Trigonometric Functions

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1 Trigonometric Functions (三角函數)

I Trigonometric Ratios (三角比) or Trigonometric Functions

i Right triangle definitions for acute angles

Given a right triangle with an acute angle being theta:

- Sine (sin) (正弦):
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
- Cosine (cos) (餘弦):
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
- Tangent (tan) (正切):
$$\tan\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
- Cotangent (cot) (餘切):
$$\cot\theta = \frac{1}{\tan\theta}$$
- Secant (sec) (正割):
$$\sec\theta = \frac{1}{\cos\theta}$$
- Cosecant (csc) (餘割):
$$\csc\theta = \frac{1}{\sin\theta}$$

ii Unit-circle definitions for generalized angles and properties

Let Γ be the ray obtained by rotating by an angle θ the positive half of the x-axis (counterclockwise rotation for $\theta > 0$ and clockwise rotation for $\theta < 0$). This ray intersects the unit circle with center $O = (0, 0)$ at the point $A = (x_A, y_A)$. Let the line extended to by Γ intersectes $x = 1$ at $B = (1, y_B)$ and $y = 1$ at $C = (x_C, 1)$. The tangent line to the unit circle at A intersects the x- and y-axes at points $D = (x_D, 0)$ and $E = (0, y_E)$.

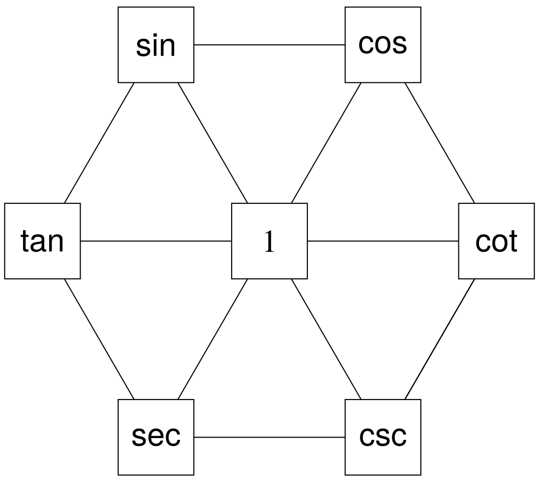
Function	Symbol	Definition	Domain	Range	Period	Odd or even	Reflectional symmetric about	Rotationally symmetric of order 2 about
Sine	sin	y_A	$\mathbb{R}$	$[-1, 1]$	2π	odd	$x \in \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\}$	
Cosine	cos	x_A	$\mathbb{R}$	$[-1, 1]$	2π	even	$x \in \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\}$	
Tangent	tan	y_B	$\mathbb{R} \setminus \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\}$	$\mathbb{R}$	π	odd		$(x, y) \in \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\} \times \{0\}$
Cotangent	cot	x_C	$\mathbb{R} \setminus \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\}$	$\mathbb{R}$	π	odd		$(x, y) \in \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\} \times \{0\}$
Secant	sec	x_D	$\mathbb{R} \setminus \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\}$	$(-\infty, -1] \cup [1, \infty)$	2π	even	$x \in \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\}$	$(x, y) \in \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\} \times \{0\}$
Cosecant	csc	y_E	$\mathbb{R} \setminus \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\}$	$(-\infty, -1] \cup [1, \infty)$	2π	odd	$x \in \left\{x \middle\vert \left(\frac{x}{\pi} - \frac{1}{2}\right) \in \mathbb{Z}\right\}$	$(x, y) \in \left\{x \middle\vert \frac{x}{\pi} \in \mathbb{Z}\right\} \times \{0\}$

iii Trigonometric functions of important angles

Radian	Angle	sin	cos	tan
0	0°	0	1	0

Radian	Angle	sin	cos	tan
$\frac{\pi}{2}$	90°	1	0	
π	180°	0	-1	0
$\frac{3\pi}{2}$	270°	-1	0	
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{3\pi}{4}$	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	60 °	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{2\pi}{3}$	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{5\pi}{6}$	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{\pi}{12}$	15°	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$
$\frac{5\pi}{12}$	75°	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$
$\frac{\pi}{10}$	18°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$
$\frac{2\pi}{10}$	36°	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$
$\frac{3\pi}{10}$	54°	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$
$\frac{4\pi}{10}$	72°	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$
	37°	≈ 0.6018	≈ 0.7986	≈ 0.7536
	53°	≈ 0.7986	≈ 0.6018	≈ 1.3270

iv　基本關係



- 名稱：左側三者為正；右側三者為餘；上面二者為弦；中間二者為切；下面二者為割。
- 餘角關係：以鉛直軸為對稱軸，位於線對稱位置者為餘角關係，即對於銳角 θ，左 (θ) = 右 $\left(\frac{\pi}{2}-\theta\right)$。
- 倒數關係：三條通過中心點的連線為倒數關係，其兩端者互為倒數，相乘為 1。
- 商數關係：六邊形周上，連續三個頂點形成的連線，其兩端者相乘等於中間者。

- 平方關係：圖中有三個倒正三角形，其在上方兩頂點之二者之平方和等於在下方頂點者。

v　奇變偶不變，正負看象限

今有函數 f，已知其為 sin、cos、tan、sec、csc、cot 之一，且已知 $f(\theta)$。欲求 $f(\phi)$，其中 $\phi = \pm\theta \pm n\frac{\pi}{2}$，其中 $n \in \mathbb{Z}$。

- 判斷方法：奇變偶不變，正負看象限。
- 上句：奇偶指 n 之奇偶，變指倒數，即：若 n 為奇數則令 $g(\theta) = \frac{1}{f(\theta)}$，否則令 $g(\theta) = f(\theta)$，則 $|f(\phi)| = |g(\theta)|$。
- 下句：象限指假設 $[r, \theta]$ 在第一象限時，$[r, \phi]$ 之象限。令該象限中任意角度為 ω。令 $k = \frac{f(\phi)}{g(\theta)}$。則 $k = \frac{f(\omega)}{|f((\omega)|}$，即：

象限 / f	一	二	三	四
sin	+	+	-	-
cos	+	-	-	+
tan	+	-	+	-
csc	+	+	-	-
sec	+	-	-	+
cot	+	-	+	-

vi　指數形式

$$
\begin{aligned}
\sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\
\cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
\tan x &= -i\frac{e^{2ix} - 1}{e^{2ix} + 1}, \quad x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \\
\cot x &= i\frac{e^{2ix} + 1}{e^{2ix} - 1}, \quad x \neq k\pi, k \in \mathbb{Z} \\
\sec x &= \frac{2e^{ix}}{e^{2ix} + 1}, \quad x \neq \pi + 2k\pi, k \in \mathbb{Z} \\
\csc x &= i\frac{2e^{ix}}{e^{2ix} - 1}, \quad x \neq 2k\pi, k \in \mathbb{Z}
\end{aligned}
$$

vii Power notation

$$\sin^n x := \begin{cases} (\sin x)^n, & n \geq 0 \\ \arcsin x, & n = -1 \end{cases}$$

$$\cos^n x := \begin{cases} (\cos x)^n, & n \geq 0 \\ \arccos x, & n = -1 \end{cases}$$

$$\tan^n x := \begin{cases} (\tan x)^n, & n \geq 0 \\ \arctan x, & n = -1 \end{cases}$$

$$\cot^n x := \begin{cases} (\cot x)^n, & n \geq 0 \\ \operatorname{arccot} x, & n = -1 \end{cases}$$

$$\sec^n x := \begin{cases} (\sec x)^n, & n \geq 0 \\ \operatorname{arcsec} x, & n = -1 \end{cases}$$

$$\csc^n x := \begin{cases} (\csc x)^n, & n \geq 0 \\ \operatorname{arccsc} x, & n = -1 \end{cases}$$

II Inverse trigonometric functions (反三角函數)

i Definition

Function	Symbols	Definition	Domain	Range
Inverse sine (反正弦)	$y = \arcsin x = \sin^{-1}(x) = \operatorname{asin}(x)$	$x = \sin y$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
Inverse cosine (反餘弦)	$y = \arccos x = \cos^{-1}(x) = \operatorname{acos}(x)$	$x = \cos y$	$[-1, 1]$	$[0, \pi]$
Inverse tangent (反正切)	$y = \arctan x = \tan^{-1}(x) = \operatorname{atan}(x)$	$x = \tan y$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
Inverse cotagent (反餘切)	$y = \operatorname{arccot} x = \cot^{-1}(x) = \operatorname{acot}(x)$	$x = \cot y$	$\mathbb{R}$	$(0, \pi)$
Inverse secant (反正割)	$y = \operatorname{arcsec} x = \sec^{-1}(x) = \operatorname{asec}(x)$	$x = \sec y$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
Inverse cosecant (反餘割)	$y = \operatorname{arccsc} x = \csc^{-1}(x) = \operatorname{acsc}(x)$	$x = \csc y$	$(-\infty, -1] \cup [1, +\infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

ii atan2 函數

atan2 or arctan2(y, x) : $\mathbb{R}^2 \setminus \{(0,0)\}$ 在 $x > 0$ 時返還 $\tan(\theta) = \frac{y}{x}$ 在 $(-\frac{\pi}{2}, \frac{\pi}{2})$ 中的解，在 $x < 0$、$y \geq 0$ 時返還 $\tan(\theta) = \frac{y}{x}$ 在 $(\frac{\pi}{2}, \pi)$ 中的解，在 $x < 0$、$y < 0$ 時返還 $\tan(\theta) = \frac{y}{x}$ 在 $(-\pi, -\frac{\pi}{2})$ 中的解，在 $x = 0$、$y \neq 0$ 時返還 $\frac{y}{|y|}\frac{\pi}{2}$，在 $x = y = 0$ 時未定義。

iii Trigonometric functions of inverse trigonometric functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\arcsin(x)$	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{1-x^2}},\quad \lvert x\rvert<1$	$\frac{\sqrt{1-x^2}}{x},\quad x\neq 0$	$\frac{1}{\sqrt{1-x^2}},\quad \lvert x\rvert<1$	$\frac{1}{x},\quad x\neq 0$
$\arccos(x)$	$\sqrt{1-x^2}$	x	$\frac{\sqrt{1-x^2}}{x},\quad x\neq 0$	$\frac{x}{\sqrt{1-x^2}},\quad \lvert x\rvert<1$	$\frac{1}{x},\quad x\neq 0$	$\frac{1}{\sqrt{1-x^2}},\quad \lvert x\rvert<1$
$\arctan(x)$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{1}{\sqrt{1+x^2}}$	x	$\frac{1}{x},\quad x\neq 0$	$\sqrt{1+x^2}$	$\frac{\sqrt{1+x^2}}{x},\quad x\neq 0$
$\operatorname{arccot}(x)$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{1}{x},\quad x\neq 0$	x	$\frac{\sqrt{1+x^2}}{x},\quad x\neq 0$	$\sqrt{1+x^2}$
$\operatorname{arcsec}(x)$	$\frac{\sqrt{x^2-1}}{\lvert x\rvert}$	$\frac{1}{x}$	$\sqrt{x^2-1}\,\operatorname{sgn}(x)$	$\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}},\quad \lvert x\rvert>1$	x	$\frac{\lvert x\rvert}{\sqrt{x^2-1}}$
$\operatorname{arccsc}(x)$	$\frac{1}{x}$	$\frac{\sqrt{x^2-1}}{\lvert x\rvert}$	$\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}},\quad \lvert x\rvert>1$	$\sqrt{x^2-1}\,\operatorname{sgn}(x)$	$\frac{\lvert x\rvert}{\sqrt{x^2-1}}$	x

III Identities

i 正切萬能公式

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

ii 二倍角公式

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= 1-2\sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

iii 半角公式與平方化倍角公式

$$\begin{aligned}\sin^2\frac{\theta}{2} &= \frac{1-\cos\theta}{2} \\ \cos^2\frac{\theta}{2} &= \frac{1+\cos\theta}{2} \\ \tan^2\frac{\theta}{2} &= \frac{1-\cos\theta}{1+\cos\theta} \\ \tan\frac{\theta}{2} &= \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta} \\ &= \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} \\ &= \csc\theta - \cot\theta\end{aligned}$$

iv 三倍角公式

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

v　和差角公式

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$
$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta + 1}{-\cot\alpha + \cot\beta}$$
$$\sec(\alpha+\beta) = \frac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{-\sec\alpha\sec\beta + \csc\alpha\csc\beta}$$
$$\sec(\alpha-\beta) = \frac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\sec\alpha\sec\beta + \csc\alpha\csc\beta}$$
$$\csc(\alpha+\beta) = \frac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\sec\alpha\sec\beta + \csc\alpha\csc\beta}$$
$$\csc(\alpha-\beta) = \frac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\sec\alpha\sec\beta - \csc\alpha\csc\beta}$$

vi　平方關係

$$\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} = 1 - \cos^2\theta$$
$$\cos^2\theta = \frac{1}{1+\tan^2\theta} = 1 - \sin^2\theta$$
$$\tan^2\theta = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1-\sin^2\theta}$$

vii　三角形內角正切公式

$$(\alpha+\beta+\gamma = \pi + 2k\pi, \quad k \in \mathbb{Z}) \iff (\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\cdot\tan\beta\cdot\tan\gamma)$$

viii　正餘弦函數疊合

$$(a\sin\theta + b\cos\theta)^2 \le a^2 + b^2, \quad a, b \in \mathbb{R}$$
$$\begin{aligned} a\sin x + b\cos x &= \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right) \\ &= \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}\left(\frac{a}{b}\right)\right) \end{aligned}$$

ix 和差化積公式

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

x 積化和差公式

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$
$$2\sin\alpha\sin\beta=-\cos(\alpha+\beta)+\cos(\alpha-\beta)$$

xi 連加公式

$$\sum_{k=1}^{n}\sin(k\theta)=\frac{\sin\left(\frac{n\theta}{2}\right)\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$
$$\sum_{k=1}^{n}\cos(k\theta)=\frac{\sin\left(\frac{n\theta}{2}\right)\cdot\cos\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

xii 正餘切和等於正餘割積公式

$$\tan\theta+\cot\theta=\sec\theta\csc\theta$$

xiii 正餘弦四次方和公式

$$\sin^4\theta+\cos^4\theta=1-2\sin^2\cos^2\theta=1-\frac{1}{2}\sin^2(2\theta)$$

xiv 正餘弦四次方差公式

$$\sin^4\theta-\cos^4\theta=\sin^2\theta-\cos^2\theta=-\cos(2\theta)$$

xv 正餘弦六次方和公式

$$\sin^6\theta+\cos^6\theta=1-3\sin^2\cos^2\theta=1-\frac{3}{4}\sin^2(2\theta)$$

IV 正弦連乘、餘切連加、餘割平方級數與餘切平方級數公式

$$\prod_{k=0}^{n-1}\sin\left(x+\frac{\pi k}{n}\right)=2^{1-n}\sin(nx)$$

$$\sum_{k=0}^{n-1} \cot\left(x + \frac{\pi k}{n}\right) = n \cot(nx)$$

$$\sum_{k=0}^{n-1} \csc^2\left(x + \frac{\pi k}{n}\right) = n^2 \csc^2(nx)$$

$$\sum_{k=1}^{n-1} \csc^2 \frac{\pi k}{n} = \frac{(n-1)(n+1)}{3}$$

$$\sum_{k=1}^{n-1} \cot^2 \frac{\pi k}{n} = \frac{(n-1)(n-2)}{3}$$

Proof.

$$\begin{aligned}
&\prod_{k=0}^{n-1} \sin\left(x + \frac{\pi k}{n}\right) \\
&= \prod_{k=0}^{n-1} \frac{i}{2}\left(e^{-i\left(x+\frac{\pi k}{n}\right)} - e^{i\left(x+\frac{\pi k}{n}\right)}\right) \\
&= i^n 2^{-n} \prod_{k=0}^{n-1} e^{-i\left(x+\frac{\pi k}{n}\right)} \prod_{k=0}^{n-1}(1 - e^{2i\left(x+\frac{\pi k}{n}\right)}) \\
&= i^n 2^{-n} e^{-inx} e^{-i\pi\left(\frac{n-1}{2}\right)} \prod_{k=0}^{n-1}(1 - e^{2i\left(x+\frac{\pi k}{n}\right)}) \\
&= i^n 2^{-n} e^{-inx} i^{1-n} \prod_{k=0}^{n-1}(1 - e^{2i\left(x+\frac{\pi k}{n}\right)})
\end{aligned}$$

考慮：

$$f(t) = t^n - e^{2inx}$$

$f(t) = 0$ 的根為：

$$t = e^{2i\left(x+\frac{\pi k}{n}\right)}, \quad k \in \mathbb{N}_0 \wedge k < n$$

故：

$$f(t) = \prod_{k=0}^{n-1}(t - e^{2i\left(x+\frac{\pi k}{n}\right)})$$

$$\prod_{k=0}^{n-1}(1 - e^{2i\left(x+\frac{\pi k}{n}\right)}) = 1 - e^{2inx}$$

代回：

$$\begin{aligned}
&\prod_{k=0}^{n-1} \sin\left(x + \frac{\pi k}{n}\right) \\
&= i^n 2^{-n} e^{-inx} i^{1-n}(1 - e^{2inx}) \\
&= 2^{-n} i(e^{-inx} - e^{inx}) \\
&= 2^{1-n} \sin(nx)
\end{aligned}$$

$$\sum_{k=0}^{n-1} \ln\left|\sin\left(x + \frac{\pi k}{n}\right)\right| = (1-n)\ln(2) + \ln|\sin(nx)|$$

微分兩次：

$$\sum_{k=0}^{n-1} \cot\left(x+\frac{\pi k}{n}\right) = n\cot(nx)$$

$$\sum_{k=0}^{n-1} \csc^2\left(x+\frac{\pi k}{n}\right) = n^2\csc^2(nx)$$

$$\sum_{k=1}^{n-1} \csc^2\left(x+\frac{\pi k}{n}\right) = n^2\csc^2(nx) - \csc^2(x)$$

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{\pi k}{n}\right) = \lim_{x\to 0} n^2\csc^2(nx) - \csc^2(x) = \frac{(n-1)(n+1)}{3}$$

$$\cot^2(x) = \csc^2(x) - 1$$

$$\sum_{k=1}^{n-1} \cot^2\frac{\pi k}{n} = \sum_{k=1}^{n-1} \csc^2\frac{\pi k}{n} - n + 1 = \frac{n^2-3n+2}{3}$$

□