MATH2050A HW4

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1 P84 Q4a

The sequence in the question is $(x_n = 1 - (-1)^n + \frac{1}{n})$.
Suppose it is convergent, then $\exists L \in \mathbb{R}\ s.t.$

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^*\ s.t.\ \forall n > N, |x_n - L| < \frac{\epsilon}{2}.$$

Which implies that

$$\forall n, m > N, |x_n - x_m| \leq |x_n - L| + |L - x_m| < \epsilon.$$

Let $\epsilon = 0.5, n = 2N + 2, m = 2N + 1$. Note that $n, m > N$. So we should have

$$|x_n - x_m| < 0.5$$

.

However,

$$|x_n - x_m| = |1 - (-1)^{2N+2} + \frac{1}{2N+2} - 1 + (-1)^{2N+1} - \frac{1}{2N+1}| = 2 + \frac{1}{2N+1} - \frac{1}{2N+2} > 2.$$

This is contradict with the previous $|x_n - x_m| < 0.5$.
Hence, the (x_n) is divergent.

2 P84 Q7a

We consider the sequence $a_k = (1 + \frac{1}{k})^k$ on $\mathbb{R}$.
Let $k_n = n^2$, note that $a_{k_n} = (1 + \frac{1}{n^2})^{n^2}$ is a subsequence of (a_k).

First we prove that (a_k) is increasing. Note that $a_k > 0, \forall k$.
So we inspect

$$\frac{a_{k+1}}{a_k} = (\frac{k+2}{k+1})^k (\frac{k}{k+1})^k (\frac{k+2}{k+1}) = (1 + \frac{1}{1+k})[1 - \frac{1}{(k+1)^2}]^k.$$

Note that $\frac{-1}{(k+1)^2} \geq -1$ and $k \geq 1$, so by Bernoulli's inequality, we have

$$[1 - \frac{1}{(k+1)^2}]^k \geq 1 - \frac{k}{(k+1)^2}.$$

We have

$$\frac{a_{k+1}}{a_k} \geq \frac{k+2}{k+1}(1 - \frac{k}{(k+1)^2}) = 1 + \frac{1}{(k+1)^3} \geq 1.$$

Hence, we have already proved (a_k) is increasing, then we'd prove (a_k) is bounded.
Consider $n \geq 2$. Using binomial theorem, we have

$$a_k = 2+\sum_{i=2}^{k} \binom{k}{i} \frac{1}{k^i} = 2+\sum_{i=2}^{k} \frac{k(k-1)...(k-i+1)}{k^i} \frac{1}{i!} \leq 2+\sum_{i=2}^{k} \frac{1}{i!} \leq 2+\sum_{i=2}^{k} \frac{1}{2^{i-1}} = 3-\frac{1}{2^{k-1}}.$$

Hence $a_k < 3$, so (a_k) is bounded.
So (a_k) is bounded and increasing, hence (a_k) is convergent. Which implies its subsequence a_{k_n} converges and converges to the same limit as (a_k) does.
As $\lim_{k\to\infty}(1+\frac{1}{k})^k$ has limit and is defined to be e, so

$$\lim_{n\to\infty}(1+\frac{1}{n^2})^{n^2} = e.$$

3 P84 Q8a

Claim $\lim_{n\to\infty} n^{\frac{1}{n}} = 0$.
Consider $n \geq 2$. By the definition of limit, $\forall \epsilon > 0$ (we further consider $\epsilon < 1$.)

$$\exists N \in \mathbb{N}\ s.t.\ |n^{\frac{1}{n}} - 1| < \epsilon, \forall n > N.$$

Which is equivalent to $(1-\epsilon)^n < n < (1+\epsilon)^n$.
The inequality in the left hand side holds naturally because $(1-\epsilon)^n < 1 < n$.

Hence we need to find such N s.t. $\forall n > N$

$$n < (1+\epsilon)^n.$$

Using binomial theorem, we have

$$(1+\epsilon)^n > \binom{n}{2}\epsilon^2 = \frac{n(n-1)}{2}\epsilon^2.$$

Suffice to let

$$\frac{n(n-1)}{2}\epsilon^2 > n.$$

Which is equivalent to $n > 1 + \frac{2}{\epsilon^2}$. So it suffices to let $N > 1 + \frac{2}{\epsilon^2}$.
Hence

$$\forall n > N, |n^{\frac{1}{n}} - 1| < \epsilon.$$

So we have $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
Hence

$$\lim_{n\to\infty}(3n)^{\frac{1}{2n}} = \lim_{n\to\infty}((3n)^{\frac{1}{3n}})^{\frac{3}{2}} = 1^{\frac{3}{2}} = 1.$$

So the limit in this question is 1.