

解答

A.1	$\nabla \cdot \mathbf{v} = 0$	5 pt
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A.2	證明	5 pt
$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{P}{\rho} + gz = \text{const.}$		
<p>上式稱為時變的白努力方程式。注意到是整個區域任意處皆為同個常數。</p>		

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \nabla (v^2) = -\frac{\nabla P}{\rho} - \rho g \hat{\mathbf{z}} = \nabla \left(\frac{\partial \phi}{\partial t} \right) + \frac{1}{2} \nabla (\nabla \phi)^2 \quad (2 \text{ pt})$$

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{P}{\rho} + gz \right) = 0 \quad (2 \text{ pt})$$

因此有

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{P}{\rho} + gz = \text{const. (各處)} \quad (1 \text{ pt})$$

A.3	$v(t) = \sqrt{2gh} \tanh \left(\sqrt{\frac{gh}{2L^2}} t \right)$	4 pt
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$$\phi = \int_O^A \mathbf{v} \cdot d\mathbf{r} = vL \quad (1 \text{ pt})$$

代入 A.2 的方程式可得

$$L \frac{\partial v}{\partial t} + \frac{1}{2} v^2 + \frac{P_0}{\rho} = \frac{P_0}{\rho} + gh \quad (2 \text{ pt})$$

$$\frac{1}{2L} dt = \frac{dv}{2gh - v^2}$$

答案正確 (1 pt)。

B.1	$v_z = \frac{dh}{dt} = \frac{\partial h}{\partial t} + (\mathbf{v} \cdot \nabla) h = \frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x}$	2 pt
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B.2 證明在介質交界處的動態邊界條件為

3 pt

$$\rho_1 \left(\frac{\partial \varphi_1}{\partial t} + U_1 \frac{\partial \varphi_1}{\partial x} \right) = \rho_2 \left(\frac{\partial \varphi_2}{\partial t} + U_2 \frac{\partial \varphi_2}{\partial x} \right)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{P}{\rho} = \frac{1}{2} U^2 \quad (1 \text{ pt})$$

代入

$$\nabla \phi \approx \left(U + \frac{\partial \varphi}{\partial x} \right) \hat{x} \quad (1 \text{ pt})$$

由壓力分佈連續可知 $P_1 = P_2$ (1 pt)。代入即可得證。

B.3

$$\nabla \cdot \mathbf{v} = 0$$

2 pt

$$\nabla^2 \varphi = 0$$

B.4

$$\omega = \left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} + i \frac{\sqrt{\rho_1 \rho_2} |u|}{\rho_1 + \rho_2} \right) k$$

$$U_1 \neq U_2$$

7 pt

假設壓力為 $P_i = \bar{P}_i$ ，其中 \bar{P}_i 為擾動項。界面壓力連續

$$\bar{P}_1 = \bar{P}_2 \quad (1 \text{ pt})$$

時變的白努力方程式

由 B.2 的結果代入解可得

$$-i\rho_1(\omega - kU_1)\varphi_1 = -i\rho_2(\omega - kU_2)\varphi_2 \quad (1 \text{ pt})(0)$$

由 B.1 可得

$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} \approx \frac{\partial \varphi}{\partial z} \quad (1 \text{ pt})$$

代入解可得

$$\varphi_1 = i(\omega - kU_1) \frac{h}{k}, \quad \varphi_2 = -i(\omega - kU_2) \frac{h}{k} \quad (1 \text{ pt})(1)$$

(0)(1)式解聯立方程式可得

$$\rho_1(\omega - kU_1)^2 + \rho_2(\omega - kU_2)^2 = 0 \quad (1 \text{ pt})$$

$$(\rho_1 + \rho_2)\omega^2 - 2(\rho_1 U_1 + \rho_2 U_2)k\omega + (\rho_1 U_1^2 + \rho_2 U_2^2)k^2 = 0 \quad (1 \text{ pt})$$

$$\omega = \frac{\rho_1 U_1 + \rho_2 U_2 \pm \sqrt{-\rho_1 \rho_2 (U_1 - U_2)^2}}{\rho_1 + \rho_2} \quad (1 \text{ pt})$$

答案正確 (1 pt)。

B.5	$\frac{\omega}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} + \sqrt{-\frac{\rho_1 \rho_2 u^2}{(\rho_1 + \rho_2)^2} - \frac{g}{k} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right) + \frac{\sigma k}{\rho_1 + \rho_2}}$	14 pt
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假設壓力為 $P_i = -\rho_i g z + \bar{P}_i$ ，其中 \bar{P}_i 為擾動項。

1. 界面壓力連續

$$-\rho_1 g h + \bar{P}_1 = -\rho_2 g h + \bar{P}_2 + \sigma \frac{\partial^2 h}{\partial x^2} \quad (3 \text{ pt})(2)$$

2. 時變的白努力方程式

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{P}{\rho} + g z = \text{const.} \quad (1 \text{ pt})$$

其中 $(\nabla \phi)^2 \approx U^2 + 2U \frac{\partial \phi}{\partial x}$ (1 pt)。代入 $z \rightarrow \pm \infty$ ， $\phi(x, \pm \infty, t) \rightarrow 0$ 、 $\bar{P}(x, \pm \infty, t) \rightarrow 0$ ，因此可得

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} + \frac{\bar{P}}{\rho} = 0 \quad (2 \text{ pt})$$

代入(2)式

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right) + \sigma k^2 h - (\rho_1 - \rho_2) g h \quad (2 \text{ pt})$$

代入解可得

$$-i \rho_1 (\omega - k U_1) \phi_1 = -i \rho_2 (\omega - k U_2) \phi_2 + \sigma k^2 h - (\rho_1 - \rho_2) g h \quad (1 \text{ pt})(3)$$

由 B.1 可得

$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} \approx \frac{\partial \phi}{\partial z} \quad (1 \text{ pt})$$

代入解可得

$$\phi_1 = i (\omega - k U_1) \frac{h}{k}, \quad \phi_2 = -i (\omega - k U_2) \frac{h}{k} \quad (1 \text{ pt})(4)$$

(3)(4)式解聯立方程式可得

$$\rho_1 (\omega - k U_1)^2 \frac{1}{k} + \rho_2 (\omega - k U_2)^2 \frac{1}{k} = \sigma k^2 - (\rho_1 - \rho_2) g \quad (1 \text{ pt})$$

令 $v = \omega/k$ ，則

$$(\rho_1 + \rho_2) v^2 - 2(\rho_1 U_1 + \rho_2 U_2) v + (\rho_1 U_1^2 + \rho_2 U_2^2) - \sigma k + (\rho_1 - \rho_2) \frac{g}{k} = 0 \quad (1 \text{ pt})$$

答案正確 (1 pt)。

B.6

$$u_c = \left(\frac{4g\sigma}{\rho_1^2 \rho_2^2} (\rho_2 - \rho_1) (\rho_1 + \rho_2)^2 \right)^{\frac{1}{4}}$$

4 pt

由 B.5 顯然有

$$-\frac{\rho_1 \rho_2 u^2}{(\rho_1 + \rho_2)^2} - \frac{g}{k} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right) + \frac{\sigma k}{\rho_1 + \rho_2} < 0$$

又因

$$\begin{aligned} \frac{g}{k} \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} \right) + \frac{\sigma k}{\rho_1 + \rho_2} &\geq \frac{2\sqrt{g\sigma(\rho_2 - \rho_1)}}{\rho_1 + \rho_2} \\ u_c^2 &= 2 \left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \right) \sqrt{g\sigma(\rho_2 - \rho_1)} \end{aligned}$$

C.1 瑞利－泰勒不穩定性必定為指數型成長，故微擾一定有 $e^{\gamma t}$ 的形式， 1 pt
所以只能取正根。

C.2

$$\sqrt{gk \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)}$$

3 pt

C.3

$$\omega = \sqrt{gk + \frac{\sigma}{\rho} k^3}$$

3 pt

D.1

$$c = \sqrt{\frac{\partial p}{\partial \rho}}$$

3 pt

D.2

$$\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0 \rho_0}}$$

3 pt

$$\rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} + c^2 \nabla \frac{\partial \tilde{\rho}}{\partial t} + \frac{1}{\mu_0} \mathbf{B}_0 \times \left(\nabla \times \frac{\partial \mathbf{b}}{\partial t} \right) = 0 \quad (1 \text{ pt})$$

$$\rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} - c^2 \rho_0 \nabla (\nabla \cdot \mathbf{v}) + \frac{1}{\mu_0} \mathbf{B}_0 \times \{ \nabla \times [\nabla \times (\mathbf{v} \times \mathbf{B}_0)] \} = 0 \quad (2 \text{ pt})$$

D.3 請證明行進波的色散關係為

3 pt

$$\omega^2 \mathbf{v}_0 = (c^2 + v_A^2) (\mathbf{k} \cdot \mathbf{v}_0) \mathbf{k} + (\mathbf{k} \cdot \mathbf{v}_A) [(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{v}_A) \mathbf{k} - (\mathbf{k} \cdot \mathbf{v}_0) \mathbf{v}_A]$$

此即為描述阿爾芬波的方程式。

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}$$

$$\nabla \times \mathbf{A} = i\mathbf{k} \times \mathbf{A}$$

錯誤一處扣一分。

D.4

$$v_T = v_A \cos \theta$$

2 pt

$$\omega^2 \mathbf{v}_0 = (\mathbf{k} \cdot \mathbf{v}_A) [(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{v}_A) \mathbf{k}] \quad (1 \text{ pt})$$

顯然解為

$$\omega = (\mathbf{k} \cdot \mathbf{v}_A) = k v_A \cos \theta \quad (1 \text{ pt})$$

且 $\mathbf{v}_0 \cdot \mathbf{v}_A = 0$ ，所以速度 \mathbf{v}_0 也與磁場 \mathbf{B}_0 垂直。

D.5

$$v_M = \sqrt{\frac{1}{2} \left[c^2 + v_A^2 \pm \sqrt{(c^2 + v_A^2)^2 - 4c^2 v_A^2 \cos^2 \theta} \right]}$$

7 pt

在此不使用特徵值的解法。若解法過程正確則全對。

$$\omega^2 \mathbf{v}_0 = (c^2 + v_A^2) (\mathbf{k} \cdot \mathbf{v}_0) \mathbf{k} + (\mathbf{k} \cdot \mathbf{v}_A) [(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{v}_A) \mathbf{k} - (\mathbf{k} \cdot \mathbf{v}_0) \mathbf{v}_A]$$

對 \mathbf{k} 內積得

$$\omega^2 (\mathbf{k} \cdot \mathbf{v}_0) = (c^2 + v_A^2) (\mathbf{k} \cdot \mathbf{v}_0) k^2 - (\mathbf{k} \cdot \mathbf{v}_A) (\mathbf{v}_0 \cdot \mathbf{v}_A) k^2 \quad (2 \text{ pt})(5)$$

對 \mathbf{v}_A 內積得

$$\begin{aligned} \omega^2 (\mathbf{v}_0 \cdot \mathbf{v}_A) &= (c^2 + v_A^2) (\mathbf{k} \cdot \mathbf{v}_0) (\mathbf{k} \cdot \mathbf{v}_A) - (\mathbf{k} \cdot \mathbf{v}_0) (\mathbf{k} \cdot \mathbf{v}_A) v_A^2 \\ &= c^2 (\mathbf{k} \cdot \mathbf{v}_0) (\mathbf{k} \cdot \mathbf{v}_A) \end{aligned} \quad (2 \text{ pt})(6)$$

將(6)式中 $\mathbf{v}_0 \cdot \mathbf{v}_A$ 代入(5)式得

$$\omega^4 - (c^2 + v_A^2) \omega^2 k^2 + c^2 k^2 (\mathbf{k} \cdot \mathbf{v}_A)^2 = 0 \quad (2 \text{ pt})$$

$$v_M^4 - (c^2 + v_A^2) v_M^2 + c^2 v_A^2 \cos^2 \theta = 0$$

答案正確 (1 pt)。

E.1

5 pt

$$P(z) = \begin{cases} P_1 - \rho_1 g z, & z > 0 \\ P_2 - \rho_2 g z, & z < 0 \end{cases}$$

$$P_1 - P_2 = \frac{B_2^2 - B_1^2}{2\mu_0} + \frac{[(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{z}}][(\mathbf{B}_1 + \mathbf{B}_2) \cdot \hat{\mathbf{z}}]}{2\mu_0}$$

磁場的邊界條件 $\mathbf{B}_{1\parallel} - \mathbf{B}_{2\parallel} = \mu_0 \mathbf{K} \times \hat{\mathbf{z}}$ ，其中 $\mathbf{B}_{\parallel} = \mathbf{B} - (\mathbf{B} \cdot \hat{\mathbf{z}}) \hat{\mathbf{z}}$ 。(1 pt)

$$\hat{\mathbf{z}} \times (\mu_0 \mathbf{K} \times \hat{\mathbf{z}}) = \hat{\mathbf{z}} \times (\mathbf{B}_1 - \mathbf{B}_2)$$

$$\mathbf{K} = \frac{\hat{\mathbf{z}} \times (\mathbf{B}_1 - \mathbf{B}_2)}{\mu_0} \quad (1 \text{ pt})$$

由力平衡

$$(-P_1 + P_2) + \left[\mathbf{K} \times \frac{1}{2} (\mathbf{B}_1 + \mathbf{B}_2) \right] \cdot \hat{\mathbf{z}} = 0 \quad (2 \text{ pt})$$

$$P_1 - P_2 = \frac{\{[\hat{\mathbf{z}} \times (\mathbf{B}_1 - \mathbf{B}_2)] \times (\mathbf{B}_1 + \mathbf{B}_2)\} \cdot \hat{\mathbf{z}}}{2\mu_0} = \frac{B_2^2 - B_1^2}{2\mu_0} + \frac{[(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{z}}][(\mathbf{B}_1 + \mathbf{B}_2) \cdot \hat{\mathbf{z}}]}{2\mu_0}$$

兩個答案都給分 (1 pt)。

E.2 請證明以下微分方程組成立

6 pt

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{b}) \times \mathbf{B}$$

由不可壓縮可得

$$\nabla \cdot \left(\frac{\partial \mathbf{u}}{\partial t} \right) = 0 \quad (1 \text{ pt})$$

故 $\nabla \cdot \mathbf{u} = C$ (C 對 t 而言為常數)。但是 $\mathbf{u} = \mathbf{u}_0(z) e^{i(k_x x + k_y y - \omega t)}$ ，所以若 $\nabla \cdot \mathbf{u}$ 不為零，則必定為 t 的函數。故 $C = 0$ 。(1 pt)

代入 Maxwell equation

$$\mathbf{E} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma}$$

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{1}{\sigma} \nabla \times \mathbf{J} - \frac{\partial}{\partial t} [\nabla \times (\mathbf{u} \times \mathbf{B})] = \frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \mathbf{b}) - \frac{\partial}{\partial t} [\nabla \times (\mathbf{u} \times \mathbf{B})] \\ &\approx -\frac{\partial}{\partial t} [\nabla \times (\mathbf{u} \times \mathbf{B})]\end{aligned}\quad (2 \text{ pt})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{b}}{\partial t} \quad (1 \text{ pt})$$

比對得 $\mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{B}) + C$ 。同上討論 $C = 0$ 。顯然有

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p + \mathbf{J} \times \mathbf{B} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{b}) \times \mathbf{B} \quad (1 \text{ pt})$$

E.3 請證明 E.2 可以化簡為

10 pt

$$\frac{du_z}{dz} + i\mathbf{k} \cdot \mathbf{u} = 0$$

$$\mathbf{b} = i(\mathbf{k} \cdot \mathbf{B}) \mathbf{u}$$

$$\rho \omega^2 u_j = i\bar{p} k_j - \frac{i}{\mu_0} (\mathbf{k} \cdot \mathbf{B}) b_j$$

$$\rho \omega^2 u_z = \frac{d\bar{p}}{dz} - \frac{i}{\mu_0} (\mathbf{k} \cdot \mathbf{B}) b_z$$

其中 $j = x, y$ 。並試求出參數 \bar{p} 的表達式。

$$\nabla \cdot [\mathbf{u}_0(z) e^{i(k_x x + k_y y - \omega t)}] = \frac{du_{0z}(z)}{dz} + i(k_x u_{0x} + k_y u_{0y}) = 0 \quad (1 \text{ pt})$$

$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$ ，令 $\mathbf{A} = \mathbf{b}$ (1 pt) 則有

$$\mathbf{b} = (\mathbf{B} \cdot \nabla) \mathbf{u} = i(\mathbf{B} \cdot \mathbf{k}) \mathbf{u} \quad (2 \text{ pt})$$

$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$ ，令 $\mathbf{A} = \mathbf{b}$ (1 pt) 則有

$$(\nabla \times \mathbf{b}) \times \mathbf{B} = -\nabla(\mathbf{b} \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{b} = -\nabla(\mathbf{b} \cdot \mathbf{B}) + i(\mathbf{B} \cdot \mathbf{k}) \mathbf{b} \quad (2 \text{ pt})$$

所以有

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p - \frac{1}{\mu_0} \nabla(\mathbf{b} \cdot \mathbf{B}) + \frac{i}{\mu_0} (\mathbf{k} \cdot \mathbf{B}) \mathbf{b} = -\nabla \bar{p} + \frac{i}{\mu_0} (\mathbf{k} \cdot \mathbf{B}) \mathbf{b} \quad (2 \text{ pt})$$

分量答案正確。(1 pt)

E.4

$$k' = k = \sqrt{k_x^2 + k_y^2}$$

4 pt

$$\rho\omega^2 u_j = i\bar{p}k_j + \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2 u_j \quad (1 \text{ pt})$$

$$\rho\omega^2 u_z = \frac{d\bar{p}}{dz} + \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2 u_z \quad (1 \text{ pt})$$

$$\frac{du_z}{dz} - \frac{\bar{p}k^2}{\rho\omega^2 - \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2} = 0 \quad (1 \text{ pt})$$

$$\frac{d^2 u_z}{dz^2} - k^2 u_z = 0 \quad (1 \text{ pt})$$

E.5 請證明波動的色散關係為

9 pt

$$\omega^2 = \frac{2 (\mathbf{k} \cdot \mathbf{B})^2}{\mu_0 (\rho_1 + \rho_2)} - gk \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)$$

邊界條件有兩者

1. 顯然 u_z 連續。(1 pt)

2. 壓力連續由 E.1 可知

$$P_1 + p_1 - \rho_1 g u_{z1} + \frac{(\mathbf{B}_1 + \mathbf{b}_1)^2}{2\mu_0} = P_2 + p_2 - \rho_1 g u_{z2} + \frac{(\mathbf{B}_2 + \mathbf{b}_2)^2}{2\mu_0} + \frac{[(\mathbf{b}_1 - \mathbf{b}_2) \cdot \hat{\mathbf{z}}][(\mathbf{b}_1 + \mathbf{b}_2) \cdot \hat{\mathbf{z}}]}{2\mu_0} \quad (2 \text{ pt})$$

略去 b^2 項，整理可得 $\bar{p}_1 - \rho_1 g u_z = \bar{p}_2 - \rho_2 g u_z$ 。

注意到

$$\rho\omega^2 u_j k_j = i\bar{p}k_j^2 + \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2 u_j k_j$$

$$\left[\rho\omega^2 - \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2 \right] \mathbf{k} \cdot \mathbf{u} = i\bar{p}k^2 \quad (3 \text{ pt})$$

$$\bar{p} = \frac{\rho\omega^2 - \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2}{k^2} \frac{du_z}{dz} \quad (2 \text{ pt})$$

代入 $u_{1z}(0) = -k u_{1z0}$ 、 $u_{2z}(0) = k u_{2z0}$

$$\left[-\frac{\rho_1\omega^2 - \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2}{k} - \rho_1 g \right] u_{1z0} = \left[\frac{\rho_2\omega^2 - \frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B})^2}{k} - \rho_2 g \right] u_{2z0}$$

又因 u_z 連續， $u_{1z0} = u_{2z0}$ ，整理可得

$$\omega^2 = \frac{2(\mathbf{k} \cdot \mathbf{B})^2}{\mu_0(\rho_1 + \rho_2)} - gk \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)$$

參考資料

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4. <https://iopscience.iop.org/article/10.1088/0004-637X/785/2/110/pdf>