Definition and Theorem Gallery of Set Theory, Topology, Linear Algebra, Measure Theory, Geometry, and Calculus

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December 9, 2024

Contents

Chapter 1 Definition and Theorem Gallery of Set Theory, Topology, Linear Algebra, Measure Theory, Geometry, and Calculus

1 Set Theory (集合論)

I ZFC axiom system (ZFC 公理系統) and ZF axiom system (ZF 公理系統)

The ZFC axiom system refers to the Zermelo-Fraenkel axiom system (ZF axiom system) plus the axiom of choice (AC). The most commonly used axiom systems in set theory are the ZFC axiom system or the ZF axiom system, denoted by $\vdash_{ZFC}$ and $\vdash_{ZF}$, or both denoted by $\vdash$.

i Axiom of extensionality (外延公理)

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$$

ii Axiom of regularity (正則公理)

$$\forall x\, (x \neq \emptyset \Rightarrow \exists y (y \in x \land y \cap x = \emptyset))$$

iii Axiom of separation (分類公理) or axiom schema of specification (規範公理模式)

Let φ be any formula in the language of ZFC with all free variables among $x, z, w_1, \ldots, w_n$ so that y is not free in φ. Then:

$$\forall z \forall w_1 \forall w_2 \ldots \forall w_n \exists y \forall x [x \in y \Leftrightarrow ((x \in z) \land \varphi(x, w_1, w_2, \ldots, w_n, z))]$$

This axiom can be used to prove the existence of the empty set, denoted as $\emptyset$ or $\emptyset$.
Axiom of empty set (空集公理) :

$$\exists x, \forall y, (y \notin x)$$

The $\emptyset$ is defined as the x above.

iv Pairing axiom (配對公理)

$$\forall x \forall y \exists z ((x \in z) \land (y \in z))$$

v Union axiom (聯集公理)

$$\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x[(x \in Y \wedge Y \in \mathcal{F}) \Rightarrow x \in A]$$

Although this formula doesn't directly assert the existence of $\cup\mathcal{F}$, the set $\cup\mathcal{F}$ can be constructed from A in the above using the axiom schema of specification:

$$\cup\mathcal{F} = \{x \in A \mid \exists Y(x \in Y \wedge Y \in \mathcal{F})\}$$

vi Axiom schema of replacement (替代公理模式)

Let φ be any formula in the language of ZFC with all free variables among $x, z, w_1, \ldots, w_n$ so that B is not free in φ. Then:

$$\forall A \forall w_1 \forall w_2 \ldots \forall w_n [\forall x(x \in A \Rightarrow \exists! y \, \varphi) \Rightarrow \exists B \; \forall x(x \in A \Rightarrow \exists y(y \in B \wedge \varphi))]$$

vii Axiom of infinity (無窮公理)

$$\exists X \left[\exists e(\forall z \, \neg(z \in e) \wedge e \in X) \wedge \forall y(y \in X \Rightarrow S(y) \in X)\right]$$

viii Power set axiom (冪集公理)

$$\forall A \exists P(A) \forall x(x \in P(A) \leftrightarrow x \subseteq A)$$

The $P(A)$ above is called power set (冪集) and is denoted as 2^A.

ix Axiom of choice (AC) (選擇公理) or axiom of well-ordering (良序公理)

$$\forall X \left[\emptyset \notin X \implies \exists f \colon X \to \bigcup_{A \in X} A \quad \forall A \in X \, (f(A) \in A)\right]$$

II Set operation notations

i Cardinality

$|A$ and $n(A)$ denote the cardinality of the set A.

ii Set scalar arithmetic operation

- If $\forall a \in A$, sa is defined, $sA := \{sa : \; a \in A\}$.
- If $\forall a \in A$, $a + v$ is defined, $A + v := \{a + v : \; a \in A\}$.

iii Kernel (核) of a family of sets

The kernel of a family $\mathcal{B} \neq \emptyset$ of sets is defined to be:

$$\ker(\mathcal{B}) := \bigcap_{B \in \mathcal{B}} B.$$

iv Cartesian product (笛卡爾積)

The Cartesian product of two sets A and B, denoted $A \times B$, is defined to be

$$A \times B := \{(a, b) : a \in A \wedge b \in B\}.$$

III Partially ordered set (poset) (偏序集)

A partially ordered set (poset for short) is an ordered pair $P = (X, \leq)$ consisting of a set X (called the ground set of P) and a partial order $\leq$ on X. That is, for all a, b, $c \in X$ it must satisfy:

1. Reflexivity: $a \leq a$, i.e. every element is related to itself.
2. Antisymmetry: $a \leq b \wedge b \leq a \implies a = b$, i.e. no two distinct elements precede each other.
3. Transitivity: $a \leq b \wedge b \leq c \implies a \leq c$.

When the meaning is clear from context and there is no ambiguity about the partial order, the set X itself is sometimes called a poset.

IV Upward closure

Let A be a subset of a poset X, the upward closure of A (denoted as $\uparrow A$) is defined to be:

$$\uparrow A := \{x \in X : \exists a \in A \text{ s.t. } a \leq x\}.$$

2 Topology (拓樸學)

I Topological space (拓樸空間)

A topological space consists of a set X and a topology $\mathcal{T}$ on it, denoted as $(X, \mathcal{T})$. Where the set X is the set of points in the space, and the topology $\mathcal{T}$ is the set of subsets of X that satisfies:

1. $\emptyset$, $X \in \mathcal{T}$
2. Closed under arbitrary unions: $\forall \mathcal{S} \subseteq \mathcal{T} : \bigcup_{A \in \mathcal{S}} A \in \mathcal{T}$
3. Closed under finite intersection: $\forall \mathcal{S} \subseteq \mathcal{T}$ s.t. $|\mathcal{S}| < \infty : \bigcap_{A \in \mathcal{S}} A \in \mathcal{T}$

II Homeomorphism (同胚) and isomorphism (同構)

Topological spaces X, $\mathcal{T}_X$ and Y, $\mathcal{T}_Y$ are called homeomorphic if there exists a mapping $f : X \to Y$ between them such that f is bijective and continuous and f^{-1} is continuous, written as $X \cong Y$, and f is called a homeomorphism between them.
In linear algebra, when X and Y are homeomorphic vector spaces and f is a linear map, X and Y are also called isomorphic, and a homeomorphism between them is also called an isomorphism.

III Open set (開集)

In a topological space $(X, \mathcal{T})$, an open subset $O \subseteq X$ is defined to be

$$O \in \mathcal{T}.$$

IV Open neighborhood (開鄰域)

In a topological space X, a open neighborhood of a point $P \in X$ is any open subset $O \subseteq X$ such that $P \in O$.

V Neighborhood (鄰域)

In a topological space, a subset U is a neighborhood of a point P if and only if there exists an open set O such that $P \in O \subseteq U$.

VI Limit point or cluster point or accumulation point (極限點、集積點)

Let S be a subset of a topological space X. A point x in X is a limit point, cluster point, or accumulation point of the set S if every neighborhood of x contains at least one point of S different from x itself.

VII Closure (閉包)

The closure of a subset S of points in a topological space consists of all points in S together with all limit points of S.

VIII Closed set (閉集)

A subset A of a topological space $(X, \mathcal{T})$ is called closed if its complement $X \quad A \in \mathcal{T}$.

IX Borel set (博雷爾集)

A Borel set B is any set in a topological space $(X, \mathcal{T})$ that can be formed from open sets through the operations of countable union, countable intersection, and relative complement, that is:

$$\begin{aligned} B \in & \{\bigcup_{O \in \mathcal{S}} O : \mathcal{S} \subseteq \mathcal{T}, |\mathcal{S}| < \infty\} \\ & \cup \{\bigcap_{O \in \mathcal{S}} O : \mathcal{S} \subseteq \mathcal{T}, |\mathcal{S}| < \infty\} \\ \cup \{O \quad P : O, P \in \mathcal{T}\} & \end{aligned}$$

X Filter

A filter on a set X is a family $\mathcal{B}$ of subsets of X such that:

1. $X \in \mathcal{B}$.

2. $\emptyset \in \mathcal{B}$.

3. $A \in \mathcal{B} \wedge B \in \mathcal{B} \implies A \cap B \in \mathcal{B}$.

4. $A \subseteq B \subseteq X \wedge A \in \mathcal{A} \in \mathcal{B} \implies B \in \mathcal{B}$.

XI Base or basis (基)

Given a topological space $(X, \mathcal{T})$, a base (or basis) for the topology $\mathcal{T}$ (also called a base for X if the topology is understood) is a family $\mathcal{B} \subseteq \mathcal{T}$ of open sets such that every open set of the topology can be represented as the union of some subfamily of $\mathcal{B}$.
The topology generated by a base $\mathcal{B}$, generally denoted by $\tau(\mathcal{B})$ can be defined to be follows: A subset $O \subseteq X$ is to be declared as open, if for all $x \in O$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq O$.

XII Prefilter or filter base

$\mathcal{B}$ is called a prefilter if its upward closure $\uparrow \mathcal{B}$ is a filter.

XIII Connected space (連通空間)

A topological space is said to be disconnected if it can be expressed as the union of two disjoint non-empty open sets. Otherwise, it is said to be connected.

XIV Connected set

A set $C \subseteq X$ of a topological space (X, τ) are said to be disconnected if

$$\exists A\, B \in \tau \text{ s.t. } A \cap B = \emptyset \wedge C \cap A \neq \emptyset \wedge C \cap B \neq \emptyset \wedge C \subseteq A \cup B.$$

Otherwise, it is said to be connected.

XV Compact space (緊緻空間)

A topological space X is called compact if every open cover of X has a finite subcover. That is, X is compact if for every collection C of open subsets of X such that

$$X = \bigcup_{S \in C} S,$$

there is a finite subcollection $F \subseteq C$ such that

$$X = \bigcup_{S \in F} S.$$

XVI Hausdorff space, separated space or T2 space (郝斯多夫空間、分離空間或 T2 空間)

Points x and y in a topological space X can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjointed, i.e., $U \cap V = \emptyset$.
X is a Hausdorff space if any two distinct points in X are separated by neighborhoods. This condition is the third separation axiom (after T0 and T1), which is why Hausdorff spaces are also called T2 spaces. The name separated space is also used.

XVII Metric space (度量空間或賦距空間)

Metric space is an ordered pair (M, d) where M is a set and d is a metric on M, i.e., a function $d: M \times M \to \mathbb{R}$ satisfying the following axioms for all points $x, y, z \in M$:

1. The distance from a point to itself is zero: $d(x, x) = 0$.
2. (Positivity) The distance between two distinct points is always positive:$x \neq y \implies d(x, y) > 0$.
3. (Symmetry) The distance from x to y is always the same as the distance from y to x: $d(x, y) = d(y, x)$.
4. The triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

XVIII Open ball (開球)

In a metric space (X, d), given a point a and radius r, the open ball $B(a)_{<r}$ is defined to be:

$$B(a)_{<r} := \{p \in X : d(a, p) < r\}.$$

XIX Closed ball (閉球)

In a metric space (X, d), given a point a and radius r, the closed ball $B(a)_{\leq r}$ is defined to be:

$$B(a)_{\leq r} := \{p \in X : d(a, p) \leq r\}.$$

XX Topological field (拓樸域)

A topological field is a topological space, such that addition, multiplication, the maps $a \mapsto -a$, and $a \mapsto a^{-1}$ are continuous maps with respect to the topology of the space.

XXI Ordered field (有序域)

A field $(F, +, \cdot)$ together with a total order $\leq$ on F is an ordered field if the order satisfies the following properties for all $a, b, c \in F$:

1. $a \leq b \implies a + c \leq b + c$.
2. $0 \leq a \land 0 \leq b \implies 0 \leq a \cdot b$.

As usual, we write $a < b$ for $a \leq b$ and $a \neq b$. The notations $b \geq a$ and $b > a$ stand for $a \leq b$ and $a < b$, respectively. Elements $a \in F$ with $a > 0$ are called positive.

3 Linear Algebra (線性代數)

I Norm (範數) and seminorm (半範數)

Given a vector space X over a ordered field F, a seminorm on X is a real-valued function $p: X \to \mathbb{R}$ with the following conditions, where $|s|$ denotes the usual absolute value of a scalar s:

1. Subadditivity/Triangle inequality:

$$\forall x,\, y \in X:\, p(x+y) \leq p(x) + p(y).$$

2. Absolute homogeneity:

$$\forall s \in \mathbb{R},\, x \in X:\, p(sx) = |s|p(x).$$

These conditions implies that:

1. Non-negativity: $\forall x \in X:\, p(x) \geq 0$.

2. $p(0) = 0$.

A norm on X is a seminorm $p:\, X \to \mathbb{R}$ with the following properties:

- Positive definiteness/Positiveness/Point-separating:

$$\forall x \in X:\, p(x) = 0 \implies x = 0$$

II Linear function

A linear function is a map f between two vector spaces such that

$$f(x+y) = f(x) + f(y), \quad af(x) = f(ax),$$

where $x,\, y \in \mathrm{Dom}(f)$ and a denotes a constant of some field $\mathbb{K}$ of scalars.

III Sublinear function or quasi-seminorm

Let $p:\, X \to \mathbb{R}$ be a function on a vector space X over the field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. p is called a sublinear function or quasi-seminorm if it satisfies the following conditions:

1. Subadditivity/Triangle inequality:

$$\forall x,\, y \in X:\, p(x+y) \leq p(x) + p(y).$$

2. Nonnegative homogeneity:

$$\forall x \in X,\, r \geq 0:\, p(rx) = rp(x).$$

IV Vector space (向量空間)

A vector space over a field F is a non-empty set V together with a binary operation and a binary function that satisfy the eight axioms listed below. In this context, the elements of V are commonly called vectors, and the elements of F are called scalars.
The binary operation, called vector addition or simply addition assigns to any two vectors v and w in V a third vector in V which is commonly written as $v + w$, and called the sum of these two vectors.
The binary function, called scalar multiplication, assigns to any scalar a in F and any vector v in V another vector in V, which is denoted av.
To have a vector space, the eight following axioms must be satisfied for every u, v and w in V, and a and b in F.

1. Associativity of vector addition: $u + (v + w) = (u + v) + w$.
2. Commutativity of vector addition: $u + v = v + u$.
3. Identity element of vector addition: There exists an element $0 \in V$, called the zero vector, such that $\forall v \in V : v + 0 = v$.
4. Inverse elements of vector addition: $\forall v \in V : \exists$ an element $-v \in V$, called the additive inverse of v, such that $v + (-v) = 0$.
5. Compatibility of scalar multiplication with field multiplication: $a(bv) = (ab)v$.
6. Identity element of scalar multiplication: $1v = v$, where 1 denotes the multiplicative identity in F.
7. Distributivity of scalar multiplication with respect to vector addition: $a(u + v) = au + av$.
8. Distributivity of scalar multiplication with respect to field addition: $(a + b)v = av + bv$.

When the scalar field is the real numbers, the vector space is called a real vector space, and when the scalar field is the complex numbers, the vector space is called a complex vector space. These two cases are the most common ones, but vector spaces with scalars in an arbitrary field F are also commonly considered. Such a vector space is called an F-vector space or a vector space over F.

V Topological vector space (TVS) (拓樸向量空間)

A topological vector space X is a vector space over a topological field $\mathbb{K}$, such that vector addition $X \times X \to X$ and scalar multiplication $\mathbb{K} \times X \to X$ are continuous functions.

VI Locally convex topological vector space (LCTVS) or locally convex space

A locally convex topological vector space (LCTVS) $(X, \mathcal{T}$ is a TVS whose topology is generated by a family of seminorms P on it such that:

$$\bigcap_{p \in P} \{x : p(x) = 0\} = \{0\}$$

That is, $\mathcal{T}$ is generated by a basis $\{U_{\epsilon, x}\}_{\epsilon \in \mathbb{R}_{\geq 0}, x \in X}$ of neighborhoods defined to be:

$$U_{\epsilon, 0} = \{y \in X : \forall p \in P : p(y) < \epsilon\}.$$

$$U_{\epsilon, x} = \{y \in X : y - x \in U_{\epsilon, 0}\}.$$

This definition implies that a LCTVS is necessarily a Hausdorff space (T2 space).

VII (von Neumann) bounded

Suppose X is a topological vector space (TVS) over a field $\mathbb{K}$. A subset B of X is called von Neumann bounded (or bounded) in X if for every neighborhood V of the origin, there exists a real $r > 0$ such that $B \subseteq sV$ for all scalars s satisfying $|s| \geq r$.

VIII Bounded linear operator (有界線性運算子)

A bounded linear operator is a linear transformation $L : X \to Y$ between topological vector spaces (TVSs) X and Y that maps bounded subsets of X to bounded subsets of Y. If X and Y are normed vector spaces, then L is bounded if and only if there exists some $M > 0$ such that:

$$\forall x \in X : \|Lx\|_Y \leq M\|x\|_X.$$

The smallest such M is called the operator norm of L and denoted by $\|L\|$. A bounded operator between normed spaces is continuous and vice versa.

IX Symmetric bilinear form (對稱雙線性形式)

Let V be a vector space of dimension n over a field $\mathbb{K}$. A map $B : V \times V \to \mathbb{K}$ is a symmetric bilinear form on the space if:

1. $\forall u, v \in V : B(u, v) = B(v, u)$.
2. $\forall u, v, w \in V : B(u + v, w) = B(u, w) + B(v, w)$.
3. $\forall u, v \in V, \lambda \in \mathbb{K} : B(\lambda u, v) = \lambda B(u, v)$.

X Functional

A functional is a function from a vector space into the field of real or complex numbers.

XI Linear map

A map $T : V \to W$ between vector spaces V and W over field F is linear if

$$\forall a, b \in F, v, w \in V : T(av + bw) = aT(v) + bT(w).$$

XII Invertible map

A map $T : V \to W$ is Invertible if and only if it is bijective. The unique inverse of T, called T^{-1}, is defined to be:

$$T^{-1} : W \to V; \forall v \in V : T(v) \mapsto v = T^{-1}(T(v)).$$

XIII Balanced set (平衡集) , circled set, or disk set

Let X be a vector space over a field $\mathbb{K}$ with an absolute value function $|\cdot|$. A subset S of X is called a balanced set or balanced if:

$$\forall a \in \mathbb{K} \text{ s.t. } |a| \leq 1 : aS \subseteq S.$$

XIV Hahn–Banach theorem

Let $p : X \to \mathbb{R}$ be a sublinear functional on a vector space X over the field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. If $f : M \to \mathbb{K}$ is a linear functional on a vector subspace M such that

$$\forall m \in M : f(m) \leq p(m),$$

then there exists a linear functional $F : X \to \mathbb{K}$ such that

$$\forall m \in M : F(m) = f(m),$$

$$\forall x \in X : F(x) \leq p(x).$$

XV Linear subspace (線性子空間)

A linear subspace of a vector space V over a field $\mathbb{K}$ is a nonempty subset W of V such that,

$$\forall w_1, w_2 \in W, a, b \in \mathbb{K} : aw_1 + bw_2 \in W.$$

XVI Affine space (仿射空間)

An affine space is a point set A together with a vector space $\overrightarrow{A}$, and a transitive and free action of the additive group of $\overrightarrow{A}$ on the set A. The vector space $\overrightarrow{A}$ is said to be associated to the affine space, and its elements are called vectors, translations, or sometimes free vectors.
Explicitly, the definition above means that the action is a mapping, generally denoted as an addition,

$$\begin{aligned} A \times \overrightarrow{A} &\to A \\ (a, v) &\mapsto a + v \end{aligned}$$

that has the following properties.

1. Right identity:
$$\forall a \in A,\ a + 0 = a, \quad \text{where 0 is the zero vector in } \overrightarrow{A}$$

2. Associativity:
$$\forall v, w \in \overrightarrow{A}, \forall a \in A,\ (a + v) + w = a + (v + w)$$

3. Free and transitive action:
$$\forall a \in A : \text{the mapping } \overrightarrow{A} \to A \colon v \mapsto a + v \text{ is a bijection.}$$

4. Existence of one-to-one translations
$$\forall v \in \overrightarrow{A} : \text{the mapping } \overrightarrow{A} \to A \colon v \mapsto a + v \text{ is a bijection.}$$

4 Measure Theory (測度論)

I σ-algebra

Let X be some set, and let $P(X)$ represent its power set. Then a subset $\Sigma \subseteq P(X)$ is called a σ-algebra if and only if it satisfies the following three properties:

1. $X \in \Sigma$.
2. Closed under complementation: $A \in \Sigma \implies X \quad A \in \Sigma$.
3. Closed under countable unions: $\forall \mathcal{S} \subseteq \Sigma$ s.t. $|\mathcal{S}| < \infty : \bigcup_{A \in \mathcal{S}} A \in \Sigma$.

II Measurable space (可測空間)

Consider a set X and a σ-algebra Σ on X. Then the tuple (X, Σ) is called a measurable space.

III Measure (測度)

Let X be a set and Σ be a σ-algebra over X. A set function μ from Σ to the extended real number line, defined to be $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$, is called a measure or positive measure on (X, Σ) if the following conditions hold:

1. Non-negativity:
$$\forall E \in \Sigma : \mu(E) \geq 0$$
2. $\mu(\emptyset) = 0$.
3. Countable additivity (or σ-additivity): For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ (i.e. $\forall i \neq j : E_i \cap E_j = \emptyset$),
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

If the condition of non-negativity is dropped, then μ is called a signed measure.

IV σ-finite measure (σ 有限測度)

Let (X, Σ) be a measurable space and μ be a positive measure or signed measure on it. μ is called a σ-finite measure, if:

$$\exists \{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma \text{ s.t. } \forall n \in \mathbb{N} : \mu(A_n) < \infty \wedge \bigcup_{n \in \mathbb{N}} A_n = X.$$

If μ is a σ-finite measure, the measure space (X, Σ, μ) is called a σ-finite measure space.

V Measure space (測度空間)

A measure space is a triple (X, Σ, μ), where:

1. X is a set.

2. Σ is a σ-algebra on the set X.

3. μ is a measure on (X, Σ).

VI Ring of sets

We call a family $\mathcal{R}$ of subsets of Ω a ring of sets if it has the following properties:

1. $\emptyset \in \mathcal{R}$.

2. Closed under pairwise unions: $\forall A,\, B \in \mathcal{R} :\ A \cup B \in \mathcal{R}$.

3. Closed under relative complements: $\forall A,\, B \in \mathcal{R} :\ A \quad B \in \mathcal{R}$.

VII Pre-measure (前測度)

Let $\mathcal{R}$ be a ring of subsets of a fixed set X. A set function μ_0 from $\mathcal{R}$ to the extended real number line is called a pre-measure on $(X, \mathcal{R})$ if the following conditions hold:

1. Non-negativity:
$$\forall E \in \mathcal{R} :\ \mu_0(E) \geq 0$$

2. $\mu_0(\emptyset) = 0$.

3. Countable additivity (or σ-additivity): For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{R}$ (i.e. $\forall i \neq j :\ E_i \cap E_j = \emptyset$),
$$\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k)$$

VIII Outer measure or exterior measure

Given a set X, let 2^X denote the collection of all subsets of X, including the empty set $\emptyset$. An outer measure on X is a set function $\mu : 2^X \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$.

2. Countably subadditive:
$$\forall A \subseteq X, \{B_i \subseteq X\}_{i=1}^{\infty} :\ A \subseteq \bigcup_{i=1}^{\infty} B_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

IX Carathéodory's extension theorem

Let $\mathcal{R}$ be a ring of sets on X, let $\mu : \mathcal{R} \to [0, +\infty]$ be a pre-measure on $\mathcal{R}$, and let $\sigma(\mathcal{R})$ be a σ-algebra generated by $\mathcal{R}$.
The Carathéodory's extension theorem states that there exists a measure $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ such that μ' is an extension of μ, that is,

$$\mu'\big|_{\mathcal{R}} = \mu.$$

Moreover, if μ is σ-finite, then the extension μ' is unique and also σ-finite.

X Lebesgue measure (勒貝格測度)

For any interval $I = [a, b]$ or $I = (a, b)$ that is a subset of $\mathbb{R}$, let $\ell(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the Lebesgue outer measure $\lambda(E)$ is defined to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The above definition can be generalised to higher dimensions as follows. For any n-dimensional rectangular cuboid, that is, a cuboid with rectangular faces in which all of its dihedral angles are right angles, C, which is a Cartesian product $C = \prod_{i=1}^{n} I_i$ of intervals, we define its Lebesgue outer measure $\lambda(C)$ to be

$$\lambda(C) := \prod_{i=1}^{n} \ell(I_i).$$

For any subset $E \subseteq \mathbb{R}^n$, we define its Lebesgue outer measure $\lambda(E)$ to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(C_k) : (C_k)_{k\in\mathbb{N}} \text{ is a sequence of products of intervals with } E \subseteq \bigcup_{k=1}^{\infty} C_k \right\}.$$

We say a set $E \in \mathbb{R}^n$ satisfies the Carathéodory criterion if

$$\forall A \subseteq \mathbb{R} : \lambda(A) = \lambda(A \cap E) + \lambda(A \cap (\mathbb{R}^n \quad E)).$$

The sets $E \subseteq \mathbb{R}^n$ that satisfy the Carathéodory criterion are said to be Lebesgue-measurable, with its Lebesgue measure being defined as its Lebesgue outer measure. The set of all such E forms a σ-algebra. A set $E \subseteq \mathbb{R}^n$ that does not satisfy the Carathéodory criterion is not Lebesgue-measurable. ZFC proves that such sets do exist.

XI Hausdorff measure (郝斯多夫測度)

Let X, p be a metric space. For any subset $U \subseteq X$, let $\operatorname{diam} U$ denote its diameter, that is

$$\operatorname{diam}(U) := \sup\{p(x, y) : x, y \in U\}, \quad \operatorname{diam}(\emptyset) := 0.$$

Let S be any subset of X, and $\delta > 0$ a real number. Define

$$H_\delta^d(S) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam}(U_i))^d : S \subseteq \bigcup_{i=1}^{\infty} U_i \wedge \operatorname{diam}(U_i) < \delta \right\}.$$

Note that $H_\delta^d(S)$ is monotone nonincreasing in δ since the larger δ is, the more collections of sets are permitted, making the infimum not larger. Thus, $\lim_{\delta\to 0} H_\delta^d(S)$ exists but may be infinite. Let

$$H^d(S) := \lim_{\delta\to 0} H_\delta^d(S).$$

It can be seen that $H^d(S)$ is an outer measure, or more precisely, a metric outer measure. By Carathéodory's extension theorem, its restriction to the σ-algebra of Carathéodory-measurable sets is a measure. It is called the d-dimensional Hausdorff measure of S. Due to the metric outer measure property, all Borel subsets of X are H^d measurable.

5 Geometry (幾何學)

I Euclidean vector space (歐幾里德向量空間)

A Euclidean vector space $\overrightarrow{E}$ is a finite-dimensional inner product space over the real numbers. This implies a symmetric bilinear form:

$$\overrightarrow{E} \times \overrightarrow{E} \to \mathbb{R}$$
$$(x,\, y) \mapsto \langle x,\, y\rangle$$

that the inner product $\langle x,\, x\rangle$ is always positive $\forall x \neq 0$.
The inner product of a Euclidean space is also called dot product and denoted $x \cdot y$. This is specially the case when a Cartesian coordinate system has been chosen, as, in this case, the inner product of two vectors is the dot product of their coordinate vectors.
The Euclidean norm of a vector x in a Euclidean space is:

$$\|x\| = \sqrt{x \cdot x}$$

The concept of size, area, or volume as a measure of Euclidean vector space can be defined to be Lebesgue measure.

II Euclidean (affine) space (歐幾里德 (仿射) 空間)

A Euclidean space, also known as Euclidean affine space, is an affine space over the reals such that the associated vector space is a Euclidean vector space.
The Euclidean distance $d:\ E \times E \to \mathbb{R};\ x,\, y \mapsto d(x,\, y)$ of a Euclidean space E is a metric of E, defined to be:

$$d(x,\, y) = \|\overrightarrow{xy}\|,$$

where $\overrightarrow{xy}$ is a vector in the associated Euclidean vector space such that $x + \overrightarrow{xy} = y$, and $\|\overrightarrow{xy}\|$ is the Euclidean norm of the vector $\overrightarrow{xy}$.

III Convex set (凸集)

Let S be a vector space or an affine space over some ordered field. A subset C of S is convex if, for all x and y in C, the line segment connecting x and y is included in C.

IV Affine hull (仿射包)

An affine combination is a linear combination of points where all coefficients sum up to 1.
The affine hull $\operatorname{aff}(S)$ of a set of points S in a Euclidean space is the smallest affine space that contains all the points in S, namely, the set of all affine combinations of the points in S, i.e.:

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i : k > 0,\, x_i \in S,\, \alpha_i \in \mathbb{R},\, \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

V Convex hull (凸包)

A convex combination is a linear combination of points where all coefficients are non-negative and sum up to 1.
The convex hull $\operatorname{conv}(S)$ of a set of points S in a Euclidean space is the smallest convex set that contains all the points in S, namely, the set of all convex combinations of points in S, i.e.:

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i : k > 0,\, x_i \in S,\, \alpha_i \in [0,\, 1],\, \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

VI Manifold (流形)

Let M be a Hausdorff space. If for any $x \in M$, there exists a neighborhood U_x of x that is homeomorphic to some open set of the m-dimensional Euclidean space $\mathbb{R}^m$, then M is called an m-dimensional manifold.

6 Calculus (微積分)

I Fréchet derivative (弗蘭歇導數)

Let V and W be normed vector spaces, and $U \subseteq V$ be an open subset of V. A function $f : U \to W$ is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A : V \to W$ such that

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0.$$

If there exists such an operator A, it is unique, so we write $Df(x) = A$ and call it the Fréchet derivative of f at x.
A function f that is Fréchet differentiable for any point of U is said to be C^1 if the function

$$Df : U \to B(V,\, W);\, x \mapsto Df(x)$$

is continuous.

II Gateaux derivative (加托導數)

Let V and W be locally convex topological vector spaces (LCTVSs), $U \subseteq V$ be an open subset of V, and a function $F : U \to W$. The Gateaux derivative $dF(u;\, \psi)$ of F at x in U in the direction $\psi \in V$ is defined to be

$$\begin{aligned} dF(u;\, \psi) &= \lim_{\tau \to 0} \frac{F(u + \tau\psi) - F(u)}{\tau} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau} F(u + \tau\psi) \Big|_{\tau=0} \end{aligned}$$

If the limit exists for all $\psi \in V$, then it is said that F is Gateaux differentiable at u.

III Riemann integral (黎曼積分) and Darboux integral (達布積分)

1. Partition of an interval

A partition $P(x,\, n)$ of an interval $[a, b]$ is a finite sequence of numbers of the form

$$P(x\, n) := \{x_i :\, a = x_0,\, b = x_n,\, \forall 1 \le i < j \le n :\, x_i < x_j\}_{i=0}^{n}.$$

Each $[x_i,\, x_{i+1}]$ is called a sub-interval of the partition. The mesh or norm of a partition is defined to be the length of the longest sub-interval, that is,

$$\max\left(x_{i+1} - x_i\right), \quad i \in [0,\, n-1].$$

A tagged partition $P(x,\, n,\, t)$ of an interval $[a,\, b]$ is a partition together with a choice of a sample point within each of all n sub-intervals, that is, numbers $\{t_i\}_{i=0}^{n-1}$ with $t_i \in [x_i,\, x_{i+1}]$ for each $i \in [0,\, n-1]$. The mesh of a tagged partition is the same as that of an ordinary partition.
Suppose that two partitions $P(x,\, n,\, t)$ and $Q(y,\, m,\, s)$ are both partitions of the interval $[a,\, b]$. We say that $Q(y,\, m,\, s)$ is a refinement of $P(x,\, n,\, t)$ if for each integer $i \in [0,\, n]$, there exists an integer $r(i) \in [0,\, m]$ such that $x_i = y_{r(i)}$ and that $\forall i \in [0,\, n-1] :\, \exists j \in [r(i),\, r(i+1)]$ s.t. $t_i = s_j$. That is, a tagged partition breaks up some of the sub-intervals and adds sample points where necessary, "refining" the accuracy of the partition.
We can turn the set of all tagged partitions into a directed set by saying that one tagged partition is greater than or equal to another if the former is a refinement of the latter.

2. Riemann sum

Let f be a real-valued function defined on the interval $[a,\, b]$. The Riemann sum of f with respect to the tagged partition $P(x,\, n,\, t)$ is defined to be

$$R(f,\, P) := \sum_{i=0}^{n-1} f(t_i)\left(x_{i+1} - x_i\right).$$

Each term in the sum is the product of the value of the function at a given point and the length of an interval. Consequently, each term represents the (signed) area of a rectangle with height $f(t_i)$ and width $x_{i+1} - x_i$. The Riemann sum is the (signed) area of all the rectangles.

3. Darboux sum

Lower and upper Darboux sums of f with respect to the partition $P(x,\, n)$ are two specific Riemann sums of which the tags are chosen to be the infimum and supremum (respectively) of f on each sub-interval:

$$L(f,\, P) := \sum_{i=0}^{n-1} \inf_{t \in [x_i,\, x_{i+1}]} f(t)(x_{i+1} - x_i),$$

$$U(f,\, P) := \sum_{i=0}^{n-1} \sup_{t \in [x_i,\, x_{i+1}]} f(t)(x_{i+1} - x_i).$$

4. Riemann integral

The Riemann integral of f exists and equals s if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any tagged partition $P(x, n, t)$ whose mesh is less than δ,

$$|R(f, P) - s| < \varepsilon.$$

5. Darboux integral

The Darboux integral of f exists and equals s if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P whose mesh is less than δ,

$$|U(f, P) - s| < \varepsilon \wedge |L(f, P) - s| < \varepsilon.$$

6. Integrability

A function is Riemann-integrable if and only if it is Darboux-integrable.

IV Indicator function or characteristic function

An indicator function or a characteristic function of a subset A of a set X is a function that maps elements of the subset to one, and all other elements to zero, often denoted as 1_A.

V Lebesgue integral (勒貝格積分)

Below, we will define the Lebesgue integral of measurable functions from a measure space (E, Σ, μ) into $\mathbb{R} \cup \{-\infty, \infty\}$.

1. Indicator functions

The integral of an indicator function of a measurable set S is defined to be

$$\int 1_S \, \mathrm{d}\mu = \mu(S).$$

2. Simple functions

Simple functions are finite real linear combinations of indicator functions. A simple function s of the form

$$s := \sum_k a_k 1_{S_k},$$

where the coefficients a_k are real numbers and S_k are disjoint measurable sets, is called a measurable simple function. When the coefficients a_k positive real numbers, s is called a non-negative measurable simple function. The integral of a non-negative measurable simple function $\sum_k a_k 1_{S_k}$ is defined to be

$$\int \left(\sum_k a_k 1_{S_k} \right) \mathrm{d}\mu = \sum_k a_k \int 1_{S_k} \, \mathrm{d}\mu = \sum_k a_k \, \mu(S_k).$$

whether this sum is finite or $+\infty$.

If B is a measurable subset of E and $s := \sum_k a_k 1_{S_k}$ is a non-negative measurable simple function, one defines

$$\int_B s \, \mathrm{d}\mu = \int 1_B \, s \, \mathrm{d}\mu = \sum_k a_k \, \mu(S_k \cap B).$$

3. Non-negative measurable functions

Let f be a non-negative measurable function on some measurable subset B of E. We define

$$\int_B f \, \mathrm{d}\mu = \sup\left\{\int_B s \, \mathrm{d}\mu : \forall x \in B : 0 \le s(x) \le f(x) \wedge s \text{ is a measurable simple function}\right\}.$$

4. Signed functions

Let f be a measurable function from a measure set E into $\mathbb{R} \cup \{-\infty, \infty\}$. We define

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x & \text{if } f(x) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that both f^+ and f^- are non-negative and that

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

We say that the Lebesgue integral of f exists, if

$$\min\left(\int f^+ \, \mathrm{d}\mu, \int f^- \, \mathrm{d}\mu\right) < \infty.$$

In this case we define

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu.$$

If

$$\int |f| \, \mathrm{d}\mu < \infty,$$

we say that f is Lebesgue integrable.

7 Multivariable calculus or multivariate calculus（多變數/多變量/多元微積分）

The domain of the funcitons or maps below are subsets of a Euclidean vector space. If not otherwise specified, the coordinates are the Cartesian coordinates, the norms are the Euclidean norms, and the measures are the Lebesgue measures.

I Multivariable derivatives

i Notation convention

- Zero tensor: $\mathbf{0}$ or 0 refers to the zero in the interested Euclidean tensor space V, that is, it satisfies

$$\forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}.$$

- Unit vector: $\mathbf{e}_i$ is the unit vector in the ith direction, i.e., a vector with zero norm.
- Independent variable vector: $\mathbf{x} = (x_1, x_2, \dots, x_n)$
- Vector fields: $\mathbf{F}(\mathbf{x}) = \sum_{i=1}^{n} F_i(\mathbf{x})\mathbf{e}_i$ 、$\mathbf{G}$
- Scalar fields: $A(\mathbf{x})$ 、$B(\mathbf{x})$
- Tensor fields: $f(\mathbf{x})$ 、$g(\mathbf{x})$
- Three-dimensional tensor space field: $\mathbf{T}(\mathbf{x})$
- The i-th component of the map f: f_i
- Dot product operator: $\cdot$
- Cross product operator: $\times$
- Gradient operator/Jacobian operator: ∇
- Divergence operator: $\nabla\cdot$
- Curl operator: $\nabla\times$
- Directional derivative operator: $\cdot\nabla$
- Laplace operator: ∇^2 或 Δ
- Line integral operator: $\int$
- Surface integral operator: $\iint$
- Volume integration operator: $\iiint$
- Closed line integral operator: $\oint$
- Closed surface Integral Operator: $\oiint$
- $\int \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ is used as a shorthand for $\int (\mathbf{F} \cdot \hat{\mathbf{n}})\, \mathrm{d}S$, where $\hat{n}$ is the outward pointing unit normal at almost each point on S.

ii Gradient

$$\nabla f = \left(\left(\frac{\partial f}{\partial x_1}\right)^T \quad \left(\frac{\partial f}{\partial x_2}\right)^T \quad \cdots \quad \left(\frac{\partial f}{\partial x_n}\right)^T \right)$$

The gradient of a scalar field is a vector field, the gradient of a vector field is a second-order tensor (matrix) field, and the gradient of a k-order tensor field is a $k+1$-order tensor field.
In particular, the gradient of a scalar field:

$$\nabla A = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i} e_i$$

iii Divergence

$$\nabla \cdot f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

The divergence of a vector field is a scalar field, the gradient of a second-order tensor (matrix) field is a vector field, and the divergence of a $k+1$-order tensor field is a k-order tensor field.

iv Curl

The curl is only defined on three-dimensional vector field.

$$\nabla \times \mathbf{T} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ T_1 & T_2 & T_3 \end{pmatrix}$$

The curl of a three-dimensional vector field is a three-dimensional vector field.

v Directional derivative

$$(\mathbf{f} \cdot \nabla)\mathbf{g} = \sum_{i=1}^{n} f_i \frac{\partial g}{\partial x_i}$$

vi Laplace operator

$$\nabla^2 f = \nabla \cdot (\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x_i}^2}$$

The Laplace operator applied to a tensor field is a tensor field of the same order and same dimension (but not necessarily the same field).

vii Poisson's equation（卜瓦松/帕松/泊松方程）

$$\nabla^2 A = B(\mathbf{x})$$

viii Laplace's equation（拉普拉斯方程）

$$\nabla^2 A = 0$$

A real function A with real independent variables that is second-order differentiable for all independent variables is called a harmonic function if A satisfies Laplace's equation.

ix Jacobian matrix（雅可比矩陣）

Assume that $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is a function whose first-order partial derivatives exist on $\mathbb{R}^n$. Then the Jacobian matrix of $\mathbf{F}$ is defined as an $m \times n$ matrix, denoted as J, $J(\mathbf{F}$, or $\nabla\mathbf{F}$, whose $(i,\ j)$th entry is:

$$\mathbf{J}_{ij} = \frac{\partial F_i}{\partial x_j}$$

that is,

$$\nabla\mathbf{F} = \mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{F}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{F}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix}$$

The determinant of the Jacobian matrix is called the Jacobian determinant, or Jacobian for short.

x Integral transform

Let the two spaces $\{\mathbf{x} = (x_1,\ x_2,\ \dots,\ x_n)\}$ and $\{\mathbf{y} = (y_1,\ y_2,\ \dots,\ y_n)\}$ be isomorphic, and let $\mathbf{T} : \mathbf{x} \mapsto \mathbf{y}$.

$$\int \dots \int f(\mathbf{x})\, \mathrm{d}x_1\, \mathrm{d}x_2\, \dots\, \mathrm{d}x_n = \int \dots \int f(\mathbf{T}^{-1}(\mathbf{y}))\ |\det(J(\mathbf{T}))|\ \mathrm{d}y_1\, \mathrm{d}y_2\, \dots\, \mathrm{d}y_n,$$

where $\det(J(\mathbf{T}))$ is the Jacobian determinant of $\mathbf{T}$.

xi Multi-index notation（多重指標記號）

Multiindex α is a convenient notation for partial derivatives and polynomial expansions in multiple variables. Suppose there are n variables $x_1, x_2, \dots, x_n$, then a multiindex is a vector of n non-negative integers:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \text{where } \alpha_i \in \mathbb{N}_0$$

Define:

- Norm $|\alpha|$:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

- Factorial $\alpha!$:

$$\alpha! = \alpha_1! \cdot \alpha_2! \cdot \cdots \cdot \alpha_n!$$

- Power $\mathbf{x}^\alpha$: If $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

- High-order mixed partial derivatives $D^\alpha f$:

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}},$$

xii Taylor expansion（泰勒展開式）

Assume that $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is an infinitely differentiable function, and its partial derivatives of every order exist on $\mathbb{R}^n$, then the Taylor expansion of $\mathbf{F}$ at $\mathbf{a}$ is:

$$\mathbf{F}(\mathbf{x}) = \sum_{|\alpha|\geq 0} \frac{D^\alpha \mathbf{F}(\mathbf{a})}{\alpha!}(\mathbf{x}-\mathbf{a})^\alpha,$$

First-order approximation:

$$\mathbf{F}(\mathbf{a}+\delta\mathbf{x}) \approx \mathbf{F}(\mathbf{a}) + \nabla\mathbf{F}(\mathbf{a}) \cdot \delta\mathbf{x}.$$

II Line integral（線積分）

i Scalar field line integral

For a scalar field $A : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and the path $C \in U$, the line integral of A is:

$$\int_C A\,\mathrm{d}s = \int_a^b A(\mathbf{r}(t))\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)\|\,\mathrm{d}t,$$

where $\mathbf{r} : [a,\, b] \to C$ is a one-to-one parametric function with $\mathbf{r}(a)$ and $\mathbf{r}(b)$ being the two endpoints of the path C.
A is called the integral function, C is called the integral path, and the result of the line integration does not depend on the parametric function r.

ii Vector field line integral

For a scalar field $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and the path $C \in U$, the line integral of $\mathbf{F}$ is:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)\,\mathrm{d}t$$

where $\mathbf{r} : [a,\, b] \to C$ is a one-to-one parametric function with $\mathbf{r}(a)$ and $\mathbf{r}(b)$ being the two endpoints of the path C.
$\mathbf{F}$ is called the integral function, C is called the integral path, and the result of the line integration does not depend on the parametric function $\mathbf{r}$.

III Fundamental theorem of multivariable calculus（多變數微積分基本定理）

i Gradient theorem（梯度定理）

If $\varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function and γ a differentiable curve in U which starts at a point p and ends at a point q, then,

$$\int_\gamma \nabla\varphi(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \varphi\left(\mathbf{q}\right) - \varphi\left(\mathbf{p}\right).$$

Gradient theorem is a special case of generalized Stokes theorem.

ii Divergence theorem, Gauss's theorem, or Ostrogradsky's theorem（高斯散度定理）

Suppose $V \subseteq \mathbb{R}^n$ is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). The closed, measurable set ∂V is oriented by outward-pointing normals. If F is a continuously differentiable vector field defined on a neighborhood of V, then,

$$\iiint_V (\nabla \cdot \mathbf{F})\ \mathrm{d}V = \oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Divergence theorem is a special case of generalized Stokes theorem.

iii Stokes' theorem or Kelvin–Stokes theorem（斯托克斯定理）

Let S be a positively oriented, piecewise smooth surface in $\mathbb{R}^3$ with boundary $\partial S \equiv L$. If a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined and has continuous first order partial derivatives in a region containing S, then,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_L \mathbf{F} \cdot \mathrm{d}\mathbf{L}$$

Stokes' theorem is a special case of generalized Stokes theorem.

iv Green's theorem（格林定理或綠定理）

Let S be a positively oriented, piecewise smooth surface in $\mathbb{R}^2$ with boundary $\partial S \equiv L$. If scalar function $P, (x,\, y)\, Q(x,\, y)$ are defined and has continuous first order partial derivatives in a region containing S, then,

$$\oint_L (P\mathrm{d}x + Q\mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

where the path of integration along C is counterclockwise.
Green's theorem is a special case of Stokes' theorem.

v Generalized Stokes theorem, Stokes–Cartan theorem, or fundamental theorem of multivariable calculus

The generalized Stokes theorem says that the integral of a differential form ω over the boundary $\partial\Omega$ of some orientable manifold Ω is equal to the integral of its exterior derivative d over the whole of Ω, i.e.,

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}$$