MATH2040C Homework 3

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1 Section 2.2, Q3

According to the question, $\beta = \{(1,0),(0,1)\}$.
Therefore, $T((1,0)) = (1,1,2), T((0,1)) = (-1,0,1)$.
Then we need to find $[T((1,0))]_\gamma = [(1,1,2)]_\gamma$ and $[T((0,1))]_\gamma = [(-1,0,1)]_\gamma$.
By the question, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$.

Note that $\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{1}{3}\\0\\\frac{2}{3}\end{pmatrix}$.

Hence $[T((1,0))]_\gamma = [(1,1,2)]_\gamma = (-\frac{1}{3},0,\frac{2}{3})$.

Also note that $\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-1\\1\\0\end{pmatrix}$.

Therefore, $[T]_\beta^\gamma = \begin{pmatrix}-\frac{1}{3}&-1\\0&1\\\frac{2}{3}&0\end{pmatrix}$.

Note that $\alpha = \{(1,2),(2,3)\}$. And $T(1,2) = (-1,1,4)$ and $T(2,3) = (-1,2,7)$. Then we will find $[(-1,1,4)]_\gamma$ and $[-1,2,7]_\gamma$.

Note that $\begin{pmatrix}-1\\1\\4\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{7}{3}\\2\\\frac{2}{3}\end{pmatrix}$.

Hence $[T(1,2)]_\gamma = [(-1,1,4)]_\gamma = (-\frac{7}{3},2,\frac{2}{3})$.

Also note that $\begin{pmatrix}-1\\2\\7\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{11}{3}\\3\\\frac{4}{3}\end{pmatrix}$.

Hence $[T(2,3)]_\gamma = [(-1,2,7)]_\gamma = (-\frac{11}{3},3,\frac{4}{3})$.

Therefore, $[T]_\alpha^\gamma = \begin{pmatrix}-\frac{7}{3}&-\frac{11}{3}\\2&3\\\frac{2}{3}&\frac{4}{3}\end{pmatrix}$.

Done.

2 Section 2.2, Q5

2.1 (a)

Note that $\alpha = \{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$. Hence

$$[T]_\alpha = [\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}]_\alpha = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}.$$

2.2 (b)

$$[T]_\beta^\alpha = [T(1), T(x), T(x^2)]_\alpha = [\begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & 2\end{pmatrix}]_\alpha = \begin{pmatrix}0 & 1 & 0\\2 & 2 & 2\\0 & 0 & 0\\0 & 0 & 2\end{pmatrix}.$$

2.3 (c)

The basis $\alpha = \{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$.
$[T]_\alpha^\gamma = [tr\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, tr\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, tr\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, tr\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}]_\gamma = [1,1,1,1]_\gamma = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}.$

2.4 (d)

Recall that $\beta = \{1, x, x^2\}$.
$[T]_\beta^\gamma = [T(1), T(x), T(x^2)]_\gamma = \begin{pmatrix}1 & 2 & 4\end{pmatrix}.$

2.5 (e)

The basis $\alpha = \{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$.
Because $A = \begin{pmatrix}1 & -2\\0 & 4\end{pmatrix}$, then

$$[A]_\alpha = \begin{pmatrix}1\\-2\\0\\4\end{pmatrix}.$$

2.6 (f)

Note that $f(x) = 3 - 6x + x^2$. Therefore

$$[f(x)]_\beta = \begin{pmatrix}3\\-6\\1\end{pmatrix}.$$

2.7 (g)

$\forall a \in F,, [a]_\gamma = a$.

3 Section 2.3, Q3

3.1 (a)

$[U]_\beta^\gamma = [U(1), U(x), U(x^2)]_\gamma = [(1,0,1),(1,0,-1),(0,1,0)]_\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

$[T]_\beta = [T(1), T(x), T(x^2)]_\beta = [2, 3+3x, 6x+4x^2]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$.

And $[UT]_\beta^\gamma = [UT(\beta)]_\gamma = [U(2), U(3+3x), U(6x+4x^2)]_\gamma$

$[UT]_\beta^\gamma = [(2,0,2),(6,0,0),(6,4,-6)]_\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$.

Verify that

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = [U]_\beta^\gamma [T]_\beta.$$

Done.

3.2 (b)

Because $h(x) = 3 - 2x + x^2$. $[h(x)]_\beta = [3 - 2x + x^2] = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

$[U(h(x))]_\gamma = [U(3 - 2x + x^2)]_\gamma = [(1,1,5)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

Note that

$$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = [U]_\beta^\gamma [h(x)]_\beta.$$

Hence the theorem 2.14 is verified.
Done.

4 Section 2.3, Q16

4.1 (a)

Note that $T : V \to V$. And $T^2 : V \to V$.

Given that $rank(T) = rank(T^2)$. By rank-nullity theorem,

$$rank(T) + nullity(T) = \dim V.$$

Also by rank-nullity theorem,

$$rank(T^2) + nullity(T^2) = \dim V.$$

Hence $nullity(T^2) = nullity(T)$.
It is obvious that $\forall x \in N(T), T^2(x) = T(T(x)) = T(0_V) = 0_V$.
Hence $N(T) \subset N(T^2)$.

Suppose that $\exists y \in N(T^2)$ such that $T(y) = 0_V$. Then $\dim N(T^2) > \dim N(T)$.
But $nullity(T) = \dim V - rank(T) = \dim V - rank(T^2) = nullity(T^2)$. Which arises contradiction.
Hence $\forall y \in N(T^2), T(y) = 0_V$ at the first place, then

$$N(T) = \{0_V\}.$$

Note that $0_V \in R(T)$, therefore $R(T) \cap N(T) = \{0_V\}$. Which deduces that

$$V = R(T) \bigoplus N(T).$$

4.2 (b)

Let $k = 1$. Then the proof follows the case in (a).

5 Section 2.4, Q14

We construct the transformation to be $T : V \to F^3$.
With T defined as $\forall v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in V$.

$$T(v) = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{22} \end{pmatrix}.$$

First we prove that T should be linear.
Note that $\forall v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in V, w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \in V, a \in F$,

$$T(av + w) = \begin{pmatrix} av_{11} + w_{11} \\ av_{12} + w_{12} \\ av_{22} + w_{22} \end{pmatrix} = a \begin{pmatrix} v_{11} \\ v_{12} \\ v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} \\ w_{12} \\ w_{22} \end{pmatrix} = aT(v) + T(w)$$

Hence we proved the transformation T is linear.
Note that $\beta = \{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ is a basis for V.
And $\alpha = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis for F^3.

Note that the matrix

$$[T]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible. Hence T is invertible.

Therefore, T is a isomorphism constructed by us.

Done.

6 Section 2.4, Q15

Because the V is a n-dimensional space, hence denote $\beta = \{v_1, v_2, ..., v_n\}$.

6.1 Only If

Given the condition that T is an isomorphism. And the linearity is given in the question, then we only need to prove T is invertible.

That is we have $T : V \to W$ is bijection. What we want is that $T(\beta)$ is a basis for W.

Note that $\forall y \in W$, because T is surjective, then $\exists x = \sum_{i=1}^n a_i v_i \in V$ such that $T(x) = y$. Which is

$$y = T(x) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i).$$

From this we know that $\{T(v_i), i = 1, 2, ..., n\}$ is a span of W.

Then we want to prove that $\{T(v_i), i = 1, 2, ..., n\}$ is linear indenpendent.

Suppose not, then $\exists b_1, b_2, ..., b_n$ (not all zeros) such that

$$\sum_{i=1}^n b_i T(v_i) = 0.$$

Then $T(b_1 v_1 + ... + b_n v_n) = 0$. Note that $b_1 v_1 + ... + b_n v_n$ is not zero while it is inside $N(T)$. This contradicts with T is injective.

Therefore, $\{T(v_i), i = 1, 2, ..., n\}$ is linear indenpendent at the first place.

Hence $\{T(v_i), i = 1, 2, ..., n\} = T(\beta)$ is a basis for W.

6.2 If

As the question assume that T is linear, then in order to prove that T is isomorphism, we need to prove T is invertible only.

Note that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W. Then $\forall y \in W$, $\exists b_1, b_2, ..., b_n$ such that

$$y = \sum_{i=1}^n b_i T(v_i) = T(\sum_{i=1}^n b_i v_i).$$

Hence T is surjective.

Suppse $\exists c_1, c_2, ..., c_n$ (not all zero) such that

$$T(c_1 v_1 + ... + c_n v_n) = 0.$$

Then $\sum_{i=1}^{n} c_i T(v_i) = 0$. Which deduces that $T(\beta)$ is linear dependent. It contradicts with it being a basis.

Hence $N(T) = \{0\}$. Which implies T is injective. Then T is bijection, and hence T is isomorphism.

Done.

7 Section 2.4, Q16

To prove Φ is isomorphism, i.e. linear and invertible. We first prove its linearity.

$\forall a \in F$ and $A, C \in M_{n\times n}(F)$, note that

$$\Phi(aA + C) = B^{-1}(aA + C)B = aB^{-1}AB + B^{-1}CB = a\Phi(A) + \Phi(C).$$

Hence, Φ is linear.

Then we need to prove Φ is invertible.

$\forall Y \in M_{n\times n}(F)$, Note that $BYB^{-1} \in M_{n\times n}(F)$ and

$$\Phi(BYB^{-1}) = B^{-1}BYB^{-1}B = Y.$$

Hence Φ is surjective.

Suppose $\exists A \neq A' \in M_{n\times n}(F)$ such that $\Phi(A) = \Phi(A')$.

Then

$$\Phi(A) = B^{-1}AB = B^{-1}A'B = \Phi(A').$$

Which implies $A = A'$. It contradicts with the assumption. Therefore Φ is injective at the first place.

Then we proved Φ is bijection, hence invertible.

Based on above, Φ is isomorphism.

8 Section 2.5, Q3(f)

Denote the matrix as

$$Q = [Id(\beta')]_\beta = [(9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2)]_\beta = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ 1 & 5 & 2 \end{pmatrix}.$$

Done.

9 Section 2.5, Q4

By the theorem 2.23, we know that

$$[T]'_\beta = Q^{-1}[T]_\beta Q.$$

Where $Q = [I]_{\beta'}^{\beta} = [I(\beta')]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Also, $Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

And $[T]_\beta = [T(\beta)]_\beta = [T(1,0), T(0,1)]_\beta = [\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$.

Then

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}.$$

10 Section 2.5, Q6(d)

$[L_A]_\beta = Q^{-1}AQ$.

Note that $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$. And $Q^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

Hence

$$[L_A]_\beta = Q^{-1}AQ = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$