MATH2050A HW1

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1 Q4 (Section 2.3)

Note that n appears in the denominator, so $n \neq 0$. Hence $n \in \mathbb{N}^*$.
If n is odd, $1 - \frac{(-1)^n}{n} = 1 + \frac{1}{n}$, and if n is even, $1 - \frac{(-1)^n}{n} = 1 - \frac{1}{n}$.
So we can write $S_4 = S_{41} \cup S_{42}$, where $S_{41} = \{1 + \frac{1}{n} | n = 2k+1, k \in \mathbb{N}\}$ and $S_{42} = \{1 - \frac{1}{n} | n = 2k, k \in \mathbb{N}^*\}$.

We claim that $2 = \sup S_4$.
(i) Note that $S_4 = S_{41} \cup S_{42}$.
$\forall x \in S_{42}$, by definition, $x \leq 1 < 2$, hence $x \leq 2$.
$\forall x \in S_{41}$, there exists $m \in \mathbb{N}$ and m is odd s.t. $x = 1 + \frac{1}{m}$. Note that $x \leq 2$ is equivalent to $1 \leq m$, which is true trivially.
So, $\forall x \in S_4, x \leq 2$. Hence 2 is an upper bound of S_4.

(ii)Note that $1 - \frac{(-1)^n}{n} = 1 + \frac{1}{n} = 2$ when $n = 1$, so $2 \in S_4$. $\forall L$ is an upper bound of S_4, $2 \leq L$ holds for sure, because 2 itself is an element in S_4. So 2 is the smallest upper bound of S_4.
So, by (i) and (ii), we have $2 = \sup S_4$.

We claim that $\frac{1}{2} = \inf S_4$.
(iii) Note that $S_4 = S_{41} \cup S_{42}$.
$\forall x \in S_{41}$, by definition, $x \geq 1 > \frac{1}{2}$, hence $x \geq \frac{1}{2}$.
$\forall x \in S_{42}$, there exists $m \in \mathbb{N}^*$ and m is even s.t. $x = 1 - \frac{1}{m}$. Note that $x \geq \frac{1}{2}$ is equivalent to $1 - \frac{1}{m} \geq \frac{1}{2}$,i.e. $m \geq 2$, which is true trivially.
So, $\forall x \in S_4, x \geq \frac{1}{2}$. Hence $\frac{1}{2}$ is an lower bound of S_4.

(iv) Suppose l is a lower bound of S_4. Note that $1 - \frac{(-1)^n}{n} = \frac{1}{2}$ when $n = 2$. Hence $\frac{1}{2} \in S_4$.
Because l is lower bound of S_4 as supposed, $l \leq \frac{1}{2}$.
So by (iii) and (iv), $\frac{1}{2}$ is the greatest lower bound of S_4. And we have $\frac{1}{2} = \inf S_4$.

2 Q10 (Section 2.3)

If A and B are bounded, $\forall x \in A, a1 \leq x \leq a2$. $\forall y \in B, b1 \leq y \leq b2$.
$\therefore \forall z \in A \cup B, \min\{a1, b1\} \leq z \leq \max\{a2, b2\}$, which implies $A \cup B$ is bounded.

Note that the R.H.S. of the equation, $\sup\{\sup A, \sup B\} = \max\{\sup A, \sup B\}$.

Denote $\sup\{A \cup B\} = L$. $\forall x \in A \cup B, x \leq L$. Hence $\forall x \in A, x \leq L$, L is an upper bound of A.

Hence $L = \sup\{A \cup B\} \geq \sup A$. For the same reasoning, $L = \sup\{A \cup B\} \geq \sup B$.

$$\therefore \sup\{A \cup B\} \geq \max\{\sup A, \sup B\} = \sup\{\sup A, \sup B\}$$

Since $\sup A$ and $\sup B$ are two numbers, we consider the situation that $\sup A \geq \sup B$. Then $\sup\{\sup A, \sup B\} = \sup A$.

$\forall x \in A \cup B$, if $x \in A$, by definition, $x \leq \sup A$, if $x \in B$, by definition, $x \leq \sup B \leq \sup A$. So $\forall x \in A \cup B$, $x \leq \sup A$.

So $\sup A$ is an upper bound of $A \cup B$, hence

$$\sup\{\sup A, \sup B\} = \sup(A) \geq \sup\{A \cup B\}.$$

And when $\sup A \leq \sup B$,

$$\sup\{\sup A, \sup B\} = \sup(B) \geq \sup\{A \cup B\}.$$

Combining all above, we have the desired equation proved, which is

$$\sup\{\sup A, \sup B\} = \sup\{A \cup B\}.$$

3 Q12 (Section 2.3)

From Q10 we know that $\sup\{\sup A, \sup B\} = \sup\{A \cup B\}$.

So, under the given situation, we have

$$\sup\{\sup S, \sup\{u\}\} = \sup\{S \cup \{u\}\}.$$

Note that $\{u\}$ is a one-element-set, $\sup\{u\} = u$. Because $u \leq u$ (u is upper bound). And $\forall$ upper bound L of $\{u\}$, $u \leq L$, for u itself is inside the set $\{u\}$.

Also the question gives us $s^* = \sup S$, plug this and $\sup\{u\} = u$ into the above equation, we have

$$\sup\{s^*, u\} = \sup\{S \cup \{u\}\}.$$