電磁學筆記

Classical Electrodynamics 3rd edition, John David Jackson

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目錄

第一章　基本數學

1.1　積分公式

本小節中 m 與 n 皆為正整數。

1.1.1　Gamma 函數

引進 $\Gamma(x)$ 函數

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, \mathrm{d}x$$

利用分部積分法可得

$$\int_0^\infty x^{n-1} e^{-x} \, \mathrm{d}x = -x^{n-1} e^{-x} \Big|_0^\infty + (n-1) \int_0^\infty x^{n-2} e^{-x} \, \mathrm{d}x \tag{1.1}$$

觀察右式，可發現

$$\left(-x^{n-1} e^{-x}\right)\Big|_0^\infty = 0 \tag{1.2}$$

$$\int_0^\infty x^{n-2} e^{-x} \, \mathrm{d}x = \Gamma(n-1) \tag{1.3}$$

將(1.2)(1.3)代入(1.1)，可得 $\Gamma(n)$ 的遞迴式

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

以此類推，可得 $\Gamma(n)$ 的表達式

$$\Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2) = (n-1)!$$

由此可得階乘的廣義定義

$$n! = \int_0^\infty x^n e^{-x} \, \mathrm{d}x$$

$\left(\frac{1}{2}\right)!$

以下說明 $\left(\frac{1}{2}\right)!$ 的值。

$$\left(\frac{1}{2}\right)! = \int_0^\infty x^{-\frac{1}{2}} e^{-x} \, \mathrm{d}x$$

利用變數代換

$$u = \sqrt{x}, \quad \mathrm{d}u = \frac{\mathrm{d}x}{2\sqrt{x}}$$

積分式改為

$$\left(\frac{1}{2}\right)! = \frac{1}{2} \int_0^\infty e^{-u^2} \, \mathrm{d}u = \frac{\sqrt{\pi}}{2}$$

因此 $\left(\frac{1}{2}\right)!$ 為

Formula 1.1.1

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

此式在後面的內容會經常使用到。

Euler's reflection formula

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty x^{z-1} e^{-x} \, \mathrm{d}x \int_0^\infty y^{-z} e^{-y} \, \mathrm{d}y$$

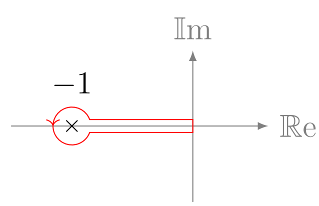
令 $x/y = u, x + y = v$，再計算 Jacobian

$$\mathrm{d}u \, \mathrm{d}v = \left| \begin{bmatrix} \frac{1}{y} & \frac{x}{y^2} \\ 1 & 1 \end{bmatrix} \right| \mathrm{d}x \, \mathrm{d}y = \frac{(1+u)^2}{v} \, \mathrm{d}x \, \mathrm{d}y$$

代入可得

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty e^{-v} \, \mathrm{d}v \int_0^\infty \frac{u^z}{1+u} \, \mathrm{d}u = \int_0^\infty \frac{u^z}{1+u} \, \mathrm{d}u \equiv I$$

使用以下積分路徑



可得

$$\left(e^{2\pi i z} - 1\right) I = 2\pi i e^{\pi i z}$$

$$I = \frac{\pi}{\sin \pi z}$$

代回可得

Formula 1.1.2 Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

1.1.2　Beta 函數

先計算 $\Gamma(a)\Gamma(b)$

$$\Gamma(a)\Gamma(b) = \int_0^\infty x^{a-1}e^{-x}\,\mathrm{d}x \int_0^\infty y^{b-1}e^{-y}\,\mathrm{d}y$$

令 $x = uv, y = u(1-v)$，再計算 Jacobian

$$\mathrm{d}x\,\mathrm{d}y = \left|\begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}\right| \mathrm{d}u\,\mathrm{d}v = u\,\mathrm{d}u\,\mathrm{d}v$$

代回可得

$$\Gamma(a)\Gamma(b) = \int_0^\infty u^{a+b-1}e^{-u}\,\mathrm{d}u \int_0^1 v^{a-1}(1-v)^{b-1}\,\mathrm{d}v$$

Definition 1.1.1

定義 Beta 函數 $B(a,b)$ 為

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x \tag{1.4}$$

與 Gamma 函數 $\Gamma(x)$ 的關係為

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

令(1.4)中的 $x = \sin^2\theta$ 可得

$$B(a,b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2b-1}\theta\,\mathrm{d}\theta$$

反過來表示 a, b

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

1.1.3　三角函數

Formula 1.1.3 三角函數積分公式

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\,\mathrm{d}\theta = \frac{(2^n n!)^2}{(2n+1)!} = \frac{(2n)!!}{(2n+1)!!}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos^{2n}\theta\,\mathrm{d}\theta = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

1.1.4　指數

$$\int_0^{\infty} x^{2n} e^{-ax^2}\,\mathrm{d}x = \frac{1\cdot 3\ldots(2n-1)}{2^{2n+1}}\sqrt{\frac{\pi}{a^{2n+1}}} = \frac{(2n)!}{2^{2n+1}n!}\sqrt{\frac{\pi}{a^{2n+1}}}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2}\,\mathrm{d}x = \frac{n!}{2a^{n+1}}$$

$$\int f(x)\,e^{-x}\,\mathrm{d}x = -e^{-x}\sum_{n=0}^{\infty} f^{(n)}(x)$$

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,\mathrm{d}x = 2\pi\delta(k-k')$$

1.2　伴隨勒讓得多項式與球諧函數

1.2.1　勒讓得多項式 Legendre polynomial

■ **微分方程式**

Definition 1.2.1 微分方程式

勒讓得多項式 $P_l(x)$ 符合

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-x^2\right)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\right] + l(l+1)P_l(x) = 0$$

等價於

$$\left(1-x^2\right)\frac{\mathrm{d}^2P_l\left(x\right)}{\mathrm{d}x^2}-2x\frac{\mathrm{d}P_l\left(x\right)}{\mathrm{d}x}+l\left(l+1\right)P_l\left(x\right)=0$$

代入 $x=\cos\theta$ 可得

Corollary 1.2.1 微分方程式

勒讓得多項式 $P_l\left(\cos\theta\right)$ 符合

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_l\left(\cos\theta\right)}{\mathrm{d}\theta}\right)+l\left(l+1\right)P_l\left(\cos\theta\right)=0$$

■ **洛巨德公式 (Rodrigues formula)**

$$P_n\left(x\right)=\frac{1}{2^nn!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n$$

■ **級數關係**

$$\frac{1}{\sqrt{1-2tx+t^2}}=\sum_{n=1}^{\infty}P_n\left(x\right)t^n$$

其中各項為

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\sqrt{1-2tx+t^2}}=\frac{x-t}{\left(1-2tx+t^2\right)^{3/2}}$$

$$\left(1-2tx+t^2\right)\sum_{n=1}^{\infty}nP_n\left(x\right)t^{n-1}=\sum_{n=1}^{\infty}xP_n\left(x\right)t^n-\sum_{n=1}^{\infty}P_n\left(x\right)t^{n+1}$$

■ **遞迴關係**

$$\left(n+1\right)P_{n+1}\left(x\right)=\left(2n+1\right)xP_n\left(x\right)-nP_{n-1}\left(x\right)$$

1.2.2　伴隨勒讓得多項式 Associated Legendre polynomial

■ **微分方程式**

Definition 1.2.2 微分方程式

伴隨勒讓得多項式 $P_l^m\left(x\right)$ 符合

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-x^2\right)\frac{\mathrm{d}P_l^m\left(x\right)}{\mathrm{d}x}\right]+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]P_l^m\left(x\right)=0$$

等價於

$$\left(1-x^2\right)\frac{\mathrm{d}^2P_l^m\left(x\right)}{\mathrm{d}x^2}-2x\frac{\mathrm{d}P_l^m\left(x\right)}{\mathrm{d}x}+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]P_l^m\left(x\right)=0$$

代入 $x=\cos\theta$ 可得

Corollary 1.2.2 微分方程式

伴隨勒讓得多項式 $P_l^m(x)$ 符合

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_l^m(\cos\theta)}{\mathrm{d}\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P_l^m(\cos\theta)=0$$

■ **洛巨德公式**

伴隨勒讓得多項式的洛巨德公式為

$$P_l^m(x)=(-1)^m\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x)$$

特殊範圍下的定義

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

■ **遞迴關係**

$$\left(1-x^2\right)\frac{\mathrm{d}P_l^m}{\mathrm{d}x}=-\sqrt{1-x^2}P_l^{m+1}-mxP_l^m$$

等價於

$$\frac{\mathrm{d}P_l^m(\cos\theta)}{\mathrm{d}\theta}=P_l^{m+1}(\cos\theta)+m\cot\theta P_l^m(\cos\theta)$$

■ **正交性**

$$\int_{-1}^{1}P_k^m(x)P_l^m(x)\,\mathrm{d}x=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{kl}$$

■ **加法定理 (The addition theorem)**

$$P_l(\cos\gamma)=P_l(\cos\theta)P_l(\cos\theta')+2\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\theta',\phi')P_l^m(\theta,\phi)\cos\left[m(\phi-\phi')\right]$$

1.2.3　球諧函數 spherical harmonic

$$Y_l^m(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$$

$$Y_l^{m*}(\theta,\phi)=(-1)^mY_l^{-m}(\theta,\phi)$$

■ **正交性**

$$\int_S Y_l^m(\theta,\phi)\,Y_{l'm'}^*(\theta,\phi)\,\mathrm{d}\Omega=\delta_{ll'}\delta_{mm'}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^{m*}(\theta',\phi')\,Y_l^m(\theta,\phi)=\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

1.2.4　展開式

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_l^{m*}(\theta',\phi')\,Y_l^m(\theta,\phi)$$

1.3　亥姆霍茲方程式

1.3.1　齊次亥姆霍茲方程式 Homogeneous Helmholtz equation

$$\nabla^2 A(\boldsymbol{r})+k^2A(\boldsymbol{r})=0$$

其解為

$$A(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}j_l(kr)+B_{lm}y_l(kr)\right]Y_l^m(\theta,\phi)$$

其中 $j_l(x), y_l(x)$ 為球型巴賽爾函數 (spherical Bessel function)、$Y_l^m(\theta,\phi)$ 為球諧函數 (spherical harmonics)。

1.3.2　非齊次亥姆霍茲方程式 Inhomogeneous Helmholtz equation

$$\nabla^2 A(\boldsymbol{r})+k^2A(\boldsymbol{r})=-f(\boldsymbol{r})$$

其解為

$$A(\boldsymbol{r})=\int G(\boldsymbol{r},\boldsymbol{r}')\,f(\boldsymbol{r}')\,d^3\boldsymbol{r}'$$

其中函數 $G(\boldsymbol{r},\boldsymbol{r}')$ 稱為格林函數 (Green function)

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

1.4　橢圓積分

1.4.1　第一類完全橢圓積分 complete elliptic integral of the first kind

$$K\left(k\right)=\int_{0}^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^{2}\sin^{2}\theta}}\,\mathrm{d}\theta$$

■ $K\left(1-k\right)$

在電磁學電容電感的章節中常用到 $K\left(1-k\right)$，其中 $0<k\ll 1$。

$$K\left(1-k\right)\approx\int_{0}^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\left(1-2k\right)\sin^{2}\theta}}\,\mathrm{d}\theta=I_{1}+I_{2}$$

個別計算可得

$$I_{1}=\lim_{\substack{\epsilon\to 0\\ \epsilon\gg\sqrt{2k}}}\int_{0}^{\frac{\pi}{2}-\epsilon}\frac{1}{\sqrt{1-\left(1-2k\right)\sin^{2}\theta}}\,\mathrm{d}\theta=\int_{0}^{\frac{\pi}{2}-\epsilon}\frac{1}{\cos\theta}\,\mathrm{d}\theta=\ln\left(\frac{2}{\epsilon}\right)$$

$$\begin{aligned}I_{2}&=\lim_{\substack{\epsilon\to 0\\ \epsilon\gg\sqrt{2k}}}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\left(1-2k\right)\sin^{2}\theta}}\,\mathrm{d}\theta\\&=\lim_{\substack{\epsilon\to 0\\ \epsilon\gg\sqrt{2k}}}\int_{-\epsilon}^{0}\frac{1}{\sqrt{\theta^{2}+2k}}\,\mathrm{d}\theta=\ln\left(\frac{\sqrt{\epsilon^{2}+2k}+\epsilon}{\sqrt{2k}}\right)\end{aligned}$$

相加可得

Formula 1.4.1 $K\left(1-k\right)$

$$K\left(1-k\right)\approx\frac{1}{2}\ln\left(\frac{8}{k}\right)$$

1.4.2　第二類完全橢圓積分 complete elliptic integral of the second kind

$$E\left(k\right)=\int_{0}^{\frac{\pi}{2}}\sqrt{1-k^{2}\sin^{2}\theta}\,\mathrm{d}\theta$$

由泰勒展開式

$$\left(1-x\right)^{\frac{1}{2}}=1-\sum_{n=1}^{\infty}\frac{\left(2n-1\right)!!}{\left(2n\right)!!}\frac{x^{n}}{2n-1}$$

代回原式

$$E\left(k\right)=\int_{0}^{\frac{\pi}{2}}\left[1-\sum_{n=1}^{\infty}\frac{\left(2n-1\right)!!}{\left(2n\right)!!}\frac{k^{2n}}{2n-1}\sin^{2n}\theta\right]\mathrm{d}\theta$$

將積分值 (見1.1.3) 代入可得

$$E(k) = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{k^{2n}}{2n-1}\right\}$$

1.4.3 導數與微分方程式

■ **第一類完全橢圓積分**

$$\frac{\mathrm{d}K(k)}{\mathrm{d}k} = \int_0^{\frac{\pi}{2}} \frac{k\sin^2\theta}{\left(1-k^2\sin^2\theta\right)^{3/2}}\,\mathrm{d}\theta$$

Lemma 1.4.1 *$E(k)$*

$E(k)$ 的另種形式為

$$E(k) = \left(1-k^2\right)\int_0^{\frac{\pi}{2}} \left(1-k^2\sin^2\theta\right)^{-3/2}\mathrm{d}\theta$$

Proof. 令上式為 I。
由泰勒展開式

$$(1-x)^{-3/2} = \sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n)!!}x^n$$

將積分值 (見 section 1.1.3) 代入

$$\int_0^{\frac{\pi}{2}}\left(1-k^2\sin^2\theta\right)^{-3/2}\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n)!!}k^{2n}\sin^{2n}\theta\,\mathrm{d}\theta = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n+1)!!}{(2n)!!}\right]^2\frac{k^{2n}}{2n+1}$$

代回可得

$$I = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left\{\left[\frac{(2n+1)!!}{(2n)!!}\right]^2\frac{1}{2n+1} - \left[\frac{(2n-3)!!}{(2n-2)!!}\right]^2\frac{1}{2n-3}\right\}k^{2n}\right\}$$

後項的大括號內整理得

$$\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left[2n+1-\frac{(2n)^2}{2n-1}\right] = \left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{1}{2n-1}$$

$$I = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{k^{2n}}{2n-1}\right\}$$

比對可得 $I = E(k)$。 □

利用 Lemma 1.4.1，注意到

$$\frac{E\left(k\right)}{1-k^{2}}-K\left(k\right)=\int_{0}^{\frac{\pi}{2}}\frac{k^{2}\sin^{2}\theta}{\left(1-k^{2}\sin^{2}\theta\right)^{3/2}}\,\mathrm{d}\theta$$

所以有

$$\boxed{\frac{\mathrm{d}K\left(k\right)}{\mathrm{d}k}=\frac{E\left(k\right)}{k\left(1-k^{2}\right)}-\frac{K\left(k\right)}{k}}$$

微分方程式為

$$\frac{\mathrm{d}}{\mathrm{d}k}\left[k\left(1-k^{2}\right)\frac{\mathrm{d}K\left(k\right)}{\mathrm{d}k}\right]=kK\left(k\right)$$

■ **第二類完全橢圓積分**

$$\frac{\mathrm{d}E\left(k\right)}{\mathrm{d}k}=-\int_{0}^{\frac{\pi}{2}}\frac{k\sin^{2}\theta}{\sqrt{1-k^{2}\sin^{2}\theta}}\,\mathrm{d}\theta$$

注意到

$$K\left(k\right)-E\left(k\right)=\int_{0}^{\frac{\pi}{2}}\frac{k^{2}\sin^{2}\theta}{\sqrt{1-k^{2}\sin^{2}\theta}}\,\mathrm{d}\theta$$

所以有

$$\frac{\mathrm{d}E\left(k\right)}{\mathrm{d}k}=\frac{E\left(k\right)-K\left(k\right)}{k}$$

微分方程式為

$$-\left(1-k^{2}\right)\frac{\mathrm{d}}{\mathrm{d}k}\left(k\frac{\mathrm{d}E\left(k\right)}{\mathrm{d}k}\right)=kE\left(k\right)$$

第二章

Problem 2.0.1 Problem 5.32

A circular loop of mean radius a is made of wire having a circular cross section of radius b, with b for this problem. a. The sketch shows the relevant dimensions and coordinates

(a) Using (5.37), the expression for the vector potential of a filamentary circular loop, and appropriate approximations for the elliptic integrals, show that the vector potential at the point P near the wire is approximately

$$A_\phi = \frac{\mu_0 I}{2\pi}\left(\ln\frac{8a}{\rho} - 2\right)$$

where ρ is the transverse coordinate shown in the figure and corrections are of order $\rho\cos\phi/a$ and $(\rho/a)^2$.

(b) Since the vector potential of part a is, apart from a constant, just that outside a straight circular wire carrying a current I, determine the vector potential inside the wire ($\rho < b$) in the same approximation by requiring continuity of A and its radial derivative at $\rho = b$, assuming that the current is uniform in density inside the wire:

$$A_\phi = \frac{\mu_0 I}{4\pi}\left(1 - \frac{\rho^2}{b^2}\right) + \frac{\mu_0 I}{2\pi}\left(\ln\frac{8a}{b} - 2\right)$$

(c) Use (5.149) to find the magnetic energy, hence the self-inductance,

$$L = \mu_0 a\left(\ln\frac{8a}{b} - \frac{7}{4}\right)$$

Are the corrections of order b/a or $(b/a)^2$? What is the change in L if the current is assumed to flow only on the surface of the wire (as occurs at high frequencies when the skin depth is small compared to (b)?

$$A_\phi(r,\theta)=\frac{\mu_0 Ia}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\,\mathrm{d}\phi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\phi'}}$$

注意到

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\cos^2\theta}}\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\frac{k^2}{2}-\frac{k^2}{2}\cos 2\theta}}\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\pi}\frac{1}{\sqrt{1-\frac{k^2}{2}-\frac{k^2}{2}\cos\theta}}\,\mathrm{d}\theta$$

$$E(k)=\frac{1}{2}\int_0^{\pi}\sqrt{1-\frac{k^2}{2}-\frac{k^2}{2}\cos\theta}\,\mathrm{d}\theta$$

相減可得

$$\begin{aligned}\left(1-\frac{k^2}{2}\right)K(k)-E(k)&=\frac{1}{4}\int_0^{2\pi}\frac{\frac{k^2}{2}\cos\theta}{\sqrt{1-\frac{k^2}{2}-\frac{k^2}{2}\cos\theta}}\,\mathrm{d}\theta\\&=\frac{k^2}{8\sqrt{1-\frac{k^2}{2}}}\int_0^{2\pi}\frac{\cos\theta}{\sqrt{1-\frac{k^2}{2-k^2}\cos\theta}}\,\mathrm{d}\theta\end{aligned}$$

觀察原式可令

$$\frac{k^2}{2-k^2}=\frac{2ar\sin\theta}{a^2+r^2}$$

解得

$$k^2=\frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$$

代入得

$$A_\phi(r,\theta)=\frac{\mu_0 Ia}{4\pi}4\sqrt{\frac{1-\frac{k^2}{2}}{a^2+r^2}}\frac{(2-k^2)\,K(k)-2E(k)}{k^2}$$

$$A_\phi(r,\theta)=\frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\frac{(2-k^2)\,K(k)-2E(k)}{k^2}$$

在 $\rho\ll a$ 的條件下

$$k^2=\frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}=\frac{4a(a+\rho\cos\phi)}{(2a+\rho\cos\phi)^2+\rho^2\sin^2\phi}=\frac{4a(a+\rho\cos\phi)}{4a^2+4a\rho\cos\phi+\rho^2}\approx 1-\frac{\rho^2}{4a^2}$$

$$k\approx 1-\frac{\rho^2}{8a^2}$$

代回可得

$$A_\phi(r,\theta)\approx\frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{8a}{\rho}\right)-2\right]$$

2.1 延遲勢 Retarded Potential

2.1.1 電荷移動的電場

由 Jackson Eq.(6.55)

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q}{4\pi\epsilon_0}\int\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\left[\rho\left(\boldsymbol{r}',t'\right)\right]_{\text{ret}}+\frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho\left(\boldsymbol{r}',t'\right)}{\partial t'}\right]_{\text{ret}}-\frac{1}{c^2R}\left[\frac{\partial\boldsymbol{J}\left(\boldsymbol{r}',t'\right)}{\partial t'}\right]_{\text{ret}}\right\}\mathrm{d}^3\boldsymbol{r}' \tag{2.1}$$

代入 $\rho(\boldsymbol{r},t)=q\delta\left(\boldsymbol{r}-\boldsymbol{r}'\left(t_r\right)\right)$。注意到

$$\int\delta\left(\boldsymbol{r}-\boldsymbol{r}'\left(t_r\right)\right)\mathrm{d}^3\boldsymbol{r}=\sum_{\boldsymbol{r}-\boldsymbol{r}'(t_r)=0}\frac{1}{\left|\frac{\partial(\boldsymbol{r}-\boldsymbol{r}'(t_r))}{\partial\boldsymbol{r}}\right|}$$

由 $t_r=t-\frac{|\boldsymbol{r}-\boldsymbol{r}'(t_r)|}{c}$。所以

$$\frac{\partial\left(\boldsymbol{r}-\boldsymbol{r}'\left(t_r\right)\right)}{\partial\boldsymbol{r}}=1-\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{c}$$

在(2.1)中 $\boldsymbol{r}'$ 為 dummy index，與 t,t' 無關。所以可寫為

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q}{4\pi\epsilon_0}\int\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\left[\rho\left(\boldsymbol{r}',t'\right)\right]_{\text{ret}}+\frac{\partial}{\partial t'}\left[\frac{\hat{\boldsymbol{R}}}{cR}\rho\left(\boldsymbol{r}',t'\right)\right]_{\text{ret}}-\frac{\partial}{\partial t'}\left[\frac{\boldsymbol{J}\left(\boldsymbol{r}',t'\right)}{c^2R}\right]_{\text{ret}}\right\}\mathrm{d}^3\boldsymbol{r}'$$

積分可得 Jackson Eq.(6.58)

$$\boldsymbol{E}=\frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\boldsymbol{R}}}{\kappa R^2}\right]_{\text{ret}}+\frac{\partial}{c\partial t}\left[\frac{\hat{\boldsymbol{R}}}{\kappa R}\right]_{\text{ret}}-\frac{\partial}{c^2\partial t}\left[\frac{\boldsymbol{v}}{\kappa R}\right]_{\text{ret}}\right\} \tag{2.2}$$

其中 $\kappa=1-\boldsymbol{v}\cdot\hat{\boldsymbol{R}}/c$。

由 Jackson Eq.(6.60)，Feynman 表達式為

$$\boldsymbol{E}=\frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\boldsymbol{R}}}{R^2}\right]_{\text{ret}}+\frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\boldsymbol{R}}}{R^2}\right]_{\text{ret}}+\frac{\partial^2}{c^2\partial t^2}[\hat{\boldsymbol{R}}]_{\text{ret}}\right\} \tag{2.3}$$

由 Griffith Eq.(10.72)

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q}{4\pi\epsilon_0}\frac{R}{(\hat{\boldsymbol{R}}\cdot\boldsymbol{u})^3}\left[\left(c^2-v^2\right)\boldsymbol{u}+\hat{\boldsymbol{R}}\times(\boldsymbol{u}\times\boldsymbol{a})\right] \tag{2.4}$$

其中 $\boldsymbol{u}=c\hat{\boldsymbol{R}}-\boldsymbol{v}$。以下將證明三式為等價的。

由上式可知必須求出 $\partial\boldsymbol{R}/\partial t$, $\partial R/\partial t$, $\partial\kappa/\partial t$ 。另外注意 $\boldsymbol{v}=\partial\boldsymbol{r}'/\partial t_r$, $\boldsymbol{a}=\partial\boldsymbol{v}/\partial t_r$ ，因此在計算物理量對時間微分 $\partial\boldsymbol{A}/\partial t$ ，我皆拆成

$$\frac{\partial\boldsymbol{A}}{\partial t}=\frac{\partial\boldsymbol{A}}{\partial t_r}\frac{\partial t_r}{\partial t}$$

再藉由算出 $\partial t/\partial t_r$ 減少計算。

以下我們分別計算上述所需的物理量：

(1) $\partial t/\partial t_r$

t 與 t_r 的關係，由 $R=c\,(t-t_r)$ 平方可得

$$r^2+r'^2-2\boldsymbol{r}\cdot\boldsymbol{r}'=c^2\,(t-t_r)^2$$

兩邊微分

$$2r'v-2\boldsymbol{r}\cdot\boldsymbol{v}=2c^2\,(t-t_r)\left(\frac{\partial t}{\partial t_r}-1\right)$$

$$\frac{\partial t}{\partial t_r}=1-\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{c}=\kappa$$

(2) $\partial\boldsymbol{R}/\partial t_r=-\boldsymbol{v}$

(3) $\partial R/\partial t_r=\partial\,|\boldsymbol{r}-\boldsymbol{r}'\,(t_r)|/\partial t_r=-\hat{\boldsymbol{R}}\cdot\boldsymbol{v}$

(4) $\partial\kappa/\partial t_r$

$$\frac{\partial\kappa}{\partial t_r}=-\frac{1}{c}\frac{\partial}{\partial t_r}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{R}}{R}\right)=-\frac{1}{c}\left[\frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{R}-v^2}{R}+\frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{R^3}\right]$$

接下來處理(2.2)中各項

$$\frac{\partial}{\partial t_r}\left(\frac{\boldsymbol{R}}{\kappa R^2}\right)=-\frac{\boldsymbol{v}}{\kappa R^2}+\frac{2\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa R^3}\boldsymbol{R}+\frac{\boldsymbol{R}}{\kappa^2R^2}\frac{1}{c}\left[\frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{R}-v^2}{R}+\frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{R^3}\right]$$

$$\frac{\partial}{\partial t_r}\left(\frac{\boldsymbol{v}}{\kappa R}\right)=\frac{\dot{\boldsymbol{v}}}{\kappa R}+\frac{\boldsymbol{v}}{\kappa R^2}\boldsymbol{v}\cdot\hat{\boldsymbol{R}}+\frac{\boldsymbol{v}}{\kappa^2 R}\frac{1}{c}\left[\frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{R}-v^2}{R}+\frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{R^3}\right]$$

代入得

$$\begin{aligned}\boldsymbol{E}=\frac{q}{4\pi\epsilon_0R^2\kappa^3}\Bigg\{&\frac{\kappa^2\boldsymbol{R}}{R}-\frac{\kappa}{c}\left(\boldsymbol{v}-\frac{2\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{R}\boldsymbol{R}\right)+\frac{c\boldsymbol{R}-\boldsymbol{v}R}{c^3}\left[\frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{R}-v^2}{R}+\frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{R^3}\right]\\&-\frac{\kappa R}{c^2}\left(\dot{\boldsymbol{v}}+\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{R}\boldsymbol{v}\right)\Bigg\}\end{aligned}$$

注意到

$$\frac{c\boldsymbol{R}-\boldsymbol{v}R}{c^3}\frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{R}}{R}-\frac{\kappa R}{c^2}\dot{\boldsymbol{v}}=\boldsymbol{R}\times[(c\hat{\boldsymbol{R}}-\boldsymbol{v})\times\dot{\boldsymbol{v}}]$$

大括弧中剩餘項為

$$\begin{aligned}&\frac{-\kappa^2+2\kappa}{R}\boldsymbol{R}+\frac{c\hat{\boldsymbol{R}}-\boldsymbol{v}}{c^3}\left[-v^2+\frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{R^2}\right]-\left[\frac{\kappa}{c}+\frac{\kappa(\boldsymbol{v}\cdot\boldsymbol{R})}{c^2}\right]\boldsymbol{v}\\=\;&\frac{-\kappa^2+2\kappa}{R}\boldsymbol{R}+\frac{c\hat{\boldsymbol{R}}-\boldsymbol{v}}{c^3}\left[-v^2+\frac{c^2(1-\kappa)^2}{R^2}\right]-\frac{\kappa}{c}(1+1-\kappa)\boldsymbol{v}\\=\;&\left(1-\frac{v^2}{c^2}\right)\left(\frac{\boldsymbol{R}}{R}-\frac{\boldsymbol{v}}{c}\right)\end{aligned}$$

整理可得(2.4)。

整理(2.3)大括弧內的項。

$$\frac{\partial^2 \hat{\boldsymbol{R}}}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r}\left(\frac{\boldsymbol{R}}{R}\right)\right] = -\frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}}{\kappa R}\right) + \frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa R^2}\boldsymbol{R}\right)$$

其中

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa R^2}\boldsymbol{R}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{1}{\kappa}-1\right)\frac{\boldsymbol{R}}{R^2}\right]$$

代入可得

$$\frac{\boldsymbol{R}}{R^3} + \frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{R}}{R^3}\right) + \frac{\partial}{c\partial t}\left(\frac{\hat{\boldsymbol{R}}}{\kappa R}\right) - \frac{\partial}{c^2\partial t}\left(\frac{\boldsymbol{v}}{\kappa R}\right) - \frac{\partial}{\partial t}\left(\frac{1}{R}\frac{\boldsymbol{R}}{R^2}\right)$$

又其中三項和為

$$\frac{\boldsymbol{R}}{R^3} + \frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{R}}{R^3}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{R}}{R^2}\right) = \frac{\boldsymbol{R}}{R^3} - \frac{\boldsymbol{R}}{cR^3}\left(-\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa}\right) = \frac{\boldsymbol{R}}{R^3}\left(1+\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa c}\right) = \frac{\boldsymbol{R}}{\kappa R^3}$$

代入比對得(2.3)。

2.1.2 電荷移動的磁場

由 Jackson Eq.(6.56)

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\left\{[\boldsymbol{J}(\boldsymbol{r}',t')]_{\mathrm{ret}} \times \frac{\hat{\boldsymbol{R}}}{R^2} + \left[\frac{\partial \boldsymbol{J}(\boldsymbol{r}',t')}{\partial t'}\right]_{\mathrm{ret}} \times \frac{\hat{\boldsymbol{R}}}{cR}\right\}\mathrm{d}^3\boldsymbol{r}' \tag{2.5}$$

同上節，積分可得 Jackson Eq.(6.59)

$$\boldsymbol{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\boldsymbol{v}\times\hat{\boldsymbol{R}}}{\kappa R^2}\right]_{\mathrm{ret}} + \frac{\partial}{c\partial t}\left[\frac{\boldsymbol{v}\times\hat{\boldsymbol{R}}}{\kappa R}\right]_{\mathrm{ret}}\right\}$$

其中

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{v}\times\boldsymbol{R}}{\kappa}\frac{1}{R}\right] = \frac{1}{R}\frac{\partial}{\partial t}\left[\frac{\boldsymbol{v}\times\boldsymbol{R}}{\kappa}\right] + \frac{\boldsymbol{v}\times\boldsymbol{R}}{\kappa R^2}\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{R}}}{\kappa}$$

代入可得 Jackson Eq.(6.61)，Heaviside 表達式

$$\boldsymbol{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\boldsymbol{v}\times\hat{\boldsymbol{R}}}{\kappa^2 R^2}\right]_{\mathrm{ret}} + \frac{1}{c[R]_{\mathrm{ret}}}\frac{\partial}{\partial t}\left[\frac{\boldsymbol{v}\times\hat{\boldsymbol{R}}}{\kappa}\right]_{\mathrm{ret}}\right\}$$