OCTOBER 19, 2016 MATH215

PARTITIONING

Let $A = \left[\begin{array}{cc|c} 7 & -1 & 2 \\ \hline 0 & 1 & -3 \\ 12 & 11 & 4 \end{array}\right]_{3\times 3} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

· Where the $\boxed{\text{blocks}}$ or $\boxed{\text{submatrices}}$

$A_{11} = [7, -1]$ $A_{12} = [2]$

$A_{21} = \begin{bmatrix} 0 & 1 \\ 12 & 11 \end{bmatrix}$ $A_{22} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

If A & B are partitioned as: $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m\times n}$ & $B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & & \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{bmatrix}_{m\times n}$

Then $A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} + \cdots & A_{1n}+B_{1n} \\ \vdots & & \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} + \cdots & A_{mn}+B_{mn} \end{bmatrix}$ · Here A_{ij} & B_{ij} are conformable for addition.

$\alpha A = \begin{bmatrix} \alpha A_1 & \alpha A_2 & \cdots & \alpha A_{1n} \\ \vdots & \vdots & & \\ \alpha A_{m1} & \alpha A_{m2} & \cdots & \alpha A_{mn} \end{bmatrix}$ ***Multiplication:** If $m=n$ and we denote $AB=C$, then $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

That is $C = \begin{bmatrix} \sum_{k=1}^{n} A_{1k}B_{k1} & \cdots & \sum_{k=1}^{n} A_{1k}B_{kn} \\ \vdots & & \vdots \\ \sum_{k=1}^{n} A_{nk}B_{k1} & \cdots & \sum_{k=1}^{n} A_{nk}B_{kn} \end{bmatrix}$

columns

rows.

Ex. If $A = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 2 & 1 & 0 \\ \hline 0 & 5 & 3 \end{array}\right]_{3\times 3} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ $B = \left[\begin{array}{c|c} 4 & 1 \\ -1 & 0 \\ \hline -2 & 5 \end{array}\right]_{3\times 2} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

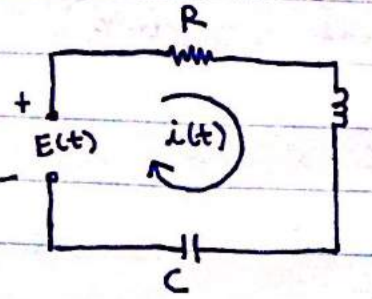
Then:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{12} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

where $A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}[-2] = \begin{bmatrix} -1 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

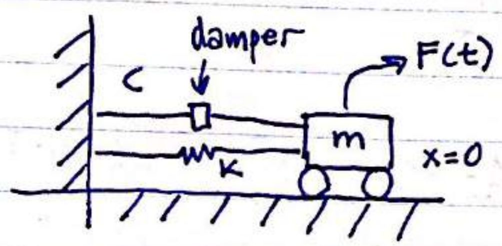
So that $AB = \begin{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} & \begin{bmatrix} -5 \\ 2 \end{bmatrix} \\ [-11] & [15] \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 7 & 2 \\ -11 & 15 \end{bmatrix}_{3\times 2}$

Circuit:



$Li'' + Ri' + \frac{1}{C} i(t) = E'(t)$

Mass-Spring System:



By Newton's 2nd Law: $mx'' + cx' + kx = F(t)$

$\Downarrow$

$$x'' + ax' + bx = f(t)$$

$a = \frac{c}{m}$, $b = \frac{k}{m}$, $f(t) = \frac{F(t)}{m}$

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10.3 Transpose Matrix

Def. Given any $m \times n$ matrix

$A : [a_{ij}]$, the transpose of A. denoted by A^T and read as "A-transpose"

is defined by $A^T = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n \times m}$

- This $m \times n$ matrix is obtained by interchanging the rows & columns of A.

The ij-entry in A becomes ji-entry in A^T. That is $a^T_{ij} = a_{ji}$

- A^T is NOT the T^{th} power of A.

Ex. If $A = \begin{bmatrix} -2 & 3 & 0 \\ 1 & \frac{1}{2} & 7 \end{bmatrix}_{2\times 3}$, then $A^T = \begin{bmatrix} -2 & 1 \\ 3 & \frac{1}{2} \\ 0 & 7 \end{bmatrix}_{3 \times 2}$

· Theorem 10.3.1 (Properties of Transpose)

(i) $(A^T)^T = A$
(ii) $(A+B)^T = A^T + B^T$
(iii) $(\alpha A)^T = \alpha A^T$
(iv)* $(AB)^T = B^T A^T$

Proof (iv): Let $AB = C = [C_{ij}]$. Then $C_{ij}{}^T = C_{ji} = \sum_{k=1}^{n} a_{jk} \cdot b_{ki}$

Due to usual scalar multiplication but NOT $AB = BA$ $\rightarrow \sum_{k=1}^{n} b_{ki} \cdot a_{jk} = \sum_{k=1}^{n} b_k{}^T a_{kj}{}^T$

$C^T = B^T A^T$ or $(AB)^T = B^T A^T$

⊕ $A = [a_{ij}] \Rightarrow A^T = [a_{ij}{}^T] = [a_{ji}]$
$B = [b_{ij}] \Rightarrow B^T = [b_{ij}{}^T] = [b_{ji}]$

· Dimensions in (iv) $[(m\times n)(n\times p)]^T = (n\times p)^T (m\times n)^T$
$\Rightarrow [(m\times p)]^T = (p\times n)(n\times m)$
$\Rightarrow p\times m = p\times m$

It follows that from (iv) that $(ABC)^T = C^T B^T A^T$
Let $D = AB$ Then $(DC)^T = C^T D^T$
$= C^T (AB)^T = C^T B^T A^T$

· $(ABCD)^T = D^T C^T B^T A^T$, and so on.

Exercise: Verify $(AB)^T = (BA)^T$ if $A = [2\ 1]$, $B = \begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix}$

$AB = \ldots = [-5, 5] \Rightarrow (AB)^T = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $B^T = \begin{bmatrix} -3 & 1 \\ 0 & 5 \end{bmatrix}$
$\Rightarrow B^T A^T = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

Dot product in a Matrix Form. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then $\vec{x}\cdot\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{j=1}^{n} x_j y_j$

In a matrix form: $\vec{x}\cdot\vec{y} = [x_1, x_2, \dots, x_n]\cdot\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \dots x_ny_n = \sum_{j=1}^{n} x_jy_j$

$\hookrightarrow B^TA^T = [\sum b_{1k}^T a_{kj}^T]$

<u>Def:</u> An <u>square</u> matrix is said to be <u>symmetric</u> if $A^T = A$.

It is said to be skew-symmetric (or anti-symmetric) if $A^T = -A$.

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<u>10.4 Determinants</u>

- The determinants of an <u>$n\times n$</u> matrix $A = [a_{ij}]$ is defined by the cofactor expansion:

$$\det(A) = \sum_{j=1}^{n} a_{jk}A_{jk}.$$

where the summation is carried out on j for any fixed value of k $(1 \leq k \leq n)$ <u>or</u> on k for any fixed value of j $(1 \leq j \leq n)$.
A_{jk} is called the cofactor of a_{jk} element and is defined by $A_{jk} = (-1)^{j+k} M_{jk}$.

where M_{jk} is called the minor of a_{jk}, namely, the determinant of the $(n-1)(n-1)$ matrix when the row & column containing a_{jk} (the jth row and kth column) are struck out.

- Another notation for determinant of A is $|A|$ ← Not absolute value.

<u>Examples</u> (i) If $A = [-5]$, then $|A| = -5$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\det(A)$ or $|A| = \sum a_{1k}A_{1k}$ (ie $j=1$)

$= a_{11}A_{11} + a_{12}A_{12}$

$= a_{11}\left((-1)^{1+1}M_{11}\right) + a_{12}\left((-1)^{1+2}M_{12}\right) = a_{11}(+1)|a_{22}| + a_{12}(-1)|a_{21}|$

$= a_{11}a_{22} - a_{12}a_{21}$ <u>OR</u> fix $k=2$, i.e.

$$|A| = \sum_{j=1}^{2} a_{j2}A_{j2} = a_{12}A_{12} + a_{22}A_{22}$$
$$= a_{12}\left((-1)^{1+2}M_{12}\right) + a_{22}\left((-1)^{2+2}M_{22}\right)$$
$$= a_{12}(-1)|a_{21}| + a_{22}(+1)|a_{11}|$$
$$= -a_{12}a_{21} + a_{22}a_{11}$$

Here $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ Let $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ then
$|A| = (2)(3) - (4)(-1) = 6+4 = 10$

Ex. Find $|A|$ if $A = \begin{bmatrix} \boxed{2 \quad 5 \quad 4} \\ 3 \quad 1 \quad 2 \\ 5 \quad 4 \quad 6 \end{bmatrix}$

<u>Sol.</u> $|A| = \sum_{k=1}^{3} a_{1k}A_{1k} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= a_{11}\left((-1)^{1+1}M_{11}\right) + a_{12}\left((-1)^{1+2}M_{12}\right) + a_3\left((-1)^{1+3}M_{13}\right)$

$= a_{11}\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - a_{12}\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + a_{13}\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 2(6-8) - 5(18-10) + 4(12-5) = -16$

· If $A = [a_{ij}]$ is an upper or lower triangular $n \times n$ matrix, then

$|A| = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$ (Product of diagonal)

· <u>Properties of Determinants</u>

?

(i) If $A \xrightarrow[c_j = c_j + d_{ik}]{r_j = r_j + \alpha r_k} B$ then $|A| = |B|$ — row / column — Row echelon ops

(ii) If $A \xrightarrow[c_j \leftrightarrow c_k]{r_j \leftrightarrow r_k} B$, then $|A| = -|B|$ — Row switch.

(iii) If A_{nn} is a triangular matrix then $|A| = \prod_{i=1}^{n} a_{ii}$

Sol
$$A=\begin{bmatrix}2&1&3\\4&2&1\\6&-3&4\end{bmatrix}\ \begin{matrix}r_2=r_2-2r_1\\r_3=r_3-3r_1\end{matrix}=\begin{bmatrix}2&1&3\\0&0&-5\\0&-6&-5\end{bmatrix}\ \underset{r_3}{\overset{r_2}{\updownarrow}}=-\begin{vmatrix}2&1&3\\0&-6&-5\\0&0&-5\end{vmatrix}=-((2)(-6)(-5))$$
$$=-60.$$

Additional Properties of Determinants

(iv) If all elements (or entries) of any row or any column are zero, then $|A|=0$

Ex. $\begin{vmatrix}-5&0&1/7\\9&0&0\\-1&0&-3\end{vmatrix}=\underset{a_{12}}{0}A_{12}+\underset{a_{22}}{0}A_{22}+\underset{a_{32}}{0}A_{32}=0$

(v) If $A\xrightarrow[c_i=\alpha c_i]{r_i=\alpha r_i}B$, then $|A|=\alpha|B|$

Here we scale only one row or only one column. ★ONLY FOR DETERMINANTS.

Ex. $\begin{vmatrix}-\frac{7}{3}&\frac{14}{3}\\2&3\end{vmatrix}=\begin{vmatrix}\frac{7}{3}(-1)&\frac{7}{3}(2)\\2&3\end{vmatrix}=\frac{7}{3}\begin{vmatrix}-1&2\\2&3\end{vmatrix}=\frac{7}{3}(-3-4)=\dots$

(vi) $|\alpha A|=\alpha^n|A|$ where A is $n\times n$

(vii) If a matrix A, $r_i=\alpha r_k$ or $c_i=\alpha c_k$ then $|A|=0$
$\alpha\in\mathbb{R}$, including $\alpha=1$

Ex. $\begin{vmatrix}2&4\\1&2\end{vmatrix}=2\cdot2-4\cdot1=0$

(viii) If, in A, any row (or column) is a linear combination of other rows (or columns), then $|A|=0$

Symbollically, in A if $\begin{Bmatrix}r_i=\alpha r_j+\beta r_k\\c_i=\alpha c_j+\beta c_k\end{Bmatrix}$, then $|A|=0$

(ix) If any $\boxed{\text{one}}$ row or column a of A is written as $a = b+c$, then $|A|_a = |A|_b + |A|_c$

a=b+c

Ex. $$\begin{vmatrix} 3 & -8 & 2 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} 1+2 & -11+3 & 1+1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix}}_{|A|_a} = \underbrace{\begin{vmatrix} 1 & -11 & 1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix}}_{|A|_b} + \underbrace{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix}}_{|A|_c}$$

unchanged

(x) $|A^T| = |A|$

(xi) $|AB| = |A|\,|B|$

(xii) $|A+B| \neq |A| + |B|$ ((in general))

* Determinant is <u>not</u> linear, i.e, $|\alpha A + \beta B| = \alpha|A| + \beta|B|$

10.5 RANK << Different approach than textbook >>.

<u>Def:</u> The maximum number of linearly independant <u>rows</u> in a matrix A is called "row rank of A".

The " " " " " <u>columns</u> "" is called "column rank of A"

· If A is a m×n zero matrix, then row rank of A = column rank of A = 0.

<u>Th. 10.5.2</u> (Rank & Linearly Dependance)
For any matrix A, the number of L. ind. row vectors <u>is equal to</u> the number of L ind. column vectors and these in turn equal the rank of A.

· Notation rank of A = rank(A) or r(A)

Th 10.5.1 (Elementary Row (or Column) Operations)

· Row (or column) equivalent matrices have the same rank. That is elementary row/(or column) operations do not alter the rank of a matrix.

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Q. How to evaluate the rank(A)?

A. Convert A into row echelon form (REF) [or column echelon form (CEF)]. Then, the number of non-zero rows [or columns] is rank(A).

Ex. Find Rank(A) where $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & -4 & 6 \\ 3 & 1 & 5 \end{bmatrix}$ $r_2 = r_2 + 2r_1$, $r_3 = r_3 - 3r_1$ $\rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

Thus rank(A) = 2. In this example, the 2^{nd} row = (-2) 1st row

· Row (and column) vector space of a matrix A.

· Let A be a $m \times n$ matrix with rows denoted by $r_1, r_2, \dots, r_m$. and columns " " $c_1, c_2, \dots, c_n$. Then, the row vector space (or row space) of A is defined by span $\{r_1, r_2, \dots, r_m\}$

· The column space of A is defined by span $\{c_1, c_2, \dots, c_n\}$.

· The dimension of the row (and column) spaces of A is equal to the number of L ind. rows (and columns) in A.

Ex. How many L. ind. vectors are in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$ $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3] \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 7 \\ 0 & 1 & 1 \end{bmatrix}$ Then,

$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 7 \\ 0 & 1 & 1 \end{bmatrix}$ $r_2 = r_2 - 2r_1 \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & 5 \\ 0 & 1 & 1 \end{bmatrix}$ $r_2 = r_2 - 5r_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Since rank(A) = 2, there are 2 l ind. vectors in the set.

Thus $\dim\left[\text{span}\{\vec{U}_1, \vec{U}_2, \vec{U}_3\}\right] = 2$

- Dimension of row space of A is 2 and the basis for that row space of A is given. $[1\ -1\ 1]$ & $[0\ 1\ 1]$

10.5.2 Application of Rank to the System $A\vec{x} = \vec{C}$ where A is $m \times n$. Then there is 1. no solution iff $(A|\vec{c}) \neq r(A)$

2. A unique sol. iff $r(A|\vec{c}) = r(A) = n$

3. An $(n-r)$ parameter family of sol's iff $r(A|\vec{c}) = r(A) = r$ is less than n.

Ex. Consider the system $\begin{cases} 2x - y = 3 \\ x + y + 2z = 1 \end{cases}$ Sol. Here $m = 2$ & $n = 3$

$$[A|\vec{c}] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \begin{array}{c} r_1 \\ \updownarrow \\ r_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & -1 & 0 & 3 \end{array}\right] r_2 = r_2 - 2r_1 \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \end{array}\right]$$ So, $r(A|\vec{c}) = r(A) = \overset{r}{2} < n = 3$

Thus, the system has $\underset{(3-2)}{1}$-parameter family of sol.

Let $z = \alpha$, then $y =$ ___, $z =$ ___

Ex. Consider the system $\begin{cases} x + y + 3z = 2 \\ 3x + y \qquad = 1 \\ 4x + 2y + 3z = 1 \end{cases}$ Sol. $[A|\vec{c}] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 2 & 3 & 1 \end{array}\right] \begin{array}{l} \\ r_2 = r_2 - 3r_1 \\ r_3 = r_3 - 4r_1 \end{array}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -2 & -9 & -3 \\ 0 & -2 & -9 & -5 \end{array}\right] r_3 = r_3 - r_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -2 & -9 & -3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$\left.\begin{array}{l} r(A) = 2 \\ r(A|\vec{c}) = 3 \end{array}\right\}$ Since $r(A|\vec{c}) \neq r(A)$, the sys is inconsistent

Th 10.5.4 (Homogenous Case Where A is m×n)

- If A is m×n, then $A\vec{x} = \vec{0}$.

1. is consistent.
2. Admits the trivial sol. $\vec{x} = \vec{0}$
3. " " the unique sol. $\vec{x} = \vec{0}$ iff $r(A) = n$.
4. Admits an $(n-r)$ - parameter family of sol's of non-trivial sol's in addition to the trivial sol, iff $r(A) = r < n$.

Th. 10.5.5 (Homog. Case Where A is n×n)

- If A is n×n, then the sys $A\vec{x} = \vec{0}$ → Admits non-trivial sol's, in addition to the trivial sol, iff $\det(A) = 0$.

Sys. $A\vec{x} = \vec{0}$ has a unique sol. $\vec{x} = \vec{0}$ iff $\det(A) \neq 0$.

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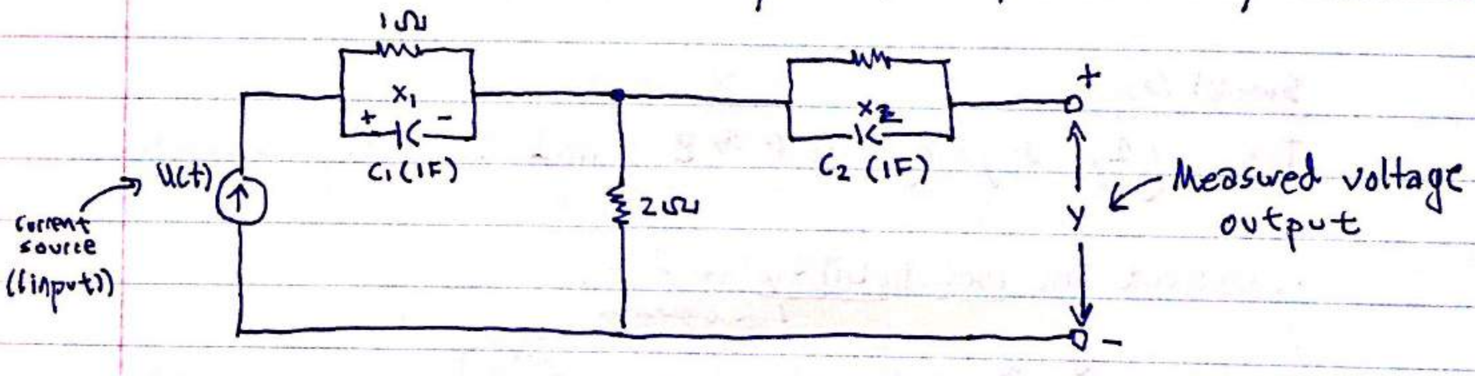
* Application to rank of a matrix
↳ Control System, Controllability (& probably) observability.



- For $i = 1, 2$, x_i is the voltage across the capacitor with capacitance C_i
- $u(t)$ is the current source at time t «u = input to network»
- $y(t)$ is the measured voltage at " " «y = Output».

Q1. Does u have effect on x_2 or can u control x_2?

A1: Controllability provides an answer.

Q2. Can the initial voltage x_i be observed from y?

A2. Observability provides an answer.

· Let $\vec{x}(0) = \vec{x}_0$ be the initial voltage vector. ($t=0$)

· Let $\vec{x}(t)$ be the voltage vector at time t.

Here $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Controllability: (roughly speaking) In order to be able to do whatever you want with given initial voltage of the system (or network) under the control input, the system must be controllable.

* Consider the n-dimensional, p-input system (or state equation)

$$\vec{x}' = A\vec{x} + B\vec{u}$$

where $\vec{x} \in \mathbb{R}^n$, i.e. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, A is an $n\times n$ matrix, B is $n\times p$ matrix and $\vec{u} \in \mathbb{R}^p$

Special Case:

Take $p=1$, i.e., single input ⇒ B is $n\times 1$ and $u(t)$ is a scalar.

· Construct the controllability matrix.

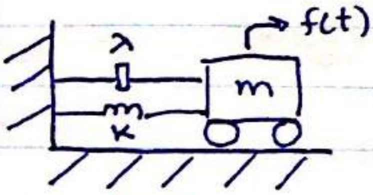
$$\tilde{C}(A,B) \text{ or } \tilde{C} \quad \tilde{C} = \Big[\overbrace{\underset{n\times 1}{B} \quad \underset{\substack{(n\times n)(n\times 1)\\(n\times 1)}}{AB} \quad \underset{n\times 1}{A^2B} \quad \underset{n\times 1}{A^{n-1}B}}^{n \text{ column}}\Big]_{n\times n}$$

full row rank.

Fact: System ① or the matrix pair (A,B) is controllable iff $\text{rank}(\tilde{C}) = n$

Remark: Controllability means that the system is fully controllable.

__Example__ : Consider the mass-spring-damper system



m = mass of body

λ = damping coefficient

k = stiffness

$x = x(t)$ is the mass position

$x' = \frac{dx}{dt}$ " " " velocity.

$x'' = \frac{d^2x}{dt}$ is the acceleration of mass

By Newton's 2^{nd} Law, the motion of the mass is given by the 2^{nd}-order DE

$$mx'' + \lambda x' + kx = f(t)$$

$$m \Rightarrow x'' + \frac{\lambda}{m}x' + \frac{k}{m}x = u(t)$$ where $u(t) = \frac{f(t)}{m}$ is the system input.

Determine if the system is controllable or not.

__Sol__ $\vec{x} = ?$ $A = ?$ $B = ?$ $u = ?$ Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Then
$$x_1' = \qquad\qquad + x_2$$
$$x_2' = -\frac{k}{m}x_1 - \frac{\lambda}{m}x_2 + u(t)$$
or
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\lambda}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t)$$

So $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\lambda}{m} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

★ Column rank: Take transpose → find row rank.

Construct the controllability matrix.

$$\tilde{C} = [B \;\; \underline{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{\lambda}{m} \end{bmatrix}_{2\times 2}$$

$\text{rank}(\tilde{C}) = 2$ ← full row rank

↳ So it's controllable.

10.6 Inverse Matrix, Cramer's Rule, Factorization

· Inverse

Motivation : If we have the scalar equation $ax = b$, then $x = \frac{1}{a} b$, where $a \neq 0$. Here, $\frac{1}{a}$ is called the multiplication inverse of a.

Q: If we have $A\vec{x} = \vec{b}$, then can we find $\vec{x}$ in the same way?
A: The answer is "yes" IF we could find the "multiplication inverse of the matrix A.

Def: A $n \times n$ matrix A is said to be nonsingular or invertible if there exists a matrix B such that $BA = AB = I_n$.

The matrix $B_{n \times n}$ is said to be the multiplication inverse of A.

If B and C are both multiplication inverse of A.

· If B and C are both multiplication inverses of A, then $B = BI = \ldots$
$\ldots B(AC) = (BA)C = IC = C$
⇒ That is, if a matrix is nonsingular, then it has at most one inverse.

· Notation If A is a nonsingular matrix, its inverse is denoted by A^{-1}, and we call it A-inverse

· If A^{-1} is the inverse of A, then A is the inverse of A^{-1}.
That is the $(A^{-1})^{-1} = A$

Ex. The matrices $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{bmatrix}$ are inverses of each other because:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\begin{bmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

· <u>Def</u> An $n \times n$ matrix is said to be <u>singular</u> if it does not have a multiplication inverse.

<u>Theorem</u> If A & B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

<u>Proof:</u> $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$

It follows that $A_1, A_2, \ldots, A_k$ are all nonsingular, then $(A_1A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1}A_1^{-1}$

· Remember that $I_n = \text{diag}(1, 1, \ldots, 1)$ or $I_n = [\delta_{ij}]_{n \times n}$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

* Important : Inverse and determinant.

If $A^{-1}A = 1$, then $\det(A^{-1}A) = \det(I) = 1$

But $\det(A^{-1}A) = \det(A^{-1})\det(A)$ so that $\det(A^{-1})\det(A) = 1$

=> That is, if a matrix A has an inverse A^{-1}, then it is <u>necessary</u> that $\det(A) \neq 0$.

Moreover, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- A is singular iff $\det(A) = 0$
- Q: How to find A^{-1} if A is nonsingular.
- Finding A^{-1} of a nonsingular matrix A. We need the following lemma

Lemma: Let A be $n \times n$ matrix if A_{jk} denotes the cofactor of a_{jk} for $k = 1, 2, \dots, n$, then $a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

I thought row operations don't affect determinant

$A = \begin{bmatrix} 2 & 6 \\ 4 & 10 \end{bmatrix} R_2 \to R_2 - 2R_1 \to \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix} \Rightarrow |A| = 2(-2) = -4$

In RREF, $\begin{bmatrix} 2 & 6 \\ 4 & 10 \end{bmatrix} \div 2 \to \begin{bmatrix} 1 & 3 \\ 4 & 10 \end{bmatrix} R_2 \to R_2 - 4R_1 \to \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \div 2 \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \to |A| = 1$

The Adjoint of A... Def. next lecture.

MATH215 Nov. 9, 2016

The Adjoint of a Matrix

Def. Let $A = [a_{ij}]$ be a $n \times n$ matrix. We define a new matrix called the adjoint of A by:

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{n3} \end{bmatrix}_{n \times n} = [A_{jk}]_{n \times n}$$

where A_{jk} is the cofactor of the a_{jk} element in A.

One can show that $A(\operatorname{adj} a) = \det(A) I_n$

Exercise: You may take $n=2$

- If A is nonsingular (or invertible) then $\det(A) \neq 0$ and we may write (from ⊛) $A\left(\frac{1}{\det A} \operatorname{adj} A\right) = I_n$

Thus, $\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A}$, when $\det(A) = 0$.

Ex. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Then the cofactor matrix is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \text{adj}A$ where

$A_{ij} = (-1)^{i+j} M_{ij} \Rightarrow \begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

↳ Transposed.

Thus if $A_{2\times 2}$ is invertible then $A^{-1} = \frac{1}{\det(A)} \text{adj}A = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

if $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ $|A| = 2 - 12 = -10 \neq 0$ So A is invertible $A^{-1} = -\frac{1}{10} \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}$

Ex. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ Find $|A|$, adj A, & A^{-1}

Sol $|A| = 5$ (exercise) $\text{adj}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = $ ↓

$$\begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^{T} \Rightarrow \text{adj} A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$ So that, $A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{5}[\ \ ]$

Th 10.6.2 (Solution of $A\vec{x} = \vec{c}$)

- If A is $n\times n$ and $|A| \neq 0$, then $A\vec{x} = \vec{c}$ admits the unique solution $\vec{x} = A^{-1}\vec{c}$ How? $A^{-1}(A\vec{x} = \vec{c}) = A^{-1}A\vec{x} \Rightarrow \underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}c \Rightarrow \vec{x} = A^{-1}\vec{c}$

Ex. Solve the linear system

$$2x + y + 2z = 1$$
$$3x + 2y + 2z = 0 \quad A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \vec{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$x + 2y + 3z = 4$$

exercise

From the last example $|A| = 5 \neq 0$, Thus $\vec{x} = A^{-1}\vec{c} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}$

Properties of Inverse

(1.) If A & B are of same dimension and non-singular, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$ See last lecture (proved)

(2.) If A is nonsingular, then $(A^T)^{-1} = (A^{-1})^T$

Proof:

Method I $(AA^{-1})^T = I^T = I$ Also $(AA^{-1})^T = (A^{-1})^T A^T = I^T = I$

$\Rightarrow (A^{-1})^T$ is the inverse of A^T

Method II. $A^{-1}A = AA^{-1} = I$ *Take "transpose" then,* $A^T(A^{-1})^T = \ldots$
$\ldots (A^{-1})^T A^T = I^T = I$

That is $(A^{-1})^T$ is the inverse of A^T. Thus, $(A^{-1})^T = (A^T)^{-1}$

(3.) If A is nonsingular, then $(A^{-1})^{-1} = A$ and $(A^m)^n = A^{mn}$ for any (+ive, -ive, 0) integers m and n.

For instance, if $m=-2$, $n=3$, then $(A^{-2})^3 = A^{-2}A^{-2}A^{-2} = A^{-6} = \ldots$
$= (A^6)^{-1} = (A^{-1})^6 = A^{-1}A^{-1}A^{-1}\ldots A^{-1}$ 6 times.

(4.) If A is nonsingular, then (1) $AB=AC$ implies $B=C$ why?
$A^{-1}(AB=AC) = A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B=C$

only one inverse

(2) $AB=0$ Implies $B=0$
zero matrix

(3) $BA=0$ implies $B=0$

ECE250 Nov 10, 2016

Cramer's Rule [Solve $A\vec{x} = \vec{b}$]

<u>Theorem:</u> Let A be a nonsingular $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$. Let A be the matrix obtained by replacing the <u>ith column</u> by $\vec{b}$.

- If $\vec{x}$ is the unique solution of $A\vec{x} = \vec{b}$, then $x_i = \dfrac{|A_i|}{|A|}$, $i = 1, 2, \ldots, n$

  ith component in $\vec{x}$

<u>Proof:</u> Since $\vec{x} = A^{-1}\vec{b} = \left(\frac{1}{|A|} \operatorname{adj} A\right)\vec{b}$

it follows that $x_i = \dfrac{1}{|A|}(b_1 A_{1i} + b_2 A_{2i} + \ldots + b_n A_{ni}) = \dfrac{|A_i|}{|A|}$

Example: Use the Cramer's Rule to solve the system

$$x + 2y + z = 5$$
$$2x + 2y + z = 6$$
$$x + 2y + 3z = 9$$

<u>Sol.</u> Here

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{So } |A| = \left| \quad \right| = -4.$$

$$|A_1| = \begin{array}{c} \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} \\ \vec{b} \end{array} = -4 \qquad |A_2| = \begin{array}{c} \vec{b} \\ \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} \end{array} = -4 \qquad |A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 3 & 9 \end{vmatrix} = -8$$

Thus, $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{-4}{-4} = 1$ $\quad x_2 = \dfrac{|A_2|}{|A|} = \dfrac{-4}{-4} = 1$ $\quad x_3 = \dfrac{|A_3|}{|A|} = \dfrac{-8}{-4} = 2$

10.6.4 (Evaluation of A^{-1} by ERO)

Consider system $A\vec{x} = \vec{c}$ or $A\vec{x} = I\vec{c}$ $A \in \mathbb{R}^{n\times n}$

Using a sequence of E.R.O's leads to $I_n \vec{x} = A^{-1}\vec{b}$ or $\boxed{\vec{x} = A^{-1}\vec{b}}$

Ex. Use the E.R.O's to find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$

Sol. $[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 = r_2 - 3r_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -2 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$

Augmented Matrix

$\xleftrightarrow{r_3 \leftrightarrow r_2} \left[\begin{array}{ccc|ccc} 1 & \boxed{-1} & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & \boxed{5} & -2 & -3 & 1 & 0 \end{array}\right] \xrightarrow[r_3 = r_3 - 5r_2]{r_1 = r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 8 & -3 & 1 & -5 \end{array}\right] \xrightarrow{r_3 = \frac{1}{8} r_3}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \boxed{-2} & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \end{array}\right] \xrightarrow{r_2 = r_2 + 2r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -6/8 & 2/8 & -2/8 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \end{array}\right]$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \quad \underbrace{\begin{matrix} 1 & 0 & 1 \\ -6/8 & 2/8 & -2/8 \\ -3/8 & 1/8 & -5/8 \end{matrix}}_{A^{-1}}$$

Then, $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -6/8 & 2/8 & -2/8 \\ -3/8 & 1/8 & -5/8 \end{bmatrix}$

Remark: If an ERO is applied to I, then the new equivalent matrix (to I) is called elementary matrix and denoted by E. Here if an ERO #i is applied to I, we get E_i.

Now: $I = E_6 E_5 \dots E_2 E_1$ Then $A = E_1^{-1} E_2^{-1} \dots E_5^{-1} E_6^{-1}$ and $A^{-1} = E_6 E_5 \dots E_2 E_1 I$.

MATH 215 10.6.5 LU Factorization [or LU Decomposition]

This is an alternative method to solve $A\vec{x} = \vec{c}$, where $A \in \mathbb{R}^{n \times n}$

Decompose the matrix $A_{3\times 2}$ as follows:

$$A = LU = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{\text{lower tri matrix}} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}_{\text{upper tri matrix}}$$

- In Doolittle's decomposition or method, we can see $l_{ii} = 1$ for all i.

The system $A\vec{x} = \vec{c}$ then becomes $(LU)\vec{x} = \vec{c} \Rightarrow L(U\vec{x}) = \vec{c}$

Let $U\vec{x} = \vec{y}$ Then $L\vec{y} = \vec{c}$ so that, to solve $A\vec{x} = \vec{c}$
(i) determine L & U, (ii) Solve $L\vec{y} = \vec{c}$ for $\vec{y}$ and (iii) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$

Ex. Use the Doolittle's Method to Solve $\begin{cases} 2x_1 + x_2 = 4 \\ 2x_1 + 2x_2 = 1 \end{cases}$

Sol.

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} \end{bmatrix}$$

So that $U_{11} = 2$, $U_{12} = 1$ $l_{21}U_{11} = 8$ $l_{21} = 8/2 = 4$
and $l_{21}U_{12} + U_{22} = 2 \Rightarrow U_{22} = 2 - (4)(1) = -2$

Thus, $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ * Check $LU = A$

Solve $L\vec{y} = \vec{c} \Rightarrow \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow y_1 = 4,\ 4y_1 + y_2 = 1 \Rightarrow y_2 = -15$

That is $\vec{y} = \begin{bmatrix} 4 \\ -15 \end{bmatrix}$ Finally, solve $U\vec{x} = \vec{y}$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 = 15/2 \\ 2x_1 + x_2 = 4 \rightarrow x_1 = 3/4 \end{array}$$

So $\vec{x} = \left[\frac{3}{4}, \frac{15}{2}\right]^T$

Chapter 11: The Eigenvalue Problem

Def: Suppose that A is a $n \times n$ matrix. A $\boxed{\text{non-zero}}$ vector $\vec{v} \in \mathbb{R}^n$ satisfying $A\vec{v} = \lambda\vec{v}$ is called an $\boxed{\text{eigenvector}}$ of A and the scalar λ is called an eigenvalue of A. The pair $\lambda, \vec{v}$ is called an eigenpair of A.

- Finding Eigenvalues & Eigenvectors of Matrices

↳ Suppose that λ is an eigenvalue of $A_{n \times n}$. Then, there exists a non-zero, eigenvector $\vec{v} \in \mathbb{R}^n$ such that
$A\vec{v} = \lambda\vec{v} = \lambda I_n \vec{v}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \neq \vec{0}$.

$$\text{or } \overbrace{(A - \lambda I_n)\vec{v} = \vec{0} \sim ① \text{ or } (\lambda I_n - A)\vec{v} = 0}^{①}$$

Eq. ① is a homogenous system of n equations in the unknowns $v_1, v_2, \ldots, v_n$. We learned that this homogenous system has a non-zero (or nontrivial) sol. $\vec{v}$ iff $|A - \lambda I_n| = 0$ $|\lambda I_n - A| = 0$ ②

Eq. ② is solved for the eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A_{n \times n}$.

Eq ② is called the "characteristic equation of A". The LHS of Eq. ② (i.e. $|A - \lambda I|$) is called the characteristic polynomial and denoted by $p(\lambda) = |A - \lambda I_n|$.

If λ^* is an eigenvalue of A, then all non-zero sol's of the homog. sol.

$(A - \lambda^* I)\vec{v} = \vec{0}$ or $(\lambda^* I - A)\vec{v} = \vec{0}$ are eigenvectors of A corresponding to λ^*.

Def: Let λ be an eigenvalue of $A_{n\times n}$. Then the set containing the $\boxed{\text{zero}}$ vector $\boxed{\text{and}}$ all eigenvectors of A corresponding to λ is called the eigenspace of λ.

Remark The eigenspace of any eigenvalue λ must contain at least one non-zero vector. Hence, the dimension of the eigenspace must be at least one.

Example ① Let $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ Find the eigenvalues of A and corresponding eigenvectors and eigenspaces.

Sol. $A - \lambda I_2 = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$

Then $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 6 = \lambda^2 - 9\lambda + 14$
$= (\lambda-2)(\lambda-7) = 0$

· So that, the roots of $p(\lambda) = (\lambda-2)(\lambda-7)$ are $\lambda_1 = 2$ & $\lambda_2 = 7$ which are the eigenvalues of A.

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Eigenvectors (e. vectors)

Solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v} \neq \vec{0}$

(1) $\lambda = 2 \Rightarrow (A - 2I) = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

So we have... $\begin{cases} 3V_1 + 2V_2 = 0 \\ 3V_1 + 2V_2 = 0 \end{cases} \Rightarrow \boxed{V_1 = -\frac{2}{3} V_2}$

Choose $V_2 = \alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$. Then $\vec{v}_1 = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \alpha \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$, $\alpha \neq 0$

e. vector corresponding to $\lambda_1 = 2$

There is only $\boxed{\text{one}}$ linearly indep. e. vector

In fact we can write $\vec{v}_1 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$

a line in $\mathbb{R}^2$

The corresponding eigenspace is span $\left\{\begin{bmatrix} -1 \\ 3/2 \end{bmatrix}\right\}$ $\boxed{\text{OR}}$ $\left\{\alpha\begin{bmatrix} -1 \\ 3/2 \end{bmatrix}, \alpha \in \mathbb{R}\right\}$

has 0

- Clearly, the dimension of the eigenspace is 1 & Only one Lind. e.vector.
- One of the e.vector is $\vec{v}_1 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ i.e., when $\alpha = 1$
- A basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 3/2 \end{bmatrix}\right\} \leftarrow \dim = 1$

(ii) $\lambda_2 = 7$ Then $(A - 7I)\vec{V} = \vec{0}$ gives $\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow V_1 - V_2 = 0 \Rightarrow \boxed{V_1 = V_2}$

Choose $V_2 = \alpha_1$, $\alpha \in \mathbb{R} \setminus \{0\}$

- Corresponding e. vector is $\vec{V}_2 = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\alpha \neq 0$ — a line in $\mathbb{R}^2$.

Eigenspace corresponding to $\lambda_2 = 7$ is span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ or $\left\{\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}\right\}$

One linearly ind. e. vector corresponding to $\lambda_2 = 7$ is $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A basis for the e. space is $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ so $\dim = 1$

⊛ <u>Important</u>: The two eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ & $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independant.

<u>Example 2.</u> Consider $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

<u>Sol.</u>

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

$$= -(\lambda+1)^2(\lambda-8) = 0$$

So, $\underbrace{\lambda_1 = -1, \lambda_2 = -1}_{\text{Algebraic multiplicity of -1 is 2}}$, $\underbrace{\lambda_3 = 8}_{\text{alg. mult of 8 is 1.}}$

Corresponding e. vectors (i) $\lambda_1, \lambda_2 = -1$

Solve $(\underbrace{A-(-1)I}_{A+I})\vec{v} = \vec{0}$ for $\vec{v} \neq \vec{0}$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Single Eq. } V_1 + \frac{1}{2}V_2 + V_3 = 0$$

Take $V_2 = \alpha$, $V_3 = \beta$ $\alpha, \beta \in \mathbb{R}$ such that $\vec{v} \neq \vec{0}$.

Then $V_1 = -\frac{1}{2}\alpha - \beta$ That is $\vec{v} = \begin{bmatrix} -\frac{1}{2}\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

To find the corresponding e. vectors, choose $\alpha = 1, \beta = 0$ to get

$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and choose $\alpha = 0, \beta = 1$ to get $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Clearly, $\vec{v}_1$ & $\vec{v}_2$ are linearly independant.

Thus, the eigenspace corresp. to $\lambda_1, \lambda_2 = -1$ is span $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← a plane in $\mathbb{R}^3$

or $\left\{ \alpha \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \alpha \beta \in \mathbb{R} \right\}$ $\alpha, \beta \in \mathbb{R}$

A basis for this e. space is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim = 2$.

(ii) $\lambda_3 = 8 = \ldots \Rightarrow \vec{v}_3 = \alpha \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$, $\alpha \neq 0$, e. space is span $\left\{ \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \right\}$ a line in $\mathbb{R}^3$.

A basis is $\left\{ \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim = 1$

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* Eigenvalues & Eigenvectors $A\vec{v}=\lambda\vec{v}$ $A\in\mathbb{R}^{n\times n}$ $\vec{v}\in\mathbb{R}^n$, $\vec{v}\neq\vec{0}$

λ is a scalar. It can be real or generally complex, i.e, $\lambda=\alpha\pm i\beta$

Def. The trace of $A=[a_{ij}]_{n\times n}$ is defined by $tr(A)=\sum_{i=1}^{n} a_{ii}$

Remarks: (i) For an $n\times n$ matrix A, the characteristic poly $p(\lambda)=|A-\lambda I|$ is of degree n.

(ii) $|A-\lambda I| = |(-1)(\lambda I - A)| = (-1)^n|\lambda I - A|$

(iii) The total number of eigenvalues of $A_{n\times n}$ is n. The eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$. They might be repeated.

(iv) The constant term in $p(\lambda)=|A-\lambda I|$ is $|A|=\prod_{I=1}^{n}\lambda_i$

(v) $tr(A)=\sum_{i=1}^{n}\lambda_i$

(vi) The eigenvalues of A^k are $\lambda_1^b, \lambda_2^b, \dots, \lambda_n^k$

Proof: Take $k=2$

- Let λ be an e. value of A. Then $A\vec{v}=\lambda\vec{v}$ So $A(A\vec{v}=\lambda\vec{v})$ gives $A^2\vec{v}=A\lambda\vec{v}=\lambda(A\vec{v})=\lambda(\lambda\vec{v})$

That is, $A^2\vec{v}=\lambda^2\vec{v}$ So by definition, λ^2 is an eigenvalue of A^2.

(vii) The matrix A is invertible iff all eigenvalues are non-zero.
Proof H.W.

(viii) If A is invertible, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$ Proof: H.W

(ix) If λ is an eigenvalve of A, the λ is also an eigenvalve of A^T.

Proof: $|A-\lambda I| = |(A-\lambda I)^T| = |A^T-(\lambda I)^T| = |A^T-\lambda I|$

Same poly.

OR: $A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$ Take transpose => $A^T\vec{v} = \lambda(I\vec{v})^T = \lambda I\vec{v} = \lambda\vec{v}$

Theorem: Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are or distinct e. valves of $A_{n\times n}$. (i.e, $\lambda_i \neq \lambda_j$ for all $i \neq j$) with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$. Then the set of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is linearly independant.

11.2.2 Application to $\overbrace{\text{elementary}}^{\text{single singulor values.}}$ singularities in phase plane.

Consider the system of 1st-order DE: $\begin{aligned} x' &= ax+by \\ y' &= cx+dy \end{aligned}$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

or $\vec{x} = A\vec{x}$

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11.2.2 Application to Elementary Singularities in phase plane

Consider the system of 1st-order DEs

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{—— (1)}$$

$\vec{x}' = A\vec{x}$ $\quad A_{2\times 2}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

· Def (Equilibrium Solution)

An equilibrium solution is a sol. that does not change within time.

So,

· an equil. sol is a constant sol., say $\vec{x}_A = \begin{pmatrix} x_* \\ y_* \end{pmatrix}$

· also, the derivative of this solution is $\vec{x}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

· Finding the equil. sol of system (1)

Set $\begin{cases} x' = 0 \\ y' = 0 \end{cases}$ Then, solve the algebraic homogenous sys. $\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$

or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ —— (2) for $\begin{pmatrix} x \\ y \end{pmatrix}$

* An equil point is written as an order pair (x_*, y_*)

· Equil. point is also called a singular point,

$$\frac{x'}{y'} = \frac{0}{0}$$

We are interested in investigating how the sol. of (1) behaves near a singular point (of (1)) when time evolves.

* We need to find the eigen values and eigenvectors of A.
* The solution of sys (1) is given by

$$\vec{x} = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t}$$

where · λ_1 & λ_2 are the e. values of A.
$\vec{V}_1$ & $\vec{V}_2$ " " e. vectors of A.

· C_1 & C_2 are arbitrary constants that can be determined if initial conditions are given.

· Stability of singular points

Roughly speaking by stability we mean if the sol. of ① starts close to the singular point, it stays close to that point, or it goes to that point.

· We can determine the stability of singular points based on the sign of λ_1 & λ_2. (or λ's).

· We have 3 cases:

Case I: λ's are real of same sign
(i) λ's $< 0 \Rightarrow$ stable node
(ii) λ's $> 0 \Rightarrow$ unstable node

Case II: λ's are real and of opposite signs.
$\Rightarrow$ singular point is saddle.

Case III: λ's are complex numbers say $\lambda = \alpha \pm i\beta$
(i) $\alpha < 0 \Rightarrow$ singular pt is stable focus (or spiral).

(ii) $\alpha > 0 \Rightarrow$ singular pt is unstable focus (or spiral)

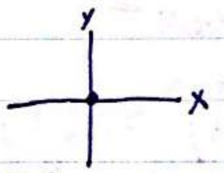
(iii) $\alpha = 0 \Rightarrow$ singular pt is center.

Ex: Consider the system $\begin{cases} x' = x + 3y \\ y' = 5x + 3y \end{cases}$ — ④

Singular pt. Set $\begin{cases} x' = 0 \\ y' = 0 \end{cases}$ Solve the system $\begin{matrix} x + 3y = 0 \\ 5 + 3y = 0 \end{matrix}$ for x_*, y_*

$\Rightarrow A = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$, $|A| \neq 0$ So the sys. has unique trivial (or zero) sol.

That is $(x_*, y_*) = (0,0)$



E. values $|A - \lambda I| = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 6$ $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda) = 15 = \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 6$

i.e λs are real with opposite signs, So the singular pt $(0,0)$ is saddle

· Eigenvectors $\lambda_1 = -2 \Rightarrow$ Solve $(A - (-2)I)\vec{v} = \vec{0}$

We get $\vec{v}_1 = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Choose $\alpha = 1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\lambda_2 = 6$, Solve $(A - 6I)\vec{v} = 0 \Rightarrow \vec{v}_2 = \beta\begin{pmatrix} \frac{3}{5} \\ 1 \end{pmatrix}$ Choose $\beta = 5$

$\Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ So that the sol. of ④ is $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

The eigenvalues and eigenvectors are plugged into equation. The next part is just setting C1 and C2 to zero.

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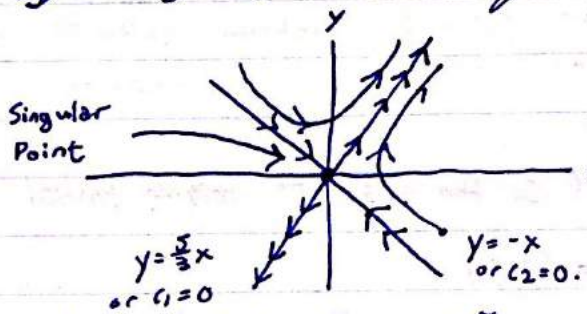
Stability: Saddle singular point

Geometrically: • If $C_1 = 0$, then $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_2 3e^{6t} \\ C_2 5e^{6t} \end{pmatrix}$ or $x(t) = C_2 3e^{6t}$, $y(t) = C_2 5e^{6t}$.

$\Rightarrow y(t) = \cancel{\frac{3}{5}x(t)}\ \frac{5}{3}x(t)$ ← a line equation in $\mathbb{R}^2$



phase plane or "portrait"

You find these x(t) and y(t) values from setting C's to zero but what do they represent?

If $C_2 = 0$, then $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-2t} \\ -C_1 e^{-2t} \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$

$\Downarrow$

~~y=x~~ $y = -x$

11.3 Symmetric Matrices

$A^T_{n \times n} = A$

11.3.1 Eigenvalue Problem $A\vec{x} = \lambda\vec{x}$

Th. 11.3.1 (Real Eigenvalues)

• If A is symmetric, then all of its eigenvalues are real.

the converse of this is not ~~time~~ true.

E.x. $A = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda = -5$.

Both real, but A is not symmetric.

Th. 11.3.2 (Dimension of eigenspace).

• If an eigenvalue of a symmetric matrix A is of multiplicity k, then the eigenspace corresponding to λ is of dimension k.

Th. 11.3.3 (Orthogonality of Eigenvectors)

- If A is symmetric, then the corresponding eigenvectors ~~eigenvectors~~ to distinct eigenvalues are orthogonal.

Proof: Next Lecture

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Proof: Let $\vec{x}_j$ and $\vec{x}_k$ be eigenvectors corresponding to distinct eigenvalues λ_j and λ_k, respectively. Then $A\vec{x}_j = \lambda_j \vec{x}_j$ (1a) and $A\vec{x}_k = \lambda_k \vec{x}_k$ (1b)

Remember that $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$ and $(AB)^T = B^T A^T$

Now, pre-dotting $\vec{x}_k$ into each side of (1a) and post-dotting $\vec{x}_j$ into each side of (1b) gives us $\vec{x}_k\cdot(A\vec{x}_j) = \vec{x}_k\cdot(\lambda\vec{x}_j)$, $(A\vec{x}_b)\cdot\vec{x}_j = (\lambda_k\vec{x}_k)\cdot\vec{x}_j$

$$\Rightarrow \vec{x}_k^T(A\vec{x}_j) \overset{2a}{=} \lambda_j \vec{x}_k^T x_j \;,\quad (A\vec{x}_k)^T\vec{x}_j = \lambda_k \vec{x}_k^T\vec{x}_j$$

$$, \Rightarrow \vec{x}_k^T A^T \vec{x}_j = \lambda_k \vec{x}_k^T \vec{x}_j$$

Matrices Eigenval

$$, \Rightarrow \underset{?}{\underline{\underline{\vec{x}_k^T A \vec{x}_j}}} = \lambda_k \vec{x}_k^T \vec{x}_j \quad (2b)$$

From 2a and 2b $\lambda_j \vec{x}_k^T x_j = \lambda_k \vec{x}_k^T \vec{x}_j \Rightarrow \overbrace{(\lambda_j - \lambda_k)}^{\neq 0} \vec{x}_k^T \vec{x}_j = 0$

Since $\lambda_j \neq \lambda_k$, then $\vec{x}_k^T\vec{x}_j = \vec{x}_k\cdot\vec{x}_j = 0$ What's going on here??

Ex. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$
- Here $A^T = A$
- E. values $p(\lambda) = |A - \lambda I| = \dots$ (HW).

$= -(\lambda-4)(\lambda-1)^2 = 0 \Rightarrow \lambda_1 = 4 \quad \lambda_2 = \lambda_3 = 1$

Eigenvectors: (H.W.)

$$\vec{x}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\beta - \gamma \\ \gamma \\ \beta \end{bmatrix}$$

• A symmetric, then

(i) in accordance with Th. 11.3.1, λ's are real

(ii) in accordance with Th. 11.3.2, $\lambda_1 = 4$ is of multiplicity 1 and its eigenspace span$\{[1,1,1]^T\}$ is of dim 1.

$\lambda_2 = \lambda_3 = 1$ (or $\lambda_2 = 1$) is of multiplicity of 2. and its eigenspace span $\{[-1,0,1]^T, [-1,1,0]^T\}$ is of dim 2.

Here, span $\{[1,1,1]^T\}$ is a line (in $\mathbb{R}^3$) passing through the origin and span $\{[-1,0,1]^T, [-1,1,0]^T\}$ is a plane (in $\mathbb{R}^3$) passing through the origin.

They come from the same eigenvalue.

(iii) In accordance with Th. 11.3.3 $\vec{x}_1 \cdot \vec{x}_2 = 0$ for all ~~the~~ choices of α, β, γ. That is, the line L is orthogonal to plane P.

(remember L & P come from distinct e. values.

Notice that the eigenvectors $[-1,0,1]^T$ and $[-1,1,0]^T$ are l. ind. and a basis for the eigenspace P, $\{[-1,0,1]^T, [-1,1,0]^T\}$ is not orthogonal. In fact, this does not violate Th. 11.33 because the vectors come from same eigenvalue.

In fact, one can still get orthogonal basis for a different choices of β, γ. How?

Let $\vec{x}_2 = \beta \vec{w}_1 + \gamma \vec{w}_2$ where $\vec{w}_1 = [-1,0,1]$ $\beta = 1, \gamma = 0$

What's going on here????

and $\vec{w}_2 = \beta\begin{bmatrix}-1\\0\\1\end{bmatrix} + \gamma\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\beta, \gamma = ?$

We want to choose β & γ such that $\vec{w}_1 \cdot \vec{w}_2 = 0$

This implies that $\vec{w}_1 \cdot \vec{w}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-\beta-\gamma\\ \gamma\\ \beta\end{bmatrix} = (\beta+\gamma) + 0\gamma + \beta$
$= 2\beta + \gamma = 0.$

Choose $\beta = 1$. Then $\gamma = -2$.

Thus, $\vec{w}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ The eigenvectors $\vec{w}_1$ & $\vec{w}_2$ constitute an orthogonal basis for eigenspace. corresp. $\lambda_2 = 1$

- The 3 e.vectors $\vec{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $(\alpha = 1)$ $w_1 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ as $w_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ constitute an orthogonal basis for the 3-space (or $\mathbb{R}^3$).

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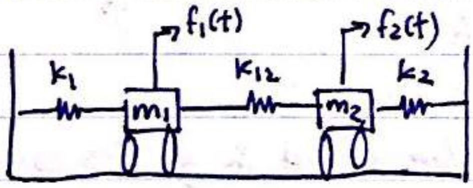
<u>Application</u>

- Consider the free viberation two-mass spring system:



Here
- m_1, m_2 are mass
- k_1, k_2, k_{12} are the stiffness of springs.
- f_1 & f_2 are external (or input) forces.
- x_1 & x_2 are the positions of masses 1 & 2.

By newton's 2nd law, we have:
$$\begin{cases} m_1 x_1'' + (k_1+k_2)x_1 - k_{12}x_2 = f_1(t) \\ m_2 x_2'' - k_{12}x_1 + (k_2+k_{12})x_2 = f_2(t) \end{cases} \quad (1)$$

- Let $m_1 = m_2 = k_1 = k_2 = k_{12} = 1$
- Let $f_1(t) = f_2(t) = 0 \ \forall t$

Then, ① becomes $x_1'' + 2x_1 - x_2 = 0$ — ②
$x_2'' - x_1 + 2x_2 = 0$

Assume that we are interested in the oscillatory behaviour of the sys. That is mathematically, we assume

$$\begin{cases} x_1(t) = q_1 \sin(\omega t + \phi) \\ x_2(t) = q_2 \sin(\omega t + \phi) \end{cases} \quad \boxed{\text{or}} \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \sin(\omega t + \phi)$$

where q_i's are the amplitudes, ω is the frequency and ϕ is the phase angle. — ③

Substituting ③ into ② gives $\begin{cases} -\omega^2 q_1 + 2q_1 - q_2 = 0 \\ -\omega^2 q_2 - q_1 + 2q_2 = 0 \end{cases}$ — ④

or in a matrix form, $\underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}_{\vec{q}} = \omega^2 \underbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}_{\vec{q}}$ or $A\vec{q} = \omega^2 \vec{q}$

symmetric

An eigenvalue problem.

That is, $\lambda = \omega^2$ is the e. value of A & $\vec{q}$ its corresponding eigenvector.

· E. values: $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \dots = (\lambda - 1)(\lambda - 3) = 0$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3$

$\omega_1 = \sqrt{\lambda_1} = 1 \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{3}$

E. vectors: $\lambda_1 = 1 \Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \alpha \neq 0$

Then $\vec{x}_1(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(t + \phi_1)$

$\omega = 1$

Arbitrary Constants.

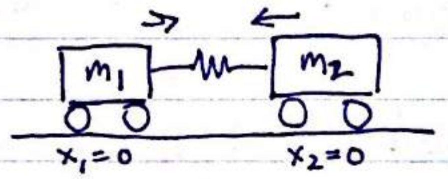
$\lambda_2 = 3 \Rightarrow \dots \Rightarrow \vec{v}_2 = \beta\begin{pmatrix}1\\-1\end{pmatrix}$ and $\vec{x}_2(t) = \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \underbrace{\beta \begin{pmatrix}1\\-1\end{pmatrix}\sin(\sqrt{3}t + \phi_2)}_{\text{Arbitrary Constants.}}$

Then the general sol of ② is $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$

$= \alpha\begin{pmatrix}1\\1\end{pmatrix}\sin(t+\phi_1) + \beta\begin{pmatrix}1\\-1\end{pmatrix}\sin(\sqrt{3}t+\phi_2)$

or $\begin{cases} x_1(t) = \alpha\sin(t+\phi_1) + \beta\sin(\sqrt{3}t+\phi_2) \\ x_2(t) = \alpha\sin(t+\phi_2) + \beta\sin(\sqrt{3}+\phi_2) \end{cases}$

For instance, if $x_1(0) = 1$, $x_1'(0) = 0$ $x_2(0) = -1$ $x_2'(0) = 0$.



then we get $\alpha = 0$, $\beta = 1$, $\phi_2 = \frac{\pi}{2}$, ϕ_1 is not important as $\alpha = 0$.

Thus, $x_1(t) = \sin(\sqrt{3}t + \frac{\pi}{2}) = \cos\sqrt{3}t$.

$x_2(t) = -\sin(\sqrt{3}t + \pi/2) = -\cos\sqrt{3}t$

Re

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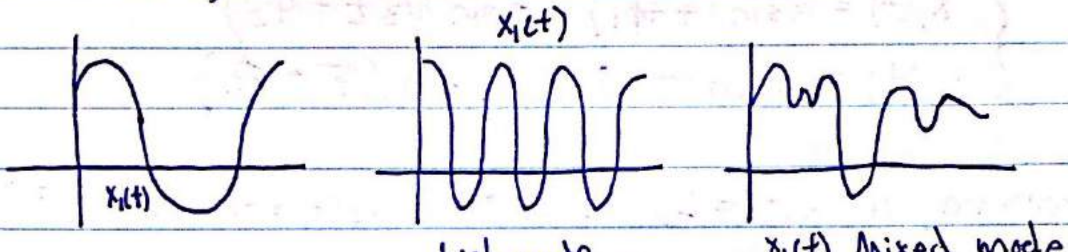
Remarks

(i) Each ~~eigenspace~~ eigenpair defines a vibration mode; the eigenvalues give the vibration frequencies ($w = \sqrt{x}$) and eigenvectors give the mode shape or configuration. $w_1 = 1$ is the low mode and $w_2 = \sqrt{3}$ is the high mode.

For instance,



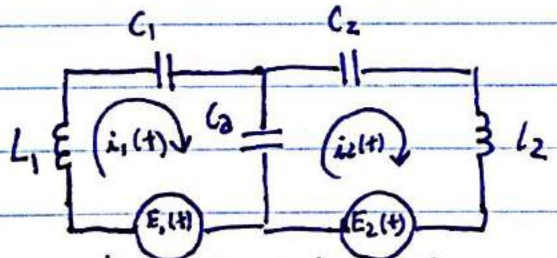
low mode high mode $x_1(t)$ Mixed mode

(ii) The frequencies are called the natural frequency (natural due to free or unforced vibration $f_1(t) = f_2(t) = 0$)

(iii) $\vec{q}_1 \cdot \vec{q}_2 = 0$ that is, the eigenvectors are orthogonal.

Electrical Circuit

* An analogue to a mass-spring system - last example.



Loop 1: $L_1 i_1' + \frac{1}{C_1}\int i_1(t)\,dt + \frac{1}{C_{12}}\int (i_1 - i_2)\,dt = E_1(t)$

2: $L_2 i_2' + \frac{1}{C_2}\int i_2\,dt + \frac{1}{C_{12}}\int (i_2 - i_1)\,dt = E_2(t)$

$$\Rightarrow \begin{cases} L_1 i_1'' + \left(\frac{1}{C_1} + \frac{1}{C_{12}}\right) i_1 - \frac{1}{C_{12}} i_2 = E_1'(t) \\ L_2 i_2'' + \left(\frac{1}{C_2} + \frac{1}{C_{12}}\right) i_2 - \frac{1}{C_{12}} i_1 = E_2'(t) \end{cases}$$

or $\begin{cases} i_1'' + \frac{1}{L_1}\left(\frac{1}{C_1} + \frac{1}{C_{12}}\right) i_1 - \frac{1}{L_1 C_{12}} i_2 = \frac{E_1'(t)}{L_1} \\ i_2'' - \frac{1}{L_2 C_{12}} i_1 + \frac{1}{L_2}\left(\frac{1}{C_2} + \frac{1}{C_{12}}\right) i_2 = \frac{E_2'(t)}{L_2} \end{cases}$

Take $L_1 = L_2 = C_1 = C_2 = C_{12} = 1$ ~~&~~ of $E_1(t) = E_2(t) = 0$.

Then we have $\begin{aligned} i_1'' + 2i_1 - i_2 = 0 \\ i_2'' - i_1 + 2i_2 = 0 \end{aligned}$ * Assume oscillatory behaviour

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \omega^2 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$ same problem.

~~Hence we have~~ How did he get those eqns from the two he had?

- Matrix Similarity (or Similar Matrices)

↳ Suppose that we have $A\vec{x} = \vec{y}$ — ① when $A \in \mathbb{R}^{n\times n}$

Set $\vec{x} = Q\vec{\tilde{x}}$ and $\vec{y} = Q\vec{\tilde{y}}$ ② where Q is an $n\times n$ invertible matrix.

Then ① becomes $A(Q\vec{\tilde{x}}) = Q\vec{\tilde{y}}$ — ③

Pre-multiplying ② by Q^{-1} gives the equivalent system

$Q^{-1}AQ\vec{\tilde{x}} = \vec{\tilde{y}}$ or $\tilde{A}\vec{\tilde{x}} = \vec{\tilde{y}}$, $\tilde{A} = Q^{-1}AQ$

Easier to work with

Def. Given any invertible matrix Q, matrices A and $\tilde{A} = Q^{-1}AQ$ are said to be similar.

Exercise: Show that if $\tilde{A}$ & A are similar then they have the same characteristic poly's hence the same eigenvalues.

Ex. $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$, $Q=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ Then $Q^{-1}AQ=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}=\tilde{A}$

$B=\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}$, $Q=\begin{bmatrix}-1/2 & -1 & 2\\ 1 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}$, $Q^{-1}BQ=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 8\end{bmatrix}$

11.4 Diagonalization

Consider the system. $\vec{x}'=Ax$, $x\in\mathbb{R}^1$, $A\in\mathbb{R}^{n\times n}$ — ①

If A is not diagonal, then system ① is coupled

Our aim is to convert ① into an uncoupled sys. by diagonalizing A, if possible.

How? Answer: Let $\vec{x}=Q\vec{y}$, where Q is an $n\times n$ invertible constant matrix.

Then $\vec{x}'=(Q\vec{y})'=Q\vec{y}'$ ⇒ that, ① gives $Qy'=AQ\vec{y}$

$\Rightarrow \vec{y}'=Q^{-1}AQ\vec{y}$ — ②

Thus, given A matrix, the idea is to try to find Q so that

$$Q^{-1}AQ=D$$

is diagonal. This leads to that system ② will be uncoupled.

Def. Let A be an $n\times n$ matrix. If there exists an invertible matrix Q and diagonal matrix D such that $Q^{-1}AQ=D$, then we say A is diagonalizable and that Q diagonalize A.

* The matrix Q is called the modal matrix of A.

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*** Vandermonde determinant**

if $n=2$ $\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_1 - \lambda_2$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} = (-1)^{\frac{n(n-1)}{2}} \prod_{\substack{j:k \\ j<k\le n}} (\lambda_j - \lambda_k)$$

$n=3$

$$\left|\quad\right| = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$$

Th. 11.4.1 (Diagonalization)

- Let A be $n \times n$ matrix

(i) A is diagonalizable iff it has n l. ind. eigenvectors.

(ii) If A has n l. ind eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_3$ and we make these the columns of Q so that $Q = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$, then $Q^{-1}AQ = D$ is diagonal~~ized~~ and the jth diagonal element of D is the jth eigenvalue of A.

Th. 11.4.2 (Distinct Eigenvalues / L. ind eigenvectors)

- If an $n \times n$ matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are l ind.

Proof: We need to show that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ — (1)

holds only if $c_1 = c_2 = \ldots = c_n = 0$

Multiplying (1) by A and ~~getting~~ noting that $A\vec{v}_j = \lambda_j \vec{v}_j$

We have $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \dots + c_n\lambda_n\vec{v}_n = \vec{0}$

Repeating the process gives $c_1\lambda^2\vec{v}_1 + \dots c_2\lambda_2^2\vec{v}_2 + \dots + c_n\lambda_n^2\vec{v}_n = \vec{0}$

$c_1\lambda_1^{n-1}v_1 + c_2\lambda_2^{n-1}\vec{v}_2 + \dots + c_n\lambda_n^{n-1}\vec{v}_n = 0$

or in a matrix form $\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & & \lambda_n^2 \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1\vec{v}_1 \\ c_2\vec{v}_2 \\ \vdots \\ c_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The determinant of the coefficient is a vondermonde determinant which is nonzero if λ_j's are distinct. So that the ~~eqn~~ sys. has a unique trivial sol.

$$c_1\vec{v}_1 = \vec{0},\ c_2\vec{v}_2 = \vec{0} \dots c_n\vec{v}_n = \vec{0}$$

Since $\vec{v}_j$'s are eigenvectors ($\neq \vec{0}$ by definition), it follows that $c_1 = c_2 = \dots = c_n = 0$ Thus, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are l. ind.

Th. 11.4.3 (Diagonizability)

· If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

~~Proof~~ * The converse of this theorem is not necessarily true.

$$\overset{A}{\overbrace{}}$$

<u>Ex.</u> Consider the sys. $\begin{cases} x' = \qquad\quad y \\ y' = -16x - 10y \end{cases}$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -16 & -10 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \lambda(A) = -8, -2 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -8 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

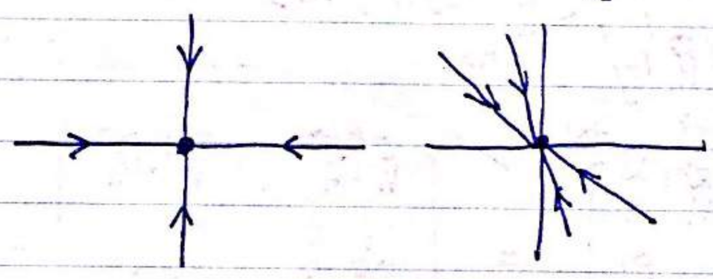
So that

$$Q = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ -8 & -2 \end{bmatrix}.$$

Then $\tilde{A} = \tilde{Q} A Q = \begin{bmatrix} -8 & 0 \\ 0 & -2 \end{bmatrix}$

and $\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ or $\begin{matrix} \vec{x}' = -8\vec{x} & \tilde{x} = A_1 e^{-8t}. \\ \vec{y}' = -2\vec{y} & \tilde{y} = A_2 e^{-2t} \end{matrix}$

Here, the eigenvectors of $\tilde{A}$ are $\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ & $\underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ = canonical basis is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



Canonical Basis

Regular Basis $\left\{ \begin{bmatrix} 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

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Th. 11.4.4 (Symmetric Matrices and Diagonalization)

Every symmetric matrix is diagonalizable

Proof: Th. 11.4.1 states that A is diagonalizable iff it has ~~a~~ n l. ind. eigenvectors and Th. 11.3.4 states that every symmetric matrix has n orthogonal [and hence] l. ind eigenvectors

* Suppose that for a symmetric matrix A, we use the $\boxed{\text{normalized}}$ eigenvectors of A to form its model matrix Q of A so that $Q = [\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}]$

Then, $Q^T Q = \underbrace{\begin{bmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{bmatrix}}_{1\times n} \underbrace{[\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]}_{n\times 1}$

because $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$

$$= \begin{bmatrix} \vec{v_1}^T\vec{v_1} & \cdots & \vec{v_1}^T\vec{v_n} \\ \vdots & & \\ \vec{v_n}^T\vec{v_1} & \cdots & \vec{v_n}^T\vec{v_n} \end{bmatrix}_{1\times 1} = \begin{bmatrix} \hat{v_1}\cdot\hat{v_1} & \cdots & \hat{v_1}\cdot\hat{v_n} \\ \vdots & & \vdots \\ \hat{v_n}\cdot\hat{v_1} & \cdots & \hat{v_n}\cdot\hat{v_n} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I_n$$

and $QQ^T = I_n$ (H.W)

Thus $\boxed{Q^{-1} = Q^T}$ — ⊛

guaranteed by <u>normalized</u> eigenvectors

<u>Remarks</u>

① The relation in ⊛ has nothing to do with $\vec{v_j}$'s being eigenvectors. It's true for any square matrix Q having columns that are orthonormal. In this case, Q is said to be an <u>orthogonal matrix.</u>

② If A is symmetric then $Q^{-1}AQ = D$ is diagonal whether or not Q has columns. that are normalized.

⊛ We'll be always interested in normalized eigenvectors.

*Jordan Normal Form (or Jordan Form)

If $A_{n\times n}$ ~~cannot~~ is not diagonalizable, it can be triangular.

In this case, a "generalized modal matrix" P can be found for A so that $P^{-1}AP = J$ is triangular. J is called "Jordan form".

It is an upper triangular matrix with main diagonal contains the eigenvalues of A and zeros above the main diagonal — except for 1's immediately above one or more diagonal elements.

Ex: Consider a 6x6 matrix A which has eigenvalues λ's & eigenvectors as follows (i) $\lambda_1, \lambda_1, \lambda_1$ and only one linearly independant eigenvector $\vec{u_1}$.

(ii) λ_2 is single with an eigenvector $\vec{v_1}$

(iii) λ_3, λ_3, and only one l ind. e. vector

In this example, there are only ③ l ind. e. vectors. So to form P we need 3 more l. ind. e. vectors.

How to get those vectors

Answer:

(i) $\lambda_1, \lambda_2, \lambda_3, \vec{u}_1 \Rightarrow$ find $\vec{u}_2$ & $\vec{u}_3$

$$\left.\begin{array}{ll} (A-\lambda_1 I)\vec{u}_1 = \vec{0} & (A\vec{u}_1 = \lambda_1 \vec{u}_1) \\ (A-\lambda_1 I)\vec{v}_2 = \vec{v}_1 & \leftarrow \text{solve for } \vec{u}_2 \\ (A-\lambda_1 I)\vec{v}_3 = \vec{v}_2 & \leftarrow \text{solve for } \vec{v}_3 \end{array}\right\} \Rightarrow \vec{u}_1, \vec{u}_2, \vec{u}_3 \text{ are l. ind vectors}$$

~~(iii) λ_2~~ $\vec{u}_2$ & $\vec{v}_3$ are called <u>generalized</u> eigenvectors.

(ii) $\lambda_2, \vec{V}_2$ ☺

(iii) $\lambda_3, \lambda_3, \vec{w}_1 \Rightarrow$ find $\vec{w}_2$

$$(A-\lambda_3 I)\vec{w}_1 = \vec{0}$$

$(A-\lambda_3 I)\vec{w}_2 = \vec{w}_1$ * $\vec{w}_1$ & $\vec{w}_2$ are l. independant

* $\vec{w}_2$ is called generalized eigenvectors

Now form the "generalized" model matrix P.

$$P = \{\vec{v}_1, \vec{u}_2, \vec{v}_3, \vec{V}_1, \vec{w}_1, \vec{w}_2\}_{6\times 6}$$

J_2

Then $P^{-1}AP = J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$ J_3

J_1

* $J_1 (3\times3)$, $J_2 (1\times1)$, $J_3 (2\times2)$ are called Jordan blocks.