# Pre-lecture brain teaser

Given $\Sigma = \{0, 1\}$, find the regular expression for the language containing all binary strings with an odd number of 0's

Formulate a **language** that describes the above problem.

# ECE-374 B: Lecture 3 - DFAs

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August 30, 2022

University of Illinois at Urbana-Champaign

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# A simple program

Program to check if an input string $w$ has odd number of 0's

```
int n = 0
While input is not finished
    read next character c
    If (c ≡'0')
        n ← n + 1
endWhile
If (n is odd) output YES
Else output NO
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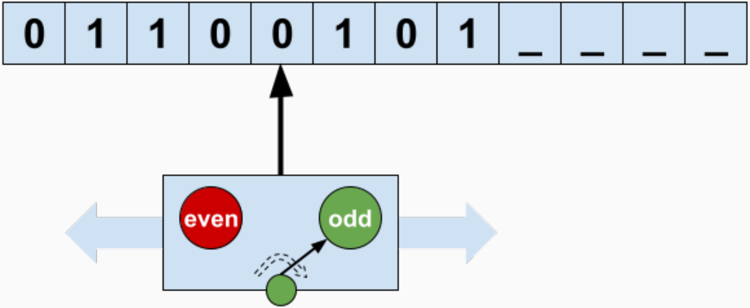
```
bit x = 0
While input is not finished
    read next character c
    If (c ≡'0')
        x ← flip(x)
endWhile
If (x = 1) output YES
Else output NO
```

# Another view




- Machine has input written on a read-only tape
- Start in specified start state
- Start at left, scan symbol, change state and move right
- Circled states are accepting
- Machine accepts input string if it is in an accepting state after scanning the last symbol.

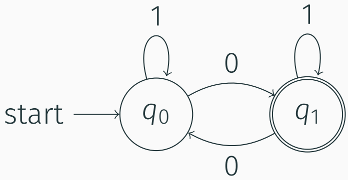
# Deterministic-finite-automata (DFA) Introduction

# DFAs also called Finite State Machines (FSMs)

- The “simplest” model for computers?
- State machines that are common in practice.
  - Vending machines
  - Elevators
  - Digital watches
  - Simple network protocols
- Programs with fixed memory

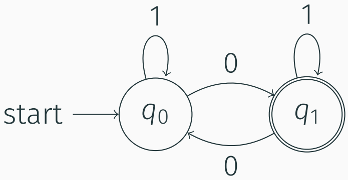
# Graphical representation of DFA

# Graphical Representation/State Machine



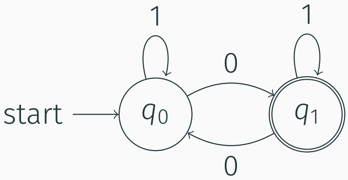

- Directed graph with nodes representing states and edge/arcs representing transitions labeled by symbols in $\Sigma$
- For each state (vertex) $q$ and symbol $a \in \Sigma$ there is exactly one outgoing edge labeled by $a$
- Initial/start state has a pointer (or labeled as $s$, $q_0$ or "start")
- Some states with double circles labeled as accepting/final states

# Graphical Representation



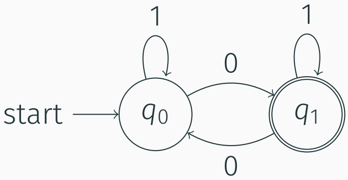

- Where does 001 lead?

# Graphical Representation



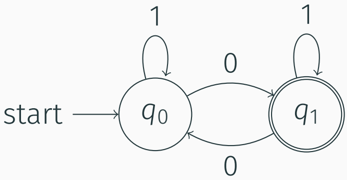

- Where does 001 lead?
- Where does 10010 lead?

# Graphical Representation

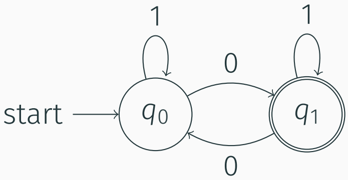



- Where does 001 lead?
- Where does 10010 lead?
- Which strings end up in accepting state?

# Graphical Representation

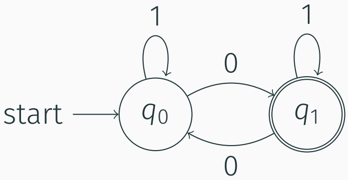



- Where does 001 lead?
- Where does 10010 lead?
- Which strings end up in accepting state?
- Every string $w$ has a unique walk that it follows from a given state $q$ by reading one letter of $w$ from left to right.




**Definition**
A DFA $M$ accepts a string $w$ iff the unique walk starting at the start state and spelling out $w$ ends in an accepting state.

# Graphical Representation




**Definition**
A DFA $M$ accepts a string $w$ iff the unique walk starting at the start state and spelling out $w$ ends in an accepting state.

**Definition**
The language accepted (or recognized) by a DFA $M$ is denote by $L(M)$ and defined as: $L(M) = \{w \mid M \text{ accepts } w\}$.

# Formal definition of DFA

# Formal Tuple Notation

**Definition**
A deterministic finite automata (DFA) $M = (Q, \Sigma, \delta, s, A)$ is a five tuple where

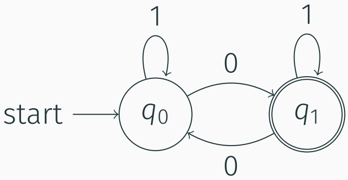
- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

Common alternate notation: $q_0$ for start state, $F$ for final states.

# DFA Notation

$$M = \Big( \quad \overbrace{Q} \quad , \quad \underbrace{\Sigma} \quad , \quad \overbrace{\delta} \quad , \quad \underbrace{s} \quad , \quad \overbrace{A} \quad \Big)$$

# Example




- $Q =$
- $\Sigma =$
- $\delta =$
- $s =$
- $A =$

# Extending the transition function to strings

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Given DFA $M = (Q, \Sigma, \delta, s, A)$, $\delta(q, a)$ is the state that $M$ goes to from $q$ on reading letter $a$

Useful to have notation to specify the unique state that $M$ will reach from $q$ on reading <u>string</u> $w$

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Transition function $\delta^* : Q \times \Sigma^* \to Q$ defined inductively as follows:

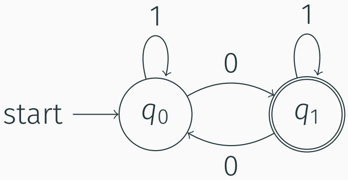
- $\delta^*(q, w) = q$ if $w = \epsilon$
- $\delta^*(q, w) = \delta^*(\delta(q, a), x)$ if $w = ax$.

# Formal definition of language accepted by M

**Definition**
The language $L(M)$ accepted by a DFA $M = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

# Example




What is:

- $\delta^*(q_1, \epsilon) =$

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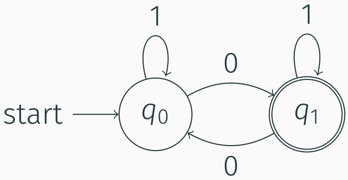


What is:

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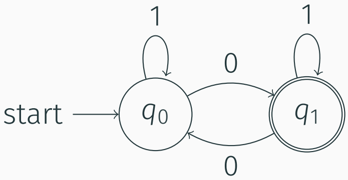


What is:

- $\delta^*(q_1, \epsilon) =$
- $\delta^*(q_0, 1011) =$
- $\delta^*(q_1, 010) =$

# Constructing DFAs: Examples

# DFAs: State = Memory

How do we design a DFA $M$ for a given language $L$? That is $L(M) = L$.

- DFA is a like a program that has fixed number of states regardless of its input size.
- The state must capture enough information from the input seen so far that it is sufficient for the suffix that is yet to be seen (note that DFA cannot go back)

# DFA Construction: Example I: Basic languages

Assume $\Sigma = \{0, 1\}$.

1. $L = \emptyset$

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Assume $\Sigma = \{0, 1\}$.

1. $L = \emptyset$

2. $L = \Sigma^*$

3. $L = \{\epsilon\}$

4. $L = \{0\}$

# DFA Construction: Example II: Length divisible by 5

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0,1\}^* \mid |w| \text{ is divisible by } 5\}$

# DFA Construction: Example III: Ends with 01

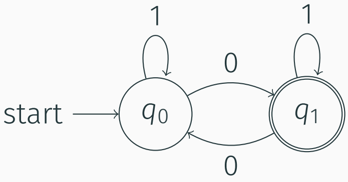
Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ ends with } 01\}$

# Complement language

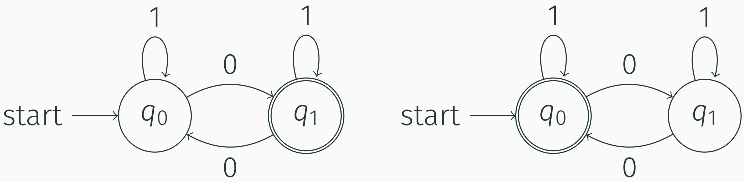
# Complement

**Question:** If $M$ is a DFA, is there a DFA $M'$ such that $L(M') = \Sigma^* \setminus L(M)$? That is, are languages recognized by DFAs closed under complement?



# Complement

Just flip the state of the states!



# Complement

**Theorem**
*Languages accepted by DFAs are closed under complement.*

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**Proof.**
Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$.

Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \bar{L}$. Why?

$\delta^*_M = \delta^*_{M'}$. Thus, for every string $w$, $\delta^*_M(s, w) = \delta^*_{M'}(s, w)$.

$\delta^*_M(s, w) \in A \Rightarrow \delta^*_{M'}(s, w) \notin Q \setminus A$.

$\delta^*_M(s, w) \notin A \Rightarrow \delta^*_{M'}(s, w) \in Q \setminus A$. $\square$

# Product Construction

# Union and Intersection

Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$?

How about intersection $L(M_1) \cap L(M_2)$?

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Idea from programming: on input string $w$

- Simulate $M_1$ on $w$
- Simulate $M_2$ on $w$
- If both accept than $w \in L(M_1) \cap L(M_2)$. If at least one accepts then $w \in L(M_1) \cup L(M_2)$.

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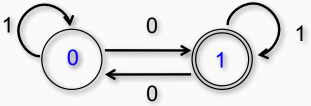
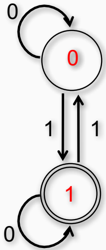
- Catch: We want a single DFA $M$ that can only read $w$ once.
- Solution: Simulate $M_1$ and $M_2$ in parallel by keeping track of states of both machines

# Example

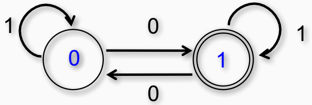



$M_1$ accepts #0 = odd




$M_2$ accepts #1 = odd

# Example

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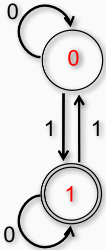


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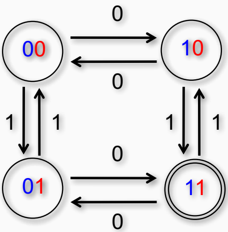


*Cross-product machine*

# Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$

**Theorem**
$L(M) = L(M_1) \cap L(M_2)$.

Create $M = (Q, \Sigma, \delta, s, A)$ where

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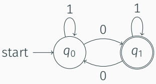
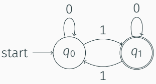
- $Q =$
- $s =$
- $\delta$ :
- $A =$

# Intersection vs Union

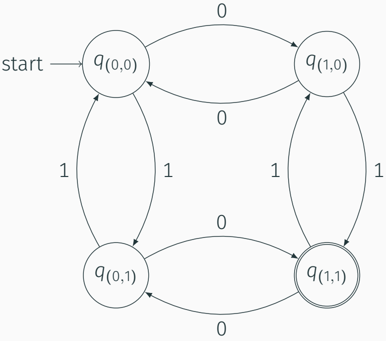
$M_1$:



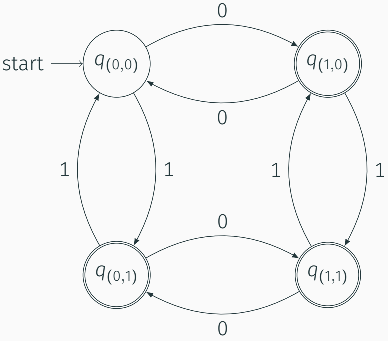

$M_2$ :




$M_1 \cap M_2$




$M_1 \cup M_2$



# Product construction for union

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$

**Theorem**
$L(M) = L(M_1) \cup L(M_2)$.

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
- $\delta : Q \times \Sigma \to Q$ where

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$

- $A =$

# Constructing regular expressions

# DFAs to regular expressions

**Personal Lemma:**
Mastering a concept means being able to do a problem in both direction.

Time to reverse problem direction and find regular expressions using DFAs.

Multiple methods but the ones I'm focusing on:

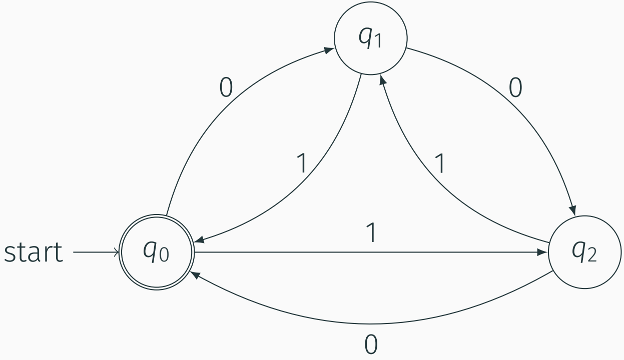
- State removal method
- Algebraic method

## State Removal method

If $q_1 = \delta(q_0, x)$ and $q_2 = \delta(q_1, y)$

then $q_2 = \delta(q_1, y) = \delta(\delta(q_0, x), y) = \delta(q_0, xy)$

# State Removal method - Example



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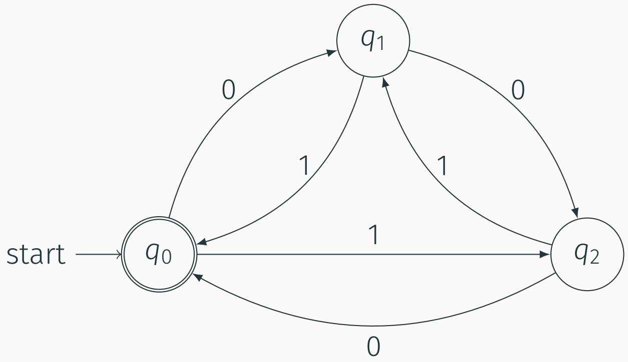


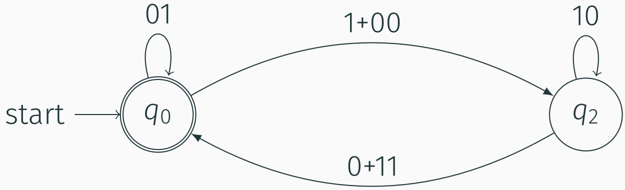

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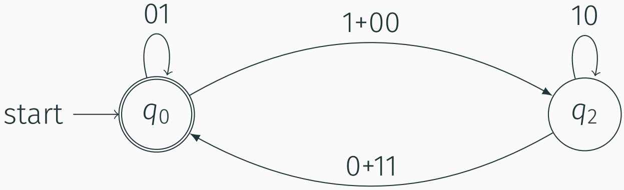

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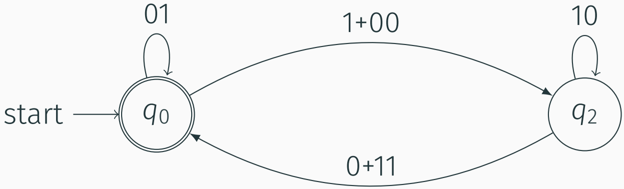
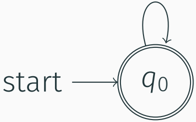
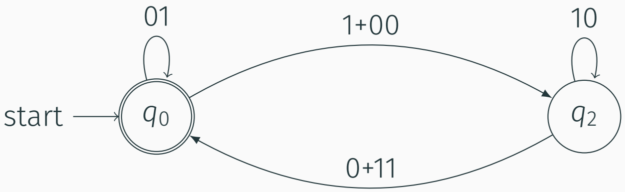
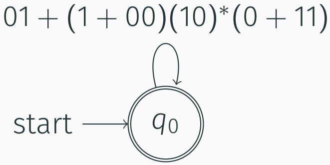



$$01 + (1 + 00)(10)^*(0 + 11)$$



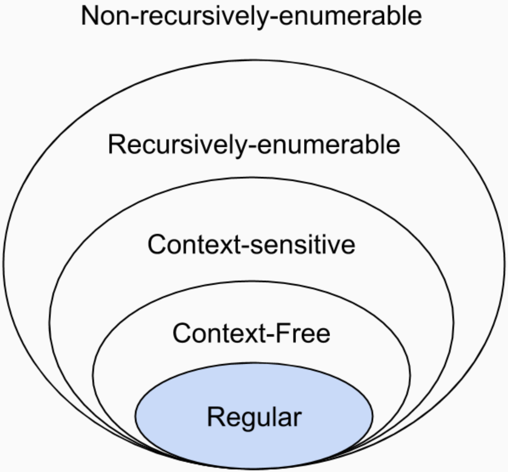



$$(01 + (1 + 00)(10)^*(0 + 11))^*$$

# DFAs and regular expressions

The thing to know right now is that DFAs and regular expressions represent the same set of languages!



## The End - Reminders

- HW 1 has been assigned. Will be due next week.
- Lab tomorrow will go over DFAs

# Extra Slides

# Algebraic method

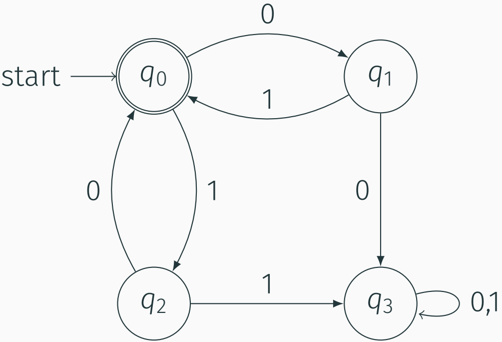
Transition functions are themselves algebraic expressions!

Demarcate states as variables.

Can rewrite $q_1 = \delta(q_0, x)$ as $q_1 = q_0 x$

Solve for accepting state.

# Algebraic method - Example



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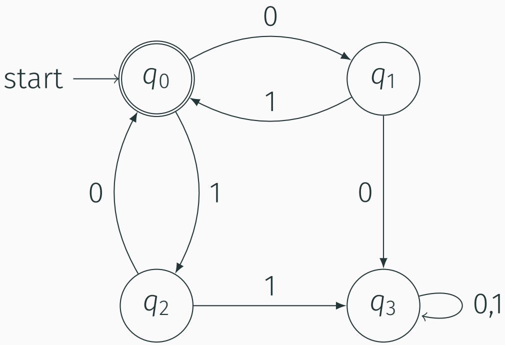


- $q_0 = \epsilon + q_1 1 + q_2 0$
- $q_1 = q_0 0$
- $q_2 = q_0 1$
- $q_3 = q_1 0 + q_2 1 + q_3 (0 + 1)$

# Algebraic method - Example

- $q_0 = \epsilon + q_1 1 + q_2 0$
- $q_1 = q_0 0$
- $q_2 = q_0 1$
- $q_3 = q_1 0 + q_2 1 + q_3(0 + 1)$

Now we simple solve the system of equations for $q_0$:

- $q_0 = \epsilon + q_1 1 + q_2 0$
- $q_0 = \epsilon + q_0 01 + q_0 10$
- $q_0 = \epsilon + q_0(01 + 10)$

**Theorem (Arden's Theorem)**
$R = Q + RP = QP^*$

# Algebraic method - Example

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- $q_0 = \epsilon + q_1 1 + q_2 0$
- $q_0 = \epsilon + q_0 01 + q_0 10$
- $q_0 = \epsilon + q_0(01 + 10)$
- $q_0 = \epsilon(01 + 10)^* = (01 + 10)^*$