MATH2050A Homework 9

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Spring, 2020

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Note that $f(x)$ is uniformly continuous on $\mathbb{R}$, i.e., $\forall \epsilon_1 > 0$, $\exists \delta_1 > 0$ s.t. $\forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta_1$, we have

$$|f(x_1) - f(x_2)| < \epsilon_1.$$

And since $g(x)$ is also uniformly continuous on $\mathbb{R}$, $\forall \epsilon_2 > 0, \exists \delta_2 > 0$ s.t. $\forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta_2$, we have

$$|g(x_1) - g(x_2)| < \epsilon_2.$$

Take $\epsilon_2 = \delta_1 > 0$. We have some $\delta_2 > 0$ s.t. $\forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta_2$, we have

$$|g(x_1) - g(x_2)| < \delta_1.$$

Note that $g(x_1), g(x_2) \in \mathbb{R}$. Then by the previous part we wrote at the beginning of the proof,

$$|f(g(x_1)) - f(g(x_2))| < \epsilon_1.$$

Hence, by the definition of uniformly continuous, $f(g(x))$ is uniformly continuous on $\mathbb{R}$.

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Note that $f(x)$ is uniformly continuous, then $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x_1, x_2 \in A, |x_1 - x_2| < \delta$, we have

$$|f(x_1) - f(x_2)| < \epsilon.$$

Let $\epsilon = 1$ and then fix a corresponding δ.

Note that A is bounded. Hence A can be covered by a union of multiple open intervals, i.e., $\exists X_1, x_2, ..., x_n \in A, B_1 = (x_1 - \delta, x_1 + \delta), B_2 = (x_2 - \delta, x_2 + \delta), ...B_n = (x_n - \delta, x_n + \delta)$ s.t.

$$A \subset B_1 \cup B_2 \cup ... \cup B_n.$$

$\forall x \in A$, $\exists m$ s.t. $x \in B_m$. Because $B_m = (x_m - \delta, x_m + \delta)$, then $|x - x_m| < \delta$, which implies

$$|f(x) - f(x_m)| < 1.$$

Then $f(x) < f(x_m) + 1$, $f(x) > f(x_m) - 1$, which implies $\forall x \in A$,

$$f(x) < \sup\{f(x_1) + 1, ...f(x_n) + 1\} = U$$

and

$$f(x) > \inf\{f(x_1) - 1, ...f(x_n) - 1\} = L.$$

Hence, f is bounded on A.