Springer Texts in Business and Economics

Norman Schofield

Mathematical Methods in Economics and Social Choice

Second Edition

Springer

Springer Texts in Business and Economics

For further volumes:
www.springer.com/series/10099

Norman Schofield

Mathematical Methods in Economics and Social Choice

Second Edition

Springer

Norman Schofield
Center in Political Economy
Washington University in Saint Louis
Saint Louis, MO, USA

ISSN 2192-4333 ISSN 2192-4341 (electronic)
ISBN 978-3-642-39817-9 ISBN 978-3-642-39818-6 (eBook)
DOI 10.1007/978-3-642-39818-6
Springer Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to the memory of Jeffrey Banks
and Richard McKelvey

Foreword

The use of mathematics in the social sciences is expanding both in breadth and depth at an increasing rate. It has made its way from economics into the other social sciences, often accompanied by the same controversy that raged in economics in the 1950s. And its use has deepened from calculus to topology and measure theory to the methods of differential topology and functional analysis.

The reasons for this expansion are several. First, and perhaps foremost, mathematics makes communication between researchers succinct and precise. Second, it helps make assumptions and models clear; this bypasses arguments in the field that are a result of different implicit assumptions. Third, proofs are rigorous, so mathematics helps avoid mistakes in the literature. Fourth, its use often provides more insights into the models. And finally, the models can be applied to different contexts without repeating the analysis, simply by renaming the symbols.

Of course, the formulation of social science questions must precede the construction of models and the distillation of these models down to mathematical problems, for otherwise the assumptions might be inappropriate.

A consequence of the pervasive use of mathematics in our research is a change in the level of mathematics training required of our graduate students. We need reference and graduate text books that address applications of advanced mathematics to a widening range of social sciences. This book fills that need.

Many years ago, Bill Riker introduced me to Norman Schofield's work and then to Norman. He is unique in his ability to span the social sciences and apply integrative mathematical reasoning to them all. The emphasis on his work and his book is on smooth models and techniques, while the motivating examples for presentation of the mathematics are drawn primarily from economics and political science. The reader is taken from basic set theory to the mathematics used to solve problems at the cutting edge of research. Students in every social science will find exposure to this mode of analysis useful; it elucidates the common threads in different fields. Speculations at the end of Chap. 5 provide students and researchers with many open research questions related to the content of the first four chapters. The answers are in these chapters. When the first edition appeared in 2004, I wrote in my Foreword that a goal of the reader should be to write Chap. 6. For the second edition of the book, Norman himself has accomplished this open task.

St. Louis, Missouri, USA
Marcus Berliant

St. Louis, Missouri, USA
2013

Marcus Berliant

Preface to the Second Edition

For the second edition, I have added a new chapter six. This chapter continues with the model presented in Chap. 3 by developing the idea of dynamical social choice. In particular the chapter considers the possibility of cycles enveloping the set of social alternatives.

A theorem of Saari (1997) shows that for any non-collegial set, $\mathbb{D}$, of decisive or winning coalitions, if the dimension of the policy space is sufficiently large, then the choice is empty under $\mathbb{D}$ for all smooth profiles in a residual subspace of $C^r(W, \Re^n)$. In other words the choice is generically empty.

However, we can define a social solution concept, known as the heart. When regarded as a correspondence, the heart is lower hemi-continuous. In general the heart is centrally located with respect to the distribution of voter preferences, and is guaranteed to be non-empty. Two examples are given to show how the heart is determined by the symmetry of the voter distribution.

Finally, to be able to use survey data of voter preferences, the chapter introduces the idea of stochastic social choice. In situations where voter choice is given by a probability vector, we can model the choice by assuming that candidates choose policies to maximise their vote shares. In general the equilibrium vote maximising positions can be shown to be at the electoral mean. The necessary and sufficient condition for this is given by the negative definiteness of the candidate vote Hessians. In an empirical example, a multinomial logit model of the 2008 Presidential election is presented, based on the American National Election Survey, and the parameters of this model used to calculate the Hessians of the vote functions for both candidates. According to this example both candidates should have converged to the electoral mean.

Saint Louis, Missouri, USA
June 13, 2013

Norman Schofield

Preface to the First Edition

In recent years, the optimisation techniques, which have proved so useful in microeconomic theory, have been extended to incorporate more powerful topological and differential methods. These methods have led to new results on the qualitative behaviour of general economic and political systems. However, these developments have also led to an increase in the degree of formalism in published work. This formalism can often deter graduate students. My hope is that the progression of ideas presented in these lecture notes will familiarise the student with the geometric concepts underlying these topological methods, and, as a result, make mathematical economics, general equilibrium theory, and social choice theory more accessible.

The first chapter of the book introduces the general idea of mathematical structure and representation, while the second chapter analyses linear systems and the representation of transformations of linear systems by matrices. In the third chapter, topological ideas and continuity are introduced and used to solve convex optimisation problems. These techniques are also used to examine existence of a "social equilibrium." Chapter four then goes on to study calculus techniques using a linear approximation, the differential, of a function to study its "local" behaviour.

The book is not intended to cover the full extent of mathematical economics or general equilibrium theory. However, in the last sections of the third and fourth chapters I have introduced some of the standard tools of economic theory, namely the Kuhn Tucker Theorem, together with some elements of convex analysis and procedures using the Lagrangian. Chapter four provides examples of consumer and producer optimisation. The final section of the chapter also discusses, in a heuristic fashion, the smooth or critical Pareto set and the idea of a regular economy. The fifth and final chapter is somewhat more advanced, and extends the differential calculus of a real valued function to the analysis of a smooth function between "local" vector spaces, or manifolds. Modem singularity theory is the study and classification of all such smooth functions, and the purpose of the final chapter to use this perspective to obtain a generic or typical picture of the Pareto set and the set of Walrasian equilibria of an exchange economy.

Since the underlying mathematics of this final section are rather difficult, I have not attempted rigorous proofs, but rather have sought to lay out the natural path of development from elementary differential calculus to the powerful tools of singularity theory. In the text I have referred to work of Debreu, Balasko, Smale, and Saari, among others who, in the last few years, have used the tools of singularity theory to develop a deeper insight into the geometric structure of both the economy and the polity. These ideas are at the heart of recent notions of "chaos." Some speculations on this profound way of thinking about the world are offered in Sect. 5.6. Review exercises are provided at the end of the book.

I thank Annette Milford for typing the manuscript and Diana Ivanov for the preparation of the figures.

I am also indebted to my graduate students for the pertinent questions they asked during the courses on mathematical methods in economics and social choice, which I have given at Essex University, the California Institute of Technology, and Washington University in St. Louis.

In particular, while I was at the California Institute of Technology I had the privilege of working with Richard McKelvey and of discussing ideas in social choice theory with Jeff Banks. It is a great loss that they have both passed away. This book is dedicated to their memory.

Saint Louis, Missouri, USA Norman Schofield

Contents

Sets, Relations, and Preferences

1

In this chapter we introduce elementary set theory and the notation to be used throughout the book. We also define the notions of a binary relation, of a function, as well as the axioms of a group and field. Finally we discuss the idea of an individual and social preference relation, and mention some of the concepts of social choice and welfare economics.

1.1 Elements of Set Theory

Let $\mathcal{U}$ be a collection of objects, which we shall call the domain of discourse, the universal set, or universe. A set B in this universe (namely a subset of $\mathcal{U}$) is a subcollection of objects from $\mathcal{U}$. B may be defined either explicitly by enumerating the objects, for example by writing

$$B = \{\text{Tom, Dick, Harry}\}, \quad \text{or}$$
$$B = \{x_1, x_2, x_3, \dots\}.$$

Alternatively B may be defined implicitly by reference to some property $P(B)$, which characterises the elements of B, thus

$$B = \bigl\{x : x \text{ satisfies } P(B)\bigr\}.$$

For example:

$$B = \{x : x \text{ is an integer satisfying } 1 \le x \le 5\}$$

is a satisfactory definition of the set B, where the universal set could be the collection of all integers. If B is a set, write $x \in B$ to mean that the element x is a member of B. Write $\{x\}$ for the set which contains only one element, x.

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_1,

If A, B are two sets write $A \cap B$ for the *intersection*: that is the set which contains only those elements which are both in A and B. Write $A \cup B$ for the *union*: that is the set whose elements are either in A or B. The *null* set or *empty* set Φ, is that subset of $\mathcal{U}$ which contains no elements in $\mathcal{U}$.

Finally if A is a subset of $\mathcal{U}$, define the *negation* of A, or complement of A in $\mathcal{U}$ to be the set $\mathcal{U} \backslash A = \overline{A} = \{x : x \text{ is in } \mathcal{U} \text{ but not in } A\}$.

1.1.1 A Set Theory

Now let Γ be a family of subsets of $\mathcal{U}$, where Γ includes both $\mathcal{U}$ and Φ, *i.e.*, $\Gamma = \{\mathcal{U}, \Phi, A, B, \ldots\}$.

If A is a member of Γ, then write $A \in \Gamma$. Note that in this case Γ is a collection or *family* of sets.

Suppose that Γ satisfies the following properties:

1. for any $A \in \Gamma$, $\overline{A} \in \Gamma$,
2. for any A, B in Γ, $A \cup B$ is in Γ,
3. for any A, B in Γ, $A \cap B$ is in Γ.

Then we say that Γ satisfies *closure* with respect to $(^{-}, \cup, \cap)$, and we call Γ a *set theory*.

For example let $2^{\mathcal{U}}$ be the set of *all* subsets of $\mathcal{U}$, including both $\mathcal{U}$ and Φ. Clearly $2^{\mathcal{U}}$ satisfies closure with respect to $(^{-}, \cup, \cap)$.

We shall call a set theory Γ that satisfies the following axioms a *Boolean algebra*.

Axioms

S1. Zero element	$A \cup \Phi = A,\ A \cap \Phi = \Phi$
S2. Identity element	$A \cup \mathcal{U} = \mathcal{U},\ A \cap U = A$
S3. Idempotency	$A \cup A = A,\ A \cap A = A$
S4. Negativity	$A \cup \overline{A} = \mathcal{U},\ A \cap \overline{A} = \Phi$
	$\overline{A} = A$
S5. Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
S6. De Morgan Rule	$\overline{A \cup B} = \overline{A} \cap \overline{B}$
	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
S7. Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
S8. Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

We can illustrate each of the axioms by Venn diagrams in the following way.

Let the square on the page represent the universal set $\mathcal{U}$. A subset B of points within $\mathcal{U}$ can then represent the set B. Given two subsets A, B the union is the hatched area, while the intersection is the double hatched area. See Fig. 1.1.

We shall use $\subset$ to mean "included in". Thus "$A \subset B$" means that every element in A is also an element of B. Thus:

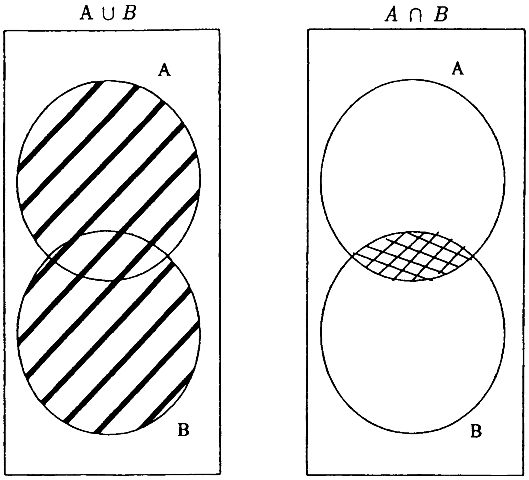


Fig. 1.1 Union

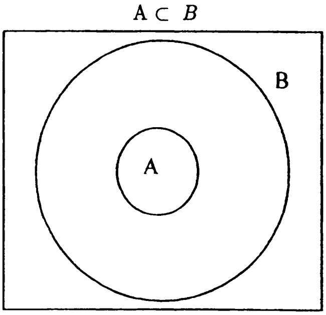


Fig. 1.2 Inclusion

Suppose now that $P(A)$ is the property that characterizes A, or that

$$A = \{x : x \text{ satisfies } P(A)\}.$$

The symbol $\equiv$ means "identical to", so that $[x \in A] \equiv$ "x satisfies $P(A)$".

Associated with any set theory is a *propositional calculus* which satisfies properties analogous with a Boolean algebra, except that we use $\wedge$ and $\vee$ instead of the symbols $\cap$ and $\cup$ for "and" and "or".

For example:

$$A \cup B = \{x : \text{"}x \text{ satisfies } P(A)\text{"} \vee \text{"}x \text{ satisfies } P(B)\text{"}\}$$
$$A \cap B = \{x : \text{"}x \text{ satisfies } P(A)\text{"} \wedge \text{"}x \text{ satisfies } P(B)\text{"}\}.$$

The analogue of "$\subset$" is "if...then" or "implies", which is written $\Rightarrow$.

Thus $A \subset B \equiv$ ["x satisfies $P(A)$" $\Rightarrow$ "x satisfies $P(B)$"].

The analogue of "=" in set theory is the symbol "$\Longleftrightarrow$" which means "if and only if", generally written "iff". For example,

$$[A = B] = \big[\text{“}x \text{ satisfies } P(A)\text{”} \Longleftrightarrow \text{“}x \text{ satisfies } P(B)\text{”}\big].$$

Hence

$$[A = B] \equiv \big[\text{“}x \in A\text{”} \Longleftrightarrow \text{“}x \in B\text{”}\big] \equiv [A \subset B \text{ and} B \subset A].$$

1.1.2 A Propositional Calculus

Let $\{\mathcal{U}, \Phi, P_1, \ldots, P_i, \ldots\}$ be a family of simple *propositions*. $\mathcal{U}$ is the *universal proposition* and always true, whereas Φ is the *null proposition* and always false. Two propositions P_1, P_2 can be combined to give a proposition $P_1 \wedge P_2$ (*i.e.*, P_1 *and* P_2) which is true iff both P_1 and P_2 are true, and a proposition $P_1 \vee P_2$ (*i.e.*, P_1 *or* P_2) which is true if either P_1 or P_2 is true. For a proposition P, the complement $\overline{P}$ in $\mathcal{U}$ is true iff P is false, and is false iff P is true.

Now extend the family of simple propositions to a family $\mathcal{P}$, by including in $\mathcal{P}$ any *propositional sentence* $S(P_1, \ldots, P_i, \ldots)$ which is made up of simple propositions combined under $^-, \vee, \wedge$. Then $\mathcal{P}$ satisfies *closure* with respect to $(^-, \vee, \wedge)$ and is called a *propositional calculus*.

Let T be the truth function, which assigns to any simple proposition, P_i, the value 0 if P_i is false and 1 if P_i is true. Then T extends to sentences in the obvious way, following the rules of logic, to give a truth function $T : \mathcal{P} \to \{0, 1\}$. If $T(S_1) = T(S_2)$ for all truth values of the constituent simple propositions of the sentences S_1 and S_2, then $S_1 = S_2$ (*i.e.*, S_1 and S_2 are identical propositions).

For example the truth values of the proposition $P_1 \vee P_2$ and $P_2 \vee P_1$ are given by the table:

$T(P_1)$	$T(P_2)$	$T(P_1 \vee P_2)$	$T(P_2 \vee P_1)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Since $T(P_1 \vee P_2) = T(P_2 \vee P_1)$ for all truth values it must be the case that $P_1 \vee P_2 = P_2 \vee P_1$.

Similarly, the truth tables for $P_1 \wedge P_2$ and $P_2 \wedge P_1$ are:

$T(P_1)$	$T(P_2)$	$T(P_1 \wedge P_2)$	$T(P_2 \wedge P_1)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Thus $P_1 \wedge P_2 = P_2 \wedge\ P_1$.

The propositional calculus satisfies commutativity of $\wedge$ and $\vee$. Using these truth tables the other properties of a Boolean algebra can be shown to be true.

For example:

(i) $P \vee \Phi = P$, $P \wedge \Phi = \Phi$.

$T(P)$	$T(\Phi)$	$T(P \vee \Phi)$	$T(P \wedge \Phi)$
0	0	0	0
1	0	1	0

(ii) $P \vee \mathcal{U} = \mathcal{U}$, $P \wedge \mathcal{U} = P$.

$T(P)$	$T(\mathcal{U})$	$T(P \vee \mathcal{U})$	$T(P \wedge \mathcal{U})$
0	1	1	0
1	1	1	1

(iii) Negation is given by reversing the truth value. Hence $\overline{\overline{P}} = P$.

$T(P)$	$T(\overline{P})$	$T(\overline{\overline{P}})$
0	1	0
1	0	1

(iv) $P \vee \overline{P} = \mathcal{U}$, $P \wedge \overline{P} = \Phi$.

$T(P)$	$T(\overline{P})$	$T(P \vee \overline{P})$	$T(P \wedge \overline{P})$
0	1	1	0
1	0	1	0

Example 1.1 Truth tables can be used to show that a propositional calculus $\mathcal{P} = (\mathcal{U}, \Phi, P_1, P_2, \ldots)$ with the operators $(^-, \vee, \wedge)$ is a Boolean algebra.

Suppose now that $S_1(A_1, \ldots, A_n)$ is a compound set (or sentence) which is made up of the sets $A_1, \ldots, A_n$ together with the operators $\{\cup, \cap, ^-\}$.

For example suppose that

$$S_1(A_1, A_2, A_3) = A_1 \cup (A_2 \cap A_3),$$

and let $P(A_1), P(A_2), P(A_3)$ be the propositions that characterise A_1, A_2, A_3. Then

$$S_1(A_1, A_2, A_3) = \big\{x : x \text{ satisfies “} S_1\big(P(A_1), P(A_2), P(A_3)\big)\text{”}\big\}$$

$S_1(P(A_1), P(A_2), P(A_3))$ has precisely the same form as $S_1(A_1, A_2, A_3)$ except that $P(A_1)$ is substituted for A_i, and $(\wedge, \vee)$ are substituted for $(\cap, \cup)$.

In the example

$$S_1\big(P(A_1), P(A_2), P(A_3)\big) = P(A_1) \vee \big(P(A_2) \wedge P(A_3)\big).$$

Since $\mathcal{P}$ is a Boolean algebra, we know [by associativity] that $P(A_1) \vee (P(A_2) \wedge P(A_3)) = (P(A_1) \vee P(A_2)) \wedge (P(A_1) \vee P(A_3)) = S_2(P(A_1), P(A_2), P(A_3))$, say.

Hence the propositions $S_1(P(A_1), P(A_2), P(A_3))$ and $S_2(P(A_1), P(A_2), P(A_3))$, are identical, and the sentence

$$\begin{aligned} S_1(A_2, A_2, A_3) &= \big\{x : x \text{ satisfies } P\big((A_1) \vee P(A_2)\big) \wedge \big(P(A_1) \vee P(A_3)\big)\big\} \\ &= (A_1 \cup A_2) \cap (A_1 \cup A_3) \\ &= S_2(A_1, A_2, A_3). \end{aligned}$$

Consequently if $\Gamma = (\mathcal{U}, \Phi, A_1, A_2, \ldots)$ is a set theory, then by exactly this procedure Γ can be shown to be a Boolean algebra.

Suppose now that Γ is a set theory with universal set $\mathcal{U}$, and X is a subset of $\mathcal{U}$. Let $\Gamma_X = (X, \Phi, A_1 \cap X, A_2 \cap X, \ldots)$. Since Γ is a set theory on $\mathcal{U}$, Γ_X must be a set theory on X, and thus there will exist a Boolean algebra in Γ_X.

To see this consider the following:

1. Since $A \in \Gamma$, then $\overline{A} \in \Gamma$. Now let $A_X = A \cap X$. To define the complement or negation (let us call it $\overline{A}_X$) of A in Γ_X we have $\overline{A}_X = \{x : x \text{ is in } X \text{ but not in } A\} = X \cap \overline{A}$. As we noted previously this is also often written $X - A$, or $X \backslash A$. But this must be the same as the complement or $A \cap X$ in X, *i.e.*, $\overline{(A \cap X)} \cap X = (\overline{A} \cup \overline{X}) \cap X = (\overline{A} \cap X) \cup (\overline{X} \cap X) = \overline{A} \cap X$.
2. If $A, B \in \Gamma$ then $(A \cap B) \cap X = (A \cap X) \cap (B \cap X)$. (The reader should examine the behaviour of union.)

A notion that is very close to that of a set theory is that of a *topology*.

Say that a family $\Gamma = (\mathcal{U}, \Phi, A_1, A_2, \ldots)$ is a *topology* on $\mathcal{U}$ iff

T1. when $A_1, A_2 \in \Gamma$ then $A_1 \cap A_2 \in \Gamma$;

T2. If $A_j \in \Gamma$ for all j belonging to some index set J (possibly infinite) then $\bigcup_{j \in J} A_j \in \Gamma$.

T3. Both $\mathcal{U}$ and Φ belong to Γ.

Axioms **T1** and **T2** may be interpreted as saying that Γ is closed under *finite intersection* and (infinite) *union*.

Let X be any subset of $\mathcal{U}$. Then the *relative topology* Γ_X induced from the topology Γ on $\mathcal{U}$ is defined by

$$\Gamma_X = (X, \Phi, A_1 \cap X, \ldots)$$

where any set of the form $A \cap X$, for $A \in \Gamma$, belongs to Γ_X.

Example 1.2 We can show that Γ_X is a topology. If $U_1, U_2 \in \Gamma_X$ then there must exist sets $A_1, A_2 \in \Gamma$ such that $U_i = A_i \cap X$, for $i = 1, 2$. But then

$$\begin{aligned} U_1 \cap U_2 &= (A_1 \cap X) \cap (A_2 \cap X) \\ &= (A_1 \cap A_2) \cap X. \end{aligned}$$

Since Γ is a topology, $A_1 \cap A_2 \in \Gamma$. Thus $U_1 \cap U_2 \in \Gamma_X$. Union follows similarly.

1.1.3 Partitions and Covers

If X is a set, a *cover* for X is a family $\Gamma = (A_1, A_2, \ldots, A_j, \ldots)$ where j belongs to an index set J (possibly infinite) such that

$$X = \cup\{A_j : j \in J\}.$$

A *partition* for X is a cover which is *disjoint*, *i.e.*, $A_j \cap A_k = \Phi$ for any distinct $j, k \in J$.

If Γ_X is a cover for X, and Y is a subset of X then $\Gamma_Y = \{A_j \cap Y : j \in J\}$ is the induced cover on Y.

1.1.4 The Universal and Existential Quantifiers

Two operators which may be used to construct propositions are the universal and existential quantifiers.

For example, "*for all* x in A it is the case that x satisfies $P(A)$." The term "for all" is the universal quantifier, and generally written as ∀.

On the other hand we may say "there exists some x in A such that x satisfies $P(A)$." The term "there exists" is the existential quantifier, generally written ∃.

Note that these have negations as follows:

$$\text{not}[\exists\ x\ s.t.\ x \text{ satisfies } P] \equiv [\forall\ x : x \text{ does not satisfy} P]$$

$$\text{not}[\forall\ x : x \text{ satisfies} P] \equiv [\exists\ x\ s.t.\ x \text{ does not satisfy } P].$$

We use $s.t.$ to mean "such that".

1.2 Relations, Functions and Operations

1.2.1 Relations

If X, Y are two sets, the *Cartesian product set* $X \times Y$ is the set of *ordered pairs* (x, y) such that $x \in X$ and $y \in Y$.

For example if we let $\Re$ be the set of real numbers, then $\Re \times \Re$ or $\Re^2$ is the set

$$\{(x, y) : x \in \Re, y \in \Re\},$$

namely the plane. Similarly $\Re^n = \Re \times \cdots \times \Re$ (n times) is the set of n-tuples of real numbers, defined by induction, *i.e.*, $\Re^n = \Re\ \times (\Re\ \times (\Re\ \times \cdots, \ldots))$.

A subset of the Cartesian product $Y \times X$ is called a *relation*, P, on $Y \times X$. If $(y, x) \in P$ then we sometimes write yPx and say that y stands in relation P to x. If it is not the case that $(y, x) \in P$ then write $(y, x) \notin P$ or not (yPx). X is called the *domain* of P, and Y is called the target or *codomain* of P.

If V is a relation on $Y \times X$ and W is a relation on $Z \times Y$, then define the relation $W \circ V$ to be the relation on $Z \times X$ given by $(z, x) \in W \circ V$ iff for some $y \in Y$, $(z, y) \in W$ and $(y, x) \in V$. The new relation $W \circ V$ on $Z \times X$ is called the composition of W and V.

The *identity relation* (or diagonal) e_X on $X \times X$ is

$$e_X = \{(x,x) : x \in X\}.$$

If P is a relation on $Y \times X$, its *inverse*, P^{-1}, is the relation on $X \times Y$ defined by

$$P^{-1} = \{(x,y) \in X \times Y : (y,x) \in P\}.$$

Note that:

$$P^{-1} \circ P = \{(z,x) \in X \times X : \exists\, y \in Y \; s.t.\; (z,y) \in P^{-1} \text{ and } (y,x) \in P\}.$$

Suppose that the *domain* of P is X, *i.e.*, for every $x \in X$ there is some $y \in Y$ s.t. $(y,x) \in P$. In this case for every $x \in X$, there exists $y \in Y$ such that $(x,y) \in P^{-1}$ and so $(x,x) \in P^{-1} \circ P$ for any $x \in X$. Hence $e_X \subset P^{-1} \circ P$. In the same way

$$P \circ P^{-1} = \{(t,y) \in Y \times Y : \exists x \in X s.t.\; (t,x) \in P \text{ and } (x,y) \in P^{-1}\}$$

and so $e_Y \subset P \circ P^{-1}$.

1.2.2 Mappings

A relation P on $Y \times X$ defines an assignment *or mapping* from X to Y, which is called ϕ_P and is given by

$$\phi_P(x) = \{y : (y,x) \in P\}.$$

In general we write $\phi : X \to Y$ for a mapping which assigns to each element of X the *set*, $\phi(x)$, of elements in Y. As above, the set Y is called the *co-domain* of ϕ.

The *domain* of a mapping, ϕ, is the set $\{x \in X : \exists\, y \in Y \; s.t.\; y \in \phi(x)\}$, and the *image* of ϕ is $\{y \in Y : \exists\, x \in X \; s.t.\; y \in \; \phi(x)\}$.

Suppose now that V, W are relations on $Y \times X, Z \times Y$ respectively. We have defined the composite relation $W \circ V$ on $Z \times X$. This defines a mapping $\phi_{W \circ V} : X \to Z$ by $z \in \phi_{W \circ V}(x)$ iff $\exists y \in Y$ such that $(y,x) \in V$ and $(z,y) \in W$. This in turn means that $y \in \phi_V(x)$ and $z \in \phi_W(y)$.

If $\phi : X \to Y$ and $\psi : Y \to Z$ are two mappings then define their *composition* $\psi \circ \phi : X \to Z$ by

$$(\psi \circ \phi)(x) = \psi[\phi(x)] = \cup\{\psi(y) : y \in \phi(x)\}.$$

Clearly $z \in \phi_{W \circ V}(x)$ iff $z \in \phi_W[\phi_V(x)]$.

Thus $\phi_{W \circ V}(x) = \phi_W[\phi_V(x)] = [(\phi_W \circ \phi_V)(x)]$, $\forall x \in X$. We therefore write $\phi_{W \circ V} = \phi_W \circ \phi_V$.

For example suppose V and W are given by

$$V : \big\{(2,3),(3,2),(1,2),(4,4),(4,1)\big\} \quad \text{and}$$
$$W : \big\{(1,4),(4,4),(4,1),(2,1),(2,3),(3,2)\big\}$$

with mappings

$$\begin{tikzcd}
1 \arrow[r, "\phi_V"] & 4 \arrow[r, "\phi_W"] \arrow[dr] & 1 \\
4 \arrow[ur] & 1 \arrow[r] \arrow[dr] & 4 \\
2 \arrow[r] \arrow[ur] & 3 \arrow[r] & 2 \\
3 \arrow[r] & 2 \arrow[r] & 3
\end{tikzcd}$$

then the composite mapping $\phi_W \circ \phi_V = \phi_{W \circ V}$ is

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 1 \\
4 \arrow[r] \arrow[ur] \arrow[dr] & 4 \\
2 \arrow[r] & 2 \\
3 \arrow[r] & 3
\end{tikzcd}$$

with relation

$$W \circ V = \big\{(3,3),(2,2),(4,2),(1,4),(4,4),(1,1),(4,1)\big\}.$$

Given a mapping $\phi : X \to Y$ then the reverse procedure to the above gives a relation, called the graph of ϕ, or graph (ϕ), where

$$\text{graph}(\phi) = \bigcup_{x \in X} \big(\phi(x) \times \{x\}\big) \subset Y \times X.$$

In the obvious way if $\phi : X \to Y$ and $\psi : Y \to Z$, are mappings, with composition $\psi \circ \phi : X \to Z$, then graph $(\psi \circ \phi) = \text{graph}(\psi) \circ \text{graph}(\phi)$.

Suppose now that P is a relation on $Y \times X$, with inverse P^{-1} on $X \times Y$, and let $\phi_P : X \to Y$ be the mapping defined by P. Then the mapping $\phi_{P^{-1}} : Y \to X$ is defined as follows:

$$\begin{aligned}
\phi_{P^{-1}}(y) &= \big\{x : (x,y) \in P^{-1}\big\} \\
&= \big\{x : (y,x) \in P\big\} \\
&= \big\{x : y \in \phi_P(x)\big\}.
\end{aligned}$$

More generally if $\phi : X \to Y$ is a mapping then the inverse mapping $\phi^{-1} : Y \to X$ is given by

$$\phi^{-1}(y) = \big\{x : y \in \phi(x)\big\}.$$

Thus

$$\phi_{P^{-1}} = (\phi_P)^{-1} : Y \to X.$$

For example let $\mathcal{Z}_4$ be the first four positive integers and let P be the relation on $\mathcal{Z}_4 \times \mathcal{Z}_4$ given by

$$P = \big\{(2,3), (3,2), (1,2), (4,4), (4,1)\big\}.$$

Then the mapping ϕ_P and inverse $\phi_{P^{-1}}$ are given by:

$$\begin{array}{lccclccc}
\phi_P\colon & 1 & \longrightarrow & 4 & \phi_{P^{-1}}\colon & 4 & \longrightarrow & 1 \\
 & & \nearrow & 1 & & 1 & \searrow & \\
 & 4 & \nearrow & & & & \searrow & 4 \\
 & 2 & \longrightarrow & 3 & & 3 & \longrightarrow & 2 \\
 & 3 & \longrightarrow & 2 & & 2 & \longrightarrow & 3
\end{array}$$

If we compose P^{-1} and P as above then we obtain

$$P^{-1} \circ P = \big\{(1,1), (1,4), (4,1), (4,4), (2,2), (3,3)\big\},$$

with mapping

$$\begin{array}{ccc}
 & \phi_{P^{-1}} \circ \phi_P & \\
1 & \longrightarrow & 1 \\
 & \nearrow\searrow & \\
4 & \longrightarrow & 4 \\
2 & \longrightarrow & 2 \\
3 & \longrightarrow & 3
\end{array}$$

Note that $P^{-1} \circ P$ contains the identity or diagonal relation $e = \{(1,1), (2,2), (3,3), (4,4)\}$ on $\mathcal{Z}_4 = \{1,2,3,4\}$. Moreover $\phi_{P^{-1}} \circ \phi_P = \phi_{(P^{-1} \circ P)}$.

The mapping $id_X : X \to X$ defined by $id_X(x) = x$ is called the *identity* mapping on X. Clearly if e_X is the identity relation, then $\phi_{e_X} = id_X$ and graph $(id_X) = e_x$.

If ϕ, ψ are two mappings $X \to Y$ then write $\psi \subset \phi$ iff for each $x \in X$, $\psi(x) \subset \phi(x)$.

As we have seen $e_X \subset P^{-1} \circ P$ and so

$$\phi_{e_X} = id_X \subset \phi_{(P^{-1} \circ P)} = \phi_{P^{-1}} \circ \phi_P = (\phi_P)^{-1} \circ \phi_P.$$

(This is only precisely true when X is the domain of P, *i.e.*, when for every $x \in X$ there exists some $y \in Y$ such that $(y, x) \in P$.)

1.2.3 Functions

If for all x in the domain of ϕ, there is exactly one y such that $y \in \phi(x)$ then ϕ is called a *function*. In this case we generally write $f : X \to Y$, and sometimes $x \xrightarrow{f} y$ to indicate that $f(x) = y$. Consider the function f and its inverse f^{-1} given by

$$\begin{tikzcd}
1 \arrow[r, "f"] & 4 \arrow[r, "f^{-1}"] \arrow[dr] & 1 & 1 \arrow[r, "f^{-1}\circ f"] \arrow[dr] & 1 \\
4 \arrow[ur] & 1 & 4 & 4 \arrow[r] \arrow[ur] & 4 \\
2 \arrow[r] & 3 \arrow[r] & 2 & 2 \arrow[r] & 2 \\
3 \arrow[r] & 2 \arrow[r] & 3 & 3 \arrow[r] & 3
\end{tikzcd}$$

Clearly f^{-1} is not a function since it maps 4 to both 1 and 4, *i.e.*, the graph of f^{-1} is $\{(1, 4), (4, 4), (2, 3), (3, 2)\}$. In this case id_X is contained in $f^{-1} \circ f$ but is not identical to $f^{-1} \circ f$. Suppose that f^{-1} is in fact a function. Then it is necessary that for each y in the image there be at most one x such that $f(x) = y$. Alternatively if $f(x_1) = f(x_2)$ then it must be the case that $x_1 = x_2$. In this case f is called $1 - 1$ or *injective*. Then f^{-1} is a function and

$$id_X = f^{-1} \circ f \quad \text{on the domain } X \text{ of } f$$

$$id_Y = f \circ f^{-1} \quad \text{on the image } Y \text{ of } f.$$

A mapping $\phi : X \to Y$ is said to be *surjective* (or called a surjection) iff every $y \in Y$ belongs to the image of ϕ; that is, $\exists\, x \in X$ s.t. $y \in \phi(x)$.

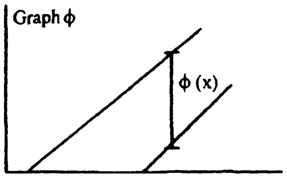
A function $f : X \to Y$ which is both injective and surjective is said to be *bijective*.

Example 1.3 Consider

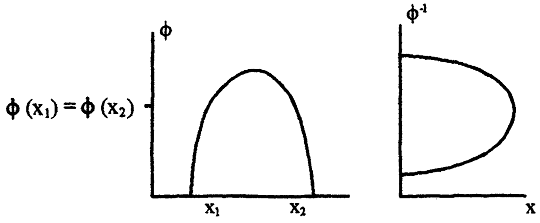
$$\begin{tikzcd}
1 \arrow[r, "\pi"] & 4 \arrow[r, "\pi^{-1}"] & 1 \\
4 \arrow[r] & 2 \arrow[r] & 4 \\
2 \arrow[r] & 3 \arrow[r] & 2 \\
3 \arrow[r] & 1 \arrow[r] & 3.
\end{tikzcd}$$

In this case the domain and image of π coincide and π is known as a *permutation*. Consider the possibilities where ϕ is a mapping $\Re \to \Re$, with graph $(\phi) \subset \Re^2$. (Remember $\Re$ is the set of real numbers.) There are three cases:

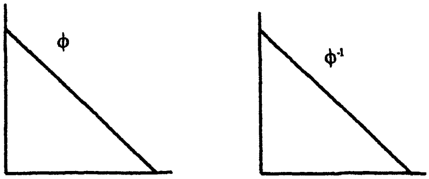
(i) ϕ is a mapping:



(ii) ϕ is a non injective function:



(iii) ϕ is an injective function:



1.3 Groups and Morphisms

We earlier defined the composition of two mappings $\phi : X \to Y$ and $\psi : Y \to X$ to be $\psi \circ \phi : X \to Z$ given by $(\psi \circ \phi)(x) = \psi[\phi(x)] = \cup \{\psi(y) : y \in \phi(x)\}$. In the case of functions $f : X \to Y$ and $g : Y \to Z$ this translates to

$$(g \circ f)(x) = g\big[f(x)\big] = \big\{g(y) : y = f(x)\big\}.$$

Since both f, g are functions the set on the right is a singleton set, and so $g \circ f$ is a function. Write $\mathcal{F}(A, B)$ for the set of functions from A to B. Thus the composition operator, $\circ$, may be regarded as a function:

$$\begin{array}{rlll} \circ : & \mathcal{F}(X, Y) \times \mathcal{F}(Y, Z) & \to & \mathcal{F}(X, Z) \\ & (f, g) & \to & g \circ f. \end{array}$$

Example 1.4 To illustrate consider the function (or matrix) F given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 & + & bx_2 \\ cx_1 & + & dx_2 \end{pmatrix}.$$

This can be regarded as a function $F : \Re^2 \to \Re^2$ since it maps $(x_1, x_2) \to (ax_1 + bx_2, cx_1 + dx_2) \in \Re^2$.

Now let

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad H = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

$F \circ H$ is represented by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \overset{H}{\to} \begin{pmatrix} e(ax_1 + bx_2) + f(cx_1 + dx_2) \\ g(ax_1 + bx_2) + h(cx_1 + dx_2) \end{pmatrix}.$$

Thus

$$(H \circ F)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ea + fc \mid eb + fd \\ \hline ga + hc \mid gb + hd \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$(H \circ F) = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc \mid eb + fd \\ \hline ga + hc \mid gb + hd \end{pmatrix}.$$

The *identity* E is the function

$$E\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since this must be true for all x_1, x_2, it follows that $a = d = 1$ and $c = b = 0$.

Thus $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Suppose that the mapping $F^{-1} : \Re^2 \to \Re^2$ is actually a matrix. Then it is certainly a function, and by Sect. 1.2.3, $F^{-1} \circ F$ must be equal to the identity function on $\Re^2$, which here we call E. To determine F^{-1}, proceed as follows:

Let $F^{-1} = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$. We know $F^{-1} \circ F = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Thus

$$\begin{array}{lcl} ea + fc = 1 & \mid & eb + fd = 0 \\ ga + hc = 0 & \mid & gb + hd = 1. \end{array}$$

If $a \neq 0$ and $b \neq 0$ then $e = -\frac{fd}{b} = \frac{1 - fc}{a}$.

Now let $|F| = (ad - bc)$, where $|F|$ is called the *determinant* of F. Clearly if $|F| \neq\ 0$, then $f = -b/|F|$. More generally, if $|F| \neq 0$ then we can solve the equations to obtain:

$$F^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \frac{1}{|F|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $|F| = 0$, then what we have called F^{-1} is not defined. This suggests that when $|F| = 0$, the inverse F^{-1} cannot be represented by a matrix, and in particular that F^{-1} is not a function. In this case we shall call F *singular*. When $|F| \neq 0$ then we shall call F *non-singular*, and in this case F^{-1} can be represented by a matrix, and thus a function. Let $M(2)$ stand for the set of 2×2 matrices, and let $M^*(2)$ be the subset of $M(2)$ consisting of non-singular matrices.

We have here defined a *composition operation*:

$$\begin{array}{lll} \circ : & M(2) \times M(2) & \rightarrow & M(2) \\ & (H, F) & \rightarrow & H \circ F. \end{array}$$

Suppose we compose E with F then

$$E \circ F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = F.$$

Finally for any $F \in M^*(2)$ it is the case that there exists a *unique* matrix $F^{-1} \in M(2)$ such that

$$F^{-1} \circ F = E.$$

Indeed if we compute the inverse $(F^{-1})^{-1}$ of F^{-1} then we see that $(F^{-1})^{-1} = F$. Thus F^{-1} itself belongs to $M^*(2)$.

$M^*(2)$ is an example of what is called a *group*.

More generally a *binary operation*, $\circ$, on a set G is a function

$$\begin{array}{llll} \circ : & G \times G & \rightarrow & G, \\ & (x, y) & \rightarrow & x \circ y. \end{array}$$

Definition 1.1 A *group* G is a set G together with a binary operation, $\circ : G \times G \rightarrow G$ which

1. is associative: $(x \circ y) \circ z = x \circ (y \circ z)$ for all x, y, z in G;
2. has an identity $e : e \circ x = x \circ e = x \ \forall\, x \in G$;
3. has for each $x \in G$ an inverse $x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = e$.

When G is a group with operation, $\circ$, write $(G, \circ)$ to signify this.

Associativity simply means that the order of composition in a sequence of compositions is irrelevant. For example consider the integers, $\mathcal{Z}$, under addition. Clearly $a + (b + c) = (a + b) + c$, where the left hand side means add b to c, and then add

a to this, while the right hand side is obtained by adding a to b, and then adding c to this. Under addition, the identity is that element $e \in \mathcal{Z}$ such that $a + e = a$. This is usually written 0. Finally the additive inverse of an integer $a \in \mathcal{Z}$ is $(-a)$ since $a + (-a) = 0$. Thus $(\mathcal{Z}, +)$ is a group.

However consider the integers under multiplication, which we shall write as "·". Again we have associativity since

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Clearly 1 is the identity since $1 \cdot a = a$. However the inverse of a is that object a^{-1} such that $a \cdot a^{-1} = 1$. Of course if $a = 0$, then no such inverse exists. For $a \neq 0$, a^{-1} is more commonly written $\frac{1}{a}$. When a is non-zero, and different from ± 1, then $\frac{1}{a}$ is not an integer. Thus $(\mathcal{Z}, \cdot)$ is not a group. Consider the set $\mathcal{Q}$ of rationals, *i.e.*, $a \in \mathcal{Q}$ iff $a = \frac{p}{q}$, where both p and q are integers. Clearly $1 \in \mathcal{Q}$. Moreover, if $a = \frac{p}{q}$ then $a^{-1} = \frac{q}{p}$ and so belongs to $\mathcal{Q}$. Although zero does not have an inverse, we can regard $(\mathcal{Q}\backslash\{0\}, \cdot)$ as a group.

Lemma 1.1 *If* $(G, \circ)$ *is a group, then the identity* e *is unique and for each* $x \in G$ *the inverse* x^{-1} *is unique. By definition* $e^{-1} = e$. *Also* $(x^{-1})^{-1} = x$ *for any* $x \in G$.

Proof

1. Suppose there exist two distinct identities, e, f. Then $e \circ x = f \circ x$ for some x. Thus $(e \circ x) \circ x^{-1} = (f \circ x) \circ x^{-1}$. This is true because the composition operation

$$\big((e \circ x), x^{-1}\big) \to (e \circ x) \circ x^{-1}$$

gives a unique answer.
By associativity $(e \circ x) \circ x^{-1} = e \circ (x \circ x^{-1})$, *etc.*
Thus $e \circ (x \circ x^{-1}) = f \circ (x \circ x^{-1})$. But $x \circ x^{-1} = e$, say.

Since e is an identity, $e \circ e = f \circ e$ and so $e = f$. Since $e \circ e = e$ it must be the case that $e^{-1} = e$.

2. In the same way suppose x has two distinct inverses, y, z, so $x \circ y = x \circ z = e$. Then

$$\begin{aligned} y \circ (x \circ y) &= y \circ (x \circ z) \\ (y \circ x) \circ y &= (y \circ x) \circ z \\ e \circ y &= e \circ z \\ y &= z. \end{aligned}$$

3. Finally consider the inverse of x^{-1}. Since $x \circ (x^{-1}) = e$ and by definition $(x^{-1})^{-1} \circ (x^{-1}) = e$ by part (2), it must be the case that $(x^{-1})^{-1} = x$. □

We can now construct some interesting groups.

Lemma 1.2 *The set $M^*(2)$ of 2×2 non-singular matrices form a group under matrix composition*, $\circ$.

Proof We have already shown that there exists an identity matrix E in $M^*(2)$. Clearly $|E| = 1$ and so E has inverse E.

As we saw in Example 1.4, when we solved $H \circ F = E$ we found that

$$H = F^{-1} = \frac{1}{|F|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

By Lemma 1.1, $(F^{-1})^{-1} = F$ and so F^{-1} must have an inverse, *i.e.*, $|F^{-1}| \neq 0$, and so F^{-1} is non-singular. Suppose now that the two matrices H, F belong to $M^*(2)$. Let

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$H = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

As in Example 1.4,

$$\begin{aligned} |H \circ F| &= \left| \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix} \right| = (ea + fc)(gb + hd) - (ga + hc)(eb + fd) \\ &= (eh - gf)(ad - bc) = |H||F|. \end{aligned}$$

Since both H and F are non-singular, $|H| \neq 0$ and $|F| \neq 0$ and so $|H \circ F| \neq 0$. Thus $H \circ F$ belongs to $M^*(2)$, and so matrix composition is a *binary operation* $M^*(2) \times M^*(2) \to M^*(2)$.

Finally the reader may like to verify that matrix composition on $M^*(2)$ is *associative*. That is to say if F, G, H are non-singular 2×2 matrices then

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

As a consequence $(M^*(2), \circ)$ is a group. □

Example 1.5 For a second example consider the addition operation on $M(2)$ defined by

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+e & f+b \\ g+c & h+d \end{pmatrix}.$$

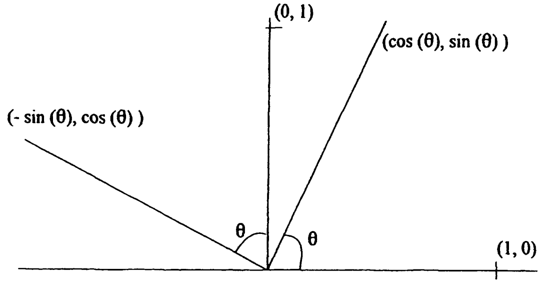


Fig. 1.3 Rotation

Clearly the identity matrix is $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and the inverse of F is

$$-F=-\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}-a&-b\\-c&-d\end{pmatrix}.$$

Thus $(M(2),+)$ is a group.

Finally consider those matrices which represent *rotations* in $\Re^2$.

If we rotate the point $(1,0)$ in the plane through an angle θ in the anticlockwise direction then the result is the point $(\cos\theta,\sin\theta)$, while the point $(0,1)$ is transformed to $(-\sin\theta,\cos\theta)$. See Fig. 1.3. As we shall see later, this rotation can be represented by the matrix

$$\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}$$

which we will call $e^{i\theta}$.

Let Θ be the set of all matrices of this form, where θ can be any angle between 0 and 360°. If $e^{i\theta}$ and $e^{i\psi}$ are rotations by θ, ψ respectively, and we rotate by θ first and then by ψ, then the result should be identical to a rotation by $\psi+\theta$. To see this:

$$\begin{aligned}&\begin{pmatrix}\cos\psi&-\sin\psi\\ \sin\psi&\cos\psi\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\\&=\left(\begin{array}{c|c}\cos\psi\cos\theta-\sin\psi\sin\theta & -\cos\psi\sin\theta-\sin\psi\cos\theta\\ \hline \sin\psi\cos\theta+\cos\psi\sin\theta & -\sin\psi\sin\theta+\cos\psi\cos\theta\end{array}\right)\\&=\begin{pmatrix}\cos(\psi+\theta)&-\sin(\psi+\theta)\\ \sin(\psi+\theta)&\cos(\psi+\theta)\end{pmatrix}=e^{i(\theta+\psi)}.\end{aligned}$$

Note that $|e^{i\theta}|=\cos^2\theta+\sin^2\theta=1$. Thus

$$\left(e^{i\theta}\right)^{-1}=\begin{pmatrix}\cos\theta&\sin\theta\\ -\sin\theta&\cos\theta\end{pmatrix}=\begin{pmatrix}\cos\theta&-\sin(-\theta)\\ \sin(-\theta)&\cos\theta\end{pmatrix}=e^{i(-\theta)}.$$

Hence the inverse to $e^{i\theta}$ is a rotation by $(-\theta)$, that is to say by θ but in the opposite direction. Clearly $E = e^{i0}$, a rotation through a zero angle. Thus $(\Theta, \circ)$ is a group. Moreover Θ is a subset of $M^*(2)$, since each rotation has a non-singular matrix. Thus Θ is a *subgroup* of $M^*(2)$.

A subset Θ of a group $(G, \circ)$ is a *subgroup* of G iff the composition operation, $\circ$, restricted to Θ is "closed", and Θ is a group in its own right. That is to say (i) if $x, y \in \Theta$ then $x \circ y \in \Theta$, (ii) the identity e belongs to Θ and (iii) for each x in Θ the inverse, x^{-1}, also belongs to Θ.

Definition 1.2 Let $(X, \circ)$ and $(Y, \cdot)$ be two sets with binary operations, $\circ$, $\cdot$, respectively. A function $f : X \to Y$ is called a *morphism* (with respect to $(\circ, \cdot)$) iff $f(x \circ y) = f(x) \cdot f(y)$, for all $x, y \in X$. If moreover f is bijective as a function, then it is called an *isomorphism.* If $(X, \circ)$, $(Y, \cdot)$ are groups then f is called a *homomorphism.*

A binary operation on a set X is one form of mathematical structure that the set may possess. When an isomorphism exists between two sets X and Y then mathematically speaking their structures are identical.

For example let *Rot* be the set of all rotations in the plane. If $rot(\theta)$ and $rot(\psi)$ are rotations by θ, ψ respectively then we can combine them to give a rotation $rot(\psi + \theta)$, *i.e.*,

$$rot(\psi) \circ rot(\theta) = rot(\psi + \theta).$$

Here $\circ$ means do one rotation then the other. To the rotation, $rot(\theta)$ let f assign the 2×2 matrix, called $e^{i\theta}$ as above. Thus

$$f : (rot, \circ) \to (\Theta, \circ),$$

where $f(rot(\theta)) = e^{i\theta}$.

Moreover

$$f\bigl(rot(\psi) \circ rot(\theta)\bigr) = f\bigl(rot(\psi + \theta)\bigr)$$

$$e^{i\psi} \circ e^{i\theta} = e^{i(\psi+\theta)}$$

Clearly the identity rotation is $rot(0)$ which corresponds to the zero matrix $e^{i\circ}$, while the inverse rotation to $rot(\theta)$ is $rot(-\theta)$ corresponding to $e^{-i\theta}$. Thus f is a morphism.

Here we have a collection of geometric objects, called rotations, with their own structure and we have found another set of "mathematical" objects namely 2×2 matrices of a certain type, which has an identical structure.

Lemma 1.3 *If $f : (X, \circ) \to (Y, \cdot)$ is a morphism between groups then*

(1) *$f(e_X) = e_Y$ where e_X, e_Y are the identities in X, Y.*
(2) *for each x in X, $f(x^{-1}) = [f(x)]^{-1}$.*

Proof

1. Since f is a morphism $f(x \circ e_X) = f(x) \cdot f(e_X) = f(x)$. By Lemma 1.2, e_Y is unique and so $f(e_X) = e_Y$.
2. $f(x \circ x^{-1}) = f(x) \cdot f(x^{-1}) = f(e_X) = e_Y$. By Lemma 1.2, $[f(x)]^{-1}$ is unique, and so $f(x^{-1}) = [f(x)]^{-1}$. □

As an example, consider the determinant function $det : M(2) \to \Re$.

From the proof of Lemma 1.3, we know that for any 2×2 matrices, H and F, it is the case that $|H \circ F| = |H|\,|F|$. Thus $det : (M(2), \circ) \to (\Re, \cdot)$ is a *morphism* with respect to matrix composition, $\circ$, in $M(2)$ and multiplication, $\cdot$, in $\Re$.

Note also that if F is non-singular then $det(F) = |F| \neq 0$, and so $det : M^*(2) \to \Re\backslash\{0\}$.

It should be clear that $(\Re\backslash\{0\}, \cdot)$ is a group.

Hence $det : (M^*(2), \circ) \to (\Re\backslash\{0\}, \cdot)$ is a *homomorphism* between these two groups. This should indicate why those matrices in $M(2)$ which have zero determinant are those without an inverse in $M(2)$.

From Example 1.4, the identity in $M^*(2)$ is E, while the multiplicative identity in $\Re$ is 1. By Lemma 1.3, $det(E) = 1$.

Moreover $|F|^{-1} = \frac{1}{|F|}$ and so, by Lemma 1.3, $|F^{-1}| = \frac{1}{|F|}$. This is easy to check since

$$|F^{-1}| = \left| \frac{1}{|F|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right| = \frac{da - bc}{|F|^2} = \frac{|F|}{|F|^2} = \frac{1}{|F|}.$$

However the determinant $det : M^*(2) \to \Re\backslash 0$ is not injective, since it is clearly possible to find two matrices, H, F such that $|H| = |F|$ although H and F are different.

Example 1.6 It is clear that the real numbers form a group $(\Re, +)$ under addition with identity 0, and inverse (to a) equal to $-a$. Similarly the reals form a group $(\Re\backslash\{0\}, \cdot)$ under multiplication, as long as we exclude 0.

Now let $\mathcal{Z}_2$ be the numbers $\{0, 1\}$ and define "addition modulo 2," written $+$, on $\mathcal{Z}_2$, by $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1$, and "multiplication modulo 2," written $\cdot$, on $\mathcal{Z}_2$, by $0 \cdot 0 = 0$, $0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.

Under "addition modulo 2," 0 is the identity, and 1 has inverse 1. Associativity is clearly satisfied, and so $(\mathcal{Z}_2, +)$ is a group. Under multiplication, 1 is the identity and inverse to itself, but 0 has no inverse. Thus $(\mathcal{Z}_2, \cdot)$ is not a group. Note that $(\mathcal{Z}_2\backslash\{0\}, \cdot)$ is a group, namely the trivial group containing only one element. Let $\mathcal{Z}$ be the integers, and consider the function

$$f : \mathcal{Z} \to \mathcal{Z}_2,$$

defined by $f(x) = 0$ if x is even, 1 if x is odd.

We see that this is a morphism $f : (\mathcal{Z}, +) \to (\mathcal{Z}_2, +)$;

1. if x and y are both even then $f(x) = f(y) = 0$; since $x + y$ is even, $f(x+y) = 0$.
2. if x is even and y odd, $f(x) = 0$, $f(y) = 1$ and $f(x) + f(y) = 1$. But $x + y$ is odd, so $f(x + y) = 1$.
3. if x and y are both odd, then $f(x) = f(y) = 1$, and so $f(x) + f(y) = 0$. But $x + y$ is even, so $f(x + y) = 0$.

Since $(\mathcal{Z}, +)$ and $(\mathcal{Z}_2, +)$ are both groups, f is a *homomorphism*. Thus $f(-a) = f(a)$.

On the other hand consider

$$f : (\mathcal{Z}, \cdot) \to (\mathcal{Z}_2, \cdot);$$

1. if x and y are both even then $f(x) = f(y) = 0$ and so $f(x) \cdot f(y) = 0 = f(xy)$.
2. if x is even and y odd, then $f(x) = 0$, $f(y) = 1$ and $f(x) \cdot f(y) = 0$. But xy is even so $f(xy) = 0$.
3. if x and y are both odd, $f(x) = f(y) = 1$ and so $f(x)f(y) = 1$. But xy is odd, and $f(xy) = 1$.

Hence f is a *morphism*. However, neither $(\mathcal{Z}, \cdot)$ nor $(\mathcal{Z}_2, \cdot)$ is a group, and so f is not a homomorphism.

A computer, since it is essentially a "finite" machine, is able to compute in binary arithmetic, using the two groups $(\mathcal{Z}_2, +)$, $(\mathcal{Z}_2 \backslash \{0\}, \cdot)$ rather than with the groups $(\Re, +)$, $(\Re \backslash \{0\}, \cdot)$.

This is essentially because the additive and multiplicative groups based on $\mathcal{Z}_2$ form what is called a *field*.

Definition 1.3

1. A group $(G, \circ)$ is *commutative* or *abelian* iff for all $a, b \in G$, $a \circ b = b \circ a$.
2. A *field* $(\mathcal{F}, +, \cdot)$ is a set together with two operations called addition $(+)$ and multiplication $(\cdot)$ such that $(\mathcal{F}, +)$ is an abelian group with zero, or identity 0, and $(\mathcal{F} \backslash \{0\}, \cdot)$ is an abelian group with identity 1. For convenience the additive inverse of an element $a \in \mathcal{F}$ is written $(-a)$ and the multiplicative inverse of a non zero $a \in \mathcal{F}$ is written a^{-1} or $\frac{1}{a}$.
   Moreover, multiplication is *distributive* over addition, *i.e.,* for all a, b, c in $\mathcal{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

To give an indication of the notion of abelian group, consider $M^*(2)$ again. As we have seen

$$H \circ F = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

However,

$$F \circ H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ea + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Thus $H \circ F \neq F \circ H$ in general and so $M^*(2)$ is non abelian. However, if we consider two rotations $e^{i\theta}$, $e^{i\psi}$ then $e^{i\psi} \circ e^{i\theta} = e^{i(\psi+\theta)} = e^{i\theta} \circ e^{i\psi}$. Thus the group $(\Theta, \circ)$ is abelian.

Lemma 1.4 *Both* $(\Re, +, \cdot)$ *and* $(\mathcal{Z}_2, +, \cdot)$ *are fields.*

Proof Consider $(\mathcal{Z}_2, +, \cdot)$ first of all. As we have seen $(\mathcal{Z}_2, +)$ and $(\mathcal{Z}_2 \backslash \{0\}, \cdot)$ are groups. $(\mathcal{Z}_2, +)$ is obviously abelian since $0 + 1 = 1 + 0 = 1$, while $(\mathcal{Z}_2 \backslash \{0\}, \circ)$ is abelian since it has one element.

To check for distributivity, note that

$$1 \cdot (1 + 1) = 1 \cdot 0 = 0 = 1 \cdot 1 + 1 \cdot 1 = 1 + 1.$$

Finally to see that $(\Re, +, \cdot)$ is a field, we note that for any real numbers, a, b, c, $\Re$, $(b + c) = ab + ac$. □

Given a field $(\mathcal{F}, +, \cdot)$ we define a new object called $\mathcal{F}^n$ where n is a positive integer as follows. Any element $x \in \mathcal{F}^n$ is of the form

$$\begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix}$$

where $x_1, \ldots, x_n$ all belong to $\mathcal{F}$.

F1. If $a \in \mathcal{F}$, and $x \in \mathcal{F}^n$ define $\alpha x \in \mathcal{F}^n$ by

$$\alpha \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \cdot \\ \alpha x_n \end{pmatrix}.$$

F2. Define addition in $\mathcal{F}^n$ by

$$x + y = \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_n + y_n \end{pmatrix}.$$

Since $\mathcal{F}$, by definition, is an abelian additive group, it follows that

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \cdot \\ x_n + y_n \end{pmatrix} = \begin{pmatrix} y_1 + x_1 \\ \cdot \\ y_n + x_n \end{pmatrix} = y + x.$$

Now let

$$\underline{0} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Clearly

$$x + \underline{0} = \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix} = x.$$

Hence $\underline{0}$ belongs to $\mathcal{F}^n$ and is an additive identity in $\mathcal{F}^n$.

Suppose we define

$$(-x) = -\begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \cdot \\ -x_n \end{pmatrix}.$$

Clearly

$$x + (-x) = \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} -x_1 \\ \cdot \\ -x_n \end{pmatrix} = \begin{pmatrix} x_1 & - & x_1 \\ x_n & - & x_n \end{pmatrix} = \underline{0}.$$

Thus for each $x \in \mathcal{F}^n$ there is an inverse, $(-x)$, in $\mathcal{F}^n$.

Finally, since F is an additive group

$$\begin{aligned} x + (y + z) &= \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 + z_1 \\ \cdot \\ y_n + z_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \cdot \\ x_n + y_n \end{pmatrix} + \begin{pmatrix} z_1 \\ \cdot \\ z_n \end{pmatrix} \\ &= (x + y) + z. \end{aligned}$$

Thus $(\mathcal{F}^n, +)$ is an abelian group, with zero $\underline{0}$.

The fact that it is possible to multiply an element $x \in \mathcal{F}^n$ by a *scalar* $a \in \mathcal{F}$ endows $\mathcal{F}^n$ with further structure. To see this consider the example of $\Re^2$.

1. If $a \in \Re$ and both x, y belong to $\Re^2$, then

$$\alpha \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right] = \alpha \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \alpha y_1 \\ \alpha x_2 + \alpha y_2 \end{pmatrix}$$

by distribution, and

$$= \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} + \begin{pmatrix} \alpha y_1 \\ \alpha y_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \alpha \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

by **F1**. Thus $\alpha(x + y) = \alpha x + \alpha y$.

2.

$$(\alpha+\beta)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}(\alpha+\beta)x_1\\(\alpha+\beta)x_2\end{pmatrix}$$

by **F1**

$$=\begin{pmatrix}\alpha x_1+\beta x_1\\\alpha x_2+\beta x_2\end{pmatrix}=\begin{pmatrix}\alpha x_1\\\alpha x_2\end{pmatrix}+\begin{pmatrix}\beta x_1\\\beta x_2\end{pmatrix}$$

by **F2**

$$=\alpha\begin{pmatrix}x_1\\x_2\end{pmatrix}+\beta\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

by **F1**. Therefore, $(\alpha+\beta)x=\alpha x+\beta x$.

3.

$$(\alpha\beta)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}(\alpha\beta)x_1\\(\alpha\beta)x_2\end{pmatrix}$$

by **F1** $=\alpha\left(\begin{smallmatrix}\beta x_1\\\beta x_2\end{smallmatrix}\right)$ by associativity and **F1**, and $=\alpha(\beta x)$ by **F1**.
Thus $(\alpha\beta)x=\alpha(\beta x)$.

4.
$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\cdot x_1\\1\cdot x_2\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Therefore $1(x)=x$.

These four properties characterise what is know as a *vector space*.

Finally, consider the operation of a matrix F on the set of elements in $\Re^2$. By definition

$$F(x+y)=\begin{pmatrix}a&b\\c&d\end{pmatrix}\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}y_1\\y_2\end{pmatrix}\right]=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}$$
$$=\begin{pmatrix}a(x_1+y_1)+b(x_2+y_2)\\c(x_1+y_1)+d(x_2+y_2)\end{pmatrix}=\begin{pmatrix}ax_1+bx_2\\cx_1+dx_2\end{pmatrix}+\begin{pmatrix}ay_1+by_2\\cy_1+dy_2\end{pmatrix}$$

by **F2**

$$=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=F(x)+F(y).$$

Hence $F:(\Re^2,+)\rightarrow(\Re^2,+)$ is a *morphism* from the abelian group $(\Re^2,+)$ into itself.

By Lemma 1.3, we know that $F(\underline{0})=\underline{0}$, and for any element $x\in\Re^2$,

$$F(-x)=F\big(-1(x)\big)=-F(x)=-1F(x).$$

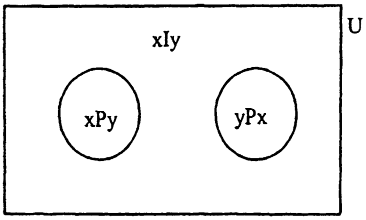


Fig. 1.4 Indifference and strict preference

A morphism between vector spaces is called a *linear transformation*. Vector spaces and linear transformations are discussed in Chap. 2.

1.4 Preferences and Choices

1.4.1 Preference Relations

A *binary relation* P on X is a subset of $X \times X$; more simply P is called a relation on X. For example let $X \equiv \Re$ (the real line) and let P be "$>$" meaning strictly greater than. The relation "$>$" clearly satisfies the following properties:

1. it is never the case that $x > x$
2. it is never the case that $x > y$ *and* $y > x$
3. it is always the case that $x > y$ and $y > z$ implies $x > z$.

These properties can be considered more abstractly. A relation P on X is:

1. *symmetric* iff $xPy \Rightarrow yPx$
   asymmetric iff $xPy \Rightarrow$ not (yPx)
   antisymmetric iff xPy and $yPx \Rightarrow x = y$
2. *reflexive* iff $(xPx)\ \forall\, x \in X$
   irreflexive iff not $(xPx)\ \forall x \in X$
3. *transitive* iff xPy and $yPz \Rightarrow xPz$
4. *connected* iff for any $x, y \in X$ either xPy or yPx.

By analogy with the relation "$>$" a relation P, which is both *irreflexive* and *asymmetric* is called a *strict preference relation*.

Given a strict preference relation P on X, we can define two new relations called I, for *indifference*, and R for *weak preference* as follows.

1. xIy iff not (xPy) and not (yPx)
2. xRy iff xPy or xIy.

By de Morgan's rule xIy iff not $(xPy \vee yPx)$. Thus for any $x, y \in X$ either xIy or xPy or yPx. Since P is asymmetric it cannot be the case that both xPy and yPx are true. Thus the propositions "xPy", "yPx", "xIy" are disjoint, and hence form a partition of the universal proposition, U.

Note that $(xPy \vee xIy) \equiv$ not (yPx) since these three propositions form a (disjoint) partition. Thus (xRy) iff not (yPx).

In the case that P is the strict preference relation "$>$", it should be clear that indifference is identical to "$=$" and weak preference to "$>$ or $=$" usually written "$\geq$".

Lemma 1.5 *If P is a strict preference relation then indifference* (I) *is reflexive and symmetric, while weak preference* (R) *is reflexive and connected. Moreover if xRy and yRx then xIy.*

Proof

1. Since not (xPx) this must imply xIx, so I is reflexive
2.

$$\begin{aligned} xIy &\iff \text{not } (xPy) \wedge \text{not } (yPx) \\ &\iff \text{not } (yPx) \wedge \text{not } (xPy) \\ &\iff yIx. \end{aligned}$$

Hence I is symmetric.

3. $xRy \iff xPy$ or xIy. Thus $xIx \Rightarrow xRx$, so R is reflexive.
4. $xRy \iff xPy$ or yIx and $yRx \iff yPx$ or yIx.
‘ Not $(xRy \vee yRx) \iff$ not $(xPy \vee yPx \vee xIy)$. But $xPy \vee yPx \vee xIy$ is always true since these three propositions form a partition of the universal set. Thus not $(xRy \vee yRx)$ is always false, and so $xRy \vee yRx$ is always true. Thus R is connected.
5. Clearly

$$\begin{aligned} xRy \quad \text{and} \quad yRx &\iff (xPy \wedge yPx) \vee xIy \\ &\iff xIy \end{aligned}$$

since $xPy \wedge yPx$ is always false by asymmetry. □

In the case that P corresponds to “$>$” then $x \geq y$ and $y \geq x \Rightarrow x = y$, so “$\geq$” is antisymmetric.

Suppose that P is a strict preference relation on X, and there exists a function $u : X \to \Re$, called a *utility function*, such that xPy iff $u(x) > u(y)$. Therefore

$$\begin{aligned} xRy \quad &\text{iff } u(x) \geq u(y) \\ xIy \quad &\text{iff } u(x) = u(y). \end{aligned}$$

The order relation “$>$” on the real line is transitive (since $x > y > z \Rightarrow x > z$). Therefore P must be transitive when it is representable by a utility function.

We therefore have reason to consider “rationality” properties, such as transitivity, of a strict preference relation.

1.4.2 Rationality

Lemma 1.6 *If P is irreflexive and transitive on X then it is asymmetric.*

Proof To show that $A \wedge B \Rightarrow C$ we need only show that $B \wedge \text{not } (C) \Rightarrow \text{not } (A)$.

Therefore suppose that P is transitive but fails asymmetry. By the latter assumption there exists $x, y \in X$ such that xPy and yPx. By transitivity this gives xPx, which violates irreflexivity. □

Call a strict preference relation, P, on X *negatively transitive* iff it is the case that, for all $x, y, z \in X$, not $(xPy) \wedge$ not $(yPz) \Rightarrow$ not (xPz). Note that $xRy \Longleftrightarrow$ not (yPx). Thus the negative transitivity of P is equivalent to the property

$$yRx \wedge zRy \Rightarrow zRx.$$

Hence R must be *transitive*.

Lemma 1.7 *If P is a strict preference relation that is negatively transitive then P, I, R are all transitive.*

Proof

1. By the previous observation, R is transitive.
2. To prove P is transitive, suppose otherwise, *i.e.*, that there exist x, y, z such that xPy, yPz but not (xPz). By definition not $(xPz) \Longleftrightarrow zRx$. Moreover yPz or $yIz \Longleftrightarrow yRz$. Thus $yPz \Longrightarrow yRz$. By transitivity of R, zRx and yRz gives yRx, or not (xPy). But we assumed xPy. By contradiction we must have xPz.
3. To show I is transitive, suppose xIy, yIz but not (xIz). Suppose xPz, say. But then xRz. Because of the two indifferences we may write zRy and yRx. By transitivity of R, zRx. But zRx and xRz imply xIz, a contradiction. In the same way if zPx, then zRx, and again xIz. Thus I must be transitive. □

Note that this lemma also implies that P, I and R combine transitively. For example, if xRy and yPz then xPz.

To show this, suppose, in contradiction, that *not* (xPz).

This is equivalent to zRx. If xRy, then by transitivity of R, we obtain zRy and so *not* (yPz). Thus xRy and *not* $(xPz) \Rightarrow$ *not* (yPz). But yPz and *not* (yPz) cannot both hold. Thus xRy and $yPz \Rightarrow xPz$. Clearly we also obtain xIy and $yPz \Rightarrow xPz$ for example.

When P is a negatively transitive strict preference relation on X, then we call it a *weak order* on X. Let $O(X)$ be the set of weak orders on X. If P is a transitive strict preference relation on X, then we call it a *strict partial order*. Let $T(X)$ be the set of strict partial orders on X. By Lemma 1.7, $O(X) \subset T(X)$.

Finally call a preference relation *acyclic* if it is the case that for any finite sequence $x_1, \ldots, x_r$ of points in X if $x_j P x_{j+1}$ for $j = 1, \ldots, r-1$ then it cannot be the case that $x_r P x_1$.

Let $A(X)$ be the set of acyclic strict preference relations on X. To see that $T(x) \subset A(X)$, suppose that P is transitive, but cyclic, *i.e.*, that there exists a finite cycle $x_1 P x_2 \ldots P x_r P x_1$. By transitivity $x_{r-1} P x_r P x_1$ gives $x_{r-1} P x_1$, and by repetition we obtain $x_2 P x_1$. But we also have $x_1 P x_2$, which violates asymmetry.

1.4.3 Choices

As we noted previously, if P is a strict preference relation on a set X, then a maximal element, or *choice*, on X is an element x such that for no $y \in X$ is it the case that yPx. We can express this another way. Since $P \subset X \times X$, there is a mapping

$$\phi_P : X \to X \quad \text{where } \phi_P(x) = \{y : yPx\}.$$

We shall call ϕ_P the *preference correspondence* of P. The *choice* of P on X is the set $C_p(X) = \{x : \phi_P(X) = \Phi\}$. Suppose now that P is a strict preference relation on X. For each subset Y of X, let

$$C_P(Y) = \bigl\{x \in Y : \phi_P(X) \cap Y = \Phi\bigr\}.$$

This defines a *choice correspondence* $C_P : 2^X \to 2^X$ from 2^X, the set of all subsets of X, into itself.

An important question in social choice and welfare economics concerns the existence of a "social" choice correspondence, C_P, which guarantees the non-emptiness of the social choice $C_P(Y)$ for each *feasible set*, Y, in X, and an appropriate social preference, P.

Lemma 1.8 *If P is an acyclic strict preference relation on a finite set X, then $C_P(Y)$ is non-empty for each subset Y of X.*

Proof Suppose that $X = \{x_1, \dots, x_r\}$. If all elements in X are indifferent then clearly $C_P(X) = X$.

So we can assume that if the cardinality $|Y|$ of Y is at least 2, then x_2Px_1 for some x_2, x_1. We proceed by induction on the cardinality of Y.

If $Y = \{x_1\}$ then obviously $x_1 = C_P(Y)$.

If $Y = \{x_1, x_2\}$ then either x_1Px_2, x_2Px_1, or x_1Ix_2 in which case $C_P(Y) = \{x_1\}, \{x_2\}$ or $\{x_1, x_2\}$ respectively. Suppose $C_P(Y) \neq \Phi$ whenever the cardinality $|Y|$ of Y is 2, and consider $Y' = \{x_1, x_2, x_3\}$.

Without loss of generality suppose that $x_2 \in C_P(\{x_1, x_2\})$, but that neither x_1 nor $x_2 \in C_P(Y')$. There are two possibilities (i) If x_2Px_1 then by asymmetry of P, not (x_1Px_2). Since $x_2 \notin C_P(Y')$ then x_3Px_2, so not (x_2Px_3). Suppose that $C_P(Y') = \Phi$. Then x_1Px_3, and we obtain a cycle $x_1Px_3Px_2Px_1$. This contradicts acyclicity, so $x_3 \in C_P(Y')$. (ii) If x_2Ix_1 and $x_3 \notin C_P(Y')$ then either x_1Px_3 or x_2Px_3. But neither x_1 nor $x_2 \in C_P(Y')$ so x_3Px_1 and x_3Px_2. This contradicts asymmetry of P. Consequently $x_3 \in C_P(Y')$.

It is clear that this argument can be generalised to the case when $|Y| = k$ and Y' is a superset of Y (*i.e.*, $Y \subset Y'$ with $|Y'| = k + 1$.) So suppose $C_P(Y) \neq \Phi$. To show $C_P(Y') \neq \Phi$ when $Y' = Y \cup \{x_{k+1}\}$, suppose $[C_P(Y) \cup \{x_{k+1}\}] \cap C_P(Y') \neq \Phi$. Then there must exist some $x \in Y$ such that xPx_{k+1}.

If $x \in C_P(Y)$ then $x_{k+1}Px$, since $x \notin C_P(Y')$ and zPx for no $z \in Y$. Hence we obtain the asymmetry $xPx_{k+1}Px$. On the other hand if $x \in Y \backslash C_P(Y)$, then there must exist a chain $x_rPx_{r-1}P \dots x_1Px$ with $r < k$, such that $x_r \in C_P(Y)$. Since $x_r \notin$

$C_P(Y')$ it must be the case that $x_{k+1}Px_r$. This gives a cycle $xPx_{k+1}Px_rP\dots x$. By contradiction, $C_P(Y') \neq \Phi$.

By induction if $C_P(Y) \neq \Phi$ then $C_P(Y') \neq \Phi$ for any superset Y' of Y. Since $C_P(Y) \neq \Phi$ whenever $|Y| = 2$, it is evident that $C_P(Y) \neq \Phi$ for any finite subset Y of X. □

If P is a strict preference relation on X and P is representable by a utility function $u : X \to \Re$ then it must be the case that all of P, I, R are transitive. To see this, we note the following:

1. xRy and yRz iff $u(x) \geq u(y) \geq u(z)$. Since "$\geq$" on $\Re$ is transitive it follows that $u(x) \geq u(z)$ and so xRz.
2. xIy and yIz iff $u(x) = u(y) = u(z)$, and thus xIz.

In this case indifference, I, is reflexive, symmetric and transitive. Such a relation on X is called an *equivalence relation*.

For any point x in X, let $[x]$ be the equivalence class of x in X, *i.e.*, $[x] = \{y : yIx\}$.

Every point in X belongs to exactly *one* equivalence class. To see this suppose that $x \in [y]$ and $x \in [z]$, then xIy and xIz. By symmetry zIx, and by transitivity zIy. Thus $[y] = [z]$.

The set of equivalence classes in X under an equivalence relation, I, is written X/I. Clearly if $u : X \to \Re$ is a utility function then an equivalence class $[x]$ is of the form

$$[x] = \{y \in X : u(x) = u(y)\},$$

which we may also write as $u^{-1}[u(x)]$.

If X is a finite set, and P is representable by a utility function then

$$C_P(X) = \{x \in X : u(x) \ = s\}$$

where s is max $[u(y) : y \in X]$, the maximum value of u on X.

Social choice theory is concerned with the existence of a *choice* under a social preference relation P which in some sense aggregates individual preferences for all members of a society $M = \{1, \dots, i, \dots, m\}$. Typically the social preference relation cannot be representable by a "social" utility function. For example suppose a society consists of n individuals, each one of whom has a preference relation P_i on the feasible set X.

Define a social preference relation P on X by xPy iff xP_iy for all $i \in M$ (P is called the *strict Pareto rule*).

It is clear that if each P_i is transitive, then so must be P. As a result, P must be acyclic. If X is a finite set, then by Lemma 1.8, there exists a choice $C_P(X)$ on X.

The same conclusion follows if we define xQy iff xR_jy $\forall$ $j \in M$, and xP_iy for some $i \in M$.

If we assume that each individual has negatively transitive preferences, then Q will be transitive, and will again have a *choice*. Q is called the *weak Pareto rule*. Note that a point x belongs to $C_Q(X)$ iff it is impossible to move to another point y

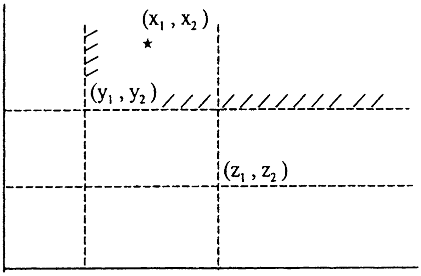


Fig. 1.5 The preference relation Q

which makes nobody "worse off", but makes some members of the society "better off". The set $C_Q(X)$ is called the *Pareto set*. Although the social preference relation Q has a choice, there is no social utility function which represents Q. To see this suppose the society consists of two individuals $1, 2$ with transitive preferences xP_1yP_1z and zP_2xP_2y.

By the definition xQy, since both individuals prefer x to y. However conflict of preference between y and z, and between x and z gives yIz and xIz, where I is the social indifference rule associated with Q. Consequently I is not transitive and there is no "social utility function" which represents Q. Moreover the elements of X cannot be partitioned into disjoint indifference equivalence classes.

To see the same phenomenon geometrically define a preference relation P on $\Re^2$ by

$$(x_1, x_2)P(y_1, y_2) \quad \Longleftrightarrow \quad x_1 > y_1 \wedge x_2 > y_2.$$

From Fig. 1.5 (x_1, x_2) P (y_1, y_2). However (x_1, x_2) I (z_1, z_2) and (y_1, y_2) I (z_1, z_2). Again there is no social utility function representing the preference relation Q. Intuitively it should be clear that when the feasible set is "bounded" in some way in $\Re^2$, then the preference relation Q has a *choice*. We shall show this more generally in a later chapter. (See Lemma 3.9 below.)

In Fig. 1.5, we have represented the preference Q in $\Re^2$ by drawing the set preferred to the point (y_1, y_2), say, as a subset of $\Re^2$.

An alternative way to describe the preference is by the graph of ϕ_P. For example, suppose X is the unit interval [0, 1] in $\Re$, and let the horizontal axis be the *domain* of ϕ_P, and the vertical axis be the *co-domain* of ϕ_P. In Fig. 1.6, the preference P is identical to the relation $>$ on the interval (so yPx iff $y > x$). The graph of ϕ_P is then the shaded set in the figure. Note that yPx iff $xP^{-1}y$. Because P is irreflexive, the diagonal $e_X = \{(x, x) : x \in X\}$ cannot belong to graph (ϕ_P). To find graph (ϕ_P^{-1}) we simply "reflect" graph (ϕ_P) in the diagonal.

The shaded set in Fig. 1.7 represents graph (ϕ_P^{-1}). Because P is asymmetric, it is impossible for both yPx and xPy to be true. This means that graph$(\phi_P) \cap$ graph$(\phi_P^{-1}) = \Phi$ (the empty set). This can be seen by superimposing Figs. 1.6 and 1.7. A preference of the kind illustrated in Fig. 1.6 is often called *monotonic* since increasing values of x are preferred.

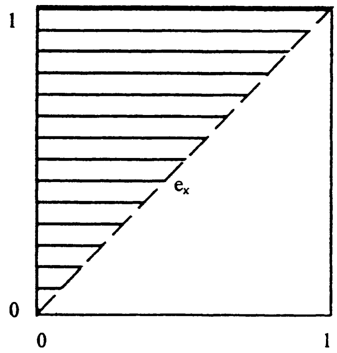


Fig. 1.6 The graph of the preference P

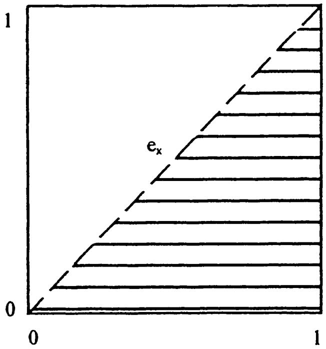


Fig. 1.7 The graph of the inverse preference

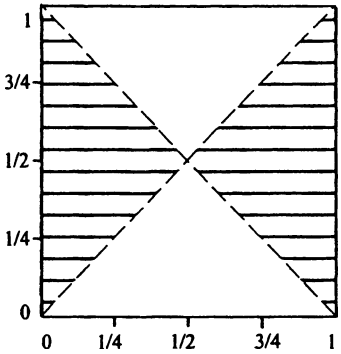


Fig. 1.8 Representation of the preference P

To illustrate a transitive, but non-monotonic strict preference, consider Fig. 1.8 which represents the preference yPx iff $x < y < 1 - x$, for $x \leq \frac{1}{2}$, or $1 - x < y < x$, for $x > \frac{1}{2}$. For example if $x = \frac{1}{4}$, then $\phi_P(x)$ is the interval $(\frac{1}{4}, \frac{3}{4})$, namely all points between $\frac{1}{4}$, and $\frac{3}{4}$, excluding the end points.

It is obvious that P represents a utility function

$$u(x) = x \quad \text{if } x \leq \frac{1}{2}$$

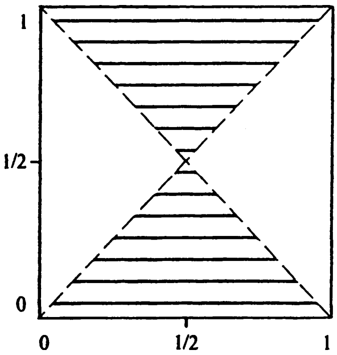


Fig. 1.9 A Euclidean preference

$$u(x) = 1 - x \quad \text{if } \frac{1}{2} < x \leq 1.$$

Clearly the choice of P is $C_P(x) = \frac{1}{2}$. Such a most preferred point is often called a "bliss point" for the preference. Indeed a preference of this kind is usually called "Euclidean", since the preference is induced from the distance from the bliss point. In other words, yPx iff $|y - \frac{1}{2}| < |x - \frac{1}{2}|$. Note again that this preference is transitive and of course acyclic. The fact that the P is asymmetric can be seen from noting that the shaded set in Fig. 1.8 (graph ϕ_P) and the shaded set in Fig. 1.9 (graph ϕ_P^{-1}) do not intersect.

Figure 1.10 represents a more complicated asymmetric preference. Here

$$\begin{aligned}\phi_P(x) &= \left(x, x + \frac{1}{2}\right) \quad \text{if } x \leq \frac{1}{2}\\ &= \left(\frac{1}{2}, x\right) \cup \left(0, x - \frac{1}{2}\right) \quad \text{if } x > \frac{1}{2}.\end{aligned}$$

Clearly there is a cycle, say $\frac{1}{4}\ P\ \frac{1}{8}\ P\ \frac{11}{16}\ P\ \frac{1}{4}$. Moreover the choice $C_P(X)$ is empty. This example illustrates that when acyclicity fails, then it is possible for the choice to be empty.

To give an example where P is both acyclic on the interval, yet no choice exists, consider Fig. 1.11. Define $\phi_P(x) = (x, x + \frac{1}{2})$ if $x \leq \frac{1}{2}$ and $\phi_P(x) = (\frac{1}{2}, x)$ if $x > \frac{1}{2}$.

P is still asymmetric, but we cannot construct a cycle. For example, if $x = \frac{1}{4}$ then yPx for $y \in (\frac{1}{4}, \frac{3}{4})$ but if zPy then $z > \frac{1}{4}$. Note however that $\phi_P(\frac{1}{2}) = (\frac{1}{2}, 1)$ so $C_P(x) = \Phi$.

This example shows that Lemma 1.8 cannot be extended directly to the case that X is the interval. In Chap. 3 below we show that we have to impose "continuity" on P to obtain an analogous result to Lemma 1.8.

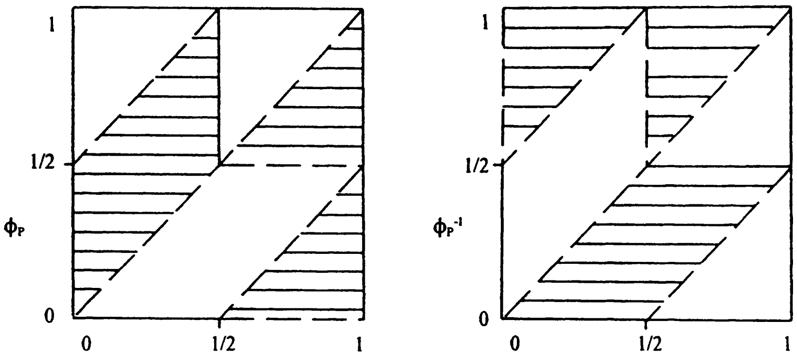


Fig. 1.10 An empty Choice Set

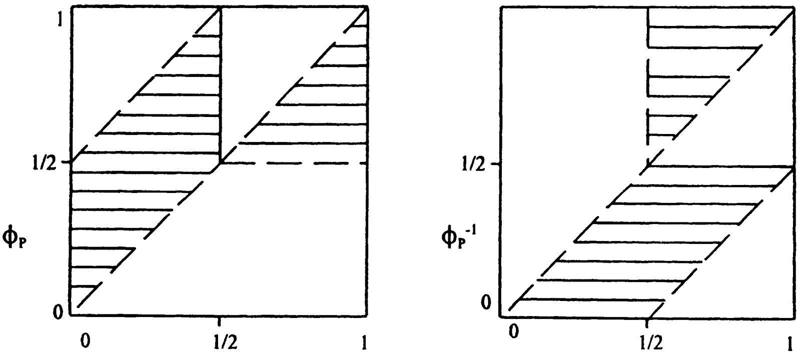


Fig. 1.11 An acyclic preference with empty choice

1.5 Social Choice and Arrow's Impossibility Theorem

The discussion following Lemma 1.8 showed that even the weak Pareto rule, Q, did not give rise to transitive indifference, and thus could not be represented by a "social utility function". However Q does give rise to transitive strict preference. We shall show that any rule that gives transitive strict preference must be "oligarchic" in the same way that Q is oligarchic. In other words any rule that gives transitive strict preference must be based on the Pareto (or unanimity) choice of some subset, say Θ of the society, M. Arrow's Theorem (Arrow 1951) shows that if it is desired that indifference be transitive, then the rule must be "dictatorial", in the sense that it obeys the preference of a single individual.

1.5.1 Oligarchies and Filters

The literature on social choice theory is very extensive and technical, and this section will not attempt to address its many subtleties. The general idea to examine the possible rationality properties of a "social choice rule"

$$\sigma : S(X)^M \longrightarrow S(X).$$

Here $S(X)$ stands for the set of strict preference relations on the set X and $M = \{1, \ldots, i, \ldots\}$ is a society. Usually M is a finite set of cardinality m. Sometimes however M will be identified with the set of integers $\mathcal{Z}$. $S(X)^M$ is the set of strict preference profiles for this society. For example if $|M| = m$, then a profile $\pi = \{P_1, \ldots, P_m\}$ is a list of preferences for the members of the society. We use $A(X)^M$, $T(X)^M$, $O(X)^M$ for profiles whose individual preferences are acyclic, strict partial orders or weak orders, respectively. Social choice theory is based on *binary* comparisons. This means that if two profiles π^1 and π^2 agree on a pair of alternatives $\{x, y\}$, say, then the social preferences $\sigma(\pi^1)$ and $\sigma(\pi^2)$ also agree on $\{x, y\}$. A key idea is that of a *decisive coalition*. Say a subset $A \subset M$ is *decisive* under the rule σ iff for any profile $\pi = (P_1, \ldots, P_m)$ such that xP_iy for all $i \in A$ then $x(\sigma(\pi))y$. That is to say whenever A is decisive, and its members agree that x is preferred to y then the social preference chooses x over y. The set of decisive coalitions under the rule is written $\mathcal{D}_\sigma$, or more simply, $\mathcal{D}$. To illustrate this idea, suppose $M = \{1, 2, 3\}$ and $\mathcal{D}_\sigma$ comprises any coalition with at least two members. It is easy to construct a profile $\pi \in A(x)^M$ such that σ is not even acyclic. For example, choose a profile π on the alternatives $\{x, y, z\}$ such that

$$xP_1yP_1z, \qquad yP_2zP_2x, \qquad zP_3xP_3y.$$

Since both 1 and 2 prefer y to z we must have $y\sigma(\pi)z$. But in the same way we find that $z\sigma(\pi)x$ and $x\sigma(\pi)y$, giving a social preference cycle on $\{x, y, z\}$. In general restricting the image of σ so that it lies in $A(X)$, $T(X)$, and $O(X)$ imposes constraints on $\mathcal{D}_\sigma$. We now examine these constraints.

Lemma 1.9 *If $\sigma : T(X)^M \longrightarrow T(X)$, and M, A, B all belong to $\mathcal{D}_\sigma$, then $A \cap B \in \mathcal{D}_\sigma$.*

Outline of Proof Partition M into the four sets $V_1 = A \cap B$,
$V_2 = A \backslash B$, $V_3 = B \backslash A$, $V_4 = M \backslash (A \cup B)$ and suppose that each individual has preferences on the alternatives $\{x, y, z\}$ as follows:

$i \in V_1$: zP_ixP_iy

$i \in V_2$: xP_iy, *with preferences for z unspecified*

$i \in V_3$: zP_ix, *with preferences for y unspecified*

$i \in V_4$: *completely unspecified.*

Now $A \backslash B = \{i \in A, \text{ but } i \notin B\}$, so $V_1 \cup V_2 = A$. Since A is decisive and every individual in A prefers x to y we obtain $x\sigma(\pi)y$. In the same way $V_1 \cup V_3 = B$, and B is decisive, so $z\sigma(\pi)x$. Since we require $\sigma(\pi)$ to be transitive, it is necessary that $z\sigma(\pi)y$. Since individual preferences are assumed to belong to $T(X)$, we require that $zP_i y$ for all $i \in V_1$. We have not however specified the preferences for the rest of the society. Thus $V_1 = A \cap B$ must be decisive for $\{x, z\}$ in the sense that V_1 can choose between x and z, independently of the rest of the society. But this must be true for every pair of alternatives. Thus $A \cap B \in \mathcal{D}_\sigma$. □

In general it could be possible for $\mathcal{D}_\sigma$ to be empty. However it is usual to assume that σ satisfies the *strict Pareto rule*. That is to say for any x, y if $xP_i y$, for all $i \in M$ then $x\sigma(\pi)y$. This simply means that $M \in \mathcal{D}_\sigma$. Moreover, this implies that $\Phi \notin \mathcal{D}_\sigma$. To see this, suppose that $\Phi \in \mathcal{D}_\sigma$ and consider a profile with $xP_i y \ \forall \ i \in M$. Since nobody prefers y to x, and the empty set is decisive, we obtain $y\sigma(\pi)x$. But by the Pareto rule, we have $x\sigma(\pi)y$. We assume however that $\sigma(\pi)$ is always a strict preference relation, and so $y\sigma(\pi)x$ cannot occur. Finally if $A \in \mathcal{D}_\sigma$ then any set B which contains A must also be decisive. Thus Lemma 1.9 can be interpreted in the following way.

Lemma 1.10 *If $\sigma : T(X)^M \longrightarrow T(X)$ and σ satisfies the strict Pareto rule, then $\mathcal{D}_\sigma$ satisfies the following conditions*:

D1. (*monotonicity*) *$A \subset B$ and $A \in \mathcal{D}_\sigma$ implies $B \in \mathcal{D}_\sigma$.*

D2. (*identity*) *$A \in \mathcal{D}_\sigma$ and $\Phi \notin \mathcal{D}_\sigma$.*

D3. (*closed under intersection*) *$A, B \in \mathcal{D}_\sigma$ implies $A \cap B \in \mathcal{D}_\sigma$.*

A collection $\mathcal{D}$ of subsets of M which satisfy **D1**, **D2**, and **D3** is called a *filter*.

Note also that when M is finite, then $\mathcal{D}_\sigma$ must also be finite. By repeating Lemma 1.9 for each pair of coalitions, we find that $\Theta = \cap\{A_i : A_i \in \mathcal{D}_\sigma\}$ must be non-empty and also decisive. This set Θ is usually called the *oligarchy*. In the case that σ is simply the strict Pareto rule, then the oligarchy is the whole society, M.

However, any rule that gives a transitive strict preference relation must be equivalent to the Pareto rule based on some oligarchy, possibly a strict subset of M.

For example, majority rule for the society $M = \{1, 2, 3\}$ defines the decisive coalitions $\{1, 2\}, \{1, 3\}, \{2, 3\}$. These three coalitions contain no oligarchy. We can immediately infer that this rule cannot be transitive. In fact, as we have seen, it is not even acyclic. Below, we explore this further. If we require the rule always to satisfy the condition of negative transitivity then the oligarchy will consist of a single individual, when M is finite.

Lemma 1.11 *If $\sigma : T(X)^M \longrightarrow O(X)$ and $M \in \mathcal{D}_\sigma$ and $A \subset M$ and $A \notin \mathcal{D}_\sigma$, then $M \backslash A \in \mathcal{D}_\sigma$.*

Proof Since $A \notin \mathcal{D}_\sigma$, we can find a profile π and a pair $\{x, y\}$ such that $yP_i x$ for all $i \in A$, yet not $(y\sigma(\pi)x)$. Let us write this latter condition as xRy, where R stands for weak social preference.

Suppose now there is an alternative z such that xP_iz for all $i \in M\backslash A$, and that yP_iz for all $i \in M$. By the Pareto condition ($M \in \mathcal{D}_\sigma$) we obtain yPz (where P stands for $\sigma(\pi)$). By negative transitivity of P, xRy and yPz implies xPz (see Lemma 1.7). However we have not specified the preferences of A on $\{x, z\}$. But we have shown that if the members of $M\backslash A$ prefer x to z, then so does the society. It then follows that $M\backslash A$ must be decisive. □

It follows from this lemma that if $\sigma : T(X)^M \longrightarrow O(X)$ and $M \in \mathcal{D}_\sigma$, then whenever $A \in \mathcal{D}_\sigma$, there is some proper subset B (such that $B \subset A$ yet $B \neq A$) with $B \in \mathcal{D}_\sigma$. To see this consider any proper subset C of A with $C \notin \mathcal{D}_\sigma$. By Lemma 1.11, $M\backslash C \in \mathcal{D}_\sigma$. But since $O(X)$ belongs to $T(X)$, we can use the property **D3** of Lemma 1.10 to infer that $A \cap (M\backslash C) \in \mathcal{D}_\sigma$. But $A \cap (M\backslash C) = A\backslash C$, and since C is a proper subset of A, $A\backslash C \neq \Phi$. Hence $A\backslash C \in \mathcal{D}_\sigma$. In the case M has finite cardinality, we can repeat this argument to show that there must be some individual i such that $\{i\} \in \mathcal{D}_\sigma$. But then i is a dictator in the sense that, for any x, y if π is a profile with xP_iy then $x\sigma(\pi)y$.

Arrow's Impossibility Theorem *If $\sigma : O(X)^M \longrightarrow O(X)$ and $M \in \mathcal{D}_\sigma$, with $|M|$ finite, then there is a dictator $\{i\}$, say, such that $\{i\} \in \mathcal{D}_\sigma$.*

It is obvious that if $\mathcal{D}_\sigma$ is non-empty, then all the coalitions in $\mathcal{D}_\sigma$ must contain the dictator $\{i\}$. In particular $\{i\} = \cap\{M_i : i \in \mathcal{D}_\sigma\}$, and $\{i\} \in \mathcal{D}_\sigma$.

A somewhat similar result holds when M is a "space" rather than a finite set. In this case there need not be a dictator in M. However in this case, the filter $\mathcal{D}_\sigma$ defines an "invisible dictator". That is to say, we can imagine the coalitions in $\mathcal{D}_\sigma$ becoming smaller and smaller, so that they define the invisible dictator in the limit. See Kirman and Sondermann (1972).

1.5.2 Acyclicity and the Collegium

As Lemma 1.8 demonstrated, if P is an acyclic preference relation on a finite set, then the choice for P will be non-empty. Given Arrow's Theorem, it is therefore useful to examine the properties of a social choice rule that are compatible with acyclicity. To this end we introduce the notion of the *Nakamura number* for a rule, σ (Nakamura 1979).

Definition 1.4 Let $\mathcal{D}$ be a family of subsets of the finite set M. The collegium $K(\mathcal{D})$ is the intersection

$$\cap\{A_i : A_i \in \mathcal{D}\}.$$

That is to say $K(\mathcal{D})$ is the largest set in M such that $K(\mathcal{D}) \subset A$ for all $A \in \mathcal{D}$.

If $K(\mathcal{D})$ is empty then $\mathcal{D}$ is said to be *non-collegial*. Otherwise $\mathcal{D}$ is *collegial*.

If σ is a social choice rule, and $\mathcal{D}_\sigma$ its family of decisive coalitions, then $K(\sigma) = K(\mathcal{D}_\sigma)$ is the collegium for σ. Again σ is called collegial or non-collegial depending on whether $K(\sigma)$ is non-empty or empty. The Nakamura number of a non-collegial family $\mathcal{D}$ is written $k(\mathcal{D})$ and is the cardinality of the smallest non-collegial subfamily of $\mathcal{D}$. That is, there exists some subfamily $\mathcal{D}'$ of $\mathcal{D}$ with $|\mathcal{D}'| = k(\mathcal{D})$ such that $K(\mathcal{D}') = \Phi$. Moreover if $\mathcal{D}''$ is a subfamily of $\mathcal{D}$ with $|\mathcal{D}''| \leq k(\mathcal{D}) - 1$ then $K(\mathcal{D}'') \neq \Phi$.

In the case $\mathcal{D}$ is collegial define $k(\mathcal{D}) = \infty$. For a social choice rule define $k(\sigma) = k(\mathcal{D}_\sigma)$, where $\mathcal{D}_\sigma$ is the family of decisive coalitions for σ.

Example 1.7

(i) To illustrate this definition, suppose $\mathcal{D}$ consists of the four coalitions, $\{A_1, A_2, A_3, A_4\}$ where $A_1 = \{2, 3, 4\}$, $A_2 = \{1, 3, 4\}$, $A_3 = \{1, 2, 4, 5\}$ and $A_4 = \{1, 2, 3, 5\}$. Of course $\mathcal{D}$ will be monotonic, so supersets of these coalitions will be decisive. It is evident that $A_1 \cap A_2 \cap A_3 = \{4\}$ and so if $\mathcal{D}' = \{A_1, A_2, A_3\}$ then $K(\mathcal{D}') \neq \Phi$. However $K(\mathcal{D}') \cap A_4 = \Phi$ and so $K(\mathcal{D}) = \Phi$. Thus $k(\mathcal{D}) = 4$.

(ii) An especially interesting case is of a q-majority rule where each individual has one vote, and any coalition with at least q voters (out of m) is decisive. In this case it is easy to show then that $k(\sigma) = 2 + [\frac{q}{m-q}]$ where $[\frac{q}{m-q}]$ is the greatest integer strictly less than $\frac{q}{m-q}$. In the case that $m = 4$ and $q = 3$, then we find that $[\frac{3}{1}] = 2$, so $k(\sigma) = 4$.

On the other hand for all other simple majority rules where $m = 2s + 1$ or $2s$ and $q = s + 1$ (and s is integer) then

$$\left[\frac{q}{m-q}\right] = \left[\frac{s+1}{s}\right] \quad \text{or} \quad \left[\frac{s+1}{s-1}\right]$$

depending on whether m is odd or even. In both cases $[\frac{q}{m-q}] = 1$. Thus $k(\sigma) = 3$ for any simple majority rule with $m \neq 4$.

(iii) Finally, observe that for any simple majoritarian rule, if M_1, M_2 both belong to $\mathcal{D}$, then $A_1 \cap A_2 \neq \Phi$. So in general, any non-collegial subfamily of $\mathcal{D}$ must include at least three coalitions. Consequently any majoritarian rule, σ, has $k(\sigma) \geq 3$.

The Nakamura number allows us to construct social preference cycles.

Nakamura Lemma *Suppose that σ is a non-collegial voting rule, with Nakamura number $k(\sigma) = k$. Then there exists an acyclic profile $\pi = (P_1, \dots, P_n)$ for the society M, on a set $X = \{x_1, \dots, x_k\}$ of cardinality k, such that $\sigma(\pi)$ is cyclic on W, and the choice of $\sigma(\pi)$ is empty.*

Proof We wish to construct a cycle by considering k different decisive coalitions, $A_1, \dots, A_k$ and assigning preferences to the members of each coalition such that

$$\begin{aligned} &x_i P_i x_2 \quad \text{for all } i \in A_1 \\ &\vdots \\ &x_{k-1} P_i x_k \quad \text{for all } i \in A_{k-1} \\ &x_k P_i x_1 \quad \text{for all } i \in A_k. \end{aligned}$$

We now construct such a profile, π. Let $\mathcal{D}_k = \{A_1, \dots, A_{k-1}\}$. By the definition of the Nakamura number, this must be collegial. Hence there exists some individual $\{k\}$, say, with $k \in A_1 \cap \cdots \cap A_{k-1}$. We can assign k the acyclic preference profile $x_1 P_k x_2 P_k \dots P_k x_k$.

In the same way, for each subfamily $\mathcal{D}_j = \{A, \dots, A_{j-1}, A_{j+1}, \dots, A_k\}$ there exists a collegium containing individual j, to whom we assign the preference

$$x_{j+1} P_j x_{j+2} \dots x_k P_j x_1 \dots x_j P_j x_j.$$

We may continue this process to assign acyclic preferences to each member of the various collegia of subfamilies of $\mathcal{D}$, so as to give the required cyclic social preference. □

Lemma 1.12 *A necessary condition for a social choice rule σ to be acyclic on the finite set X of cardinality at least $m = |M|$, for each acyclic profile π on X, is that σ be collegial.*

Proof Suppose σ is not collegial. It is easy to show that the Nakamura number $k(\sigma)$ will not exceed m. By the Nakamura Theorem there is an acyclic profile π on a set X of cardinality m, such that $\sigma(\pi)$ is cyclic on X. Thus acyclicity implies that σ must be collegial. □

Note that this lemma emphasizes the size of the set of alternatives. It is worth observing here that in the previous proofs of Arrow's Theorem, the cardinality of the set of alternatives was implicitly assumed to be at least 3.

These techniques using the Nakamura number can be used to show that a simple social rule will be acyclic whenever it is collegial. Say a social choice rule is *simple* iff whenever $x\sigma(\pi)y$ for the profile π, then xP_iy for all i in some coalition that is decisive for σ.

Note that a social choice rule need not, in general, be simple. If σ is simple then all the information necessary to analyse the rule is contained in $\mathcal{D}_\sigma$.

Lemma 1.13 *Let σ be a simple choice rule on a finite set X:*

(i) *If σ is dictatorial, then $\sigma(\pi) \subset O(X)$ for all $\pi \in O(X)^M$*
(ii) *If σ is oligarchic, then $\sigma(\pi) \subset T(X)$ for all $\pi \in T(X)^M$*
(iii) *If σ is collegial, then $\sigma(\pi) \subset A(X)$ for all $\pi \in A(X)^M$.*

Proof

(i) If i is a dictator, then xI_iy implies that x and y are socially indifferent. Because P_i belongs to $O(X)$ so must $\sigma(\pi)$.

(ii) In the same way, if Θ is the oligarchy and $x\sigma(\pi)y$ then xP_iy for all i in Θ. Thus $\sigma(\pi)$ must be transitive.

(iii) If there is a cycle $x_1\sigma(\pi)\ldots, x_k\sigma(\pi)x_1$ then each of these social preferences must be supported by a decisive coalition. Since the collegium is non-empty, there is some individual, i, say, who has such a cyclic preference. This contradicts the assumption that $\pi \in A(X)^M$.

□

Another way of expressing part (iii) of this lemma is to introduce the idea of a prefilter. Say $\mathcal{D}$ is a *prefilter* if and only if it satisfies **D1** (monotonicity) and **D2** (identity) introduced earlier, and also non-empty intersection (so $K(\mathcal{D}) \neq \Phi$). Clearly if σ is simple, and $\mathcal{D}_\sigma$ is a prefilter, then σ is acyclic and consistent with the Pareto rule.

In Chap. 3 we shall further develop the notion of social choice using the notion of the Nakamura number, in the situation where X has a geometric structure.

Further Reading

The first version of the impossibility theorem can be found in

Arrow, K. J. (1951). *Social choice and individual values*. New York: Wiley.

The ideas of the filter and Nakamura number are in

Kirman, A. P., & Sondermann, D. (1972). Arrow's Impossibility Theorem, many agents and invisible dictators. *Journal of Economic Theory*, *5*, 267–278.

Nakamura, K. (1979). The vetoers in a simple game with ordinal preferences. *International Journal of Game Theory*, *8*, 55–61.

Linear Spaces and Transformations 2

2.1 Vector Spaces

We showed in Sect. 1.3 that when $\mathcal{F}$ was a field, the n-fold product set $\mathcal{F}^n$ had an additional operation defined on it, which was induced from addition in $\mathcal{F}$, so that $(\mathcal{F}^n, +)$ became an abelian group with zero $\underline{0}$. Moreover we were able to define a product $\cdot : \mathcal{F} \times \mathcal{F}^n \to \mathcal{F}^n$ which takes (α, x) to a new element of $\mathcal{F}^n$ called (αx). Elements of $\mathcal{F}^n$ are known as *vectors*, and elements of $\mathcal{F}$ as *scalars*. The properties that we discovered in $\mathcal{F}^n$ characterise a *vector space*. A vector space is also known as a linear space.

Definition 2.1 A *vector space* $(V, +)$ is an abelian additive group with zero $\underline{0}$, together with a field $(\mathcal{F}, +, \cdot)$ with zero 0 and identity 1. An element of V is called a *vector* and an element of $\mathcal{F}$ a *scalar*. Moreover for any $\alpha \in \mathcal{F}$, $v \in V$ there is a scalar multiplication $(\alpha, v) \to \alpha v \in V$ which satisfies the following properties:

V1. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$, for any $\alpha \in \mathcal{F}$, $v_1, v_2 \in V$.
V2. $(\alpha + \beta)v = \alpha v + \beta v$, for any $\alpha, \beta \in \mathcal{F}$, $v \in V$.
V3. $(\alpha\beta)v = \alpha(\beta v)$, for any $\alpha, \beta \in \mathcal{F}$, $v \in V$.
V4. $1 \cdot v = v$, for $1 \in \mathcal{F}$, and for any $v \in V$.

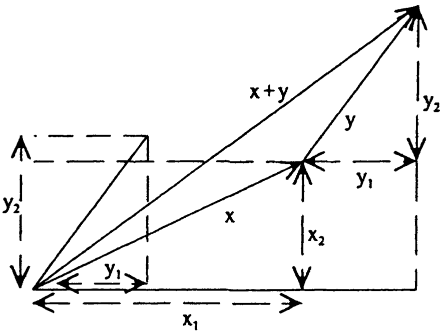
Call V a vector space over the field $\mathcal{F}$. From the previous discussion the set $\Re^n$ becomes an abelian group $(\Re^n, +)$ under addition. We shall frequently

write

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_2,

Fig. 2.1 Vector addition



for a vector in $\Re^n$, where $x_1, \dots, x_n$ are called the coordinates of x. Vector addition as in Fig. 2.1 is then defined by

$$x + y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

A vector space over $\Re$ is called a *real vector space*.

For example $(\mathcal{Z}_2, +, \cdot)$ is a field and so $(\mathcal{Z}_2)^n$ is a vector space over the field $\mathcal{Z}_2$. It may not be possible to represent each vector in a vector space by a list of coordinates. For example, consider the set of all functions with domain X and image in $\Re$. Call this set $\Re^X$. If $f, g \in \Re^X$, define $f + g$ to be that function which maps $x \in X$ to $f(x) + g(x)$. Clearly there is a zero function $\underline{0}$ defined by $\underline{0}(x) = \underline{0}$, and each f has an inverse $(-f)$ defined by $(-f)(x) = -(f(x))$. Finally for $\alpha \in \Re$, $f \in \Re^X$, define $\alpha f : X \to \Re$ by $(\alpha f)(x) = \alpha(f(x))$. Thus $\Re^X$ is a vector space over $\Re$.

Definition 2.2 Let $(V, +)$ be a vector space over a field, $\mathcal{F}$. A subset V' of V is called a *vector subspace* of V if and only if

1. $v_1, v_2 \in V' \Rightarrow v_1 + v_2 \in V'$, and
2. if $\alpha \in F$ and $v \in V'$ then $\alpha v' \in V'$.

Lemma 2.1 *If $(V, +)$ is a vector space with zero $\underline{0}$ and V' is a vector subspace, then, for each $v \in V'$, the inverse $(-v) \in V'$, and $\underline{0} \in V'$, so $(V', +)$ is a subgroup of $(V, +)$.*

Proof Suppose $v \in V'$. Since $\mathcal{F}$ is a field, there is an identity 1, with additive inverse -1. But by **V2**, $(1-1)v = 1 \cdot v + (-1)v = 0 \cdot v$, since $1 - 1 = 0$. Now $(1+0)v = 1 \cdot v + 0 \cdot v$, and so $0 \cdot v = \underline{0}$. Thus $(-1)v = (-v)$. Since V' is a vector subspace, $(-1)v \in V'$, and so $(-v) \in V'$. But then $v + (-v) = \underline{0}$, and so $\underline{0} \in V'$. □

From now on we shall simply write V for a vector space, and refer to the field only on occasion.

Definition 2.3 Let $V' = \{v_1, \ldots, v_r\}$ be a set of vectors in the vector space V. A vector v is called a *linear combination* of the set V' iff v can be written in the form

$$v = \sum_{i=1}^{r} \lambda_i v_i$$

where each $\lambda_i, i = 1, \ldots, r$ belongs to the field $\mathcal{F}$. The *span* of V', written $\text{Span}(V')$ is the set of vectors which are linear combinations of the set V'. If $V'' = \text{Span}(V')$, then V' is said *to span* V''.

For example, suppose

$$V' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Since we can solve the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any $(x, y) \in \Re^2$, by setting $\alpha = \frac{1}{3}(2y - x)$ and $\beta = \frac{1}{3}(2x - y)$, it is clear that V' is a span for $\Re^2$.

Lemma 2.2 *If V' is a finite set of vectors in the vector space, V, then* $\text{Span}(V')$ *is a vector subspace of V.*

Proof We seek to show that for any $\alpha, \beta \in \mathcal{F}$ and any $u, w \in \text{Span}(V')$, then $\alpha u + \beta w \in \text{Span}(V')$. By definition, if $V' = \{v_1, \ldots, v_r\}$, then $u = \sum_{i=1}^{r} \eta_i v_i$ and $w = \sum_{i=1}^{r} \mu_i v_i$, where $\eta_i, \mu_i \in \mathcal{F}$ for $i = 1, \ldots, r$. But then $\alpha u + \beta w = \alpha \sum_{i=1}^{r} \eta_i v_i + \beta \sum_{i=1}^{r} \mu_i v_i = \sum_{i=1}^{r} \lambda_i v_i$, where $\lambda_i = \alpha \eta_i + \beta \mu_i \in \mathcal{F}$, for $i = 1, \ldots, r$. Thus $\alpha u + \beta w \in \text{Span}(V')$.

Note that, by this lemma, the zero vector $\underline{0}$ belongs to $\text{Span}(V')$. □

Definition 2.4 Let $V' = \{v_1, \ldots, v_r\}$ be a set of vectors in V. V' is called a *frame* iff $\sum_{i=1}^{r} \alpha_i v_i = \underline{0}$ implies that $\alpha_i = 0$ for $i = 1, \ldots, r$. (Here each α_i belongs to the field $\mathcal{F}$.) In this case the set V' is called a *linearly independent set*. If V' is not a frame, the vectors in V' are said to be linearly dependent. Say a vector is *linearly dependent* on $V' = \{v_1, \ldots, v_r\}$ iff $v \in \text{Span}(V')$.

Note that if V' is a frame, then

1. $\underline{0} \notin V'$ since $\alpha \underline{0} = \underline{0}$ for every non-zero $\alpha \in F$.
2. If $v \in V'$ then $(-v) \notin V'$, otherwise $1 \cdot v + 1(-v) = \underline{0}$ would belong to V', contradicting (1).

Lemma 2.3

1. *V' is not a frame iff there is some vector $v \in V'$ which is linearly dependent on $V' \backslash \{v\}$.*
2. *If V' is a frame, then any subset of V' is a frame.*
3. *If V' spans V'', but V' is not a frame, then there exists some vector $v \in V'$ such that $V''' = V' \backslash \{v\}$ spans V''.*

Proof Let $V' = \{v_1, \dots, v_r\}$ be the set of vectors in the vector space V.

1. Suppose V' is not a frame. Then there exists an equation $\sum_{j=1}^{r} \alpha_j v_j = 0$, where, for at least one k, it is the case that $\alpha_k \neq 0$. But then $v_k = -\frac{1}{\alpha_k}(\sum_{j \neq k} \alpha_j v_j)$. Let $v_k = v$. Then v is linearly dependent on $V' \backslash \{v\}$. On the other hand suppose that v_1, say, is linearly dependent on $\{v_2, \dots, v_r\}$. Then $v_1 = \sum_{j=2}^{r} \alpha_j v_j$, and so $\underline{0} = -v_1 + \sum_{j=2}^{r} \alpha_j v_j = \sum_{j=1}^{r} \alpha_j v_j$ where $\alpha_1 = -1$. Since $\alpha_1 \neq 0$, V' cannot be a frame.
2. Suppose V'' is a subset of V', but that V'' is not a frame. For convenience let $V'' = \{v_1, \dots, v_k\}$ where $k \le r$. Then there is a non-zero solution
$$\underline{0} \neq \sum_{j=1}^{k} \alpha_j v_j.$$
Since V'' is a subset of V', this implies that V' cannot be a frame. Thus if V' is a frame, so is any subset V''.
3. Suppose that V' is not a frame, but that it spans V''. By part (1), there exists a vector v_1, say, in V' such that v_1 belongs to $\text{Span}(V' \backslash \{v_1\})$. Thus $v_1 = \sum_{j=2}^{r} \alpha_j v_j$. Since V' is a span for V'', any vector v in V'' can be written
$$\begin{aligned} v &= \sum_{j=1}^{r} \beta_j v_j \\ &= \beta_1 \left(\sum_{j=2}^{r} \alpha_j v_j \right) + \sum_{j=2}^{r} \beta_j v_j. \end{aligned}$$
Thus v is a linear combination of $V' \backslash \{v_1\}$ and so $V'' = \text{Span}(V' \backslash \{v_1\})$.

Let $V''' = V' \backslash \{v_1\}$ to complete the proof. □

Definition 2.5 A *basis* for a vector space V is a frame V' which spans V.

For example, we previously considered

$$V' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

and showed that any vector in $\Re^2$ could be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{2y-x}{3}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left(\frac{2x-y}{3}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus V' is a span for $\Re^2$. Moreover if $(x, y) = (0, 0)$ then $\lambda_1 = \lambda_2 = 0$ and so V' is a frame. Hence V' is a basis for $\Re^2$. If $V' = \{v_1, \dots, v_n\}$ is a basis for a vector space V then any vector $v \in V$ can be written

$$v = \sum_{j=1}^{n} \alpha_j v_j$$

and the elements $(\alpha_1, \dots, \alpha_n)$ are known as the *coordinates* of the vector v, *with respect to* the basis V'.

For example the *natural basis* for $\Re^n$ is the set $V' = \{e_1, \dots, e_n\}$ where $e_i = (0, \dots, 1, \dots, 0\}$ with a 1 in the ith position.

Lemma 2.4 $\{e_1, \dots, e_n\}$ *is a basis for* $\Re^n$.

Proof We can write any vector x in $\Re^n$ as $\{x_1, \dots, x_n\}$. Clearly

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \cdot \end{pmatrix} + \cdots x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

If $x = 0$ then $x_1 = \cdots = x_n = 0$ and so $\{e_1, \dots, e_n\}$ is a frame, as well as a span, and thus a basis for $\Re^n$. □

However a single vector x will have different coordinates depending on the basis chosen. For example the vector (x, y) has coordinates (x, y) in the basis $\{e_1, e_2\}$ but coordinates $(\frac{2y-x}{3}, \frac{2x-y}{3})$ with respect to the basis $\{\binom{1}{2}, \binom{2}{1}\}$.

Once the basis is chosen, the coordinates of any vector with respect to that basis are unique. □

Lemma 2.5 *Suppose* $V' = \{v_1, \dots, v_n\}$ *is a basis for* V. *Let* $v = \sum_{i=1}^{n} \alpha_i v_i$. *Then the coordinates* $(\alpha_1, \dots, \alpha_n)$, *with respect to the basis, are unique.*

Proof If the coordinates were not unique then it would be possible to write $v = \sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} \alpha_i v_i$ with $\beta_i \neq \alpha_i$ for some i.

But $\underline{0} = v - v = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i$.

Since V' is a frame, $\alpha_i - \beta_i = 0$ for $i = 1, \dots, n$. Thus $\alpha_i = \beta_i$ for all i, and so the coordinates are unique. □

Note in particular that with respect to *any* basis $\{v_1, \dots, v_n\}$ for V, the unique zero vector $\underline{0}$ always has coordinates $(0, \dots, 0)$.

Definition 2.6 A space V is *finitely generated* iff there exists a span V', for V, which has a finite number of elements.

Lemma 2.6 *If V is a finitely generated vector space, then it has a basis with a finite number of elements.*

Proof Since V is finitely generated, there is a finite set $V_1 = \{v_1, \dots, v_n\}$ which spans V. If V_1 is a frame, then it is a basis. If V_1 is linearly dependent, then by Lemma 2.3(3) there is a vector $v \in V_1$, such that $\mathrm{Span}(V_2) = V$, where $V_2 = V_1 \backslash \{v\}$. Again if V_2 is a frame, then it is a basis. If there were no subset $V_r = \{v_1, \dots, v_{n-r+1}\}$ of V_1 which was a frame, then V_1 would have to be the empty set, implying that V was an empty set. But this contradicts $\underline{0} \in V$. □

Lemma 2.7 *If V is a finitely generated vector space, and V_1 is a frame, then there is a basis V_2 for V which includes V_1.*

Proof Let $V_1 = \{v_1, \dots, v_r\}$. If $\mathrm{Span}(V_1) = V$ then V_1 is a basis. So suppose that $\mathrm{Span}(V_1) \neq V$. Then there exists an element $v_{r+1} \in V$ which does not belong to $\mathrm{Span}(V_1)$. We seek to show that $V_2 = V_1 \cup \{v_{r+1}\}$ is a frame. Consider $\underline{0} = \alpha_{r+1} v_{r+1} + \sum_{i=1}^{r} \alpha_i v_i$.

If $\alpha_{r+1} = 0$, then the linear independence of V_1 implies that $\alpha_i = 0$, for $i = 1, \dots, r$. Thus V_2 is a frame. If $\alpha_{r+1} \neq 0$, then

$$v_{r+1} = -\frac{1}{\alpha_{r+1}} \left(\sum_{i=1}^{r} \alpha_i v_i \right).$$

But this implies that v_{r+1} belongs to $\mathrm{Span}(V_1)$ and therefore that $V = \mathrm{Span}(V_1)$. Thus V_2 is a frame. If V_2 is a span for V, then it is a basis. If V_2 is not a span, reiterate this process. Since V is finitely generated, there must be some frame $V_{n-r+1} = \{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ which is a span, and thus a basis for V. □

These two lemmas show that if V is a finitely generated vector space, and $\{v_1, \dots, v_m\}$ is a span then some subset $\{v_1, \dots, v_n\}$, with $n \leq m$, is a basis. A basis is a *minimal* span.

On the other hand if $X = \{v_1, \dots, v_r\}$ is a frame, but not a span, then elements may be added to X in such a way as to preserve linear independence, until this "superset" of X becomes a basis. Consequently a basis is a *maximal* frame. These two results can be combined into one theorem.

Exchange Theorem *Suppose that V is a finitely generated vector space. Let $X = \{x_1, \dots, x_n\}$ be a frame and $Y = \{y_1, \dots, y_n\}$ a span. Then there is some subset Y' of Y such that $X \cup Y'$ is a basis for V.*

Proof By induction, let $X_S = \{x_1, \dots, x_s\}$, for each $s = 1, \dots, n$, and let $X_0 = \phi$. □

We know already from Lemma 2.6 that there is some subset Y_0 of Y such that $X_0 \cup Y_0$ is a basis for V. Suppose for some $s < m$, there is a subset Y_s of Y such that $X_s \cup Y_s$ is a basis.

Let $Y_s = \{y_1, \dots, y_t\}$. Now $x_{s+1} \notin \text{Span}(X_s \cup Y_s)$ since $X_s \cup Y_s$ is a basis. Thus $x_{s+1} = \sum_1^s \alpha_1 x_1 + \sum_1^t \beta_i y_i$. But $X_{s+1} = \{x_1, \dots, x_{s+1}\}$ is a frame, since it is a subset of X.

Thus at least one $\beta_j \neq 0$. Let $Y_{s+1} = Y_s \backslash \{y_j\}$, so $Y_j \notin \text{Span}(X_{s+1} \cup Y_{s+1})$ and so $X_{s+1} \cup Y_{s+1} = \{x_1, \dots, x_{s+1}\} \cup \{y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_t\}$ is a basis for V.

Thus if there is some subset Y_s of Y such that $X_s \cup Y_s$ is a basis, there is a subset Y_{s+1} of Y such that $X_{s+1} \cup Y_{s+1}$ is a basis.

By induction, there is a subset $Y_m = Y'$ of Y such that $X_m \cup Y_m = X \cup Y'$ is a basis. □

Corollary 2.8 *If $X = \{x_1, \dots, x_m\}$ is a frame in a vector space V, and $Y = \{y_1, \dots, y_n\}$ is a span for V, then $m \leq n$.*

Lemma 2.9 *If V is a finitely generated vector space, then any two bases have the same number of vectors, where this number is called the dimension of V, and written* $\dim(V)$.

Proof Let X, Y be two bases with m, n number of elements. Consider X as a frame and Y as a span. Thus $m \leq n$. However Y is also a frame and X a span. Thus $n \leq m$. Hence $m = n$. □

If V' is a vector subspace of a finitely generated vector space V, then any basis for V' can be extended to give a basis for V. To see this, there must exist some finite set $V'' = \{v_1, \dots, v_r\}$ of vectors all belonging to V' such that $\text{Span}(V'') = V'$. Otherwise V could not be finitely generated. As before eliminate members of V'' until a frame is obtained. This gives a basis for V'. Clearly $\dim(V') \leq \dim(V)$. Moreover if V' has a basis $V''' = \{v_1, \dots, v_r\}$ then further linear independent vectors belonging to $V \backslash V'$ can be added to V''' to give a basis for V.

As we showed in Lemma 2.3, the vector space $\Re^n$ has a basis $\{e_1, \dots, e_n\}$ consisting of n elements. Thus dim $(\Re^n) = n$.

If V^m is a vector subspace of $\Re^n$ of dimension m, where of course $m \leq n$, then in a certain sense V^m is identical to a copy of $\Re^m$ through the origin $\underline{0}$. We make this more explicit below.

2.2 Linear Transformations

In Chap. 1 we considered a morphism from the abelian group $(\Re^2, +)$ to itself. A morphism between vector spaces is called a *linear transformation*.

Definition 2.7 Let V, U be two vector spaces of dimension n, m respectively, over the same field $\mathcal{F}$. Then a *linear transformation* $T : V \to U$ is a function from V to U with domain V, such that

1. for any $a \in \mathcal{F}$, any $v \in V, T(\alpha v) = \alpha(T(v))$
2. for any $v_1, v_2 \in V,\ T(v_1 + v_2) = T(v_1) + T(v_2)$.

Note that a linear transformation is simply a morphism between $(V, +)$ and $(U, +)$ which respects the operation of the field $\mathcal{F}$. We shall show that any linear transformation T can be represented by an array of the form

$$M(T) = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}$$

consisting of $n \times m$ elements in F. An array such as this is called an n by m (or $n \times m$) *matrix*. The set of $n \times m$ matrices we shall write as $M(n, m)$.

2.2.1 Matrices

For convenience we shall consider finitely generated vector spaces over $\Re$, so that we restrict attention to linear transformations between $\Re^n$ and $\Re^m$, for any integers n and m. Now let $V = \{v_1, \dots, v_n\}$ be a basis for $\Re^n$ and $U = \{u_1, \dots, u_m\}$ a basis for $\Re^m$.

Since V is a basis for $\Re^n$, any vector $x \in \Re^n$ can be written as $x = \sum_{j=1}^n x_j v_j$, with coordinates $(x_1, \dots, x_n)$.

If T is a linear transformation, then $T(\alpha v_1 + \beta v_2) = T(\alpha v_1) + T(\beta v_2) = \alpha T(v_1) + \beta T(v_2)$. Therefore

$$T(x) = T\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j T(v_j).$$

Since each $T(v_j)$ lies in $\Re^m$ we can write $T(v_j) = \sum_{i=1}^m a_{ij} u_i$, where $(a_{1j}, a_{2j}, \dots, a_{mj})$ are the coordinates of $T(v_j)$ with respect to the basis U for $\Re^m$.

Thus

$$T(x) = \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij} u_i = \sum_{i=1}^m y_i u_i$$

where the ith coordinate, y_i, of $T(x)$ is equal to $\sum_{j=1}^n a_{ij} x_j$.

We obtain a set of linear equations:

$$\begin{array}{llllll} y_1 & = a_{11}x_1 & + a_{12}x_2 & + \cdots a_{1j}x_j & + \cdots a_{1n}x_n \\ \vdots & & \vdots & & \\ y_i & = a_{i1}x_1 & + a_{i2}x_2 & + \cdots a_{ij}x_j & + \cdots a_{in}x_n \\ \vdots & & \vdots & & \\ y_m & = a_{m1}x_1 & + a_{m2}x_2 & + \cdots a_{mj}x_j & + \cdots a_{mn}x_n. \end{array}$$

This set of equations is more conveniently written

$$\begin{array}{cc} \text{row } i & \begin{pmatrix} a_{11} & \dots & a_{ij} & \dots & a_{1n} \\ \vdots & & & & \\ a_{i1} & & a_{ij} & & a_{in} \\ \vdots & & & & \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{pmatrix}. \\ & j\text{th column} \end{array}$$

or as $M(T)x = y$, where $M(T)$ is the $n \times m$ array whose ith row is $(a_{i1}, \dots, a_{in})$ and whose jth column is $(a_{1j}, \dots, a_{mj})$. This matrix is commonly written as (a_{ij}) where it is understood that $i = 1, \dots, m$ and $j = 1, \dots, n$.

Note that the operation of $M(T)$ on x is as follows: to obtain the ith coordinate, y_i, take the ith row vector $(a_{11}, \dots, a_{1n})$ and form the *scalar product* of this with the column vector $(x_1, \dots, x_n)$, where this scalar product is defined to be $\sum_{j=1}^{n} a_{ij} x_j$.

The coefficients of $T(v_j)$ with respect to the basis $(u_1, \dots, u_m)$ are $(a_{1j}, \dots, a_{mj})$ and these turn up as the jth column of the matrix. Thus we could write the matrix as

$$M(T) = \bigl(T(v_1) \dots T(v_j) \dots T(v_n)\bigr)$$

where $T(v_j)$ is the column of coordinates in $\Re^m$. Suppose now that $W = \{w_1, \dots, w_p\}$ is a basis for $\Re^p$ and $S : \Re^m \to \Re^p$ is a linear transformation. Then to represent S as a matrix with respect to the two sets of bases, U and W, for each $i = 1, \dots, m$, we need to know

$$S(u_i) = \sum_{k=1}^{p} b_{ki} w_k.$$

Then as before S is represented by the matrix

$$M(S) = \begin{pmatrix} b_{11} & \dots & b_{1i} & \dots & b_{1m} \\ \vdots & & b_{ki} & & \vdots \\ b_{p1} & \dots & b_{pi} & \dots & b_{pm} \end{pmatrix}$$

where the ith column is the column of coordinates of $S(u_i)$ in $\Re^p$.

We can compute the composition

$$(S \circ T) : \Re^n \xrightarrow{T} \Re^m \xrightarrow{S} \Re^p.$$

The question is how should we compose the two matrices $M(S)$ and $M(T)$ so that the result "corresponds" to the matrix $M(S \circ T)$ which represents $S \circ T$.

First of all we show that $S \circ T : \Re^n \to \Re^p$ is a linear transformation, so that we know that it can be represented by an $(n \times p)$ matrix.

Lemma 2.10 *If $T : \Re^n \to \Re^m$ and $S : \Re^m \to \Re^p$ are linear transformations, then $S \circ T : \Re^n \to \Re^p$ is a linear transformation.*

Proof Consider $\alpha, \beta \in \Re$, $v_1, v_2 \in \Re^n$. Then

$$\begin{aligned}(S \circ T)(\alpha v_1 + \beta v_2) &= S\big[T(\alpha v_1 + \beta v_2)\big] \\ &= S\big(\alpha T(v_1) + \beta T(v_2)\big) \quad \text{since } T \text{ is linear} \\ &= \alpha S\big(T(v_1)\big) + \beta S\big(T(v_2)\big) \quad \text{since } S \text{ is linear} \\ &= \alpha (S \circ T)(v_1) + \beta (S \circ T)(v_2).\end{aligned}$$

Thus $S \circ T$ is linear.

By the previous analysis, $(S \circ T)$ can be represented by an $(n \times p)$ matrix whose jth column is $(S \circ T)(v_j)$. Thus

$$\begin{aligned}(S \circ T)(v_j) &= S\left(\sum_{i=1}^{m} a_{ij} u_i\right) \\ &= \sum_{i=1}^{m} a_{ij} S(u_i) \\ &= \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{p} b_{ki} w_k \\ &= \sum_{k=1}^{p} \left(\sum_{i=1}^{m} a_{ij} b_{ki}\right) w_k.\end{aligned}$$

Thus the kth entry in the jth column of $M(S \circ T)$ is $\sum_{i=1}^{m} b_{ki} a_{ij}$.

Thus $(S \circ T)$ can be represented by the matrix

$$M(S \circ T) = k\text{th row} \left(\begin{array}{ccc} & \longleftarrow n \longrightarrow & \\ \cdots & \sum_{i=1}^{m} b_{ki} a_{ij} & \cdots \\ & & \\ & j\text{th column} & \end{array}\right) p$$

The jth column in this matrix can be obtained more simply by operating the matrix $M(S)$ on the jth column vector $T(v_j)$ in the matrix $M(T)$.

Thus $M(S \circ T) = (M(S)(T(v_1)) \ldots M(S)(T(v_n))) = M(S) \circ M(T)$.

$$\begin{array}{l}k\text{th row of}\\ p\text{ rows}\end{array}\begin{pmatrix} b_{k1}\ldots & b_{ki}\ldots & b_{km}\\ \longleftarrow & m\text{ columns} & \longrightarrow\end{pmatrix}\begin{pmatrix}\longleftarrow n\longrightarrow\\ a_{ij}\\ \vdots\\ a_{ij}\\ \vdots\\ a_{mj}\\ j\text{th column}\end{pmatrix} m\text{ rows}$$

$$= M(S)\circ M(T). \qquad \square$$

Thus the "natural" method of matrix composition corresponds to the composition of linear transformations.

Now let $L(\Re^n,\Re^n)$ stand for the set of linear transformations from $\Re^n$ to $\Re^n$. As we have shown, if S,T belong to this set then $S\circ T$ is also a linear transformation from $\Re^n$ to $\Re^n$. Thus composition of functions $(\circ)$ is a binary operation $L(\Re^n,\Re^n)\times L(\Re^n,\Re^n)\to L(\Re^n,\Re^n)$.

Let $M : L(\Re^n,\Re^n)\to M(n,n)$ be the mapping which assigns to any linear transformation $T:\Re^n\to\Re^n$ the matrix $M(T)$ as above. Note that M is dependent on the choice of bases $\{v_1,\ldots,v_n\}$ and $\{u_1,\ldots,u_n\}$ for the domain and codomain, $\Re^n$. There is in general no reason why these two bases should be the same.

Now let $\circ$ be the method of matrix composition which we have just defined. Thus the mapping M satisfies

$$M(S\circ T)=M(S)\circ M(T)$$

for any two linear transformations, S and T. Suppose now that we are given a linear transformation, $T\in L(\Re^n,\Re^n)$. Clearly the matrix $M(T)$ which represents T with respect to the two bases is unique, and so M is a function.

On the other hand suppose that T,S are both represented by the same matrix $A=(a_{ij})$.

By definition $T(v_j)=S(v_j)=\sum_{i=1}^m a_{ij}u_i$ for each $j=1,\ldots,n$.

But then $T(x)=S(x)$ for any $x\in\Re^n$, and so $T=S$. Thus M is injective.

Moreover if A is any matrix, then it represents a linear transformation, and so M is surjective. Thus we have a bijective morphism

$$M:\bigl(L\bigl(\Re^n,\Re^n\bigr),\circ\bigr)\to\bigl(M(n,n),\circ\bigr).$$

As we saw in the case of 2 × 2 matrices, the subset of non-singular matrices in $M(n,n)$ forms a group. We repeat the procedure for the more general case.

2.2.2 The Dimension Theorem

Let $T:V\to U$ be a linear transformation between the vector spaces V,U of dimension n,m respectively over a field $\mathcal{F}$. The transformation is characterised by two subspaces, of V and U.

Definition 2.8

1. the *kernel* of a transformation $T : V \to U$ is the set $\operatorname{Ker}(T) = \{x \in V : T(x) = \underline{0}\}$ in V.
2. The *image* of the transformation is the set $\operatorname{Im}(T) = \{y \in U : \exists\ x \in\ V$ s.t. $T(x) = y\}$.

Both these sets are vector subspaces of U, V respectively. To see this suppose $v_1, v_2 \in \operatorname{Ker}(T)$, and $\alpha, \beta \in F$. Then $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2) = \underline{0} + \underline{0} = \underline{0}$. Hence $\alpha v_1 + \beta v_2 \in \operatorname{Ker}(T)$.

If $u_1, u_2 \in \operatorname{Im}(T)$ then there exists $v_1, v_2 \in V$ such that $T(v_1) = u_1, T(v_2) = u_2$. But then

$$\begin{aligned}\alpha\mu_1 + \beta\mu_2 &= \alpha T(v_1) + \beta T(v_2)\\ &= T(\alpha v_1 + \beta v_2).\end{aligned}$$

Since V is a vector space, $\alpha v_1 + \beta v_2 \in V$ and so $\alpha u_1 + \beta u_2 \in \operatorname{Im}(T)$.

By the exchange theorem there exists a basis $k_1, \dots, k_p$ for $\operatorname{Ker}(T)$, where $p = \dim \operatorname{Ker}(T)$ and a basis $u_1, \dots, u_s$ for $\operatorname{Im}(T)$ where $s = \dim(\operatorname{Im}(T))$. Here p is called the *kernel rank* of T, often written $kr(T)$, and s is the *rank* of T, or $rk(T)$.

The Dimension Theorem *If $T : V \to U$ is a linear transformation between vector spaces over a field $\mathcal{F}$, where dimension $(V) \neq n$, then the dimension of the kernel and image of T satisfy the relation*

$$\dim\bigl(\operatorname{Im}(T)\bigr) + \dim\bigl(\operatorname{Ker}(T)\bigr) = n.$$

Proof Let $\{u_1, \dots, u_s\}$ be a basis for $\operatorname{Im}(T)$ and for each $i = 1, \dots, s$, let v_i be the vector in V^n such that $T(v_i) = u_i$.

Let v be any vector in V. Then

$$T(v) = \sum_{i=1}^{s} \alpha_i u_i, \quad \text{for } T(v) \in \operatorname{Im}(T).$$

So

$$\begin{aligned}T(v) &= \sum_{i=1}^{s} \alpha_i T(v_i)\\ &= T\left(\sum_{i=1}^{s} \alpha_i v_i\right), \quad \text{and}\\ T\left(v - \sum_{i=1}^{s} \alpha_i v_i\right) &= \underline{0},\end{aligned}$$

the zero vector in U, *i.e.*, $v - \sum_{i=1}^{s} \alpha_i v_i \in$ kernel T. Let $\{k_1, \dots, k_p\}$ be the basis for $\operatorname{Ker}(T)$.

Then $v - \sum_{i=1}^{s} \alpha_i v_i = \sum_{j=1}^{p} \beta_j k_j$, or $v = \sum_{i=1}^{s} \alpha_i v_i + \sum_{j=1}^{p} \beta_j k_j$. Thus $(v_1, \dots, v_s, k_1, \dots, k_p)$ is a span for V.

Suppose we consider

$$\sum_{i=1}^{s} \alpha_i v_i + \sum_{j=1}^{p} \beta_j k_j = \underline{0}. \qquad (*)$$

Then, since $T(k_j) = 0$ for $j = 1, \dots, p$,

$$\begin{aligned} T\left(\sum_{i=1}^{s} \alpha_i v_i + \sum_{j=1}^{p} \beta_j k_j\right) &= \sum_{i=1}^{s} \alpha_i T(v_i) + \sum_{j=1}^{p} \beta_j T(k_j) \\ &= \sum_{i=1}^{s} \alpha_i T(v_i) = \sum_{i=1}^{s} \alpha_i u_i = \underline{0}. \end{aligned}$$

Now $\{u_i, \dots, u_s\}$ is a basis for $\text{Im}(T)$, and hence these vectors are linearly independent. So $\alpha_i = 0, i = 1, \dots, s$. Therefore (*) gives $\sum_{j=1}^{p} \beta_j k_j = \underline{0}$.

However $\{k_1, \dots, k_p\}$ is a basis for $\text{Ker}(T)$ and therefore a frame, so $\beta_j = 0$ for $j = 1, \dots, p$. Hence $\{v_1, \dots, v_s, k_1, \dots, k_p\}$ is a frame, and therefore a basis for V. By the exchange theorem the dimension of V is the unique number of vectors in a basis. Therefore $s + p = n$. □

Note that this theorem is true for general vector spaces. We specialise now to vector spaces $\Re^n$ and $\Re^m$.

Suppose $\{v_1, \dots, v_n\}$ is a basis for $\Re^n$. The coordinates of v_j with respect to this basis are $(0, \dots, 1, \dots, 0)$ with 1 in the jth place. As we have noted the image of v_j under the transformation T can be represented by the jth column $(a_{ij}, \dots, a_{mj})$ in the matrix $M(T)$, with respect to the original basis $(e_1, \dots, e_m)$, say, for $\Re^m$. Call the n different column vectors of this matrix $a_1, \dots, a_j, \dots, a_n$.

Then the equation $M(T)(x) = y$ is identical to the equation $\sum_{j=1}^{n} x_j a_j = y$ where $x = (x_1, \dots, x_n)$.

Clearly any vector y in the image of $M(T)$ can be written as a linear combination of the columns $A = \{a_1, \dots, a_n\}$. Thus $\text{Span}(A) = \text{Im}(M(T))$. Suppose now that A is not a frame. In this case a_n, say, can be written as a linear combination of $\{a_1, \dots, a_{n-1}\}$, *i.e.*, $\sum_{j=1}^{n} k_{1j} a_j = \underline{0}$ and $k_{1n} \neq 0$. Then the vector $k_1 = (k_{11}, \dots, k_{1n})$ satisfies $M(T)(k_1) = \underline{0}$. Thus k_1 belongs to $\text{Ker}(M(T))$.

Eliminate a_n, say, and proceed in this way. After p iterations we will have obtained p kernel vectors $\{k_1, \dots, k_p\}$ and the remaining column vectors $\{a_1, \dots, a_{n-p}\}$ will form a frame, and thus a basis for the image of $M(T)$.

Consequently $\dim(\text{Im}(M(T))) = n - p = n - \dim(\text{Ker}(M(T)))$. The number of linearly independent columns in the matrix $M(T)$ is called the *rank* of $M(T)$, and is clearly the dimension of the image of $M(T)$. In particular if $M_1(T)$ and $M_2(T)$ are two matrix representations with respect to different bases, of the linear transformation T, then $\text{rank}\, M_1(T) = \text{rank}\, M_2(T) = \text{rank}(T)$.

Thus $\operatorname{rank}(T)$ is an invariant, in the sense of being independent of the particular bases chosen for $\Re^n$ and $\Re^m$.

In the same way the kernel rank of T is an invariant; that is, for any matrix representation $M(T)$ of T we have $\ker\operatorname{rank}(M(T)) = \ker\operatorname{rank}(T)$.

In general if $y \in \operatorname{Im}(T)$, x_0 satisfies $T(x_0) = y$, and k belongs to the kernel, then

$$T(x_0 + k) = T(x_0) + T(k) = y + \underline{0} = y.$$

Thus if x_0 is a solution to the equation $T(x_0) = y$, the point $x_0 + k$ is also a solution. More generally $x_0 + \operatorname{Ker}(T) = \{x_0 + k : k \in \operatorname{Ker}(T)\}$ will also be the set of solutions. Thus for a particular $y \in \operatorname{Im}(T)$, $T^{-1}(y) = \{x : T(x) = y\} = x_0 + \operatorname{Ker}(T)$.

By the dimension theorem $\dim\operatorname{Ker}(T) = n - \operatorname{rank}(T)$. Thus $T^{-1}(y)$ is a geometric object of "dimension" $\dim \operatorname{Ker}(T) = n - \operatorname{rank}(T)$.

We defined T to be *injective* iff $T(x_0) = T(x)$ implies $x_0 = x$. Thus T is injective iff $\operatorname{Ker}(T) = \{\underline{0}\}$. In this case, if there is a solution to the equation $T(x_0) = y$, then this solution is unique.

Suppose that $n \leq m$, and that the n different column vectors of the matrix are linearly independent. In this case $\operatorname{rank}(T) = n$ and so $\dim\operatorname{Ker}(T) = 0$. Thus T must be *injective*. In particular if $n < m$ then not every $y \in \Re^m$ belongs to the image of T, and so not every equation $T(x) = y$ has a solution. Suppose on the other hand that $n > m$. In this case the maximum possible rank is m (since n vectors cannot be linearly independent in $\Re^m$ when $n > m$). If $\operatorname{rank}(T) = m$, then there must exist a kernel of dimension $(n - m)$.

Moreover $\operatorname{Im}(T) = \Re^m$, and so for every $y \in \Re^m$ there exists a solution to this equation $T(x) = y$. Thus T is *surjective*. However the solution is not unique, since $T^{-1}(y) = x + \operatorname{Ker}(T)$ is of dimension $(n - m)$ as before.

Suppose now that $n = m$, and that $T : \Re^n \to \Re^n$ has maximal rank n. Then T is both injective and surjective and thus an *isomorphism*. Indeed T will have an inverse *function* $T^{-1} : \Re^n \to \Re^n$. Moreover T^{-1} is linear. To see this note that if $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$ then $T(x_1) = y_1$ and $T(x_2) = y_2$ so $T(x_1 + x_2) = y_1 + y_2$. Thus $T^{-1}(y_1 + y_2) = x_1 + x_2 = T^{-1}(y_1) + T^{-1}(y_2)$. Moreover if $x = T^{-1}(\alpha y)$ then $T(x) = \alpha y$. If $\alpha \neq 0$, then $\frac{1}{\alpha}T(x) = T(\frac{1}{\alpha}x) = y$ or $\frac{1}{\alpha}x = T^{-1}(y)$. Hence $x = \alpha T^{-1}(y)$. Thus $T^{-1}(\alpha y) = \alpha T^{-1}(y)$. Since T^{-1} is linear it can be represented by a matrix $M(T^{-1})$. As we know $M : (L(\Re^n, \Re^n), \circ) \to (M(n,n), \circ)$ is a bijective morphism, so M maps the identity linear transformation, Id, to the identity *matrix*

$$M(\mathrm{Id}) = I = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

When T is an isomorphism with inverse T^{-1}, then the representation $M(T^{-1})$ of T^{-1} is $[M(T)]^{-1}$. We now show how to compute the *inverse matrix* $[M(T)]^{-1}$ of an isomorphism.

2.2.3 The General Linear Group

To compute the inverse of an $n \times n$ matrix A, we define, by induction, the determinant of A. For a 1×1 matrix (a_{11}) define det $(A_{11}) = a_{11}$, and for a 2×2 matrix $A = \binom{a_{11}\ a_{12}}{a_{21}\ a_{22}}$ define det $A = a_{11}a_{22} - a_{21}a_{12}$.

For an $n \times n$ matrix A define the (i, j)th *cofactor* to be the determinant of the $(n-1) \times (n-1)$ matrix $A(i, j)$ obtained from A by removing the ith row and jth column, then multiplying by $(-1)^{i+j}$. Write this cofactor as A_{ij}. For example in the 3×3 matrix, the cofactor in the $(1, 1)$ position is

$$A_{11} = det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{22}a_{33} - a_{32}a_{23}.$$

The $n \times n$ matrix (A_{ij}) is called the *cofactor matrix.*

The determinant of the $n \times n$ matrix A is then $\sum_{j=1}^{n} a_{1j}A_{1j}$. The determinant is also often written as $|A|$.

This procedure allows us to define the determinant of an $n \times n$ matrix. For example if $A = (a_{ij})$ is a 3×3 matrix, then

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}). \end{aligned}$$

An alternative way of defining the determinant is as follows. A permutation of n is a bijection $s : \{1, \dots, n\} \to \{1, \dots, n\}$, with degree $d(s)$ the number of exchanges needed to give the permutation.

Then $|A| = \sum_s (-1)^{d(s)} \Pi_{i=1}^{n} a_{is(i)} = a_{11}a_{22}a_{33} \cdots + \cdots$ where the summation is over all permutations. The two definitions are equivalent, and it can be shown that

$$\begin{aligned} |A| &= \sum_{j=1}^{n} a_{ij}A_{ij} \quad \text{(for any } i = 1, \dots, n) \\ &= \sum_{i=1}^{n} a_{ij}A_{ij} \quad \text{(for any } j = 1, \dots, n) \end{aligned}$$

while

$$\begin{aligned} 0 &= \sum_{i=1}^{n} a_{ij}A_{ik} \quad \text{if } j \neq k \\ &= \sum_{j=1}^{n} a_{ij}A_{kj} \quad \text{if } i \neq k. \end{aligned}$$

Thus

$$(a_{ij})\ (A_{jk})^t = \left(\sum_{j=1}^{n} a_{ij} A_{kj}\right)$$
$$= \begin{pmatrix} |A| & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & |A| \end{pmatrix} = |A| I.$$

Here $(A_{jk})^t$ is the $n \times n$ matrix obtained by transposing the rows and columns of (A_{jk}). Now the matrix A^{-1} satisfies $A \circ A^{-1} = I$, and if A^{-1} exists then it is unique. Thus $A^{-1} = \frac{1}{|A|}(A_{ij})^t$.

Suppose that the matrix A is non-singular, so $|A| \neq 0$. Then we can construct an inverse matrix A^{-1}.

Moreover if $A(x) = y$ then $y = A^{-1}(x)$ which implies that A is both injective and surjective. Thus $\text{rank}(A) = n$ and the column vectors of A must be linearly independent.

As we have noted, however, if A is not injective, with $\text{Ker}(A) \neq \{0\}$, then $\text{rank}(A) < n$, and the column vectors of A must be linearly dependent. In this case the inverse A^{-1} is not a function and cannot therefore be represented by a matrix and so we would expect $|A|$ to be zero.

Lemma 2.11 *If A is an $n \times n$ matrix with* $\text{rank}(A) < n$ *then* $|A| = 0$.

Proof Let A' be the matrix obtained from A by adding a multiple (α) of the kth column of A to the jth column of A. The jth column of A' is therefore $a_j + \alpha a_k$. This operation leaves the jth column of the cofactor matrix unchanged. Thus

$$\begin{aligned} |A'| &= \sum_{i=1}^{n} a'_{ij} A_{ij} \\ &= \sum_{i=1}^{n} (a_{ij} + \alpha a_{ik}) A_{ij} \\ &= \sum_{i=1}^{n} a_{ij} A_{ij} + \alpha \sum_{i=1}^{n} a_{ik} A_{ij} \\ &= |A| + 0 = |A|. \end{aligned}$$

Suppose now that the columns of A are linearly dependent, and that $a_j = \sum_{k \neq j} \alpha_k a_k$ for example. Let A' be the matrix obtained from A by substituting $a'_j = 0 = a_j - \sum_{k \neq j} \alpha_k a_k$ for the jth column.

By the above $|A'| = \sum_{i=1}^{n} a'_{ij} A_{ij} = 0 = |A|$. □

Suppose now that A, B are two non-singular matrices (a_{ij}), (b_{ki}). The composition is then $B \circ A = (\sum_{i=1}^{m} b_{ki} a_{ij})$ with determinant

$$|B \circ A| = \sum_{s} (-1)^{d(s)} \prod_{k=1}^{n} \circ \left(\sum_{i=1}^{m} b_{ki} a_{is(k)} \right).$$

This expression can be shown to be equal to

$$\sum_{s} (-1)^{d(s)} \prod_{i=1}^{n} a_{is(i)} \sum_{s} (-1)^{d(s)} \prod_{i=1}^{n} b_{ks(k)} = |B||A| \neq 0.$$

Hence the composition $(B \circ A)$ has an inverse $(B \circ A)^{-1}$ given by $A^{-1} \circ B^{-1}$.

Now let $(GL(\Re^n, \Re^n), \circ)$ be the set of invertible linear transformations, with $\circ$ composition of functions, and let $M^*(n, n)$ be the set of non-singular $n \times n$ matrices. Choice of bases $\{v_1, \dots, v_n\}, \{u_1, \dots, u_n\}$ for the domain and codomain defines a morphism

$$M : \big(GL\big(\Re^n, \Re^n\big), \circ\big) \to \big(M^*(n, n), \circ\big).$$

Suppose now that T belongs to $GL(\Re^n, \Re^n)$. As we have seen this is equivalent to $|M(T)| \neq 0$, so the image of M is precisely $M^*(n, n)$. Moreover if $|M(T)| \neq 0$ then $|M(T^{-1})| = \frac{1}{|M(T)|}$ and $M(T^{-1})$ belongs to $M^*(n, n)$. On the other hand if $S, T \in GL(\Re^n, \Re^n)$ then $S \circ T$ also has rank n, and has inverse $T^{-1} \circ S^{-1}$ with rank n.

The matrix $M(S \circ T)$ representing $T \circ S$ has inverse

$$\begin{aligned} M\big(T^{-1} \circ S^{-1}\big) &= M\big(T^{-1}\big) \circ M\big(S^{-1}\big) \\ &= \big[M(T)\big]^{-1} \circ \big[M(S)\big]^{-1}. \end{aligned}$$

Thus M is an *isomorphism* between the two groups $(GL(\Re^n, \Re^n, \circ))$ and $(M^*(n, n), \circ)$.

The group of invertible linear transformations is also called the *general linear group*.

2.2.4 Change of Basis

Let $L(\Re^n, \Re^m)$ stand for the set of linear transformations from $\Re^n$ to $\Re^m$, and let $M(n, m)$ stand for the set of $n \times m$ matrices. We have seen that the choice of bases for $\Re^n, \Re^m$ defines a *function*.

$$M : L\big(\Re^n, \Re^m\big) \to M(n, m)$$

which take a linear transformation T to its representation $M(T)$. We now examine the relationship between two representations $M_1(T)$, $M_2(T)$ of a single linear transformation.

Basis Change Theorem *Let $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_m\}$ be bases for $\Re^n, \Re^m$ respectively.*

Let T be a linear transformation which is represented by a matrix $A = (a_{ij})$ with respect to these bases. If $V' = \{v'_1, \ldots, v'_n\}$, $U' = \{u'_1, \ldots, u'_m\}$ are new bases for $\Re^n, \Re^m$ then T is represented by the matrix $B = Q^{-1} \circ A \circ P$, where P, Q are respectively $(n \times n)$ and $(m \times m)$ invertible matrices.

Proof For each $v'_k \in V' = \{v'_1, \ldots, v'_n\}$ let $v'_k = \sum_{i=1}^n b_{ik} v_i$ and $b_k = (b'_{1k}, \ldots, b_{nk})$.

Let $P = (b_1, \ldots, b_n)$ where the kth column of P is the column of coordinates of b_k. With respect to the new basis V', v'_k has coordinates $e_k = (0, \ldots, 1, \ldots, 0)$ with a 1 in the kth place.

But then $P(e_k) = b_k$ the coordinates of v'_k with respect to V.

Thus P is the matrix that transforms coordinates with respect to V' into coordinates with respect to V. Since V is a basis, the columns of P are linearly independent, and so rank $P = n$, and P is invertible.

In the same way let $u'_k = \sum_{i=1}^m c_{ik} u_i$, $c_k = (c_{1k}, \ldots, c_{mk})$ and $Q = (c_1, \ldots, c_m)$ the matrix with columns of these coordinates.

Hence Q represents change of basis from U' to U. Since Q is an invertible $m \times m$ matrix it has inverse Q^{-1} which represents change of basis from U to U'.

Thus we have the diagram

$$\begin{tikzcd}
\{v_1, \ldots, v_n\} \arrow[r, "A"] & \{u_1, \ldots, u_m\} \arrow[d, shift right, "Q^{-1}"'] \\
\{v'_1, \ldots, v'_n\} \arrow[r, "B"] \arrow[u, "P"] & \{u'_1, \ldots, u'_m\} \arrow[u, shift right, "Q"']
\end{tikzcd}$$

from which we see that the matrix B, representing the linear transformation $T : \Re^n \to \Re^m$ with respect to the new bases is given by $B = Q^{-1} \circ A \circ P$. □

Isomorphism Theorem *Any linear transformation $T : \Re^n \to \Re^m$ of rank r can be represented, by suitable choice of bases for $\Re^n$ and $\Re^m$, by an $n \times m$ matrix*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{where } I_r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \textit{ is the } (r \times r) \textit{ identity matrix.}$$

In particular

1. *if $n < m$ and T is injective then there is an isomorphism $S : \Re^m \to \Re^m$ such that $S \circ T(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ with $(n - m)$ zero entries, for any vector $(x_1, \ldots, x_n)$ in $\Re^n$*

2. *if $n \geq m$ and T is surjective then there are isomorphisms $\Re : \Re^n \to \Re^n$, $S : \Re^m \to \Re^m$ such that $S \circ T \circ R(x_1, \dots, x_n) = (x_1, \dots, x_m)$. If $n = m$, then $S \circ T \circ R$ is the identity isomorphism.*

Proof Of necessity $\text{rank}(T) = r \leq \min(n, m)$. If $r < n$, let $p = n - r$ and choose a basis $k_1, \dots, k_p$ for $\text{Ker}(T)$. Let $V = \{v_1, \dots, v_n\}$ be the original basis for $\Re^n$. By the exchange theorem there exists $r = (n - p)$ different members $\{v_1, \dots, v_r\}$ say of V such that $V' = \{v_1, \dots, v_r, k_1, \dots, k_p\}$ is a basis for $\Re^n$.

Choose V' as the new basis for $\Re^n$, and let P be the basis change matrix whose columns are the column vectors in V'. As in the proof of the dimension theorem the image of the vectors $v_1, \dots, v_{n-p}$ under T provide a basis for the image of T. Let $U = \{u_1, \dots, u_m\}$ be the original basis of $\Re^m$. By the exchange theorem there exists some subset $U' = \{u_1, \dots, u_{m-r}\}$ of U such that $U'' = \{T(v_1), \dots, T(v_r), u_1, \dots, u_{m-r}\}$ form a basis for $\Re^m$. Note that $T(v_1), \dots, T(v_r)$ are represented by the r linearly independent columns of the original matrix A representing T. Now let Q be the matrix whose columns are the members of U''. By the basis change theorem, $B = Q^{-1} \circ A \circ P$, where B is the matrix representing T with respect to these new bases. Thus we obtain

$$\begin{tikzcd}
\{v_1, \dots, v_n\} \arrow[r, "A"] & \{u_1, \dots, u_m\} \arrow[d, "Q^{-1}"] \\
\{v_1, \dots, v_r, k_1 \dots\} \arrow[u, "P"] \arrow[r, "B"] & \left\{T(v_1) \dots T(v_r), u_1, \dots, u_{m-r}\right\}.
\end{tikzcd}$$

With respect to these new bases, the matrix B representing T has the required form: $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

1. If $n < m$ and T is injective then $r = n$. Hence P is the identity matrix, and so $B = Q^{-1} \circ A$.

   But Q^{-1} is an $m \times m$ invertible matrix, and thus represents an isomorphism $\Re^n \longrightarrow \Re^n$, while

$$B \begin{pmatrix} x_1 \\ x_n \end{pmatrix} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \\ 0 \end{pmatrix}.$$

   Write a vector $x = \sum_{i=1}^{n} x_i v_i$ as $(x_1, \dots, x_n)$, and let S be the linear transformation $\Re^m \to \Re^m$ represented by the matrix Q^{-1}. Then $S \circ T(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$.

2. If $n \geq m$ and T is surjective then $\text{rank}(T) = m$, and $\dim \text{Ker}(T) = n - m$. Thus $B = (I_m \ 0) = Q^{-1} \circ A \circ P$. Let S, R be the linear transformations represented by Q^{-1} and P respectively.

Then $S \circ T \circ R(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$. If $n = m$ then $S \circ T \circ R$ is the identity transformation. □

Suppose now that V, U are the two bases for $\Re^n, \Re^m$ as in the basis theorem. A linear transformation $T : \Re^n \longrightarrow \Re^m$ is represented by a matrix $M_1(T)$ with respect to these bases. If V', U' are two new bases, then T will be represented by the matrix $M_2(T)$, and by the basis theorem

$$M_2(T) = Q^{-1} \circ M_1(T) \circ P$$

where Q, P are non-singular $(m \times m)$ and $(n \times n)$ matrices respectively. Since $M_1(T)$ and $M_2(T)$ represent the same linear transformation, they are in some sense equivalent. We show this more formally.

Say the two matrices $A, B \in M(n,m)$ are *similar* iff there exist non singular square matrices $P \in M^*(n,n)$ and $Q \in M^*(m,m)$ such that $B = Q^{-1} \circ A \circ P$, and in this case write $B \sim A$.

Lemma 2.12 *The similarity relation $(\sim)$ on $M(n,m)$ is an equivalence relation.*

Proof

1. To show that $\sim$ is reflexive note that $A = I_m^{-1} \circ A \circ I_n$ where I_m, I_n are respectively the $(m \times m)$ and $(n \times n)$ identity matrices.
2. To show that $\sim$ is symmetric we need to show that $B \sim A$ implies that $A \sim B$. Suppose therefore that $B = Q^{-1} \circ A \circ P$.
Since $Q \in M^*(m,m)$ it has inverse $Q^{-1} \in M^*(m,m)$.
Moreover $(Q^{-1})^{-1} \circ Q^{-1} = I_m$, and thus $Q = (Q^{-1})^{-1}$. Thus

$$\begin{aligned} Q \circ B \circ P^{-1} &= \left(Q \circ Q^{-1}\right) \circ A \circ \left(P \circ P^{-1}\right) \\ &= A \\ &= \left(Q^{-1}\right)^{-1} \circ B \circ \left(P^{-1}\right). \end{aligned}$$

Thus $A \sim B$.
3. To show $\sim$ is transitive, we seek to show that $C \sim B \sim A$ implies $C \sim A$. Suppose therefore that $C = R^{-1} \circ B \circ S$ and $B = Q^{-1} \circ A \circ P$, where $R, Q \in M^*(m,m)$ and $S, P \in M^*(n,n)$. Then

$$\begin{aligned} C &= \left(R^{-1} \circ Q^{-1}\right) \circ A \circ P \circ S \\ &= (Q \circ R)^{-1} \circ A \circ (P \circ S). \end{aligned}$$

Now $(M^*(m,m), \circ)$, $(M^*(n,n), \circ)$ are both groups and so $Q \circ R \in M^*(m,m)$, $P \circ S \in M^*(n,n)$. Thus $C \sim A$. □

The isomorphism theorem shows that if there is a linear transformation $T : \Re^n \longrightarrow \Re^m$ of rank r, then the $(n \times m)$ matrix $M_1(T)$ which represents T, with respect to some pair of the bases, is similar to an $n \times m$ matrix

$$B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad i.e., \; M(T) \sim B.$$

If S is a second linear transformation of rank r then $M_1(S) \sim B$.

By Lemma 2.12, $M_1(S) \sim M_1(T)$.

Suppose now that U', V' are a second pair of bases for $\Re^n, \Re^m$ and let $M_2(S), M_2(T)$ represent S and T. Clearly $M_2(S) \sim M_2(T)$.

Thus if S, T are linear transformations $\Re^m \longrightarrow \Re^n$ we may say that S, T are equivalent iff for any choice of bases the matrices $M(S), M(T)$ which represent S, T are similar.

For any linear transformation $T \in L(\Re^n, \Re^m)$ let $[T]$ be the equivalence class $\{S \in L(\Re^n, \Re^m) : S \sim T\}$. Alternatively a linear transformation S belongs to $[T]$ iff $\text{rank}(S) = \text{rank}(T)$. Consequently the equivalence relation partitions $L(\Re^n, \Re^m)$ into a finite number of distinct equivalence classes where each class is classified by its rank, and the rank runs from 0 to $\min(n, m)$.

2.2.5 Examples

Example 2.1 To illustrate the use of these procedures in the solution of linear equations, consider the case with $n < m$ and the equation $A(x) = y$ where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 0 & 3 \\ -1 & -4 & 5 \\ 3 & 2 & -1 \end{pmatrix} \quad \text{and} \quad y_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \qquad y_2 = \begin{pmatrix} 0 \\ 5 \\ -5 \\ 5 \end{pmatrix}.$$

To find $\text{Im}(A)$, we first of all find $\text{Ker}(A)$. The equation $A(x) = \underline{0}$ gives four equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 5x_1 + 0 + 3x_3 &= 0 \\ -x_1 - 4x_2 + 5x_3 &= 0 \\ 3x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

with solution $k = (x_1, x_2, x_3) = (-3, 7, 5)$.

Thus $\text{Ker}(A) \supset \{\lambda k \in \Re^3 : \lambda \in \Re\}$. Hence $\dim \text{Im}(A) \leq 2$. Clearly the first two columns (a_1, a_2) of A are linearly independent and so $\dim \text{Im}(A) = 2$. However $y_2 = a_1 + a_2$. Thus a particular solution to the equation $A(x) = y_2$ is $x_0 = (1, 1, 0)$.

The full set of solutions to the equation is

$$x_0 + \mathrm{Ker}(A) = \big\{(1,1,0) + \lambda(-3,7,5) : \lambda \in \Re\big\}.$$

To see whether $y_1 \in \mathrm{Im}(A)$ we need only attempt to solve the equation $y_1 = \alpha a_1 + \beta a_2$. This gives

$$\begin{aligned} -1 &= \alpha - \beta \\ 1 &= 5\alpha \\ -1 &= -\alpha - 4\beta \\ 1 &= 3\alpha + 2\beta. \end{aligned}$$

From the first two equations $\alpha = \frac{1}{5}$, $\beta = \frac{6}{5}$, which is incompatible with the fourth equation. Thus y_1 cannot belong to $\mathrm{Im}(A)$.

Example 2.2 Consider now an example of the case $n > m$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \end{pmatrix} : \Re^5 \longrightarrow \Re^2.$$

Obviously the first two columns are linearly independent and so $\dim \mathrm{Im}(A) \geq 2$. Let $\{a_i : i = 1, \dots, 5\}$ be the five column vectors of the matrix and consider the equation

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $k_1 = (1, -1, -1, 0, 0)$ belongs to $\mathrm{Ker}(A)$. On the other hand

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $k_2 = (1, 1, 0, -3, 0)$ and $k_3 = (1, 1, 0, 0, -3)$ both belong to $\mathrm{Ker}(A)$.

Consequently the rank of A has its maximal value of 2, while the kernel is three-dimensional. Hence for any $y \in \Re^2$ there is a set of solutions of the form $x_0 + \mathrm{Span}\{k_1, k_2, k_3\}$ to the equation $A(x) = y$.

Change the bases of $\Re^5$ and $\Re^2$ to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}$$

and $\binom{2}{1}$, $\binom{1}{2}$ respectively, then

$$\begin{aligned} B &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Example 2.3 Consider the matrix

$$Q = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -1 & -4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}.$$

Since $|Q| = 5$ we can compute its inverse. The cofactor matrix (Q_{ij}) of Q is

$$\begin{pmatrix} 0 & -5 & -20 & 10 \\ 1 & 1 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

and thus

$$Q^{-1} = \frac{1}{|Q|}(Q_{ij})^t = \begin{pmatrix} 0 & \frac{1}{5} & 0 & 0 \\ -1 & \frac{1}{5} & 0 & 0 \\ -4 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

Example 2.4 Let $T : \Re^3 \to \Re^4$ be the linear transformation represented by the matrix A of Example 2.1, with respect to the standard bases for $\Re^3, \Re^4$. We seek to change the bases so as to represent T by a diagonal matrix

$$B = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}.$$

By Example 2.1, the kernel is spanned by $(-3, 7, 5)$, and so we choose a new basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad k = \begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix}$$

with basis change matrix $P = (e_1, e_2, k)$. Note that $|P| = 5$ and P is non-singular. Thus $\{e_1, e_2, k\}$ form a basis for $\Re^3$. Now $\text{Im}(A)$ is spanned by the first two columns a_1, a_2, of A. Moreover $A(e_1) = a_1$ and $A(e_2) = a_2$. Thus choose

$$a_1 = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 3 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ 0 \\ -4 \\ 2 \end{pmatrix}, \quad e_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_4' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

as the new basis for $\Re^4$. Let $Q = (a_1, a_2, e_3', e_4')$ be the basis change matrix. The inverse Q^{-1} is computed in Example 2.3. Thus we have $(B) = Q^{-1} \circ A \circ P$.

To check that this is indeed the case we compute:

$$\begin{aligned} Q^{-1} \circ A \circ P &= \begin{pmatrix} 0 & \frac{1}{5} & 0 & 0 \\ -1 & \frac{1}{5} & 0 & 0 \\ -4 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 5 & 0 & 3 \\ -1 & -4 & 5 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

as required.

2.3 Canonical Representation

When considering a linear transformation $T : \Re^n \longrightarrow \Re^n$ it is frequently convenient to change the basis of $\Re^n$ to a new basis $V = \{v_1, \dots, v_n\}$ such that T is now represented by a matrix

$$M_2(T) = P^{-1} \circ M_1(T) \circ P.$$

In this case it is generally not possible to obtain $M_2(T)$ in the form $\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as before.

Under certain conditions however $M_2(T)$ can be written in a diagonal form

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \cdot & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are known as the *eigenvalues*.

More explicitly, a vector x is called an *eigenvector* of the matrix A iff there is a solution to the equation $A(x) = \lambda x$ where λ is a real number. In this case, λ is called the *eigenvalue* associated with the eigenvector x. (Note that we assume $x \neq \underline{0}$.)

2.3.1 Eigenvectors and Eigenvalues

Suppose that there are n linearly independent eigenvectors $\{x_1, \dots, x_n\}$ for A, where (for each $i = 1, \dots, n$) λ_i is the eigenvalue associated with x_i. Clearly the eigenvector x_i belongs to $\operatorname{Ker}(A)$ iff $\lambda_i = 0$. If $\operatorname{rank}(A) = r$ then there would a subset $\{x_1, \dots, x_r\}$ of eigenvectors which form a basis for $\operatorname{Im}(A)$, while $\{x_1, \dots, x_n\}$ form a basis for $\Re^n$. Now let Q be the $(n \times n)$ matrix representing a basis change from the new basis to the original basis. That is to say the ith column, v_i, of Q is the coordinate of x_i with respect to the original basis.

After transforming, the original becomes

$$Q^{-1} \circ A \circ Q = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \lambda_r & \vdots \\ 0 & & 0 \end{pmatrix} = \wedge,$$

where $\operatorname{rank} \wedge = \operatorname{rank} A = r$.

In general we can perform this diagonalisation only if there are enough eigenvectors, as the following lemma indicates.

Lemma 2.13 *If A is an $n \times n$ matrix, then there exists a non-singular matrix Q, and a diagonal matrix $\wedge$ such that $\wedge = Q^{-1}AQ$ iff the eigenvectors of A form a basis for $\Re^n$.*

Proof

1. Suppose the eigenvectors form a basis, and let Q be the eigenvector matrix. By definition, if v_i is the ith column of Q, then $A(v_i) = \lambda_i v_i$, where λ_i is real. Thus $AQ = Q\wedge$. But since $\{v_1, \dots, v_n\}$ is a basis, Q^{-1} exists and so $\wedge = Q^{-1}AQ$.
2. On the other hand if $\wedge = Q^{-1}AQ$, where Q is non-singular then $AQ = Q\wedge$. But this is equivalent to $A(v_1) = \lambda_i v_i$ for $i = 1, \dots, n$ where λ_i is the ith diagonal entry in $\wedge$, and v_i is the ith column of Q.
   Since Q is non-singular, the columns $\{v_1, \dots, v_n\}$ are linearly independent, and thus the eigenvectors form a basis for $\Re^n$. □

If there are n distinct (real) eigenvalues then this gives a basis, and thus a diagonalisation.

Lemma 2.14 *If $\{v_1, \dots, v_m\}$ are eigenvectors corresponding to distinct eigenvalues $\{\lambda_1, \dots, \lambda_m\}$, of a linear transformation $T : \Re^n \to \Re^n$, then $\{v_1, \dots, v_m\}$ are linearly independent.*

Proof Since v_1 is assumed to be an eigenvector, it is non-zero, and thus $\{v_1\}$ is a linearly independent set. Proceed by induction.

Suppose $V_k = \{v_1, \ldots, v_k\}$, with $k < m$, are linearly independent. Let v_{k+1} be another eigenvector and suppose

$$v = \sum_{r=1}^{k+1} a_r v_r = \underline{0}.$$

Then $\underline{0} = T(v) = \sum_{r=1}^{k+1} a_r T(v_r) = \sum_{r=1}^{k+1} a_r \lambda_r v_r$.

If $\lambda_{k+1} = 0$, then $\lambda_i \neq 0$ for $i = 1, \ldots, k$ and by the linear independence of V_k, $a_r\lambda_r = 0$, and thus $a_r = 0$ for $r = 1, \ldots, k$.

Suppose $\lambda_{k+1} \neq 0$. Then

$$\lambda_{k+1} v = \sum_{r=1}^{k+1} \lambda_{k+1} a_r v_r = \sum_{r=1}^{k+1} a_r \lambda_r v_r = \underline{0}.$$

Thus $\sum_{r=1}^{k} (\lambda_{k+1} - \lambda_r) a_r v_r = \underline{0}$.

By the linear independence of V_k, $(\lambda_{k+1} - \lambda_r)a_r = 0$ for $r = 1, \ldots, k$.

But the eigenvalues are distinct and so $a_r = 0$, for $r = 1, \ldots, k$.

Thus $a_{k+1}v_{k+1} = \underline{0}$ and so $a_r = 0, r = 1, \ldots, k+1$. Hence

$$V_{k+1} = \{v_1, \ldots, v_{k+1}\}, \quad k < m,$$

is linearly independent.

By induction V_m is a linearly independent set. □

Having shown how the determination of the eigenvectors gives a diagonalisation, we proceed to compute eigenvalues.

Consider again the equation $A(x) = \lambda x$. This is equivalent to the equation $A'(x) = 0$, where

$$A' = \begin{pmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdot & \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & & & a_{nn} - \lambda \end{pmatrix}.$$

For this equation to have a non zero solution it is necessary and sufficient that $|A'| = 0$. Thus we obtain a polynomial equation (called the characteristic equation) of degree n in λ, with n roots $\lambda_1, \ldots, \lambda_n$ not necessarily all real. In the 2×2 case for example this equation is $\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{21}a_{12}) = 0$. If the roots of this equation are λ_1, λ_2 then we obtain

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda(\lambda_1 + \lambda_2) \ + \lambda_1\lambda_2.$$

Hence

$$\lambda_1\lambda_2 = (a_{11}a_{22} - a_{21}a_{22}) = |A|$$
$$\lambda_1 + \lambda_2 = a_{11} + a_{22}.$$

The sum of the diagonal elements of a matrix is called the *trace* of A. In the 2×2 case therefore

$$\lambda_1\lambda_2 = |A|, \qquad \lambda_1 + \lambda_2 = a_{11} + a_{22} = \text{trace}(A).$$

In the 3×3 case we find

$$\begin{aligned}&(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\\ &\quad = \lambda^3 - \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3\\ &\quad = \lambda^3 - \lambda^2(\text{trace}\, A) + \lambda(A_{11} + A_{22} + A_{33}) - |A| = 0,\end{aligned}$$

where A_{ii} is the ith diagonal cofactor of A. Suppose all the roots are non-zero (this is equivalent to the non-singularity of the matrix A). Let

$$\wedge = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

be the diagonal eigenvalue matrix, with $| \wedge | = \lambda_1\lambda_2\lambda_3$.

The cofactor matrix of $\wedge$ is then

$$\begin{pmatrix} \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix}.$$

Thus we see that the sum of the diagonal cofactors of A and $\wedge$ are identical. Moreover trace $(A) =$ trace $(\wedge)$ and $| \wedge | = |A|$.

Now let $\sim$ be the equivalence relation defined on $L(\Re^n, \Re^n)$ by $B \sim A$ iff there exist basis change matrices P, Q and a diagonal matrix $\wedge$ such that

$$\wedge = P^{-1}AP = Q^{-1}BQ.$$

On the set of matrices which can be diagonalised, $\sim$ is an equivalence relation, and each class is characterised by n invariants, namely the trace, the determinant, and $(n - 2)$ other numbers involving the cofactors.

2.3.2 Examples

Example 2.5 Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned}(2-\lambda)\bigl[(1-\lambda)(-2-\lambda)\bigr] - 1(-2) - (-2(1-\lambda) &= -\lambda\bigl(\lambda^2 - \lambda - 2\bigr) \\ &= -\lambda(\lambda - 2)(\lambda + 1) \\ &= 0.\end{aligned}$$

Hence $(\lambda_1, \lambda_2, \lambda_3) = (0, 2, -1)$. Note that $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A) = 1$ and

$$\lambda_2\lambda_3 = -2 = A_{11} + A_{22} + A_{33}.$$

Eigenvectors corresponding to these eigenvalues are

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Let P be the basis change matrix given by these three column vectors. The inverse can be readily computed, to give

$$P^{-1}AP = \begin{pmatrix} 1 & -1 & -1 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose we now compute $A^2 = A \circ A : \Re^3 \to \Re^3$. This can easily be seen to be

$$\begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix}.$$

The characteristic function of A^2 is $(\lambda^3 - 5\lambda^2 + 4\lambda)$ with roots $\mu_1 = 0$, $\mu_2 = 4$, $\mu_3 = 1$.

In fact the eigenvectors of A^2 are x_1, x_2, x_3, the same as A, but with eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$. In this case $\text{Im}(a) = \text{Im}(A^2)$ is spanned by $\{x_2, x_3\}$ and $\text{Ker}(A) = \text{Ker}(A^2)$ has basis $\{x_1\}$.

More generally consider a linear transformation $A : \Re^n \to \Re^n$. Then if x is an eigenvector with a non-zero eigenvalue λ, $A^2(x) = A \circ A(x) = A[\lambda x] = \lambda A(x) = \lambda^2 x$, and so $x \in \text{Im}(A) \cap \text{Im}(A^2)$.

If there exist n *distinct* real roots to the characteristic equation of A, then a basis consisting of eigenvectors can be found. Then A can be diagonalized, and $\text{Im}(A) = \text{Im}(A^2)$, $\text{Ker}(A) = \text{Ker}(A^2)$.

Example 2.6 Let

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 3 & -7 \\ 5 & -3 & 1 \end{pmatrix}$$

Then $\text{Ker}(A)$ has basis $\{(1, 2, 1)\}$, and $\text{Im}(A)$ has basis $\{(3, 1, 5), (-1, 3, -3)\}$. The eigenvalues of A are $0, 0, 7$. Since we cannot find three linearly independent eigenvectors, A cannot be diagonalised. Now

$$A^2 = \begin{pmatrix} 3 & -3 & 3 \\ -29 & 29 & -29 \\ 17 & -17 & 17 \end{pmatrix}$$

and thus $\text{Im}(A^2)$ has basis $\{(3, -29, 17)\}$. Note that

$$\begin{pmatrix} 3 \\ -29 \\ 17 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix} \in \text{Im}(A)$$

and so $\text{Im}(A^2)$ is a *subspace* of $\text{Im}(A)$.

Moreover $\text{Ker}(A^2)$ has basis $\{(1, 2, 1), (1, -1, 0)\}$ and so $\text{Ker}(A)$ is a subspace of $\text{Ker}(A^2)$.

This can be seen more generally. Suppose $f : \Re^n \to \Re^n$ is linear, and $x \in \text{Ker}(f)$. Then $f^2(x) = f(f(x)) = \underline{0}$, and so $x \in \text{Ker}(f^2)$. Thus $\text{Ker}(f) \subset \text{Ker}(f^2)$. On the other hand if $v \in \text{Im}(f^2)$ then there exists $w \in \Re^n$ such that $f^2(w) = v$. But $f(w) \in \Re^n$ and so $f(f(w)) = v \in \text{Im}(f)$. Thus $\text{Im}(f^2) \subset \text{Im}(f)$.

2.3.3 Symmetric Matrices and Quadratic Forms

Given two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\Re^n$, let $\langle x, y \rangle = \sum_{i=1}^n x_i y_i \in \Re$ be the scalar product of x and y. Note that $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \langle x, \lambda y \rangle$ for any real λ. (We use $\langle -, - \rangle$ to distinguish the scalar product from a vector in $\Re^2$. However the notations (x, y) or $x \cdot y$ are often used for scalar product.)

An $n \times n$ matrix $A = (a_{ij})$ may be regarded as a map $A^* : \Re^n \times \Re^n \to \Re$, where $A^*(x, y) = \langle x, A(y) \rangle$.

A^* is linear in both x and y and is called *bilinear*. By definition $\langle x, A(y) \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} y_j$.

Call an $n \times n$ matrix A *symmetric* iff $A = A^t$ where $A^t = (a_{ji})$ is obtained from A by exchanging rows and columns.

In this case $\langle A(x), y\rangle = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ji}\ x_i)y_j = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\ a_{ij}y_j$, since $a_{ij} = a_{ji}$ for all i, j.

Hence $\langle A(x), y\rangle = \langle x, A(y)\rangle$ for any $x, y \in \Re^n$ whenever A is symmetric.

Lemma 2.15 *If A is a symmetric $n \times n$ matrix, and x, y are eigenvectors of A corresponding to distinct eigenvalues then $\langle x, y\rangle = 0$, i.e., x and y are orthogonal.*

Proof Let $\lambda_1 \neq \lambda_2$ be the eigenvalues corresponding to the distinct eigenvectors x, y. Now

$$\begin{aligned}\big\langle A(x), y\big\rangle &= \big\langle x, A(y)\big\rangle \\ &= \langle \lambda_1 x, y\rangle = \langle x, \lambda_2 y\rangle \\ &= \lambda_1\langle x, y\rangle = \lambda_2\langle x, y\rangle.\end{aligned}$$

Here $\langle A(x), y\rangle = \langle x, A(y)\rangle$ since A is symmetric. Moreover $\langle x, \lambda y\rangle = \sum_{i=1}^{n} x_i(\lambda y_i) = \lambda\langle x, y\rangle$. Thus $(\lambda_1 - \lambda_2)\langle x, y\rangle = 0$. If $\lambda_1 \neq \lambda_2$ then $\langle x, y\rangle = 0$. □

Lemma 2.16 *If there exist n distinct eigenvalues to a symmetric $n \times n$ matrix A, then the eigenvectors $X = \{x_1, \dots, x_n\}$ form an orthogonal basis for $\Re^n$.*

Proof Directly by Lemmas 2.14 and 2.15.

We may also give a brief direct proof of Lemma 2.16 by supposing that $\sum_{i=1}^{n} \alpha_i x_i = 0$. But then for each $j = i, \dots, n$,

$$0 = \langle x_j, \underline{0}\rangle = \sum_{i=1}^{n} \alpha_i\langle x_j, x_i\rangle = \alpha_j\langle x_j, x_j\rangle.$$

But since $x_j \neq 0$, $\langle x_j, x_j\rangle > 0$ and so $\alpha_j = 0$ for each j. Thus X is a frame. Since the vectors in X are mutually orthogonal, X is an orthogonal basis for $\Re^n$.

For a symmetric matrix the roots of the characteristic equation will all be real. To see this in the 2×2 case, consider the characteristic equation

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda(a_{11} + a_{22}) = (a_{11}a_{22} - a_{21}a_{12}).$$

The roots of this equation are $\frac{-b+\sqrt{b^2-4c}}{2}$ with real roots iff $b^2 - 4c \geq 0$.

But this is equivalent to

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12}) = (a_{11} - a_{22})^2 + 4(a_{12})^2 \geq 0,$$

since $a_{12} = a_{21}$.

Both terms in this expression are non-negative, and so λ_1, λ_2 are real.

In the case of a symmetric matrix, A, let E_λ be the set of eigenvectors associated with a particular eigenvalue, λ, of A together with the zero vector. Suppose x_1, x_2 belong to E_λ. Clearly $A(x_1 + x_2) = A(x_1) + A(x_2) = \lambda(x_1 + x_2)$ and so $x_1 +$

$x_2 \in E_\lambda$. If $x \in E_\lambda$, then $A(\alpha x) = \alpha A(x) = \alpha(\lambda x) = \lambda(\alpha x)$ and $\alpha x \in E_\lambda$ for each non-zero real number, α.

Since we also now suppose that for each eigenvalue, λ, the *eigenspace* E_λ contains $\underline{0}$, then E_λ will be a vector subspace of $\Re^n$. If $\lambda = \lambda_1 = \cdots = \lambda_r$ are repeated roots of the characteristic equation, then, in fact, the eigenspace, E_λ, will be of dimension r, and we can find r mutually orthogonal vectors in E_λ, forming a basis for E_λ.

Suppose now that A is a symmetric $n \times n$ matrix. As we shall show we may write $\wedge = P^{-1}AP$ where P is the $n \times n$ basis change matrix whose columns are the n linearly independent eigenvectors of A.

Now *normalise* each eigenvector x_j by defining $z_j = \frac{1}{\|x_j\|}(x_{1j}, \dots, x_{nj})$ where $\|x_j\| = \sqrt{\sum (x_{kj})^2} = \sqrt{\langle x_j, x_j\rangle}$ is called the *norm* of x_j.

Let $Q = (z_1, \dots, z_n)$ be the $n \times n$ matrix whose columns consist of $z_1, \dots, z_n$. Now

$$Q^t Q = \begin{pmatrix} z_{11} & z_{21} & z_{n1} \\ z_{1j} & & z_{nj} \\ z_{1n} & & z_{nn} \end{pmatrix} \begin{pmatrix} z_{11} & z_{1j} & z_{1n} \\ z_{21} & z_{2j} & z_{2n} \\ \vdots & \vdots & \vdots \\ z_{n1} & z_{nj} & z_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \langle z_1, z_1\rangle & \dots & \langle z_1, z_n\rangle \\ \langle z_2, z_1\rangle & & \\ \vdots & \dots & \langle z_n, z_n\rangle \end{pmatrix}$$

since the (i, k)th entry in $Q^t Q$ is $\sum_{r=1}^n z_{ri} z_{rk} = \langle z_i, z_k\rangle$.

But $(z_i, z_k) = \langle \frac{x_i}{\|x_i\|}, \frac{x_k}{\|x_k\|}\rangle = \frac{1}{\|x_i\|\|x_k\|}\langle x_i, x_k\rangle = 0$ if $i \neq k$. On the other hand $\langle z_i, z_i\rangle = \frac{1}{\|x_i\|^2}\langle x_i, x_i\rangle = 1$, and $Q^t Q = I_n$, the $n \times n$ identity matrix. Thus $Q^t = Q^{-1}$.

Since $\{z_1, \dots, z_n\}$ are eigenvectors of A with real eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ we obtain

$$\wedge = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_r & \\ 0 & & 0 \end{pmatrix} = Q^t A Q$$

where the last $(n - r)$ columns of Q correspond to the kernel vectors of A.

When A is a symmetric $n \times n$ matrix the function $A^* : \Re^n \times \Re^n \to \Re$ given by $A^*(x, y) = \langle x, A(y)\rangle$ is called a *quadratic form*, and in matrix notation is given by

$$(x_1, \dots, x_n) \begin{pmatrix} a_{ij} \end{pmatrix} \begin{pmatrix} y_1 \\ \\ y_n \end{pmatrix}$$

Consider

$$\begin{aligned} A^*(x,x) &= \langle x, A(x)\rangle \\ &= \langle x, Q \wedge A^t(x)\rangle \\ &= \langle Q^t(x), \wedge\, Q^t(x)\rangle. \end{aligned}$$

Now $Q^t(x) = (x'_1 \ldots, x'_n)$ is the coordinate representation of the vector x with respect to the new basis $\{z_1, \ldots, z_n\}$ for $\Re^n$. Thus

$$A^*(x,x) = (x'_1, \ldots, x'_n)\begin{pmatrix} \lambda_1 & & \\ & \lambda_r & \\ & & 0 \end{pmatrix}\begin{pmatrix} x'_1 \\ x'_n \end{pmatrix} = \sum_{i=1}^{r} \lambda_i (x'_i)^2.$$

Suppose that rank $A = r$ and all eigenvalues of A are non-negative. In this case, $A^*(x,x) = \sum_{i=1}^{n} |\lambda_i|(x_i)^2 \geq 0$. Moreover if x is a non-zero vector then $Q^t(x) \neq 0$, since Q^t must have rank n.

Define the nullity of A^* to be $\{x : A^*(x,x) = 0\}$. Clearly if x is a non-zero vector in Ker(A) then it is an eigenvector with eigenvalue 0. Thus the nullity of A^* is a vector subspace of $\Re^n$ of dimension at least $n-r$, where $r = \text{rank}(A)$. If the nullity of A^* is $\{\underline{0}\}$ then call A^* *non-degenerate*. If all eigenvalues of A are strictly positive (so that A^* is non-degenerate) then $A^*(x,x) > 0$ for all non-zero $x \in \Re^n$. In this case A^* is called *positive definite*. If all eigenvalues of A are non-negative but some are zero, then A^* is called *positive semi-definite*, and in this case $A^*(x,x) > 0$, for all x in a subspace of dimension r in $\Re^n$. Conversely if A^* is non-degenerate and all eigenvalues are strictly negative, then A^* is called *negative definite*. If the eigenvalues are non-positive, but some are zero, then A^* is called *negative semi-definite*.

The *index* of the quadratic form A^* is the maximal dimension of the subspace on which A^* is negative definite. Therefore index (A^*) is the number of strictly negative eigenvalues of A.

When A has some eigenvalues which are strictly positive and some which are strictly negative, then we call A^* a *saddle*.

We have not as yet shown that a symmetric $n \times n$ matrix has n real roots to its characteristic equation. We can show however that any (symmetric) quadratic form can be diagonalised.

Let $A = (a_{ij})$ and $\langle x, A(x)\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$. If $a_{ii} = 0$ for all $i = 1, \ldots, n$ then it is possible to make a linear transformation of coordinates such that $a_{ij} \neq 0$ for some j. After relabelling coordinates we can take $a_{11} \neq 0$. In this case the quadratic form can be written

$$\begin{aligned} \langle x, A(x)\rangle &= a_{11}x_1^2 + 2a_{12}x_1x_2, \ldots \\ &= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 \cdots\right)^2 + \left(a_{22} - \frac{a_{12}^2}{a_{11}}^2\right)(x_2 + \cdots)^2 \ + \cdots \\ &= \sum_{i=1}^{n} \alpha_i y_i^2. \end{aligned}$$

Here each y_i is a linear combination of $\{x_1, \dots, x_n\}$. Thus the transformation $x \to P(x) = y$ is non-singular and has inverse Q say.

Letting $x = Q(y)$ we see the quadratic form becomes

$$\begin{aligned}\big\langle x, A(x)\big\rangle &= \big\langle Q(y), A \circ Q(y)\big\rangle \\ &= \big\langle y, Q^t A Q(y)\big\rangle \\ &= \big\langle y, D(y)\big\rangle,\end{aligned}$$

where D is a diagonal matrix with real diagonal entries $(\alpha_1, \dots, \alpha_n)$. Note that $D = Q^t A Q$ and so $\operatorname{rank}(D) = \operatorname{rank}(A) = r$, say. Thus only r of the diagonal entries may be non zero. Since the symmetric matrix, A, can be diagonalised, not only are all its eigenvalues real, but its eigenvectors form a basis for $\Re^n$. Consequently $\wedge = P^{-1}AP$ where P is the $n \times n$ basis change matrix whose columns are these eigenvectors. Moreover, if λ is an eigenvalue with multiplicity r (*i.e.*, λ occurs as a root of the characteristic equation r times) then the eigenspace, E_λ, has dimension r. □

2.3.4 Examples

Example 2.7 To give an illustration of this procedure consider a matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

representing the quadratic form $x_2^2 + 2x_1x_3$. Let

$$P_1(x) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

giving the quadratic form $z_1^2 - 2(z_2 - \frac{1}{2}z_3)^2 + \frac{1}{2}z_3^2$ and

$$P_2(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Then $\langle x, A(x)\rangle = \langle y, D(y)\rangle$, where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad \text{and} \quad A = P_1^t P_2^t D P_2 P_1.$$

Consequently the matrix A can be diagonalised. A has characteristic equation $(1-\lambda)(\lambda^2-1)$ with eigenvalues $1, 1, -1$.

Then normalized eigenvectors of A are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},$$

corresponding to the eigenvalues $1, 1, -1$.

Thus A^* is a non-degenerate saddle of index 1. Let Q be the basis change matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1\\0&2&0\\1&0&-1\end{pmatrix}.$$

Then

$$Q^tAQ = \frac{1}{2}\begin{pmatrix}1&0&1\\0&2&0\\1&0&-1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&2&0\\1&0&-1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

As a quadratic form

$$(x_1,x_2,x_3)A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = (x_1,x_2,x_3)\begin{pmatrix}x_3\\x_2\\x_1\end{pmatrix} = x_1x_3 + x_2^2.$$

We can also write this as

$$\begin{aligned}&(x_1,x_2,x_3)\frac{1}{2}\begin{pmatrix}1&0&1\\0&\sqrt{2}&0\\1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&1\\0&\sqrt{2}&0\\1&0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ &= \frac{1}{2}\big(x_1+x_3, \sqrt{2x_2}, x_1-x_3\big)\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x_1+x_3\\ \sqrt{2x_2}\\ x_1-x_3\end{pmatrix}\\ &= \frac{1}{2}(x_1+x_3)^2 + 2x_2^2 \ -(x_1-x_3)^2.\end{aligned}$$

Note that A is positive definite on the subspace $\{(x_1,x_2,x_3)\in\Re^3 : (x_1=x_3)\}$ spanned by the first two eigenvectors.

We can give a geometric interpretation of the behaviour of a matrix A with both positive and negative eigenvalues. For example

$$A = \begin{pmatrix}1&2\\2&1\end{pmatrix}$$

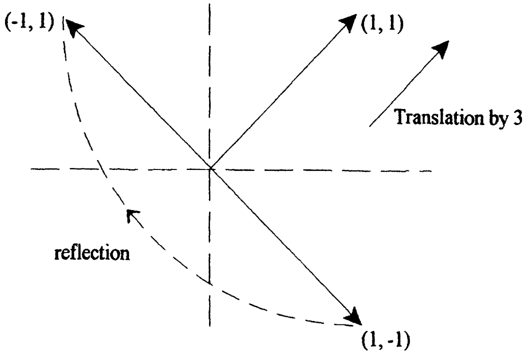


Fig. 2.2 A translation followed by a reflection

has eigenvectors

$$z_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad z_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

corresponding to the eigenvalues 3, -1 respectively. Thus A maps the vector z_1 to $3z_1$ and z_2 to $-z_2$. The second operation can be regarded as a reflection of the vector z_2 in the line $\{(x, y) : x - y = 0\}$, associated with the first eigenvalue. The first operation $z_1 \rightarrow 3z_1$ is a translation of z_1 to $3z_1$. Consider now any point $x \in \Re^2$. We can write $x = \alpha z_1 + \beta z_2$. Thus $A(x) = 3\alpha z_1 - \beta z_2$. In other words A may be decomposed into two operations: a translation in the direction z_1, followed by a reflection about z_1 as in Fig. 2.2.

2.4 Geometric Interpretation of a Linear Transformation

More generally suppose A has real roots to the characteristic equation and has eigenvectors $\{x_1, \dots, x_s, z_1, \dots, z_t, k_1, \dots, k_p\}$.

The first s vectors correspond to positive eigenvalues, the next t vectors to negative eigenvalues, and the final p vectors belong to the kernel, with zero eigenvalues.

Then A may be described in the following way:

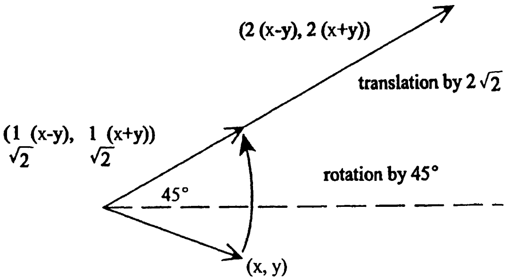
1. collapse the kernel vectors on to the image spanned by $\{x_1, \dots, x_s, z_1, \dots, z_t\}$.
2. translate each x_i to $\lambda_i x_i$.
3. reflect each z_j to $-z_j$, and then translate to $- \mid \mu_j \mid z_j$ (where μ_j is the negative eigenvalue associated with z_j).

These operations completely describe a symmetric matrix or a matrix, A, which is diagonalisable. When A is non-symmetric then it is possible for A to have complex roots.

For example consider the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Fig. 2.3 A translation followed by a rotation



As we have seen this corresponds to a rotation by θ in an anticlockwise direction in the plane $\Re^2$. To determine the eigenvalues, the characteristic equation is $(\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + (\cos^2\theta + \sin^2\theta) = 0$. But $\cos^2\theta + \sin^2\theta = 1$. Thus $\lambda = 2\cos\frac{\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} = \cos\theta \pm\ i\sin\theta$.

More generally a 2 × 2 matrix with complex roots may be regarded as a transformation $\lambda e^{i\theta}$ where λ corresponds to a translation by λ and $e^{i\theta}$ corresponds to rotation by θ.

Example 2.8 Consider $A = \left(\begin{smallmatrix} 2 & -2 \\ 2 & 2 \end{smallmatrix}\right)$ with trace $(A) = \mathrm{tr}(A) = 4$ and $|A| = 8$.

As we have seen the characteristic equation for A is $(\lambda^2 - (\text{trace}\,A)) + |A| = 0$, with roots $\frac{\text{trace}(A)\pm\sqrt{(\text{trace}\,A)^2-4|A|}}{2}$. Thus the roots are $2 \pm \frac{1}{2}\sqrt{16-32} = 2 \pm 2i = 2\sqrt{2}[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}]$ where $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$ and so $\theta = 45°$. Thus

$$A : \begin{pmatrix} x \\ y \end{pmatrix} \to 2\sqrt{2}\begin{pmatrix} x\cos 45 & -y\sin 45 \\ x\sin 45 & +y\cos 45 \end{pmatrix}.$$

Consequently A first sends (x, y) by a translation to $(2\sqrt{2}x, 2\sqrt{2}y)$ and then rotates this vector through an angle 45°.

More abstractly if A is an $n \times n$ matrix with two complex conjugate eigenvalues $(\cos\theta + i\sin\theta)$, $(\cos\theta - i\sin\theta)$, then there exists a two dimensional eigenspace E^θ such that $A(x) = \lambda e^{i\theta}(x)$ for all $x \in E_\theta$, where $\lambda e^{i\theta}(x)$ means rotate x by θ within E_θ and then translate by λ.

In some cases a linear transformation, A, can be given a *canonical form* in terms of rotations, translations and reflections, together with a collapse onto the kernel. What this means is that there exists a number of distinct *eigenspaces*

$$\{E_1, \ldots, E_p, X_1, \ldots, X_s, K\}$$

where A maps

1. E_j to E_j by rotating any vector in E_j through an angle θ_j;
2. X_j to X_j by translating a vector x in X_j to $\lambda_j x$, for some non-zero real number λ_j;

3. the kernel K to $\{0\}$.

In the case that the dimensions of these eigenspaces sum to n, then the canonical form of the matrix A is

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & \wedge & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $e^{i\theta}$ consists of p different 2×2 matrices, and $\wedge$ is a diagonal $s \times s$ matrix, while 0 is an $(n-r) \times (n-r)$ zero matrix, where $r = \text{rank}(A) = 2p + s$.

However, even when all the roots of the characteristic equation are real, it need not be possible to obtain a diagonal, canonical form of the matrix.

To illustrate, in Example 2.6 it is easy to show that the eigenvalue $\lambda\ = 0$ occurs twice as a root of the characteristic equation for the non-symmetric matrix A, even though the kernel is of dimension 1. The eigenvalue $\lambda = 7$ occurs once. Moreover the vector $(3, -29, 17)$ clearly must be an eigenvector for $\lambda = 7$, and thus span the image of A^2. However it is also clear that the vector $(3, -29, 17)$ does not span the image of A. Thus the eigenspace E_7 does not provide a basis for the image of A, and so the matrix A cannot be diagonalised.

However, as we have shown, for any *symmetric* matrix the dimensions of the eigenspaces sum to n, and the matrix can be expressed in canonical, diagonal, form.

In Chap. 4 below we consider smooth functions and show that "locally" such a function can be analysed in terms of the canonical form of a particular symmetric matrix, known as the Hessian.

Topology and Convex Optimisation 3

3.1 A Topological Space

In the previous chapter we introduced the notion of the scalar product of two vectors in $\Re^n$. More generally if a scalar product is defined on some space, then this permits the definition of a *norm*, or length, associated with a vector, and this in turn allows us to define the *distance* between two vectors. A distance function or *metric* may be defined on a space, X, even when X admits no norm. For example let X be the surface of the earth. Clearly it is possible to say what is the shortest distance, $d(x, y)$, between two points, x, y, on the earth's surface, although it is not meaningful to talk of the "length" of a point on the surface. More general than the notion of a metric is that of a *topology*. Essentially a topology on a space is a mathematical notion for making more precise the idea of "nearness". The notion of topology can then be used to precisely define the property of continuity of a function between two topological spaces. Finally continuity of a preference gives proof of existence of a social choice and of an economic equilibrium in a world that is bounded.

3.1.1 Scalar Product and Norms

In Sect. 2.3 we defined the Euclidean *scalar product* of two vectors

$$x = \sum_{i=1}^{n} x_i e_i, \quad \text{and} \quad y = \sum_{i=1}^{n} y_i e_i \quad \text{in } \Re^n,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis, to be

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i .$$

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_3,

More generally suppose that $\{v_1, \ldots, v_n\}$ is a basis for $\Re^n$, and $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ are the coordinates of x, y with respect to this basis. Then

$$\langle x, y\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \langle v_i, v_j\rangle,$$

where $\langle v_i, v_j\rangle$ is the scalar product of v_i and v_j. Thus

$$\langle x, y\rangle = (x_1, \ldots, x_n)\begin{pmatrix} \vdots \\ a_{ij} \\ \vdots \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where $A = (a_{ij}) = \langle v_i, v_j\rangle_{i=1,\ldots,n; j=1,\ldots,n}$. If we let $v_i = \sum_{k=1}^{n} v_{ik} e_k$, then clearly $\langle v_i, v_i\rangle = \sum_{k=1}^{n}(v_{ik})^2 > 0$. Moreover $\langle v_i, v_j\rangle = \langle v_j, v_i\rangle$. Thus the matrix A is symmetric. Since A must be of rank n, it can be diagonalized to give a matrix $\wedge = Q^t A Q$, all of whose diagonal entries are positive. Here Q represents the orthogonal basis change matrix and $Q^t(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$ gives the coordinates of x with respect to the basis of eigenvectors of A. Hence

$$\begin{aligned}\langle x, y\rangle &= \big\langle x, A(y)\big\rangle = \big\langle x, Q \wedge Q^t(y)\big\rangle \\ &= \big\langle Q^t(x), \wedge Q^t(y)\big\rangle \\ &= \sum_{i=1}^{n} \lambda_i x'_i y'_i.\end{aligned}$$

Thus a scalar product is a non-degenerate positive definite quadratic form. Note that the scalar product is *bilinear* since

$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle \quad \text{and} \quad \langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle,$$

and *symmetric* since

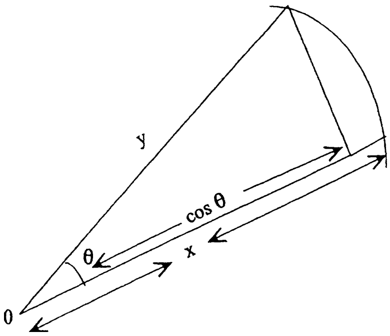
$$\langle x, y\rangle = \big\langle x, A(y)\big\rangle = \big\langle y, A(x)\big\rangle = \langle y, x\rangle.$$

We shall call the scalar product given by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ the *Euclidean* scalar product.

We define the *Euclidean norm*, $\| \; \|_E$, by $\|x\|_E = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$. Note that $\|x\|_E \geq 0$ if and only if $x = (0, \ldots, 0)$. Moreover, if $a \in \Re$, then

$$\|ax\|_E = \sqrt{\sum_{i=1}^{n} a^2 x_i^2} = |a|\,\|x\|_E.$$

Fig. 3.1 Projection



Lemma 3.1 *If* $x, y \in \Re^n$, *then* $\|x + y\|_E \leq \|x\|_E + \|y\|_E$.

Proof For convenience write $\| x \|_E$ as $\|x\|$. Now

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle.$$

But the scalar product is symmetric. Therefore

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle.$$

Furthermore $(\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$. Thus $\|x + y\| \leq \|x\| + \|y\|$ iff $\langle x, y\rangle \leq \|x\|\|y\|$. To show this note that $\sum_{i<j}(x_i y_j - x_j y_i)^2 \geq 0$. Thus $\sum_{i<j}(x_i^2 y_j^2 + x_j^2 y_i^2) \geq 2\sum_{i<j} x_i y_i x_j y_j$. Add $\sum_{i=1}^n x_i^2 y_i^2$ to both sides. This gives $(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2) \geq (\sum_{i=1}^n x_i y_i)^2$. Therefore $\|x\|^2 \ \|y\|^2 \geq \langle x, y\rangle^2$ and so $\frac{\langle x,y\rangle}{\|x\|\|y\|} \leq 1$, or $\|x + y\| \leq \|x\| + \|y\|$. □

In this lemma we have shown that

$$-1 \leq \frac{\langle x, y\rangle}{\|x\| \ \|y\|} \leq 1.$$

This ratio can be identified with $\cos\theta$, where θ is the angle between the two vectors x, y. In the case of unit vectors, $\langle x, y\rangle$ can be identified with the perpendicular projection of y onto x as in Fig. 3.1.

The property $\|x + y\| \leq \|x\| + \|y\|$ is known as the *triangle inequality* (see Fig. 3.2).

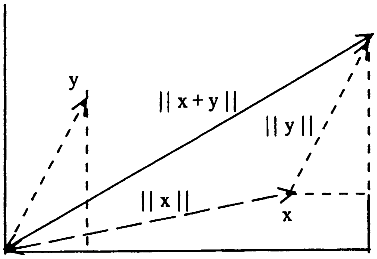
Definition 3.1 Let X be a vector space over the field $\Re$. A *norm*, $\| \ \|$, on X is a mapping $\| \ \| : X \to \Re$ which satisfies the following three properties:

N1. $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ iff $x = \underline{0}$.

N2. $\|ax\| = |a|\|x\|$ for all $x \in X$, and $a \in \Re$.

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Fig. 3.2 Triangle inequality



There are many different norms on a vector space. For example if A is a non-degenerate positive definite symmetric matrix, we could define $\| \ \|_A$ by $\|x\|_A = \sqrt{\langle x, Ax\rangle}$.

The *Cartesian* norm is $\|x\|_c = \|(x_1,\dots,x_n)\|_c = \max\{|x_1|,\dots,|x_n|\}$.

Clearly $\|x\|_c \geq 0$ and $\|x\|_c = 0$ iff $x_i = 0$ for all $i = 1,\dots,n$. Moreover $\|ax\|_c = \max\{|ax_1|,\dots,|ax_n|\} = \max\{|a|\ |x_1|,\dots,|a|\ |x_n|\} = |a|\ |x_i|$, for some i.

Thus $\|ax\|_c = |a| \max\{|x_1|,\dots,|x_n|\} = |a|\ \|x\|_c$.

Finally, $\|x + y\|_c = |x_i + y_i|$ for some $i \leq |x_i| + |y_i| \leq \max\{|x_1|,\dots,|x_n|\} + \max\{|y_1|,\dots,|y_n|\} = \|x\|_c + \|y\|_c$. Define the *city block* norm $\|x\|_B$ to be $\|x\|_B = \sum_{i=1}^n |x_i|$. Clearly $\|x + y\|_B = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n (|x_i| + |y_i|) = \|x\|_B + \|y\|_B$. If $\| \ \|$ is a norm on the vector space X, the *distance function* or *metric* d on X induced by $\| \ \|$ is the function $d : X \times X \to \Re : d(x,y) = \|x - y\|$.

Note that $d(x,y) \geq 0$ for all $x, y \in X$ and that $d(x,y) = 0$ iff $x - y = 0$, *i.e.*, $x = y$. Moreover,

$$\begin{aligned} d(x,y) + d(y,z) &= \|x - y\| + \|y - z\| \geq \big\|(x-y) + (y-z)\big\| \\ &= \|x - z\| = d(x,z). \end{aligned}$$

Hence $d(x,z) \leq \ d(x,y) + d(y,z)$.

Definition 3.2 A *metric* on a *set* X is a function $d : X \times X \to \Re$ such that
D1. $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$
D2. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Note that a metric d may be defined on a set X even when X is not a vector space. In particular d may be defined without reference to a particular norm. At the beginning of the chapter for example we mentioned that the surface of the earth, S^2, admits a metric d, where the distance between two points x, y on the surface is measured along a great circle through x, y. A second metric on S^2 is obtained by defining $d(x,y)$ to be the angle, θ, subtended at the centre of the earth by the two radii to x, y (see Fig. 3.3).

Any set X which admits a metric, d, we shall call *metrisable*, or a *metric space*. To draw attention to the metric, d, we shall sometimes write (X,d) for a metric space.

In a metric space (X,d) the *open ball* at x of radius r in X is

$$B_d(x,r) = \big\{y \in X : d(x,y) < r\big\},$$

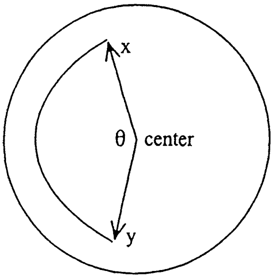


Fig. 3.3 Spherical metric

and the closed ball centre x of radius r is

$$\operatorname{Clos} B_d(x,r)=\{y\in X: d(x,y)\le r\}.$$

The *sphere* of radius r at x is

$$S_d(x,r)=\{y\in X: d(x,y)=r\}.$$

In $\Re^n$, the Euclidean sphere of radius r is therefore

$$S(x,r)=\left\{y:\sum_{i=1}^{n}(x_i-y_i)^2=r_i^2\right\}.$$

For convenience a sphere in $\Re^n$ is often written as S^{n-1}. Here the superfix is $n-1$ because as we shall see the sphere in $\Re^n$ is $(n-1)$-dimensional, even though it is not a vector space. If (X,d) is a metric space, say a set V in X is *d-open* iff for any $x\in V$ there is some radius r_x (dependent on x) such that

$$B_d(x,r_x)\subset V.$$

Lemma 3.2 *Let Γ_d be the family of all sets in X which are d-open. Then Γ_d satisfies the following properties*:

T1. *If $U,V\in\Gamma_d$, then $U\cap V\in\Gamma_d$.*

T2. *If $U_j\in\Gamma_d$ for all j belonging to an index set J (which is possibly infinite), then $\bigcup_{j\in J}U_j\in\Gamma_d$.*

T3. *Both X and the empty set, Φ, belong to Γ_d.*

Proof Clearly X and Φ are d-open. If U and $V\in\Gamma_d$, but $U\cap V=\Phi$ then $U\cap V$ is d-open. Suppose on the other hand that $x\in U\cap V$. Since both U and V are open, there exist r_1,r_2 such that $B_d(x,r_1)\subset U$ and $B_d(x_1,r_2)\subset V$. Let $r=\min(r_1,r_2)$. By definition

$$B_d(x,r)=B_d(x,r_1)\cap B_d(x,r_2)\subset U\cap V.$$

Thus there is an open ball, centre x, of radius r contained in $U\cap V$.

Finally suppose $x \in \bigcup_{j \in J} U_j = U$. Since x belongs to at least one U_j, say U_1, there is an open ball $B = B(x, r_1)$ contained in U_1. Since U_1 is open so is U. □

Note that by **T1** the *finite* intersection of open sets is an open set, but *infinite* intersection of open sets need not be open. To see this consider a set of the form

$$I = (a, b) = \{x \in \Re : a < x < b\}.$$

For any $x \in I$ it is possible to find an ε such that $a + \varepsilon < x < b - \varepsilon$. Hence the open ball $B(x, \varepsilon) = \{y : x - \varepsilon < y < x + \varepsilon\}$ belongs to I, and so I is open.

Now consider the family $\{U_r : r = 1, \ldots, \infty\}$ of sets of the form $U_r = (-\frac{1}{r}, \frac{1}{r})$.

Clearly the origin, 0, belongs to each U_r, and so $0 \in \cap\, U_r = U$. Suppose that U is open. Since $0 \in U$, there must be some open ball $B(0, \varepsilon)$ belonging to U. Let r_0 be an integer such that $r_0 > \frac{1}{\varepsilon}$, so $\frac{1}{r_0} < \varepsilon$. But then $U_{r_0} = (-\frac{1}{r_0}, \frac{1}{r_0})$ is strictly contained in $(-\varepsilon, \varepsilon)$.

Therefore the ball $B(0, \varepsilon) = \{y \in \Re : |y| < \varepsilon\} = (-\varepsilon, \varepsilon)$ is not contained in U_{r_0}, and so cannot be contained in U. Hence U is not open.

3.1.2 A Topology on a Set

We may define a *topology* on a set X to be any collection of sets in X which satisfies the three properties **T1**′, **T2**′, **T3**′.

Definition 3.3 A *topology* Γ on a set X is a collection of sets in X which satisfies the following properties:

T1′ If $U, V \in \Gamma$ then $U \cap V \in \Gamma$.
T2′ If J is any index set and $U_j \in \Gamma$ for each $j \in J$, then $\bigcup_J U_j \in \Gamma$.
T3′ Both X and the empty set belong to Γ.

A set X which has a topology Γ is called a *topological space*, and written (X, Γ). The sets in Γ are called Γ-open, or simply *open*. An open set in Γ which contains a point x is called a *neighbourhood* (or nbd.) of x.

A *base* $\mathcal{B}$ for a topology Γ is a collection of Γ-open sets such that any member U of Γ can be written as a union of members of $\mathcal{B}$. Equivalently if x belongs to a Γ-open set U, then there is a member V of the base such that $x \in V \subset U$.

By Lemma 3.2, the collection Γ_d of sets which are open with respect to the metric d comprise a topology called the *metric topology* Γ_d on X. We also said that a set U belonged to Γ_d iff each point $x \in U$ had a neighbourhood $B_d(x, r)$ which belonged to U. Thus the family of sets

$$\mathcal{B} = \left\{B_d(x, r) : x \in X, r > 0\right\}$$

forms a base for the metric topology Γ_d on X.

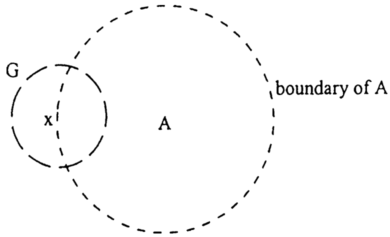


Fig. 3.4 Accumulation point

Consider again the metric topology on $\Re$. As we have shown any set of the form (a, b) is open. Indeed a set of the form $(-\infty, a)$ or (b, ∞) is also open.

In general if U is an open set (in the topology Γ for the topological space X), then the complement $\overline{U_X} = X \backslash U$ of U in X is called *closed*.

Thus in $\Re$, the set $[a, b] = \{x \in \Re : a \leq x \leq b\}$ is closed since it is the complement of the open set $(-\infty, a) \cup (b, \infty)$. Note that the sets $(-\infty, a]$ and $[b, \infty)$ are also closed since they are complements of the open sets (a, ∞) and $(-\infty, b)$ respectively.

If A is any set in a topological space, (X, Γ), then define the open set, $\text{Int}(A)$, called the *interior* of A, by $x \in \text{Int}(A)$ iff x is in A and there exists an open set G containing x such that $G \subset A$.

Conversely, define the closed set, $\text{Clos}(A)$, or *closure* of A, by $x \in \text{Clos}(A)$ iff x is in X and for any open set G containing x, $G \cap A$ is non empty.

Clearly $\text{Int}(A) \subset A \subset \text{Clos}(A)$. (See Exercise 3.1 at the end of the book.) The *boundary* of A, written δA, is $\text{Clos}(A) \cap \text{Clos}(\overline{A}_X)$, where $A_X = X \backslash A$.

For example, if A is an open set, the complement $(\overline{A}_X)$ of A in X is a closed set containing δA (*i.e.*, $\text{Clos}(\overline{A}_X) = \overline{A}_X$). The closure, $\text{Clos}(A)$, on the other hand is the closed set which intersects $(\overline{A}_X)$ precisely in δA. Clearly if x belongs to the boundary of A, then any neighbourhood of x intersects both A and its complement.

A point x is an *accumulation* or *limit point* of a set A if any open set U containing x also contains points of A different from x. If A is closed then it contains its limit points. If A is a subset of X and $\text{Clos}(A) = X$ then call A *dense* in X. Note that this means that any point $x \in X$ either belongs to A or, if it belongs to $X \backslash A$, has the property that for any neighborhood U of x, $U \cap A \neq \Phi$. Thus if $x \in X \backslash A$ it is an accumulation point of A. If A is dense in X a point outside A may be 'approximated arbitrarily closely' by points inside A. See Fig. 3.4.

For example the set of non-integer real numbers $\mathcal{Z}$ is dense (as well as open) in $\Re$. The set of rational numbers $\mathcal{Q} = \{\frac{p}{q} : p, q \in \mathcal{Z}\}$ is also dense in $\Re$, but not open, since any neighbourhood of a rational number must contain irrational numbers. For each rational $q \in \mathcal{Q}$, note that $\Re \backslash \{q\}$ is open, and dense. Thus

$$\Re \backslash \mathcal{Q} = \bigcap_{q \in \mathcal{Q}} \{\Re \backslash \{q\}\} :$$

the set of irrationals is the "countable" intersection of open dense sets. Moreover $\Re \backslash \mathcal{Q}$ is itself dense. Such a set is called a *residual* set, and is in a certain sense "more dense" than a dense set.

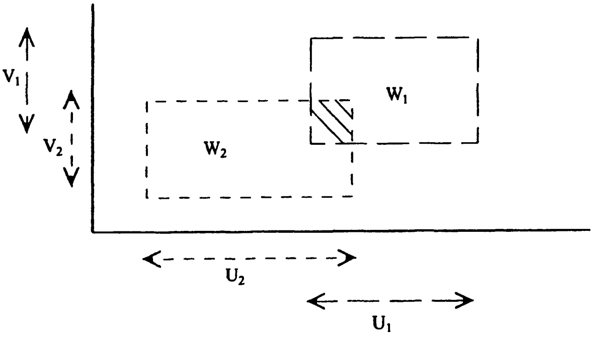


Fig. 3.5 Open sets in the product topology

We now consider different topologies on a set X.

Definition 3.4

1. If (X, Γ) is a topological space, and Y is a subset of X, the *relative topology* Γ_Y on Y is the topology whose open sets are $\{U \cap Y : U \in \Gamma\}$.
2. If (X, Γ) and $(Y, \mathcal{S})$ are topological spaces the *product topology* $\Gamma \times \mathcal{S}$ on the set $X \times Y$ is the topology whose base consists of all sets of the form $\{U \times V : U \in \Gamma, V \in \mathcal{S}\}$.

We already introduced the relative topology in Sect. 1.1.2, and showed that this formed a topology. To show that $\Gamma \times \mathcal{S}$ is a topology for $X \times Y$ we need to show that the union and intersection properties are satisfied. Suppose that

$$W_i = U_i \times V_i$$

for $i = 1, 2$, where $U_i \in \Gamma$, $V_i \in \mathcal{S}$. Now $W_1 \cap W_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. But $U_1 \cap U_2 \in \Gamma$, $V_1 \cap V_2 \in \mathcal{S}$, since $\mathcal{S}$ and Γ are topologies. Thus $W_1 \cap W_2 \in \Gamma \times \mathcal{S}$. Suppose now that $x \in W_1 \cup W_2$. Then x belongs either to $U_1 \times V_1$ or $U_2 \times V_2$ (or both). In either case x belongs to a member of the base for $\Gamma \times \mathcal{S}$.

Another way of expressing the product topology is that W is open in the product topology iff for any $x \in W$ there exist open sets, $U \in \Gamma$ and $V \in \mathcal{S}$, such that $x \in U \times V$ and $U \times V \subset W$.

For example consider the metric topology Γ induced by the norm $\| \ \|$ on $\Re$. This gives the product topology Γ^n on $\Re^n$, where U is open in Γ^n iff for each $x = (x_1, \dots, x_n) \in U$ there exists an open interval $B(x_i, r_i)$ of radius r_i about the ith coordinate x_i such that

$$B(x_1, r_1) \times \cdots \times B(x_n, r_n) \subset U.$$

Consider now the *Cartesian norm* on $\Re^n$, where

$$\|x\|_C = \max\{|x_1|, \dots, |x_n|\}.$$

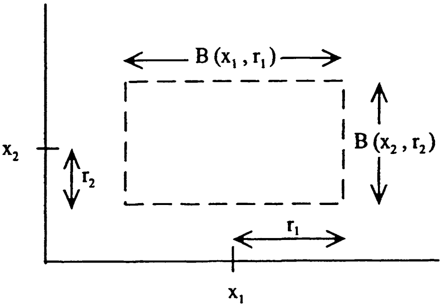


Fig. 3.6 The product topology in $\Re^2$

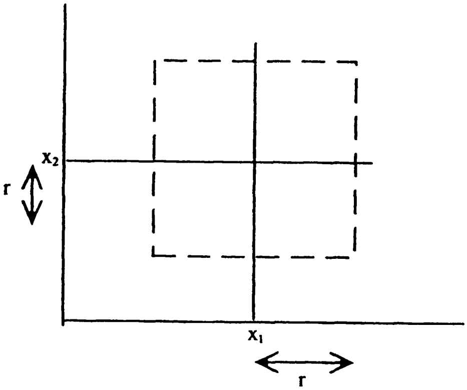


Fig. 3.7 A Cartesian open ball of radius r in $\Re^2$

This induces a *Cartesian metric*

$$d_C(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

A *Cartesian* open ball of radius r about x is then the set

$$B_C(x,r) = \{y \in \Re^n : |y_i - x_i| < r \ \forall i = 1, \ldots, n\}.$$

A set U is open in the *Cartesian topology* Γ_C for $\Re^n$ iff for every $x \in U$ there exists some $r > 0$ such that the ball $B_C(x,r) \subset U$. See Fig. 3.7.

Suppose now that U is an open set in the product topology Γ^n for $\Re^n$. At any point $x \in U$, there exist $r_1 \ldots r_n$ all > 0 such that

$$B = B(x_1, r_1) \times, \ldots, B(x_n, r_n) \subset U.$$

Now let $r = \min(r_1, \ldots, r_n)$. Then clearly the Cartesian ball $B_C(x,r)$ belongs to the product ball B, and hence to U. Thus U is open in the Cartesian topology.

On the other hand if U is open in the Cartesian topology, for each point x in U, the Cartesian ball $B_C(x,r)$ belongs to U. But this means the product ball $B(x_1, r) \times, \ldots, B(x_n, r)$ also belongs to U. Hence U is open in the product topology.

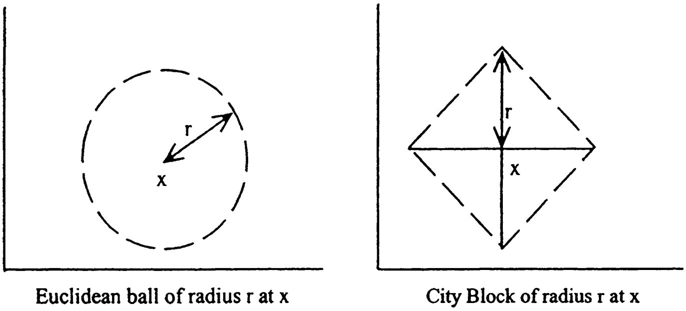


Fig. 3.8 Balls under different metrics

We have therefore shown that a set U is open in the product topology Γ^n on $\Re^n$ iff it is open in the Cartesian topology Γ_C on $\Re^n$. Thus the two topologies are identical.

We have also introduced the *Euclidean* and *city block norms* on $\Re^n$. These norms induce metrics and thus the *Euclidean topology* Γ_E and *city block topology* Γ_B on $\Re^n$. As before a set U is open in Γ_E (resp. Γ_B) iff for any point $x \in U$, there is an open neighborhood

$$B_E(x,r) = \left\{ y \in \Re^n : \sum_{i=1}^{n} (y_i - x_i)^2 < r^2 \right\},$$

resp. $B_B(x,r) = \{y \in \Re^n : \sum |y_i - x_i| < r\}$ of x which belongs to U.

(The reason we use the term "city block" should be obvious from the nature of a ball under this metric, so displayed in Fig. 3.8.)

In fact these three topologies Γ_C, Γ_E and Γ_B on $\Re^n$ are all identical. We shall show this in the case $n = 2$.

Lemma 3.3 *The Cartesian, Euclidean and city block topologies on* $\Re^2$ *are identical.*

Proof Suppose that U is an open set in the Euclidean topology Γ_E for $\Re^2$. Thus at $x \in U$, there is an $r > 0$, such that the set

$$B_E(x,r) = \left\{ y \in \Re^2 : \sum_{i=1}^{2} (y_i - x_i)^2 < r^2 \right\} \subset U.$$

From Fig. 3.9, it is obvious that the city block ball $B_B(x,r)$ also belongs to $B_E(x,r)$ and thus U. Thus U is open in Γ_B.

On the other hand the Cartesian ball $B_C(x, \frac{r}{2})$ belongs to $B_B(x,r)$ and thus to U. Hence U is open in Γ_C.

Finally the Euclidean ball $B_E(x, \frac{r}{2})$ belongs to $B_C(x, \frac{r}{2})$. Hence if U is open in Γ_C it is open in Γ_E.

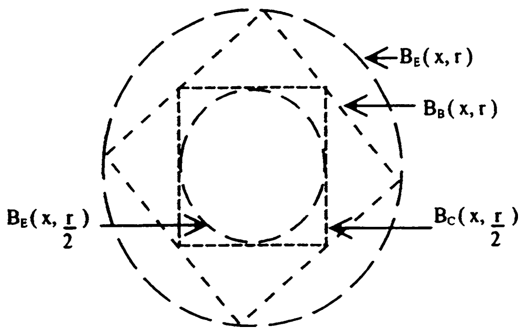


Fig. 3.9 Balls under different metrics

Thus U open in $\Gamma_E \Rightarrow U$ open in $\Gamma_B \Rightarrow U$ open in $\Gamma_C \Rightarrow U$ open in Γ_E. Consequently all three topologies are identical in $\Re^2$. □

Suppose that Γ_1 and Γ_2 are two topologies on a space X. If any open set U in Γ_1 is also an open set in Γ_2 then say that Γ_2 is *as fine as* Γ_1 and write $\Gamma_1 \subset \Gamma_2$. If $\Gamma_1 \subset \Gamma_2$ and $\Gamma_2 \subset \Gamma_1$ then Γ_1 and Γ_2 are identical, and we write $\Gamma_1 = \Gamma_2$. If $\Gamma_1 \subset \Gamma_2$ but Γ_2 contains an open set that is not open in Γ_1 then say Γ_2 is *finer* than Γ_1. We also say Γ_1 is *coarser* than Γ_2.

If d_1 and d_2 are two metrics on a space X, then under some conditions the topologies Γ_1 and Γ_2 induced by d_1 and d_2 are identical. Say the metrics d_1 and d_2 are *equivalent* iff for each $\varepsilon > 0$ there exist $\eta_1 > 0$ and $\eta_2 > 0$ such that $d_1(x, y) < \eta_1 \Rightarrow d_2(x, y) < \varepsilon$, and $d_2(x, y) < \eta_2 \Rightarrow d_1(x, y) < \varepsilon$. Another way of expressing this is that $B_1(x, \eta_1) \subset B_2(x, \varepsilon)$, and $B_2(x, \eta_2) \subset B_2(x, \varepsilon)$, where $B_i(x, r) = \{y : d_i(x, y) < r\}$ for $i = 1$ or 2.

Just as in Lemma 3.3, the Cartesian, Euclidean and city block metrics on $\Re^n$ are equivalent. We can use this to show that the induced topologies are identical.

We now show that equivalent metrics induce identical topologies.

If $f : X \to \Re$ is a function, and V is a set in X, define $\sup(f, V)$, the *supremum* (from the Latin, supremus) of f on V to be the smallest number $M \in \Re$ such that $f(x) \leq M$ for all $x \in V$.

Similarly define $\inf(f, V)$, the *infimum* (again from the Latin, infimus) of f on V to be the largest number $m \in \Re$ such that $f(x) \geq m$ for all $x \in V$.

Let $d : X \times X \to \Re$ be a metric on X. Consider a point, x, in X, and a subset of X. Then define the distance from x to V to be $d(x, V) = \inf(d(x, -), V)$, where $d(x, -) : V \to \Re$ is the function $d(x, -)(y) = d(x, y)$.

Suppose now that U is an open set in the topology Γ_1 induced by the metric d_1. For any point $x \in U$ there exists $r > 0$ such that $B_1(x, r) \subset U$, where $B_1(x, r) = \{y \in X : d_1(x, y) < r\}$. Since we assume the metrics d_1 and d_2 are equivalent, there must exist $s > 0$, say, such that

$$B_2(x, s) \subset B_1(x, r)$$

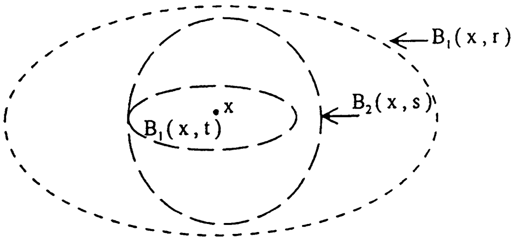


Fig. 3.10 Balls centered at x

where $B_2(x,s) = \{y \in X : d_2(x,y) < s\}$. Indeed one may choose $s = d_2(x, \overline{B_1(x,r)})$ where $\overline{B_1(x,r)}$ is the complement of $B_1(x,r)$ in X (see Fig. 3.10). Clearly the set U must be open in Γ_2 and so Γ_2 is as fine as Γ_1. In the same way, however, there exists $t > 0$ such that $B_1(x,t) \subset B_2(x,s)$, where again one may choose $t = d_1(x, \overline{B_2(x,s)})$. Hence if U is open in Γ_2 it is open in Γ_1. Thus Γ_1 is as fine as Γ_2. As a consequence Γ_1 and Γ_2 are identical.

Thus we obtain the following lemma.

Lemma 3.4 *The product topology, Γ^n, Euclidean topology Γ_E, Cartesian topology Γ_C and city block topology Γ_B are all identical on $\Re^n$.*

As a consequence we may use, as convenient, any one of these three metrics, or any other equivalent metric, on $\Re^n$ knowing that topological results are unchanged.

3.2 Continuity

Suppose that (X, Γ) and $(Y, \mathcal{S})$ are two topological spaces, and $f : X \to Y$ is a function between X and Y. Say that f is *continuous* (with respect to the topologies Γ and $\mathcal{S}$) iff for any set U in $\mathcal{S}$ (*i.e.*, U is $\mathcal{S}$-open) the set $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is Γ-open.

This definition can be given an alternative form in the case when X and Y are metric spaces, with metrics d_1, d_2, say.

Consider a point x_0 in the domain of f. For any $\varepsilon > 0$ the ball

$$B_2\bigl(f(x_0), \varepsilon\bigr) = \bigl\{y \in Y : d_2\bigl(f(x_0), y\bigr) < \varepsilon\bigr\}$$

is open. For continuity, we require that the inverse image of this ball is open. That is to say there exists some δ, such that the ball

$$B_1(x_0, \delta) = \bigl\{x \in X : d_1(x_0, x) < \delta\bigr\}$$

belongs to $f^{-1}(B_2(f(x_0), \varepsilon))$. Thus $x \in B_1(x_0, \delta) \Rightarrow f(x) \in B_2(f(x_0), \varepsilon)$.

Therefore say f is continuous at $x_0 \in X$ iff for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f\bigl(B_1(x_0, \delta)\bigr) \subset B_2\bigl(f(x_0), \varepsilon\bigr).$$

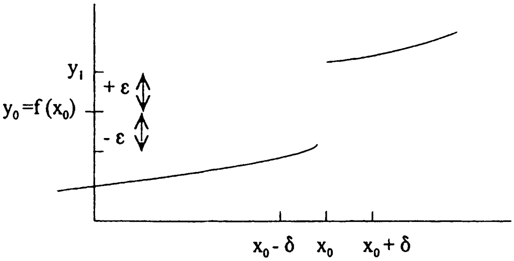


Fig. 3.11 A discontinuity

In the case that X and Y have norms $\| \|_X$ and $\| \|_Y$, then we may say that f is *continuous at* x_0 iff for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\|x - x_0\|_x < \delta \Rightarrow \left\| f(x) - f(x_0) \right\|_y < \varepsilon.$$

Then f is continuous on X iff f is continuous at every point x in its domain.

If X, Y are vector spaces then we may check the continuity of a function $f : X \to Y$ by using the metric or norm form of the definition.

For example suppose $f : \Re \to \Re$ has the graph given in Fig. 3.11. Clearly f is not continuous. To see this formally let $f(x_0) = y_0$ and choose ε such that $|y_1 - y_0| > \varepsilon$. If $x \in (x_0 - \delta, x_0)$ then $f(x) \in (y_0 - \varepsilon, y_0)$.

However for any $\delta > 0$ it is the case that $x \in (x_0, x_0 + \delta)$ implies $f(x) > y_0 + \varepsilon$. Thus there exists no $\delta > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \quad \Rightarrow \quad f(x) \in (y_0 - \varepsilon, y_0 + \varepsilon).$$

Hence f is not continuous at x_0.

We can give an alternative definition of continuity. A *sequence* of points in a space X is a collection of points $\{x_k : k \in \mathcal{Z}\}$, indexed by the positive integers $\mathcal{Z}$. The sequence is written (x_k). The sequence (x_k) in the metric space, X, has a *limit*, x, iff $\forall \varepsilon > 0\ \exists k_0 \in \mathcal{Z}$ such that $k > k_0$ implies $\|x_k - x\|_X < \varepsilon$.

In this case write $x_k \to x$ as $k \to \infty$, or $\text{Lim}_{k\to\infty} x_k = x$.

Note that x is then an accumulation point of the sequence $\{x_1, \ldots\}$.

More generally $(x_n) \to x$ iff for any open set G containing x, all but a finite number of points in the sequence (x_k) belong to G.

Thus say f is continuous at x_0 iff

$$x_k \to x_0 \quad \text{implies } f(x_k) \to f(x_0).$$

Example 3.1 Consider the function $f : \Re \to \Re$ given by

$$f : x \to \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Now $x \sin \frac{1}{x} = \frac{\sin y}{y}$ where $y = \frac{1}{x}$. Consider a sequence (x_k) where $\text{Lim}_{k\to\infty} x_k = 0$. We shall write this limit as $x \to 0$. $\text{Lim}_{x\to 0} x \sin \frac{1}{x} = \frac{\sin y}{y} = 0$ since $|\sin y|$ is bounded above by 1, and $\text{Lim}_{y\to\infty} \frac{1}{y} = 0$. Thus $\text{Lim}_{x\to 0} f(x) = 0$. But $f(0) = 0$, and so $x_k \to 0$ implies $f(x_k) \to 0$. Hence f is continuous at 0. On the other hand suppose $g(x) = \sin \frac{1}{x}$. Clearly $g(x)$ has no limit as $x \to 0$. To see this observe that for any sequence $(x_k) \to 0$ it is impossible to find a neighborhood G of some point $y \in [-1, 1]$ such that $g(x_k) \in G$ whenever $k > k_o$.

Any linear function between finite-dimensional vector spaces is continuous. Thus the set of continuous functions contains the set of linear functions, when the domain is finite-dimensional. To see this suppose that $f : V \to W$ is a linear transformation between normed vector spaces. (Note that V and W may be infinite-dimensional.) Let $\| \; \|_v$ and $\| \; \|_w$ be the norms on V, W respectively.

Say that f is *bounded* iff $\exists\ B > 0$ such that $\|f(x)\|_w \leq B\|x\|_v$ for all $x \in V$. Suppose now that

$$\left\| f(x) - f(x_0) \right\|_w < \varepsilon.$$

Now

$$\left\| f(x) - f(x_0) \right\|_w = \left\| f(x - x_0) \right\|_w \leq B\|x - x_0\|_v,$$

since f is linear and bounded. Choose $\delta = \frac{\varepsilon}{B}$. Then

$$\|x - x_0\|_v < \delta \Rightarrow \left\| f(x) - f(x_0) \right\|_w \leq B\|x - x_0\|_v.$$

Thus if f is linear and bounded it is continuous.

Lemma 3.5 *Any linear transformation $f : V \to W$ is bounded and thus continuous if V is finite-dimensional* (*of dimension n*).

Proof Use the Cartesian norm $\| \; \|_c$ on V, and let $\| \; \|_w$ be the norm on W. Let $e_1 \ldots e_n$ be a basis for V,

$$\|x\|_c = \sup\{|x_i| : i = 1, \ldots, n\}, \quad \text{and}$$
$$e = \sup_n\{\left\| f(e_i) \right\|_w : i = 1, \ldots, n\}.$$

Now $f(x) = \sum_{i=1} x_i f(e_i)$. Thus $\|f(x)\|_w \leq \sum_i^n \|f(x_i e_i)\|_w$, by the triangle inequality, and $n \leq \sum_{i=1}^n |x_i| \|f(e_i)\|_w$, since $\|ay\|_w = |a|\ \|y\|_w \leq n\, e\|x\|_c$. Thus f is bounded, and hence continuous with respect to the norms $\| \; \|_c$, $\| \; \|_w$. But for any other norm $\| \; \|_v$ it can be shown that there exists $B' > 0$ such that $\|x\|_c \leq B'\|x\|_v$. Thus f is bounded, and hence continuous, with respect to the norms $\| \; \|_v$, $\| \; \|_w$. □

Consider now the set $L(\Re^n, \Re^m)$ of linear functions from $\Re^n$ to $\Re^m$. Clearly if $f, g \in L(\Re^n, \Re^m)$ then the sum $f + g$ defined by $(f + g)(x) = f(x) + g(x)$ is also linear, and for any $\alpha \in \Re$, αf, defined by $(\alpha f)(x) = \alpha(f(x))$ is linear.

Hence $L(\Re^n, \Re^m)$ is a vector space over $\Re$. Since $\Re^n$ is finite dimensional, by Lemma 3.5 any member of $L(\Re^n, \Re^m)$ is bounded. Therefore for any $f \in L(\Re^n, \Re^m)$ we may define

$$\|f\| = \sup_{x\in\Re^n}\left\{\frac{\|f(x)\|}{\|x\|} : \|x\| \neq 0\right\}.$$

Since f is bounded this is defined. Moreover $\|\ \| = 0$ only if f is the zero function. By definition $\|f\|$ is the real number such that $\|f\| \leq B$ for all B such that $\|f(x)\| \leq B\|x\|$. In particular $\|f(x)\| \leq \|f\|\ \|x\|$. If $f, g \in L(\Re^n, \Re^m)$, then $\|(f + g)(x)\| = \|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\| \leq \|f\|\ \|x\| + \|g\|\ \|x\| = (\|f\| + \|g\|)\|x\|$. Thus $\|f + g\| \leq \|f\| + \|g\|$.

Hence $\|\ \|$ on $L(\Re^n, \Re^m)$ satisfies the triangle inequality, and so $L(\Re^n, \Re^m)$ is a normed vector space. This in turn implies that $L(\Re^n, \Re^m)$ has a metric and thus a topology. It is often useful to use the metrics

$$d_1(f, g) = \sup\bigl\{\bigl\|f(x) - g(x)\bigr\| : x \in \Re^n\bigr\}, \quad \text{or}$$
$$d_2(f, g) = \sup\bigl\{\bigl|f_i(x) - g_i(x)\bigr| : i = 1, \dots, m, x \in \Re^n\bigr\}$$

where $f = (f_1, \dots, f_m)$, $g = (g_1, \dots, g_m)$. We write $\mathcal{L}^1(\Re^n, \Re^m)$ and $\mathcal{L}^2(\Re^n, \Re^m)$ for the set $L(\Re^n, \Re^m)$ with the topologies induced by d_1 and d_2 respectively. Clearly these two topologies are identical.

Alternatively, choose bases for $\Re^n$ and $\Re^m$ and consider the matrix representation function

$$M : \bigl(L\bigl(\Re^n, \Re^m\bigr), +\bigr) \to \bigl(M(n, m), +\bigr).$$

On the right hand side we add matrices element by element under the rule $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$. With this operation $M(n, m)$ is also a vector space. Clearly we may choose a basis for $M(n, m)$ to be $\{E_{ij} : i = 1, \dots, n;\ j = 1, \dots, m\}$ where E_{ij} is the elementary matrix with 1 in the ith column and jth row.

Thus $M(n, m)$ is a vector space of dimension nm. Since M is a bijection, $L(\Re^n, \Re^m)$ is also of dimension nm.

A norm on $M(n, m)$ is given by

$$\|A\| = \sup\bigl\{|a_{ij}| : i = 1, \dots, n;\ j = 1, \dots, m\bigr\},$$

where $A = (a_{ij})$.

This in turn defines a metric and thus a topology on $M(n, m)$. Finally this defines a *topology* on $L(\Re^n, \Re^m)$ as follows. For any open set U in $M(n, m)$, let $V = M^{-1}(U)$ and call V open. The base for the topology on $L(\Re^n, \Re^m)$ then consists of all sets of this form. One can show that the topology induced on $L(\Re^n, \Re^m)$ in this way is independent of the choice of bases. We call this the induced topology on $L(\Re^n, \Re^m)$ and write $\mathcal{L}(\Re^n, \Re^m)$ for this topological space. If we consider the norm topology on $M(n, m)$ and the induced topology $\mathcal{L}(\Re^n, \Re^m)$ then the representation map is continuous. Moreover the two topologies induced by the metrics d_1 and d_2 on $L(\Re^n, \Re^m)$ are identical to the induced topology $\mathcal{L}(\Re^n, \Re^m)$. Thus M is also continuous when these metric topologies are used for its domain. (Exercise 3.3 at the end of the book is devoted to this observation.)

If V is an infinite dimensional vector space and $f \in L(V, W)$, then f need not be continuous or bounded. However the subset of $L(V, W)$ consisting of bounded, and thus continuous, maps in $L(V, W)$ admits a norm and thus a topology. So we may write this subset as $\mathcal{L}(V, W)$.

Now let $C_o(\Re^n, \Re^m)$ be the set of continuous functions from $\Re^n$ to $\Re^m$. We now show that $C_o(\Re^n, \Re^m)$ is a vector space.

Lemma 3.6 *$C_o(\Re^n, \Re^m)$ is a vector space over $\Re$.*

Proof Suppose that f, g are both continuous maps. At any $x_0 \in \Re^n$, and any $\varepsilon > 0$, $\exists\ \delta_1, \delta_2 > 0$ such that

$$\|x - x_0\| < \delta_1 \Rightarrow \left\| f(x) - f(x_0) \right\| < \left(\frac{1}{2}\right)\varepsilon$$

$$\|x - x_0\| < \delta_2 \Rightarrow \left\| g(x) - g(x_0) \right\| < \left(\frac{1}{2}\right)\varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then $\|x - x_0\| < \delta \Rightarrow \|(f+g)(x) - (f+g)(x_0)\| = \|f(x) - f(x_0) + g(x) - g(x_0)\| \leq \|f(x) - f(x_0)\| + \|g(x) - g(x_0)\| < (\frac{1}{2})\varepsilon + (\frac{1}{2})\varepsilon = \varepsilon$. Thus $f + g \in C_o(\Re^n, \Re^m)$.

Also for any $\alpha \in \Re$, any $\varepsilon > 0\ \exists\ \delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \left\| f(x) - f(x_0) \right\| < \frac{\varepsilon}{|\alpha|}.$$

Therefore $\|(\alpha f)(x) - (\alpha f)(x_0)\| = |\alpha|\ \|f(x) - f(x_0)\| < \varepsilon$. Thus $\alpha f \in C_o(\Re^n, \Re^m)$. Hence $C_o(\Re^n, \Re^m)$ is a vector space. □

Since $[L(\Re^n, \Re^m)]$ is closed under addition and scalar multiplication, it is a *vector subspace* of dimension nm of the vector space $C_o(\Re^n, \Re^m)$. Note however that $C_o(\Re^n, \Re^m)$ is an *infinite*-dimensional vector space (a *function* space).

Lemma 3.7 *If (X, Γ), $(Y, \mathcal{S})$ and $(Z, \mathcal{R})$ are topological spaces and $C_o((X, \Gamma), (Y, \mathcal{S}))$, $C_o((Y, \mathcal{S}), (Z, \mathcal{R}))$ and $C_o((X, \Gamma), (Z, \mathcal{R}))$ are the sets of functions which are continuous with respect to these topologies, then the composition operator, $\circ$, maps $C_o((X, \Gamma), (Y, \mathcal{S})) \times C_o((Y, \mathcal{S}), (Z, \mathcal{R}))$ to $C_o((X, \Gamma), (Z, \mathcal{R}))$.*

Proof Suppose $f : (X, \Gamma) \to (Y, \mathcal{S})$ and $g : (Y, \mathcal{S}) \to (Z, \mathcal{R})$ are continuous. We seek to show that $g \circ f : (X, \Gamma) \to (Z, \mathcal{R})$ is continuous. Choose any open set U in Z. By continuity of g, $g^{-1}(U)$ is an $\mathcal{S}$-open set in Y. But by the continuity of

f, $f^{-1}(g^{-1}(U))$ is a Γ-open set in X. However $f^{-1}g^{-1}(U) = (g \circ f)^{-1}(U)$. Thus $g \circ f$ is continuous. □

Note therefore that if $f \in L(\Re^n, \Re^m)$ and $g \in L(\Re^m, \Re^k)$ then $g \circ f \in L(\Re^n, \Re^k)$ will also be continuous.

3.3 Compactness

Let (X, Γ) be a topological space. An *open cover* for X is a family $\{U_j : j \in J\}$ of Γ-open sets such that $X = \bigcup_{j \in J} U_j$.

If $U = \{U_j : j \in J\}$ is an open cover for X, *a subcover* of U is an open cover U' of X where $U' = \{U_j : j \in J'\}$ and the index set J' is a subset of J. The subcover is called *finite* if J' is a finite set (*i.e.*, $|J'|$, the number of elements of J', is finite).

Definition 3.5 A topological space (X, Γ) is called *compact* iff any open cover of X has a finite subcover. If Y is a subset of the topological space (X, Γ) then Y is compact iff the topological space (Y, Γ_Y) is compact. Here Γ_Y is the topology induced on Y by Γ. (See Sect. 1.1.3.)

Say that a family $C_J = \{C_j : j \subset J\}$ of closed sets in a topological space (X, Γ) has the *finite intersection property* (*FIP*) iff whenever J' is a finite subset of J then $\bigcap_{j \in J'} C_j$ is non-empty.

Lemma 3.8 *A topological space (X, Γ) is compact iff whenever C_J is a family of closed sets with the finite intersection property then $\bigcap_{j \in J} C_j$ is non-empty.*

Proof We establish first of all that $U_J = \{U_j : j \in J\}$ is an open cover of X iff the family $C_J = \{C_j = X \backslash U_j : j \in J\}$ of closed sets has empty intersection. Now

$$\begin{aligned}\bigcup_J U_j &= \bigcup_J (X \backslash C_j) = \bigcup_J (X \cap \overline{C_j}) \\ &= X \cap \left(\bigcup_j \overline{C}_j\right) = X \cap \left(\overline{\bigcap_J C_j}\right).\end{aligned}$$

Thus

$$\bigcup_J U_j = X \quad \text{iff} \; \bigcap_J C_j = \Phi.$$

To establish necessity, suppose that X is compact and that the family C_J has the *FIP*. If C_J in fact has empty intersection then $U_J = \{X \backslash C_j : j \in J\}$ must be an open cover. Since X is compact there exists a finite set $J' \subset J$ such that $U_{J'}$ is a cover. But then $C_{J'}$ has empty intersection contradicting *FIP*. Thus C_J has non-empty intersection. To establish sufficiency, suppose that any family C_J, satisfying *FIP*, has non-empty intersection. Let $U_J = \{U_j : j \in J\}$ be an open cover for X. If U_J has no finite subcover, then for any finite J', the family $C_{J'} = \{X \backslash U_j : j \in J'\}$ must have non-empty intersection. By the *FIP*, the family C_J must have non-empty intersection, contradicting the assumption that U_J is a cover. Thus $(X, \varGamma)$ is compact. □

This lemma allows us to establish conditions under which a preference relation P on X has a non-empty choice $C_P(X)$. (See Lemma 1.8 for the case with X finite.) Say the preference relation P on the topological space X is lower demi-continuous (LDC) iff the inverse correspondence $\phi_P^{-1} : X \to X : x \to \{y \in X : xPy\}$ is open for all x in X.

Lemma 3.9 *If X is a compact topological space and P is an acyclic and lower demi-continuous preference on X, then there exists a choice $\overline{x}$ of P in X.*

Proof Suppose on the contrary that there is no choice. Thus for all $x \in X$ there exists some $y \in X$ such that yPx. Thus $x \in \phi_P^{-1}(y)$. Hence $U = \{\phi_P^{-1}(y) : y \in X\}$ is a cover for X. Moreover for each $y \in X$, $\phi_P^{-1}(y)$ is open. Since X is compact, there exists a finite subcover of U. That is to say there exists a finite set A in X such that $U' = \{\phi_P^{-1}(y) : y \in A\}$ is a cover for X. In particular this implies that for each $x \in A$, there is some $y \in A$ such that $x \in P_P^{-1}(y)$, or that yPx. But then $C_P(A) = \{x \in A : \phi_P(x) = \varPhi\} = \varPhi$. Now P is acyclic on X, and thus acyclic on A. Hence, by the acyclicity of P and Lemma 1.8, $C_P(A) \neq \varPhi$. By the contradiction $U = \{\phi_P^{-1}(y) : y \in X\}$ cannot be a cover. That is to say there is some $\overline{x} \in X$ such that $\overline{x} \in \phi_P^{-1}(y)$ for no $y \in X$. But then $yP\overline{x}$ for no $y \in X$, and $\overline{x} \in C_P(X)$, or $\overline{x}$ is the choice on X. □

This lemma can be used to show that a continuous function $f : X \to \Re$ from a compact topological space X into the reals attains its bounds. Remember that we defined the supremum and infimum of f on a set Y to be

1. $\sup(f, Y) = M$ such that $f(x) \leq M$ for all $x \in Y$ and if $f(x) \leq M'$ for all $x \in Y$ then $M' \geq M$
2. $\inf(f, Y) = m$ such that $f(x) \geq m$ for all $x \in Y$ and if $f(x) \geq m'$ for all $x \in Y$ then $m \geq m'$.

Say f attains its upper bound on Y iff there is some x_s in Y such that $f(x_s) = \sup(f, Y)$. Similarly say f attains its lower bound on Y iff there is some x_i in Y such that $f(x_i) = \inf(f, Y)$.

Given the function $f : X \to \Re$, define a preference P on $X \times X$ by xPy iff $f(x) > f(y)$. Clearly P is acyclic, since $>$ on $\Re$ is acyclic. Moreover for any $x \in X$,

$$\phi_P^{-1}(x) = \big\{y : f(y) < f(x)\big\}$$

is open, when f is continuous.

To see this let $U = (-\infty, f(x))$. Clearly U is an open set in $\Re$. Moreover $f(y)$ belongs to the open interval $(-\infty, f(x))$ iff $y \in \phi_P^{-1}(x)$. But f is continuous, and so $f^{-1}(U)$ is open in X. Since $y \in f^{-1}(U)$ iff $y \in \phi_P^{-1}(x)$, $\phi_P^{-1}(x)$ is open for any $x \in X$.

Weierstrass Theorem *Let (X, Γ) be a topological space and $f : X \to \Re$ a continuous real-valued function. If Y is a compact subset of X, then f attains its bounds on Y.*

Proof As above, for each $x \in Y$, define $U_x = (-\infty, f(x))$. Then $\phi_P^{-1}(x) = \{y \in Y : f(y) < f(x)\} = f^{-1}(U_x) \cap Y$ is an open set in the induced topology on Y. By Lemma 3.9 there exists a choice $\overline{x}$ in Y such that $\phi_P^{-1}(\overline{x}) = \Phi$. But then $f(y) > f(\overline{x})$ for no $y \in Y$. Hence $f(y) \leq f(\overline{x})$ for all $y \in Y$. Thus $f(\overline{x}) = \sup(f, Y)$.

In the same way let Q be the relation on X given by xQy iff $f(x) < f(y)$. Then there is a choice $\overline{\overline{x}} \in Y$ such that $f(y) < f(\overline{\overline{x}})$ for no $y \in Y$. Hence $f(y) \geq f(\overline{\overline{x}})$ for all $y \in Y$, and so $f(\overline{\overline{x}}) = \inf(f, Y)$. Thus f attains its bounds on Y. □

We can develop this result further.

Lemma 3.10 *If $f : (X, \Gamma) \to (Z, \mathcal{S})$ is a continuous function between topological spaces, and Y is a compact subset of X, then*

$$f(Y) = \left\{ f(y) \ \in Z : y \in Y \right\}$$

is compact.

Proof Let $\{W_\alpha\}$ be an open cover for $f(Y)$. Then each member W_α of this cover may be expressed in the form $W_\alpha = U_\alpha \cap f(Y)$ where U_α is an open set in Z. For each α, let $V_\alpha = f^{-1}(U_\alpha) \cap Y$. Now each V_α is open in the induced topology on Y. Moreover, for each $y \in Y$, there exists some W_α such that $f(y) \in W_\alpha$. Thus $\{V_\alpha\}$ is an open cover for Y. Since Y is compact, $\{V_\alpha\}$ has a finite subcover $\{V_\alpha : \alpha \in J\}$, and so $\{f(V_\alpha) : \alpha \in J\}$ is a finite subcover of $\{W_\alpha\}$. Thus $f(Y)$ is compact. □

Now a real-valued continuous function f is bounded on a compact set, Y (by the Weierstrass Theorem). So $f(Y)$ will be contained in $[f(\overline{\overline{x}}, f(\overline{x}))]$ say, for some $\overline{\overline{x}} \in Y$. Since $f(Y)$ must also be compact, this suggests that a closed set of the form $[a, b]$ must be compact.

For a set $Y \subset \Re$ define $\sup(Y) = \sup(\text{id}, Y)$, the *supremum* of Y, and $\inf(Y) = \inf(\text{id}, Y)$, the *infimum* of Y. Here $\text{id} : \Re \to \Re$ is the identity on $\Re$. The set $Y \subset \Re$ is *bounded above* (or below) iff its supremum (or infimum) is finite. The set is *bounded* iff it is both bounded above and below. Thus a set of the form $[a, b]$, say with $-\infty < a < b < +\infty$ is bounded.

Heine Borel Theorem *A closed bounded interval, $[a, b]$, of the real line is compact.*

Sketch of Proof Consider a family $C = \{[a, c_i], [d_j, b] : i \in I, j \in J\}$ of subsets of $[a, b]$ with the *finite intersection property*. Suppose that neither I nor J is empty. Let $d = \sup(\{d_j : j \in J\})$ and suppose that for some $k \in I$, $c_k < d$. Then there exists $i \in I$ and $j \in J$ such that $[a, c_i] \cap [d_j, b] = \Phi$, contradicting the finite intersection property. Thus $c_i \geq d$, and so $[a, c_i] \cap [d, b] \neq \Phi$, for all $i \in I$. Hence the family C has non empty intersection. By Lemma 3.8, $[a, b]$ is compact □

Definition 3.6 A topological space (X, Γ) is called *Hausdorff* iff any two distinct points $x, y \in X$ have Γ-open neighbourhoods U_x, U_y such that $U_x \cap U_y = \Phi$.

Lemma 3.11 *If (X, d) is a metric space then (X, Γ_d) is Hausdorff, where Γ_d is the topology on X induced by the metric d.*

Proof For two points $x \neq y$, let $\varepsilon = d(x, y) \neq 0$. Define $U_x = B(x, \frac{\varepsilon}{3})$ and $U_x = B(y, \frac{\varepsilon}{3})$.

Clearly, by the triangle inequality, $B(x, \frac{\varepsilon}{3}) \cap B(y, \frac{\varepsilon}{3}) = \Phi$. Otherwise there would exist a point z such that $d(x, z) < \frac{\varepsilon}{3}$, $d(z, y) < \frac{\varepsilon}{3}$, which would imply that $d(x, y) < d(x, z) + d(z, y) = \frac{2\varepsilon}{3}$. By contradiction the open balls of radius $\frac{\varepsilon}{3}$ do not intersect. Thus (x, Γ_d) is Hausdorff. □

A Hausdorff topological space is therefore a natural generalisation of a metric space.

Lemma 3.12 *If (X, Γ) is a Hausdorff topological space, then any compact subset Y of X is closed.*

Proof We seek to show that $X \backslash Y$ is open, by showing that for any $x \in X \backslash Y$, there exists a neighbourhood G of x and an open set H containing Y such that $G \cap H = \Phi$. Let $x \in X \backslash Y$, and consider any $y \in Y$. Since X is Hausdorff, there exists a neighbourhood $V(y)$ of y and a neighbourhood $U(y)$, say, of x such that $V(y) \cap U(y) = \Phi$. Since the family $\{V(y) : y \in Y\}$ is an open cover of Y, and Y is compact, there exists a finite subcover $\{V(y) : y \in A\}$, where A is a finite subset of Y.

Let $H = \bigcup_{y \in A} V(y)$ and $G = \bigcap_{y \in A} U(y)$.

Suppose that $G \cap H \neq \Phi$. Then this implies there exists $y \in A$ such that $V(y) \cap U(y)$ is non-empty. Thus $G \cap H = \Phi$. Since A is finite, G is open. Moreover Y is contained in H. Thus $X \backslash Y$ is open and Y is closed. □

Lemma 3.13 *If (X, Γ) is a compact topological space and Y is a closed subset of X, then Y is compact.*

Proof Let $\{U_\alpha\}$ be an open cover for Y, where each $U_\alpha \subset X$. Then $\{V_\alpha = U_\alpha \cap Y\}$ is also an open cover for Y.

Since $X \backslash Y$ is open, $\{X \backslash Y\} \cup \{V_\alpha\}$ is an open cover for X.

Since X is compact there is a finite subcover, and since each $V_\alpha \subset Y$, $X \backslash Y$ must be a member of this subcover. Hence the subcover is of the form $\{X \backslash Y\} \cup \{V_j : j \in J\}$. But then $\{V_j : j \in J\}$ is a finite subcover of $\{V_\alpha\}$ for Y. Hence Y is compact. □

Tychonoff's Theorem *If* (X, Γ) *and* $(Y, \mathcal{S})$ *are two compact topological spaces then* $(X \times Y, \Gamma \times \mathcal{S})$, *with the product topology, is compact.*

Proof To see this we need only consider a cover for $X \times Y$ of the form $\{U_\alpha \times V_\beta\}$ for $\{U_\alpha\}$ an open cover for X and $\{V_\beta\}$ an open cover for Y. Since both X and Y are compact, both $\{U_\alpha\}$ and $\{V_\alpha\}$ have finite subcovers $\{U_j\}_{j \in J}$ and $\{V_k\}_{k \in K}$, and so $\{U_j \times V_k : (j, k) \in J \times K\}$ is a finite subcover for $X \times Y$. □

As a corollary of this theorem, let $I_k = [a_k, b_k]$ for $k = 1, \dots, n$ be a family of closed bounded intervals in $\Re$. Each interval is compact by the Heine Borel Theorem. Thus the closed *cube* $I^n = I_1 \times I_2, \dots, I_n$ in $\Re^n$ is compact, by Tychonoff's Theorem. Say that a set $Y \subset \Re^n$ is bounded iff for each $y \in Y$ there exists some finite number $K(y)$ such that $\|x - y\| < K(y)$ for all $x \in Y$. Here $\| \ \|$ is any convenient norm on $\Re^n$. If Y is bounded then clearly there exists some closed cube $I^n \subset \Re^n$ such that $Y \subset I^n$. Thus we obtain:

Lemma 3.14 *If* Y *is a closed bounded subset of* $\Re^n$ *then* Y *is compact.*

Proof By the above, there is a closed cube I^n such that $Y \subset I^n$. But I^n is compact by Tychonoff's Theorem. Since Y is a closed subset of I^n, Y is compact, by Lemma 3.13. □

In $\Re^n$ a compact set Y is one that is both closed and bounded. To see this note that $\Re^n$ is certainly a metric space, and therefore Hausdorff. By Lemma 3.12, if Y is compact, then it must be closed. To see that it must be bounded, consider the unbounded closed interval $A = [0, \infty)$ in $\Re$, and an open cover $\{U_k = (k-2, k); k = 1, \dots, \infty\}$. Clearly $\{(-1, 1), (0, 2), (1, 3), \dots\}$ cover $[0, \infty)$. A finite subcover must be bounded above by K, say, and so the point K does not belong to the subcover. Hence $[0, \infty)$ is non-compact.

Lemma 3.15 *A compact subset* Y *of* $\Re$ *contains its bounds.*

Proof Let $s = \sup(\mathrm{id}, Y)$ and $i = \inf(\mathrm{id}, Y)$ be the supremum and infimum of Y. Here $\mathrm{id} : \Re \to \Re$ is the identity function. By the discussion above, Y must be bounded, and so i and s must be finite. We seek to show that Y contains these bounds, *i.e.*, that $i \in Y$ and $s \in Y$. Suppose for example that $s \notin Y$. By Lemma 3.12, Y must be closed and hence $\Re \backslash Y$ must be open. But then there exists a neighbourhood $(s - \varepsilon, s + \varepsilon)$ of s in $\Re \backslash Y$, and so $s - \frac{\varepsilon}{2} \notin Y$. But this implies that $y \leq s - \frac{\varepsilon}{2}$ for all $y \in Y$, which contradicts the assumption that $s = \sup(\mathrm{id}, Y)$. Hence $s \in Y$. A similar argument shows that $i \in Y$. Thus $Y = [i, y_1] \cup, \dots, \cup [y_r, s]$ say, and so Y contains its bounds. □

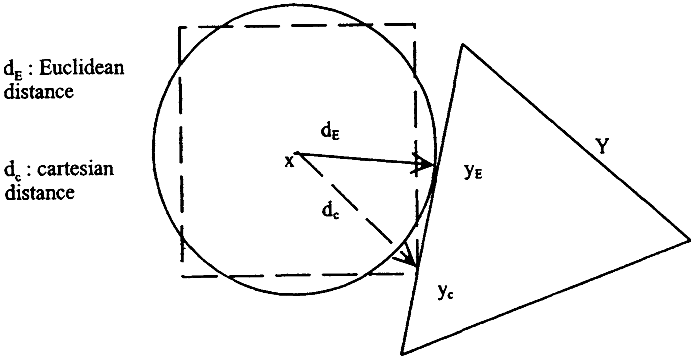


Fig. 3.12 Nearest points under different metrics

Lemma 3.16 *Let (X, Γ) be a topological space and $f : X \to \Re$ a continuous real-valued function. If $Y \subset X$ is compact then there exist points x_0 and x_1 in Y such that $f(x_0) \leq f(y) \leq f(x_1)$ for all $y \in Y$.*

Proof By Lemma 3.10, $f(Y)$ is compact. By Lemma 3.15, $f(Y)$ contains its infimum and supremum. Thus there exists $x_0, x_1 \in Y$ such that

$$f(x_0) \leq f(y) \leq f(x_1) \quad \text{for all } y \in Y.$$

Note that $f(Y)$ must be bounded, and so $f(x_0)$ and $f(x_1)$ must be finite. □

We have here obtained a second proof of the Weierstrass Theorem that a continuous real-valued function on a compact set attains its bounds, and shown moreover that these bounds are finite. A useful application of this theorem is that if Y is a compact set in $\Re^n$ and $x \notin Y$ then there is some point y in Y which is nearest to x. Remember that we defined the *distance* from a point x in a metric space (X, d) to a subset Y of X to be $d(x, Y) = \inf(f_x, Y)$ where $f_x : Y \to \Re$ is defined by $f_x(y) = d(x, -)(y) = d(x, y)$. See Fig. 3.12.

Lemma 3.17 *Suppose Y is a subset of a metric space (X, d) and $x \in X$. Then the function $f_x : Y \to \Re$ given by $f_x(y) = d(x, y)$ is continuous.*

Proof Consider $y_1, y_2 \in Y$ and suppose that $d(x, y_1) \geq d(x, y_2)$. Then $|d(x, y_1) - d(x, y_2)| = d(x, y_1) - d(x, y_2)$.

By the triangle inequality $d(x, y_1) \leq d(x, y_2) + d(y_2, y_1)$. Hence $|d(x, y_1) - d(x, y_2)| \leq d(y_1, y_2)$ and so $d(y_1, y_2) < \varepsilon \Rightarrow |d(x, y_1) - d(x, y_2)| < \varepsilon$, for any $\varepsilon > 0$.

Thus for any $\varepsilon > 0$, $d(y_1, y_2) < \varepsilon \Rightarrow |f_x(y_1) - f_x(y_2)| < \varepsilon$, and so f_x is continuous. □

Lemma 3.18 *If Y is a compact subset of a compact metric space X and $x \in X$, then there exists a point $y_0 \in Y$ such that $d(x, Y) = d(x, y_0) < \infty$.*

Proof By Lemma 3.17, the function $d(x, -) : Y \to \Re$ is continuous. By Lemma 3.16, this function attains its lower and upper bounds on Y. Thus there exists $y_0 \in Y$ such that $d(x, y_0) = \inf(d(x, -), Y) = d(x, Y)$, where $d(x, y_0)$ is finite. □

The point y_0 in Y such that $d(x, y_0) = d(x, Y)$ is the nearest point in Y to x.

Note of course that if $x \in Y$ then $d(x, Y) = 0$.

More importantly, when Y is compact $d(x, Y) = 0$ *if and only if* $x \in Y$. To see this necessity, suppose that $d(x, Y) = 0$. Then by Lemma 3.18, there exists $y_0 \in Y$ such that $d(x, y_0) = 0$. By the definition of a metric $d(x, y_0) = 0$ iff $x = y_0$ and so $x \in Y$. The point $y \in Y$ that is nearest to x is dependent on the metric of course, and may also not be unique.

3.4 Convexity

3.4.1 A Convex Set

If x, y are two points in a vector space, X, then the *arc*, $[x, y]$, is the set $\{z \in X : \exists\ \lambda \in [0, 1] \text{ s.t. } z = \lambda x + (1 - \lambda)y\}$. A point in the arc $[x, y]$ is called a *convex combination* of x and y. If Y is a subset of X, then the *convex hull*, $\text{con}(Y)$, of Y is the smallest set in X that contains, for every pair of points x, y in Y, the arc $[x, y]$. The set Y is called *convex* iff $\text{con}(Y) = Y$. The set Y is *strictly convex* iff for any $x, y \in Y$ the combination $\lambda x + (1 - \lambda)y$, for $\lambda \in (0, 1)$, belongs to the interior of Y.

Note that if Y is a vector subspace of the real vector space X then Y must be convex. For then if $x, y \in Y$ both $\lambda, (1 - \lambda) \in \Re$ and so $\lambda x + (1 - \lambda)y \in Y$.

Definition 3.7 Let Y be a real vector space, or a convex subset of a real vector space, and let $f : Y \to \Re$ be a function. Then f is said to be

1. *convex* iff $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$
2. *concave* iff $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda) f(y)$
3. *quasi-concave* iff $f(\lambda x + (1 - \lambda)y) \geq \min[f(x), f(y)]$ for any $x, y \in Y$ and any $\lambda \in [0, 1]$.

Suppose now that $f : Y \to \Re$ and consider the preference $P \subset Y \times Y$ induced by f. For notational convenience from now on we regard P as a correspondence $P : Y \to Y$. That is define P by

$$P(x) = \{y \in Y : f(y) > f(x)\}.$$

If f is quasi-concave then when $y_1, y_2, \in P(x)$,

$$f\bigl(\lambda y_1 + (1 - \lambda)y_2\bigr) \geq \min\bigl[f(y_1), f(y_2)\bigr] > f(x).$$

Hence $\lambda y_1 + (1-\lambda)y_2 \in P(x)$. Thus for all $x \in Y$, $P(x)$ is convex.

We shall call a preference correspondence $P : Y \to Y$ *convex* when Y is convex and P is such that $P(x)$ is convex for all $x \in Y$.

When a function $f : Y \to \Re$ is quasi-concave then the strict preference correspondence P defined by f is convex. Note also that the weak preference $R : Y \to Y$ given by

$$R(x) = \bigl\{y \in Y : f(y) \geq f(x)\bigr\}$$

will also be convex.

If $f : Y \to \Re$ is a concave function then it is quasi-concave. To see this consider $x, y \in Y$, and suppose that $f(x) \leq f(y)$. By concavity,

$$\begin{aligned} f\bigl(\lambda x + (1-\lambda)y\bigr) &\geq \lambda f(x) + (1-\lambda)f(y) \\ &\geq \lambda f(x) + (1-\lambda)f(x) \\ &\geq \min\bigl[f(x), f(y)\bigr]. \end{aligned}$$

Thus f is quasi-concave. Note however that a quasi-concave function need be neither convex nor concave. However if f is a linear function then it is convex, concave and quasi-concave. There is a partial order $>$ on $\Re^n$ given by $x > y$ iff $x_i > y_i$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. A function $f : \Re^n \to \Re$ is *weakly monotonically increasing* iff $f(x) \geq f(y)$ for any $x, y \in \Re^n$ such that $x > y$. A function $f : \Re^n \to \Re$ has *decreasing returns to scale* iff f is weakly monotonically increasing and concave. A very standard assumption in economic theory is that feasible production of an output has decreasing returns to scale of inputs, and that consumers' utility or preference has decreasing returns to scale in consumption. We shall return to this point below.

3.4.2 Examples

Example 3.2 (i) Consider the set $X_1 = \{(x_1, x_2) \in \Re^2 : x_2 \geq x_1\}$. Clearly if $x_2 \geq x_1$ and $x_2' \geq x_1'$ then $\lambda x_2 + (1-\lambda)x_2' \geq \lambda x_1 + (1-\lambda)x_1'$, for $\lambda \in [0, 1]$. Thus $\lambda(x_1, x_2) + (1-\lambda)(x_1', x_2') = \lambda x_1 + (1-\lambda)x_1', \lambda x_2 + (1-\lambda)x_2' \in X_1$. Hence X_1 is convex.

On the other hand consider the set $X_2 = \{(x_1, x_2) \in \Re^2 : x_2 \geq {x_1}^2\}$.

As Fig. 3.13a indicates, this is a strictly convex set. However the set $X_3 = \{(x_1, x_2) \in \Re^2 : |x_2| \geq {x_1}^2\}$ is not convex. To see this suppose $x_2 < 0$. Then $(x_1, x_2) \in X_3$ implies that $x_2 \leq -{x_1}^2$. But then $-x_2 \geq {x_1}^2$. Clearly $(x_1, 0)$ belongs to the convex combination of (x_1, x_2) and $(x_1, -x_2)$ yet $(x_1, 0) \notin X_3$.

(ii) Consider now the set $X_4 = \{(x_1, x_2) : x_2 > {x_1}^3\}$. As Fig. 3.13b shows it is possible to choose (x_1, x_2) and (x_1', x_2') with $x_1 < 0$, so that the convex combination of (x_1, x_2) and (x_1', x_2') does not belong to X_4. However $X_5 = \{(x_1, x_2) \in \Re^2 : x_2 \geq {x_1}^3$ and $x_1 \geq 0\}$ and $X_6 = \{(x_1, x_2) \in \Re^2 : x_2 \leq {x_1}^3$ and $x_1 \leq 0\}$ are both convex sets.

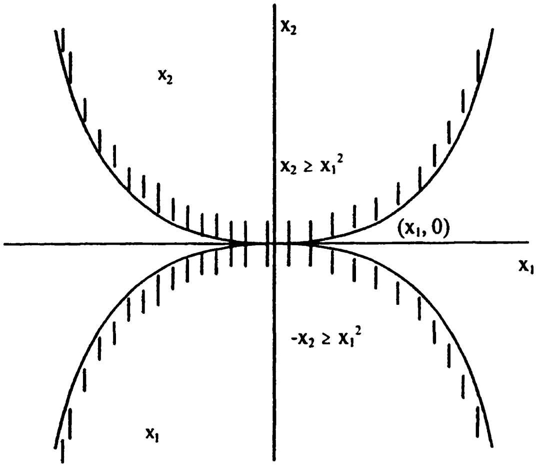


Fig. 3.13a Example 3.2(i)

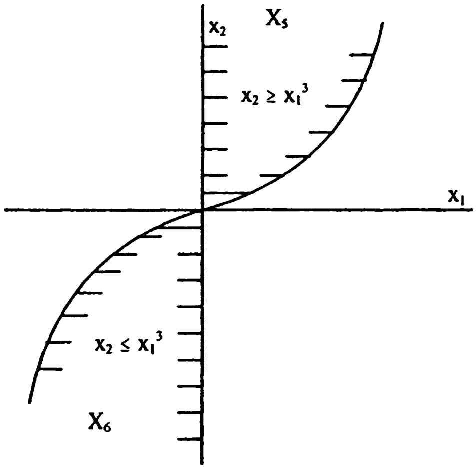


Fig. 3.13b Example 3.2(ii)

(iii) Now consider the set $X_7 = \{(x_1, x_2) \in \Re^2 : x_1 x_2 \geq 1\}$. From Fig. 3.13c it is clear that the restriction of the set X_7 to the positive quadrant $\Re^2_+ = \{(x_1, x_2) \in \Re^2 : x_1 \geq 0$ and $x_2 \geq 0\}$ is strictly convex, as is the restriction of x_7 to the negative quadrant $\Re^2_- = \{(x_1, x_2) \in \Re^2 : x_1 \leq 0$ and $x_2 \leq 0\}$. However if $(x_1, x_2) \in X_7 \cap \Re^2_+$ then $(-x_1, -x_2) \in X_7 \cap \Re^2_-$. Clearly the origin $(0, 0)$ belongs to the convex hull of (x_1, x_2) and $(-x_1, -x_2)$, yet does not belong to X_7. Thus X_7 is not convex.

(iv) Finally a set of the form

$$X_8 = \left\{(x_1, x_2) \in \Re^2_+ : x_2 \leq {x_1}^{\alpha} \text{ for } \alpha \in (0, 1)\right\}$$

is also convex. See Fig. 3.13d.

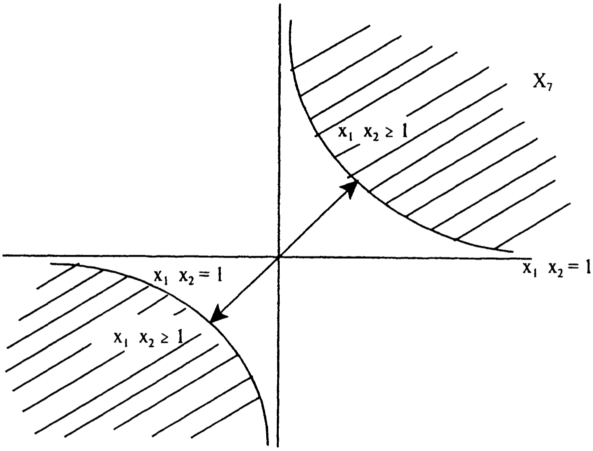


Fig. 3.13c Example 3.2(iii)

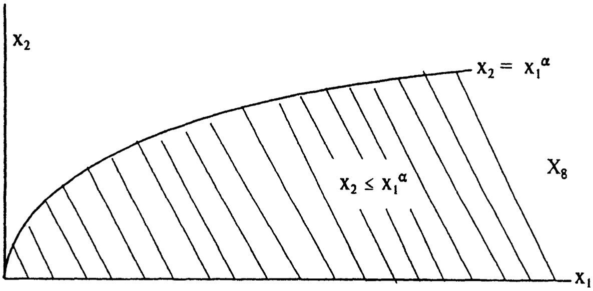


Fig. 3.13d Example 3.2(iv)

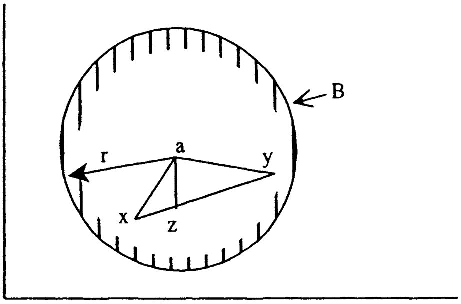


Fig. 3.13e The Euclidean ball

Example 3.3 (i) Consider the set $B = \{(x_1, x_2) \in \Re^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq r^2\}$. See Fig. 3.13e. This is the closed ball centered on $(a_1, a_2) = a$, of radius r. Suppose that $x, y \in B$ and $z = \lambda x + (1 - \lambda)y$ for $\lambda \in [0, 1]$.

Let $\| \ \|$ stand for the Euclidean norm. Then x, y both satisfy $\|x - a\| \leq r$, $\|y - a\| \leq r$. But $\|z - a\| \leq \lambda\|x - a\| + (1 - \lambda)\|y - a\|$. Thus $\|z - a\| \leq r$ and so $z \in B$. Hence B is convex. Moreover B is a closed and bounded subset of $\Re^2$ and is thus compact. For a general norm on $\Re^n$, the closed ball $B = \{x \in \Re^n : \|x - a\| \leq r\}$

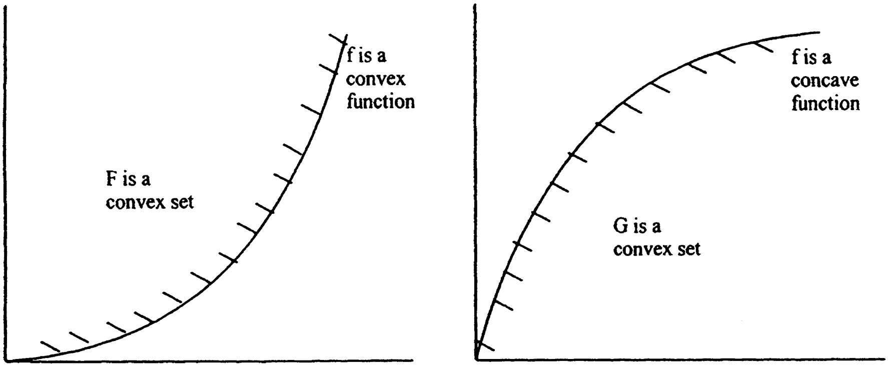


Fig. 3.14 Convex and concave functions

will be compact and convex. In particular, if the Euclidean norm is used, then B is strictly convex.

(ii) In the next section we define the hyperplane $H(\rho,\alpha)$ *normal* to a vector ρ in $\Re^n$ to be $\{x \in \Re^n : \langle \rho, x\rangle = \alpha\}$ where α is some real number. Suppose that $x, y \in H(\rho, x)$.

Now

$$\begin{aligned}\left\langle \rho, \lambda x + (1-\lambda)y\right\rangle &= \left\langle \lambda(\rho,x) + (1-\lambda)(\rho,y)\right\rangle \\ &= \alpha, \quad \text{whenever } \lambda \in [0,1].\end{aligned}$$

Thus $H(\rho,\alpha)$ is a convex set. We also define the closed half-space $H_+(\rho,\alpha)$ by $H_+(\rho,\alpha) = \{x \in \Re^n : \langle\rho,x\rangle \geq \alpha\}$. Clearly if $x, y \in H_+(\rho,\alpha)$ then $\langle \rho, \lambda x + (1-\lambda)y\rangle = (\lambda\langle\rho,x\rangle + (1-\lambda)\langle\rho,y\rangle) \geq \alpha$ and so $H_+(\rho,\alpha)$ is also convex.

Notice that if B is the compact convex ball in $\Re^n$ then there exists some $\rho \in \Re^n$ and some $\alpha \in \Re$ such that $B \subset H_+(\rho,\alpha)$.

If A and B are two convex sets in $\Re^n$ then $A \cap B$ must also be a convex set, while $A \cup B$ need not be. For example the union of two disjoint convex sets will not be convex.

We have called a function $f : Y \to \Re$ convex on Y iff $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for $x, y \in Y$.

Clearly this is equivalent to the requirement that the set $F = \{(z,x) \in \Re \times Y : z \geq f(x)\}$ is convex. (See Fig. 3.14.)

To see this suppose (z_1,x_1) and $(z_2,x_2) \in F$.

Then $\lambda(z_1,x_1) + (1-\lambda)(z_2,x_2) \in F$ iff $\lambda z_1 + (1-\lambda)z_2 \geq f(\lambda x_1 + (1-\lambda)x_2)$. But $(f(x_1),x_1)$ and $(f(x_2),x_2) \in F$, and so $\lambda z_1 + (1-\lambda)z_2 \geq \lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$ for $\lambda \in [0,1]$.

In the same way f is concave on Y iff $G = \{(z,x) \in \Re \times Y : z \leq f(x)\}$ is convex. If $f : Y \to \Re$ is concave then the function $(-f) : Y \to \Re$, given by $(-f)(x) = -f(x)$, is convex and vice versa.

To see this note that if $z \le f(x)$ then $-z \ge -f(x)$, and so $G = \{(z,x) \in \Re \times Y : z \le f(x)\}$ is convex implies that $F = \{(z,x) \in \Re \times Y : z \ge (-f)(x)\}$ is convex.

Finally f is quasi-concave on Y iff, for all $z \in \Re$, the set $G(z) = \{x \in Y : z \le f(x)\}$ is convex.

Notice that $G(x)$ is the image of G under the projection mapping $p_z : \Re \times Y \to Y : (z,x) \to x$. Since the image of a convex set under a projection is convex, clearly $G(z)$ is convex for any z whenever G is convex. As we know already this means that a concave function is quasi-concave. We now apply these observations.

Example 3.4

(i) Let $f : \Re \to \Re$ by $x \to x^2$.

As Example 3.2(i) showed, the set $F = \{(x,z) \in \Re \times \Re : z \ge f(x) = x^2\}$ is convex. Hence f is convex.

(ii) Now let $f : \Re \to \Re$ by $x \to x^3$. Example 3.2(ii) showed that the set $F = \{(x,z) \in \Re_+ \times \Re : z \ge f(x) = x^3\}$ is convex and so f is convex on the convex set $\Re_+ = \{x \in \Re : x \ge 0\}$.

On the other hand $F = \{(x,z) \in \Re_- \times \Re : z \le f(x) = x^3\}$ is convex and so f is *concave* on the convex set $\Re_- = \{x \in \Re : x \le 0\}$.

(iii) Let $f : \Re \to \Re$ by $x \to \frac{1}{x}$. By Example 3.2(iii) the set $F = \{(x,z) \in \Re_+ \times \Re : z \ge f(x) = \frac{1}{x}\}$ is convex, and so f is *convex* on $\Re_+$ and *concave* on $\Re_-$.

(iv) Let $f(x) = x^\alpha$ where $0 < \alpha < 1$. Then $F = \{(x,z) \in \Re_+ \times \Re : z \le f(x) = x^\alpha\}$ is convex, and so f is concave.

(v) Consider the exponential function $\exp : \Re \to \Re : x \to e^x$. Figure 3.15a demonstrates that the exponential function is convex. Another way of showing this is to note that $e^x > f(x)$ for any geometric function $f : x \to x^r$ for $r > 1$, for any $x \in \Re_+$.

Since the geometric functions are convex, so is e^x. On the other hand as Fig. 3.15b shows the function $\log_e : \Re_+ \to \Re$, inverse to *exp*, is concave.

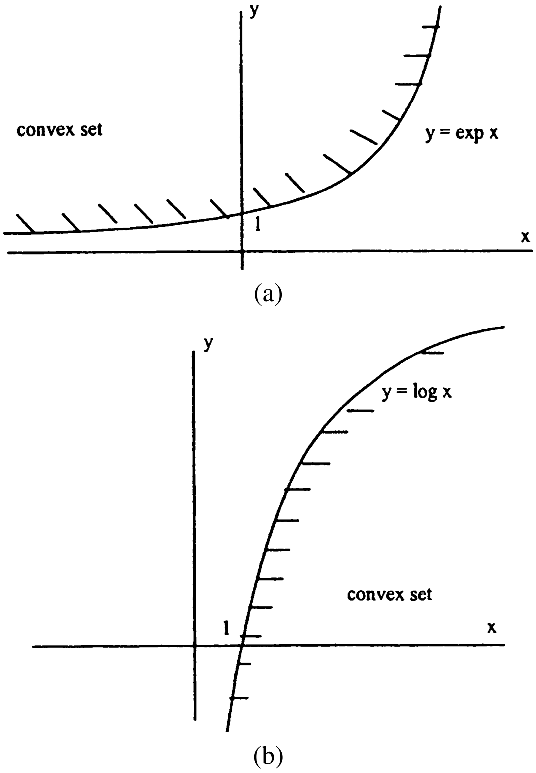
(vi) Consider now $f : \Re^2 \to \Re : (x,y) \to xy$. Just as in Example 3.2(iii) the set $\{(x,y) \in \Re^2_+ : xy \ge t\} = \Re^2_+ \cap f^{-1}[t,\infty)\}$ is convex and so f is a quasi-concave function on $\Re^2_+$. Similarly f is quasi-concave on $\Re^2_-$. However f is not quasi-concave on $\Re^2$.

(vii) Let $f : \Re^2 \to \Re : (x_1,x_2) \to r^2 - (x_1 - a_1)^2 - (x_2 - a_2)^2$. Since the function $g(x) = x^2$ is convex, $(-g)(x) = -x^2$ is concave, and so clearly f is a concave function. Moreover it is obvious that f has a supremum in $\Re^2$ at the point $(x_1,x_2) = (a_1,a_2)$. On the other hand the functions in Examples 3.4(iv) to (vi) are monotonically increasing.

3.4.3 Separation Properties of Convex Sets

Let X be a vector space of dimension n with a scalar product $\langle,\rangle$. Define $H(\rho,\alpha) = \{x \in X : \langle \rho, x\rangle = \alpha\}$ to be the hyperplane in X *normal* to the vector $\rho \in \Re^n$. It should be clear that $H(\rho,\alpha)$ is an $(n-1)$ dimensional plane displaced some distance from the origin. To see this suppose that $x = \lambda\rho$ belongs to $H(\rho,\alpha)$. Then $\langle \rho, \lambda\rho\rangle =$

Fig. 3.15 The exponential and logarithmic functions



$\lambda\|\rho\|^2 = \alpha$. Thus $\lambda = \frac{\alpha}{\|\rho\|^2}$. Hence the length of x is $\|x\| = |\lambda|\ \|\rho\| = \frac{|\alpha|}{\|\rho\|}$ and so $x = \pm\, \frac{|\alpha|}{\|\rho\|^2}\rho$.

Clearly if $y = \lambda\rho + y_0$ belongs to $H(\rho,\alpha)$ then $\langle\rho, y\rangle = \langle\rho, \lambda\rho + y_0\rangle = \alpha + \langle\rho, y_0\rangle$ and so $\langle\rho, y_0\rangle = 0$.

Thus any vector y in $H(\rho,\alpha)$ can be written in the form $y = \lambda\rho + y_0$ where y_0 is orthogonal to ρ. Since there exist $(n-1)$ linearly independent vectors $y_1, \dots, y_{n-1}$, all orthogonal to ρ, any vector $y \in H(\rho,\alpha)$ can be written in the form

$$y = \lambda\rho + \sum_{i=1}^{n-1} a_i y_i,$$

where $\lambda\rho$ is a vector of length $\frac{|\alpha|}{\|\rho\|}$. Now let $\{\rho^{\perp}\} = \{x \in \Re^n : \langle\rho, x\rangle = 0\}$. Clearly $\{\rho^{\perp}\}$ is a vector subspace of $\Re^n$, of dimension $(n-1)$ through the origin.

Thus $H(\rho,\alpha) = \lambda\rho + \{\rho^{\perp}\}$ has the form of an $(n-1)$-dimensional vector subspace displaced a distance $\frac{|\alpha|}{\|\rho\|}$ along the vector ρ. Clearly if ρ_1 and ρ_2 are colinear vectors (*i.e.*, $\rho_2 = a\rho_1$ for some $a \in \Re$) then $\{\rho^{\perp}\} = \{(\rho_2)^{\perp}\}$.

Suppose that $\frac{\alpha_1\rho_1}{\|\rho\|^2} = \frac{\alpha_2\rho_2}{\|\rho\|^2}$, then both $H(\rho_1, \alpha_1)$ and $H(\rho_2, \alpha_2)$ contain the same point and are thus identical. Thus $H(\rho_1, \alpha_1) = H(a\rho_1, a\alpha_1) = H(\frac{\rho_1}{\|\rho_2\|}, \frac{\alpha_1}{\|\rho_1\|})$.

The hyperplane $H(\rho, \alpha)$ separates X into two *closed* half-spaces:

$$H_+(\rho, \alpha) = \{x \in X : \langle \rho, x \rangle \geq \alpha\}, \quad \text{and} \quad H_-(\rho, \alpha) = \{x \in X : \langle \rho, x \rangle \leq \alpha\}.$$

We shall also write

$$H_+^0(\rho, \alpha) = \{x \in X : \langle \rho, x \rangle > \alpha\}, \quad \text{and} \quad H_-^0(\rho, \alpha) = \{x \in X : \langle \rho, x \rangle < \alpha\}$$

for the open half-spaces formed by taking the interiors of $H_+(\rho, \alpha)$ and $H_-(\rho, \alpha)$, in the case $\rho \neq 0$.

Lemma 3.19 *Let Y be a non-empty compact convex subset of a finite dimensional real vector space X, and let x be a point in $X \backslash Y$. Then there is a hyperplane $H(\rho, \alpha)$ through a point $y_0 \in Y$ such that*

$$\langle \rho, x \rangle < \alpha = \langle \rho, y_0 \rangle \leq \langle \rho, y \rangle \quad \textit{for all } y \in Y.$$

Proof As in Lemma 3.17 let $f_x : Y \to \Re$ be the function $f_x(y) = \|x - y\|$, where $\| \ \|$ is the norm induced from the scalar product $\langle , \rangle$ in X.

By Lemma 3.18 there exists a point $y_0 \in Y$ such that $\|x - y_0\| = \inf\,(f_x, Y) = d(x, Y)$. Thus $\|x - y_0\| \leq \|x - y\|$ for all $y \in Y$. Now define $\rho = y_0 - x$ and $\alpha = \langle \rho, y_0 \rangle$. Then

$$\langle \rho, x \rangle = \langle \rho, y_0 \rangle - \langle \rho, (y_0 - x) \rangle = \langle \rho, y_0 \rangle - \|\rho\|^2 < \langle \rho, y_0 \rangle.$$

Suppose now that there is a point $y \in Y$ such that $\langle \rho, y_0 \rangle > \langle \rho, y \rangle$. By convexity, $w = \lambda y + (1 - \lambda) y_0 \in Y$, where λ belongs to the interval $(0, 1)$. But

$$\begin{aligned} \|x - y_0\|^2 - \|x - w\|^2 &= \|x - y_0\|^2 - \|x - \lambda y - y_0 + \lambda y_0\|^2 \\ &= 2\lambda \langle \rho, (y_0 - y) \rangle - \lambda^2 \|y - y_0\|^2. \end{aligned}$$

Now $\langle \rho, y_0 \rangle > \langle \rho, y \rangle$ and so, for sufficiently small λ, the right hand side is positive. Thus there exists a point w in Y, close to y_0, such that $\|x - y_0\| > \|x - w\|$. But this contradicts the assumption that y_0 is the nearest point in Y to x. Thus $\langle \rho, y \rangle \geq \langle \rho, y_0 \rangle$ for all $y \in Y$. Hence $\langle \rho, x \rangle < \alpha = \langle \rho, y_0 \rangle \leq \langle \rho, y \rangle$ for all $y \in Y$. □

Note that the point y_0 belongs to the hyperplane $H(\rho, \alpha)$, the set Y belongs to the closed half-space $H_+\langle \rho, \alpha \rangle$, while the point x belongs to the *open* half-space

$$H_-^0(\rho, \alpha) = \{z \in X : \langle \rho, z \rangle < \alpha\}.$$

Thus the hyperplane *separates* the point x from the compact convex set Y (see Fig. 3.16).

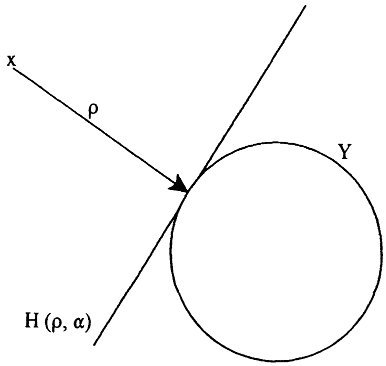


Fig. 3.16 A separating hyperplane

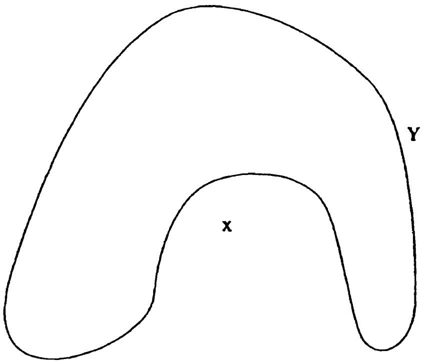


Fig. 3.17 Non-existance of a separating hyperplane

While convexity is necessary for the proof of this theorem, the compactness requirement may be weakened to Y being closed. Suppose however that Y is an open set. Then it is possible to choose a point x outside Y, which is, nonetheless, an accumulation point of Y such that $d(x, Y) = 0$.

On the other hand if Y is compact but not convex, then a situation such as Fig. 3.17 is possible. Clearly no hyperplane separates x from Y.

If A and B are two sets, and $H(\rho, \alpha) = H$ is a hyperplane such that $A \subseteq H_-(\rho, \alpha)$ and $B \subseteq H_+(\rho, \alpha)$ then say that H *weakly separates* A and B. If H is such that $A \subset H_-(\rho, \alpha)$ and $B \subset H_+(\rho, \alpha)$ then say H *strongly separates* A and B. Note in the latter case that it is necessary that $A \cap B = \Phi$.

In Lemma 3.19 we found a hyperplane $H(\rho, \alpha)$ such that $\langle \rho, x \rangle < \alpha$. Clearly it is possible to find $\alpha_- < \alpha$ such that $\langle \rho, x \rangle < \alpha_-$.

Thus the hyperplane $H(\rho, \alpha_-)$ *strongly* separates x from the compact convex set Y.

If Y is convex but not compact, then it is possible to find a hyperplane H that weakly separates X from Y.

We now extend this result to the separation of convex sets.

Separating Hyperplane Theorem *Suppose that A and B are two disjoint non-empty convex sets of a finite dimensional vector space X. Then there exists a hyperplane H that weakly separates A from B. If both A and B are compact then H strongly separates A from B.*

Proof Since A and B are convex the set $A - B = \{w \in X : w = a - b$ where $a \in A, b \in B\}$ is also convex.

To see this suppose $a_1 - b_1$ and $a_2 - b_2 \in A - B$. Then $\lambda(a_1 - b_1) + (1-\lambda)(a_2 - b_2) = [\lambda a_1 + (1-\lambda)a_2] + [\lambda b_1 + (1-\lambda)b_2] \in A - B$.

Now $A \cap B = \Phi$. Thus there exists no point in both A and B, and so $\underline{0} \notin A - B$. By Lemma 3.19, there exists a hyperplane $H(-\rho, \underline{0})$ weakly separating $\underline{0}$ from $A - B$, *i.e.*, $\langle \rho, \underline{0} \rangle \leq \langle \rho, w \rangle$ for all $w \in B - A$. But then $\langle \rho, a \rangle \leq \langle \rho, b \rangle$ for all $a \in A, b \in B$.

Choose $\alpha \in [\sup_{a\in A}\langle \rho, a\rangle, \inf_{b\in B}\langle \rho, b\rangle]$. In the case that A, B are non-compact, it is possible that

$$\sup_{a\in A}\langle \rho, a\rangle = \inf_{b\in B}\langle \rho, b\rangle.$$

Thus $\langle \rho, a\rangle \leq \alpha \leq \langle \rho, b\rangle$ and so $H(\rho, \alpha)$ weakly separates A and B.

Consider now the case when A and B are compact.

The function $\rho^* : X \to \Re$ given by $\rho^*(x) = \langle \rho, x\rangle$ is clearly continuous. By Lemma 3.16, since both A and B are compact, there exist points $\overline{a} \in A$ and $\overline{b} \in B$ such that

$$\langle \rho, \overline{a}\rangle = \sup_{a\in A}\langle \rho, a\rangle \quad \text{and} \quad \langle \rho, \overline{b}\rangle = \inf_{b\in B}\langle \rho, b\rangle.$$

If $\sup_{a\in A}\langle \rho, a\rangle = \inf_{b\in B}\langle \rho, b\rangle$, then $\langle \rho, \overline{a}\rangle = \langle \rho, \overline{b}\rangle$, and so $\overline{a} = \overline{b}$, contradicting $A \cap B = \Phi$.

Thus $\langle \rho, \overline{a}\rangle < \langle \rho, \overline{b}\rangle$ and we can choose α such that $\langle \rho, a\rangle \leq \langle \rho, \overline{a}\rangle < \alpha < \langle \rho, \overline{b}\rangle \leq \langle \rho, b\rangle$ for all a, b in A, B.

Thus $H(\rho, \alpha)$ strongly separates A and B. (See Fig. 3.18b.) □

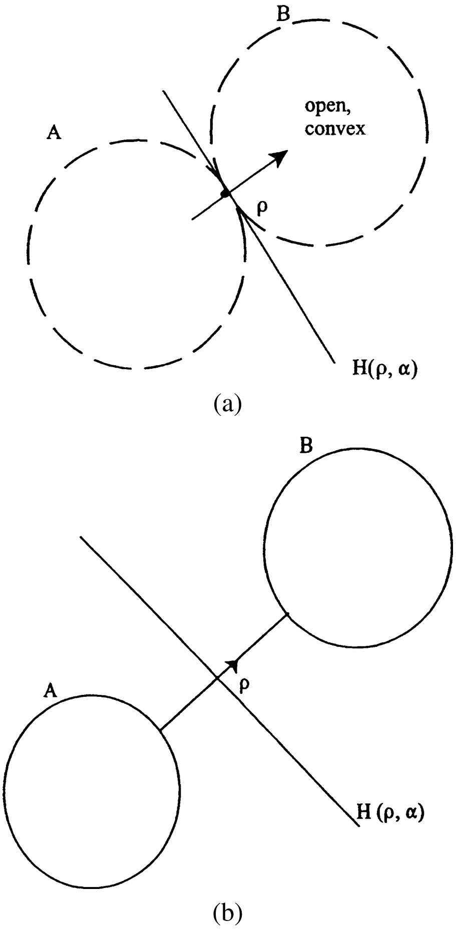
Example 3.5 Hildenbrand and Kirman (1976) have applied this theorem to find a price vector which supports a final allocation of economic resources. Consider a society $M = \{1, \ldots, m\}$ in which each individual i has an initial *endowment* $e_i = (e_{i1}, \ldots, e_{in}) \in \Re^n$ of n commodities. At price vector $p = (p_1, \ldots, p_n)$, the budget set of individual i is

$$B_i(p) = \bigl\{x \in \Re^n : \langle p, x\rangle \leq \langle p, e_i\rangle\bigr\}.$$

Each individual has a preference relation P_i on $\Re^n \times \Re^n$, and at the price vector p the *demand* $D_i(p)$ by i is the set

$$\bigl\{x \in B_i(p) : yP_ix \text{ for no } y \in B_i(p)\bigr\}.$$

Fig. 3.18 **a** Weak separation. **b** Strong separation



Let $f_i = (f_{i1}, \ldots, f_{in}) \in \Re^n$ be the final allocation to individual i, for $i = 1, \ldots, m$. Suppose there exists a price vector $p = (p_1, \ldots, p_n)$ with the property $(*)$ $x P_i f_i \Rightarrow \langle p, x \rangle > \langle p, e_i \rangle$. Then this would imply that $f_i \in B_i(p) \Rightarrow f_i \in D_i(p)$. If property $(*)$ holds at some price vector p, for each i, then $f_i \in D_i(p)$ for each i.

To show existence of such a price vector, let

$$\pi_i = P_i(f_i) - e_i \in \Re^n.$$

Here as before $P_i(f_i) = \{x \in \Re^n : x P_i f_i\}$. Suppose that there exists a hyperplane $H(p, 0)$ strongly separating 0 from π_i. In this case $0 < \langle p, x - e_i \rangle$ for all $x \in P_i(f_i)$. But this is equivalent to $\langle p, x \rangle > \langle p, e_i \rangle$ for all $x \in P_i(f_i)$.

Let $\pi = \text{Con}[\bigcup_{i \in N} \pi_i]$ be the convex hull of the sets π_i, $i \in M$. Clearly if $\underline{0} \notin \pi$ and there is a hyperplane $H(p, 0)$ strongly separating $\underline{0}$ from π, then p is a price vector which supports the final allocation $f_1, \ldots, f_n$.

3.5 Optimisation on Convex Sets

A key notion underlying economic theory is that of the maximisation of an objective function subject to one or a number of constraints. The most elementary case of such a problem is the one addressed by the Weierstrass Theorem: if $f : X \to \Re$ is a continuous function, and Y is a compact constraint set then there exists some point $\overline{y}$ such that $f(\overline{y}) = \sup(f, Y)$. Here $\overline{y}$ is a *maximum* point of f on Y.

Using the Separating Hyperplane Theorem we can extend this analysis to the optimisation of a convex preference correspondence on a compact convex constraint set.

3.5.1 Optimisation of a Convex Preference Correspondence

Suppose that Y is a compact, convex constraint set in $\Re^n$ and $P : \Re^n \to \Re^n$ is a preference correspondence which is convex (*i.e.*, $P(x)$ is convex for all $x \in \Re^n$). A choice for P on Y is a point $\overline{y} \in Y$ such that $P(\overline{y}) \cap Y = \Phi$.

We shall say that P is *non-satiated* in $\Re^n$ iff for no $y \in \Re^n$ is it the case that $P(y) = \Phi$. A sufficient condition to ensure non-satiation for example is the assumption of monotonicity, *i.e.*, $x > y$ (where as before this means $x_i > y_i$, for each of the coordinates $x_i, y_i, i = 1, \dots, n$) implies that $x \in P(y)$.

Say that P is *locally non-satiated* in $\Re^n$ iff for each $y \in \Re^n$ and any neighbourhood U_y of y in $\Re^n$, then $P(y) \cap U_y \neq \Phi$.

Clearly monotonicity implies local non-satiation implies non-satiation.

Suppose that y belongs to the interior of the compact constraint set Y. Then there is a neighbourhood U_y of y within Y. Consequently $P(y) \cap U_y \neq \Phi$ and so y cannot be a choice from Y. On the other hand, since Y is compact it is closed, and so if y belongs to the boundary δY of Y, it belongs to Y itself. By definition if $y \in \delta Y$ then any neighbourhood U_y of y intersects $\Re^n \backslash Y$. Thus when $P(y) \subset \Re^n \backslash Y$, y will be a choice from Y. Alternatively if y is a choice of P from Y, then y must belong to the boundary of P.

Lemma 3.20 *Let Y be a compact, convex constraint set in $\Re^n$ and let $P : \Re^n \to \Re^n$ be a preference correspondence which is locally non-satiated, and is such that, for all $x \in \Re^n$, $P(x)$ is open and convex. Then $\overline{y}$ is a choice of P from Y iff there is a hyperplane $H(p, \alpha)$ through $\overline{y}$ in Y which separates Y from $P(\overline{y})$ in the sense that*

$$\langle p, y\rangle \le \alpha = \langle p, \overline{y}\rangle < \langle p, x\rangle \quad \textit{for all } y \in Y \textit{ and all } x \in P(\overline{y}).$$

Proof Suppose that the hyperplane $H(p, \alpha)$ contains $\overline{y}$ and separates Y from $P(\overline{y})$ in the above sense. Clearly $\overline{y}$ must belong to the boundary of Y. Moreover $\langle p, y\rangle < \langle p, x\rangle$ for all $y \in Y, x \in P(\overline{y})$. Thus $Y \cap P(\overline{y}) = \Phi$ and so $\overline{y}$ is the choice of P from Y.

On the other hand suppose that $\overline{y}$ is a choice. Then $P(\overline{y}) \cap Y = \Phi$.

Moreover the local non-satiation property, $P(\overline{y}) \cap U_{\overline{y}} \neq \Phi$ for $U_{\overline{y}}$ a neighborhood of $\overline{y}$ in $\Re^n$, guarantees that $\overline{y}$ must belong to the boundary of Y. Since Y and $P(\overline{y})$ are both convex, there exists a hyperplane $H(p,\alpha)$ through $\overline{y}$ such that

$$\langle p, y\rangle \leq \alpha = \langle p, \overline{y}\rangle \leq \langle p, x\rangle$$

for all $y \in Y$, all $x \in P(\overline{y})$. But $P(\overline{y})$ is open, and so the last inequality can be written $\langle p, \overline{y}\rangle < \langle p, x\rangle$. □

Note that if either the constraint set, Y, or the correspondence P is such that $P(y)$ is strictly convex, for all $y \in \Re^n$, then the choice $\overline{y}$ is unique.

If $f : \Re^n \to \Re$ is a concave or quasi-concave function then application of this lemma to the preference correspondence $P : \Re^n \to \Re$, where $P(x) = \{y \in \Re^n : f(y) > f(x)\}$, characterises the maximum point $\overline{y}$ of f on Y. Here $\overline{y}$ is a maximum point of f on Y if $f(\overline{y}) = \sup(f, Y)$. Note that local non-satiation of P requires that for any point x in $\Re^n$, and any neighbourhood U_x of x in $\Re^n$, there exists $y \in U_x$ such that $f(y) > f(x)$.

The vector $p = (p_1, \dots, p_n)$ which characterises the hyperplane $H(p,\alpha)$ is called in economic applications the vector of *shadow prices*. The reason for this will become clear in the following example.

Example 3.6 As an example suppose that optimal use of $(n-1)$ different inputs $(x_1, \dots, x_{n-1})$ gives rise to an output y, say, where $y = y(x_1, \dots, x_{n-1})$. Any n vector $(x_1, \dots, x_{n-1}, x_n)$ is *feasible* as long as $x_n \leq y(x_1, \dots, x_{n-1})$. Here x_n is the output. Write $g(x_1, \dots, x_{n-1}, x_n) = y(x_1, \dots, x_{n-1}) - x_n$. Then a vector $x = (x_1, \dots, x_{n-1}, x_n)$ is feasible iff $g(x) \geq 0$.

Suppose now that $y = y(x_1, \dots, x_{n-1})$ is a concave function in $x_1, \dots, x_{n-1}$. Then clearly the set $G = \{x \in \Re^n : g(x) \geq 0\}$ is a convex set. Now let $\pi(x_1, \dots, x_{n-1}, x_n) = -\sum p_i x_i + p_n x_n$ be the *profit function* of the producer, when prices for inputs and outputs are given *exogenously* by $(-p_1, \dots, -p_{n-1}, p_n)$. Again let $P : \Re^n \to \Re^n$ be the preference correspondence $P(x) = \{z \in \Re^n : \pi(z) > \pi(x)\}$. Since for each x, $P(x)$ is convex, and locally non-satiated, there is a choice $\overline{x}$ and a hyperplane $H(\rho,\alpha)$ separating $P(\overline{x})$ from G.

Indeed it is clear from the construction that $P(\overline{x}) \subset H^0_+(\rho,\alpha)$ and $G \subset H_-(\rho,\alpha)$.

Moreover the hyperplane $H(\rho,\alpha)$ must coincide with the set of points $\{x \in \Re^n : \pi(x) = \pi(\overline{x})\}$. Thus the hyperplane $H(\rho,\alpha)$ has the form

$$\left\{x \in \Re^n : \langle p, x\rangle = \pi(\overline{x})\right\}$$

and so may be written $H(p, \pi(x))$. Note that the intercept on the x_n axis is $\frac{\pi(\overline{x})}{p_n}$ while the distance of the hyperplane from the origin is $\frac{\pi(\overline{x})}{\|p\|} = \frac{\pi(\overline{x})}{\sqrt{\sum p_i^2}}$. Thus the intercept gives the profit measured in units of output, while the distance from the

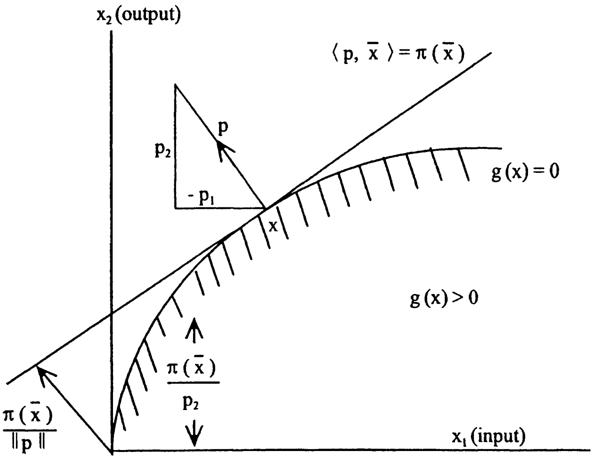


Fig. 3.19 Separation from a feasible set

origin of the profit hyperplane gives the profit in terms of a normalized price vector ($\|p\|$).

Figure 3.19 illustrates the situation with one input (x_1) and one output (x_2).

Precisely the same analysis can be performed when optimal production is characterised by a general production function $F : \Re^n \to \Re$.

Here $x_1, \dots, x_m$ are inputs, with prices $-p_1, \dots, -p_m$ and $x_{m+1}, \dots, x_n$ are outputs with prices $p_{m+1}, \dots, p_n$. Let $p = (-p_1, \dots, -p_m, -p_{m+1}, \dots, p_n) \in \Re^n$.

Define F so that a vector $x \in \Re^n$ is *feasible* iff $F(x) \geq 0$. Note that we also need to restrict all inputs and outputs to be non-negative. Therefore define

$$\Re^n_+ = \{x : x_i \geq 0 \text{ for } i = 1, \dots, n\}.$$

Assume that the feasible set (or *production set*)

$$G = \left\{x \in \Re^n_+ : F(x) \geq 0\right\} \quad \text{is convex.}$$

As before let $P : \Re^n \to \Re^n$ where $P(x) = \{z \in \Re^n : \pi(z) > \pi(x)\}$. Then the point $\overline{x}$ is a choice of P from G iff $\overline{x}$ maximises the profit function

$$\pi(x) = \sum_{j=1}^{n-m} p_{m+j}\, x_{m+j} - \sum_{j=1}^{m} p_j x_j.$$

By the previous example $\overline{x}$ is a choice iff the hyperplane $H(p, \pi(\overline{x}))$ separates $P(\overline{x})$ and G: *i.e.*, $P(\overline{x}) \subset H^0_+(p, \pi(\overline{x}))$ and $G \subset H_-(p, \pi(\overline{x}))$.

In the next chapter we shall use this optimality condition to characterise the choice $\overline{m}$ more fully in the case when F is "smooth".

Example 3.7 Consider now the case of a *consumer* maximising a preference correspondence $P : \Re^n \to \Re^n$ subject to a budget constraint $B(p)$ which is dependent on a set of exogenous prices $p_1, \dots, p_n$.

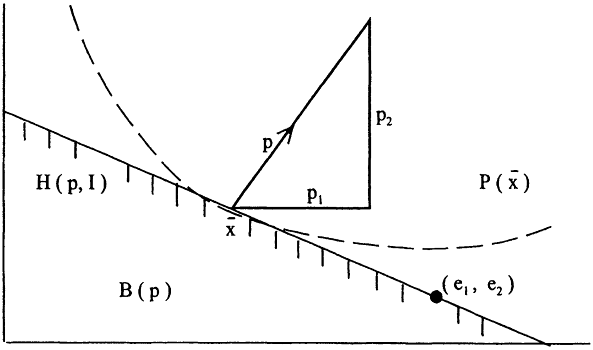


Fig. 3.20 Example 3.7

For example the consumer may hold an initial set of endowments $(e_1, \dots, e_n)$, so let

$$I = \sum_{i=1}^{n} p_i e_i = \langle p, e \rangle.$$

The budget set is then

$$B(p) = \left\{ x \in \Re^n_+ : \langle p, x \rangle \leq I \right\},$$

where for convenience we assume the consumer only buys a non-negative amount of each commodity. Suppose that P is monotonic, and $P(x)$ is open, convex for all $x \in \Re^n$. As before $\overline{x}$ is a choice from $B(p)$ iff there is a hyperplane $H(\rho, \alpha)$ separating $P(\overline{x})$ from $B(p)$.

Under these conditions the choice must belong to the upper boundary of $B(p)$ and so satisfy $(p, \overline{x}) = (p, e) = I$. Thus the hyperplane has the form $H(p, I)$, and so the optimality condition is $P(\overline{x}) \subset H^0_+(p, I)$ and $B(p) \subset H_-(p, I)$; *i.e.*, $(p, x) \leq I = (p, \overline{x}) < (p, y)$ for all $x \in B(p)$ and all $y \in P(\overline{x})$.

Figure 3.20 illustrates the situation with two commodities x_1 and x_2.

In the next chapter we use this optimality condition to characterise a choice when preference is given by a smooth utility function $f : \Re^n \to \Re$.

In the previous two examples we considered

1. optimisation of a profit function, which is determined by exogenously given prices, subject to a fixed production constraint, and
2. optimisation of a fixed preference correspondence, subject to a budget constraint, which is again determined by exogenous prices.

Clearly at a given price vector each producer will "demand" a certain input vector and supply a particular output vector, so that the combination is his choice in the environment determined by p. In the same way a consumer will respond to a price vector p by demanding optimal amounts of each commodity, and possibly supplying other commodities such as labor, or various endowments. In contrast to Example 3.7, regard all prices as positive, and consider a commodity x_j demanded by an agent i as an input to be negative, and positive when supplied as an output.

Let $\overline{x}_{ij}(p)$ be the optimal demand or supply of commodity j by agent i at the price vector p, with m agents and n commodities, then *market equilibrium* of supply and demand in the economy occurs when $\sum_{i=1}^{m}\sum_{j=1}^{n} \overline{x}_{ij}(p) = 0$. A price vector which leads to market equilibrium in demand and supply is called an *equilibrium price vector.*

Example 3.8 To give a simple example, consider two agents. The first agent controls a machine which makes use of labor, x, to produce a single output y. Regard $x \in (-\infty, 0]$ and consider a price vector $p \in \Re^2_+$, where $p = (w, r)$ and w is the price of labor, and r the price of the output. An output is feasible from Agent One iff $F(x, y) \geq 0$.

Agent Two is the only supplier of labor, but is averse to working. His preference is described by a quasi-concave utility function $f : \Re^2 \to \Re$ and we restrict attention to a domain

$$D = \{(x, y) \in \Re^2 : x \leq 0, y \geq 0\}.$$

Assume that f is monotonic, *i.e.*, if $x_1 < x_2$ and $y_1 < y_2$ then $f(x_1, y_1) < f(x_2, y_2)$. The budget constraint of Agent Two at (w, r) is therefore

$$B(w, r) = \{(x_1, y_2) \in D : ry_2 \leq w|x|\},$$

where $|x|$ is the quantity of labor supplied, and y_2 is the amount of commodity y consumed. Profit for Agent One is $\pi(x, y) = ry - wx$, and we shall assume that this agent then consumes an amount $y_1 = \frac{\pi(x,y)}{r}$ of commodity y.

For equilibrium of supply and demand at prices $(\overline{w}, \overline{r})$

1. $\overline{y} = \overline{y}_1 + \overline{y}_2$;
2. $(\overline{x}, \overline{y})$ maximises $\pi(x, y) = \overline{r}y - \overline{w}x$ subject to $F(x, y) \geq 0$;
3. $(\overline{x}, \overline{y}_2)$ maximises $f(x, y_2)$ subject to $\overline{r}y_2 = \overline{w}x$.

At any point $(x, y) \in D$, and vector (w, r) define

$$P(x, y) = \{(x', y') \in D : f(x', y' - y_1) > f(x, y - y_1)\}$$

where as above $y_1 = \frac{ry - wx}{r}$ is the amount of commodity y consumed by the producer. Thus $P(x, y)$ is the preference correspondence of Agent One displaced by the distance y_1 up the y-axis.

Figure 3.21 illustrates that it is possible to find a vector (w, r) such that $H = H((w, r), \pi(\overline{x}, \overline{y}))$ separates $P(\overline{x}, \overline{y})$ and the production set

$$G = \{(x, y) \in D : F(x, y) \geq 0\}.$$

As in Example 3.6, the intersect of H with the y-axis is $\overline{y}_1 = \frac{\pi(\overline{x}, \overline{y})}{r}$, the consumption of y by Agent One.

The hyperplane H is the set

$$\{(x, y) : ry + wx = r\overline{y}_1\}.$$

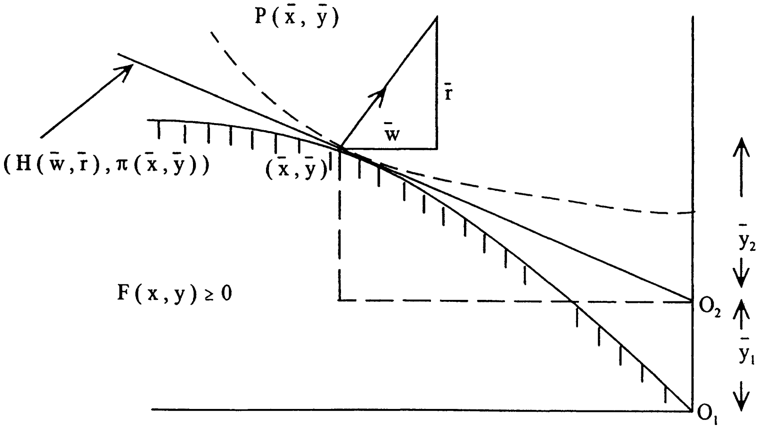


Fig. 3.21 Example 3.8

Hence for all $(x, y) \in H$, $(x, y - \overline{y}_1)$ satisfies $r(y - \overline{y}_1) + wx = 0$. Thus H is the boundary of the second agent's budget set $\{(x, y_2) : ry_2 + wx = 0\}$ displaced up the y-axis by $\overline{y}_1$. Consequently $\overline{y} = \overline{y}_1 + \overline{y}_2$ and so $(\overline{x}, \overline{y})$ is a market equilibrium.

Note that the hyperplane separation satisfies:

$$py - rx \leq \pi(\overline{x}, \overline{y}) < ry' - wx'$$

for all $(x, y) \in G$, and all $(x', y') \in P(\overline{x}, \overline{y})$.

As above $(x', y') \in P(\overline{x}, \overline{y})$ iff $f(x', y' - \overline{y}_1) > f(\overline{x}, \overline{y}_2)$. So the right hand side implies that if $(x', y') \in P(\overline{x}, \overline{y})$, then $ry' - wx' > \pi(\overline{x}, \overline{y})$.

Since $y'_2 = y' - \overline{y}_1, ry'_2 - wx' > 0$ or $(x', y' - \overline{y}_1) \in H^0_+((\overline{w}, \overline{r}), 0)$ and so $(x', y' - \overline{y}_1)$ is infeasible for Agent Two.

Finally $(\overline{x}, \overline{y})$ maximises $\pi(x, y)$ subject to $(x, y) \in G$, and so $(\overline{x}, \overline{y}) \in G$ and so $(\overline{x}, \overline{y})$ results from optimising behaviour by both agents at the price vector (w, r).

As this example illustrates it is possible to show existence of a market equilibrium in economies characterised by compact convex constraint sets, and convex preference correspondences. To do this in a general context however requires more powerful mathematical tools, which we shall introduce in Sect. 3.8 below. Before this however we consider one further application of the hyperplane separation theorem to a situation where we wish to optimise a concave objective function subject to a number of concave constraints.

3.6 Kuhn-Tucker Theorem

Here we consider a family of constraints in $\Re^n$. Let $g = (g_1, \ldots, g_m) : \Re^n \to \Re^m$. As before let $\Re^m_+ = \{(y_1, \ldots, y_m) : y_i \geq 0$ for $i = 1, \ldots, n\}$. A point x is *feasible* iff $x \in \Re^n_+$ and $g(x) \in \Re^m_+$ (*i.e.*, $g_i(x) \geq 0$ for $i = 1, \ldots, m$). Let $f : \Re^n \to \Re$ be the objective function. Say that $x^* \in \Re^n$ is an *optimum* of the constrained optimisation problem (f, g) iff x^* solves the problem: maximise $f(x)$ subject to the feasibility constraint $g(x) \in \Re^m_+$, $x \in \Re^n_+$.

Call the problem (f, g) *solvable* iff there is some $x \in \Re^n_+$ such that $g_i(x) > 0$ for $i = 1, \dots, m$. The *Lagrangian* to the problem (f, g) is:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) = f(x) + \big(\lambda, g(x)\big)$$

where $x \in \Re^n_+$ and $\lambda = (\lambda_1, \dots, \lambda_m) \in \Re^m_+$. The pair $(x^*, \lambda^*) \in \Re^{n+m}_+$ is called a *global saddle point* for the problem (f, g) iff

$$L\big(x, \lambda^*\big) \le L\big(x^*, \lambda^*\big) \le L(x^*, \lambda)$$

for all $x \in \Re^n_+$, $\lambda \in \Re^m_+$.

Kuhn-Tucker Theorem 1 *Suppose $f, g_1, \dots, g_m : \Re^n \to \Re^m$ are concave functions for all $x \in \Re^n_+$. Then if x^* is an optimum to the solvable problem $(f, g) : \Re^n \to \Re^{m+1}$ there exists a $\lambda^* \in \Re^m_+$ such that (x^*, λ^*) is a saddle point for (f, g).*

Proof Let $A = \{y \in \Re^{m+1} : \exists\ x \in \Re^n_+ : y \le (f, g)(x)\}$. Here $(f, g)(x) = (f(x), g(x), \dots, g_m(x))$. Thus $y = (y_1, \dots, y_{m+1}) \in A$ iff $\exists x \in \Re^n_+$ such that

$$\begin{aligned} y_1 &\le f(x) \\ y_{j+1} &\le g_j(x) \quad \text{for } j = 1, \dots, m. \end{aligned}$$

Let x^* be an optimum and

$$B = \big\{z = (z_1, \dots, z_{m+1}) \in \Re^m : z_1 > f\big(x^*\big) \text{ and } (z_2, \dots, z_{m+1}) > 0\big\}.$$

Since f, g are concave, A is convex. To see this suppose $y_1, y_2 \in A$. But since both f and g are concave $af(x_1) + (1-a)f(x_2) \le f(ax_1 + (1-a)x_2)$ and similarly for g, for any $a \in [0, 1]$. Thus

$$ay_1 + (1-a)y_2 \le a(f, g)(x_1) + (1-a)(f, g)(x_2) \le (f, g)\big(ax_1 + (1-a)x_2\big).$$

Since $x_1, x_2 \in \Re^n_+$, $ax_1 + (1-a)x_2 \in \Re^n_+$, and so $ay_1 + (1-a)y_2 \in \Re^n_+$.

Clearly B is convex, since $az_{11} + (1-a)z_{12} > f(x^*)$ if $a \in [0, 1]$ and $z_{11}, z_{12} > f(x^*)$.

To see $A \cap B = \Phi$, consider $x \in \Re^n$ such that $g(x) < 0$. Then $(y_2, \dots, y_{m+1}) \le g(x) < \underline{0} \le (z_2, \dots, z_{m+1})$.

If $g(x) \in \Re^m_+$ then x is feasible. In this case $y_1 \le f(x) \le f(x^*) < z_1$.

By the separating hyperplane theorem, there exists $(p_1, \ldots, p_{m+1}) \in \Re^{m+1}$ and $\alpha \in \Re$ such that $H(p, \alpha) = \{w \in \Re^{m+1} : \sum_{j=1}^{m+1} w_j p_j = \alpha\}$ separates A and B, *i.e.*, $\sum_{j=1}^{m+1} p_j y_j \leq \alpha \leq \sum_{j=1}^{m+1} p_j z_j$ for any $y \in A$ and $z \in B$.

Moreover $p \in \Re_+^{m+1}$. By the definition of A, for any $y \in A$, $\exists\ x \in \Re_+^n$ such that $y \leq (f, g)(x)$.

Thus for any $x \in \Re_+^n$,

$$p_1 f(x) + \sum_{j=2}^{m} p_j g_j(x) \leq \sum_{j=1}^{m+1} p_j z_j.$$

Since $(f(x^*), 0, \ldots, 0)$ belongs to the boundary of B,

$$p_1 f(x) + \sum_{j=2}^{m} p_j g_j(x) \leq p_1 f\bigl(x^*\bigr).$$

Suppose $p_1 = 0$. Since $p \in \Re_+^{m+1}$, there exists $p_j > 0$.

Since the problem is solvable, $\exists\ x \in \Re_+^n$ such that $g_j(x) > 0$. But this gives $\sum_{j=2}^{m} p_j g_j(x) > 0$, contradicting $p_1 = 0$. Hence $p_1 > 0$.

Let $\lambda_j^* = \frac{p_{j+1}}{p_1}$ for $j = 1, \ldots, m$.

Then $L(x, \lambda^*) = f(x) + \sum_{j=1}^{m} \lambda_j^* g_j(x) \leq f(x^*)$ for all $x \in \Re_+^n$, where $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \Re_+^m$. Since x^* is feasible, $g(x^*) \in \Re_+^m$, and $\langle \lambda^*, g(x^*) \rangle \geq 0$. But $f(x^*) + \langle \lambda^*, g(x^*) \rangle \leq f(x^*)$ implying $\langle \lambda^*, g(x^*) \rangle \leq 0$. Thus $\langle \lambda^*, g(x^*) \rangle = 0$. Clearly $\langle \lambda, g(x^*) \rangle \geq 0$ if $\lambda \in \Re_+^m$. Thus $L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda)$ for any $x \in \Re_+^n, \lambda \in \Re_+^m$. □

Kuhn-Tucker Theorem 2 *If the pair (x^*, λ^*) is a global saddle point for the problem (f, g), then x^* is an optimum.*

Proof By the assumption

$$L\bigl(x, \lambda^*\bigr) \leq L\bigl(x^*, \lambda^*\bigr) \leq L\bigl(x^*, \lambda\bigr)$$

for all $x \in \Re_+^n, \lambda \in \Re_+^m$.

Choose $\lambda = (\lambda_1^*, \ldots, 2\lambda_i^*, \ldots, \lambda_m^*)$.

Then $L(x^*, \lambda) \geq L(x^*, \lambda^*)$ implies $g_i(x^*)\lambda_i^* \geq 0$. If $\lambda_i^* \neq 0$ then $g_i(x^*) \geq 0$, and so $\langle \lambda^*, g(x^*) \rangle \geq 0$.

On the other hand, $L(x^*, \lambda^*) \leq L(x^*, 0)$ implies $\langle \lambda^*, g(x^*) \rangle \leq 0$. Thus $\langle \lambda^*, g(x^*) \rangle = 0$. Hence $f(x) + \langle \lambda^*, g(x) \rangle \leq f(x^*) \leq f(x^*) + \langle \lambda, g(x^*) \rangle$. If x is feasible, $g(x) \geq 0$ and so $\langle \lambda^*, g(x) \rangle \geq 0$. Thus $f(x) \leq f(x) + \langle \lambda^*, g(x) \rangle \leq f(x^*)$ for all $x \in \Re_+^n$, whenever $g(x) \in \Re_+^m$. Hence x^* is an optimum for the problem (f, g). Note that for a concave optimisation problem (f, g), x^* is an optimum for (f, g) iff (x^*, λ^*) is a global saddle point for the Lagrangian $L(x, \lambda), \lambda \in \Re_+^m$.

Moreover (x^*, λ^*) are such that $\langle \lambda^*, g(x^*) \rangle$ *minimises* $\langle \lambda, g(x) \rangle$ for all $\lambda \in \Re^m_+$, $x \in \Re^n_+$, $g(x) \in \Re^m_+$. Since $\langle \lambda^*, g(x^*) \rangle = 0$ this implies that if $g_i(x^*) > 0$ then $\lambda^*_i = 0$ and if $\lambda^*_i > 0$ then $g_i(x^*) = 0$.

The coefficients $(\lambda^*_1, \ldots, \lambda^*_m)$ are called *shadow prices*. If the optimum is such that $g_i(x^*) > 0$ then the shadow price $\lambda^*_1 = 0$. In other words if the optimum does not lie in the boundary of the ith constraint set $\partial B_i = \{x : g_i(x) = 0\}$, then this constraint is *slack*, with zero shadow price. If the shadow price is non zero then the constraint cannot be slack, and the optimum lies on the boundary of the constraint set.

In the case of a single constraint, the assumption of non-satiation was sufficient to guarantee that the constraint was not slack.

In this case

$$f(x) + \frac{p_2}{p_1} g(x) \le f\left(x^*\right) \le f\left(x^*\right) + \lambda g\left(x^*\right)$$

for any $x \in \Re_+$, and $\lambda \in \Re_+$, where $\frac{p_2}{p_1} > 0$.

The Kuhn-Tucker theorem is of particular use when objective and constraint functions are smooth. In this case the Lagrangean permits computation of the optimal points of the problem. We deal with these procedures in Chap. 4. □

3.7 Choice on Compact Sets

In Lemma 3.9 we showed that when a preference relation is acyclic and lower demicontinuous (LDC) on a compact space, X, then P admits a choice. Lemma 3.20 gives a different kind of result making use of compactness and convexity of the constraint set, and convexity and openness of the image $P(x)$ at each point x. We now present a class of related results using compactness, convexity and lower demicontinuity. These results are essentially all based on the Brouwer Fixed Point Theorem (Brouwer 1912) and provide the technology for proving existence of a market equilibrium. We first introduce the notion of retraction and contractibility, and then show that any continuous function on the closed ball in $\Re^n$ admits a fixed point.

Definition 3.8 Let X be a topological space.

(i) Suppose Y is a subset of X. Say Y has the *fixed point property* iff whenever $f : Y \to X$ is a continuous function (with respect to the induced topology on Y) such that $f(Y) = \{f(y) \in X : y \in Y\} \subset Y$, then there exists a point $\overline{x} \in Y$ such that $f(\overline{x}) = \overline{x}$.

(ii) If Y is a *topological space*, and there exists a bijective continuous function $h : X \to Y$ such that h^{-1} is also continuous then h is called a *homeomorphism* and X, Y are said to be homeomorphic (see Sect. 1.2.3 for the definition of bijective).

(iii) If Y is a topological space and there exist continuous functions $g : Y \to X$ and $h : X \to Y$ such that $h \circ g : Y \to X \to Y$ is the identity on Y, then h is called an r-map (for g).

(iv) If $Y \subset Z \subset X$ and $g = \mathrm{id} : Y \to Y$ is the (continuous) identity map and $h : Z \to Y$ is an r-map for g then h is called a *retraction* (of Z on Y) and Y is called a *retract* of Z.

(v) If $Y \subset X$ and there exists a continuous function $f : Y \times [0,1] \to Y$ such that $f(y,0) = y$ (so $f(\ ,0)$ is the identity on Y) and $f(y,1) = y_0 \in Y$, for all $y \in Y$, then Y is said to be *contractible*.

(vi) Suppose that $Y \subset Z \subset X$ and there exists a continuous function $f : Z \times [0,1] \to Z$ such that $f(z,0) = z\ \forall z \in Z$, $f(y,t) = y\ \forall y \in Y$ and $f(z,1) = h(z)$ where $h : Z \to Y$ is a retraction, then f is called a strong retraction of Z on Y, and Y is called a *deformation retract* of Z.

To illustrate the idea of contractibility, observe that the closed *Euclidean* ball in $\Re^n$ of radius ξ, centered at x, namely

$$B^n = \mathrm{clos}\big(B_d(x,\xi)\big) = \big\{y \in \Re^n : d(x,y) \le \xi\big\},$$

is obviously strictly convex and compact. Moreover the center, $\{x\}$, is clearly a deformation retract of B^n. To see this let $f(y,t) = (1-t)y + tx$ for $y \in B^n$. Clearly $f(y,1) = h(y) = \{x\}$ so $h : B^n \to \{x\}$ and $h(x) = x$. Since f is continuous, this also implies that B^n is contractible. A continuous function such as $f : B^n \times [0,1] \to B^n$, or more generally $f : Z \times [0,1] \to Z$, is often called a *homotopy* and written $f_t : Z \to Z$ where $f_t(z) = f(z,t)$. The homotopy f_t between the identity and the retraction f_1 of Z on Y means that Z and Y are "topologically" equivalent (in some sense). Thus the ball B^n and the point $\{x\}$ are topologically equivalent. More generally, if Y is contractible and there is a strong retraction of Z on Y, then Z is also contractible.

Lemma 3.21 *Let X be a topological space. If Y is contractible and $Y \subset Z \subset X$ such that Y is a deformation retract of Z, then Z is contractible.*

Proof Let $g : Z \times [0,1] \to Z$ be the strong retraction of Z on Y, and let $f : Y \times [0,1] \to Y$ be the contraction of Y to $y \in Y$. Define

$$r : Z \times \left[0, \frac{1}{2}\right] \to Z \text{ by } r(x,t) = g(z,2t)$$

$$r' : Z \times \left[\frac{1}{2}, 1\right] \to Z \text{ by } r'(z,t) = f\big(g(z,1), 2t-1\big).$$

To see this define a contraction s of Z onto y_0. Note that $r(z,\frac{1}{2}) = r'(z,\frac{1}{2})$, since $g(z,1) = f(g(z,1),0)$. This follows because g is a strong retraction and so $g(z,1) \in Y$, $g(y,1) = f(y,0) = y$ if $y \in Y$. Clearly $s : Z \times [0,1] \to Z$ (defined by $s(z,t) = r(z,t)$ if $t < \frac{1}{2}$, $s(z,t) = r'(z,t)$ if $t \ge \frac{1}{2}$) is continuous and is the identity at $t = 0$. Finally if $t = 1$, then $s(z,1) = f(g(z,1),1) = y_0$. □

Lemma 3.22 *If Z is a (non-empty) compact, convex set in $\Re^n$, then it is contractible.*

Proof For any x in the interior of Z there exists some $\xi > 0$ such that $B^n = \text{clos}(B_d(x, \xi))$ is contained in Z. (Indeed ξ can be chosen so that B^n is contained in the interior of Z.) As observed in Example 3.3, the closed ball, B^n, is both compact and strictly convex. By Lemma 3.17, the distance function $d(z, -) : B^n \to \Re$ is continuous for each $z \in Z$, and so there exists a point $\overline{y}(z)$ in B^n, say, such that $d(z, \overline{y}(z)) < d(z, y)) \; \forall y \in B^n$. Then $d(z, \overline{y}(z)) = d(z, B^n)$, the distance between z and B^n. Indeed $d(z, \overline{y}(z)) = 0$ iff $z \in B^n$. Moreover for each $z \in Z$, $\overline{y}(z)$ is unique. Define the function $f : Z \times [0, 1] \to Z$ by $f(z, t) = tz + (1 - t)\overline{y}(z)$. Since $\overline{y}(z) \in B^n \subset Z$ for each z, and Z is convex, $f(z, t) \in Z$ for all $t \in (0, 1]$. Clearly if $z \in B^n$ then $f(z, t) = z$ for all $t \in [0, 1]$ and $f(-, 1) = h : Z \to B^n$ is a retraction. Thus f is a strong retraction, and B^n is a deformation retract of Z. By Lemma 3.21, Z is contractible. □

Note that compactness of Z is not strictly necessary for the validity of this lemma.

Lemma 3.23 *If Z is contractible to z_0 and $Y \subset Z$ is a retract of Z by $h : Z \to Y$ then Y is contractible to $h(z_0)$.*

Proof Let $f : Z \times [0, 1] \to Z$ be the contraction of Z on z_0, and let $h : Z \to Y$ be the retraction. Clearly $h \circ f : Z \times [0, 1] \to Z \to Y$. If $y \in Y$, then $f(y, 0) = y$ and $h(y) = y$ (because h is a retraction). Thus $h \circ f(y, 0) = y$. Moreover $f(z, 1) = z_0 \; \forall \in Z$, so $h \circ f(z, 1) = h(z_0)$. □

Clearly being a deformation retract of Z is a much stronger property than being a retract of Z. Both of these properties are useful in proving that any compact convex set has the fixed point property, and that the sphere is neither contractible nor has the fixed point property.

Remember that the sphere of radius ξ in $\Re^n$, with center x, is

$$S^{n-1} = \text{Boundary}\big(\text{clos}\big(B_d(x, \xi)\big)\big) = \big\{y \in \Re^n : d(x, y) = \xi\big\}.$$

Now let $x_0 \in S^{n-1}$ be the north pole of the sphere. We shall give an intuitive argument why $D = S^{n-1} \backslash \{x_0\}$ is contractible, but S^{n-1} is not contractible.

Example 3.9 Let $D = S^{n-1} \backslash \{x_0\}$ and let Z be a copy of D which is flattened at the south pole. Let D_0 be the flattened disc round the South Pole, x_S. Clearly D_0 is homeomorphic to an $(n - 1)$ dimensional ball B^{n-1} centered on x_S. Then Z is homeomorphic to the object $D_0 \times \; [0, 1)$. There is obviously a strong retraction of D onto D_0. This retraction may be thought of as the function that moves any point $z \in S^{n-1} \backslash \{x_0\}$ down the lines of longitude to D_0. Since D_0 is compact, convex it is contractible to x_S and thus, by Lemma 3.20, there is a contraction $f : D \times [0, 1] \to D$ to x_S.

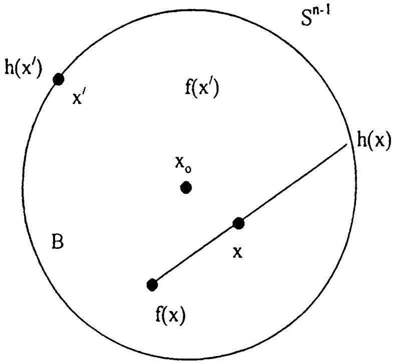


Fig. 3.22 The retraction of B^n on S^{n-1}

To indicate why S^{n-1} cannot be contractible, let us suppose without loss of generality, that $g : S^{n-1} \times [0, 1] \to S^{n-1}$ is a contraction of S^{n-1} to x_S, and that g extends the contraction $f : D \times [0, 1] \to D$ (*i.e.*, $g(z, t) = f(z, t)$ whenever $z \in D$). Now $f(y, t)$ maps each point $y \in D$ to a point further south on the longitudinal line through y. If we require $g(y, t) = f(y, t)$ for each $y \in \mathcal{D}$, and we require g to be continuous at $(x_0, 0)$ then it is necessary for $g(x_0, t)$ to be an equatorial circle in S^{n-1}. In other words if g is a function it must fail continuity at $(x_0, 0)$. While this is not a formal argument, the underlying idea is clear: The sphere S^{n-1} contains a hole, and it is topologically different from the ball B^n.

Brouwer's Theorem *Any compact, convex set in $\Re^n$ has the fixed point property.*

Proof We prove first that the ball $B^n \equiv B$ has the fixed point property. Suppose otherwise: that is there exists a continuous function $f : B \to B$ with $x \neq f(x)$ for all $x \in B$.

Since $f(x) \neq x$, construct the arc from $f(x)$ to x and extend this to the boundary of B. Now label the point where the arc and the boundary of B intersect as $h(x)$. Since the boundary of B is S^{n-1}, we have constructed a function $h : B \to S^{n-1}$. It is easy to see that h is continuous (because f is continuous). Moreover if $x' \in S^{n-1}$ then $h(x') \in S^{n-1}$. Since $S^{n-1} \subset B$, it is clear that $h : B \to S^{n-1}$ is a retraction. (See Fig. 3.22.) By Lemma 3.23, the contractibility of B to its center, x_0 say, implies that S^{n-1} is contractible to $h(x_0)$. But Example 3.9 indicates that S^{n-1} is not contractible. The contradiction implies that any continuous function $f : B \to B$ has a fixed point.

Now let Y be any compact convex set in $\Re^n$. Then there exists for some ξ and $y_0 \in Y$, a closed ξ-ball, centered at y_0, such that Y is contained in $B = \text{clos}(B_d(y_0, \xi))$. As in the proof of Lemma 3.22, there exists a strong retraction $g : B \times [0, 1] \to B$; so Y is a deformation retract of B. (See Fig. 3.23.) In particular $g(-, 1) = h : B \to Y$ is a retraction. If $f : Y \to Y$ is continuous, then $f \circ h : B \to Y \to Y \subset B$ is continuous and has a fixed point. Since the image of $f \circ h$ is in Y, this fixed point, y_1, belongs to Y. Hence $f \circ h(y_1) = y_1$ for some $y_1 \in Y$. But $h = \text{id}$ (the identity) on Y, so $h(y_1) = y_1$ and thus $y_1 = f(y_1)$. Consequently $y_1 \in Y$ is a fixed point of f. Thus Y has the fixed point property. □

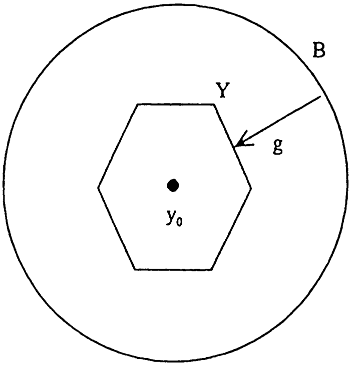


Fig. 3.23 The strong retraction of B on Y

Example 3.10 The standard compact, convex set in $\Re^n$ is the $(n-1)$-simplex Δ^{n-1} defined by

$$\Delta = \Delta^{n-1} = \left\{ x = (x_1, \dots, x_n) \in \Re^n : \sum_{i=1}^{n} x_i = 1, \text{ and } x_i \geq 0 \; \forall \, i \right\}.$$

Δ has n vertices $\{x_i^0\}$, where $x_i^0 = (0, \dots, 1, \dots, 0)$ with a 1 in the ith entry. An *edge* between x_i^0 and x_j^0 is the arc $\langle\langle x_i^0, x_j^0 \rangle\rangle$ or convex set of the form

$$\left\{ x \in \Re^n : x = \lambda x_i^0 + (1-\lambda) x_j^0 \text{ for } \lambda \in [0,1] \right\}.$$

An s-dimensional face of Δ is simply the convex hull of $(s+1)$ different vertices. Note in particular that there are n different $(n-2)$ dimensional faces. Such a face is opposite the vertex x_i^0, so we may label this face Δ_i^{n-2}. These n different faces have empty intersection. However any subfamily, $\mathcal{F}$, of this family of n faces (where $\mathcal{F}$ has cardinality at most $(n-1)$), does have a non-empty intersection. In fact if $\mathcal{F}$ has cardinality $(n-1)$ then the intersection is a vertex.

Brouwer's Theorem allows one to derive further results on the existence of choice.

Lemma 3.24 *Let $Q : \Delta \to \Re^n$ be an LDC correspondence from the $(n-1)$ dimensional simplex to $\Re^n$, such that $Q(x)$ is both non-empty and convex, for each $x \in \Delta$. Then there exists a continuous selection, f, for Q, namely a continuous function $f : \Delta \to \Re^n$ such that $f(x) \in Q(x)$ for all $x \in \Delta$.*

Proof Since $Q(x) \neq \Phi$, $\forall x \in \Delta$, then for each $x \in \Delta, x \in Q^{-1}(y)$ for some $y \in \Re^n$. Hence $\{Q^{-1}(y) : y \in \Re^n\}$ is a cover for Δ. Since Q is LDC, $Q^{-1}(y)$ is open, $\forall y \in \Re^n$, and thus the cover is an open cover. Δ is compact. As in the proof of Lemma 3.9, there is a finite index set, $A = \{y_1, \dots, y_k\}$ of points in $\Re^n$ such that $\{Q^{-1}(y_i) : y_i \in A\}$ covers Δ. Define $\alpha_i : \Delta \to \Re$ by $\alpha_i(x) = d(x, \Delta \backslash Q^{-1}(y_i))$ for $i = 1, \dots, k$, and let $g_i : \Delta \to \Re$ be given by $g_i(x) = \alpha_i(x) / \sum_{j=1}^{k} \alpha_j(x)$. As before,

d is the distance operator, so $\alpha_i(x)$ is the distance from x to $\Delta\backslash Q^{-1}(y_i)$. $\{g_i\}$ is known as a *partition of unity* for Q. Clearly $\sum g_i(x) = 1$ for all $x \in \Delta$, and $g_i(x) = 0$ iff $x \in \Delta\backslash Q^{-1}(y_i)$ (since $\Delta\backslash Q^{-1}(y_i)$ is closed and thus compact). Finally define $f : \Delta \to \Re^n$ by $f(x) = \sum_{i=1}^{k} g_i(x)y_i$. By the construction, $g_i(x) = 0$ iff $y_i \notin Q(x)$, thus $f(x)$ is a convex combination of points all in $Q(x)$. Since $Q(x)$ is convex, $f(x) \in Q(x)$. By Lemma 3.17, each α_i is continuous. Thus f is continuous. □

Lemma 3.25 *Let $P : \Delta \to \Delta$ be an LDC correspondence such that for all $x \in \Delta$, $P(x)$ is convex and $x \notin$ Con $P(x)$, the convex hull of $P(x)$. Then the choice $C_P(\Delta)$ is non empty.*

Proof Suppose $C_P(\Delta) = \Phi$. Then $P(x) \neq \Phi$ $\forall x \in \Delta$. By Lemma 3.24, there exists a continuous function $f : \Delta \to \Delta$ such that $f(x) \in P(x)$ $\forall x \in \Delta$. By Brouwer's Theorem, there exists a fixed point $x_0 \in \Delta$ such that $x_0 = f(x_0)$. This contradicts $x \notin$ Con $P(x)$, $\forall x \in \Delta$. Thus $C_P(\Delta) \neq \Phi$. □

These two lemmas are stated for correspondences with domain the finite dimensional simplex, Δ. Clearly they are valid for correspondences with domain a (finite dimensional) compact convex space. However both results can be extended to (infinite dimensional) topological vector spaces. The general version of Lemma 3.24 is known as Michael's Selection Theorem (Michael 1956). However it is necessary to impose conditions on the domain and codomain spaces. In particular it is necessary to be able to construct a partition of unity. For this purpose we can use a condition call "paracompactness" rather than compactness. Paracompactness of a space X requires that there exist, at any point $x \in X$, an open set U_x containing x, such that for any open cover $\{U_i\}$ of X, only finitely many of the open sets of the cover intersect U_x. To construct the continuous selection it is also necessary that the codomain Y of the correspondence has a norm, and is complete (essentially this means that a limit of a convergent sequences of points is contained in Y). A complete normed topological vector space Y is called a *Banach* space. We also need Y to be "separable" (ie Y contains a countable dense subset.) If Y is a separable *Banach* space we say it is admissible.

Michael's Selection Theorem employs a property, called lower hemi-continuity.

Definition 3.9 A correspondence $Q : X \to Y$ between the topological spaces, X and Y, is *lower hemi-continuous* (LHC) if whenever U is open in Y, then the set $\{x \in X : Q(x) \cap U \neq \Phi\}$ is open in X.

Michael's Selection Theorem *Suppose $Q : X \to Y$ is a lower hemi-continuous correspondence from a paracompact, Hausdorff topological space X into the admissible space Y, such that $Q(x)$ is non-empty closed and convex, for all $x \in X$. Then there exists a continuous selection $f : X \to Y$ for Q.*

Lemma 3.24 also provides the proof for a useful intersection theorem known as the Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem.

Before stating this theorem, consider an arbitrary collection $\{x_1, \ldots, x_k\}$ of distinct points in $\Re^n$. Then clearly the convex hull, Δ, of these points can be identified with a $(k-1)$-dimensional simplex. Let $S \subset \{1, \ldots, k\}$ be any index set, and let Δ_S be the simplex generated by this collection of $s-1$ points (where $s = |S|$).

KKM Theorem *Let $R : X \to Y$ be a correspondence between a convex set X contained in a Hausdorff topological vector space Y such that $R(x) \neq \Phi$ for all $x \in X$. Suppose that for at least one point $x_0 \in X$, $R(x_0)$ is compact. Suppose further that $R(x)$ is closed for all $x \in X$. Finally for any set $\{x_1, \ldots, x_k\}$ of points in X, suppose that*

$$\mathrm{Con}\{x_1, \ldots, x_k\} \subset \bigcup_{i=1}^{k} R(x_i).$$

Then $\bigcap_{x \in X} R(x)$ is non empty.

Proof By Lemma 3.8, since $R(x_0)$ is compact, $\bigcap_{x \in X} R(x)$ is non-empty iff $\bigcap_{i=1}^{k} R(x_i) \neq \Phi$ for any finite index set. So let $K = \{1, \ldots, k\}$ and let Δ be the $(k-1)$-dimensional simplex spanned by $\{x_1, \ldots, x_k\}$. Define $P : \Delta \to \Delta$ by $P(x) = \{y \in \Delta : x \in \Delta \backslash R(y)\}$ and define $Q : \Delta \to \Delta$ by $Q(x) = \mathrm{Con}\, P(x)$, the convex hull of $P(x)$.

But $P^{-1}(y) = \{x \in \Delta : y \in P(x)\} = \Delta \backslash R(y)$ is an open set, in Δ, and so P is LDC. Thus Q is LDC. Now suppose that $\bigcap_{i \in K} R(x_i) = \Phi$.

Thus for each $x \in \Delta$ there exists $x_i (i \in K)$ such that $x \notin R(x_i)$. But then $x \in \Delta - R(x_i)$ and so $x \in P^{-1}(x_i)$. In particular, for each $x \in \Delta$, $P(x)$, and thus $Q(x)$, is non-empty. Moreover $\{Q^{-1}(x_i) : i \in K\}$ is an open cover for Δ. As in the proof of Lemma 3.24, there is a partition of unity for Q. (We need Y to be Hausdorff for this construction.) In particular there exists a continuous selection $f : \Delta \to \Delta$ for Q. By Brouwer's Theorem, f has a fixed point $x_0 \in \Delta$. Thus $x_0 \in \mathrm{Con}\ P(x_0)$, and so $x_0 \in \mathrm{Con}\{y_1, \ldots, y_k\}$ where $y_i \in P(x_0)$ for $i \in K$. But then $x_0 \in \Delta \backslash R(y_i)$ for $i \in K$, and so $x_0 \notin R(y_i)$ for $i \in K$.

Hence

$$\mathrm{Con}\{y_1, \ldots, y_k\} \not\subset \bigcup_{i=1}^{k} R(y_i).$$

This contradicts the hypothesis of the Theorem. Consequently $\bigcap_{i \in K} R(x_i) \neq \Phi$ for any finite vertex set K. By compactness $\bigcap_{x \in X} R(x) \neq \Phi$. □

We can immediately use the KKM theorem to prove a fixed point theorem for a *correspondence* P from a compact convex set X to a Hausdorff topological vector space, Y. In particular X need not be finite dimensional.

Browder Fixed Point Theorem *Let $Q : X \to X$ be a correspondence where X is a compact convex subset of the Hausdorff topological vector space, Y. Suppose further that Q is LDC, and that $Q(x)$ is convex and non-empty for all $x \in X$. Then there exists $x_0 \in X$ such that $x_0 \in Q(x_0)$.*

Proof Suppose that $x \notin Q(x)$ $\forall x \in X$. Define $R : X \to X$ by $R(x) = X\backslash Q^{-1}(x)$. Since Q is LDC, $R(x)$ is closed and thus compact $\forall x \in X$. To use KKM, we seek to show that Con $\{x_1, \dots, x_k\} \subset \bigcup_{i=1}^{k} R(x_i)$ for any finite index set, $K = \{1, \dots, k\}$.

We proceed by contradiction. That is, suppose that there exists x_0 in X with $x_0 \in$ Con $\{x_1, \dots, x_k\}$ but $x_0 \notin R(x_i)$ for $i \in K$. Then $x_0 \in Q^{-1}(x_i)$, so $x_i \in Q(x_0), \forall i \in K$. But then $x_0 \in$ Con $Q(x_0)$. Since $Q(x)$ is convex $\forall$ $x \in X$, this implies that $x_0 \in Q(x_0)$, a contradiction. Consequently $x \in R(x_i)$ for some $i \in K$. By the KKM Theorem, $\bigcap_{x \in X} R(x) \neq \Phi$.

Thus $\exists$ $x_0 \in X$ with $x_0 \in R(x)$, and so $x_0 \in X\backslash Q^{-1}(x)$ $\forall x \in X$. Thus $x_0 \notin Q^{-1}(x)$ and $x \notin Q(x_0)$ $\forall x \in X$. This contradicts the assumption that $Q(x) \neq \Phi$ $\forall x \in$ X. Hence $\exists$ $x_0 \in X$ with $x_0 \in Q(x_0)$. □

Ky Fan Theorem *Let $P : X \to X$ be an LDC correspondence where X is a compact convex subset of the Hausdorff topological vector space, Y. If $x \notin$* Con *$P(x)$, $\forall x \in X$, then the choice $C_P(X_0) \neq \Phi$ for any compact convex subset, X_0 of X.*

Proof Define $Q : X \to X$ by $Q(x)$ = Con $P(x)$. If $Q(x) \neq \Phi$ for all $x \in X$, then by the Browder fixed point theorem, $\exists$ $x_0 \in X$ with $x_0 \in Q(x_0)$. This contradicts $x \notin$ Con $P(x)$ $\forall$ $x \in X$. Hence $Q(x_0) = \Phi$ for some $x_0 \in X$. Thus $C_P(X) = \{x \in X : P(x) = \Phi\}$ is non-empty. The same inference is valid for any compact, convex subset X_0 of X. □

3.8 Political and Economic Choice

The results outlined in the previous section are based on the intersection property of a family of closed sets. With compactness, this result can be extended to the case of a correspondence $R : X \to X$ to show that $\bigcap_{x \in X} R(x) \neq \Phi$. If we regard R as derived from an LDC correspondence $P : X \to X$ by $R(x) = X\backslash P^{-1}(x)$ then $R(x)$ can be interpreted as the set of points "no worse than" or "at last as good as" x.

But then the choice $C_P(X) = \bigcap_{x \in X} R(x)$, since such a choice must be at least as good as any other point. The finite-dimensional version (Lemma 3.25) of the proof that the choice is non-empty is based simply on a fixed point argument using Brouwer's Theorem. To extend the result to an infinite dimensional topological vector space we reduced the problem to one on a finite dimensional simplex, Δ, spanned by $\{x_1, \dots, x_k : x_i \in X\}$ and then showed essentially that $\bigcap_{x \in \Delta} R(x)$ is non empty. By compactness then $\bigcap_{x \in X} R(x)$ is non-empty. There is, in fact, a related infinite dimensional version of Brouwer's Fixed Point Theorem, known as Schauder's fixed point theorem for a continuous function $f : X \to Y$, where Y is a compact subset of the convex Banach space X.

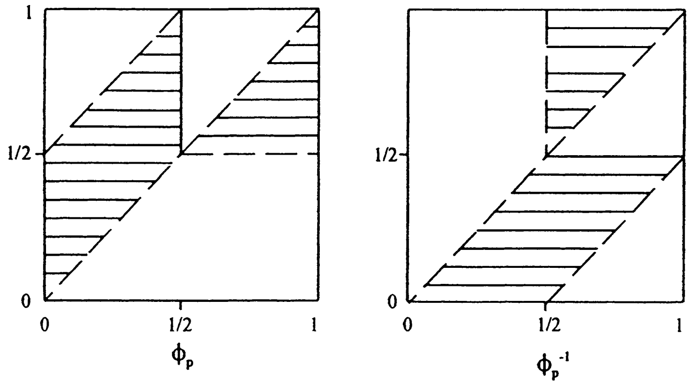


Fig. 3.24 Illustration of the Ky Fan Theorem

One technique for proving existence of an equilibrium price vector in an exchange economy (as discussed in Sect. 3.5) is to construct a continuous function $f : \Delta \to \Delta$, where Δ is the price simplex of feasible price vectors, and show that f has a fixed point (using either the Brouwer or Schauder fixed point theorems).

An alternative technique is to use the Ky Fan Theorem to prove existence of an equilibrium price vector. This technique permits a proof even when preferences are not representable by utility functions. More importantly, perhaps, it can be used in the infinite dimensional case.

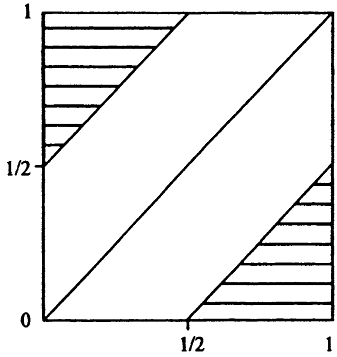
Example 3.11 To illustrate the Ky Fan Theorem with a simple example, consider Fig. 3.24, which reproduces Fig. 1.11 from Chap. 1.

It is evident that the inverse preference, P^{-1}, is not LDC: for example, $P^{-1}(\frac{3}{4}) = (\frac{1}{4}, \frac{1}{2}] \cup (\frac{3}{4}, 1]$ which is not open. As we saw earlier the choice of P on the unit interval is empty. In fact, to ensure existence of a choice we can require simply that P be lower hemi-continuous. This we can do by deleting the segment $(\frac{1}{2}, 1)$ above the point $\frac{1}{2}$. If the choice were indeed empty, then by Michael's Selection Theorem we could find a continuous selection $f : [0, 1] \to [0, 1]$ for P. By Brouwer's fixed point theorem f has a fixed point, x_0, say, with $x_0 \in P(x_0)$. By inspection the fixed point must be $x_0 = \frac{1}{2}$. If we require P to be irreflexive (since it is a strict preference relation) then this means that $\frac{1}{2} \notin P(\frac{1}{2})$ and so the choice must be $C_P([0, 1]) = \{\frac{1}{2}\}$. Notice that the preference displayed in Fig. 3.24 cannot be represented by a utility function. This follows because the implicit indifference relation is intransitive. See Fig. 3.25.

The Ky Fan Theorem gives a proof of the existence of a choice for a "spatial voting game". Remember a social choice procedure, σ is *simple* iff it is defined by a family $\mathcal{D}$ of decisive coalitions, each a subset of the society M. In this case if $\pi = (P_1, \dots, P_m)$ is a profile on the topological space X, then the social preference is given by

$$x\sigma(\pi)y \quad \text{iff } x \in \bigcup_{A \in \mathcal{D}} \bigcap_{i \in A} P_i(y) = P_{\mathcal{D}}(y).$$

Fig. 3.25 The graph of indifference



Here we use $P_i : X \twoheadrightarrow X$ to denote the preference correspondence of individual i. For coalition A, then $x \in P_A(y) = \bigcap_{i \in A} P_i(y)$ means every member of A prefers x to y.

Thus $x \in \bigcup_{A \in \mathcal{D}} P_A(y)$ means that for some coalition $A \in \mathcal{D}$, all members of A prefer x to y. Finally we write $x \in P_{\mathcal{D}}(y)$ for the condition that x is socially preferred to y. The choice for $\sigma(\pi)$ on X is then

$$C_{\sigma(\pi)}(X) = \{x : P_{\mathcal{D}}(x) = \Phi\}.$$

Nakamura Theorem *Suppose X is a compact convex topological vector space of dimension n. Suppose that $\pi = (P_1, \dots, P_m)$ is a profile on X such that each preference $P_i : X \twoheadrightarrow X$ is* (i) *LDC; and* (ii) semi-convex, *in the sense that $x \notin$ Con $P_i(x)$ for all $x \in X$. If σ is simple and has Nakamura number $k(\sigma)$, and if $n \leq k(\sigma) - 2$ then the choice $C_{\sigma(\pi)}(X)$ is non-empty.*

Proof For any point x, $y \in P_{\mathcal{D}}^{-1}(x)$ means $x \in P_{\mathcal{D}}(y)$ and so $x \in P_i(y)$ $\forall i \in A$, some $A \in \mathcal{D}$. Thus $y \in P_i^{-1}(x)$ $\forall i \in A$ or $y \in \bigcap_{i \in A} P_i^{-1}(x)$ or $y \in \bigcup_{\mathcal{D}} \bigcap_A P_i^{-1}(x)$. But each P_i is LDC and so $P_i^{-1}(x)$ is open, for all $x \in X$. Finite intersection of open sets is open, and so $P_{\mathcal{D}}$ is LDC.

Now suppose that $P_{\mathcal{D}}$ is not semi-convex (that is $x \in \text{Con}\, P_{\mathcal{D}}(x)$ for some $x \in X$). Since X is n-dimensional and convex, this means it is possible to find a set of points $\{x_1, \dots, x_{n+1}\}$ such that $x \in \text{Con}\,\{x_1, \dots, x_{n+1}\}$ and such that $x_j \in P_{\mathcal{D}}(x)$ for each $j = 1, \dots, n+1$. Without loss of generality this means there exists a subfamily $\mathcal{D}' = \{A_1, \dots, A_{n+1}\}$ of $\mathcal{D}$ such that $x_j \in P_{A_j}(x)$. Now $n + 1 \leq k(\sigma) - 1$, and by the definition of the Nakamura number, the collegium $K(\mathcal{D}')$ is non-empty. In particular there is some individual $i \in A_j$, for $j = 1, \dots, n+1$. Hence $x_j \in P_i(x)$ for $j = 1, \dots, n+1$. But then $x \in \text{Con}\, P_i(x)$. This contradicts the semi-convexity of individual preference. Consequently $P_{\mathcal{D}}$ is semi-convex. The conditions of the Ky Fan Theorem are satisfied, and so $P_{\mathcal{D}}(\overline{x}) = \Phi$ for some $\overline{x} \in X$. Thus $C_{\sigma(\pi)}(x) \neq \Phi$. □

It is worth observing that in finite dimensional spaces the Ky Fan Theorem is valid with the continuity property weakened to lower hemi-continuity (LHC). Note first that if P is LDC then it is LHC; this follows because $\{x \in X : P(x) \cap V \neq \Phi\} = \bigcup_{y \in V} (P^{-1}(y) \cap X)$ is the union of open sets and thus open.

Moreover (as suggested in Example 3.11) if P is LHC and the choice is non empty, then the correspondence $x \to \operatorname{Con} P(x)$ has a continuous selection f (by Michael's Selection Theorem). By the Brouwer Fixed Point Theorem, there is a fixed point x_o such that $x_o \in \operatorname{Con} P(x_0)$. This violates semi-convexity of P. Thus the Nakamura Theorem is valid when preferences are LHC. The finite dimensional version of the Ky Fan Theorem can be used to show existence of a Nash equilibrium (Nash 1950).

Definition 3.10

(i) A *Game* $G = \{(P_i, X) : i \in M\}$ for society M consists of a *strategy space*, X_i, and a strict preference correspondence $P_i : X \to X$ for each $i \in M$, where $X = \Pi_i\ X_i = X_1 \times \cdots \times X_m$ is the joint strategy space.

(ii) In a game G, the Nash improvement correspondence for individual i is defined by

$$\begin{aligned} \hat{P}_i : X \to X \quad & \text{where } y \in \hat{P}_i(x) \text{ iff } y \in P_i(x) \text{ and} \\ & y = \left(x_1, \ldots, x_{i-1}, x_i^*, \ldots x_m\right), \\ & x = (x_1, \ldots, x_{i-1},\ x_i, \ldots, x_m). \end{aligned}$$

(iii) The overall Nash improvement correspondence is

$$\hat{P} = \bigcup_{i \in M} P_i : X \to X.$$

(iv) A point $\overline{x} \in X$ is a Nash Equilibrium for the game G iff $\hat{P}(\overline{x}) = \Phi$.

Bergstrom (1975, 1992) showed the following.

Bergstrom's Theorem *Let $G = \{(P_i, X)\}$ be a game, and suppose each $X_i \subset \Re^n$ is a non-empty compact, convex subset of $\Re^n$. Suppose further that for all $i \in M$, $\hat{P}_i$ is both semi-convex and LHC. Then there exists a Nash equilibrium for G.*

Proof Since each $\hat{P}_i$ is LHC, it follows easily that $\hat{P} : X \to X$ is LHC. To see that $\hat{P}$ is semi-convex, suppose that $y \in \operatorname{Con}\ \hat{P}(x)$, then $y = \sum \lambda_i y^i$, $\sum_{i \in N} \lambda_i = 1$, $\lambda_i \geq 0$, $\forall i \in\ N$ and $y^i \in \hat{P}_i(x)$. By the definition of $\hat{P}_j$, $y - x = \sum_{i \in M} \lambda_i z^i$ where $z^i = y^i - x$.

This follows because y^i and x only differ on the ith coordinate, and so $z^i = (0, \ldots, z_0^i, \ldots)$ where $z_0^i \in X_i$. Moreover, $\lambda_i \neq 0$ iff $z^i \neq 0$ because $\hat{P}_i$ is semi-convex.

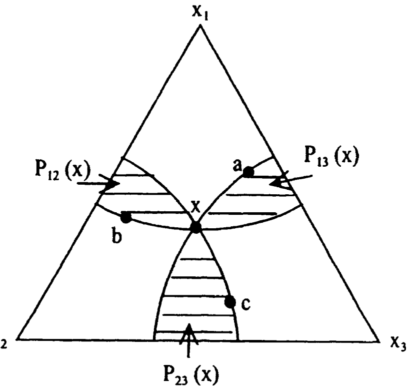


Fig. 3.26 Failure of semi-convexity

Clearly $\{z^i : i \in M, \lambda_i \neq 0\}$ is a linearly independent set, so $y = x$ iff $\lambda_i = 0$ $\forall i \in M$. But then $y^i = x$ $\forall i \in M$, which again violates semi-convexity of P_i. Thus $y \neq x$ and so $\hat{P}$ is semi-convex. By the Ky Fan Theorem, for X finite dimensional, the choice of $\hat{P}$ on X is non-empty. Thus the Nash Equilibrium is non-empty. □

Although the Nakamura Theorem guarantees existence of a social choice for a social choice rule, σ, for any semi-convex and LDC profile in dimension at most $k(\sigma) - 2$, it is easy to construct situations in higher dimensions with empty choice. The example we now present also describes a game with no Nash equilibrium.

Example 3.12 Consider a simply voting procedure with $M = \{1, 2, 3\}$ and let $\mathcal{D}$ consist of any coalition with at least two voters. Let X be a compact convex set in $\mathfrak{R}^2$, and construct preferences for each i in M as follows. Each i has a "bliss point" $x_i \in X$ and a preference P_i on X such that for $y, x \in P_i(y)$ iff $\|x - x_i\| < \|y - y_i\|$. The preference is clearly LDC and semi-convex (since $P_i(x)$ is a convex set and $x \notin P(x)$ for all $x \in X$). Now let $\Delta = \text{Con}\{x_1, x_2, x_3\}$ the 2-dimensional simplex in X (for convenience suppose all x_i are distinct and in the interior of X). For each $A \subset M$ let $P_A(x) = \bigcap_{i \in A} P_i(x)$ as before.

In particular the choice for the strict Pareto rule is $C_{P_M}(X) = C_M(X) = \Delta$. This can be seen by noting that if $x \in C_M(X)$ iff there is no point $y \in M$ such that $\|y - x_i\| < \|x - x_i\|$ $\forall i \in M$. Clearly this condition holds exactly at those points in Δ. For this reason preferences of this kind are called "Euclidean preferences".

Now consider a point in the interior of X. At x the preferred sets for the three coalitions ($\mathcal{D}' = \{1, 2\}, \{1, 3\}, \{2, 3\}$) do not satisfy the semi-convexity property. Figure 3.26 shows that

$$x \in \text{Con}\left\{P_{12}(x), P_{13}(x), P_{23}(x)\right\}.$$

While $P_{\mathcal{D}'}$ is LDC, it violates semi-convexity. Thus the Ky Fan Theorem cannot be used to guarantee existence of a choice. To illustrate the connection with Walker's Theorem, note also that there is a voting cycle. That is 1 prefers a to b to c, 2 prefers b to c to a, and 3 prefers c to a to b. The reason the cycle can be constructed is that

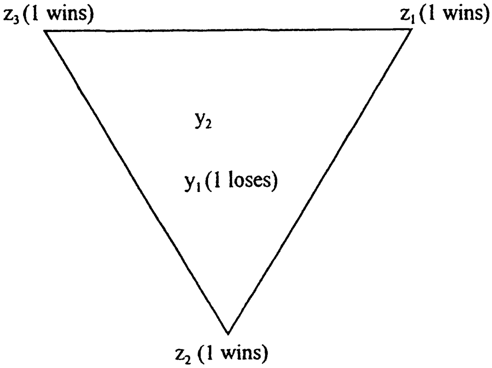


Fig. 3.27 Empty Nash equilibrium

the Nakamura number is 3, and dimension $(X) = 2$, thus $n = k(\sigma) - 1$. In fact there is no social choice in X. This voting cycle also defines a game with no Nash equilibrium.

Let $\{1, 2\}$ be the two parties, and let the "strategy spaces" be $X_1 = X_2 = \Delta$. Say party i adopts a winning position $y_i \in X_i$ over party $j's$ position, $y_j \in X_j$ iff $y_i \in P_{\mathcal{D}'}(y_j)$, so party i gains a majority over party j. This induces a Nash improvement correspondence $\hat{P}_j$, for $j = 1, 2$.

For example $\hat{P}_1 : Y \twoheadrightarrow Y$ where $Y = X_1 \times X_2$ is defined by $(y_1^*, y_2) \in \hat{P}_1(y_1, y_2)$ iff $y_1^* \in P_{\mathcal{D}'}(y_2)$, so y_1^* is a winning position over y_2 but $y_1 \notin P_{\mathcal{D}'}(y_2)$, so y_1 is not a winning position. In other words party 1 prefers to move from y_1 to y_1^*, so as to win. It is evident if we choose y_1, y_2 and the points $\{z_1, z_2, z_3\}$ as in Fig. 3.27, then $z_i \in P_{\mathcal{D}'}(y_2)$ so $(z_i, y_2) \in \hat{P}_1(y_1, y_2)$ for $i = 1, 2, 3$ and $y_1 \in \mathrm{Con}\{z_1, z_2, z_3\}$.

Thus $(y_1, y_2) \in \mathrm{Con}\, \hat{P}_1(y_1, y_2)$. Because of the failure of semi-convexity of both $\hat{P}_1$ and $\hat{P}_2$, a Nash equilibrium cannot be guaranteed. In fact the Nash equilibrium is empty.

We now briefly indicate how the Ky Fan Theorem can be used to show existence of an equilibrium price vector in an exchange economy. First of all each individual i initially holds a vector of endowments $e_i \in \Re^n$. A price vector $p \in \Delta^{n-1}$ belongs to the $(n-1)$-dimensional simplex: that is $p = (p_1, \dots, p_n)$ such that $\sum_{i=1}^n p_i = 1$.

An *allocation* $x \in X = \prod_{i \in N} X_i \subset (\Re^n)^m$ where X_i is i's consumption set in $\Re_+^n$ (here $x_i \in \Re_+^n$ iff $x_{ij} \geq 0$, $j = 1, \dots, n$). At the price vector p, the ith budget set is

$$B_i(p) = \left\{ x_i \in \Re_+^n : \langle p, x \rangle \leq \langle p, e_i \rangle \right\}.$$

At price p, the *demand vector* $\overline{x} = (\overline{x}_1, \dots, \overline{x}_m)$ satisfies the *optimality condition* $\hat{P}_i(\overline{x}) \cap \{x \in X : x_i \in B_i(p)\} = \Phi$ for each i.

As before $\hat{P}_i : X \twoheadrightarrow X$ is the Nash improvement correspondence (as in Definition 3.10).

As we discussed earlier in Sect. 3.5.1, an equilibrium price vector $\overline{p}$ is a price vector $\overline{p} = (p_1, \dots, p_n)$ such that the demand vector $\overline{x}$ satisfies the optimality condition at $\overline{p}$ and such that total demand does not exceed supply. This latter condition requires that $\sum_{i\in M}\overline{x}_i \leq \sum_{i\in M} e_i$ (the two terms are both vectors in $\Re^n$). That is, if we use the suffix of x_j to denote commodity j, then $\sum_{i\in M}(\overline{x}_{ij} - e_{ij}) \leq 0$ for each $j = 1, \dots, n$.

Note also that a transformation $p > \lambda p$, for a real number $\lambda > 0$, does not change the budget set.

This follows because $B_i(p) = \{x_i \in \Re^n_+ : \langle p, x\rangle \leq \langle p, e_i\rangle\} = B_i(\lambda p)$.

Consequently if $\overline{p}$ is an equilibrium price vector, then so is $\lambda\overline{p}$. Without loss of generality, then, we can normalize the price vector, p, so that $\|p\| = 1$ for some norm on $\Re^n$. We may do this for example by assuming that $\sum_{j=1}^n p_j = 1$ and that $p_j \geq 0\ \forall j$.

For this reason we let Δ^{n-1} represent the set of all price vectors.

A further point is worth making. Since we assume that $p_j \geq 0\ \forall j$ it is possible for commodity j to have zero price. But then the good must be in excess supply. To ensure this, we require the equilibrium price vector $\overline{p}$ and i's demand $\overline{x}_i$ at $\overline{p}$ to satisfy the condition $\langle \overline{p}, \overline{x}_i\rangle = \langle \overline{p}, \overline{e}_i\rangle$ As we noted in Sect. 3.5.1, this can be ensured by assuming that preference is locally non-satiated in X. That is if we let $\overline{P}_i(x) \subset X_i$ be the projection of $\hat{P}_i$ onto X_i at x, then for any neighborhood U of x_i in X_i there exists $x_i' \in U$ such that $x_i' \in \overline{P}_i(x)$.

To show existence of a price equilibrium we need to define a price adjustment mechanism.

To this end define:

$$\hat{P}_0 : \Delta \times X \to \Delta \quad \text{by}$$

$$\hat{P}_0(p, x) = \left\{ p' \in \Delta : \left\langle p' - p \sum_{i\in M} (x_i - e_i) \right\rangle > 0 \right\}. \tag{**}$$

Now let $X^* = \Delta \times X$ and define $P_0^* : X^* \to X^*$ by $(p', x) \in P_0^*(p, y)$ iff $x = y$ and $p' \in \hat{P}_0(p, x)$.

In the same way for each $i \in M$ extend $\hat{P}_i : X \to X$ to $P_i^* : X^* \to X^*$ by letting $(p', x) \in P_i^*(p, y)$ iff $p' = p$ and $x \in \hat{P}_i(y)$. This defines an *exchange game* $G_e = \{(P_i^*, X^*) : i = 0, \dots, m\}$.

Bergstrom (1992) has shown (under the appropriate conditions of semi-convexity, LHC and local monotonicity for each $\hat{P}_i$) that there is a Nash equilibrium to G_e. Note that $e \in (\Re^n)^m$ is the initial vector of endowments. We can show that the Nash equilibria comprise a non-empty set $\{(\overline{p}, \overline{x}) \in \Delta \times X\}$ where $\overline{x} = (\overline{x}_1, \dots, \overline{x}_m)$ is a vector of final demands for the members of M, and $\overline{p}$ is an equilibrium price vector.

A Nash equilibrium $(\overline{p}, \overline{x}) \in \Delta \times X$ satisfies the following properties:

(i) Since $P_i^*(\overline{p}, \overline{x}) = \Phi$ for each $i \in N$, it follows that $\overline{x}_i$ is i's demand. Moreover by local monotonicity we have $\langle \overline{p}, \overline{x}_i\rangle = \langle \overline{p}, e_i\rangle$, $\forall i \in M$. Thus

$$\sum_{i\in M} \langle \overline{p}, \overline{x}_i - e_i\rangle = 0. \tag{*}$$

(ii) Now $P_0^*(\overline{p}, \overline{x}) = \Phi$ so $\langle p - \overline{p}, \sum_{i\in M}(\overline{x}_i - e_i)\rangle \leq 0\ \forall p \in \Delta$ (See (**)).

Suppose that $\sum_{i \in M}(\overline{x}_i - e_i) > 0$. Then it is clearly possible to find $p \in \Delta$ such that $p_j > 0$, for some j, with $\langle p, \sum_{i \in M}(x_i - e_i)\rangle > 0$. But this violates (*) and (**). Consequently $\sum_{i \in M}(\overline{x}_{ij} - e_{ij}) \leq 0$ for $j = 1, \ldots, n$.

Thus $\overline{x} \in (\Re^n)^m$ satisfies the feasibility constraint $\sum_{i \in M} \overline{x}_i \leq \sum_{i \in M} e_i$.

Hence $(\overline{p}, \overline{x})$ is a *free-disposal* equilibrium, in the sense that total demand may be less than total supply. Bergstrom (1992) demonstrates how additional assumptions on individual preference are sufficient to guarantee equality of demand and supply. Section 4.4, below, discusses this more fully.

Further Reading

The reference for the Kuhn-Tucker Theorems is:

Kuhn, H. W., & Tucker, A. W. (1950). Non-linear programming. In *Proceedings: 2nd Berkeley symposium on mathematical statistics and probability*, Berkeley: University of California Press.

A useful further reference for economic applications is:

Heal, E. M. (1973). *Theory of economic planning*. Amsterdam: North Holland.

The classic references on fixed point theorems and the various corollaries of the theorems are:

Brouwer, L. E. J. (1912). Uber Abbildung von Mannigfaltikeiten. *Mathematische Annalen*, *71*, 97–115.

Browder, F. E. (1967). A new generalization of the Schauder fixed point theorem. *Mathematische Annalen*, *174*, 285–290.

Browder, F. E. (1968). The fixed point theory of multivalued mappings in topological vector spaces. *Mathematische Annalen*, *177*, 283–301.

Fan, K. (1961). A generalization of Tychonoff's fixed point theorem. *Mathematische Annalen*, *142*, 305–310.

Knaster, B., Kuratowski, K., & Mazerkiewicz, S. (1929). Ein Beweis des Fixpunktsatze fur n-dimensionale Simplexe. *Fundamenta Mathematicae*, *14*, 132–137.

Michael, E. (1956). Continuous selections I. *Annals of Mathematics*, *63*, 361–382.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences of the United States of America*, *36*, 48–49.

Schauder, J. (1930). Der Fixpunktsatze in Funktionalräumn. *Studia Mathematica*, *2*, 171–180.

A useful discussion of the relationships between the theorems can be found in:

Border, K. (1985). *Fixed point theorems with applications to economics and game theory*. Cambridge: Cambridge University Press.

Hildenbrand, W., & Kirman, A. P. (1976). *Introduction to equilibrium analysis*. Amsterdam: Elsevier.

The proof of the Browder fixed point theorem by KKM is given in:

Yannelis, N., & Prabhakar, N. (1983). Existence of maximal elements and equilibria in linear topological spaces. *Journal of Mathematical Economics*, *12*, 233–245.

Applications of the Fan Theorem to show existence of a price equilibrium can be found in:

Aliprantis, C., & Brown, D. (1983). Equilibria in markets with a Riesz space of commodities. *Journal of Mathematical Economics*, *11*, 189–207.

Bergstrom, T. (1975). *The existence of maximal elements and equilibria in the absence of transitivity* (Typescript). Washington University in St. Louis.

Bergstrom, T. (1992). When non-transitive relations take maxima and competitive equilibrium can't be beat. In W. Neuefeind & R. Riezman (Eds.), *Economic theory and international trade*, Berlin: Springer.

Shafer, W. (1976). Equilibrium in economies without ordered preferences or first disposal. *Journal of Mathematical Economics*, *3*, 135–137.

Shafer, W., & Sonnenschein, H. (1975). Equilibrium in abstract economies without ordered preferences. *Journal of Mathematical Economics*, *2*, 345–348.

The application of the idea of the Nakamura number to existence of a voting equilibrium can be found in:

Greenberg, J. (1979). Consistent majority rules over compact sets of alternatives. *Econometrica*, *41*, 285–297.

Schofield, N. (1984). Social equilibrium and cycles on compact sets. *Journal of Economic Theory*, *33*, 59–71.

Strnad, J. (1985). The structure of continuous-valued neutral monotonic social functions. *Social Choice and Welfare*, *2*, 181–195.

Finally there are results on existence of a joint political economic equilibrium, namely an outcome $(\overline{p}, \overline{t}, \overline{x}, \overline{y})$, where $(\overline{p}, \overline{x})$ is a market equilibrium, $\overline{t}$ is an equilibrium tax schedule voted on under a rule $\mathcal{D}$ and $\overline{y}$ is an allocation of publicly provided goods.

Konishi, H. (1996). Equilibrium in abstract political economies: with an application to a public good economy with voting. *Social Choice and Welfare*, *13*, 43–50.

Differential Calculus and Smooth Optimisation 4

Under certain conditions a continuous function $f : \Re^n \to \Re^m$ can be approximated at each point x in $\Re^n$ by a linear function $df(x) : \Re^n \to \Re^m$, known as the differential of f at x. In the same way the differential df may be approximated by a bilinear map $d^2 f(x)$. When all differentials are continuous then f is called smooth. For a smooth function f, Taylor's Theorem gives a relationship between the differentials at a point x and the value of f in a neighbourhood of a point. This in turn allows us to characterise maximum points of the function by features of the first and second differential. For a real-valued function whose preference correspondence is convex we can virtually identify critical points (where $df(x) = 0$) with maxima.

In the maximisation problem for a smooth function on a "smooth" constraint set, we seek critical points of the Lagrangian, introduced in the previous chapter. In particular in economic situations with exogenous prices we may characterise optimum points for consumers and producers to be points where the differential of the utility or production function is given by the price vector. Finally we use these results to show that for a society the set of Pareto optimal points belongs to a set of generalised critical points of a function which describes the preferences of the society.

4.1 Differential of a Function

A function $f : \Re \to \Re$ is *differentiable* at $x \in \Re$ if $\text{Lim}_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists (and is neither $+\infty$ nor $-\infty$). When this limit exists we shall write it as $\frac{df}{dx}|_x$. Another way of writing this is that as $(x_n) \to x$ then

$$\frac{f(x_n) - f(x)}{x_n - x} \to \left.\frac{df}{dx}\right|_x,$$

the *derivative* of f at x.

This means that there is a real number $\lambda(x) = \frac{df}{dx}|_x \in \Re$ such that $f(x) = \lambda(x)h + \epsilon|h|$, where $\epsilon \to 0$ as $h \to 0$.

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_4,

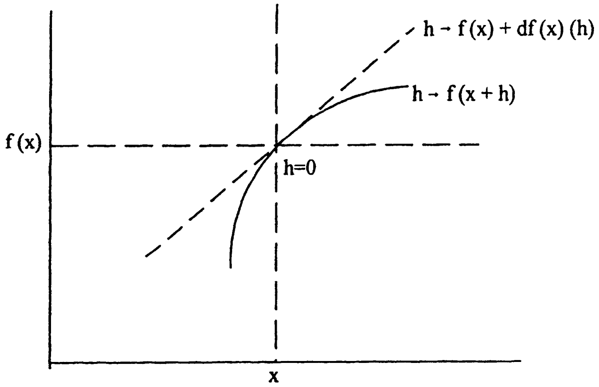


Fig. 4.1 The differential

Let $df(x)$ be the linear function $\Re \to \Re$ given by $df(x)(h) = \lambda(x)h$. Then the map $\Re \to \Re$ given by

$$h \to f(x) + df(x)(h) = g(x+h)$$

is a "first order approximation" to the map

$$h \to f(x+h).$$

In other words the maps $h \to g(x+h)$ and $h \to f(x+h)$ are "tangent" to one another where "tangent" means that

$$\frac{|f(x+h) - g(x+h)|}{|h|}$$

approaches 0 as $h \to 0$. Note that the map $h \to f(x) + df(x)(h) = g(x+h)$ has a straight line graph, and so $df(x)$ is a "linear approximation" to the function f at x. See Fig. 4.1.

Example 4.1

1. Suppose $f : \Re \to \Re : x \to x^2$. Then $\text{Lim}_{h \to 0} \frac{f(x+h)-f(x)}{h} = \text{Lim}_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \text{Lim}_{h \to 0} \frac{2hx+h^2}{h} = 2x + \text{Lim}_{h \to 0} h = 2x$. Similarly if $f : \Re \to \Re : x \to x^r$ then $df(x) = rx^{r-1}$.
2. Suppose $f : \Re \to \Re : x \to \sin x$. Then

$$\begin{aligned}
&\operatorname*{Lim}_{h \to 0}\left(\frac{\sin(x+h) - \sin x}{h}\right) \\
&\quad = \operatorname*{Lim}_{h \to 0}\left(\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\right) \\
&\quad = \operatorname*{Lim}_{h \to 0} \frac{\sin x}{h}\left(\frac{-h^2}{2}\right) + \operatorname*{Lim}_{h \to 0} \frac{\cos x}{h}(h) \\
&\quad = \cos x.
\end{aligned}$$

3. $f : \Re \to \Re : x \to e^x$.

$$\underset{h\to 0}{\text{Lim}} \frac{e^{x+h} - e^x}{h} = \underset{h\to 0}{\text{Lim}} \frac{e^x}{h}\left[1 + h + \frac{h^2}{2}\cdots - 1\right] = e^x.$$

4. $f : \Re \to \Re : x \to x^4$ if $x \geq 0$, x^2 if $x < 0$. Consider the limit as h approaches 0 from above (*i.e.*, $h \to 0_+$). Then

$$\underset{h\to 0_+}{\text{Lim}} \frac{f(0+h) - f(0)}{h} = \frac{h^4 - 0}{h} = h^3 = 0.$$

The limit as h approaches 0 from below is

$$\underset{h\to 0_-}{\text{Lim}} \frac{f(0+h) - f(0)}{h} = \frac{h^3 - 0}{h} = h^2 = 0.$$

Thus $df(0)$ is defined and equal to 0.

5. $f : \Re \to \Re$, by

$$\begin{aligned}
&x \to -x^2 \quad x \leq 0\\
&x \to (x-1)^2 - 1 \quad 0 < x \leq 1\\
&x \to -x \quad x > 1\\
&\underset{x\to 0}{\text{Lim}}\, f(x) = 0\\
&\underset{x\to 0_+}{\text{Lim}}\, f(x) = 0.
\end{aligned}$$

Thus f is continuous at $x = 0$.

$$\begin{aligned}
&\underset{x\to 1_-}{\text{Lim}}\, f(x) = -1\\
&\underset{x\to 1_+}{\text{Lim}}\, f(x) = -1.
\end{aligned}$$

Thus f is continuous at $x = 1$.

$$\begin{aligned}
\underset{x\to 0_-}{\text{Lim}}\, df(x) &= \underset{x\to 0_-}{\text{Lim}}\, (-2x) = 0\\
\underset{x\to 0_+}{\text{Lim}}\, df(x) &= \underset{x\to 0_+}{\text{Lim}}\, 2(x-1) = -2\\
\underset{x\to 1_-}{\text{Lim}}\, df(x) &= \underset{x\to 1_-}{\text{Lim}}\, 2(x-1) = 0\\
\underset{x\to 1_+}{\text{Lim}}\, df(x) &= \underset{x\to 1_+}{\text{Lim}}\, (-1) = -1.
\end{aligned}$$

Hence $df(x)$ is not continuous at $x = 0$ and $x = 1$. See Fig. 4.2.

To extend the definition to the higher dimension case, we proceed as follows:

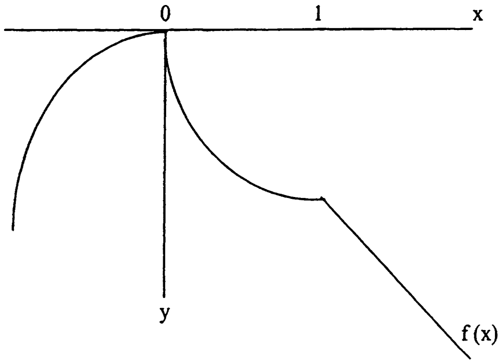


Fig. 4.2 Example 4.1.5

Definition 4.1 Let X, Y be two normed vector spaces, with norms $\| \|_X$, $\| \|_Y$, and suppose $f, g : X \to Y$. Then say f and g are *tangent* at $x \in X$ iff

$$\operatorname*{Lim}_{\|h\|_X \to 0} \frac{\|f(x+h) - g(x+h)\|_Y}{\|h\|_X} = 0.$$

If there exists a linear map $df(x) : X \to Y$ such that the function $g : X \to Y$ given by

$$g(x+h) = f(x) + df(x)(h)$$

is tangent to f at x, then f is said to be differentiable at x, and $df(x)$ is called the *differential* of f at x.

In other words $df(x)$ is the differential of f at x iff there is a linear approximation $df(x)$ to f at x, in the sense that

$$f(x+h) - f(x) = df(x)(h) + \|h\|_X \mu(h)$$

where $\mu : X \to Y$ and $\|\mu(h)\|_Y \to 0$ as $\|h\|_X \to 0$.

Note that since $df(x)$ is a linear map from X to Y, then its image is a vector subspace of Y, and $df(x)(\underline{0})$ is the origin, $\underline{0}$, in Y.

Suppose now that f is defined on an open ball U in X.

For some $x \in U$, consider an open neighborhood V of x in U. The image of the map

$$h \to g(x+h) \quad \text{for each } h \in U$$

will be of the form $f(x) + df(x)(h)$, which is to say a linear subspace of Y, but translated by the vector $f(x)$ from the origin.

If f is differentiable at x, then we can regard $df(x)$ as a linear map from X to Y, so $df(x) \in L(X, Y)$, the set of linear maps from X to Y. As we have shown in Sect. 3.2 of Chap. 3, $L(X, Y)$ is a normed vector space, when X is finite dimensional. For example, for $k \in L(X, Y)$ we can define $\|k\|$ by

$$\|k\| = \sup\left\{ \left\|k(x)\right\|_Y : x \in X \text{ s.t. } \|x\|_X = 1 \right\}.$$

Let $\mathcal{L}(X, Y)$ be $L(X, Y)$ with the topology induced from this norm.

When $f : U \subset X \to Y$ is continuous we shall call f a C^0-map. If f is C^0, and $df(x)$ is defined at x, then $df(x)$ will be linear and thus continuous, in the sense that $df(x) \in \mathcal{L}(X, Y)$.

Hence we can regard df as a map

$$df : U \to \mathcal{L}(X, Y).$$

It is important to note here that though the map $df(x)$ may be continuous, the map $df : U \to \mathcal{L}(X, Y)$ need not be continuous at x. However when f is C^0, and the map

$$df : U \to \mathcal{L}(X, Y)$$

is continuous for all $x \in U$, then we shall say that f is a C^1-differentiable map on U. Let $C_0(U, Y)$ be the set of maps from U to Y which are continuous on U, and let $C_1(U, Y)$ be the set of maps which are C^1-differentiable on U. Clearly $C_1(U, Y) \subset C_0(U, Y)$. If f is a differentiable map, then $df(x)$, since it is linear, can be represented by a matrix. Suppose therefore that $f : \Re^n \to \Re$, and let $\{e_1, \dots, e_n\}$ be the standard basis for $\Re^n$. Then for any $h \in \Re^n$, $h = \sum_{i=1}^n h_i e_i$ and so $df(x)(h) = \sum_{i=1}^n h_i df(x)(e_i) = \sum_{i=1}^n h_i \alpha_i$ say.

Consider the vector $(0, \dots, h_i, \dots, 0) \in \Re^n$.

Then by the definitions

$$\begin{aligned}\alpha_i &= df(x)(0, \dots, e_i, \dots, 0)\\ &= \operatorname*{Lim}_{h_i \to o} \left\{ \frac{f(x_1, \dots, x_i + h_i, \dots,) - f(x_1, \dots, x_i, \dots, x_n)}{h_i} \right\} = \left.\frac{\partial f}{\partial x_i}\right|_x,\end{aligned}$$

where $\frac{\partial f}{\partial x_i}|_x$ is called the *partial derivative* of f at x with respect to the ith coordinate, x_i.

Thus the linear function $df(x) : \Re \to \Re$ can be represented by a "row vector" or matrix

$$Df(x) = \left(\left.\frac{\partial f_j}{\partial x_1}\right|_x, \dots, \left.\frac{\partial f}{\partial x_n}\right|_x \right).$$

Note that this representation is dependent on the particular choice of the basis for $\Re^n$. This matrix $Df(x)$ can also be regarded as a vector in $\Re^n$, and is then called the *direction gradient* of f at x. The ith coordinate of $Df(x)$ is the partial derivative of f with respect to x_i at x.

If h is a vector in $\Re^n$ with coordinates $(h_1, \dots, h_n)$ with respect to the standard basis, then

$$df(x)(h) = \sum_{i=1}^n h_i \left.\frac{\partial f}{\partial x_i}\right|_x = \langle Df(x), h_i \rangle$$

where $\langle Df(x), h \rangle$ is the scalar product of h and the direction gradient $Df(x)$.

In the same way if $f : \Re^n \to \Re^m$ and f is differentiable at x, then $df(x)$ can be represented by the $n \times m$ matrix

$$Df(x) = \left(\frac{\partial f_j}{\partial x_i}\right)_x, \quad i = 1, \ldots, n;\ j = 1, \ldots, m$$

where $f(x) = f(x_1, \ldots, x_n) = (f_1(x), \ldots, f_j(x), \ldots, f_m(x))$. This matrix is called the *Jacobian* of f at x. We may define the *norm* of $Df(x)$ to be

$$\|Df(x)\| = \sup\left\{\left|\frac{\partial f_j}{\partial x_i}\right|_x : i = 1, \ldots, n;\ j = 1, \ldots, m\right\}.$$

When f has domain $U \subset \Re^n$, then continuity of $Df : U \to M(n,m)$, where $M(n,m)$ is the set of $n \times m$ matrices, implies the continuity of each partial derivative

$$U \to \Re : x \to \left.\frac{\partial f_j}{\partial x_i}\right|_x .$$

Note that when $f : \Re \to \Re$ then $\frac{\partial f}{\partial x}|_x$ is written simply as $\frac{df}{dx}|_x$ and is a real number.

Then the linear function $df(x) : \Re \to \Re$ is given by $df(x)(h) = (\frac{df}{dx}|_x)h$.

To simplify notation we shall not distinguish between the linear function $df(x)$ and the real number $\frac{df}{dx}|_x$ when $f : \Re \to \Re$.

Suppose now that $f : U \subset X \to Y$ and $g : V \subset Y \to Z$ such that $g \circ f : U \subset X \to Z$ exists. If f is differentiable at x, and g is differentiable at $f(x)$ then $g \circ f$ is differentiable at x and is given by

$$d(g \circ f)(x) = dg\big(f(x)\big) \circ df(x).$$

In terms of Jacobian matrices this is $D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$, or $\frac{\partial g_k}{\partial x_i} = \sum_{j=1}^m \frac{\partial g_k}{\partial f_j}\frac{\partial f_j}{\partial x_i}$, *i.e.*,

$$k\text{th row} \begin{pmatrix} \frac{\partial g_k}{\partial f_1} & \cdots & \frac{\partial g_k}{\partial f_m} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_m}{\partial x_i} \\ i\text{th column} \end{pmatrix}$$

This is also known as the *chain-rule*.

If Id : $\Re^n \to \Re^n$ is the identity map then clearly the Jacobian matrix of Id must be the identity matrix.

Suppose now that $f : U \subset \Re^n \to V \subset \Re^n$ is differentiable at x, and has an inverse $g = f^{-1}$ which is differentiable. Then $g \circ f = \text{Id}$ and so $\text{Id} = D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$. Thus $D(f^{-1})(f(x)) = [Df(x)]^{-1}$.

In particular, for this to be the case $Df(x)$ must be an $n \times n$ matrix of rank n. When this is so, f is called a *diffeomorphism.*

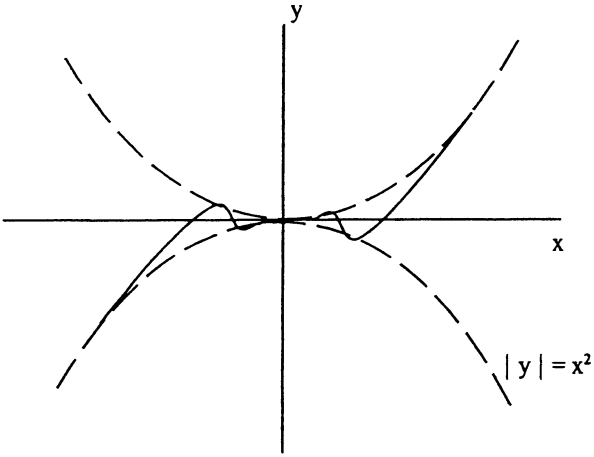


Fig. 4.3 Example 4.2

On the other hand suppose $f : X \to \Re$ and $g : Y \to \Re$, where f is differentiable at $x \in X$ and g is differentiable at $y \in Y$.

Let $fg : X \times Y \to \Re : (x, y) \to f(x)g(y)$. From the chain rule, $d(fg)(x, y) \times (h, k) = f(x)\ dg(y)(k) + g(y)\ df(x)(h)$, and so $d(fg)(x, y) = f(x)dg(y) + g(y)df(x)$.

Hence fg is differentiable at the point $(x, y) \in X \times Y$.

When $X = Y = \Re$, and $(fg)(x) = f(x)g(x)$ then $d(fg)(x) = fdg(x) + gdf(x)$, where this is called the *product rule*.

Example 4.2 Consider the function $f : \Re \to \Re$ given by $x \to x^2 \sin\frac{1}{x}$ if $x \neq 0$; 0 if $x = 0$. See Fig. 4.3.

We first of all verify that f is continuous. Let $g(x) = x^2$, $h(x) = \sin\frac{1}{x} = \rho[m(x)]$ where $m(x) = \frac{1}{x}$ and $\rho(y) = \sin(y)$. Since m is continuous at any non zero point, both h and g are continuous. Thus f is continuous at $x \neq 0$. (Compare with Example 3.1.)

Now $\text{Lim}_{x\to 0} x^2 \sin\frac{1}{x} = \text{Lim}_{y\to+\infty} \frac{\sin y}{y^2}$. But $-1 \leq \sin y \leq 1$, and so $\text{Lim}_{y\to+\infty} \frac{\sin y}{y^2} = 0$.

Hence $x_n \to 0$ implies $f(x_n) \to 0 = f(0)$. Thus f is also continuous at $x = 0$.

Consider now the differential of f. By the product rule, since $f = gh$,

$$d(gh)(x) = x^2 dh(x) + \left(\sin\frac{1}{x}\right) dg(x).$$

Since $h(x) = \rho[m(x)]$, by the chain rule,

$$\begin{aligned} dh(x) &= d\rho\big(m(x)\big) \cdot\ dm(x) \\ &= \cos\left[\big(m(x)\big)\left(-\frac{1}{x^2}\right)\right] \\ &= -\frac{1}{x^2}\cos\frac{1}{x}. \end{aligned}$$

Thus

$$\begin{aligned} df(x) &= d(gh)(x) \\ &= x^2\left[-\frac{1}{x^2}\cos\frac{1}{x} + 2x\sin\frac{1}{x}\right] \\ &= -\cos\frac{1}{x} + 2x\sin\frac{1}{x}, \end{aligned}$$

for any $x \neq 0$.

Clearly $df(x)$ is defined and continuous at any $x \neq 0$. To determine if $df(0)$ is defined, let $k = \frac{1}{h}$. Then

$$\begin{aligned} \operatorname*{Lim}_{h\to 0_+} \frac{f(0+h) - f(0)}{h} &= \operatorname*{Lim}_{h\to 0_+} \frac{h^2 \sin\frac{1}{h}}{h} \\ &= \operatorname*{Lim}_{h\to 0_+} h\sin\frac{1}{h} \\ &= \operatorname*{Lim}_{k\to +\infty} \frac{\sin k}{k}. \end{aligned}$$

But again $-1 \leq \sin k \leq 1$ for all k, and so $\text{Lim}_{k\to+\infty} \frac{\sin k}{k} = 0$. In the same way $\text{Lim}_{h\to 0_-} \frac{h^2 \sin\frac{1}{h}}{h} = 0$. Thus $\text{Lim}_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$, and so $df(0) = 0$. Hence $df(0)$ is defined and equal to zero.

On the other hand consider $(x_n) \to 0_+$. We show that $\text{Lim}_{x_n\to 0_+}\ df(x_n)$ does not exist. By the above $\text{Lim}_{x\to 0_+} df(x) = \text{Lim}_{x\to 0_+}[2x\sin\frac{1}{x} - \cos\frac{1}{x}]$. While $\text{Lim}_{x\to 0_+} 2x\sin\frac{1}{x} = 0$, there is no limit for $\cos\frac{1}{x}$ as $x \to 0_+$ (see Example 3.1). Thus the function $df : \Re \to \mathcal{L}(\Re, \Re)$ is not continuous at the point $x = 0$.

The reason for the discontinuity of the function df at $x = 0$ is that in any neighbourhood U of the origin, there exist an "infinite" number of non-zero points, x', such that $df(x') = 0$. We return to this below.

4.2 C^r-Differentiable Functions

4.2.1 The Hessian

Suppose that $f : X \to Y$ is a C^1-differentiable map, with domain $U \subset X$. Then as we have seen $df : U \to \mathcal{L}(X, Y)$ where $\mathcal{L}(X, Y)$ is the topological vector space of linear maps from X to Y with the norm

$$\|k\| = \sup\left\{ \left\|k(x)\right\|_Y : x \in X \text{ s.t. } \|x\|_X = 1\right\}.$$

Since both U and $\mathcal{L}(X, Y)$ are normed vector spaces, and df is continuous, df may itself be differentiable at a point $x \in U$. If df is differentiable, then its derivative at x is written $d^2 f(x)$, and will itself be a linear approximation of the map df from X to $\mathcal{L}(X, Y)$. If df is C^1, then df will be continuous, and $d^2 f(x)$ will also be a continuous map. Thus $d^2 f(x) \in \mathcal{L}(X, \mathcal{L}(X, Y))$.

When $d^2f : U \to \mathcal{L}(X, \mathcal{L}(X, Y))$ is itself continuous, and f is C^1-differentiable, then say f is C^2-differentiable. Let $C_2(U, Y)$ be the set of C^2-differentiable maps on U. In precisely the same way say that f is C^r-differentiable iff f is C^{r-1}-differentiable, and the rth derivative $df : U \to \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X, \ldots)))$ is continuous.

The map is called *smooth* or C^∞ if $d^r f$ is continuous for all r.

Now the second derivative $d^2f(x)$ satisfies $d^2f(x)(h)(k) \in Y$ for vectors $h, k \in X$. Moreover $d^2f(x)(h)$ is a linear map from X to Y and $d^2f(x)$ is a linear map from X to $\mathcal{L}(X, Y)$.

Thus $d^2f(x)$ is linear in both factors h and k. Hence $d^2f(x)$ may be regarded as a map

$$H(x) : X \times X \to Y$$

where $H(x)(h, k) = d^2f(x)(h)(k) \in Y$.

Moreover $d^2f(x)$ is linear in both h and k, and so $H(x)$ is linear in both h and k. As in Sect. 2.3.3, we say $H(x)$ is bilinear.

Let $L^2(X; Y)$ be the set of *bilinear maps* $X \times X \to Y$. Thus $H \in L^2(X; Y)$ iff

$$H(\alpha_1 h_1 + \alpha_2 h_2, k) = \alpha_1 H(h_1, k) + \alpha_2 H(h_2, k)$$

$$H(h, \beta_1 k_1 + \beta_2 k_2) = \beta_1 H(h, k_1) + \beta_2 H(h, k_2)$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in Re, h, h_1, h_2, k, k_1, k_2 \in X$.

Since X is a finite-dimensional normed vector space, so is $X \times X$, and thus the set of bilinear maps $L^2(X; Y)$ has a norm topology. Write $\mathcal{L}^2(X, Y)$ when the set of bilinear maps has this topology. The continuity of the second differential $d^2f : U \to \mathcal{L}(X, \mathcal{L}(X, Y))$ is equivalent to the continuity of the map $H : U \to \mathcal{L}^2(X; Y)$, and we may therefore regard d^2f as a map $d^2f : U \to \mathcal{L}^2(X; Y)$. In the same way we may regard $d^r f$ as a map $d^r f : U \to \mathcal{L}^r(X; Y)$ where $\mathcal{L}^r(X; Y)$ is the set of maps $X^r \to Y$ which are linear in each component, and is endowed with the norm topology.

Suppose now that $f : \Re^n \to \Re$ is a C^2-map, and consider a point $x = (x_1, \ldots, x_n)$ where the coordinates are chosen with respect to the standard basis. As we have seen the differential $df : U \to \mathcal{L}(\Re^n, \Re)$ can be represented by a continuous function

$$Df : x \to \left(\left.\frac{\partial f}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial f}{\partial x_n}\right|_x \right).$$

Now let $\partial f_j : U \to \Re$ be the continuous function $x \to \frac{\partial f}{\partial x_j}|_x$. Clearly the differential of ∂f_j will be

$$x \to \left(\left.\frac{\partial}{\partial x_1}(\partial f_j)\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}(\partial f_j)\right|_x \right);$$

write $\frac{\partial}{\partial x_i}(\partial f_j)|_x = \partial f_{ji} = \frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})|_x$.

Then the differential $d^2 f(x)$ can be represented by the matrix array

$$(\partial f_{ji})_x = \begin{pmatrix} \frac{\partial}{\partial x_1}(\frac{\partial f}{\partial x_1}) & \dots & \frac{\partial}{\partial x_1}(\frac{\partial f}{\partial x_n}) \\ \vdots & \vdots & \\ \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_1}) & \dots & \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_n}) \end{pmatrix}_x .$$

This $n \times n$ matrix we shall also write as $Hf(x)$ and call the *Hessian matrix* of f at x. Note that $Hf(x)$ is dependent on the particular basis, or coordinate system for X.

From elementary calculus it is known that

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)\bigg|_x = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\bigg|_x$$

and so the matrix $Hf(x)$ is *symmetric*.

Consequently, as in Sect. 2.3.3, $Hf(x)$ may be regarded as a *quadratic form* given by

$$\begin{aligned} D^2 f(x)(h,k) &= \langle h, Hf(x)(k) \rangle \\ &= (h_1, \dots, h_n)\left(\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)\begin{pmatrix} k_1 \\ k_n \end{pmatrix}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_i \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) k_j . \end{aligned}$$

As an illustration if $f : \Re^2 \to \Re$ is C^2 then $D^2 f(x) : \Re^2 \times \Re^2 \to \Re$ is given by

$$\begin{aligned} D^2 f(x)(h,h) &= \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial {x_1}^2 \partial x_2} \\ \frac{\partial^2 f}{\partial {x_1}^2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} \end{pmatrix} \begin{pmatrix} h_1 \\ \\ h_2 \end{pmatrix} \\ &= \left({h_1}^2 \, \frac{\partial^2 f}{\partial {x_1}^2} + 2 h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + {h_2}^2 \, \frac{\partial^2 f}{\partial {x_2}^2} \right)\bigg|_x . \end{aligned}$$

In the case that $f : \Re \to \Re$ is C^2, then $\frac{\partial^2 f}{\partial x^2}|_x$ is simply written as $\frac{d^2 f}{dx^2}|_x$, a real number. Consequently the second differential $D^2 f(x)$ is given by

$$\begin{aligned} D^2 f(x)(h,h) &= h\left(\frac{d^2 f}{dx^2}\bigg|_x\right)h \\ &= h^2 \frac{d^2 f}{dx^2}\bigg|_x . \end{aligned}$$

We shall not distinguish in this case between the linear map $D^2 f(x) : \Re^2 \to \Re$ and the real number $\frac{d^2 f}{dx^2}|_x$.

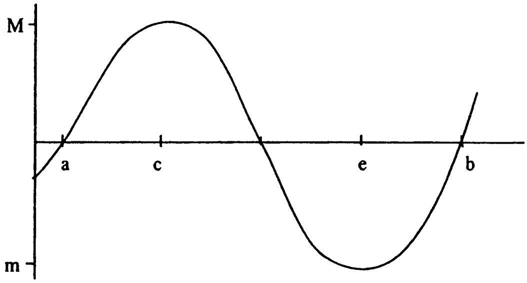


Fig. 4.4 Rolle's Theorem

4.2.2 Taylor's Theorem

From the definition of the derivative of a function $f : X \to Y$, $df(x)$ is the linear approximation to f in the sense that $f(x+h) - f(x) = df(x)(h) + \|h\|_x \mu(h)$ where the "error" $\|h\|_x \mu(h)$ approaches zero as h approaches zero. Taylor's Theorem is concerned with the "accuracy" of this approximation for a small vector h, by using the higher order derivatives. Our principal tool in this is the following. If $f : U \subset X \to \Re$ and the convex hull $[x, x+h]$ of the points x and $x+h$ belongs to U, then there is some point $z \in [x, x+h]$ such that $f(x+h) = f(x) + df(z)(h)$.

To prove this result we proceed as follows.

Lemma 4.1 (Rolle's Theorem) *Let $f : U \to \Re$ where U is an open set in $\Re$ containing the compact interval $I = [a, b]$, and $a < b$. Suppose that f is continuous and differentiable on U, and that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $df(c) = 0$.*

Proof From the Weierstrass Theorem (and Lemma 3.16) f attains its upper and lower bounds on the compact interval, I. Thus there exists finite $m, M \in \Re$ such that $m \leq f(x) \leq M$ for all $x \in I$.

If f is constant on I, so $m = f(x) = M, \forall x \in I$, then clearly $df(x) = 0$ for all $x \in I$.

Then there exists a point c in the interior (a, b) of I such that $df(c) = 0$. Suppose that f is not constant. Since f is continuous and I is compact, there exist points $c, e \in I$ such that $f(c) = M$ and $f(e) = m$. Suppose that neither c nor e belong to (a, b). In this case we obtain $a = e$ and $b = c$, say. But then $M = m$ and so f is the constant function. When f is not the constant function either c or e belongs to the interior (a, b) of I.

1. Suppose $c \in (a, b)$. Clearly $f(c) - f(x) \geq 0$ for all $x \in I$. Since $c \in (a, b)$ there exists $x \in I$ s.t. $x > c$, in which case $\frac{f(x)-f(c)}{x-c} \leq 0$. On the other hand there exists $x \in I$ s.t. $x < c$ and $\frac{f(x)-f(c)}{x-c} \geq 0$. By the continuity of df at x, $df(c) = \text{Lim}_{x \to c_+} \frac{f(x)-f(c)}{x-c} = \text{Lim}_{x \to c_-} \frac{f(x)-f(c)}{x-c} = 0$. Since $c \in (a, b)$ and $df(x) = 0$ we obtain the result. See Fig. 4.4.
2. If $e \in (a, b)$ then we proceed in precisely the same way to show $df(e) = 0$. Thus there exists some point $c \in (a, b)$, say, such that $df(c) = 0$. □

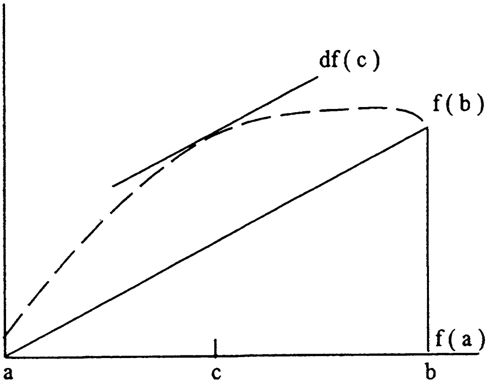


Fig. 4.5 Lemma 4.2

Note that when both c and e belong to the interior of I, then these maximum and minimum points for the function f are critical points in the sense that the derivative is zero.

Lemma 4.2 *Let $f : \Re \to \Re$ where f is continuous on the interval $I = [a, b]$ and df is continuous on (a, b). Then there exists a point $c \in (a, b)$ such that $df(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof Let $g(x) = f(b) - f(x) - k(b - x)$ and $k = \frac{f(b)-f(a)}{b-a}$. Clearly, $g(a) = g(b) = 0$. By Rolle's Theorem, there exists some point $c \in (a, b)$ such that $dg(c) = 0$. But $dg(c) = k - df(c)$. Thus $df(c) = \frac{f(b)-f(a)}{b-a}$. See Fig. 4.5. □

Lemma 4.3 *Let $f : \Re \to \Re$ be continuous and differentiable on an open set containing the interval $[a, a + h]$. Then there exists a number $t \in (0, 1)$ such that*

$$f(a + h) = f(a) + df(a + th)(h).$$

Proof Put $b = a + h$. By the previous lemma there exists $c \in (a, a + h)$ such that

$$df(c) = \frac{f(b) - f(a)}{b - a}.$$

Let $t = \frac{c-a}{b-a}$. Clearly $t \in (0, 1)$ and $c = a + th$. But then $df(a + th) : \Re \to \Re$ is the linear map given by $df(a + th)(h) = f(b) - f(a)$, and so $f(a + h) = f(a) + df(a + th)(h)$. □

Mean Value Theorem *Let $f : U \subset X \to \Re$ be a differentiable function on U, where U is an open set in the normed vector space X. Suppose that the line segment*

$$[x, x + h] = \{z : z = x + th \text{ where } t \in [0, 1]\}$$

belongs to U. *Then there is some number* $t \in (0, 1)$ *such that*

$$f(x+h) = f(x) + df(x+th)(h).$$

Proof Define $g : [0, 1] \to \Re$ by $g(t) = f(x + th)$. Now g is the composition of the function

$$\rho : [0, 1] \to U : t \to x + th$$

with $f : [x, x + h] \to \Re$.

Since both ρ and f are differentiable, so is g. By the chain rule,

$$\begin{aligned} dg(t) &= df\big(\rho(t)\big) \circ d\rho(t) \\ &= df(x+th)(h). \end{aligned}$$

By Lemma 4.3, there exists $t \in (0, 1)$ such that $dg(t) = \frac{g(1)-g(0)}{1-0}$. But $g(1) = f(x + h)$ and $g(0) = f(x)$. Hence $df(x + th)(h) = f(x + h) - f(x)$. □

Lemma 4.4 *Suppose* $g : U \to \Re$ *is a* C^2*-map on an open set* U *in* $\Re$ *containing the interval* $[0, 1]$. *Then there exists* $\theta \in (0, 1)$ *such that*

$$g(1) = g(0) + dg(0) + \frac{1}{2}d^2g(\theta).$$

Proof (Note here that we regard $dg(t)$ and $d^2g(t)$ as real numbers.) Now define $k(t) = g(t) - g(0) - tdg(0) - t^2[g(1) - g(0) - dg(0)]$.

Clearly $k(0) = k(1) = 0$, and so by Rolle's Theorem, there exists $\theta \in (0, 1)$ such that $dk(\theta) = 0$. But $dk(t) = dg(t) - dg(0) - 2t[g(1) - g(0) - dg(0)]$. Hence $dk(0) = 0$.

Again by Rolle's Theorem, there exists $\theta' \in (0, \theta)$ such that $d^2k(\theta') = 0$. But $d^2k(\theta') = d^2g(\theta') - 2[g(1) - g(0) - dg(0)]$.

Hence $g(1) - g(0) - dg(0) = \frac{1}{2}d^2g(\theta')$ for some $\theta' \in (0, 1)$. □

Lemma 4.5 *Let* $f : U \subset X \to \Re$ *be a* C^2*-function on an open set* U *in the normed vector space* X. *If the line segment* $[x, x + h]$ *belongs to* U, *then there exists* $z \in (x, x + h)$ *such that*

$$f(x+h) = f(x) + df(x)(h) + \frac{1}{2}d^2f(z)(h,h).$$

Proof Let $g : [0, 1] \to \Re$ by $g(t) = f(x + th)$. As in the mean value theorem, $dg(t) = df(x + th)(h)$. Moreover $d^2g(t) = d^2f(x + th)(h, h)$.

By Lemma 4.4, $g(1) = g(0) + dg(0) + \frac{1}{2}d^2g(\theta')$ for some $\theta' \in (0, 1)$.

Let $z = x + \theta' h$. Then $f(x + h) = f(x) + df(x)(h) + \frac{1}{2}d^2f(h, h)$. □

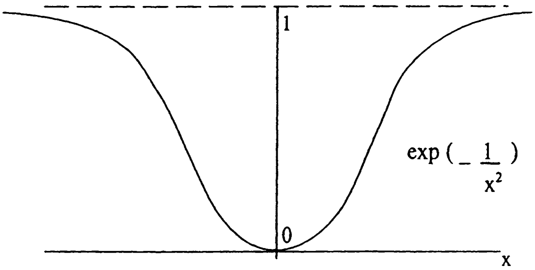


Fig. 4.6 The flat function

Taylor's Theorem *Let $f: U \subset X \to \Re$ be a smooth (or C^∞-) function on an open set U in the normed vector space X. If the line segment $[x, x+h]$ belongs to U, then $f(x+h) = f(x) + \sum_{r=1}^{n} \frac{1}{r!} d^r f(x)(h, \dots, h) + R_n(h)$ where the error term $R_n(h) = \frac{1}{(n+1)!} d^{n+1} f(z)(h, \dots, h)$ and $z \in (x, x+h)$.*

Proof By induction on Lemma 4.5, using the mean value theorem. □

The *Taylor series of f at x to order k* is

$$\big[f(x)\big]_k = f(x) + \sum_{r=1}^{k} \frac{1}{r!} d^r f(x)(h, \dots, h).$$

When f is C^∞, then $[f(x)]_k$ exists for all k. In the case when $X = \Re^n$ and the error term $R_k(h)$ approaches zero as $k \to \infty$, then the Taylor series $[f(x)]_k$ will converge to $f(x+h)$.

In general however $[f(x)]_k$ need not converge, or if it does converge then it need not converge to $f(x+h)$.

Example 4.3 To illustrate this, consider the *flat* function $f: \Re \to \Re$ given by

$$\begin{aligned} f(x) &= \exp\left(-\frac{1}{x^2}\right), \quad x \neq 0, \\ &= 0 \quad x = 0. \end{aligned}$$

Now $Df(x) = -\frac{2}{x^3} \exp(-\frac{1}{x^2})$ for $x \neq 0$. Since $y^{\frac{3}{2}} e^{-y} \to 0$ as $y \to \infty$, we obtain $Df(x) \to 0$ as $x \to 0$. However

$$Df(0) = \operatorname*{Lim}_{h \to 0} \frac{f(0+h) - f(0)}{h} = \operatorname*{Lim}_{h \to 0} \frac{1}{h} \exp\left(-\frac{1}{h^2}\right) = 0.$$

Thus f is both continuous and C^1 at $x = 0$. In the same way f is C^r and $D^r f(0) = 0$ for all $r > 1$. Thus the Taylor series is $[f(0)]_k = 0$. However for small $h > 0$ it is evident that $f(0+h) \neq 0$. Hence the Taylor series cannot converge to f.

These remarks lead directly to classification theory in differential topology, and are beyond the scope of this work. The interested reader may refer to Chillingsworth (1976) for further discussion.

4.2.3 Critical Points of a Function

Suppose now that $f : U \subset \Re^n \to \Re$ is a C^2-map. Once a coordinate basis is chosen for $\Re^n$, then $D^2 f(x)$ may be regarded as a quadratic form. In matrix notation this means that

$$D^2 f(x)(h,h) = h^t Hf(x)h.$$

As we have seen in Chap. 2 if the Hessian matrix $Hf(x) = (\partial f_{ji})_x$ has all its eigenvalues positive, then $D^2 f(x)(h,h) > 0$ for all $h \in \Re^n$, and so $Hf(x)$ will be *positive definite*.

Conversely $Hf(x)$ is negative definite iff all its eigenvalues are strictly negative.

Lemma 4.6 *If $f : U \subset \Re^n \to \Re$ is a C^2 map on U, and the Hessian matrix $Hf(x)$ is positive (negative) definite at x, then there is a neighbourhood V of x in U s. t. $Hf(y)$ is positive (negative) definite for all $y \in V$.*

Proof If $Hf(x)$ is positive definite, then as we have seen there are n different algebraic relationships between the partial derivatives $\partial f_{ji}(x)$ for $j = 1, \dots, n$ and $i = 1, \dots, n$, which characterise the roots $\lambda_1(x), \dots, \lambda_n(x)$ of the characteristic equation

$$\left| Hf(x) - \lambda(x)I \right| = 0.$$

But since f is C^2, the map $D^2 f : U \to L^2(\Re^n; \Re)$ is continuous. In particular this implies that for each i, j the map $x \to \frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})|_x = \partial f_{ji}(x)$ is continuous. Thus if $\partial f_{ji}(x) > 0$ then there is a neighbourhood V of x in U such that $\partial f_{ji}(y) > 0$ for all $y \in V$. Moreover if

$$C(x) = C\big(\partial f_{ji}(x) : i = 1, \dots, n; j = 1, \dots, n\big)$$

is an algebraic sentence in $\partial f_{ji}(x)$ such that $C(x) > 0$ then again there is a neighbourhood V of x in U such that $C(y) > 0$ for all $y \in V$.

Thus there is a neighborhood V of x in U such that $\lambda_i(x) > 0$ for $i = 1, \dots, n$ implies $\lambda_i(y) > 0$ for $i = 1, \dots, n$, and all $y \in V$. Hence $Hf(x)$ is positive definite at $x \in U$ implies that $Hf(y)$ is positive definite for all y in some neighborhood of x in U. The same argument holds if $Hf(x)$ is negative definite at x. □

Definition 4.2 Let $f : U \subset \Re^n \to \Re$ where U is an open set in $\Re^n$. A point x in U is called

1. a *local strict maximum* of f in U iff there exists a neighbourhood V of x in U such that $f(y) < f(x)$ for all $y \in V$ with $y \neq x$;
2. a *local strict minimum* of f in U iff there exists a neighbourhood V of x in U such that $f(y) > f(x)$ for all $y \in V$ with $y \neq x$;
3. a *local maximum* of f in U iff there exists a neighbourhood V of x in U such that $f(y) \leq f(x)$ for $y \in V$;

4. a *local minimum* of f in U iff there exists a neighbourhood V of x in U such that $f(y) \geq f(x)$ for all $y \in V$.
5. Similarly a *global* (strict) *maximum* (or minimum) on U is defined by requiring $f(y) < (\leq, >, \geq) f(x)$ respectively on U.
6. If f is C^1-differentiable then x is called a *critical point* iff $df(x) = 0$, the zero map from $\Re^n$ to $\Re$.

Lemma 4.7 *Suppose that $f : U \subset \Re^n \to \Re$ is a C^2-function on an open set U in $\Re^n$. Then f has a local strict maximum (minimum) at x if*

1. *x is a critical point of f and*
2. *the Hessian $Hf(x)$ is negative (positive) definite.*

Proof Suppose that x is a critical point and $Hf(x)$ is negative definite. By Lemma 4.5

$$f(y) = f(x) + df(x)(h) + \frac{1}{2}d^2 f(z)(h, h)$$

whenever the line segment $[x, y] \in U$, $h = y - x$ and $z = x + \theta h$ for some $\theta \in (0, 1)$.

Now by the assumption there is a coordinate base for $\Re^n$ such that $Hf(x)$ is negative definite. By Lemma 4.6, there is a neighbourhood V of x in U such that $Hf(y)$ is negative definite for all y in V. Let $N_\epsilon(x) = \{x + h : \|h\| < \epsilon\}$ be an ϵ-neighborhood in V of x. Let $S_{\frac{\epsilon}{2}}(\underline{0}) = \{h \in \Re^n : \|h\| = \frac{1}{2}\epsilon\}$.

Clearly any vector $x + h$, where $h \in S_{\frac{\epsilon}{2}}(\underline{0})$ belongs to $N_\epsilon(x)$, and thus V. Hence $Hf(z)$ is negative definite for any $z = x + \theta h$, where $h \in S_{\frac{\epsilon}{2}}(\underline{0})$, and $\theta \in (0, 1)$. Thus $Hf(z)(h, h) < 0$, and any $z \in [x, x + h]$.

But also by assumption $df(x) = 0$ and so $df(x)(h) = 0$ for all $h \in \Re^n$. Hence $f(x + h) = f(x) + \frac{1}{2}d^2 f(z)(h, h)$ and so $f(x + h) < f(x)$ for $h \in S_{\frac{\epsilon}{2}}(\underline{0})$. But the same argument is true for any h satisfying $\|h\| < \frac{\epsilon}{2}$. Thus $f(y) < f(x)$ for all y in the open ball of radius $\frac{\epsilon}{2}$ about x. Hence x is a local strict maximum. The same argument when $Hf(x)$ is positive definite shows that x must be a local strict minimum. □

In Sect. 2.3 we defined a quadratic form $A^* : \Re^n \times \Re^n \to \Re$ to be non-degenerate iff the *nullity* of A^*, namely $\{x : A^*(x, x) = 0\}$, is $\{\underline{0}\}$. If x is a critical point of a C^2-function $f : U \subset \Re^n \to \Re$ such that $d^2 f(x)$ is non-degenerate (when regarded as a quadratic form), then call x a *non-degenerate critical point*.

The dimension (s) of the subspace of $\Re^n$ on which $d^2 f(x)$ is negative definite is called the *index* of f at x, and x is called a critical point of index s.

If x is a non-degenerate critical point, then when any coordinate system for $\Re^n$ is chosen, the Hessian $Hf(x)$ will have s eigenvalues which are negative, and $n - s$ which are positive.

For example if $f : \Re \to \Re$, then only three cases can occur at a critical point

1. $d^2 f(x) > 0 : x$ is a local minimum;

2. $d^2 f(x) < 0 : x$ is a local maximum;
3. $d^2 f(x) = 0 : x$ is a degenerate critical point.

If $f : \Re^2 \to \Re$ then a number of different cases can occur. There are three non-degenerate cases:

1. $Hf(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, say, with respect to a suitable basis; x is a local minimum since both eigenvalues are positive. Index $= 0$.
2. $Hf(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; x is a local maximum, both eigenvalues are negative. Index $= 2$.
3. $Hf(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; x is a *saddle point* or index 1 non-degenerate critical point.

In the degenerate cases, one eigenvalue is zero and so det $(Hf(x)) = 0$.

Example 4.4 Let $f : \Re^2 \to \Re : (x, y) \to xy$. The differential at (x, y) is $Df(x, y) = (y, x)$. Thus $H = Hf(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Clearly $(0, 0)$ is the critical point. Moreover $|H| = -1$ and so $(0, 0)$ is non-degenerate. The eigenvalues λ_1, λ_2 of the Hessian satisfy $\lambda_1 + \lambda_2 = 0, \lambda_1\lambda_2 = -1$. Thus $\lambda_1 = 1, \lambda_2 = -1$. Eigenvectors of $Hf(x, y)$ are $v_1 = (1, 1)$ and $v_2 = (1, -1)$ respectively. Let $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ be the normalized eigenvector (basis change) matrix, so $P^{-1} = P$. Then

$$\wedge = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider a vector $h = (h_1, h_2) \in \Re^2$. In matrix notation, $h^t Hh = h^t P \wedge P^{-1}h$. Now

$$P(h) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} h_1 + h_2 \\ h_1 - h_2 \end{pmatrix}.$$

Thus $h^t Hh = \frac{1}{2}[(h_1 + h_2)^2 - (h_1 - h_2)^2] = 2h_1h_2$.

It is clear that $D^3 f(0, 0) = 0$. Hence from Taylor's Theorem,

$$f(0 + h_1, 0 + h_2) = f(0) + Df(0)(h) + \frac{1}{2}D^2 f(0)(h, h)$$

and so $f(h_1, h_2) = \frac{1}{2}h^t Hh = h_1h_2$.

Suppose we make the basis change represented by P. Then with respect to the new basis $\{v_1, v_2\}$ the point (x, y) has coordinates $(\frac{1}{\sqrt{2}}(h_1 + h_2), \frac{1}{\sqrt{2}}(h_1 - h_2))$.

Thus f can be represented in a neighbourhood of the origin as

$$(h_1, h_2) \to \frac{1}{\sqrt{2}}(h_1 + h_2)\frac{1}{\sqrt{2}}(h_1 - h_2) = \frac{1}{2}\left(h_1{}^2 - h_2{}^2\right).$$

Notice that with respect to this new coordinate system

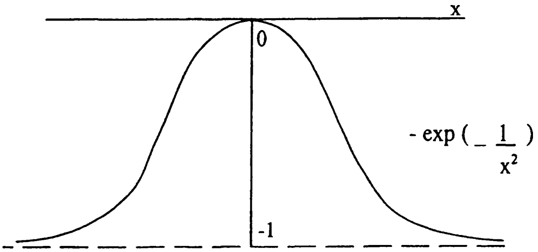


Fig. 4.7 The flat function

$$Df(h_1, h_2) = (h_1, -h_2), \quad \text{and}$$
$$Hf(h_1, h_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the eigenspace $E_1 = \{(x, y) \in \Re^2 : x = y\}$ the Hessian has eigenvalue 1 and so f has a local minimum at 0, when restricted to E_1.

Conversely in the eigenspace $E_{-1} = \{(x, y) \in \Re^2 : x + y = 0\}$, f has a local maximum at 0.

More generally when $f : \Re^2 \to \Re$ is a quadratic function in x, y, then at a critical point f can be represented, after a suitable coordinate change, either as

1. $(x, y) \to x^2 + y^2$ an index 0, minimum
2. $(x, y) \to -x^2 - y^2$ an index 2, maximum
3. $(x, y) \to x^2 - y^2$ an index 1 saddle point.

Example 4.5 Let $f : \Re^3 \to \Re : (x, y, z) \to x^2 + 2y^2 + 3z^2 + xy + xz$. Therefore $Df(x, y, z) = (2x + y + z, 4y + x, 6z + x)$.

Clearly $(0, 0, 0)$ is the only critical point.

$$H = Hf(x, y, z) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 6 \end{pmatrix}$$

$|H| = 38$ and so $(0, 0, 0)$ is non-degenerate. It can be shown that the eigenvalues of the matrix are strictly positive, and so H is positive definite and $(0, 0, 0)$ is a minimum of the function. Thus f can be written in the form $(u, v, w) \to au^2 + bv^2 + cw^2$, where $a, b, c > 0$ and (u, v, w) are the new coordinates after a (linear) basis change.

Notice that Lemma 4.7 does not assert that a local strict maximum (or minimum) of a C^1-function must be a critical point where the Hessian is negative (respectively positive) definite.

For example consider the "flat" function $f : \Re \to \Re$ given by $f(x) = -\exp(-\frac{1}{x^2})$, and $f(0) = 0$. As we showed in Example 4.3, $df(0) = d^2f(0) = 0$.

Yet clearly $0 > -\exp(-\frac{1}{a^2})$ for any $a \neq 0$. Thus 0 is a global strict maximum of f on $\Re$, although d^2f is not negative definite. See Fig. 4.7.

A local maximum or minimum must however be a critical point. If the point is a local maximum for example then the Hessian can have no positive eigenvalues. Consequently $D^2 f(x)(h,h) \leq 0$ for all h, and so the Hessian must be negative semi-definite. As the flat function indicates, the Hessian may be identically zero at the local maximum.

Lemma 4.8 *Suppose that $f : U \subset \Re^n \to \Re$ is a C^2-function on an open set u in $\Re^n$. Then f has a local maximum or minimum at x only if x is a critical point of f.*

Proof Suppose that $df(x) \neq 0$. Then we seek to show that x can be neither a local maximum nor minimum at x.

Since $df(x)$ is a linear map from $\Re^n$ to $\Re$ it is possible to find some vector $h \in \Re^n$ such that $df(x)(h) > 0$.

Choose h sufficiently small so that the line segment $[x, x+h]$ belongs to U.

Now f is C^1, and so $df : U \to \mathcal{L}(\Re^n, \Re)$ is continuous. In particular since $df(x) \neq 0$ then for some neighbourhood V of x in U, $df(y) \neq 0$ for all $y \in V$. Thus for all $y \in V, df(y)(h) > 0$ (see Lemma 4.18 for more discussion of this phenomenon).

By the mean value theorem there exists $t \in (0,1)$ such that $f(x+h) = f(x) + df(x+th)(h)$. By choosing h sufficiently small, the vector $x+th \in V$. Hence $f(x+h) > f(x)$. Consequently x cannot be a local maximum.

But in precisely the same way if $df(x) \neq 0$ then it is possible to find h such that $df(x)(h) < 0$. A similar argument can then be used to show that $f(x+h) < f(x)$ and so x cannot be a local minimum. Hence if x is either a local maximum or minimum of f then it must be a critical point of f. □

Lemma 4.9 *Suppose that $f : U \subset \Re^n \to \Re$ is a C^2-function on an open set U in $\Re^n$. If f has a local maximum at x then the Hessian $d^2 f$ at the critical point must be negative semi-definite* (*i.e.*, $d^2 f(x)(h,h) \leq 0$ *for all* $h \in \Re^n$).

Proof We may suppose that x is a critical point. Suppose further that for some coordinate basis at x, and vector $h \in \Re^n$, $d^2 f(x)(h,h) > 0$. From Lemma 4.6, by the continuity of $d^2 f$ there is a neighbourhood V of x in U such that $d^2 f(x')(h,h) > 0$ for all $x' \in V$.

Choose an ϵ-neighbourhood of x in V, and choose $\alpha > 0$ such that $\|\alpha h\| = \frac{1}{2}\epsilon$. Clearly $x + \alpha h \in V$. By Taylor's Theorem, there exists $z = x + \theta \alpha h, \theta \in (0,1)$, such that $f(x+\alpha h) = f(x) + df(x)(h) + \frac{1}{2} d^2 f(z)(\alpha h, \alpha h)$. But $d^2 f(z)$ is bilinear, so $d^2 f(z)(\alpha h, \alpha h) = \alpha^2 d^2 f(z)(h,h) > 0$ since $z \in V$. Moreover $df(x)(h) = 0$. Thus $f(x+\alpha h) > f(x)$.

Moreover for *any* neighbourhood U of x it is possible to choose ϵ sufficiently small so that $x + \alpha h$ belongs to U. Thus x cannot be a local maximum. □

Similarly if f has a local minimum at x then x must be a critical point with positive semi-definite Hessian.

Example 4.6 Let $f:\Re^2\to\Re:(x,y)\to x^2y^2-4yx^2+4x^2$; $Df(x,y)=(2xy^2-8xy+8x,2x^2y-4x^2)$. Thus (x,y) is a critical point when

1. $x(2y^2-8y+8)=0$ and
2. $2x^2(y-2)=0$.

Now $2y^2-8y+8=2(y-2)^2$. Thus (x,y) is a critical point *either* when $y=2$ or $x=0$.

Let $S(f)$ be the set of critical points. Then $S(f)=V_1\cup V_2$ where

$$V_1=\{(x,y)\in\Re^2:x=0\}$$
$$V_2=\{(x,y)\in\Re^2:y=2\}.$$

Now

$$Hf(x,y)=\begin{pmatrix}2(y-2)^2 & -4x(2-y)\\ -4x(2-y) & 2x^2\end{pmatrix}$$

and so when $(x,y)\in V_1$ then $Hf(x,y)=\left(\begin{smallmatrix}\mu^2 & 0\\ 0 & 0\end{smallmatrix}\right)$, and when $(x,y)\in V_2$, then $Hf(x,y)=\left(\begin{smallmatrix}0 & 0\\ 0 & \tau^2\end{smallmatrix}\right)$.

For suitable μ and τ, any point in $S(f)$ is degenerate. On $V_1\backslash\{(0,0)\}$ clearly a critical point is not negative semi-definite, and so such a point cannot be a local maximum. The same is true for a point on $V_2\backslash\{(0,0)\}$.

Now $(0,0)\in V_1\cap V_2$, and $Hf(0,0)=(0)$. Lemma 4.9 does not rule out $(0,0)$ as a local maximum. However it should be obvious that the origin is a local minimum.

Unlike Examples 4.4 and 4.5 no linear change of coordinate bases transforms the function into a quadratic canonical form.

To find a local maximum point we therefore seek all critical points. Those which have negative definite Hessian must be local maxima. Those points remaining which do not have a negative semi-definite Hessian cannot be local maxima, and may be rejected. The remaining critical points must then be examined.

A C^2-function $f:\Re^n\to\Re$ with a non-degenerate Hessian at every critical point is called a *Morse function*. Below we shall show that any Morse function can be represented in a *canonical* form such as we found in Example 4.4. For such a function, local maxima are precisely those critical points of index n. Moreover, any smooth function with a degenerate critical point can be "approximated" by a Morse function.

Suppose now that we wish to maximise a C^2-function on a compact set K. As we know from the Weierstrass theorem, there does exist a maximum. However, Lemmas 4.8 and 4.9 are now no longer valid and it is possible that a point on the boundary of K will be a local or global maximum but not a critical point. However, Lemma 4.7 will still be valid, and a negative definite critical point will certainly be a local maximum.

A further difficulty arises since a local maximum need not be a global maximum. However, for concave functions, local and global maxima coincide. We discuss maximisation by smooth optimisation on compact convex sets in the next section.

4.3 Constrained Optimisation

4.3.1 Concave and Quasi-concave Functions

In the previous section we obtained necessary and sufficient conditions for a critical point to be a local maximum on an open set U. When the set is not open, then a local maximum need not be a critical point. Previously we have defined the differential of a function only on an open set. Suppose now that $Y \subset \Re^n$ is compact and therefore closed, and has a boundary ∂Y. If df is continuous at each point in the interior, Int (Y) of Y, then we may extend df over Y by defining $df(x)$, at each point x in the boundary, ∂Y of Y, to be the limit $df(x_k)$ for any sequence, (x_k), of points in Int (Y), which converge to x. More generally we shall say a function $f : Y \subset \Re^n \to \Re$ is C^1 on the *admissible* set Y if $df : Y \to \mathcal{L}(\Re^n, \Re)$ is defined and continuous in the above sense at each $x \in Y$. We now give an alternative definition of the differential of a C^1-function $f : Y \to \Re$. Suppose that Y is convex and both x and $x + h$ belong to Y. Then the arc $[x, x + h] = \{z \in \Re^n : z = x + \lambda h, \lambda \in [0, 1]\}$ belongs to Y.

Now $df(x)(h) = \lim_{\lambda \to 0_+} \frac{f(x+\lambda h) - f(x)}{\lambda}$ and thus $df(x)(h)$ is often called the *directional derivative* of f at x in the direction h.

Finding maxima of a function becomes comparatively simple when f is a concave or quasi-concave function (see Sect. 3.4 for definitions of these terms). Our first result shows that if f is a concave function then we may relate $df(x)(y - x)$ to $f(y)$ and $f(x)$.

Lemma 4.10 *If $f : Y \subset \Re^n \to \Re$ is a concave C^1-function on a convex admissible set Y then*

$$df(x)(y - x) \geq f(y) - f(x).$$

Proof Since f is concave

$$f\big(\lambda y + (1 - \lambda)x\big) \geq \lambda f(y) + (1 - \lambda) f(x)$$

for any $\lambda \in [0, 1]$ whenever $x, y \in Y$. But then $f(x + \lambda(y - x) - f(x)) \geq \lambda[f(y) - f(x)]$, and so

$$\begin{aligned} df(x)(y - x) &= \operatorname*{Lim}_{\lambda \to 0_+} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &\geq f(y) - f(x). \end{aligned}$$ □

This enables us to show that for a concave function, f, a critical point of f must be a global maximum when Y is open.

First of all call a function $f : Y \subset \Re^n \to \Re$ *strictly quasi-concave* iff Y is convex and for all $x, y \in Y$

$$f\big(\lambda y + (1 - \lambda)x\big) > \min\big(f(x), f(y)\big) \quad \text{for all } \lambda \in (0, 1).$$

Remember that f is quasi-concave if

$$f\big(\lambda y + (1-\lambda)x\big) \geq \min\big(f(x), f(y)\big) \quad \text{for all } \lambda \in [0,1].$$

As above let $P(x;Y) = \{y \in Y : f(y) > f(x)\}$ be the preferred set of a function f on the set Y. A point $x \in Y$ is a global maximum of f on Y iff $P(x;Y) = \Phi$. When there is no chance of misunderstanding we shall write $P(x)$ for $P(x;Y)$. As shown in Sect. 3.4, if f is (strictly) quasi-concave then, $\forall x \in Y$, the preferred set, $P(x)$, is (strictly) convex.

Lemma 4.11

1. *If $f : Y \subset \Re^n \to \Re$ is a concave or strictly quasi-concave function on a convex admissible set, then any point which is a local maximum of f is also a global maximum.*
2. *If $f : U \subset \Re^n \to \Re$ is a concave C^1-function where U is open and convex, then any critical point of f is a global maximum on U.*

Proof

1. Suppose that f is concave or strictly quasi-concave, and that x is a local maximum but not a global maximum on Y. Then there exists $y \in Y$ such that $f(y) > f(x)$.

   Since Y is convex, the line segment $[x,y]$ belongs to Y. For any neighbourhood U of x in Y there exists some $\lambda^* \in (0,1)$ such that, for $\lambda \in (0,\lambda^*)$, $z = \lambda y + (1-\lambda)x \in U$. But by concavity

   $$f(z) \geq \lambda f(y) + (1-\lambda)f(x) > f(x).$$

   Hence in any neighbourhood U of x in Y there exists a point z such that $f(z) > f(x)$. Hence x cannot be a local maximum. Similarly by strict quasi-concavity

   $$f(z) > \min\big(f(x), f(y)\big) = f(x),$$

   and so, again, x cannot be a local maximum. By contradiction a local maximum must be a global maximum.
2. If f is C^1 and U is open then by Lemma 4.8, a local maximum must be a critical point. By Lemma 4.10, $df(x)(y-x) \geq f(y) - f(x)$. Thus $df(x) = 0$ implies that $f(y) - f(x) \leq 0$ for all $y \in Y$. Hence x is a global maximum of f on Y. □

Clearly if x were a critical point of a concave function on an open set then the Hessian $d^2 f(x)$ must be negative semi-definite. To see this, note that by Lemma 4.11, the critical point must be a global maximum, and thus a local maximum. By Lemma 4.9, $d^2 f(x)$ must be negative semi-definite. The same is true if f is quasi-concave.

Lemma 4.12 *If $f : U \subset \Re^n \to \Re$ is a quasi-concave C^2-function on an open set, then at any critical point, x, $d^2 f(x)$ is negative semi-definite.*

Proof Suppose on the contrary that $df(x)=0$ and $d^2f(x)(h,h)>0$ for some $h\in\Re^n$. As in Lemma 4.6, there is a neighbourhood V of x in U such that $d^2f(z)(h,h)>0$ for all z in V.

Thus there is some $\lambda^*\in(0,1)$ such that, for all $\lambda\in(0,\lambda^*)$, there is some z in V such that

$$\begin{aligned} f(x+\lambda h)&=f(x)+df(x)(\lambda h)+\lambda^2d^2f(z)(h,h),\quad\text{and}\\ f(x-\lambda h)&=f(x)+df(x)(-\lambda h)+(-\lambda h)^2d^2f(z)(h,h),\end{aligned}$$

where $[x-\lambda h,x+\lambda h]$ belongs to U. Then $f(x+\lambda h)>f(x)$ and $f(x-\lambda h)>f(x)$. Now $x\in[x-\lambda h,x+\lambda h]$ and so by quasi-concavity,

$$f(x)\geq\min\bigl(f(x+\lambda h),f(x-\lambda h)\bigr).$$

By contradiction $d^2f(x)(h,h)\leq 0$ for all $h\in\Re^n$. □

For a concave function, f, on a convex set Y, $d^2f(x)$ is negative semi-definite not just at critical points, but at every point in the interior of Y.

Lemma 4.13

1. *If $f:U\subset\Re^n\to\Re$ is a concave C^2-function on an open convex set U, then $d^2f(x)$ is negative semi-definite for all $x\in U$.*
2. *If $f:Y\subset\Re^n\to\Re$ is a C^2-function on an admissible convex set Y and $d^2f(x)$ is negative semi-definite for all $x\in Y$, then f is concave.*

Proof

1. Suppose there exists $x\in U$ and $h\in\Re^n$ such that $d^2f(x)(h,h)>0$. By the continuity of d^2f, there is a neighbourhood V of x in U such that $d^2f(z)(h,h)>0$, for $z\in V$. Choose $\theta\in(0,1)$ such that $x+\theta h\in V$. Then by Taylor's theorem there exists $z\in(x,x+\theta h)$ such that

$$\begin{aligned} f(x+\theta h)&=f(x)+df(x)(\theta h)+\frac{1}{2}d^2f(z)(\theta h,\theta h)\\ &>f(x)+df(x)(\theta h).\end{aligned}$$

But then $df(x)(\theta h)<f(x+\theta h)-f(x)$. This contradicts $df(x)(y-x)\geq f(y)-f(x)$, $\forall x,y$ in U. Thus $d^2f(x)$ is negative semi-definite.

2. If $x,y\in Y$ and Y is convex, then the arc $[x,y]\subset Y$. Hence there is some $z=\lambda x+(1-\lambda)y$, where $\lambda\in(0,1)$, such that

$$\begin{aligned} f(y)&=f(x)+df(x)(y-x)+d^2f(z)(y-x,y-x)\\ &\leq f(x)+df(x)(y-x).\end{aligned}$$

But in the same way $f(x)-f(z)\leq df(z)(x-z)$ and $f(y)-f(z)\leq df(z)(y-z)$. Hence

$$f(z)\geq\lambda\bigl[f(x)-df(z)(x-z)\bigr]+(1-\lambda)\bigl[f(y)-df(z)(y-z)\bigr].$$

Now $\lambda df(z)(x-z)+(1-\lambda)df(z)(y-z)=df(z)[\lambda x+(1-\lambda)y-z]=df(z)(0)=0$, since $df(z)$ is linear.

Hence $f(z)\geq\lambda f(x)+(1-\lambda)f(y)$ for any $\lambda\in[0,1]$ and so f is concave. □

We now extend the analysis to a quasi-concave function and characterise the preferred set $P(x;Y)$.

Lemma 4.14 *Suppose $f:Y\subset\Re^n\to\Re$ is a quasi-concave C^1-function on the convex admissible set Y.*

1. *If $f(y)\geq f(x)$ then $df(x)(y-x)\geq 0$.*
2. *If $df(x)(y-x)>0$, then there exists some $\lambda^*\in(0,1)$ such that $f(z)>f(x)$ for any*

$$z=\lambda y+(1-\lambda)x \quad \textit{where } \lambda\in\big(0,\lambda^*\big).$$

Proof

1. By the definition of quasi-concavity $f(\lambda y+(1-\lambda)x)>f(x)$ for all $\lambda\in[0,1]$. But then, as in the analysis of a concave function,

$$f\big(x+\lambda(y-x)\big)-f(x)\geq 0$$

and so $df(x)(y-x)=\mathrm{Lim}_{\lambda\to 0_+}\frac{f(x+\lambda(y-x))-f(x)}{\lambda}\geq 0$.

2. Now suppose $f(x)\geq f(z)$ for all z in the line segment $[x,y]$. Then $df(x)(y-x)=\mathrm{Lim}_{\lambda\to 0_+}\frac{f(x+\lambda)(y-x)-f(x)}{\lambda}\leq 0$, contradicting $df(x)(y-x)>0$.

Thus there exists $z^*=\lambda^*y+(1-\lambda^*)x$ such that $f(z^*)>f(x)$. But then for all $z\in(x,\lambda^*y+(1-\lambda^*)x)$, $f(z)>f(x)$. □

The property that $f(y)\geq f(x)\Rightarrow df(x)(y-x)\geq 0$ is often called *pseudo-concavity*.

We shall also say that $f:Y\subset\Re^n\to\Re$ is *strictly pseudo-concave* iff for any x,y in Y, with $y\neq x$ then $f(y)\geq f(x)$ implies that $df(x)(y-x)>0$. (Note we do not require Y to be convex, but we do require it to be admissible.)

Lemma 4.15 *Suppose $f:Y\subset\Re^n\to\Re$ is a strictly pseudo-concave function on an admissible set Y.*

1. *Then f is strictly quasi-concave when Y is convex.*
2. *If x is a critical point, then it is a global strict maximum.*

Proof

1. Suppose that f is not strictly quasi-concave. Then for some $x,y\in Y$, $f(\lambda^*y+(1-\lambda^*)x)\leq\min\,(f(x),f(y))$ for some $\lambda^*\in(0,1)$. Without loss of generality suppose $f(x)\leq f(y)$, and $f(\lambda y+(1-\lambda)x)\leq f(x)$ for all $\lambda\in(0,1)$. Then $df(x)(y-x)=\mathrm{Lim}_{\lambda\to 0_+}\frac{f(\lambda y+(l-\lambda)x)-f(x)}{\lambda}\leq 0$. But by strict pseudo-concavity, we require that $df(x)(y-x)\geq 0$. Thus f must be strictly quasi-concave.

2. If $df(x)=0$ then $df(x)(y-x)=0$ for all $y \in U$. Hence $f(y) < f(x)$ for all $y \in U$, $y \neq x$. Thus x is a global strict maximum. □

As we have observed, when f is a quasi-concave function on Y, the preferred set $P(x;Y)$ is a convex set in Y. Clearly when f is continuous then $P(x;Y)$ is open in Y. As we might expect from the Separating Hyperplane Theorem, $P(x;Y)$ will then belong to an "open half space". To see this note that Lemma 4.14 establishes (for a quasi-concave C^1 function f) that the weakly preferred set

$$R(x;Y) = \{y \in Y : f(y) \geq f(x)\}$$

belongs to the closed half-space

$$H(x;Y) = \{y \in Y : df(x)(y-x) \geq 0\}.$$

When Y is open and convex the boundary of $H(x;Y)$ is the hyperplane $\{y \in Y : df(x)(y-x)=0\}$ and $H(x;Y)$ has relative interior $\overset{0}{H}(x;Y) = \{y \in Y : df(x)(y-x) > 0\}$. Write $H(x), R(x), P(x)$ for $H(x;Y), R(x;Y), P(x;Y)$, *etc.*, when Y is understood.

Lemma 4.16 *Suppose $f : U \subset \Re^n \to \Re$ is C^1, and U is open and convex.*

1. *If f is quasi-concave, with $df(x) \neq 0$ then $P(x) \subset \overset{0}{H}(x)$,*
2. *If f is concave or strictly pseudo-concave then $P(x) \subset \overset{0}{H}(x)$ for all $x \in U$. In particular if x is a critical point, then $P(x) = \overset{0}{H}(x) = \Phi$.*

Proof

1. Suppose that $df(x) \neq 0$ but that $P(x) \not\subset \overset{0}{H}(x)$. However both $P(x)$ and $\overset{0}{H}(x)$ are open sets in U. By Lemma 4.14, $R(x) \subset H(x)$, and thus the closure of $P(x)$ belongs to the closure of $\overset{0}{H}(x)$ in U. Consequently there must exist a point y which belongs to $P(x)$ yet $df(x)(y-x)=0$, so y belongs to the boundary of $\overset{0}{H}(x)$. Since $P(x)$ is open there exists a neighbourhood V of y in $P(x)$, and thus in $R(x)$. Since y is a boundary point of $\overset{0}{H}(x)$ in any neighbourhood V of y there exists z such that $z \notin H(x)$. But this contradicts $R(x) \subset H(x)$. Hence $P(x) \subset \overset{0}{H}(x)$.
2. Since a concave or strictly pseudo-concave function is a quasi-concave function, (1) establishes that $P(x) \subset \overset{0}{H}(x)$ for all $x \in U$ such that $df(x) \neq 0$. By Lemmas 4.11 and 4.15, if $df(x)=0$ then x is a global maximum. Hence $P(x) = \overset{0}{H}(x) = \Phi$. □

Lemma 4.8 shows that if $\overset{0}{H}(x) \neq \Phi$, then x cannot be a global maximum on the open set U.

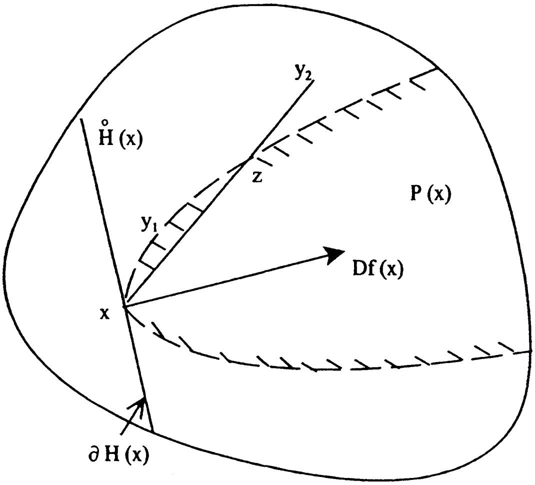


Fig. 4.8 Illustration of Lemma 4.16

Moreover for a concave or strictly pseudo-concave function $P(x)$ is empty if $\overset{0}{H}(x)$ is empty (*i.e.*, when $df(x) = 0$).

Figure 4.8 illustrates these observations. Let $P(x)$ be the preferred set of a quasi-concave function f (at a non-critical point x).

Point y_1 satisfies $f(y_1) = f(x)$ and thus belongs to $R(x)$ and hence $H(x)$. Point $y_2 \in H(x) \backslash P(x)$ but there exists an open interval (x, z) belonging to $[x, y_2]$ and to $P(x)$.

We may identify the linear map $df(x) : \Re^n \rightarrow \Re$ with a vector $Df(x) \in \Re^n$ where $df(x)(h) = \langle Df(x), h \rangle$ the scalar product of $Df(x)$ with h. $Df(x)$ is the *direction gradient*, and is *normal* to the indifference surface at x, and therefore to the hyperplane $\partial H(x) = \{y \in Y : df(x)(y - x) = 0\}$.

To see this intuitively, note that the indifference surface $I(x) = \{y \in Y : f(y) = f(x)\}$ through x, and the hyperplane $\partial H(x)$ are tangent at x. Just as $df(x)$ is an approximation to the function f, so is the hyperplane $\partial H(x)$ an approximation to $I(x)$, near to x.

As we shall see in Example 4.7, a quasi-concave function, f, may have a critical point x (so $\overset{0}{H}(x) = \Phi$) yet $P(x) \neq \Phi$. For example, if Y is the unit interval, then f may have a degenerate critical point with $P(x) \neq \Phi$. Lemma 4.16 establishes that this cannot happen when f is concave and Y is an open set. The final Lemma of this section extends Lemma 4.16 to the case when Y is admissible.

Lemma 4.17 *Let $f : Y \subset \Re^n \rightarrow \Re$ be C^1, and let Y be a convex admissible set.*

1. *If f is quasi-concave on Y, then $\forall x \in Y$, $\overset{0}{H}(x; Y) \neq \Phi$ implies $P(x; y) \neq \Phi$. If x is a local maximum of f, then $\overset{0}{H}(x; Y) = \Phi$.*
2. *If f is a strictly pseudo-concave function on an admissible set Y, and x is a local maximum, then it is a global strict maximum.*

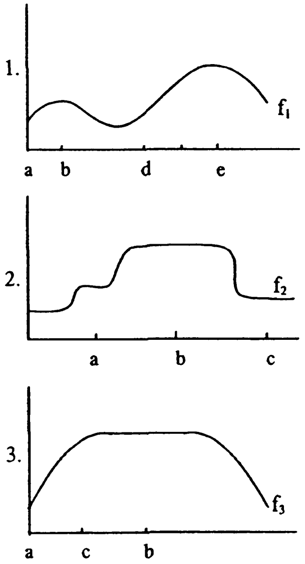


Fig. 4.9 Illustration of Lemma 4.17

3. *If f is concave C^1 or strictly pseudo-concave on Y, then $P(x;Y) \subset \overset{0}{H}(x : Y)\ \forall x \in Y$. Hence $\overset{0}{H}(x;Y) = \Phi \Rightarrow P(x;Y) = \Phi$.*

Proof

1. If $\overset{0}{H}(x;Y) \neq \Phi$ then $df(x)(y-x) > 0$ for some $y \in Y$. Then by Lemma 4.14(2), in any neighbourhood U of x in Y there exists z such that $f(z) > f(x)$. Hence x cannot be a local maximum, and indeed $P(x) \neq \Phi$.
2. By Lemma 4.15, f must be quasi-concave. By (1), if x is a local maximum then $df(x)(y-x) \leq 0$ for all $y \in Y$. By definition this implies that $f(y) < f(x)$ for all $y \in Y$ such that $y \neq x$. Thus x is a global strict maximum.
3. If f is concave and $y \in P(x;Y)$, then $f(y) > f(x)$. By Lemma 4.10, $df(x)(y-x) \geq f(y) - f(x) > 0$. Thus $y \in \overset{0}{H}(x)$.
4. If f is strictly pseudo-concave then $f(y) > f(x)$ implies $df(x)(y-x) > 0$, and so $P(x;Y) \subset \overset{0}{H}(x;Y)$. □

Example 4.7 These results are illustrated in Fig. 4.9.

1. For the general function f_1, b is a critical point and local maximum, but not a global maximum (e). On the compact interval $[a,d]$, d is a local maximum but not a global maximum.
2. For the quasi-concave function f_2, a is a degenerate critical point but neither a local nor global maximum, while b is a degenerate critical point which is also a global maximum.

   Point c is a critical point which is also a local maximum. However on $[b,c]$, c is not a global maximum.

3. For the concave function f_3, clearly b is a degenerate (but negative semi-definite) critical point, which is also a local and global maximum. Moreover on the interval $[a, c]$, c is the local and global maximum, even though it is not a critical point. Note that $df_3(c)(a-c) < 0$.

Lemma 4.17 suggests that we call any point x in an admissible set Y a *generalized critical point in Y* iff $\overset{0}{H}(x; Y) = \Phi$, of course if $df(x) = 0$, then $\overset{0}{H}(x; Y) = \Phi$, but the converse is not true when x is a boundary point.

Lemma 4.17 shows that (i) for a quasi-concave C^1-function, a global maximum is a local maximum is a generalised critical point; (ii) for a concave C^1-or strictly pseudo-concave function a critical point is a generalised critical point is a local maximum is a global maximum.

4.3.2 Economic Optimisation with Exogenous Prices

Suppose now that we wish to find the maximum of a quasi-concave C^1-function $f : Y \to \Re$ subject to a constraint $g(x) \geq 0$ where g is also a quasi-concave C^1-function $g : Y \to \Re$.

As we know from the previous section, when $P_f(x) = \{y \in Y : f(y) > f(x)\}$ and $df(x) \neq 0$, then $P_f(x) \subset H_f(x) = \{y \in Y : df(x)(x-y) \geq 0\}$.

Suppose now that $H_g(x) = \{y \in Y : dg(x)(x-y) \geq 0\}$ has the property that $H_g(x) \cap \overset{0}{H}_f(x) = \Phi$, and x satisfies $g(x) = 0$.

In this case, there exists no point y such that $g(y) \geq 0$ and $f(y) > f(x)$.

A condition that is sufficient for the disjointness of the two half-spaces $\overset{0}{H}_f(x)$ and $H_g(x)$ is clearly that $\lambda dg(x) + df(x) = 0$ for some $\lambda > 0$.

In this case if $df(x)(v) > 0$, then $dg(x)(v) < 0$, for any $v \in \Re^n$.

Now let $L = L_\lambda(f, g)$ be the Lagrangian $f + \lambda g : Y \to \Re$. A *sufficient* condition for x to be a solution to the optimisation problem is that $dL(x) = 0$.

Note however that this is not a necessary condition. As we know from the previous section it might well be the case for some point x on the boundary of the admissible set Y that $dL(x) \neq 0$ yet there exists no $y \in Y$ such that

$$dg(x)(x-y) \geq 0 \quad \text{and} \quad df(x)(x-y) > 0.$$

Figure 4.10 illustrates such a case when

$$Y = \left\{(x, y) \in \Re^2 : x \geq 0, y \geq 0\right\}.$$

We shall refer to this possibility as the *boundary problem*.

As we know we may represent the linear maps $df(x), dg(x)$ by the direction gradients, or vectors normal to the indifference surfaces, labelled $Df(x), Dg(x)$.

When the maps $df(x), dg(x)$ satisfy $df(x) + \lambda dg(x) = 0$, $\lambda > 0$, then the direction gradients are *positively dependent* and satisfy $Df(x) + \lambda Dg(x) = 0$.

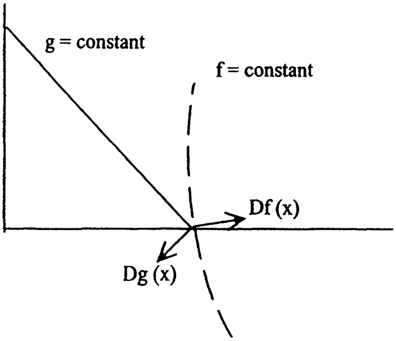


Fig. 4.10 Optimisation of f on a constraint, g

In the example $Df(x)$ and $Dg(x)$ are not positively dependent, yet x is a solution to the optimisation problem.

It is often common to make some boundary assumption so that the solution does not belong to the boundary of the feasible or admissible set Y.

In the more general optimisation problem $(f, g) : Y \to \Re^{m+1}$, the Kuhn Tucker theorem implies that a global saddle point (x^*, λ^*) to the Lagrangian $L_\lambda(f, g) = f + \sum \lambda_i g_i$ gives a solution x^* to the optimisation problem. Aside from the boundary problem, we may find the global maxima of $L_\lambda(f, g)$ by finding the critical points of $L_\lambda(f, g)$.

Thus we must choose x^* such that

$$df(x^*) + \sum_{i=1}^{m} \lambda_i \, dg_i(x^*) = 0.$$

Once a coordinate system is chosen this is equivalent to finding x^*, and coefficients $\lambda_1, \dots, \lambda_m$ all non-negative such that

$$Df(x^*) + \sum_{i=1}^{m} \lambda_i Dg_i(x^*) = 0.$$

The Kuhn Tucker Theorem also showed that if x^* is such that $g_i(x^*) > 0$, then $\lambda_i = 0$ and if $g_i(x^*) = 0$ then $\lambda_i > 0$.

Example 4.8 Maximise the function $f : \Re \to \Re$:

$$x \to x^2 : x \geq 0$$

$$x \to 0 : x < 0$$

subject to $g_1(x) = x \geq 0$ and $g_2(x) = 1 - x \geq 0$. Now $L_\lambda(x) = x^2 + \lambda_1 x + \lambda_2(1 - x)$; $\frac{\partial L}{\partial x} = 2x + \lambda_1 - \lambda_2 = 0$, $\frac{\partial L}{\partial \lambda_1} = x = \frac{\partial L}{\partial \lambda_2} = 1 - x = 0$.

Clearly these equations have no solution. By inspection the solution cannot satisfy $g_1(x) = 0$. Hence choose $\lambda_1 = 0$ and solve

$$L_\lambda(x) = x^2 + \lambda(1 - x).$$

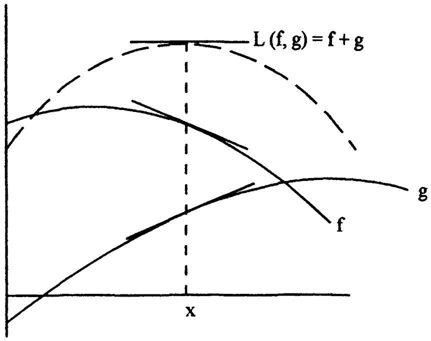


Fig. 4.11 The Lagrangian $L(f, g)$

Then $\frac{\partial L}{\partial x} = 2x - \lambda$, $\frac{\partial L}{\partial \lambda} = 1 - x = 0$. Thus $x = 1$ and $\lambda = 2$ is a solution.

Suppose now that $f, g_1, \dots, g_m$ are all concave functions on the convex admissible set $Y = \{x \in \Re^n : x_i \geq 0, i = 1, \dots, n\}$. Obviously if $z = \alpha y + (1 - \alpha)x$, then

$$\begin{aligned} L_\lambda(f, g)(z) &= f(z) + \sum_{i=1}^{m} \lambda_i g_i(z) \\ &\geq \alpha f(y) + (1 - \alpha) f(x) + \sum_{i=1}^{m} \lambda_i \big[\alpha g_i(y) + (1 - \alpha) g_i(x)\big] \\ &= \alpha L_\lambda(f, g)(y) + (1 - \alpha) L_\lambda(f, g)(x). \end{aligned}$$

Thus $L_\lambda(f, g)$ is a concave function. By Lemma 4.11, x^* is a global maximum of $L_\lambda(f, g)$ iff $dL(f, g)(x^*) = 0$ (aside from the boundary problem).

For more general functions, to find the global maximum of the Lagrangian $L_\lambda(f, g)$, and thus the optimum to the problem (f, g), we find the critical points of $L_\lambda(f, g)$. Those critical points which have negative definite Hessian will then be local maxima of $L_\lambda(f, g)$. However we still have to examine the local maxima when the Hessian of the Lagrangian is negative semi-definite to find the global maxima. Even in this general case, any solution x^* to the problem (f, g) must be a global maximum for a suitably chosen Lagrangian $L_\lambda(f, g)$, and thus must satisfy the first order condition

$$Df\big(x^*\big) + \sum_{i=1}^{m} \lambda_i Dg_i\big(x^*\big) = 0$$

(again, subject to the boundary problem).

Example 4.9 Maximise $f : \Re^2 \to \Re : (x, y) \to xy$ subject to the constraint $g(x, y) = 1 - x^2 - y^2 \geq 0$. We seek a solution to the first order condition:

$$DL(x, y) = Df(x, y) + \lambda Dg(x, y) = 0.$$

Thus $(y, x) + \lambda(-2x, -2y) = 0$ or $\lambda = \frac{y}{2x} = \frac{x}{2y}$ so $x^2 = y^2$.

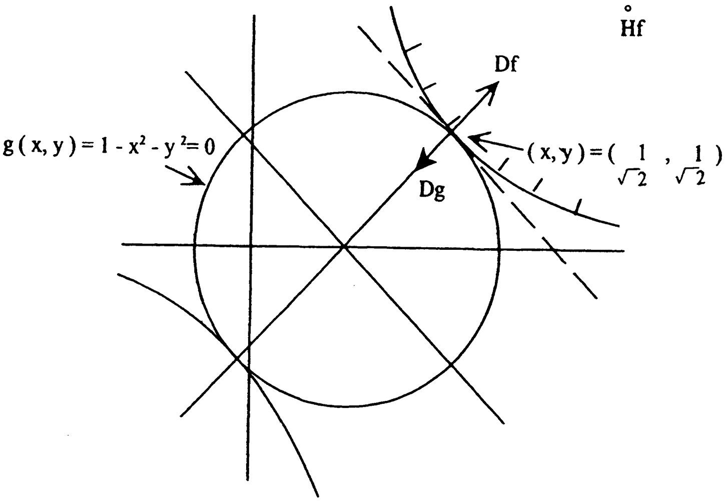


Fig. 4.12 Example 4.9

For $x = -y$, $\lambda < 0$ and so $Df(x, y) = |\lambda| Dg(x, y)$, (corresponding to a minimum of f on the feasible set $g(x, y) \geq 0$).

Thus we choose $x = y$ and $\lambda = \frac{1}{2}$. For (x, y) on the boundary of the constraint set we require $1 - x^2 - y^2 = 0$. Hence $x = y = \pm\frac{1}{\sqrt{2}}$.

The Lagrangian is therefore $L = xy + \frac{1}{2}(1 - x^2 - y^2)$ with differential (with respect to x, y)

$$DL(x, y) = (y - x, x - y) \quad \text{and Hessian}$$

$$HL(x, y) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues of HL are -2, 0 corresponding to eigenvectors $(1, -1)$ and $(1, 1)$ respectively. Hence HL is negative semi-definite, and so for example the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a local maximum for the Lagrangian. See Fig. 4.12.

As we have observed in Example 3.4, the function $f(x, y) = xy$ is not quasi-concave on $\Re^2$, and hence it is not the case that $P_f \subset H_f$. However on $\Re^2_+ = \{(x, y) \in \Re^2 : x \geq 0, y \geq 0\}$, f is quasi-concave, and so the optimality condition $H_g(x, y) \cap H_f(x, y) = \Phi$ is sufficient for an optimum.

Note also that $Df(x, y) = (y, x)$ and so the origin $(0, 0)$ is a critical point of f. However setting $DL(x, y) = 0$ at $(x, y) = (0, 0)$ requires $\lambda = 0$. In this case however

$$HL(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

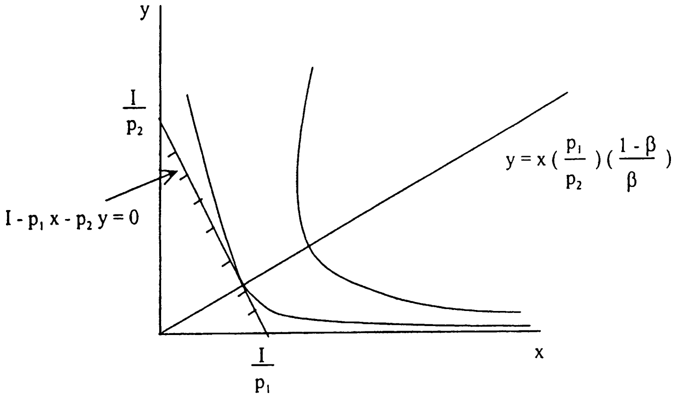


Fig. 4.13 Optimising on a budget constraint

and as in Example 4.4, HL is non-degenerate with eigenvalues $+1$, -1. Hence HL is not negative semi-definite, and so $(0, 0)$ cannot be a local maximum for the Lagrangian.

However if we were to maximise $f(x, y) = -xy$ on the feasible set $\Re^2_+$ subject to the same constraint then L would be maximized at $(0, 0)$ with $\lambda = 0$.

Example 4.10 In Example 3.7 we examined the maximisation of a convex preference correspondence of a *consumer* subject to a budget constraint of the form $B(p) = \{x \in \Re^n_+ : \langle p, x\rangle \leq \langle p, e\rangle = I\}$, given by an exogenous price vector $p \in \Re^n_+$, and initial endowment vector $e \in \Re^n_+$.

Suppose now that the preference correspondence is given by a utility function:

$$f : \Re^2_+ \to \Re : (x, y) \to \beta \log x + (1 - \beta) \log y, \quad 0 < \beta < 1.$$

Clearly $Df(x, y) = (\frac{\beta}{x}, \frac{1-\beta}{y})$, and so

$$Hf(x, y) = \begin{pmatrix} \frac{-\beta}{x^2} & 0 \\ 0 & \frac{-(1-\beta)}{y^2} \end{pmatrix}$$

is negative definite. Thus f is concave on $\Re^2$. The budget constraint is

$$g(x, y) = I - p_1x - p_2y \geq 0$$

where p_1, p_2 are the given prices of commodities x, y. The first order condition on the Lagrangian is: $Df(x, y) + \lambda Dg(x, y) = 0$, *i.e.*, $(\frac{\beta}{x}, \frac{1-\beta}{y}) + \lambda(-p_1, -p_2) = 0$ and $\lambda > 0$. Hence $\frac{p_1}{p_2} = \frac{\beta}{x} \cdot \frac{y}{1-\beta}$. See Fig. 4.13.

Now f is concave and has no critical point within the constraint set. Thus (x, y) maximises L_λ iff

$$y = x\left(\frac{p_1}{p_2}\right)\left(\frac{1-\beta}{\beta}\right) \quad \text{and} \quad g(x, y) = 0.$$

Thus $y = \frac{I - p_1 x}{p_2}$ and so $x = \frac{I\beta}{p_1}$, $y = \frac{I(1-\beta)}{p_2}$, and $\lambda = \frac{1}{I}$, is the marginal utility of income.

Now consider a situation where prices vary. Then optimal consumption $(x^*, y^*) = (d_1(p_1, p_2), d_2(p_1, p_2))$ where $d_i(p_1, p_2)$ is the *demand* for commodity x or y. As we have just shown, $d_1(p_1, p_2) = \frac{I\beta}{p_1}$, and $d_2(p_1, p_2) = \frac{I(1-\beta)}{p_2}$.

Suppose that all prices are increased by the same proportion *i.e.*, $(p'_1, p'_2) = \alpha(p_1, p_2), \alpha > 0$.

In this exchange situation $I' = p'_1 e_1 + p'_2 e_2 = \alpha I = \alpha(p_1 e_1 + p_2 e_2)$.

Thus $x' = \frac{I'\beta}{p'_1} = \frac{\alpha I \beta}{\alpha p_1} = x'$, and $y' = y$.

Hence $d_i(\alpha p_1, \alpha p_2) = d_i(p_1, p_2)$, for $i = 1, 2$. The demand function is said to be *homogeneous* in prices.

Suppose now that income is obtained from supplying labor at a wage rate w say.

Let the supply of labor by the consumer be $e = 1 - x_3$, where x_3 is leisure time and enters into the utility function.

Then $f : \Re^3 \to \Re : (x_1 x_2 x_3) \to \sum_{i=1}^{3} a_i \log x_i$ and the budget constraint is $p_1 x_1 + p_2 x_2 \leq (1 - x_3)w$, or $g(x_1, x_2, x_3) = w - (p_1 x_1 + p_2 x_2 + w x_3) \geq 0$. The first order condition is $(\frac{a_1}{x_1}, \frac{a_2}{x_2}, \frac{a_3}{x_3}) = \lambda(p_1, p_2, w), \lambda > 0$.

Clearly the demand function will again be homogeneous, since $d(p_1, p_2, w) = d(\alpha p_1, \alpha p_2, \alpha w)$.

For the general consumer optimisation problem, we therefore *normalise* the price vector. In general, in an n-commodity exchange economy let

$$\Delta = \left\{ p \in \Re^n_+ : \|p\| = 1 \right\}$$

be the *price simplex*. Here $\| \ \|$ is a convenient norm on $\Re^n$.

If $f : \Re^n_+ \to \Re$ is the utility function, let

$$D^* f(x) = \frac{Df(x)}{\|Df(x)\|} \in \Delta.$$

Suppose then that $x^* \in \Re^n$ is a maximum of $f : \Re^n_+ \to \Re$ subject to the budget constraint $\langle p, x \rangle \leq I$.

As we have seen the first order condition is

$$Df(x) + \lambda Dg(x) = 0,$$

where $Dg(x) = -p = (-p_1, \ldots, -p_n)$, $p \in \Delta$, and

$$Df(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Thus $Df(x) = \lambda(p_1, \ldots, p_n) = \lambda p \in \Re^n_+$. But then $D^* f(x) = \frac{p}{\|p\|} \in \Delta$.

Subject to boundary problems, a *necessary* condition for optimal consumer behavior is that $D^* f(x) = \frac{p}{\|p\|}$.

As we have seen the optimality condition is that $\frac{\partial f}{\partial x_i} / \frac{\partial f}{\partial x_j} = \frac{p_i}{p_j}$, for the ith and jth commodity, where $\frac{\partial f}{\partial x_i}$ is often called the *marginal utility* of the ith commodity.

Now any point y on the boundary of the budget set satisfies

$$\langle p, y\rangle = I = \frac{1}{\lambda}\big\langle Df\big(x^*\big), x^*\big\rangle.$$

Hence $y \in H(p, I)$, the hyperplane separating the budget set from the preferred set at the optimum x^*, iff $\langle Df(x^*), y - x^*\rangle = 0$.

Consider now the problem of maximisation of a profit function by a *producer*

$$\pi(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \sum_{j=1}^{n-m} p_{m+j}x_{m+j} - \sum_{j=1}^{m} p_j x_j,$$

where $(x_1, \dots, x_m) \in \Re$ are inputs, $(x_{m+1}, \dots, x_n)$ are outputs and $p \in \Re^n_+$ is a non-negative price vector.

As in Example 3.6, the set of *feasible* input-output combinations is given by the production set $G = \{x \in \Re^n_+ : F(x) \geq 0\}$ where $F : \Re^n_+ \to \Re$ is a smooth function and $F(x) = 0$ when x is on the upper boundary or *frontier* of the production set G.

At a point x on the boundary, the vector which is normal to the surface $\{x : F(x) = 0\}$ is

$$DF(x) = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)_x.$$

The first order condition for the Lagrangian is that

$$D\pi(x) + \lambda DF(x) = 0$$

or $(-p_1, \dots, -p_m, p_{m+1}, \dots, p_n) + \lambda(\frac{\lambda F}{\lambda x_1}, \dots, \frac{\partial F}{\partial x_n}) = 0$.

For example with two inputs (x_1 and x_2) and one output (x_3) we might express maximum possible output y in terms of x_1 and x_2, *i.e.*, $y = g(x_1, x_2)$. Then the feasible set is

$$\big\{x \in \Re^3_+ : F(x_1, x_2, x_3) = g(x_1, x_2) - x_3 \geq 0\big\}.$$

Then $(-p_1, -p_2, p_3) + \lambda(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, -1) = 0$ and so

$$p_1 = p_3\frac{\partial g}{\partial x_1}, \qquad p_2 = p_3\frac{\partial g}{\partial x_2} \quad \text{or} \quad \frac{p_1}{p_2} = \frac{\partial g}{\partial x_1}\Big/\frac{\partial g}{\partial x_2}.$$

Here $\frac{\partial g}{\partial x_j}$ is called the *marginal product* (with respect to commodity j for $j = 1, 2$).

For fixed $\overline{x} = (\overline{x}_1, \overline{x}_2)$ consider the locus of points in $\Re^2_+$ such that $y = g(\overline{x})$ is a constant. If $(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2})_{\overline{x}} \neq 0$ at $\overline{x}$, then by the implicit function theorem (discussed in the next chapter) we can express x_2 as a function $x_2(x_1)$ of x_1, only, near $\overline{x}$.

In this case $\frac{\partial g}{\partial x_1} + \frac{dx_2}{dx_1}\frac{\partial g}{\partial x_2} = 0$ and so $\frac{\partial g}{\partial x_1}/\frac{\partial g}{\partial x_2}|_{\overline{x}} = \frac{dx_2}{dx_1}|_x = \frac{p_1}{p_2}$.

The ratio $\frac{\partial g}{\partial x_1}/\frac{\partial g}{\partial x_2}|_{\overline{x}}$ is called the *marginal rate of technical substitution* of x_2 for x_1 at the point $(\overline{x}_1, \overline{x}_2)$.

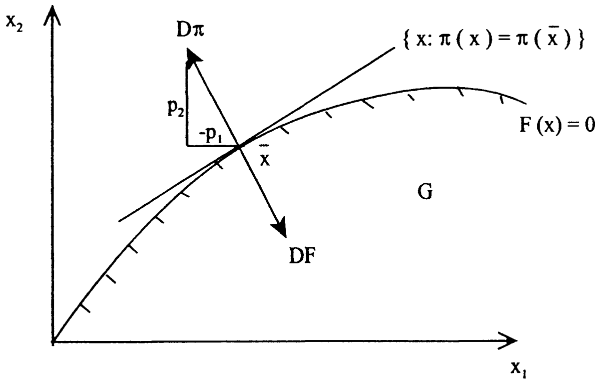


Fig. 4.14 Maximising profit on a production set

Example 4.11 There are two inputs K (capital) and L (labor), and one output, Y, say.

Let $g(K, L) = [dK^{-\rho} + (1-d)L^{-\rho}]^{\frac{-1}{\rho}}$. The feasibility constraint is

$$F(K, L, Y) = g(K, L) - Y \geq 0.$$

Let $-v$, $-w$, p be the prices of capital, labor and the output. For optimality we obtain:

$$(-v, -w, p) + \lambda\left(\frac{\partial F}{\partial K}, \frac{\partial F}{\partial L}, \frac{\partial F}{\partial Y}\right) = 0.$$

On the production frontier, $g(K, L) = Y$ and so $p = -\lambda\frac{\partial F}{\partial Y} = \lambda$ since $\frac{\partial F}{\partial L} = -1$. Now let $X = [dk^{-\rho} + (1-d)L^{-\rho}]$.
Then $\frac{\partial F}{\partial K} = (-\frac{1}{\rho})[-\rho dk^{-\rho-1}]\, X^{-\frac{1}{\rho}-1}$.
Now $Y = X^{-\frac{1}{\rho}}$ so $Y^{1+\rho} = X^{-\frac{1}{\rho}-1}$. Thus $\frac{\partial F}{\partial K} = d(\frac{Y}{K})^{1+\rho}$.
Similarly $\frac{\partial F}{\partial L} = (1-d)(\frac{Y}{L})^{1+\rho}$. Thus $\frac{r}{w} = \frac{\partial F}{\partial K}/\frac{\partial F}{\partial L} = \frac{d}{1-d}(\frac{L}{K})^{1+\rho}$.

In the case just of a single output, where the production frontier is given by a function

$$x_{n+1} = g(x_1, \ldots, x_n) \quad \text{and} \quad (x_1, \ldots, x_n) \in \Re^n_+$$

is the input vector, then clearly the constraint set will be a convex set if and only if g is a concave function. (See Example 3.3.) In this case the solution to the Lagrangian will give an optimal solution. See Fig. 4.14. However when the constraint set is not convex, then some solutions to the Lagrangian problem may be local minima. See Fig. 4.15 for an illustration.

As with the consumer, the optimum point on the production frontier is unchanged if all prices are multiplied by a positive number.

For a general consumer let $d : \Re^n_+ \to \Re^n_+$ be the demand map where

$$d(p_1, \ldots, p_n) = \big(x_1^*(p), \ldots, x_n^*(p)\big) = x^*(p)$$

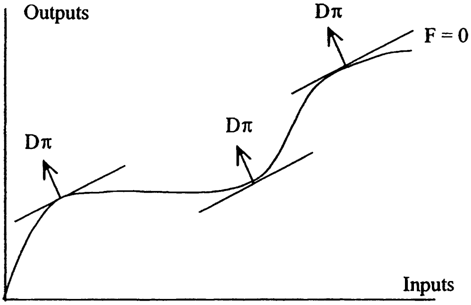


Fig. 4.15 A non-convex production set

and $x^*(p)$ is any solution to the maximisation problem on the budget set

$$B(p_1,\dots,p_n)=\left\{x\in\Re^n_+ : \langle p,x\rangle\le I\right\}.$$

In general it need not be the case that d is single-valued, and so it need not be a function.

As we have seen, d is homogeneous in prices and so we may regard d as a correspondence

$$d:\Delta\to\Re^n_+.$$

Similarly for a producer let $s:\Delta\to\Re^n_+$ where $s(p_1,\dots,p_n)=(-x_1^*(p),\dots,-x_m^*(p),x_{m+1}^*(p)\dots)$ be the supply correspondence. (Here the first m values are negative because these are inputs.)

Now consider a society $\{1,\dots,i,\dots,m\}$ and commodities named $\{1\dots j\dots n\}$. Let $e_i\in\Re^n_+$ be the initial endowment vector of agent i, and $e=\sum_{i=1}^m e_i$ the total endowment of the society. Then a price vector $p^*\in\Delta$ is a *market-clearing price equilibrium* when $e+\sum_{i=1}^m s_i(p^*)=\sum_{i=1}^m d_i(p^*)$ where $s_i(p^*)\in\Re^n_+$ belongs to the set of optimal input-output vectors at price p^* for agent i, and $d_i(p^*)$ is an optimal demand vector for consumer i at price vector p^*.

As an illustration, consider a two person, two good exchange economy (without production) and let e_{ij} be the initial endowment of good j to agent i. Let $(f_1,f_2):\Re^2_+\to\Re^2$ be the C^1-utility functions of the two players.

At $(p_1,p_2)\in\Delta$, for optimality we have

$$\left(\frac{\partial f_i}{\partial x_{i1}},\frac{\partial f_i}{\partial x_{i2}}\right)=\lambda_i(p_1,p_2),\qquad \lambda_i>0.$$

But $x_{1j}+x_{2j}=e_{1j}+e_{2j}$ for $j=1$ or 2, in market equilibrium. Thus $\frac{\partial f_i}{\partial x_{ij}}=-\frac{\partial f_i}{\partial x_{kj}}$ when $i\neq k$. Hence $\frac{1}{\lambda_1}(-\frac{\partial f_1}{\partial x_{11}},\frac{\partial f_1}{\partial x_{12}})=(p_1,p_2)=\frac{1}{\lambda_2}(\frac{\partial f_2}{\partial x_{11}},-\frac{\partial f_2}{\partial x_{12}})$ or $(\frac{\partial f_1}{\partial x_{11}},\frac{\partial f_1}{\partial x_{12}})+\lambda(\frac{\partial f_2}{\partial x_{11}},\frac{\partial f_2}{\partial x_{12}})=0$, for some $\lambda>0$. See Fig. 4.16.

As we shall see in the next section this implies that the result $(x_{11},x_{12},x_{21},x_{22})$ of optimal individual behaviour at the market-clearing price equilibrium is a *Pareto optimal outcome* under certain conditions.

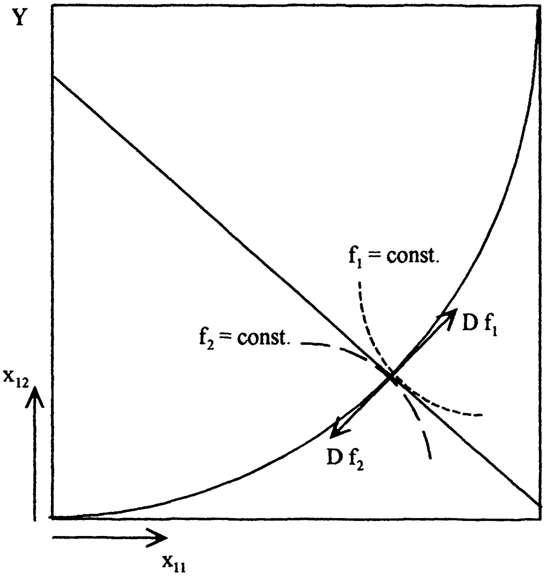


Fig. 4.16 Two person, two good exchange economy

4.4 The Pareto Set and Price Equilibria

4.4.1 The Welfare and Core Theorems

Consider a society $M = \{1, \dots, m\}$ of m individuals where the preference of the ith individual in M on a convex admissible set Y in $\Re^n$ is given by a C^1-function $u_i : Y \subset \Re^n \to \Re$. Then $u = (u_1, \dots, u_m) : Y \subset \Re^n \to \Re^m$ is called a C^1-*profile* for the society.

A point $y \in Y$ is said to be *Pareto preferred* (for the society M) to $x \in Y$ iff $u_i(y) > u_i(x)$ for all $i \in M$. In this case write $y \in P_M(x)$ and call $P_M : Y \to Y$ the *Pareto correspondence*. The (global) Pareto set for M on Y is the choice $P(u_1, \dots, u_m) = \{y \in Y : P_M(y) = \Phi\}$. We seek to characterise this set.

In the same way as before we shall let

$$H_i(x) = \{y \in Y : du_i(x)(y - x) > 0\}$$

where $H_i : Y \to Y$ for each $i = 1, \dots, m$. (Notice that $H_i(x)$ is open $\forall x \in Y$.)

Given a correspondence $P : Y \to Y$ the inverse correspondence $P^{-1} : Y \to Y$ is defined by

$$P^{-1}(x) = \{y \in Y : x \in P(y)\}.$$

In Sect. 3.3 we said a correspondence $P : Y \to Y$ (where Y is a topological space) is *lower demi-continuous* (LDC) iff for all $x \in Y$, $P^{-1}(x)$ is open in Y.

Clearly if $u_i : W \to \Re$ is continuous then the preference correspondence $P_i : Y \to Y$ given by $P_i(x) = \{y : u_i(y) > u_i(x)\}$ is LDC, since

$$P_i^{-1}(y) = \{x \in Y : u_i(y) > u(x)\}$$

is open.

We show now that when $u_i : Y \to \Re$ is C^1 then $H_i : Y \to Y$ is LDC and as a consequence if $H_i(x) \neq \Phi$ then $P_i(x) \neq \Phi$.

This implies that if x is a global maximum of u_i on Y (so $P_i(x) = \Phi$) then $H_i(x) = \Phi$ (so x is a generalised critical point).

Lemma 4.18 *If $u_i : Y \to \Re$ is a C^1-function on the convex admissible set Y, then $H_i : Y \to Y$ is lower demi-continuous and if $H_i(x) \neq \Phi$ then $P_i(x) \neq \Phi$.*

Proof Suppose that $H_i(x) \neq \Phi$. Then there exists $y \in Y$ such that $du_i(x)(y - x) > 0$. Let $h = y - x$.

By the continuity of $du_i : Y \to \mathcal{L}(\Re^n, \Re)$ there exists a neighbourhood U of x in Y, and a neighbourhood V of h in $\Re^n$ such that $du_i(z)(h') > 0$ for all $z \in U$, for all $h' \in V$.

Since $y \in H_i(x)$, we have $x \in H_i^{-1}(y)$. Now $h = y - x$. Let

$$U' = \left\{x' \in U : y - x' \in V\right\}.$$

For all $x' \in U'$, $du_i(x')(y - x') > 0$. Thus $U' \subset H_i^{-1}(y)$. Hence $H_i^{-1}(y)$ is open. This is true at each $y \in Y$, and so H_i is LDC.

Suppose that $H_i(x) \neq \Phi$ and $h \in H_i(x)$. Since H_i is LDC it is possible to choose $\lambda \in (0, 1)$, by Taylor's Theorem, such that

$$u_i(x + \lambda h) = u_i(x) + du_i(z)(\lambda h),$$

where $du_i(z)(h) > 0$, and $z \in (x, x + \lambda h)$. Thus $u_i(x + \lambda h) > u_i(x)$ and so $P_i(x) \neq \Phi$. □

When $u = (u_1, \ldots, u_m) : Y \to \Re^m$ is a C^1-profile then define the correspondence $H_M : Y \to Y$ by $H_M(x) = \bigcap_{i \in M} H_i(x)$ *i.e.*, $y \in H_M(x)$ iff $du_i(x)(y - x) > 0$ for all $i \in M$.

Lemma 4.19 *If $(u_1, \ldots, u_m) : Y \to \Re^m$ is a C^1-profile, then $H_M : Y \to Y$ is lower demi-continuous. If $H_M(x) \neq \Phi$ then $P_M(x) \neq \Phi$.*

Proof Suppose that $H_M(x) \neq \Phi$. Then there exists $y \in H_i(x)$ for each $i \in M$. Thus $x \in H_i^{-1}(y)$ for all $i \in M$. But each $H_i^{-1}(y)$ is open; hence $\exists$ an open neighbourhood U_i of x in $H_i^{-1}(y)$; let $U = \bigcap_{i \in M} U_i$. Then $x' \in U$ implies that $x' \in H_M^{-1}(y)$. Thus H_M is LDC. As in the proof of Lemma 4.18 it is then possible to choose $h \in \Re^n$ such that, for all i in M,

$$u_i(x + h) = u_i(x) + du_i(z)(h)$$

where z belongs to U, and $du_i(z)(h) > 0$. Thus $x + h \in P_M(x)$ and so $P_M(x) \neq \Phi$. □

The set $\{x : H_M(x) = \Phi\}$ is called the *critical Pareto set*, and is often written as Θ_M, or $\Theta(u_1, \dots, u_m)$. By Lemma 4.19, $\Theta(u_1, \dots, u_m)$ *contains* the Pareto set $P(u_1, \dots, u_m)$.

Moreover we can see that Θ_M must be closed in Y. To see this suppose that $H_M(x) \neq \Theta$, and $y \in H_M(x) \neq \Phi$. Thus $x \in H_M^{-1}(y)$. But H_M is LDC and so there is a neighbourhood U of x in Y such that $x' \in H_M^{-1}(y)$ for all $x' \in U$. Then $y \in H_M(x')$ for all $x' \in U$, and so $H_M(x') \neq \Phi$ for all $x' \in U$.

Hence the set $\{x \in Y : H_M(x) \neq \Phi\}$ is open and so the critical Pareto set is closed.

In the same way, the Pareto correspondence $P_M : Y \to Y$ is given by $P_M(x) = \bigcap_{i \in M} P_i(x)$ where $P_i(x) = \{y : u_i(y) > u_i(x)\}$ for each $i \in M$. Since each P_i is LDC, so must be P_M, and thus the Pareto set $P(u_1, \dots, u_m)$ must also be closed.

Suppose now that $u_1, \dots, u_m$ are all concave C^1-or strictly pseudo-concave functions on the convex set Y.

By Lemma 4.17, for each $i \in M$, $P_i(x) \subset H_i(x)$ at each $x \in Y$.

If $x \in \Theta(u_1, \dots, u_m)$ then

$$\bigcap_{i \in M} P_i(x) \subset \bigcap_{i \in M} H_i(x) = \Phi$$

and so x must also belong to the (global) Pareto set. Thus if $u = (u_1, \dots, u_m)$ with each u_i concave C^1 or strictly pseudo-concave, then the global Pareto set $P(u)$ and the critical Pareto set $\Theta(u)$ coincide. In this case we may more briefly say the preference profile represented by u is *strictly convex*.

A point in $P(u_1, \dots, u_m)$ is the precise analogue, in the case of a family of functions, of a maximum point for a single function, while a point in $\Theta(u_1, \dots, u_m)$ is the analogue of a critical point of a single function $u : Y \to \Re$. In the case of a family or profile of functions, a point x belongs to the critical Pareto set $\Theta_M(u)$, when a generalised Lagrangian $L(u_1, \dots, u_m)$ has differential $dL(x) = 0$.

This allows us to define a Hessian for the family and determine which critical Pareto points are global Pareto points.

Suppose then that $u = (u_1, \dots, u_m) : Y \to \Re^m$ where Y is a convex admissible set in $\Re^n$ and each $u_i : Y \to \Re$ is a C^1-function.

A generalised Lagrangian $L(\lambda, u)$ for u is a *semipositive* combination $\sum_{i=1}^m \lambda_i u_i$ where each $\lambda_i \geq 0$ but not all $\lambda_i = 0$.

For convenience let us write

$$\begin{aligned}
\Re^m_+ &= \left\{x \in \Re^m : x_i \geq 0 \text{ for } i \in M\right\} \\
\overset{0}{\Re^m_+} &= \left\{x \in \Re^m : x_i > 0 \text{ for } i \in M\right\}, \quad \text{and} \\
\overline{\Re^m_+} &= \Re^m_+ \backslash \{0\}.
\end{aligned}$$

Thus $\lambda \in \overline{\Re^m_+}$ iff each $\lambda_i \geq 0$ but not all $\lambda_i = 0$. Since each $u_i : Y \to \Re$ is a C^1-function, the differential at x is a linear map $du_i(x) : \Re^n \to \Re$. Once a coordinate basis for $\Re^n$ is chosen, $du_i(x)$ may be represented by the row vector

$$Du_i(x) = \left(\frac{\partial u_i}{\partial x}\bigg|_x, \ldots, \frac{\partial u_i}{\partial x_n}\bigg|_x \right).$$

Similarly the profile $u : Y \to \Re^m$ has differential at x represented by the $(n \times m)$ Jacobian matrix

$$Du(x) = \begin{pmatrix} Du_1(x) \\ \vdots \\ Du_m(x) \end{pmatrix} : \Re^n \to \Re^m.$$

Suppose now that $\lambda \in \Re^m$. Then define $\lambda \cdot Du(x) : \Re^n \to \Re$ by

$$\big(\lambda \cdot Du(x)\big)(v) = \big\langle \lambda, Du(x)(v) \big\rangle$$

where $\langle \lambda, Du(x)(v) \rangle$ is the scalar product of the two vectors λ, $Du(x)(v)$ in $\Re^m$.

Lemma 4.20 *The gradient vectors* $\{Du_i(x) : i \in M\}$ *are linearly dependent and satisfy the equation*

$$\sum_{i=1}^{m} \lambda_i Du_i(x) = 0$$

iff $[\operatorname{Im} Du(x)]^\perp$ *is the subspace of* $\Re^m$ *spanned by* $\lambda = (\lambda_1, \ldots, \lambda_m)$.

Here $\lambda \in [\operatorname{Im} Du(x)]^\perp$ *iff* $\langle \lambda, w \rangle = 0$ *for all* $w \in \operatorname{Im} Du(x)$.

Proof

$$\begin{aligned} \lambda \in \big[\operatorname{Im} Du(x)\big]^\perp \quad &\Leftrightarrow \quad \langle \lambda, w \rangle = 0 \quad \forall\, w \in \operatorname{Im} Du(x) \\ &\Leftrightarrow \quad \big\langle \lambda, Du(x)(v) \big\rangle = 0 \quad \forall\, v \in \Re^n \\ &\Leftrightarrow \quad \big(\lambda \cdot Du(x)\big)(v) = 0 \quad \forall\, v \in \Re^n \\ &\Leftrightarrow \quad \lambda \cdot Du(x) = 0. \end{aligned}$$

But $\lambda \cdot Du(x) = 0 \Leftrightarrow \sum_{i=1}^m \lambda_i Du_i(x) = 0$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$. □

Theorem 4.21 *If* $u : Y \to \Re^m$ *is a* C^1*-profile on an admissible convex set and* x *belongs to the interior of* Y, *then* $x \in \Theta(u_1, \ldots, u_m)$ *iff there exists* $\lambda \in \overline{\Re^m_+}$ *such that* $dL(\lambda, u)(x) = 0$.

If x *belongs to the boundary of* Y *and* $dL(\lambda, u)(x) = 0$, *for* $\lambda \in \overline{\Re^m_+}$, *then* $x \in \Theta(u_1, \ldots, u_m)$.

Proof Pick a coordinate basis for $\Re^n$. Suppose that there exists $\lambda \in \overline{\Re^m_+}$ such that

$$L(\lambda, u)(x) = \sum_{i=1}^{m} \lambda_i u_i(x) \in \Re,$$

satisfies $\sum_{i=1}^m \lambda_i Du_i(x) = 0$ (that is to say $DL(\lambda, u)(x) = 0$).

By Lemma 4.20 this implies that

$$\lambda \in \bigl[\mathrm{Im}\bigl(Du(x)\bigr)\bigr]^{\perp}.$$

However suppose $x \notin \Theta(u_1, \dots, u_m)$. Then there exists $v \in \Re^n$ such that $Du(x)(v) = w \in \overset{0}{\Re^m_+}$, *i.e.*, $\langle Du_i(x), v\rangle = w_i > 0$ for all $i \in M$, where $w = (w_1, \dots, w_m)$. But $w \in \mathrm{Im}\, Du(x)$ and $w \in \overset{0}{\Re^m_+}$.

Moreover $\lambda \in \overline{\Re^m_+}$ and so $\langle \lambda, w\rangle > 0$ (since not all $\lambda_i = 0$).

This contradicts $\lambda \in [\mathrm{Im}(Du(x))]^{\perp}$, since $\langle \lambda, w\rangle \neq 0$. Hence $x \in \Theta(u_1, \dots, u_m)$. Thus we have shown that for any $x \in Y$, if $DL(\lambda, u)(x) = 0$ for some $\lambda \in \overline{\Re^m_+}$, then $x \in \Theta(u_1, \dots, u_m)$. Clearly $DL(\lambda, u)(x) = 0$ iff $DL(\lambda, u)(x) = 0$, so we have proved sufficiency.

To show necessity, suppose that $\{Du_i(x) : i \in M\}$ are linearly independent. If x belongs to the interior of Y then for a vector $h \in \Re^n$ there exists a vector $y = x + \theta h$, for θ sufficiently small, so that $y \in Y$ and $\forall i \in M$, $\langle Du_i(x), h\rangle > 0$. Thus $x \notin \Theta(u_1, \dots, u_m)$.

So suppose that $DL(\lambda, u)(x) = 0$ where $\lambda \neq 0$ but $\lambda \notin \overset{0}{\Re^m_+}$. Then for at least one i, $\lambda_i < 0$. But then there exists a vector $w \in \overset{0}{\Re^m_+}$ where $w = (w_1, \dots, w_m)$ and $w_i > 0$ for each $i \in M$, such that $\langle \lambda, w\rangle = 0$. By Lemma 4.20, $w \in \mathrm{Im}(Du(x))$. Hence there exists a vector $h \in \Re^n$ such that $Du(x)(h) = w$.

But $w \in \overset{0}{\Re^m_+}$, and so $\langle Du_i(x), h\rangle > 0$ for all $i \in M$. Since x belongs to the interior of Y, there exists a point $y = x + \alpha h$ such that $y \in H_i(x)$ for all $i \in M$. Hence $x \notin \Theta(u_1, \dots, u_m)$.

Consequently if x is an interior point of Y then $x \in \Theta(u_1, \dots, u_m)$ implies that $dL(\lambda, u)(x) = 0$ for some semipositive λ in $\overline{\Re^m_+}$. □

Example 4.12 To illustrate, we compute the Pareto set in $\Re^2$ when the utility functions are

$$u_1(x_1, x_2) = {x_1}^{\alpha} x_2 \quad \text{where } \alpha \in (0, 1), \quad \text{and}$$

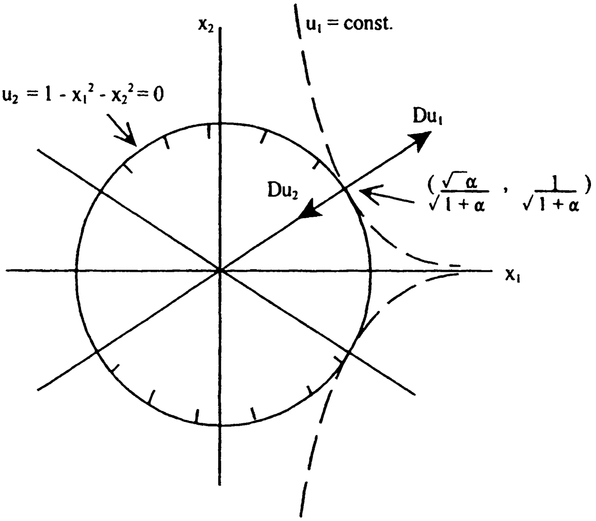
$$u_2(x_1, x_2) = 1 - {x_1}^2 - {x_2}^2.$$

We maximise u_1 subject to the constraint $u_2(x_1, x_2) \geq 0$.

As in Example 4.9, the first order condition is

$$\bigl(\alpha {x_1}^{\alpha-1} x_2, {x_1}^{\alpha}\bigr) + \lambda(-2x_1, -2x_2) = 0.$$

Fig. 4.17 Example 4.12


x₂
u₁ = const.
u₂ = 1 - x₁² - x₂² = 0
Du₁
Du₂
$(\frac{\sqrt{\alpha}}{\sqrt{1+\alpha}}, \frac{1}{\sqrt{1+\alpha}})$
x₁

Hence $\lambda = \frac{\alpha {x_1}^{\alpha-1} x_2}{2x_1} = \frac{{x_1}^{\alpha}}{2x_2}$, so $\alpha {x_2}^2 = {x_1}^2$, or $x_1 = \pm\sqrt{\alpha}x_2$.

If $x_1 = -\sqrt{\alpha}x_2$ then $\lambda = \frac{{x_1}^{\alpha}}{\sqrt{\alpha}}2(-x_1) < 0$, and so such a point does not belong to the critical Pareto set. Thus $(x_1, x_2) \in \Theta(u_1, u_2)$ iff $x_1 = x_2\sqrt{\alpha}$. Note that if $x_1 = x_2 = 0$ then the Lagrangian may be written as

$$L(\lambda, u)(0,0) = \lambda_1 u_1(0,0) + \lambda_2 u_2(0,0)$$

where $\lambda_1 = 0$ and λ_2 is any positive number. In the positive quadrant $\Re^2_+$, the critical Pareto set and global Pareto set coincide.

Finally to maximise u_1 on the set $\{(x_1, x_2) : u_2(x_1, x_2) \geq 0\}$ we simply choose λ such that $u_2(x_1, x_2) = 0$.

Thus ${x_1}^2 + {x_2}^2 = \alpha {x_2}^2 + {x_2}^2 = 1$ or $x_2 = \frac{1}{\sqrt{1+\alpha}}$ and so $(x_1, x_2) = (\frac{\sqrt{\alpha}}{\sqrt{1+\alpha}}, \frac{1}{\sqrt{1+\alpha}})$.

In the next chapter we shall examine the critical Pareto set $\Theta(u_1, \dots, u_m)$ and demonstrate that the set belongs to a singularity "manifold" which can be topologically characterised to be of "dimension" $m - 1$. This allows us then to examine the price equilibria of an exchange economy.

Note that in the example of a two person exchange economy studied in Sect. 4.3.2, we showed that the result of individual optimising behaviour led to an outcome, x, such that

$$Du_1(x) + \lambda Du_2(x) = 0$$

where $\lambda > 0$. As we have shown here this "market clearing equilibrium" must belong to the critical Pareto set $\Theta(u_1, u_2)$. Moreover, when both u_1 and u_2 represent strictly convex preferences, then this outcome belongs to the global Pareto set. We develop this in the following theorem.

The Welfare Theorem for an Exchange Economy *Consider an exchange economy where each individual in the society $M=\{1,\dots,i,\dots,m\}$ has initial endowment $e_i\in\Re^n_+$.*

1. *Suppose that the demand $x_i^*(p)\in\Re^n_+$ of agent i at each price $p\in\Delta$ is such that $x_i^*(p)$ maximises the C^1-utility function $u_i:\Re^n\to\Re$ on the budget set $B_i(p)=\{x_i\in\Re^n_+:\langle p,x_i\rangle\leq\langle p,e_i\rangle\}$ and satisfies*
   (a) $Du_i(x_i^*(p))=\lambda_i p,\ \lambda_i>0,$
   (b) $\langle p,x_i^*(p)\rangle=\langle p,e_i\rangle.$
2. *Suppose further that p^* is a market clearing price equilibrium in the sense that $\sum_{i=1}^m x_i^*(p^*)=\sum_{i=1}^m e_i\in\Re^n$.*

Then $x^*=(x_1^*(p^*),\dots,x_m^*(p))\in\Theta(u_1,\dots,u_m)$.

Moreover if each u_i is either concave, or strictly pseudo-concave then x^* belongs to the Pareto set, $P(u_1,\dots,u_m)$.

Proof We need to define the set of outcomes first of all. An outcome, x, is a vector

$$x=(x_1,\dots,x_i,\dots,x_m)\in\left(\Re^n_+\right)^m=\Re^{nm}_+,$$

where $x_i=(x_{i1},\dots,x_{in})\in\Re^n_+$ is an *allocation* for agent i. However there are n resource constraints

$$\sum_{i=1}^m x_{ij}=\sum_{i=1}^m e_{ij}=e_{\cdot j},$$

for $j=1,\dots,n$, where $e_i=(e_{i1},\dots,e_{in})\in\Re^n_+$ is the initial endowment of agent i.

Thus the set, Y, of feasible allocations to the members of M is a hyperplane of dimension $n(m-1)$ in $\Re^{nm}_+$ through the point $(e_1,\dots,e_m)$.

As coordinates for any point $x\in Y$ we may choose

$$x=(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n},\dots,x_{(m-1)1},\dots,x_{(m-1)n})$$

where it is implicit that the bundle of commodities available to agent m is

$$x_m=(x_{m1},\dots,x_{mn})$$

where $x_{mj}=e_{\cdot j}-\sum_{i=1}^{m-1}x_{ij}$.

Now define $u_i^*:Y\to\Re$, the extended utility function of i on Y by

$$u_i^*(x)=u_i(x_{i1},\dots,x_{in}).$$

For $i\in M$, it is clear that the direction gradient of i on Y is

$$Du_i^*(x)=\left(0,\dots,0,\frac{\partial u_i}{\partial x_i},\dots,\frac{\partial u_i}{\partial x_{in}},0,\dots\right)=\big(\dots,0,\dots,Du_i(x),\dots,0\big).$$

For agent m, $\frac{\partial u_m^*}{\partial x_{ij}} = -\frac{\partial u_m}{\partial x_{mj}}$ for $i = 1, \dots, m-1$; thus

$$\begin{aligned} Du_m^*(x) &= -\left(\frac{\partial u_m}{\partial x_{m1}}, \dots, \frac{\partial u_m}{\partial x_{mn}}, \dots\right) \\ &= -\big(Du_m(x), \dots, \dots, Du_m(x)\big). \end{aligned}$$

If p^* is a market-clearing price equilibrium, then by definition

$$\sum_{i=1}^{m} x_i^*(p^*) = \sum_{i=1}^{m} e_i.$$

Thus $x^*(p^*) = (x_1^*(p^*), \dots, x_{m-1}^*(p^*))$ belongs to Y. But each x_i^* is a critical point of $u_i : \Re_+^n \to \Re$ on the budget set $B_i(p^*)$ and $Du_i(x_i^*(p^*)) = \lambda_i p^*$.

Thus the Jacobian for $u^* = (u_1^*, \dots, u_m^*) : Y \to \Re^m$ at $x^*(p^*)$ is

$$Du^*\big(x^*\big) = \begin{bmatrix} \lambda_1 p^* & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 p^* & & & \\ \vdots & \vdots & & & \lambda_{m-1} p^* \\ -\lambda_m p^* & & -\lambda_m p^* & & -\lambda_m p^* \end{bmatrix}$$

Hence $\frac{1}{\lambda_1} Du_1^*(x^*) + \frac{1}{\lambda_2} Du_2^*(x^*) \cdots + \frac{1}{\lambda_m} Du_m^*(x^*) = 0$. But each $\lambda_i > 0$ for $i = 1, \dots, m$. Then $dL(\mu, u^*)(x^*(p^*)) = 0$ where $L(\mu, u^*)(x) = \sum_{i=1}^m \mu_i u_i^*(x)$ and $\mu_i = \frac{1}{\lambda_i}$ and $\mu \in \overline{\Re_+^m}$.

By Theorem 4.21, $x^*(p^*)$ belongs to the critical Pareto set.

Clearly, if for each i, $u_i : \Re_+^n \to \Re$ is concave C^1- or strictly pseudo-concave then $u_1^* : Y \to \Re$ will be also. By previous results, the critical and global Pareto set will coincide, and so $x^*(p^*)$ will be Pareto optimal. □

One can also show that the *competitive allocation*, $x^*(p^*) \in \Re_+^{nm}$, constructed in this theorem is Pareto optimal in a very simple way. By definition $x^*(p^*)$ is characterised by the two properties:

1. $\sum_{i=1}^m x_i^*(p^*) = \sum_{i=1}^m e_i$ in $\Re^n$ (feasibility)
2. If $u_i(x_i) > u_i(x_i^*(p^*))$ then $\langle p^*, x_i \rangle > \langle p^*, e_i \rangle$ (by the optimality condition for agent i).

But if $x_i^*(p^*)$ is not Pareto optimal, then there exists a vector $x = (x_1, \dots, x_m) \in \Re_+^{nm}$ such that $u_i(x_i) > u_i(x_i^*(p^*))$ for $i = 1, \dots, m$. By (2), $\langle p^*, x_i \rangle > \langle p^*, e_i \rangle$ for each i and so

$$\sum_{i=1}^{m} \langle p^*, x_i \rangle = \left\langle p^*, \sum_{i=1}^{m} x_i \right\rangle > \left\langle p^*, \sum_{i=1}^{m} e_i \right\rangle.$$

But if $x \in \Re_+^{nm}$ is feasible then $\sum_{i=1}^m x_i \le \sum_{i=1}^m e_i$ which implies $\langle p^*, \sum_{i=1}^m x_i \rangle \le \langle p^*, \sum_{i=1}^m e_i \rangle$.

By contradiction $x^*(p^*)$ must belong to the Pareto optimal set.

The observation has an immediate extension to a result on existence of a *core* of an economy.

Definition 4.3 Let $e=(e_1,\dots,e_m)\in\Re^{nm}_+$ be an initial endowment vector for a society M. Let $\mathcal{D}$ be any family of subsets of M, and let $P=(P_1,\dots,P_m)$ be a profile of preferences for society M, where each $P_i:\Re^n_+\to\Re^n_+$ is a preference correspondence for i on the ith consumption space $X_i\subset\Re^n_+$.

An *allocation* $x\in\Re^{nm}_+$ is S-feasible (for $S\subset M$) iff $x=(x_{ij})\in\Re^{nm}_+$ and $\sum_{i\in S}x_{ij}=\sum_{i\in S}e_{ij}$ for each $j=1,\dots,n$.

Given e and P, an allocation $x\in\Re^{nm}_+$ belongs to the $\mathcal{D}$-*core* of (e,P) iff $x=(x_1,\dots,x_m)$ is M-feasible and there exists no coalition $S\in\mathcal{D}$ and an allocation $y\in\Re^{nm}_+$ such that y is S-feasible and of the form $y=(y_1,\dots,y_m)$ with $y_i\in P_i(x_i),\forall i\in S$.

To clarify this definition somewhat, consider the set Y from the proof of the welfare theorem. $Y=Y_M$ is a hyperplane of dimension $n(m-1)$ through the endowment point $e\in\Re^{nm}_+$. For any coalition $S\in\mathcal{D}$ of cardinality s, there is a hyperplane Y_S, say, of dimension $n(s-1)$ through the endowment point e, consisting of S-feasible trades among the members of S. Clearly $Y_S\subset Y_M$. If $x\in Y_M$ but there is some $y\in Y_S$ such that every member of S prefers y to x, then the members of S could refuse to accept the allocation x. If there is no such point x, then x is "unbeaten", and belongs to the $\mathcal{D}$-*core* of the economy described by (e,P).

Core Theorem *Let $\mathcal{D}$ be any family of subsets of M. Suppose that $p^*\in\Delta$ is a market clearing price equilibrium for the economy (e,P) and $x^*(B)\in\Re^{nm}_+$ is the demand vector, where*

$$x^*\bigl(p^*\bigr)=\bigl(x_1^*(p),\dots,x_m^*(p)\bigr)\in Y_M,\quad \textit{and}\quad P_i\bigl(x_i^*\bigl(p^*\bigr)\bigr)\cap B_i\bigl(p^*\bigr)=\Phi.$$

Then $x^*(p^*)$ belongs to the $\mathcal{D}$-core, for the economy (e,P).

Proof Suppose that $x^*(p^*)$ is not in the core. Then there is some $y\in Y_S$ such that $y=(y_i:i\in S)$ and $y\in P_i(x_i)$ for each $i\in S$. Now $x_i^*(p^*)$ is a most preferred point for i on $B_i(p^*)$ so $\langle p^*,y_i\rangle>\langle p^*,e_i\rangle$ for all $i\in S$.

Hence

$$\Bigl\langle p^*,\sum_{i\in S}y_i\Bigr\rangle=\sum_{i\in S}\bigl\langle p^*,y_i\bigr\rangle>\sum_{i\in S}\bigl\langle p^*,e_i\bigr\rangle.$$

However if $y\in Y_S$, then $\sum_{i\in S}y_i=\sum_{i\in S}e_i\in\Re^n_+$, which implies $\langle p^*,\sum_{i\in S}(y_i-e_i)\rangle=0$. By contradiction, $x^*(p^*)$ must be in the core. □

The Core Theorem shows, even if a price mechanism is not used, that if a market clearing price equilibrium, p^* does exist, then the core is non-empty. This means in essence that the competitive allocation, $x^*(p^*)$, is Pareto optimal for every coalition $S\in\mathcal{D}$.

By the results of Sect. 3.8, a market clearing price equilibrium p^* will exist under certain conditions on preference. In particular suppose preference is representable by smooth utility functions that are concave or strictly pseudo-concave and monotonic in the individual consumption spaces. Then the conditions of the Welfare Theorem will be satisfied, and there will exist a market clearing price equilibrium, p^*, and a competitive allocation, $x^*(p^*)$, at p^* which belongs to the Pareto set for the society M. Indeed, since the Core Theorem is valid when $\mathcal{D}$ consists of all subsets of N, the two results imply that $x^*(p^*)$ will then belong to the critical Pareto set Θ_S, associated with each coalition $S \in M$. This in turn suggests for any S, there is a solution $x^* = x^*(p^*)$ to the Lagrangian problem $dL_S(\mu, u)(x^*) = 0$, where $L_S(\mu, u^*)(x) = \sum_{i \in S} \mu_i u_i^*(x)$ and $\mu_i \geq 0 \; \forall i \in S$.

Here $u_i^* : Y_S \to \Re$ is the extended utility function for i on Y_S.

It is also possible to use the concept of a core in the more general context considered in Sect. 3.8, where preferences are defined on the full space $X = \Pi_i X_i \in \Re_+^{nm}$. In this case, however, a price equilibrium may not exist if the induced social preference violates convexity or continuity. It is then possible for the $\mathcal{D}$-core to be empty.

Note in particular that the model outlined in this section implicitly assumes that each economic agent chooses their demand so as to optimize a utility function on the budget set determined by the price vector. Thus prices are treated as exogenous variables. However, if agents treat prices as strategic variables then it may be rational for them to compute their effect on prices, and thus misrepresent their preferences. The economic game then becomes much more complicated than the one analyzed here.

A second consideration is whether the price equilibria are unique, or even locally unique. If there exists a continuum of pure equilibria, then prices may move around chaotically.

A third consideration concerns the attainment of the price equilibrium. In Sect. 3.8 we constructed an abstract preference correspondence for an "auctioneer" so as to adjust the price vector to increase the value of the excess supply of the commodities. We deal with these considerations in the next section and in Chap. 5.

4.4.2 Equilibria in an Exchange Economy

The Welfare Theorem gives an important insight into the nature of competitive allocations. The coefficients μ_i of the Lagrangean $L(\mu, u)$ of the social optimisation problem turn out to be inverse to the coefficients λ_i in the individual optimisation problems, where λ_i is equal to the marginal utility of income for the ith agent. This in turn suggests that it is possible for an agent to transform his utility function from u_i to u_i' in such a way as to decrease λ_i and thus increase μ_i, the "weight" of the ith agent in the social optimisation problem. This is called the *problem of preference manipulation* and is an interesting research problem with applications in trade theory.

Secondly the weights μ_i can be regarded as functionally dependent on the initial endowment vector $(e_1, \ldots, e_m) \in \Re_+^{nm}$. Thus the question of market equilibrium could be examined in terms of the functions $\mu_i : \Re_+^{nm} \to \Re$, $i = 1, \ldots, m$.

It is possible that one or a number of agents could destroy or exchange commodities so as to increase their weights. This is termed the problem of *resource manipulation* or the *transfer paradox* (see Gale 1974, and Balasko 1978).

Example 4.13 To illustrate these observations consider a two person ($i = 1, 2$) exchange economy with two commodities ($j = 1, 2$).

As in Example 4.10, assume the preference of the ith agent is given by a utility function $f_i : \Re^2_+ \to \Re : f_i(x, y) = \beta_i \log x + (1 - \beta_i) \log y$ where $0 < \beta_i < 1$.

Let the initial endowment vector of i be $e_i = (e_{i1}, e_{i2})$. At the price vector $p = (p_1, p_2)$, demand by agent i is $d_i(p_1, p_2) = (\frac{I_i \beta_i}{p_1}, \frac{I_i(1-\beta_i)}{p_2})$ where $I = p_1 e_{i1} + p_2 e_{i2}$ is the value at p of the endowment.

Thus agent i "desires" to change his initial endowment from e_i to e'_i:

$$(e_{i1}, e_{i2}) \to \left(e'_{i1}, e'_{i2}\right) = \left(\beta_i e_{i1} + \frac{\beta_i e_{i2} p_2}{p_1}, (1 - \beta_i) e_{i2} + (1 - \beta_i) \frac{e_{i1} p_1}{p_2}\right).$$

Another way of interpreting this is that i optimally divides expenditure between the first and second commodities in the ratio $\beta : (1 - \beta)$. Thus agent i offers to sell $(1 - \beta) e_{i1}$ units of commodity 1 for $(1 - \beta) e_{i1} p_1$ monetary units and buy $(1 - \beta_i) e_{i1} \frac{p_1}{p_2}$ units of the second commodity, and offers to sell $\beta_i e_{i2}$ units of the second commodity and buy $\beta_i e_{i2} \frac{p_2}{p_1}$ units of the first commodity.

At the price vector (p_1, p_2) the amount of the first commodity on offer is $(1 - \beta_i) e_{11} + (1 - \beta_2) e_{21}$ and the amount on request is $\beta_1 e_{12} p^*_2 + \beta_2 e_{22} p^*_2$; where p^*_2 is the ratio $p_2 : p_1$ of relative prices. For (p_1, p_2) to be a market-clearing price equilibrium we require

$$e_{11}(1 - \beta_1) + e_{21}(1 - \beta_2) = p^*_2(e_{12}\beta_1 + e_{22}\beta_2).$$

Clearly if all endowments are increased by a multiple $\alpha > 0$, then the equilibrium relative price vector is unchanged. Thus p^*_2 is uniquely determined and so the final allocations $(e'_{11}, e'_{12}), (e'_{21}, e'_{22})$ can be determined.

As we showed in Example 4.10, the coefficients λ_i for the individual optimisation problems satisfy $\lambda_i = \frac{1}{I_i}$, where λ_i is the marginal utility of income for agent i.

By the previous analysis, the weights μ_i in the social optimisation problem satisfy $\mu_i = I_i$. After some manipulation of the price equilibrium equation we find

$$\frac{\mu_i}{\mu_k} = \frac{e_{i1}(e_{i2} + \beta_k e_{k2}) + e_{i2}(1 - \beta_k) e_{k1}}{e_{k1}(e_{k2} + \beta_i e_{i2}) + e_{k2}(1 - \beta_i) e_{i1}}.$$

Clearly if agent i can increase the ratio $\mu_i : \mu_k$ then the relative utility of i vis-à-vis k is increased. However since the relative price equilibrium is uniquely determined in this example, it is not possible for agent i, say, to destroy some of the initial endowments (e_{i1}, e_{i2}) so as to bring about an advantageous final outcome. The interested reader is referred to Balasko (1978).

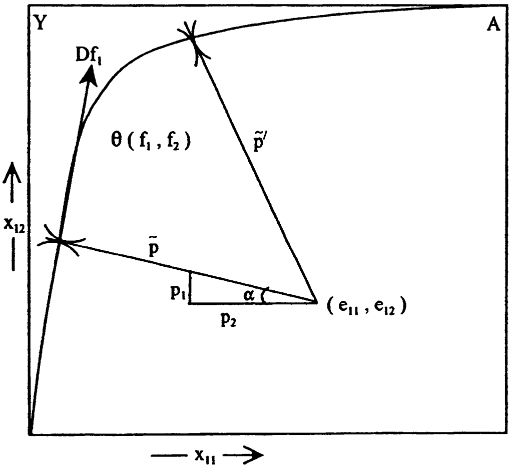


Fig. 4.18 Multiple price equilibria in Example 4.13

In this example the (relative) price equilibrium is unique, but this need not always occur. Consider the two person, two commodity case illustrated below in Fig. 4.18. As in the Welfare Theorem, the set of feasible outcomes Y is the subset of $\Re^4_+ = (x_{11}, x_{12}, x_{21}, x_{22})$ such that $x_{11} + x_{21} = e_{\cdot 1}$; $x_{12} + x_{22} = e_{\cdot 2}$, and this is a two-dimensional hyperplane through the point $(e_{11}, e_{12}, e_{21}, e_{22})$. Thus Y can be represented in the usual two-dimensional *Edgeworth* box where point A, the most preferred point for agent 1, satisfies $(x_{11}, x_{12}) = (e_{\cdot 1}, e_{\cdot 2})$.

The price ray $\tilde{p}$ is that ray through (e_{11}, e_{12}) where $\tan \alpha = \frac{p_1}{p_2}$. Clearly (p_1, p_2) is an equilibrium price vector if the price ray intersects the critical Pareto set $\Theta(f_1, f_2)$ at a point (x_{11}, x_{12}) in Y where $\tilde{p}$ is tangential to the indifference curves for f_1 and f_2 through (x_{11}, x_{12}). At such a point we then have $Df_1(x_{11}, x_{12}) + \mu Df_2(x_{11}, x_{12}) = 0$.

As Fig. 4.18 indicates there may well be a second price ray $\tilde{p}'$ which satisfies the tangency property. Indeed it is possible that there exists a family of such rays, or even an open set V in the price simplex such that each p in V is a market clearing equilibrium. We now explore the question of local uniqueness of price equilibria.

To be more formal let $X = \Re^n_+$ be the commodity or consumption space. An *initial endowment* is a vector $e = (e_1, \ldots, e_m) \in X^m$. A C^r utility function is a C^r-function $u = (u_1, \ldots, u_m) : X \to \Re^m$.

Let $C_r(X, \Re^m)$ be the set of C^r-profiles, and endow $C_r(X, \Re^m)$ with a topology in the following way. (See the next chapter for a more complete discussion of the *Whitney* topology.)

A neighbourhood of $f \in C_r(X, \Re^m)$ is a set

$$\left\{g \in C_r\left(X, \Re^m\right) : \left\|d^k g(x) - d^k f(x)\right\| < \epsilon(x)\right\} \quad \text{for } k = 0, \ldots, r$$

where $\epsilon(x) > 0$ for all $x \in X$. (We use the notation that $d^0 g = g$ and $d^1 g = dg$.)

Write $C^r(X, \Re^m)$ for $C_r(X, \Re^m)$ with this topology. A property K is called *generic* iff it is true for all profiles which belong to a *residual* set in $C^r(X, \Re^m)$. Here *residual* means that the set is the countable intersection of open dense sets in $C^r(X, \Re^m)$.

If a property is generic then we may say that almost all profiles in $C^r(X, \Re^m)$ have that property.

A smooth *exchange economy* is a pair $(e, u) \in X^m \times C^r(X, \Re^m)$. As before the feasible outcome set is

$$Y = \left\{ (x_1, \dots, x_m) \in X^m : \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \right\}$$

and a price vector p belongs to the simplex $\Delta = \{p \in X : \|p\| = 1\}$, where Δ is an object of dimension $n - 1$.

As in the welfare theorem, the demand by agent i at $p \in \Delta$ satisfies : $x_i^*(p)$ maximises u_i on

$$B(p) = \big\{ x_i \in X : \langle p, x_i \rangle \leq \langle p, e_i \rangle \big\}.$$

As we have observed, under appropriate boundary conditions, we may assume $\forall\, i \in M, x_i^*(p)$ satisfies

1. $\langle p, x_i^*(p) \rangle = \langle p, e_i \rangle$
2. $D^* u_i(x_i^*(p)) = p \in \Delta$. Say $(x^*(p^*), p^*) = (x_1^*(p^*), \dots, x_m^*(p^*), p^*) \in X^m \times \Delta$ is a market or *Walrasian equilibrium* iff $x^*(p^*)$ is the competitive allocation at p^* and satisfies

$$\sum_{i=1}^m x_i^*\big(p^*\big) = \sum_{i=1}^m e_i \in \Re_+^n.$$

The economy (e, u) is *regular* iff (e, u) is such that the set of Walrasian equilibria is finite.

Debreu-Smale Theorem on Generic Existence of Regular Economies *There is a residual set U in $C^r(X, \Re^m)$ such that for every profile $u \in U$, there is a dense set $V \in X^m$ with the property that (e, u) is a regular economy whenever $(e, u) \in V \times U$.*

The proof of this theorem is discussed in the next chapter (the interested reader might also consult Smale 1976). However we can give a flavor of the proof here.

Consider a point $(e, x, p) \in X^m \times X^m \times \Delta$. This space is of dimension $2nm + (n - 1)$.

Now there are n resource restrictions

$$\sum_{i=1}^m x_{ij} = \sum_{i=1}^m e_{ij}$$

for $j = 1, \dots, n$, together with $(m - 1)$ budget restrictions $\langle p, x_i \rangle = \langle p, e_i \rangle$ for $i = 1, \dots, m - 1$.

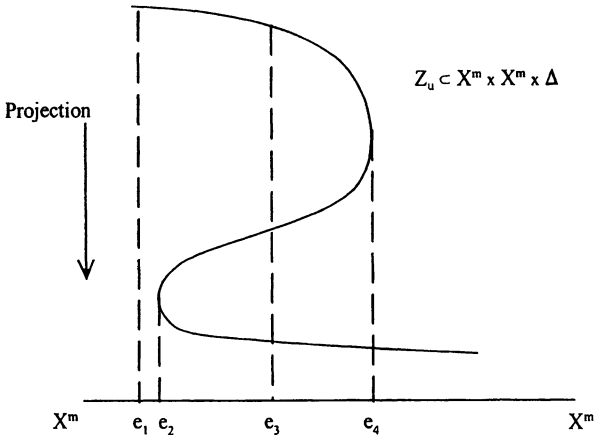


Fig. 4.19 Finite Walrasian equilibria

Note that the budget restriction for the mth agent is redundant.

Let $\Gamma = \{(e, x, p) \in X^m \times X^m \times \Delta\}$ be the set of points satisfying these various restrictions. Then Γ will be of dimension $2nm + (n-1) - [n+m-1] = 2nm - m$.

However, we also have m distinct vector equations $D^*u_i(x) = p$, $i = 1, \dots, m$.

Since these vectors are normalised, each one consists of $(n-1)$ separate equations. Chapter 5 shows that singularity theory implies that for every profile u in a residual set, each of these constraints is independent. Together these $m(n-1)$ constraints reduce the dimension of Γ by $m(n-1)$. Hence the set of points in Γ satisfying the first order optimality conditions is a smooth object Z_u of dimension

$$2nm - m - m(n-1) = nm.$$

Now consider the projection

$$Z_u \subset X^m \times \left(X^m \times \Delta\right) \to X^m : (e, x, p) \to e,$$

and note that both Z_u and X^m have dimension nm.

A regular economy (e, u) is one such that the projection map *proj* : $Z_u \to X^m$: $(e, x, p) \to e$ has differential with maximal rank nm. Call e a *regular value* in this case. From singularity theory it is known that for all u in a residual set U, the set of regular values of the projection map is dense in X^m. Thus when $u \in U$, and e is regular, the set of Walrasian equilibria for (e, u) will be finite. Figure 4.19 illustrates this. At e_1 there is only one Walrasian equilibrium, while at e_3 there are three. Moreover in a neighbourhood of e_3 the Walrasian equilibria move continuously with e. At e_4 the Walrasian equilibrium set is one-dimensional. As e moves from the right past e_2 the number of Walrasian equilibria drops suddenly from 3 to 1, and displays a discontinuity. Note that the points (x, p) satisfying $(e, x, p) \in (proj)^{-1}(e)$ need not be Walrasian equilibria in the classical sense, since we have considered only the first order conditions. It is clearly the case that if there is non-convex preference, then the first order conditions are not sufficient for equilibrium. However, Smale's

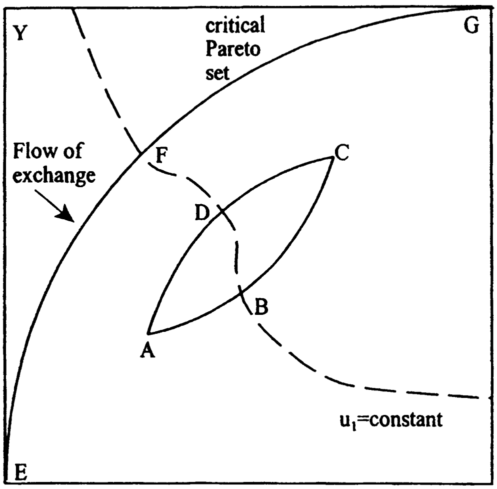


Fig. 4.20 Two agent, two commodity exchange economy

theorem shows the existence of extended Walrasian equilibria. The same difficulty occurs in the proof that a Walrasian equilibrium gives a Pareto optimal point in Y.

Let $\overset{0}{\Theta}(u) = \overset{0}{\Theta}(u_1, \dots, u_m)$ be the set of points satisfying the first order condition $dL(\lambda, u) = 0$ where $\lambda \in \overline{\Re^m_+}$. Suppose that we solve this with $\lambda_1 \neq 0$.

Then we may write $Du_1(x) + \sum_{i=2}^{m} \frac{\lambda_i}{\lambda_1} Du_i(x) = 0$.

Clearly there are $(m-1)$ degrees of freedom in this solution and indeed $\overset{0}{\Theta}(u_1, \dots, u_m)$ can be shown to be a geometric object of dimension $(m-1)$ "almost always" (see Chap. 5). However $\overset{0}{\Theta}(u)$ will contain points that are the "social" equivalents of the minima of a real-valued function.

Note, by Theorem 4.21, that $\overset{0}{\Theta}(u)$ and the critical Pareto set $\Theta(u)$ coincide, except for boundary points. If the boundary of the space is smooth, then it is possible to define a Lagrangian which characterises the boundary points in $\Theta(u)$.

For example consider Fig. 4.20, of a two agent two commodity exchange economy.

Agent 1 has non-convex preference, and the critical Pareto set consists of three components ABC, ADC and EFG.

On ADC although the utilities satisfy the first order condition, there exist nearby points that both agents prefer. For example, both agents prefer a nearby point y to x. See Fig. 4.21.

In Fig. 4.22 from an initial endowment such as $e = (e_{11}, e_{12})$, there exists three Walrasian extended equilibria, but at least one can be excluded. Note that if e is the initial endowment vector, then the Walrasian equilibrium B which is accessible by exchange along the price vector may be Pareto inferior to a Walrasian equilibrium, F, which is not readily accessible.

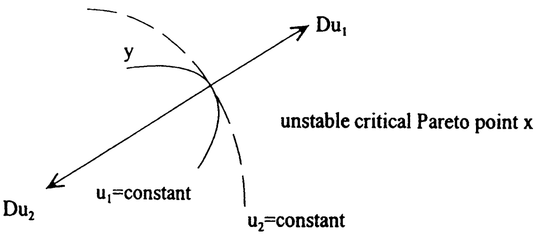


Fig. 4.21 Unstable critical Paret point

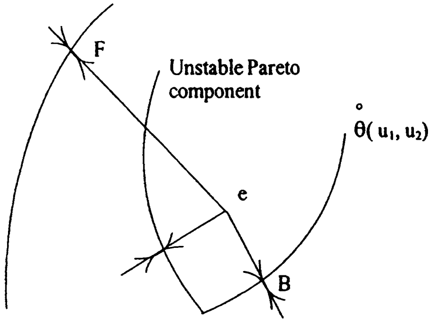


Fig. 4.22 Unstable Pareto component

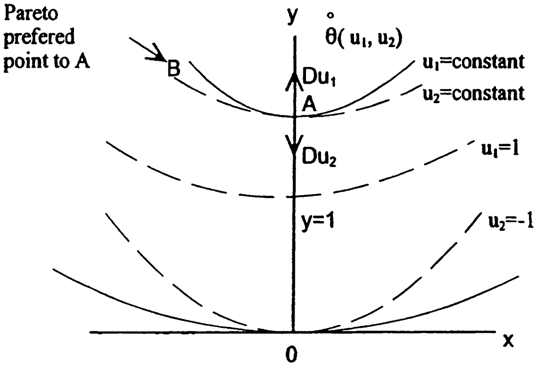


Fig. 4.23 Example 4.14

Example 4.14 Consider the example due to Smale (as in Fig. 4.23). Let $Y = \Re^2$ and suppose

$$u_1(x, y) = y - x^2$$
$$u_2(x, y) = \frac{-y}{x^2 + 1}.$$

Then

$$Du_1(x, y) = (-2x, 1)$$
$$Du_2(x, y) = \left(\frac{2xy}{(x^2 + 1)^2}, \frac{-1}{x^2 + 1}\right).$$

Let $DL_\lambda(x, y) = \lambda_1(-2x, 1) + \lambda_2(\frac{2xy}{(x^2+1)^2}, \frac{-1}{x^2+1})$.

Clearly one solution will be $x = 0$, in which case $\lambda_1(0, 1) + \lambda_2(0, -1) = 0$ or $\lambda_1 = \lambda_2 = 1$.

The Hessian for L at $x = 0$ is then

$$\begin{aligned} HL(0, y) &= D^2u_1(0, y) + D^2u_2(0, y) \\ &= \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2y & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

which is negative semi-definite for $2(y - 1) < 0$ or $y < 1$.

Further Reading

For a lucid account of economic equilibrium theory see:

Arrow, K. J., & Hahn, F. H. (1971). *General competitive analysis*. Edinburgh: Oliver and Boyd.

Hildenbrand, W., & Kirman, A. P. (1976). *Introduction to equilibrium analysis*. Amsterdam: North Holland.

For the ideas of preference or resource manipulation, see:

Balasko, Y. (1978). The transfer problem and the theory of regular economies. *International Economic Review*, *19*, 687–694.

Gale, D. (1974). Exchange equilibrium and coalitions. *Journal of Mathematical Economics*, *1*, 63–66.

Guesnerie, R., & Laffont, J.-J. (1978). Advantageous reallocations of initial resources. *Econometrica*, *46*, 687–694.

Safra, Z. (1983). Manipulation by reallocating initial endowments. *Journal of Mathematical Economics*, *12*, 1–17.

For a general introduction to the application of differential topology to economics see:

Smale, S. (1976). Dynamics in general equilibrium theory. *American Economic Review*, *66*, 288–294. Reprinted in S. Smale (1980) *The Mathematics of Time*. Springer: Berlin.

Singularity Theory and General Equilibrium 5

In the last section of the previous chapter we introduced the critical Pareto set $\Theta(u)$ of a smooth profile for a society, and the notion of a regular economy. Both ideas relied on the concept of a singularity of a smooth function $f : X \to \Re^m$, where a singularity is a point analogous to the critical point of a real-valued function.

In this chapter we introduce the fundamental result in singularity theory, that the set of singularity points of a smooth profile almost always has a particular geometric structure. We then go on to use this result to discuss the Debreu-Smale Theorem on the generic existence of regular economies. No attempt is made to prove these results in full generality. Instead the aim is to provide a geometric understanding of the ideas. Section 5.4 uses an example of Scarf (1960) to illustrate the idea of an excess demand function for an exchange economy. The example provides a general way to analyse a smooth adjustment process leading to a Walrasian equilibrium. Sections 5.5 and 5.6 introduce the more abstract topological ideas of structural stability and chaos in dynamical systems.

5.1 Singularity Theory

In Chap. 4 we showed that when $f : X \to \Re$ was a C^2-function on a normed vector space, then knowledge of the first and second differential of f at a critical point, x, gave information about the local behavior (near x) of the function. In this section we discuss the case of a differentiable function $f : X \to Y$ between general normed vector spaces, and consider regular points (where the differential has maximal rank) and singularity points (where the differential has non-maximal rank). For both kinds of points we can locally characterise the behavior of the function.

5.1.1 Regular Points: The Inverse and Implicit Function Theorem

Suppose that $f : X \to Y$ is a function between normed vector spaces and that for all x' in a neighbourhood U of the point x the differential $df(x')$ is defined. By the results of Sect. 3.2, if $df(x')$ is bounded, or if X is finite-dimensional, then

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_5,

$df(x')$ will be a *continuous* linear function. Suppose now that X and Y have the same finite dimension (n) and $df(x')$ has rank n at all $x' \in U$. Then we know that $df(x')^{-1} : Y \to X$ is a linear map and thus continuous. We shall call f a C^2-*diffeomorphism* on U in this case.

In general, when Y is an infinite-dimensional vector space, then even if $df(x')^{-1}$ exists it need not be continuous. However when X and Y are *complete* normed vector spaces then the existence of $df(x')^{-1}$ is sufficient to guarantee that $df(x')^{-1}$ is continuous.

Essentially a normed vector space X is *complete* iff any "convergent" sequence (x_k) does indeed converge to a point in X. More formally, a *Cauchy sequence* is a sequence (x_k) such that for any $\epsilon > 0$ there exists some number $k(\epsilon)$ such that $r, s > k(\epsilon)$ implies that $\|x_r - x_s\| < \epsilon$. If (x_k) is a sequence with a limit x_0 in X then clearly (x_k) must be a Cauchy sequence. On the other hand a Cauchy sequence need not in general converge to a point in the space X. If *every* Cauchy sequence has a limit in X, then X is called *complete*. A complete normed vector space is called a *Banach* space. Clearly $\Re^n$ is a Banach space. Suppose now that X, Y are normed vector spaces of the same dimension, and $f : U \subset X \to Y$ is a C^r-differentiable function on U, such that $df(x)$ has a continuous inverse $[df(x)]^{-1}$ at x. We call f a C^r-diffeomorphism at x. Then we can show that f has an inverse $f^{-1} : f(U) \to U$ with differential $df^{-1}(f(x)) = [df(x)]^{-1}$. Moreover there exists a neighbourhood V of x in U such that f is a C^r-diffeomorphism on V. In particular this means that f has an inverse $f^{-1} : f(V) \to V$ with continuous differential $df^{-1}(f(x')) = [df(x')]^{-1}$ for all $x' \in V$, *and* that f^{-1} is C^r-differentiable on V. To prove the theorem we need to ensure that $[df(x)]^{-1}$ is not only linear but continuous, and it is sufficient to assume X and Y are Banach spaces.

Inverse Function Theorem *Suppose $f : U \subset X \to Y$ is C^r-differentiable, where X and Y are Banach spaces of dimension n. Suppose that the linear map $df(x) : X \to Y$, for $x \in U$, is an isomorphism with inverse $[df(x)]^{-1} : Y \to X$. Then there exist open neighbourhoods V of x in U and V' of $f(x)$ such that $f : V \to V'$ is a bijection with inverse $f^{-1} : V' \to V$. Moreover f^{-1} is itself C^r-differentiable on V', and for all $x' \in V$, $df^{-1}(f(x')) = [df(x')]^{-1}$. f is called a local C^r-diffeomorphism, at x.*

Outline of Proof Let $t = df(x) : X \to Y$. Since $[df(x)]^{-1}$ exists and is continuous, $t^{-1} : Y \to X$ is linear and continuous.

It is possible to choose a neighbourhood V' of $f(x)$ in $f(U)$ and a closed ball V_x in U centered at x, such that, for each $y \in V'$, the function $g_y : V_x \subset U \subset X \to V_x \subset X$ defined by $g_y(x') = x' - t^{-1}[f(x') - y]$ is continuous. By Brouwer's Theorem, each g_y has a fixed point.

That is to say for each $y \in V'$, there exists $x' \in V_x$ such that $g_y(x') = x'$. But then $t^{-1}[f(x') - y] = 0$. Since, by hypothesis, t^{-1} is an isomorphism, its kernel $= \{0\}$, and so $f(x') = y$. Thus for each $y \in V'$ we establish $g_y(x') = x'$ is equivalent to $f(x') = y$. Define $f^{-1}(y) = g_y(x') = x'$, which gives the inverse function on V'. To show f^{-1} is differentiable, proceed as follows.

Note that $dg_y(x') = \mathrm{Id} - t^{-1} \circ df(x')$ is independent of y. Now $dg_y(x')$ is a linear and continuous function from X to X and is thus bounded. Since X is Banach, it is possible to show that $\mathcal{L}(X, X)$, the topological space of linear and continuous maps from X to X, is also Banach. Thus if $u \in \mathcal{L}(X, X)$, so is $(\mathrm{Id} - u)^{-1}$. This follows since $(\mathrm{Id} - u)^{-1}$ converges to an element of $\mathcal{L}(X, X)$.

Now $dg_y(x') \in \mathcal{L}(X, X)$ and so $(\mathrm{Id} - dg_y(x'))^{-1} \in \mathcal{L}(X, X)$. But then $t^{-1} \circ df(x')$ has a continuous linear inverse. Now $t^{-1} \circ df(x') : X \to Y \to X$ and t^{-1} has a continuous linear inverse. Thus $df(x')$ has a continuous linear inverse, for all $x' \in V$. Let V be the interior of V_x. By the construction the inverse of $df(x')$, for $x' \in V$, has the required property. □

This is the fundamental theorem of differential calculus. Notice that the theorem asserts that if $f : \Re^n \to \Re^n$ and $df(x)$ has rank n at x, then $df(x')$ has rank n for all x' in a neighbourhood of x.

Example 5.1 (i) For a simple example, consider the function $\exp : \Re \to \Re_+ : x \to e^x$. Clearly for any finite $x \in \Re$, $d(\exp)(x) = e^x \neq 0$, and so the rank of the differential is 1. The inverse $\phi : \Re_+ \to \Re$ must satisfy

$$d\phi(y) = \left[d(\exp)(x)\right]^{-1} = \frac{1}{e^x}$$

where $y = \exp(x) = e^x$. Thus $d\phi(y) = \frac{1}{y}$.

Clearly ϕ must be the function $\log_e : y \to \log_e y$.

(ii) Consider sin: $(0, 2\pi) \to [-1, +1]$.

Now $d(\sin)(x) \neq 0$. Hence there exist neighbourhoods V of x and V' of $\sin x$ and an inverse $\phi : V' \to V$ such that

$$d\phi(y) = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - y^2}}.$$

This inverse ϕ is only locally a function. As Fig. 5.1 makes clear, even when $\sin x = y$, there exist distinct values x_1, x_2 such that $\sin(x_1) = \sin(x_2) = y$. However $d(\sin)(x_1) \neq d(\sin)(x_2)$.

The figure also shows that there is a neighbourhood V' of y such that $\phi : V' \to V$ is single-valued and differentiable on V'. Suppose now $x = \frac{\pi}{2}$. Then $d(\sin)(\frac{\pi}{2}) = 0$. Moreover there is no neighbourhood V of $\frac{\pi}{2}$ such that $\sin : (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon) = V \to V'$ has an inverse function.

Note one further consequence of the theorem. For h small, we may write

$$\begin{aligned} f(x + h) &= f(x) + df(x) \circ \left[df(x)\right]^{-1}\big(f(x + h) - f(x)\big) \\ &= f(x) + df(x)\psi(h), \end{aligned}$$

where $\psi(h) = d[f(x)]^{-1}(f(x + h) - f(x))$. Now by a linear change of coordinates we can diagonalise $df(x)$. So that in the case $f = (f_1, \dots, f_n) : \Re^n \to \Re^n$ we can ensure $\frac{\partial f_i}{\partial x_j}|_x = \partial_{ij}$ where $\partial_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Hence $f(x + h) = f(x) +$

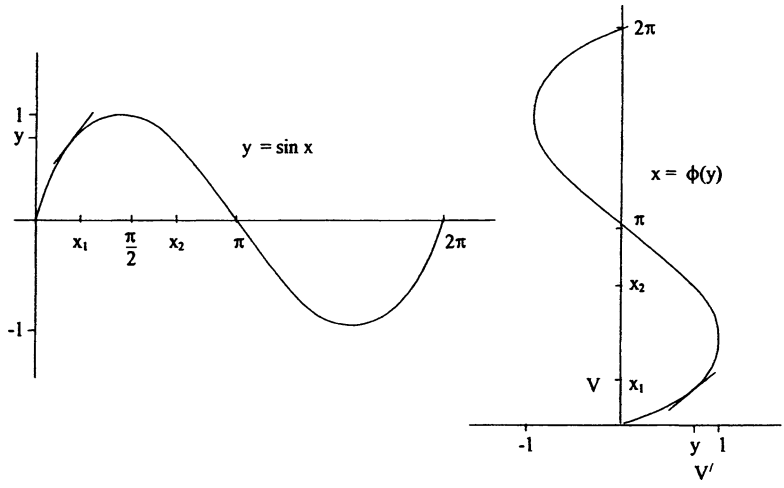


Fig. 5.1 Example 5.1

$(\psi_1(h), \ldots, \psi_n(h))$. There is therefore a C^r-diffeomorphic change of coordinates ϕ near x such that $\phi(\underline{0}) = x$ and

$$f\big(\phi(h_1, \ldots, h_n)\big) = f(x) + (h_1, \ldots, h_n).$$

In other words by choosing coordinates appropriately f may be represented by its linear differential.

Suppose now that $f : U \subset \Re^n \to \Re^m$ is a C^1-function. The maximal rank of df at a point x in U is $\min(n, m)$. If indeed $df(x)$ has maximal rank then x is called a *regular point* of f, and $f(x)$ a *regular value*. In this case we write $x \in S_0(f)$. The inverse function theorem showed that when $n = m$ and $x \in S_0(f)$ then f could be regarded as an identity function near x.

In particular this means that there is a neighbourhood U of x such that $f^{-1}[f(x)] \cap U = \{x\}$ is an isolated point.

In the case that $n \neq m$ we use the inverse function theorem to characterise f at regular points.

Implicit Function Theorem for Vector Spaces

1. (Surjective Version). *Suppose that* $f : U \subset \Re^n \to \Re^m$, $n \geq m$, *and rank* $(df(x)) = m$, *with* $f(x) = \underline{0}$ *for convenience. If* f *is* C^r*-differentiable at* x, *then there exists a* C^r*-diffeomorphism* $\phi : \Re^n \to \Re^n$ *on a neighbourhood of the origin such that* $\phi(\underline{0}) = x$, *and* $f \circ \phi(h_1, \ldots, h_n) = (h_1, \ldots, h_m)$.
2. (Injective Version). *If* $f : U \subset \Re^n \to \Re^m$, $n \leq m$, *rank* $(df(x)) = n$, *with* $f(\underline{0}) = y$, *and* f *is* C^r*-differentiable at* x *then there exists a* C^r*-diffeomorphism* $\psi : \Re^m \to \Re^m$ *such that* $\psi(y) = \underline{0}$ *and*

$$\psi f(h_1, \ldots, h_n) = (h_1, \ldots, h_n, 0, \ldots, 0).$$

Proof

1. Now $df(x) = [B\ C]$, with respect to some coordinate system, where B is an $(m \times m)$ non singular matrix and C is an $(n-m) \times m$ matrix. Define $F : \Re^n \to \Re^n$ by

   $$F(x_1, \dots, x_n) = \big(f(x_1, \dots, x_n), x_{m+1}, \dots, x_n\big).$$

   Clearly $DF(x)$ has rank n, and by the inverse function theorem there exists an inverse ϕ to F near x. Hence $F \circ \phi(h_1, \dots, h_n) = (h_1, \dots, h_n)$. But then $f \circ \phi(h_1, \dots, h_n) = (h_1, \dots, h_m)$.
2. Follows similarly. □

As an application of the theorem, suppose $f : \Re^m \times \Re^{n-m} \to \Re^m$. Write x for a vector in $\Re^m$, and y for a vector in $\Re^{n-m}$, and let $df(x, y) = [B\ C]$ where B is an $m \times m$ matrix and C is an $(n-m) \times m$ matrix. Suppose that B is invertible, at $(\overline{x}, \overline{y})$, and that $f(\overline{x}, \overline{y}) = 0$. Then the implicit function theorem implies that there exists an open neighbourhood U of $\overline{y}$ in $\Re^{n-m}$ and a differentiable function $g : U \to \Re^m$ such that $g(y') = x'$ and $f(g(y'), y') = 0$ for all $y' \in V$.

To see this define

$$F : \Re^m \times \Re^{n-m} \to \Re^m \times \Re^{n-m}$$

by $F(x, y) = (f(x, y), y)$.

Clearly $dF(\overline{x}, \overline{y}) = \left[\begin{smallmatrix} B & C \\ O & I \end{smallmatrix}\right]$ and so $dF(\overline{x}, \overline{y})$ is invertible. Thus there exists a neighbourhood V of $(\overline{x}, \overline{y})$ in $\Re^n$ on which F has a diffeomorphic inverse G. Now $F(\overline{x}, \overline{y}) = (0, \overline{y})$. So there is a neighbourhood V' of $(0, \overline{y})$ and a neighbourhood V of $(\overline{x}, \overline{y})$ s.t. $G : V \subset \Re^n \to V' \subset \Re^n$ is a diffeomorphism.

Let $g(y')$ be the x coordinate of $G(0, y')$ for all y' such that $(0, y') \in V'$.

Clearly $g(y')$ satisfies $G(0, y') = (g(y'), y')$ and so $F \circ G(0, y') = F(g(y'), y) = (f(g(y', y')), y') = (0, y')$.

Now if $(x', y') \in V'$ then $y' \in U$ where U is open in $\Re^{n-m}$. Hence for all $y' \in U$, $g(y')$ satisfies $f(g(y'), y') = 0$. Since G is differentiable, so must be $g : U \subset \Re^{n-m} \to \Re^m$. Hence $x' = g(y')$ *solves* $f(x', y') = 0$.

Example 5.2

1. Let $f : \Re^3 \to \Re^2$ where

   $$f_1(x, y, z) = x^2 + y^2 + z^2 - 3$$
   $$f_2(x, y, z) = x^3y^3z^3 - x + y - z.$$

   At $(x, y, z) = (1, 1, 1)$, $f_1 = f_2 = 0$.

   We seek a function $g : \Re \to \Re^2$ such that $f(g_1(z), g_2(z), z) = 0$ for all z in a neighbourhood of 1.

   Now

   $$df(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \\ 3x^2y^3z^3 - 1 & 3x^3y^2z^3 + 1 & 3x^3y^3z^2 - 1 \end{pmatrix}$$

and so

$$df(1,1,1)=\begin{pmatrix}2&2&2\\2&4&2\end{pmatrix}.$$

The matrix

$$\begin{pmatrix}2&2\\2&4\end{pmatrix}$$

is non-singular. Hence there exists a diffeomorphism $G:\Re^3\to\Re^3$ such that $G(0,0,1)=(1,1,1)$, and $G(0,0,z')=(g_1(z'),g_2(z'),z')$ for z' near 1.

2. In a simpler example consider $f:\Re^2\to\Re:(x,y)\to(x-a)^2+(y-b)^2-25=0$, with $df(x,y)=(2(x-a),2(y-b))$.

   Now let $F(x,y)=(x,f(x,y))$, where $F:\Re^2\to\Re^2$, and suppose $y\neq b$. Then

$$\begin{aligned}dF(x,y)&=\begin{pmatrix}1&0\\\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}\end{pmatrix}\\&=\begin{pmatrix}1&0\\2(x-a)&2(y-b)\end{pmatrix}\end{aligned}$$

   with inverse

$$dG(x,y)=\frac{1}{2(y-b)}\begin{pmatrix}2(y-b)&0\\-2(x-a)&1\end{pmatrix}.$$

   Define $g(x')$ to be the y-coordinate of $G(x',0)$. Then $F\circ G(x',0)=F(x',g(x'))=F(x',f(x',y'))=(x',0)$, and so $y'=g(x')$ for $f(x',y')=0$ and y' sufficiently close to y. Note also that

$$\left.\frac{dg}{dx}\right|_{x'}=\left.\frac{dG_2}{\partial x}\right|_{(x',g')}=-\frac{(x'-a)}{(y'-b)}.$$

In Example 5.2(1), the “solution” $g(z)=(x,y)$ to the smooth constraint $f(x,y,z)=0$ is, in a topological sense, a one-dimensional object (since it is given by a single constraint in $\Re^2$).

In the same way in Example 5.2(2) the solution $y=g(x)$ is a one-dimensional object (since it is given by a single constraint in $\Re^2$).

More specifically say that an open set V in $\Re^n$ is an *r-dimensional smooth manifold* iff there exists a diffeomorphism

$$\phi:V\subset\Re^n\to U\subset\Re^r.$$

When $f:\Re^n\to\Re^m$ and rank $(df(x))=m\leq n$ then say that f is a *submersion* at x. If rank $(df(x))=n\leq m$ then say f is an *immersion* at x.

One way to interpret the implicit function theorem is as follows:

(1) When f is a submersion at x, then the inverse $f^{-1}(f(x))$ of a point $f(x)$ "looks like" an object of the form $\{x, h_{m+1}, \dots, h_n\}$ and so is a smooth manifold in $\Re^n$ of dimension $(n-m)$.

(2) When f is an immersion at x, then the image of an (n-dimensional) neighborhood U of x "looks like" an n-dimensional manifold, $f(u)$, in $\Re^m$. These observations can be generalized to the case when $f : X^n \to Y^m$ is "smooth", and X, Y are themselves smooth manifolds of dimension n, m respectively. Without going into the formal details, X is a smooth manifold of dimension n if it is a paracompact topological space and for any $x \in X$ there is a neighborhood V of x and a smooth "chart", $\phi : V \subset X \to U \subset \Re^n$. In particular if $x \in V_i \cap V_j$ for two open neighborhoods, V_i, V_j of x then

$$\phi_i \circ \phi_j^{-1} : \phi_j(V_i \cap V_j) \subset \Re^n \to \phi_i(V_i \cap V_j) \subset \Re^n$$

is a diffeomorphism. A smooth structure on X is an *atlas*, namely a family $\{(\phi_i, V_i)\}$ of charts such that $\{V_i\}$ is an open cover of X. The purpose of this definition is that if $f : X^n \to Y^m$ then there is an induced function near a point x given by

$$f_{ij} = \psi_i \circ f \circ \phi_j^{-1} : \Re^n \to \phi_j^{-1}(V_j) \to Y \to \Re^m.$$

Here (ϕ_j, V_j) is a chart at x, and (ψ_i, V_i) is a chart at $f(x)$. If the induced functions $\{f_{ij}\}$ at every point x are differentiable then f is said to be differentiable, and the "induced" differential of f is denoted by df. The charts thus provide a convenient way of representing the differential df of f at the point x. In particular once (ϕ_j, V_j) and (ψ_i, V_i) are chosen for x and $f(x)$, then $df(x)$ can be represented by the Jacobian matrix $Df(x) = (\partial f_{ij})$. As before $Df(x)$ will consist of n columns and m rows. Characteristics of the Jacobian, such as rank, will be independent of the choices for the charts (and thus coordinates) at x and $f(x)$. (See Chillingsworth (1976), for example, for the details.)

If the differential df of a function $f : X^n \to Y^m$ is defined and continuous then f is called C^1. Let $C_1(X, Y)$ be the collection of such C^1-maps. Analogous to the case of functions between real vector spaces, we may also write $C_r(X, Y)$ for the class of C^r-differentiable functions between X and Y.

The implicit function theorem also holds for members of $C_1(X, Y)$.

Implicit Function Theorem for Manifolds *Suppose that $f : X^n \to Y^m$ is a C^1-function between smooth manifolds of dimension n, m respectively.*

1. *If $n \geq m$ and f is a submersion at x (i.e., rank $df(x) = m$) then $f^{-1}(f(x))$ is (locally) a smooth manifold in X of dimension $(n-m)$. Moreover, if Z is a manifold in Y^n of dimension r, and f is a submersion at each point in $f^{-1}(Z)$, then $f^{-1}(Z)$ is a submanifold of X of dimension $n-m+r$.*
2. *If $n \leq m$ and f is an immersion at x (i.e., rank $df(x) = n$) then there is a neighbourhood U of x in X such that $f(U)$ is an n-dimensional manifold in Y and in particular $f(U)$ is open in Y.*

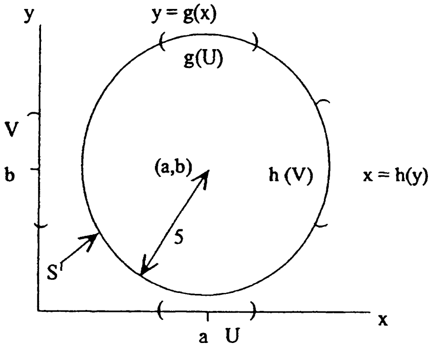


Fig. 5.2 Example 5.3

The proof of this theorem is considerably beyond the scope of this book, but the interested reader should look at Golubitsky and Guillemin (1973, p. 9) or Hirsch (1976, p. 22). This theorem is a smooth analogue of the isomorphism theorem for linear functions given in Sect. 2.2. For a linear function $T : \Re^n \rightarrow \Re^m$ when $n \geq m$ and T is surjective, then $T^{-1}(y)$ has the form $x_0 + K$ where K is the $(n-m)$-dimensional kernel. Conversely if $T : \Re^n \rightarrow \Re^m$ and $n \leq m$ when T is injective, then image (T) is an n-dimensional subspace of $\Re^m$. More particularly if U is an n-dimensional open set in $\Re^n$ then $T(U)$ is also an n-dimensional open set in $\Re^m$.

Example 5.3 To illustrate, consider Example 5.2(2) again. When $y \neq b$, df has rank 1 and so there exists a "local" solution $y' = g(x')$ such that $f(x', g(x')) = 0$. In other words

$$f^{-1}(0) = \left\{ \left(x', g\left(x'\right)\right) \in \Re^2 : x' \in U \right\},$$

which essentially is a copy of U but deformed in $\Re^2$. Thus $f^{-1}(0)$ is "locally" a one-dimensional manifold. Indeed the set $S^1 = \{(x, y) : f(x, y) = 0\}$ itself is a 1-dimensional manifold in $\Re^2$.

If $y \neq b$, and $(x, y) \in S^1$ then there is a neighbourhood U of x and a diffeomorphism $g : S^1 \rightarrow \Re : (x', y') \rightarrow g(y')$ and this parametrises S^1 near (x, y).

If $y = b$, then we can do the same thing through a local solution $x' = h(y')$ satisfying $f(h(y'), y') = 0$.

5.1.2 Singular Points and Morse Functions

When $f : X^n \rightarrow Y^m$ is a C^1-function between smooth manifolds, and rank $df(x)$ is maximal $(= \min(n, m))$ then as before write $x \in S_0(f)$.

The set of *singular points* of f is $S(f) = X \backslash S_0(f)$. Let $z = \min(n, m)$ and say that x is a *corank r singularity*, or $x \in S_r(f)$, if rank $(df(x)) = z - r$.

Clearly $S(f) = \bigcup_{r>1} S_r(f)$.

In the next section we shall examine the corank r singularity sets of a C^1-function and show that they have a nice geometric structure. In this section we consider the case $m = 1$.

In the case of a C^2-function $f : X^n \to \Re$, either x will be regular (in $S_0(f)$) or a critical point (in $S_1(f)$) where $df(x) = 0$. We call a critical point of f *non-degenerate* iff $d^2f(x)$ is non-singular. A C^2-function all of whose critical points are non degenerate is called a *Morse* function. A Morse function, f, near a critical point has a very simple representation.

A local system of coordinates at a point x in X is a smooth assignment

$$y \xrightarrow{\phi} (h_1, \dots, h_n)$$

for every y in some neighbourhood U of x in X.

Lemma 5.1 (Morse) *If $f : X^n \to \Re$ is C^2 and x is a non-degenerate critical point of index k, then there exists a local system of coordinates (or chart (ϕ, V)) at x such that f is given by*

$$y \xrightarrow{\phi} (h_1, \dots, h_n) \xrightarrow{g} f(x) - \sum_{i=1}^{k} {h_i}^2 + \sum_{i=k+1}^{n} {h_i}^2.$$

As before the *index* of the critical point is the number of negative eigenvalues of the Hessian Hf at x. The C^2-function g has Hessian

$$Hg(0) = \begin{pmatrix} -2 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & -2 & & & \\ & & & & 2 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix} \begin{matrix} \uparrow \\ k \\ \downarrow \\ \\ \\ \\ \\ \end{matrix}$$

with k negative eigenvalues. Essentially the Morse lemma implies that when x is a non-degenerate critical point of f, then f is topologically equivalent to the function g with a similar Hessian at the point.

By definition, if f is a Morse function then all its critical points are non-degenerate. Moreover if $x \in S_1(f)$ then there exists a neighbourhood V of x such that x is the only critical point of f in V.

To see this note that for $y \in V$,

$$df(y) = dg(h_1, \dots, h_n) = (-2h_1, \dots, 2h_n) = 0$$

iff $h_1 = \cdots = h_n = 0$, or $y = x$. Thus each critical point of f is isolated, and so $S_1(f)$ is a set of isolated points and thus a zero-dimensional object.

As we shall see almost any smooth function can be approximated arbitrarily closely by a Morse function.

To examine the regular points of a differentiable function $f : X \to \Re$, we can use the Sard Lemma.

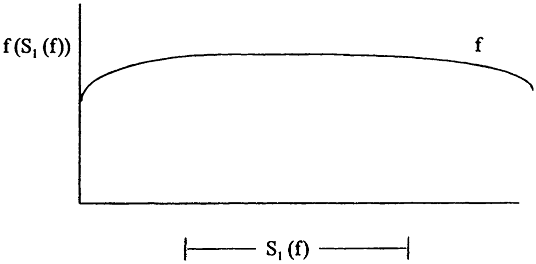


Fig. 5.3 Illustration of Sard's Lemma

First of all a set V in a topological space X is called *nowhere dense* if its closure, $clos(V)$, contains no non-empty open set. Alternatively $X \backslash clos(V)$ is dense.

If X is a complete metric space then the union of a countable collection of closed nowhere dense sets is nowhere dense. This also means that a residual set (the intersection of a countable collection of open dense sets) is dense. (See Sect. 3.1.2). A set V is of *measure zero* in X iff for any $\epsilon > 0$ there exists a family of cubes, with volume less than ϵ, covering V. If V is closed, of measure zero, then it is nowhere dense.

Lemma 5.2 (Sard) *If $f : X^n \to \Re$ is a C^r-map where $r \geq n$, then the set $f(S_1(f))$ of critical values of f has measure zero in $\Re$. Thus $f(S_0(f))$, the set of regular values of f, is the countable intersection of open dense sets and thus is dense.*

To illustrate this consider Fig. 5.3. f is a quasi-concave C^1-function $f : \Re \to \Re$. The set of critical points of f, namely $S_1(f)$, clearly does not have measure zero, since $S_1(f)$ has a non-empty interior. Thus f is not a Morse function. However $f(S_1(f))$ is an isolated point in the image.

Example 5.4 To illustrate the Morse lemma let $Z = S^1 \times S^1$ be the *torus* (the skin of a doughnut) and let $f : Z \to \Re$ be the height function.

Point s, at the bottom of the torus, is a minimum of the function, and so the index of $s = 0$. Let $f(s) = 0$.

Then near s, f can be represented by

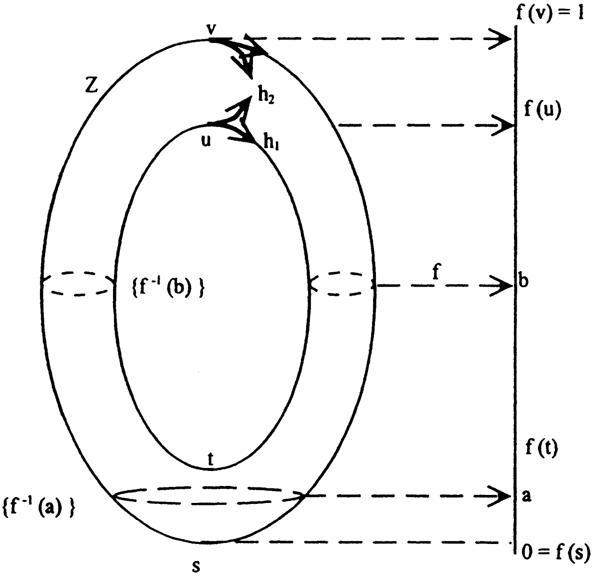
$$(h_1, h_2) \to 0 + {h_1}^2 + {h_2}^2.$$

Note that the Hessian of f at s is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and so is positive definite.

The next critical point, t, is obviously a saddle, with index 1, and so we can write $(h_1, h_2) \to f(t) + {h_1}^2 - {h_2}^2$. Clearly $Hf(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$.

Suppose now that $a \in (f(s), f(t))$. Clearly a is a regular value, and so any point $x \in Z$ satisfying $f(x) = a$ is a regular point, and f is a submersion at x. By the implicit function theorem $f^{-1}(a)$ is a one-dimensional manifold. Indeed it is a single copy of the circle, S^1. See Fig. 5.4.

Fig. 5.4 Critical points on Z



The next critical point is the saddle, u, near which f is represented as

$$(h_1, h_2) \to f(u) - {h_1}^2 + {h_2}^2.$$

Now for $b \in (f(t), f(u))$, $f^{-1}(b)$ is a one-dimensional manifold, but this time it is *two* copies of S^1. Finally v is a local maximum and f is represented near v by $(h_1, h_2) \to f(u) - {h_1}^2 - {h_2}^2$. Thus the index of v is 2.

We can also use this example to introduce the idea of the Euler characteristic $\chi(X)$ of a manifold X. If X has dimension, n, let $c_i(X, f)$ be the number of critical points of index i, of the function $f : X \to \Re$ and let

$$\chi(X, f) = \sum_{i=0}^{n} (-1)^i c_i(X, f).$$

For example the height function, f, on the torus Z has

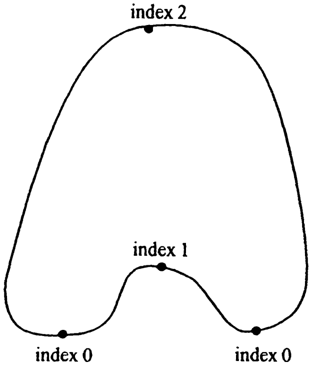
(i) $c_0(Z, f) = 1$, since s has index 0
(ii) $c_1(Z, f) = 2$, since both t and u have index 1
(iii) $c_2(Z, f) = 1$, since v has index 2.

Thus $\chi(Z, f) = 1 - 2 + 1 = 0$. In fact, it can be shown that $\chi(X, f)$ is independent of f, when X is a compact manifold. It is an *invariant* of the smooth manifold X, labelled $(\chi(X))$. Example 5.4 illustrates the fact that $\chi(Z) = 0$.

Example 5.5

(1) The sphere S^1. It is clear that the height function $f : S^1 \to \Re$ has an index 0 critical point at the bottom and an index 1 critical point at the top, so $\chi(S^1) = 1 - 1 = 0$.

Fig. 5.5 Critical points in S^2



(2) The sphere S^2 has an index 0 critical point at the bottom and an index 2 critical point at the top, so $\chi(S^2) = c_0 + c_2 = 1 + 1 = 2$. See Fig. 5.5.

It is possible to deform the sphere, S^2, so as to induce a saddle, but this creates an index 0 critical point. In this case $c_0 = 2, c_1 = 1$, and $c_2 = 1$ as in Fig. 5.5. Thus $\chi(S^2) = 2 - 1 + 1 = 2$ again.

(3) More generally, $\chi(S^n) = 0$ if n is odd and $= 2$ if n is even.

(4) To compute $\chi(B^n)$ for the closed n-ball, take the sphere S^n and delete the top hemisphere. The remaining bottom hemisphere is diffeomorphic to B^n. By this method we have removed the index n critical point at the top of S^n.

For $n = 2k + 1$ odd, we know

$$\chi\left(S^{2k+1}\right) = \sum_{i=0}^{2k}(-1)^i c_i\left(S^n\right) - c_n\left(S^n\right) = 0,$$

so $\chi(B^{2k+1}) = \chi(S^{2k+1}) + 1 = 1$. For $n = 2k$, even we have

$$\sum_{i=0}^{2k-1}(-1)c_i\left(S^n\right) + c_n\left(S^n\right) = 2, \quad \text{so } \chi\left(B^{2k}\right) = \chi\left(S^{2k}\right) - 1 = 1.$$

5.2 Transversality

To examine the singularity set $S(f)$ of a smooth function $f : X \to Y$ we introduce the idea of *transversality*.

A linear manifold V in $\Re^n$ of dimension v is of the form $x_0 + K$, where K is a vector subspace of $\Re^n$ of dimension v. Intuitively if V and W are linear manifolds in $\Re^n$ of dimension v, w then typically they will not intersect if $v + w < n$.

On the other hand if $v + w \geq n$ then $V \cap W$ will typically be of dimension $v + w - n$.

For example two lines in $\Re^2$ will intersect in a point of dimension $1 + 1 - 2$.

Another way of expressing this is to define the codimension of V in $\Re^n$ to be $n - v$. Then the codimension of $V \cap W$ in W will typically be $w - (v + w - n) = n - v$, the same codimension.

Suppose now that $f : X^n \to Y^m$ where X, Y are vector spaces of dimension n, m respectively. Let Z be a z-dimensional linear manifold in Y. Say that f is *transversal* to Z iff for all $x \in X$, either (i) $f(x) \notin Z$ or (ii) the image of $df(x)$, regarded as a vector subspace of Y^m, together with Z span Y. In this case write $f \overset{T}{\cap} Z$. The same idea can be extended to the case when X, Y, Z, are all manifolds. Whenever $f \overset{T}{\cap} Z$, then if $x \in f^{-1}(Z)$, f will be a submersion at x, and so $f^{-1}(Z)$ will be a smooth manifold in X of codimension equal to the codimension of Z in Y. Another way of interpreting this is that the number of constraints which determine Z in Y will be equal to the number of constraints which determine $f^{-1}(Z)$ in X. Thus $\dim(f^{-1}(Z)) = n - (m - z)$.

In the previous chapter we put the Whitney C^s-topology on the set of C^s-differentiable maps $X^n \to Y^m$, and called this $C^s(X, Y)$. In this topological space a residual set is dense. The fundamental theorem of singularity theory is that transversal intersection is generic.

Thom Transversality Theorem *Let X^n, Y^m be manifolds and Z^z a submanifold of Y. Then the set*

$$\left\{f \in C^s(X, Y) : f \overset{T}{\cap} Z\right\} = \overset{T}{\cap} (X, Y; Z)$$

is a residual (and thus dense) set in the topological space $C^s(X, Y)$.

Note that if $f \in \overset{T}{\cap} (X, Y; Z)$ then $f^{-1}(Z)$ will be a manifold in X of dimension $n - m + z$.

Moreover if $g \in C^s(X, Y)$ but g is not transversal to Z, then there exists some C^s-map, as near to g as we please in the C^s topology, which is transversal to Z. Thus transversal intersection is typical or *generic*.

Suppose now that $f : X^n \to Y^m$, and corank $df(x) = r$, so rank $df(x) = \min(n, m) - r$. In this case we said that $x \in S_r(f)$, the corank r singularity set of f. We seek to show that $S_r(f)$ is a manifold in X, and compute its dimension.

Suppose X^n, Y^m are vector spaces, with dimension n, m respectively. As before $\mathcal{L}(X, Y)$ is the normed vector space of linear maps from X to Y. Let $\mathcal{L}_r(X, Y)$ be the subset of $\mathcal{L}(X, Y)$ consisting of linear maps with corank r.

Lemma 5.3 *$\mathcal{L}_r(X, Y)$ is a submanifold of $\mathcal{L}(X, Y)$ of codimension $(n - z + r)(m - z + r)$ where $z = \min(n, m)$.*

Proof Choose bases such that a corank r linear map S is represented by a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where rank $(A) = k = z - r$.

Let U be a neighbourhood of S in $\mathcal{L}(X, Y)$, such that for all $S' \in U$, corank $(S') = r$.

Define $F : U \to \mathcal{L}(\Re^{n-k}, \Re^{m-k})$ by $F(S') = D' - C(A')^{-1}B'$.

Now $S' \in F^{-1}(0)$ iff rank $(S') = k$. The codimension of 0 in $\mathcal{L}(\Re^{n-k}, \Re^{m-k})$ is $(n-k)(m-k)$. Since F is a submersion $F^{-1}(0) = \mathcal{L}_r(X, Y)$ is of codimension $(n-z+r)(m-z+r)$. □

Now if $f \in C^s(X, Y)$, then $df \in C^{s-1}(X, \mathcal{L}(X, Y))$. If $df(x) \in \mathcal{L}_r(X, Y)$ then $x \in S_r(f)$. By the Thom Transversality Theorem, there is a residual set in $C^{s-1}(X, \mathcal{L}(X, Y))$ such that $df(x)$ is transversal to $\mathcal{L}_r(X, Y)$. But then $S_r(f) = df^{-1}(\mathcal{L}_r(X, Y))$ is a submanifold of X of codimension $(n-z+r)(m-z+r)$. Thus we have:

The Singularity Theorem *There is the residual set V in $C^s(X, Y)$ such that for all $f \in V$, the corank r singularity set of f is a submanifold in X of codimension $(n-z+r)(m-z+r)$.*

If $f : X^n \to \Re$ then codim $S_1(f) = (n-1+1)(1-1+1) = n$. Hence generically the set of critical points will be zero-dimensional, and so critical points will be isolated. Now a Morse function has isolated, non-degenerate critical points. Let $M^s(X)$ be the set of Morse functions on X with the topology induced from $C^s(X, \Re)$.

Morse Theorem *The set $M^s(X)$ of C^s-differentiable Morse functions (with non-degenerate critical points), is an open dense set in $C^s(X, \Re)$. Moreover if $f \in M^s(X)$, then the critical points of f are isolated.*

More generally if $f : X^n \to Y^m$ with $m \leq n$ then in the generic case, $S_1(f)$ is of codimension $(n-m+1)(n-m+1) = n-m+1$ in X and so $S_1(f)$ will be of dimension $(m-1)$.

Suppose now that $n > 2m-4$, and $n \geq m$. Then $2n - 2m + 4 > n$ and so $r(n-m+r) > n$ for $r \geq 2$. But codimension $(S_r(f)) = r(n-m+r)$, and since codimension $(S_r(f)) >$ dimension X, $S_r(f) = \Phi$ for $r \geq 2$.

Submanifold Theorem *If Z^z is a submanifold of Y^m and $z < m$ then Z is nowhere dense in Y (here $z = \dim(Z)$ and $m = \dim(Y)$).*

In the case $n \geq m$, the singularity set $S(f)$ will generically consist of a union of the various co-rank r singularity submanifolds, for $r \geq 1$. The highest dimension of these is $m-1$. We shall call an $S(f)$ a *stratified manifold* of dimension $(m-1)$. Note also that $S(f)$ will then be nowhere dense in X. We also require the following theorem.

Morse Sard Theorem *If $f : X^n \to Y^m$ is a C^s-map, where $s > n-m$, then $f(S(f))$ has measure zero in Y and $f(S_0(f))$ is residual and therefore dense in Y.*

We are now in a position to apply these results to the critical Pareto set. Suppose then that $u = (u_1, \dots, u_m) : X^n \to \Re^m$ is a smooth profile on the manifold of feasible states X.

Say $x \in \overset{0}{\Theta}(u_1, \dots, u_m) = \overset{0}{\Theta}(u)$ iff $dL(\lambda, u)(x) = 0$, where $L(\lambda, u) = \sum_{i=1}^m \lambda_i u_i$ and $\lambda \in \overline{\Re^m_+}$.

By Theorem 4.21, the critical Pareto set, $\Theta(u)$, contains $\overset{0}{\Theta}(u)$ but possibly also contains further points on the boundary of X. However we shall regard $\overset{0}{\Theta}$ as the differential analogue of the Pareto set. By Lemma 4.19, $\overset{0}{\Theta}(u)$ must be closed in X. Moreover when $n \geq m$ and $x \in \overset{0}{\Theta}(u)$ then the differentials $\{du_i(x) : i \in M\}$ must be linearly dependent. Hence $\overset{0}{\Theta}(u)$ must belong to $S(u)$. But also $S(u)$ will be nowhere dense in X. Thus we obtain the following theorem (Smale 1973 and Debreu 1970).

Pareto Theorem *There exists a residual set U in $C^1(X, \Re^m)$, for* $\dim(X) \geq m$, *such that for any profile $u \in U$, the closed critical Pareto set $\overset{0}{\Theta}(u)$ belongs to the nowhere dense stratified singularity manifold $S(u)$ of dimension $(m-1)$. Moreover if dim$(X) > 2m-4$, then $\Theta(u)$ is itself a manifold of dimension $(m-1)$, for all $u \in C^1(X, \Re^m)$.*

As we have already observed this result implies that $\Theta(u)$ can generally be regarded as an $(m-1)$ dimensional object parametrised by $(m-1)$ coefficients $(\frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_m}{\lambda_1})$ say. Since points in $\overset{0}{\Theta}(u)$ are characterised by first order conditions alone, it is necessary to examine the Hessian of L to find the Pareto optimal points.

5.3 Generic Existence of Regular Economies

In this section we outline a proof of the Debreu-Smale Theorem on the Generic Existence of Regular Economies (see Debreu 1970, and Smale 1974a).

As in Sect. 4.4, let $u = (u_1, \dots, u_m) : X^m \to \Re^m$ be a smooth profile, where $X = \Re^n_+$, the commodity space facing each individual. Let $e = (e_1, \dots, e_m) \in X^m$ be an initial endowment vector. Given u, define the *Walras manifold* to be the set

$$Z_u = \left\{(e, x, p) \in X^m \times X^m \times \Delta\right\}$$

(where Δ is the price simplex) such that (x, p) is a *Walrasian equilibrium* for the economy (e, u). That is, $(x, p) = (x_1, \dots, x_m, p) \in X^m \times \Delta$ satisfies:

1. individual optimality: $D^*u_i(x_i) = p$ for $i \in M$,
2. individual budget constraints: $\langle p, x_i \rangle = \langle p, e_i \rangle$ for $i \in M$,
3. social resource constraints: $\sum_{i=1}^m x_{ij} = \sum_{i=1}^m e_{ij}$ for each commodity $j = 1, \dots, n$.

Note that we implicitly assume that each individual's utility function, u_i, is defined on a domain $X_i \equiv X \subset \Re^n_+$, so that the differential $du_i(x_i)$ at $x_i \in X_i$ can be represented by a vector $Du_i(x_i) \in \Re^n$. As we saw in Chap. 4, we may normalize

Du_i and p so the optimality condition for i becomes $D^*u_i(x) = p$ for $p \in \Delta$. For the space of normalized price vectors, we may identify Δ with $\{p \in \Re^n_+ : \|p\| = 1\}$. Observe that $\dim(\Delta) = n - 1$.

We seek to show that there is a residual set U in $C^s(X, \Re^m)$ such that the Walras manifold is a smooth manifold of dimension mn.

Now define the Debreu projection

$$\pi : Z_u \subset X^m \times X^m \times \Delta \to X^m : (e, x, p) \to e.$$

Note that both Z_u and X^m will then have dimension mn.

By the Morse Sard Theorem the set

$$V = \left\{e \in X^m : d\pi \text{ has rank } nm \text{ at}(e, x, p)\right\}$$

is dense in X^m.

Say the economy (e, u) is *regular* if $\pi(e, x, p) = e$ is a regular value of π (or rank $d\pi = mn$) for all $(x, p) \in X^m \times \Delta$ such that $((e, x, p) \in Z_u)$.

When e is a regular value of π, then by the inverse function theorem,

$$\pi^{-1}(e) = \left\{(e, x, p)^1, (e, x, p)^2, \ldots, (e, x, p)^k\right\}$$

is a zero-dimensional object, and thus will consist of a finite number of isolated points. Thus for each $e \in V$, the Walrasian equilibria associated with e will be finite in number. Moreover there will exist a neighbourhood N of e in V such that the Walrasian equilibria move continuously with e in N.

Proof of the Generic Regularity of the Debreu Projection Define $\psi_u : X^m \times \Delta \to \Delta^{m+1}$ where $u \in C^r(X, \Re^m)$ by $\psi_u(x, p) = (D^*u_1(x_1), \ldots, D^*u_m(x_m), p)$ where $x = (x_1, \ldots, x_m)$ and $u = (u_1, \ldots, u_m)$. Let I be the diagonal $\{(p, \ldots, p)\}$ in Δ^{m+1}. If $(x, p) \in \psi_u^{-1}(I)$ then for each i, $D^*u_i(x_i) = p$ and so the first order individual optimality conditions are satisfied. By the Thom Transversality Theorem there is a residual set (in fact an open dense set) of profiles U such that ψ_u is transversal to I for each $u \in U$. But then the codimension of $\psi_u^{-1}(I)$ in $X^m \times \Delta$ equals the codimension of I in Δ^{m+1}.

Now Δ and I are both of dimension $(n - 1)$ and so codimension (I) in Δ^{m+1} is $(m + 1)(n - 1) - (n - 1) = m(n - 1)$. Thus $\dim(X^m \times \Delta) - \dim(\psi_u^{-1}(I)) = m(n - 1)$ and $\dim(\psi_u^{-1}(I)) = mn + (n - 1) - m(n - 1) = n + m - 1$, for all $u \in U$.

Now let $e \in X^m$ be the initial endowment vector and

$$Y(e) = \left\{(x_1, \ldots, x_m) \in X^m : \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i\right\}$$

be the set of feasible outcomes, a hyperplane in $\Re^{nm}_+$ of dimension $n(m - 1)$. For each i, let $B_i(p) = \{x_i \in X : \langle p, x_i\rangle = \langle p, e_i\rangle\}$, be the hyperplane through the boundary of the ith budget set at the price vector p.

Define

$$\begin{aligned}
&\Sigma(e)=\bigl\{(x,p)\in X^m\times\Delta : x\in Y(e), x_i\in B_i(p),\ \forall i\in M\bigr\}, \quad \text{and}\\
&\Gamma=\bigl\{(e,x,p): e\in X^m, (x,p)\in\Sigma(e)\bigr\}.
\end{aligned}$$

As discussed in Chap. 4, $Y(e)$ is characterized by n linear equations, while the budget restrictions induce a further $(m-1)$ linear restraints (the mth budget restraint is redundant). Thus the dimension of Γ is $2mn+(n-1)-n-(m-1)=2mn-m$. (In fact, if Δ is taken to be the $(n-1)$ dimensional simplex, then Γ will be a linear manifold of dimension $2mn-m$. More generally, Γ will be a submanifold of $X^m\times X^m\times\Delta$ of dimension $2mn-m$. At each point the projection is a regular map (*i.e.*, the rank of the differential of $(e,x,p)\to(x,p)$ is maximal).

To see this define $\phi : X^m\times X^m\times\Delta\to\Re^n\times\Re^{m-1}$ by

$$\phi(e,x,p)=\left(\sum_{i=1}^m x_i-\sum_{i=1}^m e_i, \langle p,x_1\rangle-\langle p,e_1\rangle,\dots,\langle p,x_{m-1}\rangle-\langle p,e_{m-1}\rangle\right).$$

Clearly if $\phi(e,x,p)=\underline{0}$ then $x\in Y(e)$ and $x_i\in B_i(p)$ for each i. But $\underline{0}$ is of codimension $n+m-1$ in $\Re^n\times\Re^{m-1}$; thus $\phi^{-1}(\underline{0})$ is of the same codimension in $X^{2m}\times\Delta$. Thus $\dim(X^{2m}\times\Delta)-\dim\phi^{-1}(\underline{0})=n+m-1$ and $\dim\phi^{-1}(\underline{0})=2nm+(n-1)-(n+m-1)=2mn-m$ (giving the dimension of Γ). In a similar fashion, for $(x,p)\in\Sigma(e)$, $\phi(e,x,p)=\underline{0}$, and so

$$\dim\bigl(X^m\times\Delta\bigr)-\dim\phi^{-1}(\underline{0})=n+m-1.$$

Thus $\Sigma(e)$ is a submanifold of $X^m\times\Delta$ of dimension

$$nm+(n-1)-(n+m-1)=mn-m.$$

Finally define $Z_u=\{(e,x,p)\in\Gamma:\psi_u(x,p)\in I\}$. For each $u\in U$, Z_u is a submanifold of $X^{2m}\times\Delta$ of dimension mn.

To see this, let $f_u(e,x,p)=\psi_u(x,p)$. Then

$$f_u:\Gamma\to X^m\times X^m\times\Delta\to X^m\times\Delta\overset{\psi_u}{\to}\Delta^{m+1}.$$

As we observed for all $u\in U$, ψ_u is transversal to I in Δ^{m+1}. But the codimension of I in Δ^{m+1} is $m(n-1)$. Since f_u will be transversal to I,

$$\dim(\Gamma)-\dim\bigl(f_u^{-1}(I)\bigr)=m(n-1).$$

Hence $\dim(f_u^{-1}(I))=mn$. Clearly $Z_u=f_u^{-1}(I)$.

Thus for all $u\in U$, the Debreu projection $\pi: Z_u\to X^m$ will be a C^1-map between manifolds of dimension mn. The Morse Sard Theorem gives the result. □

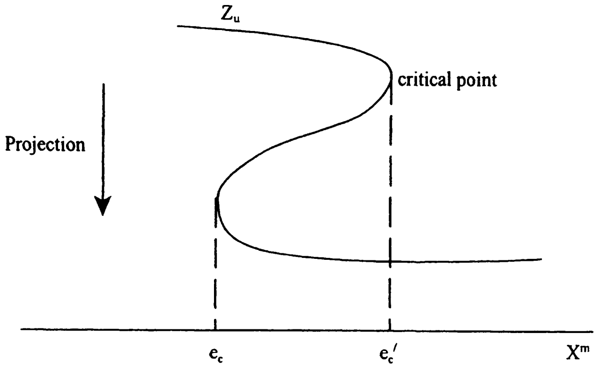


Fig. 5.6 The Debreu map

Thus we have shown that for each smooth profile u in an open dense set U, there exists an open dense set V of initial endowments such that (e, u) is a regular economy for all $e \in V$.

The result is also related to the existence of a demand function for an economy.

A demand function for i (with utility u_i) is the function

$$f_i : \Delta \times \Re_+ \to X$$

where $f_i(p, I)$ is that $x_i \in X$ which maximizes u_i on

$$B_i(p, I) = \{x \in X : \langle p, x\rangle = I\}.$$

Now define $\phi_i : X \to \Delta \times \Re_+$ by $\phi_i(x) = (D^*u_i(x), \langle D^*u_i(x), x\rangle)$.

But the optimality condition is precisely that $D^*u_i(x) = p$ and $\langle D^*u_i(x), x\rangle = \langle p, x\rangle = I$. Thus when ϕ_i has maximal rank, it is locally invertible (by the inverse function theorem) and so locally defines a demand function.

On the other hand if f_i is a C^1-function then ϕ_i must be locally invertible (by f_i). If this is true for all the agents, then $\psi_u : X^m \times \Delta \to \Delta^{m+1}$ must be transversal to I. Consequently if $u = (u_1, \dots, u_m)$ is such that each u_i defines a C^1-demand function $f_i : \Delta \times \Re_+ \to X$ then $u \in U$, the open dense set of the regular economy theorem.

As a final note suppose that $u \in U$ and e is a critical value of the Debreu projection. Then it is possible that $\pi^{-1}(e) = e \times W$ where W is a continuum of Walrasian equilibria. Another possibility is that there is a continuum of singular or *catastrophic* endowments C, so that as the endowment vector crosses C the number of Walrasian equilibria changes suddenly. As we discussed in Sect. 4.4, at a "catastrophic" endowment, stable and unstable Walrasian equilibria may merge (see Balasko 1975).

Another question is whether for every smooth profile, u, and every endowment vector, e, there exists a Walrasian equilibrium (x, p). This is equivalent to the requirement that for every u the projection $Z_u \to X^m$ is *onto* X^m.

In this case for each $e \in X^m$ there will exist some $(e, x, p) \in Z_u$. This is clearly necessary if there is to exist a market clearing price equilibrium p for the economy (e, u).

The usual general equilibrium arguments to prove existence of a market clearing price equilibrium typically rely on convexity properties of preference (see Chap. 3). However weaker assumptions on preference permit the use of topological arguments to show the existence of an extended Walrasian equilibrium (where only the first order conditions are satisfied).

When the market-clearing price equilibrium does exist, it is useful to consider a price adjustment process (or "auctioneer") to bring about equilibrium.

5.4 Economic Adjustment and Excess Demand

To further develop the idea of a demand function and price adjustment process, we consider the following famous example of Scarf (1960).

Example 5.6 There are three individuals $i \in M = \{1, 2, 3\}$ and three commodities. Agent i has utility $u_i(x_1^i, x_2^i, x_3^i) = \min(x_i^i, x_j^i)$ where $x^i = (x_1^i, x_2^i, x_3^3) \in \Re_+^3$ is the ith commodity space.

At income I, and price vector $p = (p_1, p_2, p_3)$, i demands equal amounts of x_i^i, x_j^j and zero of x_k^i: thus $(p_i + p_j)x = I$, so $x_i^i = I(p_i + p_j)^{-1} = x_j^i$.

Suppose the initial endowment e_i of agent i is 1 unit of the ith good, and nothing of the jth and kth. Then $I = p_i$ and so $i's$ demand function f_i has the form $f_i(p) = (f_{ii}(p), f_{ij}(p), f_{ik}(p)) = (\frac{p_i}{p_i+p_j}, \frac{p_i}{p_i+p_j}, 0) \in \Re^3$.

The *excess demand* function by i is $\xi_i(p) = f_i(p) - e_i$.

Since $e_i = (e_{ii}, e_{ij}, e_{ik}) = (1, 0, 0)$ this gives

$$\xi_i(p) = \left(\frac{-p_j}{p_i + p_j}, \frac{p_i}{p_i + p_j}, 0\right) = (\xi_{ii}, \xi_{ij}, \xi_{ik}).$$

Suppose now the other two consumers are described by cyclic permutation of subscripts, *e.g.*, j has 1 unit of the jth good and utility $u_j(x_j^j, x_k^j, x_i^j) = \min(x_j^j, x_k^j)$, *etc.*, then the total excess demand at p is

$$\xi(p) = \sum_{i=1}^{3} \xi_i(p) \in \Re^3.$$

For example, the excess demand in commodity j is:

$$\xi_j = \xi_{1j} + \xi_{2j} + \xi_{3j} = \frac{p_i}{p_i + p_j} - \frac{p_k}{p_j + p_k}.$$

Since each i chooses $f_i(p)$ to maximize utility subject to $\langle p_1, f_i(p)\rangle = I = \langle p, e_i\rangle$ we expect $\sum_{i=1}^{3} \langle p, f_i(p) - e_i\rangle = 0$.

To see this note that

$$\begin{aligned}\langle p, \xi(p)\rangle &= \left\langle p, \left(\frac{p_3}{p_3 + p_1} - \frac{p_2}{p_1 + p_2}, \frac{p_1}{p_1 + p_2} - \frac{p_3}{p_2 + p_3}, \frac{p_2}{p_2 + p_3} - \frac{p_1}{p_1 + p_3}\right)\right\rangle \\ &= 0.\end{aligned}$$

The equation $\langle p, \xi(p)\rangle = 0$ is known as Walras' Law. To interpret it, suppose we let Δ be the simplex in $\Re^3$ of price vectors such that $\|p\| = 1$, and $p_i \geq 0$. Walras' Law says that the excess demand vector $\xi(p)$ is orthogonal to the vector p. In other words $\xi(p)$ may be thought of as a tangent vector in Δ. (This is easier to see if we identify Δ with a quadrant of the sphere, S^2.)

We may therefore consider a *price adjustment process*, which changes the price vector $p(t)$, at time t by the differential equation

$$\frac{dp(t)}{dt} = \xi(p) \qquad (*)$$

This adjustment process is a vector field on Δ: that is at every p there exists a rule that changes $p(t)$ by the equation $\frac{dp(t)}{dt} = \xi(p)$.

If at a vector p^*, the excess demand $\xi(p^*) = 0$ then $\frac{dp(t)}{dt}|_{p^*} = 0$, and the price adjustment process has a stationary point. The flow on Δ can be obtained by integrating the differential equation. It is easy to see that if p^* satisfies $p_1^* = p_2^* = p_3^*$ then $\xi(p^*) = 0$, so there clearly is a price equilibrium where excess demand is zero.

The price adjustment equation (*) does not result in a flow into the price equilibrium.

To see this, compute the scalar product

$$\begin{aligned}
&\big\langle (p_2p_3, p_1p_3, p_1p_2), \xi(p)\big\rangle \\
&\quad = -p_3\frac{({p_1}^2 - {p_2}^2)}{p_1 + p_2} + p_2\frac{({p_3}^2 - {p_1}^2)}{p_1 + p_3} + p_1\frac{({p_1}^2 - {p_3}^2)}{p_2 + p_3} \\
&\quad = p_3(p_1 - p_2) + p_2(p_3 - p_1) + p_1(p_2 - p_1) \\
&\quad = 0.
\end{aligned}$$

But if $\xi(p) = \frac{dp}{dt}$ then we obtain $p_2p_3\frac{dp}{dt} + p_1p_3\frac{dp}{dt} + p_1p_2\frac{dp}{dt} = 0$.

The obvious solution to this equation is that $p_1(t)p_2(t)p_3(t) = \text{constant}$.

In other words when the adjustment process satisfies (*) then the price vector p (regarded as a function of time, t,) satisfies the equation $p_1(t)p_2(t)p_3(t) = \text{constant}$. The flow through any point $p = (p_1, p_2, p_3)$, other than the equilibrium price vector p^*, is then homeomorphic to a circle S^1, inside Δ.

Just to illustrate, consider a vector p with $p_3 = 0$.

Then $\xi(p) = (\frac{-p_2}{p_1+p_2}, \frac{p_1}{p_1+p_2}, 0)$.

Because we have drawn the flow on the simplex $\Delta = \{p \in \Re^3_+ : \sum p_i = 1\}$ the flow $\frac{dp}{dt}(t) = \xi(p)$ is discontinuous in p at the three vertices of Δ.

However in the interior of Δ the flow is essentially circular (and anti-clockwise). See Fig. 5.7.

To examine the nature of the flow given by the differential equation $\frac{dp}{dt}(t) = \xi(p)$, define a Lyapunov function at $p(t)$ to be $L(p(t)) = \frac{1}{2}\sum_{i=1}^{3}(p_i(t) - p_i^*)^2$ where $p^* = (p_1^*, p_2^*, p_3^*)$ is the equilibrium price vector satisfying $\xi(p^*) = 0$. Since

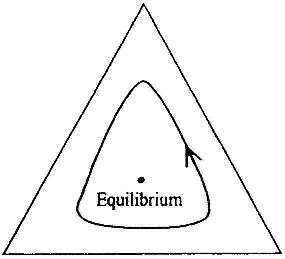


Fig. 5.7 Flow on the price simplex

$p^* \in \Delta$, we may choose $p^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then

$$\begin{aligned}\frac{dL}{dt} &= \sum_{i=1}^{3} -\bigl(p_i(t) - p_i^*\bigr)\frac{dp_i}{dt} \\ &= \sum_{i=1}^{3} \xi_i\bigl(p(t)\bigr)p_i(t) - \sum_{i=1}^{3} p_i^*\xi_i\bigl(p(t)\bigr) \\ &= -\frac{1}{3}\sum_{i=1}^{3} \xi_i\bigl(p(t)\bigr).\end{aligned}$$

(This follows since $\langle p, \xi(p)\rangle = 0$.)

If $\xi_i(p(t)) > 0$ for $i = 1, 2, 3$ then $\frac{dL}{dt} < 0$ and so the Lyapunov distance $L(p(t))$ of $p(t)$ from p^* decreases as $t \to \infty$. In other words if $\delta p(t) = p(t) - p^*$ then the distance $\|\delta p(t)\| \to 0$ as $t \to \infty$. The equilibrium p^* is then said to be *stable.*

If on the contrary $\xi_i(p(t)) < 0 \ \forall i$, then $\frac{dL}{dt} > 0$ and $\|\delta p(t)\|$ increases as $t \to \infty$. In this case p^* is called *unstable.*

However it is easy to show that the equilibrium point p^* is neither stable nor unstable. To see this consider the price vector $p(t) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. It is then easy to show that $\xi = (0, \frac{3}{10}, \frac{-3}{10})$ so the flow through p (locally) keeps the distance $L(p(t))$ constant. To see how $L(p(t))$ behaves near $p(t)$, consider the points $p(t - \delta t) = (\frac{2}{3}, \frac{1}{6} - \frac{1}{20}, \frac{1}{6} + \frac{1}{10})$ and $p(t + \delta t) = (\frac{2}{3}, \frac{1}{6} + \frac{1}{10}, \frac{1}{6} - \frac{1}{10})$. After some easy arithmetic we find that $\xi(t - \delta t) = (0.195, 0.409, -0.814)$ so that $\|\delta p(t - \delta t)\|$ is increasing at $p(t - \delta t)$. On the other hand $\xi(t + \delta t) = (-0.195, 0.814, -0.409)$ so $\|\delta p(t + \delta t)\|$ is decreasing at $p(t + \delta t)$. In other words the total excess demand $\sum_{i=1}^{3} \xi_i(p(t))$ oscillates about zero as we transcribe one of the closed orbits, so the distance $\|\delta p(t)\|$ increases then decreases.

The Scarf Example gives a way to think more abstractly about the process of price adjustment in an economy.

As we have observed, the differential equation $\frac{dp}{dt} = \xi(p)$ on Δ defines a *flow* in Δ. That is if we consider any point $p^0 \in \Delta$ and then solve the equation for p, we obtain an "orbit"

$$\bigl\{p(t) \in \Delta, t \in (-\infty, \infty);\ p(0) = p^0 \text{ and } dp = \xi\bigl(p(t)\bigr)\bigr\}$$

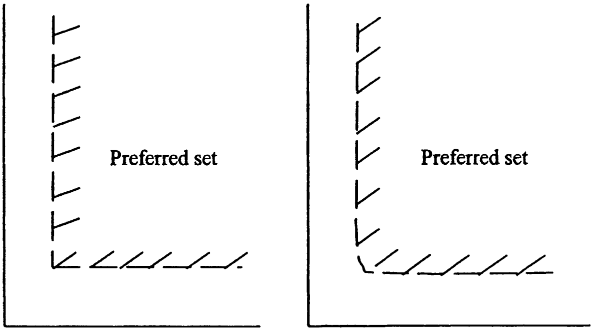


Fig. 5.8 Smoothing the Scarf profile

that commences at the point $p(0) = p^0$, and gives the past and future trajectory. Because the differential equation has a unique solution, any point p_0 can belong to only one orbit. As we saw, each orbit in the example satisfies the equation $p_1(t) \cdot p_2(t) \cdot p_3(t) =$ constant. The *phase portrait* of the differential equation is its collection of orbits.

The differential equation $\frac{dp}{dt} = \xi(p)$ assigns to each point $p \in \Delta$ a vector $\xi(p) \in \Re^n$, and so ξ may be regarded as a function $\xi : \Delta \to \Re^n$. In fact ξ is a continuous map except at the boundary of Δ. This discontinuity only occurs because Δ itself is not smooth at its vertices. If we ignore this boundary feature, then we may write $\xi \in C_0(\Delta, \Re^n)$, where C_0 as before stands for the set of continuous maps. In fact if we examine ξ as a function of p then it can be seen to be differentiable, so $\xi \in C_1(\Delta, \Re^n)$. Obviously $C_1(\Delta, \Re^n)$ has a natural metric and therefore the set $C_1(\Delta, \Re^n)$ can be given the C^1-topology. A differential equation $\frac{dp}{dt} = \xi(p)$ of this kind can thus be treated as an element of $C_1(\Delta, \Re^n)$ in which case it is called a *vector field*. $C_1(\Delta, \Re^n)$ with the C^1-topology is written $C^1(\Delta, \Re^n)$ or $\mathcal{V}^1(\Delta)$. We shall also write $\mathcal{P}(\Delta)$ for the collection of phase portraits on Δ. Obviously, once the vector field, ξ, is specified, then this defines the *phase portrait*, $\tau(\xi)$, of ξ.

In the example, ξ was determined by the utility profile u and endowment vector $e \in \Re^{3\times 3}$. As Fig. 5.8 illustrates the profile u can be smoothed by rounding each u_i without changing the essence of the example.

5.5 Structural Stability of a Vector Field

More abstractly then we can view the excess demand function ξ as a map from $C^s(X, \Re^m) \times X^m$ to the metric space of vector fields on Δ: that is

$$\xi : C^s\big(X, \Re^m\big) \times X^m \longrightarrow \mathcal{V}^1(\Delta).$$

The genericity theorem given above implies that, in fact, there is an open dense set U in $C^s(X, \Re^m)$ such that ξ is indeed a C^1 vector field on Δ. Moreover ξ is an excess demand function obtained from the individual demand functions $\{f_i\}$ as described above.

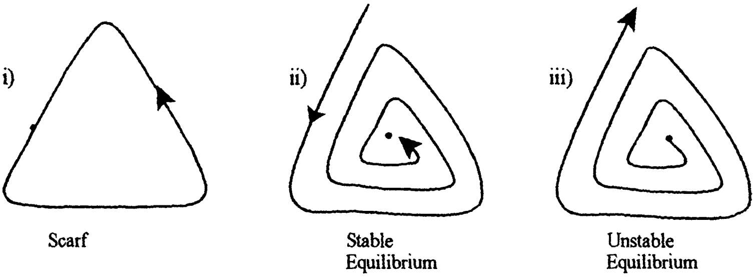


Fig. 5.9 Dissimilar phase portraits

An obvious question to ask is how ξ "changes" as the parameters $u \in C^s(X, \Re^m)$ and $e \in X^m$ change. One way to do this is to consider small perturbations in a vector field ξ and determine how the phase portrait of ξ changes.

It should be clear from the Scarf example that small perturbations in the utility profile or in e may be sufficient to change ξ so that the orbits change in a qualitative way. If two vector fields, ξ_1, and ξ_2 have phase portraits that are homeomorphic, then $\tau(\xi_1)$ and $\tau(\xi_2)$ are qualitatively identical (or similar). Thus we say ξ_1 and ξ_2 are *similar* vector fields if there is a homeomorphism $h : \Delta \to \Delta$ such that each orbit in the phase portrait $\tau(\xi_1)$ of ξ_1 is mapped by h to an orbit in $\tau(\xi_2)$.

As we saw in the Scarf example, each of the orbits of the excess demand function, ξ_1, say, comprises a closed orbit (homeomorphic to S^1). Now consider the vector field ξ_2 whose orbits approach an equilibrium price vector p^*. The phase portraits of ξ_1 and ξ_2 are given in Fig. 5.9.

The price equilibrium in Fig. 5.9(b) is *stable* since $\lim_{t\to\infty} p(t) \to p^*$. Obviously each of the orbits of ξ_2 are homeomorphic to the half open interval $(-\infty, 0]$. Moreover $(-\infty, 0]$ and S^1 are not homeomorphic, so ξ_1 and ξ_2 are not similar.

It is intuitively obvious that the vector field, ξ_2 can be obtained from ξ_1 by a "small perturbation", in the sense that $\|\xi_1 - \xi_2\| < \delta$, for some small $\delta > 0$. When there exists a small perturbation ξ_2 of ξ_1, such that ξ_1 and ξ_2 are dissimilar, then ξ_1 is called *structurally unstable*. On the other hand, it should be plausible that, for any small perturbation ξ_3 of ξ_2 then ξ_3 will have a phase portrait $\tau(\xi_3)$ homeomorphic to $\tau(\xi_2)$, so ξ_2 and ξ_3 will be similar. Then ξ_2 is called *structurally stable*. Notice that structural stability of ξ_2 is a much more general property than stability of the equilibrium point p^* (where $\xi_2(p^*) = 0$).

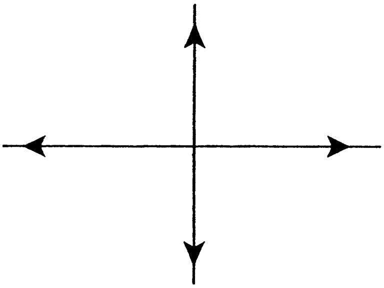
All that we have said on Δ can be generalised to the case of a smooth manifold Y. So let $\mathcal{V}^1(Y)$ be the topological space of C^1-vector fields on Y and $\mathcal{P}(Y)$ the collection of phase portraits on Y.

Definition 5.1

(1) Let $\xi_1, \xi_2 \in \mathcal{V}^1(Y)$. Then ξ_1 and ξ_2 are said to be *similar* (written $\xi_1 \sim \xi_2$) iff there is a homeomorphism $h : Y \to Y$ such that an orbit σ is the phase portrait $\tau(\xi_1)$ of ξ_1 iff $h(\sigma)$ is in the phase portrait of $\tau(\xi_2)$.

(2) The vector field ξ is *structurally stable* iff there exists an open neighborhood V of ξ in $\mathcal{V}^1(Y)$ such that $\xi' \sim \xi$ for all $\xi' \in V$.

Fig. 5.10 A source



(3) A property K of vector fields in $\mathcal{V}^1(2)$ is *generic* iff the set $\{\xi \in \mathcal{V}^1(Y) : \xi \text{ satisfies} K\}$ is residual in $\mathcal{V}^1(Y)$.

As before, a residual set, V, is the countable intersection of open dense sets, and, when $\mathcal{V}^1(Y)$ is a "Baire" space, V will itself be dense.

It was conjectured that structural stability is a generic property. This is true if the dimension of Y is 2, but is false otherwise (Smale 1966; Peixoto 1962).

Before discussing the Peixoto-Smale Theorems, it will be useful to explore further how we can qualitatively "classify" the set of phase portraits on a manifold Y. The essential feature of this classification concerns the nature of the critical or *singularity* points of the vector field on Y and how these are constrained by the topological nature of Y.

Example 5.7 Let us return to the example of the torus $Z = S^1 \times S^1$ examined in Example 5.4. We defined a height function $f : Z \to \Re$ and considered the four critical points $\{s, t, u, v\}$ of f. To remind the reader v was an index 2 critical point (a local maximum of f). Near v, f could be represented as

$$f(h_1, h_2) = f(v) - {h_1}^2 - {h_2}^2.$$

Now f defines a gradient vector field ξ where

$$\xi(h_1, h_2) = -df(h_1, h_2)$$

Looking down on v we see the flow near v induced by ξ resembles Fig. 5.10.

The field ξ may be interpreted as the law of motion under a potential energy field, f, so that the system flows from the "source", v, towards the "sink", s, at the bottom of Z.

Another way of characterizing the source, v, is by what we can call the "degree" of v. Imagine a small ball B^2 around v and consider how the map $g : S^1 \to S^1 : (h_1, h_2) \to \frac{\xi(h_1,h_2)}{\|\xi(h_1,h_2)\|}$ behaves as we consider points (h_1, h_2) on the boundary S^1 of B^2.

At point 1, $\xi(h_1', h_2')$ points "north" so $1 \to 1'$. Similarly at 2, $(\xi({h_1}^2, {h_2}^2))$ points east, so $2 \to 2'$. As we traverse the circle once, so does g. The degree of g is $+1$, and the degree of v is also $+1$. However the saddle, u, is of degree -1.

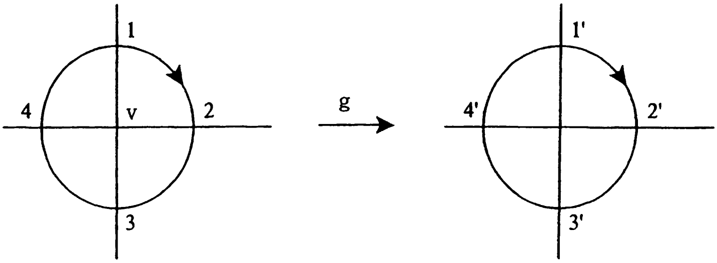


Fig. 5.11 Degree of the map g

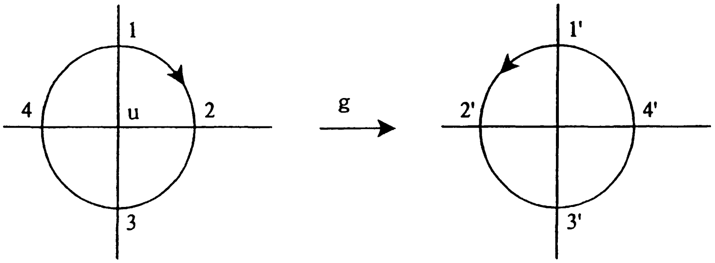


Fig. 5.12 A saddle of degree -1

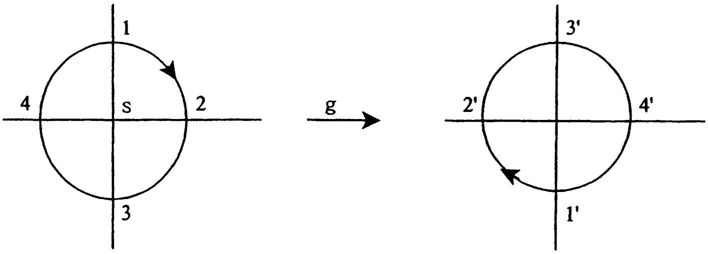


Fig. 5.13 A sink of degree $+1$

At 1, the field points north, but at 2 the field points west, so as we traverse the circle on the left in a clockwise direction, we traverse the circle on the right in an anti-clockwise direction. Because of this change of orientation, the degree of u is -1.

It can also easily be shown that the sink s has degree $+1$.

The rotation at s induced by g is clockwise. It can be shown that, in general, the Euler characteristic can be interpreted as the sum of the degrees of the critical points. Thus $\chi(Z) = 1 - 1 - 1 + 1$, since the degree at each of the two saddle points is -1, and the degree at the source and sink is $+1$.

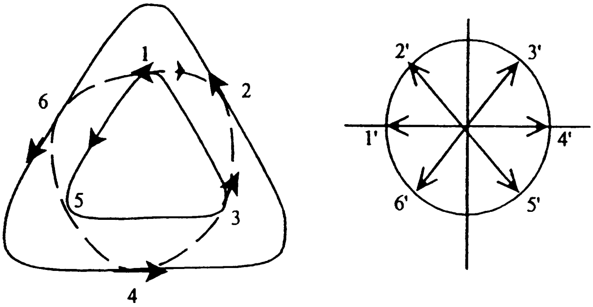


Fig. 5.14 Computing the degree

Example 5.8 It is obvious that the flow for the Scarf example is not induced by a gradient vector field. If there were a function $f : \Delta \to \Re$ satisfying $\xi(p) = -df(p)$, then the orbits of ξ would correspond to decreasing values of f. As we saw however, there are circular orbits for ξ. It is clearly impossible to have a circular flow such that f decreases round the circle. However the Euler characteristic still determines the nature of the zeros (or *singularities*) of ξ.

First we compute the degree of the singularity p^* where $\xi(p^*) = 0$.

As we saw in Example 5.5, the orbits look like smoothed triangles homeomorphic to S^1. Let S^1 be a copy of the circle (see Fig. 5.14). At 1, g points west, while at 2, g points northwest. At 3, g points northeast, at 4, g points east. Clearly the degree is $+1$ again.

As we showed, the Euler characteristic $\chi(\Delta)$ of the simplex is $+1$, and the degree of the only critical point p^* of the vector field ξ is $+1$. This suggests that again there is a relationship between the Euler characteristic $\chi(Y)$ of a manifold Y and the sum of the degrees of the critical points of any vector field ξ on Y. One technical point should be mentioned, concerning the nature of the flow on the boundary of Δ. In the Scarf example the flow of ξ was "along" the boundary, $\partial\Delta$, of Δ. In a real economy one would expect that as the price vector approaches the boundary $\partial\Delta$ (so that $p_i \to 0$ for some price p_i), then excess demand ξ_i for that commodity would rapidly increase as ($\xi_i \to \infty$). This essentially implies that the vector field ξ would point towards the interior of Δ. So now consider perturbations of the Scarf example as in Fig. 5.15.

In Fig. 5.15(a) is a perturbation where one of the circular orbits (called S) are retained; only flow commencing near to the boundary approaches S, as does any flow starting near to the zero p^* where $\xi(p^*) = 0$. In this case the boundary $\partial\Delta$ is a *repellor*; the closed orbit is an *attractor*, and the singularity point or zero, p^*, is a source (or point repellor). In Fig. 5.15(b) the flow is reversed. The closed orbit S is a repellor; p^* is a sink (or attractor), while $\partial\Delta$ is an attractor. Now consider a copy Δ' of Δ inside Δ (given by the dotted line in Fig. 5.15(b)). On the boundary of Δ' the flow points outwards. Then $\chi(\Delta')$ is still 1 and the degree of p^* is still 1. This illustrates the following theorem.

Poincaré-Hopf Theorem *Let Y be a compact smooth manifold with boundary ∂Y. Suppose $\xi \in \mathcal{V}^1(Y)$ has only a finite number of singularities, and points outwards on ∂Y. Then $\chi(Y)$ is equal to the sum of the degrees of the singularities of ξ.*

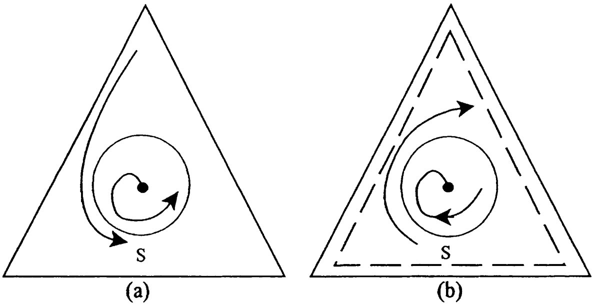


Fig. 5.15 Illustration of the Poincare Hopf Theorem

To apply this theorem, suppose ξ is the vector field given by the excess demand function. Suppose that ξ points towards the interior of Δ. Then the vector field $(-\xi)$ points *outward*. By the Debreu-Smale Theorem, we can generically assume that ξ has (at most) a finite number of singularities. Since $\chi(\Delta) = 1$, there must be at least one singularity of $(-\xi)$ and thus of ξ. Unfortunately this theorem does not allow us to infer whether or not there exists a singularity p^* which is stable (*i.e.*, an attractor).

The Poincaré-Hopf Theorem can also be used to understand singularities of vector fields on manifolds without boundary. As we have suggested, the Euler characteristic of a sphere is 2 for the even dimensional case and 0 for the odd dimensional case. This gives the following result.

The "Hairy Ball" Theorem *Any vector field ξ on S^{2n} (even dimension) must have a singularity. However there exists a vector field ξ on S^{2n+1} (odd dimension) such that $\xi(p) = 0$ for no $p \in S^{2n+1}$.*

To illustrate Fig. 5.16 shows a vector field on S^2 where the flow is circular on each of the circles of latitude, but both north and south poles are singularities. The flow is evidently non-gradient, since no potential function, f, can increase around a circular orbit.

Example 5.9 As an application, we may consider a more general type of flow, by defining at each point $x \in Y$ a *set* $h(x)$ of vectors in the tangent space at x. As discussed in Chap. 4, h could be induced by a family of utility functions $\{u_i : Y \to \Re, i \in M\}$, such that

$$h(x) = \left\{v \in \Re^n : \left\langle du_i(x), v\right\rangle > 0,\ \forall i \in M\right\}.$$

That is to say $v \in h(x)$ iff each utility function increases in the direction v. We can interpret Theorem 4.21 to assert that $h(x) = \Phi$ whenever $x \in \overset{0}{\Theta}(u_1, \dots, u_m)$, the critical Pareto set.

Suppose that $\overset{0}{\Theta} = \Phi$. Then in general we can use a selection theorem to select from $h(x)$ a non-zero vector $\xi(x)$, at every $x \in Y$, such that ξ is a continuous vector field. That is, $\xi \in \mathcal{V}^1(Y)$, but ξ has no singularities. However if $\chi(Y) \neq 0$ then any

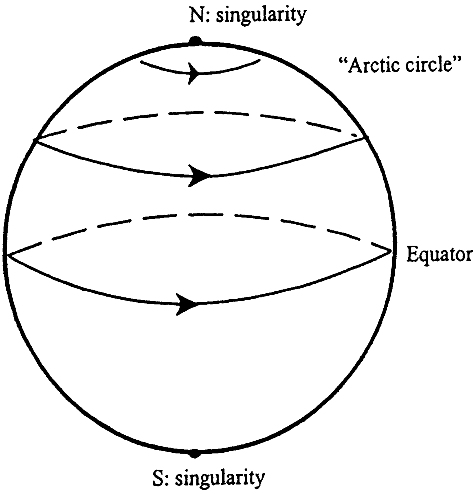


Fig. 5.16 Zeros of a field on S^2

vector field ξ on Y has critical points whose degrees sum to $\chi(Y)$, subject to the boundary condition of the Poincaré-Hopf Theorem.

Pareto Theorem *If* $\chi(Y) \neq 0$ *then* $\overset{0}{\Theta}(u) \neq \Phi$ *for any smooth profile* u *on* Y.

The Euler characteristic can be interpreted as an *obstruction* to the non-existence of equilibria, or of fixed points. For example suppose that $\chi(Y) = 0$. Then it is possible to construct a vector field ξ on Y without zeros. It is then possible to construct a function $f : Y \to Y$ which follows the trajectories of ξ a small distance, ϵ, say. But then the function f is homotopic to the identity.

That is to say, from each point x construct a path $c_x : [0,1] \to Y$ with $c_x(0) = x$ and $c_x(1) = f(x)$ whose gradient $\frac{dc}{dt}(t)$ at time t is given by the vector field ξ at the point $x' = c_x(t)$. Say that f is *induced* by the vector field, ξ. The homotopy $F : [0,1] \times Y \to Y$ is then given by $F(0,x) = x$ and $F(t,x) = c_x(t)$. Since c_x is continuous, so is F. Thus F is a homotopy between f and the identity on Y. A function $f : Y \to Y$ which is homotopic to the identity is called a *deformation* of Y.

If $\chi(Y) = 0$ then it is possible to find a vector field ξ on Y without singularities and then construct a deformation f of Y induced by ξ. Since $\xi(x) = 0$ for no x, f will not have a fixed point. Conversely if f is a deformation on Y and $\chi(Y) \neq 0$, then the homotopy between f and the identity generates a vector field ξ. Were f to have no fixed point, then ξ would have no singularity. If f and thus ξ have the right behavior on the boundary, then ξ must have at least one singularity. This contradicts the fixed point free property of f.

Lefschetz Fixed Point Theorem *If* Y *is a manifold with* $\chi(Y) = 0$ *then there exists a fixed point free deformation of* Y. *If* $\chi(Y) \neq 0$ *then any deformation of* Y *has a fixed point.*

Note that the Lefschetz Fixed Point Theorem does not imply that any function $f : S^2 \to S^2$ has a fixed point. For example the "antipodal" map $f(x) = -x$, for $\|x\| = 1$, is fixed point free. However f cannot be induced by a continuous vector field, and is therefore not a deformation. The Lefschetz fixed point theorem just stated is in fact a corollary of a deeper result: For any continuous map $f : Y \to Y$, with Y compact, there is an "obstruction" called the Lefschetz number $\lambda(f)$. If $\lambda(f) \neq 0$, then f must have a fixed point. If Y is "homotopy equivalent" to the compact ball then $\lambda(f) \neq 0$ for every continuous function on the ball, so the ball has the fixed point property. On the other hand if f is homotopic to the identity, Id, on Y then $\lambda(f) = \lambda(Id)$, and it can be shown that $\lambda(Id) = \chi(Y)$ the Euler characteristic of Y. It therefore follows that $\chi(Y)$ is an obstruction to the existence of a fixed point free deformation of the compact manifold Y.

Example 5.10 To illustrate an application of this theorem in social choice, suppose that Y is a compact manifold of dimension at most $k(\sigma) - 2$, where $k(\sigma)$ is the Nakamura number of the social choice rule, σ (see Sect. 3.8). Suppose $(u_1, \dots, u_m)$ is a smooth profile on Y. Then it can be shown (Schofield 1984) that if the choice $C_{\sigma(\pi)}(Y)$ is empty, then there exists a fixed point free deformation on Y. Consequently if $\chi(X) \neq 0$, then $C_{\sigma(\pi)}(Y)$ must be non-empty.

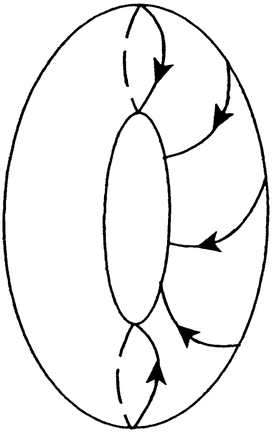
Example 5.11 It would be useful to be able to use the notion of the Lefschetz obstruction to obtain conditions under which the singularities of the excess demand function, ξ, of an economy were stable. However, as Scarf's example showed, it is entirely possible for there to be a single attractor, or a single repellor (as in Fig. 5.15) or even a situation with an infinite number of closed orbits. However, consider a more general price adjustment process as follows. At each p in the interior, Int Δ, of Δ let

$$\xi^*(p) = \{v \in \Re^n : \langle v, \xi(p) \rangle > 0\}.$$

A vector field $v \in \mathcal{V}^1(\Delta)$ is *dual* to ξ iff $v(p) \in \xi^*(p)$ for all $p \in \text{Int}\,\Delta$, and $v(p) = 0$ iff $\xi(p) = 0$ for $p \in \text{Int}\,\Delta$. It may be possible to find a vector field, v, dual to ξ which has attractors. Suppose that v is dual to ξ, and that $f : \Delta \to \Delta$ is induced by v. As we have seen, the Lefschetz number of f gives information about the singularities of v.

Dierker (1972) essentially utilized the following hypothesis: there exists a dual vector field, v, and a function $f : \Delta \to \Delta$ induced by v such that $f = f_0$ is homotopic to the constant map $f_1 : \Delta \to \{(\frac{1}{n}, \dots, \frac{1}{n})\}$ such that $\{p \in \Delta : f_t(p) = p \text{ for } t \in [0, 1]\}$ is compact. Under the assumption that the economy is regular (so the number of singularities of ξ is finite), then he showed that the number of such singularities must be odd. Moreover, if it is known that ξ only has stable singularities, then there is only one. The proof of the first assertion follows by observing that $\lambda(f_0) = \lambda(f_1)$. But f_1 is the constant map on Δ so $\lambda(f_1) = 1$. Moreover $\lambda(f_0)$ is equal to the sum of the degrees of the singularities of v, and Dierker shows that at each singularity of v, the degree is ± 1. Consequently the number of singularities must be odd. Finally if there are only stable singularities, each has degree $+1$, so it must be unique.

Fig. 5.17 A flow on the Torus



Example 5.12 As a further application of the Lefschetz obstruction, suppose, contrary to the usual assumption that negative prices are forbidden, that $p \in S^{n-1}$ rather than Δ. It is natural to suppose that $\xi(p) = \xi(-p)$ for any $p \in S^{n-1}$. Suppose now that $\xi(p) = 0$ for no $p \in S^{n-1}$. This defines an (even) spherical map $g : S^{n-1} \to S^{n-1}$ by $g(p) = \xi(p)/\|\xi(p)\|$. Thus $g(p) = g(-p)$. The degree (deg (g)) of such a g can readily be seen to be an even integer, and it follows that the Lefschetz obstruction of g is $\lambda(g) = 1 + (-1)^{n-1}$ deg (g).

Clearly $\lambda(g) \neq 0$ and so g has a fixed point $\overline{p}$ such that $g(\overline{p}) = \overline{p}$. But then $\xi(p) = \alpha p$ for some $\alpha > 0$. This violates Walras' Law, since $\langle p, \xi(p) \rangle = \alpha \|p\|^2 \neq 0$, so $\xi(p) = 0$ for some $p \in S^{n-1}$. Keenan (1992) goes on to develop some of the earlier work by Rader (1972) to show, in this extended context, that for generic, regular economies there must be an odd number of singularities.

The above examples have all considered flows on the simplex or the sphere. To return to the idea of structural stability, let us consider once again examples of a vector field on the torus.

Example 5.13 (1) For a more interesting deformation of the torus $Z = S^1 \times S^1$, consider Fig. 5.17.

The closed orbit at the top of the torus is a repellor, R, say. Any flow starting near to R winds towards the bottom closed orbit, A, an attractor. There are no singularities, and the induced deformation is fixed point free.

(2) Not all flows on the torus Z need have closed orbits. Consider the flow on Z given in Fig. 5.18. If the tangent of the angle, θ, is rational, then the orbit through x is closed, and will consist of a specific number of turns round Z. However suppose this flow is perturbed. There will be, in any neighborhood of θ, an irrational angle. The orbits of an irrational flow will not close up. To relate this to the Peixoto Theorem which follows, with rational flow there will be an infinite number of closed orbits. However the phase portrait for rational flow cannot be homeomorphic to the portrait for irrational flow. Thus any perturbations of rational flow gives a non-

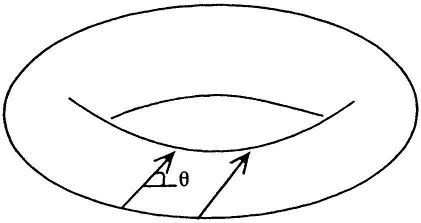


Fig. 5.18 Structurally unstable rational flow on the Torus

homeomorphic irrational flow. Clearly any vector field on the torus which gives rational flow is structurally unstable.

Structural Stability Theorem

1. *If* $\dim Y = 2$ *and* Y *is compact, then structural stability of vector fields on* Y *is generic.*
2. *If* $\dim Y \geq 3$, *then structural stability is non-generic.*

Peixoto (1962) proved part (1) by showing that structurally stable vector fields on compact Y (of dimension 2) must satisfy the following properties:

(1) there are a finite number of non-degenerate isolated singularities (that is, critical points which can be sources, sinks, or saddles)
(2) there are a finite number of attracting or repelling closed orbits
(3) every orbit (other than closed orbits) starts at a source or saddle, or winds away from a repellor and finishes at a saddle or sink, or winds towards an attractor
(4) no orbit connects saddle points.

Peixoto showed that for any vector field ξ on Y and any neighborhood V of ξ in $\mathcal{V}^1(Y)$ there was a vector field ξ' in V that satisfied the above four conditions and thus was structurally stable.

Although we have not carefully defined the terms used above, they should be intuitively clear. To illustrate, Fig. 5.19(a) shows an orbit connecting saddles, while Fig. 5.19(b) shows that after perturbation a qualitatively different phase portrait is obtained.

In Fig. 5.19(a), A and B are connected saddles, C is a repellor (orbits starting near to C leave it) and D is a closed orbit. A small perturbation disconnects A and B as shown in Fig. 5.19(b), and orbits starting near to D (either inside or outside) approach D, so it is an attractor.

The excess demand function, ξ, of the Scarf example clearly has an infinite number of closed orbits (all homeomorphic to S^1). Thus ξ cannot be structurally stable. From Peixoto's Theorem, small perturbations of ξ will destroy this feature. As we suggested, a small perturbation may change ξ so that p^* becomes a stable equilibrium (an attractor) or an unstable equilibrium (a repellor).

Smale's (1966) proof that structural stability was non-generic in three or more dimensions was obtained by constructing a diffeomorphism $f : Y^3 \to Y^3$ (with $Y = S^1 \times S^1 \times S^1$). This induced a vector field $\xi \in \mathcal{V}^1(Y)$ that had the property that for a neighborhood V of ξ in $\mathcal{V}^1(Y)$, no ξ' in V was structurally stable. In other

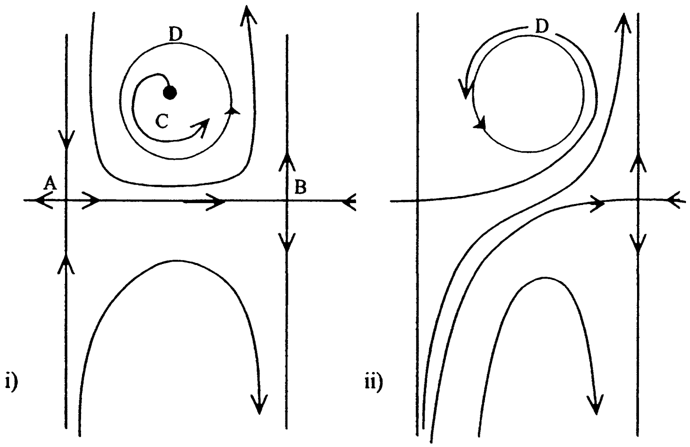


Fig. 5.19 (**i**) shows an orbit connecting saddle points, while (**ii**) shows that this property is destroyed by a small perturbation

words every ξ' when perturbed led to a qualitatively different phase portrait. We could say that ξ was *chaotic*. Any attempt to model ξ by an approximation ξ', say, results in an essentially different vector field. The possibility of chaotic flow and its ramifications will be discussed in general terms in the next section. The consequence for economic theory is immediate, however. Since it can be shown that *any* excess demand function ξ and thus any vector field can result from an economy, (u, e), it is possible that the price adjustment process is chaotic.

As we observed after Example 5.8, an economically realistic excess demand function ξ on Δ should point into the price simplex at any price vector on $\partial\Delta$. This follows because if $p_i \to 0$ then ξ_i would be expected to approach ∞. Let $\mathcal{V}_0^1(\Delta)$ be the topological space of vector fields on Δ, of the form $\frac{dp}{dt}|_p = \xi(p)$, such that $\frac{dp}{dt}|_p$ points into the interior of Δ for p near $\partial\Delta$.

The Sonnenschein-Mantel-Debreu Theorem *The map*

$$\xi : C^s\big(X, \Re^m\big) \times X^m \to \mathcal{V}_0^1(\Delta)$$

is onto if $m \geq n$.

Suppose that there are at least as many economic agents (m) as commodities. Then it is possible to construct a well-behaved economy (u, e) with monotonic, strictly convex preferences induced from smooth utilities, and an endowment vector $e \in X^m$, such that *any* vector field in $\mathcal{V}_0^1(\Delta)$ is generated by the excess demand function for the economy (u, e).

Versions of the theorem were presented in Sonnenschein (1972), Mantel (1974), and Debreu (1974). A more recent version can be found in Mas-Colell (1985). As we have discussed in this section, because the simplex Δ has $\chi(\Delta) = 1$, then the "excess demand" vector field ξ will always have at least one singularity. In fact, from the Debreu-Smale theorem, we expect ξ to generically exhibit only a finite number of singularities. Aside from these restrictions, ξ, is essentially unconstrained. If there are at least four commodities (and four agents) then it is always possible to construct (u, e) such that the vector field induced by excess demand is "chaotic".

As we saw in Sect. 5.4, the vector field ξ of the Scarf example was structurally unstable, but any perturbation of ξ led to a structurally stable field ξ', say, either with an attracting or repelling singularity. The situation with four commodities is potentially much more difficult to analyze. It is possible to find (u, e) such that the induced vector field ξ on Δ is chaotic—in some neighborhood V of ξ there is no structurally stable field. Any attempt to model ξ by ξ', say, must necessarily incorporate some errors, and these errors will multiply in some fashion as we attempt to map the phase portrait. In particular the flow generated by ξ through some point $x \in \Delta$ can be very different from the flow generated by ξ' through x. This phenomenon has been called "sensitive dependence on initial conditions."

5.6 Speculations on Chaos

It is only in the last twenty years or so that the implications of "chaos" (or failure of structural stability in a profound way) have begun to be realized. In a recent book Kauffman commented on the failure of structural stability in the following way.

"One implication of the occurrence or non-occurrence of structural stability is that, in structurally stable systems, smooth walks in parameter space must (result in) smooth changes in dynamical behavior. By contrast, chaotic systems, which are not structurally stable, adapt on uncorrelated landscapes. Very small changes in the parameters pass through many interlaced bifurcation surfaces and so change the behavior of the system dramatically."[1]

The whole point of the Debreu-Smale Theorem is that generically the Debreu map is regular. Thus there are open sets in the parameter space (of utility profiles and endowments) where the number, $E(\xi)$, of singularities of ξ is finite and constant. As the Scarf example showed, however, even though $E(\xi)$ may be constant in a neighborhood, the vector field ξ can be structurally unstable. The structurally unstable circular vector field of the Scarf example is not particularly surprising. After all, similar structurally unstable systems are common (the oscillator or pendulum is one example). These have the feature that, when perturbed, they become structurally unstable. Thus the dynamical system of a pendulum with friction is structurally stable. Its phase portrait shows an attractor, which still persists as the friction is increased or decreased. Smale's Structural Instability Theorem together with the Sonnenschien-Mantel-Debreu Theorem suggests that the vector field generated by excess demand can indeed be chaotic when there are at least four commodities and agents. This does not necessarily mean that chaotic price adjustment processes are pervasive. As in Dierker's example, even though the vector field $\xi(p) = \frac{dp}{dt}$ can be

[1] S. Kauffman, *The Origins of Order* (1993) Oxford University Press: Oxford.

chaotic, it may be possible to find a structurally stable vector field, v, dual to ξ, which is structurally stable.

This brief final section will attempt to discuss in an informal fashion, whether or not it is plausible for economies to exhibit chaotic behavior.

It is worth mentioning that the idea of structural stability is not a new one, though the original discussion was not formalized in quite the way it is today. Newton's great work *Philosophiae Naturalis Principia Mathematica* (published in 1687) grew out of his work on gravitation and planetary motion. The laws of motion could be solved precisely giving a vector field and the orbits (or phase portrait) in the case of a planet (a point mass) orbiting the sun. The solution accorded closely with Kepler's (1571–1630) empirical observations on planetary motion. However, the attempt to compute the planetary orbits for the solar system had to face the problem of perturbations. Would the perturbations induced in each orbit by the other planets cause the orbital computations to converge or diverge? With convergence, computing the orbit of Mars, say, can be done, by approximating the effects of Jupiter, Saturn perhaps, on the Mars orbit. The calculations would give a prediction very close to the actual orbit. Using the approximations, the planetary orbits could be computed far into the future, giving predictions as precise as calculating ability permitted. Without convergence, it would be impossible to make predictions with any degree of certainty. Laplace in his work "Mécanique Céleste" (published between 1799 and 1825) had argued that the solar system (viewed as a formal dynamical system) is structurally stable (in our terms). Consistent with his view was the use of successive approximations to predict the perihelion (a point nearest the sun) of Haley's comet, in 1759, and to infer the existence and location of Neptune in 1846.

Structural stability in the three-body problem (of two planets and a sun) was the obvious first step in attempting to prove Laplace's assertion. In 1885 a prize was announced to celebrate the King of Sweden's birthday. Henri-Poincaré submitted his entry "Sur le problème des trois corps et les Equations de la Dynamique." This attempted to prove structural stability in a restricted three body problem. The prize was won by Poincaré's entry, although it was later found to contain an error. Poincaré had obtained his doctorate in mathematics in Paris in 1878, had briefly taught at Caen and later became professor at Paris. His work on differential equations in the 1880s and his later work on Celestial Mechanics in the 1890s developed new qualitative techniques (in what we now call differential topology) to study dynamical equations.

In passing it is worth mentioning that since there is a natural periodicity to any rotating celestial system, the state space in some sense can be viewed as products of circles (that is tori). Many of the examples mentioned in the previous section, such as periodic (rational) or a-periodic (non-rational) flow on the torus came up naturally in celestial mechanics.

One of the notions implicitly emphasized in the previous sections of this chapter is that of *bifurcation*: namely a dynamical system on the boundary separating qualitatively different systems. At such a bifurcation, features of the system separate out in pairs. For example, in the Debreu map, a bifurcation occurs when two of the price equilibria coalesce. This is clearly linked to the situation studied by Dierker, where the number of price equilibria (in Δ) is odd. At a bifurcation, two equilibria with opposite degrees coalesce. In a somewhat similar fashion Poincaré showed that, for the three-body problem, if there is some value μ_0 (of total mass, say) such that periodic solutions exist for $\mu \leq \mu_0$ but not for $\mu > \mu_0$, then two periodic solutions must have coalesced at μ_0. However Poincaré also discovered that the bifurcation could be associated with the appearance of a new solution with period double that of the original. This phenomenon is central to the existence of a period-doubling cascade as one of the characteristics of chaos. Near the end of his *Celestial Mechanics*, Poincaré writes of this phenomenon:

"Neither of the two curves must ever cut across itself, but it must bend back upon itself in a very complex manner an infinite number of times.... Nothing is more suitable for providing us with an idea of the complex nature of the three body problem."[2]

Although Poincaré was led to the possibility of chaos in his investigations into the solar system, it appears that the system is in fact structurally stable. Arnol'd showed in 1963 that for a system with small planets, there is an open set of initial conditions leading to bounded orbits for all time. Computer simulations of the system far into time also suggests it is structurally stable.[3] Even so, there are events in the system that affect us and appear to be chaotic (perhaps catastrophic would be a more appropriate term). The impact of large asteroids may have a dramatic effect on the biosphere of the earth, and these have been suggested as a possible cause of mass extinction. The onset and behavior of the ice ages over the last 100,000 years is very possibly chaotic, and it is likely that there is a relationship between these violent climatic variations and the recent rapid evolution of human intelligence.[4]

More generally, evolution itself is often perceived as a gradient dynamical process, leading to increasing complexity. However Stephen Jay Gould has argued over a number of years that evolution is far from gradient-like: increasing complexity coexists with simple forms of life, and past life has exhibited an astonishing variety.[5] Evolution itself appears to proceed at a very uneven rate.[6]

[2] My observations and quotations are taken from D. Goroff's introduction and the text of a recent edition of Poincaré's *New Methods of Celestial Mechanics,* (1993) American Institute of Physics: New York.

[3] See I. Peterson, *Newton's Clock: Chaos in the Solar System* (1993) Freeman: New York.

[4] See W. H. Calvin, *The Ascent of Mind.* Bantam: New York.

[5] S. J. Gould, *Full House* (1996) Harmony Books: New York; S. J. Gould, *Wonderful Life* (1989) Norton: New York.

[6] N. Eldredge and S. J. Gould, "Punctuated Equilibria: An Alternative to Phyletic Gradualism," in *Models in Paleobiology* (1972), T. J. M. Schopf, ed. Norton: New York.

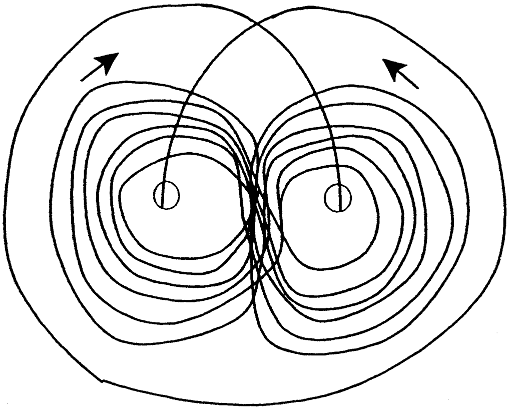


Fig. 5.20 The butterfly

"Empirical" chaos was probably first discovered by Edward Lorenz in his efforts to numerically solve a system of equations representative of the behavior of weather.[7] A very simple version is the non-linear vector equation

$$\frac{dx}{dt} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} -a(x_1 - x_2) \\ -x_1x_3 + a_2x_1 - x_2 \\ x_1x_2 - a_3x_3 \end{pmatrix}$$

which is chaotic for certain ranges of the three constants, a_1, a_2, a_3.

The resulting "butterfly" portrait winds a number of times about the left hole (A in Fig. 5.20), then about the right hole (B), then the left, *etc.* Thus the phase portrait can be described by a sequence of winding numbers ($w_l^1, w_k^1, w_l^2, w_k^2$, *etc.*). Changing the constants a_1, a_2, a_3 slightly changes the winding numbers.

Given that chaos can be found in such a simple meteorological system, it is worthwhile engaging in a thought experiment to see whether "climatic" chaos is a plausible phenomenon. Weather occurs on the surface of the earth, so the spatial context is $S^2 \times I$, where I is an interval corresponding to the depth of the atmosphere. As we know, $\chi(S^2) = \chi(S^2 \times I) = 2$ so we would expect singularities. Secondly there are temporal periodicities, induced by the distance from the sun and earth's rotation. Thirdly there are spatial periodicities or closed orbits. Chief among these must be the jet stream and the oceanic orbit of water from the southern hemisphere to the North Atlantic (the Gulf Stream) and back. The most interesting singularities are the hurricanes generated each year off the coast of Africa and channeled across the Atlantic to the Caribbean and the coast of the U.S.A. Hurricanes

[7] E.N. Lorenz, "The Statistical Prediction of Solutions of Dynamical Equations," *Proceedings Int. Symp. Num. Weather Pred* (1962) Tokyo; E. N. Lorenz "Deterministic Non Periodic Flow," *J. Atmos. Sci.* (1963): 130–141.

are self-sustaining heat machines that eventually dissipate if they cross land or cool water. It is fairly clear that their origin and trajectory is chaotic.

Perhaps we can use this thought experiment to consider the global economy. First of all there must be local periodicities due to climatic variation. Since hurricanes and monsoons, *etc.* effect the economy, one would expect small chaotic occurrences. More importantly, however, some of the behavior of economic agents will be based on their future expectations about the nature of economic growth, *etc.* Thus one would expect long term expectations to affect large scale "decisions" on matters such as fertility. The post-war "baby boom" is one such example. Large scale periodicities of this kind might very well generate smaller chaotic effects (such as, for example, the oil crisis of the 1970s), which in turn may trigger responses of various kinds.

It is evident enough that the general equilibrium (GE) emphasis on the existence of price equilibria, while important, is probably an incomplete way to understand economic development. In particular, GE theory tends to downplay the formation of expectations by agents, and the possibility that this can lead to unsustainable "bubbles".

Remember, it is a key assumption of GE that agents' preferences are defined on the commodity space alone. If, on the contrary, these are defined on commodities *and* prices, then it is not obvious that the assumptions of the Ky Fan Theorem (*cf.*, Chap. 3) can be employed to show existence of a price equilibrium. Indeed manipulation of the kind described in Chap. 4 may be possible. More generally one can imagine energy engines (very like hurricanes) being generated in asset markets, and sustained by self-reinforcing beliefs about the trajectory of prices. It is true that modern decentralised economies are truly astonishing knowledge or data-processing mechanisms. From the perspective of today, the argument that a central planning authority can be as effective as the market in making "rational" investment decisions appears to have been lost. Hayek's case, the so-called "calculation" argument, with von Mises and against Lange and Schumpeter, was based on the observation that information is dispersed throughout the economy and is, in any case, predominantly subjective. He argued essentially that only a market, based on individual choices, can possibly "aggregate" this information.[8]

Recently, however, theorists have begun to probe the degree of consistency or convergence of beliefs in a market when it is viewed as a game. It would seem that when the agents "know enough about each other", then convergence in beliefs is a consequence.[9]

[8] See F. A. Hayek, "The Use of Knowledge in Society," *American Economic Review* (1945) 55: 519–530, and the discussion in A. Gamble, *Hayek: The Iron Cage of Liberty* (1996) Westview: Boulder, Colorado.

[9] See R. J. Aumann, "Agreeing to Disagree," *Annals of Statistics* (1976) 1236–1239 and K. J. Arrow "Rationality of Self and Others in an Economic System," *Journal of Business* (1986) 59: S385–S399.

In fact the issue about the "truth-seeking" capability of human institutions is very old and dates back to the work of Condorcet.[10] Nonetheless it is possible for belief cascades or bubbles to occur under some circumstances.[11] It is obvious enough that economists writing after the Great Crash of the 1930s might be more willing than those writing today to consider the possibility of belief cascades and collapse. John Maynard Keynes' work on *The General Theory of Employment, Interest and Money* (1936) was very probably the most influential economic book of the century. What is interesting about this work is that it does appear to have grown out of work that Keynes did in the period 1906 to 1914 on the foundation of probability, and that eventually was published as the *Treatise on Probability* (1921). In the *Treatise*, Keynes viewed probability as a degree of belief. He also wrote: "The old assumptions, that all quantity is numerical and that all quantitative characteristics are additive, can no longer be sustained. Mathematical reasoning now appears as an aid in its symbolic rather than its numerical character. I, at any rate, have not the same lively hope as Condorcet, or even as Edgeworth, 'Eclairer le Science morales et politiques par le flambeau de l'Algèbre.'"[12]

Macro-economics as it is practiced today tends to put a heavy emphasis on the empirical relationships between economic aggregates. Keynes' views, as I infer from the *Treatise*, suggest that he was impressed neither by econometric relationships nor by algebraic manipulation. Moreover, his ideas on "speculative euphoria" and crashes[13] would seem to be based on an understanding of the economy grounded not in econometrics or algebra but in the qualitative aspects of its dynamics.

Obviously I have in mind a dynamical representation of the economy somewhere in between macro-economics and general equilibrium theory. The laws of motion of such an economy would be derived from modeling individuals' "rational" behavior as they process information, update beliefs and locally optimise. At present it is not possible to construct such a micro-based macro-economy because the laws of motion are unknown. Nonetheless, just as simulation of global weather systems can be based on local physical laws, so may economic dynamics be built up from local "rationality" of individual agents. In my view, the qualitative theory of dynamical systems will have a major rôle in this enterprise. The applications of this theory, as outlined in the chapter, are intended only to give the reader a taste of how this theory might be developed.

[10] See his work on the so-called Jury Theorem in his *Essai* of 1785. A discussion of Condorcet's work can be found in I. McLean and F. Hewitt, *Condorcet: Foundations of Social Choice and Political Theory* (1994) Edward Elgar: Aldershot, England.

[11] See S. Bikhchandani, D. Hirschleifer and I. Welsh, "A Theory of Fads, Fashion, Custom, and Cultural Change as Information Cascades," *Journal of Political Economy* (1992) 100: 992–1026.

[12] John Maynard Keynes, *Treatise on Probability* (1921) Macmillan: London pp. 349. The two volumes by Robert Skidelsky on *John Maynard Keynes* (1986, 1992) are very useful in helping to understand Keynes' thinking in the *Treatise* and the *General Theory*.

[13] See, for example, the work of Hyman Minsky *John Maynard Keynes* (1975) Columbia University Press: New York, and *Stabilizing an Unstable Economy* (1986) Yale University Press: New Haven.

Further Reading

A very nice though brief survey of the applications of global analysis (or differential topology) to economics is:

Debreu, G. (1976). The application to economics of differential topology and global analysis: regular differentiable economies. *The American Economic Review*, *66*, 280–287.

An advanced and detailed text on the use of differential topology in economics is:

Mas-Colell, A. (1985). *The theory of general economic equilibrium*. Cambridge: Cambridge University Press.

Background reading on differential topology and the ideas of transversality can be found in:

Chillingsworth, D. R. J. (1976). *Differential topology with a view to applications*. Pitman: London.

Golubitsky, M., & Guillemin, V. (1973). *Stable mappings and their singularities*. Berlin: Springer.

Hirsch, M. (1976). *Differential topology*. Berlin: Springer.

For the Debreu-Smale Theorem see:

Balasko, Y. (1975). Some results on uniqueness and on stability of equilibrium in general equilibrium theory. *Journal of Mathematical Economics*, *2*, 95–118.

Debreu, G. (1970). Economies with a finite number of equilibria. *Econometrica*, *38*, 387–392.

Smale, S. (1974a). Global analysis of economics IV: finiteness and stability of equilibria with general consumption sets and production. *Journal of Mathematical Economics*, *1*, 119–127.

Smale, S. (1974b). Global analysis and economics IIA: extension of a theorem of Debreu. *Journal of Mathematical Economics*, *1*, 1–14.

The Smale-Pareto theorem on the generic structure of the Pareto set is in:

Smale, S. (1973). Global analysis and economics I: Pareto optimum and a generalization of Morse theory. In M. Peixoto (Ed.), *Dynamical systems*. New York: Academic Press.

Scarf's example and the use of the Euler characteristic are discussed in Dierker's notes on topological methods.

Dierker, E. (1972). Two remarks on the number of equilibria of an economy. *Econometrica*, 951–953.

Dierker, E. (1974). *Lecture notes in economics and mathematical systems: Vol. 92. Topological methods in Walrasian economics*. Berlin: Springer.

Scarf, H. (1960). Some examples of global instability of the competitive equilibrium. *International Economic Review*, *1*, 157–172.

The Lefschetz fixed point theorem as an obstruction theory for the existence of fixed point free functions and continuous vector fields can be found in:

Brown, R. (1971). *The Lefschetz fixed point theorem*. Glenview: Scott and Foresman.

Some applications of these techniques in economics are in:

Keenan, D. (1992). Regular exchange economies with negative prices. In W. Nenuefeind & R. Riezman (Eds.), *Economic theory and international trade*. Berlin: Springer.

Rader, T. (1972). *Theory of general economic equilibrium*. New York: Academic Press.

An application of this theorem to existence of a voting equilibrium is in:

Schofield, N. (1984). Existence of equilibrium on a manifold. *Journal of Operations Research*, *9*, 545–557.

The results on structural stability are given in:

Peixoto, M. (1962). Structural stability on two-dimensional manifolds. *Topology*, *1*, 101–120.

Smale, S. (1966). Structurally stable systems are not dense. *American Journal of Mathematics*, *88*, 491–496.

A very nice and fairly elementary demonstration of Peixoto's Theorem in two dimensions, together with the much earlier classification results of Andronov and Pontrjagin, is given in:

Hubbard, J. H., & West, B. H. (1995). *Differential equations: a dynamical systems approach*. Berlin: Springer.

René Thom's early book applied these "topological" ideas to development:

Thom, R. (1975). *Structural stability and morphogenesis*. Reading: Benjamin.

The result that any excess demand function is possible can be found in:

Debreu, G. (1974). Excess demand functions. *Journal of Mathematical Economics*, *1*, 15–21.

Mantel, R. (1974). On the characterization of aggregate excess demand. *Journal of Economic Theory*, *12*, 197–201.

Sonnenschein, H. (1972). Market excess demand functions. *Econometrica*, *40*, 549–563.

The idea of chaos has become rather fashionable recently. For a discussion see:

Gleick, J. (1987). *Chaos: making a new science*. New York: Viking.

An excellent background to the work of Poincaré, Birkhoff and Smale, and applications in meteorology is:

Lorenz, E. N. (1993). *The essence of chaos*. Seattle: University of Washington Press.

For applications of the idea of chaos in various economic and social choice contexts see:

Saari, D. (1985). Price dynamics, social choice, voting methods, probability and chaos. In D. Aliprantis, O. Burkenshaw, & N. J. Rothman (Eds.), *Lecture notes in economics and mathematical systems* (Vol. 244). Berlin: Springer.

Topology and Social Choice 6

In Chap. 3 we showed the Nakamura Theorem that a social choice could be guaranteed as long as the dimension of the space did not exceed $k(\sigma)-2$. We now consider what can happen in dimension above $k(\sigma)-1$. We then go on to consider "probabilistic" social choice, where there is some uncertainty over voters' preferences.

6.1 Existence of a Choice

First we repeat some of the results presented in Chap. 3.

As we showed in Chap. 3, arguments for the existence of an equilibrium or choice are based on some version of Brouwer's fixed point theorem, which we can regard as a variant of the Fan Choice Theorem. Brouwer's theorem asserts that any continuous function $f: B \to B$ between the finite dimensional ball, B or indeed any compact convex set in $\Re^w$, has the fixed point property.

This section will consider the use of variants of the Brouwer theorem, to prove existence of an equilibrium of a general social choice mechanism. We shall argue that the condition for existence of an equilibrium will be violated if there are cycles in the underlying mechanism.

Let $W \subset \Re^w$ be the set of alternatives and, and let 2^Wbe the set of all subsets of W. A *preference correspondence*, P, on W assigns to each point $x \in W$, its *preferred set* $P(x)$. Write $P: W \twoheadrightarrow W$ to denote that the image of x under P is a set (possibly empty) in W. For any subset V of W, the restriction of P to V gives a correspondence $P_V: V \twoheadrightarrow V$. Define $P_V^{-1}: V \twoheadrightarrow V$ such that for each $x \in V$,

$$P_V^{-1}(x) = \{y : x \in P(y)\} \cap V.$$

The sets $P_V(x)$, $P_V^{-1}(x)$ are sometimes called the *upper* and *lower* preference sets of P on V. When there is no ambiguity we delete the suffix V. The *choice* of P from W is the set

$$C(W,P) = \{x \in W : P(x) = \varnothing\}.$$

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_6,

Here $\varnothing$ is the empty set. The choice of P from a subset, V, of W is the set

$$C(V,P)=\{x\in V: P_V(x)=\varnothing\}.$$

Call C_P a *choice function* on W if $C_P(V)=C(V,P)\neq\varnothing$ for every subset V of W. We now seek general conditions on W and P which are sufficient for C_P to be a choice function on W. Continuity properties of the preference correspondence are important and so we require the set of alternatives, W, to be a topological space.

Definition 6.1 Let W, Y be two topological spaces. A correspondence $P: W \twoheadrightarrow Y$ is

(i) *Lower hemi-continuous* (*lhc*) iff, for all $x\in W$, and any open set $U\subset Y$ such that $P(x)\cap U\neq\varnothing$ there exists an open neighborhood V of x in W, such that $P(x')\cap U\neq\varnothing$ for all $x'\in V$.

(ii) *Upper hemi-continuous* (*uhc*) iff, for all $x\in W$ and any open set $U\subset Y$ such that $P(x)\subset U$, there exists an open neighborhood V of x in W such that $P(x')\subset U$ for all $x'\in V$.

(iii) *Lower demi-continuous* (*ldc*) iff, for all $x\in Y$, the set

$$P^{-1}(x)=\{y\in W: x\in P(y)\}$$

is open (or empty) in W.

(iv) *Upper demi-continuous* (*udc*) iff, for all $x\in W$, the set $P(x)$ is open (or empty) in Y.

(v) *Continuous* iff P is both *ldc* and *udc*.

(vi) *Acyclic* if it is impossible to find a cycle $x_t\in P(x_{t-1}), x_{t-1}\in P(x_{t-2}),\dots,$ $x_1\in P(x_t)$.

We shall use lower demi-continuity of a preference correspondence to prove existence of a choice. In some cases, however, it is possible to make use of lower hemi-continuity. Note that if P is ldc then it is lhc.

We shall now show that if W is compact, and P is an acyclic and ldc preference correspondence $P: W\twoheadrightarrow W$, then $C(W,P)\neq\varnothing$. First of all, say a preference correspondence $P: W\twoheadrightarrow W$ satisfies the *finite maximality property* (FMP) on W iff for every finite set V in W, there exists $x\in V$ such that $P(x)\cap V=\varnothing$.

Lemma 6.1 (Walker 1977) *If W is a compact, topological space and P is an ldc preference correspondence that satisfies FMP on W, then $C(W,P)\neq\varnothing$. This follows readily, using compactness to find a finite subcover, and then using FMP.*

Corollary 6.1 *If W is a compact topological space and P is an acyclic, ldc preference correspondence on W, then $C(W,P)\neq\varnothing$.*

As Walker (1977) noted, when W is compact and P is ldc, then P is acyclic iff P satisfies FMP on W, and so either property can be used to show existence of a choice. A alternative method of proof to show that C_P is a choice function is to substitute a convexity property for P rather than acyclicity.

Definition 6.2

(i) If W is a subset of a vector space, then the *convex hull* of W is the set, Con$[W]$, defined by taking all convex combinations of points in W.

(ii) W is *convex* iff $W = \text{Con}[W]$. (The empty set is also convex.)

(iii) W is *admissible* iff W is a compact, convex subset of a topological vector space.

(iv) A preference correspondence $P: W \twoheadrightarrow W$ on a convex set W is *convex* iff, for all $x \in W$, $P(x)$ is convex.

(v) A preference correspondence $P: W \twoheadrightarrow W$ is *semi-convex* iff, for all $x \in W$, it is the case that $x \notin \text{Con}(P(x))$.

As we showed in Chap. 3, Fan (1961) has demonstrated that if W is admissible and P is ldc and semi-convex, then $C(W, P)$ is non-empty.

Choice Theorem (Fan 1961; Bergstrom 1975) *If W is an admissible subset of a Hausdorff topological vector space, and $P: W \twoheadrightarrow W$ a preference correspondence on W which is ldc and semi-convex then $C(W, P) \neq \varnothing$.*

As demonstrated in Chap. 3, the proof uses the KKM lemma due to Knaster et al. (1929). There is a useful corollary to the Fan Choice theorem. Say a preference correspondence on an admissible space W satisfies the *convex maximality property* (CMP) iff for any finite set V in W, there exists $x \in \text{Con}(V)$ such that $P(x) \cap \text{Con}(V) = \varnothing$.

Corollary 6.2 *Let W be admissible and $P: W \twoheadrightarrow W$ be ldc and semi-convex. Then P satisfies the convex maximality property.*

Numerous applications of the procedure have been made to show existence of such an economic equilibrium. Note however, that these results all depend on semi-convexity of the preference correspondences.

6.2 Dynamical Choice Functions

We now consider a *generalized preference field* $H : W \twoheadrightarrow TW$, on a smooth manifold W.

We use this notation to mean that at any $x \in W$, $H(x)$ is a *cone* in the tangent space $T_x W$ above x. That is, if a vector $v \in H(x)$, then $\lambda v \in H(x)$ for any $\lambda > 0$. If there is a smooth curve, $c : [-1, 1] \to W$, such that the differential $\frac{dc(t)}{dt} \in H(x)$, whenever $c(t) = x$, then c is called an integral curve of H. An integral curve of H from $x = c(o)$ to $y = \lim_{t \to 1} c(t)$ is called an *H-preference curve* from x to y. The preference field H is called *S-continuous* iff, for any $v \in H(x) \neq \varnothing$ then there is an integral curve, c, in a neighborhood of x with $\frac{dc(0)}{dt} = v$. The choice $C(W, H)$ of H on W is defined by

$$C(W, H) = \{x \in W : H(x) = \varnothing\}.$$

Say H is *half open* if at every $x \in W$, either $H(x) = \varnothing$ or there exists a vector $v' \in T_x W$ such that $(v' \mathbin{\boldsymbol{.}} v) > 0$ for all $v \in H(x)$. We can say in this case that there is, at x, a *direction gradient* d in the cotangent space $T_x^* W$ of linear maps from $T_x W$ to $\Re$ such that $d(v) > 0$ for all $v \in H(x)$. If H is S-continuous and half-open, then there will exist such a continuous direction gradient d $V \to T^* V$ on a neighborhood V of x.

Choice Theorem *If H is an S-continuous half open preference field, on a finite dimensional compact manifold, W, then $C(W, H) \neq \varnothing$. If H is not half open then there exists an H*-preference cycle *through $\{x_1, x_2, x_3, .x_r.x_1\}$. For each arc (x_s, x_{s+1}) there is an H*-preference curve *from x_s to x_{s+1}, with a final H*-preference curve *from x_r to x_1.*

The Choice Theorem implies the existence of a *singularity* of the field, H.

Existence of Nash Equilibrium Let $\{W_1, \ldots, W_n\}$ be a family of compact, contractible, smooth, strategy spaces with each $W_i \subset \Re^w$. A smooth profile is a function $u : W^N = W_1 \times W_2 \times \cdots \times W_n \twoheadrightarrow \Re^n$. Let $H_i : W_i \twoheadrightarrow TW_i$ be the induced i-preference field in the tangent space over W_i. If each H_i is S-continuous and half open in TW_i then there exists a *critical Nash equilibrium*, $\mathbf{z} \in W^N$ such that $H^N(\mathbf{z}) = (H_1 \times \cdots \times H_n)(\mathbf{z}) = \varnothing$.

This follows from the previous choice theorem because the product preference field, H^N, will be half-open and S-continuous Below we consider existence of local Nash equilibrium. With smooth utility functions, a local Nash equilibrium can be found by checking the second order conditions on the Hessians. We now repeat Example 3.12. from Chap. 3.

Example 6.1 To illustrate the Choice Theorem, consider the example due to Kramer (1973), with $N = \{1, 2, 3\}$. Let the preference relation $P_{\mathbb{D}} \colon W \twoheadrightarrow W$ be generated by a set of decisive coalitions, $\mathbb{D} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, so that $y \in P_{\mathbb{D}}(x)$ whenever two voters prefer y to x. Suppose further that the preferences of the voters are characterized by the direction gradients

$$\left\{du_i(x) \colon i = 1, 2, 3\right\}$$

as in Fig. 6.1.

As the figure makes evident, it is possible to find three points $\{a, b, c\}$ in W such that

$$u_1(a) > u_1(b) = u_1(x) > u_1(c)$$

$$u_2(b) > u_2(c) = u_2(x) > u_2(a)$$

$$u_3(c) > u_3(a) = u_3(x) > u_3(b).$$

That is to say, preferences on $\{a, b, c\}$ give rise to a *Condorcet cycle*. Note also that the set of points $P_{\mathbb{D}}(x)$, preferred to x under the voting rule, are the shaded "win sets" in the figure. Clearly $x \in \text{Con}\, P_{\mathbb{D}}(x)$, so $P_{\mathbb{D}}(x)$ is not semi-convex. Indeed it

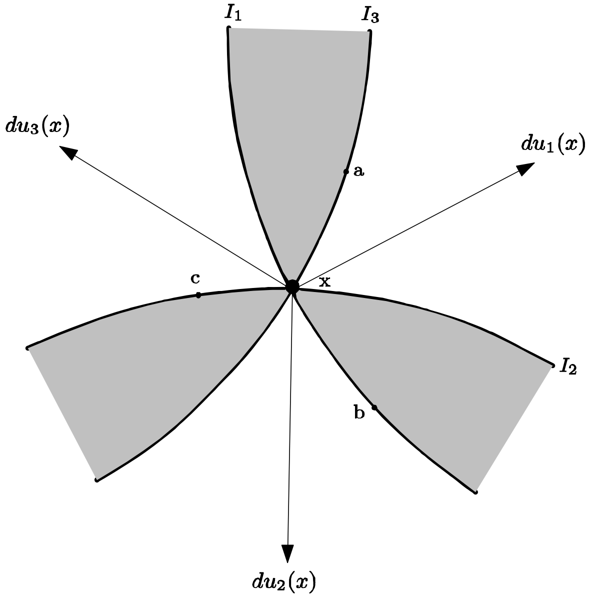


Fig. 6.1 Cycles near a point x

should be clear that in *any* neighborhood V of x it is possible to find three points $\{a', b', c'\}$ such that there is *local* voting cycle, with $a' \in P_{\mathbb{D}}(b'), b' \in P_{\mathbb{D}}(c'), c' \in P_{\mathbb{D}}(a')$. We can write this as

$$a' \to c' \to b' \to a'.$$

Not only is there a voting cycle, but the Fan theorem fails, and we have no reason to believe that $C(W, P_{\mathbb{D}}) \neq \varnothing$.

We can translate this example into one on preference fields by writing

$$H_{\mathbb{D}}(u) = \cup H_M(u) : W \twoheadrightarrow TW$$

where each $M \in \mathbb{D}$ and

$$H_M(u)(x) = \big\{v \in T_x W : \big(du_i(x) \centerdot v\big) > 0,\ \forall i \in M\big\}.$$

Figure 6.2 shows the three difference preference fields $\{H_i : i = 1, 2, 3\}$ on W, as well as the intersections H_M, for $M = \{1, 2\}$ etc.

Obviously the joint preference field $H_{\mathbb{D}} : W \twoheadrightarrow TW$ fails the half open property at x. Although $H_{\mathbb{D}}$ is S-continuous, we cannot infer that $C(W, H) \neq \varnothing$. If we define

$$Cycle(W, H) = \big\{x \ \in W : H(x) \text{ is not half open}\big\},$$

then at any point in $Cycle(W, H)$ it is possible to construct local cycles in the manner just described.

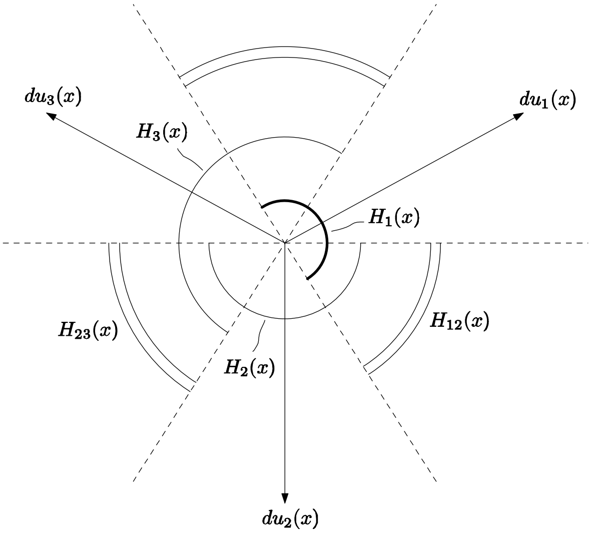


Fig. 6.2 The failure of half-openness of a preference field

The choice theorem can then be interpreted to mean that for any S-continuous field on W, then

$$Cycle(W, H) \cup C(W, H) \neq \varnothing.$$

Chilchinisky (1995) has obtained similar results for markets, where the condition that the dual is non-empty was termed *market arbitrage*, and defined in terms of global market co-cones associated with each player. Such a dual co-cone, $[H_i(u)]^*$ is precisely the set of prices in the cotangent space that lie in the dual of the preferred cone, $[H_i(u)]$, of the agent. By analogy with the above, she identifies this condition on non-emptiness of the intersection of the family of co-cones as one which is necessary and sufficient to guarantee an equilibrium.

The following Theorem implies that Fig. 6.2 is "generic." As in Chap. 4, a property is *generic* if it is true of all profiles in a residual set in $C^r(W, \Re^n)$, where this is the Whitney topology on smooth profiles, for a society of size n, on the policy space W. Now consider a non-collegial voting game, $\mathbb{D}$ with Nakamura number $\kappa(\mathbb{D})$ Then we have the following Theorem by Saari (1997).

Saari Theorem *For any non-collegial* $\mathbb{D}$, *there exists an integer* $w(\mathbb{D}) > \kappa(\mathbb{D})$ *such that* $\dim(W) > w(\mathbb{D})$ *implies that* $C(W, H_{\mathbb{D}}(u)) = \varnothing$ *for all* u *in a residual subspace of* $C^r(W, \Re^n)$.

This result was essentially proved by Saari (1997), building on earlier results by Banks (1995), McKelvey (1976, 1979), Kramer (1973), Plott (1967), Schofield (1978, 1983) and McKelvey and Schofield (1987). Although this result formally applies to voting rules, Schofield (2010) argues that it is applicable to any noncollegial social mechanism, and as a result can be interpreted to imply that chaos is

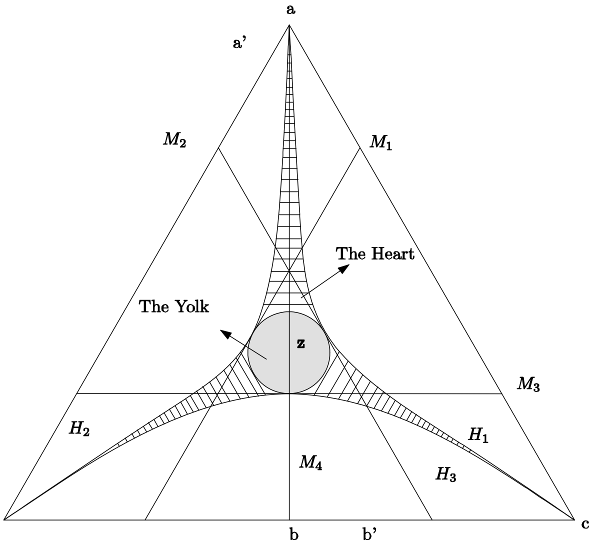


Fig. 6.3 The heart with a uniform electorate on the triangle

a generic phenomenon in coalitional systems. Since an equilibrium may not exist, we now introduce a more general social solution concept, called the *heart*.

Definition 6.3 (The Heart)

(i) If W is a topological space, then $x \in W$ is *locally covered* (under a preference correspondence $\mathbf{Q}$) iff for any neighborhood Y of x in W, there exists $y \in Y$ such that (a) $y \in \mathbf{Q}(x)$, and (b) there exists a neighborhood Y' of y, with $Y' \subseteq Y$ such that $Y' \cap \mathbf{Q}(y) \subset \mathbf{Q}(x)$.

(ii) The *heart* of $\mathbf{Q}$, written $\mathcal{H}(\mathbf{Q})$, is the set of locally uncovered points in W.

This notion can be applied to a preference correspondence $P_{\mathbb{D}}$ or to the preference field, $H_{\mathbb{D}}(u)$, in which case we write $\mathcal{H}(P_{\mathbb{D}})$ or $\mathcal{H}(H_{\mathbb{D}}(u))$. Schofield (1999) shows that the heart will belong to the Pareto set, and is lower hemi-continuous when regarded as a correspondence.

Example 6.2 To illustrate the heart, Fig. 6.3 gives a simple artificial example where the utility profile, u, is one where society has "Euclidean" preferences, based on distance, and the ideal points are uniformly distributed on the boundary of the equilateral triangle. Under majority rule, $\mathbb{D}$, the heart $\mathcal{H}(\mathbf{H}_{\mathbb{D}}(u))$, is the star-shaped figure inside the equilateral triangle (the Pareto set), and contains the "yolk" McKelvey (1986). The heart is generated by an infinite family of "median lines," such as $\{M_1, M_2, \ldots\}$. The shape of the heart reflects the asymmetry of the distribution. Inside the heart, voting trajectories can wander anywhere. Outside the heart the dual cones intersect, so any trajectory starting outside the heart converges to the heart. Thus the heart is an "attractor" of the voting process. Figure 6.4 gives a similar example, this time where preferences are defined by the pentagon, and the heart is the small centrally located ball.

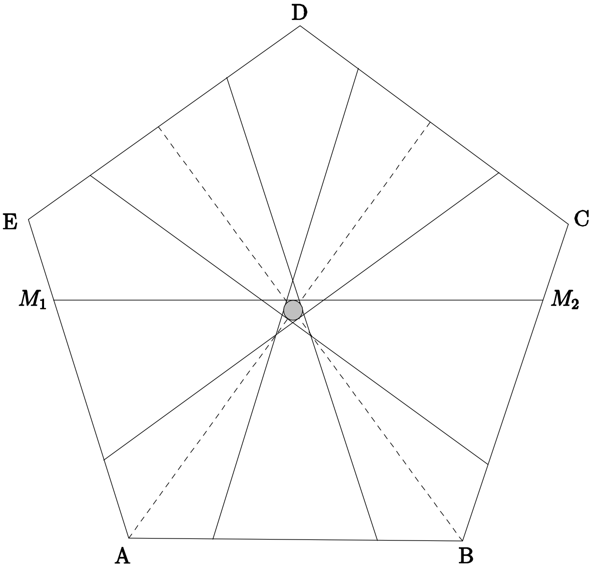


Fig. 6.4 The heart with a uniform electorate on the pentagon

6.3 Stochastic Choice

To construct such a social preference field, we first consider political choice on a compact space, W, of political proposals. This model is an extension of the standard multiparty stochastic model, modified by inducing asymmetries in terms of the preferences of voters.

We define a stochastic electoral model, $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$, which utilizes sociodemographic variables and voter perceptions of character traits. For this model we assume that voter i utility is given by the expression

$$u_{ij}(x_i, z_j) = \lambda_j + \mu_j(z_j) + (\theta_j \cdot \eta_i) + (\alpha_j \cdot \tau_i) - \beta\|x_i - z_j\|^2 + \varepsilon_j \quad (6.1)$$

$$= \left[u^*_{ij}(x_i, z_j)\right] + \varepsilon_j \quad (6.2)$$

The points $\{x_i \in W : i \in N\}$ are the preferred policies of the set, N, of voters in the political or policy space W, and $\mathbf{z} = \{z_j \in W : j \in Q\}$ are the positions of the agents/candidates. The term $\|x_i - z_j\|$ is simply the Euclidean distance between x_i and z_j. The error vector $(\varepsilon_1, \ldots, \varepsilon_j, \ldots, \varepsilon_p)$ is distributed by the iid type I extreme value distribution, as assumed in empirical multinomial logit estimation (MNL). The symbol $\boldsymbol{\theta}$ denotes a set of k-vectors $\{\theta_j : j \in Q\}$ representing the effect of the k different sociodemographic parameters (class, domicile, education, income, religious orientation, etc.) on voting for agent j while η_i is a k-vector denoting the ith individual's relevant "sociodemographic" characteristics. The compositions $\{(\theta_j \cdot \eta_i)\}$ are scalar products, called the *sociodemographic valences* for j. These scalar terms characterize the various types of the voters. See Lin et al. (1999) for work with the stochastic model.

The terms $\{(\alpha_j \centerdot \tau_i)\}$ are scalars giving voter i's perceptions and beliefs. These can include perceptions of the character traits of candidate or agent j, or beliefs about the state of the economy, etc. We let $\alpha = (\alpha_q, \dots, \alpha_1)$. A *trait score* can be obtained by factor analysis from a set of survey questions asking respondents about the traits of the agent, including 'moral', 'caring', 'knowledgeable', 'strong', 'honest', 'intelligent', etc. The perception of traits can be augmented with voter perception of the state of the economy, etc. in order to examine how anticipated changes in the economy affect each agent's electoral support.

Finally the *exogenous* valence vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ gives the general perception of the quality of the various candidates, $\{1, \dots, q\}$. This vector satisfies $\lambda_q \geq \lambda_{q-1} \geq \cdots \geq \lambda_2 \geq \lambda_1$, where $(1, \dots, q)$ label the candidates, and λ_j is the exogenous valence of agent or candidate j. In empirical multinomial logit models, the valence vector λ is given by the intercept terms for each agent. Finally $\{\mu_j(z_j)\}$ represent the *endogenous* valences of the candidates. These valences depend on the positions $\{z_j \in W : j \in Q\}$ of the agents.

In the model, the probability that voter i chooses candidate j, when party positions are given by $\mathbf{z}$ is:

$$\rho_{ij}(\mathbf{z}) = \Pr\bigl[\bigl[u_{ij}(x_i, z_j) > u_{il}(x_i, z_l)\bigr], \text{ for all } l \neq j\bigr].$$

A local Nash equilibrium (LNE) is one that locally maximizes the expectation.

A *strict local Political Nash equilibrium* (SLNE) is a vector, $\mathbf{z}$, such that each candidate, j, has chosen z_j to locally strictly maximize the expectation $\Sigma_i \rho_{ij}(\mathbf{z})$.

The type I extreme value distribution, Ψ, has a cumulative distribution the closed form

$$\Psi(h) = \exp\bigl[-\exp[-h]\bigr],$$

while its pdf has variance $\frac{1}{6}\pi^2$.

With this distribution it follows, for each voter i, and candidate, j, that

$$\rho_{ij}(\mathbf{z}) = \frac{\exp[u^*_{ij}(x_i, z_j)]}{\sum_{k=1}^{q} \exp u^*_{ik}(x_i, z_k)}. \tag{6.3}$$

This game is an example of what is known as a *Quantal response game* McKelvey and Palfrey (1995), Levine and Palfrey (2007). Note that the utility expressions $\{u^*_{ik}(x_i, z_k)\}$ can be estimated from surveys that include vote intentions. We can use the American National Election Survey (ANES for 2008) which gives gave individual perceptions of the important political policy questions. As indicated in Table 6.1, we are able to use factor analysis of these responses to construct a two dimensional policy space. ANES 2008 also gave voter perceptions of the character traits of the candidates, in terms of "moral", "caring", "knowledgeable", "strong" and "honest". We performed a factor analysis of these perceptions as shown in Table 6.2. Further details of the model can be found in Schofield et al. (2011)). Table 6.3 gives estimates of the average voter positions for the two parties . We also obtained data on those voters who declared they provided support for the candidates. these we designated as activists. Figure 6.5 gives an estimate of the voter estimated positions as well as the two presidential candidate positions in 2008.

Table 6.1 Factor loadings for economic and social policy for the 2008 election

Question	Economic policy	Social policy
Less Government services	0.53	0.12
Oppose Universal health care	0.51	0.22
Oppose Bigger Government	0.50	0.14
Prefer Market to Government	0.56	
Decrease Welfare spending	0.24	
Less government	0.65	
Worry more about Equality	0.14	0.37
Tax Companies Equally	0.28	0.10
Support Abortion		0.55
Decrease Immigration	0.12	0.25
Civil right for gays		0.60
Disagree Traditional values		0.53
Gun access	0.36	
Support Afr. Amer	0.14	0.45
Conservative v Liberal	0.30	0.60
Eigenvalue	1.93	1.83

Table 6.2 Factor loadings for candidate traits scores 2008

Question	Obama traits	McCain traits
Obama Moral	0.72	−0.01
Obama Caring	0.71	−0.18
Obama Knowledgeable	0.61	−0.07
Obama Strong	0.69	−0.13
Obama Honest	0.68	−0.09
Obama Intelligent	0.61	0.08
Obama Optimistic	0.55	0.00
McCain Moral	−0.09	0.67
McCain Cares	−0.17	0.63
McCain Knowledgeable	−0.02	0.65
McCain Strong	−0.10	0.70
McCain Honest	−0.03	0.63
McCain Intelligent	0.11	0.68
McCain Optimistic	−0.07	0.57
Eigenvalue	3.07	3.00

These survey data allow us to construct a spatial logit model of the election as in Table 6.4. This table has Obama as the baseline candidate.

Table 6.3 Estimated voter, activist and candidate positions in 2008

	Econ policy			Social policy			n
	Mean	s.e.	95 % C.I	Mean	s.e.	95 % C.I	
Activists							
Democrats	−0.20	0.09	[−0.38, −0.02]	1.14	0.11	[0.92, 1.37]	80
Republicans	1.41	0.13	[1.66, 1.16]	−0.82	0.09	[−0.99, −0.65]	40
Non-activists							
Democrats	−0.17	0.03	[−0.24, −0.11]	0.36	0.04	[0.29, 0.44]	449
Republicans	0.72	0.06	[0.60, 0.84]	−0.56	0.05	[−0.65, −0.46]	219
							788

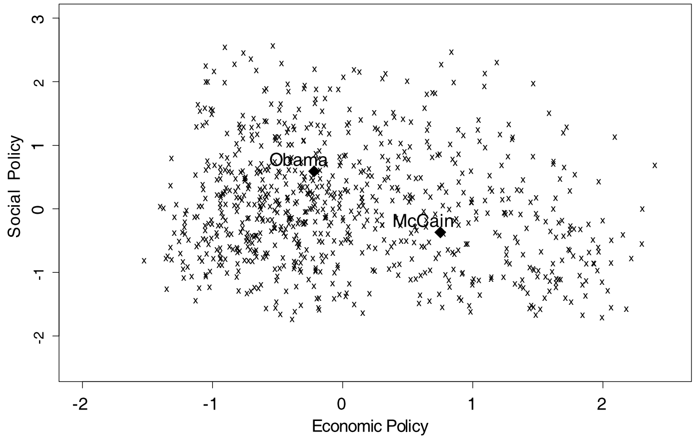


Fig. 6.5 Distribution of voter ideal points and candidate positions in the 2008 presidential election

Once the voter probabilities over a given set $\mathbf{z} = \{z_j : j \in Q\}$ are computed, then estimation procedures allow these probabilities to be computed for each possible vector $\mathbf{z}$. This allows the determination and proof of existence of *local Nash equilibrium* (LNE), namely a vector, $\mathbf{z}^* = (z_1^*, \dots, z_j^*, \dots, z_q^*)$ such that each candidate, j, chooses z_j^* to locally maximize its expected vote share, given by the expectation $V_j(\mathbf{z}^*) = \sum_i \rho_{ij}(\mathbf{z}^*)$.

Schofield (2006) shows that the first order condition for a LNE is that the *marginal electoral pull* at $\mathbf{z}^* = (z_1^*, \dots, z_j^*, \dots, z_q^*)$ is zero. For candidate j, this is defined to be

$$\frac{d\mathcal{E}_j^*}{dz_j}(z_j^*) = \left[z_j^{el} - z_j^*\right]$$

$$\text{where } z_j^{el} \equiv \sum_{i=1}^{n} \varpi_{ij} x_i$$

Table 6.4 Spatial logit models for USA 2008[a]

Variable	(1) Spatial	(2) Sp. & Traits	(3) Sp. & Dem.	(4) Full
McCain valence λ	−0.84***	−1.08***	−2.60**	−3.58***
	(7.6)	(8.3)	(2.8)	(3.4)
Spatial β	0.85***	0.78***	0.86***	0.83***
	(14.1)	(10.1)	(12.3)	(10.3)
McCain traits		1.30***		1.36***
		(7.6)		(7.15)
Obama traits		−1.02***		−1.16***
		(6.8)		(6.44)
Age			−0.01	−0.01
			(1.0)	(1.0)
Gender (F)			0.29	0.44
			(1.26)	(0.26)
African American			−4.16***	−3.79***
			(3.78)	(3.08)
Hispanic			−0.55	−0.23
			(1.34)	(0.51)
Education			0.15*	0.22***
			(2.5)	(3.66)
Income			0.03	0.01
			(1.5)	(0.50)
Working Class			−0.54*	−0.70**
			(2.25)	(2.59)
South			0.36	−0.02
			(1.5)	(0.07)
Observations	788			
log likelihood (LL)	−299	−243	−250	−207
AIC	601	494	521	438
BIC	611	513	567	494

[a] Obama is the baseline for this model

is the *weighted electoral mean* of candidate j.

Here the weights $\{\varpi_{ij}\}$ are individual specific, and defined at the vector $\mathbf{z}^*$ by:

$$[\varpi_{ij}] = \left[\frac{[\rho_{ij}(\mathbf{z}^*) - \rho_{ij}(\mathbf{z}^*)^2]}{\sum_{k \in N}[\rho_{kj}(\mathbf{z}^*) - \rho_{kj}(\mathbf{z}^*)^2]} \right] \tag{6.4}$$

Because the candidate utility functions $\{V_j : W \to \Re\}$ are differentiable, the second order condition on the Hessian of each V_j at $\mathbf{z}^*$ can then be used to determine whether $\mathbf{z}^*$ is indeed an LNE. Proof of existence of such an LNE will

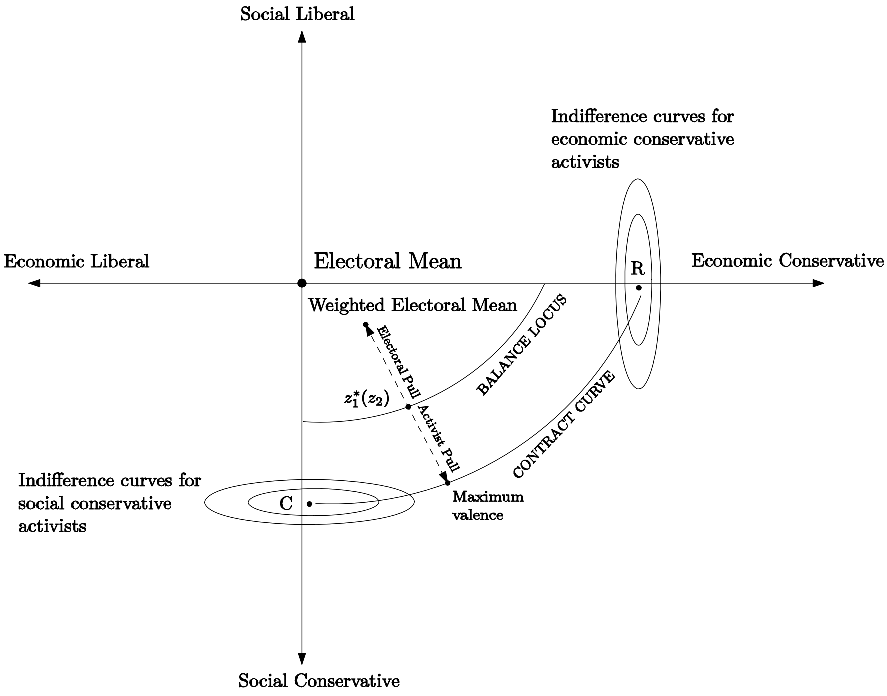


Fig. 6.6 Optimal Republican position

then follow from some version of the Choice Theorem. For example, in the simpler model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, without activists, all weights are equal to $\frac{1}{n}$, so the electoral mean $x_0 = \frac{1}{n}\sum x_i$ satisfies the first order condition, as suggested by Hinich (1977). Schofield (2007) gives the necessary and sufficient second order conditions for an LNE at the mean.

The underlying idea of this model is that each candidate, j, will be attracted to the respective weighted electoral mean, z_j^{el}, but will also be pulled away to a position preferred by the party activists. Figure 6.6 suggests the location of the weighted electoral mean for a Republican candidate. The contract curve in this figure is a heuristic estimated locus of preferred activist positions. See also Schofield et al. (2011).

Because the candidate utility functions $\{V_j : W \to \Re\}$ are differentiable, the second order condition on the Hessian of each V_j at $\mathbf{z}^*$ can then be used to determine whether $\mathbf{z}^*$ is indeed an LNE. Proof of existence of such an LNE will then follow from some version of the Choice Theorem. For example, in the simpler model $\mathbb{M}(\lambda, \beta)$, without activists, all weights are equal to $\frac{1}{n}$, so the electoral mean $x_0 = \frac{1}{n}\sum x_i$ satisfies the first order condition, as suggested by Hinich (1977). Schofield (2007) gives the necessary and sufficient second order conditions for an LNE at the mean.

The underlying idea of this model is that each candidate, j, will be attracted to the respective weighted electoral mean, z_j^{el}, but will also be pulled away to a position preferred by the party activists. Figure 6.6 suggests the location of the weighted electoral mean for a Republican candidate. The contract curve in this figure is a heuristic estimated locus of preferred activist positions. See also Schofield et al. (2011).

We can compute the necessary and sufficient second order conditions for an LNE at the electoral mean. In the case that the activist valence functions and sociodemographic terms are identically zero, we call this the *pure spatial model*, denoted $\mathbb{M}(\boldsymbol{\lambda}, \beta)$.

In this case, the first order condition is

$$\frac{dV_j(\mathbf{z})}{dz_j} = \frac{1}{n}\sum_{i\in N}\frac{d\rho_{ij}}{dz_j} \tag{6.5}$$

$$= \frac{1}{n}\sum_{i\in N}\{2\beta(x_i - z_j)\}\big[\rho_{ij} - \rho_{ij}^2\big] = 0. \tag{6.6}$$

Suppose that all z_j are identical. Then all ρ_{ij} are independent of $\{x_i\}$ and thus of i, and ρ_{ij} may be written as ρ_j. Then for each fixed j, the first order condition is

$$\frac{dV_j(\mathbf{z})}{dz_j} = 2\beta\big[\rho_j - \rho_j^2\big]\sum_{i\in N}\big[(x_i - z_j)\big] = 0. \tag{6.7}$$

The second order condition for an LNE at z* depends on the negative definiteness of the Hessian of the activist valence function. If the eigenvalues of these Hessians are negative at a balance solution, and of sufficient magnitude, then this will guarantee that a vector $\mathbf{z}^*$ which satisfies the balance condition will be a SLNE. Indeed, this condition can ensure concavity of the vote share functions, and thus of existence of a PNE.

6.3.1 The Model Without Activist Valence Functions

We now apply the Theorem to the pure spatial model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, by setting $\boldsymbol{\mu} = \boldsymbol{\theta} = \boldsymbol{\alpha} \equiv \mathbf{0}$.

As we have shown above, the *joint electoral mean* $\mathbf{z}_0$ satisfies the first order condition for a LNE. We now consider the second order condition.

Definition 6.4 (The Convergence Coefficient of the Model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$) When the space W has dimension w.

(i) Define

$$\rho_1 = \left[1 + \sum_{k=2}^{p}\exp[\lambda_k - \lambda_1]\right]^{-1}. \tag{6.8}$$

(ii) Let W be endowed with an orthogonal system of coordinate axes $(1, \ldots, s, \ldots, t, \ldots, w)$. For each coordinate axis let $\xi_t = (x_{1t}, x_{2t}, \ldots, x_{nt}) \in \mathbb{R}^n$ be the vector of the tth coordinates of the set of n voter ideal points. Let $(\xi_s, \xi_t) \in \mathbb{R}$ denote scalar product. The covariance between the sth and tth axes is denoted $(\sigma_s, \sigma_t) = \frac{1}{n}(\xi_s, \xi_t)$ and $\sigma_s^2 = \frac{1}{n}(\xi_s, \xi_s)$ is the electoral variance on the sth axis. Note that these variances and covariances are taken about the electoral means on each axis.

(iii) The symmetric $w \times w$ *electoral covariance matrix* ∇_0 is defined to be $\frac{1}{n}[(\xi_s, \xi_t)]_{t=1,\ldots,w}^{s=1,\ldots,w}$.

(iv) The *electoral variance* is

$$\sigma^2 = \sum_{s=1}^{w} \sigma_s^2 = \frac{1}{n} \sum_{s=1}^{w} (\xi_s, \xi_s) = \operatorname{trace}(\nabla_0).$$

(v) The w by w *characteristic matrix,* of agent 1 is given by

$$C_1 = 2\beta(1 - 2\rho_1)\nabla_0 - I. \tag{6.9}$$

(vi) The *convergence coefficient of the model* $\mathbb{M}(\boldsymbol{\lambda}, \beta)$ is

$$c \equiv c(\boldsymbol{\lambda}, \beta) = 2\beta[1 - 2\rho_1]\sigma^2. \tag{6.10}$$

Observe that the β-parameter has dimension L^{-2}, so that c is dimensionless. We can therefore use c to compare different models.

Note also that agent 1 is by definition the agent with the lowest valence, and ρ_1, as defined above, is the probability that a generic voter will choose this agent when all agents are located at the origin. The estimate of the probability ρ_1 depends only on the comparison functions $\{f_{kj}\}$, as given above and these can be estimated in terms of the valence differences.

The following result is proved in Schofield (2007).

The Mean Voter Valence Theorem

(i) *The joint mean z_0 satisfies the first order condition to be a LNE for the model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$.*

(ii) *The necessary and sufficient second order condition for SLNE at $\mathbf{z}_0$ is that C_1 has negative eigenvalues.*[1]

(iii) *A necessary condition for $\mathbf{z}_0$ to be a SLNE for the model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$ is that $c(\boldsymbol{\lambda}, \beta) < w$.*

(iv) *A sufficient condition for convergence to $\mathbf{z}_0$ in the two dimensional case is that $c < 1$.*

Notice that (iii) follows from (ii) since the condition of negative eigenvalues means that

$$\operatorname{trace}(C_1) = 2\beta[1 - 2\rho_1]\sigma^2 - w < 0.$$

[1] In the usual way, the condition for an LNE is that the eigenvalues are negative semi-definite.

In the case $c(\boldsymbol{\lambda}, \beta) = w$, then trace $(C_1) = 0$, which means either that all eigenvalues are zero, or at least one is positive. This degenerate situation requires examination of C_1. The additional condition $c < 1$ is sufficient to guarantee that $\det(C_1) > 0$, which ensures that both eigenvalues are negative.

The expression for C_1 has a simple form because of the assumption of a single distance parameter β. It is possible to use a model with different coefficients $\boldsymbol{\beta} = \{\beta_1, \beta_2, \ldots, \beta_w\}$ on each dimension. In this case the characteristic matrix can readily be shown to be

$$\mathbf{C}_1 = 2(1 - 2\rho_1)\boldsymbol{\beta}\nabla_0\boldsymbol{\beta} - \boldsymbol{\beta},$$

We require trace $(\mathbf{C}_1) < 0$, or

$$2(1 - 2\rho_1)\,\mathrm{trace}(\boldsymbol{\beta}\nabla_0\boldsymbol{\beta}) < \beta_1 + \beta_2 + \cdots + \beta_w.$$

The convergence coefficient in this case is

$$c(\boldsymbol{\lambda}, \boldsymbol{\beta}) = \frac{2(1 - 2\rho_1)\,\mathrm{trace}(\boldsymbol{\beta}\nabla_0\boldsymbol{\beta})}{\frac{1}{w}(\beta_1 + \beta_2 + \cdots + \beta_w)}$$

again giving the necessary condition of $c(\boldsymbol{\lambda}, \boldsymbol{\beta}) < w$.

Note that if C_1 has negative eigenvalues, then the Hessians of the vote shares for all agents are negative definite at the joint mean, $\mathbf{z}_0$. When this is true, then the joint mean is a candidate for a PNE, and this property can be verified by simulation.

When the convergence condition $c(\boldsymbol{\lambda}, \boldsymbol{\beta}) < w$ is violated the joint origin cannot be a SPNE.

In the degenerate case $c(\boldsymbol{\lambda}, \boldsymbol{\beta}) = w$ it is again necessary to examine the characteristic matrix to determine whether the joint mean can be a PNE.

Model (1) in Table 6.4 shows the coefficients in 2008 for the pure spatial model, $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, to be

$$(\lambda_{Obama}, \lambda_{McCain}, \beta) = (0, -0.84, 0.85).$$

Table 6.4 indicates, the loglikelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) are all quite acceptable, and all coefficients are significant with probability < 0.001.

Note that these parameters are estimated when the candidates are located at the estimated positions. Again, λ_{McCain} is the relative negative exogenous valence of McCain, with respect to Obama, according to the pure spatial model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$. We assume that the parameters of the model remain close to these values as we modify the candidates positions in order to determine the equilibria of the model.

According to the model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, the probability that a voter chooses McCain or Obama when both are positioned at the electoral mean, $\mathbf{z}_0$, are

$$(\rho_{McCain}, \rho_{Obama}) = \left(\frac{e^0}{e^0 + e^{0.84}}, \frac{e^{0.84}}{e^0 + e^{0.84}}\right) = (0.30, 0.70).$$

Now the covariance matrix can be estimated to be

$$\nabla_0 = \begin{bmatrix} 0.80 & -0.13 \\ -0.13 & 0.83 \end{bmatrix}.$$

Thus from Table 6.4, we obtain

$$\begin{aligned} C_{McCain} &= [2\beta(1-2\rho_{McCain})\nabla_0 = [2 \times 0.85 \times 0.4 \times \nabla_0] - I \\ &= (0.68)\nabla_0 - I \\ &= (0.68)\begin{bmatrix} 0.80 & -0.13 \\ -0.13 & 0.83 \end{bmatrix} - I = \begin{bmatrix} 0.54 & -0.09 \\ -0.09 & 0.56 \end{bmatrix} - I \\ &= \begin{bmatrix} -0.46 & -0.09 \\ -0.09 & -0.44 \end{bmatrix}, \end{aligned}$$

$$c = 2\beta(1-2\rho_{McCain})\,\text{trace}\,\nabla_0 = 2(0.85)(0.4)(1.63) = 1.1.$$

The determinant of C_{McCain} is positive and the trace negative, so both eigenvalues are negative, showing that the mean is an LNE. The lower 95 % estimate for ρ_{McCain} is 0.26, and the upper 95 % estimate for β is 0.97, so a very conservative upper estimate for $\beta(1-2\rho_{McCain})$ is $0.97 \times 0.48 = 0.47$, so the upper estimate for c is 1.53, giving an estimate for C_{McCain} of

$$\begin{aligned} &(0.94)\begin{bmatrix} 0.80 & -0.09 \\ -0.09 & 0.83 \end{bmatrix} - I \\ &= \begin{bmatrix} 0.75 & -0.13 \\ -0.13 & 0.78 \end{bmatrix} - I \\ &= \begin{bmatrix} -0.25 & -0.13 \\ -0.13 & -0.22 \end{bmatrix}, \end{aligned}$$

which still has negative eigenvalues.

We also considered a spatial model where the x and y axes had different coefficients, $\beta_1 = 0.8$, $\beta_2 = 0.92$.

Using

$$c(\boldsymbol{\lambda}, \boldsymbol{\beta}) = \frac{2(1-2\rho_{lib})\,\text{trace}(\boldsymbol{\beta}\nabla_0\boldsymbol{\beta})}{\frac{1}{w}(\beta_1 + \beta_2 + \cdots + \beta_w)}$$

with $\frac{1}{2}(\beta_1+\beta_2) = \frac{1}{2}(0.80+0.92) = 0.86$ and $\rho_{lib} = 0.25$, we find

$$\begin{aligned} c(\boldsymbol{\lambda}, \boldsymbol{\beta}) &= \frac{2(0.4)}{0.86}\,\text{trace}\begin{bmatrix} (0.80)^2(0.80) & (0.80)(0.92)(-0.13) \\ (0.80)(0.92)(-0.13) & (0.92)^2(0.83) \end{bmatrix} \\ &= (0.93)\,\text{trace}\begin{bmatrix} 0.51 & -0.09 \\ -0.09 & 0.70 \end{bmatrix} = (0.93)(1.21) = 1.23. \end{aligned}$$

For the characteristic matrix,

$$\begin{aligned}\mathbf{C}_{McCain} &= 2(1-2\rho_{McCain})\boldsymbol{\beta}\nabla_0\boldsymbol{\beta}-\boldsymbol{\beta}\\ &= 2(0.4)\begin{bmatrix}0.51 & -0.09\\ -0.09 & 0.70\end{bmatrix}-\begin{bmatrix}0.80 & 0\\ 0 & 0.92\end{bmatrix}\\ &= \begin{bmatrix}-0.41 & -0.07\\ -0.07 & -0.56\end{bmatrix}-\begin{bmatrix}0.80 & 0\\ 0 & 0.92\end{bmatrix}\\ &= \begin{bmatrix}-0.39 & -0.07\\ -0.07 & -0.36\end{bmatrix}.\end{aligned}$$

The analysis showed the Hessian for this case had negative eigenvalues, so again $\mathbf{z}_0$ is a LNE. This model is essentially the same as the model with a single β. Since these models imply the origin is an LNE but nether candidate is located there we infer that activists exert influence to move the candidate positions into opposite quadrants of the policy space. See Miller and Schofield (2003) for a devlopment of this observation.

References

Banks, J. S. (1995). Singularity theory and core existence in the spatial model. *Journal of Mathematical Economics*, *24*, 523–536.

Bergstrom, T. (1975). *The existence of maximal elements and equilibria in the absence of transitivity*. Typescript, University of Michigan.

Chichilnisky, G. (1995). Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales. *Economic Theory*, *5*, 79–107.

Fan, K. (1961). A generalization of Tychonoff's fixed point theorem. *Mathematische Annalen*, *42*, 305–310.

Hinich, M. J. (1977). Equilibrium in spatial voting: the median voter theorem is an artifact. *Journal of Economic Theory*, *16*, 208–219.

Knaster, B., Kuratowski, K., & Mazurkiewicz, S. (1929). Ein Beweis des Fixpunktsatzes fur *n*-dimensionale Simplexe. *Fundamenta Mathematicae*, *14*, 132–137.

Kramer, G. H. (1973). On a class of equilibrium conditions for majority rule. *Econometrica*, *41*, 285–297.

Levine, D., & Palfrey, T. R. (2007). The paradox of voter participation. *American Political Science Review*, *101*, 143–158.

Lin, T., Enelow, M. J., & Dorussen, H. (1999). Equilibrium in multicandidate probabilistic spatial voting. *Public Choice*, *98*, 59–82.

McKelvey, R. D. (1976). Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory*, *12*, 472–482.

McKelvey, R. D. (1979). General conditions for global intransitivities in formal voting models. *Econometrica*, *47*, 1085–1112.

McKelvey, R. D. (1986). Covering, dominance and institution free properties of social choice. *American Journal of Political Science*, *30*, 283–314.

McKelvey, R. D., & Palfrey, T. R. (1995). Quantal response equilibria in normal form games. *Games and Economic Behavior*, *10*, 6–38.

McKelvey, R. D., & Schofield, N. (1987). Generalized symmetry conditions at a core point. *Econometrica*, *55*, 923–933.

Miller, G., & Schofield, N. (2003). Activists and partisan realignment in the US. *American Political Science Review*, *97*, 245–260.

Plott, C. R. (1967). A notion of equilibrium and its possibility under majority rule. *The American Economic Review*, *57*, 787–806.

Saari, D. (1997). The generic existence of a core for q-rules. *Economic Theory*, *9*, 219–260.

Schofield, N. (1978). Instability of simple dynamic games. *Review of Economic Studies*, *45*, 575–594.

Schofield, N. (1983). Generic instability of majority rule. *Review of Economic Studies*, *50*, 695–705.

Schofield, N. (1999). The heart and the uncovered set. *Journal of Economics. Supplementum*, *8*, 79–113.

Schofield, N. (2006). Equilibria in the spatial stochastic model of voting with party activists. *Review of Economic Design*, *10*, 183–203.

Schofield, N. (2007). The Mean Voter Theorem: necessary and sufficient conditions for convergent equilibrium. *Review of Economic Studies*, *74*, 965–980.

Schofield, N. (2010). Social orders. *Social Choice and Welfare*, *34*, 503–536.

Schofield, N. (2013). The probability of a fit choice. *Review of Economic Design*, *17*, 129–150.

Schofield, N., Claassen, C., & Ozdemir, U. (2011). Empirical and formal models of the US presidential elections in 2004 and 2008. In N. Schofield & G. Caballero (Eds.), *The political economy of institutions, democracy and voting* (pp. 217–258). Berlin: Springer.

Walker, M. (1977). On the existence of maximal elements. *Journal of Economic Theory*, *16*, 470–474.

7 Review Exercises

7.1 Exercises to Chap. 1

1.1. Consider the relations:

$$P = \{(2,3), (1,4), (2,1), (3,2), (4,4)\} \quad \text{and} \quad Q = \{(1,3), (4,2), (2,4), (4,1)\}.$$

Compute $Q \circ P$, $P \circ Q$, $(P \circ Q)^{-1}$ and $(Q \circ P)^{-1}$. Let ϕ_Q and ϕ_P be the mappings associated with these two relations. Are either ϕ_Q and ϕ_P functions, and are they surjective and/or injective?

1.2. Suppose that each member i of a society $M = \{1, \dots, m\}$ has weak and strict preferences (R_i, P_i) on a finite set X of feasible states. Define the weak Pareto rule, Q, on X by xQy iff $xR_iy \forall i \in M$, and xP_jy for some $j \in M$. Show that if each R_i, $i \in M$, is transitive, then Q is transitive. Hence show that the Pareto choice set $C_Q(X)$ is non empty.

1.3. Show that the set $\Theta = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, of all 2×2 matrices representing rotations, is a subgroup of $(M^*(2), \circ)$, under matrix composition, $\circ$.

7.2 Exercises to Chap. 2

2.1. With respect to the usual basis for $\Re^3$, let $x_1 = (1,1,0)$, $x_2 = (0,1,1)$, $x_3 = (1,0,1)$. Show that $\{x_1, x_2, x_3\}$ are linearly independent.

2.2. Suppose $f : \Re^5 \to \Re^4$ is a linear transformation, with a 2-dimensional kernel. Show that there exists some vector $z \in \Re^4$, such that for any vector $y \in \Re^4$ there exists a vector $y_0 \in \text{Im}(f)$ with $y = y_0 + \lambda z$ for some $\lambda \in \Re$.

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6_7,

2.3. Find all solutions to the equations $A(x) = b_i$, for $i = 1, 2, 3$, where

$$A = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 1 & -1 & 1 \\ 1 & -1 & 4 & 6 \end{pmatrix}$$

and

$$b_1 = \begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad b_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

2.4. Find all solutions to the equation $A(x) = b$ where

$$A = \begin{pmatrix} 6 & -1 & 1 & 4 \\ 1 & 1 & 3 & -1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}.$$

2.5. Let $F : \Re^4 \to \Re^2$ be the linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 5 & -1 & 3 \\ -1 & 0 & -4 & 2 \end{pmatrix}.$$

Compute the set $F^{-1}(y)$, when $y = \binom{4}{1}$.

2.6. Find the kernel and image of the linear transformation, A, represented by the matrix

$$\begin{pmatrix} 3 & 7 & 2 \\ 4 & 10 & 2 \\ 1 & -2 & 5 \end{pmatrix}.$$

Find new bases for the domain and codomain of A so that A can be represented as a matrix

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to these bases.

2.7. Find the kernel of the linear transformation, A, represented by the matrix

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ -1 & 1 & 3 \end{pmatrix}.$$

Use the dimension theorem to compute the image of A. Does the equation $A(x) = b$ have a solution when

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}?$$

2.8. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}.$$

Is this matrix positive or negative definite or neither?

2.9. Diagonalize the matrix

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 10 & 2 \end{pmatrix}.$$

2.10. Compute the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and thus diagonalize the matrix.

7.3 Exercises to Chap. 3

3.1. Show that if A is a set in a topological space $(X, \mathcal{T})$ then the interior, $\text{Int}(A)$, of A is open and the closure, $\text{Clos}(A)$, is closed. Show that $\text{Int}(A) \subset A \subset \text{Clos}(A)$. What is the interior and what is the closure of the set $[a, b)$ in $\Re$, with the Euclidean topology? What is the boundary of $[a, b)$? Determine the limit points of $[a, b)$.

3.2. If two metrics d_1, d_2 on a space X are equivalent write $d_1 \sim d_2$. Show that $\sim$ is an equivalence relation on the set of all metrics on X. Thus show that the Cartesian, Euclidean and city block topologies on $\Re^n$ are equivalent.

3.3. Show that the set, $L(\Re^n, \Re^m)$, of linear transformations from $\Re^n$ to $\Re^m$ is a normed vector space with norm

$$\|f\| = \sup_{x \in \Re^n} \left\{ \frac{\|f(x)\|}{\|x\|} : \|x\| \neq 0 \right\},$$

with respect to the Euclidean norms on $\Re^n$ *and* $\Re^m$. *In particular verify that* $\| \ \|_L$ satisfies the three norm properties. Describe an open neighbourhood of a member f of $L(\Re^n, \Re^m)$ with respect to the induced topology on $L(\Re^n, \Re^m)$. Let $M(n, m)$ be the set of $n \times m$ matrices with the natural topology (see page 106), and let

$$M : L\big(\Re^n, \Re^m\big) \to M(n, m)$$

be the matrix representation with respect to bases for $\Re^n$ and $\Re^m$. Discuss the continuity of M with respect to these topologies for $L(\Re^n, \Re^m)$ and $M(n, m)$.

3.4. Determine, from first principles, whether the following functions are continuous on their domain:
1. $\Re_+ \to \Re : x \to \log_e x$;
2. $\Re \to \Re_+ : x \to x^2$;
3. $\Re \to \Re_+ : x \to e^x$;
4. $\Re \to \Re : x \to \cos x$;
5. $\Re \to \Re : x \to \cos \frac{1}{x}$.

3.5. What is the image of the interval $[-1, 1]$ under the function $x \to \cos \frac{1}{x}$? Is the image compact?

3.6. Determine which of the following sets are convex:
1. $X_1 = \{(x_1, x_2) \in \Re^2 : 3x_1^2 + 2x_2^2 \leq 6\}$;
2. $X_2 = \{(x_1, x_2) \in \Re^2 : x_1 \leq 2, x_2 \leq 3\}$;
3. $X_3 = \{(x_1, x_2) \in \Re^2_+ : x_1 x_2 \leq 1\}$;
4. $X_4 = \{(x_1, x_2) \in \Re^2_+ : x_2 - 3 \geq -x_1^2\}$.

3.7. In $\Re^2$, let $B_C(x, r_1)$ be the *Cartesian* open ball of radius r_1 about x, and $B_E(y, r_2)$ the *Euclidean* ball of radius r_2 about x. Show that these two sets are convex. For fixed $x, y \in \Re^2$ obtain necessary and sufficient restrictions on r_1, r_2 so that these two open balls may be strongly separated by a hyperplane.

3.8. Determine whether the following functions are convex, quasi-concave, or concave:
1. $\Re \to \Re_+ : x \to e^x$;
2. $\Re \to \Re : x \to x^7$;
3. $\Re \to \Re : (x, y) \to xy$;
4. $\Re \to \Re : x \to \frac{1}{x}$;
5. $\Re^2 \to \Re : (x, y) \to x^2 - y$.

7.4 Exercises to Chap. 4

4.1. Suppose that $f : \Re^n \to \Re^m$ and $g : \Re^m \to \Re^k$ are both C^r-differentiable. Is $g \circ f : \Re^n \to \Re^k$, a C^r-differentiable function? If so, why?

4.2. Find and classify the critical points of the following functions:

1. $\Re^2 \to \Re : (x, y) \to x^2 + xy + 2y^2 + 3;$
2. $\Re^2 \to \Re : (x, y) \to -x^2 + xy - y^2 + 2x + y;$
3. $\Re^2 \to \Re : (x, y) \to e^{2x} - 2x + 2y.$

4.3. Determine the critical points, and the Hessian at these points, of the function $\Re^2 \to \Re : (x, y) \to x^2 y$.

Compute the eigenvalues and eigenvectors of the Hessian at critical points, and use this to determine the nature of the critical points.

4.4. Show that the origin is a critical point of the function:

$$\Re^3 \to \Re : (x, y, z) \to x^2 + 2y^2 + Zz^2 + xy + xz.$$

Determine the nature of this critical point by examining the Hessian.

4.5. Determine the *set* of critical points of the function

$$\Re^2 \to \Re : (x, y) \to -x^2 y^2 + 4xy - 4x^2.$$

4.6. Maximise the function $\Re^2 \to \Re : (x, y) \to x^2 y$ subject to the constraint $1 - x^2 - y^2 = 0$.

4.7. Maximise the function $\Re^2 \to \Re : (x, y) \to a \log x + b \log y$, subject to the constraint $px + qy \leq I$, where $p, q, I \in \Re_+$.

7.5 Exercises to Chap. 5

5.1. Show that if dimension $(X) > m$, then for almost every smooth profile $u = (u_1, \ldots, u_m) : X \to \Re^m$ it is the case that Pareto optimal points in the interior of X can be parametrised by at most $(m - 1)$ strictly positive coefficients $\{\lambda_1, \ldots, \lambda_{m-1}\}$.

5.2. Consider a two agent, two good exchange economy, where the initial *endowment* of good j, by agent i is e_{ij}. Suppose that each agent, i, has utility function $u_i : (x_{i1}, x_{i2}) \to a \log x_{i1} + b \log x_{i2}$. Compute the critical Pareto set Θ, within the feasible set

$$Y = \left\{(x_{11}, x_{12}, x_{21}, x_{22}) \in \Re^4_+\right\},$$

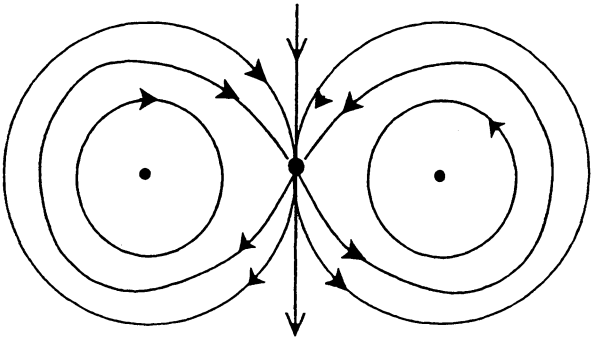


Fig. 7.1 The butterfly singularity

where the coordinates of Y satisfy

$$x_{11} + x_{21} = e_{11} + e_{21} \quad \text{and} \quad x_{12} + x_{22} = e_{12} + e_{22}.$$

What is the dimension of Y and what is the codimension of Θ in Y? Compute the market-clearing equilibrium.

5.3. Figure 7.1 shows a "butterfly singularity", A, in $\Re^2$. Compute the degree of this singularity. Show why such a singularity (though it is isolated) cannot be associated with a generic excess demand function on the two-dimensional price simplex.

Subject Index

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6,

Author Index

N. Schofield, *Mathematical Methods in Economics and Social Choice*,
Springer Texts in Business and Economics, DOI 10.1007/978-3-642-39818-6,