MATH2040C Homework 7

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1 Section 6.3, Q3(c)

For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(c) $\mathsf{V} = \mathsf{P}_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

The first thing we need to do is to find a orthonormal basis for V.

A basis for V is $\alpha = \{1, t\}$. Note that $\int_{-1}^{1} 1 \cdot t\, dt = 0$. Therefore α is an orthogonal basis. Applying the Gram-Schmidt process, we can generate an orthonormal basis $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}\}$.

Then according to Remark 16.3, we can have

$$T^*(g(t)) = \sum_{i=1}^{n} \overline{\langle T(v_i), g(t)\rangle} v_i.$$

With $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$. And $T(\sqrt{\frac{3}{2}}t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}}t..$

Therefore,

$$T^*(g(t)) = \frac{3}{2}\int_{-1}^{1} g(t)dt + \frac{3}{2}t\int_{-1}^{1}(1+3t)g(t)dt.$$

The given vector is $f(t) = 4 - 2t$. Hence the answer should be

$$T^*(4 - 2t) = 12 + 6t.$$

Done.

2 Section 6.3, Q13

Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$. Deduce that $\text{rank}(\mathsf{T}^*\mathsf{T}) = \text{rank}(\mathsf{T})$.

(b) $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^*)$. Deduce from (a) that $\text{rank}(\mathsf{T}\mathsf{T}^*) = \text{rank}(\mathsf{T})$.

(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

2.1 (a)

Note that $\forall x \in N(T)$,

$$T^*Tx = T^*(Tx) = T^*(0) = 0.$$

Therefore $x \in N(T^*T)$. Hence $N(T) \subset N(T^*T)$.
Forall $y \in N(T^*T)$, consider the norm of Ty :

$$||Ty||^2 = \langle Ty, Ty \rangle = \langle y, T^*Ty \rangle = \langle y, 0 \rangle = 0.$$

Which implies that $Ty = 0$. Therefore $y \in N(T)$. Hence $N(T^*T) \subset N(T)$.
Based on all above, $N(T^*T) = N(T)$.
Recall that $T \in \mathcal{L}(V)$. Hence $T : V \to V$. And according to $\forall y \in V$,

$$T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y \rangle} v_i.$$

We know that $T^* : V \to V$. Therefore $T^*T : V \to V$.
Applying the rank nullity theorem, we have that

$$\dim V = rank(T^*T) + \dim N(T^*T)\ ,\ \ \dim V = rank(T) + \dim N(T).$$

Using the just proved fact $N(T^*T) = N(T)$, we can simply deduce

$$rank(T^*T) = rank(T).$$

2.2 (b)

By changing name of the identity in (a), we can have $N(TT^*) = N(T^*)$ and $rank(TT^*) = rank(T^*)$.
Notice that

$$rank(T) = rank[T]_\beta = rank[T]_\beta^* = rank[T^*]_\beta = rank(T^*).$$

And then $rank(TT^*) = rank(T)$ follows.

2.3 (c)

From (b), $rank(AA^*) = rank(A)$ follows naturally.
And note that $(AA^*)^* = A^*A$, then

$$rank(AA^*) = rank(A^*A).$$

Therefore,

$$rank(AA^*) = rank(A^*A) = rank(A).$$

Done.

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3 Section 6.3, Q14

Let V be an inner product space, and let $y, z \in \mathsf{V}$. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(x) = \langle x, y\rangle z$ for all $x \in \mathsf{V}$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

First we would prove that T is linear. $\forall x_1, x_2 \in V, \forall c \in F$,

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y\rangle z = \langle cx_1, y\rangle z + \langle x_2, y\rangle z = c\langle x_1, y\rangle z + \langle x_2, y\rangle z.$$

The equalities are deduced from the properties of inner product. And hence

$$T(cx_1 + x_2) = cT(x_1) + T(x_2).$$

Therefore, T is linear.
From course not we directly construct $\forall x \in V$,

$$T^*(x) = \sum_{i=1}^{n} \overline{\langle T(v_i), x\rangle} v_i = \sum_{i=1}^{n} \overline{\langle\langle v_i, y\rangle z, x\rangle} v_i = \overline{\langle z, x\rangle} \sum_{i=1}^{n} \overline{\langle v_i, y\rangle} v_i = \overline{\langle z, x\rangle} y.$$

Recall that $y = I(y) = I^*(y) = \sum_{i=1}^{n} \overline{\langle I(v_i), y\rangle} v_i = \sum_{i=1}^{n} \overline{\langle v_i, y\rangle} v_i$. Hence the last equality holds properly.

4 Section 6.3, Q15

Definition. *Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. A function $\mathsf{T}^*: \mathsf{W} \to \mathsf{V}$ is called an* **adjoint** *of T if $\langle \mathsf{T}(x), y \rangle_2 = \langle x, \mathsf{T}^*(y) \rangle_1$ for all $x \in \mathsf{V}$ and $y \in \mathsf{W}$.*

15. Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

(a) There is a unique adjoint T^* of T, and T^* is linear.
(b) If β and γ are orthonormal bases for V and W, respectively, then $[\mathsf{T}^*]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^*$.
(c) $\text{rank}(\mathsf{T}^*) = \text{rank}(\mathsf{T})$.
(d) $\langle \mathsf{T}^*(x), y \rangle_1 = \langle x, \mathsf{T}(y) \rangle_2$ for all $x \in \mathsf{W}$ and $y \in \mathsf{V}$.
(e) For all $x \in \mathsf{V}$, $\mathsf{T}^*\mathsf{T}(x) = \mathit{0}$ if and only if $\mathsf{T}(x) = \mathit{0}$.

4.1 (a)

Define $g : V \to F$ by $g(x) = \langle T(x), y \rangle_2$. Obviously, note that g is linear for sure.

Then apply the theorem 6.8, there exists a unique vector $y' \in W$ such that

$$g(x) = \langle x, y' \rangle_1.$$

Recall g's definition, we have $\langle T(x), y \rangle_2 = \langle x, y' \rangle_1,\ \forall x \in V.$

Define that $T^* : W \to V$ by $T^*(y) = y'$.

Then we have $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$.

Hence the wanted T^* function exists, and becasue the y' is "unique" as mentioned above (for any y), then the T^* is also unique.

Then we prove the linearity of T^*. $\forall y_1, y_2 \in W$ and $\forall c \in F$,

$$\langle x, T^*(cy_1 + y_2) \rangle_1 = \langle T(x), cy_1 + y_2 \rangle_2 = \langle x, cT^*(y_1) \rangle_1 + \langle x, T^*(y_2) \rangle_1$$

$$= \langle x, cT^*(y_1) + T^*(y_2) \rangle_1, \forall x \in V.$$

Therefore $T^*(cy_1 + y_2) = cT^*(y_1) + T^*(y_2)$. Hence T^* is linear.

4.2 (b)

For such kind of problem, we need to compare the two matrices entry wise. Denote $\beta = \{v_1, v_2, \dots, v_n\}, \gamma = \{w_1, w_2, \dots, w_m\}$.

Inspect $[T^*]_\gamma^\beta = [T^*(w_1), T^*(w_2), \dots, T^*(w_m)]_\beta$.

Note that $T^*(w_j) = \sum_{i=1}^n \overline{\langle v_i, T^*(w_j)\rangle_1} v_i,\ \forall j = 1, 2, \dots, m.$

From this we know that the i-th row, j-th column of $[T^*]_\gamma^\beta$ is $\overline{\langle v_i, T^*(w_j)\rangle_1}$.

Inspect $[T]_\beta^\gamma = [T(v_1), T(v_2), \dots, T(v_n)]_\gamma$.

Note that $T(v_i) = \sum_{k=1}^m \overline{\langle w_k, T(v_i)\rangle_2} w_k,\ \forall i = 1, 2, \dots, n.$

Therefore, the i-th column, k-th row of $[T]_\beta^\gamma = \overline{\langle w_k, T(v_i)\rangle_2}$.

Hence, for $([T]_\beta^\gamma)^*$, the i-th row, j-th column is $\langle w_j, T(v_i)\rangle_2$.

What remains to be done is to show

$$\langle w_j, T(v_i)\rangle_2 = \overline{\langle v_i, T^*(w_j)\rangle_1}.$$

This is equivalent to show $\overline{\langle w_j, T(v_i)\rangle_2} = \langle v_i, T^*(w_j)\rangle_1$.

Note that the L.H.S.=$\langle T(v_i), w_j\rangle_2$=R.H.S. from the definition of T^*. Hence this is proved.

4.3 (c)

$rank(T^*) = rank([T^*]_\gamma^\beta)$ and $rank(T) = rank([T]_\beta^\gamma) = rank(([T]_\beta^\gamma)^*)$.

Followed from (b), we have $rank(T^*) = rank(T)$.

4.4 (d)

We want to prove $\langle T^*(y), x\rangle_1 = \langle y, T(x)\rangle_2, \forall y \in W, x \in V$. which is equivalent to prove $\langle x, T^*(y)\rangle_1 = \langle T(x), y\rangle_2$.

And L.H.S.=$\langle T(x), y\rangle_2$ followed from the definition of T^*.

Done.

4.5 (e)

It is suffice to prove $N(T) = N(T^*T)$. And it is obvious to see that

$$N(T) \subset N(T^*T).$$

Take any x such that $T^*Tx = 0$. We have $\langle Tx, Tx\rangle_2 = \langle x, T^*Tx\rangle_1 = 0$. Which implies that $Tx = 0$. Hence $x \in N(T)$, and $N(T^*T) = N(T)$.

Done.

5 Section 6.4, Q2(d)

For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

Let $\{1, t, t^2\}$ be a basis for $P_2(\mathbb{R})$, then apply Gram-Schmidt process upon it, we can have an orthonormal basis $\beta = \{1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6})\}$.

First, we claim that T is not self-adjoint, by the spectral theorem, T is self-adjoint iff, T is diagnoalizable, which implies that it will lead to the eigenspace decomposition of V. Note that there is only one eigenvalue of T, which is 0 and the only corresponding set of eigenvectors is $span\{1\}$. It is obvious that $E_0 \neq V$, since $t^2 \notin E_0$. Therefore T is not self-adjoint and meanwhile, it is impossible to derive a orthonormal basis of eigenvectors of T for V.

Also, T is not normal.

6 Section 6.4, Q7

Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(a) If T is self-adjoint, then T_W is self-adjoint.
(b) $W^\perp$ is T^*-invariant.
(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

6.1 (a)

Denote $\dim V = n, \dim W = m$, with $m \leq n$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V, $\beta_W = \{v_1, v_2, \ldots, v_m\}$ be an orthonormal basis for W.

$\forall y \in W$, $T_W(y) = T(y) = \sum_{i=1}^n \langle T(y), v_i\rangle v_i = T^*(y)$. Because T is a self adjoint opeartor.

From the construction rule of adjoint, we have $T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y\rangle} v_i$.

From the question, we know that W is T-invarian, then $T_W(y) \in W$. Combined with v_i's are linear independent, then

$$T_W(y) = \sum_{i=1}^m \overline{\langle T(v_i), y\rangle} v_i = \sum_{i=1}^m \overline{\langle T_W(v_i), y\rangle} v_i.$$

Note that the R.H.S. is the definition of $T_W^*(y)$. Hence

$$T_W(y) = T_W^*(y), \forall y \in W.$$

6.2 (b)

Make it clear that what we want is $T^*(W^\perp) \subset W^\perp$. By the construction of T^*, we have

$$T^*(y) = \sum_{i=1}^n \overline{\langle T(v_i), y\rangle} v_i.$$

Note that $\forall y \in W^\perp, y = \sum_{j=m+1}^n \langle y, v_i\rangle v_i$.
Therefore

$$T^*(y) = \sum_{i=1}^n v_i(\sum_{j=m+1}^n \langle y, v_j\rangle\langle v_j, T(v_i)\rangle).$$

Recall that W is T-invariant, hence $T(v_i) \in W$, then $\langle v_j, T(v_i)\rangle = 0, \forall i \leq m$.
Hence $T^*(y) = \sum_{i=m+1}^n v_i \sum_{j=m+1}^n \langle y, v_j\rangle\langle v_j, T(v_i)\rangle \in W^\perp$.

6.3 (c)

$\forall y \in W$, $(T_W)^*(y) = \sum_{i=1}^{m} \overline{\langle T_W(v_i), y\rangle} v_i$. Because $(T_W)^*(y) \in W$ assumed in question.

Note that $y \in W$, then $y \in V$. Hence we can use the original definition of T. We then have

$$(T^*)_W(y) = T^*(y) = \sum_{i=1}^{n} \overline{\langle T(v_i), y\rangle} v_i.$$

Inspect the $\langle T(v_i), y\rangle$ terms, here $y \in W$. If $i = 1, 2, \ldots, m$ then $T(v_i) \in W$, when $i = m+1, m+2, \ldots, n$ then $v_i \in W^\perp$ and hence $T(v_i) \in W^\perp$. Therefore $\langle T(v_i), y\rangle = 0, \forall i = m+1, m+2, \ldots, n$.

Hence, we have

$$(T^*)_W(y) = T^*(y) = \sum_{i=1}^{m} \overline{\langle T(v_i), y\rangle} v_i = \sum_{i=1}^{m} \overline{\langle T_W(v_i), y\rangle} v_i = (T_W)^*(y).$$

Done.

6.4 (d)

Note that as mentioned in the question, W is both $T-$ and T^*- invariant. Hence $\forall y \in W$,

$$T_W(T_W)^*(y) = T_W(T^*(y)).$$

Where $T^*(y) \in W$. Then $T_W(T_W)^*(y) = TT^*(y)$, where $TT^*(y) \in W$ as well.

On the other hand,

$$(T_W)^*T_W(y) = (T_W)^*T(y) = T^*T(y).$$

Which is valid for similar reasons.

Recall that T is normal. Hence $(T_W)^*T_W(y) = T_W(T_W)^*(y), \forall y \in W$.

Therefore, T_W is normal.

7 Section 6.4, Q9

Let T be a normal operator on a finite-dimensional inner product space V. Prove that N(T) = N(T*) and R(T) = R(T*). *Hint:* Use Theorem 6.15 and Exercise 12 of Section 6.3.

From theorem 6.15, we know that $||T(x)|| = ||T^*(x)||, \forall x \in V$.

$\forall x \in N(T)$, then $T(x) = 0$, which is equivalent to $||T(x)|| = 0$. Then $||T^*(x)|| = 0$, which is equivalent to $T^*(x) = 0$. Thus $x \in N(T^*)$. Note that each step above is revertible, then $N(T) = N(T^*)$.

Using the question 12 of section 6.3, we then have $N(T)^\perp = R(T^*)$ and $N(T^*)^\perp = R((T^*)^*) = R(T)$.

As $N(T) = N(T^\perp)$, then $R(T) = R(T^*)$.

Done.

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8 Section 6.5, Q2(c)

For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

(c) $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

By solving its characteristic polynomial, we have 2 eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 8$.

And we can have $v_1 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$.

Normalize them, we then have $v_1' = \frac{1}{\sqrt{3}} v_1, v_2' = \frac{1}{\sqrt{3}} v_2$.

Then let $P = \begin{pmatrix} v_1' & v_2' \end{pmatrix}$.

And hence the $P^* = \begin{pmatrix} -\overline{v_1'}- \\ -\overline{v_2'}- \end{pmatrix}$.

Note that $P^*P = I_2$. Therefore P is a unitary matrix.

Then

$$P^*AP = P^*(\lambda_1 v_1', \lambda_2 v_2') = \begin{pmatrix} -\overline{v_1'}- \\ -\overline{v_2'}- \end{pmatrix} (\lambda_1 v_1', \lambda_2 v_2') = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

9 Section 6.5, Q6

Let V be the inner product space of complex-valued continuous functions on $[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$

Let $h \in \mathsf{V}$, and define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(f) = hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for $0 \le t \le 1$.

9.1 Only If

We would like to prove it by contradiction. Note that now we have the condition of T being unitary opeartor. This implies that

$$||T(f)|| = ||f||, \forall f \in V.$$

Doing the inner product, we left with

$$\int_0^1 |f(t)|^2(|h(t)|^2 - 1)dt = 0.$$

Recall that all of the h, f are continuous functions defined on compact interval. On which we can apply the boundedness theorem to obtain that

$$\exists M > 0, s.t. \; -M < \int_0^1 |f(t)|^2(|h(t)|^2 - 1)dt < M$$

Suppose $|h(t)| \neq 1$ for some $t_0 \in [0,1]$ (WLOG, we suppose $h(t_0) > 1$.) we will construct a $g(t)$ s.t.

$$\int_0^1 |g(t)|^2(|h(t)|^2 - 1)dt \neq 0.$$

Note that $\exists$ a interval I such that $\forall t \in I, h(t) \geq \frac{1+h(t_0)}{2}$. Where I is defined in the following way: $\exists \delta > 0$ s.t.

$$I = [t_0 - \delta, t_0 + \delta] \;\; (t_0 \in (0,1))$$

$$I = [0, t_0 + \delta] \;\; (t_0 = 0)$$
$$I = [t_0 - \delta, 1] \;\; (t_0 = 1).$$

Applying the boundedness theorem on I, then $\int_I |f(t)|^2(|h(t)|^2 - 1)dt$ is also finite. Which deduces that $\int_{[0,1]-I} |f(t)|^2(|h(t)|^2 - 1)dt$ is a subtraction of 2 finite values, hence is also a finite number, denote it as M_1.

Note that

$$\int_I |f(t)|^2(|h(t)|^2 - 1)dt \geq (h(t_0) - 1) \int_I |f(t)|^2 dt.$$

Manipulate the value of $f(t), t \in I$ such that $\int_I |f(t)|^2 dt > \frac{|M_1|+1}{h(t_0)-1}$.
Thus we have

$$\int_0^1 |f(t)|^2(|h(t)|^2 - 1)dt > M_1 + |M_1| + 1 > 0.$$

Which is a contradiction, hence $\forall t \in [0,1], |h(t)| = 1$ at the first place.

9.2 If