Limit

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一、　A Limit for a Sequence

(一)　Glossary of Terms

1. **Sequence (數列): A sequence is a function whose domain is an interval of integers, usually denoted as $\langle a_n \rangle$, $\{a_n\}$, or (a_n), sometimes with domain as $\langle a_n \rangle_{n=1}^{m}$, $\{a_n\}_{n=1}^{m}$, or $(a_n)_{n=1}^{m}$, where the subscript n refers to the nth element of the sequence, that is, the function value when the independent variable is n.**

2. **Finite sequence (有限數列): A finite sequence is a sequence with finite terms, e.g. $\langle a_n \rangle_{n=1}^{m} = \langle a_1, a_2, \ldots, a_m \rangle$, $m \geq 1$ and m is finite.**

3. **Infinite sequence (無限數列): An infinite sequence is a sequence with infinite terms, e.g. $\langle a_n \rangle_{n=1}^{\infty} = \langle a_1, a_2, \ldots \rangle$. Unless otherwise specified, the sequences referred to below are infinite sequences.**

4. **Monotone Increasing/Increasing/Non-Decreasing Sequence (單調遞增/遞增/非遞減數列): $\langle a_n \rangle$ is a monotone increasing/increasing/non-decreasing sequence if and only if $\forall n$ such that $\exists a_n, a_{n+1} : a_n \leq a_{n+1}$.**

5. **Strictly Increasing Sequence (嚴格遞增數列): $\langle a_n \rangle$ is a strictly increasing sequence if and only if $\forall n$ such that $\exists a_n, a_{n+1} : a_n < a_{n+1}$.**

6. **Monotone Decreasing/Decreasing/Non-Increasing Sequence (單調遞減/遞減/非遞增數列): $\langle a_n \rangle$ is a monotone decreasing/decreasing/non-increasing sequence if and only if $\forall n$ such that $\exists a_n, a_{n+1} : a_n \geq a_{n+1}$.**

7. **Strictly Decreasing Sequence (嚴格遞減數列): $\langle a_n \rangle$ is a strictly decreasing sequence if and only if $\forall n$ such that $\exists a_n, a_{n+1} : a_n > a_{n+1}$.**

8. **Monotone Sequence (單調數列): $\langle a_n \rangle$ is a monotone sequence if and only if $\langle a_n \rangle$ is either a increasing sequence or a decreasing sequence.**

9. **Upper Bound (上界): M is an upper bound of $\langle a_n \rangle$ if and only if $\forall n$ such that $\exists a_n : a_n \leq M$. $\exists$ upper bound of $\langle a_n \rangle \iff \langle a_n \rangle$ is bounded-above.**

10. **Lower Bound (下界): m is an lower bound of $\langle a_n \rangle$ if and only if $\forall n$ such that $\exists a_n : a_n \geq m$. $\exists$ lower bound of $\langle a_n \rangle \iff \langle a_n \rangle$ is bounded-below.**

11. **Supremum/Least Upper Bound (最小上界): M is the supremum/least upper bound of $\langle a_n \rangle$ if and only if $\forall$ upper bound M of $\langle a_n \rangle : M_0 \leq M$, denoted as $\sup\limits_{n} a_n$.**

12. Infimum/Greatest Lower Bound (最大下界): M is the infimum/greatest lower bound of $\langle a_n \rangle$ if and only if $\forall$ lower bound m of $\langle a_n \rangle : m_0 \geq m$, denoted as $\inf_n a_n$.

13. Bounded: A sequence is bounded if and only if it is bounded-above and bounded-below.

14. Series (級數): The sum of the terms of a sequence.

15. Finite Series (有限級數): The sum of the terms of a finite sequence.

16. Infinite Series (無窮級數): The sum of the terms of an infinite sequence.

(二) Definition of a Limit for a Sequence

For a sequence $\langle a_n \rangle$, the limit of $\langle a_n \rangle$ as n approaches ∞ is defined as follows:

$$\lim_{n \to \infty} a_n = L \equiv \forall \epsilon > 0 : \exists N \in \mathbb{N} \,\text{such that}\, n \geq N \implies |a_n - L| < \epsilon.$$

In other words, as n becomes arbitrarily large, a_n gets arbitrarily close to L.

(三) Infinite Limits

$$\lim_{n \to \infty} a_n = \infty \equiv \forall M > 0, \exists N \in \mathbb{N} \,\text{such that}\, n > N \implies a_n > M.$$
$$\lim_{n \to \infty} a_n = -\infty \equiv \forall M < 0, \exists N \in \mathbb{N} \,\text{such that}\, n > N \implies a_n < M.$$

(四) Existence of a Limit

$$\exists \lim_{n \to \infty} a_n \iff \exists \,\text{finite}\, L \,\text{such that}\, \forall \epsilon > 0 : \exists N \in \mathbb{N} \,\text{such that}\, n \geq N \implies |a_n - L| < \epsilon.$$

(五) Convergence (收斂) and Divergence (發散)

A sequence $\langle a_n \rangle$ converges to L if $\exists \lim_{n \to \infty} a_n \wedge \lim_{n \to \infty} a_n = L$. If no such L exists, we say the sequence does not converge, namely, the sequence diverges.

(六) Limit Laws

Given sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ which:

$$\exists \lim_{n \to \infty} a_n \wedge \exists \lim_{n \to \infty} b_n$$

Then:

$$\lim_{x \to a} \langle a_n \rangle + \langle b_n \rangle = \lim_{x \to a} \langle a_n \rangle + \lim_{x \to a} \langle b_n \rangle$$
$$\lim_{x \to a} \langle a_n \rangle - \langle b_n \rangle = \lim_{x \to a} \langle a_n \rangle - \lim_{x \to a} \langle b_n \rangle$$
$$\text{If}\, c \,\text{is a constant} : \lim_{x \to a} c \langle a_n \rangle = c \lim_{x \to a} \langle a_n \rangle$$
$$\lim_{x \to a} \langle a_n \rangle \langle b_n \rangle = \lim_{x \to a} \langle a_n \rangle \lim_{x \to a} \langle b_n \rangle$$
$$\text{If}\, \langle b_n \rangle \neq 0 \wedge \lim_{x \to a} \langle b_n \rangle \neq 0 : \lim_{x \to a} \frac{\langle a_n \rangle}{\langle b_n \rangle} = \frac{\lim_{x \to a} \langle a_n \rangle}{\lim_{x \to a} \langle b_n \rangle}$$

(七) Special Sequences and Their Limits

$$\text{If } c \text{ is a constant}: \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\forall r \in (-1,\, 1): \lim_{n\to\infty} r^n = 0$$

$$\exists \lim_{n\to\infty} r^n \iff r \in (-1,\, 1]$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

(八) Squeeze (夾擠)/Sandwich (三明治) theorem

Given sequences $\langle a_n \rangle$, $\langle b_n \rangle$, and $\langle c_n \rangle$ which:

$$a_n \le c_n \le b_n$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L,$$

then:

$$\lim_{n\to\infty} c_n = L.$$

(九) Monotone Convergence Theorem (單調收斂定理)/Completeness of the Real Number (實數的完備性)

Proposition.
(A) For a non-decreasing and bounded-above sequence of real numbers $\langle a_n \rangle_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty} a_n = \sup_n a_n$$

(B) For a non-increasing and bounded-below sequence of real numbers $\langle a_n \rangle_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty} a_n = \inf_n a_n$$

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be the set of values of $\langle a_n \rangle$. By assumption, $\{a_n\}$ is non-empty and bounded-above by $\sup_n a_n$. Let $c = \sup_n a_n$.

$$\forall \epsilon > 0: \exists N \text{ such that } c \ge a_N > c - \epsilon,$$

since otherwise $c - \epsilon$ is a strictly smaller upper bound of $\langle a_n \rangle$, contradicting the definition of the supremum.
Then since $\langle a_n \rangle$ is non decreasing, and c is an upper bound:

$$\forall \epsilon > 0: \exists N \text{ such that } \forall n > N: |c - a_n| = c - a_n \le c - a_N = |c - a_N| < \epsilon.$$

The proof of the (B) part is analogous or follows from (A) by considering $\langle -a_n \rangle_{n\in\mathbb{N}}$. □

Theorem.
If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a monotone sequence of real numbers, i.e., if $a_n \le a_{n+1}$ for every $n \ge 1$ or $a_n \ge a_{n+1}$ for every $n \ge 1$, then this sequence has a finite limit if and only if the sequence is bounded.

Proof. "If"-direction: The proof follows directly from the proposition.
"Only If"-direction: By $(\epsilon,\, \delta)$-definition of limit, every sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ with a finite limit L is necessarily bounded. □

二、 A Limit for a Series

(一) Definition

Let:

$$S_n = \sum_{i=1}^{n} a_i,$$

where a_i are terms of a sequence. The limit of S_n, denoted as $\lim\limits_{n\to\infty} S_n$ or $\sum\limits_{i=1}^{\infty} a_i$, is defined as the following:

$$\sum_{i=1}^{\infty} a_i = L \equiv \forall \epsilon > 0 :\ \exists N \in \mathbb{N}\, \text{such that}\, n \geq N \implies |S_n - L| < \epsilon.$$

(二) Convergence (收斂) and Divergence (發散)

A series S_n converges to L if $\exists \lim\limits_{n\to\infty} S_n \wedge \lim\limits_{n\to\infty} S_n = L$. If no such L exists, we say the series does not converge, namely, the series diverges.

(三) Limit Laws

Given series $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{i=1}^{n} b_i$ which:

$$\exists \lim_{n\to\infty} \sum_{i=1}^{n} a_i \wedge \exists \lim_{n\to\infty} \sum_{i=1}^{n} b_i$$

Then:

$$\sum_{i=1}^{\infty} a_i + \langle b_n \angle = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$$

$$\sum_{i=1}^{\infty} a_i - \langle b_n \angle = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i$$

$$\text{If}\, c\, \text{is a constant} :\ \sum_{i=1}^{\infty} c a_i = c \sum_{i=1}^{\infty} a_i$$

三、 A Limit for a Function

(一) Limit at Finity

Let I be an interval containing the point a. Let $f(x)$ be a function defined on I, except possibly at a itself. The limit of $f(x)$ as x approaches a is defined as follows:

$$\lim_{x\to a} f(x) = L \equiv \forall \epsilon > 0 :\ \exists \delta > 0\, \text{such that}\, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

In other words, as x becomes arbitrarily close to a, $f(x)$ gets arbitrarily close to L.

(二) Limit at Infinity

1. **Let I be a left-bounded, right-unbounded interval with the point a being its endpoint on the left. Let $f(x)$ be a function defined on I. The limit of $f(x)$ as x approaches ∞ is defined as follows:**

$$\lim_{x\to\infty} f(x) = L \equiv \forall\epsilon > 0 : \exists M > a\,\textbf{such that}\,x > M \implies |f(x) - L| < \epsilon.$$

In other words, as x becomes arbitrarily close to ∞, $f(x)$ gets arbitrarily close to L.

2. **Let I be a right-bounded, left-unbounded interval with the point a being its endpoint on the right. Let $f(x)$ be a function defined on I. The limit of $f(x)$ as x approaches $-\infty$ is defined as follows:**

$$\lim_{x\to-\infty} f(x) = L \equiv \forall\epsilon > 0 : \exists M < a\,\textbf{such that}\,x < M \implies |f(x) - L| < \epsilon.$$

In other words, as x becomes arbitrarily close to $-\infty$, $f(x)$ gets arbitrarily close to L.

3. **Horizontal asymptote (水平漸近線):**

$$(\lim_{x\to\infty} f(x) = L \vee \lim_{x\to-\infty})lf(x) = L) \implies (y = L\ \textbf{is a horizontal asymptote of}\ f(x))$$

(三) One-side Limits

1. **Right-hand Limit: Let I be a left-open interval with the point a being its endpoint on the left. Let $f(x)$ be a function defined on I. The right-hand limit of $f(x)$ as x approaches a is defined as follows:**

$$\lim_{x\to a^+} f(x) = L \equiv \forall\epsilon > 0 : \exists\delta > 0\,\textbf{such that}\,0 < x - a < \delta \implies |f(x) - L| < \epsilon.$$

In other words, as x becomes arbitrarily close to a and is greater than a, $f(x)$ gets arbitrarily close to L.

2. **Left-hand Limit: Let I be a right-open interval with the point a being its endpoint on the right. Let $f(x)$ be a function defined on I. The left-hand limit of $f(x)$ as x approaches a is defined as follows:**

$$\lim_{x\to a^-} f(x) = L \equiv \forall\epsilon > 0 : \exists\delta > 0\,\textbf{such that}\,0 < a - x < \delta \implies |f(x) - L| < \epsilon.$$

In other words, as x becomes arbitrarily close to a and is less than a, $f(x)$ gets arbitrarily close to L.

(四) Infinite Limits

$$\lim_{x\to a} f(x) = \infty \equiv \forall M > 0, \exists\delta > 0\,\text{such that}\,0 < |x - a| < \delta \implies f(x) > M.$$
$$\lim_{x\to a} f(x) = -\infty \equiv \forall M < 0, \exists\delta > 0\,\text{such that}\,0 < |x - a| < \delta \implies f(x) < M.$$
$$\lim_{x\to\infty} f(x) = \infty \equiv \forall M > 0, \exists\delta > 0\,\text{such that}\,x > \delta \implies f(x) > M.$$

$$\lim_{x\to\infty} f(x) = -\infty \equiv \forall M > 0,\, \exists\delta > 0 \,\text{such that}\, x > \delta \implies f(x) < M.$$

$$\lim_{x\to-\infty} f(x) = \infty \equiv \forall M > 0,\, \exists\delta < 0 \,\text{such that}\, x < \delta \implies f(x) > M.$$

$$\lim_{x\to-\infty} f(x) = -\infty \equiv \forall M > 0,\, \exists\delta < 0 \,\text{such that}\, x < \delta \implies f(x) < M.$$

$a \in \mathbb{R}$, $(\left|\lim_{x\to a^+} f(x)\right| = \infty \vee \left|\lim_{x\to a^-} f(x)\right| = \infty) \implies x = a$ is the vertical asymptote (鉛直漸近線) of $f(x)$

(五) Existence of Limits

$$\exists \lim_{x\to a^+} f(x) \iff \exists\,\text{finite}\, L \,\text{such that}\, \forall\epsilon > 0 : \exists\delta > 0 \,\text{such that}\, 0 < x - a < \delta \implies |f(x) - L| < \epsilon.$$

$$\exists \lim_{x\to a^-} f(x) \iff \exists\,\text{finite}\, L \,\text{such that}\, \forall\epsilon > 0 : \exists\delta > 0 \,\text{such that}\, 0 < a - x < \delta \implies |f(x) - L| < \epsilon.$$

$$\exists \lim_{x\to a} f(x) \iff \exists \lim_{x\to a^+} f(x) \wedge \exists \lim_{x\to a^-} f(x) \wedge \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

(六) Limit Laws

Let I be an interval containing the point a. Let $f(x)$ and $g(x)$ be functions defined on I, except possibly at a itself which:

$$\exists \lim_{x\to a} f(x) \wedge \exists \lim_{x\to a} g(x).$$

Then:

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\text{If}\, c \,\text{is a constant} : \lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$$\text{If}\, g(x) \neq 0 \wedge \lim_{x\to a} g(x) \neq 0 : \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

(七) Squeeze/Sandwich theorem

Let I be an interval containing the point a. Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on I, except possibly at a itself which:

$$f(x) \leq h(x) \leq g(x)$$

and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L,$$

then:

$$\lim_{x\to a} h(x) = L.$$

(八) Continuity (連續性)

1. **If $\exists \lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x) = f(a)$, we say $f(x)$ is continuous (連續的) at $x = a$, or we say $f(x)$ is discontinuous (不連續的) at $x = a$.**

2. **If $f(x)$ is continuous at all points in the open interval (a, b), we say $f(x)$ is continuous on the open interval (a, b).**

3. **If $f(x)$ is continuous on the open interval (a, b), and $\lim_{x\to a^+} = f(a)$, we say $f(x)$ is continuous on the left-close right-open interval $[a, b)$.**

4. **If $f(x)$ is continuous on the open interval (a, b), and $\lim_{x\to b^-} = f(b)$, we say $f(x)$ is continuous on the right-close left-open interval $(a, b]$.**

5. **If $f(x)$ is continuous on the left-close right-open interval $[a, b)$ and the right-close left-open interval $(a, b]$, we say $f(x)$ is continuous on the open interval (a, b).**

6. **If $f(x)$ is continuous at all points in its domain, we say $f(x)$ is continuous function (連續函數).**

7. **If both $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)+g(x)$, $f(x)-g(x)$, and $f(x)\cdot g(x)$ are continuous at $x = a$.**

8. **If both $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \leq 0$, then $\frac{f(x)}{g(x)}$ is continuous at $x = a$.**

9. **If both $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \in \mathrm{dom}(f)$, then $f \circ g(x)$ is continuous at $x = a$, namely, $\lim_{x\to a}((f\circ g))(x)) = \lim_{x\to a}(f(g(x))) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)) = f \circ g(a)$.**

(九) Theorem: Limits Involving Quotient Functions

Let a and b be real numbers, $A = \{f(x) \mid f(x) = x \vee \ln(f(x)) \in A \vee e^{f(x)} \in A\}$, and $f(x) \in A$. Then:

$$\lim_{x\to\infty} \frac{(f(x))^a}{(f(x))^b} = \infty \quad \text{if} \quad a > b$$

$$\lim_{x\to\infty} \frac{af(x)}{bf(x)} = \frac{a}{b} \quad \text{if} \quad b \neq 0$$

$$\lim_{x\to\infty} \frac{n^{af(x)}}{bf(x)} = \infty \quad \text{if} \quad ab > 0 \wedge n > 1$$

$$\lim_{x\to\infty} \frac{af(x)}{b\log_n f(x)} = \infty \quad \text{if} \quad ab > 0 \wedge n > 1$$

$$\lim_{x\to\infty} \frac{h(x)}{g(x)} = 0 \quad \text{if} \quad \left|\Re \lim_{x\to\infty} \frac{g(x)}{h(x)}\right| + \left|i\Im \lim_{x\to\infty} \frac{g(x)}{h(x)}\right| = \infty$$