STAT3003 Problem Sheet 1

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1 Q1

We want to prove $E[\hat{\sigma}^2] = \frac{N}{N-1}\sigma^2$.
Note that $\sigma^2 = \frac{1}{N}\sum_{j=1}^{N}(u_j - \mu) = \frac{1}{N^2}[(N-1)\sum_{j=1}^{N} u_j^2 - \sum_{j=1}^{N}\sum_{k\neq j} u_j u_k]$
Hence R.H.S.$=\frac{N}{N-1}\sigma^2 = \frac{1}{N(N-1)}[\sum_{j=1}^{N}\sum_{k\neq j} u_j^2 - \sum_{j=1}^{N}\sum_{k\neq j} u_j u_k]$.
Also note that $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \frac{1}{n(n-1)}[(n-1)\sum_{i=1}^{n} Y_i^2 - \sum_{j=1}^{n}\sum_{k\neq j} Y_j Y_k]$
$= \frac{1}{n(n-1)}[(n-1)\sum_{i=1}^{n} u_i^2 Z_i^2 - \sum_{j=1}^{n}\sum_{k\neq j} u_j u_k Z_j Z_k]$.
Hence L.H.S.$=E[\hat{\sigma}^2] = \frac{1}{n}\sum_{i=1}^{n} u_i^2 E[Z_i^2] - \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{k\neq j} u_j u_k E[Z_j Z_k]$
$= \frac{1}{n}\sum_{i=1}^{n} u_i^2 E[Z_i^2] - \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{k\neq j} u_j u_k E[Z_j Z_k]$
$= \frac{1}{n}\sum_{i=1}^{n} u_i^2 \frac{n}{N} - \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{k\neq j} u_j u_k \frac{n(n-1)}{N(N-1)}$
$= \frac{1}{N}\sum_{i=1}^{n} u_i^2 - \sum_{j=1}^{n}\sum_{k\neq j} u_j u_k \frac{1}{N(N-1)} = \frac{1}{N(N-1)}[\sum_{j=1}^{N}\sum_{k\neq j} u_j^2 - \sum_{j=1}^{N}\sum_{k\neq j} u_j u_k]$=R.H.S..
Q.E.D.

2 Q2