15-381 Artificial Intelligence
Homework 4

Jordan Widjaja
jwidjaja@andrew.cmu.edu
03 December 2017

1.1: Nash Equilibrium

Let $p_1 = (U, 1-U)$ and $p_2 = (L, 1-L)$ be the probabilities of the choices for player 1 and player 2, and let P_1 and P_2 be matrices for their payoffs. We calculate the expected values for each of the players, as follows:

$$P_1 p_2 = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} L \\ 1-L \end{bmatrix} = \begin{bmatrix} 2L + (-2)(1-L) \\ L + 2(1-L) \end{bmatrix} = \begin{bmatrix} 4L-2 \\ 2-L \end{bmatrix}$$

$$p_1 \cdot P_1 p_2 = \begin{bmatrix} U \\ 1-U \end{bmatrix} \cdot \begin{bmatrix} 4L-2 \\ 2-L \end{bmatrix} = 4UL - 2U + 2 - L - 2U + UL = 5UL - 4U - L + 2$$

Take the derivative of this with respect to U and set equal to 0, then obtain the following:

$$5L - 4 = 0 \implies L = \frac{4}{5}$$

Similarly, find the expected value for player 2, as follows:

$$P_2 p_2 = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} L \\ 1-L \end{bmatrix} = \begin{bmatrix} 2(1-L) \\ L + (-1)(1-L) \end{bmatrix} = \begin{bmatrix} 2-2L \\ 2L-1 \end{bmatrix}$$

$$p_1 \cdot P_2 p_2 = \begin{bmatrix} U \\ 1-U \end{bmatrix} \cdot \begin{bmatrix} 2-2L \\ 2L-1 \end{bmatrix} = 2U - 2UL + 2L - 1 - 2UL + U = -4UL + 3U + 2L - 1$$

Again, take the first derivative with respect to L and set equal to 0, then obtain the following:

$$-4U + 2 = 0 \implies U = \frac{1}{2}$$

Therefore, our Nash Equilibrium is: $\boxed{p_1 = \left(\frac{1}{2}, \frac{1}{2}\right)}$ and $\boxed{p_2 = \left(\frac{4}{5}, \frac{1}{5}\right)}$.

To verify Nash equilibrium, we ensure first that for any mixed strategy of player 1, say $p_1' = (U', 1-U')$, we have $p_1' \cdot P_1 p_2 \leq p_1 \cdot P_1 p_2$:

$$\begin{aligned} p_1' \cdot P_1 p_2 &= 5UL - 4U - L + 2 \\ &= 5U\left(\frac{4}{5}\right) - 4U - \frac{4}{5} + 2 \\ &= \frac{3}{5} \end{aligned}$$

Also,

$$\begin{aligned} p_1 \cdot P_1 p_2 &= 5UL - 4U - L + 2 \\ &= 5\left(\frac{1}{2}\right)\left(\frac{4}{5}\right) - 4\left(\frac{1}{2}\right) - \frac{4}{5} + 2 \\ &= 2 - 2 - \frac{4}{5} + 2 \\ &= \frac{3}{5} \end{aligned}$$

Now, we ensure that for any mixed strategy of player 2, say $p_2' = (L', 1 - L')$, we have that $p_1 \cdot P_2 p_2' \leq p_1 \cdot P_2 p_2$:

$$\begin{aligned}
p_1 \cdot P_1 p_2' &= -4UL + 3U + 2L - 1 \\
&= -4\left(\frac{1}{2}\right)L + 3\left(\frac{1}{2}\right) + 2L - 1 \\
&= -2L + \frac{3}{2} + 2L - 1 \\
&= \frac{1}{2}
\end{aligned}$$

Also,

$$\begin{aligned}
p_1 \cdot P_2 p_2 &= -4UL + 3U + 2L - 1 \\
&= -4\left(\frac{1}{2}\right)\left(\frac{4}{5}\right) + 3\left(\frac{1}{2}\right) + 2\left(\frac{4}{5}\right) - 1 \\
&= \frac{1}{2}
\end{aligned}$$

Thus, the Nash Equilibrium holds.

1.2: Extensive Form to Normal Form

1. From left to right, denote the P_2 nodes as 1, 2, and 3. P_2 will look to maximize their score, so in node 1, we take path b, in node 2, we take path e, and in node 3, we take path g. Now, we have node 1 labelled as $(0, 2)$, node 2 labelled as $(2, 2)$, and node 3 labelled as $(-1, 1)$. Now that it is P_1's turn, we find the maximum value for player 1 given these values, and we find that path Y gives us this. Therefore, the solution to the game is the path $\boxed{Ye}$.

   We have the following conversion from extensive form game to normal form game:

	a/d/g	a/d/h	a/e/g	a/e/h	a/f/g	a/f/h	b/d/g	b/d/h	b/e/g
X	(2,1)	(2,1)	(2,1)	(2,1)	(2,1)	(2,1)	(0,2)	(0,2)	(0,2)
Y	(1,-1)	(1,-1)	(2,2)	(2,2)	(0,1)	(0,1)	(1,-1)	(1,-1)	(2,2)
Z	(-1,1)	(0,0)	(-1,1)	(0,0)	(-1,1)	(0,0)	(-1,1)	(0,0)	(-1,1)

	b/e/h	b/f/g	b/f/h	c/d/g	c/d/h	c/e/g	c/e/h	c/f/g	c/f/h
X	(0,2)	(0,2)	(0,2)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
Y	(2,2)	(0,1)	(0,1)	(1,-1)	(1,-1)	(2,2)	(2,2)	(0,1)	(0,1)
Z	(0,0)	(-1,1)	(0,0)	(-1,1)	(0,0)	(-1,1)	(0,0)	(-1,1)	(0,0)

2. We have the following conversion from extensive form game to normal form game:

	a/d	a/e	a/f	b/d	b/e	b/f	c/d	c/e	c/f
X	(2,-2)	(2,-2)	(2,-2)	(1,-1)	(1,-1)	(1,-1)	(-1,1)	(-1,1)	(-1,1)
Y	(3,-3)	(2,-2)	(0,0)	(3,-3)	(2,-2)	(0,0)	(3,-3)	(2,-2)	(0,0)
Z	(1,-1)	(2,-2)	(0,0)	(1,-1)	(2,-2)	(0,0)	(1,-1)	(2,-2)	(0,0)

   To solve the game, notice that the a/e, the b/d column, the a/d column, and the b/e columns are strictly dominated by the c/f column for player 2, so we can simply remove those columns.

By backward induction now, from the node connecting edges a, b, and c, we have that player 2 will take edge c because it provides the most utility for player 2, and from the next two nodes connecting edges d,e, and f, player 2 will select edge f. Player 1 will now select the node containing $(0, 0)$, so the solution is either Yf or Zf.

1.3: Electing the Median

We can create the following cases. Suppose student i selects the temperature S such that $S < T_i < T^*$ or $T_i < S < T^*$. In either case, the median will remain unchanged regardless of if he chose S or T_i, and because the utility is even less for selecting S as it is even further away from T^* than T_i, there is no gain in selecting $S < T_i$. Suppose we have that $T_i < T^* < S$ or $T^* < T_i < S$. We have that the symmetric result will also occur, such that there is no gain.