

# MATH2040C Homework 7

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## 1 Section 6.3, Q3(c)

For each of the following inner product spaces  $V$  and linear operators  $T$  on  $V$ , evaluate  $T^*$  at the given vector in  $V$ .

(c)  $V = P_1(R)$  with  $\langle f, g \rangle = \int_{-1}^1 f(t)g(t) dt$ ,  $T(f) = f' + 3f$ ,  
 $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for  $V$ .

A basis for  $V$  is  $\alpha = \{1, t\}$ . Note that  $\int_{-1}^1 1 \cdot t dt = 0$ . Therefore  $\alpha$  is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis  $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$ .

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^n \overline{\langle T(v_i), g(t) \rangle} v_i.$$

With  $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$ . And  $T(\frac{\sqrt{3}t}{\sqrt{2}}) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t$ .

Therefore,

$$T^*(g(t)) = \frac{3}{2} \int_{-1}^1 g(t) dt + \frac{3}{2} t \int_{-1}^1 (1 + 3t) g(t) dt.$$

The given vector is  $f(t) = 4 - 2t$ . Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

## 2 Section 6.3, Q13

Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ . Prove the following results.

- (a)  $N(T^*T) = N(T)$ . Deduce that  $\text{rank}(T^*T) = \text{rank}(T)$ .
- (b)  $\text{rank}(T) = \text{rank}(T^*)$ . Deduce from (a) that  $\text{rank}(TT^*) = \text{rank}(T)$ .
- (c) For any  $n \times n$  matrix  $A$ ,  $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$ .

### 2.1 (a)

Note that  $\forall x \in N(T)$ ,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore  $x \in N(T^*T)$ . Hence  $N(T) \subset N(T^*T)$ .

For all  $y \in N(T^*T)$ , consider the norm of  $Ty$  :

$$\|Ty\|^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that  $Ty = 0$ . Therefore  $y \in N(T)$ . Hence  $N(T^*T) \subset N(T)$ .

Based on all above,  $N(T^*T) = N(T)$ .

Recall that  $T \in \mathcal{L}(V)$ . Hence  $T : V \rightarrow V$ . And according to  $\forall y \in V$ ,

$$T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y \rangle} v_i.$$

We know that  $T^* : V \rightarrow V$ . Therefore  $T^*T : V \rightarrow V$ .

Applying the rank nullity theorem, we have that

$$\dim V = \text{rank}(T^*T) + \dim N(T^*T), \quad \dim V = \text{rank}(T) + \dim N(T).$$

Using the just proved fact  $N(T^*T) = N(T)$ , we can simply deduce

$$\text{rank}(T^*T) = \text{rank}(T).$$

### 2.2 (b)

By changing name of the identity in (a), we can have  $N(TT^*) = N(T^*)$  and  $\text{rank}(TT^*) = \text{rank}(T^*)$ .

Notice that

$$\text{rank}(T) = \text{rank}[T]_\beta = \text{rank}[T]_\beta^* = \text{rank}[T^*]_\beta = \text{rank}(T^*).$$

And then  $\text{rank}(TT^*) = \text{rank}(T)$  follows.

### 2.3 (c)

From (b),  $\text{rank}(AA^*) = \text{rank}(A)$  follows naturally.

And note that  $(AA^*)^* = A^*A$ , then

$$\text{rank}(AA^*) = \text{rank}(A^*A).$$

Therefore,

$$\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A).$$

Done.

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## 3 Section 6.3, Q14

Let  $V$  be an inner product space, and let  $y, z \in V$ . Define  $T: V \rightarrow V$  by  $T(x) = \langle x, y \rangle z$  for all  $x \in V$ . First prove that  $T$  is linear. Then show that  $T^*$  exists, and find an explicit expression for it.

First we would prove that  $T$  is linear.  $\forall x_1, x_2 \in V, \forall c \in F$ ,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle z = \langle cx_1, y \rangle z + \langle x_2, y \rangle z = c\langle x_1, y \rangle z + \langle x_2, y \rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore,  $T$  is linear.

From course not we directly construct  $\forall x \in V$ ,

$$T^*(x) = \sum_{i=1}^n \overline{\langle T(v_i), x \rangle} v_i = \sum_{i=1}^n \overline{\langle \langle v_i, y \rangle z, x \rangle} v_i = \overline{\langle z, x \rangle} \sum_{i=1}^n \overline{\langle v_i, y \rangle} v_i = \overline{\langle z, x \rangle} y.$$

Recall that  $y = I(y) = I^*(y) = \sum_{i=1}^n \overline{\langle I(v_i), y \rangle} v_i = \sum_{i=1}^n \overline{\langle v_i, y \rangle} v_i$ . Hence the last equality holds properly.

## 4 Section 6.3, Q15

**Definition.** Let  $T: V \rightarrow W$  be a linear transformation, where  $V$  and  $W$  are finite-dimensional inner product spaces with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ , respectively. A function  $T^*: W \rightarrow V$  is called an **adjoint** of  $T$  if  $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$  for all  $x \in V$  and  $y \in W$ .

15. Let  $T: V \rightarrow W$  be a linear transformation, where  $V$  and  $W$  are finite-dimensional inner product spaces with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ , respectively. Prove the following results.
- (a) There is a unique adjoint  $T^*$  of  $T$ , and  $T^*$  is linear.
  - (b) If  $\beta$  and  $\gamma$  are orthonormal bases for  $V$  and  $W$ , respectively, then  $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$ .
  - (c)  $\text{rank}(T^*) = \text{rank}(T)$ .
  - (d)  $\langle T^*(x), y \rangle_1 = \langle x, T(y) \rangle_2$  for all  $x \in W$  and  $y \in V$ .
  - (e) For all  $x \in V$ ,  $T^*T(x) = 0$  if and only if  $T(x) = 0$ .

### 4.1 (a)

Define  $g: V \rightarrow F$  by  $g(x) = \langle T(x), y \rangle_2$ . Obviously, note that  $g$  is linear for sure.

Then apply the theorem 6.8, there exists a unique vector  $y' \in W$  such that

$$g(x) = \langle x, y' \rangle_1.$$

Recall  $g$ 's definition, we have  $\langle T(x), y \rangle_2 = \langle x, y' \rangle_1, \forall x \in V$ .

Define that  $T^*: W \rightarrow V$  by  $T^*(y) = y'$ .

Then we have  $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$ .

Hence the wanted  $T^*$  function exists, and because the  $y'$  is "unique" as mentioned above (for any  $y$ ), then the  $T^*$  is also unique.

Then we prove the linearity of  $T^*$ .  $\forall y_1, y_2 \in W$  and  $\forall c \in F$ ,

$$\begin{aligned} \langle x, T^*(cy_1 + y_2) \rangle_1 &= \langle T(x), cy_1 + y_2 \rangle_2 = \langle x, cT^*(y_1) \rangle_1 + \langle x, T^*(y_2) \rangle_1 \\ &= \langle x, cT^*(y_1) + T^*(y_2) \rangle_1, \forall x \in V. \end{aligned}$$

Therefore  $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$ . Hence  $T^*$  is linear.

## 4.2 (b)

For such kind of problem, we need to compare the two matrices entry wise.

Denote  $\beta = \{v_1, v_2, \dots, v_n\}$ ,  $\gamma = \{w_1, w_2, \dots, w_m\}$ .

Inspect  $[T^*]_{\gamma}^{\beta} = [T^*(w_1), T^*(w_2), \dots, T^*(w_m)]_{\beta}$ .

Note that  $T^*(w_j) = \sum_{i=1}^n \overline{\langle v_i, T^*(w_j) \rangle_1} v_i$ ,  $\forall j = 1, 2, \dots, m$ .

From this we know that the i-th row, j-th column of  $[T^*]_{\gamma}^{\beta}$  is  $\overline{\langle v_i, T^*(w_j) \rangle_1}$ .

Inspect  $[T]_{\beta}^{\gamma} = [T(v_1), T(v_2), \dots, T(v_n)]_{\gamma}$ .

Note that  $T(v_i) = \sum_{k=1}^m \overline{\langle w_k, T(v_i) \rangle_2} w_k$ ,  $\forall i = 1, 2, \dots, n$ .

Therefore, the i-th column, k-th row of  $[T]_{\beta}^{\gamma}$  is  $\overline{\langle w_k, T(v_i) \rangle_2}$ .

Hence, for  $([T]_{\beta}^{\gamma})^*$ , the i-th row, j-th column is  $\langle w_j, T(v_i) \rangle_2$ .

What remains to be done is to show

$$\langle w_j, T(v_i) \rangle_2 = \overline{\langle v_i, T^*(w_j) \rangle_1}.$$

This is equivalent to show  $\overline{\langle w_j, T(v_i) \rangle_2} = \langle v_i, T^*(w_j) \rangle_1$ .

Note that the L.H.S. =  $\overline{\langle T(v_i), w_j \rangle_2}$  = R.H.S. from the definition of  $T^*$ . Hence this is proved.

## 4.3 (c)

$\text{rank}(T^*) = \text{rank}([T^*]_{\gamma}^{\beta})$  and  $\text{rank}(T) = \text{rank}([T]_{\beta}^{\gamma}) = \text{rank}(([T]_{\beta}^{\gamma})^*)$ .

Followed from (b), we have  $\text{rank}(T^*) = \text{rank}(T)$ .

## 4.4 (d)

We want to prove  $\langle T^*(y), x \rangle_1 = \langle y, T(x) \rangle_2$ ,  $\forall y \in W, x \in V$ . which is equivalent to prove  $\langle x, T^*(y) \rangle_1 = \langle T(x), y \rangle_2$ .

And L.H.S. =  $\langle T(x), y \rangle_2$  followed from the definition of  $T^*$ .

Done.

## 4.5 (e)

It is suffice to prove  $N(T) = N(T^*T)$ . And it is obvious to see that

$$N(T) \subset N(T^*T).$$

Take any  $x$  such that  $T^*Tx = 0$ . We have  $\langle Tx, Tx \rangle_2 = \langle x, T^*Tx \rangle_1 = 0$ . Which implies that  $Tx = 0$ . Hence  $x \in N(T)$ , and  $N(T^*T) = N(T)$ .

Done.

## 5 Section 6.4, Q2(d)