Lagrangian Mechanics

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1 Lagrangian Mechanics

I Configuration Space

The degrees of freedom, or parameters, that define the configuration, or position of all constituent point particles, of a physical system are called generalized coordinates, and the space defined by these coordinates is called the configuration space of the system. Each unique possible configuration of the system corresponds to a unique point in the space. It is often the case that the set of all actual configurations of the system is a manifold in the configuration space, called the configuration manifold of the system. The number of dimensions of the configuration space is equal to the degrees of freedom of the system.

II Lagrangian Mechanics

Lagrangian mechanics describes a mechanical system as a pair $(M,\, L)$ consisting of a configuration manifold M and a smooth functional $L\colon TM \to \mathbb{R}$, called a Lagrangian, where TM is the tagent bundle of M, that is, $L = L(\mathbf{q},\, \dot{\mathbf{q}},\, t)$, where the vector $\mathbf{q}$ is a point in the configuration space of the system, $\dot{\mathbf{q}} = \dfrac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}$, and t is time.

III Non-Relativistic Lagrangian

Let $\mathbf{r}_k$ denote the position of the kth particle for ${}_{k=1}^{N}$ in the position space $\mathbb{R}^3$ equipped with Cartesian coordinates as function of time, and

$$\mathbf{v}_k = \dot{\mathbf{r}_k} = \frac{\mathrm{d}\mathbf{r}_k}{\mathrm{d}t}.$$

The non-relativistic Lagrangian for a system of N particles is given by

$$L = T - V,$$

where

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k {\mathbf{v}_k}^2$$

is the total kinetic energy of the system. Each particle labeled k has mass m, and ${\mathbf{v}_k}^2 = \mathbf{v}_k \cdot \mathbf{v}_k$ is the square of the norm of its velocity, equivalent to the dot product of the velocity with itself.

V, the potential energy of the system, is the energy any one of the particles has due to all the others together with all external influences. For conservative forces (e.g. Newtonian gravity), V is a function of the vectors of the position of the particles only, that is, $V = V((\mathbf{r}_k)_{i=1}^{N})$. For non-conservative forces that can be derived from an appropriate potential (e.g. electromagnetic potential), V is a function of the vectors of the position and velocity of the particles, that is, $V = V((\mathbf{r}_k)_{i=1}^{N},\, (\mathbf{v}_k)_{i=1}^{N})$. For some time-dependent external field or force (e.g. electric field and magnetic flux density field in electromagnetodynamics), the potential changes with time, so most generally, V is a function of the vectors of the position and velocity of the particles and time, that is, $V = V((\mathbf{r}_k)_{i=1}^{N},\, (\mathbf{v}_k)_{i=1}^{N},\, t)$.

IV Lagrange's Equations of the First Kind

With these definitions, Lagrange's equations of the first kind are

$$\nabla_{\mathbf{r}_k} L - \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\dot{\mathbf{r}}_k} L + \sum_{i=1}^{C} \lambda_i \nabla_{\mathbf{r}_k} f_i = 0,$$

where k for ${}_{k=1}^{N}$ labels the particles, f_i for ${}_{i=1}^{C}$ labels C constraint equations, and λ_i for ${}_{i=1}^{C}$ labels the Lagrange multiplier for the ith constraint equation. The constraint equations can be either holonomic, that is, in the form of $f_i((\mathbf{r}_k)_{i=1}^{N}) = 0$, non-holonomic, that is, in the form of $f_i((\mathbf{r}_k)_{i=1}^{N},\, (\mathbf{v}_k)_{i=1}^{N}) = 0$, or most generally, time dependent, that is, in the form of $f_i((\mathbf{r}_k)_{i=1}^{N},\, (\mathbf{v}_k)_{i=1}^{N},\, t) = 0$.

V From Position Space to Configuration Space

In each constraint equation, one coordinate can be determined from the other coordinates. The number of independent coordinates is therefore $n = 3N - C$. We can construct a configuration space with n generalized coordinates, conveniently written as an n-vector $\mathbf{q} = ((q_k)_{k=1}^{n})$. Hence the position coordinates as functions of the generalized coordinates and time are

$$\mathbf{r}_k \colon \mathbb{R}^{n+1} \to \mathbb{R}^3;\ (\mathbf{q},\, t) \mapsto \mathbf{r}_k(x_k,\, y_k,\, z_k).$$

The vector $\mathbf{q}$ is a point in the configuration space of the system. The time derivatives of the generalized coordinates are called the generalized velocities, and for each particle labelled as k for ${}_{k=1}^{N}$, its velocity vector, namely, the total derivative of its position with respect to time, is

$$\mathbf{v}_k = \left(\dot{\mathbf{q}} \cdot \nabla_{\mathbf{q}}\right) \mathbf{r}_k + \frac{\partial \mathbf{r}_k}{\partial t}.$$

VI Euler–Lagrange Equations or Lagrange's Equations of the Second Kind

With these definitions, the Euler–Lagrange equations, or Lagrange's equations of the second kind are

$$\nabla_{\mathbf{q}} L = \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\dot{\mathbf{q}}} L.$$

The number of equations has decreased compared to Newtonian mechanics, from $3N$ to $n = 3N - C$ coupled second-order differential equations in the generalized coordinates. These equations do not include constraint forces at all, only non-constraint forces need to be accounted for.