Inequalities

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1 Inequalities

I Triangle Inequality (三角不等式)

For any normed space V with norm $\|\cdot\|$:

$$\forall a, b \in V: \ \|a\| + \|b\| \geq \|a + b\| \geq |\|a\| - \|b\||\,.$$

II Reverse Triangle Inequality (反三角不等式)

For any normed space V with norm $\|\cdot\|$:

$$\forall a, b \in V: \ \|a\| + \|b\| \geq \|a - b\| \geq |\|a\| - \|b\||\,.$$

III Jensen's Inequality (詹森不等式)

Let (Ω, μ) be a probability space, $g : \Omega \to \mathbb{R}$ be a real-valued μ-integrable function, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, Jensen's inequality states:

$$\int_\Omega \varphi \circ g(\omega) \, \mathrm{d}\mu(\omega) \geq \varphi\left(\int_\Omega g(\omega) \, \mathrm{d}\mu(\omega)\right)$$

Proof.
Since φ is convex, at each real number x, we have a non-empty set of subderivatives, which may be thought of as lines touching the graph of φ at x, but which are below the graph of φ at all points.

We define:

$$x_0 := \int_\Omega g \, \mathrm{d}\mu.$$

Because of the existence of subderivatives for convex functions, we may choose a and b such that $ax + b \leq \varphi(x)$ for all real x and $ax_0 + b = \varphi(x_0)$.

But then we have that for almost all $\omega \in \Omega$:

$$\varphi(g(\omega)) \geq ag(\omega) + b$$

Since we have a probability measure, the integral is monotone with $\mu(\Omega) = 1$ so that

$$\begin{aligned}
\int_\Omega \varphi \circ g(\omega) \, \mathrm{d}\mu(\omega) &\geq \int_\Omega ag(\omega) + b \, \mathrm{d}\mu(\omega) \\
&= a \int_\Omega g \, \mathrm{d}\mu + b \int_\Omega \mathrm{d}\mu \\
&= ax_0 + b = \varphi(x_0) \\
&= \varphi\left(\int_\Omega g \, \mathrm{d}\mu\right).
\end{aligned}$$

$\square$

IV AM-GM Inequality (算幾不等式)

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

Proof.

Lemma: Jensen's inequality.

Applying Jensen's inequality to the logarithm function, which is concave, and the arithmetic mean:

$$\log\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \sum_{i=1}^{n} \frac{1}{n} \log\left(x_i\right) = \log\left(\left(\prod_{i=1}^{n} x_i\right)^{1/n}\right)$$

Exponentiating both sides gives the desired inequality:

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

□

V Cauchy-Schwarz Inequality (柯西-施瓦茨不等式)

For any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

and

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}| \iff \mathbf{u} \parallel \mathbf{v},$$

where $\mathbf{u} \cdot \mathbf{v}$ is the standard inner product defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i \overline{v_i},$$

with $\overline{v_i}$ denoting the complex conjugate of v_i.

Proof.

Consider the complex number $z = \mathbf{u} \cdot \mathbf{v}$.

Define the function

$$f(t) = |\mathbf{u} + t\mathbf{v}|^2$$

for some real number t. Then, we have

$$\begin{aligned} f(t) &= (\mathbf{u} + t\mathbf{v}) \cdot \overline{(\mathbf{u} + t\mathbf{v})} \\ &= \left(\mathbf{u} \cdot \bar{\mathbf{u}}\right) + t\left(\mathbf{u} \cdot \bar{\mathbf{v}}\right) + t\left(\mathbf{v} \cdot \bar{\mathbf{u}}\right) + t^2\left(\mathbf{v} \cdot \bar{\mathbf{v}}\right) \\ &= |\mathbf{u}|^2 + 2t\Re(\mathbf{u} \cdot \bar{\mathbf{v}}) + t^2|\mathbf{v}|^2. \end{aligned}$$

Since $f(t) \geq 0$ for all $t \in \mathbb{R}$, the quadratic equation in t must have a non-positive discriminant. The discriminant of this quadratic is:

$$\begin{aligned} \Delta &= \left(2\Re(\mathbf{u} \cdot \bar{\mathbf{v}})\right)^2 - 4 \times |\mathbf{v}|^2 \times |\mathbf{u}|^2 \\ &= 4\left(\Re(\mathbf{u} \cdot \bar{\mathbf{v}})\right)^2 - 4|\mathbf{u}|^2|\mathbf{v}|^2. \end{aligned}$$

We require $\Delta \leq 0$. This implies:

$$\left(\Re(\mathbf{u} \cdot \bar{\mathbf{v}})\right)^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2.$$

Taking the square root of both sides and noting that $|\mathbf{u} \cdot \mathbf{v}| \geq \Re(\mathbf{u} \cdot \bar{\mathbf{v}})$:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

Equality $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$ holds if and only if the discriminant $\Delta = 0$. This happens when the quadratic equation has a double root or equivalently, when $\mathbf{u} + t\mathbf{v} = 0$ for some real t, implying $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, meaning:

$$\mathbf{u} \parallel \mathbf{v}.$$

□

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

and

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}| \iff \mathbf{u} \parallel \mathbf{v},$$

where $\mathbf{u} \cdot \mathbf{v}$ is the standard inner product defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$