Category theory, lecture 2

Theorem: A function $f: A \longrightarrow B$ is a bijection iff it is injective and surjective.

Is this a tautology? No, why?

Recall:
Def^n: A function $f: A \longrightarrow B$ is a bijection if there exists a function $g: B \longrightarrow A$ such that

$$f \circ g = id_B, \quad g \circ f = id_A.$$

Proof of theorem:
Injectivity: Say $a_1, a_2 \in A$ such that

$$f(a_1) = f(a_2)$$

then $g(f(a_1)) = g(f(a_2))$

$$\Rightarrow \quad a_1 = a_2.$$

Surjectivity: Say $b \in B$. Then $g(b)$ is such that

$$f(g(b)) = b. \qquad \square$$

Theorem: A linear transformation $f: V \longrightarrow W$ between vector spaces V, W is an isomorphism iff it is injective and surjective. $\square$

all:

Def$^{\underline{n}}$: A <u>Group</u> G is a set (also called G) along with an operation (function)

$$\circ : G \times G \longrightarrow G$$

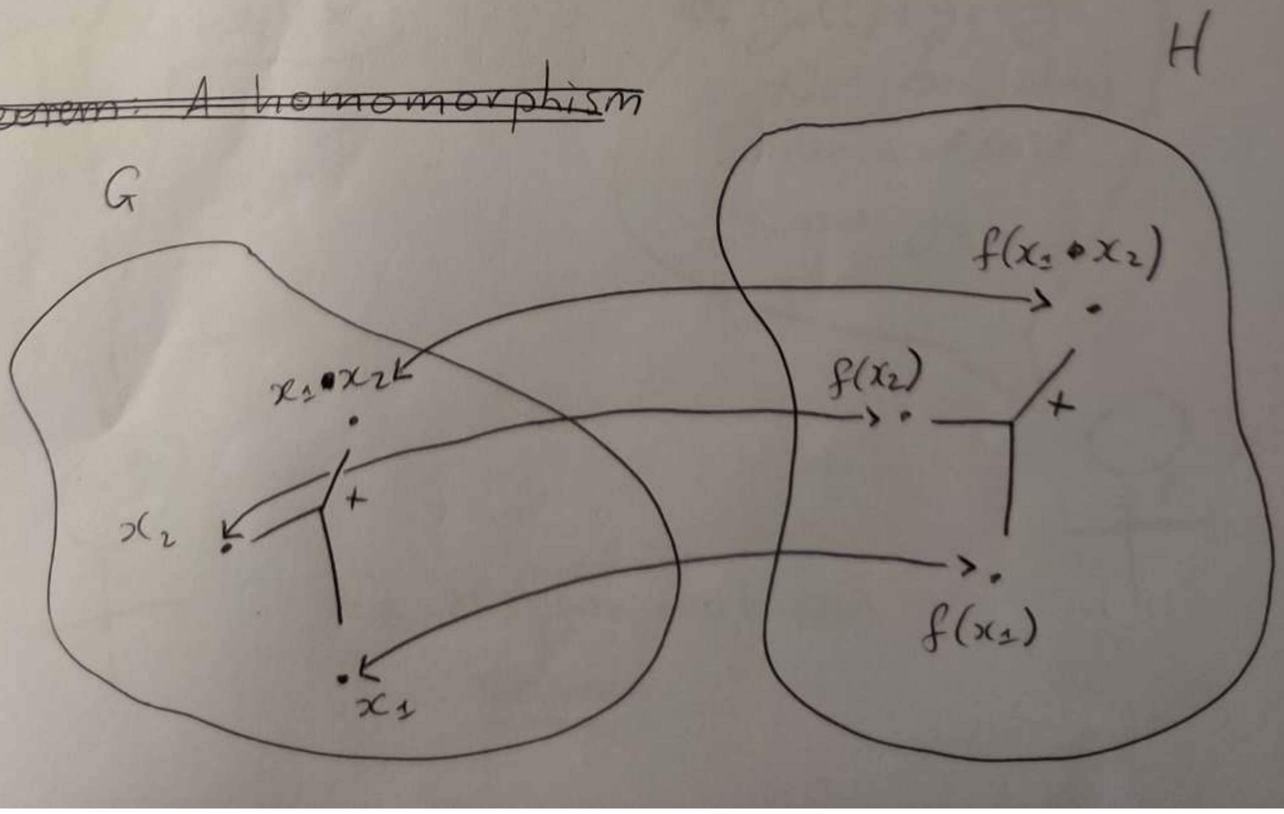
and an identity element $e \in G$ such that:

- $\forall x, y, z \in G : (x \circ y) \circ z = x \circ (y \circ z)$
- $\forall x \in G, \quad e \circ x = x \circ e = x$
- $\forall x \in G, \exists y \in G, \quad x \circ y = y \circ x = e$.

Def$^{\underline{n}}$: A function between groups $f : G \longrightarrow H$ is a <u>homomorphism</u> if

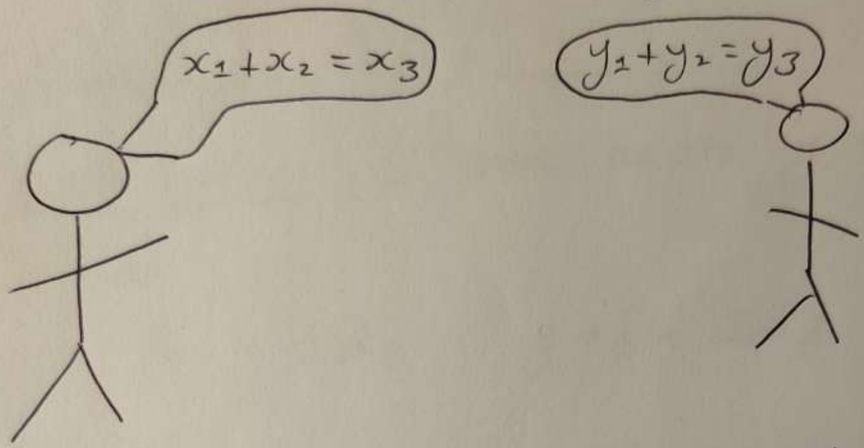
$$\forall x, y \in G, \quad f(\underset{\substack{\uparrow \\ \text{in } G}}{x \circ y}) = \underset{\substack{\uparrow \\ \text{in } H}}{f(x) \circ f(y)}$$

~~Theorem: A homomorphism~~



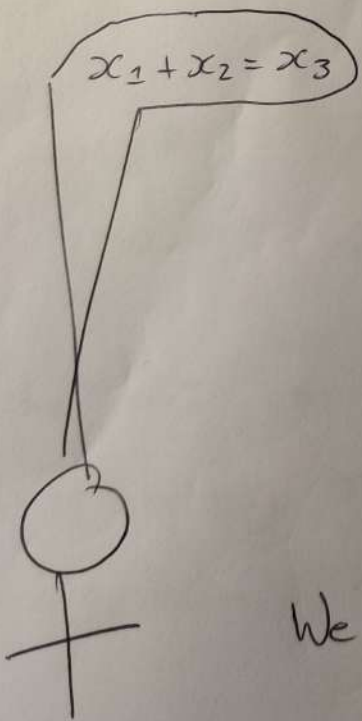
— do we distinguish between groups which are isomorphic?

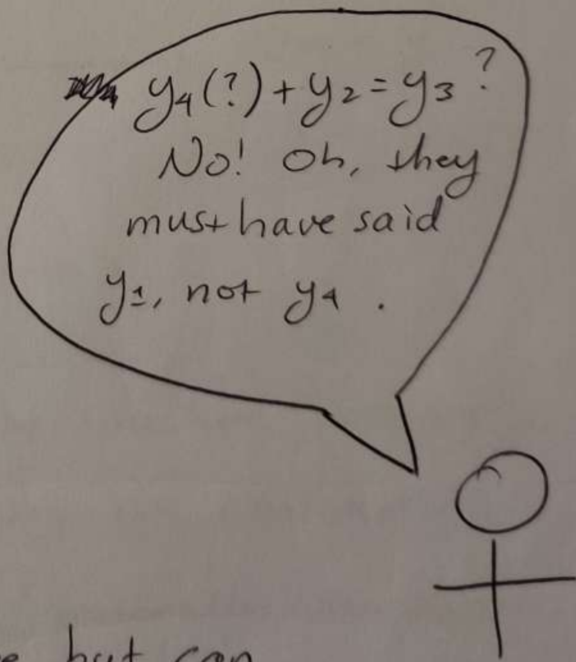
$$x_1 \longleftrightarrow y_1$$
$$x_2 \longleftrightarrow y_2$$
$$x_3 \longleftrightarrow y_3$$



We agree!

Introduce $x_4 \longleftrightarrow y_4$





We disagree, but can <u>correct</u>.

Thus, it is not the objects (or labels) which matter, but rather their <u>relationships</u> (be them x's or y's).

Def$^{\underline{n}}$: A morphism $f : A \longrightarrow B$ in a category is an <u>isomorphism</u> if there exists $g : B \longrightarrow A$ such that

$$f \circ g = Id_B, \quad g \circ f = Id_A .$$

~~Slogan Isomorphic~~

Notice: the preservation of <u>structure</u> is crucial.

Eg) Let $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$

$$(x, y) \longmapsto 2x + y .$$

jective:

If $2x + y = 2x' + y'$ then

$y = 1 \Rightarrow y' = 1 \Rightarrow x = x'$

$y = 0 \Rightarrow x = x'$.

So $(x, y) = (x', y')$.

Surjective:

$0 = f(0,0)$

$1 = f(0,1)$

$2 = f(1,0)$

$3 = f(1,1)$.

But these <u>groups</u> can be distinguished:

$$\begin{array}{ccc}
(0,1) & \longleftrightarrow & 1 \\
+ & & + \\
(0,1) & \longleftrightarrow & 1 \\
\| & & \| \\
(0,0) & \not\longleftrightarrow & 2
\end{array}$$

So the bijection <u>must</u> be structure preserving in both directions. This is why we consider homomorphisms (or more generally, morphisms).

~ we can guess definitions correctly:

Groups have: multiplication, identity element.

So ~~fun ham~~ a morphism of groups should preserve:
multiplication, identity element.

$$f(x \circ y) = f(x) \circ f(y) \qquad (f: G \longrightarrow H)$$
$$f(e) = e.$$

(However, in this case the second condition is redundant:

$\cancel{\forall x \in H, \quad x = x \cdot f(e) = f(e) \cdot x}$

$$\overrightarrow{f}(e_G) = f(e_G \circ e_G)$$
$$= f(e_G) \cdot f(e_G)$$
$$\Rightarrow f(e_G) \circ f(e_G)^{-1} = f(e_G) \cdot f(e_G) \circ f(e_G)^{-1}$$
$$\Rightarrow \underline{e_H = f(e_G)} \quad)$$

◁

Vector spaces have: addition, scalar multiplication.

$$f(x_1 + x_2) = f(x_1) + f(x_2), \quad \forall x_1, x_2 \in V$$
$$f(\lambda x_1) = \lambda f(x_1), \quad \forall x_1 \in V, \forall \lambda \in k$$

(the base field)

Categories have: Composition, identity elements.

Thus:

$\therefore$ we can guess definitions correctly:

Groups have: multiplication, identity element.

So ~~function~~ a morphism of groups should preserve: multiplication, identity element.

$$f(x \cdot y) = f(x) \cdot f(y) \qquad (f : G \longrightarrow H)$$
$$f(e) = e .$$

(However, in this case the second condition is redundant:

$$\cancel{\forall x \in H, \quad x = x f(e) = f(e) \cdot x}$$

$$\Longrightarrow$$

$$f(e_G) = f(e_G \circ e_G)$$
$$= f(e_G) \cdot f(e_G)$$
$$\Rightarrow f(e_G) \cdot f(e_G)^{-1} = f(e_G) \cdot f(e_G) \circ f(e_G)^{-1}$$
$$\Rightarrow \quad e_H = f(e_G) \quad)$$

Vector spaces have: addition, scalar multiplication.

$$f(x_1 + x_2) = f(x_1) + f(x_2), \quad \forall x_1, x_2 \in V$$
$$f(\lambda x_1) = \lambda f(x_1), \quad \forall x_1 \in V, \forall \lambda \in k$$

(the base field)

Categories have: Composition, identity elements.

Thus:

D: A <u>functor</u> $F : \mathcal{C} \longrightarrow \mathcal{D}$ between categories is a an association of an element $FC \in \mathcal{D}$ to each $c \in \mathcal{C}$, and for each pair $(X,Y) \in \mathcal{C}$ a function:

$$F_{X,Y} : \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(FX, FY)$$

such that:

- For all $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ in $\mathcal{C}$,

$$F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$$

- For all $X \in \mathcal{C}$,

$$F_{X,X}(\mathrm{id}_X) = \mathrm{id}_{FX}.$$

Examples:

-
$$\begin{aligned} F : \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto X \amalg \{*\} \\ (f : X \to Y) &\longmapsto \left(\begin{aligned} Ff : FX &\longrightarrow FY \\ x &\longmapsto f(x) \\ * &\longmapsto * \end{aligned} \right) \end{aligned}$$

-
$$\cancel{\begin{aligned} F : \mathbb{C}\text{-Vect} &\longrightarrow \mathbb{C}\text{-Vect} \\ V &\longmapsto V^{**} \end{aligned}}$$

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J: Prescrvation of identity:

$$F(id_V) : V^{**} \longrightarrow V^{**}$$
$$\psi \longmapsto \psi \circ id^*$$

Let $\psi \in V^{**}$ be arbitrary, let $\varphi \in V^*$ be arbitrary.

Then $F(id_V)(\psi)(\varphi) = (\psi \circ id^*)(\varphi)$ (1)

where $id^* : V^* \longrightarrow V^*$
$\delta \longmapsto \delta \circ id$

So (1) $= (\psi \circ id)(\varphi) = \psi(\varphi)$.

So $F(id_V) = id_{V^{**}}$.

E: Preservation of Composition:

Let $f : V \longrightarrow W$, $g : W \longrightarrow Z$.

Then
$$F(g \circ f) : V^{**} \longrightarrow Z^{**}$$

Need $\forall \psi \in V^{**}$, $F(g \circ f)(\psi) = (F(g) \circ F(f))(\psi)$

That is, for all $\varphi \in V^*$:
$$F(g \circ f)(\psi)(\varphi) = (F(g) \circ F(f))(\psi)(\varphi)$$

We calculate:

$$
\begin{aligned}
F : \mathbb{C}\text{-Vect} &\longrightarrow \mathbb{C}\text{-Vect} \\
V &\longmapsto V^{**} \\
(f : V \to W) &\longmapsto \left(\begin{aligned} f^{**} : V^{**} &\longrightarrow W^{**} \\ \psi &\longmapsto \psi(\_ \circ f) \end{aligned} \right)
\end{aligned}
$$

Composition: $f : V \to W$, $g : W \longrightarrow Z$, $\psi \in V^{**}$ $(V^{*} \longrightarrow \mathbb{C})$.

$$F(g \circ f)(\psi) = \psi(\_ \circ g \circ f)$$

$$
\begin{aligned}
(F(g) \circ F(f))(\psi) &= F(g)\big(\psi(\_ \circ f)\big) \\
&= \big(\psi(\_ \circ f) \circ\big)(\_ \circ g) \\
&= \psi(\_ \circ g \circ f)
\end{aligned}
$$

So $F(g \circ f) = F(g) \circ F(f)$. $\square$

Def: A morphism $f : A \longrightarrow B$ in a category 10
$\mathcal{C}$ is a <u>monomorphism</u> (or is <u>monic</u>) if for all
$g, h : \cancel{1}\, C \longrightarrow A$ we have:

$$f \circ g = f \circ h \Rightarrow g = h$$

(Easy to remember: injectivity is:

$$f(x) = f(y) \Rightarrow x = y$$

monomorphism is the same but x and y are now morphisms).

Lemma: In the category of sets a function $f : A \longrightarrow B$ is monic iff it is injective.

Proof: Say $f : A \longrightarrow B$ is monic. Let $x, y \in A$ be such that $f(x) = f(y)$. We consider the functions $g, h : \{\cdot\} \longrightarrow A$ given by

$$g(\cdot) = x$$
$$h(\cdot) = y.$$

then $(f \circ g)(\cdot) = (f \circ h)(\cdot) \Rightarrow f \circ g = f \circ h$

$$\Rightarrow g = h$$
$$\Rightarrow g(\cdot) = h(\cdot)$$
$$\Rightarrow x = y.$$

Say $g, h : C \longrightarrow A$ are such that

$$f \circ g = f \circ h.$$

Then $\forall x \in C$ we have:

$$(f \circ g)(x) = (f \circ h)(x)$$
$$\Rightarrow f(g(x)) = f(h(x))$$
$$\Rightarrow g(x) = h(x) \text{ by injectivity.}$$

Thus $g = h$. $\square$

Def^n: A morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is an <u>epimorphism</u> (or is <u>epic</u>) if for all $g, h : B \longrightarrow C$ we have:

$$g \circ f = h \circ f \Rightarrow g = h.$$

Lemma: A function $f : A \longrightarrow B$ is epic iff it is surjective.

Proof: ~~Say f is epic.~~ Say f is not surjective. Let $b \in B \setminus \operatorname{im} f$. Consider the two functions:

$$\begin{aligned} g : B &\longrightarrow \{\bullet, *\} \\ y &\longmapsto \bullet \end{aligned}$$

$$\begin{aligned} h : B &\longrightarrow \{\bullet, *\} \\ y &\longmapsto \begin{cases} * & , y = b \\ \bullet & , \text{else.} \end{cases} \end{aligned}$$

then $g \neq h$ but $g \circ f = h \circ f$.

Thus f is not epic.

Now say f is surjective. Let $g, h : B \longrightarrow C$ be such that $g \circ f = h \circ f$.

Let $y \in B$. Since f is surjective, $\exists\, x \in A$ such that $f(x) = y$.

then $(g \circ f)(x) = (h \circ f)(x)$

$\Rightarrow g(y) = h(y)$.

$\blacksquare$

Def^n: A morphism $f : A \longrightarrow B$ in a category is an <u>isomorphism</u> if there exists $g : B \longrightarrow A$ such that

$$f \circ g = id_B, \quad g \circ f = id_A .$$

WARNING:

$$\text{Monic} + \text{Epic} \not\Rightarrow \text{Isomorphism}.$$

$\iota : \mathbb{Z} \longrightarrow \mathbb{Q}$ in the category of rings is monic and epic but not ~~isomor~~ an isomorphism.

Def^{n}: An <u>equivalence of categories</u>

$F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor such that:

- The induced functions

$$F_f : \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(FX,FY)$$

for each morphism $f: X \longrightarrow Y$ in $\mathcal{C}$ are injective and surjective.

- F is <u>essentially surjective</u>, that is,

  for every object $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ ~~such that~~ and an isomorphism

$$f: FC \longrightarrow D.$$

Example: Let <u>$\mathbb{C}$-Vect</u> denote the category of complex finite dimensional vector spaces. ~~the~~ Let $\mathrm{Mat}(\mathbb{C})$ denote the category with objects $\mathbb{N}$ and morphism $n \longrightarrow m$ $m \times n$ matrices.

There is a functor

$$F: \mathrm{Mat}(\mathbb{C})^{op} \longrightarrow \mathbb{C}\text{-Vect}$$

$$n \longmapsto \mathbb{C}^n$$

$$(M: n \to m) \longmapsto \left(\begin{array}{c} \mathbb{C}^m \longrightarrow \mathbb{C}^n \\ z \longmapsto Mz \end{array} \right)$$

under the standard basis.

F is an equivalence of categories.

WARNING: If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of categories, then this does not imply there exists $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that

$$G \circ F = Id_{\mathcal{C}} \quad F \circ G = Id_{\mathcal{D}}.$$

Our previous example is such a thing.

We will come back to this...

Defn: Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^{op}$ the collection of ~~ob~~ objects the same as $\mathcal{C}$, and let

$$Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X).$$

Then $\mathcal{C}^{op}$ is a category.

A functor out of an opposite category is sometimes called a contravariant functor.