Normal functors

mply

In this class, we learnt about the simply typed λ -calculus, which attributes types to a subset of the terms in the untyped λ -calculus, indeed, the typing system can be seen as a restriction to the terms which may be given as input to other terms.

So how do we think of terms in the untyped setting as different to those in the typed setting?

but all of these act very differently to one-another.

Recall: The untyped I - calculus is a syntactic system, so to determine what it is the language of, we must find a piece of mathematics which is captured by this language.

Idea: Abstraction is a function, but one with its beliaviour described by variables, so maybe 1-terms are polynomials?

Clumsy idea: Identify: Variables: N) ---> NO n ---->n (this is the polynomia 1/2c"). Applications: ~ N×N -> N (min) -) [MJ(m) [NJ(n (multiplication) Abstraction: N ----> N Xx.M ~ n .---- , [M] (n) But what is (\(\gamma y . \gamma \) w? NX NX NX NX (n, m, p) | m.w but Z is: N - N and these are not equal, so this system betrays B-reduction and is thus not a system described by 2 - calculus!

part, models of the untyped λ -calculus (3) hard to find because you need an object ich is "reflexive", in the sense that maps on are in Lijection with a inelf.

Why? Because:

then $\sigma = \sigma_1 \rightarrow \sigma_2$, and $\sigma = \sigma_1$, and $\sigma = \sigma_2$, and $\sigma = \sigma_2$, for all σ_2 , including σ_2 .

that is:

So what behaves like this?

Def: Let A be a set. A functor J: Set A _____, set

is analytic if there exists a set of sets

3 Ca3 G E Int A

such that for all X & Set 4,

F(X) = IL Cax Set A(G, X)

where Int A is the set of integral functions A -> set.

Def?: A functor G: A --- > Set is integral

if G(a) = of for all but finitely many a e A,

and if for ell a e A, G(a) is finite a

Def?: Define: 0:= \$ 1:= {0}= {\$\$} 2 := 30,13 = 3 \$, 3\$3} 3:= 30,2,23=34,343,36,3\$3}}

0,1,2,... are Von- Neumann The sets integers.

Theorem:

A functor F: Set A set is analytic if and only if it is normal, that is, it preserves wide pullbacks and directed colinits. We will use this theorem to construct a model of the untyped > - calculus using analytic functors. Def-: Leb Ass be the smallest set satisfying the following:

x e A p

if XEAD is of finite cardinality, f: X->Nsoisa function, and a EA => 1323 then $(f,a) \in A\infty$. If a = +, then

There exists a bijection: q: An - Int An X An iven (f,a) EA as, construct Ff: Ass --- Set , ff(b), bedomf then Ff is integral, define $g(f,a) = (F_f,a).$ A180, g(*) = (0, *).Prof: Clearly injective. Clearly surjective. This induces an isomorphism of categories: Set 9: Set In+Am×Am ____ > Set to x .____, X.2. This is an isomorphism, so by the theorem is analytic (being normal). Next define: App: Set Int Assistan x Set Ass App(F,X)(a) = IL F(G,a) x Set (G,X) . Notice that this is analytic and thus normal. Now we define a model of λ -calculus in normal

· functors.

Oef: Let b be an untyped l-term and (xz, m, x. a list of variables containing the free variables of We define an interpretation with respect to this context as a normal functor Ix1,..., xn | E]: (Set 10) -If E=xi is a variable: [x1,..., xn | E](X2,..., Xn) = Xi. If 6 = su is an application: [x2,..., xn | su] (X2,..., Xn) = App (Set 2)-1 ([x1,...,xn] (X1,..., Xn)), [x1,...,xn | E] (X1,..., Xn) If $t = \lambda y \cdot s$ is an abstraction: Given [x1,...,xn,y | s](X1,..., xn, Y)(a) = (GS,--,Gn,H) (a) x Set A0 (G:, X2) x--x Set A0 (Gn,Xn) E IntApprox × Set Am (H, Y) We define E+: (Sex 40) --- Sex Int An X AD $(X_2,...,X_n) \longrightarrow ((H,a) \mapsto$ (G2,...,Gn) (G2,X1) (C) X Set AD (G2,X1) X --- X Set AD (G2,X1) (G2,X1) X--- X Set AD (G2,X1) Then define II x 1, m, x n | t I = (Set 2) - 1 +

Theorem: this is a model of the untyped 1-calculus.

1

re is a curiosity:

We expected 0->0 "=" o

Int Aoo x Aoo = Aoo

So what is "In+" from the perspective of the 1-calculus?

It seems like we have an embedding:

0->0 mm @0=>0

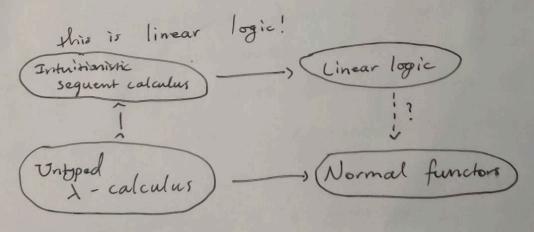
What is @?

Let's use! instead.

So the type system is:

 $O_2,..., \sigma_{n-2} + \sigma_{n-7}$ and some ! I rule ...

This looks close to Intuitionistic logic, but a subfragment, and with a modality...



This is an example of how logics are discovered