Lecture 05, Math 447

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Real Numbers

Last time:

$\mathbb{R} = \{D \subseteq \mathbb{Q} \mid D \text{ Dedekind},\ D \neq \mathbb{Q}\}$.

has structure $D_1 + D_2 = \{d_1 + d_2 \mid d_1 \in D_1\ d_2 \subset D_2\}$.

multiplication similar, adjusting for negatives.

Properties: 1) $\mathbb{Q}$ are dense in $\mathbb{R}$.

2) $\mathbb{R}$ are not countable.

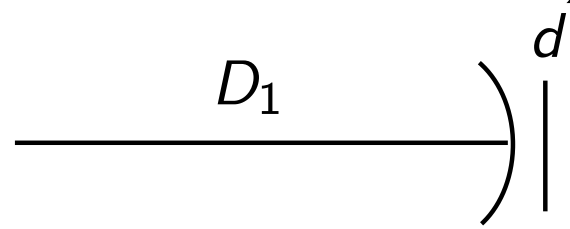
3) $\forall a \in \mathbb{R}, a > 0 \quad \exists n \in \mathbb{N} \quad n > a$. (Archimedean property)

We'll prove 3 later

ad 1) Let $D_1 < D_2 \in \mathbb{R}$ Dedekind cuts. (show $\exists$ rational in between.)

This means $D_1 \subsetneq D_2$.

This means $\exists\ d \in D_2 \quad d \notin D_1$.



This means $D_1 \subsetneq \underbrace{(-\infty, d)}_{\varphi(d)} \subsetneq D_2$.

(recall $\varphi(d) = (-\infty, d)$)

This means $D_1 < \varphi(d) < D_2$. $\square$

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Lemma: $2^{\mathbb{N}}$ is not countable

<u>Notes</u> D-cut makes $\mathbb{Q}$ complete wrt. order; only way to fill all the holes.
The other algebraic relations follow from the relations of $\mathbb{Q}$.
Density comes for free.

ad 2) **Lemma** $2^{\mathbb{N}}$ is not countable.

<u>Proof</u> Assume $2^{\mathbb{N}}$ is countable: $\exists\ \varphi\colon \mathbb{N} \to 2^{\mathbb{N}}$ which is a bijection.

Consider $\{n \in \mathbb{N} \mid n \notin \varphi(n)\} = B$ (which is $\subset \mathbb{N}$).

Since $B \in 2^{\mathbb{N}}$ and φ is a bijeciton, $\exists m\ \varphi(m) = B$.

$$\begin{cases} \text{case 1: } m \in B \Rightarrow m \notin \varphi(m) = B, \rightarrow\!\leftarrow . \\ \text{case 2: } m \notin B = \varphi(m) \Rightarrow m \in B, \rightarrow\!\leftarrow . \end{cases} \qquad \square$$

Note $\mathbb{N}$ and $2^{\mathbb{N}}$ are sets; countability of $2^{\mathbb{N}} \iff \exists$ bijection φ above.

There is no set-theoretic problem here like in Russel's antinomy.

$[0,1]$ is not countable

We want to show $\mathbb{R}$ are uncountable.

If a set is countable then every subset is either countable or finite (since every subset of $\mathbb{N}$ is either countable or finite).

Lemma Let $\varphi\colon 2^{\mathbb{N}} \to [0,1]$ be defined by

$$\varphi(A) = \sum_{k\in A} 3^{-k} = \sup\Big\{ \sum_{\substack{k\in A\\ k\le n}} 3^{-k} \;\Big|\; n \in \mathbb{N}\Big\}.$$

Claim 1) $\varphi(A)$ well-defined (sup exists).

$$\sum_{\substack{k\in A\\ k\le n}} 3^{-k} \le \sum_{1\le k\le n} 3^{-k} = \tfrac{1}{3}\Big[\sum_{k\le n-1} 3^{-k}\Big] = \tfrac{1}{3}\Big[\tfrac{1-3^{-n}}{1-1/3}\Big]$$
$$\le \tfrac{1}{3}\Big(\tfrac{1}{2/3}\Big) \le \tfrac{1}{2} \implies \text{set is bounded above.}$$

Claim 2) $A \neq B \implies \varphi(A) \neq \varphi(B) \quad (\varphi \text{ is injective}).$

This is sufficient for our goal, since then $[0,1]$ has an uncountable subset ($\mathrm{Im}(\varphi) \subset [0,1]$), and so $[0,1]$ cannot be countable.

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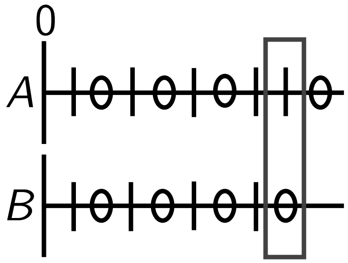
Proof continued: injectivity of φ



If $A \neq B$ there is a first place this pattern disagrees.

Let $m_0 = \min\{m \in A \mid m \notin B\}$.

$\forall j < m_0 \quad j \in A \iff j \in B$.

(Let m_0 be this first disagreement. Assume this occurs with $m_0 \in A \setminus B$; else switch A and B.)

Then $\varphi(A) \geq \sum_{\substack{j<m_0 \\ j\in A}} 3^{-j} + 3^{-m_0}$. *Denote* $\alpha_{m_0} = \sum_{\substack{j<m_0 \\ j\in A}} 3^{-j}$.

$\varphi(B) = \sup_k \sum_{\substack{j\leq k \\ j\in A}} 3^{-j}$. *Denote* $\varphi_k(B) = \sum_{\substack{j\leq k \\ j\in A}} 3^{-j}$.

<u>fix k</u> $\qquad \varphi_k(B) = \alpha_{m_0} + 0 + \sum_{\substack{m_0<j\in B \\ j\leq k}} 3^{-j}$.

$$\sum_{m_0<j\leq k} 3^{-j} \leq \sum_{j>m_0} 3^{-j} = 3^{-(m_0+1)} \sum_{j\geq 0} 3^{-j} =$$

$$= 3^{-(m_0+1)} \frac{1}{1-1/3} = 3^{-(m_0+1)} \frac{3}{2} \leq \frac{3^{-m_0}}{2}.$$

$$\varphi_k(B) \leq \alpha_{m_0} + \frac{3^{-m_0}}{2} < \alpha_{m_0} + 3^{-m_0} \leq \varphi(A).$$

$$\implies \varphi(B) = \sup_k \varphi_k(B) \leq \alpha_{m_0} + \frac{3^{-m_0}}{2} < \varphi(A). \qquad \square$$

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$\mathbb{R}$ is not countable

Thm $\mathbb{R}$ is not countable.

Proof Let $\varphi : 2^{\mathbb{N}} \to \mathbb{R}$ be given as above. Note that φ is injective.

$\mathbb{R}$ countable $\implies$ $[0,1]$ countable $\implies$ $\varphi(2^{\mathbb{N}})$ countable

$\implies$ $2^{\mathbb{N}}$ countable. This contradicts Lemma 1. $\square$

Usual proof uses function $A \mapsto \sum_{j \in A} 2^{-j}$ instead.

Not injective because $\sum_{j>k} 2^{-j} = 2^{-k}$ (so replacing a final 1 with a 0 followed by an infinite sequence of 1s gives two different sets for the same number).

One must prove the set of all infinite subsets of $\mathbb{N}$ is uncountable (harder).

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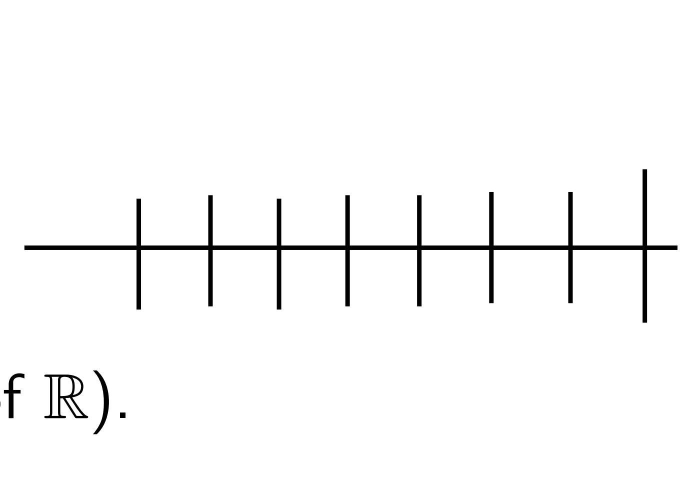
Archimedean property of $\mathbb{N}$

Claim $\forall a \in \mathbb{N} \;\; \exists n \in \mathbb{N} \quad a < n$.



<u>Proof</u> Assume not.

Then $\mathbb{N}$ are bounded from above.

Let $S = \sup \mathbb{N}$ (exists by completeness of $\mathbb{R}$).

Why does this not make any sense?

Recall $S = \sup A$ means: 1) $\forall a \in A \;\; a \leq S$

2) $\forall \epsilon > 0 \;\; \exists a \in A \quad S - \epsilon < a$.

$\exists n \quad S - \frac{1}{2} < n < S$

because we can choose $\epsilon = \frac{1}{2}$.

Then $n + 1 \leq S, \qquad n + 1 > S - \frac{1}{2} + 1 \geq S + \frac{1}{2} \;\; \rightarrow\!\times\!\leftarrow$.

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Sequences of Real Numbers

$$(x_n)_{n\geq 1} \qquad (x_n)_{n\in\mathbb{N}}$$ *what do these mean?*

Def A *sequence* of real numbers is given by a function $x : \mathbb{N} \to \mathbb{R}$.

Notation: $\boxed{x_n = x(n)}$

Def (x_n) is *monotone* if either:

a) $\forall n \quad x(n) \leq x(n+1)$ non-decreasing or "increasing"

or

b) $\forall n \quad x(n) \geq x(n+1)$ non-increasing or "decreasing".

Note: "increasing" (likewise "decreasing") does not mean *strictly* increasing b/c terms could be equal.

Next time:

Every sequence has a monotone subsequence.

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