

Polynomial Interpolation

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1 Polynomial Interpolation

Polynomial interpolation is a method of constructing a polynomial that passes through a given set of points. Given $n + 1$ data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, the goal is to find a polynomial $p(x)$ of degree at most n such that:

$$p(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n.$$

I Newton's Polynomial

Newton's polynomial interpolation uses the concept of divided differences to construct the polynomial in a recursive form. The Newton interpolating polynomial can be written as:

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

where the coefficients a_0, a_1, \dots, a_n are obtained using divided differences. The divided differences are recursively computed as follows:

$$\begin{aligned} f[x_i] &= y_i \\ f[x_i, x_{i+1}] &= \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \\ f[x_i, x_{i+1}, \dots, x_k] &= \frac{f[x_{i+1}, \dots, x_k] - f[x_i, \dots, x_{k-1}]}{x_k - x_i} \end{aligned}$$

The Newton polynomial can be succinctly written as:

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

II Lagrange Polynomial

Lagrange polynomial interpolation expresses the polynomial as a linear combination of basis polynomials. The Lagrange form of the interpolating polynomial is given by:

$$p(x) = \sum_{i=0}^n y_i L_i(x)$$

where $L_i(x)$ are the Lagrange basis polynomials, defined as:

$$L_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}$$

Each $L_i(x)$ is a polynomial that is 1 at $x = x_i$ and 0 at all other x_j ($j \neq i$).

Barycentric Form of Lagrange Interpolation:

The Barycentric form is a more efficient and numerically stable way to compute the Lagrange interpolation polynomial. The Barycentric form of the interpolating polynomial is:

$$p(x) = \frac{\sum_{i=0}^n \frac{w_i y_i}{x - x_i}}{\sum_{i=0}^n \frac{w_i}{x - x_i}}$$

where w_i are the barycentric weights, defined as:

$$w_i = \frac{1}{\prod_{\substack{0 \leq j \leq n \\ j \neq i}} (x_i - x_j)}$$