MATH2050A HW2

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1 Q5 (Section 3.1)

(i)
$\forall \epsilon > 0, \forall n > N = \lceil \frac{1}{\epsilon} \rceil$, we have

$$|\frac{n}{n^2+1} - 0| < |\frac{n}{n^2}| = \frac{1}{n} < \epsilon.$$

which implies that $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$.

(ii)
$\forall \epsilon > 0$, $\forall n > N = \lceil \frac{2}{\epsilon} \rceil$, we have

$$|\frac{2n}{n+1} - 2| = |\frac{2}{n+1}| < \frac{2}{n} < \epsilon.$$

Which implies that $\lim_{n\to\infty} \frac{2n}{n+1} = 2$.

(iii)
$\forall \epsilon > 0$, $\forall n > N = \lceil \frac{13}{4\epsilon} \rceil$, we have

$$|\frac{3n+1}{2n+5} - \frac{3}{2}| = |\frac{-6.5}{2n+5}| < \frac{13}{4n} < \epsilon.$$

Which implies that $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$.

(iv)
$\forall \epsilon > 0$, $\forall n > N = \lceil \sqrt{\frac{5}{4\epsilon}} \rceil$, we have

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| = |\frac{5}{4n^2+6}| < \frac{5}{4n^2} < \epsilon.$$

Which implies that $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$.

2 Q8 (Section 3.1)

First we need to prove that

$$\lim_{n\to\infty} (x_n) = 0 \iff \lim_{n\to\infty} (|x_n|) = 0$$

Suppose $\lim_{n\to\infty}(x_n) = 0$, by definition, we have

$$\forall\epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ |x_n| < \epsilon, \forall n > N.$$

So, $|x_n| = ||x_n| - 0| < \epsilon$. Hence we have

$$\forall\epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ ||x_n| - 0| < \epsilon, \forall n > N$$

which implies $\lim_{n\to\infty}(|x_n|) = 0$

Suppose $\lim_{n\to\infty}(|x_n|) = 0$, by definition, we have

$$\forall\epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ ||x_n| - 0| < \epsilon, \forall n > N,$$

note that $||x_n| - 0| < \epsilon$ implies $|x_n| < \epsilon$, so

$$\forall\epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ |x_n| < \epsilon, \forall n > N$$

also holds. Which is just implies $\lim_{n\to\infty}(x_n) = 0$. So from all above, we have

$$\lim_{n\to\infty}(x_n) = 0 \iff \lim_{n\to\infty}(|x_n|) = 0.$$

Let $x_n = (-1)^{(n+1)}, n \in \mathbb{N}$. Note that the $|x_n| = 1, \forall n$. So $\lim_{n\to\infty}|x_n|$ exists.

Note that $\{x_n\} = \{-1, 1\}$. Suppose x_n's limit is L, then $\forall\epsilon > 0$ there exists an $N \in \mathbb{N}$ s.t. $|x_n - L| < \epsilon, \forall n > N$.

Consider when $n > N$ and n are even numbers, $x_n = -1$, we have $|-1-L| = |L-(-1)| < \epsilon$.

Consider when $n > N$ and n are odd numbers, $x_n = 1$, we have $|1 - L| = |L - 1| < \epsilon$.

Since the ϵ can be picked as any positive number, we choose it to be $\frac{1}{4}$.

We have $L \in (-\frac{5}{4}, -\frac{3}{4})$ and $L \in (\frac{3}{4}, \frac{5}{4})$. The two sets are disjoint, so such L does not exist. So $\lim_{n\to\infty} x_n$ doesn't exist.

So we have an example showing that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).