Assignment 5

id: 519021910861

name: huidong xu

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8-1

(a) Yes. One solution would be: *Interval Scheduling* can be solved in polynomial time, and so it can also be solved in polynomial time with access to a black box for *Vertex Cover*. (It need never call the black box.) Another solution would be : *Interval Scheduling* is in NP, and anything in NP can be reduced to *Vertex Cover*. A third solution would be : we've seen in the book the reductions $IntervalScheduling \leq_P IndependentSet$ and $IndependentSet \leq_P VertexCover$, so the result follows by transitivity.

(b) This is equivalent to whether P = NP. If P = NP, then *Independent Set* can be solved in polynomial time, and so $IndependentSet \leq_P IntervalScheduling$. Conversely, if $IndependentSet \leq_P IntervalScheduling$, then since *Interval Scheduling* can be solved in polynomial time, so could *Independent Set*. But *Independent Set* is NP-complete, so solving it in polynomial time would imply P = NP.

8-3

The problem is in NP since, given a set of k counselors, we can check that they cover all the sports.

Suppose we had such an algorithm A; here is how we would solve an instance of *Vertex Cover*. Given a graph G = (V, E) and an integer k, we would define a sport S_e for each edge e, and a counselor C_v for each vertex v. C_v is qualified in sport S_e if and only if e has an endpoint equal to v.

Now, if there are k counselors that, together, are qualified in all sports, the corresponding vertices in G have the property that each edge has an end in at least one of them; so they define a vertex cover of size k. Conversly, if there is a vertex cover of size k, then this set of counselors has the property that each sport is contained in the list of qualifications of at least one of them.

Thus, G has a vertex cover of size at most k if and only if the instance of *Efficient Recruiting* that we create can be solved with at most k counselors. Moreover, the instance of *Efficient Recruiting* has size polynomial in the size of G. Thus, if we could determine the answer to the *Efficient Recruiting* instance in polynomial time, we could also solve the instance of *Vertex Cover* in polynomial time.

8-21

The problem is in *NP* because we can exhibit a choice of one element from each option set, and it can be checked there are no incompatibilities between these.

We now show that $3-SET \leq_P FullyCompatibleConfiguration$. The reduction will be very similar to the reduction that showed $3-SET \leq_P IndependentSet$. Given an instance of corresponding to the terms in the clause. We now declare an option in A_i to be incompatible with an option in A_j if the terms corresponding to these two options correspond to a variable and its negation.

If there is a satisfying assignment for the *3-SET* instance, then we can choose a term from each clause in such a way that we never choose both a variable and its negation. Thus, the corresponding selection of options will have no incompatibilities. Conversely, if there is a way to select options without incompatibilities, then there is a corresponding selection of terms from clauses so that we never choose a variable and its negation. Thus, we can set all variables as indicated by the selected terms (setting variables arbitrary when they appear in no selected term) so as to satisfy the *3-SET* instance.

8-27

The problem is in *NP* because we can exhibit a partition of the numbers into sets, and then sum the squares of the totals in each set.

We now show that *Number Partition*, which we proved NP-complete in the previous problem, is reducible to this problem. Thus, given a collection of $x_1, \ldots, x_n$, where we want to know if they can be divided into two sets with the same sum, we construct an instance of this sum-of-squares problem in which $k = 2$ and $B = \frac{1}{2}S^2 = B$. Conversely, suppose we have two sets whose total sums are S_1 and S_2 respectively. Then we have $S_1 + S_2 = S$, and $S_1^2 + S_2^2 = \frac{1}{2}S^2$. The only solution to this is $S_1 = S_2 = \frac{1}{2}S$, so these two sets form a solution to the instance of *Number Partitioning*.

8-28

The problem is in *NP* since we can exhibit a set of k nodes and check that the distance between all pairs is at least 3.

We now show $IndependentSet \leq_P StronglyIndependentSet$. Given a graph G and a number k, we construct a new graph $G^{'}$ in which we replace each edge e = (u, v) by a path of length two: we add a new node w_e, and we add edges $(u, w_e), (w_e, v)$. We also include edges between every pair of new nodes.

Now suppose that G has an independent set of size k. Then in this new graph $G^{'}$, all these k nodes are distance at least three from each other, so this is a strongly independent set of size k. Conversely, suppose $G^{'}$ has a strongly independent set of size k. Now, this set can't contain any of the new nodes, since all such nodes are within distance two of every node in the graph. Thus, it consists of nodes present in G. Moreover, no two of these nodes can be neighbors in G, since then they'd be at distance two in $G^{'}$. Thus this set of nodes forms an independent set of size k in G.

8-31

Given a set X of vertices, we can use depth-first search to determine if G - X has no cycles. Thus *undirected feedback set* is in NP.

We now show that *vertex cover* can be reduced to *undirected feedback set*. Given a graph G = (V, E) and integer k, construct a graph $G^{'} = (V^{'}, E^{''})$ in which each edge $(u, v) \in E$ is replaced by the four edges $(u, x_{uv}^{1}), (u, x_{uv}^{2}), (v, x_{uv}^{1})$, and (v, x_{uv}^{2}) for new vertices x_{uv}^{i} that appear only incident to these edges. Now, suppose that X is a vertex cover of G. Then viewing X as a subset of $V^{'}$, it is easy to verify that $G^{'} - X$ has no cycles. Conversely, suppose that Y is a feedback set of $G^{'}$ of minimum cardinality. We may choose Y so that it contains no vertex of the form X_{uv}^{i} -- for it does, then $Y \cup \{u\} - \{x_{uv}^{i}\}$ is a feed back set of no greater cardinality. Thus, we may view Y as a subset of V. For every edge $(u, v) \in E$, Y must interact the four-node cycle formed by u, v, x_{uv}^{1}, and x_{uv}^{2}; since we have chosen Y so that it contains no node of the form x_{uv}^{i}, it follows that Y contains one of u or v. Thus, Y is a vertex cover of G.