MATH2040C Homework 2

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February 4, 2021

1 Section 1.6, Q12

Given $\{u, v, w\}$ is a basis for V, then we have $\{u, v, w\}$ is linearly independent itself and $V = \text{span}\{u, v, w\}$.

Note that hence $\forall x \in V, \exists a, b, c \in \mathbb{F}$ such that $x = au + bv + cw = a(u + v + w - (v + w)) + b(v + w - w) + cw = a(u + v + w) + (b - a)(v + w) + (c - b)w$.

Hence $x \in$ span $\{u+v+w, v+w, w\}$, i.e., V=span$\{u, v, w\} \subset$ span $\{u+v+w, v+w, w\}$.

And span$\{u+v+w, v+w, w\} \subset$ span$\{u, v, w\}$ holds trivially, hence $\{u+v+w, v+w, w\}$ spans V.

Remains to prove that $\{u + v + w, v + w, w\}$ is linearly independent, suppose not, $\exists d, e, f \in \mathbb{F}$ (not all zeros) such that

$$d(u + v + w) + e(v + w) + fw = 0_V,$$

where the 0_V is the zero vector of V.

which implies

$$du + (d + e)v + (d + e + f)w = 0_V.$$

Note that as supposed d, e, f are not all zeros, hence $d, (d + e), (d + e + f)$ are not all zeros.

Contradiction with the assumption that $\{u, v, w\}$ is linearly independent.

Hence $\{u + v + w, v + w, w\}$ is linearly independent at the first place.

Q.E.D.

2 Section 1.6, Q15

We construct the basis by ourself. Let $B = D \cup A$.

$A = \{A_{mn} : m = 1, 2, ..., n; n = 1, 2, ..., n; m \neq n.\}$ where $A_{mn}[i, j] = 1$ if $(i = m, j = n)$, else it is 0.

is a set of n by n matrices having all slot being zeros unless one slot not on the main diagnoal.

And $D = \{D_1, D_2, ..., D_{n-1}\}$, where $D_k[i, j] = 1$ if $(i = j = n$ or $i = j = k)$, else is zero entry.

Note that $\forall$ n by n matrix M with $tr(M) = 0$. We have

$$M = \sum_{i,j:i\neq j} M_{ij}A_{ij} + \sum_{i=1}^{n-1} M_{ii}D_i.$$

Also note that $B = D \cup A$ is linearly independent, and from above we have

$$spanB = \{M_{n\times n} : tr(M) = 0\}.$$

Therefore B is a basis of $\{M_{n\times n} : tr(M) = 0\}$. And

$$|B| = |A| + |D| = n - 1 + n^2 - n = n^2 - 1.$$

Q.E.D.

3 Section 1.6, Q26

Consider the transform: $T : P_n(\mathbb{R}) \to \mathbb{R}$, with the function $T(f) = f(a)$.
. $\forall f_1, f_2 \in P_n(\mathbb{R})$,

$$T(f_1 + f_2) = (f_1 + f_2)(a) = f_1(a) + f_2(a) = T(f_1) + T(f_2).$$

And $\forall c \in \mathbb{R}$,

$$T(cf) = cf(a).$$

We proved that the transformation is linear.
From the rank nullity theorem,

$$\dim N(T) + \dim R(T) = \dim P_n(\mathbb{R}) = n + 1.$$

Note that $\dim R(T) = 1$, hence $\dim N(T) = n$.
Note that $N(T) = \dim\{f \in P_n(\mathbb{R}) : f(a) = 0\}$. Therefore it is n.
Done.

4 Section 1.6, Q30

Note that the $0_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. And by definition, $0_V \in W_1, W_2$.

Note that $\forall x_1, x_2 \in W_1$, $\exists a_1, a_2, b_1, b_2, c_1, c_2 \in F$ such that $x_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 \end{pmatrix}$.

Note that $x_1 + x_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 \end{pmatrix}$, which is also $\in W_1$. And $\forall a \in F$,

$$ax_1 = \begin{pmatrix} aa_1 & ab_1 \\ ac_1 & aa_1 \end{pmatrix} \in W_1.$$

Note that $\forall y_1, y_2 \in W_2$, $\exists d_1, d_2, e_1, e_2 \in F$ such that $y_1 = \begin{pmatrix} 0 & d_1 \\ -d_1 & e_1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} 0 & d_2 \\ -d_2 & e_2 \end{pmatrix}$ Hence $y_1 + y_2 = \begin{pmatrix} 0 & d_1 + d_2 \\ -d_1 + d_2 & e_1 + e_2 \end{pmatrix} \in W_2$. And $\forall a \in F$,

$$ay_1 = \begin{pmatrix} 0 & ad_1 + ad_2 \\ -ad_1 + ad_2 & ae_1 + ae_2 \end{pmatrix} \in W_2.$$

Therefore, both W_1, W_2 are subspaces of V.
Note that a basis for W_1 can be $\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\}$. Hence $\dim W_1 = 3$.
A basis for W_2 can be $\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$. Hence $\dim W_2 = 2$.
$W_1 \cap W_2 = \{\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix} : a \in F\}$, hence $\dim W_1 \cap W_2 = 1$.
$\forall \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in V$,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & c+b\\ 0 & a\end{pmatrix} + \begin{pmatrix}0 & -c\\ c & d-a\end{pmatrix}.$$

The first term is element of W_1 and the second term is element of W_2. Hence

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in W_1 + W_2.$$

It is obvious that $W_1 + W_2 \in V$, then $W_1 + W_2 = V$. Hence $\dim W_1 + W_2 = \dim V = 4$.
Done.

5 Section 2.1, Q3

Note that $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$, $T((a_1, a_2) + (b_1, b_2)) = (a_1 + b_1 + a_2 + b_2, 0, 2a_1 + 2b_1 - a_2 - b_2) = (a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) = T((a_1, a_2)) + T((b_1, b_2))$.
And also note that $\forall (a_1, a_2) \in \mathbb{R}^2, c \in \mathbb{R}$,

$$T(c(a_1, a_2)) = (ca_1 + ca_2, 0, 2ca_1 - ca_2) = c(a_1 + a_2, 0, 2a_1 - a_2) = cT((a_1, a_2).$$

Hence T is linear.
The basis for $N(T)$ is $\{(0, 0)\}$, and $\dim N(T) = 0$.
And a basis for $R(T)$ is $\{(1, 0, 2), (1, 0, -1)\}$. Hence $\dim R(T) = 2$.
Note that

$$\dim R(T) + \dim N(T) = 2 = \dim R^2$$

.

Therefore the rank nullity theorem is verified.
Note that $(1, 1, 1) \notin R(T)$, hence T is not onto (surjective).
Also note that $N(T) = (0, 0)$, then T is one-to-one (injective).
Done.

6 Section 2.1, Q5

Note that $\forall f(x), g(x) \in P_2(\mathbb{R})$, we have

$$T(f(x)+g(x)) = x(f(x)+g(x))+(f(x)+g(x))' = xf(x)+f'(x)+xg(x)+g'(x) = T(f(x))+T(g(x)).$$

Also note that $\forall f(x) \in P_2(\mathbb{R}), \forall a \in \mathbb{R}$,

$$T(af(x)) = xaf(x) + af'(x) = aT(f(x)).$$

Therefore, T is linear.

Consider the $N(T)$, notice that $xf(x) + f'(x) = 0$ does not hold if $f(x)$ is a nonzero polynomial, because the L.H.S. always have nonzero order.

Hence the nullspace of T is only $\{f(x) = 0\}$.

And the basis for it is $\{f(x) = 0\}, \dim\{f(x) = 0\} = 0$.

A basis for $R(T)$ is $\{x^3 + 2x, x, x^2 + 1\}$ and hence $\dim\{x^3 + 2x, x, x^2 + 1\} = 3$.

Note that $\dim R(T) + \dim N(T) = 0 + 3 = \dim P_2(\mathbb{R})$. Hence the rank nullity theorem is verified.

Note that $N(T) = 0$, then T is one-to-one (injective).

Also note that $h(x) = 2x^3 + x^2 + 5x + 1 \in P_3(\mathbb{R})$ but $h(x) \notin R(T)$. Hence T is not an onto (surjective).

Done.

7 Section 2.1, Q14

7.1 (a)

We prove the "only if" direction first. Suppose T is injective. Then suppose $\exists$ a linearly independent subset $S \subset V$ such that $T(S)$ is not linearly independent.

Write $S = \{s_1, s_2, ..., s_n\}$. Then $T(S) = \{T(s_1), T(s_2), ..., T(s_n)\}$, since $T(S)$ is linearly dependent. Then $\exists a_1, a_2, ..., a_n \in F$ such that

$$a_1T(s_1) + a_2T(s_2) + ...a_nT(s_n) = 0.$$

That is

$$T(a_1s_1 + a_2s_2 + ...a_ns_n) = 0.$$

Recall that T is injective, then the $N(T)$ contains only the zero vector. Then $a_1s_1 + a_2s_2 + ...a_ns_n$ is the zero vector, which contradicts with the assumption that S is linearly independent.

Therefore $\forall$ linearly independent subset $S \in V$, $T(S)$ is also linearly independent.

Then we prove the "if" part. Hence we are given that $\forall$ linearly independent subset $S \in V$, $T(S)$ is linearly independent follows.

Suppose T is not one-to-one, then $\exists u \in V$ $(u \neq 0_V)$ such that $T(u) = 0_W$.

Consider a set $\{u\}$, which is a singleton with a nonzero element, hence linearly independent, but $1 \cdot T(u) = 0_W$, which contradicts with our assumption that $\forall$ linearly independent subset $S \in V$, $T(S)$ is linearly independent follows.

Q.E.D.

7.2 (b)

Note that from (a), the "only if" part of this question is proved directly. What remains to be proved is that if $T(S)$ is linearly independent, then S is linearly independent.

Write $S = \{s_1, s_2, ..., s_n\}$

Suppose not, suppose S is linearly dependent, then $\exists a_1, a_2, ..., a_n$ (not all zero) such that

$$a_1s_1 + a_2s_2 + ... + a_ns_n = 0.$$

Then $0_W = T(a_1s_1+a_2s_2+...+a_ns_n) = a_1T(s_1)+...+a_nT(s_n)$. Which implies that $T(S)$ is linearly dependent, contradicts with our assumption. Hence S is linearly independent at the first place.

Q.E.D.

7.3 (c)

Since $\beta = \{v_1, v_2, ..., v_n\}$ is a basis, it is linearly independent.

Suppose $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is linearly dependent, then $\exists a_1, ..., a_n$ (not all zeros) such that

$$a_1T(v_1) + ... + a_nT(v_n) = 0.$$

Which implies $T(a_1v_1 + ... + a_nv_n) = 0$. By T is one-to-one, $a_1v_1 + ... + a_nv_n = 0$, which contradicts with our assumption that β is linearly independent.

Hence $T(\beta)$ is linearly independent.

What remains to prove is that $W \subset spanT(\beta)$.

Since T is surjective, then $\forall w \in W, \exists v \in V$ such that $w = T(v)$.

And $\exists b_1, ..., b_n$ such that

$$v = b_1v_1 + ... + b_nv_n.$$

Therefore,

$$w = T(b_1v_1 + ... + b_nv_n) = b_1T(v_1) + ... + b_nT(v_n) \in spanT(\beta).$$

Then we proved that $T(\beta)$ is a basis for W.

Q.E.D.

8 Section 2.1, Q17

8.1 (b)

By the rank-nullity theorem, we have

$$\dim N(T) + \dim R(T) = \dim V.$$

Which implies $\dim N(T) = \dim V - \dim R(T)$.

Recall that $\dim R(T) \leq \dim W$, then $-\dim R(T) \geq -\dim W$.

Therefore, $\dim N(T) = \dim V - \dim R(T) \geq \dim V - \dim W > 0$.

Note that being one-to-one requires that $\dim N(T) = 0$. Hence in this case, T cannot be one-to-one.

Q.E.D.

8.2 (a)

Note that T is onto iff $R(T) = W$. Which implies $\dim R(T) = \dim W$. This is impossible to achieve since

$$\dim R(T) = \dim V - \dim N(T) < \dim W - \dim N(T).$$

Hence T cannot be onto.
Q.E.D.

9 Section 2.1, Q21

9.1 (a)

$\forall a = (a_0, a_1, ...), b = (b_0, b_1, ...) \in V, \forall c \in F$, we have

$$T(a+b) = T(a_0 + b_0, ...) = (a_1 + b_1, a_2 + b_2, ...) = (a_1, a_2, ...) + (b_1, b_2, ...) = T(a) + T(b).$$

And

$$T(ca) = (ca_1, ca_2, ...) = c(a_1, a_2, ...) = cT(a).$$

Therefore T is linear.
$\forall a = (a_0, a_1, ...), b = (b_0, b_1, ...) \in V, \forall c \in F$, we have

$$U(a+b) = U(a_0+b_0, ...) = (0, a_0+b_0, a_1+b_1, ...) = (0, a_0, a_1, ...)+(0, b_0, b_1, ...) = U(a)+U(b).$$

And

$$U(ca) = (ca_0, ca_1, ...) = c(a_0, a_1, ...) = cU(a).$$

Therefore U is linear.
Q.E.D.

9.2 (b)

Note that $\forall w \in V, \exists (1, w) \in V$ such that

$$T((1, w)) = w.$$

Therefore T is surjective.
Also note that $T((1, 1, 0, ...)) = T((0, 1, 0, ...)) = (1, 0, ...)$ (the ... part are all zeros)
Therefore T is not one-to-one, not injective.
Q.E.D.

9.3 (c)

$\forall x, y \in V$ with $x \neq y$, we have $U(x) = (0, x)$ and $U(y) = (0, y)$.
$U(x) \neq U(y)$ follows since $x \neq y$. Therefore, U is one-to-one.
Also note that $(1, 0, 0, 0, ...) \in V$ (the ... part are all zeros) but obviously it does not inside the range of U.
Hence U is not onto.
Q.E.D.

10 Section 2.1, Q22

10.1 $T : \mathbb{R}^3 \to \mathbb{R}$

Denote $a = T((1,0,0)), b = T((0,1,0)), c = T((0,0,1))$. Then $\forall (x,y,z) \in \mathbb{R}^3$ we have

$$T(x,y,z) = T(x(1,0,0) + y(0,1,0) + z(0,0,1)) = ax + by + cz.$$

Which is what the question wanted.

10.2 $T : F^n \to F$

To generate it to $F^n \to F$, we need to first find a basis.

Take the set $\{e_i, i = 1, 2, ...n\}$ be the basis, where $e_i = (0, ...1, ...0)$ where the 1 is sloted at the i-th entry.

Denote $T(e_i) = a_i$. Then $\forall (x_1, x_2, ..., x_n) \in F^n$,

$$T(x_1, x_2, ..., x_n) = T(x_1 e_1 + x_2 e_2 + ... + x_n e_n) = a_1 x_1 + a_2 x_2 + ... + a_n x_n.$$

That's it.

10.3 $T : F^n \to F^m$

Statement: $\exists u_1, u_2, ..., u_n \in F^m$ such that $\forall (x_1, x_2, ..., x_n) \in F^n$, we have

$$T((x_1, x_2, ..., x_n)) = x_1 u_1 + x_2 u_2 + ... + x_n u_n.$$

Proof: denote $T(e_i) = u_i$, then $\forall (x_1, x_2, ..., x_n) \in F^n$, we have

$$T((x_1, x_2, ..., x_n)) = T(x_1 e_1 + ... + x_n e_n) = x_1 u_1 + x_2 u_2 + ... + x_n u_n.$$

Q.E.D.