解答

A.1

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$$

5 pt

A.2 證明

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2+\frac{P}{\rho}+gz=\text{const.}$$

上式稱為時變的白努力方程式。注意到是整個區域任意處皆為同個常數。

5 pt

$$\frac{\partial\boldsymbol{v}}{\partial t}+\frac{1}{2}\boldsymbol{\nabla}\left(v^2\right)=-\frac{\boldsymbol{\nabla}P}{\rho}-\rho g\hat{\boldsymbol{z}}=\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}\right)+\frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\phi\right)^2 \qquad (2\text{ pt})$$

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2+\frac{P}{\rho}+gz\right)=0 \qquad (2\text{ pt})$$

因此有

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2+\frac{P}{\rho}+gz=\text{const.}\ \textbf{(各處)} \qquad (1\text{ pt})$$

A.3

$$v\left(t\right)=\sqrt{2gh}\tanh\left(\sqrt{\frac{gh}{2L^2}}t\right)$$

4 pt

$$\phi=\int_O^A\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r}=vL \qquad (1\text{ pt})$$

代入 **A.2** 的方程式可得

$$L\frac{\partial v}{\partial t}+\frac{1}{2}v^2+\frac{P_0}{\rho}=\frac{P_0}{\rho}+gh \qquad (2\text{ pt})$$

$$\frac{1}{2L}\,\mathrm{d}t=\frac{\mathrm{d}v}{2gh-v^2}$$

答案正確 (1 pt)。

B.1

$$v_z=\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\partial h}{\partial t}+\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)h=\frac{\partial h}{\partial t}+v_x\frac{\partial h}{\partial x}$$

2 pt

B.2 證明在介質交界處的動態邊界條件為 3 pt

$$\rho_1\left(\frac{\partial\varphi_1}{\partial t}+U_1\frac{\partial\varphi_1}{\partial x}\right)=\rho_2\left(\frac{\partial\varphi_2}{\partial t}+U_2\frac{\partial\varphi_2}{\partial x}\right)$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2+\frac{P}{\rho}=\frac{1}{2}U^2 \qquad (1\text{ pt})$$

代入

$$\boldsymbol{\nabla}\phi\approx\left(U+\frac{\partial\varphi}{\partial x}\right)\hat{\boldsymbol{x}} \qquad (1\text{ pt})$$

由壓力分佈連續可知 $P_1=P_2$ (1 pt)。代入即可得證。

B.3 2 pt

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$$

$$\nabla^2\varphi=0$$

B.4 7 pt

$$\omega=\left(\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2}+i\frac{\sqrt{\rho_1\rho_2}\,|u|}{\rho_1+\rho_2}\right)k$$

$$U_1\neq\ U_2$$

假設壓力為 $P_i=\bar{P}_i$，其中 $\bar{P}_i$ 為擾動項。界面壓力連續

$$\bar{P}_1=\bar{P}_2 \qquad (1\text{ pt})$$

時變的白努力方程式
由 **B.2** 的結果代入解可得

$$-i\rho_1\left(\omega-kU_1\right)\varphi_1=-i\rho_2\left(\omega-kU_2\right)\varphi_2 \qquad (1\text{ pt})(0)$$

由 **B.1** 可得

$$\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\approx\frac{\partial\varphi}{\partial z} \qquad (1\text{ pt})$$

代入解可得

$$\varphi_1=i\left(\omega-kU_1\right)\frac{h}{k},\quad \varphi_2=-i\left(\omega-kU_2\right)\frac{h}{k} \qquad (1\text{ pt})(1)$$

(0)(1)式解聯立方程式可得

$$\rho_1\left(\omega-kU_1\right)^2+\rho_2\left(\omega-kU_2\right)^2=0 \qquad (1\text{ pt})$$

$$\left(\rho_1+\rho_2\right)\omega^2-2\left(\rho_1U_1+\rho_2U_2\right)k\omega+\left(\rho_1U_1^2+\rho_2U_2^2\right)k^2=0 \qquad (1\text{ pt})$$

$$\omega=\frac{\rho_1U_1+\rho_2U_2\pm\sqrt{-\rho_1\rho_2\left(U_1-U_2\right)^2}}{\rho_1+\rho_2} \qquad (1\text{ pt})$$

答案正確 (1 pt)。

B.5

$$\frac{\omega}{k}=\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2}+\sqrt{-\frac{\rho_1\rho_2u^2}{\left(\rho_1+\rho_2\right)^2}-\frac{g}{k}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)+\frac{\sigma k}{\rho_1+\rho_2}}$$

14 pt

假設壓力為 $P_i=-\rho_igz+\bar{P}_i$，其中 $\bar{P}_i$ 為擾動項。

1. 界面壓力連續

$$-\rho_1gh+\bar{P}_1=-\rho_2gh+\bar{P}_2+\sigma\frac{\partial^2h}{\partial x^2} \qquad (3\text{ pt})(2)$$

2. 時變的白努力方程式

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2+\frac{P}{\rho}+gz=\text{const.} \qquad (1\text{ pt})$$

其中 $\left(\boldsymbol{\nabla}\phi\right)^2\approx U^2+2U\frac{\partial\varphi}{\partial x}$ (1 pt)。代入 $z\to\pm\infty$，$\varphi\left(x,\pm\infty,t\right)\to0$、$\bar{P}\left(x,\pm\infty,t\right)\to0$，因此可得

$$\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}+\frac{\bar{P}}{\rho}=0 \qquad (2\text{ pt})$$

代入(2)式

$$\rho_1\left(\frac{\partial\varphi_1}{\partial t}+U_1\frac{\partial\varphi_1}{\partial x}\right)=\rho_2\left(\frac{\partial\varphi_2}{\partial t}+U_2\frac{\partial\varphi_2}{\partial x}\right)+\sigma k^2h-\left(\rho_1-\rho_2\right)gh \qquad (2\text{ pt})$$

代入解可得

$$-i\rho_1\left(\omega-kU_1\right)\varphi_1=-i\rho_2\left(\omega-kU_2\right)\varphi_2+\sigma k^2h-\left(\rho_1-\rho_2\right)gh \qquad (1\text{ pt})(3)$$

由 **B.1** 可得

$$\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\approx\frac{\partial\varphi}{\partial z} \qquad (1\text{ pt})$$

代入解可得

$$\varphi_1=i\left(\omega-kU_1\right)\frac{h}{k},\quad \varphi_2=-i\left(\omega-kU_2\right)\frac{h}{k} \qquad (1\text{ pt})(4)$$

(3)(4)式解聯立方程式可得

$$\rho_1\left(\omega-kU_1\right)^2\frac{1}{k}+\rho_2\left(\omega-kU_2\right)^2\frac{1}{k}=\sigma k^2-\left(\rho_1-\rho_2\right)g \qquad (1\text{ pt})$$

令 $v=\omega/k$，則

$$(\rho_1+\rho_2)\,v^2-2\,(\rho_1U_1+\rho_2U_2)\,v+\left(\rho_1U_1^2+\rho_2U_2^2\right)-\sigma k+(\rho_1-\rho_2)\,\frac{g}{k}=0 \qquad \text{(1 pt)}$$

答案正確 (1 pt)。

B.6	$$u_c=\left(\frac{4g\sigma}{\rho_1^2\rho_2^2}\,(\rho_2-\rho_1)\,(\rho_1+\rho_2)^2\right)^{\frac{1}{4}}$$	4 pt

由 **B.5** 顯然有

$$-\frac{\rho_1\rho_2u^2}{(\rho_1+\rho_2)^2}-\frac{g}{k}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)+\frac{\sigma k}{\rho_1+\rho_2}<0$$

又因

$$\frac{g}{k}\left(\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\right)+\frac{\sigma k}{\rho_1+\rho_2}\geq\frac{2\sqrt{g\sigma\,(\rho_2-\rho_1)}}{\rho_1+\rho_2}$$

$$u_c^2=2\left(\frac{\rho_1+\rho_2}{\rho_1\rho_2}\right)\sqrt{g\sigma\,(\rho_2-\rho_1)}$$

C.1	瑞利－泰勒不穩定性必定為指數型成長，故微擾一定有 $e^{\gamma t}$ 的形式，所以只能取正根。	1 pt

C.2	$$\sqrt{gk\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)}$$	3 pt

C.3	$$\omega=\sqrt{gk+\frac{\sigma}{\rho}k^3}$$	3 pt

D.1	$$c=\sqrt{\frac{\partial p}{\partial\rho}}$$	3 pt

D.2	$$\boldsymbol{v}_A=\frac{\boldsymbol{B}_0}{\sqrt{\mu_0\rho_0}}$$	3 pt

$$\rho_0\frac{\partial^2\boldsymbol{v}}{\partial t^2}+c^2\boldsymbol{\nabla}\frac{\partial\tilde{\rho}}{\partial t}+\frac{1}{\mu_0}\boldsymbol{B}_0\times\left(\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{b}}{\partial t}\right)=0 \qquad \text{(1 pt)}$$

$$\rho_0 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} - c^2 \rho_0 \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}\right) + \frac{1}{\mu_0}\boldsymbol{B}_0 \times \left\{\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\left(\boldsymbol{v}\times\boldsymbol{B}_0\right)\right]\right\} = 0 \qquad (2\text{ pt})$$

D.3 請證明行進波的色散關係為 3 pt

$$\omega^2 \boldsymbol{v}_0 = \left(c^2 + v_A^2\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\boldsymbol{k} + \left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\left[\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\boldsymbol{v}_0 - \left(\boldsymbol{v}_0\cdot\boldsymbol{v}_A\right)\boldsymbol{k} - \left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\boldsymbol{v}_A\right]$$

此即為描述阿爾芬波的方程式。

$$\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right) = \left(\boldsymbol{A}\cdot\boldsymbol{C}\right)\boldsymbol{B} - \left(\boldsymbol{A}\cdot\boldsymbol{B}\right)\boldsymbol{C}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = i\boldsymbol{k}\times\boldsymbol{A}$$

錯誤一處扣一分。

D.4 2 pt

$$v_T = v_A \cos\theta$$

$$\omega^2 \boldsymbol{v}_0 = \left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\left[\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\boldsymbol{v}_0 - \left(\boldsymbol{v}_0\cdot\boldsymbol{v}_A\right)\boldsymbol{k}\right] \qquad (1\text{ pt})$$

顯然解為

$$\omega = \left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right) = k v_A \cos\theta \qquad (1\text{ pt})$$

且 $\boldsymbol{v}_0\cdot\boldsymbol{v}_A = 0$，所以速度 $\boldsymbol{v}_0$ 也與磁場 $\boldsymbol{B}_0$ 垂直。

D.5 7 pt

$$v_M = \sqrt{\frac{1}{2}\left[c^2 + v_A^2 \pm \sqrt{\left(c^2+v_A^2\right)^2 - 4c^2 v_A^2 \cos^2\theta}\right]}$$

在此不使用特徵值的解法。若解法過程正確則全對。

$$\omega^2 \boldsymbol{v}_0 = \left(c^2 + v_A^2\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\boldsymbol{k} + \left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\left[\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\boldsymbol{v}_0 - \left(\boldsymbol{v}_0\cdot\boldsymbol{v}_A\right)\boldsymbol{k} - \left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\boldsymbol{v}_A\right]$$

對 $\boldsymbol{k}$ 內積得

$$\omega^2\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right) = \left(c^2+v_A^2\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)k^2 - \left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\left(\boldsymbol{v}_0\cdot\boldsymbol{v}_A\right)k^2 \qquad (2\text{ pt})(5)$$

對 $\boldsymbol{v}_A$ 內積得

$$\begin{aligned}\omega^2\left(\boldsymbol{v}_0\cdot\boldsymbol{v}_A\right) &= \left(c^2+v_A^2\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right) - \left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)v_A^2 \\ &= c^2\left(\boldsymbol{k}\cdot\boldsymbol{v}_0\right)\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)\end{aligned} \qquad (2\text{ pt})(6)$$

將(6)式中 $\boldsymbol{v}_0\cdot\boldsymbol{v}_A$ 代入(5)式得

$$\omega^4 - \left(c^2+v_A^2\right)\omega^2 k^2 + c^2 k^2\left(\boldsymbol{k}\cdot\boldsymbol{v}_A\right)^2 = 0 \qquad (2\text{ pt})$$

$$v_M^4-\left(c^2+v_A^2\right)v_M^2+c^2v_A^2\cos^2\theta=0$$

答案正確 (1 pt)。

E.1 5 pt

$$P\left(z\right)=\begin{cases}P_1-\rho_1 gz,\ z>0\\P_2-\rho_2 gz,\ z<0\end{cases}$$

$$P_1-P_2=\frac{B_2^2-B_1^2}{2\mu_0}+\frac{\left[\left(\boldsymbol{B}_1-\boldsymbol{B}_2\right)\cdot\hat{\boldsymbol{z}}\right]\left[\left(\boldsymbol{B}_1+\boldsymbol{B}_2\right)\cdot\hat{\boldsymbol{z}}\right]}{2\mu_0}$$

磁場的邊界條件 $\boldsymbol{B}_{1\parallel}-\boldsymbol{B}_{2\parallel}=\mu_0\boldsymbol{K}\times\hat{\boldsymbol{z}}$，其中 $\boldsymbol{B}_{\parallel}=\boldsymbol{B}-\left(\boldsymbol{B}\cdot\hat{\boldsymbol{z}}\right)\hat{\boldsymbol{z}}$。(1 pt)

$$\hat{\boldsymbol{z}}\times\left(\mu_0\boldsymbol{K}\times\hat{\boldsymbol{z}}\right)=\hat{\boldsymbol{z}}\times\left(\boldsymbol{B}_1-\boldsymbol{B}_2\right)$$

$$\boldsymbol{K}=\frac{\hat{\boldsymbol{z}}\times\left(\boldsymbol{B}_1-\boldsymbol{B}_2\right)}{\mu_0}\qquad(1\text{ pt})$$

由力平衡

$$\left(-P_1+P_2\right)+\left[\boldsymbol{K}\times\frac{1}{2}\left(\boldsymbol{B}_1+\boldsymbol{B}_2\right)\right]\cdot\hat{\boldsymbol{z}}=0\qquad(2\text{ pt})$$

$$P_1-P_2=\frac{\left\{\left[\hat{\boldsymbol{z}}\times\left(\boldsymbol{B}_1-\boldsymbol{B}_2\right)\right]\times\left(\boldsymbol{B}_1+\boldsymbol{B}_2\right)\right\}\cdot\hat{\boldsymbol{z}}}{2\mu_0}=\frac{B_2^2-B_1^2}{2\mu_0}+\frac{\left[\left(\boldsymbol{B}_1-\boldsymbol{B}_2\right)\cdot\hat{\boldsymbol{z}}\right]\left[\left(\boldsymbol{B}_1+\boldsymbol{B}_2\right)\cdot\hat{\boldsymbol{z}}\right]}{2\mu_0}$$

兩個答案都給分 (1 pt)。

E.2 請證明以下微分方程組成立 6 pt

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$$

$$\boldsymbol{b}=\boldsymbol{\nabla}\times\left(\boldsymbol{u}\times\boldsymbol{B}\right)$$

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}=-\boldsymbol{\nabla}p+\frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\boldsymbol{b}\right)\times\boldsymbol{B}$$

由不可壓縮可得

$$\boldsymbol{\nabla}\cdot\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)=0\qquad(1\text{ pt})$$

故 $\boldsymbol{\nabla}\cdot\boldsymbol{u}=C$ (C 對 t 而言為常數)。但是 $\boldsymbol{u}=\boldsymbol{u}_0\left(z\right)e^{i\left(k_xx+k_yy-\omega t\right)}$，所以若 $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ 不為零，則必定為 t 的函數。故 $C=0$。(1 pt)

代入 Maxwell equation

$$\boldsymbol{E}+\frac{\partial\boldsymbol{u}}{\partial t}\times\boldsymbol{B}=\frac{\boldsymbol{J}}{\sigma}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E}&=\frac{1}{\sigma}\boldsymbol{\nabla}\times\boldsymbol{J}-\frac{\partial}{\partial t}\left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\right]=\frac{1}{\mu_0\sigma}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})-\frac{\partial}{\partial t}\left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\right]\\&\approx-\frac{\partial}{\partial t}\left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\right]\end{aligned}\qquad(2\text{ pt})$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{b}}{\partial t}\qquad(1\text{ pt})$$

比對得 $\boldsymbol{b}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})+C$。同上討論 $C=0$。顯然有

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}=-\boldsymbol{\nabla}p+\boldsymbol{J}\times\boldsymbol{B}=-\boldsymbol{\nabla}p+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{b})\times\boldsymbol{B}\qquad(1\text{ pt})$$

E.3 請證明 D.1 可以化簡為 10 pt

$$\frac{\mathrm{d}u_z}{\mathrm{d}z}+i\boldsymbol{k}\cdot\boldsymbol{u}=0$$

$$\boldsymbol{b}=i(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{u}$$

$$\rho\omega^2u_j=i\bar{p}k_j-\frac{i}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})b_j$$

$$\rho\omega^2u_z=\frac{\mathrm{d}\bar{p}}{\mathrm{d}z}-\frac{i}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})b_z$$

其中 $j=x,y$。並試求出參數 $\bar{p}$ 的表達式。

$$\boldsymbol{\nabla}\cdot\left[\boldsymbol{u}_0(z)\,e^{i(k_xx+k_yy-\omega t)}\right]=\frac{\mathrm{d}u_{0z}(z)}{\mathrm{d}z}+i(k_xu_{0x}+k_yu_{0y})=0\qquad(1\text{ pt})$$

$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}$,令 $\boldsymbol{A}=\boldsymbol{b}$(1 pt) 則有

$$\boldsymbol{b}=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{u}=i(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{u}\qquad(2\text{ pt})$$

$\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})+\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})+(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}$,令 $\boldsymbol{A}=\boldsymbol{b}$(1 pt) 則有

$$(\boldsymbol{\nabla}\times\boldsymbol{b})\times\boldsymbol{B}=-\boldsymbol{\nabla}(\boldsymbol{b}\cdot\boldsymbol{B})+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{b}=-\boldsymbol{\nabla}(\boldsymbol{b}\cdot\boldsymbol{B})+i(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{b}\qquad(2\text{ pt})$$

所以有

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}=-\boldsymbol{\nabla}p-\frac{1}{\mu_0}\boldsymbol{\nabla}(\boldsymbol{b}\cdot\boldsymbol{B})+\frac{i}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{b}=-\boldsymbol{\nabla}\bar{p}+\frac{i}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{b}\qquad(2\text{ pt})$$

分量答案正確。(1 pt)

E.4 4 pt

$$k'=k=\sqrt{k_x^2+k_y^2}$$

$$\rho\omega^2 u_j = i\bar{p}k_j + \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2 u_j \qquad (1\text{ pt})$$

$$\rho\omega^2 u_z = \frac{\mathrm{d}\bar{p}}{\mathrm{d}z} + \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2 u_z \qquad (1\text{ pt})$$

$$\frac{\mathrm{d}u_z}{\mathrm{d}z} - \frac{\bar{p}k^2}{\rho\omega^2 - \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2} = 0 \qquad (1\text{ pt})$$

$$\frac{\mathrm{d}^2 u_z}{\mathrm{d}z^2} - k^2 u_z = 0 \qquad (1\text{ pt})$$

E.5 請證明波動的色散關係為 9 pt

$$\omega^2 = \frac{2\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2}{\mu_0\left(\rho_1+\rho_2\right)} - gk\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)$$

邊界條件有兩者

1. 顯然 u_z 連續。(1 pt)
2. 壓力連續由 **E.1** 可知

$$\begin{aligned} P_1 + p_1 - \rho_1 g u_{z1} + \frac{\left(\boldsymbol{B}_1+\boldsymbol{b}_1\right)^2}{2\mu_0} =& P_2 + p_2 - \rho_1 g u_{z2} + \frac{\left(\boldsymbol{B}_2+\boldsymbol{b}_2\right)^2}{2\mu_0} \\ &+ \frac{\left[\left(\boldsymbol{b}_1-\boldsymbol{b}_2\right)\cdot\hat{\boldsymbol{z}}\right]\left[\left(\boldsymbol{b}_1+\boldsymbol{b}_2\right)\cdot\hat{\boldsymbol{z}}\right]}{2\mu_0} \end{aligned} \qquad (2\text{ pt})$$

略去 b^2 項，整理可得 $\bar{p}_1 - \rho_1 g u_z = \bar{p}_2 - \rho_2 g u_z$。

注意到

$$\rho\omega^2 u_j k_j = i\bar{p}k_j^2 + \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2 u_j k_j$$

$$\left[\rho\omega^2 - \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2\right]\boldsymbol{k}\cdot\boldsymbol{u} = i\bar{p}k^2 \qquad (3\text{ pt})$$

$$\bar{p} = \frac{\rho\omega^2 - \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2}{k^2}\frac{\mathrm{d}u_z}{\mathrm{d}z} \qquad (2\text{ pt})$$

代入 $u_{1z}(0) = -ku_{1z0}$、$u_{2z}(0) = ku_{2z0}$

$$\left[-\frac{\rho_1\omega^2 - \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2}{k} - \rho_1 g\right]u_{1z0} = \left[\frac{\rho_2\omega^2 - \frac{1}{\mu_0}\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2}{k} - \rho_2 g\right]u_{2z0}$$

又因 u_z 連續，$u_{1z0} = u_{2z0}$，整理可得

$$\omega^2 = \frac{2\left(\boldsymbol{k}\cdot\boldsymbol{B}\right)^2}{\mu_0\left(\rho_1+\rho_2\right)} - gk\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)$$

參考資料

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