MATH2040C Homework 2

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1 Section 1.6, Q12

Given $\{u, v, w\}$ is a basis for V, then we have $\{u, v, w\}$ is linearly independent itself and $V = \text{span}\{u, v, w\}$.

Note that hence $\forall x \in V, \exists a, b, c \in \mathbb{F}$ such that $x = au + bv + cw = a(u + v + w - (v + w)) + b(v + w - w) + cw = a(u + v + w) + (b - a)(v + w) + (c - b)w$.

Hence $x \in$ span $\{u+v+w, v+w, w\}$, i.e., V=span$\{u, v, w\} \subset$ span $\{u+v+w, v+w, w\}$.

And span$\{u+v+w, v+w, w\} \subset$ span$\{u, v, w\}$ holds trivially, hence $\{u+v+w, v+w, w\}$ spans V.

Remains to prove that $\{u + v + w, v + w, w\}$ is linearly independent, suppose not, $\exists d, e, f \in \mathbb{F}$ (not all zeros) such that

$$d(u + v + w) + e(v + w) + fw = 0_V,$$

where the 0_V is the zero vector of V.

which implies

$$du + (d + e)v + (d + e + f)w = 0_V.$$

Note that as supposed d, e, f are not all zeros, hence $d, (d + e), (d + e + f)$ are not all zeros.

Contradiction with the assumption that $\{u, v, w\}$ is linearly independent.

Hence $\{u + v + w, v + w, w\}$ is linearly independent at the first place.

Q.E.D.

2 Section 1.6, Q15

We construct the basis by ourself. Let $B = D \cup A$.

$A = \{A_{mn} : m = 1, 2, ..., n; n = 1, 2, ..., n; m \neq n.\}$ where $A_{mn}[i, j] = 1$ if $(i = m, j = n)$, else it is 0.

is a set of n by n matrices having all slot being zeros unless one slot not on the main diagnoal.

And $D = \{D_1, D_2, ..., D_{n-1}\}$, where $D_k[i, j] = 1$ if $(i = j = n$ or $i = j = k)$, else is zero entry.

Note that $\forall$ n by n matrix M with $tr(M) = 0$. We have

$$M = \sum_{i,j:i\neq j} M_{ij}A_{ij} + \sum_{i=1}^{n-1} M_{ii}D_i.$$

Also note that $B = D \cup A$ is linearly independent, and from above we have

$$spanB = \{M_{n\times n} : tr(M) = 0\}.$$

Therefore B is a basis of $\{M_{n\times n} : tr(M) = 0\}$. And

$$|B| = |A| + |D| = n - 1 + n^2 - n = n^2 - 1.$$

Q.E.D.

3 Section 1.6, Q26

Consider the transform: $T : P_n(\mathbb{R}) \to \mathbb{R}$, with the function $T(f) = f(a)$.
. $\forall f_1, f_2 \in P_n(\mathbb{R})$,

$$T(f_1 + f_2) = (f_1 + f_2)(a) = f_1(a) + f_2(a) = T(f_1) + T(f_2).$$

And $\forall c \in \mathbb{R}$,

$$T(cf) = cf(a).$$

We proved that the transformation is linear.
From the rank nullity theorem,

$$\dim N(T) + \dim R(T) = \dim P_n(\mathbb{R}) = n + 1.$$

Note that $\dim R(T) = 1$, hence $\dim N(T) = n$.
Note that $N(T) = \dim\{f \in P_n(\mathbb{R}) : f(a) = 0\}$. Therefore it is n.
Done.

4 Section 1.6, Q30

Note that the $0_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. And by definition, $0_V \in W_1, W_2$.

Note that $\forall x_1, x_2 \in W_1$, $\exists a_1, a_2, b_1, b_2, c_1, c_2 \in F$ such that $x_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 \end{pmatrix}$.

Note that $x_1 + x_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 \end{pmatrix}$, which is also $\in W_1$. And $\forall a \in F$,

$$ax_1 = \begin{pmatrix} aa_1 & ab_1 \\ ac_1 & aa_1 \end{pmatrix} \in W_1.$$

Note that $\forall y_1, y_2 \in W_2$, $\exists d_1, d_2, e_1, e_2 \in F$ such that $y_1 = \begin{pmatrix} 0 & d_1 \\ -d_1 & e_1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} 0 & d_2 \\ -d_2 & e_2 \end{pmatrix}$ Hence $y_1 + y_2 = \begin{pmatrix} 0 & d_1 + d_2 \\ -d_1 + d_2 & e_1 + e_2 \end{pmatrix} \in W_2$. And $\forall a \in F$,

$$ay_1 = \begin{pmatrix} 0 & ad_1 + ad_2 \\ -ad_1 + ad_2 & ae_1 + ae_2 \end{pmatrix} \in W_2.$$

Therefore, both W_1, W_2 are subspaces of V.
Note that a basis for W_1 can be $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$. Hence $\dim W_1 = 3$.
A basis for W_2 can be $\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. Hence $\dim W_2 = 2$.
$W_1 \cap W_2 = \{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in F\}$, hence $\dim W_1 \cap W_2 = 1$.
$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c+b \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -c \\ c & d-a \end{pmatrix}.$$

The first term is element of W_1 and the second term is element of W_2. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W_1 + W_2.$$

It is obvious that $W_1 + W_2 \in V$, then $W_1 + W_2 = V$. Hence $\dim W_1 + W_2 = \dim V = 4$.
Done.

5 Section 2.1, Q3

Note that $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$, $T((a_1, a_2) + (b_1, b_2)) = (a_1 + b_1 + a_2 + b_2, 0, 2a_1 + 2b_1 - a_2 - b_2) = (a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) = T((a_1, a_2)) + T((b_1, b_2))$.
And also note that $\forall (a_1, a_2) \in \mathbb{R}^2, c \in \mathbb{R}$,

$$T(c(a_1, a_2)) = (ca_1 + ca_2, 0, 2ca_1 - ca_2) = c(a_1 + a_2, 0, 2a_1 - a_2) = cT((a_1, a_2).$$

Hence T is linear.
The basis for $N(T)$ is $\{(0, 0)\}$, and $\dim N(T) = 0$.
And a basis for $R(T)$ is $\{(1, 0, 2), (1, 0, -1)\}$. Hence $\dim R(T) = 2$.
Note that

$$\dim R(T) + \dim N(T) = 2 = \dim R^2$$

.

Therefore the rank nullity theorem is verified.
Note that $(1, 1, 1) \notin R(T)$, hence T is not onto (surjective).
Also note that $N(T) = (0, 0)$, then T is one to one (injective).
Done.

The remaining questions will be handled in handwriting.