MATH2040C Homework 7

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April 25, 2021

1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(c) $\mathsf{V} = \mathsf{P}_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V.

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^{1} 1 \cdot t\,dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^{n} \overline{\langle T(v_i), g(t)\rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\sqrt{\frac{3}{2}}t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t$..

Therefore,

$$T^*(g(t)) = \frac{3}{2}\int_{-1}^{1} g(t)dt + \frac{3}{2}t\int_{-1}^{1} (1+3t)g(t)dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N(T^*T)} = \mathsf{N(T)}$. Deduce that $\text{rank}(\mathsf{T^*T}) = \text{rank}(\mathsf{T})$.
(b) $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T^*})$. Deduce from (a) that $\text{rank}(\mathsf{TT^*}) = \text{rank}(\mathsf{T})$.
(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.
Forall $y \in N(T^*T)$, consider the norm of Ty :

$$||Ty||^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.
Based on all above, $N(T^*T) = N(T)$.