Measure Theory

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Contents

1 Measure Theory (測度論)

I σ-algebra

Let X be some set, and let $P(X)$ represent its power set. Then a subset $\Sigma \subseteq P(X)$ is called a σ-algebra if and only if it satisfies the following three properties:

1. $X \in \Sigma$.
2. Closed under complementation: $A \in \Sigma \implies X \setminus A \in \Sigma$.
3. Closed under countable unions: $\forall S \subseteq \Sigma$ s.t. $|S| < \infty : \bigcup_{A \in S} A \in \Sigma$.

II Measurable space (可測空間)

Consider a set X and a σ-algebra Σ on X. Then the tuple (X, Σ) is called a measurable space.

III Measure (測度)

Let X be a set and Σ be a σ-algebra over X. A set function μ from Σ to the extended real number line, defined to be $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$, is called a measure or positive measure on (X, Σ) if the following conditions hold:

1. Non-negativity:
$$\forall E \in \Sigma : \ \mu(E) \geq 0$$
2. $\mu(\emptyset) = 0$.
3. Countable additivity (or σ-additivity): For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ (i.e. $\forall i \neq j : \ E_i \cap E_j = \emptyset$),
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

If the condition of non-negativity is dropped, then μ is called a signed measure.

IV σ-finite measure (σ有限測度)

Let (X, Σ) be a measurable space and μ be a positive measure or signed measure on it. μ is called a σ-finite measure, if:

$$\exists \{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma \text{ s.t. } \forall n \in \mathbb{N} : \ \mu(A_n) < \infty \land \bigcup_{n \in \mathbb{N}} A_n = X.$$

If μ is a σ-finite measure, the measure space (X, Σ, μ) is called a σ-finite measure space.

V Measure space (測度空間)

A measure space is a triple (X, Σ, μ), where:

1. X is a set.
2. Σ is a σ-algebra on the set X.
3. μ is a measure on (X, Σ).

VI Ring of sets

We call a family $\mathcal{R}$ of subsets of Ω a ring of sets if it has the following properties:

1. $\varnothing \in \mathcal{R}$.
2. Closed under pairwise unions: $\forall A, B \in \mathcal{R} : A \cup B \in \mathcal{R}$.
3. Closed under relative complements: $\forall A, B \in \mathcal{R} : A \setminus B \in \mathcal{R}$.

VII Pre-measure (前測度)

Let $\mathcal{R}$ be a ring of subsets of a fixed set X. A set function μ_0 from $\mathcal{R}$ to the extended real number line is called a pre-measure on $(X, \mathcal{R})$ if the following conditions hold:

1. Non-negativity:
$$\forall E \in \mathcal{R} : \mu_0(E) \geq 0$$
2. $\mu_0(\varnothing) = 0$.
3. Countable additivity (or σ-additivity): For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{R}$ (i.e. $\forall i \neq j : E_i \cap E_j = \varnothing$),
$$\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k)$$

VIII Outer measure or exterior measure

Given a set X, let 2^X denote the collection of all subsets of X, including the empty set $\varnothing$. An outer measure on X is a set function $\mu : 2^X \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$.
2. Countably subadditive:
$$\forall A \subseteq X, \{B_i \subseteq X\}_{i=1}^{\infty} : A \subseteq \bigcup_{i=1}^{\infty} B_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

IX Carathéodory's extension theorem

Let $\mathcal{R}$ be a ring of sets on X, let $\mu : \mathcal{R} \to [0, +\infty]$ be a pre-measure on $\mathcal{R}$, and let $\sigma(\mathcal{R})$ be a σ-algebra generated by $\mathcal{R}$.

The Carathéodory's extension theorem states that there exists a measure $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ such that μ' is an extension of μ, that is,

$$\mu'\big|_{\mathcal{R}} = \mu.$$

Moreover, if μ is σ-finite, then the extension μ' is unique and also σ-finite.

X Lebesgue measure (勒貝格測度)

For any interval $I = [a, b]$ or $I = (a, b)$ that is a subset of $\mathbb{R}$, let $\ell(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the Lebesgue outer measure $\lambda(E)$ is defined to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The above definition can be generalised to higher dimensions as follows. For any n-dimensional rectangular cuboid, that is, a cuboid with rectangular faces in which all of its dihedral angles are right angles, C, which is a Cartesian product $C = \prod_{i=1}^{n} I_i$ of intervals, we define its Lebesgue outer measure $\lambda(C)$ to be

$$\lambda(C) := \prod_{i=1}^{n} \ell(I_i).$$

For any subset $E \subseteq \mathbb{R}^n$, we define its Lebesgue outer measure $\lambda(E)$ to be

$$\lambda(E) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(C_k) : (C_k)_{k\in\mathbb{N}} \text{ is a sequence of products of intervals with } E \subseteq \bigcup_{k=1}^{\infty} C_k \right\}.$$

We say a set $E \in \mathbb{R}^n$ satisfies the Carathéodory criterion if

$$\forall A \subseteq \mathbb{R} : \lambda(A) = \lambda(A \cap E) + \lambda(A \cap (\mathbb{R}^n \setminus E)).$$

The sets $E \subseteq \mathbb{R}^n$ that satisfy the Carathéodory criterion are said to be Lebesgue-measurable, with its Lebesgue measure being defined as its Lebesgue outer measure. The set of all such E forms a σ-algebra.

A set $E \subseteq \mathbb{R}^n$ that does not satisfy the Carathéodory criterion is not Lebesgue-measurable. ZFC proves that such sets do exist.

XI Hausdorff measure (郝斯多夫測度)

Let X, p be a metric space. For any subset $U \subseteq X$, let diam U denote its diameter, that is

$$\text{diam}(U) := \sup\{p(x, y) : x, y \in U\}, \quad \text{diam}(\varnothing) := 0.$$

Let S be any subset of X, and $\delta > 0$ a real number. Define

$$H^d_\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} \left(\text{diam}(U_i)\right)^d : S \subseteq \bigcup_{i=1}^{\infty} U_i \wedge \text{diam}(U_i) < \delta \right\}.$$

Note that $H_\delta^d(S)$ is monotone nonincreasing in δ since the larger δ is, the more collections of sets are permitted, making the infimum not larger. Thus, $\lim_{\delta \to 0} H_\delta^d(S)$ exists but may be infinite. Let

$$H^d(S) := \lim_{\delta \to 0} H_\delta^d(S).$$

It can be seen that $H^d(S)$ is an outer measure, or more precisely, a metric outer measure. By Carathéodory's extension theorem, its restriction to the σ-algebra of Carathéodory-measurable sets is a measure. It is called the d-dimensional Hausdorff measure of S. Due to the metric outer measure property, all Borel subsets of X are H^d measurable.

XII Radon measure (拉東測度)

Let μ be a measure on a σ-algebra of Borel sets of a Hausdorff topological space X.

- The measure μ is called inner regular or tight if, for every open set U, $\mu(U)$ equals the supremum of $\mu(K)$ over all compact subsets K of U.
- The measure μ is called outer regular if, for every Borel set B, $\mu(B)$ equals the infimum of $\mu(U)$ over all open sets U that contain B.
- The measure μ is called locally finite if every point of X has a neighborhood U for which $\mu(U)$ is finite.

The measure μ is called a Radon measure if it is inner regular and locally finite. In many situations, such as finite measures on locally compact spaces, this also implies outer regularity.