MATH2040C Homework 7

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1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(c) $\mathsf{V} = \mathsf{P}_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V.

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^{1} 1 \cdot t\, dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^{n} \overline{\langle T(v_i), g(t)\rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\sqrt{\frac{3}{2}}t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t.$.

Therefore,

$$T^*(g(t)) = \frac{3}{2}\int_{-1}^{1} g(t)dt + \frac{3}{2}t\int_{-1}^{1} (1+3t)g(t)dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$. Deduce that $\operatorname{rank}(\mathsf{T}^*\mathsf{T}) = \operatorname{rank}(\mathsf{T})$.

(b) $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{T}^*)$. Deduce from (a) that $\operatorname{rank}(\mathsf{T}\mathsf{T}^*) = \operatorname{rank}(\mathsf{T})$.

(c) For any $n \times n$ matrix A, $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.
Forall $y \in N(T^*T)$, consider the norm of Ty :

$$||Ty||^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.
Based on all above, $N(T^*T) = N(T)$.
Recall that $T \in \mathcal{L}(V)$. Hence $T : V \to V$. And according to $\forall y \in V$,

$$T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y \rangle} v_i.$$

We know that $T^* : V \to V$. Therefore $T^*T : V \to V$.
Applying the rank nullity theorem, we have that

$$\dim V = rank(T^*T) + \dim N(T^*T)\ ,\ \ \dim V = rank(T) + \dim N(T).$$

Using the just proved fact $N(T^*T) = N(T)$, we can simply deduce

$$rank(T^*T) = rank(T).$$

2.2 (b)

By changing name of the identity in (a), we can have $N(TT^*) = N(T^*)$ and $rank(TT^*) = rank(T^*)$.
Notice that

$$rank(T) = rank[T]_\beta = rank[T]_\beta^* = rank[T^*]_\beta = rank(T^*).$$

And then $rank(TT^*) = rank(T)$ follows.

2.3 (c)

From (b), $rank(AA^*) = rank(A)$ follows naturally.
And note that $(AA^*)^* = A^*A$, then

$$rank(AA^*) = rank(A^*A).$$

Therefore,

$$rank(AA^*) = rank(A^*A) = rank(A).$$

Done.

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3 Section 6.3, Q14

Let V be an inner product space, and let $y, z \in \mathsf{V}$. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(x) = \langle x, y \rangle z$ for all $x \in \mathsf{V}$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

First we would prove that T is linear. $\forall x_1, x_2 \in V, \forall c \in F$,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle z = \langle cx_1, y \rangle z + \langle x_2, y \rangle z = c\langle x_1, y \rangle z + \langle x_2, y \rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore, T is linear.
From course not we directly construct $\forall x \in V$,

$$T^*(x) = \sum_{i=1}^{n} \overline{\langle T(v_i), x \rangle} v_i = \sum_{i=1}^{n} \overline{\langle \langle v_i, y \rangle z, x \rangle} v_i = \overline{\langle z, x \rangle} \sum_{i=1}^{n} \overline{\langle v_i, y \rangle} v_i = \overline{\langle z, x \rangle} y.$$

Recall that $y = I(y) = I^*(y) = \sum_{i=1}^{n} \overline{\langle I(v_i), y \rangle} v_i = \sum_{i=1}^{n} \overline{\langle v_i, y \rangle} v_i$. Hence the last equality holds properly.

4 Section 6.3, Q15

Definition. *Let* $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ *be a linear transformation, where* V *and* W *are finite-dimensional inner product spaces with inner products* $\langle \cdot, \cdot \rangle_1$ *and* $\langle \cdot, \cdot \rangle_2$*, respectively. A function* $\mathsf{T}^*: \mathsf{W} \to \mathsf{V}$ *is called an* **adjoint** *of* T *if* $\langle \mathsf{T}(x), y \rangle_2 = \langle x, \mathsf{T}^*(y) \rangle_1$ *for all* $x \in \mathsf{V}$ *and* $y \in \mathsf{W}$.

15. Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

(a) There is a unique adjoint T^* of T, and T^* is linear.
(b) If β and γ are orthonormal bases for V and W, respectively, then $[\mathsf{T}^*]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^*$.
(c) $\text{rank}(\mathsf{T}^*) = \text{rank}(\mathsf{T})$.
(d) $\langle \mathsf{T}^*(x), y \rangle_1 = \langle x, \mathsf{T}(y) \rangle_2$ for all $x \in \mathsf{W}$ and $y \in \mathsf{V}$.
(e) For all $x \in \mathsf{V}$, $\mathsf{T}^*\mathsf{T}(x) = 0$ if and only if $\mathsf{T}(x) = 0$.

4.1 (a)

Define $g : V \to F$ by $g(x) = \langle T(x), y \rangle_2$. Obviously, note that g is linear for sure.

Then apply the theorem 6.8, there exists a unique vector $y' \in W$ such that

$$g(x) = \langle x, y' \rangle_1.$$

Recall g's definition, we have $\langle T(x), y \rangle_2 = \langle x, y' \rangle_1,\ \forall x \in V.$

Define that $T^* : W \to V$ by $T^*(y) = y'$.

Then we have $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$.

Hence the wanted T^* function exists, and becasue the y' is "unique" as mentioned above (for any y), then the T^* is also unique.

Then we prove the linearity of T^*. $\forall y_1, y_2 \in W$ and $\forall c \in F$,

$$\langle x, T^*(cy_1 + y_2) \rangle_1 = \langle T(x), cy_1 + y_2 \rangle_2 = \langle x, cT^*(y_1) \rangle_1 + \langle x, T^*(y_2) \rangle_1$$

$$= \langle x, cT^*(y_1) + T^*(y_2) \rangle_1, \forall x \in V.$$

Therefore $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$. Hence T^* is linear.

4.2 (b)

For such kind of problem, we need to compare the two matrices entry wise. Denote $\beta = \{v_1, v_2, \ldots, v_n\}, \gamma = \{w_1, w_2, \ldots, w_m\}$.

Inspect $[T^*]_\gamma^\beta = [T^*(w_1), T^*(w_2), \ldots, T^*(w_m)]_\beta$.

Note that $T^*(w_j) = \sum_{i=1}^n \overline{\langle v_i, T^*(w_j)\rangle_1} v_i, \ \forall j = 1, 2, \ldots, m.$

From this we know that the i-th row, j-th column of $[T^*]_\gamma^\beta$ is $\overline{\langle v_i, T^*(w_j)\rangle_1}$.

Inspect $[T]_\beta^\gamma = [T(v_1), T(v_2), \ldots, T(v_n)]_\gamma$.

Note that $T(v_i) = \sum_{k=1}^m \overline{\langle w_k, T(v_i)\rangle_2} w_k, \ \forall i = 1, 2, \ldots, n.$

Therefore, the i-th column, k-th row of $[T]_\beta^\gamma = \overline{\langle w_k, T(v_i)\rangle_2}$.

Hence, for $([T]_\beta^\gamma)^*$, the i-th row, j-th column is $\langle w_j, T(v_i)\rangle_2$.

What remains to be done is to show

$$\langle w_j, T(v_i)\rangle_2 = \overline{\langle v_i, T^*(w_j)\rangle_1}.$$

This is equivalent to show $\overline{\langle w_j, T(v_i)\rangle_2} = \langle v_i, T^*(w_j)\rangle_1$.

Note that the L.H.S.=$\langle T(v_i), w_j\rangle_2$=R.H.S. from the definition of T^*. Hence this is proved.

4.3 (c)

$rank(T^*) = rank([T^*]_\gamma^\beta)$ and $rank(T) = rank([T]_\beta^\gamma) = rank(([T]_\beta^\gamma)^*)$.

Followed from (b), we have $rank(T^*) = rank(T)$.

4.4 (d)

We want to prove $\langle T^*(y), x\rangle_1 = \langle y, T(x)\rangle_2, \forall y \in W, x \in V$. which is equivalent to prove $\langle x, T^*(y)\rangle_1 = \langle T(x), y\rangle_2$.

And L.H.S.=$\langle T(x), y\rangle_2$ followed from the definition of T^*.

Done.

4.5 (e)

It is suffice to prove $N(T) = N(T^*T)$. And it is obvious to see that

$$N(T) \subset N(T^*T).$$

Take any x such that $T^*Tx = 0$. We have $\langle Tx, Tx\rangle_2 = \langle x, T^*Tx\rangle_1 = 0$. Which implies that $Tx = 0$. Hence $x \in N(T)$, and $N(T^*T) = N(T)$.

Done.

5 Section 6.4, Q2(d)

For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Let $\{1, t, t^2\}$ be a basis for $P_2(\mathbb{R})$, then apply Gram-Schmidt process upon it, we can have an orthonormal basis $\beta = \{1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6})\}$.

First, we claim that T is not self-adjoint, by the spectral theorem, T is self-adjoint iff, T is diagnoalizable, which implies that it will lead to the eigenspace decomposition of V. Note that there is only one eigenvalue of T, which is 0 and the only corresponding set of eigenvectors is $span\{1\}$. It is obvious that $E_0 \neq V$, since $t^2 \notin E_0$. Therefore T is not self-adjoint and meanwhile, it is impossible to derive a orthonormal basis of eigenvectors of T for V.

Also, T is not normal.

6 Section 6.4, Q7

Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(a) If T is self-adjoint, then T_W is self-adjoint.
(b) $W^\perp$ is T^*-invariant.
(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

6.1 (a)

Denote $\dim V = n, \dim W = m$, with $m \leq n$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V, $\beta_W = \{v_1, v_2, \ldots, v_m\}$ be an orthonormal basis for W.

$\forall y \in W$, $T_W(y) = T(y) = \sum_{i=1}^{n} \langle T(y), v_i \rangle v_i = T^*(y)$. Because T is a self adjoint opeartor.

From the construction rule of adjoint, we have $T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y \rangle} v_i$.

From the question, we know that W is T-invarian, then $T_W(y) \in W$. Combined with v_i's are linear independent, then

$$T_W(y) = \sum_{i=1}^{m} \overline{\langle T(v_i), y \rangle} v_i = \sum_{i=1}^{m} \overline{\langle T_W(v_i), y \rangle} v_i.$$

Note that the R.H.S. is the definition of $T_W^*(y)$. Hence

$$T_W(y) = T_W^*(y), \forall y \in W.$$

6.2 (b)

Make it clear that what we want is $T^*(W^\perp) \subset W^\perp$. By the construction of T^*, we have

$$T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y \rangle} v_i.$$

Note that $\forall y \in W^\perp, y = \sum_{j=m+1}^{n} \langle y, v_i \rangle v_i$.

Therefore

$$T^*(y) = \sum_{i=1}^{n} v_i (\sum_{j=m+1}^{n} \langle y, v_j \rangle \langle v_j, T(v_i) \rangle).$$

Recall that W is T-invariant, hence $T(v_i) \in W$, then $\langle v_j, T(v_i) \rangle = 0, \forall i \leq m$.

Hence $T^*(y) = \sum_{i=m+1}^{n} v_i \sum_{j=m+1}^{n} \langle y, v_j \rangle \langle v_j, T(v_i) \rangle \in W^\perp$.

6.3 (c)

$\forall y \in W$, $(T_W)^*(y) = \sum_{i=1}^m \overline{\langle T_W(v_i), y\rangle} v_i$. Because $(T_W)^*(y) \in W$ assumed in question.

Note that $y \in W$, then $y \in V$. Hence we can use the original definition of T. We then have

$$(T^*)_W(y) = T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y\rangle} v_i.$$

Inspect the $\langle T(v_i), y\rangle$ terms, here $y \in W$. If $i = 1, 2, \ldots, m$ then $T(v_i) \in W$, when $i = m+1, m+2, \ldots, n$ then $v_i \in W^\perp$ and hence $T(v_i) \in W^\perp$. Therefore $\langle T(v_i), y\rangle = 0, \forall i = m+1, m+2, \ldots, n$.

Hence, we have

$$(T^*)_W(y) = T^*(y) = \sum_{i=1}^m \overline{\langle T(v_i), y\rangle} v_i = \sum_{i=1}^m \overline{\langle T_W(v_i), y\rangle} v_i = (T_W)^*(y).$$

Done.

6.4 (d)

Note that as mentioned in the question, W is both $T-$ and T^*- invariant. Hence $\forall y \in W$,

$$T_W(T_W)^*(y) = T_W(T^*(y)).$$

Where $T^*(y) \in W$. Then $T_W(T_W)^*(y) = TT^*(y)$, where $TT^*(y) \in W$ as well.

On the other hand,

$$(T_W)^* T_W(y) = (T_W)^* T(y) = T^* T(y).$$

Which is valid for similar reasons.

Recall that T is normal. Hence $(T_W)^* T_W(y) = T_W(T_W)^*(y), \forall y \in W$.

Therefore, T_W is normal.

7 Section 6.4, Q9

Let T be a normal operator on a finite-dimensional inner product space V. Prove that $\mathsf{N(T)} = \mathsf{N(T^*)}$ and $\mathsf{R(T)} = \mathsf{R(T^*)}$. *Hint:* Use Theorem 6.15 and Exercise 12 of Section 6.3.

From theorem 6.15, we know that $||T(x)|| = ||T^*(x)||, \forall x \in V$.

$\forall x \in N(T)$, then $T(x) = 0$, which is equivalent to $||T(x)|| = 0$. Then $||T^*(x)|| = 0$, which is equivalent to $T^*(x) = 0$. Thus $x \in N(T^*)$. Note that each step above is revertible, then $N(T) = N(T^*)$.

Using the question 12 of section 6.3, we then have $N(T)^\perp = R(T^*)$ and $N(T^*)^\perp = R((T^*)^*) = R(T)$.

As $N(T) = N(T^\perp)$, then $R(T) = R(T^*)$.

Done.

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8 Section 6.5, Q2(c)

For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

(c) $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

By solving its characteristic polynomial, we have 2 eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 8$.

And we can have $v_1 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$.

Normalize them, we then have $v_1' = \frac{1}{\sqrt{3}}v_1, v_2' = \frac{1}{\sqrt{3}}v_2$.

Then let $P = \begin{pmatrix} v_1' & v_2' \end{pmatrix}$.

And hence the $P^* = \begin{pmatrix} -\overline{v_1'}- \\ -\overline{v_2'}- \end{pmatrix}$.

Note that $P^*P = I_2$. Therefore P is a unitary matrix.

Then

$$P^*AP = P^*(\lambda_1 v_1', \lambda_2 v_2') = \begin{pmatrix} -\overline{v_1'}- \\ -\overline{v_2'}- \end{pmatrix} (\lambda_1 v_1', \lambda_2 v_2') = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

9 Section 6.5, Q6

Let V be the inner product space of complex-valued continuous functions on $[0, 1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$

Let $h \in \mathsf{V}$, and define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(f) = hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for $0 \le t \le 1$.

9.1 Only If

We would like to prove it by contradiction. Note that now we have the condition of T being unitary opeartor. This implies that

$$||T(f)|| = ||f||, \forall f \in V.$$

Doing the inner product, we left with

$$\int_0^1 |f(t)|^2(|h(t)|^2 - 1)dt = 0.$$

Recall that all of the h, f are continuous functions defined on compact interval. On which we can apply the boundedness theorem to obtain that

$$\exists M > 0, s.t. \ -M < \int_0^1 |f(t)|^2(|h(t)|^2 - 1)dt < M$$

Suppose $|h(t)| \neq 1$ for some $t_0 \in [0, 1]$ (WLOG, we suppose $h(t_0) > 1$.) we will construct a $g(t)$ s.t.

$$\int_0^1 |g(t)|^2(|h(t)|^2 - 1)dt \neq 0.$$

Note that $\exists$ a interval I such that $\forall t \in I, h(t) \geq \frac{1+h(t_0)}{2}$. Where I is defined in the following way: $\exists \delta > 0$ s.t.

$$I = [t_0 - \delta, t_0 + \delta] \ \ (t_0 \in (0, 1))$$

$$I = [0, t_0 + \delta] \ \ (t_0 = 0)$$
$$I = [t_0 - \delta, 1] \ \ (t_0 = 1).$$

Applying the boundedness theorem on I, then $\int_I |f(t)|^2(|h(t)|^2-1)dt$ is also finite. Which deduces that $\int_{[0,1]-I} |f(t)|^2(|h(t)|^2-1)dt$ is a subtraction of 2 finite values, hence is also a finite number, denote it as M_1.

Note that

$$\int_I |f(t)|^2(|h(t)|^2-1)dt \geq (h(t_0)-1)\int_I |f(t)|^2 dt.$$

Manipulate the value of $f(t), t \in I$ such that $\int_I |f(t)|^2 dt > \frac{|M_1|+1}{h(t_0)-1}$.

Thus we have

$$\int_0^1 |f(t)|^2(|h(t)|^2-1)dt > M_1 + |M_1| + 1 > 0.$$

Which is a contradiction, hence $\forall t \in [0,1], |h(t)| = 1$ at the first place.

9.2 If

The if part is relatively easier. Now we have the condition of

$$\forall t \in [0,1], |h(t)| = 1.$$

Note that $\forall f \in V$,

$$||T(f)||^2 = ||h(t)f(t)||^2 = \int_0^1 h(t)\overline{h(t)}f(t)\overline{f(t)}dt = \int_0^1 f(t)\overline{f(t)}dt = ||f||^2.$$

Note that this is equivalent to $TT* = T^*T = I$. Which is definitely suffice to say that T is unitary.

10 Section 6.5, Q13

Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B if and only if A and B are unitarily equivalent.

10.1 If

The ”if” part it naturally true. Because unitary matrices are invertible.

10.2 Only If

We prove this part to be not true generally using counter example.

Take $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. A, B both have 2 distint eigenvalues, hence diagnoalizable.

They are similar since $B = S^{-1}AS$, where $S = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$.

But A and B are not unitarily equivalent. That's becasue a necessary for 2 matrices being unitarily equivalent is that the sum of the square of their entries being the same. While $9 + 1 + 4 + 0 = 14 \neq 1 + 1 + 0 + 4 = 6$. Hence A, B not unitarily similar.