Exponent and Logarithm

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Contents

1 Exponent (指數) and Logarithm (對數)

I Euler's number (尤拉數)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

II Exponent (指數)

$$a^n := \begin{cases} \textbf{undefined}, & a = 0 \wedge n \in \mathbb{C} \wedge \mathfrak{R}(n) \le 0, \\ 0, & a = 0 \wedge n \in \mathbb{C} \wedge \mathfrak{R}(n) > 0, \\ 1, & a \in \mathbb{C} \wedge a \neq 0 \wedge n = 0, \\ \prod_{i=1}^{n} a, & a \in \mathbb{C} \wedge n \in \mathbb{N}, \\ e^{\mathfrak{R}(n)} \cdot (\cos \mathfrak{I}(n) + i \sin \mathfrak{I}(n)), & a = e \wedge n \in \mathbb{C}, \\ |a|^n e^{in \arg(n)}, & a, n \in \mathbb{C} \wedge \neg(a = 0 \wedge \mathfrak{R}(n) \le 0), \end{cases}$$

where

$$\arg(a) = \operatorname{atan2}\left(\frac{\mathfrak{I}(a)}{|a|}, \frac{\mathfrak{R}(a)}{|a|}\right)$$

is the principal value of the argument (輻角主值) of a,
and a is called the base (底數) and n is called the exponent (指數).

III Logarithm (對數)

$$\log_a(z) := x \text{ s.t. } a^x = z, \quad a \in \mathbb{R}_{>0} \wedge a \neq 1 \wedge z \in \mathbb{C}_{\neq 0},$$

that is,

$$\log_a(z) := \begin{cases} \ln(\|z\|) + i \arg(z), & a = e \wedge z \in \mathbb{C}_{\neq 0} \\ \dfrac{\ln(z)}{\ln(a)}, & a \in \mathbb{R}_{>0} \wedge a \neq 1 \wedge z \in \mathbb{C}_{\neq 0}, \end{cases}$$

where a is called the base (底數) and b is called the argument (真數).

IV Exponential law (指數律)

For all defined exponential terms:

$$a^r \cdot a^s = a^{r+s},$$
$$(a^r)^s = a^{rs},$$
$$(a \cdot b)^r = a^r \cdot b^r.$$

V Logarithmic law (對數律)

For all defined logarithmic terms:

$$\begin{aligned}
\log_a(r) + \log_a(s) &= \log_a(rs) \\
\log_a(r) - \log_a(s) &= \log_a\left(\frac{r}{s}\right) \\
\log_{a^m}(r^n) &= \frac{n}{m}\log_a(r) \\
\log_a(b) &= \frac{\log_c(b)}{\log_c a}
\end{aligned}$$

VI Common logarithm (常用對數)

$\log_{10}(b)$ is called common logarithm.
In basic math:

$$\log(b) = \log_{10}(b).$$

VII Natural logarithm (自然對數)

$\log_e(b)$ is called natural logarithm.

$$\ln(b) = \log_e(b).$$

In advanced math:

$$\log(b) = \log_e(b).$$

VIII Taylor expansion of natural exponential function

The Taylor expansion of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

IX Taylor expansion of natural logarithmic function

The Taylor expansion of $\ln(1+x)$ at $x = 0$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}, \quad |x| < 1.$$

X Root (根號)

$$\sqrt[y]{w} := \begin{cases} w^{\frac{1}{y}}, & w \in \mathbb{R}_{>0} \wedge y \in \mathbb{N}, \\ -\sqrt[y]{-w}, & w \in \mathbb{R}_{<0} \wedge \dfrac{y+1}{2} \in \mathbb{N}, \\ 0, & w = 0 \wedge y \in \mathbb{N}. \end{cases}$$

XI Exponential function (指數函數)

$f(x) = a^x$ where $a \neq 0$ is called an exponential function with base a, of which domain is $\mathbb{R}$ and range is $\mathbb{R}_{>0}$.

XII Scientific notation (科學記號)

Scientific notation refers to the representation of a real number in the form of $a \times 10^n$, where $1 \leq a < 10 \land n \in \mathbb{Z}$.

XIII Characteristic and mantissa (首數與尾數)

The integer part of a decimal is called the characteristic, and the remaining part is called the mantissa.

XIV Exponential growth (指數成長)

Exponential growth usually refers to a functional that satisfies $f'(x) = kf(x)$, where $k \in \mathbb{R}$, that is, $f(x) = e^{kx} + c$, where $k, c \in \mathbb{R}$.