MATH2050A Homework 6

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Spring, 2020

1 Q10 (Section 5.1)

Let $\epsilon > 0$. $\forall x_0 \in \mathbb{R}$,
If $x_0 > 0$, take $\delta_1 = \inf\{\frac{x_0}{2}, \epsilon\}$. We have

$$|f(x) - f(x_0)| = |x - x_0| < \epsilon, \quad \forall x : |x - x_0| < \delta_1.$$

If $x_0 < 0$, take $\delta_2 = \inf\{\epsilon, \frac{-x_0}{2}\}$. We have

$$|f(x) - f(x_0)| = ||x| - |x_0|| = |-x + x_0| < \epsilon, \quad \forall x : |x - x_0| < \delta_2.$$

If $x_0 = 0$, take $\delta_3 = \epsilon$. We have

$$|f(x) - f(x_0)| = |f(x)| = |x| < \epsilon, \quad \forall x : |x - x_0| < \delta_3.$$

Base on all above, we have $\forall x_0 \in \mathbb{R}, \forall \epsilon > 0$, take $\delta = \inf\{\delta_1, \delta_2, \delta_3\}$, we have

$$|f(x) - f(x_0)| < \epsilon, \quad \forall x : |x - x_0| < \delta.$$

Hence, the absolute value function is continuous at every point $x_0 \in \mathbb{R}$.

2 Q4a (Section 5.1)

Let $x_0 \in \mathbb{R}$.

2.1 Case 1: $x_0 \in \mathbb{Z}$

If $x_0 \in \mathbb{Z}$. Then $f(x_0) = x_0$ itself. Take $\epsilon = 0.5$. $\forall \delta > 0$, take $x = x_0 + \frac{\delta}{2}$ we have $x > x_0, x_0 \in \mathbb{Z}$, hence $f(x) \geq x_0 + 1 > f(x_0) = x_0$. Therefore, we have

$$f(x) - f(x_0) \geq 1, |f(x) - f(x_0)| \geq 1 > \epsilon.$$

2.2 Case 2: $x_0 \notin \mathbb{Z}$, $x_0 > 0$

If $x_0 > 0$, we know that by Archimedean's Property, $\exists n \in \mathbb{N}$ s.t.

$$0 < x_0 < n.$$

Define $N = \inf\{n \in \mathbb{N} : x_0 < n\}$. As the set is a subset of natural number, by the well-ordering principle, the definition is valid.

Note that $x_0 > N-1$. (Suppose not, $N-1$ wil be the inf of $\inf\{n \in \mathbb{N} : x_0 < n\}$) Hence $N-1 < x_0 < N$.

Let $\epsilon > 0$. Take $\delta = \inf\{\frac{|N-1-x_0|}{2}, \frac{|N-x_0|}{2}\}$, $\forall x : |x-x_0| < \delta$, which implies $N-1 < x < N$, we have

$$|f(x) - f(x_0)| = |N-1-(N-1)| = 0 < \epsilon.$$

Therefore $f(x)$ is continuous on $x_0 > 0$ with $x_0 \in \mathbb{R}$ and $x_0 \notin \mathbb{Z}$

2.3 Case 3: $x_0 \notin \mathbb{Z}$, $x_0 < 0$

Consider $-x_0$, then -$x_0 > 0$ and we can apply the argument of case 2. So $\exists N \in \mathbb{N}$ s.t.

$$N-1 < -x_0 < N.$$

which implies

$$-N < x_0 < -N+1.$$

Then $\forall \epsilon > 0$, $\forall x : |x-x_0| < \delta$, where $\delta = \inf\{\frac{|-N-x_0|}{2}, \frac{|-N+1-x_0|}{2}\}$

$$|f(x) - f(x_0)| = |-N-(-N)| = 0 < \epsilon.$$

Hence the point of continuity of $f(x)$ is $\{x \in \mathbb{R} : x \notin \mathbb{Z}\}$.

3 Q9 (Section 4.3)

Let $g(x) = xf(x)$. So we have $\lim_{x\to\infty} g(x) = L$. Define $h : (0,\infty) \to \mathbb{R}$ with $h(x) = \frac{1}{x}$. We have $\lim_{x\to\infty} h(x) = 0$.

Note that $f(x) = g(x) \cdot h(x), \forall x \in (0,\infty)$. Hence

$$\lim_{x\to\infty} f(x) = L \cdot 0 = 0.$$

4 Q13 (Section 4.3)

Let $\epsilon > 0$.

By $\lim_{x\to\infty} f(x) = L$. $\exists N$ s.t. $\forall x > N, |f(x) - L| < \epsilon$.

And by $\lim_{x\to\infty} g(x) = \infty$. There exists such K s.t. $\forall x > K$,

$$g(x) > N.$$

Therefore $\forall \epsilon > 0$, there exists K such that $\forall x > K$, hence $g(x) > N$, we have

$$|f(g(x)) - L| < \epsilon.$$

Which implies $\lim_{x\to\infty} f \circ g = L$.