MATH2040C Homework 1

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1 Section 1.2, Q13

To check if a set is a vector space, one need to check those VS's.

[VS1]: $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{V}$, note that from definition,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$(b_1, b_2) + (a_1, a_2) = (a_1 + b_1, a_2 b_2)$$

Hence $(b_1, b_2) + (a_1, a_2) = (a_1, a_2) + (b_1, b_2), \forall (a_1, a_2), (b_1, b_2) \in \mathbb{V}$. Therefore VS1 is satisfied.

[VS2]: $\forall (a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{V}$, note that by definition,

$$((a_1, a_2) + (b_1, b_2)) + (c_1, c_2) = (a_1 + b_1, a_2 b_2) + (c_1, c_2) = (a_1 + b_1 + c_1, a_2 b_2 c_2)$$

and

$$(a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = (a_1, a_2) + (b_1 + c_1, b_2 c_2) = (a_1 + b_1 + c_1, a_2 b_2 c_2)$$

$\therefore (a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = ((a_1, a_2) + (b_1, b_2)) + (c_1, c_2), \forall (a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{V}$. Therefore, VS2 is satisfied.

[VS3]: Note that an element $(0, 1) \in \mathbb{V}$. Note that $\forall (a_1, a_2) \in \mathbb{V}$,

$$(0, 1) + (a_1, a_2) = (0 + a_1, 1 \cdot a_2) = (a_1, a_2).$$

Hence VS3 is satisfied.

[VS4]: Note that $(1, 0) \in \mathbb{V}$.
And $\forall (a_1, a_2) \in \mathbb{V}, (1, 0) + (a_1, a_2) = (1 + a_1, 0) \neq (0, 1)$. Note that the $(0, 1)$ is the zero vector we defined in order to satisfy VS3.
Therefore VS4 cannot be satisfied, hence $\mathbb{V}$ is not a vector space under the operations stated in the question.

2 Section 1.2 Q21

To check if a set is a vector space, one need to check those VS's.
[VS1]: $\forall (v_1, w_1), (v_2, w_2) \in Z$, note that

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2, w_2) + (v_1, w_1).$$

Therefore, VS1 is satisfied.

[VS2]: $\forall (v_1, w_1), (v_2, w_2), (v_3, w_3) \in Z$, note that

$$((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3).$$

And

$$(v_1, w_1) + ((v_2, w_2) + (v_3, w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3)$$

Therefore $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$, which implies that VS2 is satisfied.

[VS3]: Denote 0_V is a zero vector of V and 0_W is a zero vector of W.
Note that $(0_V, 0_W) \in Z$.
And $\forall (v, w) \in Z$,

$$(0_V, 0_W) + (v, w) = (0_V + v, 0_W + w) = (v, w).$$

Therefore, VS3 is satisfied, and we also define $0_Z = (0_V, 0_W)$ as a zero vector of Z.

[VS4]: $\forall (v, w) \in Z$, note that $\exists \hat{v} \in V, \hat{w} \in W$ such that $v + \hat{v} = 0_V, w + \hat{w} = 0_W$ because V and W are themselves vector spaces.
Note that $(\hat{v}, \hat{w}) \in Z$, since $\hat{v} \in V, \hat{w} \in W$ and

$$(v, w) + (\hat{v}, \hat{w}) = (v + \hat{v}, w + \hat{w}) = (0_V, 0_W) = 0_Z.$$

Therefore, VS4 is satisfied.

[VS5]: Note that $1 \in \mathbb{F}$ and $\forall (v, w) \in Z$,

$$1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w).$$

Therefore, VS5 is satisfied.

[VS6]: Note that $\forall (v, w) \in Z, \forall a, b \in \mathbb{F}$,

$$(ab)(v, w) = (ab \cdot v, ab \cdot w) = (a)(b \cdot v, b \cdot w) = a(b(v, w)).$$

Therefore, VS6 is satisfied.

[VS7]: Note that $\forall (v_1, w_1), (v_2, w_2) \in Z, \forall a \in \mathbb{F}$,

$$a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (a \cdot v_1 + a \cdot v_2, a \cdot w_1 + a \cdot w_2) = a(v_1, w_1) + a(v_2, w_2).$$

Note that the second equailty holds for V and W themselves being vector spaces and $v_1, v_2 \in V, w_1, w_2 \in W$.

Therefore, VS7 is satisfied.

[VS8]: Note that $\forall (v, w) \in Z, \forall a, b \in \mathbb{F}$,

$$(a+b)(v, w) = ((a+b) \cdot v, (a+b) \cdot w)$$

Note that V, W are vector spaces over field $\mathbb{F}$, therefore

$$(a+b)v = a \cdot v + b \cdot v,$$

$$(a+b)w = a \cdot w + b \cdot w.$$

Hence

$$(a+b)(v, w) = (a \cdot v + b \cdot v, a \cdot w + b \cdot w) = (a \cdot v, a \cdot w) + (b \cdot v, b \cdot w) = a(v, w) + b(v, w).$$

Therefore, VS8 is satisfied.

Since the requirements are all satisfied, therefore the set Z is a vector space over $\mathbb{F}$ with the operations stated in the question.

3 Section 1.3 Q11