1 Short Questions

Part (a) (5 pts) Give an asymptotically tight solution to the following recurrence. No justification required.

$$T(n) = T(n/4) + T(3n/4) + n^3 \qquad \text{for } n \geq 4 \text{ and } T(n) = 1 \text{ for } n = 1, 2, 3.$$

Solution: Building out the recursion tree we find the work at level k is $(\frac{7}{16})^k n^3$. This results in a decreasing geometric series, which converges to $\boxed{\Theta(n^3)}$ ■

Part (b) (5 pts) Let $G = (V, E)$ be a *directed* graph. Given an edge $e = (u, v)$ describe a linear time algorithm to find the shortest cycle in G that contains e or report that there is no cycle containing e. The length of the cycle is simply the number of edges in it.

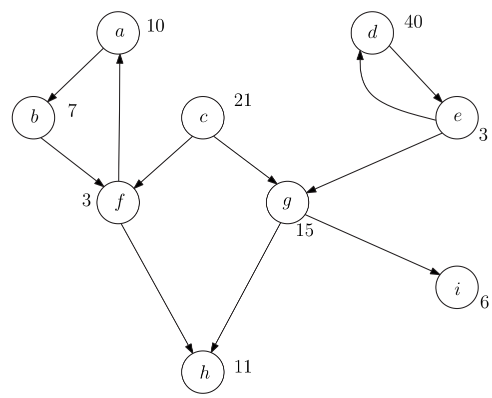
Solution: The shortest directed cycle containing e consists of e along with the shortest path $v \rightsquigarrow u$. Thus to find this shortest cycle (or report that none exists), we simply run *breadth-first search* from v to find the shortest path $v \rightsquigarrow u$ in $\boldsymbol{O(V + E)}$ ***time***. ■

Rubric: 10 points.

- 5 points for part (a): all or nothing.
- 5 points for part (b):
  - 4 points for the algorithm. 2 points for a superlinear algorithm.
  - 1 point for time analysis
  - −1 for a minor error, −2 for a major error.

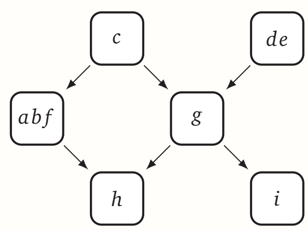
2 Directed Graphs

Let $G = (V, E)$ be a directed graph. Each vertex $v \in V$ has a weight $w(v)$ associated with it. Given a vertex $s \in V$ let $\alpha(s) = \min\{w(v) \mid s \text{ can reach } v \text{ in } G\}$ be the minimum weight among the weights of all nodes that s can reach in G. In the figure below $\alpha(a) = 3$ and $\alpha(g) = 6$.



- List the strongly connected components in the example graph and draw the meta-graph of G.

Solution: The strongly connected components are $\boxed{\{a, b, f\}, \{c\}, \{d, e\}, \{g\}, \{h\}, \{i\}}$. The meta-graph is shown below:



■

- Suppose G is DAG. Describe a linear-time algorithm that computes $\alpha(s)$ for *every* $s \in V$.

Solution: Let $v_1, \ldots, v_n$ be a topological ordering of the vertices of G. For $1 \le i \le n$, $\alpha(v_i)$ satisfies the recurrence

$$\alpha(v_i) = \min\left\{ w(v_i), \min_{v_i \to v_j \in E} \left\{\alpha(v_j)\right\} \right\}.$$

Note that when v_i is a sink, the second term is a min over an empty set, which we interpret to be ∞, so no explicit base case is necessary. Since $\alpha(v_i)$ only depends on $\alpha(v_j)$ where $i < j$ (due to the topological ordering), we can memoize this into a one-dimensional array, in order from n down to 1 (i.e., in *reverse topological order*). We return *the entire memoization structure*, since we are asked to compute $\alpha(s)$ for all $s \in V$. The running time is $\boldsymbol{O(V + E)}$.

```
ComputeAlphalpha(G):
  v_1, ..., v_n ← a topological ordering of the vertices of G
  for i from n down to 1
      α[v_i] ← w(v_i)
      for each edge v_i→v_j
          α[v_i] ← min{α[v_i], α[v_j]}
  return α
```

■

- Extend the algorithm in the previous part to the case of a general directed graph. If you cannot figure out the previous part, you can use it as a black box in this part.

Solution: Note that for a strongly-connected component S, $\alpha(s)$ is the same for all $s \in S$, since the set $\operatorname{rch}(G,s) = \{v \in V \mid s \text{ can reach } v\}$ is the same for all $s \in S$. Moreover, for any other vertex u, if u can reach any vertex of S than also u can reach the minimum-weight vertex of S.

Thus we take the strongly-connected component metagraph G^{SCC}, and for each metavertex $\bar{v}$, we set $w(\bar{v}) = \min_{u \in \operatorname{comp}(\bar{v})} w(u)$, where $\operatorname{comp}(\bar{v})$ is the strongly-connected component associated with $\bar{v}$. We then run the algorithm for the previous part on G^{SCC} with these vertex weights, and then for each vertex u in G, we will set $\alpha(u)$ to be the α-value found for the metavertex corresponding to the strongly-connected component containing u.

Forming the metagraph G^{SCC} and computing its weights takes $O(V+E)$ time, as does running the linear-time algorithm for the previous part, for a total of $\boldsymbol{O(V+E)}$ ***time***. ■

Rubric: 10 points.

- 2 points for the first part: 1 point for the vertices (i.e. the list of strongly connected components), 1 point for the edges.
- 5 points for the second part
  - 2 points max for a super-linear algorithm.
  - Scaled Dynamic programming rubric if appropriate
  - Otherwise, 4 points for the algorithm, 1 point for time analysis. −1 for minor errors, −2 for major errors.
- 3 points for the last part: 2 points for the algorithm, 1 point for time analysis. −1 for minor errors.

3 Many points further away

Let $P = \{p_1, p_2, \ldots, p_n\}$ be n points in the 2-dimesional plane where each point p_i is specified as a tuple (x_i, y_i): x_i is the x-coordinate value and y_i is the y-coordinate value for p_i. Given two points $p = (a, b)$ and $q = (c, d)$ in the plane, the L_2 distance (also called the Euclidean distance) between p and q is defined as $\sqrt{|a-c|^2 + |b-d|^2}$ (note that distance is always non-negative). Given the set P of n points and a point $q = (a, b)$, describe an $O(n)$ time algorithm to find a point $p_i \in P$ such that there are at least $n/5$ points in P that are at least as far away from q as p_i. An $O(n \log n)$ time algorithm will get you half the points (you may want to think about the slower algorithm first).

Solution (If it looks like Selection, and quacks like Selection...): We need to return a point in P whose distance from q is at most the $n/5$-th largest (i.e., $4n/5$-th smallest) distance from q. We do this by computing all of the distances from q to points in P, letting r be the distance of rank $4n/5$, and returning some point whose distance from q is smaller than r. Note that since distances are non-negative, it is equivalent to work with *squared* distances to avoid needing to take square roots.

```
THRESHOLDPOINT(P[1..n], q = (a, b)):
  allocate array A of length n
  for i ← 1 to n
      A[i] ← (x_i − a)^2 + (y_i − b)^2
  r ← SELECT(A[1..n], 4n/5)
  for i ← 1 to n
      if A[i] ≤ r
          return p_i
```

Using a linear-time selection algorithm (as shown in lecture), the algorithm takes ***O(n) time***. ■

Solution (...there might be a cleverer solution): The closest point to q *trivially* satisfies the requirement. The following algorithm scans through the list and finds this element in ***O(n) time***:

```
CLOSESTPOINT(P[1..n], q = (a, b)):
  minDist ← ∞
  closest ← NULL
  for i ← 1 to n
      dist ← (x_i − a)^2 + (y_i − b)^2
      if dist < minDist
          minDist ← dist
          closest ← p_i
  return closest
```

■

Rubric: 10 points.

- 8 points for the algorithm:
  - 5 points max for an $O(n \log n)$ algorithm.
  - 3 points max for a correct $\omega(n \log n)$ algorithm.
  - −1 for minor errors, −2 for major errors.
- 2 points for the time analysis.

4 Division

Let x and y be two positive integers with at most n digits each. Let $x \div y$ denote the maximum integer k such that $yk \leq x$. For instance $341873418723478137 \div 234334234324747 = 1458$. Describe an algorithm to compute $x \div y$ in time polynomial in n and justify its running time. You can assume that adding and subtracting (and hence comparision) of n digit numbers takes $O(n)$ time and that multiplication of two n digit numbers takes $M(n)$ time where $M(n)$ is at most $O(n^2)$. Your running time can be expressed in terms of $M(n)$. For this problem you should assume that single digit operations take $O(1)$ time but not general arithmetic operations; if you use such operations the time should be accounted for. You can assume that x and y are in binary if it helps you.

Solution: Note that $1 \leq x \div y \leq x$, so we can binary search over this space for the correct number. To divide by two, we use a bit-shift operation *shift*, that can be implemented to run in $O(n)$ time by simply copying bits over one by one. (We thus assume that x and y are given in binary.)

```
DivideViaDivideAndConquer(x, y):
  ℓ ← 1
  r ← x
  while ℓ < r
      mid ← shift(r + ℓ)  ⟨⟨= ⌊(r+ℓ)/2⌋⟩⟩
      if mid · y > x
          r ← mid − 1
      else ⟨⟨mid · y ≤ x⟩⟩
          ℓ ← mid
  return ℓ
```

In each iteration of the while loop, we perform at most two additions ($O(n)$ each), one shift ($O(n)$), one multiplication ($M(n)$), and one comparison ($O(n)$), for a total of $O(n + M(n))$ time per iteration. The number of iterations is $O(\log_2(x)) = O(n)$, for a total running time of $\boldsymbol{O(n^2 + nM(n))}$. ■

Rubric: 10 points.

- 8 points for the algorithm:
  - 6 points max for $\omega(n^2 + nM(n))$ time algorithm.
  - 2 points max for a algorithm that is exponential in n (e.g., "linear" scan)
  - −1 for a minor errors, −2 for major errors. A small number of off-by-ones collectively count as one minor error.
- 2 points for the time analysis.

5 Fire Stations

A long straight country road can be modeled as a line starting at 0. The road has n houses at locations $x_1 < x_2 < \ldots < x_n$ on the line. The city wants to build fire stations along the road such that every house is within distance D from some fire station. Fire stations cannot be built at arbitrary locations. The city has figured m potential locations on the road at $y_1 < y_2 < \ldots < y_m$. For simplicity assume that all the x and y values are distinct. The cost of building a fire station at location y_j is c_j. Describe an efficient algorithm that minimizes the total cost of building the fire stations with the constraint that each house is within a distance D of some fire station. Note that it is relatively easy to check whether a feasible solution exists.

Solution:

- We will add sentinel fire stations at locations $y_0 < x_1$ and $y_{m+1} = \infty$, each of cost ∞, and a sentinel house at location $x_{n+1} = \infty$. Let $\mathit{MinFire}(i, j)$ denote the minimum cost to cover the houses at locations $x_j, \ldots, x_n$ using potential fire stations at locations $y_i, \ldots, y_m$.
- We want to return $\mathit{MinFire}(0, 1)$.
- For $0 \le i \le m$, let $\mathit{Right}(i) = \min\left\{ j \;\middle|\; x_j > y_i + D \right\}$ be the index of the first house to the right of y_i that is not covered by y_i.

  Then *MinFire* satisfies the recurrence

$$\mathit{MinFire}(i,j) = \begin{cases} 0 & \text{if } j > n \\ \infty & \text{if } x_j < y_i - D \\ \min\left\{ \begin{array}{c} c_i + \mathit{MinFire}(i+1, \mathit{Right}(i)) \\ \mathit{MinFire}(i+1, j) \end{array} \right\} & \text{otherwise} \end{cases}$$

  (These cases correspond to: not needing to cover any more houses; the j-th house cannot be covered with the remaining stations; and choosing whether or not to open the station at y_i.)
- $\mathit{MinFire}(i, j)$ depends on entries of the form $(i + 1, k)$. We memoize into a two-dimensional array, for i in descending order in the outer loop and for j in any order in the inner loop.
- $\mathit{Right}(i)$ can be precomputed for $0 \le i \le m$ as follows:

```
KnowYourRights(x, y):
  j ← 1
  for i ← 0 to m
      while x_j < y_i + D
          j ← j + 1
      Right[i] ← j
  return Right
```

  The running time of the precomputation is $O(m + n)$ since within the nested loops, j cannot be incremented more than n times.

  With this precomputation filling each entry of the memoization structure takes $O(1)$ time, for a total of $\boldsymbol{O(mn)}$ ***time***.

For completeness, we include the iterative pseudocode on the next page.

```
ThisBurningDesire(x, y, c):
  Right ← KnowYourRights(x, y)
  for i ← m + 1 down to 0
      for j ← n + 1 down to 1
          if j > n
              MinFire[i, j] ← 0
          else if x_j < y_i − D
              MinFire[i, j] ← ∞
          else
              MinFire[i, j] ← min {c_i + MinFire[i + 1, Right[i]], MinFire[i + 1, j]}
  return MinFire[0, 1]
```

■

Rubric: 10 points. Standard Dynamic programming rubric. No penalty for slower polynomial time algorithms.

6 Shortest Paths

Let $G = (V, E)$ be a directed graph with non-negative edge lengths; $\ell(e)$ denotes the length of edge e. Suppose you have computed the shortest path distance from s to t and are not too happy about it. You have the ability to add one edge to the graph G to reduce the shortest path distance but you have to choose this edge from a given list of edges $E' = \{e_1 = (u_1, v_1), e_2 = (u_2, v_2), \ldots, e_k = (u_k, v_k)\}$ where each of these edges also has its length $\ell(e_i)$ specified to you. Design an algorithm that finds the best edge to add so that the resulting graph has the smallest shortest path distance from s to t. Ideally your algorithm's running time should be $O(k)$ plus the asymptotic time to run Dijkstra's algorithm. Slower algorithms get fewer points but incorrect algorithms get few points if at all.

Solution (Graph Modeling): We will create two copies of the graph, one representing not having taken one of the edges in E', and one representing having taken of them.

- $V^* := V \times \{\text{No}, \textsc{Yes}\}$
- $E^* := \big\{(u,a){\to}(v,a) \bigm| u{\to}v \in E, a \in \{\text{No}, \textsc{Yes}\}\big\} \cup \big\{(u,\text{No}){\to}(v,\textsc{Yes}) \bigm| u{\to}v \in E'\big\}$. For all edges, $\ell^*((u,a){\to}(v,b)) = \ell(u{\to}v)$.
- We need to find the shortest path from (s, No) to $(t, \textsc{Yes})$, and extract the edge in E^* corresponding to some $e_i \in E'$ that this path passes through.
- Since all edge lengths are non-negative, we will run Dijkstra's algorithm from (s, No).
- Building $G^* = (V^*, E^*)$ by brute force takes $O(V^* + E^*) = O(V + E + k)$ time, and running Dijkstra's algorithm on G^* takes $O(E^* + V^* \log V^*) = \boldsymbol{O(k + E + V\log V)}$ ***time***. ■

Solution (Combining Shortest Paths): For each $e_i = u_i{\to}v_i \in E'$, the length of the shortest path going through e_i has length $\mathit{dist}(s, u_i) + \ell(e_i) + \mathit{dist}(v_i, t)$. Thus if we compute $\mathit{dist}(s, v)$ and $\mathit{dist}(v, t)$ for all $v \in V$, we can then iterate over E' to find the edge that minimizes the relevant length.

```
WrinkleInTime(G, E′):
  compute dist(s, u) for all u via Dijkstra's algorithm
  compute dist(v, t) for all v via Dijkstra's algorithm on G^rev
  return arg min_{e_i∈E′} {dist(s, u_i) + ℓ(e_i) + dist(v_i, t)}
```

Finding the correct edge takes $O(k)$ time, so overall the algorithm takes $\boldsymbol{O(k + E + V\log V)}$ ***time***. ■

Rubric: 10 points.

- No penalty for citing Dijkstra's algorithm as $O(E \log V)$ (and thus getting $O(k \log V)$ plus Dijkstra's).
- 8 points max for an $O(kE \log V)$ time algorithm (e.g., overly large graph construction if using graph reduction, or making $O(k)$ calls to Dijkstra's algorithm if combining multiple shortest path computations).
- 6 points max for a polynomial time algorithm slower than $O(kE \log V)$.
- For a graph reduction solution, use the Standard graph reduction rubric.
- For a combining multiple shortest path computations solution:
  - 4 points for finding the relevant shortest paths computations
  - 4 points for combining them in the correct ways
  - 2 point for time analysis in terms of the input parameters
  - −1 for each minor error, −2 for each major error