Lecture notes, Week 1, Day 2

Goal: discuss the classification of Differential Equations. (This is based on section 1.3 in edition 8.)

We wish to classify Differential Equations so that we can organize their properties and methods of solution. When solutions cannot be readily found, classification will be an aid to solution approximation.

Ordinary and Partial Differential Equations So far, we have seen examples where the unknown function y depends on a single independent variable. In this case, only ordinary derivatives appear in the differential equation which is an Ordinary Differential Equation, such as this example

$$\frac{dy}{dt} = 3t^2 y.$$

When the unknown function y depends on more than one independent variables, partial derivatives appear in the differential equation which is a Partial Differential Equation, such as this example

$$u_{tt} = c^2 u_{xx},$$

which is called the wave equation.

Systems of Differential Equations A system is a collection of ODEs or PDEs. The dimension of the system is equal to the number of equations there, which is equal to the number of unknown functions. A very well known example is the Susceptible-Infected-Recovered system

$$\begin{aligned} S' &= -\beta SI \\ I' &= \beta SI - \gamma I \\ R' &= \gamma I. \end{aligned}$$

Here, S is the susceptible population which becomes infected after contact with infected individuals I at a per capita rate β. The infected population recovers at an average rate γ and moves into the recovered class R.

Order The order of a differential equation is the order of the highest derivative that appears in the equation. Later on we will learn about the spring-mass system

$$mu''(t) + \gamma u'(t) + ku(t) = 0,$$

which is a second-order equation. Equivalently, it can be written as a system. To do so, we need to set $u_1 = u, u_2 = u'$ which in turn implies

$$u_1' = u' = u_2, \quad \text{and} \quad mu_2' + \gamma u_2 + ku_1 = 0,$$

and by rearranging the second equation we obtain

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{k}{m}u_1 - \frac{\gamma}{m}u_2 \end{aligned}$$

Linear and Nonlinear Equations An ODE of the form

$$F(t, y, ...y^{(n)}) = 0$$

is said to be linear if F is a linear function of the variables y, y', $...y^{(n)}$. Hence, the general linear ODE of order n is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + ... + a_1(t)y' + a_0(t)y = g(t) \quad \spadesuit.$$

An equation that is not of the form $\spadesuit$ is a nonlinear equation. The so-called equation of logistic growth

$$y' = ry\left(1 - \frac{y}{K}\right),$$

is an example of a nonlinear equation. Here, r is the intrinsic growth rate and K is the environmental carrying capacity.

Solutions A solution of the ODE

$$y^{(n)} = f(t, y, y', y''...y^{(n-1)}) \quad \dagger$$

on the interval $\alpha < t < \beta$ is a function ϕ, such that ϕ', ϕ'', ..., $\phi^{(n-1)}$ exist and satisfy

$$\phi^{(n)} = f(t, \phi, \phi', \phi''...\phi^{(n-1)})$$

for every t in (α, β). Unless we state otherwise, we will assume that the function f is real valued and we are looking for real-valued solutions.

Existence, Uniqueness, Continuous Dependence on Initial Conditions One of the most important questions in the theory of differential equations is the question of existence of solutions. Not all ODEs have a solution. We are interested in the conditions on f in † that ensure existence of solutions. For those ODES that have a solution, this solution may not be unique. This is undesirable when the problem models a physical process. Finally, we are interested to know what happens if we perturb slightly the initial conditions of the ODE. Will the solution change dramatically? This is the question of continuous dependence on initial conditions.

Examples

- Determine the values of r for which the Euler equation

  $$t^2y'' + 4ty' + 2y = 0$$

  has a solution of the form $y = t^r$ for $t > 0$.
  Set $y = t^r$. It then follows $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. If we plug these into the ODE we obtain

  $$(r(r-1) + 4r + 2)\, t^r = 0 \Rightarrow r^2 + 3r + 2 = 0,$$

  after a bit of algebra and using the fact that $t > 0$. Hence, we have reduced the second-order ODE into a quadratic equation for r. Solving this purely algebraic equation yields

  $$r = -2, \quad r = -1 \Rightarrow y_1 = \frac{1}{t}, \quad y_2 = \frac{1}{t^2}.$$

- Verify that the function $y = 3t + t^2$ is a solution of the first-order linear ODE

  $$ty' - y = t^2.$$

  It holds $y' = 3 + 2t$, so

  $$ty' - y = 3t + 2t^2 - 3t - t^2 = t^2,$$

  as wanted.