MATH2040C Homework 7

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1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(c) $\mathsf{V} = \mathsf{P}_1(R)$ with $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V.

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^{1} 1 \cdot t\, dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^{n} \overline{\langle T(v_i), g(t) \rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\sqrt{\frac{3}{2}}t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t.$.

Therefore,

$$T^*(g(t)) = \frac{3}{2}\int_{-1}^{1} g(t)dt + \frac{3}{2}t\int_{-1}^{1}(1 + 3t)g(t)dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$. Deduce that $\text{rank}(\mathsf{T}^*\mathsf{T}) = \text{rank}(\mathsf{T})$.

(b) $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^*)$. Deduce from (a) that $\text{rank}(\mathsf{T}\mathsf{T}^*) = \text{rank}(\mathsf{T})$.

(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.
Forall $y \in N(T^*T)$, consider the norm of Ty :

$$||Ty||^2 = \langle Ty, Ty\rangle = \langle y, T^*Ty\rangle = \langle y, 0\rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.
Based on all above, $N(T^*T) = N(T)$.
Recall that $T \in \mathcal{L}(V)$. Hence $T : V \to V$. And according to $\forall y \in V$,

$$T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y\rangle} v_i.$$

We know that $T^* : V \to V$. Therefore $T^*T : V \to V$.
Applying the rank nullity theorem, we have that

$$\dim V = rank(T^*T) + \dim N(T^*T)\ ,\ \ \dim V = rank(T) + \dim N(T).$$

Using the just proved fact $N(T^*T) = N(T)$, we can simply deduce

$$rank(T^*T) = rank(T).$$

2.2 (b)

By changing name of the identity in (a), we can have $N(TT^*) = N(T^*)$ and $rank(TT^*) = rank(T^*)$.
Notice that

$$rank(T) = rank[T]_\beta = rank[T]^*_\beta = rank[T^*]_\beta = rank(T^*).$$

And then $rank(TT^*) = rank(T)$ follows.

2.3 (c)

From (b), $rank(AA^*) = rank(A)$ follows naturally.
And note that $(AA^*)^* = A^*A$, then

$$rank(AA^*) = rank(A^*A).$$

Therefore,

$$rank(AA^*) = rank(A^*A) = rank(A).$$

Done.

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3 Section 6.3, Q14

Let V be an inner product space, and let $y, z \in \mathsf{V}$. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(x) = \langle x, y \rangle z$ for all $x \in \mathsf{V}$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

First we would prove that T is linear. $\forall x_1, x_2 \in V, \forall c \in F$,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle z = \langle cx_1, y \rangle z + \langle x_2, y \rangle z = c\langle x_1, y \rangle z + \langle x_2, y \rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore, T is linear.
From course not we directly construct $\forall x \in V$,

$$T^*(x) = \sum_{i=1}^{n} \overline{\langle T(v_i), x \rangle} v_i = \sum_{i=1}^{n} \overline{\langle \langle v_i, y \rangle z, x \rangle} v_i = \overline{\langle z, x \rangle} \sum_{i=1}^{n} \overline{\langle v_i, y \rangle} v_i = \overline{\langle z, x \rangle} y.$$

Recall that $y = I(y) = I^*(y) = \sum_{i=1}^{n} \overline{\langle I(v_i), y \rangle} v_i = \sum_{i=1}^{n} \overline{\langle v_i, y \rangle} v_i$. Hence the last equality holds properly.

4 Section 6.3, Q15

Definition. *Let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be a linear transformation, where* V *and* W *are finite-dimensional inner product spaces with inner products* $\langle \cdot, \cdot \rangle_1$ *and* $\langle \cdot, \cdot \rangle_2$*, respectively. A function* $\mathsf{T}^*\colon \mathsf{W} \to \mathsf{V}$ *is called an* **adjoint** *of* T *if* $\langle \mathsf{T}(x), y \rangle_2 = \langle x, \mathsf{T}^*(y) \rangle_1$ *for all* $x \in \mathsf{V}$ *and* $y \in \mathsf{W}$.

15. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

(a) There is a unique adjoint T^* of T, and T^* is linear.
(b) If β and γ are orthonormal bases for V and W, respectively, then $[\mathsf{T}^*]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^*$.
(c) $\operatorname{rank}(\mathsf{T}^*) = \operatorname{rank}(\mathsf{T})$.
(d) $\langle \mathsf{T}^*(x), y \rangle_1 = \langle x, \mathsf{T}(y) \rangle_2$ for all $x \in \mathsf{W}$ and $y \in \mathsf{V}$.
(e) For all $x \in \mathsf{V}$, $\mathsf{T}^*\mathsf{T}(x) = \mathit{0}$ if and only if $\mathsf{T}(x) = \mathit{0}$.

4.1 (a)

Define $g : V \to F$ by $g(x) = \langle T(x), y \rangle_2$. Obviously, note that g is linear for sure.

Then apply the theorem 6.8, there exists a unique vector $y' \in W$ such that

$$g(x) = \langle x, y' \rangle_1.$$

Recall g's definition, we have $\langle T(x), y \rangle_2 = \langle x, y' \rangle_1,\ \forall x \in V.$

Define that $T^* : W \to V$ by $T^*(y) = y'$.

Then we have $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$.

Hence the wanted T^* function exists, and becasue the y' is "unique" as mentioned above (for any y), then the T^* is also unique.

Then we prove the linearity of T^*. $\forall y_1, y_2 \in W$ and $\forall c \in F$,

$$\langle x, T^*(cy_1 + y_2) \rangle_1 = \langle T(x), cy_1 + y_2 \rangle_2 = \langle x, cT^*(y_1) \rangle_1 + \langle x, T^*(y_2) \rangle_1$$

$$= \langle x, cT^*(y_1) + T^*(y_2) \rangle_1, \forall x \in V.$$

Therefore $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$. Hence T^* is linear.

4.2 (b)

For such kind of problem, we need to compare the two matrices entry wise. Denote $\beta = \{v_1, v_2, \ldots, v_n\}, \gamma = \{w_1, w_2, \ldots, w_m\}$.

Inspect $[T^*]_\gamma^\beta = [T^*(w_1), T^*(w_2), \ldots, T^*(w_m)]_\beta$.

Note that $T^*(w_j) = \sum_{i=1}^n \overline{\langle v_i, T^*(w_j)\rangle_1} v_i, \ \forall j = 1, 2, \ldots, m.$

From this we know that the i-th row, j-th column of $[T^*]_\gamma^\beta$ is $\overline{\langle v_i, T^*(w_j)\rangle_1}$.

Inspect $[T]_\beta^\gamma = [T(v_1), T(v_2), \ldots, T(v_n)]_\gamma$.

Note that $T(v_i) = \sum_{k=1}^m \overline{\langle w_k, T(v_i)\rangle_2} w_k, \ \forall i = 1, 2, \ldots, n.$

Therefore, the i-th column, k-th row of $[T]_\beta^\gamma = \overline{\langle w_k, T(v_i)\rangle_2}$.

Hence, for $([T]_\beta^\gamma)^*$, the i-th row, j-th column is $\langle w_j, T(v_i)\rangle_2$.

What remains to be done is to show

$$\langle w_j, T(v_i)\rangle_2 = \overline{\langle v_i, T^*(w_j)\rangle_1}.$$

This is equivalent to show $\overline{\langle w_j, T(v_i)\rangle_2} = \langle v_i, T^*(w_j)\rangle_1$.

Note that the L.H.S.=$\langle T(v_i), w_j\rangle_2$=R.H.S. from the definition of T^*. Hence this is proved.

4.3 (c)

$rank(T^*) = rank([T^*]_\gamma^\beta)$ and $rank(T) = rank([T]_\beta^\gamma) = rank(([T]_\beta^\gamma)^*)$.

Followed from (b), we have $rank(T^*) = rank(T)$.

4.4 (d)

We want to prove $\langle T^*(y), x\rangle_1 = \langle y, T(x)\rangle_2, \forall y \in W, x \in V$. which is equivalent to prove $\langle x, T^*(y)\rangle_1 = \langle T(x), y\rangle_2$.

And L.H.S.=$\langle T(x), y\rangle_2$ followed from the definition of T^*.

Done.

4.5 (e)

It is suffice to prove $N(T) = N(T^*T)$. And it is obvious to see that

$$N(T) \subset N(T^*T).$$

Take any x such that $T^*Tx = 0$. We have $\langle Tx, Tx\rangle_2 = \langle x, T^*Tx\rangle_1 = 0$. Which implies that $Tx = 0$. Hence $x \in N(T)$, and $N(T^*T) = N(T)$.

Done.

5 Section 6.4, Q2(d)

For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

Let $\{1, t, t^2\}$ be a basis for $P_2(\mathbb{R})$, then apply Gram-Schmidt process upon it, we can have an orthonormal basis $\beta = \{1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6})\}$.

First, we claim that T is not self-adjoint, by the spectral theorem, T is self-adjoint iff, T is diagnoalizable, which implies that it will lead to the eigenspace decomposition of V. Note that there is only one eigenvalue of T, which is 0 and the only corresponding set of eigenvectors is $span\{1\}$. It is obvious that $E_0 \neq V$, since $t^2 \notin E_0$. Therefore T is not self-adjoint and meanwhile, it is impossible to derive a orthonormal basis of eigenvectors of T for V.

Also, T is not normal (?)