Exponent and Logarithm

沈威宇

January 16, 2025

Contents

1 Exponent (指數) and Logarithm (對數)

I Euler's number (歐拉數/尤拉數)

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

II Natural logarithm (自然對數)

$\ln(x) = \log_e(x)$, which is called the natural logarithm of x and in advanced math also denoted as $\log(x)$, is defined to be:

$$\ln(x) := \begin{cases} \int_1^x \frac{1}{t}\,\mathrm{d}t, & x \in \mathbb{R}_{>0}, \\ \ln(|x|) + i\arg(x), & x \in \mathbb{C}_{\neq 0} \end{cases} \quad \in \mathbb{C},$$

where

$$\arg(x) := \operatorname{atan2}\left(\frac{\Im(x)}{|x|}, \frac{\Re(x)}{|x|}\right) \quad \in (-\pi, \pi]$$

is the principal value of the argument (輻角主值) of x.

III Logarithm (對數)

$$\log_a(z) := \frac{\ln(z)}{\ln(a)}, \quad a, z \in \mathbb{C}_{\neq 0} \wedge a \neq 1, \quad \in \mathbb{C},$$

where a is called the base (底數) and z is called the argument (真數).

IV Exponent (指數)

$$a^n := \begin{cases} x \text{ s.t. } n\ln(a) = \ln(x), & a, x \in \mathbb{C}_{\neq 0} \wedge n \in \mathbb{C}, \\ 0, & a = 0 \wedge n \in \mathbb{R}_{>0}. \end{cases} \quad \in \mathbb{C}$$

where a is called the base (底數) and n is called the exponent (指數). This implies,

$$\begin{aligned} a^n &= \prod_{i=1}^n a, \quad n \in \mathbb{N}, \\ e^n &= e^{\Re(n)} \cdot (\cos\Im(n) + i\sin\Im(n)), \quad n \in \mathbb{C}, \\ a^n &= |a|^n e^{ni\arg(a)}, \quad a, n \in \mathbb{C} \wedge a \neq 0. \end{aligned}$$

V Exponential law (指數律)

For all defined exponential terms:

$$\begin{aligned} a^r \cdot a^s &= a^{r+s}, \\ (a^r)^s &= a^{rs}, \\ (a \cdot b)^r &= a^r \cdot b^r. \end{aligned}$$

VI Logarithmic law (對數律)

For all defined logarithmic terms:

$$\begin{aligned}
&\log_a(r) + \log_a(s) = \log_a(rs)\\
&\log_a(r) - \log_a(s) = \log_a\left(\frac{r}{s}\right)\\
&\log_{a^m}(r^n) = \frac{n}{m}\log_a(r)\\
&\log_a(b) = \frac{\log_c(b)}{\log_c a}\\
&a^{\log_c(b)} = b^{\log_c(a)}
\end{aligned}$$

VII Common logarithm (常用對數)

$\log_{10}(b)$ is called common logarithm, in basic math also denoted as $\log(b)$

VIII Taylor expansion of natural exponential function

The Taylor expansion of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

IX Taylor expansion of natural logarithmic function

The Taylor expansion of $\ln(1 + x)$ at $x = 0$ is

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, \quad |x| < 1.$$

X Root (根號)

$$\sqrt[y]{w} := \begin{cases} w^{\frac{1}{y}}, & w \in \mathbb{R}_{>0} \wedge y \in \mathbb{N},\\ -\sqrt[y]{-w}, & w \in \mathbb{R}_{<0} \wedge \dfrac{y+1}{2} \in \mathbb{N},\\ 0, & w = 0 \wedge y \in \mathbb{N}. \end{cases}$$

XI Exponential function (指數函數)

$f(x) = a^x$ where $a \neq 0$ is called an exponential function with base a, of which the domain is $\mathbb{R}$ and the range is $\mathbb{R}_{>0}$.

XII Scientific notation (科學記號)

Scientific notation refers to the representation of a real number in the form of $a \times 10^n$, where $1 \leq |a| < 10 \wedge n \in \mathbb{Z}$.

XIII Characteristic (首數) and mantissa (尾數)

The integer part of a decimal is called the characteristic, and the remaining part is called the mantissa.

XIV Exponential growth (指數成長)

Exponential growth usually refers to a functional that satisfies $f'(x) = kf(x)$ where $k \in \mathbb{R}$, that is, $f(x) = e^{kx} + c$ where $k, c \in \mathbb{R}$.

XV 重要值

$$\log 2 \approx 0.3010, \quad \log 3 \approx 0.4771, \quad \log 7 \approx 0.8451, \quad \log 11 \approx 1.0414, \quad \ln 10 \approx 2.3026, \quad \log \mathrm{e} \approx 0.4343$$

$$\sqrt{2} \approx 1.4142, \quad \sqrt{3} \approx 1.7321, \quad \sqrt{5} \approx 2.2361, \quad \sqrt{6} \approx 2.4495, \quad \sqrt{7} \approx 2.6458, \quad \sqrt{8} \approx 2.8284$$

$$\sqrt{10} \approx 3.1623, \quad \frac{\sqrt{5}+1}{2} \approx 1.6180, \quad \frac{\sqrt{5}-1}{2} \approx 0.6180$$