Basics of Generalized Linear Models

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Generalized Linear Models

Generalized Linear Models

- We will introduce a topic that is typically taught only in a class where you are expected to know linear algebra.
- Fear not though!
- We will show some of the math behind this but this is to teach you methods that link to the modern way data analysis is done.

Why Bother?

- By learning the we can understand how to fit, linear, logistic, poisson, multinomial, data from distributions like Gamma and Inverse Gamma, longitudinal data and multivariate data.
- We will not have time to learn all of these in this class but this is a very versatile model.
- The mathematics behind these models are matrix related but we will focus on the application of them.

The Generalized Linear Model

- The generalized linear model refers to a whole family of models.
- They became popular with a book by McCullagh and Nelder (1982).
- They have 3 basic components.

Components of any GLM

1 **The Random Component** - probability distribution of the response variable. In linear regression this is the normal distribution.

2 **The Systematic Component** - fixed structure of explanatory variables usually a linear function. We have seen this as $\beta_0 + \beta_1 X_1 + \ldots$.

3 **The Link Function** - maps the systematic component onto the random component. This was $E(Y_i | X_{1i}, \ldots)$ in the linear regression case.

The Random Component

- Observations of the outcome represent a sample from a random variable.
- This random variable has a mean value and variation that depends on the distribution it follows.
- GLM uses random variables that follow an exponential family distribution.

The Systematic Component

- We use the covariates or independent variables to model to estimate the means of the random variable that our sample was drawn from.
- This is added to the variation to give use the data that we observed.

The Model

- We use

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} = \sum_{j=1}^{p} \beta_j x_{ij}$$

- where:
  - η is the linear predictor.
  - $x_1, \ldots, x_p$ are the explanatory variables.
  - $\beta_1, \ldots, \beta_p$ are the coefficients of the explanatory variables.
  - β_0 is the value of η when all the x's are 0.

What you typically will see:

- Most of the time this is written as:

$$\eta = \mathbf{X}\beta$$

- where:
  - $\eta = (\eta_1, \ldots, \eta_N)^T$ is a column vector.
  - $\beta = (\beta_0, \ldots, \beta_p)^T$ is a column vector.
  - $\mathbf{X}$ is a $N \times p$ matrix of the explanatory variables x_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, p$.

Visualizing the Matrices

- In other words:

$$\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & \dots & x_{Np} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

- This linear predictor allows the least squares regression approach to be generalized to a wide range of models.

The Link Function

- We cannot always model a direct relationship between the random and the systematic component.
- This is where the link function comes into place.
- This function allows us to specify a relationship between the linear(systematic component) and the random component.
- We essentially link η_i to $\mu_i = E(y_i)$.

The Link Function

- We have

$$g(\mu_i) = \eta_i$$

- where:
  - $g()$ is the link function.
  - μ_i represents the expected value of the random component.
  - η_i represents the linear(structural) component.

What is this link Function?

- The link function is specifically defined by how the distribution is identified as an exponential family.
- We will not go through this math however feel free to look up exponential families and try and put the distributions we talk about into this framework.

Common Link Functions:

- Some common link functions are

Random Component	Link Function	Outcome Ex Variable	planatory Variable	Model
Normal	Indentity	Continuous	Factor	ANOVA
Normal	Indentity	Continuous	Continuous	Regression
Binomial	Logit	Binary	Mixed	Logistic Regression
Multinomial	Generalized logit	Binary	Mixed	Multinomial Regression
Poisson	Log	Count	Mixed	Poisson Regression

What Does this Mean?

- The chart shows just some of the many types of models we can learn to do just from a simple concept of GLMs.
- Essentially every type of technique you have used up until this point can be structured in such a way that is represents a GLM.

Assumptions of a GLM

- The data $Y_1, Y_2, \ldots, Y_2$ are independently distributed.
- The dependent variable Y_i is from an exponential family.
  - Normal (Gaussian)
  - Bernoulli
  - Binomial
  - Multinomial
  - Exponential
  - Poisson

Assumptions of a GLM

- Linear Relationship between link function and systematic component.
- Errors are independent.
- Uses Maximum Likelihood Estimation rather than Least Squares Estimation.
- For goodness-of-fit tests need large sample sizes.

What Assumptions are not needed?

- We do **NOT** some assumptions we needed before.
  - We do **NOT** need a linear relationship between the dependent variable and the independent variables.
  - We do **NOT** need need normally distributed errors.
  - We do **NOT** need homogeneity of errors.

The Case of Linear Regression

Linear Regression as a Case

- We have previously been using linear regression and it can be easily display how we use it in this framework.
- For example in a multiple linear regression we have

$$y_i|x_{i1},\ldots,x_{ip} = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i$$

- Then we know that

$$\mu_i = E(y_i y_i|x_{i1},\ldots,x_{ip}) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}$$

- Thus we can directly relate μ_i to the systematic component.

What Link in the Linear Case?

- Thus in this case our function $g()$ is

$$g(\mu_i) = \mu_i$$

- We call this the identity link.

What do we have?

- Then we have that
  - **Random Component**: y is the outcome and is normally distributed. So we let $\epsilon_i \sim N(0\sigma^2)$.
  - **Systematic Component**: $x_1, \ldots, x_p$ are the explanatory variables. They can be categorical or continuous. We have a linear combination of these terms but we can still have x^2 or $\log(x)$ terms in here as well.
  - **Link Function**:

Identity Link

- We have the identity function:

$$\eta = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} g(E(y_i)) \qquad = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} g(E(y_i)) = E(y_i)$$

- With linear regression we have the simplest link function because we are able to model the mean directly.

The Case of Logistic Regression

Logistic Regression

- We will now move onto logistic regression.
- With logistic regression we are concerned with binary data.
- This is data that is in a format of either yes or no, 0 or 1, or some variation of that.

Binomial Distribution

- If we consider binary data we find that what we have is called the Binomial distribution.
- Let's assume that we have Y where

$$Y = \begin{cases} 1 & \text{if sucess} \\ 0 & \text{if failure} \end{cases}$$

What Does this mean?

- Then

$$\Pr(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$$

- where p is the probability that $Y = 1$.
- This leads us to

$$E(Y) = np$$

$$Var(Y) = np(1-p)$$

Regression Model for Logistic

- Recall from simple linear regression that our systematic part of our model is

$$E(Y_i|x_i) = \beta_0 + \beta_1 x_i$$

- That would mean with this type of data we have

$$p_i = \beta_0 + \beta_1 x_i$$

Why Can't we do Linear Regresion?

- The issue with this is now we can have values that fall outside of 0 and 1.
- To overcome the problem with negative values we could exponeniate:

$$p_i = \exp(\beta_0 + \beta_1 x_1)$$

- We now have values that can fall between 0 and infinity.

What about Values greater than 1?

- In order to solve the problem of values being greater than 1, we divide by 1 plus the exponential:

$$p_i = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x_1)}$$

- This new function now lies completely between 0 and 1 as needed.
- Then we solve back to where we have the systematic part.

The Systematic Part

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} p_i \left(1 + \exp(\beta_0 + \beta_1 x_i)\right) \qquad = \exp(\beta_0 + \beta_1 x_i) p_i = \exp(\beta_0 + \beta_1 x_i$$

What does this mean?

- This means we are fitting a linear regression to the logistic unit (logit) or the log odds of the probability of a success.
- This is why we refer to this as logistic regression.

The Logit

Then if we consider the logit:

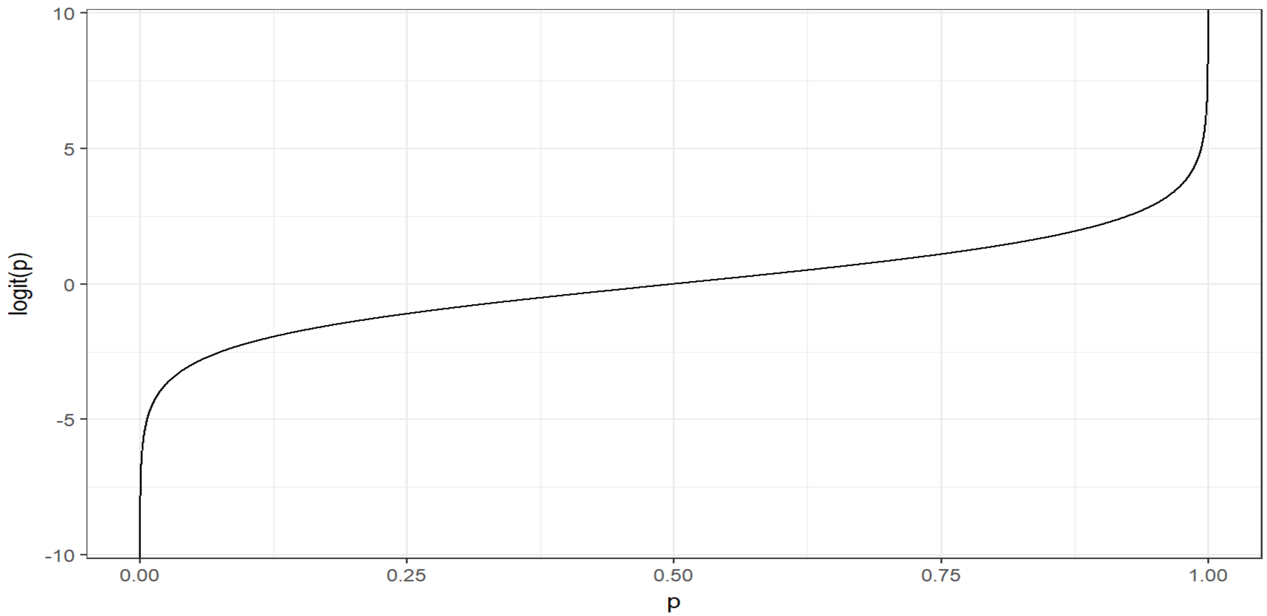
$$\text{If } p = 0 \text{then } \log\left(\frac{p}{1-p}\right) = -\infty \text{If } p = \tfrac{1}{2} \quad \text{then } \log\left(\frac{p}{1-p}\right) = 0 \text{If } p = 1 \text{then } \log\left(\frac{p}{1-\right.$$

What does the Logit imply?

- We can see that as p increases the logit does as well.
- We have that the logit can be anything between $-\infty$ and ∞, but p is between 0 and 1 as needed.

Relationship Between p and the logit

- We can see the relationship between p and the logit below.



Logistic as a GLM

- From the above work we can see that with logistic regression we have

$$logit\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_1$$

$$\text{or}$$

$$logit\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

Logistic as a GLM

- Where $E(y_i|y_{i1},\ldots,x_{ip}) = p_i$ therefore what we have is
  - **Random Component**: y is the outcome and is binomial and we assume the variance to be that of a binomial.
  - **Systematic Component**: $x_1,\ldots,x_p$ are the explanatory variables. They can be categorical or continuous.
  - We have a linear combination of these terms but we can still have x^2 or $\log(x)$ terms in here as well.

The Link Funcion:

- Where $E(y_i|y_{i1},\dots,x_{ip}) = p_i$ therefore what we have is
  - **Link Function**: We can see from above that with p_i being the mean that we have the logit as the link function:

$$\eta = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} g(E(y_i)) \qquad = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} g(p_i) = logit\,($$

Maximum Likelihood Estimation

- In linear regression we learned about least squares estimation.
- This falls apart with logistic regression when we have $p = 0$ or $p = 1$.
- Due to this we prefer a technique that can accurately estimate p no matter what.
- We will map out what this looks like right now.

Our Data

- With our data we have

$$\Pr(Y_i = 1|x_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

- Then we also have that

$$\Pr(Y_i = 0|x_i) = 1 - \Pr(Y_i = 1|x_i) \qquad = 1 - \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} = \frac{1}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

Our Data

- If we combine these together we find that:

$$\Pr(Y_i = y_i | x_i) = \frac{\exp((\beta_0 + \beta_1 x_i) \cdot y_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}, \quad y_i = 0, 1$$

The Likelihood

- The likelihood is defined as the probability of obtaining the data that was observed.

$$\Pr(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n | x_1, x_2, \ldots, x_n)$$

- Then we assumed that in our data the responses are independent from one another.
- This leads to

$$\Pr(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n | x_1, x_2, \ldots, x_n) = \Pr(Y_1 = y_1 | x_1) \cdots \Pr(Y_n = y_n | x_n)$$

The Likelihood

- Then the probability we obtain our data is

$$L = \prod_{i=1}^{n} \left[\frac{\exp((\beta_0 + \beta_1 x_i) \cdot y_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \right]$$

Maximum Likelihood

- Maximum likelihood estimates for β_0 and β_1 are found by searching for which values $\hat{\beta}_0$ and $\hat{\beta}_1$ maximize L.
- Unlike in least squares we cannot find these solutions in a closed form.
- We calculate MLEs with some sort of iterative technique.

Normal Distribution and Maximum Likelihood

- It can be shown that maximum likelihood estimators are normally distributed.
- This means in our data

$$\hat{\beta}_0 \overset{approx}{\sim} N\left(\beta_0, \widehat{Var}\left(\hat{\beta}_0\right)\right)$$

$$\hat{\beta}_1 \overset{approx}{\sim} N\left(\beta_1, \widehat{Var}\left(\hat{\beta}_1\right)\right)$$

Why do we use MLE?

- Finally we have that MLEs are the most efficient estimators out there.
- Meaning that any other consistent estimators $\tilde{\beta}_0$ and $\tilde{\beta}_1$ will have larger variances then $\hat{\beta}_0$ and $\hat{\beta}_1$.
- This means we will have the tightest confidence intervals around our MLEs and possibly show significance when other estimators would fail to.