Lecture 01, Math 447

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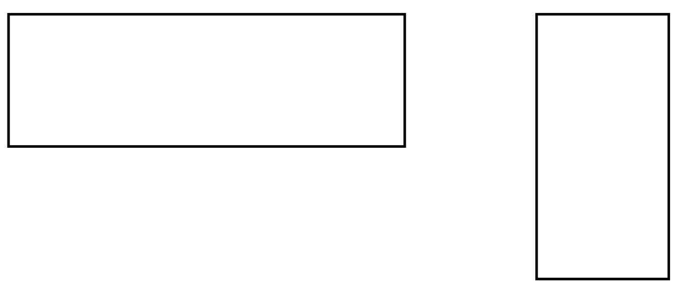
Introduction To Real Analysis: Goals

- Have fun
- Proofs - rigorous foundation of calculus
- $\mathbb{R}$ (not $\mathbb{C}$ as much – hence *real* analysis)
- Motivating problems
  - Physics
  - Fix perimeter of rectangles; maximize area.



Self-contained proofs that the tools you're used to are valid.

- Metric spaces (topological properties)

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Question: What is $\mathbb{R}$?

Answer 1) A complete ordered field (it's unique).

Answer 2) $\mathbb{R}$ is a particular subset of $2^{\mathbb{Q}}$.

$$\left(2^A = P(A) = \left\{ B \,\middle|\, \underset{B \text{ set}}{B \subset A} \right\}\right)$$

What is a set? Hard question! For

a logician (not for this class).

Russel's paradox:

$$R = \left\{ A \,\middle|\, A \notin A \right\}$$

Q: Is $R \in R$?

$$R \in R \iff R \notin R$$

Ans: R is not a set.

Plan:

$$\begin{tikzcd}
\mathbb{N} \arrow[d, bend right] \\
\mathbb{Z} \arrow[d, bend right] \\
\mathbb{Q} \arrow[uu, bend right, "\text{Grothendieck Lemma}"'] \arrow[d, bend left, "\text{different}"] \\
\mathbb{R}
\end{tikzcd}$$

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Axioms for $\mathbb{N}$ (proposed)

$\mathbb{N}$ a set, $S : \mathbb{N} \to \mathbb{N}$ function

a1) $1 \in \mathbb{N}$.

a2) $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$.

a3) If $A \subset \mathbb{N}$ is a set such that

$$1 \in A$$

$$n \in A \implies (n+1) \in A,$$

then $A = \mathbb{N}$. (domino prin.)

b1) $\mathrm{Im}(S) = \mathbb{N} \backslash \{1\}$.

b2) S injective.

b3) If $A \subset \mathbb{N}$ is a set such that

$$S(A) \subset A$$

$$1 \in A,$$

then $A = \mathbb{N}$.

Formal link: $S(n) = n + 1$.

Then a3 $=$ b3, and b1 is "close to" a1 & a2, but then there's b2...

Do these axioms produce the same version of $\mathbb{N}$?

To be precise, natural numbers is the triple $(\mathbb{N}, S, 1)$ (axioms b1-3).

The triple in a1-3 would include the function "$+1$" $n \mapsto n+1$.

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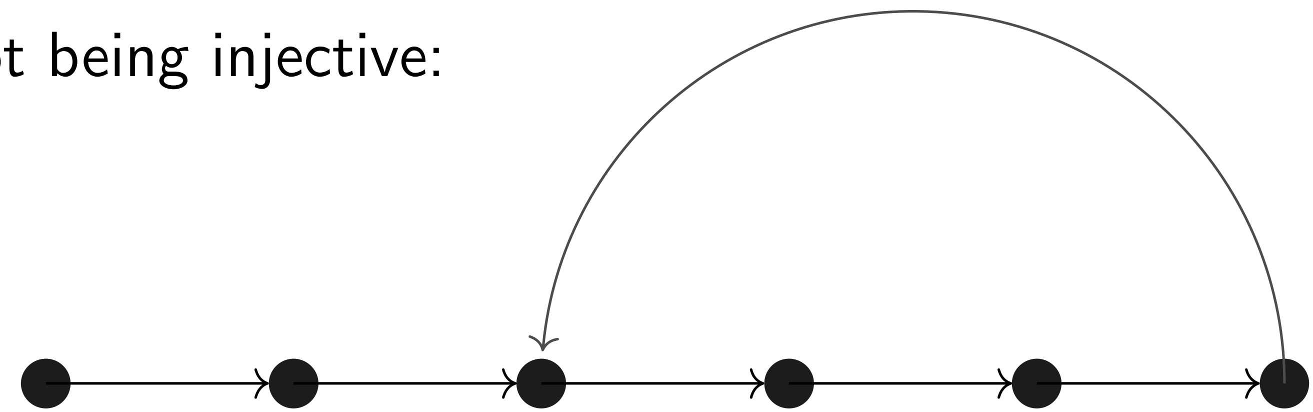
Axioms for $\mathbb{N}$

Think about S not being injective:



This model satisfies a1-3 but it's not $\mathbb{N}$. What's missing?

What can we add to a1-3 to make the axioms equivalent?

One element with two predecessors (noninjectivity) $\implies$ a loop.

How could we modify a1-3 to prevent loops?

Noninjectivity $\implies$ a loop $\implies$ the set is finite.

Axioms for $\mathbb{N}$ (really)

a0) $\mathbb{N}$ is not finite

a1) $1 \in \mathbb{N}$.

a2) $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$.

a3) If $A \subset \mathbb{N}$ is a set such that

$$1 \in A$$

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$\mathbb{N}$ a set, $S : \mathbb{N} \to \mathbb{N}$ function

b1) $\mathrm{Im}(S) = \mathbb{N} \backslash \{1\}$.

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Sharpen your skills:

1. Prove a0-3 and b1-3 are equivalent.
2. Add axiom: b0) $\mathbb{N}$ is not finite

   Prove b0) $\implies$ S is injective (b2)

How to go from $\mathbb{N}$ to $\mathbb{Z}$? Add negative numbers.
$\mathbb{Z}$ is the smallest abelian group that contains $\mathbb{N}$.
Look Up: abelian group or commutative group

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