Lecture notes, Week 4, Day 1

Goal: learn about modeling with first order ODEs. (This is based on section 2.3 in edition 8.)

Compound interest Suppose that a sum of money is deposited in a bank that pays interest at an annual rate r. The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as on the interest rate. Some financial institutions compound monthly, some weekly and some daily. Here, we assume that compounding takes place continuously, so we can set up an IVP that describes the growth of the investment.

The rate of change of the value of the investment $\frac{dS}{dt}$ is equal to the rate at which the interest accrues, which is equal to $rS(t)$. Hence,

$$\frac{dS}{dt} = rS, \quad S(0) = S_0 \Rightarrow S(t) = S_0 e^{rt}. \tag{1}$$

On the other hand, if compounding occurs at finite time intervals, say m times per year, then after t years

$$S(t) = S_0 \left(1 + \frac{r}{m}\right)^{tm}. \tag{2}$$

Therefore, as $m \to \infty$,

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{tm} = e^{\lim_{m\to\infty} tm \ln\left(1+\frac{r}{m}\right)} = e^{t \lim_{m\to\infty} m \ln\left(1+\frac{r}{m}\right)}. \tag{3}$$

The limit in the exponent has to be determined by l'Hospital's rule:

$$\lim_{m\to\infty} m \ln\left(1 + \frac{r}{m}\right) = \lim_{m\to\infty} \frac{\ln\left(1+\frac{r}{m}\right)}{\frac{1}{m}} = \lim_{m\to\infty} \frac{\frac{1}{1+\frac{r}{m}}\left(-\frac{r}{m^2}\right)}{-\frac{1}{m^2}} = \lim_{m\to\infty} \frac{r}{1+\frac{r}{m}} = r.$$

Hence, (3) becomes

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{tm} = e^{\lim_{m\to\infty} tm \ln\left(1+\frac{r}{m}\right)} = e^{rt},$$

which is identical to the e^{rt} in (1). This shows that, as expected, the continuous approximation (1) is a not bad approximation of (2) when m is large.

Returning to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to accrual of interest. If we assume that there are deposits/withdrawals that take place at a constant rate k, then

$$S' = rS + k \Rightarrow S' - rS = k, \tag{4}$$

where $k > 0$ for deposits and $k < 0$ for withdrawals. (4) is a first-order linear ODE that can be solved with the method of the integrating factors:

$$\begin{aligned} &\mu S' - r\mu S = k\mu \Rightarrow \mu' = -r\mu \Rightarrow \mu = e^{-rt} \Rightarrow \left(e^{-rt}S\right)' = ke^{-rt} = \left(-\frac{k}{r}e^{-rt}\right)' \Rightarrow \\ &e^{-rt}S = -\frac{k}{r}e^{-rt} + c_1 \Rightarrow S(t) = -\frac{k}{r} + c_1 e^{rt}, \end{aligned} \tag{5}$$

where c_1 is an arbitrary constant. If an IC $S(0) = S_0$ is given, then (5) yields

$$S(0) = c_1 - \frac{k}{r} = S_0 \Rightarrow c_1 = S_0 + \frac{k}{r},$$

so the solution to the IVP is

$$S(t) = S_0 e^{rt} + \frac{k}{r}\left(e^{rt} - 1\right), \tag{6}$$

where the first part corresponds to the return accumulated on the initial amount S_0 and the second term is due to the deposit or withdrawal at rate k. In reality, the rates r and k may depend on time t, since they usually fluctuate. In this case, the solution would be more complicated.

Example 1 A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously. (a) Determine the maximum amount that the buyer can afford to borrow. (b) Determine the total interest paid during the term of the mortgage.
Solution: (a) The amount owed by the home buyer to the bank satisfies the ODE

$$S' = rS + k,$$

where we have set $k < 0$ and it is actually $k = -12 * 800 = -9600$, t is in years, and $r = 0.09$. So, the solution is

$$S(t) = S_0 e^{0.09t} - \frac{9600}{0.09}(e^{0.09t} - 1).$$

Since the home owner wishes to pay off the mortgage in 20 years, it should hold

$$S(20) = 0 = S_0 e^{0.09*20} - \frac{9600}{0.09}(e^{0.09*20} - 1) \Rightarrow S_0 = \frac{9600}{0.09}(1 - e^{-1.8}) \Rightarrow S_0 = \$89,034.785.$$

(b) The total interest rate is

$$I = \int_0^{20} rS(t)dt = \int_0^{20} (S'(t) - k)dt = S(20) - S_0 - 20 * k = 20 * 9600 - 89,034.785 = 192,000 - 89,034.785 \Rightarrow$$
$$I = \$102,965.215.$$

Practice problem A rocket sled having initial speed of 150 mi/hr is slowed down by a channel of water. Assume that, during the braking process, the acceleration a given by $a(v) = -\mu v^2$, where v is the velocity and μ is a constant. (a) Use the relation $\frac{dv}{dt} = v\frac{dv}{dx}$ to write the equation of motion in terms of v and x. (b) If it requires a distance of 2000 ft to slow the sled to 15 mi/hr, determine the value of μ.
Solution: (a) It holds

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

from the chain rule and the definition of the derivative. Hence, based on the given information

$$a(v) = -\mu v^2 = v\frac{dv}{dx} \Rightarrow \frac{dv}{dx} = -\mu v.$$

(b) This is a separable equation that can be solved by direct integration

$$\frac{1}{v}v' = -\mu \Rightarrow \ln|v| = -\mu x + c_1 \Rightarrow v = \pm e^{c_1}e^{-\mu x} \Rightarrow v = ce^{-\mu x}.$$

When $x = 0$ it holds $v(0) = c = 150$, so the velocity as a function of x is

$$v(x) = 150e^{-\mu x}.$$

We need to have consistent distance units. 1 mile corresponds to 5280 ft. So, when $x_0 = \frac{2000}{5280}$ miles, the velocity is 15 mile/hr. This implies that

$$15 = 150e^{-\mu\frac{25}{66}} \Rightarrow e^{\mu\frac{25}{66}} = 10 \Rightarrow \frac{25}{66}\mu = \ln 10 \Rightarrow \mu = \frac{66}{25}\ln 10,$$

and this constant has units of inverse miles.