Lecture notes, Week 1, Day 3

Goal: learn about the method of integrating factors for first order linear ODES

$$y' = f(t, y).$$

Here, f depends linearly on the dependent variable y. (This is based on section 2.1 in edition 8.)

The most general first order linear equation has the form

$$y' = -p(t)y + g(t), \tag{1}$$

where p, g are given functions of the independent variable t.

Example: Solve the special case

$$y' = -ay + b,$$

where a, b are constants.
To solve this equation, we may rearrange it by writing

$$\frac{y'}{-ay+b} = 1 \Rightarrow \frac{y'}{y - \frac{b}{a}} = -a,$$

assuming that $y \neq \frac{b}{a}$. The, we recall the chain rule for logarithms $\frac{f'(t)}{f(t)} = (\ln |f(t)|)'$ and notice that $(y - \frac{b}{a})' = y'$. Hence, by integration we obtain

$$\ln \left| y - \frac{b}{a} \right| = -at + c_1 \Rightarrow y - \frac{b}{a} = \pm e^{c_1} e^{-at} \Rightarrow y = \frac{b}{a} + ce^{-at} \quad (*),$$

where c is an arbitrary constant.
What happens when $y = \frac{b}{a} \equiv y_e$? In this case, the derivative on the left-hand side of the equation is zero. Hence, we say that y_e is an equilibrium. We can retrieve the equilibrium solution from the general solution $(*)$ by setting $c = 0$.

Unfortunately, this method of solution by direct integration, does not work for (1). Instead, we will use a method due to Leibniz called the method of integrating factor. The main idea can be summarized as follows. First, we rewrite equation (1) as

$$y' + p(t)y = g(t).$$

Next, we multiply both sides of the equation with a function $\mu(t)$, which is called the integrating factor

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)g(t). \tag{2}$$

The next step is to find an appropriate $\mu(t)$ such that the left-hand side of (2) is equal to the derivative of a function. To achieve that, we recall the product rule

$$(f(t)h(t))' = f'(t)h(t) + f(t)h'(t).$$

We now notice that the left-hand side of (2) becomes a derivative if we set

$$\begin{aligned} &\mu'(t) = \mu(t)p(t) \Rightarrow \frac{\mu'(t)}{\mu(t)} = p(t) \Rightarrow \ln |\mu(t)| = \int p(t)dt + c_1 \Rightarrow \mu(t) = \pm e^{c_1} e^{\int p(t)dt} \Rightarrow \\ &\mu(t) = ce^{\int p(t)dt} \Rightarrow \mu(t) = e^{\int p(t)dt}, \end{aligned} \tag{3}$$

where we picked the simplest function $\mu(t)$ with $c = 1$. This is without loss of generality, since if we keep the constant c and plug $\mu(t)$ into (2), we can cancel it out since it appears in all three terms of the equation. Now, with $\mu(t)$ given by (3), it follows from (2)

$$\begin{aligned} &\mu(t)y'(t) + \mu'(t)y(t) = \mu(t)g(t) \Rightarrow (\mu(t)y(t))' = \mu(t)g(t) \Rightarrow \mu(t)y(t) = \int_{t_0}^{t} \mu(s)g(s)ds + c \\ &y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(s)g(s)ds + c \right). \end{aligned} \tag{4}$$

In (4), t_0 is some convenient lower limit of integration.

An alternative method of solution is the method of variation of parameters. We consider again (1) written equivalently as

$$y' + p(t)y = g(t) \quad \spadesuit.$$

We notice that if $g(t) = 0$, then

$$y' + p(t)y = 0 \Rightarrow y' = -p(t)y \Rightarrow \frac{y'}{y} = -p(t) \Rightarrow \ln|y(t)| = -\int p(t)dt + c_1 \Rightarrow y(t) = Ae^{-\int p(t)dt},$$

where A is a constant. Next, when $g(t)$ is not everywhere zero, we look for a solution of the form

$$y(t) = A(t)e^{-\int p(t)dt} \quad \clubsuit,$$

where A is now a function of the independent variable t. We substitute $\clubsuit$ into $\spadesuit$ and obtain

$$A'(t)e^{-\int p(t)dt} - p(t)A(t)e^{-\int p(t)dt} + p(t)A(t)e^{-\int p(t)dt} = g(t) \Rightarrow A'(t)e^{-\int p(t)dt} = g(t) \Rightarrow$$
$$A'(t) = g(t)e^{\int p(t)dt} \Rightarrow A(t) = \int_{t_0}^{t} g(s)e^{\int p(s)ds}ds + c.$$

If we now substitute this into $\clubsuit$, we obtain the same solution as (4)

$$y(t) = e^{-\int p(t)dt}\left(\int_{t_0}^{t} g(s)e^{\int p(s)ds}ds + c\right).$$

Problem Solve the IVP $ty' + (t+1)y = t$, with $y(\ln 2) = 1$ and $t > 0$.

Solution We write the ODE as

$$y' + \frac{t+1}{t}y = 1 \Rightarrow y' + \left(1 + \frac{1}{t}\right)y = 1.$$

We will use the method of the integrating factor

$$\mu y' + \mu\left(1 + \frac{1}{t}\right)y = \mu.$$

So, we need to solve

$$\mu' = \left(1 + \frac{1}{t}\right)\mu \Rightarrow \frac{\mu'}{\mu} = 1 + \frac{1}{t} \Rightarrow \ln|\mu| = t + \ln t + c_1,$$

since $t > 0$. This in turn implies

$$\mu(t) = \pm e^{c_1}te^t \Rightarrow \mu(t) = cte^t \Rightarrow \mu(t) = te^t.$$

In the final step we set $c = 1$. We now recall the fundamental theorem of calculus

$$\int_{t_0}^{t} f'(s)ds = f(t) - f(t_0),$$

and we apply it in what follows with $t_0 = \ln 2$

$$(te^t y)' = te^t \Rightarrow te^t y(t) - t_0e^{t_0}y(t_0) = \int_{t_0}^{t} se^s ds \Rightarrow te^t y(t) = \int_{\ln 2}^{t} se^s ds + 2\ln 2 y(\ln 2).$$

To proceed, we evaluate the integral using the method of integration by parts

$$\int_{\ln 2}^{t} se^s ds = te^t - 2\ln 2 - \int_{\ln 2}^{t} e^s ds = te^t - 2\ln 2 - e^t + 2.$$

This yields

$$te^t y(t) = te^t - 2\ln 2 - e^t + 2 + 2\ln 2 \Rightarrow te^t y(t) = te^t - e^t + 2 \Rightarrow y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}.$$