Homework 1-Math 447

Due Date: September 4 in Box folder

DEFINITION 0.1. *i) A map $\phi : G \to H$ between two groups is called group homomorphism if*

$$\phi(gh) \;=\; \phi(g)\phi(h)\,.$$

holds for all g, h and $\phi(1) = 1$. The map is called a group endomorphism if ϕ is injective. Finally group isomorphism if in addition ϕ is bijective.
ii) A subset $H \subset G$ is called a subgroup if $g, h \in H$ implies $gh \in H$, and $1 \in H$.

i) Let G be an abelian group and $T \subset G$ be a nonempty subset. Show that $G(T) = \{gh^{-1} | g, h \in T\}$ is a subgroup of G. The set $G(T)$ is called the subgroup generated by T.

ii) Let $T_1 \subset G_1$ and $T_2 \subset G_2$ be two subsets of abelian groups. Let $\phi : T_1 \to T_2$ be a map such that

$$gh^{-1} \;=\; 1 \quad \Rightarrow \quad \phi(g)\phi(h)^{-1} \;=\; 1\,.$$

Show that $\psi(gh^{-1}) = \phi(g)\phi(h)^{-1}$ is a well-defined map and a group homomorphism from $G(T_1)$ to $G(T_2)$.

iii) Let S be a semigroup with cancellation property, $\tilde{G}$ and abelian group and $\phi : S \to \tilde{G}$ be such that

$$(ac \;=\; bd) \quad \Rightarrow \quad (\phi(a)\phi(b)^{-1} \;=\; \phi(c)\phi(d)^{-1})\,.$$

Show that there exists a group homomorphism $\psi : G(S) \to \tilde{G}$ such that $\psi(s) = \phi(s)$ for $s \in S$.

iv) Let S be an abelian semigroup with the cancelation property. Let $\phi_{1,2} : S \to G_{1,2}$ be two injective maps with values in abelian groups. Show that there exists a group isomorphism between $G(\phi_1(S))$ and $G(\phi_2(S))$.

v) Let $G(\mathbb{N})$ be the Grothendieck group of the semigroup $(\mathbb{N}, +)$ (you may assume that this is an abelian semigroup with cancelation property). Show that we have a disjoint decomposition

$$G(\mathbb{N}) \;=\; \{1\} \cup \phi(\mathbb{N}) \cup \{\phi(n)^{-1} | n \in \mathbb{N}\}\,.$$

Warning: On the right hand side we use multiplicative notation instead of the more familiar $+$. **Hint:** Show that the right hand side can be made into an abelian group or show that the definition of equivalence classes in the Grothendieck construction has exactly the form above.