MATH2040C Homework 6

ZHENG Weijia (William, 1155124322)

April 9, 2021

1 Section 6.1, Q8

Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b),(c,d)\rangle = ac - bd$ on R^2.
(b) $\langle A, B\rangle = \mathrm{tr}(A+B)$ on $\mathsf{M}_{2\times 2}(R)$.
(c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

1.1 (a)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in \mathbb{R}^2$.
Let $x = (1, 10)$. Then $\langle\, x, x\rangle = \langle\,(1,10),(1,10)\rangle = 1^2 - 10^2 = -99 < 0$.
Therefore, this is not an inner product.

1.2 (b)

Suppose this is an inner product. Then $\langle\, x, x\rangle \geq 0$ should hold $\forall x \in M_{2\times 2}(R)$.
Let $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\langle\, x, x\rangle = tr(x + x) = -2 - 2 = -4 < 0.$$

Therefore, this is not an inner product.

1.3 (c)

Suppose this is an inner product. Then $\forall f, g \in P(\mathbb{R}), \overline{\langle\, g, f\rangle} = \langle\, f, g\rangle$ should hold.
Let $f(x) = x, g(x) = x^2 + x$.
Then

$$\langle\, f, g\rangle = \int_0^1 1(x^2 + x)\,dx = \frac{5}{6}.$$

$$\overline{\langle\, g, f\rangle} = \overline{\int_0^1 (2x+1)x\,dx} = \frac{7}{6}.$$

Therefore $\overline{\langle g, f\rangle} \neq \langle f, g\rangle$ for some $f, g \in P(\mathbb{R})$.
Hence, this is not an inner product.
Done.

---

2 Section 6.1, Q17

Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Note that because we have $||T(x)|| = ||x||$. Then $\forall x \in V$, with $x \neq 0$ we have

$$||T(x)|| = ||x|| > 0.$$

Therefore $x \neq 0$.
Note that $||T(0)|| = ||0|| = 0$, which implies

$$T(0) = 0.$$

Hence $N(T) = \{0\}$.
Hence, T is one-to-one.
Done.

---

3 Section 6.1, Q18

Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y\rangle' = \langle \mathsf{T}(x), \mathsf{T}(y)\rangle$ defines an inner product on V if and only if T is one-to-one.

3.1 If part

In the if part, we assume T is one-to-one and try to prove $\langle \cdot, \cdot\rangle'$ is an inner product.
One-to-one implies $N(T) = \{0\}$. Then $\forall x (\neq 0) \in V, T(x) \neq 0$. Therefore

$$\langle x, x\rangle' = \langle T(x), T(x)\rangle > 0.$$

Because $\langle \cdot, \cdot\rangle$ is an inner product.
Also note that $\forall x, y, z \in V, \forall c \in F$,

$$\langle x + z, y\rangle' = \langle T(x + z), T(y)\rangle = \langle T(x) + T(z), T(y)\rangle = \langle T(x), T(y)\rangle + \langle T(z), T(y)\rangle$$

$$= \langle x, y\rangle' + \langle z, y\rangle'.$$

Besides,

$$\langle\, cx, y\rangle' = \langle\, T(cx), T(y)\rangle = \langle\, cT(x), T(y)\rangle = c\langle\, T(x), T(y)\rangle = c\langle\, x, y\rangle'.$$

And finally,

$$\overline{\langle\, x, y\rangle'} = \overline{\langle\, T(x), T(y)\rangle} = \langle\, T(y), T(x)\rangle = \langle\, y, x\rangle'.$$

Based on all above, $\langle\,\cdot,\cdot\rangle'$ is an inner product.

3.2 Only if part

In the only if part, we have $\langle\,\cdot,\cdot\rangle'$ is already an inner product and try to prove T is injective.
Note that T is linear, then $T(0) = 0$ must hold.
Becasue $\forall x(\neq 0) \in V, \langle\, x, x\rangle' = \langle\, T(x), T(x)\rangle > 0$. Therefore $\forall x \neq 0, T(x) \neq 0$.
Therefore $N(T) = \{0\}$, follows that T is injective.
Done.

---

4 Section 6.1, Q19

Let V be an inner product space. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re\langle x, y\rangle + \|y\|^2$ for all $x, y \in$ V, where $\Re\langle x, y\rangle$ denotes the real part of the complex number $\langle x, y\rangle$.

(b) $|\,\|x\| - \|y\|\,| \leq \|x - y\|$ for all $x, y \in$ V.

4.1 (a)

Our goal is to prove $2\mathcal{R}\langle\, x, y\rangle = ||x + y||^2 - ||x||^2 - ||y||^2$ and $-2\mathcal{R}\langle\, x, y\rangle = ||x - y||^2 - ||x||^2 - ||y||^2$.
For the plus sign, note that the

$$R.H.S. = \mathcal{R}\langle\, x + y, x + y\rangle - \mathcal{R}\langle\, x, x\rangle - \mathcal{R}\langle\, y, y\rangle$$

$$= \mathcal{R}\langle\, x, y\rangle + \mathcal{R}\langle\, y, x\rangle.$$

Note that $\mathcal{R}\langle\, x, y\rangle = \mathcal{R}\langle\, y, x\rangle$ because $\overline{\langle\, x, y\rangle} = \langle\, y, x\rangle$.
Therefore $L.H.S. = R.H.S.$.
For the minus sign, note that the

$$R.H.S. = \mathcal{R}\langle\, x - y, x - y\rangle - \mathcal{R}\langle\, x, x\rangle - \mathcal{R}\langle\, y, y\rangle$$

$$= \mathcal{R}\langle\, x, -y\rangle + \mathcal{R}\langle\, -y, x\rangle.$$

Which is equal to $2\mathcal{R}\langle\, x, -y\rangle = -2\mathcal{R}\langle\, x, y\rangle = L.H.S.$.
Therefore, (a) is proved. Done.

4.2 (b)

To prove (b), it is suffice to prove $|\ ||x|| - ||y||\ |^2 \leq ||x-y||^2$.
Iff,

$$||x||^2 + ||y||^2 - 2||x|| \cdot ||y|| \leq ||x||^2 + 2\mathcal{R}\langle\, x, y\rangle + ||y||^2.$$

iff,

$$||x|| \cdot ||y|| \geq \mathcal{R}\langle\, x, y\rangle.$$

Which is bound to be true because

$$||x|| \cdot ||y|| \geq |\langle\, x, y\rangle| \geq \mathcal{R}\langle\, x, y\rangle.$$

Done.

---

5 Section 6.1, Q23

Let $\mathsf{V} = \mathsf{F}^n$, and let $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n\times n}(F)$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

5.1 (a)

Denote $T = L_A : V \to V$, note that T is linear. Note that we have a corollary that $(L_A)^* = L_{A^*}$.
Then $\langle\, x, Ay\rangle = \langle\, A^*x, y\rangle$ follows. Done.

5.2 (b)

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a standard basis for $V = F^n$, then we have a expression of A such that $\forall x \in V$,

$$B(x) = \sum_{i=1}^{n} \langle\, B(x), v_i\rangle v_i = \sum_{i=1}^{n} \langle\, x, Av_i\rangle v_i = \sum_{i=1}^{n} \langle\, A^*x, v_i\rangle v_i = A^*(x).$$

Therefore, $B = A^*$.
Done.

5.3 (c)