Bayesian Model Averaging

Hoff Chapter 9, Hoeting et al 1999, Clyde & George 2004, Liang et al 2008

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Outline

- Problems with g-priors
- Alternatives: Mixtures of g-priors
- Model Averaging
- Choice of Model

Bayes Factors

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- Posterior Probability

$$P(\mathcal{M}_\gamma \mid \mathbf{Y}) = \frac{BF[\mathcal{M}_\gamma : \mathcal{M}_0] p(\mathcal{M}_\gamma)/p(\mathcal{M}_0)}{\sum_{\mathcal{M}_\gamma \in \Gamma} BF[\mathcal{M}_\gamma : \mathcal{M}_0] p(\mathcal{M}_\gamma)/p(\mathcal{M}_0)}$$

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"Information paradox"

Resolution of Paradox

Liang et al (2008) show that paradox can be resolved with mixtures of g-priors

$$p(\boldsymbol{\beta}_\gamma \mid \phi) = \int_0^\infty \mathsf{N}(\boldsymbol{\beta}_\gamma; 0, g(\mathbf{X}_\gamma^T \mathbf{X}_\gamma)^{-1}/\phi) p(g) dg$$

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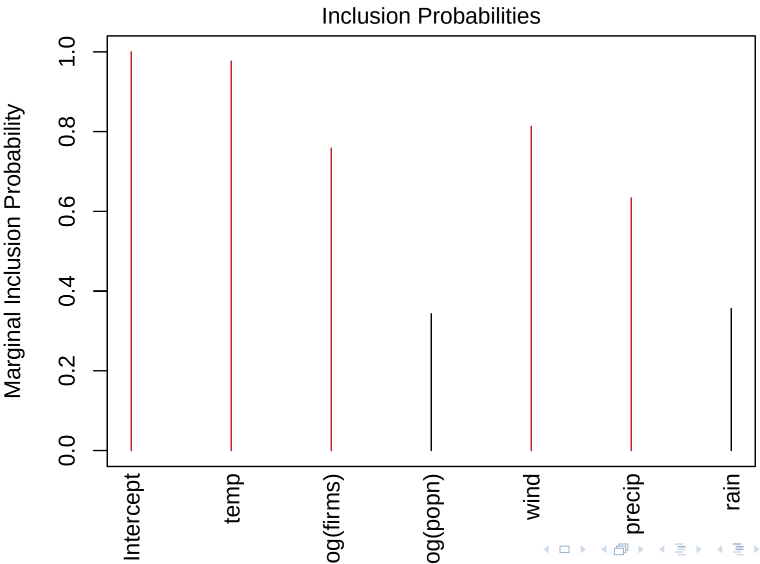
- "hyper-g/n"
- robust prior (Bayarri et al Annals of Statistics 2012)

Example

```r
library(BAS)
poll.ZS = bas.lm(log(SO2) ~ temp + log(firms) +
                             log(popn) + wind +
                             precip+ rain,
                   data=usair,
                   prior="ZS-null",
                   alpha=41,     # g = n
                   n.models=2^7,# enumerate (can omit)
                   modelprior=uniform(),
                   method="deterministic")    # fast enumera
```

use ‘prior = ”hyper-g”‘ and ‘a = 3‘ for hyper-g or ‘prior = ”hyper-g/n”‘ and ‘a=3‘ for hyper-g/n

```r
plot(poll.ZS, which=4)
```



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  with expectation expressed as a weighted average

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- Posterior Distribution of β_j

$$p(\beta_j \mid \mathbf{Y}) = p(\gamma_j = 0 \mid \mathbf{Y}) \delta_0(\beta) + \sum p(\beta_j \mid \mathbf{Y}, \mathcal{M}_\gamma) \gamma_j p(\mathcal{M}_\gamma \mid \mathbf{Y})$$

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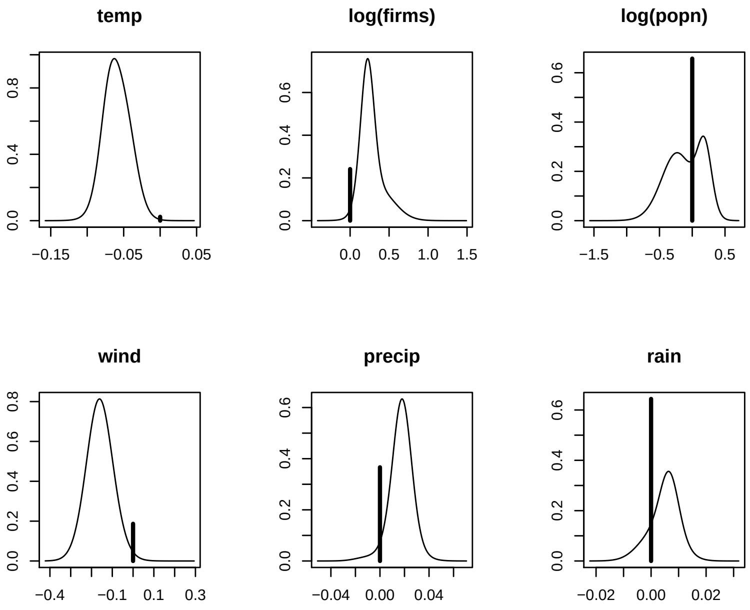
- If one model has probability 1, then BMA is equivalent to using the highest posterior probability model
- incorporates estimates from other models when there is substantial uncertainty

Coefficients under BMA

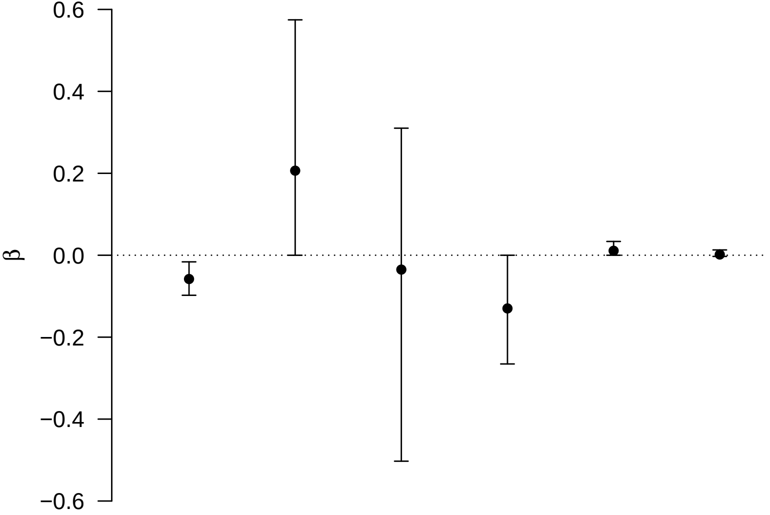
```
beta.ZS = coef(poll.ZS)
beta.ZS

##
##  Marginal Posterior Summaries of Coefficients:
##
##  Based on the top  64 models
##              post mean  post SD    post p(B != 0)
## Intercept     3.153004   0.082226   1.000000
## temp         -0.058053   0.020325   0.976833
## log(firms)    0.206384   0.177253   0.758554
## log(popn)    -0.035074   0.174760   0.342677
## wind         -0.129875   0.085195   0.813330
## precip        0.010898   0.011327   0.633639
## rain          0.001759   0.004034   0.356085
```

Posterior of Coefficients under BMA



Credible Intervals for Coefficients under BMA

```
plot(confint(beta.ZS, parm=2:7))
```



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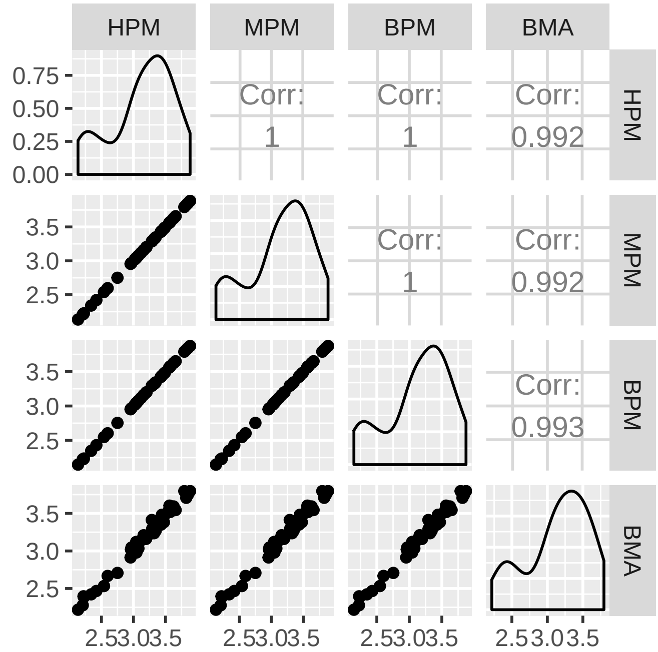
- Often contains more predictors than the HPM or Median Probability Model

Best Predictive Model

```
#BPM
BPM = predict(poll.ZS, estimator = "BPM")
BPM$bestmodel

## [1] 0 1 2 4 5 6

(poll.ZS$namesx[attr(BPM$fit, 'model') +1])[-1]

## [1] "temp"        "log(firms)" "wind"        "precip"

#HPM
HPM = predict(poll.ZS, estimator = "HPM")
HPM$bestmodel

## [1] 0 1 2 4 5
```



HPM
MPM
BPM
BMA
Corr:
1
Corr:
1
Corr:
0.992
Corr:
1
Corr:
0.992
Corr:
0.993
0.75
0.50
0.25
0.00
3.5
3.0
2.5
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- Mixtures of g priors preferred to usual g prior