MATH 441, Practice problems for Exam 1

Problem 1

Determine an interval in which the solution to

$$\ln t y' + y = \cot t, \quad y(2) = 3$$

is certain to exist.

Solution: We express the ODE in the form

$$y' + \frac{1}{\ln t} y = \frac{\cot t}{\ln t}.$$

It must hold $t \neq 1$ since we need $\ln t \neq 0$ and $t > 0$ for $\ln t$ to be defined. Also, since $\cot t = \frac{\cos t}{\sin t}$, it must also hold $t \neq \pm k\pi$, where k is an integer. We must pick an interval that contains the initial point $t = 2$, so we pick $(1, \pi)$. Under these conditions, $p(t) = \frac{1}{\ln t}$ and $g(t) = \frac{\cot t}{\ln t}$ are continuous. Then, by the existence theorem for linear ODEs it follows that the solution exist.

Problem 2

Solve the following Bernoulli equation $t^2 y' + 2ty - y^3 = 0, \quad t > 0$.

Solution: We express the ODE in the form

$$y' + \frac{2}{t} y = \frac{y^3}{t^2}.$$

Since $n = 3$, it holds

$$v = y^{1-3} = \frac{1}{y^2} \Rightarrow v' = -2y^{-3} y' \Rightarrow v' - \frac{4}{t} v = -\frac{2}{t^2}.$$

The integrating factor is $\mu = \frac{1}{t^4}$ so we get

$$\frac{1}{t^4} v = \frac{2}{5t^5} + c \Rightarrow v = \frac{2}{5t} + ct^4 \Rightarrow y^2 = \frac{1}{\frac{2}{5t} + ct^4}$$

Problem 3

Solve the IVP

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1.$$

Determine the interval where the solution is valid.

Solution: This is a separable equation, it can be solved by direct integration.

$$\begin{aligned}(3y^2 - 6y)y' = 1 + 3x^2 &\Rightarrow y^3 - 3y^2 = x + x^3 + c \Rightarrow \\ 1 - 3 = c \Rightarrow c = -2 &\Rightarrow y^3 - 3y^2 = x + x^3 - 2\end{aligned}$$

However,it must hold

$$3y^2 - 6y \neq -0 \Rightarrow y \neq 0, \quad y \neq 2.$$

Hence, we should avoid the values of x that are roots to

$$x + x^3 - 2 = 0, \quad \text{and} \quad x + x^3 + 2 = 0.$$

While these can be solved numerically, one may stop here and state that we select the interval that includes $x = 0$ and avoid all these roots.

Problem 4

A tank originally contains 200 gal of salt water with a concentration of 1 lb/gal. The tank is rinsed with fresh water flowing in at a rate of 2 gal/min. Find the time until the salt quantity in the tank is 0.01 lb.

Solution: We assume that the well-mixed fluid flows out at the same rate. Then, the volume $V(t)$ stays constant at 200 gal and the initial salt quantity is $Q(0) = 200$ lb.

$$\begin{aligned}
&Q' = -2\frac{Q}{200} = -\frac{Q}{100} \Rightarrow \frac{Q'}{Q} = -\frac{1}{100} \Rightarrow \\
&\ln Q = c - \frac{t}{100} \Rightarrow Q = Ce^{-\frac{t}{100}} \Rightarrow \\
&Q(0) = C = 200 \Rightarrow Q(t) = 200e^{-\frac{t}{100}}.
\end{aligned}$$

Let T be the time when the quantity of the salt in the tank drops to 0.01 lb. It holds

$$200e^{-\frac{T}{100}} = 0.01 \Rightarrow e^{\frac{T}{100}} = 20,000 \Rightarrow T = 100\ln(20,000) = 100(4\ln 5 + 5\ln 2).$$

Problem 5

Consider the equation

$$(x+2)\sin y + x\cos y y' = 0.$$

Show that it is not exact. Then, find an integrating factor in the form $\mu(x,y) = \mu(x)$ and solve it.

Solution: It holds

$$\begin{aligned}
&M(x,y) = (x+2)\sin y, \quad N(x,y) = x\cos y \Rightarrow \\
&M_y = (x+2)\cos y, \quad N_x = \cos y \Rightarrow \\
&M_y \neq N_x.
\end{aligned}$$

We know look for $\mu(x)$ such that

$$(\mu M)_y = (\mu N)_x.$$

It follows

$$\begin{aligned}
&\mu(x+2)\cos y = \mu_x x\cos y + \mu\cos y \Rightarrow \frac{\mu_x}{\mu} = \frac{x+1}{x} \Rightarrow \ln|\mu| = x + \ln|x| + c \Rightarrow \\
&\mu = Cxe^x \Rightarrow \mu = xe^x.
\end{aligned}$$

The ODE then becomes

$$\begin{aligned}
&e^x(x^2+2x)\sin y + x^2e^x\cos y y' = 0 \Rightarrow (x^2e^x)'\sin y + x^2e^x(\sin y)' = 0 \Rightarrow \\
&x^2e^x\sin y = C.
\end{aligned}$$