Lecture notes, Week 2, Day 2

Goal: learn about the exact equations of first order

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{1}$$

for which there is a well-defined method of solution. (This is based on section 2.6 in edition 8.)

What is the main idea behind this type of equations? Suppose that we can identify a function ψ that satisfies

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y) \tag{2}$$

and such that

$$\psi(x,y) = c \tag{3}$$

defines $y = \phi(x)$ implicitly as a differentiable function of x. Then,

$$M(x,y) + N(x,y)\frac{dy}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,\phi(x))$$

and the ODE (1) becomes

$$\frac{d}{dx}\psi(x,\phi(x)) = 0. \tag{4}$$

In this case, (1) is said to be an exact equation. Solutions are given implicitly by (3), where c is an arbitrary constant.

The following theorem provides a systematic way of of determining whether a given ODE is exact.

Theorem Let the functions M, N, M_y and N_x, where the subscripts denote partial derivatives, be continuous in the rectangular region R: $\alpha < x < \beta$, $\gamma < y < \delta$. Then, (1)

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is an exact equation in R if and only if

$$M_y(x,y) = N_x(x,y) \tag{5}$$

at each point of R. That is, there exists a function ψ satisfying (2)

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y),$$

if and only if M and N satisfy (5).

Proof:
<u>Part A</u> Let a function ψ exist that satisfies

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y).$$

Computing M_y and N_x it holds

$$M_y(x,y) = \psi_{xy}(x,y), \quad N_x(x,y) = \psi_{yx}(x,y) \tag{6}$$

and since M_y, N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This in turn implies their equality, and hence (5) follows.

Part B Assume now that (5) holds. We are looking for a function ψ that satisfies (2). Set $\psi_x(x,y) = M(x,y)$ and integrate with respect to x, holding y constant

$$\psi(x,y) = \int_{x_0}^{x} M(s,y)ds + h(y) = Q(x,y) + h(y), \tag{7}$$

where x_0 is some specified constant in $\alpha < x_0 < \beta$. The function $h(y)$ is an arbitrary differentiable function of y, playing the role of the integration constant. One may check that differentiating (7) with respect to x yields the first of equations (2).

The next step is to show that we may choose $h(y)$ so that the second of equations (2)

$$\psi_y(x,y) = N(x,y)$$

also holds. Hence, we need to differentiate (7) with respect to y

$$\psi_y(x,y) = Q_y(x,y) + h'(y) = N(x,y).$$

From this we obtain

$$h'(y) = N(x,y) - Q_y(x,y). \tag{8}$$

But, we need to confirm that the right hand side depends only on y. This is true since

$$N_x - Q_{xy} = N_x - Q_{yx} = N_x - \frac{\partial}{\partial y} Q_x = N_x - \frac{\partial}{\partial y} M = N_x - M_y = 0,$$

since (5) holds. Hence, despite appearances the right hand side of (8) depends only on y. The proof finishes by finding $h(y)$ through integration of (8) and substituting the result into (7). This yields the required function ψ.

Example: Solve the following IVP and determine approximately where the solution is valid

$$2x - y + (2y - x)y'(x) = 0, \quad y(1) = 3.$$

We begin by setting $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. It holds $M_y = -1 = N_x$. Hence, by the previous theorem, the equation is exact. We then follow the same steps as in the proof to identify the function $\psi(x,y)$ that yields the solution. So, we set $\psi_x = M = 2x - y$ which implies

$$\psi(x,y) = x^2 - xy + h(y)$$

after we integrate with respect to x. Next, we differentiate this ψ with respect to y to obtain

$$\psi_y = -x + h'(y) = N = 2y - x \Rightarrow h'(y) = 2y \Rightarrow h(y) = y^2 + c_1.$$

Plugging this into the equation for ψ yields

$$\psi(x,y) = x^2 - xy + y^2 + c_1.$$

Since the solution is given implicitly by

$$\psi(x,y) = c,$$

we can ignore c_1 since it will be absorbed in c. So, the solution is

$$y^2 - xy + x^2 = c.$$

To find c we need to use the IC

$$3^2 - 3 + 1 = c \Rightarrow c = 7.$$

Therefore,

$$y^2 - xy + x^2 = 7 \Rightarrow y^2 - xy + x^2 - 7 = 0.$$

To obtain an expression for y as a function of x, we need to solve a quadratic equation

$$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2} \Rightarrow y = \frac{x \pm \sqrt{28 - 3x^2}}{2} \Rightarrow y = \frac{x + \sqrt{28 - 3x^2}}{2}.$$

From this, it follows that $|x| < \sqrt{\frac{28}{3}}$. We took the + sign to satisfy the IC, and we omit the endpoints of the interval $|x| < \sqrt{\frac{28}{3}}$ so the derivative of y is well defined.