# Inequalities

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# 1 Inequalities

## I Triangle Inequality

$$|a| + |b| \geq |a + b|$$

*Proof.*

$$(|a| + |b|)^2 - |a + b|^2 = 2(|a||b| - ab) \geq 0$$

□

$$|a - b| \geq ||a| - |b||$$

*Proof.*

$$(|a - b|)^2 - (|a| - |b|)^2 = 2(|a||b| - ab) \geq 0$$

□

$$\|\vec{a}\| + \left\|\vec{b}\right\| \geq \left\|\vec{a} + \vec{b}\right\|$$

*Proof.*

$$(\|\vec{a}\| + \left\|\vec{b}\right\|)^2 - \left\|\vec{a} + \vec{b}\right\|^2 = 2(\|\vec{a}\|\left\|\vec{b}\right\| - \vec{a} \cdot \vec{b}) \geq 0$$

□

$$\left\|\vec{a} - \vec{b}\right\| \geq \left|\|\vec{a}\| - \left\|\vec{b}\right\|\right|$$

*Proof.*

$$\left\|\vec{a} - \vec{b}\right\|^2 - (\|\vec{a}\| - \left\|\vec{b}\right\|)^2 = 2(\|\vec{a}\|\left\|\vec{b}\right\| - \vec{a} \cdot \vec{b}) \geq 0$$

□

## II Jensen's Inequality

Let $(\Omega, \mu)$ be a probability space, $g : \Omega \to \mathbb{R}$ be a real-valued $\mu$-integrable function, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, Jensen's inequality states:

$$\int_\Omega \varphi(g(\omega))\, \mathrm{d}\mu(\omega) \geq \varphi\left(\int_\Omega g(\omega)\, \mathrm{d}\mu(\omega)\right)$$

*Proof.* Since $\varphi$ is convex, at each real number $x$, we have a non-empty set of subderivatives, which may be thought of as lines touching the graph of $\varphi$ at $x$, but which are below the graph of $\varphi$ at all points (support lines of the graph). Now, if we define:

$$x_0 := \int_\Omega g\, \mathrm{d}\mu$$

because of the existence of subderivatives for convex functions, we may choose $a$ and $b$ such that $ax + b \leq \varphi(x)$ for all real $x$ and $ax_0 + b = \varphi(x_0)$. But then we have that for almost all $\omega \in \Omega$:

$$\varphi(g(\omega)) \geq ag(\omega) + b$$

Since we have a probability measure, the integral is monotone with $\mu(\Omega) = 1$ so that

$$\begin{aligned}
\int_\Omega \varphi(g(\omega))\,\mathrm{d}\mu(\omega) &\geq \int_\Omega (ag(\omega)+b)\,\mathrm{d}\mu(\omega) \\
&= a\int_\Omega g\,\mathrm{d}\mu + b\int_\Omega \mathrm{d}\mu \\
&= ax_0 + b = \varphi(x_0) = \varphi\left(\int_\Omega g\,\mathrm{d}\mu\right)
\end{aligned}$$

as desired. □

## III AM-GM Inequality

$$\frac{\sum_{i=1}^n x_i}{n} \geq \sqrt[n]{\prod_{i=1}^n x_i}$$

*Proof.* **Lemma:**
Let $(\Omega, \mu)$ be a probability space, $g : \Omega \to \mathbb{R}$ be a real-valued $\mu$-integrable function, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, Jensen's inequality states:

$$\int_\Omega \varphi(g(\omega))\,\mathrm{d}\mu(\omega) \geq \varphi\left(\int_\Omega g(\omega)\,\mathrm{d}\mu(\omega)\right)$$

Thus, applying Jensen's inequality to the logarithm function, which is concave, and the arithmetic mean:

$$\log\left(\frac{\sum_{i=1}^n x_i}{n}\right) \geq \sum_{i=1}^n \frac{1}{n}\log\left(x_i\right) = \log\left(\sqrt[n]{\prod_{i=1}^n x_i}\right)$$

Exponentiating both sides gives the desired inequality:

$$\frac{\sum_{i=1}^n x_i}{n} \geq \sqrt[n]{\prod_{i=1}^n x_i}$$

□

## IV Cauchy-Schwarz Inequality

For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$,

$$|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

and

$$(|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|) \iff ())\mathbf{u}/\mathbf{v})$$

*Proof.* Consider the complex number $z = \mathbf{u}\cdot\mathbf{v}$ where $\mathbf{u}\cdot\mathbf{v}$ is the standard inner product defined as

$$\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^n u_i\overline{v_i},$$

with $\overline{v_i}$ denoting the complex conjugate of $v_i$.
Define the function

$$f(t) = |\mathbf{u} + t\mathbf{v}|^2$$

for some real number $t$. Then, we have

$$\begin{aligned}
f(t) &= (\mathbf{u} + t\mathbf{v}) \cdot \overline{(\mathbf{u} + t\mathbf{v})} \\
&= \left(\mathbf{u} \cdot \overline{\mathbf{u}}\right) + t\left(\mathbf{u} \cdot \overline{\mathbf{v}}\right) + t\left(\mathbf{v} \cdot \overline{\mathbf{u}}\right) + t^2\left(\mathbf{v} \cdot \overline{\mathbf{v}}\right) \\
&= |\mathbf{u}|^2 + 2t\Re(\mathbf{u} \cdot \overline{\mathbf{v}}) + t^2|\mathbf{v}|^2,
\end{aligned}$$

where $\Re(\mathbf{u} \cdot \overline{\mathbf{v}})$ denotes the real part of the complex number $\mathbf{u} \cdot \overline{\mathbf{v}}$.
Since $f(t) = |\mathbf{u} + t\mathbf{v}|^2 \geq 0$ for all $t \in \mathbb{R}$, the quadratic equation in $t$ must have a non-positive discriminant. The discriminant of this quadratic is:

$$\begin{aligned}
\Delta &= \left(2\Re(\mathbf{u} \cdot \overline{\mathbf{v}})\right)^2 - 4 \times |\mathbf{v}|^2 \times |\mathbf{u}|^2 \\
&= 4\left(\Re(\mathbf{u} \cdot \overline{\mathbf{v}})\right)^2 - 4|\mathbf{u}|^2|\mathbf{v}|^2.
\end{aligned}$$

For $f(t) \geq 0$ for all $t$, we require $\Delta \leq 0$. This implies:

$$\left(\Re(\mathbf{u} \cdot \overline{\mathbf{v}})\right)^2 \leq |\mathbf{u}|^2|\mathbf{v}|^2.$$

Taking the square root of both sides and noting that $|\mathbf{u} \cdot \mathbf{v}| \geq \Re(\mathbf{u} \cdot \overline{\mathbf{v}})$:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

This completes the proof of the Cauchy-Schwarz inequality.
Equality $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$ holds if and only if the discriminant $\Delta = 0$. This happens when the quadratic equation has a double root or equivalently, when $\mathbf{u} + t\mathbf{v} = 0$ for some real $t$, implying $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, meaning

$$\mathbf{u}/\mathbf{v}.$$

This concludes the proof of the Cauchy-Schwarz inequality along with its equality condition. $\square$