Sequences and Series

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Contents

1 Sequences and Series (數列與級數)

I Sequence (數列)

i Sequence (數列)

A sequence is a function whose domain is an interval of integers, usually denoted as $\langle a_n \rangle$, $\{a_n\}$, or (a_n), sometimes with domain as $\langle a_n \rangle_{n=1}^{m}$, $\{a_n\}_{n=1}^{m}$, or $(a_n)_{n=1}^{m}$, where the subscript n refers to the nth element of the sequence, that is, the function value when the independent variable is n.

ii Finite sequence (有限數列)

A finite sequence is a sequence with finite terms, e.g. $\langle a_n \rangle_{n=1}^{m} = \langle a_1, a_2, \dots, a_m \rangle, m \geq 1$ and m is finite.

iii Infinite sequence (無窮數列)

An infinite sequence is a sequence with infinite terms, e.g. $\langle a_n \rangle_{n=1}^{\infty} = \langle a_1, a_2, \dots \rangle$. Unless otherwise specified, the sequences referred to below are infinite sequences.

II Series (級數)

i Series (級數)

The sum of the terms of a sequence.

ii Finite Series (有限級數)

The sum of the terms of a finite sequence.

iii Infinite Series (無窮級數)

The sum of the terms of an infinite sequence.

III Arithmetic progression/sequence (等差數列) and arithmetic series (等差級數)

An arithmetic sequence is a sequence $\langle a_n \rangle = \langle a_1 + (n-1)d \rangle$.

Given a and b, $\dfrac{a+b}{2}$ is called the median of an arithmetic sequence (等差中項).

An arithmetic series is a series $S_n = \sum\limits_{i=1}^{n} a_i$, where $\langle a_n \rangle$ is an arithmetic sequence.

$$\nexists \lim_{n \to \infty} a_n, \quad d \neq 0$$

$$\lim_{n \to \infty} a_n = a_1, \quad d = 0$$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right) = \frac{n}{2}\left(2a_1 + (n-1)d\right) = na_1 + \frac{n(n-1)d}{2}$$

$$\nexists \lim_{n\to\infty} S_n, \quad a_1 \neq 0 \vee d \neq 0$$
$$\lim_{n\to\infty} S_n = 0, \quad a_1 = 0 \wedge d = 0$$

IV Geometric progression/sequence (等比/幾何數列) and geometric series (等比/幾何級數)

A geometric sequence is a sequence $\langle a_n \rangle = \langle a_1 \cdot r^{n-1} \rangle$, where $a_1 r \neq 0$.

Given a and b, $\pm\sqrt{ab}$ is called the median of an geometric sequence (等比中項).

A geometric series is a series $S_n = \sum_{i=1}^{n} a_i$, where $\langle a_n \rangle$ is a geometric sequence.

$$S_n = \frac{a_1 \left(1 - r^n\right)}{1 - r}, \quad r \neq 1$$
$$S_n = na_1, \quad r = 1$$
$$\lim_{n\to\infty} S_n = \frac{a_1}{1 - r}, \quad |r| < 1$$
$$\nexists \lim_{n\to\infty} S_n, \quad |r| \geq 1$$

V Riemann zeta function (黎曼 zeta 函數)

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x\end{aligned}$$

i Harmonic series (調和級數)

$$S_n = \sum_{n=1}^{n} \frac{1}{n}$$
$$\nexists \sum_{n=1}^{\infty} \frac{1}{n}$$

ii Basel problem (巴塞爾問題)

$$\zeta(2) = \frac{\pi^2}{6}$$

iii Other even positive integers

$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{\pi^6}{945}$$
$$\zeta(8) = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{691\pi^{12}}{638512875}$$

$$\zeta(14) = \frac{2\pi^{14}}{18243225}$$

iv Infinity

$$\lim_{n\to\infty} \zeta(n) = 1$$

VI Euler–Mascheroni constant (歐拉–馬斯克若尼常數)

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln(n)\right)\\ &= \int_1^\infty \left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right) \mathrm{d}x\end{aligned}$$

VII 冪級數（Power series）

$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\begin{aligned}\sum_{i=1}^{n} i^r &= n + \sum_{k=1}^{n-1}(n-k)((k+1)^r - k^r)\\ &= n + \sum_{k=1}^{n-1}(n-k)\sum_{j=0}^{r-1}\binom{r}{j}k^j\end{aligned}$$