MATH2040C Homework 3

ZHENG Weijia (William, 1155124322)

February 19, 2021

1 Section 2.2, Q3

According to the question, $\beta = \{(1,0),(0,1)\}$.
Therefore, $T((1,0)) = (1,1,2), T((0,1)) = (-1,0,1)$.
Then we need to find $[T((1,0))]_\gamma = [(1,1,2)]_\gamma$ and $[T((0,1))]_\gamma = [(-1,0,1)]_\gamma$.
By the question, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$.
Note that $\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{1}{3}\\0\\\frac{2}{3}\end{pmatrix}$.
Hence $[T((1,0))]_\gamma = [(1,1,2)]_\gamma = (-\frac{1}{3},0,\frac{2}{3})$.
Also note that $\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-1\\1\\0\end{pmatrix}$.
Therefore, $[T]_\beta^\gamma = \begin{pmatrix}-\frac{1}{3}&-1\\0&1\\\frac{2}{3}&0\end{pmatrix}$.

Note that $\alpha = \{(1,2),(2,3)\}$. And $T(1,2) = (-1,1,4)$ and $T(2,3) = (-1,2,7)$. Then we will find $[(-1,1,4)]_\gamma$ and $[-1,2,7]_\gamma$.
Note that $\begin{pmatrix}-1\\1\\4\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{7}{3}\\2\\\frac{2}{3}\end{pmatrix}$.
Hence $[T(1,2)]_\gamma = [(-1,1,4)]_\gamma = (-\frac{7}{3},2,\frac{2}{3})$.
Also note that $\begin{pmatrix}-1\\2\\7\end{pmatrix} = \begin{pmatrix}1&0&2\\1&1&2\\0&1&3\end{pmatrix}\begin{pmatrix}-\frac{11}{3}\\3\\\frac{4}{3}\end{pmatrix}$.
Hence $[T(2,3)]_\gamma = [(-1,2,7)]_\gamma = (-\frac{11}{3},3,\frac{4}{3})$.
Therefore, $[T]_\alpha^\gamma = \begin{pmatrix}-\frac{7}{3}&-\frac{11}{3}\\2&3\\\frac{2}{3}&\frac{4}{3}\end{pmatrix}$.
Done.

2 Section 2.2, Q5

2.1 (a)

Note that $\alpha = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. Hence

$$[T]_\alpha = [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.2 (b)

$$[T]_\beta^\alpha = [T(1), T(x), T(x^2)]_\alpha = [\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}]_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2.3 (c)

The basis $\alpha = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.
$[T]_\alpha^\gamma = [tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, tr\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}]_\gamma = [1,1,1,1]_\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$

2.4 (d)

Recall that $\beta = \{1, x, x^2\}$.
$[T]_\beta^\gamma = [T(1), T(x), T(x^2)]_\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}.$

2.5 (e)

The basis $\alpha = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.
Because $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$, then

$$[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}.$$

2.6 (f)

Note that $f(x) = 3 - 6x + x^2$. Therefore

$$[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}.$$

2.7 (g)

$\forall a \in F,, [a]_\gamma = a.$

3 Section 2.3, Q3

3.1 (a)

$[U]_\beta^\gamma = [U(1), U(x), U(x^2)]_\gamma = [(1,0,1),(1,0,-1),(0,1,0)]_\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$

$[T]_\beta = [T(1), T(x), T(x^2)]_\beta = [2, 3+3x, 6x+4x^2]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$

And $[UT]_\beta^\gamma = [UT(\beta)]_\gamma = [U(2), U(3+3x), U(6x+4x^2)]_\gamma$

$[UT]_\beta^\gamma = [(2,0,2),(6,0,0),(6,4,-6)]_\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$

Verify that

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = [U]_\beta^\gamma [T]_\beta.$$

Done.

3.2 (b)

Because $h(x) = 3 - 2x + x^2$. $[h(x)]_\beta = [3 - 2x + x^2] = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$

$[U(h(x))]_\gamma = [U(3-2x+x^2)]_\gamma = [(1,1,5)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$

Note that

$$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = [U]_\beta^\gamma [h(x)]_\beta.$$

Hence the theorem 2.14 is verified.
Done.

4 Section 2.3, Q16

4.1 (a)

Note that $T : V \to V$. And $T^2 : V \to V$.

Given that $rank(T) = rank(T^2)$. By rank-nullity theorem,

$$rank(T) + nullity(T) = \dim V.$$

Also by rank-nullity theorem,

$$rank(T^2) + nullity(T^2) = \dim V.$$

Hence $nullity(T^2) = nullity(T)$.
It is obvious that $\forall x \in N(T), T^2(x) = T(T(x)) = T(0_V) = 0_V$.
Hence $N(T) \subset N(T^2)$.

Suppose that $\exists y \in N(T^2)$ such that $T(y) = 0_V$. Then $\dim N(T^2) > \dim N(T)$.
But $nullity(T) = \dim V - rank(T) = \dim V - rank(T^2) = nullity(T^2)$. Which arises contradiction.
Hence $\forall y \in N(T^2), T(y) = 0_V$ at the first place, then

$$N(T) = \{0_V\}.$$

Note that $0_V \in R(T)$, therefore $R(T) \cap N(T) = \{0_V\}$. Which deduces that

$$V = R(T) \bigoplus N(T).$$

4.2 (b)