MATH2040C Homework 6

ZHENG Weijia (William, 1155124322)

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1 Section 6.1, Q8

Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a, b), (c, d) \rangle = ac - bd$ on R^2.
(b) $\langle A, B \rangle = \text{tr}(A + B)$ on $\mathsf{M}_{2\times 2}(R)$.
(c) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

1.1 (a)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in \mathbb{R}^2$.
Let $x = (1, 10)$. Then $\langle\, x, x\rangle = \langle\, (1, 10), (1, 10)\rangle = 1^2 - 10^2 = -99 < 0$.
Therefore, this is not an inner product.

1.2 (b)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in M_{2\times 2}(R)$.
Let $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\langle\, x, x\rangle = tr(x + x) = -2 - 2 = -4 < 0.$$

Therefore, this is not an inner product.

1.3 (c)

Suppose this is an inner product. Then $\forall f, g \in P(\mathbb{R}), \overline{\langle\, g, f\rangle} = \langle\, f, g\rangle$ should hold.
Let $f(x) = x, g(x) = x^2 + x$.
Then

$$\langle\, f, g\rangle = \int_0^1 1(x^2 + x)\,dx = \frac{5}{6}.$$

$$\overline{\langle\, g, f\rangle} = \overline{\int_0^1 (2x + 1)x\,dx} = \frac{7}{6}.$$

Therefore $\overline{\langle g, f\rangle} \neq \langle f, g\rangle$ for some $f, g \in P(\mathbb{R})$.
Hence, this is not an inner product.
Done.

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2 Section 6.1, Q17

Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Note that because we have $||T(x)|| = ||x||$. Then $\forall x \in V$, with $x \neq 0$ we have

$$||T(x)|| = ||x|| > 0.$$

Therefore $x \neq 0$.
Note that $||T(0)|| = ||0|| = 0$, which implies

$$T(0) = 0.$$

Hence $N(T) = \{0\}$.
Hence, T is one-to-one.
Done.

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3 Section 6.1, Q18

Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y\rangle' = \langle \mathsf{T}(x), \mathsf{T}(y)\rangle$ defines an inner product on V if and only if T is one-to-one.

3.1 If part

In the if part, we assume T is one-to-one and try to prove $\langle \cdot, \cdot\rangle'$ is an inner product.
One-to-one implies $N(T) = \{0\}$. Then $\forall x (\neq 0) \in V, T(x) \neq 0$. Therefore

$$\langle x, x\rangle' = \langle T(x), T(x)\rangle > 0.$$

Because $\langle \cdot, \cdot\rangle$ is an inner product.
Also note that $\forall x, y, z \in V, \forall c \in F$,

$$\langle x + z, y\rangle' = \langle T(x + z), T(y)\rangle = \langle T(x) + T(z), T(y)\rangle = \langle T(x), T(y)\rangle + \langle T(z), T(y)\rangle$$

$$= \langle x, y\rangle' + \langle z, y\rangle'.$$

Besides,

$$\langle cx, y\rangle' = \langle T(cx), T(y)\rangle = \langle cT(x), T(y)\rangle = c\langle T(x), T(y)\rangle = c\langle x, y\rangle'.$$

And finally,

$$\overline{\langle x, y\rangle'} = \overline{\langle T(x), T(y)\rangle} = \langle T(y), T(x)\rangle = \langle y, x\rangle'.$$

Based on all above, $\langle \cdot, \cdot\rangle'$ is an inner product.

3.2 Only if part

In the only if part, we have $\langle \cdot, \cdot\rangle'$ is already an inner product and try to prove T is injective.
Note that T is linear, then $T(0) = 0$ must hold.
Becasue $\forall x(\neq 0) \in V, \langle x, x\rangle' = \langle T(x), T(x)\rangle > 0$. Therefore $\forall x \neq 0, T(x) \neq 0$.
Therefore $N(T) = \{0\}$, follows that T is injective.
Done.

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4 Section 6.1, Q19

Let V be an inner product space. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re\langle x, y\rangle + \|y\|^2$ for all $x, y \in$ V, where $\Re\langle x, y\rangle$ denotes the real part of the complex number $\langle x, y\rangle$.

(b) $|\,\|x\| - \|y\|\,| \leq \|x - y\|$ for all $x, y \in$ V.

4.1 (a)

Our goal is to prove $2\mathcal{R}\langle x, y\rangle = ||x + y||^2 - ||x||^2 - ||y||^2$ and $-2\mathcal{R}\langle x, y\rangle = ||x - y||^2 - ||x||^2 - ||y||^2$.
For the plus sign, note that the

$$R.H.S. = \mathcal{R}\langle x + y, x + y\rangle - \mathcal{R}\langle x, x\rangle - \mathcal{R}\langle y, y\rangle$$

$$= \mathcal{R}\langle x, y\rangle + \mathcal{R}\langle y, x\rangle.$$

Note that $\mathcal{R}\langle x, y\rangle = \mathcal{R}\langle y, x\rangle$ because $\overline{\langle x, y\rangle} = \langle y, x\rangle$.
Therefore $L.H.S. = R.H.S.$.
For the minus sign, note that the

$$R.H.S. = \mathcal{R}\langle x - y, x - y\rangle - \mathcal{R}\langle x, x\rangle - \mathcal{R}\langle y, y\rangle$$

$$= \mathcal{R}\langle x, -y\rangle + \mathcal{R}\langle -y, x\rangle.$$

Which is equal to $2\mathcal{R}\langle x, -y\rangle = -2\mathcal{R}\langle x, y\rangle = L.H.S.$.
Therefore, (a) is proved. Done.

4.2 (b)

To prove (b), it is suffice to prove $|\ ||x|| - ||y||\ |^2 \leq ||x-y||^2$.
Iff,

$$||x||^2 + ||y||^2 - 2||x|| \cdot ||y|| \leq ||x||^2 + 2\mathcal{R}\langle x, y\rangle + ||y||^2.$$

iff,

$$||x|| \cdot ||y|| \geq \mathcal{R}\langle x, y\rangle.$$

Which is bound to be true because

$$||x|| \cdot ||y|| \geq |\langle x, y\rangle| \geq \mathcal{R}\langle x, y\rangle.$$

Done.

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5 Section 6.1, Q23

Let $\mathsf{V} = \mathsf{F}^n$, and let $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n\times n}(F)$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

5.1 (a)

Denote $T = L_A : V \to V$, note that T is linear. Note that we have a corollary that $(L_A)^* = L_{A^*}$.
Then $\langle x, Ay\rangle = \langle A^*x, y\rangle$ follows. Done.

5.2 (b)

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a standard basis for $V = F^n$, then we have a expression of A such that $\forall x \in V$,

$$B(x) = \sum_{i=1}^{n}\langle B(x), v_i\rangle v_i = \sum_{i=1}^{n}\langle x, Av_i\rangle v_i = \sum_{i=1}^{n}\langle A^*x, v_i\rangle v_i = A^*(x).$$

Therefore, $B = A^*$.
Done.

5.3 (c)

Denote $\beta = \{v_1, v_2, \dots, v_n\}$ is a orthonormal basis as the problem said. Then

$$Q := (v_1, v_2, \dots, v_n).$$

Therefore

$$Q^* = \begin{pmatrix} -v_1- \\ -v_2- \\ -\cdots- \\ -v_n- \end{pmatrix}.$$

Hence

$$Q^*Q = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n.$$

Therefore, $Q^* = Q^{-1}$. Done.

5.4 (d)

Note that

$$[U]_\beta = \begin{pmatrix} Uv_1 & Uv_2 & \dots & Uv_n \end{pmatrix}_\beta = \begin{pmatrix} A^*v_1 & A^*v_2 & \dots A^*v_n \end{pmatrix}_\beta.$$

And

$$[T]^*_\beta = \begin{pmatrix} Tv_1 & Tv_2 & \dots Tv_n \end{pmatrix}^*_\beta = \begin{pmatrix} Av_1 & Av_2 & \dots & Av_n \end{pmatrix}^*_\beta = [\begin{pmatrix} Av_1 & Av_2 & \dots & Av_n \end{pmatrix}^*]_\beta.$$

Suffice to prove that

$$\begin{pmatrix} A^*v_1 & A^*v_2 & \dots A^*v_n \end{pmatrix}_\beta = \begin{pmatrix} Av_1 & Av_2 & \dots & Av_n \end{pmatrix}^*_\beta.$$

Denote $Q = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$.
Iff

$$\begin{pmatrix} A^*Q \end{pmatrix}_\beta = \begin{pmatrix} AQ \end{pmatrix}^*_\beta.$$

Iff

$$(A^*)_\beta Q_\beta = (A_\beta Q_\beta)^*.$$

Note that Q_β is always I_n. Therefore it suffice to prove that

$$(A^*)_\beta = (A_\beta)^*.$$

Denote that the j-th column of $[A]_\beta$ is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{nj} \end{pmatrix}$.

I.e.,

$$[A_\beta]_{ij} = a_{ij}.$$

Hence the j-th row of $A^*_\beta = \begin{pmatrix} a^*_{1j} & a^*_{2j} & \dots a^*_{nj} \end{pmatrix}$. Hence

$$[A^*_\beta]_{ij} = a^*_{ji}.$$

Therefore

$$(A_\beta)^* = (A^*)_\beta.$$

Hence (d) is proved.

6 Section 6.2, Q2(g)

(g) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$

What I need to do is to 1. apply the G-S process to obtain a orthonormal basis. Compute Fourier coefficient and use Theorem 6.5 to check.

Because this is about the matrices, I adopt the Frobenius inner product

$$\langle A, B \rangle = tr(B^* A).$$

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Denote the ones inside S as w_1, w_2, w_3.
Let $v_1 = w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$.
And $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}$.
And $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$.
By checking, the v_i's are orthogonal.
To normalize them, we can have

$$S' = \{\frac{1}{6}v_1, \frac{1}{6\sqrt{2}}v_2, \frac{1}{9\sqrt{2}}v_3\} = \{v_1', v_2', v_3'\}.$$

Which is an orthonormal basis.
To calculate the Fourier coefficient of A, we calculate

$$\langle A, v_1' \rangle = 24, \langle A, v_2' \rangle = 6\sqrt{2}, \langle A, v_3' \rangle = -9\sqrt{2}.$$

By the Theorem 6.5, we

$$24\frac{1}{6}v_1 + 6\sqrt{2}\frac{1}{6\sqrt{2}}v_2 - 9\sqrt{2}\frac{1}{9\sqrt{2}}v_3 = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}.$$

Which means that the theorem is verified.
Done.

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7 Section 6.2, Q2(i)

(i) $\mathsf{V} = \operatorname{span}(S)$ with the inner product $\langle f, g \rangle = \int_0^\pi f(t)g(t)\,dt$, $S = \{\sin t, \cos t, 1, t\}$, and $h(t) = 2t + 1$

Let $v_1 = w_1 = \sin t$.
Let $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \cos t$.
Let $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = 1 - \frac{4}{\pi} \sin t$.
Let $v_4 = w_4 - \frac{\langle w_4, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_4, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle w_4, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 = t + \frac{4}{\pi} \cos t - \frac{\pi}{2}$.
To normalize them, we can just calculate and divide them by their norm.
$\because ||v_1|| = \sqrt{\frac{\pi}{2}}, \therefore v_1' = \sqrt{\frac{2}{\pi}} v_1$.
$\because ||v_2|| = \sqrt{\frac{\pi}{2}}, \therefore v_2' = \sqrt{\frac{2}{\pi}} v_2$.
$\because ||v_3|| = \sqrt{\pi - \frac{8}{\pi}}, \therefore v_3' = \frac{1}{\sqrt{\pi - \frac{8}{\pi}}} v_3$.
$\because ||v_4|| = \sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}, \therefore v_4' = \frac{1}{\sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}} v_4$.
Our $h(t) = 2t + 1$.
Note that $\langle h, v_1' \rangle = 6.609, \langle h, v_2' \rangle = -3.1915, \langle h, v_3' \rangle = 3.195, \langle h, v_4' \rangle = 0.38666$.
Then by doing the calculation, we have

$$h(t) = 1.99999t + 1.00000.$$

Which is what we expected.
Done.

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8 Section 6.2, Q6

6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin$ W, prove that there exists $y \in$ V such that $y \in$ W$^\perp$, but $\langle x, y \rangle \neq 0$. *Hint:* Use Theorem 6.6.

Using the theorem 6.6, there exists unique $y, pj_W(x)$ such that

$$x = y + pj_W(x).$$

Where $pj_W(x)$ is the component of x which is inside W. Therefore, $y \in W^\perp$.
Then we need to check

$$\langle x, y \rangle = \langle y + pj_W(x), y \rangle = \langle y, y \rangle + \langle pj_W(x), y \rangle.$$

Note that $pj_W(x) \in W$ and $y \in W^\perp$, then $\langle pj_W(x), y \rangle = 0$.
Hence $\langle x, y \rangle = \langle y, y \rangle$. Suppose $\langle y, y \rangle = 0$, then $y = 0$.
Then $x = y + pj_W(x) = pj_W(x) \in W$. Which is contradicting with $x \notin W$.
Therefore $\langle x, y \rangle \neq 0$ at the first place.

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9 Section 6.2, Q10

10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $\mathsf{W}^\perp$ that satisfies $\mathsf{N}(\mathsf{T}) = \mathsf{W}^\perp$. In addition, prove that $\|\mathsf{T}(x)\| \leq \|x\|$ for all $x \in \mathsf{V}$. *Hint:* Use Theorem 6.6 and Exercise 10 of Section 6.1. (Projections are defined in the exercises of Section 2.1.)

Inspect a orthonormal basis for W, denote it as $\{v_1, v_2, \ldots, v_k\}$ and $W^\perp$'s (orthonormal basis) as $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$.

Because these are basis, hence $\forall x \in V, \exists! a_1, \ldots, a_n$ such that

$$x = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{k} a_i v_i + \sum_{j=k+1}^{n} a_j v_j.$$

Then a feasible projection is $T(x) = \sum_{i=1}^{k} a_i v_i$.
We now prove $N(T) = W^\perp$.
$\forall x \in W^\perp, T(x) = 0$.
$\forall x \notin W^\perp, x \in V$, there exists some (not all zero, otherwise $x \in W^\perp$) $b_1, \ldots, b_k$ such that

$$T(x) = \sum_{i=1}^{k} b_i v_i \neq 0.$$

Therefore $N(T) = W^\perp$.
Further, we prove that $\forall x \in V, ||T(x)|| \leq ||x||$.
$\forall x \in W^\perp$, it has already been proved.
$\forall x \notin W^\perp$, there exists unique $c_1, c_2, \ldots, c_n$ such that

$$x = \sum_{i=1}^{n} c_1 v_1.$$

Therefore, $||x|| = \sqrt{\sum_{i=1}^{n} c_i^2}$ (because $\{v_1, v_2, \ldots, v_n\}$ is orthonormal)
Then notice that $T(x) == \sum_{i=1}^{k} c_1 v_1$ from our construction of T.
Hence $||T(x)|| = \sqrt{\sum_{i=1}^{k} c_i^2}$.
Note that $k \leq n$, then $||T(x)|| \leq ||x||$ follows naturally.
Done.

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10 Section 6.2, Q15

15. Let V be a finite-dimensional inner product space over F.

(a) *Parseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$ prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot \rangle$, then for any $x, y \in V$

$$\langle \phi_\beta(x), \phi_\beta(y)\rangle' = \langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle,$$

where $\langle \cdot, \cdot \rangle'$ is the standard inner product on F^n.

10.1 (a)

$$\langle x, y\rangle = \langle \sum_{i=1}^{n} \langle x, v_i\rangle v_i, \sum_{j=1}^{n} \langle y, v_j\rangle v_j\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \langle x, v_i\rangle v_i, \langle y, v_j\rangle v_j\rangle = \sum_{i=1}^{n} \langle \langle x, v_i\rangle v_i, \langle y, v_i\rangle v_i\rangle$$

$$= \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle} \langle v_i, v_i\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

10.2 (b)

What we need to prove is that $\sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle} = \langle [x]_\beta, [y]_\beta\rangle'$.
Inspecting the R.H.S., $([x]_\beta)_j = \langle x, v_j\rangle$ and $([y]_\beta)_j = \langle y, v_j\rangle$.
Therefore $\langle [x]_\beta, [y]_\beta\rangle' = \sum_{j=1}^{n} ([x]_\beta)_j \cdot \overline{([y]_\beta)_j} = \sum_{i=1}^{n} \langle x, v_j\rangle \cdot \overline{\langle y, v_j\rangle}$.
Hence $L.H.S. = R.H.S.$.
Done.