Basics of Generalized Linear Models

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Generalized Linear Models

Generalized Linear Models

- We will introduce a topic that is typically taught only in a class where you are expected to know linear algebra.
- Fear not though!
- We will show some of the math behind this but this is to teach you methods that link to the modern way data analysis is done.

Why Bother?

- By learning the we can understand how to fit, linear, logistic, Poisson, multinomial, data from distributions like Gamma and Inverse Gamma, longitudinal data and multivariate data.
- We will not have time to learn all of these in this class but this is a very versatile model.
- The mathematics behind these models are matrix related but we will focus on the application of them.

The Generalized Linear Model

- The generalized linear model refers to a whole family of models.
- They became popular with a book by McCullagh and Nelder (1982).
- They have 3 basic components.

Components of any GLM

1 **The Random Component** - probability distribution of the response variable. In linear regression this is the normal distribution.

2 **The Systematic Component** - fixed structure of explanatory variables usually a linear function. We have seen this as $\beta_0 + \beta_1 X_1 + \ldots$.

3 **The Link Function** - maps the systematic component onto the random component. This was $E(Y_i|X_{1i}, \ldots)$ in the linear regression case.

The Random Component

- Observations of the outcome represent a sample from a random variable.
- This random variable has a mean value and variation that depends on the distribution it follows.
- GLM uses random variables that follow an exponential family distribution.

The Systematic Component

- We use the covariates or independent variables to model to estimate the means of the random variable that our sample was drawn from.
- This is added to the variation to give use the data that we observed.

The Model

- We use

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} = \sum_{j=1}^{p} \beta_j x_{ij}$$

- where:
  - η is the linear predictor.
  - $x_1, \ldots, x_p$ are the explanatory variables.
  - $\beta_1, \ldots, \beta_p$ are the coefficients of the explanatory variables.
  - β_0 is the value of η when all the x's are 0.

What you typically will see:

- Most of the time this is written as:

$$\eta = \mathbf{X}\beta$$

- where:
  - $\eta = (\eta_1, \ldots, \eta_N)^T$ is a column vector.
  - $\beta = (\beta_0, \ldots, \beta_p)^T$ is a column vector.
  - $\mathbf{X}$ is a $N \times p$ matrix of the explanatory variables x_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, p$.

Visualizing the Matrices

- In other words:

$$\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & \dots & x_{Np} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} = \mathbf{X}\beta$$

- This linear predictor allows the least squares regression approach to be generalized to a wide range of models.

The Link Function

- We cannot always model a direct relationship between the random and the systematic component.
- This is where the link function comes into place.
- This function allows us to specify a relationship between the linear(systematic component) and the random component.
- We essentially link η_i to $\mu_i = E(y_i)$.

The Link Function

- We have

$$g(\mu_i) = \eta_i$$

- where:
    - $g()$ is the link function.
    - μ_i represents the expected value of the random component.
    - η_i represents the linear(structural) component.

What is this link Function?

- The link function is specifically defined by how the distribution is identified as an exponential family.
- We will not go through this math however feel free to look up exponential families and try and put the distributions we talk about into this framework.

Common Link Functions:

- Some common link functions are

RANDOM COMPONENT	LINK FUNCTION	OUTCOME EX VARIABLE	PLANATORY VARIABLE	MODEL
Normal	Identity	Continuous F	actor A	NOVA
Normal	Identity	Continuous C	ontinuous R	egression
Binomial	Logit	Binary	Mixed	Logistic Regression
Multinomial	Generalized logit	Binary	Mixed	Multinomial Regression
Poisson	Log	Count	Mixed	Poisson Regression

What Does this Mean?

- The chart shows just some of the many types of models we can learn to do just from a simple concept of GLMs.
- Essentially every type of technique you have used up until this point can be structured in such a way that is represents a GLM.

Assumptions of a GLM

- The data $Y_1, Y_2, \ldots, Y_2$ are independently distributed.
- The dependent variable Y_i is from an exponential family.
  - Normal (Gaussian)
  - Bernoulli
  - Binomial
  - Multinomial
  - Exponential
  - Poisson

Assumptions of a GLM

- Linear Relationship between link function and systematic component.
- Errors are independent.
- Uses Maximum Likelihood Estimation rather than Least Squares Estimation.
- For goodness-of-fit tests need large sample sizes.

What Assumptions are not needed?

- We do **NOT** some assumptions we needed before.
    - We do **NOT** need a linear relationship between the dependent variable and the independent variables.
    - We do **NOT** need need normally distributed errors.
    - We do **NOT** need homogeneity of errors.

The Case of Linear Regression

Linear Regression as a Case

- We have previously been using linear regression and it can be easily display how we use it in this framework.
- For example in a multiple linear regression we have

$$y_i|x_{i1}, \ldots, x_{ip} = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i$$

- Then we know that

$$\mu_i = E(y_i y_i|x_{i1}, \ldots, x_{ip}) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}$$

- Thus we can directly relate μ_i to the systematic component.

What Link in the Linear Case?

- Thus in this case our function $g()$ is

$$g(\mu_i) = \mu_i$$

- We call this the identity link.

What do we have?

- Then we have that
  - **Random Component**: y is the outcome and is normally distributed. So we let $\epsilon_i \sim N(0\sigma^2)$.
  - **Systematic Component**: $x_1, \ldots, x_p$ are the explanatory variables. They can be categorical or continuous. We have a linear combination of these terms but we can still have x^2 or $\log(x)$ terms in here as well.
  - **Link Function**:

Identity Link

- We have the identity function:

$$\begin{aligned} \eta &= \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} \\ g(E(y_i)) &= \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} \\ g(E(y_i)) &= E(y_i) \end{aligned}$$

- With linear regression we have the simplest link function because we are able to model the mean directly.

The Case of Logistic Regression

Logistic Regression

- We will now move onto logistic regression.
- With logistic regression we are concerned with binary data.
- This is data that is in a format of either yes or no, 0 or 1, or some variation of that.

Binomial Distribution

- If we consider binary data we find that what we have is called the Binomial distribution.
- Let's assume that we have Y where

$$Y = \begin{cases} 1 & \text{if sucess} \\ 0 & \text{if failure} \end{cases}$$

What Does this mean?

- Then

$$\Pr(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}$$

- where p is the probability that $Y = 1$.
- This leads us to

$$E(Y) = np$$

$$Var(Y) = np(1 - p)$$

Regression Model for Logistic

- Recall from simple linear regression that our systematic part of our model is

$$E(Y_i|x_i) = \beta_0 + \beta_1 x_i$$

- That would mean with this type of data we have

$$p_i = \beta_0 + \beta_1 x_i$$

Why Can't we do Linear Regresion?

- The issue with this is now we can have values that fall outside of 0 and 1.
- To overcome the problem with negative values we could exponeniate:

$$p_i = \exp(\beta_0 + \beta_1 x_1)$$

- We now have values that can fall between 0 and infinity.

What about Values greater than 1?

- In order to solve the problem of values being greater than 1, we divide by 1 plus the exponential:

$$p_i = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x_1)}$$

- This new function now lies completely between 0 and 1 as needed.
- Then we solve back to where we have the systematic part.

The Systematic Part

$$\begin{aligned} p_i &= \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \\ p_i\,(1 + \exp(\beta_0 + \beta_1 x_i)) &= \exp(\beta_0 + \beta_1 x_i) \\ p_i &= \exp(\beta_0 + \beta_1 x_i)\,(1 - p_i) \\ \log\left(\frac{p_i}{1 - p_i}\right) &= \beta_0 + \beta_1 x_i \\ logit\,(p_i) &= \beta_0 + \beta_1 x_i \end{aligned}$$

What does this mean?

- This means we are fitting a linear regression to the logistic unit (logit) or the log odds of the probability of a success.
- This is why we refer to this as logistic regression.

The Logit

Then if we consider the logit:

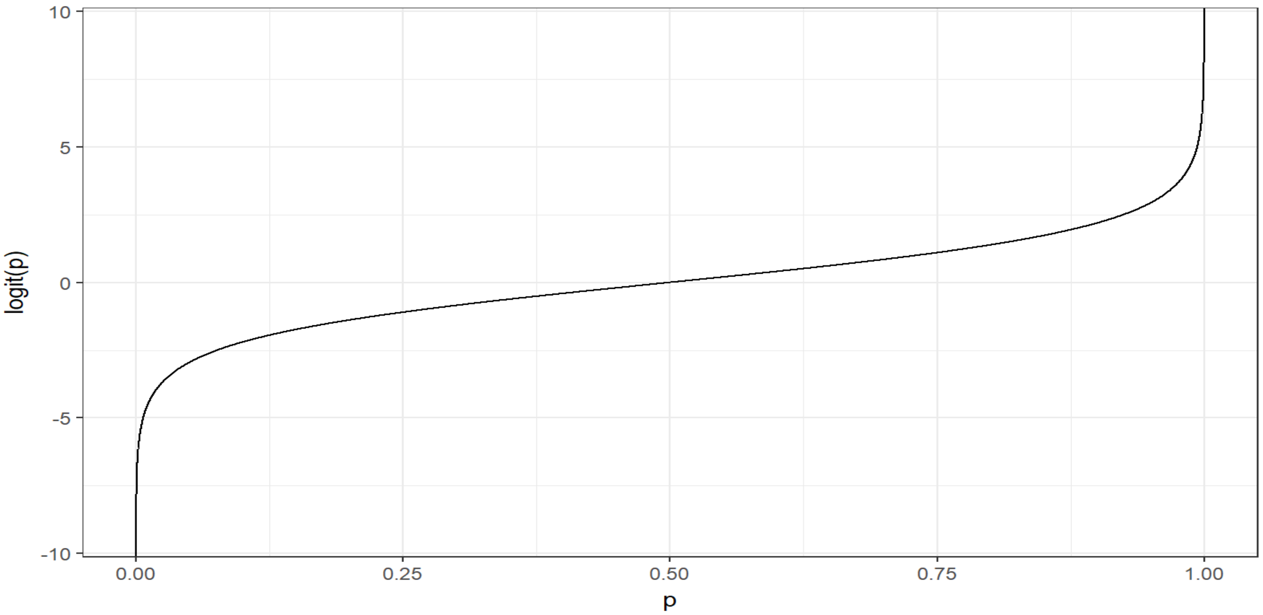
$$\begin{gathered}
\text{If } p = 0 \text{then } \log\left(\frac{p}{1-p}\right) = -\infty \\
\text{If } p = \tfrac{1}{2} \text{then } \log\left(\frac{p}{1-p}\right) = 0 \\
\text{If } p = 1 \text{then } \log\left(\frac{p}{1-p}\right) = \infty
\end{gathered}$$

What does the Logit imply?

- We can see that as p increases the logit does as well.
- We have that the logit can be anything between $-\infty$ and ∞, but p is between 0 and 1 as needed.

Relationship Between p and the logit

- We can see the relationship between p and the logit below.



Logistic as a GLM

- From the above work we can see that with logistic regression we have

$$logit\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_1$$

$$\text{or}$$

$$logit\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

Logistic as a GLM

- Where $E(y_i|y_{i1},\dots,x_{ip}) = p_i$ therefore what we have is
  - **Random Component**: y is the outcome and is binomial and we assume the variance to be that of a binomial.
  - **Systematic Component**: $x_1,\dots,x_p$ are the explanatory variables. They can be categorical or continuous.
  - We have a linear combination of these terms but we can still have x^2 or $\log(x)$ terms in here as well.

The Link Funcion:

- Where $E(y_i|y_{i1},\ldots,x_{ip}) = p_i$ therefore what we have is
    - **Link Function**: We can see from above that with p_i being the mean that we have the logit as the link function:

$$
\begin{aligned}
\eta &= \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} \\
g(E(y_i)) &= \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} \\
g(p_i) &= logit\,(p_i)
\end{aligned}
$$

Maximum Likelihood Estimation

- In linear regression we learned about least squares estimation.
- This falls apart with logistic regression when we have $p = 0$ or $p = 1$.
- Due to this we prefer a technique that can accurately estimate p no matter what.
- We will map out what this looks like right now.

Our Data

- With our data we have

$$\Pr(Y_i = 1|x_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

- Then we also have that

$$\begin{aligned}\Pr(Y_i = 0|x_i) &= 1 - \Pr(Y_i = 1|x_i) \\ &= 1 - \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \\ &= \frac{1}{1 + \exp(\beta_0 + \beta_1 x_i)}\end{aligned}$$

Our Data

- If we combine these together we find that:

$$\Pr(Y_i = y_i | x_i) = \frac{\exp((\beta_0 + \beta_1 x_i) \cdot y_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}, \quad y_i = 0, 1$$

The Likelihood

- The likelihood is defined as the probability of obtaining the data that was observed.

$$\Pr(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n | x_1, x_2, \ldots, x_n)$$

- Then we assumed that in our data the responses are independent from one another.
- This leads to

$$\Pr(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n | x_1, x_2, \ldots, x_n) = \Pr(Y_1 = y_1 | x_1) \cdots \Pr(Y_n = y_n | x_n)$$

The Likelihood

- Then the probability we obtain our data is

$$L = \prod_{i=1}^{n} \left[\frac{\exp((\beta_0 + \beta_1 x_i) \cdot y_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \right]$$

Maximum Likelihood

- Maximum likelihood estimates for β_0 and β_1 are found by searching for which values $\hat{\beta}_0$ and $\hat{\beta}_1$ maximize L.
- Unlike in least squares we cannot find these solutions in a closed form.
- We calculate MLEs with some sort of iterative technique.

Normal Distribution and Maximum Likelihood

- It can be shown that maximum likelihood estimators are normally distributed.
- This means in our data

$$\hat{\beta}_0 \overset{approx}{\sim} N\left(\beta_0, \widehat{Var}\left(\hat{\beta}_0\right)\right)$$

$$\hat{\beta}_1 \overset{approx}{\sim} N\left(\beta_1, \widehat{Var}\left(\hat{\beta}_1\right)\right)$$

Why do we use MLE?

- Finally we have that MLEs are the most efficient estimators out there.
- Meaning that any other consistent estimators $\tilde{\beta}_0$ and $\tilde{\beta}_1$ will have larger variances then $\hat{\beta}_0$ and $\hat{\beta}_1$.
- This means we will have the tightest confidence intervals around our MLEs and possibly show significance when other estimators would fail to.