MATH2050 HW3

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1 P69 Q9

We claim that the limit of $(\sqrt{n}y_n)$ is $\frac{1}{2}$. Following is to prove the claim.
Note that $y_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

$$\therefore \forall \epsilon > 0, |\sqrt{n}y_n - \frac{1}{2}| = |\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{2}| = \frac{1}{2}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{2}\frac{1}{(\sqrt{n+1}+\sqrt{n})^2}.$$

$$\therefore \forall \epsilon > 0, |\sqrt{n}y_n - \frac{1}{2}| < \frac{1}{2}\frac{1}{(2\sqrt{n})^2} = \frac{1}{8n}.$$

So, $\forall \epsilon > 0, \forall n > N > \frac{1}{8\epsilon}$, we have $|\sqrt{n}y_n - \frac{1}{2}| < \epsilon$.
Hence, we proved that $(\sqrt{n}y_n)$ is convergent and the limit is $\frac{1}{2}$.

2 P69 Q20

(i) As $L := \lim_{n\to\infty}(x_n^{1/n}) < 1$, we have

$$\forall \epsilon > 0, \exists N \ s.t. \ \forall n > N, |x_n^{1/n} - L| < \epsilon.$$

That is to say, $\forall n > N$, $L - \epsilon < x_n^{1/n} < L + \epsilon$.
Take $\epsilon = \frac{1-L}{2}$, and notice that (x_n) is a positive real sequence, we have

$$0 < x_n < (L + \frac{1-L}{2})^n = (\frac{1+L}{2})^n, \ \forall n > N.$$

So there exists a real number $r = \frac{1+L}{2}$ such that $x_n \in (0, r^n), \forall n > N$.

(ii) Consider $r = \frac{1+L}{2}$, and from (i), we have $0 < x_n < r^n$.
Note that $\forall \epsilon > 0, \forall n > N_1 = \frac{\ln \epsilon}{\ln r}, r^n < \epsilon$. Therefore $\lim_{n\to\infty} r^n = 0$.
Hence,

$$0 \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} r^n = 0.$$

Which deduces

$$\lim_{n\to\infty} x_n = 0.$$

3 P69 Q22

Of course (x_n) is convergent.
Denote the limit of (x_n) to be L. Note that $\forall \epsilon > 0$ we have an N_1 s.t.$\forall n > N_1$,

$$|x_n - L| < \frac{\epsilon}{2}.$$

From the question, we have $\forall \epsilon > 0, \exists M$ s.t. $\forall n > M$,

$$|x_n - y_n| < \frac{\epsilon}{2}.$$

Note that $\forall n > \max N_1, M$,

$$|y_n - L| < |y_n - x_n| + |x_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence we proved that (y_n) is convergent, and further, its limit is the same as x_n's.