MATH2040C Homework 6

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1 Section 6.1, Q8

8. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b),(c,d)\rangle = ac - bd$ on R^2.
(b) $\langle A, B\rangle = \operatorname{tr}(A+B)$ on $\mathsf{M}_{2\times 2}(R)$.
(c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

Figure 1: Section 6.1 Q8

1.1 (a)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in \mathbb{R}^2$.
Let $x = (1, 10)$. Then $\langle\, x, x\rangle = \langle\, (1,10),(1,10)\rangle = 1^2 - 10^2 = -99 < 0$.
Therefore, this is not an inner product.

1.2 (b)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in M_{2\times 2}(R)$.
Let $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\langle\, x, x\rangle = tr(x + x) = -2 - 2 = -4 < 0.$$

Therefore, this is not an inner product.

1.3 (c)

Suppose this is an inner product. Then $\forall f, g \in P(\mathbb{R}), \overline{\langle\, g, f\rangle} = \langle\, f, g\rangle$ should hold.
Let $f(x) = x, g(x) = x^2 + x$.
Then

$$\langle\, f, g\rangle = \int_0^1 1(x^2 + x)\,dx = \frac{5}{6}.$$

$$\overline{\langle\, g, f\rangle} = \overline{\int_0^1 (2x+1)x\,dx} = \frac{7}{6}.$$

Therefore $\overline{\langle\, g, f\rangle} \neq \langle\, f, g\rangle$ for some $f, g \in P(\mathbb{R})$.
Hence, this is not an inner product.
Done.

2 Section 6.1, Q17

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Figure 2: Section 6.1 Q17

Note that because we have $||T(x)|| = ||x||$. Then $\forall x \in V$, with $x \neq 0$ we have

$$||T(x)|| = ||x|| > 0.$$

Therefore $x \neq 0$.
Note that $||T(0)|| = ||0|| = 0$, which implies

$$T(0) = 0.$$

Hence $N(T) = \{0\}$.
Hence, T is one-to-one.
Done.