MATH2040C Homework 4

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1 Section 5.1, Q2(e)

Given that $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$.

And note that $T(1 - x + x^3) = -1 + x - x^3$. $T(1 + x^2) = -x - x^2 + x^3$. $T(1) = x^2$. $T(x + x^2) = -x - x^2$.

Hence $T(\beta) = \{-1 + x - x^3, -x - x^2 + x^3, x^2, -x - x^2\}$.

$[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Suppose β is containing T's eigenvectors, then $\exists \lambda \in F$ such that

$$T(1 + x^2) = \lambda(1 + x^2).$$

Then $\lambda + \lambda x^2 = -x - x^2 + x^3$. Note that the degree of them do not equal in any sense. Hence β is not a basis consisting of eigenvectors of T.

2 Section 5.1, Q2(f)

Given that $\beta = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\}$.

Note that $T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

$T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$,

$T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, and

$T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

Hence we deduce that β is a basis consisting of eigenvectors of T.

3 Section 5.1, Q3(d)

3.1 (i)

Given that $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, then its characteristic polynomial is

$$f_A(t) = \det \begin{pmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{pmatrix} = -t(t-1)^2.$$

Observe the $f_A(t)$'s zeros, we have A should have 2 eigenvalues: 1 and 0.

3.2 (ii)

For eigenvalue 1, the corresponding eigenvectors should be in the span of set

$$\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}.$$

For eigenvalue 0, the corresponding eigenvectors should be in the span of set

$$\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \}.$$

3.3 (iii)

In this case, the $n = 3, F = \mathbb{R}$. So $F^3 = \mathbb{R}^3$.

Note that $\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \}$ is a 3-linear-indenpendent set. Hence it is a basis of $\mathbb{R}^3$.
And by our conclusion above, these 3 vectors are eigenvectors of A.

3.4 (iv)

Let $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}$. Note that Q is invertible and $Q^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ -1 & 0 & 1 \end{pmatrix}$.
Note that

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4 Section 5.1, Q4(h)

Let β be the standard basis. Note that $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. By extracting its characteristic polynomial, it is

$$f_T(t) = (t-1)^3(t+1) = 0.$$

And note that their corresponding eigenvectors to be $\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \}$.

Note that by the diagnoalizability of $[T]_\beta$, (for its every eigenvalue: 1 and -1: algebraic multiplicity equals geometric multiplicity) we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q^{-1}[T]_\beta Q.$$

Where $Q = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$.

Regard Q as a change of basis matrix from another basis γ to our known standard basis β. Therefore, $Q = [I]_\gamma^\beta$.

Let $\gamma = \{y_1, y_2, y_3, y_4\}$. Therefore

$$[y_1, y_2, y_3, y_4]_\beta = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Hence, $y_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $y_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $y_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Note that $[T]_\gamma = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$. So γ is the ordered basis we need to find.

5 Section 5.1, Q4(e)

Let $\beta = \{1+x, -3-13x+4x^2, -3+x\}$ be a ordered basis. Then

$$[T]_\beta = [4x+4, 8x^2-26x-6, 0]_\beta = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Which is a diagnoal matrix. Hence the β is what we want to find. And the eigenvalues of T are 4,2 and 0, with corresponding eigenvectors elements of the ordered basis β.

6 Section 5.1, Q17

6.1 (a)

Note that for the identity matrix I, $T(I) = I = 1 \cdot I$, hence 1 is a eigenvalue. Also note that for a matrix X with only 2 entries on the right top and left buttom being 1 and -1, then

$$T(X) = -X.$$

Hence -1 is a eigenvalue.

Suppose there exists some eigenvalue $|\lambda| \neq 1$ such that

$$A^T = \lambda A.$$

Then we can deduce $\lambda A^T = A = \lambda^2 A$. Which implies $(1 - \lambda^2)A = 0$.

Because A is regarded as an eigenvector, hence it is not zero, so $1 - \lambda^2$ must be 0. But other than 1 and -1, it cannot be 0.

Hence 1, -1 are the only eigenvalues of A.

6.2 (b)

For eigenvalue 1, the corresponding eigenvectors are all symmetric matrices.

For eigenvalue -1, the corresponding eigenvectors are all skew symmetric matrices.

6.3 (c)

Consider $\gamma = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

Then $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q^{-1}[T]_\gamma Q,$$

where $Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Where Q can be regarded as $[I]_\beta^\gamma$. Let $\beta = \{v_1, v_2, v_3, v_4\}$.

$$[I]_\beta^\gamma = [v_1, v_2, v_3, v_4]_\gamma = [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}].$$

Hence the γ is the basis we want.

6.4 (d)

7 Section 5.1, Q18

7.1 (a)

If A is not invertible, then let $c = 0$. We have

$$\det(A + cB) = \det A = 0.$$

Since A is singular as we supposed.

If A is invertible, then note that $A = AB^{-1}B$, then

$$\det(A + cB) = \det AB^{-1}B + cB = \det(AB^{-1} + cI)\det(B).$$

Note that $\det(B) \neq 0$ and $\det(AB^{-1} + cI) = 0$ if $-c$ is the eigenvalue of AB^{-1}. And by the fundemantal theorem of algebra, there must exist such c.

Done.

7.2 (b)

Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Note that $\det(A) = 2 \neq 0$. And $\forall c \in \mathbb{C}$,

$$\det(A + cB) = \det(\begin{pmatrix} 2 & 1 + c \\ 0 & 1 \end{pmatrix}) = 2 \neq 0.$$

Therefore, A and A+cB are both invertible.

8 Section 5.2, Q3(c)

9 Section 5.2, Q8

10 Section 5.2, Q13