Physics Cup 2024 Problem 4

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1 The relationship between displacement and velocity

In this document, I use $\boldsymbol{r}$ as the displacement of the satellite, and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ as the velocity ◦ As we all know that the displacement follows Newton's gravity law

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

and we can get $r(\theta)$

$$r(\theta) = \frac{l}{1 - e\cos\theta}$$

recall the definition of the value of angular momentum $J = |\boldsymbol{J}|$

$$J = mr^2\dot{\theta} = \text{constant}$$

And the rate of the sweeping area is

$$L = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$$

so we can rewrite the newton's law as

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GMm^2}{J}\dot{\theta}\hat{\boldsymbol{r}}$$

and note that

$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot{\theta}\hat{\boldsymbol{r}}$$

By direct integration, we can get

$$\boldsymbol{v}(\boldsymbol{r}) - \frac{GMm}{J}\hat{\boldsymbol{\theta}}(\boldsymbol{r}) = \boldsymbol{c} = \text{constant} \tag{1}$$

Furthermore, it is easy to know the vector $\boldsymbol{c}$ by choosing to calculate some special points, such as the aphelion and the perihelion. I choose to use the perihelion. Denote the velocity at the point as u, by energy conservation and angular momentum

$$\frac{1}{2}mu^2 - \frac{GMm}{\frac{l}{1+e}} = \frac{1}{2}m\left(u\frac{1-e}{1+e}\right)^2 - \frac{GMm}{\frac{l}{1-e}}$$

$$u = (1+e)\sqrt{\frac{GM}{l}}$$

so we can know that

$$J = m\frac{l}{1+e}u = m\sqrt{GMl} \tag{2}$$

$$\boldsymbol{c} = e\sqrt{\frac{GM}{l}}\hat{\boldsymbol{u}} \tag{3}$$

Furthermore, we can use L to represent l

$$l = \frac{4L^2}{GM} \tag{4}$$

2 Relative velocity

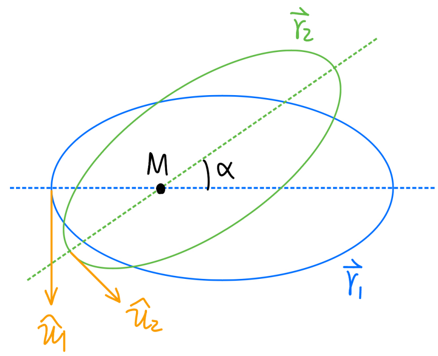

Figure 1. the trajectory of two satellites

Using (1)(2)(3)

$$\boldsymbol{v_1} - \boldsymbol{v_2} = \sqrt{\frac{GM}{l_1}}\hat{\boldsymbol{\theta}}_1 - \sqrt{\frac{GM}{l_2}}\hat{\boldsymbol{\theta}}_2 + \boldsymbol{c}_1 - \boldsymbol{c}_2$$

and

$$\boldsymbol{c}_1 - \boldsymbol{c}_2 = e_1\sqrt{\frac{GM}{l_1}}\hat{\boldsymbol{u}}_1 - e_2\sqrt{\frac{GM}{l_2}}\hat{\boldsymbol{u}}_2$$

Now, we want to know the maximum relative velocity. Since we know that $\boldsymbol{c}_1 - \boldsymbol{c}_2$ is a constant, we can choose $\sqrt{\frac{GM}{l_1}}\hat{\boldsymbol{\theta}}_1$ and $\sqrt{\frac{GM}{l_2}}\hat{\boldsymbol{\theta}}_2$ to be parallel with $\boldsymbol{c}_1 - \boldsymbol{c}_2$. So the value of the maximum relative velocity $v_{\max}$ is

$$v_{\max} = |\boldsymbol{v_1} - \boldsymbol{v_2}|_{\max} = |\boldsymbol{c}_1 - \boldsymbol{c}_2|_{\max} + \sqrt{GM}\left(\frac{1}{\sqrt{l_1}} + \frac{1}{\sqrt{l_2}}\right)$$

By the definition of α, we can know that

$$|\boldsymbol{c}_1 - \boldsymbol{c}_2|_{\max} = \sqrt{|\boldsymbol{c}_1|^2 + |\boldsymbol{c}_2|^2 + 2|\boldsymbol{c}_1||\boldsymbol{c}_2|\cos\alpha} = \sqrt{GM}\sqrt{\frac{e_1^2}{l_1} + \frac{e_2^2}{l_2} + \frac{2e_1e_2}{\sqrt{l_1l_2}}\cos\alpha}$$

$$v_{\max} = \sqrt{GM}\sqrt{\frac{e_1^2}{l_1} + \frac{e_2^2}{l_2} + \frac{2e_1e_2}{\sqrt{l_1l_2}}\cos\alpha} + \sqrt{GM}\left(\frac{1}{\sqrt{l_1}} + \frac{1}{\sqrt{l_2}}\right)$$

Finally by substituting L for l using (4), the answer is

$$v_{\max} = \frac{GM}{2}\left(\sqrt{\frac{e_1^2}{L_1^2} + \frac{e_2^2}{L_2^2} + \frac{2e_1e_2}{L_1L_2}\cos\alpha} + \frac{1}{L_1} + \frac{1}{L_2}\right)$$

as for the answer of the special case, $\alpha = 90^{\circ}$, $L_1 = L_2 = L$

$$v_{\max} = \frac{GM}{2L}\left(\sqrt{e_1^2 + e_2^2} + 2\right)$$