Describing finite colimits using the internal language of a topos

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1 Introduction

Given a surjective family of functions of sets $\{t_i : A_i \to A\}_{i=0}^{\infty}$ and a family of functions $\{g_i : A_i \to U\}_{i=0}^{\infty}$, if

$$\forall a_i \in A_i, \forall a_j \in A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{1}$$

then there exists a well defined function $f : A \to U$ which is given by "choosing a lift" $a_i \in A_i$ of $a \in A$ along some t_i and then defining $f(a) := g_i(a_i)$. In choosing a lift $a_i \in A_i$ of $a \in A$ there are two pieces of information, one is that $t_i(a_i) = a$ and the other is $g_i(a_i)$, which can be captured by the following subset of $A \times U$:

$$\left\{\vec{z} \in A \times U \mid \exists i \in \mathbb{N}, \exists a_i \in A_i, \vec{z} = \big(t_i(a_i), g_i(a_i)\big)\right\} \subseteq A \times U$$

which we denote by X. Thus there is the following diagram

$$\begin{tikzcd} & X \arrow[d, tail] & \\ & A \times U \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_R"] & \\ A & & U \end{tikzcd}$$

It then follows from Equation 1 that there exists a bijection $\hat{f} : X \xrightarrow{\sim} A$. The function $\pi_R \hat{f}^{-1}$ is then equal to f. The following Lemma generalises this description to an arbitrary elementary topos $\mathscr{E}$ by making use of the *internal language* (see [2, §D1, §D4.1]) of $\mathscr{E}$:

Lemma 1.0.1. *Let $\{t_i(a_i) : A\}_{i=0}^{\infty}$ be a finite set of terms satisfying the following sequent:*

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, t_i(a_i) = a \tag{2}$$

Let $\{g_i : A_i \to U\}_{i=0}^{\infty}$ be a set of morphisms in $\mathscr{E}$, and assume the following sequent holds for each i, j:

$$\vdash \forall a_i : A_i, \forall a_j : A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{3}$$

Then there exists a (necessarily unique) morphism $f : A \to U$ such that for each i, the following diagram commutes

$$\begin{tikzcd} A_i \arrow[d, "t_i"'] \arrow[dr, "g_i"] & \\ A \arrow[r, "f"'] & U \end{tikzcd}$$

Lemma 1.0.1 will be proved in Section 2, and constitutes the main technical contribution of this paper. The significance of Lemma 1.0.1 is emphasised by the fact that in the topos $\underline{\text{Sets}}$ both the coproduct of a collection of functions and the coequaliser of a pair of functions are examples of morphisms satisfying the hypotheses of Lemma 1.0.1 (and thus, so are all colimits) as the following examples demonstrate:

Example 1.0.1. *Say $\{g_i : A_i \to U\}_{i=0}^{\infty}$ is a family of functions, and $\iota_i : A_i \rightarrowtail \coprod_{i=0}^{\infty} A_i$ is the i^{th} inclusion map. Then there exists a unique morphism $f : \coprod_{i=0}^{\infty} A_i \to U$ such that $f\iota_i = g_i$ for each i, and this function f is given by:*

$$
\begin{aligned}
f : \coprod_{i=0}^{\infty} A_i &\to U \\
a_i &\mapsto g_i(a_i), \text{ where } \iota_i(a_i) = a_i
\end{aligned}
$$

Example 1.0.2. *Say $g_0, g_1 : A'' \to A'$ and $e : A' \twoheadrightarrow \mathrm{Coeq}(g_0, g_1)$ is the coequaliser of g_0 and g_1. Then given a morphism $g : A' \to U$ such that $gg_0 = gg_1$ there exists a unique function $f : \mathrm{Coeq}(g_0, g_1) \to U$ such that $fe = g$, and this function f is given by:*

$$
\begin{aligned}
f : \mathrm{Coeq}(g_0, g_1) &\to U \\
[a] &\mapsto g(a), \text{ where } e(a) = [a]
\end{aligned}
$$

In Section 3, Lemma 1.0.1 will be applied to show how given a finite diagram $\mathscr{J} : J \to \mathscr{E}$ in a topos $\mathscr{E}$, one can write a term t in the internal language of $\mathscr{E}$ whose *interpretation* (in the precise sense of [2, §D4.1]) is a colimit of $\mathscr{J}$.

In fact, two different ways of constructing such a term t will be shown. In both approaches, we begin with a finite diagram $\mathscr{J} : J \to \mathscr{E}$ and we choose some presentation:

$$
\begin{tikzcd}
\mathscr{J}(j) \arrow[r, "\mathrm{id}_{\mathscr{J}(j)}"] \arrow[d, "\iota_{\mathscr{J}(f)}"'] & \mathscr{J}(j) \arrow[d, "\iota_{\mathscr{J}(j)}"] & \\
\coprod_{f:j\to j'\in J} \mathscr{J}(j) \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & \coprod_{j\in J} \mathscr{J}(j) \arrow[r] & \mathrm{Coeq}(g_0, g_1) \cong \mathrm{Colim}_{\mathscr{J}}(J) \\
\mathscr{J}(j) \arrow[r, "\mathscr{J}f"'] \arrow[u, "\iota_{\mathscr{J}(f)}"] & \mathscr{J}(j') \arrow[u, "\iota_{\mathscr{J}(j')}"'] &
\end{tikzcd} \tag{4}
$$

There seems to be no way of constructing terms t_0, t_1 whose interpretations are g_0, g_1 on the nose, one possible reason for this is because such terms would need to be of type $\coprod_{j\in J} \mathscr{J} j$ which is a *dependent type* which is not available to the type theory at hand (but see Remark 1.0.2). This is the point where the two approaches split. In the first approach (given in more detail in Section 3) we give a general way of describing the coproducts which appear in Diagram 4 in the internal language of $\mathscr{E}$ as subobjects of the objects with the same names as the types $\prod_{f:j\to j'} \Omega^{\mathscr{J}(j)}$ and $\prod_{j\in J} \Omega^{\mathscr{J}(j)}$, we denote the domain of these subobjects $\coprod_{j\in J} \mathscr{J}(j)$ and $\coprod_{f:j\to j'} \mathscr{J}(j)$ respectively. We then construct two terms whose interpretations yield two morphisms

$$\prod_{f:j\to j'} \Omega^{\mathscr{J}(j)} \rightrightarrows \prod_{j\in J} \Omega^{\mathscr{J}(j)}$$

We then use Lemma A.0.1 to show that these two morphisms descend to the subobjects:

$$\coprod_{j\in J} \mathscr{J}(j) \overset{s_0}{\underset{s_1}{\rightrightarrows}} \coprod_{f:j\to j'} \mathscr{J}(j)$$

These two morphisms, thought of as function symbols, are used to define a formula whose interpretation is a subobject of the object with the same name as the type $\Omega^{\prod_{j\in J} \Omega^{\mathscr{J}(j)}}$. A morphism with codomain given by this subobject is then defined which along with this subobject forms a coequaliser of g_0 and g_1.

Remark 1.0.1. *This concept of working with "bigger" objects within which the situation of interest is embedded is a central theme of this first approach, and is most similar to the method used in [10]. As discussed in [10] there seems to be no way to write terms whose interpretations are morphisms between subobjects. This is problematic because the axioms of an elementary topos imply the existence of much more structure than what is at first clear, such as arbitrary exponentials [1, §IV 2] and all finite colimits [1, §IV 5], but the internal language does not allow one to talk about this extra structure directly. In the author's opinion, this is a drawback of the internal language of a topos. We observe that one could add special types to the internal language whose interpretations correspond to whatever extra structure one wishes to use, but then why use the internal language?*

In the second approach, we adapt the first approach to truly come up with a single term and associated morphism whose interpretations give a coequaliser of g_0 and g_1 (without the use of "interim" morphisms s_0, s_1). The downside of this approach is that it does not make use of the universal property of the coproduct and thus is more ad hoc than the previous, despite being simpler in concept.

Remark 1.0.2. *We observe that the internal language of an* ∞-topos *(which is a model of* homotopy type theory *[9]) does allow for dependent types, we leave writing finite colimits in the internal language of an* ∞*-topos as a future research question.*

2 Dealing with Surjective Families

Lemma 2.0.1. *Let* $\{t_i(a_i) : A\}_{i=0}^{\infty}$ *be a finite set of terms satisfying the following sequent:*

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, t_i(a_i) = a \tag{5}$$

Let $\{g_i : A_i \to U\}_{i=0}^{\infty}$ *be a set of morphisms in* $\mathscr{E}$*, and assume the following sequent holds for each* i, j*:*

$$\vdash \forall a_i : A_i, \forall a_j : A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{6}$$

Then there exists a (necessarily unique) morphism $f : A \to U$ *such that for each* i*, the following diagram commutes*

$$\begin{tikzcd} A_i \arrow[d, "t_i"'] \arrow[dr, "g_i"] & \\ A \arrow[r, "f"'] & U \end{tikzcd}$$

Proof. First, define the following subobject

$$[\![(z : A \times U). \bigvee_{i=0}^{n} (\exists a_i : A_i, z = \langle t_i(a_i), g_i a_i \rangle)]\!] \overset{c}{\rightarrowtail} A \times U$$

To ease notation let $\phi_i(z, a_i)$ be the formula $z = \langle t_i(a_i), g_i a_i \rangle$ and $\phi(z)$ the formula $\bigvee_{i=0}^{\infty} \exists a_i : A_i, \phi_i(z, a_i)$. Then consider the composite

$$[\![(z : A \times U).\phi(z)]\!] \overset{c}{\rightarrowtail} A \times U \overset{\pi_A}{\to} A$$

It now suffices to show that this is an isomorphism, as then the morphism f can be taken to be $\pi_U c(\pi_A c)^{-1}$. Since every elementary topos is *balanced*, that is, any morphism in a topos which is both epic and monic is an isomorphism (see [1, §IV.1 Prop 2]), it suffices to show this of $\pi_A c$. By Lemma [10, 3.4.2] it suffices to show

$$\phi(z_1) \wedge \phi(z_2) \vdash_{z_1, z_2 : A \times U} \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \Rightarrow z_1 = z_2 \tag{7}$$

and

$$\vdash_{a:A} \exists z : A \times U, \phi(z) \wedge \mathrm{fst}(z) = a \tag{8}$$

Recall that the following Sequents hold in any elementary topos:

1.
$$(\vec{x} = \vec{s}) \wedge \psi \vdash_{\vec{y}} \psi[\vec{x} := \vec{s}]$$

where $\vec{x}$ is a string of variables, $\vec{s}$ is a string of terms with the same length and type as $\vec{x}$, and no free variable in ψ becomes bound in $\psi[\vec{x} := \vec{s}]$ (this follows from the *substitution* and *equality* axioms given in [2, §4.1 Definition 1.3.1]).

2.
$$\vdash_{w:W_1 \times W_2} \langle \mathrm{fst}(w), \mathrm{scd}(w) \rangle = w$$

(See [2, §D4.1 Lemma 4.1.6])

3.
$$z_1 = z_2 \vdash_{z_1:Z, z_2:Z} \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

4.
$$\psi[x := t] \vdash_{\vec{y}} \exists x : X, \psi$$

To show that Sequent 7 holds, it suffices to show that for each i, j:

$$\exists a_i : A_i, \exists a_j : A_j, \phi_i(z_1, a_i) \wedge \phi_j(z_2, a_j) \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \vdash_{z_1, z_2 : A \times U} z_1 = z_2 \tag{9}$$

by 2 above, it suffices to show:

$$\exists a_i : A_i, \exists a_j : A_j, \phi_i(z_1, a_i) \wedge \phi_j(z_2, a_j) \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \vdash_{z_1, z_2 : A \times U} \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

ie,

$$z_1 = \langle t_i(a_i), g_i a_i \rangle \wedge z_2 = \langle t_j(a_j), g_j a_j \rangle \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

where $\Gamma = (a_i : A_i, a_j : A_j, z_1 : A \times U, z_2 : A \times U)$. By 1 above, it suffices to show

$$\mathrm{fst}(\langle t_i(a_i), g_i a_i \rangle) = \mathrm{fst}(\langle t_j(a_j), g_j a_j \rangle) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

that is,

$$t_i(a_i) = t_j(a_j) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

Since $=$ is an equivalence relation [2, §4.1 Definition 1.3.1b] and using 3 above, we have

$$z_1 = \langle t_i(a_i), g_i a_i \rangle \wedge z_2 = \langle t_j(a_j), g_j a_j \rangle \vdash_\Gamma \mathrm{scd}(z_1) = g_i a_i \wedge \mathrm{scd}(z_2) = g_j a_j$$

thus it suffices to show:

$$t_i(a_i) = t_j(a_j) \vdash_\Gamma g_i a_i = g_j a_j$$

which is exactly Sequent 6.

To show that Sequent 8 holds, by 4 above, it suffices to show

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, \langle t_i(a_i), g_i a_i \rangle = \langle t_i(a_i), g_i a_i \rangle \wedge t_i a_i = a$$

which follows from Sequent 5. $\square$

3 Finite Colimits

3.1 Coequalisers

In the topos Sets the coequaliser of functions $f, g : A \to B$ is given by ${}^{B}/_{\sim}$, where $\sim$ is the smallest equivalence relation on B such that $f(a) \sim g(a)$, for all $a \in A$. We emulate this by taking the subobject of Ω^B consisting of "equivalence classes" of $\sim$. First though, we define a term which simulates $\sim$:

Definition 3.1.1. *Given morphisms $g_0, g_1 : A \to B$ in $\mathscr{E}$, let R_{g_0,g_1} be the term*

$$\Big\{z : B \times B \,|\exists b_1, b_2 : B,\ z = \langle b_1, b_2\rangle \wedge \Big(b_1 = b_2$$
$$\vee \bigvee_{n=1}^{\infty} \bigvee_{\alpha \in \mathbb{Z}_2^n} (\exists a_1, ..., a_n, (b_1 = g_{\alpha_0} a_1)$$
$$\wedge (g_{\alpha_1+1} a_1 = g_{\alpha_1} a_2) \wedge ... \wedge (g_{\alpha_{n-1}+1} a_{n-1} = g_{\alpha_n} a_n)$$
$$\wedge (g_{\alpha_n+1} a_n = b_2))\Big)\Big\}$$

where $\mathbb{Z}_2^n$ is the set of length n sequences of elements of $\mathbb{Z}_2$.

The idea of this term is that it is a generalisation of the smallest equivalence relation $\sim$ such that for all a, $g_0(a) \sim g_1(a)$. It does this by declaring $\langle b_1, b_2\rangle$ to be in the relation if either $b_1 = b_2$, or there exists a sequence $(a_1, ..., a_n)$ which "connect" b_1 and b_2 by the images of these a_i under g_0 and g_1. A diagram representing an example of this is the following,

$$\begin{tikzcd}[column sep=small]
 & a_1 \arrow[dl] \arrow[dr] & & a_2 \arrow[dl] \arrow[dr] & & a_3 \arrow[dl] \arrow[dr] & \\
b_1 = g_0(a_1) & & g_1(a_1) = g_0(a_2) & & g_1(a_2) = g_0(a_3) & & g_1(a_3) = b_2
\end{tikzcd}$$

However, it must also be allowed for that g_0 and g_1 do not always appear in this order, due to the symmetry axiom of an equivalence relation, this is why the set $\mathbb{Z}_2^n$ is considered. The next Definition generalises "the set of elements related to b under R_{g_0,g_1}":

Definition 3.1.2. *In the setting of Definition 3.1.1, for any variable $b : B$, define the term*

$$[b]_{g_0,g_1} := \{b' : B \mid \langle b, b'\rangle \in R_{g_0,g_1}\}$$

The morphism $[\![(b : B).[b]]\!]$ factors through its image:

$$\begin{tikzcd}
 & B \arrow[dl, "{[\![(b:B).[b]]\!]}"'] \arrow[d, two heads, "s"] \\
\Omega^B & {[\![(z : \Omega^B).\exists b : B, [b] = z]\!]} \arrow[l, tail, "c"]
\end{tikzcd}$$

Since s is epic the following sequent holds:

$$\vdash_{z:Z} \exists b : B, sb = z$$

where Z is the codomain of s. Thus, by Lemma 2.0.1, if there exists a morphism $g : B \to U$ such that

$$sb_1 = sb_2 \vdash_{b_1:B, b_2:B} gb_1 = gb_2$$

that is, $gg_0 = gg_1$, then there exists a unique morphism $f : Z \to U$ such that the diagram

$$\begin{tikzcd}
B \arrow[d, "s"'] \arrow[dr, "g"] & \\
Z \arrow[r, "f"'] & U
\end{tikzcd}$$

commutes. That is, (Z, s) is the *coequaliser* of g_0 and g_1.

3.2 Finite Coproducts

Recall that in the category <u>Sets</u> the coproduct of a finite collection of sets $\{A_i\}_{i=1}^n$ is the disjoint union $\coprod_{i=1}^n A_i$ along with the inclusion maps $\{\iota_i : A_i \to \coprod_{i=1} A_n\}_{i=1}^n$. Here we emulate the disjoint union by viewing A, B as subsets of "marked" singletons in $\Omega^A \times \Omega^B$. Throughout, the notation $\{a\} = \{a' : A \mid a' = a\}$, and $\varnothing_{a:A} = \{a : A \mid \bot\}$ will be used.

Definition 3.2.1. *Let $\{A_i\}_{i=1}^n$ be a finite set of objects, then let $\iota_{A_i}(a_i)$ be the following term:*

$$\langle\langle ...\langle \varnothing_{a_0:A_0}, \varnothing_{a_1:A_1}\rangle, \varnothing_{a_2:A_2}\rangle, ...\rangle, \varnothing_{a_{i-1}:A_{i-1}}\rangle, \{a_i\}\rangle, \varnothing_{a_{i+1}:A_{i+1}}\rangle, ...\rangle, \varnothing_{a_n:A_n}\rangle$$

Each morphism $[\![(a_i : A_i).\iota_i(a_i)]\!]$ factors through the collection of images of all such morphisms:

$$\begin{tikzcd}
 & A_i \arrow[dl, "{[\![(a_i:A_i).\iota_i(a_i)]\!]}"'] \arrow[d, "s_i"] \\
\prod_{i=1}^n \Omega^{A_i} & {[\![(z : \prod_{i=1}^n \Omega^{A_i}). \bigvee_{i=1}^n \exists a_i : A_i, \iota_i(a_i) = z]\!]} \arrow[l, tail, "c"]
\end{tikzcd}$$

We suggestively denote the domain of c by $\coprod_{i=1}^n A_i$. The collection $\{s_i : A_i \to \coprod_{i=1}^n A_i\}_{i=1}^n$ forms a surjective family, that is, the following sequent holds:

$$\vdash_{z:\coprod_{i=1}^n A_i} \bigvee_{i=1}^n \exists a_i : A_i, s_i a_i = z$$

Thus, by Lemma 2.0.1, if there exists a collection of morphisms $\{g_i : A_i \to U\}_{i=1}^n$ then there exists a unique morphism $f : \coprod_{i=1}^n A_i \to U$ such that the for each i the diagram

$$\begin{tikzcd}
A_i \arrow[d, "s_i"'] \arrow[dr, "g_i"] & \\
\coprod_{i=1}^n A_i \arrow[r, "f"'] & U
\end{tikzcd}$$

commutes. This is because the hypotheses of Lemma 2.0.1, that for each i, j

$$s_i(a_i) = s_j(a_j) \vdash_{a_i:A_i, a_j:A_j} g_i(a_i) = g_j(a_j)$$

holds, is trivially satisfied as each s_i is monic. Thus, $(\coprod_{i=1}^n A_i, \{s_i\}_{i=1}^n\})$ is the *coproduct* of $\{A_i\}_{i=1}^n$.

3.3 Finite Colimits

Let $\mathscr{J} : J \to \mathscr{E}$ be a finite diagram in a topos $\mathscr{E}$. Choose a presentation

$$\begin{tikzcd}
\mathscr{J}(j) \arrow[r, "\mathrm{id}_{\mathscr{J}(j)}"] \arrow[d, "\iota_{\mathscr{J}(f)}"'] & \mathscr{J}(j) \arrow[d, "\iota_{\mathscr{J}(j)}"] & \\
\coprod_{f:j\to j'\in J} \mathscr{J}(j) \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & \coprod_{j\in J} \mathscr{J}(j) \arrow[r] & \mathrm{Coeq}(g_0, g_1) \cong \mathrm{Colim}_{\mathscr{J}}(J) \\
\mathscr{J}(j) \arrow[r, "\mathscr{J} f"'] \arrow[u, "\iota_{\mathscr{J}(f)}"] & \mathscr{J}(j') \arrow[u, "\iota_{\mathscr{J}(j')}"'] &
\end{tikzcd} \tag{10}$$

From the remainder of this Section, the notation $\coprod_{j\in J} \mathscr{J}(j)$ will mean the domain of the monic

$$[\![(z : \prod_{f:j\to j'} \Omega^{\mathscr{J}(j)}). \bigvee_{j\in J} \exists a_j : \mathscr{J} j, \iota_j(a_j) = z]\!] \rightarrowtail \prod_{f:j\to j'} \Omega^{\mathscr{J}(j)}$$

(see Definition 3.2.1 for the definition of the term $\iota_j(a_j)$) and similarly for $\coprod_{f:j\to j'} \mathscr{J}(j)$. We have the solid arrows in the following Diagram:

$$\begin{tikzcd}
\prod_{f:j\to j'} \Omega^{\mathscr{J}(j)} \arrow[r, dashed, shift left, "{[\![(w_0:\prod_{f:j\to j'} \Omega^{\mathscr{J}(j)}).t_0(w_0)]\!]}"] \arrow[r, dashed, shift right, "{[\![(w_1:\prod_{f:j\to j'} \Omega^{\mathscr{J}(j)}).t_1(w_1)]\!]}"'] & \prod_{j\in J} \Omega^{\mathscr{J}(j)} \\
\coprod_{f:j\to j'} \mathscr{J}(j) \arrow[u, tail] \arrow[r, dashed, shift left, "s_0"] \arrow[r, dashed, shift right, "s_1"'] & \coprod_{j\in J} \mathscr{J}(j) \arrow[u, tail]
\end{tikzcd} \tag{11}$$

The two horizontal dashed arrows on the top of Diagram 11 are respectively given by

$$t_0(w_0) = \iota_j(\pi_j(w_0))$$
$$t_1(w_1) = \iota_j(\Omega^{\mathscr{J}(f)}(\pi_j(w_0)))$$

s_0 and s_1 are then the morphisms induced by these terms given by applying Lemma A.0.1.

Now, for the remainder of this Section, the notation $\mathrm{Coeq}(s_0, s_1)$ will mean the domain of the monic

$$[\![(z : \Omega^{\prod_{j\in J} \mathscr{J}(j)}).\exists w : \prod_{j\in J} \mathscr{J}(j), z = [w]_{cs_0,cs_1}]\!] \rightarrowtail \Omega^{\prod_{j\in J} \Omega^{\mathscr{J}(j)}}$$

We now have the solid arrows in the following Diagram:

$$\begin{tikzcd}
 & \prod_{j\in J} \Omega^{\mathscr{J}(j)} \arrow[r, dashed, "{[\![(b:\prod_{j\in J} \Omega^{\mathscr{J}(j)}).t(b)]\!]}"] & \Omega^{\prod_{j\in J} \Omega^{\mathscr{J}(j)}} \\
\coprod_{f:j\to j'} \mathscr{J}(j) \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & \coprod_{j\in J} \mathscr{J}(j) \arrow[u, tail] \arrow[r, dashed, "s"] & \mathrm{Coeq}(s_0, s_1) \arrow[u, tail]
\end{tikzcd} \tag{12}$$

The horizontal dashed arrow on the top of Diagram 12 is given by

$$t(b) = \{z : \prod_{j\in J} \Omega^{\mathscr{J}(j)} \mid z = [b]_{cs_0,cs_1}\}$$

and s is induced by this term again given by applying Lemma A.0.1. The pair $(s, \mathrm{Coeq}(s_0, s_1))$ is the colimit of the diagram $\mathscr{J} : J \to \mathscr{E}$. Notice that this pair makes use of the function symbols s_0 and s_1 which is not desireable as these do not lie in the image of the functor $\mathscr{J}$, in the next section we show an alternative approach which truly comes up with a single formula written using only those function symbols which appear in the image of $\mathscr{J}$ whose interpretation is the colimit of J through $\mathscr{J}$.

4 Another Approach

In Section 3 it was shown how to write finite coproducts and coequalisers using the Internal Logic of a topos $\mathscr{E}$. From this, one can derive a way of writing a colimit of a finite diagram $\mathscr{J} : J \to \mathscr{E}$ using this type theory, however, this term will not be written purely using the objects and morphisms from the indexing category J as induced morphisms are also required. In this Section we show how to alter the approach used in Section 3 so that one can obtain a term t whose interpretation is a colimit of the diagram $\mathscr{J}$ and is such that t does not consist of any function symbols other than those with the same name as some morphism in J.

We begin with the following question: say there are two sets of morphisms $\{g_0^i : A_i \to B\}_{i=0}^n$ and $\{g_1^i : A_i \to B\}_{i=0}^n$ such that $\coprod_{i=0}^n g_0^i = g_0$ and $\coprod_{i=0}^n g_1^i = g_1$, then how can we describe the coequaliser of g_0 and g_1 in the Internal Logic of $\mathscr{E}$ using the morphims g_0^i and g_1^i?

Considering the above circumstance in the topos Sets the image of g_0 is given by

$$\{b \in B \mid \exists i \in \{0, ..., n\}, \exists a_i \in A_i, g_0^i(a_i) = b\}$$

and similarly for g_1. This description of the images can be substituted into the term given in Definition 3.1.1 to give an answer to the question posed in the second paragraph of this Secition:

Definition 4.0.1. *Let $\{A_i\}_{i=0}^k$ be a set of objects in a topos $\mathscr{E}$ and let $\{g_0^i : A_i \to B\}_{i=0}^k$ and $\{g_1^i : A_i \to B\}_{i=0}^k$ be two sets of morphisms. Define $R_{\{g_0^i\}_{i=0}^k, \{g_1^i\}_{i=0}^k}$ be the following term:*

$$\begin{aligned}\Big\{z : & B \times B \mid \exists b_1, b_2 : B, z = \langle b_1, b_2 \rangle \wedge \Big(b_1 = b_2 \\ & \vee \bigvee_{n=1}^{\infty} \bigvee_{\alpha \in \mathbb{Z}_2^n} (\exists z^1 ... z^n : \prod_{i=1}^{n} \Omega^{A_i}, \bigwedge_{j=1}^{n} (\bigvee_{m=0}^{k} \exists a_m^j : A_m, \iota_m(a_m^j) = a^m) \\ & \wedge (\bigvee_{m=0}^{k} b_1 = g_{\alpha_1}^m a_m^1) \\ & \wedge (\bigvee_{m=0}^{k} \bigvee_{l=0}^{k} g_{\alpha_2+1}^l a_l^1 = g_{\alpha_1}^m a_m^2) \wedge \ldots \wedge (\bigvee_{m=0}^{k} \bigvee_{l=0}^{k} g_{\alpha_{n-1}+1}^l a_l^1 = g_{\alpha_n}^m a_m^2) \\ & \wedge (\bigvee_{l=0}^{k} g_{\alpha_n}^l a_l^n = b_2) \Big\}\end{aligned}$$

Also define the following term with free variable $b : B$:

$$[b]_{\{g_0^i\}_{i=0}^n, \{g_1^i\}_{i=0}^n} := \{b' : B \mid \langle b, b' \rangle \in R_{\{g_0^i\}_{i=0}^k, \{g_1^i\}_{i=0}^k}\}$$

The morphism $[\![(b : B).[b]]\!]$ factors through its image:

$$\begin{tikzcd} & B \arrow[dl, "{[\![(b:B).[b]]\!]}"'] \arrow[d, two heads, "s"] \\ \Omega^B & {[\![(z : \Omega^B).\exists b : B, [b] = z]\!]} \arrow[l, hook', "c"] \end{tikzcd}$$

Denote the domain of the monomorphism c by Z, then for the same reason as given in Section 3.1 it then follows that (Z, s) is the coequaliser of g_0 and g_1.

5 Examples

Let $\mathscr{C}$ be the category generated by the following diagram

$$\begin{tikzcd} & X_0 & \\ X_3 \arrow[ur] \arrow[dr] & & X_1 \arrow[ul] \arrow[dl] \\ & X_2 & \end{tikzcd}$$

and let $B\mathscr{C}$ denote its nerve. Then $B\mathscr{C}$ is the colimit of the following diagram:

$$\begin{tikzcd}
 & & {[0]} \arrow[dl, "\epsilon_1^0"', "i=0"] \arrow[dr, "\epsilon_1^0", "i=0"'] & & \\
 & {[1]_0} & & {[1]_1} & \\
{[0]} \arrow[ur, "\epsilon_1^1", "i=3"'] \arrow[dr, "i=3", "\epsilon_1^1"'] & & & & {[0]} \arrow[ul, "\epsilon_1^1"', "i=1"] \arrow[dl, "i=1"', "\epsilon_1^1"] \\
 & {[1]_4} & & {[1]_3} & \\
 & & {[0]} \arrow[ul, "i=2", "\epsilon_0^1"'] \arrow[ur, "i=2"', "\epsilon_0^1"] & &
\end{tikzcd}$$

Thus there is the following commuting diagram

$$\begin{tikzcd}
{[0]} \arrow[r, "\epsilon_1^{i(\mathrm{mod}\,2)}"] \arrow[d, "\iota_i"'] & {[1]} \arrow[d, "\iota_i"] & \\
\coprod_{i\in\mathbb{Z}_4}[0] \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & \coprod_{i\in\mathbb{Z}_4}[1] \arrow[r, dashed] & \mathrm{Coeq}(g_0, g_1) \cong B\mathscr{C} \\
{[0]} \arrow[u, "\iota_i"] \arrow[r, "\epsilon_1^{i\,\mathrm{mod}\,2}"'] & {[1]} \arrow[u, "\iota_{i+1}"'] &
\end{tikzcd}$$

5.1 Example via the first approach

In accordance with Definition 3.2.1 let $a_0 : [1]$ be a variable and $\iota_0(a_0)$ be the term:

$$\langle\langle\langle\{a_0\}, \varnothing_{a_1:[1]}\rangle, \varnothing_{a_2:[1]}\rangle, \varnothing_{a_3:[1]}\rangle$$

and let $\iota_i(a_i)$ for $i = 2, 3, 4$ be defined similarly. Then as described in Section 3.2, the coproduct $\{[1]\}_{i\in\mathbb{Z}_4}$ is given by $(Z, \{s_i\}_{i\in\mathbb{Z}_4})$ where Z is the domain of the subobject:

$$[\![(z : \prod_{i\in\mathbb{Z}_4} \Omega^{[1]}).\ \bigvee_{i\in\mathbb{Z}_4} \exists a_i : [1], \iota_i(a_i) = z]\!] \rightarrowtail \prod_{i\in\mathbb{Z}_4} \Omega^{[1]}$$

and $s_i : [1] \to Z$ is the factorisation of the morphism $[\![(a_i : [1]).\iota_i(a_i)]\!]$ through Z. A subobject of Z is the collection of images of all the morphisms $\{\iota_i\epsilon_1^{i\,\mathrm{mod}2}\}_{i\in\mathbb{Z}_4}$:

$$[\![(z : \prod_{i\in\mathbb{Z}_4} \Omega^{[1]}).\ \bigvee_{i\in\mathbb{Z}_4} \exists a_i : [0], \iota_i(\epsilon_1^{i(\mathrm{mod}2)}(a_i)) = z]\!] \overset{\hat{g}_0}{\rightarrowtail} Z$$

Let g_0^i be the factorisation of $\iota_i\epsilon_1^{i(\mathrm{mod}\,2)}$ through the subobject g_0^i. The family $\{\iota_i(a_i)\}_{i\in\mathbb{Z}_4}$ form an epimorphic family, so by Lemma 2.0.1 there exists a morphism f such that for all i the diagram

$$\begin{tikzcd}
{[0]} \arrow[d, "\iota_i"'] \arrow[dr, "g_0^i"] & \\
\coprod_{i\in\mathbb{Z}_4}[0] \arrow[r, "f"'] & {[\![(z : \prod_{i\in\mathbb{Z}_4} \Omega^{[1]}).\ \bigvee_{i\in\mathbb{Z}_4} \exists a_i : [0], \iota_i(\epsilon_1^{i\,\mathrm{mod}2}(a_i)) = z]\!]}
\end{tikzcd}$$

commutes. The morphism g_0 then corresponds to $\hat{g}_0 f$. Similarly, there is a morphism $\hat{g}_1 f'$ which corresponds to g_1.

5.2 Example via the second approach

The term which one obtains is:

$$\Big\{z:\Omega^B \mid \exists b:B, \big\{b' : B \mid \langle b,b'\rangle \in \{z : B\times B \mid \exists b_1,b_2:B, z=\langle b_1,b_2\rangle \wedge \Big(b_1=b_2$$

$$\vee \bigvee_{n=1}^{\infty}\bigvee_{\alpha\in\mathbb{Z}_2^n}(\exists z^1...z^n : \prod_{i\in\mathbb{Z}_4}\Omega^{[0]]}, \bigwedge_{j=1}^{n}(\bigvee_{m\in\mathbb{Z}_4}\exists a_m^j : [0], \iota_m(a_m^j)=a^m)$$

$$\wedge(\bigvee_{m\in\mathbb{Z}_4} b_1 = \iota_{m+\alpha_1}\epsilon_1^{m \pmod 2} a_m^1)$$

$$\wedge(\bigvee_{m=0}^k\bigvee_{l=0}^k \iota_{m+\alpha_2+1}\epsilon_1^{l\pmod 2}a_l^1 = \iota_{m+\alpha_1}\epsilon_1^{m\pmod 2}a_m^2)\wedge\dots\wedge(\bigvee_{m=0}^k\bigvee_{l=0}^k \iota_{m+\alpha_{n-1}+1}\epsilon_1^{l\pmod 2}a_l^1 = \iota_{m+\alpha_n}\epsilon_1^{m\pmod 2}a_m^2)$$

$$\wedge(\bigvee_{l=0}^k \iota_{m+\alpha_n}\epsilon_1^{l\pmod 2}a_l^n = b_2)\}\} = z\Big\}$$

A Proofs

Lemma A.0.1. *If $t(a)$ is a term of type B with free variable $a:A$, and $[\![a:A \mid p(a)]\!] \rightarrowtail A$ is a subobject of A, then $p(a) \vdash_{a:A} t(a) \in \{b : B \mid q(b)\}$ if and only if there exists a (necessarily unique) morphism $g:[\![a:A\mid p(a)]\!] \to [\![b:B\mid q(b)]\!]$ such that the diagram*

$$\begin{tikzcd}
{[\![a:A \mid p(a)]\!]} \arrow[r, "g"] \arrow[d, tail] & {[\![b:B \mid q(b)]\!]} \arrow[d, tail] \\
A \arrow[r, "{[\![a:A \mid t(a)]\!]}"'] & B
\end{tikzcd}$$

commutes.

Proof. See [10]. □

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