Homework 3

Question 1

$$xy'(x) = 2y(x)$$
$$\Rightarrow y' = \frac{2y}{x}$$
$$\Rightarrow y' - \frac{2}{x}y = 0$$

Consider $\mu(x) = e^{-1\int \frac{2}{x}dx} = \frac{1}{x^2}$. and multiply $\mu(x)$ on both side:

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = 0 \Rightarrow (\frac{1}{x^2}y)' = 0 \Rightarrow \frac{1}{x^2}y = C \Rightarrow y = Cx^2$$

From the question, it give us $y(0) = 0$. Thus $0 \cdot C = 0$, and C can be any number.

It is not violate the existence and uniqueness theorem. By $y' = \frac{2y}{x}$ and $\frac{\partial f(x,y)}{\partial y} = \frac{2}{x}$ they are all continuous everywhere when $x \neq 0$. Thus the existence and uniqueness theorem will noe violate when $x = 0$, which is the question initial condition. Thus we cannot know if tere are unique solution or have a solution by the theorem.

Question 2

$$\frac{dy}{dx} = 3y^{\frac{2}{3}}$$
$$\Rightarrow y' = 3y^{\frac{2}{3}}$$
$$\Rightarrow y' + 0y = 3 \cdot y^{\frac{2}{3}}$$

Consider $\mu(x) = y^{\frac{1}{3}}$, we can know that the equation is a Bernoulli equation with $n = \frac{2}{3}$. and we can get

$$3\mu^2 \cdot \mu' = 3 \cdot \mu^2$$

- Consider $\mu \neq 0$, Which we have $y \neq 0$ and $\mu' = 1$

$$y(x) = \mu(x)^3 = (x + C)^3, \quad y(x) \neq 0$$

- The question give us $y(0) = 0$,

$$y(0) = (0 + C)^3 = 0$$

  Thus we can know $C = 0$. Thus $y(x) = x^3$, $y(x) \neq 0$

When $y(x) = 0, y' = 0, 3y^{\frac{2}{3}} = 3 \cdot 0^{\frac{2}{3}} = 0$. and we can get $y(x) = x^3$ and $y(x) = 0$.

Since $y' = f(x, y) = 3y^{2/3}$ is continuous everywhere, and $\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$ is continuous everywhere when $y \neq 0$. The existence and uniqueness theorem, for $y > 0$ and $y < 0$. and it did not contradict because the initial value the question give us $(0, 0)$ is not satisfy the assumption.

Question 3

$$2xyy' = 4x^2 + 3y^2$$

$$\Rightarrow y' = \frac{4x^2 + 3y^2}{2xy}$$

$$\Rightarrow y' - \frac{3y}{2x} = 2x \cdot y^{-1}$$

We can know the $n = -1$, Consider $\mu(x) = y^2$.and $y = \pm\sqrt{v}$

- Consider $y = \sqrt{v}$

$$y' = \frac{1}{2} \cdot \mu^{-\frac{1}{2}} \cdot \mu'$$

$$\frac{\mu^{-\frac{1}{2}} \cdot \mu'}{2} - \frac{3\mu^{\frac{1}{2}}}{2x} = 2x\mu^{-\frac{1}{2}}$$

$$\mu' - \frac{3}{x}\mu = 4x$$

Consider $u(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln|x|} = |x|^{-3}$.Then we can know $\left(x^3\mu\right)' = 4x^{-2}$ when $x > 0$.

Then we can know $x^{-3}\mu = 4\int x^{-2}dx = \frac{-4}{x} + C$.

At last we can get $y(x) = \pm x\sqrt{Cx - 4}$ where C is arbitrary constant number.

Question 4

$$xy' + 6y = 3xy^{\frac{3}{3}}$$

$$\Rightarrow y' + \frac{6}{x}y = 3y^{\frac{4}{3}}$$

We can know $n = \frac{4}{3}$. Consider $\mu(x) = y^{\frac{-1}{3}}$ $y \neq 0$. and then we can know $y' = -3\mu^{-4} \cdot \mu'$.

$$-3\mu^{-4} \cdot \mu' + \frac{6}{x} \cdot \mu^{-3} = 3 \cdot \mu^{-4}$$

$$\Rightarrow \mu' - \frac{2}{x}\mu = -1$$

Consider $u(x) = e^{-\int \frac{2}{x}dx} = |x|^{-2}$, suppose $x > 0$ we can know $u(x) = x^{-2}$. Multiplying $u(x)$ on both side.

$$\left(x^{-2}\cdot\mu\right)' = -x^{-2}$$
$$x^{-2}\cdot\mu = -\int x^{-2}dx = \frac{1}{x} + C$$
$$\mu = \left(\frac{1}{x} + c\right)x^2 = x + cx^2.$$

Then we can know $y(x) = \left(x + Cx^2\right)^{-3} = \frac{1}{x^3(Cx+1)^3}$.

Then when $y(x) = 0$, $y'(x) = 0$. $xy' + 6y^2 = 3xy^{\frac{4}{3}} = 0$. Thus we can get $y(x) = \frac{1}{x^3(Cx+1)^3}$ and $y(x) = 0$.

Question 5

(a) $V(t) = V_0 + (2-3)t = 60 - t$, when $(0 \leq t \leqslant 60)$. Thus the volume of fluid in the tank after $t\ min$ will be $60 - t$ gallen.