Lecture notes, Week 3, Day 2

Goal: learn about differences between linear and nonlinear equations and Bernoulli equations. (This is based on section 2.4 in edition 8.)

One of the most important questions in ODE theory, is the question about Existence and Uniqueness of solutions. For linear equations, the following key theorem provides the answer.

Existence and uniqueness theorem for the IVP of linear first order ODEs If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t) \tag{1}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Proof: Multiply (1) by the integrating factor

$$\mu(t) = e^{\int p(t)dt}.$$

Then, the solution becomes

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + c\right). \tag{2}$$

From this, it follows that since $p(t)$ in continuous in I, $\mu(t)$ is nonzero and continuously differentiable. Also, $g(t)$ is continuous, and so is $\mu(t)g(t)$ and therefore integrable. The integral $\int \mu(t)g(t)dt$ is in turn differentiable. We conclude that y given by (2) exists and is differentiable throughout I. By substituting (2) into (1) one may confirm that it is indeed a solution. It is actually the general solution of the ODE (1). Finally, the IC determines the constant c uniquely, so there is only one solution to the IVP. To see this, we may use definite integrals

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds} \qquad y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + c\right)$$

and by plugging in $t = t_0$ we obtain

$$\mu(t_0) = 1, \quad y(t_0) = c = y_0,$$

so the solution becomes

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0\right).$$

For nonlinear ODEs we have the following theorem.

Existence and uniqueness theorem for the IVP of nonlinear first order ODEs Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{3}$$

Remarks: (A) The hypotheses of the second theorem reduce to those of the first, if the ODE is linear

$$f(t, y) = -p(t)y + g(t) \Rightarrow \frac{\partial f}{\partial y} = -p(t),$$

since the continuity of f and $\frac{\partial f}{\partial y}$ is equivalent to the continuity of p and g.

(B) The conditions in the second theorem are sufficient, but not necessary. Actually, the existence (but not the uniqueness) of solutions can be proved if f is just continuous.

(C) A geometrical consequence of the uniqueness of solutions is that the graphs of two solutions cannot intersect each other. Otherwise, there would be two solutions satisfying the IC corresponding to the intersection point.

(D) Vertical asymptotes or other discontinuities in the solution can only occur at points of discontinuity of p or g. For a nonlinear equation, it is harder to determine the interval in which solutions exist. The solution $y = \phi(t)$ is certain to exist as long as $(t, \phi(t))$ remains in the region $\alpha < t < \beta$, $\gamma < y < \delta$.

(E) For first order linear ODEs we obtain a solution containing an arbitrary constant c, as in the first Theorem, from which all possible solutions follow by specifying values for the constant. For nonlinear equations, this may not be the case. Despite having such as constant c, not all possible solutions may follow by giving values to this constant.

Example 1: State where in the $ty-$plane the hypotheses of the second theorem are satisfied for

$$y' = \frac{1+t^2}{3y-y^2} = f(t,y).$$

f is continuous everywhere except at the lines $y = 0$ and $y = 3$. Its partial derivative with respect to y is $\frac{\partial f}{\partial y} = -\frac{1+t^2}{(3y-y^2)^2}(3-2y)$, which is also continuous everywhere except at the lines $y = 0$ and $y = 3$. Hence, the conditions of the theorem are satisfied everywhere except at $y = 0$ and $y = 3$.

Example 2: One may verify that $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ both solve the IVP

$$y' = \frac{-t + \sqrt{t^2+4y}}{2} = f(t,y), \quad y(2) = -1.$$

It holds

$$y_1' = -1, \quad \frac{-t+\sqrt{t^2+4y}}{2} = \frac{-t+\sqrt{t^2-4t+4}}{2} = \frac{-t+t-2}{2} = -1, \quad y_1(2) = -1$$

and

$$y_2' = -\frac{t}{2}, \quad \frac{-t+\sqrt{t^2+4y}}{2} = \frac{-t+\sqrt{t^2-t^2}}{2} = -\frac{t}{2}, \quad y_2(2) = -1.$$

$f(t,y)$ is continuous for $y \geq -\frac{t^2}{4}$ and $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2+4y}}$, so it is continuous for $y > -\frac{t^2}{4}$. Existence and uniqueness is guaranteed for $y > -\frac{t^2}{4}$. Hence, there is no violation of the theorem since the initial condition $(2,-1)$ does not satisfy this assumption. For the second solution it holds $y = -\frac{t^2}{4}$ which also violates the condition $y > -\frac{t^2}{4}$.

Bernoulli equations It is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation is

$$y' + p(t)y = g(t)y^n, \tag{4}$$

which is called a Bernoulli equation after Jakob Bernoulli.

Example 3 (a) Solve Bernoulli's equation when $n = 0, 1$.
If $n = 0$, then (4) becomes

$$y' + p(t)y = g(t),$$

which is a linear equation that can be solved by the method of the integrating factors

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(s)g(s)ds + c\right), \quad \mu(t) = e^{\int p(t)dt}.$$

If $n = 1$, then (4) becomes

$$y' + p(t)y = g(t)y \Rightarrow y' + (p(t) - g(t))y = 0,$$

which is a linear equation and separable equation, that can be solved by direct integration

$$y(t) = ce^{-\int (p(t)-g(t))dt}.$$

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.
Applying the chain rule yields

$$v' = (1-n)y^{-n}y'.$$

Next, we multiply (4) by y^{-n}

$$y^{-n}y' + p(t)y^{1-n} = g(t) \Rightarrow \frac{v'}{1-n} + p(t)v = g(t) \Rightarrow v' + (1-n)p(t)v = (1-n)g(t),$$

which is a linear equation in v that can be solved by the method of the integrating factor

$$v(t) = \frac{1}{\mu(t)}\left(\int \mu(s)(1-n)g(s)ds + c\right), \quad \mu(t) = e^{\int (1-n)p(t)dt}.$$

From this, one may obtain $y(t)$.