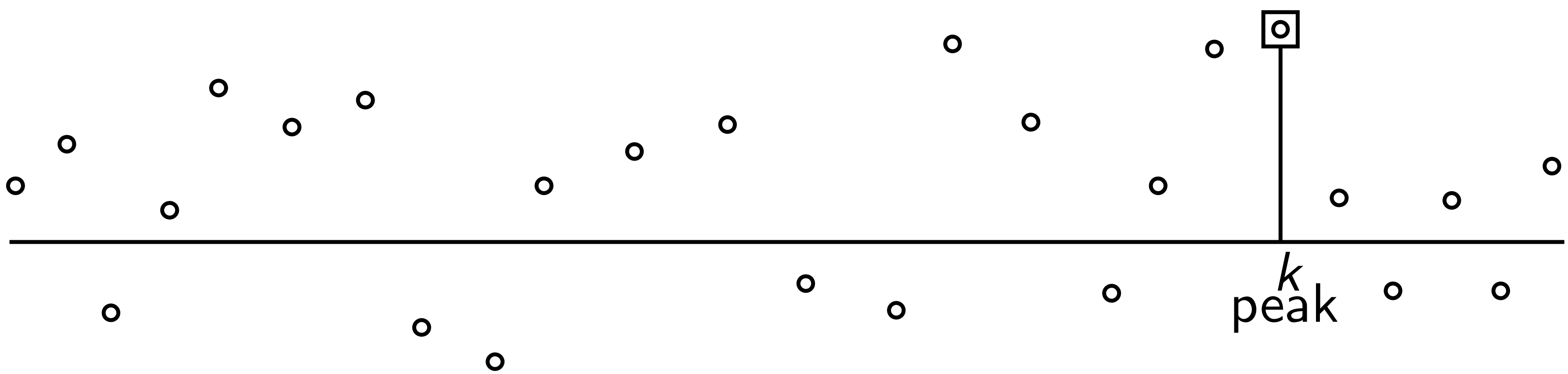
Lecture 06, Math 447

Marius Junge

University of Illinois at Urbana-Champaign

Sequences of Real Numbers



Thm I Every sequence has a monotone subsequence.

Def An element $k \in \mathbb{N}$ is called a *peak* if $\forall n \geq k \quad x_n \leq x_k$.

$$P = \{n \in \mathbb{N} \mid n \ \text{ is a peak}\}.$$

$\nearrow$ P has infinitely many elements A

$\searrow$ P has finitely many elements B

timestamp: 00:04

Case B (corrected)

See 13:09 in re corrections.

P is finite (**case B**): $\exists$ point past which there are no more peaks:

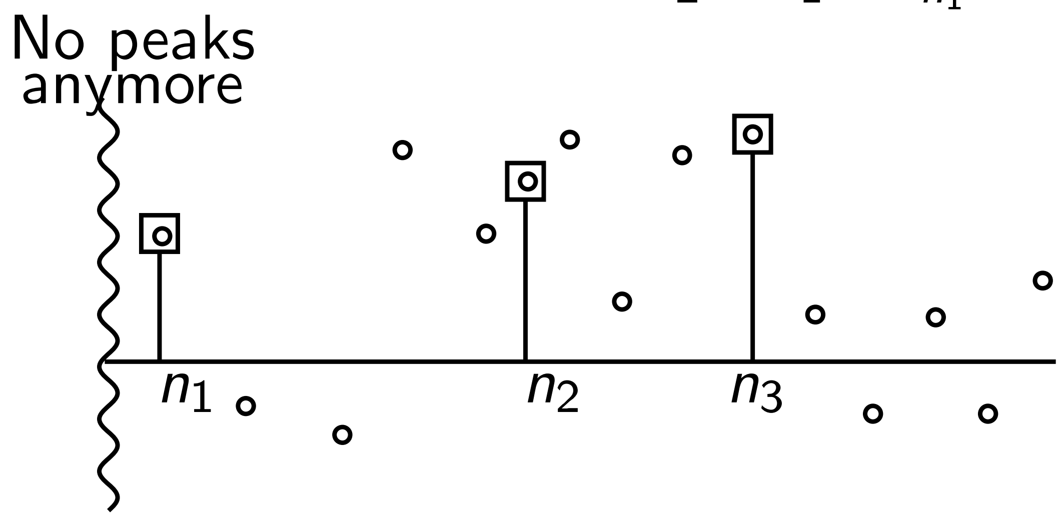
Let $N_0 \in \mathbb{N}$ such that $P \subseteq \{1, \dots, N_0\}$.

Hence for $n > N_0$, n is not a peak.

This means $\forall n > N_0 \quad \exists_{k>n} \quad x_n < x_k$.

Now construct: $n_0 = N_0 + 1$. $\exists n_1 > n_0 \quad x_{n_0} < x_{n_1}$.

$\exists n_2 > n_1 \quad x_{n_1} < x_{n_2}$.



Thus $n_k \nearrow$: $\forall k \quad x_{n_{k+1}} > x_{n_k} \implies (x_{n_k})$ is strictly increasing.

Case A

Case A: P is an infinite set, so we can enumerate it:

$$n_1 = \min P, \quad n_2 = \min P \backslash \{n_1\}, \quad n_{k+1} = \min P \backslash \{n_1, \ldots, n_k\}.$$

Then n_k is an increasing sequence of natural numbers.
(In fact $P = \{n_k \mid k \geq 1\}$.)

$$\forall k \quad \forall m \geq n_k \quad x_m \leq x_{n_k} \qquad \textit{(n_k is a peak!)}$$

In particular, $\forall k \quad x_{\underbrace{n_{k+1}}_{m}} \leq x_{n_k}$ (taking $m = n_{k+1}$).

Therefore (x_{n_k}) is a decreasing subsequence.

Case B $\implies \exists$ strictly increasing subsequence
Case A $\implies \exists$ decreasing subsequence $\blacksquare$

Limits

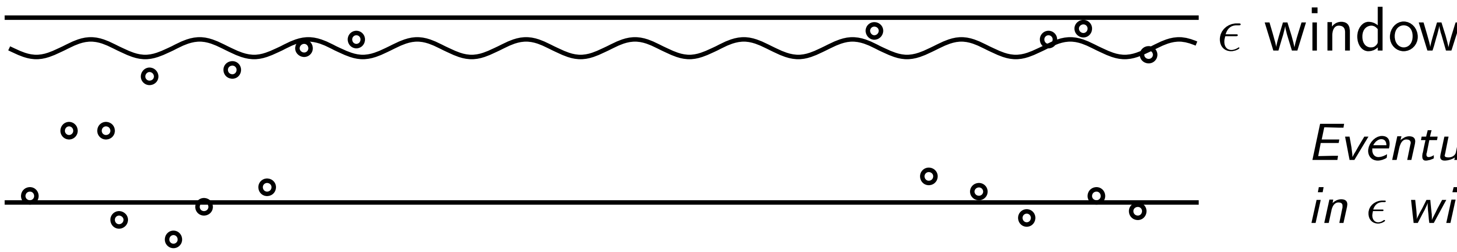
Def 1) A sequence of real numbers *converges* to $x_0 \in \mathbb{R}$ if

$$\forall \epsilon > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \underbrace{\operatorname{dist}(x_n, x_0)}_{=|x_n - x_0|} < \epsilon.$$

(We write $\lim_n x_n = x_0$*.)*

2) A sequence is called *convergent* if $\exists x_0 \ \lim_n x_n = x_0$.

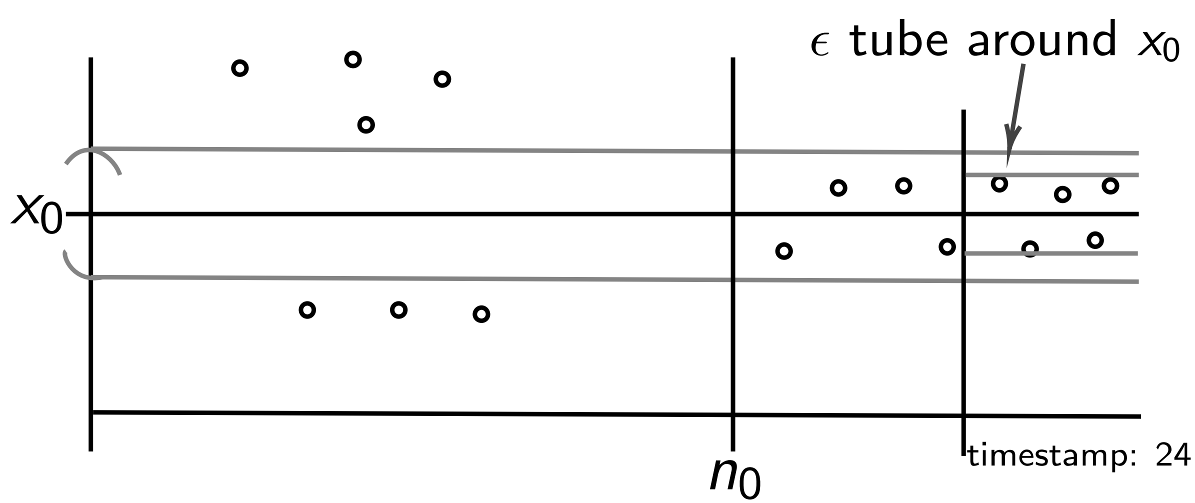
Recall $\sup\{x_n \mid n \in \mathbb{N}\}$ = smallest line that sits on top of them:



Eventually some folks in ϵ window ($\forall \epsilon$).

Limit is similar.
Two-sided picture of sup.

For each ϵ there is a lablel st. everything after will be in the tube.



timestamp: 24:26

Bounded monotone $\implies$ convergent

Prop 2 Every bounded monotone sequence is converging.

Proof WLOG assume (x_n) is increasing and bounded.

Claim $\boxed{\sup_n x_n = \lim_n x_n}$

Recall $\sup A = c$ if

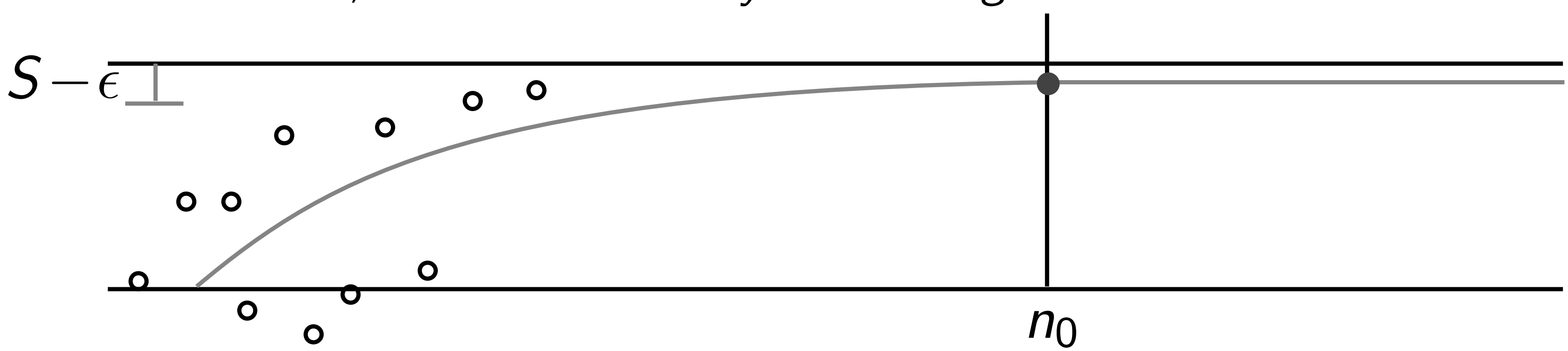
$$\forall a \in A \quad a \leq c \quad \text{and} \quad \forall \epsilon > 0 \ \exists a \in A \quad c - \epsilon < a.$$

Proof (of claim): Let $S = \sup\{x_n \mid n \in \mathbb{N}\}$. Let $\epsilon > 0$.

By definition of S we have 1) $\forall n \quad x_n \leq S$.

2) $\exists n_0 \quad S - \epsilon < x_{n_0} \leq S$.

Since monotone, once in the tube you're "caught" forever.



Now consider $n \geq n_0$. Then we have $S - \epsilon < x_{n_0} \leq x_n \leq S$.

Hence $|x_n - S| < \epsilon$. ■

timestamp: 31:49

Bounded sequences

Thm II Every bounded sequence has a convergent subsequence.

Proof Let (x_n) be a bounded sequence. By Thm I there exists a monotone subsequence $(x_{n_k})_k$. By Prop 2 this is convergent. ■

We will use Thm II to prove a continuous function on a compact interval achieves a maximum.

Thm III Every bounded interval is sequentially compact.
("sequentially compact": every sequence has convergent subsequence.)

Intermission on **continuous functions**:

$A \subset \mathbb{R}$ $\quad f : A \to \mathbb{R}$; $\quad f$ is *continuous* at x_0 if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \quad |y - x_0| < \delta \implies |f(y) - f(x_0)| < \epsilon.$$

timestamp: 40:46