MATH 215 September 8, 2016

Chapter 8: Systems of Linear Algebraic Equations

· Linear Systems (L.S.) appear in solving some ODEs, PDEs, finding the current at a node or loop in electric circuits, etc.

Def: A L.S. in m equations (eq's) and n variables $x_1, x_2, \ldots, x_n$ has the form:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \vdots \\ a_{m}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{array}\right\} (1)$$

· In $a_{ij}x_j$, $i(\& j)$ represent the equation (& variable) numbers.

· m & n are finite.

· a_{ij}'s & C_j's are real numbers for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

· Def. If all C_j's in (1) are zero, sys. (1) is said to be homogenous L.S.

· Least one of c_j's is not zero, (1) is said to be non-homogenous.

· Def. If at least one of the variables (var.) x_j's in (1) appears non-linearly in an eq., (1) is called non-linear system.

Def. If sys. (1) has at least one solution (sol.), it is said to be consistent. If sys. (1) has no sol., it's said to be inconsistent.

Example: (Ex.): (1) $\begin{cases} x_1 - \sqrt{2}\, x_2 = 6 \\ 3x_1 + e^{-1} x_2 = -\frac{1}{2} \end{cases}$ L.S. in x_1 & x_2, $m=2$, $n=2$.

(2) $\begin{cases} 2x_1 + x_2 = 0 \\ x_1 - \sin(x_2) + x_3 = 1 \end{cases}$ Nonlinear b/c $\sin(x_i)$.

Def. The set $\{s_1, s_2, \ldots, s_n\}$ is a sol. of (1) if the m eq's in (1) are satisfied when we substitute $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$ into (1).

Ex. Show that $\{s_1 = 1, s_2 = 2\}$ is a solution of the L.S.

$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 2x_2 = 5 \end{cases}$ Check: $\begin{cases} 2(1) - (2) = 0 \checkmark \\ (1) + 2(2) = 5 \checkmark \end{cases}$

Solving L.S.

Method I: Gauss Elimination / Back Substitution.

2 Steps : Step 1: Gauss Elimination Method. (to eliminate varia

Step 2. Back Substitution.

· In Step 1. We need/use the elementary equation operations. (E.E

which are ①. Multiply one equation by a non-zero constant.

Symbolic Representation { (eq. j → α eq. j) (or $E_j \Rightarrow \alpha E_j$).

② Add a multiple of one eq. to another eq.

(eq. j ⇒ eq. j + eq. k) or $E_j \Rightarrow E_j + \alpha E_k$).

Def: Two systems are equivalent if they have same sol. set

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· Use operations to turn 1 plane into another.

Remarks: ⓘ In elimination, each E.E.O. must be reversible, and must leave the solution set unchanged.

ⓘⓘ Every system produced during elimination procedure is equivalen to the original sys. Ch. 8.3.1 (Equivalent system.).

If one linear system is obtained from another system by a finite number of E.E.O's, then the two sys' are equivalent

Example. Determine if the L.S.

$$\text{Pivot}\quad \left.\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 7 \\ x_1 + 2x_2 + 3x_3 &= 9. \end{aligned}\right\} \text{ has a solution} \quad 1$$

$x_1 + x_2 - x_3 = 2$ (unchanged).

$E_2 \to E_2 + (-2)E_1 \quad E_3 \to E_3 + (-1)E_1 \Rightarrow$

$$-3x_2 + 3x_3 = 3$$
$$x_2 + 4x_3 = 7$$

$E_2 \leftrightarrow E_3 \Rightarrow$

$$x_1 + x_2 - x_3 = 2$$
pivot $x_2 + 4x_3 = 7$
$$\boxed{-3x_2} + 3x_3 = 3$$

$E_3 \to E_3 + 3E_2 \Rightarrow$

$$x_1 + x_2 - x_3 = 2$$
$$x_2 + 4x_3 = 7$$
$$15x_3 = 24$$

End of Gauss Elim. Start back sub.

$E_3 \to \frac{1}{15}E_3 \to$

$$x_1 + x_2 - x_3 = 2$$
$$x_2 + 4x_3 = 7$$
$$x_3 = \frac{8}{5}$$

: Using the back sub gives.

$x_3 = \frac{8}{5}, \; x_2 = \frac{3}{5}, \; x_1 = 1$

The sys. has a unique solution.

Example. Consider the sys.

$$x_1 + x_2 - x_3 + 3x_4 = -2$$
$$\boxed{x_1} + x_2 + 2x_3 + 2x_4 = 3$$
$$\boxed{x_1} + x_2 + 3x_3 - x_4 = 4$$

$m = 3$
$n = 4.$

$E_2 \to E_2 + (-1)E_{\cancel{2}}2.$
$E_3 \to E_3 + (-1)E_1 \Rightarrow$

$$x_1 + x_2 - x_3 + 3x_4 = 2$$
$$3x_3 - x_4 = 5$$
$$\cancel{3}4x_3 - 4x_4 = \cancel{4}6$$

$E_2 \to \frac{1}{3}E_2 \Rightarrow$

$$x_1 + x_2 - x_3 + 3x_4 = \cancel{2}-2.$$
pivot $x_3 - \frac{1}{3}x_4 = \frac{5}{3}.$
$$\boxed{3x_3} - 4x_4 = 6$$

$E_3 \to E_3 + (-\cancel{3}4)E_2 \Rightarrow$

$$x_1 + x_2 - x_3 + 3x_4 = 2$$
$$x_3 - \frac{1}{3}x_4 = \frac{5}{3}$$
$$-\frac{8}{3}x_4 = -\frac{2}{3}$$

$E_3 \to -\frac{3}{8}E_3 \Rightarrow$

$$x_1 + x_2 - x_3 + 3x_4 = -2$$
$$x_3 - \frac{1}{3}x_4 = \frac{5}{3}$$
$$x_4 = \frac{1}{4}$$

Using back substitution gi

$x_4 = \frac{1}{4}, \; x_3 = \frac{5}{3} + \frac{1}{3}\left(\frac{1}{4}\right) =$

Plugging x_3 & x_4 into eq. ① gives $x_1 + x_2 - \frac{21}{12} + \frac{3}{4} = -2.$

$$\to x_1 = -2 - x_2 + \frac{21}{12} - \frac{3}{4}$$

Let $x_2 = \alpha \; (\alpha \in \mathbb{R})$ Thus the sol. set is.

$$\left\{-2 - \alpha + \frac{21}{12} - \frac{3}{4}, \; \alpha, \; \frac{21}{12}, \; \frac{1}{4}\right\}$$

The system has infinite sol's. We also say the L.S. has 1-parameter family of sol's.

Particular sol. Choose $\alpha = 1$ Then $\{ \_, \_, \_, \_ \}$

Ex. Consider the L.S. $2x_1 + 4x_2 = 1$
$x_1 + 2x_2 = 3$

Any other cases?

$E_2 \leftrightarrow E_1$ $x_1 + 2x_2 = 3$ $E_2 \to E_2 + (-2)E_1$ $x_1 + 2x_2 = 3$ No Solut
$\boxed{2x_1} + 4x_2 = 1$ $0 = -5$.' Impossible

∴ The L.S. is inconsistent.

Th. 8.3.2 (Existence / Uniqueness for L.S.).
If $m < n$, sys ① can be consistent or inconsistent.
· If it is consistent, it cannot have a unique solution.
; it will have a p-parameter family of solutions where.
$n - m \le p \le n$. If $m \ge n$, sys ① can be consistent or.
inconsistent. If it's consistent it can ~~not~~ have unique sol or a p-parameter ~~a~~ family of sol where $1 \le p \le n$.

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Theorem 8.3.3:
· Every L.S necessarily admits a <u>unique</u> sol, an infinity of soln, or no soln.

Theorem 8.3.4: (Existence / Uniqueness for Homogenous Sys').
· Every homog. L.S. of m eq's in n unknowns is <u>consistent</u>. Either it admits the unique trivial sol. or else it admits an infinity of sol's of non-trivial sol's in addition to the trivial sol.

· If $m < n$, then there is an infinity of sol's in addition to the trivial ~~sol~~. sol.

Ex: $x_1 + x_2 + 2x_3 - 2x_4 + 2x_5 = 0$
$\boxed{3x_1} + 3x_2 + 5x_3 + 2x_5 = 0$
$m = 2 < n = 5$

$E_2 \to E_2 + (-3)E_1 \to x_1 + x_2 + 2x_3 - 2x_4 + 2x_5 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad -x_3 + 6x_4 - 4x_5 = 0$

$E_2 \to (-1)E_2 \to x_1 + x_2 + 2x_3 - 2x_4 + 2x_5 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad x_3 - 6x_4 + 4x_5 = 0$

· Using back sub. gives $x_3 = 6x_4 - 4x_5$

Choose $x_4 = \alpha_1 \in \mathbb{R}$, & $x_5 = \alpha_2 \in \mathbb{R} \Rightarrow x_3 = 6\alpha_1 - 4\alpha_2$

· Plug into equation 1 to get $x_1 = -x_2 - 2x_3 + 2x_4 - 2x_5$

$\qquad\qquad\qquad\qquad\qquad\qquad = -x_2 - 2(6\alpha_1 - 4\alpha_2) + 2\alpha_1 - 2\alpha_2$

Choose $x_2 = \alpha_3$ ~~Then $x_1 = -\alpha_3$~~, $\alpha_3 \in \mathbb{R}$ Then $x_1 = -\alpha_3 - 10\alpha_1 + 6\alpha_2$

So the 3-parameter family of sol's is

$\{-\alpha_3 - 10\alpha_1 + 6\alpha_2,\ \alpha_3,\ -,\ -,\ -\}$

8.3.3 Matrix Notation

· In the L.S. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = c_m.$$

→ Ommitting the variables x_j's leads to the rectangular arr
called augmented matrix.

$$[A : \vec{C}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & c_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & c_3 \end{bmatrix}$$

$$A = \begin{array}{cc} & \\ & \\ & \\ \text{row } m \to & \\ & \end{array}\begin{array}{c}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \\ \begin{array}{cccc} \uparrow & & & \uparrow \\ \text{column } 1 & & & \text{column } n \end{array}\end{array}, \quad \vec{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

· To find the sol., we use the <u>elementary row operations</u> (E.R.Os) which ~~are~~ are:

(a) Addition of a multiple of one row to another row ($R_j \to R_j + \alpha R$

(b) Multiplying a row by a <u>non-zero</u> constant ($R_j \to \alpha R_j$)

(c) Interchange 2 rows ($R_j \leftrightarrow R_k$)

· In the Gauss - Jordan reduction method, the final result, after applying the E.R.O's to $[A : \vec{C}]$ is a new augmented matrix, say $[A' : \vec{C}']$, in <u>reduced</u> row - echelon form (RREF).

Def: (RREF) A matrix M is said to be in RREF if it satisfies the following

1. In each row not made up entirely of zeros, the first non-zero element is a 1, ~~a~~ often called <u>Leading 1</u>

2. In any two consecutive rows not made entirely of zeros, the <u>leading 1</u> in the lower row is to the right of the <u>leading 1</u> in the upper row.

3. In a $\boxed{\text{column}}$, contains a <u>leading 1</u>, every other element in that $\boxed{\text{column}}$ is a zero.

4. All rows made up entirely of zeros are grouped together at the bottom of the matrix.

Ex.

$$M_1 = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \text{ in RREF}$$

* Rule 3 broken

$$M_2 = \left[\begin{array}{cccc|c} 1 & \boxed{-1} & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \text{ not in a RREF}$$

Method II:

The process of using E.R.O's to convert a matrix to RREF is called Gauss - Jordan reduction.

Ex. Use the Gauss - Jordan reduction to solve the L.S.

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$$-x_1 + x_2 - x_3 + 3x_4 = 0$$
$$3x_1 + x_2 - x_3 - x_4 = 0$$
$$2x_1 - x_2 - 2x_3 - x_4 = 1$$

Sol: $[A\,\vdots\,\vec{c}\,] = \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{E_1 \to (-1)E_1 \text{ pivot}} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ \boxed{3} & 1 & -1 & -1 & 0 \\ \boxed{2} & -1 & -2 & -1 & 1 \end{array}\right]$

$R_2 \to R_2 + (-3)R_1$
$R_3 \to R_3 + (-2)R_1$

$\Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 1 \end{array}\right] R_2 \to \frac{1}{4}R_2 \Rightarrow \left[\begin{array}{cccc|c} 1 & \boxed{-1} & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & \boxed{1} & -4 & 5 & 1 \end{array}\right]$

$R_1 \to R_1 + R_2 \quad R_3 \to R_3 + (-1)R_2$

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 1 \end{array}\right] R_3 \to -\frac{1}{3}R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & \boxed{-1} & 2 & 0 \\ 0 & 0 & 1 & -1 & -\frac{1}{3} \end{array}\right] R_2 \to R_2 + R_3.$

$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & -\frac{1}{3} \end{array}\right] = [A'\,\vdots\,\vec{c}\,'].$

The Equivalent Sys. Is.

$$x_1 - x_4 = 0$$
$$x_2 + x_4 = -\frac{1}{3}$$
$$x_3 - x_4 = -\frac{1}{3}$$

Back Sub $\Rightarrow x_3 = -\frac{1}{3} + x_4$ Choose $x_4 = \alpha$, $\alpha \in \mathbb{R}$

$x_2 = -\frac{1}{3} - x_4 = -\frac{1}{3} - \alpha \qquad x_1 = x_4 = \alpha$

· Thus the 1-parameter family of sol's is $\{\alpha, -\frac{1}{3} - \alpha, -\frac{1}{3} + \alpha, \alpha\}$

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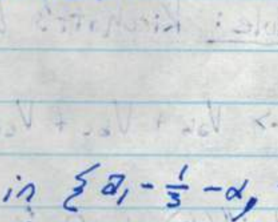
Application: I. Electrical Circuits II. Particular Sol's of ODEs.

I. Consider the circuit in the figure:

We want to find the current i_1, i_2, i_3.



* Rule: Kirchoff's Laws

Law 1: At every node the sum of incoming currents = the sum of the out coming currents.

Law 2: Around every closed loop, the algebraic sum of the voltage gains = the algebraic sum of the voltage drops.

L.S. By Law 1: At node A: $i_1 - i_2 + i_3 = 0$

B: $-i_1 + i_2 - i_3 = 0$

By Law 2: $\begin{cases} \text{Top Loop: } 4i_1 + 2i_2 = 8 \\ \text{Bottom Loop: } 2i_2 + 5i_3 = 9 \quad (5i_3 = 2i_3 + 3i_3). \end{cases}$

The system can be described by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right] \xrightarrow{E.R.OS.} \overset{\text{RREF}}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]} \quad \begin{array}{l} i_1 = 1 \\ i_2 = 2 \\ i_3 = 1 \\ 0 = 0 \end{array} \quad \begin{array}{l} i_1 = 1, \\ i_2 = 2, \\ i_3 = 1 \end{array}$$

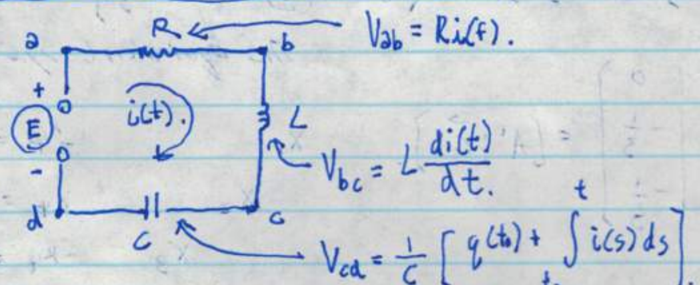
II: Particular Sol's of ODE's . Consider the RLC circuit.

*E: voltage source

R: resistor.

L: inductor

C: capacitor.



Rule: Kirchoff's Law of Voltages. $\sum_{i=1}^{4} V_i = 0$.

$\to V_{ab} + V_{bc} + V_{cd} + V_{da} = 0 \Rightarrow$ Current Version a ODE is.

$$Li''(t) + Ri'(t) + \frac{1}{C}i(t) = E'(t) \quad \left[\ ' := \frac{d}{dt},\ '' := \frac{d^2}{dt^2}\right.$$

$$\div L \Rightarrow \boxed{i'' + ai' + bi = f(t)} \quad (1) \quad \text{where } a = \frac{R}{L},\ b = \frac{1}{LC},\ f(t) = \frac{E'(t)}{L}.$$

The general solution of the DE (1) is $i(t) = i_h(t) + i_p(t)$.
where (in this example) $i_h(t)$ is the sol. of the homogenous DE

$i'' + ai' + bi = 0$ and $i_p(t)$ is the particular soln of the DE (1)

Ex. In the RLC, let $L = 20\,(H)$, $R = 80\,(\Omega)$ $C = 10^{-2}F$,
$E(t) = 50\sin(t)$ $\boxed{\text{Find } i(t)}$

$i(0)=4$ } They give this to find c_1 and c_2
$i'(0)=7$ } derive $i(t) = i_h(t) + i_c(t)$

ODE ② gives $20i'' + 80i' + \frac{1}{10^{-2}} i = 100\cos(2t)$, or
$i'' + 4i' + 5i = 5\cos(2t)$. — ②

The general sol of ② is $i(t) = i_h(t) + i_p(t)$ where (in this example
$i_h = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$.

where c_1 & c_2 are constants to be found, and: $\boxed{i_p(t) = A\cos(2t) + B\sin(2t)}$
where A & B are two constants to be found.

* We want to find A & B. Since $i_p(t)$ is a sol. of ② then...
$i_p'' + 4i_p' + 5i_p \cancel{+5} = 5\cos(2t)$.
$\hookrightarrow -4A\cos(2t) - 4B\sin(2t) + 4\overbrace{(-2A\sin(2t) + 2B\cos 2t)}^{i_p'}$
$+ 5\underbrace{(A\cos(2t) + B\sin(2t))}_{i_p} = \underbrace{5\cos(2t)}_{\text{R.H.S of ②}}$

<u>Finding the L.S.</u> Coefficients of $\cos(2t)$: $-4A + 8B + 5A = 5 \leftarrow$ RHS
" " $\sin(2t)$: $-4B - 8A + 5B = 0$ or $\begin{cases} A + 8B = 5 \\ -8A + \cancel{8}B = 0 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 8 & 5 \\ -8 & 1 & 0 \end{array}\right]$$

· Solve the system for A & B and plug these values into $i_p(t)$.

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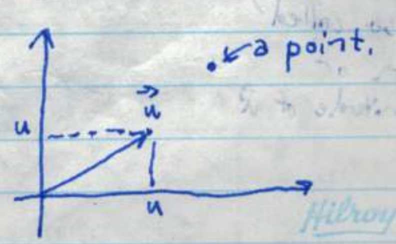
9.4 n-space.

$$\mathbb{R}^n = \left\{ \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \overset{\text{such that}}{;} u_1, u_2, \dots, u_n \in \mathbb{R} \right\}$$

$u_1, u_2, \dots, u_n$ are called the components of $\vec{u}$

For simplicity, we write the <u>vector</u> $\vec{u}$ as follows $\vec{u}(u_1, u_2, \dots, u_n)$

If $n=2$, $\mathbb{R}^2 = \{ \vec{u} = (u_1, u_2) ; u_1, u_2 \in \mathbb{R} \}$



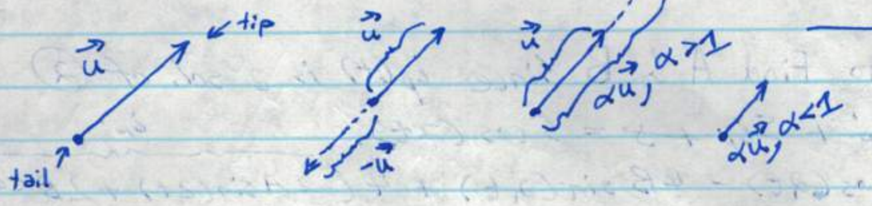
* If $\vec{u}, \vec{v} \in \mathbb{R}^2$, $\underset{\text{scalar}}{\alpha} \in \mathbb{R}$ we define $\vec{u} = (u_1, u_2)$,
$\vec{v}(v_1, v_2)$ * addition $\vec{u} + \vec{v} = (u_1, v_2) + (v_1, v_2)$
$= (u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2$

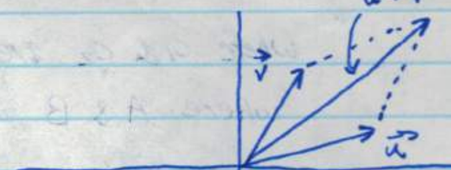
Scalar Multiplication: $\alpha\vec{u} = \alpha(u_1, u_2) = (\alpha u_1, \alpha u_2) \in \mathbb{R}^2$

Zero vector: $\vec{0} = (0,0)$ Negative vector: $-\vec{u} = \overset{\alpha}{(-1)}\vec{u} = (-u_1, -u_2)$.

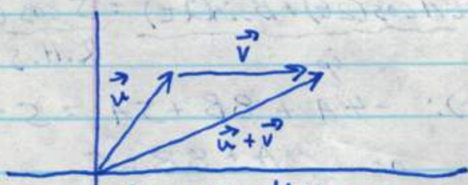
<u>Subtraction:</u> $\vec{u} - \vec{v} = \vec{u} + \overset{\alpha}{(-1)}\vec{v} = \cancel{(u_1 - v_1, u_2 - v_2)}\ (u_1, u_2) + (-v_1, -v_2)$
$= (u_1 - v_1, u_2 - v_2) \in \mathbb{R}^2$.

<u>Geometric Representations:</u>





parallelogram addition.



triangular addition.

* Generalize the operations to $\mathbb{R}^n$.

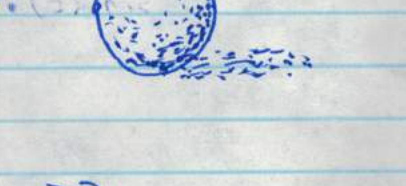
<u>· Properties:</u>
1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \vec{0}$.
5. $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$
6. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
7. $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$
8. $1\vec{u} = \vec{u}$
9. $0\vec{u} = \vec{0}$
10. $(-1)\vec{u} = -\vec{u}$
11. $\alpha\vec{0} = \vec{0}$.

9.5 Dot Product, Norm, and Angles for n-space.

*Norm Def: If $\vec{u} \in \mathbb{R}^n$, the norm of $\vec{u}$ is defined by

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\sum_{j=1}^{n} u_j^2}$$

Also called length or magnitude of $\vec{u}$



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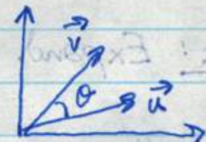
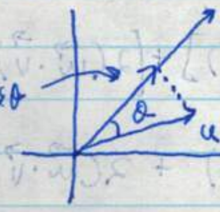
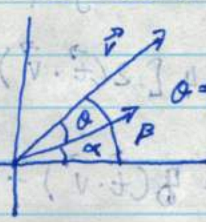
Dot Product Def: The dot product of nonzero vector $\vec{u}$ & $\vec{v}$ is defined as follows $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta)$

where θ is the angle between $\vec{u}$ and $\vec{v}$



The dot product of $\vec{u}, \vec{v} \in \mathbb{R}^2$ in terms of the vector components is $\boxed{\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2}$

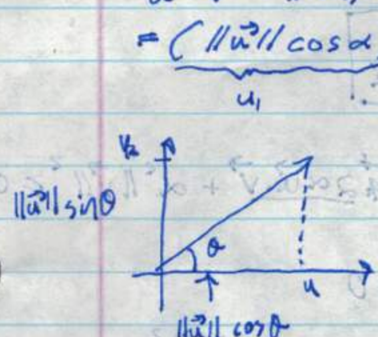




Proof:

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \|\vec{u}\|\,\|\vec{v}\|\cos(\beta-\alpha) = \|\vec{u}\|\|\vec{v}\|[\cos\beta\cos\alpha + \sin$

$= \underbrace{(\|\vec{u}\|\cos\alpha)}_{u_1}\underbrace{(\|\vec{v}\|\cos\beta)}_{v_1} + (\|\vec{u}\|\sin\alpha)(\|\vec{v}\|\sin\beta)$.



$= u_1 v_1 + u_2 v_2$ if $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

$= \sum_{j=1}^{n} u_j v_j$ $\qquad \|u\|^2 = \vec{u}\cdot\vec{u} = \sum_{j=1}^{n} u_j^2$

$\Rightarrow \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$

9.5.2 Properties of Dot Product Sept. 21, 2016.

If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ & $\alpha, \beta \in \mathbb{R}$ then

Properties

1 $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ commutative. (2a)

2. $\vec{u}\cdot\vec{u} > 0$ $\forall \vec{u} \neq \vec{0}$; $= 0$ for $\vec{u} = \vec{0}$ } non-negative. (2b)

3. $(\alpha\vec{u} + \beta\vec{v})\cdot\vec{w} = \alpha(\vec{u}\cdot\vec{w}) + \beta(\vec{v}\cdot\vec{w})$ linear (2c)

Proof (2a); $\vec{u}, \vec{v} \in \mathbb{R}^2$ $\left.\begin{array}{l}\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 \\ \vec{v}\cdot\vec{u} = v_1 u_1 + v_2 u_2\end{array}\right]$ = Proof (2b) n (2c) Exercise.

Ex. Find the angle between $(3,4)$ & $(-1,7)$.

Sol: $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$ $\quad \vec{u}\cdot\vec{v} = (3,4)\cdot(-1,7) = -3 + 28 = 25$

$\|\vec{u}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

Hilroy

$\|\vec{v}\| = \sqrt{(-1)^2 + 7^2} = \sqrt{50} \Rightarrow \theta = \cos^{-1}\left(\frac{\cancel{25}^{\,5}}{\cancel{5}\sqrt{50}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}.$

Ex: Expand the dot product $(4\vec{t} + \vec{u}) \cdot (2\vec{v} - \vec{w})$

Sol. Using properties in 2 gives $(4\vec{t} + \vec{u}) \cdot (2\vec{v} - \vec{w}) = 4[\vec{t} \cdot (2\vec{v} - \vec{w})] + [\vec{u} \cdot (2\vec{v} - \vec{w})]$

$= 4[2(\vec{t} \cdot \vec{v}) - (\vec{t} \cdot \vec{w})] + [2(\vec{u} \cdot \vec{v}) - (\vec{u} \cdot \vec{w})]$

$= 8(\vec{t} \cdot \vec{v}) - 4(\vec{t} \cdot \vec{w}) + 2(\vec{u} \cdot \vec{v}) - (\vec{u} \cdot \vec{w})$

· Schwartz inequality $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

Proof:

~~Proof:~~ $(\vec{u} + \alpha\vec{v}) \cdot (\vec{u} + \alpha\vec{v}) \geq 0$ $\boxed{\alpha = -\frac{\vec{u}\vec{v}}{\|v\|^2}}$

$\Rightarrow \|\vec{u}\|^2$ Remember $= \|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$ $\Rightarrow \|\vec{u}\|^2 + 2\alpha\vec{u} \cdot \vec{v} + \alpha^2 \|\vec{v}\|^2 \geq 0$

$\Rightarrow \|\vec{u}\|^2 - 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}(\vec{u} \cdot \vec{v}) + \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^{\cancel{4}\,2}} \cancel{\|\vec{v}\|^2} \geq 0$

$\Rightarrow \|\vec{u}\|^2 - 2 \cdot \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} + \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} \geq 0 \Rightarrow \|\vec{u}\|^2 - \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} \geq 0$

$(\vec{u} \cdot \vec{v})^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$ $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|.$

9.5.3 Properties of the norm

* $\vec{u} \in \mathbb{R}^n$, $\|u\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\sum u_i^2}$ — 4 $\alpha \in \mathbb{R}$

Properties.

(1) Scaling: $\|\alpha\vec{u}\| = |\alpha| \|\vec{u}\|$ — 5a

(2) Non-negative $\begin{cases} \|\vec{u}\| > 0 \;\; \forall \vec{u} \neq \vec{0} \\ \phantom{\|\vec{u}\|} = 0 \text{ for } \vec{u} = \vec{0}. \end{cases}$ — 5b

(3) Triangle Ineq. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ — 5c

Proof: (5a) $\|\alpha\vec{u}\| = \sqrt{\alpha\vec{u}\cdot\alpha\vec{u}} = \sqrt{a^2(\vec{u}\cdot\vec{u})} = \sqrt{a^2}\sqrt{\|\vec{u}\|^2} = |\alpha|\,\|\vec{u}\|$

Proof (5c): $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + 2(\vec{u}\vec{v}) + \|\vec{v}\|^2$

$\leq \|\vec{u}\|^2 + 2|\vec{u}\vec{v}| + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2.$

if st vectors verify Schwartz inequality

$= (\|\vec{u}\| + \|\vec{v}\|)^2 \Rightarrow \|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

4.3.4 Orthogonality

Def: The 2 vectors $\vec{u}$ & $\vec{v}$ are said to be orthogonal if $\vec{u}\cdot\vec{v} = 0$

Def: The set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ said to be orthogonal if $\vec{u}_i \cdot \vec{v}_j = 0 \quad \forall i \neq j \quad i,j = 1, 2, \dots, k.$

* The zero vector is orthogonal to all vectors including itself.

Def: If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ such that $\vec{u}\cdot\vec{v} = 0$ (i.e. $\vec{u}$ & $\vec{v}$ are orthogonal)
then $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \frac{\pi}{2}$ and we say $\vec{u}$ & $\vec{v}$ are perpendicular

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Ex: $\vec{u} = (2,-1)$, $\vec{v} = (3,6)$ $\vec{u}\cdot\vec{v} = (2)(3) + (-1)(6) = 0$

9.5.5 Normalization

Def: If $\vec{v} \neq \vec{0}$, then the normalized vector of $\vec{u}$ is $\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u}$

· The length of $\hat{u} = 1$, Why? A: $\|\hat{u}\| = \left\|\underbrace{\frac{1}{\|\vec{u}\|}}_{>0}\vec{u}\right\| = \frac{1}{\cancel{\|\vec{u}\|}}\cancel{\|\vec{u}\|} = 1$ scaling prop

Def: A vector of unit length is called a unit vector.

Def: A set of vectors is said to be orthonormal if it is orthogonal & each vector has length 1.

Ex. $\vec{u}_1 = (0, 0, 1)$, $\vec{u}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ $\vec{u}_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

① $\vec{u}_i \cdot \vec{u}_j = 0 \quad \forall i, j = 1, 2, 3$ $\quad \vec{u}_2 \cdot \vec{u}_3 = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0$
$= 0.$

S, the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthogonal.

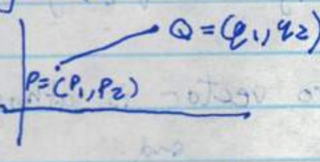
$\|\vec{u}_1\| = \sqrt{0^2 + 0^2 + 1^2} = 1 \quad \|\vec{u}_2\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\|\vec{u}_3\| = \quad\quad = 1$. Thus, the set is orthonormal.

Application to dot product.

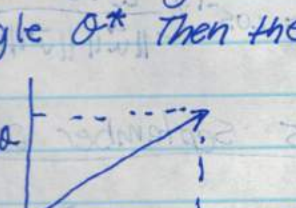
- If a constant vector force $\vec{F}$ acts to move an object on a straight surface in the $\vec{PQ}$ direction, then the work (done to move this object).

$\vec{W} = \vec{F} \cdot \vec{PQ}$ ← Displacement.



$\Rightarrow \vec{PQ} = (q_1 - p_1, q_2 - p_2)$ * If we have the magnitude of the force $\|\vec{F}\|$ to moving an object on a plane with an angle θ^* Then the vector force is

$\vec{F} = \left(\|\vec{F}\| \cos\theta^*, \|\vec{F}\| \sin\theta^*\right)$



9.6 Vector Space

Def: A non-empty set S of objects (often called vectors) is said to be a vector space if the following properties are satisfied:

(a) (i) S is closed under the addition operation, i.e. if $\vec{u}$ & $\vec{v} \in S$ then $\vec{u} + \vec{v} \in S$

(ii) $\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \forall \vec{u}, \vec{v} \in S$

(iii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \quad \forall \vec{u}, \vec{v}, \vec{w} \in S$

(iv) S contains a unique zero vector such that (s.t.) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$ $\forall \vec{u} \in S$

(v) for each $\vec{u} \in S$, there is a unique negative inverse vector $-\vec{u} \in S$ s.t. $\vec{u}+(-\vec{u})=\vec{0}$

(b) (i) S closed under scalar mult., i.e, if $\vec{u} \in S, \alpha \in \mathbb{R}$ then $\alpha\vec{u} \in S$

(ii) $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ $\forall \vec{u}, \vec{v} \in S$

(iii) $(\alpha+\beta)\vec{u} = \alpha\vec{u}+\beta\vec{u}$, $\forall \vec{u} \in S$ $\alpha, \beta \in \mathbb{R}$

(iv) $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$, $\forall u \in S$ $\alpha, \beta \in \mathbb{R}$

(v) $1\vec{u}=\vec{u}$ $\forall \vec{u} \in S$

* Ex: $S=\mathbb{R}^n$ with classical addition and scalar mult. form a vector called Euclidean (vector) space.

Ex: Let $S=\{$set of 1st-order polynomials$\}$ That is if $p(x) \in S$ then $p(x)=a_0+a_1x$
i.e. $S=\{p(x)=a_0+a_1x + a_0, a_1 \in \mathbb{R}\}$

Show that S with classical addition & scalar ~~unit~~ mult. forms a vector space called 1st-order poly space.

Sol: (a) If $p(x) \in S$, $q(x) \in S$ then $p(x)=a_0+a_1x$
$q(x)=b_0+b_1x$.

Then $p(x)+q(x)=\overline{(a_0+b_0)}+\overline{(a_1+b_1)}x \in S$ So S is closed under this addition

(ii) $p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x = (b_0+a_0)+(b_1+a_1)x$
$=(b_0+b_1x)+(a_0+a_1x) = q(x)+p(x)$

(iii) $(p(x)+q(x)) + r(x) = [(a_0+a_1x)+(b_0+b_1x)] + (c_0+c_1x)$
$= (a_0+b_0)+(a_1+b_1)x + c_0 + c_1x$
$= a_0+(b_0+c_0)+[a_1+(b_1+c_1)]x$ $= p(x)+[q(x)+r(x)]$
$=(a_0+a_1x)[(b_0+c_0)+(b_1+c_1)x]$

*Ex: $S = \mathbb{R}^2$ with the following operation:

$\vec{u} = (u_1, u_2)$, $\vec{v}(v_1, v_2)$

$\vec{u} + \vec{v} = (u_1, u_2) + (v_1, v_2) = (u_1+v_1, u_2+1)$

$\alpha \vec{u} = (\alpha u_1, \alpha u_2)$.

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$S = \{p(x);\ p(x) = a_0 + a_1 x,\ a_0, a_1 \in \mathbb{R}\}$.

(iv) If $0 \in S$, then $0(x) = 0 + 0x$ if $p(x) \in S$ then $p(x) + 0(x) = (a_0 + a_1 x)$

$= (a_0 + 0) + (a_1 + 0)x$

$= a_0 + a_1 x = p(x)$

(v) $p(x) + (-p(x)) = \dots = 0 \leftarrow 0(x) = 0 + 0x$

$\in S$

$-(a_0 + a_1 x) = -\underset{\in\mathbb{R}}{a_0} - \underset{\in\mathbb{R}}{a_1} x$

(b) (i) If $p(x) \in S$ and $\alpha \in \mathbb{R}$, then $\alpha p(x) = \alpha \cdot (a_0 + a_1 x) =$

$= \underset{\in\mathbb{R}}{(\alpha a_0)} + \underset{\in\mathbb{R}}{(\alpha a_1)} x \in S$

So S is closed under scalar multiplication.

Ex. Let $S = \mathbb{R}^2$ and the 2 operations of addition & scalar multiplication are defined as follows

(i) If $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + 1)$

(ii) If $\vec{u} \in S$ & $\alpha \in \mathbb{R}$, then $\alpha \vec{u} = (\alpha u_1, \alpha u_2)$.

Determine if S equipped with the above operation forms a v. space

Solution: (a) (i) Let $\vec{u}, \vec{v} \in S$ then $\vec{u} = (u, v)$, $\vec{v} = (v_1, v_2)$

and $\vec{w} + \vec{v} = (\underbrace{u_1 + v_1}, \underbrace{u_2 + 1}) \in S = \mathbb{R}^2$

(ii) $\vec{u}+\vec{v} \overset{?}{=} \vec{v}+\vec{u}$ LHS: $\vec{u}+\vec{v} = (u_1+v_1, u_2+1)$

RHS... $\vec{v}+\vec{u} = (v_1+u_1, v_2+1)$ So, $\vec{u}+\vec{v} \neq \vec{v}+\vec{u}$

So S' does not form a vector space.

- Given a vector space S' and an inner product $\vec{u}\cdot\vec{v}$ (for $\vec{u},\vec{v}, \in S'$) if this (given) dot product satisfies the properties of the dot product, then S' is called "inner product space".

- Similarly, if the vector space S' is equipped with a norm $\|\vec{u}\|$ (for $\vec{u}\in S$) and this norm satisfies the properties of the norm, then S' is called "normed vector space".

- If the vector space S' is equipped with $\vec{u}\cdot\vec{v}$ & $\|\vec{u}\|$ (for $\vec{u},\vec{v}\in S'$) and then ($\vec{u}\cdot\vec{v}$ & $\|\vec{u}\|$) satisfy the properties of dot product + norm the S' is called "normed inner product space".

- Some type of dot product: * $\vec{u}\cdot\vec{v} = \sum_{j=1}^{n} u_j v_j$

  * $\vec{u}\cdot\vec{v} = \sum_{j=1}^{n} w_j u_j v_j$

Here, w_j, are fixed constants called "weights"

* For an integrable function vector space.

We may define $u(x)\cdot v(x) = \langle u(x), v(x)\rangle = \int_0^1 w(x)\,u(x)\,v(x)\,dx$.

or $= \int_0^1 u(x)v(x)\,dx$

<u>Norms</u>

* $\|\vec{u}\| = \sqrt{\sum_{j=1}^{n} u_j^2}$ * $\|\vec{u}\| = \sqrt{\sum_{j=1}^{n} w_j u_j^2}$ * $\|\vec{u}\| = |u_1| + |u_2| + \dots + |u_n|$

9.7 Span and Subspace September 28, 2016.

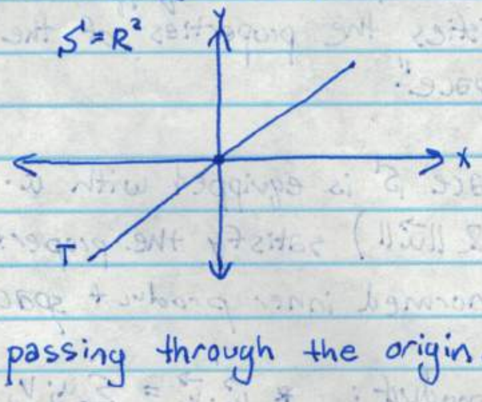
*** Subspaces**

• Def : Let $(S, +, *)$ denote a vector space. S with addition $(+)$ and scalar mult. $(*)$. A non-empty set T (we write $T \subseteq S$) is said to be a <u>subspace of S</u> (write $(T, +, *)$)

If it contains the zero vector of S and for all $\vec{u}, \vec{v} \in S$ and $\alpha \in \mathbb{R}$, then (i) $\vec{u} + \vec{v} \in T$ (i.e. T is closed under + operation).

(ii) $\alpha \vec{u} \in T$ (i.e. T is closed under $*$).

• Ex. Show that $T = \{ (x,y) \mid 2x - 3y = 0 \}$ is a subspace of $\mathbb{R}^2$



* T is a line passing through the origin. So $(0,0) \in T$

<u>Check</u> $2(0) - 3(0) = 0 \rightarrow (0,0) \in T$ Therefore, T is not empty. Let $\vec{x}_1 = (x_1, y_1)$ $\vec{x}_2 = (x_2, y_2)$

$\in T \Rightarrow 2x_1 - 3y_1 = 0$ $\quad \in T \Rightarrow 2x_2 - 3y_2 = 0$

(i) Is $\vec{x}_1 + \vec{x}_2 \in T$? (ii) Is $\alpha \vec{x}_1 \in T$?

<u>Let's see</u> (i) $\vec{x}_1 + \vec{x}_2 = (x_1 + x_2, y_1 + y_2)$

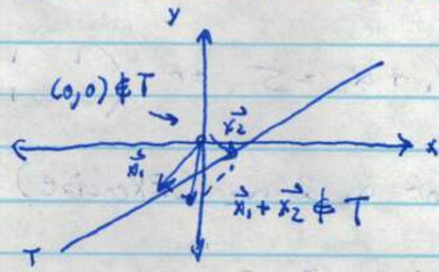
$$2(x_1 + x_2) - 3(y_1 + y_2) = 2x_1 + 2x_2 - 3y_1 - 3y_2$$
$$= (2x_1 - 3y_1) + (2x_2 - 3y_2) = 0 + 0 = 0$$

So, $\vec{x}_1 + \vec{x}_2 \in T$

(ii) $\alpha\vec{x}_1 = (\alpha x_1, \alpha y_1) \Rightarrow 2(\alpha x_1) - 3(\alpha y_1) = \alpha(2x_1 - 3y_1) = \alpha 0 = 0$

So, $\alpha\vec{x}_1 \in T$. Thus, T is a subspace of $\mathbb{R}^2$

Ex: Determine if $T = \{(x,y) \mid 2x - 3y = 1\}$ is a subspace of $\mathbb{R}^2$.



Choose $\vec{x}_1, \vec{x}_2 \in T \Rightarrow 2x_1 - 3y_1 = 1$ Now $\vec{x}_1 + \vec{x}_2 = (x_1 + x_2, y_1 + y_2)$

$2x_2 - 3y_2 = 1 \Rightarrow 2(x_1 + y_1) - 3(x_2 + y_2) = (2x_1 - 3y_1) + (2x_2 - 3$

$= 1 + 1 = 2 \neq 1$

$\alpha\vec{x}_1 = (\alpha x_1, \alpha y_1)$

If $\alpha = 0 \in \mathbb{R}$, then $\alpha\vec{x} = (0,0) \notin T$ ...

Remark

① A vector space S (or subspace T) must have at least one vector. For example, for $\alpha = 0 \in \mathbb{R}$, & $x \in S$ (or $\in T$), we have $0\vec{x} = \vec{0} \in S$ (or $\in T$).

(ii) $T = \{\vec{0}\}$ is a subspace of R^n. This is called trivial (or zero) subspace.

(iii) R^n is a subspace of itself. Here $S = R^n = T$

Every vector space is subspace of itself

• Span Def: Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_K\}$ We say that $\vec{v}$ is a linear combination of the vectors of B if there exist scalar numbers $\alpha_1, \alpha_2, \ldots, \alpha_K \in \mathbb{R}$. such that

$\vec{v} = \alpha_1\vec{v}_1, \alpha_2\vec{v}_2 + \ldots + \alpha_K\vec{v}_K$ —— ⊛

or $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_K\vec{v}_K = \vec{v}$

⊛ is a system in k unknowns $\alpha_1, \alpha_2, \ldots, \alpha_K$

Ex: Show that $\vec{w} = (3,5)$ is a linear combination of $\vec{u} = (1,2)$ & $\vec{v} = (1,1)$.

<u>Solution.</u> $\vec{w} = \alpha_1 \vec{u} + \alpha_2 \vec{v}$ or $\alpha_1 \vec{u} + \alpha_2 \vec{v} = \vec{w} \Rightarrow \alpha_1 (1,2) + \alpha_2 (1,1) = (3,$

$\alpha_1 + \alpha_2 = 3 \quad 2\alpha_1 + \alpha_2 = 5$ or $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right)$

Solve this system for α_1 & α_2 (Exercise) $\alpha_1 = 2, \alpha_2 = 1$

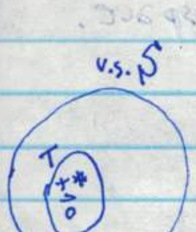
We can write $\vec{w} = 2\vec{u} + \vec{v}$

<u>Def:</u> If $B = \{\vec{v}_1, \dots, \vec{v}_2\}$ are vectors in a <u>vetor space</u> S, then the set of <u>all</u> possible linear combinations of these vectors (in B) $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_K \vec{v}_K$ $\boxed{\alpha_1, \dots, \alpha_K \in R}$ is called the span of $B = \{\vec{v}_1, \dots, \vec{v}_K\}$ and denoted by

$$\text{Span } B \text{ or Span } \{\vec{v}_1, \cdots, \vec{v}_K\}$$

- The set B (in the definition) is called the spanning (or generating) set of Span B.

v.s. S



- <u>Theorem 9.7.1 (span & Subspace)</u>

If $\{\vec{v}_1, \dots, \vec{v}_K\}$ are vectors in a vector space S, then span $\{\vec{v}_1, \dots, \vec{v}_K\}$ is itself a vector space (or a subspace) of S

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Ex (1): Determine the span of $\{\vec{u} = (1,2)\}$ in R^2

a line in R^2

<u>Sol:</u> The lin. combination of $\vec{u}$ is $\vec{u} = \alpha \vec{u}_1 \Rightarrow (u_1, u_2) = u_1 (1,2) = (\alpha,$

$\Rightarrow \begin{array}{l} \alpha_1 = u_1 \\ \alpha_2 = \frac{1}{2} u_2 \end{array}$

- In this example $\vec{u}_1$ spans a subspace (L) in $\mathbb{R}^2$
- The spanning (or generating) set of span $\{\vec{u}\} = L$ is $\vec{u}_1$

Ex. 2 $\vec{u_1} = (1,2)$ & $\vec{u_2} = (3,1)$ in $\mathbb{R}^2$

Sol: Linear Combination $\vec{u} = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2}$ Solve for α_1 & α_2.

$(u_1, u_2) = \alpha_1 (1,2) + \alpha_2 (3,1)$

$= (\alpha_1, 2\alpha_2) + (3\alpha_2, \alpha_2)$

$(u_1, u_2) = (\alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2)$

$\Rightarrow \alpha_1 + 3\alpha_2 = u_1$

$2\alpha_1 + \alpha_3 = u_2$

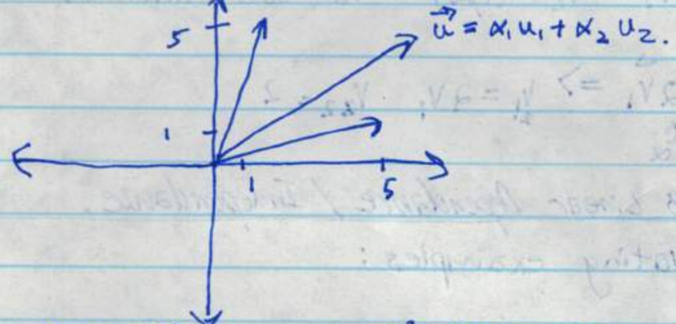
$\Rightarrow \alpha_1 + 3\alpha_2 = u_1$

$2\alpha_1 + \alpha_2 = u_2$

Solve for α_1 & α_2 $\left(\begin{array}{cc|c} 1 & 3 & u_1 \\ 2 & 1 & u_2 \end{array}\right)$

$\Rightarrow \alpha_1 = -\frac{1}{13} u_1 + \frac{3}{5} u_2$ $\alpha_2 = \frac{1}{5}(6u_1 - u_2)$ For any $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$ there exist $\alpha_1 + \alpha_2$.

Geometrically:



In this example, $\{\vec{u_1}, \vec{u_2}\}$ spans $\mathbb{R}^2$. We write span $\{\vec{u_1}, \vec{u_2}\} = \mathbb{R}^2$

Ex. ③ Determine the span of $\{\underset{u_1}{\overset{\vec{v_1}}{(1,0)}}, \underset{u_2}{\overset{\vec{v_2}}{(1,2)}}, \underset{u_3}{\overset{\vec{v_3}}{(1,1)}}\}$ in $\mathbb{R}^2$

Sol: Let $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$ Linear Comb. $\vec{u} = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{v_3}$

$\Rightarrow \left\{ \begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 = u_1 \\ 2\alpha_2 + \alpha_3 = u_2 \end{array} \right\}$ Solve for $\alpha_1, \alpha_2, \alpha_3$.

$\Rightarrow \alpha_3 = u_2 - 2\alpha_2$ $\alpha_1 = u_1 - \alpha_2 - \overbrace{(u_2 - 2\alpha_2)}^{\alpha_3}$

Choose $\alpha_2 = \alpha^*$, $\alpha^* \in \mathbb{R}^2$

- System is consistent - it has infinity of solutions.

In this example: Span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \mathbb{R}^2$

Notice that $\vec{v_1} = 2\vec{v_3} - \vec{v_2}$ $\Rightarrow$ In this case, we write

span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ = span $\{\vec{v_2}, \vec{v_3}\} = \mathbb{R}^2$

Ex. Find the span $\{1, 2x_1, -x^2\}$ in $\mathbb{P}_2(x)$ ← vector space of 2^{nd}-order polynomial.

Solution Let $p(x) = a + bx + cx^2 \in \mathbb{P}_2(x)$. — $a, b, c \in \mathbb{R}$

Linear Combination

$$p(x) = \alpha_1(1) + \alpha_2(2x) + \alpha_3(-x^2)$$

$$\Rightarrow \begin{cases} \alpha_1 = a \\ 2\alpha_2 = b \\ -\alpha_3 = c \end{cases} \Rightarrow \begin{array}{l} \alpha_1 = a \\ \alpha_2 = b/2 \\ \alpha_3 = -c \end{array}$$

Thus, span $\{1, 2x, -x^2\} = \mathbb{P}_2(x)$ entire space.

$\vec{v}_1 = (1,1)$ $\vec{v}_2 = (1,2)$ Can we find $\alpha \in \mathbb{R}$ such that $\vec{v}_2 = \alpha\vec{v}_1$

$\vec{v}_2 = \underset{\alpha}{2}\vec{v}_1 \Rightarrow v_{21} = 2v_1$ $v_{22} = 2$

9.8 Linear Dependance / Independance.

· Motivating examples:

① Consider $\vec{v}_1(1,2)$, $\vec{v}_2(2,4)$ Notice that: $\vec{v}_2 = 2\vec{v}_1$ or $2\vec{v}_1 - \vec{v}_2 = \vec{0}$

So, $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$

Infinite solns = L.D.

· $2\vec{v}_1 - \vec{v}_2 = \vec{0}$ can be written of $\overbrace{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0} \quad \alpha_1 = 2, \alpha_2 = -1}^{\text{we have infinite solns.}}$

In this example $\vec{v}_2$ depends on $\vec{v}_1$

② Consider $\vec{v}_1(1,2)$, $\vec{v}_2 = (0,1)$ In this example, neither $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$, nor $\vec{v}_2$ " " " " $\vec{v}_1$

$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = 0$ is L.I.

So if we write $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$

then the only solution is $\alpha_1 = \alpha_2 = 0$ (the trivial soln.).

Def: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be "linearly dependent" (l.d.) if there exist scalars $\alpha_1, \dots, \alpha_k$ (not all zero) such that
$\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = \vec{0} \longleftarrow$ (the homo sys has infinite solns).
or $\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k$

Def: A set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be "linearly independant" (l.i.).
of the only sol. to $\alpha_1 \vec{v}_1 + \dots \alpha_k \vec{v}_k = \vec{0}$ is $\alpha_1 = \dots = \alpha_k = 0$ (the trivial s

* Test for (l.d.) or (l.i.)

We have $\{\vec{v}_1, \dots, \vec{v}_k\}$

Write $\vec{\alpha}_1 v_1 + \dots + \alpha_k \vec{v}_k = \vec{0}$ — (*)

Solve this homog. system (*) for $\alpha_1, \dots, \alpha_k$.

(i) If (*) has a unique sol. ($\alpha_1 = \dots = \alpha_k = 0$), then the set is l.i.

(ii) If * has infinity of soln's (which include the trivial sol), then the set is l.d.

* This is the summary of Theoram 9.8.1

Ex: Determine if the set $\{\vec{v}_1 = (2,3), \vec{v}_2 = (1,6)\}$ is (l.d) or (l.i)

Sol. $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0} \Rightarrow \begin{matrix} 2\alpha_1 + \alpha_2 = 0 \\ 3\alpha_1 + 6\alpha_2 = 0 \end{matrix}$ or $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 6 & 0 \end{array}\right)$

$\overset{\text{H.W.}}{\sim \dots \sim} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$ So, $\alpha_1 = \alpha_2 = 0$.

Thus, $\{\vec{v}_1, \vec{v}_2\}$ is l.i.

Ex

Ex Determine if $\{\overset{P_1(x)}{\overline{2x}}, \overset{P_2(x)}{\overline{-5}}, \overset{P_3(x)}{\overline{10-4x}}\}$, in $\mathbb{P}_1(x)$
is (l.i.) or (l.d.)

Soln. $\alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) = 0$ ← zero polynomial! $0(x) = 0 + 0x$.

$\Rightarrow \alpha_1(2x) + \alpha_2(-5) + \alpha_3(10-4x) = 0 + 0x$

Constant Coefficient: $-5\alpha_2 + 10\alpha_3 = 0$

Coeff. of x: $2\alpha_1 \quad -4\alpha_3 = 0$

Solve for $\alpha_1, \alpha_2, \alpha_3$ $\quad \alpha_2 = 2\alpha_3 \quad \alpha_1 = 2\alpha_3$, $\alpha_3 = \alpha^* \in \mathbb{R}$

So the system has infinite sol's.

Thus, the set is l.d.

Here, if $\alpha_3 = 1$, then $\alpha_2 = 2$, $\alpha_1 = 2$.

Then we can write $2P_1(x) + 2P_2(x) + P_3(x) = 0$.

or $P_3(x) = -2(P_1(x) + P_2(x))$.

$p_1(x) = -P_2(x) - \frac{1}{2}P_3(x) \ldots$

Theorem 9.8.3

· A set containing zero vector is linearly dependant.

Proof: Consider the set. $\{\vec{v}_1, \ldots, \vec{v}_j, \ldots \vec{v}_k\}$

if $\vec{v}_k$ is 0
then α_k is
not necess. 0.

If for instance, $\vec{v}_k = \vec{0}$ $\qquad \vec{v}_k = \vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_{k-1}$

Here, $\alpha_1 = \ldots, \alpha_{k-1} = 0$, and $\alpha_k \neq 0$.

$\alpha_k \vec{v}_k = -(\alpha_1 \vec{v}_1 + \ldots + \alpha_{k-1}\vec{v}_{k-1}) \Rightarrow \alpha_k 0 = 0\vec{v}_1 + \ldots + 0\vec{v}_{k-1}$

$\alpha_k \in \mathbb{R} \quad \alpha_1 = \ldots = \alpha_{k-1} = 0$

$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k \overset{\swarrow 0}{\vec{v}_k} = 0$. ① or $\alpha_k\vec{v}_k = \vec{0} = 0\vec{v}_1 + \dots + 0\vec{v}_{k-1}$

but α_k is any real number $\Rightarrow$ we may take $\alpha_k = 1 \neq 0$

So sys. ① has sol $\alpha_1 = 0, \dots, \alpha_{k-1} = 0, \alpha_k = 1$

This is not the trivial soln. Thus the set is l.d.

OR we can write $0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_k + 1\vec{v}_k = \vec{0}$ or any $\alpha^* \in \mathbb{R}$.

MATH 215 October 4, 2016

Th. 9.8.5: (Orthogonal Sets): Every finite orthogonal set of $\boxed{\text{non-zero}}$ vectors is l.i.

Proof: Consider the set $\{\vec{v}_1, \dots, \vec{v}_k\}$

Let $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k = \vec{0}$ — ①

Dot $\vec{v}_1$ into both sides of ① to get $v_1 \cdot (\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k) = \vec{v}_1 \cdot$

$\Rightarrow \alpha_1(\vec{v}_1 \cdot \vec{v}_2) + \alpha_2 \overset{0}{\cancel{(\vec{v}_1 \cdot \vec{v}_2)}} + \dots + \alpha_k \overset{0}{\cancel{(\vec{v}_1 \cdot \vec{v}_k)}} = \vec{0}$ ← By linearity property of dot product

~~$\Rightarrow \alpha_1 \|\vec{v}_1\|$~~ $\|\vec{u}_1\| \neq 0$ because $\vec{v}_1 \neq \vec{0}$ ~~because~~

$\Rightarrow \alpha_1 \|\vec{v}_1\|^2 + \vec{0} + \dots + \vec{0} = \vec{0} \Rightarrow \alpha_1 = 0$.

Dot $\vec{v}_0$ into ① to get $\alpha_k = 0$ Repeat the dot product until we get $\alpha_k = 0$.

So, the homog. sys. ① has only trivial sol: $\alpha_1 = 0 = \dots = \alpha_k$.

Thus the set is l.i.

9.9 Bases, Expansions, Dimensions

9.9.1 Bases & Expansions.

Linear combo of other fcns.

Motivation In Calculus $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

$\hookrightarrow = \sum_{j=0}^{\infty} a_j x^j$

where $a_j = \frac{f^{(j)}(0)}{j!}$, $j = 0, 1, \dots$

That is, $f(x)$ is written or ((expanded)) as a linear comb. of other base fcn's $\{1, x, x^2, \dots x^n, \dots\}$.

An example: $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n} + \dots = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$

$* a_j = \frac{1}{j!}$ | Here $\{1, x, \dots, x^n, \dots\}$ is a base vectors (or polynomials) in a poly. space.

Q. Can we expand a vector, say $\vec{w}$, in terms of base vectors $\vec{v}_1 \dots \vec{v}_k$,

i.e. $\vec{w} = \alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k$ $\boxed{\text{A: We study "Basis"}}$

Def.
Basis: The set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be a basis for a vector space S if

(i) $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a spanning set for S,

(ii) $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.i.

Remark: This definition combines Def. 9.9.1 & Th. 9.9.1 in the textbook.

Remark: We can define a basis for a subspace T of S

?!

Ex: Let $\{\vec{v}_1=(1,-3,0),\ \vec{v}_2=(3,0,4),\ \vec{v}_3=(11,-10,2)\}$

· Let T be a subspace of $S=\mathbb{R}^3$ given by

$T=\text{Span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ Find a basis for T.

WRONG

Solution. We should (if possible) find the smallest set that spans T.

$$\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3=\vec{0} \Rightarrow \left.\begin{aligned} \alpha_1+3\alpha_2+11\alpha_3&=0\\ -3\alpha_1 \qquad -6\alpha_3&=0\\ +4\alpha_2+2\alpha_3&=0 \end{aligned}\right\} \begin{aligned}\alpha_1&=\\ \alpha_2&=\\ \alpha_3&=\end{aligned}$$

That is the set is l.i. In Fact, $\vec{v}_3=2\vec{v}_1+3\vec{v}_2$.
an expansion of $\vec{v}_3$.

Moreover $\{\vec{v}_1,\vec{v}_2\}$ is l.i. because $\alpha_1\vec{v}+\alpha_2\vec{v}_2=0$ has a unique sol. $\alpha_1=\alpha_2=0$.

Hence, $\{\vec{v}_1,\vec{v}_2\}$ is l ind. and spans T.

· Thus, it is a basis for T.

· Fact: In $\mathbb{R}^k$, the set $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k,\vec{v}_{k+1}\}$ is l.d.

Exercise: Given $\{\vec{v}_1=(1,-1),\ \vec{v}_2=(-3,-2),\ \vec{v}_3=(0,2)\}$
Find a basis for $\mathbb{R}^2$.
vector space, S"

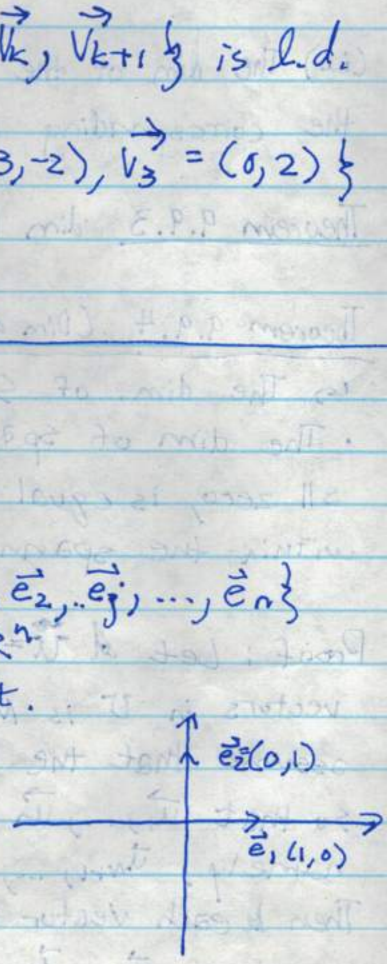
~~EGT~~ MATH 215 Oct 5, 2016.

You'll get basis for $\mathbb{R}^2$: $\{\vec{v}_2\cdot\vec{v}_3\}$

· Standard Basis for $\mathbb{R}^n$

Def: The standard basis for $\mathbb{R}^n$ is $\{\vec{e}_1,\vec{e}_2,..\vec{e}_j,\dots,\vec{e}_n\}$
where $\vec{e}_j=(0,0,\dots,0,1,0,\dots,0)\in\mathbb{R}^n$
j th component.

In $\mathbb{R}^2$, the standard basis $\{\vec{e}_1\cdot\vec{e}_2\}$



· Every vector $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ can be written as a linear combination of the standard basis of $\mathbb{R}^n$

$\vec{v} = (v_1, v_2, \ldots, v_n) = v_1\vec{e_1} + v_2\vec{e_2} + \ldots + v_n\vec{e_n}$

· Standard basis for $\mathbb{P}_n(x)$ is $\{1, x, x^2, \ldots, x^n\}$

· Dimension

Def: If a vector space S has a basis with n vectors, then we say that the dimension of S is n and we write $\dim S = n$

· Remark

(i) If a vector space S has a basis with an arbitrary large no. of vectors, we say that S is "infinite-dimensional

Ex: Function vector space: $F = \{1, x, x^2, \ldots, x^n, \ldots\}$

(ii) The dim of the trivial v. space $S = \{\vec{0}\}$ is zero because the corresponding basis is empty.

$\vec{0} = \alpha\vec{0}$

This is l. dep. with all vectors (including itself) of S

Theorem 9.9.3 $\dim \mathbb{R}^n = n$

Theorem 9.9.4 (Dim of $\mathrm{Span}\{u_1, \ldots, u_k\}$)

↳ The dim. of $\mathrm{Span}\{u_1, \ldots, u_k\}$.

· The dim of $\mathrm{span}\{u_1, \ldots, u_k\}$, where the $\vec{u_j}$'s are not all zero, is equal to the largest no. of l. ind vectors within the spanning set $\{\vec{u_1}, \ldots, \vec{u_k}\}$.

Proof: Let ~~A~~ $\vec{U} = \{\vec{u_1}, \ldots, \vec{u_k}\}$ let the largest no. of l ind. vectors in U is N, where $1 \le N \le k$. Without loss of generality, assume that the l ind. N vectors have been numbered. so that $\vec{u_1}, \ldots, \vec{u_n}$. Then the remaining vectors of U, namely, $\vec{u}_{N+1}, \ldots, \vec{u_k}$. can be written as a l. comb of $\vec{u_1}, \ldots, \vec{u_N}$. Then ~~h~~ each vector in span $U = \mathrm{Span}\{\vec{u_1}, \ldots, u_N\}$ can be written as a l. comb. of $\vec{u_1}, \ldots, \vec{u_N}$. Thus $\{\vec{u_1}, \ldots, u_N\}$ is l ind. ....

and spans Span $\mathcal{V}$. Therefore, by 9.9.3, dim {Span $\mathcal{V}$} = N, the largest no. of l ind. vectors in Span$\{\vec{u}_1, \cdots, \vec{u}_N, \vec{u}_K\}$.

See example and exercise after the definition of Basis.

~~ECE~~ MATH 215 Oct 6, 2016

9.9.3 Orthogonal Bases

· Suppose that $\{\vec{v}_1 \cdot \vec{v}_2, \ldots, \vec{v}_K\}$ is an orthogonal basis for a vector space S, that is, it is both a "basis for S" and "orthogonal" (i.e. $\vec{v}_i \cdot \vec{v}_j = 0 \ \forall i \neq j$).

Suppose that we want to expand a vector $\vec{\omega} \in S$ in terms of $\vec{v}_j$'s, i.e, $\vec{\omega} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_K \vec{v}_K$ — ①

Q. How to find α_j, $j = 1, 2, \ldots k$?

A. Dot ① with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_K$ in turn to get $\vec{\omega} \cdot \vec{v}_1 = \alpha_1(\vec{v}_1 \cdot \vec{v}_1) + \alpha_2 \cancel{(\vec{v}_2 \cdot \vec{v}_1)}^{0} + \ldots + \alpha_K \cancel{(\vec{v}_K \cdot \vec{v}_1)}^{0}$

$$\vec{\omega} \cdot \vec{v}_2 = \alpha_1 \cancel{(\vec{v}_1 \cdot \vec{v}_2)}^{0} + \alpha_2 (\vec{v}_2 \cdot \vec{v}_2) + \ldots + \alpha_K \cancel{(\vec{v}_K \cdot \vec{v}_1)}^{0}$$

$$\vdots$$

$$\vec{\omega} \cdot \vec{v}_K = \alpha_1 \cancel{(\vec{v}_1 \vec{v}_K)}^{0} + \alpha_2 \cancel{(\vec{v}_2 \vec{v}_K)}^{0} + \ldots + \alpha_K (\vec{v}_K \cdot \vec{v}_K)$$

Sys. ② is "uncoupled" in k eq's & k unknowns $\alpha_1, \ldots, \alpha_K$. The sol of ② is:

$$\alpha_1 = \frac{\vec{\omega} \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}, \quad \alpha_2 = \frac{\vec{\omega} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}, \quad \ldots, \quad \alpha_K = \frac{\vec{\omega} \cdot \vec{v}_K}{\vec{v}_K \cdot \vec{v}_K}$$

Thus if $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_K\}$ is an orthogonal basis, then.

$$\vec{\omega} = \left(\frac{\vec{\omega} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \cdot \vec{v}_1 + \left(\frac{\vec{\omega} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2 + \ldots + \left(\frac{\vec{\omega} \cdot \vec{v}_K}{\vec{v}_K \cdot \vec{v}_K}\right) \vec{v}_K$$

· If, moreover, $\|\vec{v}_j\|^2 = (\vec{v}_j \cdot \vec{v}_j) = 1$ i.e, $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_K)$ is orthonormal, then $\vec{\omega} = (\vec{\omega} \cdot \vec{v}_1)\vec{v}_1 + (\vec{\omega} \cdot \vec{v}_2)\vec{v}_2 + \ldots + (\vec{\omega} \cdot \vec{v}_K)\vec{v}_K$

Ex. Expand $\vec{w} = (-1,5)$ in terms of $\{\vec{v_1} = (2,0), \vec{v_2} = (0,3)\}$

Sol. The set is orthogonal basis (Exercise)

$$\vec{w} = \left(\frac{\vec{w}\cdot\vec{v}}{\vec{v_1}\cdot\vec{v_1}}\right)\vec{v_1} + \left(\quad\right)\vec{v_2}$$

$\vec{v_1}\cdot\vec{v_1} = 4$, $\vec{w}\cdot\vec{v_1} = -2$

$$\vec{w} = \frac{-2}{4}\vec{v_1} + \frac{15}{9}\vec{v_2} = -\frac{1}{2}\vec{v_1} + \frac{5}{3}\vec{v_2}$$

$\vec{v_2}\cdot\vec{v_2} = 9$, $\vec{w}\cdot\vec{v_2} = 15$

Not orthogonal on exam? How do we construct orthogonal set from non-ortho... If we are short of vector (1 vector) only v_1 or v_2. R_2 ~~≠~~ 1 vector. What is basis? If non-zero it is linearly ind. Can we span R_2 by 1 vector? No... inconsistent system.

9.10 Best Approximation

- Let S be a normed inner product v. space (i.e., a ~~nor~~ v. space with norm & inner (dot) product defined).

- let the norm be the natural norm.

(i.e. $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$) . Let $\dim S = N$

. Let $\{\vec{u_1}, \vec{u_2}, \ldots, u_N\}$ be an orthonormal $\boxed{\text{basis for } S}$ and $\vec{w} \in S$.

Then $\vec{w} = \sum_{j=1}^{N} \alpha_j \vec{u_j} = \alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \ldots + \alpha_N\vec{u_N}$

where $\alpha_j = \vec{w}\cdot\vec{u_j}$, for $j = 1, 2, \ldots, N$

Q. If $\dim S > N$, ~~where~~ what is the best approximation of $\vec{w}$ in terms of the vectors of the set $\{\vec{u_1}, \vec{u_2}, \ldots \vec{u_N}\}$

A: The following Theorem gives the answer.

Th. 9.10.1: (Best Approximation)

Let $\vec{w}$ be any vector in a "normed inner product v. space" S with natural norm and let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N\}$ be an orthonormal $\boxed{\text{set in } S}$. Then the best approximation

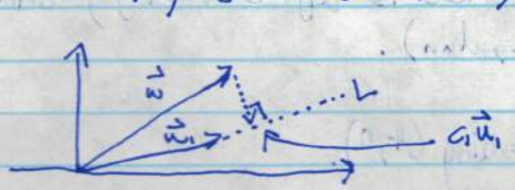
$$\vec{w} \approx c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_N\vec{u}_N = \sum_{j=1}^{N} c_j \vec{u}_j \text{ is obtained when}$$

$$c_j = \vec{w}\cdot\vec{u}_j, \; j = 1, 2, \ldots, N$$

Ex: $S = \mathbb{R}^2$, $\vec{w} = (1,1)$, $\{\vec{u}_1 = \frac{1}{13}(12,5)\}$

Here $N=1$ $\dim S = 2$

$\vec{w} \approx (\vec{w}\cdot\vec{u})\vec{u}_1$



MATH215 October 17, 2015

Chapter 10: Matrices & Linear Equations.

10.2 Matrices & Matrix Algebra.

Def: An $m \times n$ matrix is a rectangular array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

1st column   jth column

- m = no. of rows
- n = no. of columns.

$m \times n$ is the size or dimension of the matrix A,

Notation: $A = [a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$ $i = 1, 2, \ldots, m$ $j = 1, 2, \ldots, n$.

Def: Two matrices $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$ are equal iff $a_{ij} = b_{ij}$ $\forall i,j$.

Def: • If $m=n$, then $A_{n\times n}$ is called a square matrix.

Def: A square matrix $U = [u_{ij}]_{n\times n}$ is said to be "upper triangular" if the entries below the main diagonal are all zero that is $u_{ij} = 0$ $\forall i > j$

Def: A square matrix $L = [l_{ij}]_{n\times n}$ is said to be "lower triangular" if the entries above the main diagonal are all zero, that is, $l_{ij} = 0, \forall i < j$

Ex: Upper triangular matrix. Ex. Lower Triangular Matrix.

$$\begin{bmatrix} 10 & -2 & 0 \\ 0 & 0 & 19 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Def A matrix $D = (d_{ij})_{n\times n}$ is said to be "diagonal" if it is both upper lower triangular - that is $d_{ij} = 0 \ \forall i \neq j$, and denoted by $D = \operatorname{diag}(d_{11}, d_{22}, \dots, d_{nn})$.

Ex: $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $D = \operatorname{diag}(4, 0)$.

Def. Matrix Addition, Scalar Multiplication Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $\alpha \in \mathbb{R}$.

We define the addition by $A+B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$.

and the scalar multiplication by $\alpha A = \alpha [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$.

Ex. Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 6 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} -1 & 9 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{2\times 3}$

Find $A+B$, $-2A+3B$.

Sol. $A+B = \begin{bmatrix} 1 & 12 & 5 \\ 1 & -1 & 9 \end{bmatrix}$ $-2A = \begin{bmatrix} -4 & -6 & -10 \\ 0 & 2 & -12 \end{bmatrix}$ $3B = \begin{bmatrix} -3 & 27 & 0 \\ 3 & 0 & 9 \end{bmatrix}$

$\Rightarrow -2A + 3B = \begin{bmatrix} -7 & 21 & -10 \\ 3 & 2 & -3 \end{bmatrix}$

Properties:

Theorem 10.2.1

· If $A, B, \& C$ are $m \times n$ matrices, $0 = [0_{ij}]_{m \times n}$ (zero matrix) and $\alpha, \beta \in \mathbb{R}$, then $A+B = B+A$.

$(A+B)+C = A+(B+C)$, $A+0 = A$, $A+(-A) = 0$ ~~$\alpha(B+A)$~~

$\alpha(\beta A) = (\alpha\beta)A$, $(\alpha+\beta)A = \alpha A + \beta A$, $\alpha(A+B) = \alpha A + \alpha B$, $1A = A$.

$0A = 0 \quad \alpha 0 = 0.$

zero $\in \mathbb{R}$

· Remark: Any non-empty set of all $m \times n$ matrices is a vector space called "matrix space" S_{mm} or $M_{m \times n}$ $= \{A : A \text{ is } m \times n \text{ matrix}\}$

~~A matrix~~ · An example of a matrix subspace or matrix space is $T_{n \times n} = \{A : A \text{ is an upper (or) lower triangular}\} \subseteq S_{n \times n}$.

Must be same dimension.

Def: Matrix Multiplication. Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

We define the $m \times p$ matrix $C = AB$ whose ij-entry defined by $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

$i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, p.$

Here A time $B = C$

$m \times n = n \times p = m \times p.$

Ex. Evaluate AB if $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}_{2 \times 3}$ & $B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -4 & 0 \\ -2 & 0 & 3 \end{bmatrix}_{3 \times 3}$ $c_{11} = \sum_{k=1}^{3} a_{1k} b_{k1}$

Sol. $AB = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & -4 & 0 \\ -2 & 0 & 3 \end{bmatrix} = \left[\overbrace{(2)(4)+(1)(0)+(0)(-2)}^{c_{11}} \quad \overbrace{(2)(1)+(1)(-4)+}^{c_{12}} \right.$

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$$= \begin{bmatrix} \overbrace{(2)(4)+(1)(0)+(0)(-2)}^{c_{11} = \sum_{k=1}^{3} a_{1k}b_{k1}} & \overbrace{(2)(1)+(1)(-4)+(0)(0)}^{c_{12} = \sum_{k=1}^{3} a_{1k}b_{k2}} & \overbrace{(2)(0)+(1\cdot 0)+0\cdot 5}^{c_{13} = \sum_{k=1}^{3} a_{1k}b_{k3}} \\ \underbrace{(-1)(4)+(3)(0)+(1)(-2)}_{c_{21} = \sum_{k=1}^{3} a_{2k}b_{k1}} & \underbrace{(-1)(1)+(3)(-4)+(1)(0)}_{c_{22}} & \underbrace{(-1)(0)+(3)(0)+(1)(3)}_{c_{23}.} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -2 & 0 \\ -6 & -13 & 3 \end{bmatrix}_{2\times 3}.$$

Q: Can we evaluate BA? A: No, because the matrices are not conformabl for multiplication. Here $\underset{3\times 3}{B}\,\underset{2\times 3}{A}$.

Ex: Let $A = [2 \;\; 3]_{1\times 2}$, $B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}_{2\times 1}$ $AB = [-2+15] = [13]_{1\times 1}$

$BA = \begin{bmatrix} -2 & -3 \\ 10 & 15 \end{bmatrix}_{2\times 2}.$

In general $AB \neq BA$. If $A_{m\times n}$: $B_{n\times m}$ then $AB = (m\times n)(n\times m) = (m\times$
$BA = (n\times m)(m\times n) = (n\times n)$

We say A is pre-multiplied by B
or " " B is post-multiplied by A.

· The algebraic linear system in m eq's & n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= c_1 \\ a_{21}x_2 + a_{22}x_2 + \dots + a_{2n}x_n &= c_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= c_m \end{aligned}$$

· Can be written in a vector (or matrix) form as

$$A\vec{x} = \vec{c}$$

where the coefficient matrix is.

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$, The variable vector is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ We usually write $\vec{x} = (x_1, x_2, \dots, x_n)$.