

# MATH2040C Homework 6

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## 1 Section 6.1, Q8

Provide reasons why each of the following is not an inner product on the given vector spaces.

- (a)  $\langle (a, b), (c, d) \rangle = ac - bd$  on  $\mathbb{R}^2$ .
- (b)  $\langle A, B \rangle = \text{tr}(A + B)$  on  $M_{2 \times 2}(R)$ .
- (c)  $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t) dt$  on  $P(R)$ , where  $'$  denotes differentiation.

### 1.1 (a)

Suppose this is an inner product. Then  $\langle x, x \rangle \geq 0$  should hold  $\forall x \in \mathbb{R}^2$ .  
Let  $x = (1, 10)$ . Then  $\langle x, x \rangle = \langle (1, 10), (1, 10) \rangle = 1^2 - 10^2 = -99 < 0$ .  
Therefore, this is not an inner product.

### 1.2 (b)

Suppose this is an inner product. Then  $\langle x, x \rangle \geq 0$  should hold  $\forall x \in M_{2 \times 2}(R)$ .  
Let  $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . Then

$$\langle x, x \rangle = \text{tr}(x + x) = -2 - 2 = -4 < 0.$$

Therefore, this is not an inner product.

### 1.3 (c)

Suppose this is an inner product. Then  $\forall f, g \in P(\mathbb{R}), \overline{\langle g, f \rangle} = \langle f, g \rangle$  should hold.  
Let  $f(x) = x, g(x) = x^2 + x$ .  
Then

$$\langle f, g \rangle = \int_0^1 1(x^2 + x) dx = \frac{5}{6}.$$

$$\overline{\langle g, f \rangle} = \overline{\int_0^1 (2x + 1)x dx} = \frac{7}{6}.$$

Therefore  $\overline{\langle g, f \rangle} \neq \langle f, g \rangle$  for some  $f, g \in P(\mathbb{R})$ .  
Hence, this is not an inner product.  
Done.

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## 2 Section 6.1, Q17

Let  $T$  be a linear operator on an inner product space  $V$ , and suppose that  $\|T(x)\| = \|x\|$  for all  $x$ . Prove that  $T$  is one-to-one.

Note that because we have  $\|T(x)\| = \|x\|$ . Then  $\forall x \in V$ , with  $x \neq 0$  we have

$$\|T(x)\| = \|x\| > 0.$$

Therefore  $x \neq 0$ .

Note that  $\|T(0)\| = \|0\| = 0$ , which implies

$$T(0) = 0.$$

Hence  $N(T) = \{0\}$ .

Hence,  $T$  is one-to-one.

Done.

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## 3 Section 6.1, Q18

Let  $V$  be a vector space over  $F$ , where  $F = \mathbb{R}$  or  $F = \mathbb{C}$ , and let  $W$  be an inner product space over  $F$  with inner product  $\langle \cdot, \cdot \rangle$ . If  $T: V \rightarrow W$  is linear, prove that  $\langle x, y \rangle' = \langle T(x), T(y) \rangle$  defines an inner product on  $V$  if and only if  $T$  is one-to-one.

### 3.1 If part

In the if part, we assume  $T$  is one-to-one and try to prove  $\langle \cdot, \cdot \rangle'$  is an inner product.  
One-to-one implies  $N(T) = \{0\}$ . Then  $\forall x (\neq 0) \in V, T(x) \neq 0$ . Therefore

$$\langle x, x \rangle' = \langle T(x), T(x) \rangle > 0.$$

Because  $\langle \cdot, \cdot \rangle$  is an inner product.

Also note that  $\forall x, y, z \in V, \forall c \in F$ ,

$$\begin{aligned} \langle x + z, y \rangle' &= \langle T(x + z), T(y) \rangle = \langle T(x) + T(z), T(y) \rangle = \langle T(x), T(y) \rangle + \langle T(z), T(y) \rangle \\ &= \langle x, y \rangle' + \langle z, y \rangle'. \end{aligned}$$

Besides,

$$\langle cx, y \rangle' = \langle T(cx), T(y) \rangle = \langle cT(x), T(y) \rangle = c\langle T(x), T(y) \rangle = c\langle x, y \rangle'.$$

And finally,

$$\overline{\langle x, y \rangle'} = \overline{\langle T(x), T(y) \rangle} = \langle T(y), T(x) \rangle = \langle y, x \rangle'.$$

Based on all above,  $\langle \cdot, \cdot \rangle'$  is an inner product.

### 3.2 Only if part

In the only if part, we have  $\langle \cdot, \cdot \rangle'$  is already an inner product and try to prove  $T$  is injective.

Note that  $T$  is linear, then  $T(0) = 0$  must hold.

Because  $\forall x (\neq 0) \in V, \langle x, x \rangle' = \langle T(x), T(x) \rangle > 0$ . Therefore  $\forall x \neq 0, T(x) \neq 0$ .

Therefore  $N(T) = \{0\}$ , follows that  $T$  is injective.

Done.

## 4 Section 6.1, Q19

Let  $V$  be an inner product space. Prove that

- (a)  $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re \langle x, y \rangle + \|y\|^2$  for all  $x, y \in V$ , where  $\Re \langle x, y \rangle$  denotes the real part of the complex number  $\langle x, y \rangle$ .
- (b)  $|\|x\| - \|y\|| \leq \|x - y\|$  for all  $x, y \in V$ .

### 4.1 (a)

Our goal is to prove  $2\Re \langle x, y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2$  and  $-2\Re \langle x, y \rangle = \|x - y\|^2 - \|x\|^2 - \|y\|^2$ .

For the plus sign, note that the

$$\begin{aligned} R.H.S. &= \mathcal{R} \langle x + y, x + y \rangle - \mathcal{R} \langle x, x \rangle - \mathcal{R} \langle y, y \rangle \\ &= \mathcal{R} \langle x, y \rangle + \mathcal{R} \langle y, x \rangle. \end{aligned}$$

Note that  $\mathcal{R} \langle x, y \rangle = \mathcal{R} \langle y, x \rangle$  because  $\overline{\langle x, y \rangle} = \langle y, x \rangle$ .

Therefore  $L.H.S. = R.H.S.$ .

For the minus sign, note that the

$$\begin{aligned} R.H.S. &= \mathcal{R} \langle x - y, x - y \rangle - \mathcal{R} \langle x, x \rangle - \mathcal{R} \langle y, y \rangle \\ &= \mathcal{R} \langle x, -y \rangle + \mathcal{R} \langle -y, x \rangle. \end{aligned}$$

Which is equal to  $2\mathcal{R} \langle x, -y \rangle = -2\mathcal{R} \langle x, y \rangle = L.H.S.$ .

Therefore, (a) is proved. Done.

## 4.2 (b)

To prove (b), it is suffice to prove  $||x|| - ||y|||^2 \leq ||x - y||^2$ .

Iff,

$$||x||^2 + ||y||^2 - 2||x|| \cdot ||y|| \leq ||x||^2 + 2\mathcal{R}\langle x, y \rangle + ||y||^2.$$

iff,

$$||x|| \cdot ||y|| \geq \mathcal{R}\langle x, y \rangle.$$

Which is bound to be true because

$$||x|| \cdot ||y|| \geq |\langle x, y \rangle| \geq \mathcal{R}\langle x, y \rangle.$$

Done.

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## 5 Section 6.1, Q23

Let  $V = F^n$ , and let  $A \in M_{n \times n}(F)$ .

- (a) Prove that  $\langle x, Ay \rangle = \langle A^*x, y \rangle$  for all  $x, y \in V$ .
- (b) Suppose that for some  $B \in M_{n \times n}(F)$ , we have  $\langle x, Ay \rangle = \langle Bx, y \rangle$  for all  $x, y \in V$ . Prove that  $B = A^*$ .
- (c) Let  $\alpha$  be the standard ordered basis for  $V$ . For any orthonormal basis  $\beta$  for  $V$ , let  $Q$  be the  $n \times n$  matrix whose columns are the vectors in  $\beta$ . Prove that  $Q^* = Q^{-1}$ .
- (d) Define linear operators  $T$  and  $U$  on  $V$  by  $T(x) = Ax$  and  $U(x) = A^*x$ . Show that  $[U]_\beta = [T]_\beta^*$  for any orthonormal basis  $\beta$  for  $V$ .

### 5.1 (a)

Denote  $T = L_A : V \rightarrow V$ , note that  $T$  is linear. Note that we have a corollary that  $(L_A)^* = L_{A^*}$ .

Then  $\langle x, Ay \rangle = \langle A^*x, y \rangle$  follows. Done.

### 5.2 (b)

Let  $\beta = \{v_1, v_2, \dots, v_n\}$  be a standard basis for  $V = F^n$ , then we have a expression of  $A$  such that  $\forall x \in V$ ,

$$B(x) = \sum_{i=1}^n \langle B(x), v_i \rangle v_i = \sum_{i=1}^n \langle x, Av_i \rangle v_i = \sum_{i=1}^n \langle A^*x, v_i \rangle v_i = A^*(x).$$

Therefore,  $B = A^*$ .

Done.

### 5.3 (c)

Denote  $\beta = \{v_1, v_2, \dots, v_n\}$  is a orthonormal basis as the problem said. Then

$$Q := (v_1, v_2, \dots, v_n).$$

Therefore

$$Q^* = \begin{pmatrix} -v_1 - \\ -v_2 - \\ \dots - \\ -v_n - \end{pmatrix}.$$

Hence

$$Q^*Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n.$$

Therefore,  $Q^* = Q^{-1}$ . Done.

### 5.4 (d)

Note that

$$[U]_\beta = (Uv_1 \ Uv_2 \ \dots \ Uv_n)_\beta = (A^*v_1 \ A^*v_2 \ \dots \ A^*v_n)_\beta.$$

And

$$[T]_\beta^* = (Tv_1 \ Tv_2 \ \dots \ Tv_n)_\beta^* = (Av_1 \ Av_2 \ \dots \ Av_n)_\beta^* = [(Av_1 \ Av_2 \ \dots \ Av_n)^*]_\beta.$$

Suffice to prove that

$$(A^*v_1 \ A^*v_2 \ \dots \ A^*v_n)_\beta = (Av_1 \ Av_2 \ \dots \ Av_n)_\beta^*.$$

Denote  $Q = (v_1 \ v_2 \ \dots \ v_n)$ .

Iff

$$(A^*Q)_\beta = (AQ)_\beta^*.$$

Iff

$$(A^*)_ \beta Q_\beta = (A_\beta Q_\beta)^*.$$

Note that  $Q_\beta$  is always  $I_n$ . Therefore it suffice to prove that

$$(A^*)_ \beta = (A_\beta)^*.$$

Denote that the  $j$ -th column of  $[A]_\beta$  is  $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{nj} \end{pmatrix}$ .

I.e.,

$$[A_\beta]_{ij} = a_{ij}.$$

Hence the  $j$ -th row of  $A_\beta^* = (a_{1j}^* \ a_{2j}^* \ \dots \ a_{nj}^*)$ . Hence

$$[A_\beta^*]_{ij} = a_{ji}^*.$$

Therefore

$$(A_\beta)^* = (A^*)_ \beta.$$

Hence (d) is proved.

## 6 Section 6.2, Q2(g)

$$(g) \quad V = M_{2 \times 2}(R), \quad S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}, \text{ and}$$

$$A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$$

What I need to do is to 1. apply the G-S process to obtain a orthonormal basis. Compute Fourier coefficient and use Theorem 6.5 to check.

Because this is about the matrices, I adopt the Frobenius inner product

$$\langle A, B \rangle = \text{tr}(B^* A).$$

Denote the ones inside  $S$  as  $w_1, w_2, w_3$ .

$$\text{Let } v_1 = w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}.$$

$$\text{And } v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}.$$

$$\text{And } v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}.$$

By checking, the  $v_i$ 's are orthogonal.

To normalize them, we can have

$$S' = \left\{ \frac{1}{6} v_1, \frac{1}{6\sqrt{2}} v_2, \frac{1}{9\sqrt{2}} v_3 \right\} = \{v'_1, v'_2, v'_3\}.$$

Which is an orthonormal basis.

To calculate the Fourier coefficient of  $A$ , we calculate

$$\langle A, v'_1 \rangle = 24, \langle A, v'_2 \rangle = 6\sqrt{2}, \langle A, v'_3 \rangle = -9\sqrt{2}.$$

By the Theorem 6.5, we

$$24 \frac{1}{6} v_1 + 6\sqrt{2} \frac{1}{6\sqrt{2}} v_2 - 9\sqrt{2} \frac{1}{9\sqrt{2}} v_3 = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}.$$

Which means that the theorem is verified.

Done.

## 7 Section 6.2, Q2(i)

$$\text{Let } v_1 = w_1 = \sin t.$$

$$\text{Let } v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \cos t.$$

$$\text{Let } v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = 1 - \frac{4}{\pi} \sin t.$$

$$\text{Let } v_4 = w_4 - \frac{\langle w_4, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_4, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle w_4, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 = t + \frac{4}{\pi} \cos t - \frac{\pi}{2}.$$

To normalize them, we can just calculate and divide them by their norm.

$$\because \|v_1\| = \sqrt{\frac{\pi}{2}}, \therefore v'_1 = \sqrt{\frac{2}{\pi}}v_1.$$

$$\because \|v_2\| = \sqrt{\frac{\pi}{2}}, \therefore v'_2 = \sqrt{\frac{2}{\pi}}v_2.$$

$$\because \|v_3\| = \sqrt{\pi - \frac{8}{\pi}}, \therefore v'_3 = \frac{1}{\sqrt{\pi - \frac{8}{\pi}}}v_3.$$

$$\because \|v_4\| = \sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}, \therefore v'_4 = \frac{1}{\sqrt{\frac{\pi^3}{12} - \frac{8}{\pi}}}v_4.$$

Our  $h(t) = 2t + 1$ .

Note that  $\langle h, v'_1 \rangle = 6.609$ ,  $\langle h, v'_2 \rangle = -3.1915$ ,  $\langle h, v'_3 \rangle = 3.195$ ,  $\langle h, v'_4 \rangle = 0.38666$ .

Then by doing the calculation, we have

$$h(t) = 1.99999t + 1.00000.$$

Which is what we expected.

Done.

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