

$$\begin{tikzcd}
 & & A_i \arrow[r, "j"] & \mathrm{coker}(d_{i-1}) \\
G \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & \mathrm{im}(d_{i-1}) \arrow[ur, "\iota"] \arrow[r, "k_i"'] & \mathrm{ker}(d_i) \arrow[u, "\gamma"'] &
\end{tikzcd}$$

Then $j\gamma k_i g_1 = j\gamma k_i g_2 = 0 \Rightarrow j\iota\, g_1 = j\gamma g_2 = 0$, by commutativity.
So by the uniqueness of the universal property of $\ker(j)$, $\iota\, g_1 = \iota\, g_2 \Rightarrow g_1 = g_2$, as $\iota$ is monic. $\square$.

So it makes sense to make the following definition,

**Def$^{n}$**

Given a cocomplex $A_\cdot$ in an abelian category $\mathcal{A}$, the <u>$i^{th}$ cohomology object</u>, denoted $H^i(A_\cdot)$ is $\mathrm{Coker}(k_{i-1})$, where $k_{i-1} : \mathrm{im}(d_{i-1}) \to \ker(d_i)$ is as given above.

Lemma 2.1: $h^i$ is a functor.

Proof: (Note: the following proves that $h^{i+1}$ is a functor, but the only difference is the labelling).
Given a morphism of chain complexes $f: A_\cdot \to B_\cdot$ there is an induced map

$$\begin{tikzcd}
 & & & & \mathrm{Coker}(d_{i-1}) \arrow[dr, dashed, "p_{i+1}^A"] & & & \\
\cdots \arrow[r] & A_{i-1} \arrow[rr, "d_{i-1}"] \arrow[dr, dashed, "\pi_{i-1}^A"] \arrow[ddd, "f_{i-1}"'] & & A_i \arrow[ur, "j_i^A"] \arrow[rr, "d_i"] \arrow[ddd, "f_i"] & & A_{i+1} \arrow[r] \arrow[ddd, "f_{i+1}"] & \cdots & \\
 & & \mathrm{im}(d_{i-1}) \arrow[ur, "l_i^A"] \arrow[rr, dashed, "k_i^A"] \arrow[ddd, "\varphi_i^1"] & & \ker(d_i) \arrow[ul, "\gamma_i^A"'] \arrow[rr, "\ell_i^A"] \arrow[ddd, "\varphi_i^2"] & & h^i(A_\cdot) \arrow[ddd, "h^i(f)"] & \\
 & & & & \mathrm{coker}(e_i) \arrow[dr, dashed] & & & \\
\cdots \arrow[r] & B_i \arrow[rr, "e_{i-1}"] \arrow[dr, dashed] & & B_{i+1} \arrow[ur, "j_i^B"] \arrow[rr, "e_i"] & & B_{i+2} \arrow[r] & \cdots & \\
 & & \mathrm{im}(e_{i-1}) \arrow[ur, "l_i^B"] \arrow[rr, dashed, "k_i^B"] & & \ker(e_i) \arrow[ul, "\gamma_i^B"'] \arrow[rr, "\ell_i^B"] & & h^i(B_\cdot) &
\end{tikzcd}$$

$\varphi_i^1$: $j_i^B e_{i-1} f_{i-1} = j_i^B f_i d_{i-1} = j_i^B f_i l_i^A \pi_{i-1}^A$, and $j_i^B e_{i-1} f_{i-1} = 0 f_{i-1} = 0$. So $j_i^B f_i l_i^A \pi_{i-1}^A = 0$, so since $\gamma_i^A$ is an epi, $j_i^B f_i l_i^A = 0$, so by UP of $im(e_{i-1})$, $\varphi_1$ exists and is such that $l_i^B \varphi_i = f_i l_i^A$.

$\varphi_i^2$: $e_i f_i \gamma_i^A = f_{i+1} d_i \gamma_i^A = f_{i+1} 0 = 0$. So by UP of $ker(e_i)$, $\varphi_2$ exists, and is such that $l_i^B \varphi_2 = f_i l_i^A$.

$h^i(f)$: $l_i^B k_i^B \varphi_1 = l_i^B \varphi_1 = f_i l_i^A = f_{i+1} l_i^A k_i^A = l_i^B \varphi_2 k_i^B$, so since $l_i^B$ is monic, $k_i^B \varphi_1 = \varphi_2 k_i^A$.
So $\ell_i \varphi_2 k_i^A = \ell_i k_i^B \varphi_1 = 0 \varphi_1 = 0$. So by the UP of $h^i(A_\cdot)$, which, recall, is coker$(k_i^A)$, there exists $h^i(f)$.

Functoriality follows easily but with a lot of drawing, so the proof is omitted here. □

Def$^n$

A <u>short exact sequence</u> in an abelian category is a sequence of morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

with $gf = 0$, such that the induced map $im(f) \to ker(g)$ is an isomorphism.

# Section 3 : Injective objects, resolutions, and basic properties.

**Def$^n$** Let $\mathcal{A}$ be an arbitrary abelian category. Then an object $I \in \mathcal{A}$ is <u>injective</u> if for every monic $U \rightarrowtail V$, and morphism $U \longrightarrow I$, there exists a map $V \dashrightarrow I$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, tail] \arrow[dr] & V \arrow[d, dashed] \\ & I \end{tikzcd}$$

**Def$^n$** Let $A$ be an object of an abelian category $\mathcal{A}$. Then an <u>injective resolution for $A$</u>, is an exact sequence,

$$0 \longrightarrow A \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots$$

Such that all $I_i$ are injective.

Notation: Write $0 \to A \to I_\bullet$ or $A \to I_\bullet$ for an injective resolution of $A$.

Terminolog: A category $\mathcal{A}$ <u>has enough injectives</u> if for every object $C \in \mathcal{C}$, there exists an injective object $I$ and a mono $C \hookrightarrow I$.

**Lemma 3.1**, If $\mathcal{A}$ is an abelian category with enough injectives, then every object admits an injective resolution.

Proof: The top row induced by the maps beneath,

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & A \arrow[r, "f"] & I_1 \arrow[rr, "i_1"] \arrow[dr, "j_1"'] & & I_2 \arrow[rr, "i_2"] \arrow[dr, "j_2"'] & & I_3 \arrow[r, "i_3"] \arrow[dr, "j_3"'] & \cdots \\ & & & \mathrm{Coker}(f) \arrow[ur, "e_1"'] & & \mathrm{Coker}(i_1) \arrow[ur, "e_2"'] & & \mathrm{Coker}(i_2) \end{tikzcd}$$

Exists as $\mathcal{A}$ has enough injectives.

is exact.

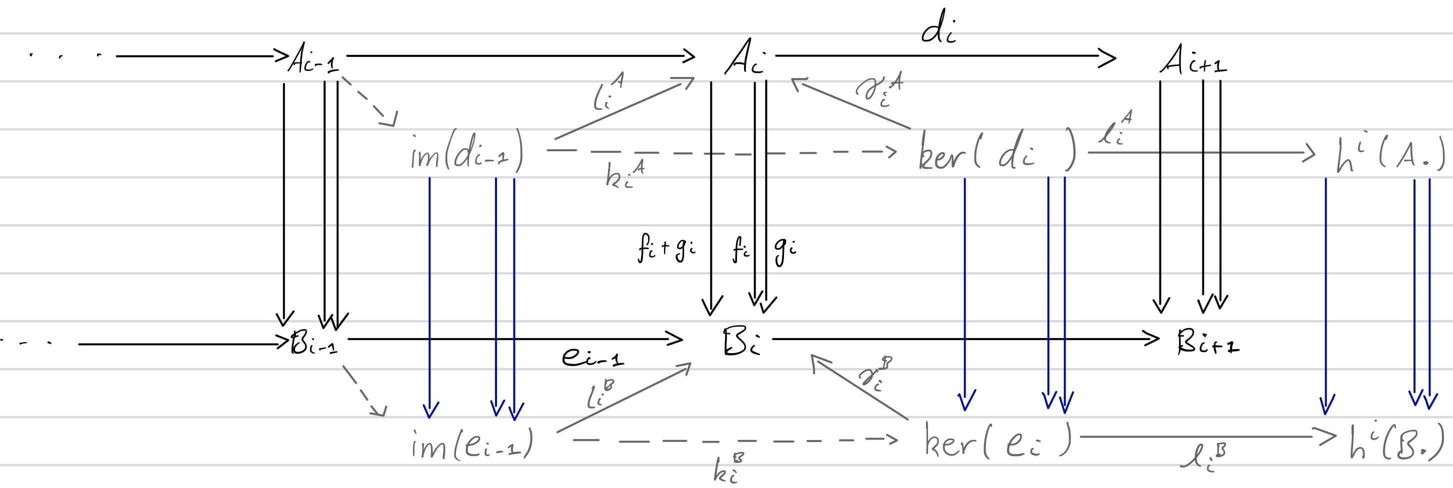
This is because $\forall k \in \mathbb{N}$, $\ker(i_k) = \ker(e_k j_k) = \ker(j_k)$, as $e_k$ is monic, and $\ker(j_k) = \ker(\mathrm{coker}(i_k)) = \mathrm{im}(i_k)$. $\square$

**Def$^n$**, Let $A_\bullet$ and $B_\bullet$ be two cocomplexes in an abelian category, and let $f_\bullet, g_\bullet : A_\bullet \longrightarrow B_\bullet$ be morphisms of cocomplexes between them. Then $f_\bullet$ and $g_\bullet$ are <u>chain homotopic</u> if there exists a collection of morphisms $t_i : A_i \longrightarrow B_{i-1}$ such that $f_i - g_i = e_{i-1} t_i + t_{i+1} d_i$.

The relevance of this definition is the fact that chain homotopic maps induce the same map on homology. The following lemma will be required to prove this,

**Lemma 3.2**, $h^i$ is an additive functor. PTO for proof.

Proof: Let $f, g : A \longrightarrow B$ be two morphisms in $\mathcal{A}$. Consider the following diagram,




In order to remove clutter, some labels have been removed. From left to right, the blue arrows are $(f_i+g_i)|_{\mathrm{im}(d_{i-1})}$, $f_i|_{\mathrm{im}(d_{i-1})}$, $g_i|_{\mathrm{im}(d_{i-1})}$, $(f_i+g_i)|_{\ker(d_i)}$, $f_i|_{\ker(d_i)}$, $g_i|_{\ker(d_i)}$, $h^i(f+g)$, $h^i(f)$, and $h^i(g)$, respectively.

Now, $l_i^B(f_i+g_i)|_{\mathrm{im}(d_{i+1})} = (f_i+g_i)l_i^A = f_i l_i^A + g_i l_i^A = l_i^B f_i|_{\mathrm{im}(d_{i-1})} + l_i^B g_i|_{\mathrm{im}(d_{i-1})} = l_i^B(f_i|_{\mathrm{im}(d_{i-1})} + g_i|_{\mathrm{im}(d_{i-1})})$.
So since $l_i^B$ is monic, $(f_i+g_i)|_{\mathrm{im}(d_{i+1})} = f_i|_{\mathrm{im}(d_{i+1})} + g_i|_{\mathrm{im}(d_{i-1})}$.

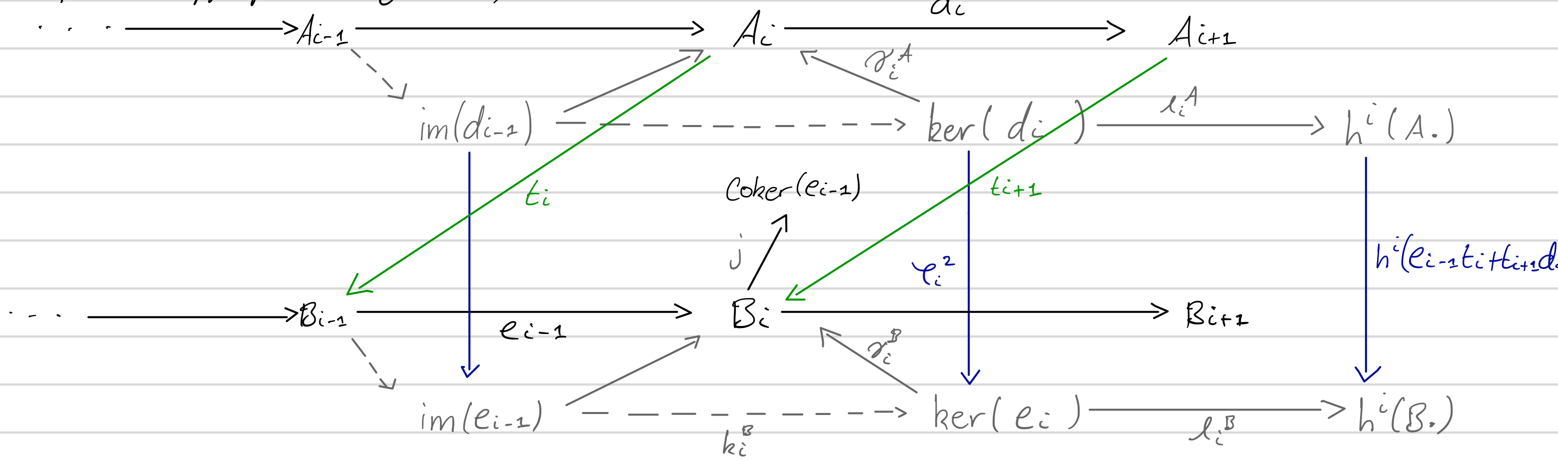
An exactly similar argument shows that $(f_i+g_i)|_{\ker(d_i)} = f_i|_{\ker(d_i)} + g_i|_{\ker(d_i)}$.

It then follows from uniqueness of the UP of $h^i(A)$ that $h^i(f)+h^i(g) = h^i(f+g)$. $\square$

Lemma 3.3, Let $f., g. : A. \longrightarrow B.$ be chain homotopic morphisms of cocomplexes. Then for all $i$, $h^i(f.) = h^i(g.)$.

(PTO for proof).

Proof: The appropriate diagram is,



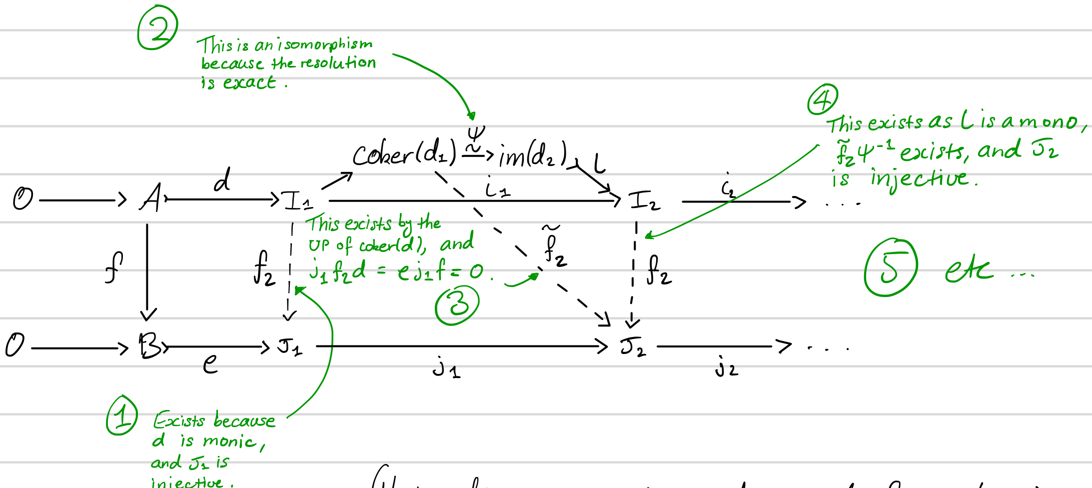

Since $h^i$ is additive, if $f_i - g_i = e_{i-1}t_i + t_{i+1}d_i$, then $h^i(f) - h^i(g) = h^i(e_{i-1}t_i + t_{i-1}d_i)$. So it remains to show that $h^i(e_{i-1}t_i + t_{i+1}d_i) = 0$.
Since $\ell_i^A$ is epic, it suffices to show that $h^i(e_{i-1}t_i + t_{i+1}d_i)\ell_i^A = 0$, for which, it suffices to show $\ell_i^B \varphi_i^2 = 0$, by commutativity.
To show this, it suffices to show that there exists a map $\ker(d_i) \xrightarrow{w} \operatorname{im}(e_{i+1})$ such that $\varphi_i^2 = k_{i-1}^B w$, and to show this, it suffices to show that $j\gamma_{i-1}^B \varphi_i^2 = 0$, because then $\gamma_i^B k_i^B w = \gamma_i^B \varphi_i^2$, and $\gamma_i^B$ is monic, so this would imply that $k_i^B w = \varphi_i^2$.

Now, $j(e_{i-1}t_i + t_{i+1}d_i)\gamma_i^A = je_{i-1}t_i\gamma_i^A + jt_{i+1}d_i\gamma_i^A = 0+0$, as $je_{i-1} = 0$ and $d_i\gamma_i^A = 0$.
So by the UP of $\operatorname{im}(e_{i-1})$, there exists a map $\delta : \ker(d_i) \longrightarrow \operatorname{im}(e_{i-1})$ such that $\gamma_i^B k_i^B w = \gamma_i^B \varphi_i^2$, as require

Thus chain homotopic maps induce the same map on homology. □

**Lemma 8.4** Let $A, B \in \mathcal{A}$, $\mathcal{A}$ abelian, and $A \to I_\cdot$, $B \to J_\cdot$ injective resolutions. Then a function $f : A \to B$ induces a morphism of cocomplexes on the resolutions. Moreover, any two choices of extension are chain homotopic.

Proof: This can be done in the following way (please read in the order given by the labels).




(Note: this argument can be made formal using induction).

Say $g.$ was a different choice of extension. Then a chain homotopy can be constructed in the following way, first define $t_0 : I_0 \longrightarrow A$ and $t_1 : I_1 \longrightarrow I_0$. Again, please read in the order as labelled.

$$\begin{tikzcd}
 & & \mathrm{im}(d) \arrow[dr, "\iota_1"] & & \mathrm{coker}(d) \arrow[dr, dashed, "\pi"] \arrow[dddl, dashed, "\varphi"'] & & \\
0 \arrow[r] & A \arrow[rr, "d"] \arrow[dd, "f-f"'] & & I_0 \arrow[ur, "j_1"] \arrow[rr, "i_1"] \arrow[dd, "f_1-g_1"] \arrow[ddll, dashed] & & I_1 \arrow[r] \arrow[dd, "f_2-g_2"] \arrow[ddll, dashed, "t_1"'] & {} \\
 & & & & & & \\
0 \arrow[r] & B \arrow[rr, "e"'] & & J_0 \arrow[rr] & & J_1 \arrow[r] & {}
\end{tikzcd}$$

③ This map is induced by the UP of $\mathrm{coker}(d)$, and the fact that $j_1 \iota_1 = 0$. Also, this map is monic by lemma 1.7.

① Set $t_0$ to be the zero map.

② This exists by the UP of $\mathrm{coker}(d)$, since $(f_1-g_1)d = ef - ef = 0$.

④ This exists as $\varphi$ exists, $\pi$ is monic, and $J_1$ is injective.

and the inductive step,

$$\begin{tikzcd}
 & & & & \mathrm{im}(i_{k-1}) \arrow[dr] & & \mathrm{Coker}(i_{k-1}) \arrow[dr, dashed, "\pi"] \arrow[dddl, dashed] & & \\
\cdots \arrow[r] & I_{k-2} \arrow[rr, "i_{k-2}"] \arrow[dd] & & I_{k-1} \arrow[rr, "i_{k-1}"] \arrow[dd, "f_{k-2}-g_{k-2}"] \arrow[ddll, "t_{k-2}"'] & & I_k \arrow[ur] \arrow[rr, "i_k"] \arrow[dd, "f_{k-1}-g_{k-1}"] \arrow[ddll, "t_{k-1}"'] & & I_{k+1} \arrow[r] \arrow[dd, "f_k-g_k"] \arrow[ddll, dashed, "\varphi"'] & {} \\
 & & & & & & & & \\
\cdots \arrow[r] & J_{k-2} \arrow[rr, "j_{k-2}"'] & & J_{k-1} \arrow[rr, "j_{k-1}"'] & & J_k \arrow[rr, "j_k"'] & & J_{k+1} \arrow[r] & {}
\end{tikzcd}$$

① Exists by UP of $\mathrm{Coker}(i_{k-1})$, and from calculation ⊗.

② Exists and is monic for the same reason as in the base case.

③ Exists as $\pi$ is monic and $J_k$ is injective.

⊗ Consider $f_{k-1} - g_{k-1} - j_{k-1}t_{k-1}$. Then $(f_{k-1}-g_{k-1}-j_{k-1}t_{k-1})i_{k-1} = f_{k-1}i_{k-1} - g_{k-1}i_{k-1} - j_{k-1}t_{k-1}i_{k-1}$
$= j_{k-1}f_{k-2} - j_{k-1}g_{k-2} - j_{k-1}t_{k-1}i_{k-1} = j_{k-1}(j_{k-2}t_{k-2} + t_{k-1}i_{k-1} - t_{k-1}i_{k-1}) = j_{k-1}j_{k-2}t_{k-2} = 0.$

Holds by inductive hypothesis.

Define $t_k := \varphi$ and check that $f_{k-1} - g_{k-1} = t_{k+1}i_k + j_{k-1}t_{k-1}$, which is clear by construction. □

Note: it's an easy exercise to show that additive functors map zero objects to zero objects, so applying an additive, left exact functor to an injective resolution gives a cochain complex. In light of this, the following notation is defined.

**Notation:** Given an injective resolution $0 \to A \to I.$ of an object $A$ in an abelian category $\mathcal{A}$, denote the cocomplex given by applying an additive functor $F : \mathcal{A} \to \mathcal{B}$ ($\mathcal{B}$ abelian) by $F(A \to I.)$.

Motivated for now by the blind trust that it's a good idea, the next aim is to study the following construction,

Construction: Let $\mathcal{A}$ be an abelian category with enough injectives, and choose (using choice) an injective resolution for each object $A \in \mathcal{A}$. Consider each object to be associated to its chosen resolution. Next, let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive, left exact functor, where $\mathcal{B}$ is abelian. Then for each object $A \in \mathcal{A}$, define $R^iF(A) := h^i(F(A \to I.))$.

The following all follow from the work already done,

Lemma 3.5

a) $R^iF$ is an additive functor for all $i$. (Hartshorne, theorem 1.1A, (a), first part).

b) If $A \in \mathcal{A}$ and $A \to I.$, $A \to J.$ are two injective resolutions of $A$, then $R^i(F(A \to I.))$ is naturally isomorphic to $R^i(F(A \to J.))$. (Hartshorne, theorem 1.1A, (a), second part).

Proof:

a) It was shown in lemma 2.1 that $h^i$ is a functor, and it was shown in lemma 3.2 that $h^i$ is additive. Clearly, the composition of additive functors is additive, and so $R^iF$ is an additive functor.

b) A map of cochain complexes $f.: (A \to I.) \to (A \to J.)$ can be induced by the identity morphism $A \to A$, by lemma 3.4. Similarly, $g.: (A \to J.) \to (A \to I.)$ can be induced. Then, $f.g.$ and $g.f.$ both are extensions of the identity map $id: A \to A$, so again by 3.4, $f.g.$ (resp $g.f.$) is chain homotopic to $id: (A \to I.) \to (A \to I.)$ (resp $id: (A \to J.) \to (A \to J.)$). Applying the functor $F$ to all of this shows that $F(f.)F(g.)$ is chain homotopic to $F(id) = id$, and similarly, $F(g.)F(f.)$ is chain homotopic to id. (Note: it is crucial here that $F$ be left exact, in order to get cochain complexes from the injective resolutions, and that $F$ be additive, to ensure chain homotopies are sent to chain homotopies).
Lastly, applying lemma 3.3 gives $R^iF(A \to I.) \cong R^iF(A \to J.)$. Naturality follows from horizontally composing the natural transformation involved in naturality of $h^i$. ◻

Note: Now that it has been established that $R^iF(A \to I.)$ is independent (up to natural isomorphism) of the choice of injective resolution, the notation $R^iF(A)$ will be used. Also, now a resolution may be picked after choosing the object (in order to avoid the one use of choice briefly mentioned earlier).

Lemma 3.6. $R^0F$ is naturally isomorphic to $F$. (Hartshorne, theorem 1.1A, (b)).

Proof: Let $A \in \mathcal{A}$, and pick an injective resolution $A \to I.$ By definition, $R^iF$ concerns the following cocomplex $0 \longrightarrow F(I_1) \xrightarrow{F(i_1)} F(I_2) \xrightarrow{F(i_2)} \cdots$ Now, since $F$ is left exact, applying $F$ to the short exact sequence $0 \longrightarrow A \xrightarrow{d} I_1 \longrightarrow \mathrm{coker}(d) \longrightarrow 0$ gives the exact sequence $0 \longrightarrow F(A) \xrightarrow{F(d)} F(I_1) \longrightarrow F(\mathrm{coker}(d))$, showing that $F(d)$ is monic.

So, by definition, $R^0F(A) = h(F(I.))$, which is defined by the following diagram

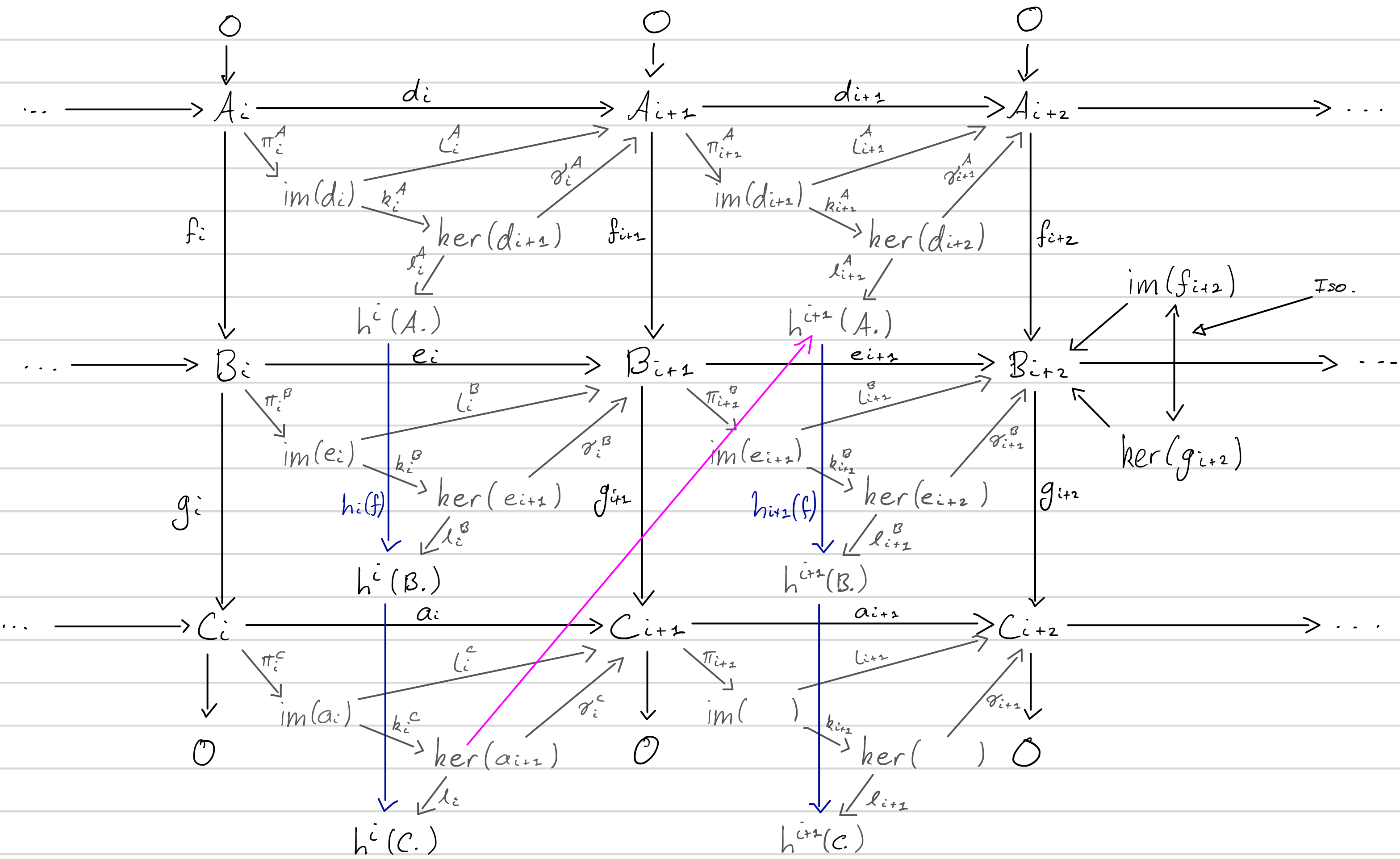
$$\begin{tikzcd}
 & 0 \arrow[r, "0"] & F(I_1) \arrow[r, "F(i_1)"] & F(I_2) \arrow[r] & \cdots \\
0 \cong & im(0) \arrow[ur, hook] \arrow[rr, "k"] & & ker(F(i_1)) \arrow[r] & Coker(k) = h^i(F(A))
\end{tikzcd}$$

Then, $k$ is a $0$ map, and cokernels of $0$ maps are isomorphisms. So $h^i(F(A)) \cong ker(F(i_1))$.
So by exactness of $0 \to F(A) \to F(I) \to F(im(i_1))$, $ker(F(i_1)) \cong im(F(d))$, which in turn is isomorphic to $F(A)$, as $F(d)$ is monic. $\square$ (Note: naturality is easy, so the proof has been omitted).

Lemma 3.7, If $I \in \mathcal{A}$ is injective, then $R^iF(I) = 0$. (Hartshorne, theorem 1.1A, (e)).
Proof: Since $R^iF(I)$ is independent (up to isomorphism) of the choice of injective resolution, any such resolution can be chosen. The fact that $R^iF(I) = 0$ then follows from the fact that $0 \to I \to I \to 0$ is an injective resolution of $I$. $\square$

A very important (near central) property of $R^iF$ (hereby called the right derived functors of $F$) is that any short exact sequence in $\mathcal{A}$ gives rise to a long exact sequence in the derived functors.

However, proving this without an embedding theorem is very difficult. For fun, here is just one ugly diagram inevitably crossed when soldiering on with diagrams alone,



There is no reason to believe this is as bad as they get either.

Also, more to the point, why continue with diagrams anyway? Surely at this stage, the diagrams are too unwieldy to gain any intuition from. Are there even any examples of abelian categories of interest where these theorems must be proved purely diagramatically anyway?

Lastly, with the time and effort required to find these diagrams, one could have proved a suitable embedding theorem, which comes with much more, anyway.

First though, a few results must be proved in the particular setting of abelian groups. These are presented next.

(Note: this is kind of a lie. In fact the above theorem can be proved purely digramatically by first establishing some preliminary results, namely, the nine lemma, snake lemma, etc. In fact doing so reveals the above lemma as a corollary of these theorems, an understanding of this theorem the embedding argument perhaps obfuscates, still, the embedding approach has been taken here).

## Section 4, proofs of some theorems in the category of abelian groups.

First and foremost, it must be shown that the category of abelian groups has enough injectives. As usual, some preliminary results are required first.

**Lemma 4.1 (Baer's criterion).**
An abelian group $A$ is injective if and only if for every ideal $a\mathbb{Z}$ of $\mathbb{Z}$, any map $a\mathbb{Z} \longrightarrow A$ can be extended to a map $\mathbb{Z} \longrightarrow A$.

Proof: If $A$ is injective, then for any ideal $\langle p \rangle$ of $\mathbb{Z}$, subgroup $B \subseteq A$, and map $\langle a \rangle \longrightarrow B$, there is the following diagram

$$\begin{tikzcd} a\mathbb{Z} \arrow[r, tail] \arrow[dr] & \mathbb{Z} \arrow[d, dashed] \\ & A \end{tikzcd}$$

where the existence of the dashed line follows immediately from injectivity of $A$.

Conversely, let $C \overset{m}{\rightarrowtail} B$ be an injective map, $C \xrightarrow{\varphi} A$ an arbitrary map. Then define the set
$$\Sigma := \{\alpha : C' \longrightarrow A \mid C \subseteq C' \subseteq B,\ \alpha \text{ extends } \varphi\}.$$
Define the partial order $\alpha_1 \leq \alpha_2$ if $\alpha_2$ extends $\alpha_1$ on $\Sigma$.

Zorn's lemma then implies the existence of a maximal extension $\alpha : C' \longrightarrow A$. It remains to show that $C' = B$.

Suppose for a contradiction that $\exists b \in B \setminus C'$. Then consider the ideal $R := \{n \in \mathbb{Z} \mid n \cdot b \in C'\}$.
By assumption, the map $R \xrightarrow{\times b} C' \xrightarrow{\alpha} A$ extends to a map $\mathbb{Z} \xrightarrow{f} A$.
Now consider the subgroup $X$ of $B$ generated by $\{C', \mathbb{Z}b\}$ and define the map $X \longrightarrow A$ defined by $c + n \cdot b \mapsto \alpha(c) + f(n)$. This is easily verified to be well defined, and extends $\alpha$, contradicting the existence of $b$.
Thus $C' = B$. $\square$

**Lemma 4.2:** Let $\{I_i\}_{i \in I}$ be a set of injective objects in the abelian category $Ab$. Then $\prod_{i \in I} A_i$ is injective.

Proof: Let $U \rightarrowtail M$ be a mono, and let $U \longrightarrow \prod_{i \in I} A_i$ be an arbitrary morphism. Then for each projection map $\pi_i : \prod_{i \in I} A_i \longrightarrow A_i$, there exists a map $\varphi_i : M \longrightarrow A_i$ such that the diagram

$$\begin{tikzcd} U \arrow[r, tail] \arrow[d] & M \arrow[ddl, "\varphi_i"] \\ \prod_{i \in I} A_i \arrow[d, "\pi_i"'] & \\ A_i & \end{tikzcd}$$

commutes. Taking the product of all such maps (note: in this special case of abelian groups, in fact all products exist) gives a suitable map. $\square$

Note: there was nothing special about the ring $\mathbb{Z}$ used in this proof, and in fact the statement holds for arbitrary modules, but the definition of injective in the case of abelian groups was defined, and so this proof was given here.

**Lemma 4.3: The abelian group $\mathbb{Q}/\mathbb{Z}$ is injective.**

Proof: The map $\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ induced by $1 \longmapsto \left[\frac{e(a)}{a}\right]$ extends arbitrary $e: a\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$, so Baer's criterion is satisfied. $\square$

**Lemma 4.4, Ab has enough injectives.**

Let $A \in Ab$. Let $\prod_{f \in Hom(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ be the product of a copy of $\mathbb{Q}/\mathbb{Z}$ for each morphism $A \longrightarrow \mathbb{Q}/\mathbb{Z}$. By lemma 4.1 and lemma 4.3, this is injective. Claim: $\prod_{f \in Hom(A, \mathbb{Q}/\mathbb{Z})} f$ (the product of all maps $A \longrightarrow \mathbb{Q}/\mathbb{Z}$) is injective.

Proof of claim: Let $a \in A \setminus \{0\}$. Then there exists a map $a\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$ such that $f(a) \neq 0$. Then, since $\mathbb{Q}/\mathbb{Z}$ is injective (lemma 4.3), and $a\mathbb{Z} \hookrightarrow A$ is injective, there exists a map $A \xrightarrow{\tilde{f}} \mathbb{Q}/\mathbb{Z}$ which extends $f$. This map is such that $\tilde{f}(a) \neq 0$, and so $\left(\prod_{f \in Hom(A, \mathbb{Q}/\mathbb{Z})} f\right)(a)$ is non-zero in at least one entry. Thus this map is injective. $\square$

**Corollary 4.5, Every abelian group admits an injective resolution.**

Next, a proof of the snake lemma for abelian groups will be presented. Later in these notes, a proof of the snake lemma for arbitrary abelian categories will be presented, but the proof of that result will require this statement, so this is presented first.

**Lemma 4.6 (the snake lemma).**
**Given a commutative diagram of abelian groups with exact rows as follows,**

$$\begin{tikzcd} & A \arrow[r, "a"] \arrow[d, "c"] & B \arrow[r, "b"] \arrow[d, "d"] & C \arrow[r] \arrow[d, "e"] & 0 \\ 0 \arrow[r] & D \arrow[r, "f"'] & E \arrow[r, "g"'] & F & \end{tikzcd}$$

**there is a morphism $ker(e) \longrightarrow coker(c)$ such that the following sequence is exact**

$$ker(c) \longrightarrow ker(d) \longrightarrow ker(e) \longrightarrow coker(c) \longrightarrow coker(d) \longrightarrow coker(e)$$

Proof: let $\gamma \in ker(e)$. Then as $b$ is surjective, there exists $\beta \in B$ such that $b(\beta) = \gamma$. Since $\gamma \in ker(e)$ and the diagram commutes, $eb(\beta) = gd(\beta) = 0$. So by exactness of the bottom row, $\exists\, \delta \in D$ such that $f(\delta) = d(\beta)$. Then project $\delta$ into $coker(c)$. This defines a way of obtaining an element $[\delta] \in coker(c)$ from an element $\gamma \in ker(C)$. Define this to be the function, it remains to check well definedness.

Since $f$ is injective, there is only one choice for $\delta$, so the only choice made in the procedure is the choice of $\beta$. Say a different choice $\beta'$ was made. Then $b(\beta-\beta')=0$, So by exactness of the top row, $\exists \alpha \in A$ such that $a(\alpha)=\beta-\beta'$. Also, $gd(\beta)=gd(\beta')=0$, so by exactness of the bottom row, there exists $\delta_1, \delta_2 \in D$ such that $f(\delta_1)=d(\beta)$ and $f(\delta_2)=d(\beta')$.
$f(\delta_1-\delta_2)=d(\beta-\beta')$, and $\delta_1-\delta_2$ is the unique such (by injectivity). Thus, $c(\alpha)=\delta_1-\delta_2$, so $\delta_1-\delta_2 \in \mathrm{im}(c)$. This proves well definedness.

The fact that this map is a homomorphism is an easy check. Exactness is also easy (albeit tedious). $\square$.

Section 5, embedding theorems.

The references for this section are

[Mit], Barry Mitchell, "Theory of Categories".

[Murf], Daniel Murfet, "Diagram Chasing in Abelian Categories", which can be found on his webpage "therisingsea.org".

Throughout, the word "embedding" means a faithful functor which maps distinct objects to distinct objects.

There are at least two embedding theorems one may concern oneself with.

1) The Group Valued embedding theorem.

2) The full embedding theorem.

The statements of these are as follows.

1) Any small, abelian category $\mathcal{A}$ admits an exact embedding into the category of abelian groups. ([Mit], Chapter 4, thm 2.6).

2) Any small, abelian category $\mathcal{A}$ admits an exact embedding into the category of $R$-modules, for an appropriate ring $R$, such that $\mathcal{A}$ is a full subcategory of $R$-mod. ([Mit], Chapter 7, thm 7.2).

These theorems are of little use without the following two propositions,

Prop 5.1, Let $\mathcal{A}$ be an (arbitrary) abelian category, and let $S \subseteq \mathcal{A}$ be a set of objects from $\mathcal{A}$. Then there exists a small, abelian subcategory of $\mathcal{A}$ containing all the objects in $S$.

and

Prop 5.2, let $\mathcal{A}$ be an abelian category, and $F: \mathcal{A} \to Ab$ an exact embedding. Then $F$ preserves and reflects monomorphisms, epimorphisms, commutative diagrams, limits and colimits of finite diagrams, and exact sequences. (Note: reflects means if $F(f)$ has a property, then $f$ does).

Neither of these propositions are difficult, proof sketches are outlined now,

Prop 5.1:

Proof idea: given a set of objects $\{A_i\}_{i\in I}$ in an abelian category, to construct the smallest abelian category containing $\{A_i\}_{i\in I}$, first collate for each morphism $f: A_i \to A_j$, both $\ker(f)$ and $\operatorname{coker}(f)$, as well as the objects $A_i \oplus A_j$ for each $(i,j) \in I^2$. Since these are indexed by sets, this collection is a set. Then repeat this process countably infinitely many more times. The rest of the proof is formalism. □

Prop 5.2:

Proof: let $m: A \to B$ be a monic. Then the sequence $0 \to A \to B \to \mathrm{Coker}(A) \to 0$ is exact, so by left exactness, $0 \to FA \to FB \to F\mathrm{Coker}(A)$ is exact. So $F(m)$ is monic. A similar argument shows that $F$ preserves epimorphisms.

To see that $F$ reflects monomorphisms, assume $Fm$ is monic. Then if $fg_1 = fg_2 \Rightarrow F(f)F(g_1) = F(f)F(g_2)$, which implies $F(g_1) = F(g_2)$ since $F(f)$ is monic. Then $g_1 = g_2$ by faithfulness of $F$. A similar argument shows that $F$ reflects epimorphisms.

All functors preserve commuting diagrams, all exact functors preserve exact sequences. Reflection of commuting diagrams comes easily from faithfulness, and it is easy to see that exact functors preserve kernels and cokernels, and thus exact sequences.

Exactness preserves 0, kernels, and cokernels, and so preserves equalisers and coequalisers. Exactness also preserves binary products and coproducts, and faithfulness reflects them, thus all finite limits and colimits are preserved and reflected. □

Loosely speaking, there are two common types of arguments used involving diagrams, those which express a particular property of an already existing morphism (eg, a particular diagram commutes, a particular arrow is a monic/epi/iso, etc), and those which insist on the existence of a morphism which is not present a priori. Due to the two propositions, the first embedding theorem is ideal for the first style of argument, and the second embedding theorem for the second. However, the second embedding theorem is significantly harder to prove than the first one, and the full strength of the second one is not needed. After all, in a situation where one is looking at a diagram wishing upon the existence of a particular morphism, why go overboard and use the fact that <u>all</u> morphisms entirely between two objects in the image of the embedding are mapped onto by some morphism in the domain? In fact there is theory [Murf] describing when a morphism in the image of the embedding is mapped onto, the theory of so called "walks in a category". This full theory is not presented and proved here, but will be applied and referenced.

Interesting note: In my conversations with Daniel Murfet, he made it clear to me that in fact for all professional uses he has made of embedding theorems thus far, only the first embedding theorem along with the theory of walks has been enough. This is good because it means the mathematician who wishes to hand check the theory him/her self is given a much less daunting task. The full embedding theorem is still an incredible theorem and interesting for its own sake too though.

The following is an example, the snake lemma.

**Lemma 5.3** (Snake lemma for arbitrary abelian categories.)
The statement of the lemma is the same as lemma 4.6 but with the obvious changes made to suit arbitrary abelian categories.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker(c) \arrow[r] \arrow[d] & \ker(d) \arrow[r] \arrow[d] & \ker(e) \arrow[d, "h"] & \\
 & A \arrow[r, "a"] \arrow[d, "c"] & B \arrow[r, "b"] \arrow[d, "d"] & C \arrow[r] \arrow[d, "e"] & 0 \\
0 \arrow[r] & D \arrow[r, "f"'] \arrow[d, "\ell"'] & E \arrow[r, "g"'] \arrow[d] & F \arrow[d] & \\
 & \mathrm{Coker}(c) \arrow[r] \arrow[d] & \mathrm{Coker}(d) \arrow[r] \arrow[d] & \mathrm{Coker}(e) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

*Proof.* Consider the diagram on the right which has exact columns, exact middle two rows. The collection of all objects in this diagram is finite and therefore a set. So by proposition 5.1, there exists a small, abelian category $\mathcal{A}'$ containg all of these objects. By the first embedding theorem, there exists an exact, faithful functor $F: \mathcal{A}' \to Ab$ which sends distinct objects to distinct objects.
By prop 5.2, the embedding of this diagram preserves all commutativity and exactness. The notation for the embedded diagram will be the same as the non-embedded diagram. Now, by the snake lemma for abelian groups, (lemma 4.6), there exists a morphism $\ker(e) \longrightarrow \mathrm{Coker}(c)$ making the appropriate sequence exact. Since the functor $F$ reflects exact sequences, it remains only to show that there exists a morphism in $\mathcal{A}'$ which maps onto this morphism, which can be done with a walk.

A <u>walk</u> $W: A \longrightarrow B$ is simply a sequence of morphisms in $\mathcal{A}$ $(\alpha_1, \dots, \alpha_n)$ such that
- $\alpha_i : A_i \to A_{i+1}$, or $\alpha_i : A_{i+1} \to A_i$, and
- $A_1 = A$, $A_{n+1} = B$.

Given a walk in the category of abelian groups, each element of the walk is a homomorphism, so in particular is a function, so in particular is a relation. Every relation admits an inverse relation, so take the inverse relation of each function in the walk of the form $A_{i+1} \to A_i$. Composing all the relations in this new walk gives a new relation. If this new relation is a function, then the original walk is said to be a <u>function walk</u>. (Actually, function walks require something a bit stronger than this, but there are technical lemmas which assert that this amounts to the same thing, so details have been left out here for the sake of simplicity).

The sequence $(h, b, d, f, \ell)$ is a walk in $\mathcal{A}$, and $(Fh, Fb, Fd, Ff, F\ell)$ is a function walk.

Then, theorem 13 in [Murf] proves that this is sufficient to ensure that there exists exactly one morphism in $\mathcal{A}$ which maps onto this function walk, which, is the desired connecting morphism. $\square$

## Section 6, proofs for abelian categories using the embedding theorem.

To avoid this document from becoming excessive in size, the following theorems are stated without proof.

Lemma 6.1, Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor, with $\mathcal{A}$ & $\mathcal{B}$ abelian. Assume further that $\mathcal{A}$ has enough injectives. Suppose there is the following exact sequence in $\mathcal{A}$ with all $I_i$ injective,

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{m-1} \longrightarrow I_m \longrightarrow M \longrightarrow 0$$

Then there are isomorphisms $\rho^n : R^nF(M) \longrightarrow R^{n+m+1}F(A)$ for $n \geq 1$, and if $F$ is left exact, then there is an exact sequence

$$F(I_m) \longrightarrow F(M) \longrightarrow R^{m+1}F(A) \longrightarrow 0.$$

See [Murf], proposition 51.

Note: the proof of this result requires the a connecting morphism on homologies, the existence of which will be proved. However, the proof of the existence of the connecting morphisms requires the following result, which will not be proved.

Lemma 6.2, Let $C.$ be a cochain complex in $\mathcal{A}$. There always exists a unique morphism $\mathrm{coker}(d_{n-1}) \longrightarrow \ker(d_{n+1})$, making the following diagram commute

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] & C_n \arrow[r, "d_n"] \arrow[d] & C_{n+1} \arrow[r, "d_{n+1}"] & C_{n+2} \\
& & \mathrm{coker}(d_{n-1}) \arrow[r] & \ker(d_{n+1}) \arrow[u] &
\end{tikzcd}$$

Moreover, there is an exact sequence

$$0 \longrightarrow h^n(C) \longrightarrow \mathrm{coker}(d_{n-1}) \longrightarrow \ker(d_{n+1}) \longrightarrow h^{i+1}(C) \longrightarrow 0$$

For a proof, see lemma 25 of [Murf].

Note: the only external result that this lemma requires is the nine lemma, suitably stated for arbitrary abelian categories. This one is easy though as no nonsense about walks in categories is needed, only the first embedding theorem, prop 5.2, and the nine lemma which is specific to abelian groups.

Lemma 6.3, For each short exact sequence of chain complexes $0 \to A. \to B. \to C. \to 0$ in an arbitrary abelian category $\mathcal{A}$, there exists a connecting morphism $h^i(C) \longrightarrow h^{i+1}(A)$ so that the sequence

$$\cdots \to h^i(A) \longrightarrow h^i(B.) \to h^i(C.) \longrightarrow h^{i+1}(A.) \to \cdots \qquad \text{is exact.}$$

Proof: this result will make use of the snake lemma, which makes use of an embedding theorem. Since $0 \to A_i \to B_i \to C_i \to 0$ is exact for all $i$, it follows that

$$0 \to \ker(d_i) \to \ker(e_i) \to \ker(f_i) \quad \text{and} \quad \operatorname{coker}(d_i) \to \operatorname{coker}(e_i) \to \operatorname{coker}(f_i) \to 0$$

are both exact, where $d_i, e_i, f_i$ are the morphisms $A_i \to A_{i+1}$, $B_i \to B_{i+1}$, $C_i \to C_{i+1}$ respectively. Then, by lemma 6.2, there is the following commutative diagram with exact columns, and two middle rows exact,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & h^i(A) \arrow[r] \arrow[d] & h^i(B) \arrow[r] \arrow[d] & h^i(C) \arrow[d] & \\
 & \operatorname{coker}(d_{i-1}) \arrow[r] \arrow[d] & \operatorname{coker}(e_{i-1}) \arrow[r] \arrow[d] & \operatorname{coker}(f_{i-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(d_{i+1}) \arrow[r] \arrow[d] & \ker(e_{i+1}) \arrow[r] \arrow[d] & \ker(f_{i+1}) \arrow[d] & \\
 & h^{i+1}(A) \arrow[r] \arrow[d] & h^{i+1}(B) \arrow[r] \arrow[d] & h^{i+1}(C) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So then the snake lemma implies the existence of the desired connecting morphism. □

Lemma 6.1 (Hartshorne, proposition 1.2A).
Let $F: \mathcal{A} \to \mathcal{B}$ be an additive, left exact functor with $\mathcal{A}$ and $\mathcal{B}$ abelian categories. Assume further that $\mathcal{A}$ has enough injectives. Let

$$0 \to A \xrightarrow{\varepsilon} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots$$

be an exact sequence where each $J_i$ is such that $R^jF(J_i) = 0 \ \forall j > 0$.
Then $R^iF(A) \cong h^i(F(J_\cdot)) \ \forall i$.

Proof: By lemma 3.6, $R^0F(A) \cong F(A)$, and using a similar argument to the proof of that lemma, $h^0(F(J_\cdot)) \cong F(A)$, which proves the base case.
For $n=1$, let $\mu_1$ be the cokernel $J_0 \to \operatorname{coker}(A \to J_0)$, and for $n>1$, let $\mu_n$ be the cokernel $J_{n-1} \to \operatorname{coker}(J_{n-2} \to J_{n-1})$, which gives rise to the following long exact sequence,

$$0 \to A \to J_1 \to \cdots \to J_{n-1} \to \operatorname{coker}(J_{n-2} \to J_{n-1}) \to 0.$$

By the UP of $\operatorname{coker}(J_{n-2} \to J_{n-1})$ and the fact that $J_{n-2} \to J_{n-1} \to J_n = 0$, there exists a unique map $e: \operatorname{coker}(J_{n-2} \to J_{n-1}) \to J_n$, such that

$$\begin{tikzcd}
 & \operatorname{coker}(J_{n-2} \to J_{n-1}) \arrow[dr, "e"] & \\
J_{n-1} \arrow[ur, "\mu_n"] \arrow[rr] & & J_n
\end{tikzcd}$$

commutes. (PTO)

Then there is the following diagram with exact rows

$$\begin{tikzcd}
 & F(J_{n-1}) \arrow[r, equal] \arrow[d, "F(\mu_n)"'] & F(J_{n-1}) \arrow[r] \arrow[d, "F(J_{n-1}\to J_n)"'] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(\mathrm{coker}(J_{n-2}\to J_{n-1})) \arrow[r, "F(e)"'] & F(J_n) \arrow[r, "F(J_n \to J_{n+1})"'] & F(J_{n+1}) &
\end{tikzcd}$$

Note: the bottom row is exact because $0 \longrightarrow \ker(J_n \to J_{n+1}) \longrightarrow J_n \longrightarrow \mathrm{coker}(J_n \to J_{n+1}) \longrightarrow 0$ is exact, $\ker(J_n \to J_{n+1}) \cong \mathrm{coker}(J_{n-2} \to J_n)$ (by exactness), and $F$ is left exact.

It follows from the snake lemma that the sequence

$$0 \longrightarrow \mathrm{coker}(F(\mu_n)) \longrightarrow \mathrm{coker}(F(J_{n-2} \to J_n)) \longrightarrow F(J_{n+1})$$

is exact. However, from lemma 6.2, the sequence

$$0 \longrightarrow h^n(F(J_.)) \longrightarrow \mathrm{coker}(Fd_{n-1}) \longrightarrow \ker(Fd_{n+1})$$

is exact, and so $h^n(F(J_.))$ is a kernel of $\mathrm{coker}(Fd_{n-1}) \longrightarrow \ker(Fd_{n+1})$. Since $\ker(Fd_{n+1}) \rightarrowtail FJ_n$ is monic, $h^n(F(J_.))$ is also a kernel of the composite $\mathrm{coker}(Fd_{n-1}) \longrightarrow \ker(Fd_{n+1}) \longrightarrow FJ_n$.
Thus, $h^n(F(J_.)) \cong \mathrm{coker}(F(\mu_n))$. PTO.
There is also an isomorphism $R^nF(A) \cong \mathrm{coker}(F(\mu_n))$, by lemma 6.1. The result then follows. $\square$