

MATH2040C Homework 6

ZHENG Weijia (William, 1155124322)

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1 Section 6.1, Q8

Provide reasons why each of the following is not an inner product on the given vector spaces.

- (a) $\langle (a, b), (c, d) \rangle = ac - bd$ on \mathbb{R}^2 .
- (b) $\langle A, B \rangle = \text{tr}(A + B)$ on $M_{2 \times 2}(R)$.
- (c) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t) dt$ on $P(R)$, where $'$ denotes differentiation.

1.1 (a)

Suppose this is an inner product. Then $\langle x, x \rangle \geq 0$ should hold $\forall x \in \mathbb{R}^2$.

Let $x = (1, 10)$. Then $\langle x, x \rangle = \langle (1, 10), (1, 10) \rangle = 1^2 - 10^2 = -99 < 0$.

Therefore, this is not an inner product.

1.2 (b)

Suppose this is an inner product. Then $\langle x, x \rangle \geq 0$ should hold $\forall x \in M_{2 \times 2}(R)$.

Let $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\langle x, x \rangle = \text{tr}(x + x) = -2 - 2 = -4 < 0.$$

Therefore, this is not an inner product.

1.3 (c)

Suppose this is an inner product. Then $\forall f, g \in P(\mathbb{R}), \overline{\langle g, f \rangle} = \langle f, g \rangle$ should hold.

Let $f(x) = x, g(x) = x^2 + x$.

Then

$$\langle f, g \rangle = \int_0^1 1(x^2 + x) dx = \frac{5}{6}.$$

$$\overline{\langle g, f \rangle} = \overline{\int_0^1 (2x + 1)x dx} = \frac{7}{6}.$$

Therefore $\overline{\langle g, f \rangle} \neq \langle f, g \rangle$ for some $f, g \in P(\mathbb{R})$.
Hence, this is not an inner product.
Done.

2 Section 6.1, Q17

Let T be a linear operator on an inner product space V , and suppose that $\|T(x)\| = \|x\|$ for all x . Prove that T is one-to-one.

Note that because we have $\|T(x)\| = \|x\|$. Then $\forall x \in V$, with $x \neq 0$ we have

$$\|T(x)\| = \|x\| > 0.$$

Therefore $x \neq 0$.

Note that $\|T(0)\| = \|0\| = 0$, which implies

$$T(0) = 0.$$

Hence $N(T) = \{0\}$.

Hence, T is one-to-one.

Done.

3 Section 6.1, Q18

Let V be a vector space over F , where $F = \mathbb{R}$ or $F = \mathbb{C}$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $T: V \rightarrow W$ is linear, prove that $\langle x, y \rangle' = \langle T(x), T(y) \rangle$ defines an inner product on V if and only if T is one-to-one.

3.1 If part

In the if part, we assume T is one-to-one and try to prove $\langle \cdot, \cdot \rangle'$ is an inner product.
One-to-one implies $N(T) = \{0\}$. Then $\forall x (\neq 0) \in V, T(x) \neq 0$. Therefore

$$\langle x, x \rangle' = \langle T(x), T(x) \rangle > 0.$$

Because $\langle \cdot, \cdot \rangle$ is an inner product.

Also note that $\forall x, y, z \in V, \forall c \in F$,

$$\begin{aligned} \langle x + z, y \rangle' &= \langle T(x + z), T(y) \rangle = \langle T(x) + T(z), T(y) \rangle = \langle T(x), T(y) \rangle + \langle T(z), T(y) \rangle \\ &= \langle x, y \rangle' + \langle z, y \rangle'. \end{aligned}$$

Besides,

$$\langle cx, y \rangle' = \langle T(cx), T(y) \rangle = \langle cT(x), T(y) \rangle = c\langle T(x), T(y) \rangle = c\langle x, y \rangle'.$$

And finally,

$$\overline{\langle x, y \rangle'} = \overline{\langle T(x), T(y) \rangle} = \langle T(y), T(x) \rangle = \langle y, x \rangle'.$$

Based on all above, $\langle \cdot, \cdot \rangle'$ is an inner product.

3.2 Only if part

In the only if part, we have $\langle \cdot, \cdot \rangle'$ is already an inner product and try to prove T is injective.

Note that T is linear, then $T(0) = 0$ must hold.

Because $\forall x (\neq 0) \in V, \langle x, x \rangle' = \langle T(x), T(x) \rangle > 0$. Therefore $\forall x \neq 0, T(x) \neq 0$.

Therefore $N(T) = \{0\}$, follows that T is injective.

Done.

4 Section 6.1, Q19

Let V be an inner product space. Prove that

- (a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re \langle x, y \rangle + \|y\|^2$ for all $x, y \in V$, where $\Re \langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$.
- (b) $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in V$.

4.1 (a)

Our goal is to prove $2\Re \langle x, y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2$ and $-2\Re \langle x, y \rangle = \|x - y\|^2 - \|x\|^2 - \|y\|^2$.

For the plus sign, note that the

$$\begin{aligned} R.H.S. &= \mathcal{R} \langle x + y, x + y \rangle - \mathcal{R} \langle x, x \rangle - \mathcal{R} \langle y, y \rangle \\ &= \mathcal{R} \langle x, y \rangle + \mathcal{R} \langle y, x \rangle. \end{aligned}$$

Note that $\mathcal{R} \langle x, y \rangle = \mathcal{R} \langle y, x \rangle$ because $\overline{\langle x, y \rangle} = \langle y, x \rangle$.

Therefore $L.H.S. = R.H.S.$.

For the minus sign, note that the

$$\begin{aligned} R.H.S. &= \mathcal{R} \langle x - y, x - y \rangle - \mathcal{R} \langle x, x \rangle - \mathcal{R} \langle y, y \rangle \\ &= \mathcal{R} \langle x, -y \rangle + \mathcal{R} \langle -y, x \rangle. \end{aligned}$$

Which is equal to $2\mathcal{R} \langle x, -y \rangle = -2\mathcal{R} \langle x, y \rangle = L.H.S.$.

Therefore, (a) is proved. Done.

4.2 (b)

To prove (b), it is suffice to prove $||x|| - ||y|||^2 \leq ||x - y||^2$.

Iff,

$$||x||^2 + ||y||^2 - 2||x|| \cdot ||y|| \leq ||x||^2 + 2\mathcal{R}\langle x, y \rangle + ||y||^2.$$

iff,

$$||x|| \cdot ||y|| \geq \mathcal{R}\langle x, y \rangle.$$

Which is bound to be true because

$$||x|| \cdot ||y|| \geq |\langle x, y \rangle| \geq \mathcal{R}\langle x, y \rangle.$$

Done.

5 Section 6.1, Q23

Let $V = F^n$, and let $A \in M_{n \times n}(F)$.

- (a) Prove that $\langle x, Ay \rangle = \langle A^*x, y \rangle$ for all $x, y \in V$.
- (b) Suppose that for some $B \in M_{n \times n}(F)$, we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in V$. Prove that $B = A^*$.
- (c) Let α be the standard ordered basis for V . For any orthonormal basis β for V , let Q be the $n \times n$ matrix whose columns are the vectors in β . Prove that $Q^* = Q^{-1}$.
- (d) Define linear operators T and U on V by $T(x) = Ax$ and $U(x) = A^*x$. Show that $[U]_\beta = [T]_\beta^*$ for any orthonormal basis β for V .

5.1 (a)

Denote $T = L_A : V \rightarrow V$, note that T is linear. Note that we have a corollary that $(L_A)^* = L_{A^*}$.

Then $\langle x, Ay \rangle = \langle A^*x, y \rangle$ follows. Done.

5.2 (b)

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a standard basis for $V = F^n$, then we have a expression of A such that $\forall x \in V$,

$$B(x) = \sum_{i=1}^n \langle B(x), v_i \rangle v_i = \sum_{i=1}^n \langle x, Av_i \rangle v_i = \sum_{i=1}^n \langle A^*x, v_i \rangle v_i = A^*(x).$$

Therefore, $B = A^*$.

Done.

5.3 (c)

Denote $\beta = \{v_1, v_2, \dots, v_n\}$ is a orthonormal basis as the problem said. Then

$$Q := (v_1, v_2, \dots, v_n).$$

Therefore

$$Q^* = \begin{pmatrix} -v_1 - \\ -v_2 - \\ \dots - \\ -v_n - \end{pmatrix}.$$

Hence

$$Q^*Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n.$$

Therefore, $Q^* = Q^{-1}$. Done.

5.4 (d)

Note that

$$[U]_\beta = (Uv_1 \ Uv_2 \ \dots \ Uv_n)_\beta = (A^*v_1 \ A^*v_2 \ \dots \ A^*v_n)_\beta.$$

And

$$[T]_\beta^* = (Tv_1 \ Tv_2 \ \dots \ Tv_n)_\beta^* = (Av_1 \ Av_2 \ \dots \ Av_n)_\beta^* = [(Av_1 \ Av_2 \ \dots \ Av_n)^*]_\beta.$$

Suffice to prove that

$$(A^*v_1 \ A^*v_2 \ \dots \ A^*v_n)_\beta = (Av_1 \ Av_2 \ \dots \ Av_n)_\beta^*.$$

Denote $Q = (v_1 \ v_2 \ \dots \ v_n)$.

Iff

$$(A^*Q)_\beta = (AQ)_\beta^*.$$

Iff

$$(A^*)_ \beta Q_\beta = (A_\beta Q_\beta)^*.$$

Note that Q_β is always I_n . Therefore it suffice to prove that

$$(A^*)_ \beta = (A_\beta)^*.$$

Denote that the j -th column of $[A]_\beta$ is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{nj} \end{pmatrix}$.

I.e.,

$$[A_\beta]_{ij} = a_{ij}.$$

Hence the j -th row of $A_\beta^* = (a_{1j}^* \ a_{2j}^* \ \dots \ a_{nj}^*)$. Hence

$$[A_\beta^*]_{ij} = a_{ji}^*.$$

Therefore

$$(A_\beta)^* = (A^*)_ \beta.$$

Hence (d) is proved.

6 Section 6.2, Q2(g)

$$(g) \quad V = M_{2 \times 2}(R), \quad S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}, \text{ and} \\ A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$$

What I need to do is to 1. apply the G-S process to obtain a orthonormal basis. Compute Fourier coefficient and use Theorem 6.5 to check.

Because this is about the matrices, I adopt the Frobenius inner product

$$\langle A, B \rangle = \text{tr}(B^* A).$$

Denote the ones inside S as w_1, w_2, w_3 .

$$\text{Let } v_1 = w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}.$$

$$\text{And } v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}.$$

$$\text{And } v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}.$$

By checking, the v_i 's are orthogonal.

To normalize them, we can have

$$S' = \left\{ \frac{1}{6}v_1, \frac{1}{6\sqrt{2}}v_2, \frac{1}{9\sqrt{2}}v_3 \right\} = \{v'_1, v'_2, v'_3\}.$$

Which is an orthonormal basis.

To calculate the Fourier coefficient of A , we calculate

$$\langle A, v'_1 \rangle = 24, \langle A, v'_2 \rangle = 6\sqrt{2}, \langle A, v'_3 \rangle = -9\sqrt{2}.$$

By the Theorem 6.5, we

$$24 \frac{1}{6}v_1 + 6\sqrt{2} \frac{1}{6\sqrt{2}}v_2 - 9\sqrt{2} \frac{1}{9\sqrt{2}}v_3 = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}.$$

Which means that the theorem is verified.

Done.

7 Section 6.2, Q2(i)

$$(i) \quad V = \text{span}(S) \text{ with the inner product } \langle f, g \rangle = \int_0^\pi f(t)g(t) dt, \\ S = \{\sin t, \cos t, 1, t\}, \text{ and } h(t) = 2t + 1$$

Let $v_1 = w_1 = \sin t$.

Let $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \cos t$.

Let $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = 1 - \frac{4}{\pi} \sin t$.

Let $v_4 = w_4 - \frac{\langle w_4, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_4, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle w_4, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 = t + \frac{4}{\pi} \cos t - \frac{\pi}{2}$.

To normalize them, we can just calculate and divide them by their norm.

$$\because \|v_1\| = \sqrt{\frac{\pi}{2}}, \therefore v'_1 = \sqrt{\frac{2}{\pi}} v_1.$$

$$\because \|v_2\| = \sqrt{\frac{\pi}{2}}, \therefore v'_2 = \sqrt{\frac{2}{\pi}} v_2.$$

$$\because \|v_3\| = \sqrt{\pi - \frac{8}{\pi}}, \therefore v'_3 = \frac{1}{\sqrt{\pi - \frac{8}{\pi}}} v_3.$$

$$\because \|v_4\| = \sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}, \therefore v'_4 = \frac{1}{\sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}} v_4.$$

Our $h(t) = 2t + 1$.

Note that $\langle h, v'_1 \rangle = 6.609$, $\langle h, v'_2 \rangle = -3.1915$, $\langle h, v'_3 \rangle = 3.195$, $\langle h, v'_4 \rangle = 0.38666$.

Then by doing the calculation, we have

$$h(t) = 1.99999t + 1.00000.$$

Which is what we expected.

Done.

8 Section 6.2, Q6

6. Let V be an inner product space, and let W be a finite-dimensional subspace of V . If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$. *Hint:* Use Theorem 6.6.

Using the theorem 6.6, there exists unique $y, pj_W(x)$ such that

$$x = y + pj_W(x).$$

Where $pj_W(x)$ is the component of x which is inside W . Therefore, $y \in W^\perp$.

Then we need to check

$$\langle x, y \rangle = \langle y + pj_W(x), y \rangle = \langle y, y \rangle + \langle pj_W(x), y \rangle.$$

Note that $pj_W(x) \in W$ and $y \in W^\perp$, then $\langle pj_W(x), y \rangle = 0$.

Hence $\langle x, y \rangle = \langle y, y \rangle$. Suppose $\langle y, y \rangle = 0$, then $y = 0$.

Then $x = y + pj_W(x) = pj_W(x) \in W$. Which is contradicting with $x \notin W$.

Therefore $\langle x, y \rangle \neq 0$ at the first place.