STAT3003 Problem Sheet 2

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1 Q1

Note that $L = 4$, $N = N_1+N_2+N_3+N_4 = 225$, with $N_1 = 64, N_2 = 43, N_3 = 92, N_4 = 26$. And $\hat{p_1} = \frac{2}{7}, \hat{p_2} = \frac{1}{3}, \hat{p_3} = \frac{8}{21}, \hat{p_4} = \frac{1}{3}$.

Hence we can have the

$$\hat{p_{st}} = \frac{1}{N}\sum_{i=1}^{L} N_i\hat{p_i} = 0.3393.$$

Also, by

$$V\hat{a}r(\hat{p_{st}}) = \frac{1}{N^2}\sum_{i=1}^{L} N_i^2(1-\frac{n_i}{N_i})\frac{1}{n_i-1}\hat{p_i}(1-\hat{p_i}) = \frac{190.979}{225^2} = 3.7724 * 10^{-3}.$$

Therefore, $\sqrt{V\hat{a}r(\hat{p_{st}})} = 0.0614$.

Our goal is to find

$$(\ \hat{p_{st}} \pm t_{df,1-\frac{\alpha}{2}} * \sqrt{V\hat{a}r(\hat{p_{st}})}\ \ ).$$

By Satterthwaite's Approximation, we have

$$df \approx \frac{(\sum_{i=1}^{L} k_i s_i^2)^2}{\sum_{i=1}^{L}\frac{(k_i s_i^2)^2}{n_i-1}}, k_i = \frac{N_i(N_i-n_i)}{N^2 n_i},$$

and hence $df = 45$.

Then $t_{df,1-\frac{\alpha}{2}} = t_{45,0.975} = 2.0141$. Therefore, then result should be

$$(\ 0.3393 \pm 2.0141 * 0.0614\ )$$

Done.

2 Q2

Note that $L = 3, c = 500, c_0 = 0$.

Using

$$n_i = \frac{(c-c_0)\frac{N_i S_i}{\sqrt{c_i}}}{\sum_{i=1}^{L} N_i S_i\sqrt{c_i}}.$$

Note that we have the σ_i are given, hence by $\hat{\sigma_i}^2 = \frac{N_i}{N_i-1}\sigma_i^2$. Then we have $s_i = \hat{\sigma_i}$. With $s_1 = 1.5067, s_2 = 1.8134, s_3 = 1.8235$.

Which gives that $n_1 = 18.1510, n_2 = 7.9581, n_3 = 3.8247$. By doing rounding, we have $n_1 = 18, n_2 = 8, n_3 = 4$. Under this case, we have the cost to be 506.

For saving the cost, we need to reduce the sample size in North America by 1.

Then we have $n_1 = 17, n_2 = 8, n_3 = 4$. Under this case, the cost is 497, which is okay.

And by

$$Var(\bar{Y_{st}}) = \frac{1}{N^2}\sum_{i=1}^{L} N_i^2 \frac{\sigma_i^2}{n_i}(\frac{N_i - n_i}{N_i - 1}) = 0.08825 < 0.1.$$

Which can satisfy the corporation's requirement. Hence they can be happy.

Done.

3 Q3

Recall that the function we want to optimize is

$$f(n_1, n_2, ..., n_L) = \frac{1}{N^2}\sum_{i=1}^{L} N_i^2 \frac{\sigma_i^2}{n_i}(\frac{N_i - n_i}{N_i - 1}).$$

And the constraint is

$$g(n_1, n_2, ..., n_L) = c - (c_0 + c_1 n_1 + ... + c_L n_L) = 0.$$

The lagrangian is

$$L = f - \lambda g.$$

Note that

$$\frac{\partial f}{\partial n_j} = \frac{1}{N^2}\frac{\partial (N_j^3\ \sigma_j^2)}{n_j(N_j - 1)} = -\frac{N_j^3 \sigma_j^2}{N^2(N_j - 1)n_j^2}.$$

And

$$\frac{\partial g}{\partial n_j} = -c_j.$$

Therefore we can have

$$\lambda c_j = \frac{1}{n_j^2}\frac{N_j^3 \sigma_j^2}{N^2(N_j - 1)}.$$

Which implies

$$n_j = \frac{1}{\sqrt{\lambda}}\sqrt{\frac{N_j}{N_j - 1}}\frac{N_j \sigma_j}{\sqrt{c_j} N}.$$

Recall the constraint g=0. We can have

$$\sum_{j=1}^{L} c_j n_j = (c - c_0) = \sum_{j=1}^{L} \frac{1}{\sqrt{\lambda}}\sqrt{\frac{N_j}{N_j - 1}}\frac{N_j \sigma_j}{\sqrt{c_j} N}.$$

Hence

$$\frac{1}{\sqrt{\lambda}} = \frac{c - c_0}{\sum_{j=1}^{L}\sqrt{\frac{N_j}{N_j - 1}}\frac{N_j \sigma_j}{\sqrt{c_j} N}}.$$

And therefore,

$$n_j = \frac{c - c_0}{\sum_{j=1}^{L} \sqrt{\frac{N_j}{N_j - 1}} \frac{N_j \sigma_j}{\sqrt{c_j} N}} \sqrt{\frac{N_j}{N_j - 1} \frac{N_j \sigma_j}{\sqrt{c_j} N}} = \frac{(c - c_0)(\sqrt{\frac{N_i}{N_i - 1}} \frac{N_i \sigma_i}{\sqrt{c_i}})}{\sum_{i=1}^{L} \sqrt{\frac{N_i}{N_i - 1}} N_i \sigma_i \sqrt{c_i}}.$$

Which is want we want.

Done.

4 Q4

By the formulae $\hat{\mu} = \frac{\hat{\tau}}{M}$, we can investigate the $\hat{\tau}$ and then go from $\hat{\tau}$ to get $\hat{\mu}$.
Note that $M = 3500, N = 108, n = 25$.
Note that

$$\hat{\tau} = N \frac{1}{n} \sum_{i=1}^{n} Y_i = 151014.24.$$

And

$$t_{n-1,1-\frac{\alpha}{2}} = t_{24,0.975} = 2.06.$$

Also, we have

$$\hat{Var}(\hat{\tau}) = \hat{Var}(N\bar{Y}) = N^2 (1 - \frac{n}{N}) \frac{\hat{\sigma_c}^2}{n}.$$

Where the $\hat{\sigma_c}^2$ is the sample variance of cluster totals. Hence $\hat{\sigma_c}^2 = 149422.2016$. And

$$\hat{Var}(\hat{\tau}) = 53576824.61.$$

Therefore $\sqrt{\hat{Var}(\hat{\tau})} = 7319.6192$. Hence the 95% CI for τ is

$$(\ 151014.24 \pm 2.06 * 7319.6192\ )$$

. Therefore 95% CI for μ is $(\ 43,147 \pm 4.3081\ )$.

5 Q5

Note that $\hat{\mu} = \frac{N}{M} \bar{Y}$. Hence

$$Var(\hat{\mu}) = Var(\bar{Y}) \frac{N^2}{M^2} = \frac{N^2}{M^2} \frac{N - n}{N - 1} \frac{1}{n} \sigma_c^2.$$

Hence the width $2d = 2t_{n-1,1-\frac{\alpha}{2}} \sqrt{Var(\hat{\mu})} \leq 4$. Solving this inequality, we then have

$$n \geq \frac{1}{\frac{1}{N} + \frac{4(N-1)M^2}{N^3 \sigma_c^2 t_{n-1,1-\frac{\alpha}{2}}^2}}.$$

Now we plug in $M = 3500$, and σ_c^2 can be used by 155648.1267 and N be pluged in by 100.

Then we can see the minimal n such that the above inequality hold is $n = 57$.

Done.

6 Q6

Using the formulae $\hat{\tau} = N\frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i$ and $\hat{Y}_i = \frac{M_i}{m_i}\sum_{j=1}^{m_i} Y_{ij}$.
We have the numbers of $N = 300, n = 4, m_i = 3,\ \forall i = 1, 2, 3$.
$M_1 = 18, M_2 = 12, M_3 = 9, M_4 = 14$.
Hence $\hat{Y_1} = 18, \hat{Y_2} = 12, \hat{Y_3} = 12, \hat{Y_4} = \frac{28}{3}$.
Therefore

$$\hat{\tau} = \frac{300}{4}(18 + 12 + 12 + \frac{28}{3}) = 3850.$$

And then we need to find $\sqrt{\hat{Var}(\hat{\tau})}$.
Note that $\hat{\sigma_c}^2 = 13.4444$
Using formulae $\hat{Var}(\hat{\tau}) = N(N-n)\frac{1}{n}\hat{\sigma_c^2} + \frac{N}{n}\sum_{i=1}^{n} M_i(M_i - m_i)\frac{1}{m_i}\hat{\sigma_i^2}$, we can have

$$\hat{Var}(\hat{\tau}) = 315050.$$

Then we have $\sqrt{\hat{Var}(\hat{\tau})} = 561.28$.
Hence the standard error is 561.28.

7 Q7

It should be obvious. Note that $\hat{b}$ (stands for number of retired people per block) $= \frac{\hat{\tau}}{N}$.
Then,

$$\hat{b} = \frac{\hat{\tau}}{N} = \frac{3850}{300} = 12.8333.$$

And the estimated variance should be

$$\hat{Var}(\hat{b}) = \hat{Var}(\frac{\hat{\tau}}{N}) = \frac{1}{N^2}\hat{Var}(\hat{\tau}) = \frac{1}{90000} * 315050 = 3.5.$$

Hence the standard error should be $\sqrt{3.5} = 1.87$.
Done.