MATH2040C Homework 4

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1 Section 5.1, Q2(e)

Given that $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$.

And note that $T(1 - x + x^3) = -1 + x - x^3$. $T(1 + x^2) = -x - x^2 + x^3$. $T(1) = x^2$. $T(x + x^2) = -x - x^2$.

Hence $T(\beta) = \{-1 + x - x^3, -x - x^2 + x^3, x^2, -x - x^2\}$.

$[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Suppose β is containing T's eigenvectors, then $\exists \lambda \in F$ such that

$$T(1 + x^2) = \lambda(1 + x^2).$$

Then $\lambda + \lambda x^2 = -x - x^2 + x^3$. Note that the degree of them do not equal in any sense. Hence β is not a basis consisting of eigenvectors of T.

2 Section 5.1, Q2(f)

Given that $\beta = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\}$.

Note that $T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

$T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$,

$T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, and

$T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

Hence we deduce that β is a basis consisting of eigenvectors of T.

3 Section 5.1, Q3(d)

3.1 (i)

Given that $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, then its characteristic polynomial is

$$f_A(t) = \det \begin{pmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{pmatrix} = -t(t-1)^2.$$

Observe the $f_A(t)$'s zeros, we have A should have 2 eigenvalues: 1 and 0.

3.2 (ii)

For eigenvalue 1, the corresponding eigenvectors should be in the span of set

$$\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}.$$

For eigenvalue 0, the corresponding eigenvectors should be in the span of set

$$\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \}.$$

3.3 (iii)

In this case, the $n = 3, F = \mathbb{R}$. So $F^3 = \mathbb{R}^3$.

Note that $\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \}$ is a 3-linear-indenpendent set. Hence it is a basis of $\mathbb{R}^3$.
And by our conclusion above, these 3 vectors are eigenvectors of A.

3.4 (iv)

Let $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}$. Note that Q is invertible and $Q^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ -1 & 0 & 1 \end{pmatrix}$.
Note that

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4 Section 5.1, Q4(h)

Let β be the standard basis. Note that $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. By extracting its characteristic polynomial, it is

$$f_T(t) = (t-1)^3(t+1) = 0.$$

And note that their corresponding eigenvectors to be $\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\}$.

Note that by the diagnoalizability of $[T]_\beta$, (for its every eigenvalue: 1 and -1: algebraic multiplicity equals geometric multiplicity) we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q^{-1}[T]_\beta Q.$$

Where $Q = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$.

Regard Q as a change of basis matrix from another basis γ to our known standard basis β. Therefore, $Q = [I]_\gamma^\beta$.

Let $\gamma = \{y_1, y_2, y_3, y_4\}$. Therefore

$$[y_1, y_2, y_3, y_4]_\beta = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Hence, $y_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $y_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $y_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Note that $[T]_\gamma = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$. So γ is the ordered basis we need to find.

5 Section 5.1, Q4(e)

Let $\beta = \{1+x, -3-13x+4x^2, -3+x\}$ be a ordered basis. Then

$$[T]_\beta = [4x+4, 8x^2-26x-6, 0]_\beta = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Which is a diagnoal matrix. Hence the β is what we want to find. And the eigenvalues of T are 4,2 and 0, with corresponding eigenvectors elements of the ordered basis β.

6 Section 5.1, Q17

6.1 (a)

Note that for the identity matrix I, $T(I) = I = 1 \cdot I$, hence 1 is a eigenvalue. Also note that for a matrix X with only 2 entries on the right top and left buttom being 1 and -1, then

$$T(X) = -X.$$

Hence -1 is a eigenvalue.

Suppose there exists some eigenvalue $|\lambda| \neq 1$ such that

$$A^T = \lambda A.$$

Then we can deduce $\lambda A^T = A = \lambda^2 A$. Which implies $(1-\lambda^2)A = 0$.

Because A is regarded as an eigenvector, hence it is not zero, so $1 - \lambda^2$ must be 0. But other than 1 and -1, it cannot be 0.

Hence 1, -1 are the only eigenvalues of A.

6.2 (b)

For eigenvalue 1, the corresponding eigenvectors are all symmetric matrices.

For eigenvalue -1, the corresponding eigenvectors are all skew symmetric matrices.

6.3 (c)

Consider $\gamma = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

Then $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q^{-1}[T]_\gamma Q,$$

where $Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Where Q can be regarded as $[I]_\beta^\gamma$. Let $\beta = \{v_1, v_2, v_3, v_4\}$.

$$[I]_\beta^\gamma = [v_1, v_2, v_3, v_4]_\gamma = [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}].$$

Hence the γ is the basis we want.

6.4 (d)

Note that for 1 as an eigenvalue, its corresponding eigenvectors are

$$E_1 = \{e_{1ij}, i \le j\}.$$

Where the e_{1ij} are defined as a matrix with i-j's slot to be 1 and j-i's slot also being 1, while others remains 0.

And for -1 as an eigenvalue, its corresponding eigenvectors are

$$E_{-1} = \{e_{2ij}, i < j\}.$$

Where e_{2ij} is defined as a matrix with i-j's slot being 1 and j-i slot being -1.

Then, take $\gamma = \{e_{ij}, 1 \le i, j \le n\}$. Take $[E_1, E_{-1}]_\gamma$ as a change of order matrix.

If we regard the E_1, E_{-1} as a basis of $M_{n\times n}(\mathbb{R})$, then we can regard

$$[E_1, E_{-1}]_\gamma = [I(v_1), I(v_2), ..., I(v_{n^2})]_\gamma.$$

Note that the eigenvectors inside E_1, E_{-1} are all linear indenpendent, and there are $\frac{n(n-1)}{2} + n + \frac{n(n-1)}{2} = n^2 = \dim M_{n\times n}(\mathbb{R})$. Hence $[E_1, E_{-1}]$ is a basis.

Than note that

$$\begin{pmatrix} \lambda_1 & ... & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & \lambda_n \end{pmatrix} = ([E_1, E_{-1}]_\beta^\gamma)^{-1} [T]_\gamma [E_1, E_{-1}]_\beta^\gamma$$

where $|\lambda_i| = 1, \forall i$.

Deifine the $\beta = \{v_1, v_2, ..., v_{n^2}\} = \{$ every column of $E_1, E_{-1}\}$.

Hence the β is the basis we want to find.

7 Section 5.1, Q18

7.1 (a)

If A is not invertible, then let $c = 0$. We have

$$\det(A + cB) = \det A = 0.$$

Since A is singular as we supposed.

If A is invertible, then note that $A = AB^{-1}B$, then

$$\det(A + cB) = \det AB^{-1}B + cB = \det(AB^{-1} + cI)\det(B).$$

Note that $\det(B) \ne 0$ and $\det(AB^{-1} + cI) = 0$ if $-c$ is the eigenvalue of AB^{-1}. And by the fundemantal theorem of algebra, there must exist such c.

Done.

7.2 (b)

Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Note that $\det(A) = 2 \neq 0$. And $\forall c \in \mathbb{C}$,

$$\det(A + cB) = \det(\begin{pmatrix} 2 & 1+c \\ 0 & 1 \end{pmatrix}) = 2 \neq 0.$$

Therefore, A and A+cB are both invertible.

8 Section 5.2, Q3(c)

Note that $V = \mathbb{R}^3$.

Define $\gamma = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$ and hence $T(\gamma) = \{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\}$ And we can see that the characteristic polynomial of $[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

$$f_T(t) = (2 - t)(t^2 + 1).$$

Which only have one eigenvalue in $\mathbb{R}$. Hence it only have not enough eigenvalues, hence not diagnoalizabile.

9 Section 5.2, Q8

$1 \leq \gamma_T(\lambda_1) = \dim(E_{\lambda_1}) = n - 1$.

$1 \leq \gamma_T(\lambda_2) = \dim(E_{\lambda_2})$.

Take $n - 1$ orthogonal and eigenvectors from E_{λ_1} denote them as $v_1, v_2, ..., v_{n-1}$.

And then take 1 eigenvector from E_{λ_2}, make it orthogonal to $v_1, v_2, ..., v_{n-1}$ and denote it as w.

Consider $Q = \begin{pmatrix} v_1 & v_2 & ... & v_n & w \end{pmatrix}$. Note that $Q^{-1} = Q^T$.

Because $Q^T Q = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \\ w \end{pmatrix} \begin{pmatrix} v_1 & v_2 & ... & v_n & w \end{pmatrix} = I$. Because every v_i, w are orthogonal.

Then, note that

$$Q^{-1} A Q = Q^T [\lambda_1 v_1, \lambda_1 v_2, ..., \lambda_1 v_{n-1}, \lambda_2 w] = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \\ w \end{pmatrix} [\lambda_1 v_1, \lambda_1 v_2, ..., \lambda_1 v_{n-1}, \lambda_2 w]$$

.

Which is that $Q^{-1}AQ = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_1 & ... & ... \\ ... & ... & ... & ... \\ 0 & ... & ... & \lambda_2 \end{pmatrix}$.

Therefore, A is diagnoalizabile.

10 Section 5.2, Q13

10.1 (a)

Consider matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. We know that A has 2 eigenvalues 2 and -1, with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

And $A^T = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, while having a eigenvalue 2, its corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So the eigenbasis E_2 of A and A^T are not the same for sure.

10.2 (b)

$\forall \lambda$, where λ is a eigenvalue of A (hence also an eigenvalue of A^T).
We have $\dim(E_\lambda) = \dim N(A - \lambda I) = n - rank(A - \lambda I)$.
Note that $\forall$ matrix M, we have $rank(M) = rank(M^T)$.
Then $rank((A - \lambda I)^T) = rank(A^T - \lambda I)$. Then note that

$$\dim(E_\lambda) = n - rank(A^T - \lambda I) = \dim N(A^T - \lambda I) = \dim(E'_\lambda).$$

Done.

10.3 (c)

If A is diagnoalizabile, then $\forall 1 \leq i \leq k$, A's eigenvalue λ_i's algebraic multiplicity $m_i = \dim(E_{\lambda_i})$.

Note that A^T shares the same eigenvalues with A and their characteristic polynomials are also the same. Hence all the m_i's are still.

By the result from (b),

$$\dim(E_{\lambda_i}) = \dim(E'_{\lambda_i}), \forall 1 \leq i \leq k$$

where E'_λ is the eigenspace of λ_i of A^T.

Then we have

$$m_i = \dim(E'_{\lambda_i}), \forall 1 \leq i \leq k.$$

Hence A^T is diagnoalizabile.