Homework 3

Question 1

Consider $y \in (X \backslash \bar{B}_\varepsilon(x))$. Since $d(x,y) > \varepsilon$, Then $\delta = d(x,y) - \varepsilon > 0$.

$$\begin{aligned} 0 < \delta - d(y,m) \\ &= d(x,y) - \varepsilon - d(y,m) \\ &\leq d(x,m) - \varepsilon \end{aligned}$$

$\forall m \in B_\delta(y)$ there will have $B_\delta(y \subseteq X \backslash \bar{B}_\varepsilon(x))$. Then we can know $m \in (X \backslash \bar{B}_\varepsilon(x))$. Then $X \backslash \bar{B}_\varepsilon(x)$ is open. Then $\bar{B}_\varepsilon(x)$ is closed.

Question 2

Consider $x_m \in Lim(x_n)$, we can know there exists $n_j, x_m = Lim_j x_n j$. Since (x_{nj}) is convergent subsequence. When $x = \lim_{n\to\infty} x_m$, There exists $m, x = Lim_j x_m$. We can conclude that $x \in Lim(x_n)$. Thus $Lim(x_n)$ is closed.

Question 3

Idea From Tianyue Cao

Consider $\{(x_n, y_n)\}_1^{\infty}$ be an arbitrary Cauchy sequence that converges to a point (x, y). There will be two cases, when $f(x) < 0$ or $g(y) < f(x)$.

- Case 1:When $f(x) < 0$.

  Consider $\varepsilon = -f(x) > 0$. We can know $\lim_{n\to\infty} f\left(x_n\right) = f(x)$. By definition of limit, There will exists N such that $\forall n > N$.

  $$\begin{gathered} |f(x_n) - f(x)| < \varepsilon \\ f(x_n) < f(x) + \varepsilon = 0 \end{gathered}$$

  Since $(x_n, y_n) \in \{(x, y) \mid 0 \leq f(x) \leq g(y)\}$, we have $f(x_n) \leq 0$, which is a contradiction.

- Case 2: When $g(y) < f(x)$.

  Since both f and g are continuous, we can know $\lim_{n\to\infty} f\left(x_n\right) = f(x)$ and the $\lim_{n\to\infty} g\left(y_n\right) = g(y)$. By the definiton of limit,

  - There will exists N_x such that for all $m > N_x$, We can know $\left|f\left(x_m\right) - f(x)\right| < \frac{\epsilon}{2}$. Then we can know $f\left(x_m\right) < f(x) + \frac{\epsilon}{2}$.
  - There will exists N_y such that for all $m > N_y$, We can know $\left|g\left(y_m\right) - g(y)\right| < \frac{\epsilon}{2}$. Then we can know $g\left(y_m\right) > g(y) - \frac{\epsilon}{2}$.

  Then we can know $g\left(y_n\right) > g(y) - \frac{\epsilon}{2} = f(x) + \epsilon - \frac{\epsilon}{2} = f(x) + \frac{\epsilon}{2} > f\left(x_n\right)$. Since $m > max\{N_x, N_y\}$.

  Then there will have a contradiction. Since, $(x_m, y_m) \in \{(x, y) \mid 0 \leq f(x) \leq g(y)\}$, We will have $f(x_n) \leq g(x_n)$.

Then we can know $0 \leq f(x) \leq g(y)$ implies that $(x, y) \in \{(x, y) \mid 0 \leq f(x) \leq g(y)\}$. We show all cauchy sequences in $\{(x, y) \mid 0 \leq f(x) \leq g(y)\}$ converge to a point in $\{(x, y) \mid 0 \leq f(x) \leq g(y)\}$. Thus it is closed.

Question 4

($\Rightarrow$) We know that S is oper under $d(x,y)$, Thus it will also open under $d_f(x,y)$. and the f is strictly increasing. Consider $\forall x \in S, B_\varepsilon(x) \subseteq S$.

$$\begin{aligned} d(x,y) < \varepsilon &\Rightarrow y \in S \\ x - \varepsilon < y < x + \varepsilon &\Rightarrow y \in S \quad (1) \\ a < b &\Leftrightarrow f(a) < f(b) \quad (2) \end{aligned}$$

Since we know that S is also open under $d_f(x,y)$. we can combine two equation.

$$f(x-\varepsilon) < f(y) < f(x+\varepsilon) \Rightarrow y \in S$$

Consider $m = f(x+\varepsilon) - f(x) > 0$, and $n = f(x) - f(x-\varepsilon) > 0$.

$$\begin{aligned} d_f(x,y) < min\{m,n\} \\ &\Rightarrow -m < f(x) - f(y) < n \\ &\Rightarrow f(x) - f(x+\varepsilon) < f(x) - f(y) < f(x) - f(x-\varepsilon) \\ &\Rightarrow f(x-\varepsilon) < f(y) < f(x+\varepsilon) \\ &\Rightarrow y \in S \end{aligned}$$

($\Leftarrow$): We know f^{-1} is also increasing and continuous. By the prove above we can know S is opern under $d_f(x,y) = |f(x) - f(y)|$, Then S is open under $(d_f)_{f^{-1}}(x,y) = \left|f^{-1}(f(x)) - f^{-1}(f(y))\right| = d(x,y)$.

Thus by conclusion, we know the statement is true.