MATH2040C Homework 7

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1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(c) $\mathsf{V} = \mathsf{P}_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V.

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^{1} 1 \cdot t\, dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^{n} \overline{\langle T(v_i), g(t)\rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\sqrt{\frac{3}{2}}t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t..$

Therefore,

$$T^*(g(t)) = \frac{3}{2}\int_{-1}^{1} g(t)dt + \frac{3}{2}t\int_{-1}^{1}(1+3t)g(t)dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4-2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$. Deduce that $\text{rank}(\mathsf{T}^*\mathsf{T}) = \text{rank}(\mathsf{T})$.

(b) $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^*)$. Deduce from (a) that $\text{rank}(\mathsf{T}\mathsf{T}^*) = \text{rank}(\mathsf{T})$.

(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.
Forall $y \in N(T^*T)$, consider the norm of Ty :

$$||Ty||^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.
Based on all above, $N(T^*T) = N(T)$.
Recall that $T \in \mathcal{L}(V)$. Hence $T : V \to V$. And according to $\forall y \in V$,

$$T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y \rangle} v_i.$$

We know that $T^* : V \to V$. Therefore $T^*T : V \to V$.
Applying the rank nullity theorem, we have that

$$\dim V = rank(T^*T) + \dim N(T^*T)\ , \ \dim V = rank(T) + \dim N(T).$$

Using the just proved fact $N(T^*T) = N(T)$, we can simply deduce

$$rank(T^*T) = rank(T).$$

2.2 (b)

By changing name of the identity in (a), we can have $N(TT^*) = N(T^*)$ and $rank(TT^*) = rank(T^*)$.
Notice that

$$rank(T) = rank[T]_\beta = rank[T]_\beta^* = rank[T^*]_\beta = rank(T^*).$$

And then $rank(TT^*) = rank(T)$ follows.

2.3 (c)

From (b), $rank(AA^*) = rank(A)$ follows naturally.
And note that $(AA^*)^* = A^*A$, then

$$rank(AA^*) = rank(A^*A).$$

Therefore,

$$rank(AA^*) = rank(A^*A) = rank(A).$$

Done.

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3 Section 6.3, Q14

Let V be an inner product space, and let $y, z \in V$. Define $T\colon V \to V$ by $T(x) = \langle x, y\rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

First we would prove that T is linear. $\forall x_1, x_2 \in V, \forall c \in F$,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y\rangle z = \langle cx_1, y\rangle z + \langle x_2, y\rangle z = c\langle x_1, y\rangle z + \langle x_2, y\rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore, T is linear.
From course not we directly construct $\forall x \in V$,

$$T^*(x) = \sum_{i=1}^{n} \overline{\langle T(v_i), x\rangle} v_i = \sum_{i=1}^{n} \overline{\langle\langle v_i, y\rangle z, x\rangle} v_i = \overline{\langle z, x\rangle} \sum_{i=1}^{n} \overline{\langle v_i, y\rangle} v_i = \overline{\langle z, x\rangle} y.$$

Recall that $y = I(y) = I^*(y) = \sum_{i=1}^{n} \overline{\langle I(v_i), y\rangle} v_i = \sum_{i=1}^{n} \overline{\langle v_i, y\rangle} v_i$. Hence the last equality holds properly.

4 Section 6.3, Q15

Definition. *Let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be a linear transformation, where* V *and* W *are finite-dimensional inner product spaces with inner products* $\langle \cdot, \cdot \rangle_1$ *and* $\langle \cdot, \cdot \rangle_2$*, respectively. A function* $\mathsf{T}^*\colon \mathsf{W} \to \mathsf{V}$ *is called an* **adjoint** *of* T *if* $\langle \mathsf{T}(x), y \rangle_2 = \langle x, \mathsf{T}^*(y) \rangle_1$ *for all* $x \in \mathsf{V}$ *and* $y \in \mathsf{W}$.

15. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

(a) There is a unique adjoint T^* of T, and T^* is linear.
(b) If β and γ are orthonormal bases for V and W, respectively, then $[\mathsf{T}^*]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^*$.
(c) $\operatorname{rank}(\mathsf{T}^*) = \operatorname{rank}(\mathsf{T})$.
(d) $\langle \mathsf{T}^*(x), y \rangle_1 = \langle x, \mathsf{T}(y) \rangle_2$ for all $x \in \mathsf{W}$ and $y \in \mathsf{V}$.
(e) For all $x \in \mathsf{V}$, $\mathsf{T}^*\mathsf{T}(x) = \mathit{0}$ if and only if $\mathsf{T}(x) = \mathit{0}$.