Function

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Contents

1 Function (函數)

I Definition and notation

A function is formed by three sets, the domain (定義域) X, the codomain (對應域) Y, and the graph R that satisfy the three following conditions:

$$R \subseteq \{(x, y) \mid x \in X, y \in Y\}$$

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

A function f that satisfies the above is denoted as:

$$f : X \to Y.$$

in which the domain X is also denoted as D_f.

The range (值域), denoted as R_f or $f(X)$, is defined as:

$$\{y \mid \exists x \in X, (x, y) \in R\}.$$

If $x \in X$ and $(x, y) \in R$, we write $y = f(x)$, in which $f(x)$ is called the image of x under f, x is called the independent variable (自變數/獨立變數), and y is called the dependent variable (應變數/依賴變數); and the function f is also denoted as:

$$f : X \to Y; x \mapsto y.$$

II Properties

Consider function f:

$$f : X \to Y; x \mapsto y.$$

i Injection (單射)/Injective function/One-to-one (一對一) function

$$\forall a, b \in X \text{ s.t. } f(a) = f(b) : a = b$$

III Many-to-one (多對一) function

$$\exists a \neq b \in X : f(a) = f(b)$$

i Surjection (滿射/蓋射)/Surjective function/Onto function

$$f(X) = Y$$

ii Bijection (對射)/Bijective function/One-to-one (一對一) function/One-to-one correspondence (一一對應)

Injective and surjective function.

iii (Monotone) Increasing (（單調）遞增)/Non-Decreasing (非遞減) function

f is increasing on $I \subseteq X$ if and only if

$$\forall a, b \in I : a < b \implies f(a) \leq f(b).$$

f is increasing if and only if

$$\forall a, b \in X : a < b \implies f(a) \leq f(b).$$

iv Strictly increasing (嚴格遞增) function

f is strictly increasing on $I \subseteq X$ if and only if

$$\forall a, b \in I : a < b \implies f(a) < f(b).$$

f is strictly increasing if and only if

$$\forall a, b \in X : a < b \implies f(a) < f(b).$$

v (Monotone) Decreasing (（單調）遞減)/Non-Increasing (非遞增) function

f is decreasing on $I \subseteq X$ if and only if

$$\forall a, b \in I : a < b \implies f(a) \geq f(b).$$

f is decreasing if and only if

$$\forall a, b \in X : a < b \implies f(a) \geq f(b).$$

vi Strictly decreasing (嚴格遞減) function

f is strictly decreasing on $I \subseteq X$ if and only if

$$\forall a, b \in I : a < b \implies f(a) > f(b).$$

f is strictly decreasing if and only if

$$\forall a, b \in X : a < b \implies f(a) > f(b).$$

vii Monotone (單調) function

f is monotone on $I \subseteq X$ if and only if it is either monotone increasing or monotone decreasing on I.

f is monotone if and only if it is either monotone increasing or monotone decreasing.

IV Transformation

i Translation (平移)

For any function $f : \mathbb{R} \to \mathbb{R}$, shifting $y = f(x)$ right by h units and up by k units on the xy coordinate plane yields $y = f(x - h) + k$.

ii Scaling (伸縮/縮放/拉伸)

For any function $f: \mathbb{R} \to \mathbb{R}$, on the xy coordinate plane, expand $y = f(x)$ vertically by a times the original value with the x axis as the reference line, and expand $y = af\left(\frac{x}{b}\right)$ horizontally by b times the original value with the y axis as the reference line, to obtain $y = af\left(\frac{x}{b}\right)$.

V Common ways to define functions

i Function composition (函數合成)

For two functions $f: X \to Y$ and $g: V \to W$ such that $g(V) \subseteq X$, the composition of them, denoted as $(f \circ g)$, is defined as:

$$(f \circ g): V \to Y;\ x \mapsto= f(g(x))$$

ii Inverse function (反函數)

For a bijective function $f: X \to Y$, the inverse of it, denoted as f^{-1}, is defined as:

$$f^{-1}: Y \to X;\ f(x) \mapsto x$$

iii Power notation

For a bijective function $f: X \to X$, f^0 is defined by:

$$f^0: X \to X;\ x \mapsto x,$$

$f^n(x)$ for any $n \in \mathbb{N}$ is defined by:

$$f^n: X \to X;\ x \mapsto f\left(f^{n-1}(x)\right),$$

and $f^{-n}(x)$ for any $n \in \mathbb{N}$ is defined by:

$$f^{-n}: X \to X;\ x \mapsto f^{-1}\left(f^{-n+1}(x)\right).$$

iv Piecewise function (分段函數)

A piecewise function is a function defined in the form:

$$f(x) = \begin{cases} f_1(x), & x \in A_1, \\ f_2(x), & x \in A_2, \\ \vdots \\ f_n(x), & x \in A_n \end{cases},$$

where

$$\bigcup_{i=1}^{n} A_i = D_f \land \forall i \neq j \land i, j \in \mathbb{N} \land i, j \leq n:\ A_i \cap A_j = \varnothing.$$