# MATH2050A Homework 8

ZHENG Weijia William, 1155124322

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## 1 Q2 P148

(i)
$\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{2}$. Note that

$$|f(x_1) - f(x_2)| = |\frac{1}{x_1^2} - \frac{1}{x_2^2}| = |x_1 - x_2|\frac{1}{|x_1x_2|}\frac{|x_1 + x_2|}{|x_1x_2|} \leq |x_1 - x_2||\frac{1}{x_1} + \frac{1}{x_2}| \leq 2\delta = \epsilon,$$

as $x_1, x_2 \in [1, +\infty)$ and $|x_1 - x_2| < \delta$.
Therefore, $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, +\infty)$.

(ii)
Note that $\forall x_1, x_2 \in [1, +\infty)$,

$$|f(x_1) - f(x_2)| = |x_1 - x_2|\frac{1}{|x_1x_2|}\frac{|x_1 + x_2|}{|x_1x_2|}.$$

Note that if $x_1 = \frac{1}{n}, x_2 = \frac{1}{1+n}, n = 1, 2, 3...$ This implies

$$|f(x_1) - f(x_2)| = 2n + 1 \geq 1, \forall n.$$

Choose $\epsilon = 1, \forall \delta > 0, \exists N\ s.t.\ \frac{1}{N} < \delta$. Choose $x_1 = \frac{1}{N}, x_2 = \frac{1}{N+1}$. Then note that $|x_1 - x_2| = \frac{1}{N} - \frac{1}{N+1} < \frac{1}{N} < \delta$. And $|f(x_1) - f(x_2)| \geq 1 = \epsilon$.
Therefore, $f(x)$ is not uniformly continuous on $[0, +\infty)$.

## 2 Q6 P148

Note that both $f(x)$ and $g(x)$ are uniformly continuous on A, then $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ s.t. $|f(x_1) - f(x_2)| < \epsilon$ and $|g(x_1) - g(x_2)| < \epsilon$.
Also as if $f$ and $g$ are both bounded on A, then $\exists M_1, M_2$ s.t. $|f(x)| < M_1$ and $|g(x)| < M_2$, $\forall x \in A$.
Then, note that $\forall \epsilon > 0$,

$$|f(x_1)g(x_1) - f(x_2)g(x_2)| = |f(x_1)[g(x_1) - g(x_2)] + [f(x_1) - f(x_2)]g(x_2)|$$

$< |f(x_1)[g(x_1) - g(x_2)]| + |[f(x_1) - f(x_2)]g(x_2)| < (M_1 + M_2)\epsilon.\ \forall x_1, x_2 \in A.$
Therefore, if $f, g$ are bounded on A, then the product $fg$ is uniformly continuous on A.

# 3 Q7 P148

(i)

As given $f(x) = x$ and $g(x) = \sin x$.

Note that $\forall \epsilon > 0$, take $\delta = \epsilon$, we have $|f(x_1) - f(x_2)| = |x_1 - x_2| < \epsilon$, as $|x_1 - x_2| < \delta$. Therefore, $f(x)$ is uniformly continuous on $\mathbb{R}$.

Note that $\forall \epsilon > 0$, take $\delta = \epsilon$, we have

$$|g(x_1) - g(x_2)| = |\sin x_1 - \sin x_2| = |2\cos\frac{x_1 + x_2}{2}\sin\frac{x_1 - x_2}{2}| < |x_1 - x_2| < \delta = \epsilon.$$

as $|x_1 - x_2| < \delta$. Therefore, $g(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

(ii)

Let $h(x) = f(x)g(x) = x\sin x$. Note that $\forall \epsilon > 0, \forall x_1, x_2 \in \mathbb{R}, |h(x_1) - h(x_2)| = |x_1 \sin x_1 - x_2 \sin x_2|$. Note that if $x_1 = 2n\pi + \frac{1}{n}, x_2 = 2n\pi, n = 1, 2, 3...$, then

$$|h(x_1) - h(x_2)| = |(2n\pi + \frac{1}{n})\sin\frac{1}{n}| \geq |2n\pi \sin\frac{1}{n}|.$$

According to the inequality

$$\frac{\sin x}{x} > \frac{\pi}{2}\ , x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Now we fix $\epsilon = 4$.

We know that (by Archimedean's property) $\exists N\ s.t. \frac{1}{N} < \frac{2}{\pi}$ and $\frac{1}{N} < \delta, \forall \delta > 0$. Choose $x_1 = 2N\pi + \frac{1}{N}, x_2 = 2N\pi$ we have

$$|h(x_1) - h(x_2)| \geq 2\pi\frac{2}{\pi} = 4 = \epsilon.$$

Therefore $h(x) = f(x)g(x)$ is not uniformly continuous on $\mathbb{R}$.