Heat conduction in a sphere

Heat conduction in a solid is governed by the conservation law

$$\rho c \frac{\partial T}{\partial t} = -\nabla \cdot \boldsymbol{q},$$

where T is the temperature ($[T] = \mathrm{K}$), ρ the mass density ($[\rho] = \mathrm{kg/m^3}$), c the specific heat ($[c] = \mathrm{J/(kg\,K)}$) and $\boldsymbol{q}$ the heat flux vector ($[\boldsymbol{q}] = \mathrm{J/(m^2\,s)}$). The heat flux vector satisfies Fourier's law, i.e.,

$$\boldsymbol{q} = -\kappa \nabla T,$$

with κ the thermal conductivity of the material ($[\kappa] = \mathrm{J/(m\,s\,K)}$). We assume that the parameters ρ, c and κ are constant. We would like to compute the spherically symmetric temperature $T = T(r,t)$ in a sphere of radius R with centre at the origin. The sphere has a uniform initial temperature T_0. Due to symmetry, there is no heat flux at the origin. At the surface $r = R$ the sphere loses heat due to convection along its surface. This gives rise to a heat flux vector $\boldsymbol{q}_{\mathrm{conv}}(t)$ given by

$$\boldsymbol{q}_{\mathrm{conv}}(t) = h(T - T_{\mathrm{amb}}(t))\boldsymbol{e}_r, \quad T_{\mathrm{amb}}(t) = T_0 + A\sin(\omega t),$$

with h the heat transfer coefficient ($[h] = \mathrm{J/(m^2\,s\,K)}$) and $T_{\mathrm{amb}}(t)$ the ambient temperature ($[T_{\mathrm{amb}}] = \mathrm{K}$), that is varying periodically with amplitude A ($[A] = \mathrm{K}$) and frequency ω ($[\omega] = \mathrm{s^{-1}}$).

1.a Show that $T(r,t)$ satisfies the following initial-boundary-value problem (IBVP):

$$\frac{\partial T}{\partial t} = \frac{a}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right), \quad 0 < r < R, \quad t > 0, \tag{1a}$$

$$T(r,0) = T_0, \quad 0 < r < R, \tag{1b}$$

$$\frac{\partial T}{\partial r}(0,t) = 0, \quad hT(R,t) + \kappa \frac{\partial T}{\partial r}(R,t) = h(T_0 + A\sin\omega t), \quad t > 0, \tag{1c}$$

where $a := \kappa/(\rho c)$ is the thermal diffusivity ($[a] = \mathrm{m^2/s}$).

It is convenient to rewrite the IBVP (1) in dimensionless form. Therefore, we introduce the variables

$$r^* := \frac{r}{R}, \quad t^* := \frac{a}{R^2}t, \quad u(r^*, t^*) := \frac{T(r,t) - T_0}{A}.$$

1.b Show that $u(r^*, t^*)$ satisfies the IBVP

$$\frac{\partial u}{\partial t^*} = \frac{1}{(r^*)^2}\frac{\partial}{\partial r^*}\left((r^*)^2 \frac{\partial u}{\partial r^*}\right), \quad 0 < r^* < 1, \quad t^* > 0, \tag{2a}$$

$$u(r^*, 0) = 0, \quad 0 < r^* < 1, \tag{2b}$$

$$\frac{\partial u}{\partial r^*}(0, t^*) = 0, \quad u(1, t^*) + \frac{1}{\mathrm{Bi}}\frac{\partial u}{\partial r^*}(1, t^*) = \sin(\omega^* t^*), \quad t^* > 0, \tag{2c}$$

with $\mathrm{Bi} := hR/\kappa$ the Biot number and $\omega^* := \omega R^2/a$ the dimensionless frequency.

In the following we will omit the superscript $*$. We would like to compute numerical solutions of the IBVP (2). Therefore, we choose a time step $\Delta t > 0$ and a spatial grid size $\Delta r = 1/(N-1)$ and define the following space-time grid in $[0,1] \times [0,\infty)$:

$$r_j := (j-1)\Delta r \quad (j = 1, 2, \ldots, N), \quad t_n := n\Delta t \quad (n = 0, 1, 2, \ldots).$$

We adopt the method of lines approach, i.e., we first discretize the space derivatives, resulting in a system of ODEs, and subsequently apply a time integration method to this system. For space discretization, we employ the finite volume method. To that purpose we rewrite equation (2a) in the form

$$\frac{\partial u}{\partial t} = \nabla \cdot \boldsymbol{f}(u),$$

with $\boldsymbol{f}(u)$ a dimensionless flux. We denote the semi-discrete approximation of $u(r_j, t)$ by $u_j(t)$. As a result of the spatial discretization procedure we obtain for $\mathbf{u}(t) := \big(u_1(t), u_2(t), \ldots, u_N(t)\big)^{\mathrm{T}}$ an ODE system of the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{A}\mathbf{u} + \mathbf{b}(t). \tag{3}$$

We will apply several time integration methods to (3).

1.c Show that, at the interior grid points r_j ($j = 2, 3, \ldots, N-1$), the semi-discrete approximation $u_j(t)$ satisfies a differential equation of the form

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{1}{\Delta r^2}\big(\alpha_{j+1/2}(u_{j+1}(t) - u_j(t)) - \beta_{j-1/2}(u_j(t) - u_{j-1}(t))\big).$$

Determine the coefficients $\alpha_{j+1/2}$ and $\beta_{j-1/2}$.

1.d Give the differential equation for the variable $u_1(t)$ at the centre of the sphere and for the variable $u_N(t)$ at the boundary $r = 1$.

1.e Give the ϑ-method for the ODE system (3) and implement it in a MATLAB or PYTHON-script. Compute several numerical solutions of (2) for $\vartheta = 0, \frac{1}{2}, 1$. Choose several values for Bi and ω^* and discuss the results.

1.f Give the derivation of the two-stage exponential Runge-Kutta scheme for (3) and give the algorithm for its implementation. Compute several numerical solutions and discuss the results.

1.g Investigate the order of convergence of all time integration methods using Richardson extrapolation.

Traffic flow on a high way

Consider a highway without slip-roads and exits. Let x be the coordinate along this highway, $\rho(x,t)$ the number of cars per unit of length and $u(x,t)$ the average speed of cars. Since the number of cars is conserved, $\rho(x,t)$ satisfies the so-called continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0. \tag{4}$$

For the velocity we take the model

$$u(\rho) = u_{\max}\,\mathrm{e}^{-\alpha\rho/\rho_{\max}}, \tag{5}$$

with $\rho_{\max}$ the maximum car density, $u_{\max}$ the maximum velocity and $0 < \alpha < 2$. Thus, the velocity varies from $u_{\max}\,\mathrm{e}^{-\alpha}$, in bumper-to-bumper traffic, to $u_{\max}$ on an empty highway. In order to make equation (4) dimensionless, we introduce the variables

$$x^* := \frac{x}{L}, \quad t^* := \frac{u_{\max} t}{L}, \quad q(x^*, t^*) := -\frac{\rho(x,t)}{\rho_{\max}}, \tag{6}$$

with L a characteristic distance along the highway.

2.a Show that q satisfies the conservation law

$$\frac{\partial q}{\partial t^*} + \frac{\partial f(q)}{\partial x^*} = 0, \tag{7}$$

with flux function $f(q) = q\,\mathrm{e}^{\alpha q}$. Plot the flux function $f(q)$ for $q \in [-1, 0]$, and choose several values of the parameter α.

In the following we will omit the superscript $*$. The corresponding Riemann problem has the piecewise constant initial condition

$$q(x,0) = \begin{cases} q_\ell & \text{if} \quad x \leq 0, \\ q_\mathrm{r} & \text{if} \quad x > 0. \end{cases} \tag{8}$$

2.b Determine the similarity solutions of the Riemann problem (7) and (8). Verify the entropy condition.
HINT: Make a distinction between the cases $q_\ell \geq q_\mathrm{r}$ and $q_\ell < q_\mathrm{r}$.

2.c Verify that the similarity solutions from exercise (2.b) are weak solutions.

2.d Determine the Godunov numerical flux $F(q_\ell, q_\mathrm{r})$. Give the Godunov scheme. What is the stability condition?

2.e Implement the Godunov scheme in a MATLAB or PYTHON-script and compute several numerical solutions for $0 < x < 1$, $0 < t < 1$, subject to the following initial conditions:

$$q(x,0) = \begin{cases} -1 & \text{if} \quad x \leq 0.2, \\ 0 & \text{if} \quad x > 0.2 \end{cases}, \quad q(x,0) = \begin{cases} -0.5 & \text{if} \quad x \leq 0.7, \\ -1 & \text{if} \quad x > 0.7. \end{cases}$$

Choose several values for α and compare your results with the exact solution. Discuss the results.

2.f Implement the slope limiter method and repeat the numerical computations from exercise (2.e). Discuss the results.

Dam break problem

We would like to compute numerical solutions of the shallow water equations (SWE) on the spatial domain $(-100, 100)$, subject to the following initial conditions:

$$u(x,0) = 0, \quad h(x,0) = \begin{cases} 100 & \text{if} \quad x \leq 0, \\ 50 & \text{if} \quad x > 0. \end{cases} \tag{9}$$

You can find on CANVAS the incomplete MATLAB-script `swe-incomplete.m` and the functions `solveG.m`, `g.m`, `dg.m` to compute numerical solutions of the SWE.

3.a Give the Godunov numerical flux for the SWE.

3.b Give the stability condition for the Godunov scheme.

3.c Complete the script and compute several numerical solutions for the SWE. Discuss the results.

Waves in a basin

Consider the following IBVP for the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \nabla^2 u, \quad (x,y) \in \Omega := (-2,2) \times (-2,2), t > 0, \tag{10a}$$

$$u(x,y,0) = \mathrm{e}^{-\alpha(x^2+y^2)}, \quad \frac{\partial u}{\partial t}(x,y,0) = 0, \quad (x,y) \in \Omega \tag{10b}$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, t > 0, \tag{10c}$$

where a is the wave speed and $\alpha > 0$.

4.a Give a derivation of the staggered leapfrog scheme for equation (10a).

4.b Derive the stability condition for this scheme.

4.c Determine the numerical dispersion relation for the scheme. What can you conclude from this?

4.c Implement the staggered leapfrog scheme and compute numerical solutions for several values of $\Delta x = \Delta y$ and Δt. First take $a = 1$ and $\alpha = 5$, and subsequently choose several other values of a and α. Discuss the results.

TIP: You can make animations of all your numerical results using the MATLAB-command `movie`.