# Homework1

## Question 1

From the question we know $T$ is nonempty subset, Suppose $t$ in $T$ and $g, h = t$ . Then $gh^{-1} = tt^{-1} = 1$. Which show $1 \in G(T)$.

Suppose $g_1, h_1^{-1}$ in $G(T)$ and $(g_1, h_1 \ in \ T)$ and $g_2, h_2^{-1}$ in $G(T)$ and $(g_2, h_2 \ in \ T)$. Then we can know $g_1h_1^{-1}g_2h_2^{-1} = g_1g_2h_1^{-1}h_2^{-1}$.

Since $g_1, g_2 \in T$. $h_1^{-1}h_2^{-1} = (h_2h_1)^{-1}$ and $h_1, h_2 \in T$. Then we can know $g_1g_2 \in T$ and $h_2h_1 \in T$. Thus $g_1h_1^{-1}g_2h_2^{-1} = g_1g_2h_1^{-1}h_2^{-1} = g_1g_2(h_2h_1)^{-1} \in G(T)$. Which we show that if $gh \in \ subgroup$.

Thus we can know $G(T)$ is a subgroup of $G$.

## Question 2

Consider $g_1, h_1, g_2, h_2 \in T_1$ s.t. $g_1, h_1^{-1} = g_2h_2^{-1}$

$$\psi(g_1h_1^{-1}) = \phi(g_1)\phi(h_1)^{-1} \tag{1}$$
$$\psi(g_2h_2^{-1}) = \phi(g_2)\phi(h_2)^{-1} \tag{2}$$

Now we need to prove (1) = (2), Since $\phi(g_1), \phi(h_1)^{-1}$ are group element.

$$\begin{aligned}
&\phi(g_1)\phi(h_1)^{-1}(\phi(g_2)\phi(h_2)^{-1})^{-1} \\
&= \phi(g_1)\phi(h_1)^{-1}\phi(h_2)\phi(g_1)^{-1} \\
&= \phi(g_1)\phi(h_2)\phi(h_1)^{-1}\phi(g_2)^{-1} \\
&= \phi(g_1h_2)\phi(g_2h_1)^{-1} \\
&= \phi(g_1h_2)\phi(g_2h_1)^{-1} \\
&= 1
\end{aligned}$$

Thus we can know (1) and (2) have same inverse, thus we can say (1) = (2), and $\psi(gh^{-1}) = \phi(g)\phi(h)^{-1}$ is a well-define.

Let $g$ be an arbitrary element in $T_1$. $\psi(gg^{-1}) = \psi(1) = \psi(g)\psi(g)^{-1} = 1$. which is $\phi(1) = 1$.

Let $g, h$ be elements in $G(T_1)$. $g = s_1t_1^{-1}, h = s_2t_2^{-1}$. $s_1, t_1, s_2, t_2 \in T_1$.

$$
\begin{aligned}
\psi(gh) &= \psi(s_1t_1^{-1}s_2t_2^{-1}) \\
&= \psi(s_1s_2(t_2t_1)^{-1}) \\
&= \phi(s_1s_2)\phi(t_2t_1)^{-1} \\
&= \phi(s_1)\phi(s_2)(\phi(t_1)\phi(t_2))^{-1} \\
&= \phi(s_1)\phi(s_2)\phi(t_1)^{-1}\phi(t_2)^{-1} \\
&= \phi(s_1)\phi(t_1)^{-1}\phi(s_2)\phi(t_2)^{-1} \\
&= \psi(s_1t_1^{-1})\psi(s_1t_2^{-1}) \\
&= \psi(g)\psi(h)
\end{aligned}
$$

Thus we can know it is a group homomorphism from $G(T_1)$ to $G(T_2)$.

# Question 3

Suppose $g_1, g_2, h_1, h_2 \in S$ and $g_1h_2 = g_2h_1$, we can find
$\phi\left(g_1\right)\phi(h_1)^{-1} = \phi\left(g_2\right)\phi(h_2)^{-1} \Rightarrow \psi\left(g_1h_1^{-1}\right) = \phi\left(g_1\right)\phi(h_1)^{-1} = \phi\left(g_2\right)\phi(h_2)^{-1} = \psi\left(g_2h_2^{-1}\right)$

Thus $\psi$ is well define.

$$
\begin{aligned}
\psi\left(g_1h_1^{-1}g_2h_2^{-1}\right) = \psi\left((g_1g_2)(h_1h_2)^{-1}\right) &= \phi\left(g_1g_2\right)\phi\left(h_1h_2\right) \\
&= \phi\left(g_1\right)\phi\left(g_2\right)\phi(h_1)^{-1}\phi(h_2)^{-1} \\
&= \psi\left(g_1h_1^{-1}\right)\psi\left(g_2h_2^{-1}\right) \\
&= \psi(1) = \psi\left(gg^{-1}\right) = \phi(g)\phi(g)^{-1}
\end{aligned}
$$

Which is equal to 1. and from

$$
\begin{aligned}
\psi(s) &= \psi(s \cdot 1) \\
&= \phi(s)\phi(1) \\
&= \phi(s)
\end{aligned}
$$

We can know that for all $s \in S, \psi(s) = \phi(s)$. Thus it is also homomorphism.

# Question 4

Consider $g, h$ in $G(\phi_1(S))$. There exists $s_1, s_2, s_3, s_4 \in S$. s.t. $g = \phi_1(s_1)\phi(s_2)^{-1}$,
$h = \phi_1(s_3)\phi_1(s_4)^{-1}$. Define $\psi(\phi_1(s_1)\phi_1(s_2)^{-1}) = \phi_2(s_1)\phi_2(s_2)^{-1}$. By the question 2 we know it is well define and homomorphism.

Surjective is clearly, since each $\phi_2(s_1)\phi_2(s_2)^{-1}$ in $G(\phi_2(S))$ we can find $\psi(\phi_1(s_1)\phi_1(s_2)^{-1})$ in $G(\phi_1)(S)$

$$\psi(\phi_1(s_1)\phi_1(s_2)^{-1}) = \psi(\phi_1(s_3)\phi_1(s_4)^{-1})$$

We times $\psi(\phi_1(s_3)\phi_1(s_4)^{-1})^{-1}$

$$\psi(\phi_1(s_1)\phi_1(s_2)^{-1}\phi_1(s_4)\phi_1(s_3)^{-1} = 1$$
$$\phi_1(s_1)\phi_1(s_2)^{-1}\phi_1(s_4)\phi_1(s_3)^{-1} = 1$$
$$\phi_1(s_1)\phi_1(s_2)^{-1} = \phi_1(s_3)\phi_1(s_4)^{-1}$$

Thus we can know it also injective. Thus it is bijective. Since it is also homomorphism, so it is isomorphism.

# Question 5

Consider function $f(z) : Z \to G(N)$ where

$$f(0) = 1$$
$$f(n) = \phi(n) \ for \ all \ n \ in \ N$$
$$f(-n) = \phi(n)^{-1} \ for \ all \ n \ in \ N$$

$f$ is a bijection, $f(a)f(b) = f(a+b)$. For all $a, b \in Z$, $f(a+b) = f(a) + f(b)$ closure for all $a, b, c \in Z$.

$(f(a)f(b))f(c) = f(a+b)f(c) = f(a+b+c) = f(a)f(b+c) = f(a)(f(b)f(c))$ thus we know it is closure.

$(a, b) \cdot (0, 0) = (a, b)$ and $(0, 0) \cdot (a, b) = (a, b)$, we can know it have indentity which is $(0, 0)$.

From $(a, b) \cdot (c, d) = (a+c, b+d) = (c+a, d+b) = (c, d) + (a, b)$, we can know that it have commutative.

From
$((a, b) \cdot (c, d)) \cdot (e, f) = (a+c, b+d) \cdot (e, f) = (a+c+e, b+d+f) = (a, b) + ((c, d) + (e, f))$
. we can know it also have associative.

Since $f(a) + f(-1) = f(0) = 1$, Thus $\phi(a)$ and $\phi(-a)$ is inverse.

$f(a)f(b) = f(a+b) = f(b+a) = f(b)f(a)$, thus it also follow abelian property.

Thus we can know the right hand side can be made into an abelian group. Thus we prove that it is a disjoint decomposition.