MATH2050A Homework 7

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1 Q7 P134

Let $f : [0, 1] \to \mathbb{R}$ be $\forall x \in [0, 1]$:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

Note that $\forall x \in \mathbb{Q}$, we can find a sequence (x_n) s.t. $\lim_{n\to\infty} x_n = x$ and $x_n \notin \mathbb{Q}$, which implies

$$\lim_{n\to\infty} f(x_n) = -1 < f(x) = 1.$$

Hence $\forall x \in \mathbb{Q}$, f is discontinuous. Also note that $\forall x \notin \mathbb{Q}$, we can find a sequence (x_n) s.t. $\lim_{n\to\infty} x_n = x$ and $x_n \in \mathbb{Q}$, which implies

$$\lim_{n\to\infty} f(x_n) = 1 > f(x) = -1.$$

Therefore $\forall x \notin \mathbb{Q}$, f is discontinuous.

Consider $|f|$, which is $|f(x)| = 1, \forall x \in [0, 1]$. So $|f|$ is constant on its domain and continuous.

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According to the definition, $h(x) = \sup\{f(x), g(x)\}$.

$\forall x \in \mathbb{R}$, if $f(x) \leq g(x)$, then $h(x) = \sup\{f(x), g(x)\} = g(x)$.

Note that $\frac{1}{2}(h(x) + g(x)) + \frac{1}{2}|f(x) - g(x)| = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}(g(x) - f(x)) = g(x)$.

Hence $h(x) = \frac{1}{2}(h(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|, \forall x \in \{x \in \mathbb{R} : f(x) \leq g(x)\}$

If $f(x) > g(x)$, then $h(x) = \sup\{f(x), g(x)\} = f(x)$. Note that $\frac{1}{2}(h(x)+g(x))+\frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}(f(x) - g(x)) = f(x)$.

Hence $h(x) = \frac{1}{2}(h(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|, \forall x \in \{x \in \mathbb{R} : f(x) > g(x)\}$

So, $h(x) = \frac{1}{2}(h(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|, \forall x \in \mathbb{R}$. Then (i) is proved.

Note that by the question, $f(x)$ and $g(x)$ are both continuous at c, hence $f(x) - g(x)$ is continuous at c.

Therefore $|f(x) - g(x)|$ is continuous at c. Then $h(x) = \frac{1}{2}(h(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|, \forall x \in \mathbb{R}$ is continuous at c. So (ii) is proved.