

# **Glossary of Advanced Set Theory, Topology, Linear Algebra, Functional Analysis, Measure Theory, and Geometry**

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# Chapter 1 Glossary of Advanced Set Theory, Topology, Linear Algebra, Functional Analysis, Measure Theory, and Geometry

## 1 Set Theory (集合論)

### I ZFC axiom system (ZFC 公理系統) and ZF axiom system (ZF 公理系統)

The ZFC axiom system refers to the Zermelo-Fraenkel axiom system (ZF axiom system) plus the axiom of choice (AC). The most commonly used axiom systems in set theory are the ZFC axiom system or the ZF axiom system, denoted by  $\vdash_{ZFC}$  and  $\vdash_{ZF}$ , or both denoted by  $\vdash$ .

#### i Axiom of extensionality (外延公理)

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$$

#### ii Axiom of regularity (正則公理)

$$\forall x (x \neq \emptyset \Rightarrow \exists y (y \in x \wedge y \cap x = \emptyset))$$

#### iii Axiom of separation (分類公理) or axiom schema of specification (規範公理模式)

Let  $\varphi$  be any formula in the language of ZFC with all free variables among  $x, z, w_1, \dots, w_n$  so that  $y$  is not free in  $\varphi$ . Then:

$$\forall z \forall w_1 \forall w_2 \dots \forall w_n \exists y \forall x [x \in y \Leftrightarrow ((x \in z) \wedge \varphi(x, w_1, w_2, \dots, w_n, z))]$$

This axiom can be used to prove the existence of the empty set, denoted as  $\emptyset$  or  $\varnothing$ .

Axiom of empty set (空集公理) :

$$\exists x, \forall y, (y \notin x)$$

The  $\emptyset$  is defined as the  $x$  above.

#### iv Pairing axiom (配對公理)

$$\forall x \forall y \exists z ((x \in z) \wedge (y \in z))$$

## v Union axiom (聯集公理)

$$\forall \mathcal{F} \exists A \forall Y \forall x [(x \in Y \wedge Y \in \mathcal{F}) \Rightarrow x \in A]$$

Although this formula doesn't directly assert the existence of  $\cup \mathcal{F}$ , the set  $\cup \mathcal{F}$  can be constructed from  $A$  in the above using the axiom schema of specification:

$$\cup \mathcal{F} = \{x \in A \mid \exists Y (x \in Y \wedge Y \in \mathcal{F})\}$$

## vi Axiom schema of replacement (替代公理模式)

Let  $\varphi$  be any formula in the language of ZFC with all free variables among  $x, z, w_1, \dots, w_n$  so that  $B$  is not free in  $\varphi$ . Then:

$$\forall A \forall w_1 \forall w_2 \dots \forall w_n [\forall x (x \in A \Rightarrow \exists! y \varphi) \Rightarrow \exists B \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \varphi))]$$

## vii Axiom of infinity (無窮公理)

$$\exists X [\exists e (\forall z \neg (z \in e) \wedge e \in X) \wedge \forall y (y \in X \Rightarrow S(y) \in X)]$$

## viii Power set axiom (冪集公理)

$$\forall A \exists P(A) \forall x (x \in P(A) \leftrightarrow x \subseteq A)$$

The  $P(A)$  above is called power set (冪集) and is denoted as  $2^A$ .

## ix Axiom of choice (AC) (選擇公理) or axiom of well-ordering (良序公理)

$$\forall X \left[ \emptyset \notin X \Rightarrow \exists f : X \rightarrow \bigcup_{A \in X} A \quad \forall A \in X (f(A) \in A) \right]$$

# II Set operation notations

## i Cardinality

$|A$  and  $n(A)$  denote the cardinality of the set  $A$ .

## ii Set scalar arithmetic operation

- If  $\forall a \in A, sa$  is defined,  $sA := \{sa : a \in A\}$ .
- If  $\forall a \in A, a + v$  is defined,  $A + v := \{a + v : a \in A\}$ .

## iii Kernel (核) of a family of sets

The kernel of a family  $\mathcal{B} \neq \emptyset$  of sets is defined to be:

$$\ker(\mathcal{B}) := \bigcap_{B \in \mathcal{B}} B.$$

#### iv Cartesian product (笛卡爾積)

The Cartesian product of two sets  $A$  and  $B$ , denoted  $A \times B$ , is defined to be

$$A \times B := \{(a, b) : a \in A \wedge b \in B\}.$$

### III Partially ordered set (poset) (偏序集)

A partially ordered set (poset for short) is an ordered pair  $P = (X, \leq)$  consisting of a set  $X$  (called the ground set of  $P$ ) and a partial order  $\leq$  on  $X$ . That is, for all  $a, b, c \in X$  it must satisfy:

1. Reflexivity:  $a \leq a$ , i.e. every element is related to itself.
2. Antisymmetry:  $a \leq b \wedge b \leq a \implies a = b$ , i.e. no two distinct elements precede each other.
3. Transitivity:  $a \leq b \wedge b \leq c \implies a \leq c$ .

When the meaning is clear from context and there is no ambiguity about the partial order, the set  $X$  itself is sometimes called a poset.

### IV Upward closure

Let  $A$  be a subset of a poset  $X$ , the upward closure of  $A$  (denoted as  $\uparrow A$ ) is defined to be:

$$\uparrow A := \{x \in X : \exists a \in A \text{ s.t. } a \leq x\}.$$

## 2 Topology (拓樸學)

### I Topological space (拓樸空間)

A topological space consists of a set  $X$  and a topology  $\mathcal{T}$  on it, denoted as  $(X, \mathcal{T})$ . Where the set  $X$  is the set of points in the space, and the topology  $\mathcal{T}$  is the set of subsets of  $X$  that satisfies:

1.  $\emptyset, X \in \mathcal{T}$
2. Closed under arbitrary unions:  $\forall \mathcal{S} \subseteq \mathcal{T} : \bigcup_{A \in \mathcal{S}} A \in \mathcal{T}$
3. Closed under finite intersection:  $\forall \mathcal{S} \subseteq \mathcal{T} \text{ s.t. } |\mathcal{S}| < \infty : \bigcap_{A \in \mathcal{S}} A \in \mathcal{T}$

### II Homeomorphism (同胚) and isomorphism (同構)

Topological spaces  $X, \mathcal{T}_X$  and  $Y, \mathcal{T}_Y$  are called homeomorphic if there exists a mapping  $f : X \rightarrow Y$  between them such that  $f$  is bijective and continuous and  $f^{-1}$  is continuous, written as  $X \cong Y$ , and  $f$  is called a homeomorphism between them.

In linear algebra, when  $X$  and  $Y$  are homeomorphic vector spaces and  $f$  is a linear map,  $X$  and  $Y$  are also called isomorphic, and a homeomorphism between them is also called an isomorphism.

### III Open set (開集)

In a topological space  $(X, \mathcal{T})$ , an open subset  $O \subseteq X$  is defined to be

$$O \in \mathcal{T}.$$

### IV Open neighborhood (開鄰域)

In a topological space  $X$ , a open neighborhood of a point  $P \in X$  is any open subset  $O \subseteq X$  such that  $P \in O$ .

### V Neighborhood (鄰域)

In a topological space, a subset  $U$  is a neighborhood of a point  $P$  if and only if there exists an open set  $O$  such that  $P \in O \subseteq U$ .

### VI Limit point (極限點), cluster point, or accumulation point (集積點)

Let  $S$  be a subset of a topological space  $X$ . A point  $x$  in  $X$  is a limit point, cluster point, or accumulation point of the set  $S$  if every neighborhood of  $x$  contains at least one point of  $S$  different from  $x$  itself.

### VII Closure (閉包)

The closure of a subset  $S$  of points in a topological space consists of all points in  $S$  together with all limit points of  $S$ .

### VIII Closed set (閉集)

A subset  $A$  of a topological space  $(X, \mathcal{T})$  is called closed if its complement  $X \setminus A \in \mathcal{T}$ .

### IX Borel set (博雷爾集)

A Borel set  $B$  is any set in a topological space  $(X, \mathcal{T})$  that can be formed from open sets through the operations of countable union, countable intersection, and relative complement, that is:

$$\begin{aligned} B \in & \left\{ \bigcup_{O \in \mathcal{S}} O : \mathcal{S} \subseteq \mathcal{T}, |\mathcal{S}| < \infty \right\} \\ & \cup \left\{ \bigcap_{O \in \mathcal{S}} O : \mathcal{S} \subseteq \mathcal{T}, |\mathcal{S}| < \infty \right\} \\ & \cup \{O \setminus P : O, P \in \mathcal{T}\} \end{aligned}$$

### X Filter

A filter on a set  $X$  is a family  $\mathcal{B}$  of subsets of  $X$  such that:

1.  $X \in \mathcal{B}$ .
2.  $\emptyset \in \mathcal{B}$ .



$$3. A \in \mathcal{B} \wedge B \in \mathcal{B} \implies A \cap B \in \mathcal{B}.$$

$$4. A \subseteq B \subseteq X \wedge A \in \mathcal{A} \in \mathcal{B} \implies B \in \mathcal{B}.$$

## XI Base or basis (基)

Given a topological space  $(X, \mathcal{T})$ , a base (or basis) for the topology  $\mathcal{T}$  (also called a base for  $X$  if the topology is understood) is a family  $\mathcal{B} \subseteq \mathcal{T}$  of open sets such that every open set of the topology can be represented as the union of some subfamily of  $\mathcal{B}$ .

The topology generated by a base  $\mathcal{B}$ , generally denoted by  $\tau(\mathcal{B})$  can be defined to be follows: A subset  $O \subseteq X$  is to be declared as open, if for all  $x \in O$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq O$ .

## XII Prefilter or filter base

$\mathcal{B}$  is called a prefilter if its upward closure  $\uparrow \mathcal{B}$  is a filter.

## XIII Connected space (連通空間)

A topological space is said to be disconnected if it can be expressed as the union of two disjoint non-empty open sets. Otherwise, it is said to be connected.

## XIV Connected set

A set  $C \subseteq X$  of a topological space  $(X, \tau)$  are said to be disconnected if

$$\exists A, B \in \tau \text{ s.t. } A \cap B = \emptyset \wedge C \cap A \neq \emptyset \wedge C \cap B \neq \emptyset \wedge C \subseteq A \cup B.$$

Otherwise, it is said to be connected.

## XV Compact space (緊緻空間)

A topological space  $X$  is called compact if every open cover of  $X$  has a finite subcover. That is,  $X$  is compact if for every collection  $C$  of open subsets of  $X$  such that

$$X = \bigcup_{S \in C} S,$$

there is a finite subcollection  $F \subseteq C$  such that

$$X = \bigcup_{S \in F} S.$$

## XVI Hausdorff space, separated space or T2 space (郝斯多夫空間、分離空間或 T2 空間)

Points  $x$  and  $y$  in a topological space  $X$  can be separated by neighborhoods if there exists a neighborhood  $U$  of  $x$  and a neighborhood  $V$  of  $y$  such that  $U$  and  $V$  are disjoint, i.e.,  $U \cap V = \emptyset$ .

$X$  is a Hausdorff space if any two distinct points in  $X$  are separated by neighborhoods. This condition is the third separation axiom (after T0 and T1), which is why Hausdorff spaces are also called T2 spaces. The name separated space is also used.

## XVII Metric space (度量空間或賦距空間)

Metric space is an ordered pair  $(M, d)$  where  $M$  is a set and  $d$  is a metric on  $M$ , i.e., a function  $d : M \times M \rightarrow \mathbb{R}$  satisfying the following axioms for all points  $x, y, z \in M$ :

1. The distance from a point to itself is zero:  $d(x, x) = 0$ .
2. (Positivity) The distance between two distinct points is always positive:  $x \neq y \implies d(x, y) > 0$ .
3. (Symmetry) The distance from  $x$  to  $y$  is always the same as the distance from  $y$  to  $x$ :  $d(x, y) = d(y, x)$ .
4. The triangle inequality:  $d(x, z) \leq d(x, y) + d(y, z)$ .

## XVIII Open ball (開球)

In a metric space  $(X, d)$ , given a point  $a$  and radius  $r$ , the open ball  $B(a)_{<r}$  is defined to be:

$$B(a)_{<r} := \{p \in X : d(a, p) < r\}.$$

## XIX Closed ball (閉球)

In a metric space  $(X, d)$ , given a point  $a$  and radius  $r$ , the closed ball  $B(a)_{\leq r}$  is defined to be:

$$B(a)_{\leq r} := \{p \in X : d(a, p) \leq r\}.$$

## XX Topological field (拓樸域)

A topological field is a topological space, such that addition, multiplication, the maps  $a \mapsto -a$ , and  $a \mapsto a^{-1}$  are continuous maps with respect to the topology of the space.

## XXI Ordered field (有序域)

A field  $(F, +, \cdot)$  together with a total order  $\leq$  on  $F$  is an ordered field if the order satisfies the following properties for all  $a, b, c \in F$ :

1.  $a \leq b \implies a + c \leq b + c$ .
2.  $0 \leq a \wedge 0 \leq b \implies 0 \leq a \cdot b$ .

As usual, we write  $a < b$  for  $a \leq b$  and  $a \neq b$ . The notations  $b \geq a$  and  $b > a$  stand for  $a \leq b$  and  $a < b$ , respectively. Elements  $a \in F$  with  $a > 0$  are called positive.

## 3 Linear Algebra (線性代數)

### I Topological vector space (TVS) (拓樸向量空間)

A topological vector space  $X$  is a vector space over a topological field  $\mathbb{K}$ , such that vector addition  $X \times X \rightarrow X$  and scalar multiplication  $\mathbb{K} \times X \rightarrow X$  are continuous functions.

## II Locally convex topological vector space (LCTVS) or locally convex space

A locally convex topological vector space (LCTVS)  $(X, \mathcal{T})$  is a TVS whose topology is generated by a family of seminorms  $P$  on it such that:

$$\bigcap_{p \in P} \{x : p(x) = 0\} = \{0\}$$

That is,  $\mathcal{T}$  is generated by a basis  $\{U_{\epsilon, x}\}_{\epsilon \in \mathbb{R}_{\geq 0}, x \in X}$  of neighborhoods defined to be:

$$U_{\epsilon, 0} = \{y \in X : \forall p \in P : p(y) < \epsilon\}.$$

$$U_{\epsilon, x} = \{y \in X : y - x \in U_{\epsilon, 0}\}.$$

This definition implies that a LCTVS is necessarily a Hausdorff space (T2 space).

## III Symmetric bilinear form (對稱雙線性形式)

Let  $V$  be a vector space of dimension  $n$  over a field  $\mathbb{K}$ . A map  $B : V \times V \rightarrow \mathbb{K}$  is a symmetric bilinear form on the space if:

1.  $\forall u, v \in V : B(u, v) = B(v, u).$
2.  $\forall u, v, w \in V : B(u + v, w) = B(u, w) + B(v, w).$
3.  $\forall u, v \in V, \lambda \in \mathbb{K} : B(\lambda u, v) = \lambda B(u, v).$

## IV Balanced set (平衡集), circled set, or disk set

Let  $X$  be a vector space over a field  $\mathbb{K}$  with an absolute value function  $|\cdot|$ . A subset  $S$  of  $X$  is called a balanced set or balanced if:

$$\forall a \in \mathbb{K} \text{ s.t. } |a| \leq 1 : aS \subseteq S.$$

## V Affine space (仿射空間)

An affine space is a point set  $A$  together with a vector space  $\vec{A}$ , and a transitive and free action of the additive group of  $\vec{A}$  on the set  $A$ . The vector space  $\vec{A}$  is said to be associated to the affine space, and its elements are called vectors, translations, or sometimes free vectors.

Explicitly, the definition above means that the action is a mapping, generally denoted as an addition,

$$\begin{aligned} A \times \vec{A} &\rightarrow A \\ (a, v) &\mapsto a + v \end{aligned}$$

that has the following properties.

1. Right identity:

$$\forall a \in A, a + 0 = a, \quad \text{where } 0 \text{ is the zero vector in } \vec{A}$$

2. Associativity:

$$\forall v, w \in \vec{A}, \forall a \in A, (a + v) + w = a + (v + w)$$

3. Free and transitive action:

$\forall a \in A$  : the mapping  $\vec{A} \rightarrow A : v \mapsto a + v$  is a bijection.

4. Existence of one-to-one translations

$\forall v \in \vec{A}$  : the mapping  $\vec{A} \rightarrow A : v \mapsto a + v$  is a bijection.

## VI Tangent space (切空間)

Suppose that  $M$  is a  $C^k$  differentiable manifold (with smoothness  $k \geq 1$ ) and that  $x \in M$ . Pick a coordinate chart  $\varphi : U \rightarrow \mathbb{R}^n$ , where  $U$  is an open subset of  $M$  containing  $x$ . Suppose further that two curves  $\gamma_1, \gamma_2 : (-1, 1) \rightarrow M$  with  $\gamma_1(0) = \gamma_2(0) = x$  are given such that both  $\varphi \circ \gamma_1, \varphi \circ \gamma_2 : (-1, 1) \rightarrow \mathbb{R}^n$  are differentiable. We call these differentiable curves initialized at  $x$ . Then  $\gamma_1$  and  $\gamma_2$  are said to be equivalent at 0 if and only if the derivatives of  $\varphi \circ \gamma_1$  and  $\varphi \circ \gamma_2$  at 0 coincide. This defines an equivalence relation on the set of all differentiable curves initialized at  $x$ , and equivalence classes of such curves are called tangent vectors of  $M$  at  $x$ . The tangent space of  $M$  at  $x$ , denoted by  $T_x M$ , is then defined as the set of all tangent vectors at  $x$ , which does not depend on the choice of coordinate chart  $\varphi$ .

## VII Tangent bundle (切叢)

A tangent bundle  $TM$  of a smooth manifold  $M$  is defined to be

$$TM = \{(x, v) : x \in M \wedge v \in T_x M\},$$

where  $T_x M$  is the tangent space at  $x$ .

# 4 Functional Analysis (泛函分析)

## I Functional (泛函)

A functional is a function from a vector space into the field of real or complex numbers.

## II Norm (範數/模長) and seminorm

Given a vector space  $X$  over a ordered field  $F$ , a seminorm on  $X$  is a real-valued function  $p : X \rightarrow \mathbb{R}$  with the following conditions, where  $|s|$  denotes the usual absolute value of a scalar  $s$ :

1. Subadditivity/Triangle inequality:

$$\forall x, y \in X : p(x + y) \leq p(x) + p(y).$$

2. Absolute homogeneity:

$$\forall s \in \mathbb{R}, x \in X : p(sx) = |s|p(x).$$

These conditions implies that:

1. Non-negativity:  $\forall x \in X : p(x) \geq 0$ .

2.  $p(0) = 0$ .

A norm on  $X$  is a seminorm  $p : X \rightarrow \mathbb{R}$  with the following properties:

- Positive definiteness/Positiveness/Point-separating:

$$\forall x \in X : p(x) = 0 \implies x = 0$$

### III Sublinear function or quasi-seminorm

Let  $p : X \rightarrow \mathbb{R}$  be a function on a vector space  $X$  over the field  $\mathbb{K}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ .  $p$  is called a sublinear function or quasi-seminorm if it satisfies the following conditions:

1. Subadditivity/Triangle inequality:

$$\forall x, y \in X : p(x + y) \leq p(x) + p(y).$$

2. Nonnegative homogeneity:

$$\forall x \in X, r \geq 0 : p(rx) = rp(x).$$

### IV Absolutely continuous function (絕對連續函數)

Function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is absolutely continuous on  $I$  if for every positive number  $\varepsilon$ , there is a positive number  $\delta$  such that, for every finite sequence of pairwise disjoint sub-intervals  $(x_k, y_k)$  of  $I$  with  $x_k < y_k \in I$  and cardinality  $N$ ,

$$\sum_{k=1}^N (y_k - x_k) < \delta \implies \sum_{k=1}^N |f(y_k) - f(x_k)| < \varepsilon.$$

The collection of all absolutely continuous functions on  $I$  is denoted  $AC(I)$ .

### V Bounded linear operator (有界線性運算子)

A bounded linear operator is a linear transformation  $L : X \rightarrow Y$  between topological vector spaces (TVSs)  $X$  and  $Y$  that maps bounded subsets of  $X$  to bounded subsets of  $Y$ . If  $X$  and  $Y$  are normed vector spaces, then  $L$  is bounded if and only if there exists some  $M > 0$  such that:

$$\forall x \in X : \|Lx\|_Y \leq M\|x\|_X.$$

The smallest such  $M$  is called the operator norm of  $L$  and denoted by  $\|L\|$ . A bounded operator between normed spaces is continuous and vice versa.

### VI Von Neumann bounded or bounded

Suppose  $X$  is a topological vector space (TVS) over a field  $\mathbb{K}$ . A subset  $B$  of  $X$  is called von Neumann bounded (or bounded) in  $X$  if for every neighborhood  $V$  of the origin, there exists a real  $r > 0$  such that  $B \subseteq sV$  for all scalars  $s$  satisfying  $|s| \geq r$ .

## VII Hahn–Banach theorem

Let  $p : X \rightarrow \mathbb{R}$  be a sublinear functional on a vector space  $X$  over the field  $\mathbb{K}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ . If  $f : M \rightarrow \mathbb{K}$  is a linear functional on a vector subspace  $M$  such that

$$\forall m \in M : f(m) \leq p(m),$$

then there exists a linear functional  $F : X \rightarrow \mathbb{K}$  such that

$$\forall m \in M : F(m) = f(m),$$

$$\forall x \in X : F(x) \leq p(x).$$

## 5 Measure Theory (測度論)

### I $\sigma$ -algebra

Let  $X$  be some set, and let  $P(X)$  represent its power set. Then a subset  $\Sigma \subseteq P(X)$  is called a  $\sigma$ -algebra if and only if it satisfies the following three properties:

1.  $X \in \Sigma$ .
2. Closed under complementation:  $A \in \Sigma \implies X \setminus A \in \Sigma$ .
3. Closed under countable unions:  $\forall \mathcal{S} \subseteq \Sigma$  s.t.  $|\mathcal{S}| < \infty : \bigcup_{A \in \mathcal{S}} A \in \Sigma$ .

### II Measurable space (可測空間)

Consider a set  $X$  and a  $\sigma$ -algebra  $\Sigma$  on  $X$ . Then the tuple  $(X, \Sigma)$  is called a measurable space.

### III Measure (測度)

Let  $X$  be a set and  $\Sigma$  be a  $\sigma$ -algebra over  $X$ . A set function  $\mu$  from  $\Sigma$  to the extended real number line, defined to be  $[-\infty, \infty]$  or  $\mathbb{R} \cup \{-\infty, \infty\}$ , is called a measure or positive measure on  $(X, \Sigma)$  if the following conditions hold:

1. Non-negativity:

$$\forall E \in \Sigma : \mu(E) \geq 0$$

2.  $\mu(\emptyset) = 0$ .

3. Countable additivity (or  $\sigma$ -additivity): For all countable collection  $\{E_k\}_{k=1}^{\infty}$  of pairwise disjoint sets in  $\Sigma$  (i.e.  $\forall i \neq j : E_i \cap E_j = \emptyset$ ),

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

If the condition of non-negativity is dropped, then  $\mu$  is called a signed measure.

## IV $\sigma$ -finite measure ( $\sigma$ 有限測度)

Let  $(X, \Sigma)$  be a measurable space and  $\mu$  be a positive measure or signed measure on it.  $\mu$  is called a  $\sigma$ -finite measure, if:

$$\exists \{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma \text{ s.t. } \forall n \in \mathbb{N} : \mu(A_n) < \infty \wedge \bigcup_{n \in \mathbb{N}} A_n = X.$$

If  $\mu$  is a  $\sigma$ -finite measure, the measure space  $(X, \Sigma, \mu)$  is called a  $\sigma$ -finite measure space.

## V Measure space (測度空間)

A measure space is a triple  $(X, \Sigma, \mu)$ , where:

1.  $X$  is a set.
2.  $\Sigma$  is a  $\sigma$ -algebra on the set  $X$ .
3.  $\mu$  is a measure on  $(X, \Sigma)$ .

## VI Ring of sets

We call a family  $\mathcal{R}$  of subsets of  $\Omega$  a ring of sets if it has the following properties:

1.  $\emptyset \in \mathcal{R}$ .
2. Closed under pairwise unions:  $\forall A, B \in \mathcal{R} : A \cup B \in \mathcal{R}$ .
3. Closed under relative complements:  $\forall A, B \in \mathcal{R} : A \setminus B \in \mathcal{R}$ .

## VII Pre-measure (前測度)

Let  $\mathcal{R}$  be a ring of subsets of a fixed set  $X$ . A set function  $\mu_0$  from  $\mathcal{R}$  to the extended real number line is called a pre-measure on  $(X, \mathcal{R})$  if the following conditions hold:

1. Non-negativity:

$$\forall E \in \mathcal{R} : \mu_0(E) \geq 0$$

2.  $\mu_0(\emptyset) = 0$ .

3. Countable additivity (or  $\sigma$ -additivity): For all countable collection  $\{E_k\}_{k=1}^{\infty}$  of pairwise disjoint sets in  $\mathcal{R}$  (i.e.  $\forall i \neq j : E_i \cap E_j = \emptyset$ ),

$$\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k)$$

## VIII Outer measure or exterior measure

Given a set  $X$ , let  $2^X$  denote the collection of all subsets of  $X$ , including the empty set  $\emptyset$ . An outer measure on  $X$  is a set function  $\mu : 2^X \rightarrow [0, \infty]$  such that

1.  $\mu(\emptyset) = 0$ .
2. Countably subadditive:

$$\forall A \subseteq X, \{B_i \subseteq X\}_{i=1}^{\infty} : A \subseteq \bigcup_{i=1}^{\infty} B_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

## IX Carathéodory's extension theorem

Let  $\mathcal{R}$  be a ring of sets on  $X$ , let  $\mu : \mathcal{R} \rightarrow [0, +\infty]$  be a pre-measure on  $\mathcal{R}$ , and let  $\sigma(\mathcal{R})$  be a  $\sigma$ -algebra generated by  $\mathcal{R}$ .

The Carathéodory's extension theorem states that there exists a measure  $\mu' : \sigma(\mathcal{R}) \rightarrow [0, +\infty]$  such that  $\mu'$  is an extension of  $\mu$ , that is,

$$\mu'|_{\mathcal{R}} = \mu.$$

Moreover, if  $\mu$  is  $\sigma$ -finite, then the extension  $\mu'$  is unique and also  $\sigma$ -finite.

## X Lebesgue measure (勒貝格測度)

For any interval  $I = [a, b]$  or  $I = (a, b)$  that is a subset of  $\mathbb{R}$ , let  $\ell(I) = b - a$  denote its length. For any subset  $E \subseteq \mathbb{R}$ , the Lebesgue outer measure  $\lambda(E)$  is defined to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k \in \mathbb{N}} \text{ is a sequence of intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The above definition can be generalised to higher dimensions as follows. For any  $n$ -dimensional rectangular cuboid, that is, a cuboid with rectangular faces in which all of its dihedral angles are right angles,  $C$ , which is a Cartesian product  $C = \prod_{i=1}^n I_i$  of intervals, we define its Lebesgue outer measure  $\lambda(C)$  to be

$$\lambda(C) := \prod_{i=1}^n \ell(I_i).$$

For any subset  $E \subseteq \mathbb{R}^n$ , we define its Lebesgue outer measure  $\lambda(E)$  to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(C_k) : (C_k)_{k \in \mathbb{N}} \text{ is a sequence of products of intervals with } E \subseteq \bigcup_{k=1}^{\infty} C_k \right\}.$$

We say a set  $E \in \mathbb{R}^n$  satisfies the Carathéodory criterion if

$$\forall A \subseteq \mathbb{R}^n : \lambda(A) = \lambda(A \cap E) + \lambda(A \cap (\mathbb{R}^n \setminus E)).$$

The sets  $E \subseteq \mathbb{R}^n$  that satisfy the Carathéodory criterion are said to be Lebesgue-measurable, with its Lebesgue measure being defined as its Lebesgue outer measure. The set of all such  $E$  forms a  $\sigma$ -algebra.

A set  $E \subseteq \mathbb{R}^n$  that does not satisfy the Carathéodory criterion is not Lebesgue-measurable. ZFC proves that such sets do exist.



## XI Hausdorff measure (郝斯多夫測度)

Let  $X, p$  be a metric space. For any subset  $U \subseteq X$ , let  $\text{diam } U$  denote its diameter, that is

$$\text{diam}(U) := \sup\{p(x, y) : x, y \in U\}, \quad \text{diam}(\emptyset) := 0.$$

Let  $S$  be any subset of  $X$ , and  $\delta > 0$  a real number. Define

$$H_\delta^d(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(U_i))^d : S \subseteq \bigcup_{i=1}^{\infty} U_i \wedge \text{diam}(U_i) < \delta \right\}.$$

Note that  $H_\delta^d(S)$  is monotone nonincreasing in  $\delta$  since the larger  $\delta$  is, the more collections of sets are permitted, making the infimum not larger. Thus,  $\lim_{\delta \rightarrow 0} H_\delta^d(S)$  exists but may be infinite. Let

$$H^d(S) := \lim_{\delta \rightarrow 0} H_\delta^d(S).$$

It can be seen that  $H^d(S)$  is an outer measure, or more precisely, a metric outer measure. By Carathéodory's extension theorem, its restriction to the  $\sigma$ -algebra of Carathéodory-measurable sets is a measure. It is called the  $d$ -dimensional Hausdorff measure of  $S$ . Due to the metric outer measure property, all Borel subsets of  $X$  are  $H^d$  measurable.

## XII Radon measure (拉東測度)

Let  $\mu$  be a measure on a  $\sigma$ -algebra of Borel sets of a Hausdorff topological space  $X$ .

- The measure  $\mu$  is called inner regular or tight if, for every open set  $U$ ,  $\mu(U)$  equals the supremum of  $\mu(K)$  over all compact subsets  $K$  of  $U$ .
- The measure  $\mu$  is called outer regular if, for every Borel set  $B$ ,  $\mu(B)$  equals the infimum of  $\mu(U)$  over all open sets  $U$  that contain  $B$ .
- The measure  $\mu$  is called locally finite if every point of  $X$  has a neighborhood  $U$  for which  $\mu(U)$  is finite.

The measure  $\mu$  is called a Radon measure if it is inner regular and locally finite. In many situations, such as finite measures on locally compact spaces, this also implies outer regularity.

## 6 Geometry (幾何學)

### I Convex set (凸集)

Let  $S$  be a vector space or an affine space over some ordered field. A subset  $C$  of  $S$  is convex if, for all  $x$  and  $y$  in  $C$ , the line segment connecting  $x$  and  $y$  is included in  $C$ .

## II Affine hull (仿射包)

An affine combination is a linear combination of points where all coefficients sum up to 1.

The affine hull  $\text{aff}(S)$  of a set of points  $S$  in a Euclidean space is the smallest affine space that contains all the points in  $S$ , namely, the set of all affine combinations of the points in  $S$ , i.e.:

$$\text{aff}(S) = \left\{ \sum_{i=1}^k \alpha_i x_i : k > 0, x_i \in S, \alpha_i \in \mathbb{R}, \sum_{i=1}^k \alpha_i = 1 \right\}$$

## III Convex hull (凸包)

A convex combination is a linear combination of points where all coefficients are non-negative and sum up to 1.

The convex hull  $\text{conv}(S)$  of a set of points  $S$  in a Euclidean space is the smallest convex set that contains all the points in  $S$ , namely, the set of all convex combinations of points in  $S$ , i.e.:

$$\text{conv}(S) = \left\{ \sum_{i=1}^k \alpha_i x_i : k > 0, x_i \in S, \alpha_i \in [0, 1], \sum_{i=1}^k \alpha_i = 1 \right\}$$

## IV Manifold (流形)

Let  $M$  be a Hausdorff space. If for any  $x \in M$ , there exists a neighborhood  $U_x$  of  $x$  that is homeomorphic to some open set of the  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , then  $M$  is called an  $m$ -dimensional manifold.

## V Curvilinear coordinate

A curvilinear coordinate is a coordinate system for Euclidean space such that there exists an invertible map from a Cartesian coordinate system to it.