MATH2050A Homework 5

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1 Q3(b) (Section 3.5)

Let $x_n = n + \frac{(-1)^n}{n}$. We want to show (x_n) is not a Cauchy sequence.
Take $\epsilon = 1$, $\forall N \in \mathbb{N}$, choose $m = 2N+1, n = 2N+2$. We have

$$|x_m - x_n| = |2N+1+\frac{-1}{2N+1} - (2N+2) - \frac{-1}{2N+2}| = 1 + \frac{1}{(2N+1)(2N+2)} > 1 = \epsilon.$$

$\therefore$ (x_n) is not a Cauchy sequence.

2 Q5 (Section 3.5)

Note that

$$|x_{n+1} - x_n| = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

$\forall \epsilon > 0$, $\exists N > \frac{1}{\epsilon^2}$ s.t. $\forall n > N$ we have

$$|x_{n+1} - x_n - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}} < \epsilon.$$

$\therefore \lim |x_{n+1} - x_n| = 0$.
Then we would show that (x_n) is unbounded. $\forall M > 0, M \in \mathbb{N}$, choose $u = M^2 + 1$, we have

$$x_u = \sqrt{M^2+1} > M.$$

So (x_n) is unbounded. Also note that (x_n) is increasing, so (x_n) is divergent and hence not a Cauchy sequence.

3 Q9 (Section 3.5)

Let $m, n \in \mathbb{N}, m > n$, then we have

$$|x_m - x_n| \leq |x_m - x_{m-1}| + ... + |x_{n+1} - x_n| < r^{m-1} + ...r^n = \frac{r^n - r^m}{1-r} \leq \frac{r^n}{1-r}.$$

$\forall \epsilon > 0, \exists N > \frac{\ln \epsilon(1-r)}{\ln r}$ s.t. $\forall n > N$,

$$|x_m - x_n| < \epsilon.$$

$\therefore$ (x_n) is a Cauchy sequence.