Lecture notes, Week 2, Day 1

Goal: learn about the separable equations of first order

$$\frac{dy}{dx} = f(x,y). \tag{1}$$

Here, f depends on the independent variable x and the dependent variable $y(x)$ (the unknown function). (This is based on section 2.2 in edition 8.)

If equation (1) is linear, then we can use direct integration, the method of integrating factors, or the method of variation of parameters to solve it. If it is nonlinear, then no general method for solving it exists. Here we will learn how to solve a subclass of first order equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \tag{2}$$

If we set $N(x,y) = 1$ and $M(x,y) = -f(x,y)$, then (1) becomes (2).
We now make the additional assumption that $M(x,y) = M(x)$ and $N(x,y) = N(y)$. Equation (2) then takes the form

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{3}$$

Such an equation is called separable. This is the form of the subclass that we will attempt to solve. Let $H_1(x)$ and $H_2(y)$ be antiderivatives of M and N, respectively. This implies that

$$H_1'(x) = M(x), \quad H_2'(y) = N(y). \tag{4}$$

We substitute these into (3) and obtain

$$\frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} = 0,$$

and if we use the chain rule

$$\frac{dH_2(y(x))}{dx} = \frac{dH_2}{dy}\frac{dy}{dx}$$

it holds

$$\frac{d}{dx}\left(H_1(x) + H_2(y(x))\right) = 0. \tag{5}$$

Integrating this yields the solution in implicit form

$$H_1(x) + H_2(y(x)) = c, \tag{6}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies (6) is a solution of (3).
Assume now an IC is assigned: $y(x_0) = y_0$. Equation (6) yields

$$c = H_1(x_0) + H_2(y_0).$$

We may in turn substitute this into (6) to obtain

$$\begin{aligned}
&H_1(x) + H_2(y(x)) = H_1(x_0) + H_2(y_0) \Rightarrow \\
&H_1(x) - H_1(x_0) + H_2(y) - H_2(y_0) = 0 \Rightarrow \\
&\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0,
\end{aligned} \tag{7}$$

where equation (4) and the fundamental theorem of calculus

$$\int_{x_0}^{x} M(s)ds = \int_{x_0}^{x} H_1'(s)ds = H_1(x) - H_1(x_0), \qquad \int_{y_0}^{y} N(s)ds = \int_{y_0}^{y} H_2'(s)ds = H_2(y) - H_2(y_0)$$

were used to obtain equation (7). Equation (7) is an implicit solution of the solution of (3) that also satisfies the IC $y(x_0) = y_0$. A caveat is that one may not always be able to solve the implicit formula for the solution explicitly, namely to write a formula for y as a function of x.

If the right hand side of (1) $y' = f(x, y)$ can be written as a function of the ratio $\frac{y}{x}$ only, then the equation is called homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Specifically, we let

$$y'(x) = g\left(\frac{y}{x}\right) \tag{8}$$

and set $v = \frac{y}{x} \Rightarrow y = xv$. Next, it holds

$$y' = v + xv'$$

and this in turn implies from (8)

$$\begin{aligned} &v + xv' = g(v) \Rightarrow xv' = g(v) - v \Rightarrow x\frac{v'}{g(v) - v} = 1 \Rightarrow \frac{v'}{g(v) - v} = \frac{1}{x} \Rightarrow \\ &\frac{1}{g(v) - v}\frac{dv}{dx} = \frac{1}{x} \Rightarrow -\frac{1}{x} + \frac{1}{g(v) - v}\frac{dv}{dx} = 0, \end{aligned} \tag{9}$$

which is separable. We assume here that $x \neq 0$ and $g(v) - v \neq 0$, so we can freely divide. We notice that if $g(v) = v$ then the equation would be trivial:

$$v + xv' = g(v) = v \Rightarrow xv' = 0.$$

Example Consider the ODE

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\frac{y}{x}}. \tag{10}$$

Hence, the equation is homogeneous. Next, we set $v = \frac{y}{x}$ which yields $y = xv \Rightarrow y' = v + xv'$. Then,we obtain from (10)

$$\begin{aligned} &v + xv' = \frac{1 + 3v^2}{2v} \Rightarrow xv' = \frac{1 + 3v^2}{2v} - v \Rightarrow xv' = \frac{1 + 3v^2}{2v} - \frac{2v^2}{2v} \Rightarrow \\ &xv' = \frac{1 + v^2}{2v} \Rightarrow x\frac{2v}{1 + v^2}\frac{dv}{dx} = 1 \Rightarrow \frac{2v}{1 + v^2}\frac{dv}{dx} = \frac{1}{x} \Rightarrow -\frac{1}{x} + \frac{2v}{1 + v^2}v' = 0. \end{aligned} \tag{11}$$

Now, this is a separable equation like (3) with

$$M(x) = -\frac{1}{x}, \quad N(v) = \frac{2v}{1 + v^2}.$$

From these expressions it is actually easy to see (by using the derivative of ln) that

$$H_1(x) = -\ln|x|, \quad H_2(v) = \ln(1 + v^2).$$

Hence, by rewriting the equation in terms of these antiderivatives and applying the chain rule, we obtain the general solution implicitly

$$\begin{aligned} &\frac{dH_1}{dx} + \frac{dH_2}{dv}\frac{dv}{dx} = 0 \Rightarrow H_1(x) + H_2(v) = c_1 \Rightarrow \ln(1 + v^2) - \ln|x| = c_1 \Rightarrow 1 + v^2 = e^{c_1}|x| = \pm e^{c_1}x = cx \Rightarrow \\ &v^2 = cx - 1 \Rightarrow \frac{y^2}{x^2} = cx - 1 \Rightarrow y^2 = cx^3 - x^2. \end{aligned} \tag{12}$$