

MATH2040C Homework 7

ZHENG Weijia (William, 1155124322)

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1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V , evaluate T^* at the given vector in V .

(c) $V = P_1(R)$ with $\langle f, g \rangle = \int_{-1}^1 f(t)g(t) dt$, $T(f) = f' + 3f$,
 $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V .

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^1 1 \cdot t dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^n \overline{\langle T(v_i), g(t) \rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\frac{\sqrt{3}t}{\sqrt{2}}) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t$.

Therefore,

$$T^*(g(t)) = \frac{3}{2} \int_{-1}^1 g(t) dt + \frac{3}{2}t \int_{-1}^1 (1 + 3t)g(t) dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V . Prove the following results.

- (a) $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.
- (b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.
- (c) For any $n \times n$ matrix A , $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.

For all $y \in N(T^*T)$, consider the norm of Ty :

$$\|Ty\|^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.

Based on all above, $N(T^*T) = N(T)$.

Recall that $T \in \mathcal{L}(V)$. Hence $T : V \rightarrow V$. And according to $\forall y \in V$,

$$T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y \rangle} v_i.$$

We know that $T^* : V \rightarrow V$. Therefore $T^*T : V \rightarrow V$.

Applying the rank nullity theorem, we have that

$$\dim V = \text{rank}(T^*T) + \dim N(T^*T), \quad \dim V = \text{rank}(T) + \dim N(T).$$

Using the just proved fact $N(T^*T) = N(T)$, we can simply deduce

$$\text{rank}(T^*T) = \text{rank}(T).$$

2.2 (b)

By changing name of the identity in (a), we can have $N(TT^*) = N(T^*)$ and $\text{rank}(TT^*) = \text{rank}(T^*)$.

Notice that

$$\text{rank}(T) = \text{rank}[T]_\beta = \text{rank}[T]_\beta^* = \text{rank}[T^*]_\beta = \text{rank}(T^*).$$

And then $\text{rank}(TT^*) = \text{rank}(T)$ follows.

2.3 (c)

From (b), $\text{rank}(AA^*) = \text{rank}(A)$ follows naturally.

And note that $(AA^*)^* = A^*A$, then

$$\text{rank}(AA^*) = \text{rank}(A^*A).$$

Therefore,

$$\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A).$$

Done.

3 Section 6.3, Q14

Let V be an inner product space, and let $y, z \in V$. Define $T: V \rightarrow V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

First we would prove that T is linear. $\forall x_1, x_2 \in V, \forall c \in F$,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle z = \langle cx_1, y \rangle z + \langle x_2, y \rangle z = c\langle x_1, y \rangle z + \langle x_2, y \rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore, T is linear.

From course not we directly construct $\forall x \in V$,

$$T^*(x) = \sum_{i=1}^n \overline{\langle T(v_i), x \rangle} v_i = \sum_{i=1}^n \overline{\langle \langle v_i, y \rangle z, x \rangle} v_i = \overline{\langle z, x \rangle} \sum_{i=1}^n \overline{\langle v_i, y \rangle} v_i = \overline{\langle z, x \rangle} y.$$

Recall that $y = I(y) = I^*(y) = \sum_{i=1}^n \overline{\langle I(v_i), y \rangle} v_i = \sum_{i=1}^n \overline{\langle v_i, y \rangle} v_i$. Hence the last equality holds properly.

4 Section 6.3, Q15

Definition. Let $T: V \rightarrow W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. A function $T^*: W \rightarrow V$ is called an **adjoint** of T if $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$ for all $x \in V$ and $y \in W$.

15. Let $T: V \rightarrow W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

- (a) There is a unique adjoint T^* of T , and T^* is linear.
- (b) If β and γ are orthonormal bases for V and W , respectively, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$.
- (c) $\text{rank}(T^*) = \text{rank}(T)$.
- (d) $\langle T^*(x), y \rangle_1 = \langle x, T(y) \rangle_2$ for all $x \in W$ and $y \in V$.
- (e) For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$.

4.1 (a)

Define $g: V \rightarrow F$ by $g(x) = \langle T(x), y \rangle_2$. Obviously, note that g is linear for sure.

Then apply the theorem 6.8, there exists a unique vector $y' \in W$ such that

$$g(x) = \langle x, y' \rangle_1.$$

Recall g 's definition, we have $\langle T(x), y \rangle_2 = \langle x, y' \rangle_1, \forall x \in V$.

Define that $T^*: W \rightarrow V$ by $T^*(y) = y'$.

Then we have $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$.

Hence the wanted T^* function exists, and because the y' is "unique" as mentioned above (for any y), then the T^* is also unique.

Then we prove the linearity of T^* . $\forall y_1, y_2 \in W$ and $\forall c \in F$,

$$\begin{aligned} \langle x, T^*(cy_1 + y_2) \rangle_1 &= \langle T(x), cy_1 + y_2 \rangle_2 = \langle x, cT^*(y_1) \rangle_1 + \langle x, T^*(y_2) \rangle_1 \\ &= \langle x, cT^*(y_1) + T^*(y_2) \rangle_1, \forall x \in V. \end{aligned}$$

Therefore $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$. Hence T^* is linear.

4.2 (b)

For such kind of problem, we need to compare the two matrices entry wise.

Denote $\beta = \{v_1, v_2, \dots, v_n\}$, $\gamma = \{w_1, w_2, \dots, w_m\}$.

Inspect $[T^*]_{\gamma}^{\beta} = [T^*(w_1), T^*(w_2), \dots, T^*(w_m)]_{\beta}$.

Note that $T^*(w_j) = \sum_{i=1}^n \overline{\langle v_i, T^*(w_j) \rangle_1} v_i$, $\forall j = 1, 2, \dots, m$.

From this we know that the i-th row, j-th column of $[T^*]_{\gamma}^{\beta}$ is $\overline{\langle v_i, T^*(w_j) \rangle_1}$.

Inspect $[T]_{\beta}^{\gamma} = [T(v_1), T(v_2), \dots, T(v_n)]_{\gamma}$.

Note that $T(v_i) = \sum_{k=1}^m \overline{\langle w_k, T(v_i) \rangle_2} w_k$, $\forall i = 1, 2, \dots, n$.

Therefore, the i-th column, k-th row of $[T]_{\beta}^{\gamma}$ is $\overline{\langle w_k, T(v_i) \rangle_2}$.

Hence, for $([T]_{\beta}^{\gamma})^*$, the i-th row, j-th column is $\langle w_j, T(v_i) \rangle_2$.

What remains to be done is to show

$$\langle w_j, T(v_i) \rangle_2 = \overline{\langle v_i, T^*(w_j) \rangle_1}.$$

This is equivalent to show $\overline{\langle w_j, T(v_i) \rangle_2} = \langle v_i, T^*(w_j) \rangle_1$.

Note that the L.H.S. = $\overline{\langle T(v_i), w_j \rangle_2}$ = R.H.S. from the definition of T^* . Hence this is proved.

4.3 (c)

$\text{rank}(T^*) = \text{rank}([T^*]_{\gamma}^{\beta})$ and $\text{rank}(T) = \text{rank}([T]_{\beta}^{\gamma}) = \text{rank}(([T]_{\beta}^{\gamma})^*)$.

Followed from (b), we have $\text{rank}(T^*) = \text{rank}(T)$.

4.4 (d)

We want to prove $\langle T^*(y), x \rangle_1 = \langle y, T(x) \rangle_2$, $\forall y \in W, x \in V$. which is equivalent to prove $\langle x, T^*(y) \rangle_1 = \langle T(x), y \rangle_2$.

And L.H.S. = $\langle T(x), y \rangle_2$ followed from the definition of T^* .

Done.

4.5 (e)

It is suffice to prove $N(T) = N(T^*T)$. And it is obvious to see that

$$N(T) \subset N(T^*T).$$

Take any x such that $T^*Tx = 0$. We have $\langle Tx, Tx \rangle_2 = \langle x, T^*Tx \rangle_1 = 0$. Which implies that $Tx = 0$. Hence $x \in N(T)$, and $N(T^*T) = N(T)$.

Done.

5 Section 6.4, Q2(d)

For each linear operator T on an inner product space V , determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(d) $V = P_2(\mathbb{R})$ and T is defined by $T(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

Let $\{1, t, t^2\}$ be a basis for $P_2(\mathbb{R})$, then apply Gram-Schmidt process upon it, we can have an orthonormal basis $\beta = \{1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6})\}$.

First, we claim that T is not self-adjoint, by the spectral theorem, T is self-adjoint iff, T is diagonalizable, which implies that it will lead to the eigenspace decomposition of V . Note that there is only one eigenvalue of T , which is 0 and the only corresponding set of eigenvectors is $\text{span}\{1\}$. It is obvious that $E_0 \neq V$, since $t^2 \notin E_0$. Therefore T is not self-adjoint and meanwhile, it is impossible to derive a orthonormal basis of eigenvectors of T for V .

Also, T is not normal.

6 Section 6.4, Q7

Let T be a linear operator on an inner product space V , and let W be a T -invariant subspace of V . Prove the following results.

- (a) If T is self-adjoint, then T_W is self-adjoint.
- (b) W^\perp is T^* -invariant.
- (c) If W is both T - and T^* -invariant, then $(T_W)^* = (T^*)_W$.
- (d) If W is both T - and T^* -invariant and T is normal, then T_W is normal.

6.1 (a)

Denote $\dim V = n$, $\dim W = m$, with $m \leq n$. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an orthonormal basis for V , $\beta_W = \{v_1, v_2, \dots, v_m\}$ be an orthonormal basis for W .

$\forall y \in W$, $T_W(y) = T(y) = \sum_{i=1}^n \langle T(y), v_i \rangle v_i = T^*(y)$. Because T is a self adjoint operator.

From the construction rule of adjoint, we have $T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y \rangle} v_i$.

From the question, we know that W is T -invariant, then $T_W(y) \in W$. Combined with v_i 's are linear independent, then

$$T_W(y) = \sum_{i=1}^m \overline{\langle T(v_i), y \rangle} v_i = \sum_{i=1}^m \overline{\langle T_W(v_i), y \rangle} v_i.$$

Note that the R.H.S. is the definition of $T_W^*(y)$. Hence

$$T_W(y) = T_W^*(y), \forall y \in W.$$

6.2 (b)