MATH2040C Homework 1

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1 Section 1.2, Q13

To check if a set is a vector space, one need to check those VS's.

[VS1]: $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{V}$, note that from definition,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$(b_1, b_2) + (a_1, a_2) = (a_1 + b_1, a_2 b_2)$$

Hence $(b_1, b_2) + (a_1, a_2) = (a_1, a_2) + (b_1, b_2), \forall (a_1, a_2), (b_1, b_2) \in \mathbb{V}$. Therefore VS1 is satisfied.

[VS2]: $\forall (a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{V}$, note that by definition,

$$((a_1, a_2) + (b_1, b_2)) + (c_1, c_2) = (a_1 + b_1, a_2 b_2) + (c_1, c_2) = (a_1 + b_1 + c_1, a_2 b_2 c_2)$$

and

$$(a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = (a_1, a_2) + (b_1 + c_1, b_2 c_2) = (a_1 + b_1 + c_1, a_2 b_2 c_2)$$

$\therefore (a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = ((a_1, a_2) + (b_1, b_2)) + (c_1, c_2), \forall (a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{V}$. Therefore, VS2 is satisfied.

[VS3]: Note that an element $(0, 1) \in \mathbb{V}$. Note that $\forall (a_1, a_2) \in \mathbb{V}$,

$$(0, 1) + (a_1, a_2) = (0 + a_1, 1 \cdot a_2) = (a_1, a_2).$$

Hence VS3 is satisfied.

[VS4]: Note that $(1, 0) \in \mathbb{V}$.

And $\forall (a_1, a_2) \in \mathbb{V}, (1, 0) + (a_1, a_2) = (1 + a_1, 0) \neq (0, 1)$. Note that the $(0, 1)$ is the zero vector we defined in order to satisfy VS3.

Therefore VS4 cannot be satisfied, hence $\mathbb{V}$ is not a vector space under the operations stated in the question.

2 Section 1.2 Q21

To check if a set is a vector space, one need to check those VS's.
[VS1]: $\forall (v_1, w_1), (v_2, w_2) \in Z$, note that

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2, w_2) + (v_1, w_1).$$

Therefore, VS1 is satisfied.

[VS2]: $\forall (v_1, w_1), (v_2, w_2), (v_3, w_3) \in Z$, note that

$$((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3).$$

And

$$(v_1, w_1) + ((v_2, w_2) + (v_3, w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3)$$

Therefore $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$, which implies that VS2 is satisfied.

[VS3]: Denote 0_V is a zero vector of V and 0_W is a zero vector of W.
Note that $(0_V, 0_W) \in Z$.
And $\forall (v, w) \in Z$,

$$(0_V, 0_W) + (v, w) = (0_V + v, 0_W + w) = (v, w).$$

Therefore, VS3 is satisfied, and we also define $0_Z = (0_V, 0_W)$ as a zero vector of Z.

[VS4]: $\forall (v, w) \in Z$, note that $\exists \hat{v} \in V, \hat{w} \in W$ such that $v + \hat{v} = 0_V, w + \hat{w} = 0_W$ because V and W are themselves vector spaces.
Note that $(\hat{v}, \hat{w}) \in Z$, since $\hat{v} \in V, \hat{w} \in W$ and

$$(v, w) + (\hat{v}, \hat{w}) = (v + \hat{v}, w + \hat{w}) = (0_V, 0_W) = 0_Z.$$

Therefore, VS4 is satisfied.

[VS5]: Note that $1 \in \mathbb{F}$ and $\forall (v, w) \in Z$,

$$1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w).$$

Therefore, VS5 is satisfied.

[VS6]: Note that $\forall (v, w) \in Z, \forall a, b \in \mathbb{F}$,

$$(ab)(v, w) = (ab \cdot v, ab \cdot w) = (a)(b \cdot v, b \cdot w) = a(b(v, w)).$$

Therefore, VS6 is satisfied.

[VS7]: Note that $\forall (v_1, w_1), (v_2, w_2) \in Z, \forall a \in \mathbb{F}$,

$$a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (a \cdot v_1 + a \cdot v_2, a \cdot w_1 + a \cdot w_2) = a(v_1, w_1) + a(v_2, w_2).$$

Note that the second equailty holds for V and W themselves being vector spaces and $v_1, v_2 \in V, w_1, w_2 \in W$.

Therefore, VS7 is satisfied.

[VS8]: Note that $\forall (v, w) \in Z, \forall a, b \in \mathbb{F}$,

$$(a + b)(v, w) = ((a + b) \cdot v, (a + b) \cdot w)$$

Note that V, W are vector spaces over field $\mathbb{F}$, therefore

$$(a + b)v = a \cdot v + b \cdot v,$$

$$(a + b)w = a \cdot w + b \cdot w.$$

Hence

$$(a + b)(v, w) = (a \cdot v + b \cdot v, a \cdot w + b \cdot w) = (a \cdot v, a \cdot w) + (b \cdot v, b \cdot w) = a(v, w) + b(v, w).$$

Therefore, VS8 is satisfied.

Since the requirements are all satisfied, therefore the set Z is a vector space over $\mathbb{F}$ with the operations stated in the question.

3 Section 1.3 Q11

$\forall n \geq 1$ and n being an integer, note that $f_1(x) = x^n + 1 \in W$ and $f_2(x) = -x^n \in W$.

Given that $n \geq 1$, suppose that W is a subspace of $P(\mathbb{F})$. Then W is a vector space itself, which implies that

$$f_1(x) + f_2(x) = 1 \in W.$$

Note that 1 is of degree 0, and $1 \neq 0$. Hence by definition of W, $1 = f_1(x) + f_2(x) \notin W$.

This is violating the requirements of being a vector space, because the addition defined on W, which is supposed to be a vector space, should have range W.

Therefore, W is not a subspace of $P(F)$ at the first place.

4 Section 1.3 Q19

First, we prove the "if" direction.

Given that $W_1 \subset W_2$ or $W_2 \subset W_1$, we would prove $W_1 \cup W_2$ is a subspace of V.

Suppse the case is that $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$. From the condition we knwo that W_2 itself is a subspace of V. Therefore $W_1 \cup W_2 = W_2$ is a subspace of V.

Then, suppse the case is that $W_2 \subset W_1$, then $W_1 \cup W_2 = W_1$. From the condition we knwo that W_2 itself is a subspace of V. Therefore $W_1 \cup W_2 = W_1$ is a subspace of V.

The "if" direction is proved.

We would prove the "only if" part then. Now we assume $W_1 \cup W_2$ is a subspace of V and try to deduce that $W_1 \subset W_2$ or $W_1 \subset W_2$.

Assume it's not the case, neither of W_1 and W_2 can be empty set.

Then $\exists x_1 \in W_1$ such that $x_1 \notin W_2$, and $\exists x_2 \in W_2$ such that $x_2 \notin W_1$. Note that W_1, W_2 are subspaces of $W_1 \cup W_2$, hence the zero vector of $W_1 \cup W_2$'s (denoted as 0_{12}) is also W_1's (denoted as 0_1) and W_2's (denoted as 0_2).

In short,

$$0_{12} = 0_1 = 0_2.$$

Note that $x_1 + x_2 \in W_1 \cup W_2$, since both $x_1, x_2 \in W_1 \cup W_2$.

(i) Suppose $x_1 + x_2 \in W_1$. As W_1 itself is a vector space, $\exists y_1 \in W_1$ such that $x_1 + y_1 = 0_1$. Then

$$y_1 + x_1 + x_2 = 0_1 + x_2 = 0_2 + x_2 = x_2 \in W_1.$$

Which contradicts with our assumption that $x_2 \notin W_1$.

(ii) Suppose $x_1 + x_2 \in W_2$. As W_2 itself is a vector space, $\exists y_2 \in W_2$ such that $x_2 + y_2 = 0_2$. Then

$$x_1 + x_2 + y_2 = x_1 + 0_2 = x_1 + 0_1 = x_1 \in W_2.$$

Which contradicts with our assumption that $x_1 \notin W_2$.

Therefore, $\forall x_1 \in W_1, x_1 \in W_2$ or $\forall x_2 \in W_2, x_2 \in W_1$ must hold at the first place. Which is by definition $W_1 \subset W_2$ or $W_2 \subset W_1$.

5 Section 1.3 Q31

5.1 (a)

First we prove the "if" part, which assumes $v \in W$

$\forall v + x \in v + W$, where $x \in W$, note that $v + x \in W$ since W is a vector space. Hence $v + W \subset W$.

$\exists y \in W$ such that $y + v = 0_W$, where 0_W is the zero vector of W. $\forall x \in W$,

$$x = x + 0_W = x + (y + v) = (x + y) + v.$$

Note that $x + y \in W$, hence $x = (x + y) + v \in W$. Therefore, $W \subset v + W$. Then $v + W = W$, hence $v + W$ is a subspace.

Then we would prove the "only if" part, which assumes $v + W$ is a subspace of V. Note that $\forall x \in W, v + x \in v + W$.

Because $x \in W$ and W is a vector space, $\exists y \in W$ such that $x + y = 0_W$, where 0_W is the zero vector of W. Also note that $v + y \in v + W$.

Then $v + x + (v + y) = v + (x + y + v)\ v + W$, since $v + W$ is a subspace. Hence $x + y + v \in W$. Recall that $x + y = 0_W$, then $v \in W$.

Therefore, we proved the (a) part.

5.2 (b)

We prove the "if" part first, which assumes $v_1 - v_2 \in W$.
Note that $\forall v_1 + w_1 \in v_1 + W$, since $v_1 - v_2 \in W$ as assumed and $w_1 \in W$, we have

$$v_1 - v_2 + w_1 \in W.$$

Therefore, there exists an element in v_2+W, which is $v_2+v_1-v_2+w_1 = v_1+w_1 \in v_2+W$. Hence $v_1 + W \subset v_2 + W$.

Note that $\forall v_2 + w_2 \in v_2 + W$, since $v_1 - v_2 \in W$ as assumed and $w_2 \in W$, we have

$$v_1 - v_2 - w_2 \in W.$$

Therefore, there exists an element in v_1+W, which is $v_1-(v_1-v_2-w_2) = v_2+w_2 \in v_1+W$. Hence $v_2 + W \subset v_1 + W$. Therefore, $v_2 + W = v_1 + W$.

We prove the "only if" part then, which assumes $v_1 + W = v_2 + W$.

Note that $\forall v_1 + w_1 \in v_1 + W$, $v_1 + w + 1 \in v_2 + W$. Therefore $\exists w_2 \in W$ such that $v_2 + w_2 = v_1 + w_1$. Which implies

$$v_1 - v_2 = w_2 - w_1.$$

Note that $w_2, w_1 \in W$, and $1, -1 \in \mathbb{F}$, then $w_2 - w_1 = v_1 - v_2 \in W$.
Therefore we proved the (b) part.

5.3 (c)

From (b) part, since $v_1 + W = v_1^{'} + W$ and $v_2 + W = v_2^{'} + W$, we have $v_1 - v_1^{'} \in W$ and $v_2 - v_2^{'} \in W$.

Hence

$$v_1 - v_1^{'} + v_2 - v_2^{'} = v_1 + v_2 - v_1^{'} - v_2^{'} \in W.$$

By definition, $(v_1+W)+(v_2+W) = (v_1+v_2)+W$. and $(v_1^{'}+W)+(v_2^{'}+W) = (v_1^{'}+v_2^{'})+W$. Combine with the equation two lines above and result from (b),

$$(v_1 + W) + (v_2 + W) = (v_1^{'} + W) + (v_2^{'} + W).$$

By definition, $a(v_1+W) = av_1+W$, $a(v_1^{'}+W) = av_1^{'}+W$. With $v_1-v_1^{'} \in W$ and $a \in \mathbb{F}$, $a(v_1 - v_1^{'}) = av_1 - av_1^{'} \in W$.
Therefore $a(v_1 + W) = av_1 + W = av_1^{'} + W = a(v_1^{'} + W)$.
Therefore we proved the (c) part.

5.4 (d)

To prove S is a vector space as stated. One need to check the VS's.

Note that V is a vector space, and W is a subspace of V over field $\mathbb{F}$. And also note that the operations are well defined that fits the requirements.

[VS1]: $\forall v_1 + W, v_2 + W \in S$,

$$v_1 + W + (v_2 + W) = (v_1 + v_2) + W = (v_2 + v_1) + W = v_2 + W + (v_1 + W).$$

Therefore, VS1 is satisfied.

[VS2]: $\forall v_1 + W, v_2 + W, v_3 + W \in S$,
Note that $(v_1 + W + (v_2 + W)) + v_3 + W = (v_1 + v_2 + v_3) + W = v_1 + W + (v_2 + v_3) + W = v_1 + W + (v_2 + W + v_3 + W)$.
Therefore, VS2 is satisfied.

[VS3]: Since V is a vector space, denote the zero vector of V as 0_V. Note that $0_V + W \in S$

$$\forall v_1 + W \in S, v_1 + W + (0_V + W) = (v_1 + 0_V) + W = v_1 + W.$$

Hence $0_V + W$ is the zero vector of S. And the VS3 is satisfied.

[VS4]: Since V is a vector space, $\forall v_1 \in V$,
$\exists u_1 \in V$ such that $v_1 + u_1 = 0_V$. For $v_1 + W \in S$, $v_1 + W + (u_1 + W) = (v_1 + u_1) + W = 0_V + W$. Which is the zero vector of S.
Therefore, VS4 is satisfied.

[VS5]: Note that $\forall v_1 + W \in S$, $1 \cdot (v_1 + W) = 1 \cdot v_1 + W = v_1 + W$.
Therefore, VS5 is satisfied.

[VS6]: Note that $\forall v_1 + W \in S, \forall a, b \in \mathbb{F}$,

$$(ab)(v_1 + W) = ab \cdot v_1 + W = a(b \cdot v_1 + W) = a(b(v_1 + W)).$$

Therefore, VS6 is satisfied.

[VS7]: Note that $\forall v_1 + W, v_2 + W \in S, \forall a \in \mathbb{F}$,
$a(v_1 + W + (v_2 + W)) = a((v_1 + v_2) + W) = (av_1 + av_2) + W = av_1 + W + (av_2 + W) = a(v_1 + W) + a(v_2 + W)$.
Therefore, VS7 is satisfied.

[VS8]: Note that $\forall v + W \in S, \forall a, b \in \mathbb{F}$,

$$(a+b)(v+W) = (a+b)v + W = (av + bv) + W = av + W + (bv + W) = a(v + W) + b(v + W).$$

Therefore, VS8 is satisfied.
Hence, S is proved to be a vector space with operations defined in (c).

6 Section 1.4 Q10

Note that the set of all symmetric 2 by 2 matrices is $S = \{\begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{F}\}$.

Note that $\forall \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in S$,

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = aM_1 + bM_3 + cM_2.$$

Hence $S \subset Span\{M_1, M_2, M_3\}$.

Also note that $\forall x \in Span\{M_1, M_2, M_3\}$, $\exists a_1, a_2, a_3 \in \mathbb{F}$ such that

$$a_1M_1 + a_2M_2 + a_3M_3 = x.$$

Observe that

$$x = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}.$$

By the definition of S, $x \in S$. Therefore $Span\{M_1, M_2, M_3\} \subset S$, which deduces

$$S = Span\{M_1, M_2, M_3\}.$$

7 Section 1.4 Q14

Note that $\forall x \in Span(S_1 \cup S_2)$, $\exists a_1, a_2, ..., a_m, b_1, b_2, ..., b_n \in \mathbb{F}$ and $u_1, u_2, ..., u_m \in S_1, v_1, v_2, ..., v_m \in S_2$ such that

$$x = a_1u_1 + ...a_mu_m + b_1v_1 + ...b_nv_n.$$

Note that $a_1u_1 + ...a_mu_m \in Span(S_1)$ and $b_1v_1 + ...b_nv_n \in Span(S_2)$. Which implies

$$x = a_1u_1 + ...a_mu_m + b_1v_1 + ...b_nv_n \in Span(S_1) + Span(S_2).$$

Hence $Span(S_1 \cup S_2) \subset Span(S_1) + Span(S_2)$.

Note that $\forall y \in Span(S_1) + Span(S_2)$, $\exists a_1, a_2, ..., a_m, b_1, b_2, ..., b_n \in \mathbb{F}$ and $u_1, u_2, ..., u_m \in S_1, v_1, v_2, ..., v_m \in S_2$ s.t.

$$y = a_1u_1 + ...a_mu_m + b_1v_1 + ...b_nv_n.$$

That y is a linear combination fo elements in $Span(S_1 \cup S_2)$. Hence $y \in Span(S_1 \cup S_2)$. Therefore $Span(S_1) + Span(S_2) \subset Span(S_1 \cup S_2)$. Which deduces

$$Span(S_1) + Span(S_2) = Span(S_1 \cup S_2).$$

8 Section 1.4 Q15

Note that $\forall x \in Span(S_1 \cap S_2)$, $\exists a_1, a_2, ..., a_m \in \mathbb{F}$ and $u_1, u_2, ..., u_m \in S_1 \cap S_2$ such that

$$x = a_1 u_1 + ... + a_m u_m.$$

Note that $u_i \in S_1, \forall i$ and $u_i \in S_2, \forall i$. Then $x \in Span(S_1)$ and $x \in Span(S_2)$. That is $x \in Span(S_1) \cap Span(S_2)$. Which deduces that

$$Span(S_1 \cap S_2) \subset Span(S_1) \cap Span(S_2).$$

Let $S_1 = \{(1,0,0),(1,1,0)\}$ and $S_2 = \{(0,0,1),(0,1,1)\}$. Note that $Span(S_1 \cap S_2) = \emptyset$ while

$$Span(S_1) \cap Span(S_2) = \{(0,\lambda,0), \lambda \in \mathbb{F}\}.$$

Obviously, in this case, $Span(S_1 \cap S_2)$ and $Span(S_1) \cap Span(S_2)$ are not equal.

9 Section 1.5 Q15

We first prove the "if" part.
If $u_1 = 0$, note that

$$1 \cdot u_1 + 0 \cdot u_2 + ... + 0 \cdot u_n = 0.$$

Hence $S = \{u_1, u_2, ..., u_n\}$ is linear dependent.
If $u_{k+1} \in Span(\{u_1, u_2, ..., u_k\})$ for some $k(1 \leq k < n)$. Then $\exists a_1, a_2, ..., a_k$ such that

$$u_{k+1} = a_1 u_1 + ... + a_k u_k.$$

Hence notice that

$$a_1 u_1 + ... + a_k u_k - u_{k+1} + ... + 0 \cdot u_n = 0.$$

Hence $S = \{u_1, u_2, ..., u_n\}$ is linear dependent. Thus the "if" direction is proved.

We then prove the "only if" part.
Assume S is linearly dependent. Suppose $u_1 \neq 0$ and $\forall 1 \leq k < n$,

$$u_{k+1} \notin Span(\{u_1, ..., u_k\}).$$

By the definition of S being linear dependent, $\exists a_1, ..., a_n$ (not all zero) such that

$$0 = a_1 u_1 + ... a_n u_n.$$

Which contradicts with $u_n \notin Span(\{u_1, ..., u_n - 1\})$ that we assumed.
Hence, $u_1 = 0$ or $\exists 1 \leq k < n$, such that $u_{k+1} \in Span(\{u_1, ..., u_k\})$ at the first place.

10 Section 1.5 Q18

Suppse S is linearly dependent. Then $\exists$ distinct $u_1, u_2, ..., u_n \in S$ and $a_1, ..., a_n \in \mathbb{F}$ (not all zero) such that

$$a_1u_1 + ... + a_nu_n = 0.$$

WLOG, let $a_1 \neq 0$, and u_1 having the largest degree among those non-zero terms, denoted as $\deg(u_1)$.

If $\deg(u_1) > 0$, then degree of the L.H.S. is $\deg(u_1) \neq \deg(0) = 0$. Contradiction arises.

Else if $\deg(u_1) = 0$, according to u_1 is having the largest degree and those polynomials do not have same degree.

Then the L.H.S.$= a_1u_1$, which is a nonzero constant, hence not equal to 0 on R.H.S. Contradiction arises.

Hence, S is linearly independent at the first place.