1. For each of the following languages assume that the alphabet is $\{\mathtt{0}, \mathtt{1}\}$. Give a regular expression that describes that language, and briefly argue why your expression is correct.

(a) All strings that that end in $\mathtt{01}$ *and* contain $\mathtt{000}$ as a substring.

Solution: $\boxed{(\mathtt{0}+\mathtt{1})^*\mathtt{000}(\mathtt{0}+\mathtt{1})^*\mathtt{01}+(\mathtt{0}+\mathtt{1})^*\mathtt{0001}}$

The first part is the case where the $\mathtt{000}$ substring and the 01 at the end occur separately. The second part covers the case where they overlap, i.e. when the $\mathtt{0001}$ at the end contributes to both the $\mathtt{000}$ substring and the $\mathtt{01}$ termination. ■

(b) All strings that do not contain the *subsequence* $\mathtt{101}$.

Solution: $\boxed{\mathtt{0}^*\mathtt{1}^*\mathtt{0}^*}$

To not contain the subsequence $\mathtt{101}$, there can only be one run of $\mathtt{1}$s if we were to have any $\mathtt{1}$ at all. Before and after this run of $\mathtt{1}$, we may have any number of $\mathtt{0}$s. ■

(c) A finite language $L = \{w_1, w_2, \ldots, w_k\}$ where each w_i is a binary string.

Solution: If $L = \emptyset$ then the regular expression is $\boxed{\emptyset}$ otherwise if $L = \{w_1, w_2, \ldots, w_k\}$ for some $k \geq 1$, the regular expression is $\boxed{w_1 + w_2 + \ldots + w_k}$. ■

(d) The complement $\bar{L}$ of a finite language $L = \{w_1, w_2, \ldots, w_k\}$. You may want to consider the parameter $h = \max_{1 \leq i \leq k} |w_i|$, the length of the longest string in L.

Solution: If $L = \emptyset$ then $\overline{L} = \Sigma^*$ and hence a regular expression for $\overline{L}$ is $\boxed{(\mathtt{0}+\mathtt{1})^*}$.

If $L \neq \emptyset$ but finite let $h = \max_{w \in L} |w|$. Note that $h \geq 0$ (and can be 0 if $L = \{\varepsilon\}$). Consider $\overline{L}_h = \left\{w \notin L \;\middle|\; |w| \leq h\right\}$. We can see that $\overline{L} = \overline{L}_h \cup \left\{w \;\middle|\; |w| > h\right\}$. Since $\overline{L}_h$ is a finite language we can construct regular expression for it as in the preceding part. Let r_h be the regular expression for $\overline{L}_h$. Then the regular expression for $\overline{L}$ is $\boxed{r_h + (\mathtt{0}+\mathtt{1})^{h+1}(\mathtt{0}+\mathtt{1})^*}$ since $(\mathtt{0}+\mathtt{1})^{h+1}(\mathtt{0}+\mathtt{1})^*$ is a regular expression for all strings with length at least $h+1$. ■

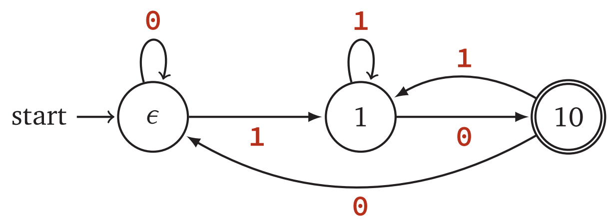
Rubric:

+ 2.5: Part (a) correct regex.
+ 2.5: Part (b) correct regex.
+ 2.5: Part (c) correct regex.
+ 2.5: Part (d) correct regex.

2. (a) Let L be the set of all strings in $\{\mathtt{0},\mathtt{1}\}^*$ that end in $\mathtt{10}$. Describe a DFA over the alphabet $\Sigma = \{\mathtt{0},\mathtt{1}\}$ that accepts the language L. Argue that your machine accepts every string in L and nothing else, by explaining what each state in your DFA *means*.

You may either draw the DFA or describe it formally, but the states Q, the start state s, the accepting states A, and the transition function δ must be clearly specified.

Solution: Since this is a relatively simple DFA, we can opt to draw it out explicity as follows.



State	Meaning
ϵ	seen no symbols yet, or only read **0** or last two symbols read are **00**.
1	Last symbol read was **1**.
10	Last two symbols read are **10**.

■

(b) Now consider some finite alphabet Σ and a string $x = a_1 a_2 \ldots a_k$ of length k where $a_1, a_2, \ldots, a_k \in \Sigma$. Let $L_x = \{w \in \Sigma^* \mid w \text{ ends in string } x\}$. Describe in formal tuple notation a DFA M that accepts L_x. Do not try to optimize number of states. How many states does your DFA have as a function of k?

Note: We changed s to x in the problem description above to avoid confusion with the start state.

Solution: $M = (Q, \Sigma, s, \delta, A)$ for L as follows.

$$
\begin{aligned}
Q &:= \{q_w \mid w \in \Sigma^*, 0 \leq |w| \leq k\} \\
s &:= q_\varepsilon \\
A &:= \{q_x\} \\
\delta(q_w, a) &:= \begin{cases} q_{w\cdot a} & \text{if } |w| < k \\ q_{z\cdot a} & \text{if } |w| = k, \text{ where } w = bz \text{ for } b \in \Sigma,\ z \in \Sigma^* \end{cases}
\end{aligned}
$$

For each string w of length at most k we have a state q_w, hence the total number of states is $\sum_{i=0}^{k} |\Sigma|^i$ which is exponential in k if $|\Sigma| \geq 2$. The state q_w has the following meaning: if $|w| < k$ then w is the string seen so far. If $|w| = k$ then w is the string of the last k characters seen so far. It is not hard to check that the transition function maintains this claimed behavior of M. The only accept state is q_x since we want to end in x. ■

Solution: Here is an alternate solution for a DFA that has only $k+1$ states $q_0, q_1, \dots, q_k$. The meaning of q_i is that the string seen so far ends in $a_1a_2\dots a_i$ *and* moreover there is no $j > i$ such that the string seen so far ends in $a_1a_2\dots a_j$. No that q_0 corresponds to ε.

$$\begin{aligned} Q &:= \{q_i \mid 0 \le i \le k\} \\ s &:= q_0 \\ A &:= \{q_k\} \\ \delta(q_i, a) &:= q_j \text{ where } j = \max\{h \mid a_1a_2\dots a_h = a_{i+2-h}\dots a_i a\} \end{aligned}$$

Intuitively, j is the length of the longest prefix of x that is also a suffix of $a_1a_2\dots a_i a$.

It is harder to convince yourself of the correctness of this but try to prove it by induction. ■

(c) **Not to submit.** Let $L = \{w_1, w_2, \dots, w_k\}$ be a finite language. Describe a DFA M that accepts L. Describe the states and justify why your DFA accepts L.

Solution: Let h be the length of the longest string in L. We create a DFA $M = (Q, \Sigma, s, \delta, A)$ for L as follows.

$$\begin{aligned} Q &:= \{q_w \mid 0 \le |w| \le h\} \cup \{\text{Fail}\} \\ s &:= q_\varepsilon \\ A &:= \{q_w \mid w \in L\} \\ \delta(q_w, a) &:= \begin{cases} q_{w\cdot a} & \text{if } |w| < h \\ \text{Fail} & \text{otherwise} \end{cases} \\ \delta(\text{Fail}, a) &:= \text{Fail} \quad \forall a \in \Sigma \end{aligned}$$

For each string w of length at most h we have a state q_w and also an additional state Fail, hence the total number of states is $1 + \sum_{i=0}^{n} |\Sigma|^i$ which is exponential in h if $|\Sigma| \ge 2$. Each state represents the string read thus far. The fail state is reached when we have read more than h symbols since such a string cannot be in L. The accept states correspond precisely to the strings $w \in L$. ■

Rubric:

- 5 points: Part (a).
  - + 3: Correct DFA.
  - + 2: Correct explanation.
- 5 points: Part (b).
  - + 3: Correct DFA.
  - + 2: Correct explanation.

3. Let L_1, L_2, and L_3 be regular languages over Σ accepted by DFAs $M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$, $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$, and $M_3 = (Q_3, \Sigma, \delta_3, s_3, A_3)$, respectively.

   (a) Describe a DFA $M = (Q, \Sigma, \delta, s, A)$ in terms of M_1, M_2, and M_3 that accepts $L = \{w \mid w$ is in at least two of $\{L_1, L_2, L_3\}\}$. Formally specify the components Q, δ, s, and A for M in terms of the components of M_1, M_2, and M_3.

   Solution: Informally, we will use the product construction to make our DFA, where the accept states are those where at least two of the component states are accept states in their respective component DFAs. This corresponds to the following formal description

$$
\begin{aligned}
Q &:= Q_1 \times Q_2 \times Q_3, \\
s &:= (s_1, s_2, s_3), \\
A &:= \big\{(q_1, q_2, q_3) \in Q \;\big|\; \text{At least two of: } q_1 \in A_1,\ q_2 \in A_2,\ q_3 \in A_3\big\} \\
&:= (A_1 \times A_2 \times Q_3) \cup (A_1 \times Q_2 \times A_3) \cup (Q_1 \times A_2 \times A_3), \\
\delta((q_1, q_2, q_3), a) &:= \big(\delta_1(q_1, a),\ \delta_2(q_2, a),\ \delta_3(q_3, a)\big).
\end{aligned}
$$

■

   (b) Prove by induction your solution is correct.

   Solution: We begin by defining the extended transition function (similar to the definition in the notes)

$$
\delta^*(q, w) := \begin{cases} q & \text{if } w = \epsilon, \\ \delta^*(\delta(q, a), x) & \text{if } w = ax. \end{cases}
$$

   We let δ^* denote the extended transition function for M and $\delta_1^*, \delta_2^*, \delta_3^*$ as the extended transition functions for M_1, M_2, and M_3 respectively. We now prove the following claim for use later

   Claim 1. *Let $(q_1, q_2, q_3) \in Q$. Then for all $w \in \Sigma^*$*

$$
\delta^*((q_1, q_2, q_3), w) = (\delta_1^*(q_1, w),\ \delta_2^*(q_2, w),\ \delta_3^*(q_3, w)).
$$

   Proof: *We proceed by induction on $|w|$. For our base case, $|w| = 0$, so $w = \epsilon$. Thus, by the definition of δ^*, we have that $\delta^*((q_1, q_2, q_3), \epsilon) = (q_1, q_2, q_3) = (\delta_1^*(q_1, \epsilon),\ \delta_2^*(q_2, \epsilon),\ \delta_3^*(q_3, \epsilon))$.*

   We now proceed to our inductive step, where we assume that $|w| > 0$ and that the claim holds for every string x such that $|x| < |w|$. Since $|w| > 0$ we may assume that $w = ax$ for some symbol a and string x where $|x| < |w|$. Thus,

$$
\begin{aligned}
&\delta^*((q_1, q_2, q_3), ax) \\
&= \delta^*(\delta((q_1, q_2, q_3), a), x) && \text{by the definition of } \delta^* \\
&= \delta^*((\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a)), x) && \text{by the definition of } \delta \\
&= \big(\delta_1^*(\delta_1(q_1, a), x),\ \delta_2^*(\delta_2(q_2, a), x),\ \delta_3^*(\delta_3(q_3, a), x)\big) && \text{by induction hypothesis} \\
&= \big(\delta_1^*(q_1, ax),\ \delta_2^*(q_2, ax),\ \delta_3^*(q_3, ax)\big) && \text{by definitions of } \delta_1^*, \delta_2^*, \delta_3^*.
\end{aligned}
$$

   This completes our induction and proves the claim. □

Now we show that our DFA M accepts the language L. Consider some string $w = a_1 \dots a_n$ accepted by our new DFA M. If we let $s = (s_1, s_2, s_3)$ be the starting state of M, then $\delta^*(s, w) \in A$. Applying the claim, $\delta^*(s, w) = (\delta_1^*(s_1, w),\ \delta_2^*(s_2, w),\ \delta_3^*(s_3, w))$, and by the definition of A, then at least two of $\delta_1^*(s_1, w) \in A_1, \delta_2^*(s_2, w) \in A_2, \delta_3^*(s_3, w) \in A_3$. Without loss of generality, we write that that $\delta_1^*(s_1, w) \in A_1, \delta_2^*(s_2, w) \in A_2$. Thus, $w \in L_1, L_2$ (as w is accepted by M_1 and M_2), and therefore $w \in L$ (as w is in at least two of $\{L_1, L_2, L_3\}$).

For the reverse direction, take some $w \in L$, and so without loss of generality, we have that $w \in L_1, L_2$. Then, $\delta_1^*(s_1, w) \in A_1, \delta_2^*(s_2, w) \in A_2$, and by our claim, $(\delta_1^*(s_1, w),\ \delta_2^*(s_2, w),\ \delta_3^*(s_3, w)) = \delta^*((s_1, s_2, s_3), w)$, meaning that $\delta^*((s_1, s_2, s_3), w) \in A$ from the definition of A. Therefore, w is accepted by M.

By both of our previous statements, $L = L(W)$ as desired.

Rubric:

+ 4: Part (a), correct construction.
- 6 points: Part (b).
  + 5: Standard induction rubric (included at end).
  + 1: Rest of proof (using result from induction).

Rubric (induction): For problems worth 10 points:

+ 1 for explicitly considering an *arbitrary* object
+ 2 for a valid **strong** induction hypothesis
  - **Deadly Sin!** Automatic zero for stating a weak induction hypothesis, unless the rest of the proof is *perfect*.
+ 2 for explicit exhaustive case analysis
  - No credit here if the case analysis omits an infinite number of objects. (For example: all odd-length palindromes.)
  - −1 if the case analysis omits an finite number of objects. (For example: the empty string.)
  - −1 for making the reader infer the case conditions. Spell them out!
  - No penalty if cases overlap (for example:
+ 1 for cases that do not invoke the inductive hypothesis ("base cases")
  - No credit here if one or more "base cases" are missing.
+ 2 for correctly applying the *stated* inductive hypothesis
  - No credit here for applying a *different* inductive hypothesis, even if that different inductive hypothesis would be valid.
+ 2 for other details in cases that invoke the inductive hypothesis ("inductive cases")
  - No credit here if one or more "inductive cases" are missing.