Lecture notes, Week 1, Day 1

Goal: introduce some basic mathematical models and solve them. (These notes cover material from sections 1.1 and 1.2 of edition 8.)

You may wonder about the reason why ODEs are used. This is because the description of many processes, many physical, chemical and biological phenomena involves relations or statements of the rates at which things happen. Mathematically, we then describe these relations as equations and use derivatives to express the rates. Equations containing derivatives are called Differential Equations. A differential equation that describes some physical phenomenon is called a mathematical model.

Problem 1 A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.
Solution: First, we recall the volume of a sphere of radius r

$$V = \frac{4\pi}{3} r^3$$

and the surface area of a sphere of radius r

$$A = 4\pi r^2.$$

Based on the statement of the problem, it holds

$$\frac{dV}{dt} = -k4\pi r^2,$$

where $k > 0$ is a proportionality constant. The minus sign exists because the as the raindrop evaporates, its volume decreases. This equation is correct, but it does not provide the final answer. To obtain it, we need to express r in terms of V. In this way, we will have time t as the independent variable and the volume $V = V(t)$ as the dependent variable. It holds

$$r^3 = \frac{3V}{4\pi} \Rightarrow r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}},$$

which if we plug into the previous equation yields

$$\frac{dV}{dt} = -k4\pi \left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}.$$

We may simplify this ODE, by combining all constants into one

$$\frac{dV}{dt} = -\kappa V^{\frac{2}{3}}.$$

It was great obtaining the ODE, but how do we solve it? Since differentiation and integration are inverse processes, we may recall some tricks from integration to help us solve ODEs. One of the best-known methods of solving ODEs, is the method of separation of variables. Specifically, we separate the unknown variable V and its derivative from the independent variable t and any constants that are assumed to be known. In the previous problem, we proceed as follows. We may express the derivative as $V' = \frac{dV}{dt}$. Then,

$$V' = -\kappa V^{\frac{2}{3}} \Rightarrow \frac{V'}{V^{\frac{2}{3}}} = -\kappa.$$

Now, we may use the chain rule

$$\frac{d}{dt} f(g(t)) \equiv (f(g(t)))' = f'(g(t)g'(t)$$

to integrate the left hand side. The chain rule for powers of functions becomes

$$\frac{d}{dt}(V^a(t)) = aV^{a-1}(t)V'(t) \Rightarrow \frac{V'}{V^{1-a}} = \frac{1}{a}\frac{d}{dt}(V^a(t)),$$

with $1-a=\frac{2}{3} \Rightarrow a=\frac{1}{3}$. Hence,

$$3\frac{d}{dt}V^{\frac{1}{3}}(t)=\frac{V'}{V^{\frac{2}{3}}}=-\kappa \Rightarrow \frac{d}{dt}V^{\frac{1}{3}}(t)=-\frac{\kappa}{3} \Rightarrow V^{\frac{1}{3}}(t)=-\frac{\kappa}{3}t+c,$$

where c is a constant of integration. After some algebraic manipulations we obtain

$$V(t)=\left(c-\frac{\kappa t}{3}\right)^3. \quad \clubsuit$$

We have found infinitely many solutions of the ODE, corresponding to the infinitely many values that the arbitrary constant c may have. This is typical when solving an ODE, since integration brings in an arbitrary constant.

Assume now we would like to solve the following practical problem. In this problem, an initial condition will be given that will help us determine the integration constant. Problems with initial conditions are called Initial Value Problems (IVP). For instance, in the previous problem, they could have given us the volume of the drop at time $t=0$, $V(0)=V_0$. Then, it would be

$$V(0)=c^3=V_0 \Rightarrow V(t)=\left(V_0^{\frac{1}{3}}-\frac{\kappa t}{3}\right)^3.\diamondsuit$$

Equation $\clubsuit$ is called the general solution of the problem. Equation $\diamondsuit$ gives the solution to the IVP.