Lecture 04, Math 447

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What is $\mathbb{R}$?

$\boxed{\text{A1}}$ $\mathbb{R}$ is the (unique) totally ordered complete field s.t. $\mathbb{Q}$ is dense.

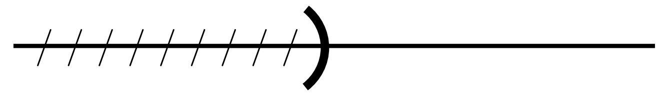
$\boxed{\text{A2}}$ Construction...

$\mathbb{Q} = G(\mathbb{Q}^+, +, 0)$
$\mathbb{Q}^+ = G(\mathbb{N}, \cdot)$

$\rightarrow$ totally ordered field.

Def A Dedekind cut is a subset $A \subseteq \mathbb{Q}$ such that

1) $a \in A, \quad b \leq a \Rightarrow b \in A$



2) $\forall a \in A \; \exists \; a' > a \quad a' \in A$

"Example":

1) Define $\quad R = \{D \subset \mathbb{Q} \mid D \text{ Dedekind}\}$.

We have $\varphi : \mathbb{Q} \rightarrow R \quad$ defined as $\quad \varphi(q) = \{a \mid a < q\}$.

φ is injective.

2) $A_{\sqrt{2}} = \{x \in \mathbb{Q} \mid x^2 < 2 \text{ or } x \leq 0\}$ is Dedekind.

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Proof of "Example" 2

Trivial for $x \leq 0$. Let $0 < \frac{p}{q} \in A_{\sqrt{2}}$. Let $\frac{p'}{q'} < \frac{p}{q}$.

Claim: $(\frac{p'}{q'})^2 < 2$. Monotonicity $\Rightarrow (\frac{p'}{q'})^2 < (\frac{p}{q})^2 < 2$. ✓

Claim: let $\frac{p}{q}$ such that $(\frac{p}{q})^2 < 2$ $(\iff p^2 < 2q^2)$.

then $\boxed{\exists\ \frac{p'}{q'} > \frac{p}{q} \quad p'^2 < 2(q')^2.}$

Intuitively, we want a number still $< \sqrt{2}$ just a little bigger than $\frac{p}{q}$.
How would we do that? ...... Take decimal expansions. Since $\frac{p}{q} < \sqrt{2}$, at some point the expansion must reflect this and you have some room.

$\frac{p'}{q'} = \frac{p}{q} + \frac{d}{q^m} < \sqrt{2}$.

$(\frac{p}{q} + \frac{d}{q^m})^2 < 2$.

Need $(pq^{m-1} + d)^2 < 2(q^{2m})$.

...find d and make m large enough so this works....

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Proof continued

Take $d = 1$: $$p^2 q^{2(m-1)} + 2pq^{(m-1)} + 1 < 2q^{2m}.$$

$$p^2 + 2pq^{-(m-1)} + \frac{1}{q^{2(m-1)}} < 2q^2.$$

Now, $\frac{1}{q^{2(m-1)}} < \frac{1}{q^{m-1}}$ so

$$p^2 + 2pq^{-(m-1)} + \frac{1}{q^{2(m-1)}} < p^2 + \frac{2p+1}{q^{m-1}}.$$

Solution: Assume $q > 1$, choose m so that $\frac{2p+1}{q^{m-1}} < 2q^2 - p^2$.

Then $\frac{p'}{q'} = \frac{p}{q} + \frac{1}{q^m}$ works. $\square$

Prove for homework: Claim $\boxed{\sup D_{\sqrt{2}} = \sqrt{2}}$

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R is complete

Def $D_{1,2} \in R$. Define $D_1 \leq D_2$ if $D_1 \subseteq D_2$.

Lemma 1 R is complete.

Proof Let $S \subseteq R$ and $D \in R$ $\forall s \in S \quad s \leq D \quad (s \subseteq D)$.

Define $\sup S = \bigcup_{s \in S} s$ Note $\bigcup_{s \leq D} s \subseteq D$.

D is a D-cut: 1) $a \in D$ $b \leq a \Rightarrow b' \in D$

2) $\forall a \in D \; \exists \; a' > a \quad a \in D$.

Obviously $\sup S \in R$ (check two cndtns above) and $\sup S \leq D$.

Claim sup S is the smallest upper bound.

Indeed let $D' \in R$ such that $\forall s \quad s \leq D'$.

$\implies \forall s \quad s \subseteq D' \implies \bigcup_{s \in S} s \subseteq D' \implies \sup S \leq D'$.

note correction.

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Definition of $\mathbb{R}$ (as a set)

R cannot be the real numbers... just a bit too big:

$\mathbb{Q}$ is a Dedekind cut ($\in R$) that corresponds to ∞ ($\notin \mathbb{R}$).

Def $\mathbb{R} = \{D \subseteq \mathbb{Q} \mid D \neq \mathbb{Q}\}$.

If $D \neq \mathbb{Q}$ take $q \notin D$. Then no $q' > q$ can be in D, so also

$$\mathbb{R} = \{D \in R \mid \exists\ q \in \mathbb{Q} \quad D < \varphi(q)\}.$$

Claim $\mathbb{R}$ is complete.

If $D \in \mathbb{R}$ then $\exists q_0$ such that $D < \varphi(q_0)$.

$\implies \sup D < \varphi(q_0) \implies \sup D \neq \infty \implies \sup D \in \mathbb{R}$.

Rest of proof is the same for $\mathbb{R}$ as for R.

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Field operations of $\mathbb{R}$

$\mathbb{R}^+ = \{D \in \mathbb{R} \mid (-\infty, 0] \subseteq D\}$.

Define $$D_1 + D_2 = \bigcup\{d_1 + d_2 \mid d_1 \in D_1^+ \; d_2 \in D_2^+\}.$$
$$D_1 D_2 = \bigcup\{d_1 d_2 \mid d_1 \in D_1^+ \; d_2 \in D_2^+\}.$$

Since negative #s complicate multiplication slightly,
assume each $D \in \mathbb{R}^+$ is $\subset \mathbb{Q}^+$ (they start at 0, not $-\infty$).
View as $\mathbb{R}^+ = \{D \subset \mathbb{Q}^+ \mid D \cup (-\infty, 0]$ is Dedekind$\}$.

Properties: $$D_1 + (D_2 + D_3) = (D_1 + D_2) + D_3$$
$$D_1(D_2 + D_3) = D_1 D_2 + D_1 D_3$$
etc. (associativity, commutativity...)

$\mathbb{R}^+$ semigroup under $+$ with cancellation (lacks "$-$"s and 0).

Final answer: $\mathbb{R} = G(\mathbb{R}^+)$.

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$\mathbb{R}$ is totally ordered

$\mathbb{R}^+$ totally ordered: D_1 and D_2 D-cuts $\implies D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Lemma $\mathbb{R}$ is totally ordered.

<u>Proof</u> D_1 and $D_2 \in \mathbb{R}$.

Case 1: $D_1 \subseteq D_2$. Then done.

Case 2: $D_1 \not\subseteq D_2$. Then $\exists d_1 \in D_1 \quad d_1 \notin D_2$

$\implies D_2 < d_1 \implies D_2 \subseteq D_1$. $\square$

(this is a sketch; stated for $\mathbb{R}$ but proven for $\mathbb{R}^+$.)

Next time:

$\mathbb{R}$ uncountable

$\mathbb{Q}$ dense in $\mathbb{R}$.

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