Logarithm and Exponent

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Contents

1 Logarithm (對數) and Exponent (指數)

I Definition

In the following definitions, $\arg\left(x \in \mathbb{C}_{\neq 0}\right)$ represents the argument (輻角) of x. If we replace $\arg(x)$ with the principal argument (輻角主值) of x, we get the definition of the principal branch (主分支) of them.

i Positive integer exponent (正整數指數)

The exponent of $a \in \mathbb{R}$ to the power of $n \in \mathbb{N}$, denoted as a^n, is defined as

$$a^n := \prod_{k=1}^{n} a.$$

ii Euler's number (歐拉數/尤拉數)

The Euler's number, denoted as e, is defined as

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

iii Natural logarithm (自然對數)

The natural logarithm of x, denoted as $\ln(x)$ or $\log_e(x)$, is defined as:

$$\ln(x) := \begin{cases} \displaystyle\int_1^x \frac{1}{t}\,\mathrm{d}t, & x \in \mathbb{R}_{>0}, \\ \ln|x| + i\arg(x), & x \in \mathbb{C}_{\neq 0}, \end{cases}$$

iv Logarithm (對數)

The logarithm of x to the base a, denoted as $\log_a(x)$, is defined as

$$\log_a(x) := \frac{\ln(x)}{\ln(a)}, \quad a, x \in \mathbb{C}_{\neq 0} \wedge a \neq 1,$$

where a is called the base (底數) and x is called the argument (真數).

v Common logarithm (常用對數)

$\log_{10}(x)$ is called the common logarithm of x, also denoted as $\log(x)$.

vi Exponent (指數)

The exponent of a to the exponent or power of n, denoted as a^n, is defined as

$$a^n := \begin{cases} e^{n\ln(a)}, & a \in \mathbb{C}_{\neq 0} \wedge n \in \mathbb{C}, \\ 0, & a = 0 \wedge n \in \mathbb{R}_{>0}, \end{cases}$$

where a is called the base (底數) and n is called the exponent (指數) or power (冪次).

vii Root (根號)

The yth root (y 次方根) of $w \in \mathbb{C}$, denoted as $\sqrt[y]{w}$ is defined as the principal branch of $w^{\frac{1}{y}}$, where $\sqrt{}$ is called the radical symbol, radical sign, root symbol, or surd (根號).

viii Scientific notation (科學記號)

Scientific notation refers to the representation of a real number in the form of $a \times 10^n$, where $1 \leq |a| < 10 \wedge n \in \mathbb{Z}$.

ix Characteristic (首數) and mantissa (尾數)

Given a common logarithm $\log(x)$, the characteristic of it is defined as $\lfloor \log(x) \rfloor$, and the mantissa of it is defined as $\log(x) - \lfloor \log(x) \rfloor$.

x Logarithmic function (對數函數)

$f(x) = k \log_a(x)$ where $a > 0 \wedge a \neq 1 \wedge k \in \mathbb{R}_{\neq 0}$ is called a logarithmic function with base a, of which the domain is $\mathbb{R}_{>0}$ and the range is $\mathbb{R}$.

xi Exponential function (指數函數)

$f(x) = ka^x$ where $a \neq 0 \wedge k \in \mathbb{R}_{\neq 0}$ is called an exponential function with base a, of which the domain is $\mathbb{R}$ and the range is $\mathbb{R}_{>0}$.

xii Natural exponential function (自然指數函數)

e^x is called the natural exponential function.

xiii Exponential growth (指數成長)

Exponential growth usually refers to a functional that satisfies $f'(x) = kf(x)$ where $k \in \mathbb{R}$, that is, $f(x) = e^{kx} + c$ where $k, c \in \mathbb{R}$.

II Laws

i Logarithmic laws (對數律)

$$
\begin{aligned}
&\log_a(r) + \log_a(s) = \log_a(rs) \\
&\log_a(r) - \log_a(s) = \log_a\left(\frac{r}{s}\right) \\
&\log_{a^m}(r^n) = \frac{n}{m}\log_a(r) \\
&\log_a(b) = \frac{\log_c(b)}{\log_c a} \\
&a^{\log_c(b)} = b^{\log_c(a)}
\end{aligned}
$$

ii Exponential laws (指數律)

$$
\begin{aligned}
&a^r \cdot a^s = a^{r+s}, \\
&(a^r)^s = a^{rs}, \\
&(a \cdot b)^r = a^r \cdot b^r
\end{aligned}
$$

iii Derivative function of natural exponential function

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} e^x &= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x \cdot e^{\Delta x} - e^x}{\Delta x} \\
&= e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x \cdot \lim_{\Delta x \to 0} \frac{\lim_{n \to \infty} \sum_{i=0}^{n} \frac{(\Delta x)^i}{i!} - 1}{\Delta x} \\
&= e^x \cdot \lim_{\Delta x \to 0} \frac{\lim_{n \to \infty} \sum_{i=1}^{n} \frac{(\Delta x)^i}{i!}}{\Delta x} \\
&= e^x \cdot \lim_{\Delta x \to 0} \sum_{i=1}^{n} \frac{(\Delta x)^{i-1}}{i!} \\
&= e^x
\end{aligned}
$$

□

iv Derivative function of natural logarithmic function

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$$

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) &= \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\ln\left(\frac{x+\Delta x}{x}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x} \\
&= \lim_{\frac{\Delta x}{x} \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} \cdot \frac{1}{x} \\
&= \frac{1}{x}
\end{aligned}
$$

□

v Taylor expansion of logarithmic function

The Taylor expansion of the logarithmic function $k\log_a(x)$ at $x = b$ is

$$k\log_a(x) = k\log_a(b) + \sum_{n=1}^{\infty} \frac{k(-1)^{n+1}}{n(\ln a)b^n}(x-b)^n, \quad \left|\frac{x-b}{b}\right| < 1.$$

vi Taylor expansion of exponential function

The Taylor expansion of the exponential function ka^x at $x = b$ is

$$ka^x = \sum_{n=0}^{\infty} \frac{ka^b(\ln a)^n}{n!}(x-b)^n.$$

vii Inverse

For any $a > 0 \land a \neq 1 \land b \neq 0$, the logarithmic function $k\log_a\left(\frac{x}{b}\right)$ and the exponential function $ba^{\frac{x}{k}}$ are inverses of each other, that is, they are symmetric about $y = x$.

viii Tangent

In the xy plane, $a = e^{\left(e^{-1}\right)}$ if and only if $y = a^x$ and $y = x$ are tangent if and only if $y = \log_a x$ and $y = x$ are tangent if and only if $y = a^x$ and $y = \log_a x$ are tangent. When $a = e^{\left(e^{-1}\right)}$, $y = a^x$, $y = a^x$, and $y = x$ are tangent at (e, e), and any two of them do not meet at any point other than (e, e).

Proof.

$$\frac{\mathrm{d}a^x}{\mathrm{d}x} = \ln(a)a^x = 1$$

$$x = a^x = \frac{1}{\ln(a)}$$

$$a^{\frac{1}{\ln(a)}} = \frac{1}{\ln(a)}$$

$$\ln(a)\frac{1}{\ln(a)} = 1 = -\ln\left(\ln(a)\right)$$

$$a = e^{\left(e^{-1}\right)}$$

□

ix Self-power function

$$\arg\min(x^x) = \frac{1}{e}, \quad \min(x^x) = \left(e^{-1}\right)^{\left(e^{-1}\right)}$$

III Values

$$\log 2 \approx 0.3010, \quad \log e \approx 0.4343, \quad \log 3 \approx 0.4771, \quad \log \pi \approx 0.4971, \quad \log 4 \approx 0.6021, \quad \log 5 \approx 0.6990,$$

$$\log 6 \approx 0.7782, \quad \log 7 \approx 0.8451, \quad \log 8 \approx 0.9031, \quad \log 9 \approx 0.9542, \quad \log 11 \approx 1.0414, \quad \log 12 \approx 1.0792.$$

$$\ln 2 \approx 0.6931, \quad \ln 3 \approx 1.0986, \quad \ln \pi \approx 1.1447, \quad \ln 4 \approx 1.3863, \quad \ln 5 \approx 1.6094, \quad \ln 6 \approx 1.7918,$$

$$\ln 7 \approx 1.9459, \quad \ln 10 \approx 2.3026, \quad \ln 11 \approx 2.3980, \quad \ln 12 \approx 2.4849, \quad \frac{1}{\ln 2} \approx 1.4427.$$

$$\sqrt{2} \approx 1.4142, \quad \sqrt{e} = 1.6487, \quad \sqrt{3} \approx 1.7321, \quad \sqrt{\pi} \approx 1.7725, \quad \sqrt{5} \approx 2.2361,$$

$$\sqrt{6} \approx 2.4495, \quad \sqrt{7} \approx 2.6458, \quad \sqrt{10} \approx 3.1623, \quad \sqrt{11} \approx 3.3166, \quad \sqrt{13} \approx 3.6056,$$

$$\sqrt{14} \approx 3.7417, \quad \sqrt{15} \approx 3.8730, \quad \sqrt{17} \approx 4.1231, \quad \sqrt{19} \approx 4.3590, \quad \sqrt{21} \approx 4.5826,$$

$$\frac{\sqrt{5}+1}{2} \approx 1.6180, \quad \frac{\sqrt{5}-1}{2} \approx 0.6180, \quad \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659, \quad \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.2588,$$

$$\frac{\sqrt{10+2\sqrt{5}}}{4} \approx 0.9511, \quad \frac{\sqrt{10-2\sqrt{5}}}{4} \approx 0.5878.$$

$$\langle a_n = n^2 \rangle_{n=1}^{20} = 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400.$$

$$\langle a_n = n^2 \rangle_{n=21}^{30} = 441, 484, 529, 576, 625, 676, 729, 784, 841, 900.$$

$$\langle a_n = n^2 \rangle_{n=31}^{40} = 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600.$$

$$\langle a_n = n^{-1} \rangle_{n=1}^{15} = 1, 0.5, 0.\bar{3}, 0.25, 0.2, 0.1\bar{6}, 0.\overline{142857}, 0.125, 0.\bar{1}, 0.1, 0.\overline{09}, 0.08\bar{3}, 0.\overline{076923}, 0.0\overline{714285}, 0.0\bar{6}.$$

$$\frac{1}{\sqrt{2}} \approx 0.7071, \quad \frac{1}{\sqrt{3}} \approx 0.5774, \quad \frac{1}{\sqrt{5}} \approx 0.4772, \quad \frac{1}{2\sqrt{2}} \approx 0.3536, \quad \frac{1}{2\sqrt{3}} \approx 0.2887, \quad \frac{1}{2\sqrt{5}} \approx 0.2236.$$

$$e \approx 2.7183, \quad \frac{1}{e} \approx 0.3680, \quad e^e \approx 15.1543, \quad e^{\left(e^{-1}\right)} \approx 1.4447, \quad \left(e^{-1}\right)^e \approx 0.0660, \quad \left(e^{-1}\right)^{\left(e^{-1}\right)} \approx 0.6922.$$