Polynomial Interpolation

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Polynomial interpolation is a method of constructing a polynomial that passes through a given set of points. Given $n+1$ data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, the goal is to find a polynomial $p(x)$ of degree at most n such that:

$$p(x_i) = y_i \quad \text{for} \quad i = 0, 1, \dots, n.$$

I Newton's Polynomial

Newton's polynomial interpolation uses the concept of divided differences to construct the polynomial in a recursive form. The Newton interpolating polynomial can be written as:

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where the coefficients $a_0, a_1, \dots, a_n$ are obtained using divided differences. The divided differences are recursively computed as follows:

$$f[x_i] = y_i$$
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$
$$f[x_i, x_{i+1}, \dots, x_k] = \frac{f[x_{i+1}, \dots, x_k] - f[x_i, \dots, x_{k-1}]}{x_k - x_i}$$

The Newton polynomial can be succinctly written as:

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

II Lagrange Polynomial

Lagrange polynomial interpolation expresses the polynomial as a linear combination of basis polynomials. The Lagrange form of the interpolating polynomial is given by:

$$p(x) = \sum_{i=0}^{n} y_i L_i(x)$$

where $L_i(x)$ are the Lagrange basis polynomials, defined as:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

Each $L_i(x)$ is a polynomial that is 1 at $x = x_i$ and 0 at all other x_j $(j \ne i)$.

Barycentric Form of Lagrange Interpolation:

The Barycentric form is a more efficient and numerically stable way to compute the Lagrange interpolation polynomial. The Barycentric form of the interpolating polynomial is:

$$p(x) = \frac{\sum_{i=0}^{n} \frac{w_i y_i}{x - x_i}}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}}$$

where w_i are the barycentric weights, defined as:

$$w_i = \frac{1}{\prod_{\substack{0 \le j \le n \\ j \ne i}} (x_i - x_j)}$$