Lecture notes, Week 2, Day 3

Goal: learn about to use integrating factors to convert equations into exact equations. (This is based on section 2.6 in edition 8.)

We consider again an equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{1}$$

which is not exact. Then, we multiply it with a function μ

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0 \tag{2}$$

and choose μ such that the resulting equation is exact

$$(\mu M)_y = (\mu N)_x.$$

This follows from the theorem you proved last time. It can be written equivalently as

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{3}$$

If a function μ satisfying (3) can be found, then (2) is exact. However, we are faced with the problem that (3) is as hard to solve as the original equation. Hence, we will focus on simple cases when μ is a function of only x or only y.

To fix ideas, assume $\mu = \mu(x)$, then (3) becomes

$$-N\mu_x + (M_y - N_x)\mu = 0 \Rightarrow \mu_x = \frac{M_y - N_x}{N}\mu. \tag{4}$$

If now the right hand-side $\frac{M_y - N_x}{N}\mu$ depends on x only, then $\mu(x)$ can be found by solving (4).

Similarly, if $\mu = \mu(y)$ then (3) becomes

$$M\mu_y + (M_y - N_x)\mu = 0 \Rightarrow \mu_y = \frac{N_x - M_y}{M}\mu. \tag{5}$$

If now the right hand-side $\frac{N_x - M_y}{M}\mu$ depends on y only, then $\mu(y)$ can be found by solving (5).

Example 1: Find an integrating factor and solve the given equation

$$e^{2x} + y - 1 - y' = 0. \quad \clubsuit$$

We notice that $M = e^{2x} + y - 1 \Rightarrow M_y = 1$ and $N = -1 \Rightarrow N_x = 0$. Hence, it is not an exact equation. We then consider the ratio

$$\frac{N_x - M_y}{M} = \frac{-1}{e^{2x} + y - 1},$$

which depends on both x and y, hence a μ that depends only on y cannot be found. We also consider that ratio

$$\frac{M_y - N_x}{N} = \frac{1}{-1} = -1,$$

which does not depend on y, and since it is constant, we may conclude that it trivially depends only on x. Hence, we may look for an integrating factor of the form $\mu = \mu(x)$ that satisfies (4)

$$\mu_x = -\mu \Rightarrow \frac{\mu_x}{\mu} = -1 \Rightarrow \ln|\mu| = -x + c_1 \Rightarrow \mu = ce^{-x},$$

and wlog we set $\mu = e^{-x}$. Next, ♣ becomes

$$e^x + e^{-x}y - e^{-x} - e^{-x}y' = 0 \Rightarrow e^x - e^{-x} = e^{-x}y' - e^{-x}y \Rightarrow (e^x + e^{-x})' = (e^{-x}y)' \Rightarrow$$
$$e^x + e^{-x} + c = e^{-x}y \Rightarrow y = 1 + e^{2x} + ce^x. \tag{6}$$

Example 2: Solve the ODE

$$y + (2x - ye^y)y' = 0.$$

Solution: It holds

$$M(x,y) = y \Rightarrow M_y = 1$$
$$N(x,y) = 2x - ye^y \Rightarrow N_x = 2.$$

Since, $M_y \neq N_x$ it follows that the equation is not exact. Hence, we will convert it into an exact one by using an appropriate integrating factor μ. We will solve for a function μ such that

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

Since $\frac{M_y - N_x}{M} = -\frac{1}{y}$, we may look for an integrating factor that depends on y only. Then, $\mu_x = 0$ and it holds

$$\mu_y = \frac{1}{y}\mu \Rightarrow \ln|\mu| = \ln|y| + c_1 \Rightarrow \mu = y.$$

Then,

$$y^2 + (2xy - y^2e^y)y' = 0 \Rightarrow \psi_x = y^2 \text{ and } \psi_y = 2xy - y^2e^y.$$

Therefore,

$$\psi = xy^2 + h(y) \Rightarrow \psi_y = 2xy + h'(y) = 2xy - y^2e^y \Rightarrow h'(y) = -y^2e^y.$$

Integrating by parts twice yields

$$h(y) = (-y^2 + 2y - 2)e^y + c_2,$$

hence the solution is given implicitly by

$$\psi(x,y) = xy^2 + (-y^2 + 2y - 2)e^y = C.$$

Practice Problem:
Determine whether $2xy^2 + 2y + (2x^2y + 2x)y' = 0$ is exact. If it is exact, find the solution.
Solution: The goal is to find $\psi(x,y)$ such that

$$\frac{d\psi(x,y(x))}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx} = M(x,y) + N(x,y)y'(x) = 0, \tag{7}$$

where in the first equality we have used the chain rule. In our case,

$$\begin{aligned} M(x,y) &= 2xy^2 + 2y \\ N(x,y) &= 2x^2y + 2x. \end{aligned}$$

We proved a theorem in class that states that an equation is exact if and only if

$$M_y(x,y) = N_x(x,y).$$

In our case

$$\begin{aligned} M_y &= 4xy + 2 \\ N_x &= 4xy + 2, \end{aligned}$$

hence it holds $M_y = N_x$ which implies that the equation is exact. It must hold

$$\psi_x = M = 2xy^2 + 2y \Rightarrow \psi = x^2y^2 + 2xy + h(y),$$

after integration with respect to x. Next, we differentiate with respect to y:

$$\psi_y = 2x^2 y + 2x + h'(y)$$

and equate this with $N(x, y)$

$$2x^2 y + 2x + h'(y) = 2x^2 y + 2x \Rightarrow h'(y) = 0 \Rightarrow h(y) = c_1.$$

This yields

$$\psi = x^2 y^2 + 2xy + c_1$$

and therefore the solutions lie on the curve $\psi(x, y) = c_2 \Rightarrow x^2 y^2 + 2xy + c_1 = c_2 \Rightarrow x^2 y^2 + 2xy = c$. **Note**: in general, we omit the constant of integration when we integrate $h'(y)$ because it is absorbed in c. We know that $\psi = c$ since (7) shows that its derivative is zero along solutions.