

STAT3003 Problem Sheet 2

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1 Q1

Note that $L = 4$, $N = N_1 + N_2 + N_3 + N_4 = 225$, with $N_1 = 64$, $N_2 = 43$, $N_3 = 92$, $N_4 = 26$.
And $\hat{p}_1 = \frac{2}{7}$, $\hat{p}_2 = \frac{1}{3}$, $\hat{p}_3 = \frac{8}{21}$, $\hat{p}_4 = \frac{1}{3}$.

Hence we can have the

$$\hat{p}_{st} = \frac{1}{N} \sum_{i=1}^L N_i \hat{p}_i = 0.3393.$$

Also, by

$$\hat{Var}(\hat{p}_{st}) = \frac{1}{N^2} \sum_{i=1}^L N_i^2 \left(1 - \frac{n_i}{N_i}\right) \frac{1}{n_i - 1} \hat{p}_i (1 - \hat{p}_i) = \frac{190.979}{225^2} = 3.7724 * 10^{-3}.$$

Therefore, $\sqrt{\hat{Var}(\hat{p}_{st})} = 0.0614$.

Our goal is to find

$$(\hat{p}_{st} \pm t_{df, 1-\frac{\alpha}{2}} * \sqrt{\hat{Var}(\hat{p}_{st})}).$$

By Satterthwaite's Approximation, we have

$$df \approx \frac{(\sum_{i=1}^L k_i s_i^2)^2}{\sum_{i=1}^L \frac{(k_i s_i^2)^2}{n_i - 1}}, k_i = \frac{N_i(N_i - n_i)}{N^2 n_i},$$

and hence $df = 45$.

Then $t_{df, 1-\frac{\alpha}{2}} = t_{45, 0.975} = 2.0141$. Therefore, then result should be

$$(0.3393 \pm 2.0141 * 0.0614)$$

Done.

2 Q2

Note that $L = 3$, $c = 500$, $c_0 = 0$.

Using

$$n_i = \frac{(c - c_0) \frac{N_i S_i}{\sqrt{c_i}}}{\sum_{i=1}^L N_i S_i \sqrt{c_i}}.$$

Note that we have the σ_i are given, hence by $\hat{\sigma}_i^2 = \frac{N_i}{N_i - 1} \sigma_i^2$. Then we have $s_i = \hat{\sigma}_i$. With $s_1 = 1.5067$, $s_2 = 1.8134$, $s_3 = 1.8235$.

Which gives that $n_1 = 18.1510, n_2 = 7.9581, n_3 = 3.8247$. By doing rounding, we have $n_1 = 18, n_2 = 8, n_3 = 4$. Under this case, we have the cost to be 506.

For saving the cost, we need to reduce the sample size in North America by 1.

Then we have $n_1 = 17, n_2 = 8, n_3 = 4$. Under this case, the cost is 497, which is okay.

And by

$$Var(\bar{Y}_{st}) = \frac{1}{N^2} \sum_{i=1}^L N_i^2 \frac{\sigma_i^2}{n_i} \left(\frac{N_i - n_i}{N_i - 1} \right) = 0.08825 < 0.1.$$

Which can satisfy the corporation's requirement. Hence they can be happy.
Done.

3 Q3

Recall that the function we want to optimize is

$$f(n_1, n_2, \dots, n_L) = \frac{1}{N^2} \sum_{i=1}^L N_i^2 \frac{\sigma_i^2}{n_i} \left(\frac{N_i - n_i}{N_i - 1} \right).$$

And the constraint is

$$g(n_1, n_2, \dots, n_L) = c - (c_0 + c_1 n_1 + \dots + c_L n_L) = 0.$$

The lagrangian is

$$L = f - \lambda g.$$

Note that

$$\frac{\partial f}{\partial n_j} = \frac{1}{N^2} \frac{\partial (N_j^3 \sigma_j^2)}{n_j (N_j - 1)} = - \frac{N_j^3 \sigma_j^2}{N^2 (N_j - 1) n_j^2}.$$

And

$$\frac{\partial g}{\partial n_j} = -c_j.$$

Therefore we can have

$$\lambda c_j = \frac{1}{n_j^2} \frac{N_j^3 \sigma_j^2}{N^2 (N_j - 1)}.$$

Which implies

$$n_j = \frac{1}{\sqrt{\lambda}} \sqrt{\frac{N_j}{N_j - 1} \frac{N_j \sigma_j}{\sqrt{c_j} N}}.$$

Recall the constraint $g=0$. We can have

$$\sum_{j=1}^L c_j n_j = (c - c_0) = \sum_{j=1}^L \frac{1}{\sqrt{\lambda}} \sqrt{\frac{N_j}{N_j - 1} \frac{N_j \sigma_j}{\sqrt{c_j} N}}.$$

Hence

$$\frac{1}{\sqrt{\lambda}} = \frac{c - c_0}{\sum_{j=1}^L \sqrt{\frac{N_j}{N_j - 1} \frac{N_j \sigma_j}{\sqrt{c_j} N}}}.$$

And therefore,

$$n_j = \frac{c - c_0}{\sum_{j=1}^L \sqrt{\frac{N_j}{N_j-1} \frac{N_j \sigma_j}{\sqrt{c_j} N}}} \sqrt{\frac{N_j}{N_j-1} \frac{N_j \sigma_j}{\sqrt{c_j} N}} = \frac{(c - c_0) \left(\sqrt{\frac{N_i}{N_i-1} \frac{N_i \sigma_i}{\sqrt{c_i}}} \right)}{\sum_{i=1}^L \sqrt{\frac{N_i}{N_i-1} \frac{N_i \sigma_i}{\sqrt{c_i}}}}.$$

Which is what we want.

Done.

4 Q4

By the formulae $\hat{\mu} = \frac{\hat{\tau}}{M}$, we can investigate the $\hat{\tau}$ and then go from $\hat{\tau}$ to get $\hat{\mu}$.

Note that $M = 3500, N = 108, n = 25$.

Note that

$$\hat{\tau} = N \frac{1}{n} \sum_{i=1}^n Y_i = 151014.24.$$

And

$$t_{n-1, 1-\frac{\alpha}{2}} = t_{24, 0.975} = 2.06.$$

Also, we have

$$\hat{Var}(\hat{\tau}) = \hat{Var}(N\bar{Y}) = N^2 \left(1 - \frac{n}{N}\right) \frac{\hat{\sigma}_c^2}{n}.$$

Where the $\hat{\sigma}_c^2$ is the sample variance of cluster totals. Hence $\hat{\sigma}_c^2 = 149422.2016$. And

$$\hat{Var}(\hat{\tau}) = 53576824.61.$$

Therefore $\sqrt{\hat{Var}(\hat{\tau})} = 7319.6192$. Hence the 95% CI for τ is

$$(151014.24 \pm 2.06 * 7319.6192)$$

. Therefore 95% CI for μ is $(43, 147 \pm 4.3081)$.

5 Q5

Note that $\hat{\mu} = \frac{N}{M} \bar{Y}$. Hence

$$Var(\hat{\mu}) = Var(\bar{Y}) \frac{N^2}{M^2} = \frac{N^2}{M^2} \frac{N-n}{N-1} \frac{1}{n} \sigma_c^2.$$

Hence the width $2d = 2t_{n-1, 1-\frac{\alpha}{2}} \sqrt{Var(\hat{\mu})} \leq 4$. Solving this inequality, we then have

$$n \geq \frac{1}{\frac{1}{N} + \frac{4(N-1)M^2}{N^3 \sigma_c^2 t_{n-1, 1-\frac{\alpha}{2}}}}.$$

Now we plug in $M = 3500$, and σ_c^2 can be used by 149422.2016 and N be plugged in by 100.

Then we can see the minimal n such that the above inequality hold is $n = 39$.

Done.

6 Q6

Using the formulae $\hat{\tau} = N \frac{1}{n} \sum_{i=1}^n \hat{Y}_i$ and $\hat{Y}_i = \frac{M_i}{m_i} \sum_{j=1}^{m_i} Y_{ij}$.

We have the numbers of $N = 300, n = 4, m_i = 3, \forall i = 1, 2, 3$.

$M_1 = 18, M_2 = 12, M_3 = 9, M_4 = 14$.

Hence $\hat{Y}_1 = 18, \hat{Y}_2 = 12, \hat{Y}_3 = 12, \hat{Y}_4 = \frac{28}{3}$.

Therefore

$$\hat{\tau} = \frac{300}{4} (18 + 12 + 12 + \frac{28}{3}) = 3850.$$

And then we need to find $\sqrt{\hat{Var}(\hat{\tau})}$.

Using formulae $\hat{Var}(\hat{\tau}) = N(N-n) \frac{1}{n} \hat{\sigma}_c^2 + \frac{N}{n} \sum_{i=1}^n M_i(M_i - m_i) \frac{1}{m_i} \hat{\sigma}_i^2$, we can have

$$\hat{Var}(\hat{\tau}) = 234904.815.$$

Then we have $\sqrt{\hat{Var}(\hat{\tau})} = 484.6698$.

Hence the standard error is 484.6698.

7 Q7

It should be obvious. Note that \hat{b} (stands for number of retired people per block) = $\frac{\hat{\tau}}{N}$.
Then,

$$\hat{b} = \frac{\hat{\tau}}{N} = \frac{3850}{300} = 12.8333.$$

And the estimated variance should be

$$\hat{Var}(\hat{b}) = \hat{Var}(\frac{\hat{\tau}}{N}) = \frac{1}{N^2} \hat{Var}(\hat{\tau}) = \frac{1}{90000} * 234904.815 = 2.61.$$

Hence the standard error should be $\sqrt{2.61} = 1.6156$.

Done.