Homework 2

Question 1

$\mathbb{N}$ is not bounded.

Consider the $\mathbb{N}$ is bounded, then we can find a supremum $\alpha \in \mathbb{Z}$, and since $\alpha - 1 < \alpha$. $\alpha - 1$ will not be upperbound of $\mathbb{N}$.

Then we can find

$$\beta > \alpha - 1 \quad \beta \in \mathbb{N}$$
$$\Rightarrow \beta + 1 > \alpha$$

Since $1, \beta \in \mathbb{N}$, $\beta + 1 \in \mathbb{N}$. Then we find a contradiction. Thus $\mathbb{N}$ is not bounded.

Question 2

Again, we can prove by contradiction.

Consider that $a > b$. and we know $\mathbb{N}$ is not bounded. We can know there exist $\alpha \in \mathbb{N} \wedge \alpha > N$ such that

$$a \leq b + \frac{1}{\alpha} < b + (a - b) = a$$

Then we find a contradiction. Thus $a \leq b$.

Question 3

Again, we can prove by contradiction.

Since (x_n) is a convergent sequence, consider the it have a limit x, where $x > b$. There exists a positive number α, where $\alpha = x - b > 0$.

Since, $a \leq x_n \leq b$, it cannot have $|x_n - x| < \alpha$, where $n \in \mathbb{N}$. Which is a contradiction, and thus $a \leq x \leq b$.

Question 4

(The Idea is From Tianyue Cao)

Clearly, all monotone function $f : \mathbb{N} \to \mathbb{N}$ without upperbound and the fucntion is increasing. So in this question we need to prove there exists a monotone function $g : \mathbb{N} \to \mathbb{N}$ with $y = lim_{m \to \infty} x_{g(m)}$.

Consider for all $m \in \mathbb{N}$, there have a $g(m)$, where $|y_m - x_{g(m)}| < |y - y_m|$. Now we will use induction.

- Base Case: Since $\left\{x_{f_1(n)}\right\}_1^\infty$ converges to y_1.

$$\left|y_1 - x_{f_1(n)}\right| < |y - y_1|$$

Where $N \in \mathbb{N} \wedge n \geq N$. Thus $g(1) = f_1(N)$.

- Induction Hypothesis: Assume when $\alpha < k$, from $g(1)$ to $g(\alpha)$ is increasing function and all follow $\left|y_m - x_{f_m(n)}\right| < |y - y_m|$.

- Induction Case:

Since $\left\{x_{f_1(n)}\right\}_1^\infty$ converges to y_{k+1}, There will have

$$\left|y_{k+1} - x_{f_{k+1}(n)}\right| < |y - y_{k+1}|$$

Where $N_{new} \in \mathbb{N} \wedge n \geq N_{new}$.

Since f_{k+1} is monotone increasing and unbounded above,

$$f_{k+1}(n) > g(k) + 1$$

Where $N_{new2} \in \mathbb{N} \wedge n > N_{new2}$.

Consider, $g(k+1) = f_{k+1}(N)$, where $N = max\{N_{new1}, N_{new2}\}$.

$$g(k+1) > g(k)$$

Thus we can know the construction g is a monotone increasing.

Also, we can know

$$\left|y_{k+1} - x_{g(k+1)}\right| < |y - y_{k+1}|$$

Thus we can know the construction g is also follow $\left|y_m - x_{g(m)}\right| < |y - y_m|$.

Now we start to prove,for all $\varepsilon > 0$ $y = \lim_{m\to\infty} x_{g(m)}$, There exists $N_{new3} \in N$ s.t. $|y - y_{N_{new3}}| < \frac{\epsilon}{2}$

$$\left|y - x_{g(N_{new3})}\right| \leq |y - y_{N_{new3}}| + \left|y_{N_{new3}} - x_{g(N_{new3})}\right| \leq 2\,|y - y_{N_{new3}}|$$

Where is less that ε . And by conclusion $y = \lim_{m\to\infty} x_{g(m)}$ for $\varepsilon > 0$.

Thus we can know $y \in Lim(X_n)$.

Question 5

Consider s_n is a convergent sequence, and $lims_n = s$. Then consider $\varepsilon = 1$. $n > N$ and it implies $|s_n - s| < 1$. By the triangle inequality, we can know $|s_n| - |s| < 1 = |s_n| < |s| + 1$.

Consider $M = max\{|s| + s, |s_1|, \ldots, |s_N|\}$. we will get $|s_n| \leq M$ for all $n \in N$.

Which implies convergent sequence is bounded.