Lecture notes, Week 3, Day 3

Goal: learn about modeling with first order ODEs. (This is based on section 2.3 in edition 8.)

Mathematical models are useful because they offer a convenient way, less expensive and time-consuming, to obtain predictions for various physical, biological, and social processes. They are only an approximation and a simplification of the actual situation, so their output should be interpreted with care.

Concentration and mixing problems Assume a fluid (such as water) flows into a tank or lake at a constant rate r_i (volume/time) and that it contains a substance (such as salt, a dye or a pollutant) whose concentration (amount/volume) is $\gamma(t)$. We also assume that the fluid gets perfectly mixed and is flowing out of the tank at a rate r_o (volume/time). If $Q(t)$ is the amount of the substance in the tank, then the concentration at time t is $c(t) = \frac{Q(t)}{V(t)}$, where $V(t)$ is the volume of the fluid at time t. It is given by

$$V(t) = V_0 + (r_i - r_o)t, \tag{1}$$

where $V(0) = V_0$ is the initial volume of the fluid. Then, the rate of change $\frac{dQ}{dt}$ of the quantity $Q(t)$ is equal to the amount that flows in minus the amount that flows out per time

$$\frac{dQ}{dt} = r_i\gamma(t) - r_o\frac{Q(t)}{V(t)} \tag{2}$$

If $r_i = r_o$ then from (1) we obtain $V(t) = V_0$ and (2) is simplified

$$\frac{dQ}{dt} = r_i\gamma(t) - \frac{r_o}{V_0}Q(t). \tag{3}$$

Example 1 A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb salt per gallon is entering at a rate of 3 gal/min, and the mixture flows out of the tank at a rate of 2 gal/min. (a) Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. (b) Find the concentration (lb/gal) of salt in the tank when it is on the point of over flowing. (c) Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.
Solution: (a) From (1) we obtain

$$V(t) = V_0 + (r_i - r_0)t = 200 + (3-2)t = 200 + t.$$

The From (2) it holds

$$Q' = r_i\gamma(t) - r_o\frac{Q(t)}{V(t)} \Rightarrow Q' = 3*1 - 2*\frac{Q}{200+t} \Rightarrow Q' = 3 - \frac{2Q}{200+t} \Rightarrow Q' + \frac{2Q}{200+t} = 3.$$

We will solve it using the method of the integrating factor.

$$\begin{aligned}
&\mu Q' + \frac{2\mu}{200+t}Q = 3\mu \Rightarrow \mu' = \frac{2\mu}{200+t} \Rightarrow \frac{\mu'}{\mu} = \frac{2}{200+t} \Rightarrow \ln|\mu| = 2\ln(200+t) + c_1 \Rightarrow \\
&\mu = (200+t)^2 \Rightarrow (200+t)^2Q' + 2(200+t)Q = 3(200+t)^2 \Rightarrow (200+t)^2Q = (200+t)^3 + c_2 \\
&Q = 200 + t + \frac{c_2}{(200+t)^2}.
\end{aligned}$$

The initial amount of salt in the tank was

$$Q(0) = 100 = 200 + \frac{c_2}{200^2} \Rightarrow c_2 = -100 * 200^2.$$

Hence, the amount of salt at time t is

$$Q = 200 + t - 100\left(\frac{200}{200+t}\right)^2. \tag{4}$$

The tank overflows at the time t_o when

$$V(t_o) = 500 \Rightarrow 200 + t_o = 500 \Rightarrow t_o = 300 \text{ min.}$$

So the solution (4) is valid for $t < 300$ min.

(b) When the tank is on the verge of overflowing, the concentration is

$$\frac{Q(t_o)}{V(t_o)} = \frac{500 - 16}{500} = \frac{484}{500} \quad \text{lb/gal.}$$

(c) If the tank had infinite capacity it would be

$$c(t) = \frac{Q(t)}{V(t)} = -\frac{100}{200+t}\left(\frac{200}{200+t}\right)^2 + \frac{200+t}{200+t} \Rightarrow \lim_{t\to\infty} c(t) = 1 \quad \text{lb/gal.}$$