MATH2040C Homework 5

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1 Section 5.4, Q2(e)

Let $w = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$. Note that $w \in W$, because w is symmetric.

Note that $T(w) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$, which is not symmetric, hence not belongs to W.

Therefore, by definition, W is not a T-invariant subspace of V.

Done.

2 Section 5.4, Q4

$\forall g(t)$ belongs to polynomials, it can be expressed as for some $a_i, i = 0, 1, 2, ..., n$,

$$g(t) = \sum_{i=0}^{n} a_i t^i.$$

Note that $\forall w \in W$, we have

$$g(T)(w) = \sum_{i=0}^{n} a_i T^i(w).$$

Because W is itself a subspace, and note that $T^i(w) \in W, \forall i$. Then

$$\forall w \in W, g(T)(w) \in W.$$

Done.

3 Section 5.4, Q6(d)

Note that $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $T^2(z) = 3 \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

And hence $T^k(z) = 3^{k-1} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \forall k \geq 1$.

Recall that $span\{z, T(z), T^2(z), \dots\}$ is the T-cyclic subspace of V generated by z.

Claim that $\{z, T(z)\}$ is a ordered basis for $span\{z, T(z), T^2(z), \dots\}$.

Note that $\forall u \in span\{z, T(z), T^2(z), \dots\}$, if $u = z$ or $u = T(z)$, then u are elements inside the basis set.

If $u = T^k(z)$ for some $k \geq 2$, notice that $T^k(z) == 3^{k-1}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 3^{k-1}T(z)$.

Therefore, $\{z, T(z)\}$ spans $span\{z, T(z), T^2(z), \dots\}$.

Done.

4 Section 5.4, Q19

According to the question, $A = \begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & ... & 0 & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix}$.

Hence its characteristic polynomial is $\det(A - tI)$, which is

$$\det \begin{pmatrix} -t & 0 & ... & 0 & -a_0 \\ 1 & -t & ... & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & ... & -t & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} - t \end{pmatrix}$$

We expand the massive matrix through the first column.

Then the value is

$$-t \det \begin{pmatrix} -t & 0 & ... & 0 & -a_0 \\ 1 & -t & ... & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & ... & -t & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} - t \end{pmatrix} + (-1) \det \begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & -t & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & ... & -t & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} - t \end{pmatrix}.$$

Observe the second term recursively, we note that it is simply equal to $(-1)^k a_0$.

We keep spliting determinants from the left term, then we have

$$\det(A - tI) = (-t)^{k-2} \det \begin{pmatrix} -t & -a_{k-2} \\ 1 & -a_{k-1} - t \end{pmatrix} + (-1)^k (a_0 + a_1 t + a_2 t^2 + a_{k-3} t^{k-3}).$$

$$= (-t)^{k-2}(t^2 + t a_{k-1} + a_{k-2}) + (-1)^k (a_0 + a_1 t + a_2 t^2 + a_{k-3} t^{k-3}).$$

$$= (-1)^k (t^k + t^{k-1} a_{k-1} + t^{k-2} a_{k-2}) + (-1)^k (a_0 + a_1 t + a_2 t^2 + a_{k-3} t^{k-3}).$$

$$= (-1)^k (a_0 + a_1 t + a_2 t^2 + ... a_{k-3} t^{k-3} + a_{k-2} t^{k-2} + a_{k-1} t^{k-1} + t^k).$$

Done.

5 Section 5.4, Q23

As suggested by the question, we would like to use Mathematical Induction to tackle this problem.

Under the circumstance stated in the problem (note that those v_i's are eigenvectors of T corresponding to distict eigenvalues), let mathematical statement $P(k)$ indicates that:

$P(k)$: If $v_1 + v_2 + v_3 + \dots v_k \in W$, then $\forall i = 1, 2, \dots k, v_i \in W$.

We would prove the base case first: note that $P(1)$ holds trivially.

Hence we can suppose that $\exists$ an integer h such that $P(k)$ holds.

In order to prove $P(k+1)$ holds, we need to prove $w_1 + w_2 + w_3 + \dots w_k + w_{k+1} \in W$ implies $\forall i = 1, 2, \dots k, k+1, v_i \in W$. Where w_j's are all eigenvectors of T corresponding to distict eigenvalues.

Suppose not, i.e., we suppose $\exists i = 1, 2, ..., k, k+1$ such that $w_i \notin W$.

WLOG, we let $w_1 \notin W$.

Note that the $P(k)$ holds, which implies that if $\exists i = 1, 2, ..., k$ such that $w_i \notin W$, then $w_1 + w_2 + \dots w_k \notin W$.

Inspect w_{k+1}, if $w_{k+1} \in W$, then $w_1 + w_2 + \dots w_k \notin W$. Which is impossible, hence $w_{k+1} \notin W$.

Repeating this process by grouping w_1 with other $k-1$ items among $w_2, w_3, ..., w_{k+1}$, we can deduce that

$$\forall i = 1, 2, ..., k+1, w_i \notin W.$$

As supposed $w_1 + w_2 + \dots w_k + w_{k+1} \in W$. Hence $T(w_1 + w_2 + \dots w_k + w_{k+1}) \in W$.

Which is $\sum_{i=1}^{k+1} \lambda_i w_i \in W$, subtract this with $w_1 + ...w_{k+1}$, we have

$$\sum_{i=2}^{k+1} (\lambda_i - \lambda_1) w_i \in W.$$

We let T affect on it and subtract the first term out. Keep doing this process on and on, we eventually get

$$\prod_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \cdot w_{k+1} \in W.$$

Note that the λ_i's are all distict, hence the coefficient is nonzero. Hence $w_{k+1} \in W$.

Which is contradicting with $w_{k+1} \notin W$.

Therefore, $\forall i = 1, 2, ..., k, k+1$, $v_i \in W$ at the first place.

Hence $P(k)$ holds forall possible integer k under proper condition.

Done.

6 Section 5.4, Q24

Let $T : V \to V$, and W be any T-invariant subspace of V.

We choose the ordered basis for V properly, such that each element in the order basis set β is a eigenvector of T.

Let $\beta = \{v_{1,1}, v_{1,2}, \dots v_{1,m_1}, \dots v_{k,1}, v_{k,2}, \dots v_{k,m_k}\}$. Let $m_1 + m_2 + \dots m_k = m$.

Note that any (subspace of V)'s basis can be represented with elements from β. Then denote W's basis set as γ.

Let $\dim \gamma = l$, then we have $l \leq m$. Where $\gamma = \{w_1, w_2, \dots w_l\}$, those w_a's are equal to some $v_{i,j}$.

WLOG, reorder γ such that $\gamma = \{v_{1,1}, \dots v_{1,q_1}, \dots v_{t,1}, \dots v_{t,q_t}\}$.

Where $v_{j,1}, v_{j,2}, \dots v_{j,q_j}$ are all corresponding to eigenvalue λ_j.

Therefore we have $q_1 + \dots q_k = l$. And the q_i eigenvectors are corresponding to one distict eigenvalue.

Note that here the q_i are λ_i's geometric multiplicities (because they are number of corresponding eigenvectors), then we can have

$$\forall i = 1, 2, \dots k, \mu_{T_W}(\lambda_i) = \gamma_{T_W}(\lambda_i).$$

Therefore, $T_W : W \to W$ must be diagnoalizable.

Done.

7 Section 7.1, Q7(a)

Note that $\forall x \in N(U)$, we have

$$Ux = 0.$$

Therefore

$$U \cdot Ux = U^2(x) = 0.$$

Hence $N(U) \subset N^2(U)$.

Suppose $\exists$ an integer $k \geq 1$ such that $N^k(U) \subset N^{k+1}(U)$.

Note that $\forall x \in N^k(U)$,

$$U^{k+1}(x) = U \cdot U^k(x) = U(0) = 0.$$

Therefore $x \in N^{k+1}(U)$.

So, based on all above, we have

$$N(U) \subset N^2(U) \subset N^3(U) \subset \dots \subset N^k(U) \subset N^{k+1}(U) \subset \dots.$$

Done.

8 Section 7.1, Q7(c)

Note that $\dim V - rank(U^m) = \dim ker(U^m)$ and $\dim V - rank(U^{m+1}) = \dim ker(U^{m+1})$ hold because of rank-nullity theorem.

Under the condition of $rank(U^m) = rank(U^{m+1})$, then

$$\dim ker(U^m) = \dim ker(U^{m+1}).$$

Recall that $ker(U^m) \subset ker(U^{m+1})$, then we can deduce that $ker(U^m) = ker(U^{m+1})$.

Let's inspect our situation at this stage, now we have $ker(U^m) = ker(U^{m+1})$, and we want to go one more step: to prove

$$ker(U^{m+1}) = ker(U^{m+2}).$$

Because $ker(U^{m+1}) \subset ker(U^{m+2})$, we then have

$$ker(U^{m+2}) = ker(U^{m+1}) \cup [ker(U^{m+2}) - ker(U^{m+1})].$$

It is suffice to prove that the set $ker(U^{m+2}) - ker(U^{m+1})$ is an empty set.

Let $x \in ker(U^{m+2}) - ker(U^{m+1})$, then we have

$$U^{m+1}x \neq 0,\ U^{m+2}x = 0.$$

Hence $U(U^{m+1}x) = 0$ and $U^{m+1}x \in kerU$. And $U^{m+1}x \in R(U^{m+1}) \subset R(U^m)$.
$\therefore \exists y \in V$ such that

$$U^m y = U^{m+1}x.$$

Hence $U^{m+1}y = U(U^{m+1}x) = U^{m+2}x = 0$.
$\because ker(U^m) = ker(U^{m+1}) \therefore U^m y = 0$.
This implies that $U^{m+1}x = 0$. This contradicts with the previous statement.
Therefore

$$ker(U^{m+1}) = ker(U^{m+2}).$$

By doing this repeatly, we can have $\forall k \geq m$,

$$ker(U^m) = ker(U^k).$$

Done.

9 Section 7.1, Q7(b)

Using the result of (c), since $\forall k \geq m$,

$$ker(U^m) = ker(U^k).$$

Then we have $\forall k \geq m$,

$$rank(U^k) = \dim V - \dim ker(U^k) = \dim V - \dim ker(U^m) = rank(U^m).$$

Done.

10 Section 7.1, Q7(d)