Study material for Partial Differential Equations (3341)

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Contents

Chapter 1

General introduction (week 1)

This document provides the material to be used when studying the subject. It can be completed by the material in the cited references, in particular [2]. The chapters are reflecting (more-or-less) respects the weekly planning. Please, be aware that this document is subject to changes, so you may want to check weekly for new versions on Blackboard. Feel free to contact the teachers for additional explanations, information, etc.

1.1 Classification of partial differential equations

We start by presenting some basic notions that will be used throughout this lecture. First of all, we mention that a **partial differential equation** is an equation in which the unknown is a function in several variables, and the equation involves the unknown function itself and its partial derivatives up to a certain order. In this respect, we immediately define the **order** of the partial differential equation, which is the highest order of the partial derivatives in the equation. Note that the partial derivatives may also be mixed, and, actually, the unknown function itself may even be missing in the equation. Also, the partial differential equation may be **non-linear** or **linear**, depending on whether or not the unknown function appears non-linearly in the equation.
Examples: Below the unknown function is u, the variables $t, x, \ldots$ are independent, and f is assumed given. Note that x is a scalar variable, in contrast to $\vec{x}$.

- $\partial_t u + \partial_x u = f$ is a linear, first order equation;
- $\partial_t u + \partial_x(u^2) = u$ and $\partial_t u + u\partial_x u = f$ are non-linear, first order equations;
- $\partial_t u - \partial_{xx} u = u$, $\partial_{xx} u + \partial_{yy} u = 0$, or $\partial_{tt} u - \partial_{xx} u = f$ are linear, second order equations;
- $\partial_t u - \partial_{xx}(u^2) + \partial_x u = 0$, $\partial_{tt} u - \partial_{xx} u = u^2$ are non-linear, second order equations;
- $\partial_t u - \partial_{xxt} u = u$ and $\partial_t u + \partial_x f(u) - \partial_{xx} u - \partial_{xxt} u = 0$ are third order equations (the first is linear, the second non-linear);

1.1.1 Three major classes of partial differential equations

Excepting the first week, in this course we focus on second order partial differential equations. For these, three major classes can be distinguished (the variables being self-understood):

- **Parabolic equations**, with the **heat/diffusion equation** as prominent example

$$\partial_t u - \Delta u = 0;$$

- **Elliptic equations**, with **Laplace/Poisson equation** as prominent examples

$$-\Delta u = 0, \text{ respectively } -\Delta u = f;$$

- **Hyperbolic equations**, with the **wave equation** as prominent example

$$\partial_{tt} u = \Delta u.$$

In the above, Δ stands for the **Laplacian/Laplace operator**

$$\Delta u := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}.$$

This operator is w.r.t. the space variable $\vec{x} = (x_1, \dots, x_k)$ in the d-dimensional space. For two dimensions ($d = 2$, thus $\vec{x} = (x, y)$) one has (using different notations)

$$\Delta u := \partial_{xx} u + \partial_{yy} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and analogously for $d = 3$.

Remark 1 *For the case of two independent variables (here $\vec{x} = (x, y)$, but replacing e.g. y by t is straightforward), the general form of a linear, second order equation is*

$$a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + au = 0. \tag{1.1}$$

*We assume the coefficients $a_{11}, \dots, a$ given and that at least one of the coefficients a_{11}, a_{12} and a_{22} are non-zero. The factor 2 in the second term is taken for convenience. In this case, the classification of the equation is equivalent to the type of the plane curve associated to the **characteristic polynomial**, which is the quadratic form in the two variables λ and μ*

$$p(\lambda, \mu) = a_{11}\lambda^2 + 2a_{12}\lambda\mu + a_{22}\mu^2 + a_1\lambda + a_2\mu + a.$$

The detalis will be worked out through an assignment.

1.2 Boundary and initial conditions

With reference to the heat equation, as introduced in the lecture, we have seen that, to find a solution, one needs to know more than the equation itself. Specifically, if the domain Ω is bounded, one needs to know what happens at the boundary $\partial\Omega$, or to prescribe the **boundary conditions**. From now on, $\vec{n}$ will denote the unit normal vector to the boundary $\partial\Omega$, pointing outward Ω (i.e. the unit vector that is orthogonal to the boundary $\partial\Omega$ and points outside Ω). We assume that the boundary is smooth, so that such unit normal vectors are defined in any particular point at the boundary. Here we use three major types of conditions (the functions u_D, q_N, or h being given), but this is by far not exhaustive:

- **Dirichlet**: $u = u_D$ (i.e. u takes a prescribed value);
- **Neumann**: $-\partial_{\vec{n}} u = -\vec{n} \cdot \nabla u = q_N$ (i.e. the **normal derivative** of u takes a prescribed value).
- **Robin**: $-\partial_{\vec{n}} u + u = h$ (a combination of the Dirichlet and the Neumann type of boundary conditions).
- **Flux**: $\vec{n} \cdot \vec{q} = h$ (i.e. the **normal flux** takes a prescribed value). The flux depends on the equation, or, more precise, the physical process that is being modelled through this equation.

Note that the boundary conditions can be combined, in the sense that for parts of the boundary $\partial\Omega$ one can prescribe Dirichlet boundary conditions, and for other parts Neumann. In the above, if the values at the boundary are 0 (e.g. if $u_D \equiv 0$ - overall), the boundary condition is called **homogeneous**. Also, observe that the first three boundary conditions are linear w.r.t. u, but one may encounter situations with nonlinear boundary conditions.

For time-dependent equations, one have to specify **initial conditions**, or what happens at $t = 0$. Their number depends on the order of time derivatives. More precise, for the heat equation, or for the first order wave equation one prescribes the initial value of u, namely

$$u(\vec{x}, 0) = u_0(\vec{x}) \text{ for } \vec{x} \in \Omega.$$

For the second order wave equation, involving the term $\partial_{tt} u$, one also prescribes the initial value for the time deirvative of u,

$$\partial_t u(\vec{x}, 0) = u_1(\vec{x}) \text{ for } \vec{x} \in \Omega.$$

More details about how many conditions and where to be imposed will be given later. In analogy with ordinary differential equations, for any given partial differential equation one can speak about a **problem**, if sufficient initial and boundary conditions are imposed. For example, for the Laplace equation posed in a bounded domain $\Omega \subset \mathbb{R}^d$, one needs to impose boundary conditions everywhere at $\partial\Omega$,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = u_D & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We speak about an **boundary value problem**. If, instead, one deals with the heat equation in a bounded domain, one also needs initial conditions,

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\ u = u_D & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

Now we speak about an **initial-boundary value problem**.
Clearly, in the examples above one may choose other types of boundary conditions, or prescribe e.g. Dirichlet boundary conditions at one part of the boundary, and Neumann on the complement of the first part. We finish this part with the following

Remark 2 *In some cases, the boundary conditions need to be **compatible**. Without entering into details, consider the elliptic problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\partial_{\vec{n}} u = g & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Here the functions f and g are assumed known. Then, to guarantee the existence of a solution, the functions f and g needs to be chosen s.t. they satisfy the condition

$$\int_{\partial\Omega} g(\vec{x}) d\sigma = \int_{\Omega} f(\vec{x}) d\vec{x} \tag{1.5}$$

(explain why!). As we will see later, in this case we have multiple solutions.

Exercise set 1 *"Warming up"*

1. ***Ordinary differential equations***

   (a) *Linear, first order equations. Use integrating factors for finding the general solution to the equations below ($a \in \mathbb{R}$ is a constant). For the last two examples, find the solution that satisfies the initial condition:*

      i. $u'(t) + atu(t) = 0$;
      ii. $u'(t) + atu(t) = t$;
      iii. $t^2u'(t) + 2tu(t) = t^2$ *met* $u(1) = 1$;
      iv. $t^2u'(t) + 2tu(t) = t^2$ *met* $u(0) = 1$;

(b) *Linear, second order equations with constant coefficients. Find the solution to the equations below, including the particular solution satisfying the initial conditions:*

   i. $u''(t) - 3u'(t) + 2u(t) = 0$, *with* $u(0) = 1$ *en* $u'(1) = 1$*;*

   ii. $u''(t) - 2u'(t) + 2u(t) = 0$, *with* $u(0) = 1$ *en* $u'(0) = 0$*;*

   iii. $u''(t) + 4u'(t) + 4u(t) = 0$, *met* $u'(0) = 0$ *en* $u(1) = 1$*;*

2. ***Partial integration, Gauß/Divergece Theorem***
Let $\Omega \subset \mathbb{R}^d$ *(*$d = 2$ *of* $d = 3$*) be a bounded domain with smooth boundary* $\partial\Omega$ *and* $\vec{\nu} = (\nu_1, \ldots, \nu_d)^T \in \mathbb{R}^d$ *the unit normal to* $\partial\Omega$ *pointing outwards* Ω*.*

   (a) *Veriify that, for any* $u \in C^1(\Omega)$ *and* $v \in C^2(\Omega)$*, one has*
   $$\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u\Delta v.$$

   (b) *Use the Gauß Theorem to prove that, for any* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$*, one has*
   $$\int_\Omega \Delta u(\vec{x})d\vec{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{\nu}} dS.$$

   (c) *Prove that for any* $u, v \in C^1(\Omega) \cap C(\bar{\Omega})$ *one has*
   $$\int_\Omega \frac{\partial u}{\partial x_k} v d\vec{x} = \int_{\partial\Omega} uv\nu_k dS - \int_\Omega u\frac{\partial v}{\partial x_k} d\vec{x}, \quad k = 1, \ldots, d.$$
   Hint: Consider the vector-valued function $\vec{F} : \bar{\Omega} \to \mathbb{R}^d, \vec{F} = (f_1, \ldots, f_d)^T$ *s.t.* $f_i \equiv 0$ *(i.e.* $f_i(x) = 0$ *for all* $x \in \bar{\Omega}$*) for all* $i \neq k$ *and take* $f_k = uv$*.*

   (d) *Let* $\Omega = (a, b)$ *be a finite, open interval. Recall that, for any* $u, v \in C^1([a, b])$ *one has*
   $$\int_a^b u'vdx = u(b)v(b) - u(a)v(a) - \int_a^b uv'dx.$$
   Explain why this situation is similar to the one in $\mathbb{R}^d$*.*

   (e) *Prove that, for any* $u \in C^1(\Omega) \cap C(\bar{\Omega})$ *and* $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$*, one has*
   $$\int_\Omega \nabla u \nabla v d\vec{x} = \int_{\partial\Omega} u\partial_{\vec{\nu}} v dS - \int_\Omega u\Delta v d\vec{x}.$$
   Hint: Use Gauß Theorem for the vector-valued function $\vec{F} = u\nabla v$*.*

3. ***A bit of analysis:***
Prove the 2^{nd} *"Vanishing lemma":*

Lemma 1 *Let* $\Omega \subset \mathbb{R}^d$ *be a domain and* $f : \Omega \to \mathbb{R}$ *a continuous, positive function (*$f(\vec{x}) \geq 0$ *for all* $\vec{x} \in \Omega$*). If* $\int_\Omega f(\vec{x})\, d\vec{x} = 0$ *then one has* $f \equiv 0$ *(this means* f *is 0 overall in* Ω*, or* $f(\vec{x}) = 0$ *for all* $\vec{x} \in \Omega$*).*

Chapter 2

The method of characteristics (week 1)

We present here the **method of characteristics**, which can be employed to solve certain categories of partial differential equations (PDEs). More precisely, the idea is assume a certain dependency on the independent variables (in the examples below, the time t and the space x, or x, y, etc.) to reduce the PDE to an ordinary differential equation (ODE) that can be solved explicitly. This general idea to reduce the PDE to an ODE will be applied more often in this lecture, and in various situations (characteristics, similarity, travelling waves).

2.1 The linear wave equation

Here we start with the *method of characteristics* (MOC) applied to the first order hyperbolic equation (*the wave equation*)

$$\partial_t u + a\partial_x u = 0, \text{ where } x \in \mathbb{R} \text{ and } t > 0. \tag{2.1}$$

Here $a \in \mathbb{R}$ is a given number. Further, as we will see later, to solve this equation one also needs to know the *initial condition*, namely the value of u at $t = 0$

$$u(x, 0) = u_0(x), \text{ for } x \in \mathbb{R}. \tag{2.2}$$

Again, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Clearly, the initial condition can be stated for another $t = t_0$ but, for the ease of presentation, we will always assume that the initial time is at $t = 0$.

Remark 3 *From a geometric point of view, the gradient of u in the $x - -t$ plane is $(\partial_x u, \partial_t u)$. This means that, if u solves (2.1), its gradient is perpendicular to the vector $\vec{V} = (a, 1)$. On the other hand, we know (Calculus 2!) that the gradient of u in a given point is perpendicular to its level set thorugh that point. This means that* *along the curves having $\vec{V}$ as tangent vector, the solution u must be constant. This suggests the construction given below, called the method of characteristics. However, the argument involving level sets remains limited to the case of a homogeneous right-hand side, therefore we introduce the method in a different way.*

2.1.1 MOC for the linear wave equation

The idea in the MOC is to assume that x is a function of t, and to choose this function properly so that the PDE reduces to an ODE. More exactly, if $x = x(t)$ (the precise function will be specified later), one can define the new function $v : [0, \infty) \to \mathbb{R}$, $v(t) = u(x(t), t)$. Then, with this choice one has for all $t \geq 0$

$$v'(t) = \frac{du}{dt}(x(t), t) = x'(t)\partial_x u(x(t), t) + \partial_t u(x(t), t), \tag{2.3}$$

where in the above we have applied the chain rule. After comparing this with the left hand side of (2.1), it looks that choosing the function $x(\cdot)$ s.t. $x'(t) = a$ allows writing

$$v'(t) = \left(\partial_t u + a\partial_x u\right)\big|_{(x(t),t)} . \tag{2.4}$$

Observe that the expression on the right was evaluated in the point (x, t) where $x = x(t)$. Therefore, using (2.1) one obtains that $v'(t) = 0$ for all $t \geq 0$. In other words, the function $v(\cdot)$ is constant in time, which means that $u(x(t), t)$ is constant along the *characteristic* curve in the $x - t$ plane. This curve is defined as

$$\left\{(x, t) \in \mathbb{R} \times [0, \infty), x = x(t)\right\}, \tag{2.5}$$

where $x(\cdot)$ is a function satisfying $x'(t) = a$.

Solving the latter ODE gives the following line as characteristic curve in the $x - t$ plane

$$x(t) = x_0 + at. \tag{2.6}$$

Here $t \geq 0$, and x_0 is the point where the characteristic intersects the x-axis.

Since $v'(t) = 0$ for all $t \geq 0$ one has that $v(t) = v(0)$, and recalling the definition of v it follows that

$$u(x(t), t) = v(t) = v(0) = u(x(0), 0) = u_0(x_0), \tag{2.7}$$

with $x_0 = x(t) - at$. In other words, for finding $u(x, t)$ one has to find the characteristic through the point (x, t), determine the point x_0 where this characteristic intersects the x-axis and then set $u(x, t) = u_0(x_0)$. In this case,

$$u(x, t) = u_0(x - at). \tag{2.8}$$

In this case, the solution is simply a translation of the initial data u_0. One can say that the initial data travels in time with a fixed velocity. This is, in fact, the simplest example of a *travelling wave* (TW). In general, for a given equation depending on both space and time, a TW is a solution $u(x,t)$ that can be written as

$$u(x,t) = v(x - at),$$

for a properly chosen $a \in \mathbb{R}$. Extensions to the multi-dimensional case are possible.

Remark 4 *Note that the characteristic in* (2.6) *is a line due to the fact that the factor multiplying $\partial_x u$ is a constant a. Later examples will include non-constant factors (and it is possible even to take u-dependent factors as well), and the characteristics will not remain straight lines. Also, the fact that v remains constant (and, consequently, that u remains constant along the characteristic) is due to the fact that the right-hand side in* (2.1) *is 0. Also this aspect will be changed in further examples.*

Remark 5 *Another important aspect here is the fact that the characteristic does intersect the x-axis, so an appropriate initial value can be found. This situation could even be extended to the case where the function u is assumed known along a curve, but one has to be sure that any characteristic does intersect that curve to provide the value of u. If the characteristics would be parallel to the curve along which u is known, than it becomes impossible to find the appropriate value of u. Finally, a crucial aspect in applying the MOC is that, for each point (x,t) where the solution needs to be known, there exists a unique characteristic passing through it. Combined with the observation before that the characteristic does intersect the x-axis (or any other curve along which u is given), the existence of a characteristic through (x,t) is needed to obtain the value $u(x,t)$. Uniqueness is equally important: if there were two different characteristics through (x,t), and they would intersect the x-axis in two different points, which of the two values of u_0 should be used for $u(x,t)$?*

2.2 The method of characteristics

We can extend this to a more general situation, to solve the equation

$$\partial_t u + a(x,t)\partial_x u = b(x,t,u), \text{ where } x \in \mathbb{R} \text{ and } t > 0, \tag{2.9}$$

with given initial condition $u_0 : \mathbb{R} \to \mathbb{R}$. Here $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^3 \to \mathbb{R}$ are given, sufficiently smooth functions (continuously differentiable). Repeating the previous ideas, we seek the characteristics as solutions to the Cauchy problem

$$\left\{ \begin{array}{rcl} x'(t) & = & a(x(t),t), \quad \text{for } t \geq 0, \\ x(0) & = & x_0. \end{array} \right. \tag{2.10}$$

Observe that, for every $x_0 \in \mathbb{R}$, (2.10) has a unique solution at least locally (explain why!). This leads to the following

Definition 1 *A characteristic curve of the equation in* (2.9) *is any curve in the* $x-t$ *plane defined as*

$$\{(x,t) \in \mathbb{R} \times [0,\infty), \ \textit{with } x = x(t)\}, \tag{2.11}$$

where the function $x(\cdot)$ *solves* (2.10) *with an arbitrary initial value* $x_0 \in \mathbb{R}$.

Note that, compared to (2.1), here the characteristic curve is not necessarily a line.

As before, we observe that the function $v(t) = u(x(t),t)$ satisfies for all $t \geq 0$

$$v'(t) = \frac{du}{dt}(x(t),t) = \partial_t u + a(x(t),t)\partial_x u = b(x(t),t,u(x(t),t)) = b(x(t),t,v(t)). \tag{2.12}$$

the function v solves the Cauchy problem

$$\begin{cases} v'(t) &=& b(x(t),t,v(t)), \quad \text{for } t \geq 0, \\ v(0) &=& u_0(x_0). \end{cases} \tag{2.13}$$

Again, due to the assumptions on b, (2.13) has a unique solution, at least locally, and this also gives the solution u along the characteristic.

From a practical point of view, with $(\tilde{x},\tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$, to find $u(\tilde{x},\tilde{t})$ solving (2.9) with the given initial data u_0, one can apply the following procedure:

1. Find the (general) characteristics by solving (2.10), for arbitrary $x_0 \in \mathbb{R}$;
2. Solve (2.13), again, for arbitrary $x_0 \in \mathbb{R}$;
3. Find the characteristic passing through $(\tilde{x},\tilde{t})$. In other words, find $x_0 \in \mathbb{R}$ s.t. the solution to (2.10) satisfies $x(\tilde{t}) = \tilde{x}$. With this, set $u(\tilde{x},\tilde{t}) = u_0(x_0)$.

In the above we have used $(\tilde{x},\tilde{t})$ to identify the point in the $x-t$ plane only to avoid any confusion between x as a real number, and the function $x(\cdot)$ solving (2.10). If there is no confusion, one may simply use again x and t.

Remark 6 *The function* a *in* (2.9) *does not depend on the unknown* u. *In other words, the expression on the left is linear w.r.t.* u. *This is essential to guarantee the existence and uniqueness of a characteristic curve passing through a given point* (x,t). *Of course, one can apply the same method also if the function* a *does depend on* u, *but only for points* (x,t) *for which the existence and uniqueness of a characteristic curve through them is guaranteed. As explained, if there is no characteristic through* (x,t), *or if there are more than one characteristics, no unique value can be assigned to* $u(x,t)$. *Hence, a solution* ***in classical sense****, namely a function* u *that has continuous first order partial derivatives and satisfies the equation in each point, cannot be found. Such problems are considered in the general of hyperbolic conservation laws. An example in this sense is the equation*

$$\partial_t u + \partial_x f(u) = 0,$$

with f *a given, non-linear function. Discussing such problems, including the concept of solution for them, exceeds the framework of this course.*

Remark 7 *It is straightforward to extend the MOC to problems involving multiple spatial dimensions. An example in this sense is*

$$\partial_t u(x,y,t) + a(x,y,t)\partial_x u(x,y,t) + b(x,y,t)\partial_y u(x,y,t) = c(x,y,t,u),$$

with given functions a, b, and given initial data u_0. Now the characteristics are triplets in $\mathbb{R}^3$

$$\{(x,y,t)/t \geq 0, x = x(t), y = y(t)\},$$

where the functions $x(\cdot)$ and $y(\cdot)$ solve the system of ODE's

$$\begin{cases} x'(t) &= a(x,y,t), \\ y'(t) &= b(x,y,t), \end{cases}$$

for $t \geq 0$ and with the initial data (x_0, y_0). Convince yourself that $v(t) = u(x(t), y(t), t)$ solving the equation $v'(t) = c(x(t), y(t), t, v)$ also provides the solution u.

2.3 Examples

We apply the steps explained before to find the solution for some equations. We omit the details, which you are asked to work out yourself. Note that u_0 is assumed known.

Example 1 *With given u_0, find u solving the equation*

$$\partial_t u + x\partial_x u = 0, \text{ for } x \in \mathbb{R}, t \geq 0,$$

and satisfying $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$.

Solution. The characteristic curves are solutions to

$$x'(t) = x \text{ for } t \geq 0,$$

and with $x(0) = x_0$. Clearly, this gives $x(t) = x_0 e^t$. Observe that these curves are not lines anymore, since the factor multiplying $\partial_x u$ is not constant. With this, let $v(t) = u(x(t), t) = u\big(x_0 e^t, t\big)$. It solves $v'(t) = 0$, thus $v(t) = v(0) = u_0(x_0)$.
Given now $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$, one has

$$u(\tilde{x}, \tilde{t}) = v(\tilde{t}) = u_0\big(x_0 e^{\tilde{t}}\big),$$

where x_0 is s.t. $x(t) = \tilde{x}$. This gives $x_0 = \tilde{x}e^{-\tilde{t}}$, and thus $u(\tilde{x}, \tilde{t}) = u_0\big(\tilde{x}e^{-\tilde{t}}\big)$. Leaving the $\tilde{}$'s out, we obtain the solution

$$u(x,t) = u_0\big(xe^{-t}\big).$$

Example 2 *We modify now the previous example by considering a non-zero right-hand side,*

$$\partial_t u + x\partial_x u = x, \text{ for } x \in \mathbb{R}, t \geq 0.$$

As before, we seek u satisfying the equation and the initial condition $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$.

Solution. Since the left-hand side of the equation was not changed, the characteristics remain the same, $x(t) = x_0 e^t$. The difference is for $v(t) = u(x(t), t) = u\big(x_0 e^t, t\big)$, which now solves $v'(t) = x(t)$, thus $v'(t) = x_0 e^t$, and with the initial condition $v(0) = u(x(0), 0) = u_0(x_0)$. Observe that the function v will not be constant anymore. One gets $v(t) = u_0(x_0) + x_0\big(e^t - 1\big)$.
For $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$ we have

$$u(\tilde{x}, \tilde{t}) = u_0(x_0) + x_0\big(e^{\tilde{t}} - 1\big),$$

and x_0 is s.t. $x(\tilde{t}) = \tilde{x}$. As before, $x_0 = \tilde{x}e^{-\tilde{t}}$, yielding

$$u(\tilde{x}, \tilde{t}) = u_0\big(\tilde{x}e^{-\tilde{t}}\big) + \tilde{x}e^{-\tilde{t}}\big(e^{\tilde{t}} - 1\big).$$

In a simpler writing,

$$u(x, t) = u_0\big(xe^{-t}\big) + xe^{-t}\big(e^t - 1\big) = u_0\big(xe^{-t}\big) + x\big(1 - e^{-t}\big).$$

Please, check that this is, indeed, the solution!

Exercise set 2 *"First order equations, MOC"*

1. *The function $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is given below. For each of these functions, find the solution to the first order equation that satisfies the given initial conditions (do not forget to check your answers!):*

$$\begin{cases} \dfrac{\partial u}{\partial t} + 3x^{\frac{4}{3}} \dfrac{\partial u}{\partial x} = f, & \text{voor } x > 0, t > 0, \\ u(x, 0) = x^{\frac{1}{3}}, & \text{voor } x > 0. \end{cases}$$

   Consider three cases: $f(x, t, u) = 0$, $f(x, t, u) = 1$ and $f(x, t, u) = u$.

2. *With the functions $f : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ given below, solve the first order equations with the given initial condition. This includes verifying the results!*

$$\begin{cases} \dfrac{\partial u}{\partial t} + 3x^{\frac{2}{3}} \dfrac{\partial u}{\partial x} = f, & \text{for } x \in \mathbb{R}, t > 0, \\ u(x, 0) = x^{\frac{1}{3}}, & \text{for } x \in \mathbb{R}. \end{cases}$$

   Consider three situations: a) $f(x, t, u) = 0$, b) $f(x, t, u) = 1$ and c) $f(x, t, u) = u$.

Chapter 3

The diffusion equation in unbounded domains: similarity solutions (week 2)

Below we consider the diffusion/heat equation in an unbounded interval, although many of the aspects discussed here extend to multiple spatial dimensions. More precisely, we start by studying the equation

$$\partial_t u = D\partial_{xx} u, \tag{3.1}$$

for times $t > 0$ and where $x \in (0, \infty)$ (first) or $x \in \mathbb{R}$ (later). The **diffusion coefficient** $D > 0$ is assumed known and, of course, initial conditions and, when applicable, boundary conditions are prescribed.

Before discussing in detail the (solutions) to (3.1), we mention some equations that, after a transformation, can be reduced to (3.1). These are summarised below, and the details will be worked out in the instruction.

A.) *Given, time-dependent source term, $f = f(t)$*
For a given function $f : [0, \infty) \to \mathbb{R}$, we consider the equation

$$\partial_t u = D\partial_{xx} u + f(t). \tag{3.2}$$

Here, one can consider the new unknown $v(x, t) = u(x, t) - \int_0^t f(z)dz$, satisfying (3.1).
Q: Can one proceed in the same way if f also depends on x?

B.) *A linear, u-dependent source term λu*
For a given $\lambda \in \mathbb{R}$, we consider the equation

$$\partial_t u = D\partial_{xx} u + \lambda u. \tag{3.3}$$

In this case, the unknown $v(x, t) = e^{-\lambda t} u(x, t)$ will satisfy (3.1).
Task: Extend this to the case when $\lambda : [0, \infty) \to \mathbb{R}$ is a given function of t.

C.) *An advection term* $a\partial_x u$
For a given $a \in \mathbb{R}$, we consider the equation

$$\partial_t u + a\partial_x u = D\partial_{xx} u. \tag{3.4}$$

In this case, one can apply a transformation inspired by the method of characteristics (see Chapter 2) and consider the unknown $v(x,t) = u(x+at,t)$, which satisfies (3.1). It is worth mentioning that, if the domain where the equation (3.1) (and thus also u) is defined is not the entire $\mathbb{R}$, then one has to account for the fact that the space–time domain has been modified through the transformation. More precisely, the point (x,t) for v is the point $(x+at,t)$ for u, so this latter point needs to be in the domain where u is defined. For example, if the equation is defined for $x > 0$ instead of $x \in \mathbb{R}$, and if $a < 0$, then the new x coordinate $x + at$ can become negative, and hence leave the domain of definition for u.
Task: Extend this to the case when $a : [0,\infty) \to \mathbb{R}$ is a given function of t.

D.) *A time-dependent diffusion coefficient,* $D = D(t)$
Given the continuous function $D : [0,\infty) \to (0,\infty)$ (why strictly positive?), we consider the equation

$$\partial_t u = D(t)\partial_{xx} u. \tag{3.5}$$

Here, one needs to change the time variable. More precisely, one can work with the new variable $\tau \geq 0$. This can be seen as a function of t, namely

$$\tau : [0,\infty) \to [0,\infty), \qquad \tau(t) = \int_0^t D(z)dz.$$

Since D is continuous and positive, the function τ is differentiable and bijective (from $[0,\infty)$ to its range), so the inverse function $t = t(\tau)$ is defined, continuous and differentiable. Therefore, one can consider the new unknown $v(x,\tau) = u(x,t(\tau))$. Then, v satisfies an equation that is similar to (3.1), but now in the variables τ and x,

$$\partial_\tau v(x,\tau) = \partial_{xx} v(x,\tau).$$

3.1 Similarity solution: general ideas

Given $D > 0$, we construct similarity solutions to (3.1). These are defined for all $t > 0$ and x in either the half-space $[0,\infty)$, or for the entire $\mathbb{R}$. We start with the problem in the quadrant $[0,\infty) \times [0,\infty)$,

$$\begin{cases} \partial_t u &= D\partial_{xx} u, & \text{for } x > 0, t > 0, \\ u(x,0) &= 0, & \text{for} x > 0, \\ u(0,t) &= 1, & \text{for } t > 0. \end{cases} \tag{3.6}$$

In fact, this can be extended straightforwardly to constant initial and boundary conditions, i.e. with u_0 and u_b (given constants in $\mathbb{R}$) instead of 0, respectively 1 in $(3.6)_2$ and $(3.6)_3$. Then, if u is the solution to $(3.6)_1$ with the initial and boundary conditions u_0 and u_b, one can consider the new unknown function

$$v(x,t) = \frac{u(x,t) - u_0}{u_b - u_0},$$

which solves precisely (3.6).

As mentioned, a good way to find solutions to a partial differential equation is to reduce it to an ordinary one. The similarity solution to (3.6) is obtained in a similar fashion. More precisely, we use the **similarity transformation**

$$\eta = xt^{-\frac{1}{2}} \qquad (= x/\sqrt{t}), \tag{3.7}$$

and seek the function $v : [0,\infty) \to \mathbb{R}$ s.t. $u(x,t) = v(\eta)$ is a solution to (3.6). Observe that, if $x \geq 0$ and $t > 0$, then $\eta \geq 0$, therefore the domain of v is $[0,\infty)$. With this, we state the following

Definition 2 *A function $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a **similarity solution** to the problem* (3.6) *if a function $v : [0,\infty) \to \mathbb{R}$ exists s.t. $u(x,t) = v(\eta)$, with $\eta = xt^{-\frac{1}{2}}$.*

Remark 8 *The problem* (3.6) *is defined in the quadrant $[0,\infty) \times [0,\infty)$. Consequently, for η in* (3.7) *one has $\eta \geq 0$. Therefore, the function v appearing in Definition 2 is defined on $[0,\infty)$. This situation will change later, when x is taken in the entire $\mathbb{R}$. Then, one also has $\eta \in \mathbb{R}$, and $v : \mathbb{R} \to \mathbb{R}$.*

Clearly, after introducing the similarity transformation in (3.7), the number of independent variables reduces from two, x and t, to one, η. Therefore, when looking for a similarity solution, the function v only depend on one variable and thus it solves an ordinary differential equation. One may wonder why the particular transformation in (3.7). For the diffusion equation, this is suggested by the argument below. However, this is not always true. For other problems, one may use a more general transformation,

$$\eta = xt^{\beta},$$

and determine $\beta \in \mathbb{R}$ s.t., by using only derivatives w.r.t. η, the variables x and t can be completely eliminated and the partial differential equation reduces to an ordinary one. This strategy also works for the diffusion equation, as will be seen in the instructions. Without being rigorous, we mention that the uniqueness of a solution to (3.6) is not guaranteed unless one assumes a controlled growth of u as $x \to \infty$, namely that

$$\lim_{x\to\infty} e^{-Cx^2}|u(x,t)| = 0,$$

for some $C > 0$ and for all $t > 0$. Then, this uniqueness also gives an argument for the choice of the similarity transformation (3.7). More precisely, if $u(x,t)$ is a solution to (3.6), then, for any $a > 0$, one may define the function

$$u_a : [0,\infty) \times [0,\infty) \to \mathbb{R}, u_a(x,t) = u(ax, a^2 t).$$

These functions are remarkable, as they are all solutions to (3.6)! This means that uniqueness does not hold true, unless for any $a > 0$ one has that

$$u_a(x,t) = u(x,t)$$

for all $x \geq 0$ and $t \geq 0$. For $t > 0$, one may take $a = 1/\sqrt{t}$ to obtain that

$$u(x,t) = u_{\frac{1}{\sqrt{t}}}(x,t) = u\big(\frac{x}{\sqrt{t}}, 1\big).$$

This shows thatm in fact, the solution u satisfies $u(x,t) = v(\eta)$ for all x and t, where $\eta = x/\sqrt{t}$ and $v(\eta) = u(x/\sqrt{t}, 1)$.
Task: Verify that u_a solves, indeed (3.6).

3.2 Finding the similarity solution to (3.6)

As stated above, we seek a similarity solution, in the sense of definition 2. In other words, we seek the function $v = v(\eta)$ s.t. $u(x,t) = v(\eta)$ solves the problem in (3.6). As mentioned, since $x, t \geq 0$, the same holds for the variable η. Moreover, $x = 0$ implies $\eta = 0$ and $t \searrow 0$ gives $\eta \to \infty$ if $x > 0$. Recalling the initial and boundary conditions in (3.6), this gives

$$\text{Initial condition:} \quad \lim_{\eta\to\infty} v(\eta) = \lim_{t\searrow 0} u(x,t) = 0 \text{ (with } x > 0), \tag{3.8}$$

$$\text{Boundary condition:} \quad v(0) = u(0,t) = 1 \text{ (with } t > 0). \tag{3.9}$$

We simplify (3.8) to $v(+\infty) = 0$, but this should always be interpreted as a limit.
We continue now to identify the equation for v. We use the chain rule to write

$$\partial_t u(x,t) = \frac{d}{dt} v(\eta) = -\frac{1}{2} x t^{-\frac{3}{2}} v'(\eta) = -\frac{1}{2}\frac{\eta}{t} v'(\eta), \tag{3.10}$$

and

$$\partial_x u(x,t) = \frac{d}{dx} v(\eta) = \frac{1}{\sqrt{t}} v'(\eta), \text{ implying } \partial_{xx} u(x,t) = \frac{1}{t} v''(\eta). \tag{3.11}$$

To have u solving $(3.6)_1$ in the entire quadrant, the function v has to solve the ordinary differential equation

$$-\frac{1}{2}\frac{\eta}{t} v'(\eta) = D\frac{1}{t} v''(\eta),$$

for all $\eta > 0$. Clearly, t can be eliminated from the above and one ends up with the problem

$$\begin{cases} v''(\eta) + \dfrac{\eta}{2D} v'(\eta) = 0, & \text{for } \eta > 0, \\ v(0) = 1, \ v(+\infty) = 0. \end{cases} \tag{3.12}$$

One can find immediately (multiply the equation by $e^{\frac{\eta^2}{4D}}$ and integrate the resulting) that

$$v'(\eta) = Ae^{-\frac{\eta^2}{4D}},$$

implying that

$$v(\eta) = B + A\int_0^\eta e^{-\frac{s^2}{4D}} ds.$$

The constants $A, B \in \mathbb{R}$ need to be determined s.t. the function v satisfies the conditions in (3.12). First of all, since $v(0) = 1$ one gets $B = 1$. To obtain A, we apply the substitution $z = s/(2\sqrt{D})$ in the above to rewrite

$$v(\eta) = 1 + 2A\sqrt{D}\int_0^{\frac{\eta}{2\sqrt{D}}} e^{-z^2} dz. \tag{3.13}$$

Observe that e^{-z^2} is not a function with well-known primitives, but we have the equality

$$\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}. \tag{3.14}$$

With this, letting $\eta \to \infty$ in (3.13) one gets

$$0 = v(\infty) = 1 + 2A\sqrt{D}\int_0^\infty e^{-z^2} dz = 1 + 2A\sqrt{D}\frac{\sqrt{\pi}}{2},$$

giving $A = -\dfrac{1}{\sqrt{\pi D}}$ and

$$v(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{\eta}{2\sqrt{D}}} e^{-z^2} dz. \tag{3.15}$$

This can be written as

$$v(\eta) = 1 - erf(\frac{\eta}{2\sqrt{D}}), \tag{3.16}$$

where erf is the so-called **error function**

$$erf : \mathbb{R} \to \mathbb{R}, \qquad erf(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-z^2} dz. \tag{3.17}$$

As an "intermezzo", we present some of the properties of the error function.

Proposition 1 *The function err is $C^\infty(\mathbb{R})$, strictly increasing in $\mathbb{R}$ and bounded,*

$$-1 < erf(\eta) < 0 \text{ if } \eta < 0 \text{ and } 0 < erf(\eta) < 1 \text{ if } \eta > 0.$$

Using the above, we obtain the **similarity solution** to the diffusion problem in (3.6),

$$u(x,t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-z^2} dz = 1 - erf\left(\frac{x}{2\sqrt{Dt}}\right). \tag{3.18}$$

This also gives immediately the solution for other (but constant!) initial and boundary conditions than 0, respectively 1. Clearly, given $u_0, u_b \in \mathbb{R}$, the solution $\tilde{u}$ to $(3.6)_1$, but satisfying

$$\tilde{u}(x,0) = u_0, \text{ and } \tilde{u}(0,t) = u_b$$

for $x > 0$, respectively $t > 0$, reads

$$\tilde{u}(x,t) = u_0 + (u_b - u_0)u(x,t) = u_b + (u_0 - u_b)\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-z^2} dz. \tag{3.19}$$

3.3 Properties of the similarity solution to (3.6)

Several properties of the similarity solution can be deduced from the explicit form in (3.19). We assume that $u_b > u_0$, the case $u_b < u_0$ being analogous, and $u_b = u_0$ trivial.

Proposition 2 *Let $\tilde{u}$ be the similarity solution with initial and boundary conditions $u_0 < u_b$. One has*

1. Monotonicity*: $\tilde{u}(\cdot, t)$ is strictly decreasing in x for all $t > 0$, while $\tilde{u}(x, \cdot)$ is strictly increasing in t for all $x > 0$.*

2. Level curves*: For any $C > 0$, $\tilde{u}$ is constant along the parabola $t = Cx^2$ in the $x - t$ plane strictly decreasing in x for all $t > 0$, while $\tilde{u}(x, \cdot)$ is strictly increasing in t for all $x > 0$.*

3. Smoothness*: $\tilde{u}$ is a C^∞ function (i.e. all mixed derivatives, of any order, exist and are continuous) everywhere in $[0, \infty) \times [0, \infty)$ $\{(0,0)\}$. In other words, $\tilde{u}$ is smooth everywhere excepting the origin.*

4. Maximum principle*: $\tilde{u}$ is bounded by u_0 and u_b. Moreover, if $x > 0$ and $t > 0$ one has*

   $$u_0 < u(x,t) < u_b.$$

   In other words, the solution $\tilde{u}$ attains its extreme values at the boundary of the domain of definition.

5. Linearity*: If $\tilde{u}$ and $\tilde{v}$ solve the equation $(3.6)_1$ but with the initial and boundary conditions u_0 and v_0, respectively u_b and v_b, then $\tilde{w} = \tilde{u} - \tilde{v}$ also solves the equation $(3.6)_1$ and satisfies the initial and boundary conditions $u_0 - v_0$, respectively $u_b - v_b$.*

6. Ordering*: As above, let $\tilde{u}$ and $\tilde{v}$ solve the equation* $(3.6)_1$ *but with the initial and boundary conditions* u_0 *and* v_0*, respectively* u_b *and* v_b*. Assume that the data is ordered, namely that*

$$u_0 \geq v_0 \qquad \text{and} \qquad u_b \geq v_b.$$

Then, the solutions $\tilde{u}$ and $\tilde{v}$ are also ordered, namely

$$\tilde{u}(x,t) \geq \tilde{v}(x,t), \qquad \text{for all } x, t \geq 0.$$

If at least one of the two inequalities in the data is strict, then the inequality is strict in any point $(x,t) \in (0,\infty) \times (0,\infty)$. As for the maximum principle, equality can only hold initially, or at the boundary.

Task: Give a proof of Proposition 2.
We note that the proof is elementary. It uses the form of the similarity solutions in (3.18) and (3.19) and the properties of erf presented in Proposition 1. We also remark that, since $erf(\eta) \in [0,1)$ for all $\eta > 0$, the solution in (3.19) is a convex combination of the initial and boundary data,

$$\tilde{u}(x,t) = u_0 \; erf\big(\frac{x}{2\sqrt{Dt}}\big) + u_b \left(1 - erf\big(\frac{x}{2\sqrt{Dt}}\big)\right). \tag{3.20}$$

Finally, we note that, though obtained by explicit calculations for a specific problem, the properties in Proposition 2 are representative for a large class of problems. In particular, in the absence of a source term, the solution to a parabolic problem, even nonlinear, will attain its extreme values on the boundary of the space-time domain. Also, solutions will be ordered in this case, and smooth.

3.4 The similarity solution to (3.1) in for the entire $\mathbb{R}$

We consider now the case when $x \in \mathbb{R}$. This means that there is no prescribed value vor u along $x = 0$. Again, $D > 0$ is given and we construct similarity solutions to (3.1), defined for all $t > 0$ but now $x \in \mathbb{R}$,

$$\left\{ \begin{array}{rcll} \partial_t u &=& D\partial_{xx} u, & \text{for } x \in \mathbb{R}, t > 0, \\ u(x,0) &=& \left\{ \begin{array}{ll} 1, & \text{for } x < 0, \\ 0, & \text{for } x > 0. \end{array} \right. & \end{array} \right. \tag{3.21}$$

We note that the initial condition consists of two constant states (left and right to $x_0 = 0$) and is not continuous. In fact, the value $u(0,0)$ remains unspecified, but we will see that the solution satisfies $u(0,t) = \frac{1}{2}$ for all $t > 0$. Obviously, a jump can be translated at some arbitrary ppoint $x_0 = a \in \mathbb{R}$ instead of $x_0 = 0$. If this is the case, the problem can be reduced to (3.21) by applying the transformation $v(x,t) = u(x-a,t)$. One can derive the solution following the ideas used in section 3.2. This includes the uniqueness argument for seeking a solution $u(x,t) = v(\eta)$ with $\eta = x/\sqrt{t}$, but now the growth should be controlled for $x \to -\infty$ as well. Observe that, in this case, the boundary condition $u(0,t) = 1$ is replaced by an initial condition for $x < 0$. Clearly, if $x < 0$, letting $t \searrow 0$ means that $\eta = x/\sqrt{t} \to -\infty$. For the function v, the condition (3.9) (at $\eta = 0$) will be replaced by the condition

$$\lim_{\eta \to -\infty} v(\eta) = \lim_{t \searrow 0} u(x,t) = 1 \text{ (with } x < 0). \tag{3.22}$$

To find the similarity solution, one needs to solve first the problem

$$\begin{cases} v''(\eta) + \dfrac{\eta}{2D} v'(\eta) = 0, \quad \text{for } \eta \in \mathbb{R}, \\ v(-\infty) = 1, \; v(+\infty) = 0. \end{cases} \tag{3.23}$$

Following the steps in section 3.2, one obtains

$$v(\eta) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{\frac{\eta}{2\sqrt{D}}} e^{-z^2} dz = \frac{1}{2}\left[1 - erf(\frac{\eta}{2\sqrt{D}})\right], \tag{3.24}$$

with $\eta \in \mathbb{R}$. Using this, one obtains the similarity solution

$$u(x,t) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-z^2} dz = \frac{1}{2}\left[1 - erf(\frac{x}{2\sqrt{Dt}})\right]. \tag{3.25}$$

We consider now a more general case, where, for given $a, u_-, u_+ \in \mathbb{R}$, the initial condition is changed to

$$u(x,0) = \begin{cases} u_-, & \text{for } x < a, \\ u_+, & \text{for } x > a. \end{cases} \tag{3.26}$$

Then, the similarity solution is

$$u(x,t) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2} erf(\frac{x-a}{2\sqrt{Dt}}) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{\sqrt{\pi}} \int_0^{\frac{x-a}{2\sqrt{Dt}}} e^{-z^2} dz. \tag{3.27}$$

Task: Apply the steps in Section 3.2 to obtain the solution in (3.24). Then, verify that u in (3.27) satisfies the initial condition.

We conclude this part with the observation that the similarity osolution obtained for $x \in \mathbb{R}$ satisfy the same properties as the ones for the half-space, as given in Proposition 2. This can be proved in a similar fashion.

Exercise set 3 *"Parabolic problems in unbounded domains; similarity solutions"*

1. *Work out the tasks listed in this text.*
2. ***Tricks to reduce some equations to the standard heat equation***
   For the equations below, apply the suggested transformations to bring them to the standard form. $D > 0$ is a given diffusion coefficient.

(a) *Given $a \in \mathbb{R}$ and with $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ solving*

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

find the problem solved by $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defined by $v(x, t) = u(x + at, t)$.

(b) *Extend the situation above by considering a function $a : [0, \infty) \to R$ instead of a real constant a. Do so by defining $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ as $v(x, t) = u\big(x + \int_0^t a(s)ds, t\big)$.*

(c) *Given $\lambda \in \mathbb{R}$ and with u solving*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda u,$$

show that v defined by $v(x, t) = u(x, t)e^{-\lambda t}$ satisfies the heat equation (hence without the last term, the "reaction term").

(d) *Generalise this approach for the case where $\lambda : [0, \infty) \to R$ is a given function and u the solution to*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda(t)u.$$

Find an appropriate function v that depends on u but solves the standard heat equation.

(e) *Given $f : [0, \infty) \to \mathbb{R}$ and with u solving*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(t),$$

find an appropriate function v, depending on u and f, and solving the heat equation.

(f) *One can rescale the time or the space to reduce the diffusion coefficient. In this sense, let u solve*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}.$$

Define $\tau = Dt$ and the function v by $v(x, \tau) = u\big(x, \frac{\tau}{D}\big) = u(x, t)$. Show that v solves

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

Generalize this approach for the case when $D : [0, \infty) \to \mathbb{R}$ is a given function satisfying $D(t) \geq D_0 > 0$ for all $t \geq 0$, by considering $\tau = \int_0^t D(s)ds$ and $v(x, \tau) = u(x, t)$.

3. ***Solving parabolic equations in unbounded domains***
Find a solution to the problems below.

(a) *The reaction-diffusion problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \dfrac{\partial^2 u}{\partial x^2} + \sin t, \quad -\infty < x < \infty, t > 0, \\ u(x,0) &= \begin{cases} 0, & x < 1, \\ 1, & x \geq 1. \end{cases} \end{cases}$$

Note that the initial condition is discontinuous at $x = 1$!

i. *Using an appropriate function f and the transformation $v(x,t) = u(x,t) + f(t)$, transform the equation into*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

ii. *Determine the similarity solution (gelijkvormigheidsoplossing) v for the transformed problem (in v) and next determine u.*

iii. *Determine the limits $\lim_{x\to-\infty} u(x,t)$ and $\lim_{x\to\infty} u(x,t)$ for $t > 0$.*

(b) *Consider the reaction-diffusion problem in the 1^e quadrant*

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \dfrac{\partial^2 u}{\partial x^2} + tu, \quad 0 < x < \infty, t > 0, \\ u(x,0) &= 1, \quad x > 0, \\ u(0,t) &= 0, \quad t > 0. \end{cases}$$

i. *Find an appropriate function v, depending on u, satisfying the standard heat/diffusion equation*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

ii. *Determine the similarity solution (gelijkvormigheidsoplossing) v of the standard problem and then u.*

(c) *Let $n \in \mathbb{N}$ be a natural number, $D > 0$ and let u_n be the solution of the diffusion problem*

$$\begin{cases} \dfrac{\partial u_n}{\partial t} &= D\dfrac{\partial^2 u_n}{\partial x^2}, \quad -\infty < x < \infty, t > 0, \\ u_n(x,0) &= \begin{cases} 0, & x < -\frac{1}{n}, \text{ or } x > \frac{1}{n}, \\ \dfrac{n}{2}, & -\frac{1}{n} < x < \frac{1}{n}. \end{cases} \end{cases}$$

i. *Determine the similarity solution u_n.*

ii. *Fix now $x \in \mathbb{R}$ and $t > 0$. Determine the limit $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Observe that this limit defines a function u still solving the heat equation. What is the initial condition for u?*

Hint: Use the Mean Value Theorem (middelwaardestelling) for integrals: let $f : [a,b] \to \mathbb{R}$ be continuous, then there exists a $\xi \in (a,b)$ s.t. $\int_a^b f(z)dz = (b-a)f(\xi)$.

Chapter 4

The diffusion equation in unbounded domains: fundamental solution and solutions with general initial conditions (week 3)

We continue studying the diffusion equation in domains, and, for given $D > 0$, consider the equation

$$\partial_t u = D\partial_{xx} u, \tag{4.1}$$

for times $t > 0$ and where $x \in \mathbb{R}$. In Section 3.4, the concept of similarity solution was introduced, and obtained the solution in (3.27), satisfying the initial condition (3.26). Since (4.1) is a linear equation, any linear combination of solutions remains a solution. This idea can be used to determine the similarity solution satisfying the initial condition

$$\Phi_n(x,0) = \begin{cases} 0, & \text{if } x < -\frac{1}{n}, \text{ or } x > \frac{1}{n}, \\ \frac{n}{2}, & \text{if } \quad -\frac{1}{n} < x < \frac{1}{n}, \end{cases} \tag{4.2}$$

where $n \in \mathbb{N}_0$ is any natural number (see also `Exercise B3` in the second set of instruction problems, and observe that the name was changed from u to Φ). Clearly, the solution can be written as the sum of two solutions to (4.1), $\Phi_n = \Phi_n^{(1)} + \Phi_n^{(2)}$, and satisfying the initial conditions

$$\Phi_n^{(1)}(x,0) = \begin{cases} 0, & \text{if } x < -\frac{1}{n}, \\ \frac{n}{2}, & \text{if } -\frac{1}{n} < x, \end{cases} \quad \text{respectively } \Phi_n^{(2)}(x,0) = \begin{cases} 0, & \text{if } x < \frac{1}{n}, \\ -\frac{n}{2}, & \text{if } \frac{1}{n} < x. \end{cases}$$

Before writing the solution, observe that, for the initial condition one has

$$\int_{\mathbb{R}} \Phi_n(x,0)\; dx = 1,$$

uniformly w.r.t. n, and that the initial conditions has a support (i.e. the closure of the set where the function is non-zero) that is shrinking to a point when $n \to \infty$. Using the solution in (3.27), one immediately obtains the solution

$$\Phi_n(x,t) = \frac{n}{2\sqrt{\pi}} \int_{\frac{x-\frac{1}{n}}{2\sqrt{Dt}}}^{\frac{x+\frac{1}{n}}{2\sqrt{Dt}}} e^{-z^2} dz. \tag{4.3}$$

Task: If not done yet, verify that Φ_n in (4.3) solves (4.1) and the initial condition.

Observe that, for any $t > 0$, the solution Φ_n is smooth, although the initial condition has two jump-type discontinuities. We are interested in the limit of Φ_n when $n \to \infty$. To find the limit function, we use the mean value theorem for integrals

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, then there exists at least one point $c \in [a,b]$ such that $(b-a)f(c) = \int_a^b f(x)\; dx$.*

Note that the point c may depend on the function f and on the interval $[a,b]$.
We apply Theorem 1 to the n-dependent family of solutions in (4.3). For fixed $t > 0$ and $x \in \mathbb{R}$, a $z_n \in \left[\frac{x-\frac{1}{n}}{2\sqrt{Dt}}, \frac{x+\frac{1}{n}}{2\sqrt{Dt}}\right]$ exists s.t.

$$\left(\frac{x+\frac{1}{n}}{2\sqrt{Dt}} - \frac{x-\frac{1}{n}}{2\sqrt{Dt}}\right) e^{-z_n^2} = \int_{\frac{x-\frac{1}{n}}{2\sqrt{Dt}}}^{\frac{x+\frac{1}{n}}{2\sqrt{Dt}}} e^{-z^2} dz,$$

implying that

$$\Phi_n(x,t) = \frac{1}{2\sqrt{\pi Dt}} e^{-z_n^2}.$$

Since $z_n \in \left[\frac{x-\frac{1}{n}}{2\sqrt{Dt}}, \frac{x+\frac{1}{n}}{2\sqrt{Dt}}\right]$, when $n \to \infty$ one gets

$$\lim_{n\to\infty} z_n = \frac{x}{2\sqrt{Dt}},$$

and, due to the continuity of Φ_n,

$$\lim_{n\to\infty} \Phi_n(x,t) = \frac{1}{2\sqrt{\pi Dt}}\; e^{-\frac{x^2}{4Dt}}. \tag{4.4}$$

The limit holds for any $t > 0$ and $x \in \mathbb{R}$.

4.1 The fundamental solution for equation (4.1)

The limit function obtained before is important for equation 4.1:

Definition 3 *The function*

$$\Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R}, \Phi(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-\frac{x^2}{4Dt}}$$

is called the ***fundamental solution*** *to the equation* (4.1).

We observe that Φ was constructed as the limit $n \to \infty$ of solutions Φ_n to the equation in (4.1), therefore we associate it to this equation. However, it is not clear why it can be called "(fundamental) solution" to this equation. This is justified by

Proposition 3 *For the function Φ introduced in Definition 3 one has*

1. $\Phi(x,t) = -\partial_x u(x,t)$, *where u is the similarity solution in* (3.25)*;*
2. $\partial_t \Phi = D \partial_{xx} \Phi$ *for all $x \in \mathbb{R}$ and $t > 0$;*
3. $\int_{\mathbb{R}} \Phi(x,t)\, dx = 1$ *for all $t > 0$;*
4. $\lim_{t \searrow 0} \Phi(x,t) = \begin{cases} 0, & \text{if } x \neq 0, \\ +\infty, & \text{if } x = 0. \end{cases}$

Proof. The first point can be shown by direct calculation. We omit the details here. Moreover, we observe that this point immediately implies the second one, since u is smooth for any $t > 0$ and $x \in \mathbb{R}$, so one can change the order of the partial derivatives in any mixed partial derivative of higher order. Then one gets

$$\partial_t \Phi - D\partial_{xx}\Phi = -\partial_{tx} u + D\partial_{xxx} u = -\partial_x \big(\partial_t u - D\partial_{xx} u\big) = 0,$$

everywhere in $\mathbb{R} \times (0, \infty)$.
For point 3 we use the substitution $z = \frac{x}{2\sqrt{Dt}}$ to obtain, for any $t > 0$, that

$$\int_{\mathbb{R}} \Phi(x,t) \underset{\cdot}{\mathrm{x}} = \frac{1}{2\sqrt{\pi D t}} \int_{\mathbb{R}} e^{-\frac{x^2}{4Dt}}\, dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\, dz = 1.$$

Finally, for $x \neq 0$ one has that (e.g. by L'Hôpital's rule)

$$\lim_{t \searrow 0} \frac{1}{2\sqrt{\pi D t}}\, e^{-\frac{x^2}{4Dt}} = 0,$$

whereas

$$\lim_{t \searrow 0} \Phi(0,t) = \lim_{t \searrow 0} \frac{1}{2\sqrt{\pi D t}} = +\infty,$$

$\square$

Remark 9 *Note that, if a function $u(x,t)$ models e.g. the concentration of a solute in water, and assuming that the water occupies the domain Ω (bounded or unbounded), the integral $M(t) = \int_\Omega u(x,t)\ dx$ will denote the total solute mass at time t in the entire Ω. Following from Proposition 3, the total mass of $\Phi(\cdot, t)$ is 1,*

$$\int_{\mathbb{R}} \Phi(x,t)\ dx = 1 \qquad \textit{for all } t > 0.$$

We can pass the above to the limit as $t \searrow 0$, yielding

$$\lim_{t \searrow 0} \int_{\mathbb{R}} \Phi(x,t)\ dx = 1.$$

In view of the limit shown at point 4, one has that the "initial condition" for the fundamental solution Φ is the Dirac δ distribution.

Following from point 2 of Proposition 3, Φ is a solution to (4.1). This justifies the word "solution" in the name "fundamental solution" (why "fundamental" will become clear later). The concept of fundamental solution can be extended to other linear partial differential equations, written in a general form

$$Lu = 0,$$

where L is a differential operator involving partial derivatives.

Definition 4 *A function $\Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is called a fundamental solution to the linear equation $Lu = 0$ if*

- *Φ is a solution to the equation, $L\Phi = 0$;*
- *$\int_{\mathbb{R}} \Phi(x,t)\ dx = 1$ for all $t > 0$;*
- *$\lim\limits_{t \searrow 0} \Phi(x,t) = 0$ for all $x \neq 0$.*

4.2 Solving (4.1) with general initial conditions

The name "fundamental" stems from the fact that such solutions can be used very well to construct solutions to the given equation, but satisfying a particular initial condition. We only consider here the diffusion problem

$$\left\{ \begin{array}{rcll} \partial_t u(x,t) & = & D\partial_{xx} u(x,t), & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) & = & u_0(x), & \text{for } x \in \mathbb{R}. \end{array} \right. \tag{4.5}$$

Here $D > 0$ and the function $u_0 : \mathbb{R} \to \mathbb{R}$ are assumed known. Before doing so, we discuss briefly the concept of ***convolution***.

4.2.1 Convolution

We restrict ourselves to the case of one spatial dimension. Given the functions $f, g : \mathbb{R} \to \mathbb{R}$, their convolution is the function $f * g : \mathbb{R} \to \mathbb{R}$ defined as

$$f * g(x) = \int_{\mathbb{R}} f(y) g(x - y)\ dy, \tag{4.6}$$

whenever the integral makes sense. With respect to this last aspect, we mention that the convolution is well defined if e.g. f has a compact (i.e. bounded and closed) support, and g is locally integrable (i.e. integrable for any compact subset of $\mathbb{R}$). In a more general setting, the convolution is well defined if one function is integrable and the other one measurable and bounded.
There are several properties related to convolution. We summarise those that we will use below in the following result.

Proposition 4 *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be s.t. the convolutions $f * g$, $f * h$, etc. are well defined over $\mathbb{R}$. Then, one has*

1. *Commutativity: $f * g = g * f$;*
2. *Multiplication with the Dirac δ: if f is continuous and has a compact support, then $\delta * f = f$;*
3. *Linearity: for $\alpha, \beta \in \mathbb{R}$ one has $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$;*
4. *Smoothness: $f * g$ is as smooth as the smoothest of f and g;*
5. *Differentiation: if $f \in C^1(\mathbb{R})$ and g is locally integrable, then $f * g$ is continuously differentiable and one has $(f * g)' = f' * g$.*

We also give an important result related to the Dirac δ distribution, which will be needed for the initial condition. We consider an ***approximation of identity*** sequence, i.e. a sequence of functions $\delta_n : \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}_0$) satisfying the following conditions:

- $\delta_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} \delta_n(x)\ dx = 1$ for all $n \in \mathbb{N}_0$;
- For any $\varepsilon > 0$ one has $\lim\limits_{n \to \infty} \int_{-\varepsilon}^{\varepsilon} \delta_n(x)\ dx = 1$.

Observe that, from a formal point of view, $\delta_n \to \delta$ as $n \to \infty$, and that the functions Φ_n in (4.2) form an approximation of the unity. We have

Proposition 5 *Let $\{\delta_n, n \in \mathbb{N}_0\}$ be an approximation of identity sequence and $f : \mathbb{R} \to \mathbb{R}$ a function that is bounded and continuous. Then, for all $x \in \mathbb{R}$, one has*

$$\lim_{n \to \infty} (\delta_n * f)(x) = f(x).$$

This can be extended to any argument x where the function f is continuous. One needs that f is bounded and integrable. In fact, this result is a justification of the formal limit $\delta_n \to \delta$, since $\delta * f = f$ (see above).
We conclude this subsection with the Dominated Convergence Theorem, which will be used below.

Theorem 2 *Let $\{f_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}\}$ be a sequence of (Lebesgue) integrable functions over $\mathbb{R}$ and assume that the sequence converges in pointwise sense to $f : \mathbb{R} \to \mathbb{R}$, namely*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for a.e. } x \in \mathbb{R}.$$

Assume further that the integrable function $g : \mathbb{R} \to \mathbb{R}$ exists ($\int_{\mathbb{R}} g(x)\ dx < \infty$) s.t.

$$|f_n(x)| \leq g(x) \quad \text{for all } x \in \mathbb{R},$$

uniformly w.r.t. $n \in \mathbb{N}_0$. Then f is (Lebesgue) integrable and one has

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\ dx = \int_{\mathbb{R}} f(x)\ dx.$$

4.2.2 The solution by convolution

Here we assume that u_0 is a given, continuous function and having a compact support. We show that the solution to the problem in (4.5) will be obtained as the convolution of the fundamental solution Φ and the initial condition u_0. Before doing so, we give an intuitive argument concerning the initial condition. Recalling (4.4), in which Φ is obtained as the limit $n \to \infty$ of the functions Φ_n in (4.3), we use the same reasoning for the initial condition. We have

$$\big(\Phi_n(\cdot, 0) * u_0(\cdot)\big)(x) = \int_{\mathbb{R}} \Phi_n(y, 0) u_0(x - y)\ dy = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} u_0(y)\ dy = u_0(y_n),$$

where $y_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is the intermediate point in Theorem 1. Here $\cdot$ means that, for Φ_n, the convolution is taken w.r.t. the first variable. Taking $n \to \infty$, since u_0 is continuous one has

$$\lim_{n\to\infty} \big(\Phi_n(\cdot, 0) * u_0(\cdot)\big)(x) = u_0(x).$$

Recalling point 2 of Proposition 4 and points 3 and 4 of Proposition 3, this suggests on one hand that, indeed, the limit $\lim\limits_{t\searrow 0} \Phi(\cdot, t) = \delta$, and that the solution u to (4.5) is given by the convolution $\Phi * u_0$,

$$u(x, t) = \int_{\mathbb{R}} \Phi(y, t) u_0(x - y)\ dy = \frac{1}{2\sqrt{\pi D t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4Dt}}\ u_0(x - y)\ dy. \tag{4.7}$$

Indeed, we have

Theorem 3 *Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a given, continuous function that is bounded. Then the function $u = \Phi * u_0$ in* (4.7) *is the solution to the problem* (4.5).

Proof: We need to prove two results, namely that u satisfies the equation for all $x \in \mathbb{R}$ and $t > 0$, and that $\lim_{t \searrow 0} u(x,t) = u_0(x)$ for all $x \in \mathbb{R}$. To prove the former, we recall that the fundamental solution Φ, introduced in Definition 3, satisfies the equation for all $x \in \mathbb{R}$ and $t > 0$, as follows from Proposition 3, point 2. Then, one has

$$\partial_t u - D\partial_{xx} u = (\partial_t \Phi - D\partial_{xx}\Phi) * u_0 = 0,$$

where we have used point 5 in Proposition 4. Observe that the function Φ is smooth in the set $\mathbb{R} \times (0, \infty)$, which allows to interchange the (partial) differentiation (in either t or x) with the integral in the convolution. This interchange is also justified by the result in [1] (Theorem 6, p. 809), as Φ is smooth and for any partial derivative, the relevant integrals exist. For example, given $x \in \mathbb{R}$ and $t > 0$, to show that

$$\partial_x(\Phi * u_0)(x,t) = (\partial_x \Phi(\cdot, t) * u_0(\cdot))(x,t),$$

one needs to prove that the integrals

$$\int_{\mathbb{R}} \Phi(x-y,t) u_0(y)\ dy, \qquad \text{and} \qquad \int_{\mathbb{R}} \partial_x \Phi(x-y,t) u_0(y)\ dy$$

do exist (note that we have used the equality $f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)\ dy = \int_{\mathbb{R}} f(y)g(x-y)\ dy$, see Proposition 4, point 1) and that a function $g : R \to \mathbb{R}$ exists s.t.

$$\int_{\mathbb{R}} g(y)\ dy < \infty, \qquad \text{and} \qquad \left|\partial_{xx}\Phi(x-y,t)u_0(y)\right| \leq g(y) \text{ for all } y \in \mathbb{R}.$$

In doing so, one uses the boundedness of u_0 and the properties of Φ.
To prove that u does satisfy the initial condition, we write

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4Dt}}\ u_0(x-y)\ dy = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\ u_0(x - 2z\sqrt{Dt})\ dz,$$

where we have used the substitution $z = \frac{y}{2\sqrt{Dt}}$. We now consider any monotonically decreasing sequence of times $\{t_n, n \in \mathbb{N}_0\}$ converging to $t = 0$, and show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\ u_0(x - 2z\sqrt{Dt_n})\ dz = u_0(x)$$

for any $x \in \mathbb{R}$. To do so, we use the dominated convergence theorem, Theorem 2. Specifically, for fixed $t > 0$ and $x \in \mathbb{R}$, we take $f_n : \mathbb{R} \to \mathbb{R}, f_n(z) = \frac{1}{\sqrt{\pi}} e^{-z^2}\ u_0(x - 2z\sqrt{Dt_n})$ and observe that, since $t_n \searrow 0$,

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} e^{-z^2}\ u_0(x - 2z\sqrt{Dt_n}) = \frac{1}{\sqrt{\pi}} e^{-z^2}\ u_0(x).$$

Furthermore, since u_0 is bounded, an $M > 0$ exists s.t. $|u_0(x)| \leq M$ for all $x \in \mathbb{R}$, and thus

$$|f_n(z)| \leq \frac{M}{\sqrt{\pi}} e^{-z^2} =: g(z).$$

The function g is integrable, and one has $\int_{\mathbb{R}} g(z)\, dz = M$. We can now apply Theorem 2 to obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(z)\, dz = \int_{\mathbb{R}} f(z)\, dz = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\, u_0(x)\, dz = u_0(x).$$

Clearly, this gives $\lim\limits_{n\to\infty} u(x, t_n) = u_0(x)$ for any sequence $t_n \searrow 0$, implying that $\lim\limits_{t\searrow 0} u(x,t) = u_0(x)$ for any $x \in \mathbb{R}$, and the theorem is proven. □

A direct consequence of Theorem 3 is the following.

Corollary 1 *Let $u = \Phi * u_0$ solve the problem in (4.5), as given in Theorem 3. Then*

1. *Smoothing: one has $u \in C^\infty(\mathbb{R} \times (0, \infty))$ even if u_0 is not so smooth;*

2. *Maximum principle: if u_0 is a bounded, continuous initial condition s.t. $u_0(x) \geq 0$ for all $x \in \mathbb{R}$, then $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$. More general, if some $m, M \in \mathbb{R}$ exist s.t. $m \leq u_0(x) \leq M$ for any $x \in \mathbb{R}$, then $m \leq u(x,t) \leq M$, for all $x \in \mathbb{R}$ and $t > 0$;*

3. *Infinite speed of propagation: if, next to the above u_0 is not 0 everywhere (thus an $x \in \mathbb{R}$ exists s.t. $u_0(x) > 0$), then $u(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$;*

4. *Comparison principle: let $u_0^{(1)}, u_0^{(2)}$ be two bounded, continuous initial conditions s.t. $u_0^{(1)}(x) \geq u_0^{(2)}(x)$ for all $x \in \mathbb{R}$, and let $u^{(1)}$ and $u^{(2)}$ be the corresponding solutions. Then, one has*

$$u^{(1)}(x,t) \geq u^{(2)}(x,t) \qquad \textit{for all } x \in \mathbb{R} \textit{ and } t > 0.$$

   Moreover, if $u_0^{(1)} \not\equiv u_0^{(2)}$ (i.e. the initial conditions are not identical), then the inequality above is strict for all $x \in \mathbb{R}$ and $t > 0$.

The proof follows directly, from the properties of Φ. We omit it here.
We conclude this section with the remark that fundamental solutions are not restricted to one spatial dimension, but can be extended to the general case $\mathbb{R}^n$.

Exercise set 4 *"Parabolic problems: fundamental solution and solutions with general initial conditions"*

1. *Given the constants $D > 0$, $V \in \mathbb{R}$ and the continuous function $g : [0, \infty) \to \mathbb{R}$, consider the initial value problem*

$$\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} + g(t) u \qquad \textit{in } \mathbb{R} \times (0, \infty), \tag{4.8}$$

$$u(x, 0) = \delta \qquad \textit{for } x \in \mathbb{R}, \tag{4.9}$$

where δ is the Dirac distribution). Find the fundamental solution of the problem above.

Hint: You may apply appropriate transformations for bringing first the equation to the standard form. then you can use directly the fundamental solution for the heat equation.

2. *Consider the heat equation in the semi-infinite interval:*

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \dfrac{\partial^2 u}{\partial x^2}, \qquad x > 0, t > 0, \\ u(x,0) &= 0, \qquad x > 0, \\ u(0,t) &= t^{-\frac{1}{2}}, \qquad t > 0. \end{cases}$$

Seek solutions in the form:

$$u(x,t) = t^{-\alpha} f(\eta) \quad \text{with} \quad \eta = x t^{-\beta}.$$

   (a) *Determine the constants $\alpha, \beta \in \mathbb{R}$ that allow eliminating the variables t and x, and bringing the original equation determine to an ordinary differential equation in f and the variable η. Which problem solves f? What are the boundary values?*

   (b) *Rewrite the equation for f in a form involving only derivatives of $f(\eta)$ and of $\eta f(\eta)$, which may be integrated once.*

   (c) *Use the initial condition for u to determine the behavior of $f(\eta)$ and of $\eta f(\eta)$ as $\eta \to \infty$.*

   (d) *Use this behavior to conclude that the derivative $f'(\eta)$ vanishes at ∞, and to conclude that the integration constant in the first order equation above is 0.*

   (e) *Determine $f = f(\eta)$ and afterwards $u = u(x,t)$.*

3. *With given $D > 0$ and $k \in \mathbb{R}$, consider the reaction-diffusion problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \dfrac{\partial^2 u}{\partial x^2} + ku, \qquad x > 0, t > 0, \\ u(x,0) &= 1, \qquad x > 0, \\ u(0,t) &= 0, \qquad t > 0. \end{cases}$$

   (a) *Apply an appropriate transformation of u into v for bringing the equation for u to the standard type*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

   What are the boundary and initial conditions then?

   (b) *Find the similarity solution v for the transformed problem and afterwards determine u.*

   (c) *Determine the limits $\lim_{t\to\infty} u(x,t)$ for $x > 0$, depending on k.*

Chapter 5

The diffusion equation in bounded domains (week 4)

Until now we have considered the diffusion equation defined in an unbounded spatial interval: $(0, \infty)$, or the entire $\mathbb{R}$. Now we consider the multi-dimensional case, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a **bounded domain**. This means that a constant $C > 0$ exists s.t. $\|\vec{x}\| \leq C$ for all $\vec{x} \in \Omega$, where $\| \cdot \|$ is e.g. the Euclidean norm (any other norm making no difference). Since Ω is bounded, one can speak about its boundary, which we denote as $\partial\Omega$.

For such cases, we need to define the ***boundary conditions***, which can be of different type. In Section 1.2 we have presented Dirichlet, Neumann, Robin, or flux boundary conditions. We mention that these can be applies on either the entire boundary $\partial\Omega$, or on parts of it. In the latter case, we assume that $\partial\Omega$ is the union of several disjoint boundary parts, say Γ_D, Γ_N, Γ_F, etc. s.t. $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_F \cup \ldots$.

We first leave out the flux boundary conditions, thus $\partial\Omega = \Gamma_D \cup \Gamma_N$, and one of the two boundary parts may even be void. Given $D > 0$ and the functions $u_0 : \Omega \to \mathbb{R}$, $u_D : \Gamma_D \times (0, \infty) \to \mathbb{R}$, $q_N : \Gamma_N \times (0, \infty) \to \mathbb{R}$, we study the problem

$$\left\{ \begin{array}{rcll} \partial_t u(\vec{x}, t) & = & D\Delta u(\vec{x}, t), & \text{for } \vec{x} \in \Omega \text{ and } t > 0, \\ u(\vec{x}, 0) & = & u_0(\vec{x}), & \text{for } \vec{x} \in \Omega, \\ u(\vec{x}, t) & = & u_D(\vec{x}, t), & \text{for } \vec{x} \in \Gamma_D \text{ and } t > 0, \\ \vec{\nu} \cdot \nabla u(\vec{x}, t) & = & q_N(\vec{x}, t), & \text{for } \vec{x} \in \Gamma_N \text{ and } t > 0. \end{array} \right. \tag{5.1}$$

5.1 Energy estimates

For simplicity we start with the problem in (5.1), but where we consider homogeneous Dirichlet boundary conditions,

$$\left\{ \begin{array}{rcll} \partial_t u(\vec{x}, t) & = & D\Delta u(\vec{x}, t), & \text{for } \vec{x} \in \Omega \text{ and } t > 0, \\ u(\vec{x}, 0) & = & u_0(\vec{x}), & \text{for } \vec{x} \in \Omega, \\ u(\vec{x}, t) & = & 0, & \text{for } \vec{x} \in \partial\Omega \text{ and } t > 0. \end{array} \right. \tag{5.2}$$

In other words, $u_D \equiv 0$, $\Gamma_D = \partial\Omega$ and $\Gamma_N = \emptyset$, while u_0 is a given function having the smoothness we need below, and $D > 0$. For this problem, we derive so-called ***energy estimates***

Before deriving the estimates, we mention that the term "energy" is inspired by natural processes. For example, consider a spring with elasticity constant k, subject to elastic deformation. If u denotes its displacement, then the elastic energy stored in this spring is given by

$$E_e = \frac{1}{2} k u^2. \tag{5.3}$$

Continuing, assume that we have a one-dimensional object (e.g. a long, thin beam) occupying a certain segment $[a, b]$ on the real axis, and that this object is deformed elastically subject to some work performed on it. One can veiw this as a sequence of infinitesimal beams. Let $u(x)$ be the displacement caused by this work. Then, the strain energy stored in this beam is given by

$$E_e = \int_a^b \frac{1}{2} k [u(x)]^2 \, dx, \tag{5.4}$$

where k is the stiffness of the beam.

Consider now an object of mass m moving on a line, and let $u(t)$ be its position at time t. Then, the object has a velocity $\partial_t u(t)$, and its kinetic energy at time t is

$$E_k(t) = \frac{1}{2} m [\partial_t u(t)]^2. \tag{5.5}$$

Finally, it $u(\vec{x}, t)$ stands for the concentration of a solute inside a solvent object occupying the domain Ω, then the concentration gradient $\nabla u(\vec{x}, t)$ indicates the chemical potential at location $\vec{x} \in \Omega$ and time $t > 0$. The chemical energy stored in the system is

$$E_c(t) = \int_\Omega D \|\nabla u(\vec{x}, t)\|^2 \, d\vec{x}, \tag{5.6}$$

where D is the diffusion coefficient.

Below we use expressions like the ones in (5.4)–(5.6) to define "energies" associated to the problem in (5.2), and show that these energies remain bounded in time, or, at least have a bounded growth in time. More precisely, we consider $E : [0, \infty) \to \mathbb{R}$,]

$$E(t) = \frac{1}{2} \int_\Omega u^2(\vec{x}, t) d\vec{x} + D \int_0^t \int_\Omega \|\nabla u(\vec{x}, t)\|^2 d\vec{x} \tag{5.7}$$

as the energy associated to the problem in (5.2). As will be seen below, this expression appears naturally. We have

Proposition 6 *Let $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ be a sufficiently smooth solution to* (5.2). *Then the energy E in* (5.7) *is conserved, i.e. $E(t) = E(0)$ for all $t \geq 0$.*

Proof: Since u is a solution, one has for all $\vec{x} \in \Omega$ and $t > 0$ that

$$\partial_t u(\vec{x}, t) - D\Delta u(\vec{x}, t) = 0.$$

Let now $t > 0$ be fixed. Multiplying the above by $u(\vec{x}, t)$ and integrating over Ω gives

$$\int_\Omega u(\vec{x}, t)\partial_t u(\vec{x}, t) d\vec{x} - \int_\Omega D u(\vec{x}, t)\Delta u(\vec{x}, t) d\vec{x} = 0. \tag{5.8}$$

For the first integral in the above one gets

$$\int_\Omega u(\vec{x}, t)\partial_t u(\vec{x}, t) d\vec{x} = \frac{1}{2}\int_\Omega \partial_t [u(\vec{x}, t)]^2 d\vec{x} = \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(\vec{x}, t) d\vec{x}. \tag{5.9}$$

For the second integral we use a corollary of the Gauß Divergence Theorem (see Problem B6, Exercises week 1, or Proposition 9.1.3 [2]) to obtain

$$\int_\Omega u(\vec{x}, t)\Delta u(\vec{x}, t) d\vec{x} = \int_{\partial\Omega} u(\vec{x}, t)\partial_{\vec{\nu}} u(\vec{x}, t) d\sigma - \int_\Omega \nabla u(\vec{x}, t) \cdot \nabla u(\vec{x}, t) d\vec{x}, \tag{5.10}$$

where $\vec{\nu}$ is the unit outward normal to Ω, and $\partial_{\vec{nu}}$ denotes the directional derivative in the direction of $\vec{nu}$. Observe now that the first integral on the right vanishes since $u \equiv 0$ on $\partial\Omega$, for all $t > 0$. Therefore, (5.10) becomes

$$\int_\Omega u(\vec{x}, t)\Delta u(\vec{x}, t) d\vec{x} = -\int_\Omega \|\nabla u(\vec{x}, t)\|^2 d\vec{x}. \tag{5.11}$$

Using (5.9) and (5.11) into (5.8), one obtains

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(\vec{x}, t) d\vec{x} + D\int_\Omega \|\nabla u(\vec{x}, t)\|^2 d\vec{x} = 0. \tag{5.12}$$

This equality holds for any $t > 0$. Observe that the sum on the left is nothing but the derivative of the energy in (5.7), thus $E'(t) = 0$ for all $t > 0$. This means that the energy $E(\cdot)$ is conserved in time, so

$$E(t) = E(0) = \frac{1}{2}\int_\Omega u_0^2(\vec{x}, t) d\vec{x},$$

which concludes the proof. □

Observe that (5.12), expressing nothing but the derivative of the energy E, is obtained from the equation in $(5.2)_1$, and we have also used the boundary condition $(5.2)_3$. This means that the specific form of the energy E is linked to the problem. Clearly, if homogeneous Neumann boundary conditions are being used in (5.2) instead of the Dirichlet ones, i.e. $-\vec{\nu} \cdot \nabla u = 0$ on $\partial\Omega$ instead of $u = 0$ there, or combinations of homogeneous Dirichlet and Neumann boundary conditions, the boundary integral in (5.10) will still vanish, and (5.10) remains unchanged. In other words, also in this case, the energy E in (5.7) will remain constant. Although other problems may lead to different expressions of E, the terms appearing in the one used here are representative. Proposition 6 has several important consequences. First of all, it immediately provides the uniqueness of a solution, the existence being shown later, and for special situations where it can be computed explicitly. One has

Corollary 2 *The problem in* (5.2) *has at most one smooth solution.*

For the proof, one can argue by considering two solutions $u^{(1)}$ and $u^{(2)}$, and show that for the difference $u = u^{(1)} - u^{(2)}$ the energy in (5.7) will remain 0 in time.

Task: Work out the details.

A second result following from Proposition 6 is related to ***stability***, in the sense that (small) perturbations in the data will have similar (or smaller) effects on the solution. In this case, we consider different initial conditions, $u_0^{(1)}$ and $u_0^{(2)}$

Corollary 3 *Let* $u^{(1)}$ *and* $u^{(2)}$ *be two solutions for the problem in* (5.2)*, obtained for the initial conditions* $u_0^{(1)}$*, respectively* $u_0^{(2)}$*. Then, for all* $t \geq 0$ *one has*

$$\int_\Omega \left|u^{(1)}(\vec{x},t) - u^{(2)}(\vec{x},t)\right|^2 d\vec{x} \leq \int_\Omega \left|u_0^{(1)}(\vec{x}) - u_0^{(2)}(\vec{x})\right|^2 d\vec{x}. \tag{5.13}$$

Also in this case, one can proceed as in the proof of Proposition 6, and obtain estimates for the difference $u = u^{(1)} - u^{(2)}$.

Task: Work out the details, again.

We conclude this part by considering the diffusion problem with a source term,

$$\left\{\begin{array}{rcll} \partial_t u(\vec{x},t) & = & D\Delta u(\vec{x},t) + f(\vec{x},t), & \text{for } \vec{x} \in \Omega \text{ and } t > 0, \\ u(\vec{x},0) & = & u_0(\vec{x}), & \text{for } \vec{x} \in \Omega, \\ u(\vec{x},t) & = & 0, & \text{for } \vec{x} \in \partial\Omega \text{ and } t > 0, \end{array}\right. \tag{5.14}$$

where $f : \Omega \times (0,\infty) \to \mathbb{R}$ is a given source term. In this case, the source may impact the energy of the system, and one cannot expect that this remains constant. However, one may still give estimates for the energy, and show that it stays bounded in time. We give first some inequalities that will be used below.

Cauchy-Schwarz inequality: Let $f, g : \Omega \to \mathbb{R}$ be two square integrable functions (e.g., for f this means that it is measurable and $\int_\Omega f^2(\vec{x})d\vec{x} < \infty$; in function spaces terminology, $f, g \in L^2(\Omega)$). One has

$$\left|\int_\Omega f(\vec{x})g(\vec{x})d\vec{x}\right|^2 \leq \left(\int_\Omega f^2(\vec{x})d\vec{x}\right)\left(\int_\Omega g^2(\vec{x})d\vec{x}\right). \tag{5.15}$$

Clearly, in the above one may replace f and g by their absolute values. Also, one may consider time-space domains (like $\Omega \times (0,T)$ for some $T > 0$), and, correspondingly, iterated integrals $\int_0^T \int_\Omega \quad d\vec{x}\, dt$, and the inequality still remains valid.
To simplify writing, we use the notations

$$(f,g)_\Omega = \int_\Omega f(\vec{x})g(\vec{x})d\vec{x}, \text{ and } \|f\|_\Omega = \left[\int_\Omega f^2(\vec{x})d\vec{x}\right]^{\frac{1}{2}}, \tag{5.16}$$

representing the inner product, respectively the norm in $L^2(\Omega)$.

Proof: For the proof, we assume that $g \not\equiv 0$ (the case $g \equiv 0$ being trivial) and use the inequality $0 \leq \int_\Omega (f(\vec{x}) - Ag(\vec{x}))^2 d\vec{x}$, holding for any $A \in \mathbb{R}$. Then, one gets

$$0 \leq \|f - Ag\|_\Omega^2 = \|f\|_\Omega^2 - 2A(f,g)_\Omega + A^2\|g\|_\Omega^2.$$

Taking $A = \dfrac{1}{\|g\|_\Omega^2}(f,g)_\Omega$ in the above yields

$$\frac{2}{\|g\|_\Omega^2}\left|(f,g)_\Omega\right|^2 \leq \|f\|_\Omega^2 + \frac{1}{\|g\|_\Omega^4}\left|(f,g)_\Omega\right|^2\|g\|_\Omega^2.$$

After multiplication by $\|g\|_\Omega^2$, the inequality (5.15) follows immediately. □

Using ***Young's inequality*** (the inequality of means)

$$|ab| \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2, \tag{5.17}$$

valid for any $a, b \in \mathbb{R}$ and $\varepsilon > 0$, one immediately obtains

$$\left|(f,g)_\Omega\right| \leq \frac{1}{2\varepsilon}\|f\|_\Omega^2 + \frac{\varepsilon}{2}\|g\|_\Omega^2. \tag{5.18}$$

We assume for simplicity that the source f in (5.14) satisfies the bound

$$\int_\Omega f^2(\vec{x},t)d\vec{x} \leq M_f, \tag{5.19}$$

for some $M_f > 0$, uniformly in time. With this, we are now in the position to prove the following

Proposition 7 *Let $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be a sufficiently smooth solution to* (5.14)*. Then the energy E in* (5.7) *has is bounded in time. Specifically, a constant $C > 0$ exists s.t. for all $t > 0$ one has*

$$E(t) \leq C\ e^t.$$

Proof: We proceed as in the proof of Proposition 6 to end up with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(\vec{x},t)d\vec{x} + D\int_\Omega \|\nabla u(\vec{x},t)\|^2 d\vec{x} = \int_\Omega f(\vec{x},t)u(\vec{x},t)d\vec{x}. \tag{5.20}$$

We employ now the inequality (5.18) with, say, $\varepsilon = 1$ and obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(\vec{x},t)d\vec{x} + D\int_\Omega \|\nabla u(\vec{x},t)\|^2 d\vec{x} \leq \frac{1}{2}\int_\Omega f^2(\vec{x},t)d\vec{x} + \frac{1}{2}\int_\Omega u^2(\vec{x},t)d\vec{x}.$$

Using the boundedness of f stated in (5.19) and the definition of E in (5.7), this yields

$$E'(t) \leq \frac{M_f}{2} + E(t),$$

for all $t > 0$. After multiplication by e^{-t}, this inequality can be rewritten as

$$\big(E(t)e^{-t}\big)' \le \frac{M_f}{2}e^{-t},$$

for all $t > 0$. Integrating in time gives

$$E(t)e^{-t} - E(0) \le \frac{M_f}{2}\big(1 - e^{-t}\big),$$

leading to

$$E(t) \le \left[E(0) + \frac{M_f}{2}\right]e^{t}, \tag{5.21}$$

which concludes the proof. □

Remark 10 *The steps in estimating the energy E carried out above are valid for many situations. For the present case, an improved result can be obtained if the integral on the right in* (5.20) *is estimated as follows*

$$\left|\int_\Omega f(\vec{x},t)u(\vec{x},t)d\vec{x}\right| \le \|f(\cdot,t)\|\|u(\cdot,t)\| \le \sqrt{M_f}\|u(\cdot,t)\|,$$

where we have used (5.19). *Recalling the definition of the energy in* (5.7), *from* (5.20) *one gets*

$$E'(t) \le \sqrt{M_f}\sqrt{2E(t)}, \qquad \text{for all } t \ge 0.$$

Observe that $E(t) \ge 0$ for all t. Assume that $E(t) > 0$ (the case $E(t) = 0$ being trivial - why?), then from the above one gets

$$\frac{d}{dt}\sqrt{E(t)} \le \frac{1}{2}\sqrt{2M_f}.$$

Finally, after integration and using (5.17) *we end up with*

$$0 \le E(t) \le 2E(0) + M_f t^2, \qquad \text{for all } t \ge 0,$$

which is better than the one in Proposition 7 since it does not grow exponentially in t.

5.2 Comparison principle, uniqueness of a solution

In this section we follow Chapter 8.1 in [2] and show that the solutions to the problem in (5.1) are ordered. More precisely, for given $D > 0$, we consider two solutions to the same equation, but with different initial and boundary data $u_0^{(k)} : \Omega \to \mathbb{R}$, $u_D^{(k)} : \Gamma_D \times [0,\infty) \to \mathbb{R}$ and $q_N^{(k)} : \Gamma_N \times [0,\infty) \to \mathbb{R}$, $k = 1, 2$:

$$\left\{\begin{array}{rcll} \partial_t u^{(k)}(\vec{x},t) &=& D\Delta u^{(k)}(\vec{x},t), & \text{for } \vec{x} \in \Omega \text{ and } t > 0,\\ u^{(k)}(\vec{x},0) &=& u_0^{(k)}(\vec{x}), & \text{for } \vec{x} \in \Omega,\\ u^{(k)}(\vec{x},t) &=& u_D^{(k)}(\vec{x},t), & \text{for } \vec{x} \in \Gamma_D \text{ and } t > 0,\\ \vec{\nu}\cdot\nabla u^{(k)}(\vec{x},t) &=& q_N^{(k)}(\vec{x},t), & \text{for } \vec{x} \in \Gamma_N \text{ and } t > 0. \end{array}\right. \tag{5.22}$$

We assume that the solutions are sufficiently smooth everywhere, i.e. twice continuously differentiable w.r.t. the spatial variables and once in time ($u \in C^{2,1}(\Omega \times [0,\infty))$). We have the following

Theorem 4 (Comparison principle) *Let $u^{(k)}$ ($k = 1, 2$) be the solutions to the problem in* (5.22) *and assume that the initial and boundary conditions are ordered,*

$$u_0^{(1)}(\vec{x}) \leq u_0^{(2)}(\vec{x}), \quad u_D^{(1)}(\vec{x},t) \leq u_D^{(2)}(\vec{x},t) \quad \text{and} \quad q_N^{(1)}(\vec{x},t) \leq q_N^{(2)}(\vec{x},t),$$

for all $\vec{x} \in \Omega$, respectively $(\vec{x},t) \in \Gamma_D \times [0,\infty)$, or $(\vec{x},t) \in \Gamma_N \times [0,\infty)$. Then, the solutions are ordered everywhere,

$$u^{(1)}(\vec{x},t) \leq u^{(2)}(\vec{x},t) \quad \text{for all } x \in \Omega \text{ and } t > 0.$$

Proof. We consider the difference $u = u^{(1)} - u^{(2)}$ and show that $u(\vec{x},t) \leq 0$ for all $\vec{x} \in \Omega$ and $t \geq 0$. We prove this by showing $u \leq 0$ everywhere in $\Omega \times [0,\infty)$. To this aim, we consider the following functions

$$j, J : \mathbb{R} \to \mathbb{R}, \quad j(z) = \begin{cases} 0, & \text{if } z < 0, \\ z, & \text{if } z \geq 0, \end{cases} \qquad \text{and } J(z) = \begin{cases} 0, & \text{if } z < 0, \\ \frac{1}{2}z^2, & \text{if } z \geq 0. \end{cases} \tag{5.23}$$

Observe that these functions are continuous and one has $J'(z) = j(z)$ for all $z \in \mathbb{R}$. Also, $j'(z) = 0$ if $z < 0$, and $j'(z) = 1$ if $z > 0$.
With this, we show that $J(u(\vec{x},t)) = 0$ for all $x \in \Omega$ and $t \geq 0$, implying that $u \leq 0$ everywhere, and also the conclusion of the theorem. We start by observing that u solves the equation $(5.22)_1$, and that $u \leq 0$ on $\Gamma_D \times (0,\infty)$ and $\vec{\nu} \cdot \nabla u \leq 0$ on $\Gamma_N \times (0,\infty)$, which is due to the assumptions on the boundary data.
We let now $t > 0$ be fixed, arbitrary. Multiplying $(5.22)_1$ by $j(u(\vec{x},t))$ and integrating over Ω gives

$$\int_\Omega j(u(\vec{x},t))\partial_t u(\vec{x},t) d\vec{x} = D \int_\Omega j(u(\vec{x},t))\Delta u(\vec{x},t) d\vec{x}. \tag{5.24}$$

For the first integral, since $J' = j$, we use the chain rule to obtain

$$\int_\Omega j(u(\vec{x},t))\partial_t u(\vec{x},t) d\vec{x} = \int_\Omega \partial_t J(u(\vec{x},t)) d\vec{x} = \frac{d}{dt}\int_\Omega J(u(\vec{x},t)) d\vec{x}. \tag{5.25}$$

For the second integral, as for see (5.10) we use (a corollary of) the Gauß Divergence Theorem to obtain

$$\int_\Omega j(u(\vec{x},t))\Delta u(\vec{x},t) d\vec{x} = \int_{\partial\Omega} j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) d\sigma - \int_\Omega \nabla j(u(\vec{x},t)) \cdot \nabla u(\vec{x},t) d\vec{x}. \tag{5.26}$$

For the boundary integral in the above we have

$$\int_{\partial\Omega} j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) d\sigma = \int_{\Gamma_D} j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) d\sigma + \int_{\Gamma_N} j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) d\sigma.$$

We use the boundary conditions for u to analyse both integrals on the right. For the first, since $u(\vec{x},t) \leq 0$ on Γ_D, one has $j(u(\vec{x},t)) = 0$ there. For the second, the sign of u is unkonwn on Γ_N. However, we know that $j(u(\vec{x},t)) \geq 0$, by the definition of j in (5.23). Next to this, due to the boundary conditions for u, on Γ_N we have that $\partial_{\vec{\nu}} u(\vec{x},t) = (q_N^{(1)} - q_N^{(2)})(\vec{x},t) \leq 0$, by the assumption on the boundary data. This implies that, on Γ_N one has $j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) \leq 0$, and therefore the last integral in the above is non-positive. We therefore obtain that

$$\int_{\partial\Omega} j(u(\vec{x},t))\partial_{\vec{\nu}} u(\vec{x},t) d\sigma \leq 0. \tag{5.27}$$

For the last integral in (5.24) we note that, by the chain rule, one has $\nabla j(u(\vec{x},t)) = j'(u(\vec{x},t)\nabla u(\vec{x},t)$ for a.e. value of u, i.e. whenever $u \neq 0$. Also, $j'(u) \geq 0$, where, in this case, j' should be interpreted in a *weak sense*. With this, one has

$$\int_\Omega \nabla j(u(\vec{x},t)) \cdot \nabla u(\vec{x},t) d\vec{x} = \int_\Omega j'(u(\vec{x},t)) \|\nabla u(\vec{x},t)\|^2 d\vec{x} \geq 0. \tag{5.28}$$

Here we used the notation $\|\nabla u\|$ for the norm of the vector giving the gradient of u,

$$\|\nabla u(\vec{x},t)\| = \left[\sum_{k=1}^{d} |\partial_{x_k} u(\vec{x},t)|^2\right]^{\frac{1}{2}}.$$

Using (5.25)–(5.28) into (5.24) gives

$$\frac{d}{dt}\int_\Omega J(u(\vec{x},t)) d\vec{x} \leq 0.$$

Therefore, for all $t > 0$ one has

$$\int_\Omega J(u(\vec{x},t)) d\vec{x} \leq \int_\Omega J(u(\vec{x},0)) d\vec{x}.$$

At $t = 0$ one has $u(\vec{x},0) = u_0^{(1)}(\vec{x}) - u_0^{(2)}(\vec{x}) \leq 0$ for all $\vec{x} \in \Omega$. By the definition of J in (5.23), this gives $J(u(\vec{x},0)) ==$ for all $\vec{x} \in \Omega$, and therefore

$$\int_\Omega J(u(\vec{x},t)) d\vec{x} \leq 0.$$

On the other hand, by definition $J(u) \geq 0$ for any argument u. This gives

$$\int_\Omega J(u(\vec{x},t)) d\vec{x} \geq 0,$$

and, therefore

$$\int_\Omega J(u(\vec{x},t)) d\vec{x} = 0 \tag{5.29}$$

for all $t \geq 0$. Using now the second *Vanishing Lemma* (see Part D, Exercises week 1) we get that $J(u(\vec{x},t)) = 0$ for all $\vec{x} \in \Omega$ and $t > 0$. This implies that $u(\vec{x},t) \leq 0$ everywhere, and the proof is completed. □

An immediate consequence of Theorem (4) is the uniqueness of a solution. Observe that this has already been stated in Corollary (2), and proved by energy estimates. We state this result below, and show how this follows from Theorem 4. We have

Corollary 4 *The problem in* (5.22) *has at most one (smooth) solution.*

Proof. We let $u^{(1)}$ and $u^{(2)}$ be two solutions for (5.22), i.e. they satisfy the equation, as well as the same initial and boundary conditions. In analogy with the situation in Theorem 4, we have

$$u_0^{(1)} = u_0^{(2)} = u_0, \quad u_D^{(1)} = u_D^{(2)} = u_D, \quad \text{and } q_N^{(1)} = q_N^{(2)} = q_N.$$

This mean that, on one hand,

$$u_0^{(1)} \leq u_0^{(2)} \quad u_D^{(1)} \leq u_D^{(2)}, \quad \text{and } q_N^{(1)} \leq q_N^{(2)},$$

and, on the other hand,

$$u_0^{(1)} \geq u_0^{(2)} \quad u_D^{(1)} \geq u_D^{(2)}, \quad \text{and } q_N^{(1)} \geq q_N^{(2)}.$$

One can use twice Theorem 4 to conclude that $u^{(1)} \leq u^{(2)}$, as well as $u^{(1)} \geq u^{(2)}$, and uniqueness is proved. □

Task: Work out the details.

5.3 Boundedness, maximum principle

We use here the comparison principle stated in Theorem 4 to prove that, under certain assumptions, the solution u of (5.22) is bounded by the extreme values of the initial and boundary data. This can be resumed as:

The solution of (5.22) *attains its extreme values on the parabolic boundary of the cylinder* $\Omega \times [0,\infty)$, *namely* $(\Omega \times \{0\}) \cup (\partial\Omega \times (0,\infty))$.

The result is called "maximum principle" although it concerns the minimum as well.

Lemma 2 (Maximum principle) *Let u be a solution to the problem*

$$\begin{cases} \partial_t u &= \Delta u & \text{in } \Omega \times (0,\infty), \\ u &= u_D & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= u_0 & \text{in } \Omega. \end{cases} \tag{5.30}$$

Assume that the functions u_D and u_0 are bounded, namely there exists $m, M \in \mathbb{R}$ s.t. $m \leq u_D(\vec{x},t) \leq M$ for all $\vec{x} \in \partial\Omega$ and $t > 0$, respectively $m \leq u_0(\vec{x}) \leq M$ for all $\vec{x} \in \Omega$. Then for the solution u the same bounds hold,

$$m \leq u(\vec{x},t) \leq M \text{ for all } \vec{x} \in \Omega, t > 0.$$

Proof. We use here the comparison principle stated in Theorem 4. In this sense, observe that the constant function $u_m \equiv m$ (i.e. $u_m(\vec{x},t) = m$ for all $\vec{x} \in \Omega$ and $t > 0$) is a solution to the equation in (5.30). Moreover, by the assumptions on the initial and boundary data one has $m \leq u_D$ (on
$partial\Omega \times (0,\infty)$) and $m \leq u_0$ (in Ω). Applying theorem 4 with $u^{(1)} = u_m$ and $u^{(2)} = u$ gives that $m \leq u(\vec{x},t)$ for all $\vec{x} \in \Omega$ and $t > 0$.
The proof for the upper bound is analogous. □

Task: Work out the details for the upper bound.

Remark 11 *The result is obtained for Dirichlet type boundary conditions. However, it can be extended to other types of boundary conditions, including mixed ones (as considered in Theorem 4). For example, the proof remains almost unchanged if homogeneous Neumann boundary conditions are assumed instead of Dirichlet ones,*

$$-\vec{\nu} \cdot \nabla u = 0 \text{ on } \partial\Omega \times (0,\infty).$$

In this case, the extreme values are taken at $t = 0$. Also, adding a "source term" f is possible, but this may change the extreme values. For example, consider homogeneous Dirichlet and/or Neumann boundary conditions and assume that the initial data u_0 is positive. Assume that the source term is also positive ($f \geq 0$ overall), and consider the equation

$$\partial_t u = \Delta u + f \text{ in } \Omega \times (0,\infty)$$

with the conditions mentioned above. Then, for the solution u one can prove that $u \geq 0$ (a "minimum principle"). However, it is not necessary true that u has a maximal value.

Exercise set 5 *"Parabolic problems, bounded domains"*

1. ***Comparison principle, uniqueness, energy estimates***
   In this section we let Ω be a bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}_0$) and $D > 0$ a given diffusion coefficient. Further, we let $u_0 : \Omega \to [0,\infty)$ be a positive initial condition.

   (a) *Let $u_D : \partial\Omega \times [0,\infty) \to [0,\infty)$ be a positive boundary conditions, and $f : \Omega \times (0,\infty) \to [0,\infty)$ be a source term that is assumed (for simplicity) continuous on the closure of the parabolic cylinder $\Omega \times [0,\infty)$. Consider the diffusion problem with source*

$$(P) \begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + f & \text{for } x \in \Omega, t > 0, \\ u(x,t) = u_D(x,t), & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases}$$

   and assume that it admits a solution that is sufficiently smooth.

i. *Prove that* $u(x,t) \geq 0$ *for all* $x \in \Omega$ *and* $t \geq 0$.

ii. *Use energy estimates to prove that* (P) *has at most one solution. More precisely, prove that if* $u = u_1 - u_2$ *is the difference of two solutions to* (P)*, the associated energy is 0.*

(b) *We derive different energy estimates for the diffusion problem with zero source and homogeneous Dirichlet boundary conditions,*

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u & \text{for } x \in \Omega, t > 0, \\ u(x,t) = 0, & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases}$$

Assume that it admits a solution that is sufficiently smooth. With $p \geq 2$ *and for* $t \geq 0$ *we define the "energy"*

$$E^p(t) = \frac{1}{p}\int_\Omega u^p(x,t)dx + \frac{4(p-1)D}{p^2}\int_0^t \int_\Omega \left|\nabla\left(u^{\frac{p}{2}}(x,s)\right)\right|^2 dxds.$$

i. *Explain why the first integral in the definition of* E *makes sense even for non-integer values of* p *(i.e. why is* u^p *well defined).*

ii. *Show that* $(E^p)'(t) = 0$ *for all* $t > 0$.
Hint: Observe that $\partial_t(u^p) = pu^{p-1}\partial_t u$ *and* $\nabla\left(u^{\frac{p}{2}}(x,t)\right) = \frac{p}{2}\left(u^{\frac{p}{2}-1}(x,t)\right)\nabla u$.

iii. *Which condition has to be fulfilled by* u_0 *so that the energy* $E^p(t)$ *is finite? Think at* L^p *spaces.*

2. ***Fourier series (recap/training)***

(a) *With* $L > 0$ *given, check that the trigonometric functions*

$$\left\{\cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \;/k = 0, 1, \dots\right\}$$

are orthogonal w.r.t. the inner product on $(-L, L)$*:*

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{p\pi x}{L}\right)\,dx = \begin{cases} 0, & \text{if } k \neq p, \\ 2L, & \text{if } k = p = 0, \\ L, & \text{if } k = p > 0, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right)\,dx = \begin{cases} 0, & \text{if } k \neq p, \\ L, & \text{if } k = p > 0, \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right) = 0, \text{ for all } k, p.$$

Hint: Use the identities $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ *and* $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$.

(b) *Below $f : \mathbb{R} \to \mathbb{R}$ is a given, periodic function with period $2L$. Determine its Fourier coefficients in the following cases:*

i. $f(x) = \begin{cases} -L - x, & \textit{if } -L < x < 0, \\ \ \ L - x, & \textit{if } 0 \leq< x < L; \end{cases}$

ii. $f(x) = \begin{cases} x + L, & \textit{if } -L < x < 0, \\ L - x, & \textit{if } 0 \leq< x < L; \end{cases}$

iii. $f(x) = x$, *and f is even;*

iv. $f(x) = x$, *and f is odd;*

Chapter 6

Separation of variables for the diffusion equation (week 5)

By now we have discussed the existence of a solution in unbounded intervals (similarity solution, as well as by means of a fundamental solution). Also, the boundedness of a solution and the comparison principle were addressed. For bounded domains, we have discussed mainly qualitative aspects, namely uniqueness, boundedness, comparison, and energy estimates. The existence is left open, as, for general situations, one needs to work in abstract spaces. Here we restrict to cases where a solution can be computed explicitly, or is obtained as the limit of convergent series of functions. This implicitly gives the existence.
We discuss below the ***separation of variables***. This is a technique suited for solving certain types of linear partial differential equations. In a nutshell, the idea is to seek a solution in *separated form*. More precisely, if the partial differential equation is in d spatial variables ($d \geq 1$) and possibly also depends on the time t, then one seeks a solution $u = u(x_1, \ldots, x_d, t)$ in the form of a product of $d+1$ functions in one variable:

$$u(x_1, \ldots, x_d, t) = X_1(x_1)X_2(x_2)\ldots X_d(x_d)T(t).$$

If u does not depend on t, then the corresponding factor T is left out. The idea is to determine the functions $X_1, \ldots, X_d$ and T separately, by solving simpler, ordinary differential equations.
An example of a function that admits a separated form is $u(x_1, x_2, t) = t^2(x_1 - 1)\sin(x_2)$. Clearly, not all functions can be written in such a form. A simple (counter)example is $u(x, t) = x + t$.

6.1 Separation of variables for the diffusion/heat equation

With given $D > 0$ and $L > 0$, here we consider parabolic equations like

$$\partial_t u = D\partial_{xx}u, \text{ for } x \in (0, L), t > 0. \tag{6.1}$$

Note that we work in one spatial variable, x. In the spirit of the above, we seek solutions in a ***separated form***, $u(x,t) = X(x)T(t)$. In other words, we seek the functions X and T s.t. $u = XT$ solves the given equation. Moreover, the boundary conditions should be satisfied.

Example 3 *We start with an example involving homogeneous boundary conditions,*

$$\begin{cases} \partial_t u(x,t) &=& D\partial_{xx} u^(x,t), & \text{for } x \in (0,L) \text{ and } t > 0, \\ u(x,0) &=& u_0(x), & \text{for } x \in (0,L), \\ u(0,t) &=& u(L,t) \;=\; 0, & \text{for } t > 0. \end{cases} \tag{6.2}$$

Before providing the solution strategy, we observe the following:

1. The method can be applied also to other equations, as long as these are linear and have the proper form. An example in this sense is (with $Q, C \in \mathbb{R}$)

$$\partial_t u + Q\partial_x u = D\partial_{xx} u + Cu, \text{ for } x \in (0,L), t > 0.$$

2. The boundary conditions stated above are homogeneous. If the conditions are inhomogeneous, then one has to "homogenize" them, namely subtract a properly chosen function from the solution s.t. the resulting takes the value 0 at the boundary. More precisely, if the boundary values are constant in time, one can take $\bar{u}(x) = Ax + B$ with $B = u(0,t)$ and $A \in \mathbb{R}$ s.t. $AL + B = u(L,t)$. Then, the function $v = u - \bar{u}$ satisfies homogeneous boundary conditions, and an unchanged equation. In other words, v still solves the same equation as u. Other choices for $\bar{u}$ are possible, e.g. $\bar{u}(x) = x^2 + Ax + B$ (with properly chosen A and B), but then $v = u - \bar{u}$ will satisfy a different equation (e.g. $\partial_t v = D\partial_{xx} v + 2D$). The problem that may appear is that the new equation does not have a form that is suitable for the separation of variables technique.

3. Also, one consider other types of boundary values, e.g. Neumann ones at, say, $x = L$, $\partial_x u(L,t) = u_r$. If also $u(0,t) = u_\ell$, then $\bar{u}(x,t) = u_\ell + u_r x$ is a good choice to homogenise the boundary conditions, as $v = u - \bar{u}$ satisfies the homogeneous boundary conditions $v(0,t) = 0$ and $\partial_x v(L,t) = 0$. Moreover, the equation satisfied by v is the same as the one for u.

4. Further, note that, after homogenising the boundary conditions, one should not forget that the initial condition is also changing. We have $v(x,0) = u(x,0) - \bar{u}(x) = u_0(x) - \bar{u}(x)$.

5. Finally, we mention that, as announced, we seek a solution u in separated form, i.e. $u(x,t) = X(x)T(t)$. Naturally, one may wonder whether other solutions exist, possibly in a different form. The answer is negative: if a solution in this form has been found, it is the only solution to the given problem. this follows from the uniqueness result, see Corollaries 2 and 4.

Before presenting the method, we observe that the reason to work with homogeneous boundary conditions will be explained later. For now one, recall that boundary conditions need to be homogenised, with notable exceptions that will be given later.
We return now to the problem in (6.2), and assume that its solution can be written in a separated form, $u(x,t) = X(x)T(t)$. Using this in $(6.2)_1$ gives

$$T'(t)X(x) = DT(t)X''(x), \qquad \text{for } x \in (0,L) \text{ and } t > 0.$$

We assume that neither T nor X become 0 in $(0,L)$, respectively $(0,\infty)$ and divide by XT to obtain the ***separated equation***

$$\frac{1}{D}\frac{T'}{T} = \frac{X"}{X}, \text{ for } x \in (0,L), t > 0. \tag{6.3}$$

Observe that the expression on the left depends only on t, while the one on the right on x. Since x and t are independent variables, this equality can only be valid if both expressions are constant, say $\beta \in \mathbb{R}$. This gives

$$\frac{1}{D}\frac{T'}{T} = \frac{X"}{X} = \beta, \text{ for } x \in (0,L), t > 0. \tag{6.4}$$

Note that, in fact, we end up with two different equations,

$$X"(x) = \beta X(x), \text{ for } x \in (0,L), \qquad \text{and } T'(t) = \beta D T(t), \text{ for } t > 0. \tag{6.5}$$

We solve them separately, using the boundary conditions and the initial condition in (6.2). The easiest is to consider the former. Since $u(0,t) = 0$ for all $t > 0$, this gives $X(0)T(t) = 0$. If $X(0) \neq 0$, then $T(t) = 0$ for all t and this leads to a ***trivial solution*** $u \equiv 0$ to the equation, also satisfying the boundary conditions, but not the initial one. We therefore rule out this case, and conclude that $X(0) = 0$.
In an analogous manner we obtain $X(L) = 0$, so X will be the solution to the problem

$$\begin{cases} X"(x) - \beta X(x) &= 0, \qquad \text{for } x \in (0,L), \\ X(0) = X(L) &= 0. \end{cases} \tag{6.6}$$

This is a second order ordinary differential equation with constant coefficients. To solve it, we need the solutions to the characteristic equation

$$r^2 - \beta = 0.$$

For $\beta > 0$ we get $r_{1,2} = \pm\sqrt{\beta}$. With this,

$$X(x) = ae^{\sqrt{\beta}x} + be^{-\sqrt{\beta}x}.$$

Using the boundary conditions $X(0) = X(L) = 0$ gives $a = b = 0$. Hence $X \equiv 0$ (trivial solution) and further $u \equiv 0$, which does not necessarily solve the original problem.

Analogously, one rules out the case $\beta = 0$. Therefore, the only case yielding a non-trivial solution for x is when $\beta < 0$. We take $\beta = -\lambda^2$ for values of $\lambda > 0$ that will be obtained below. With this, $r_{1,2} = \pm\lambda\imath$ and $X(x) = a\cos(\lambda x) + b\sin(\lambda x)$.
To find the values of a and b we use again the boundary conditions for X. Since $X(0) = 0$, one gets $a = 0$. Therefore, $X(x) = b\sin(\lambda x)$. Note that we avoid trivial solutions, so $b \neq 0$, and from $X(L) = o$ we obtain

$$\sin(\lambda L) = 0.$$

This gives a sequence of solutions, $\lambda_k = \frac{k\pi}{L}$ $(k \in \mathbb{N}_0)$, and the corresponding solutions

$$X_k(x) = \sin\Big(\frac{k\pi x}{L}\Big), \qquad \text{for } x \in (0, L) \tag{6.7}$$

and with $k = 1, 2, \ldots$. The coefficients b_k, $k \in \mathbb{N}_0$ are left out as, later, we use linear combinations involving such functions.
We now consider the factor T in the expression of u. Taking $\lambda_k = \frac{k\pi}{L}$ gives $\beta_k = -\Big(\frac{k\pi}{L}\Big)^2$, and, from (6.5) one gets

$$T'(t) = -D\Big(\frac{k\pi}{L}\Big)^2 T(t), \text{ for } t > 0,$$

and with $k = 1, 2, \ldots$. This gives the sequence of solutions

$$T_k(t) = e^{-\left(\frac{k\pi}{L}\right)^2 Dt}, \text{ for } t > 0. \tag{6.8}$$

Again, we did not take any factor $T(0)$, for the same reason mentioned for X_k when leaving out b_k.
With this, we actually found a family of functions $(k \in \mathbb{N}_0)$

$$u_k(x,t) = X_k(x)T_k(t) = e^{-\left(\frac{k\pi}{L}\right)^2 Dt}\sin\Big(\frac{k\pi x}{L}\Big), \text{ for } x \in (0, L) \text{ and } t > 0, \tag{6.9}$$

all satisfying the equation in $(6.2)_1$ and the boundary conditions $(6.2)_3$. In fact, any linear combination of the functions u_k have the same property, as the equation and the boundary conditions are linear. However, the initial condition is not satisfied unless $u(x,0) = u_0(x) = \sin\big(\frac{k\pi x}{L}\big)$ for some $k \in \mathbb{N}$. This suggest seeking the solution u to (6.2) as a series of functions

$$u(x,t) = \sum_{k=1}^{\infty} b_k u_k(x,t) = \sum_{k=1}^{\infty} b_k e^{-\left(\frac{k\pi}{L}\right)^2 Dt}\sin\Big(\frac{k\pi x}{L}\Big), \tag{6.10}$$

where the coefficients b_k are determined such that u satisfies the intial condition, $u(x,0) = u_0(x)$.

Before determining the coefficients, we mention that, since we use series of functions u_k multiplied by some coefficients, the coefficients b_k in (6.7) and the factors $T_k(0)$ in (6.8) are superfluous. Observe now that, using (6.10) in $u(x,0) = u_0(x)$ gives

$$u_0(x) = \sum_{k=1}^{\infty} b_k u_k(x,0) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right),$$

for all $x \in (0, L)$, which is a (Fourier) sinus series.
More precisely, in view of the orthogonality of the family of functions $\{\sin\left(\frac{k\pi x}{L}\right), k \in \mathbb{N}_0\}$ w.r.t. the inner product $(f,g) := \int_0^L f(x)g(x)\,dx$, from the above we get that

$$\int_0^L u_0(x)\sin\left(\frac{k\pi x}{L}\right)\,dx = \sum_{p=0}^{\infty} b_p \int_0^L \sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\,dx$$

for all $k \in \mathbb{N}_0$. Note that this identity is formal, as we have interchanged the integration and the summation. We will make this rigorous in Section 6.2 below. Now, by the orthogonality mentioned above (see Exercises week 4, part B) one gets

$$b_k = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{k\pi x}{L}\right)\,dx \tag{6.11}$$

for all $k \in \mathbb{N}_0$. Using the coefficients in (6.11) in (6.10) gives the solution to (6.2).

Remark 12 *The coefficients b_k obtained in* (6.11) *are nothing but the standard Fourier coefficients for the function u_0, after being extended by in an odd manner to $[-L, 0]$, i.e. $u_0(-x) = -u_0(x)$ for all $x \in [0, L]$, and further by periodicity on intervals of length $2L$. Since u_0 is even, the cosinus coefficients are vanishing, and for b_k one gets*

$$b_k = \frac{1}{L}\int_{-L}^{L} u_0(x)\sin\left(\frac{k\pi x}{L}\right)\,dx,$$

which, since both u_0 and $\sin\left(\frac{k\pi x}{L}\right)$ are odd, is the same as the expression in (6.11).

Remark 13 *Observe that, before seeking u in separated form, we mentioned that the boundary conditions need to be homogeneous. This is explained by the fact that, eventually, we find u as a series of functions u_k, all satisfying the boundary conditions $u_k(0,t) = 0$, respectively $u_k(L,t) = 0$. Clearly, if $u_k(0,t) = 0$ for all $k > 0$, then all partial sums in the series in* (6.10) *will satisfy the same condition, so the same holds for u. For the same reasons, u satisfies the equation too.*
If the boundary conditions in $(6.2)_3$ *are non-homogeneous, say $u(0,t) = 1$, then it is unclear what boundary conditions to choose in* (6.6). *If these are still taken homogeneous, then the function u given in* (6.10) *will also satisfy homogeneous boundary conditions and, therefore, will not be a solution to* (6.2). *On the other hand, if non-zero values are taken in* (6.6), *say, $X_k(0) = 1$ after determining the functions X_k and T_k, the functions u_k satisfy $u_k(0,t) = T_k(t)$. However, this does not guarantee that $u(0,t) = 1$, as the*

first equality in (6.10) *gives that* $u(0,t) = \sum_{k=1}^{\infty} b_k T_k(t)$, *so an additional restriction would be needed for the coefficients* b_k. *This is why, before applying the separation of variables, one needs to homogenise the boundary conditions.*

We mention that, in Example 3, we use the specific sinus functions as they satisfy the equation and the boundary conditions for X_k, see (6.7). If u solves a problem like (6.2), but with different (homogeneous) boundary conditions, or with a different equation, one will obtain different functions X_k. In this sense, we discuss another example briefly.

Example 4 *We consider a slight modification of* (6.2)*, in which the Dirichlet boundary conditions in* $(6.2)_3$ *are replaced by Neumann ones,*

$$\begin{cases} \partial_t u(x,t) &=& D\partial_{xx} u^(x,t), & \text{for } x \in (0,L) \text{ and } t > 0, \\ u(x,0) &=& u_0(x), & \text{for } x \in (0,L), \\ \partial_x u(0,t) &=& \partial_x u(L,t) \;=\; 0, & \text{for } t > 0. \end{cases} \tag{6.12}$$

Assuming that u can be found in a separated form $u(x,t) = X(x)T(t)$, we follow the same steps as in Example 3. In this case, the problem in (6.6) becomes

$$\begin{cases} X"(x) - \beta X(x) &=& 0, & \text{for } x \in (0,L), \\ X'(0) = X'(L) &=& 0. \end{cases} \tag{6.13}$$

In this case, $\beta > 0$ will lead to the trivial solution $X \equiv 0$, so, to obtain non-trivial ones, one needs that $\beta = -\lambda^2$ for some $\lambda \geq 0$. Again, this gives the solution $X(x) = a\cos(\lambda x) + b\sin(\lambda x)$, with $a, b \in \mathbb{R}$ given by the boundary conditions. Since $X'(0) = 0$, one gets that either $\lambda = 0$, or $b = 0$. In the former case we get $X_0(x) = 1$ (the constant is left out for the reason explained in Example 3). If $\lambda > 0$, $X'(L) = 0$ gives

$$\sin(\lambda L) = 0.$$

Hence, $\lambda_k = \frac{k\pi}{L}$ ($k = 0, 1, \ldots$; it is a pure coincidence that the λ values are the same as before!) and

$$X_k(x) = \cos\Big(\frac{k\pi x}{L}\Big), \qquad \text{for } x \in (0,L). \tag{6.14}$$

The rest follows as before, in the sense that the T-family is given by

$$T_k(t) = e^{-\left(\frac{k\pi}{L}\right)^2 Dt}, \text{ for } t > 0, \ (k \in \mathbb{N}), \tag{6.15}$$

and, as basis functions ($k \in \mathbb{N}$),

$$u_k(x,t) = e^{-\left(\frac{k\pi}{L}\right)^2 Dt} \cos\Big(\frac{k\pi x}{L}\Big), \text{ for } x \in (0,L) \text{ and } t > 0, \tag{6.16}$$

all satisfying the equation in $(6.2)_1$ and the boundary conditions $\partial_x u(0,t) = 0$ and $\partial_x u(L,t) = 0$. We therefore seek u in the form

$$u(x,t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k u_k(x,t) = \sum_{k=0}^{\infty} b_k e^{-\left(\frac{k\pi}{L}\right)^2 Dt} \cos\Big(\frac{k\pi x}{L}\Big), \tag{6.17}$$

where, again, the coefficients a_k are determined such that $u(x,0) = u_0(x)$. Observe a factor $\frac{1}{2}$ multiplying a_0, which is taken commonly in Fourier series, to have the same expression for all cosinus coefficients (recall the orthogonality relation for the cosinus series).
As before, since we work with cosinus functions, one considers now an even extension of the initial condition u_0, and, by periodicity, to any interval of length $2L$. In this case, we use the orthogonality of the functions $\{\cos\left(\frac{k\pi x}{L}\right), k \in \mathbb{N}\}$ (see Exercises week 4, part B) and obtain

$$a_k = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{k\pi x}{L}\right)\,dx, \text{ if } k \in \mathbb{N}. \tag{6.18}$$

6.2 Separation of variables: convergence

We discussed previously the *separation of variables* method. With the exception of some particular cases, this is a method based on series of functions. The solution is constructed in a formal manner, and the natural question one may ask is whether this can be made mathematically rigorous. This is done below.

6.2.1 Preliminary results

We start with some notions and results, the latter being stated without proof. We restrict to the case of Fourier series for periodic functions with period $2L$. For details you may consult [2] and [3], or the lecture notes for the course *Fourier analysis* given by Jochen Schütz. We mention that a piecewise continuous function also has bounded variation.

$$f : [-L, L] \to \mathbb{R}, \quad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{k\pi x}{L}\right) + b_k\sin\left(\frac{k\pi x}{L}\right)\right], \tag{6.19}$$

where $a_k, k \in \mathbb{N}$ and $b_k, k \in \mathbb{N}_0$ are the Fourier coefficients of f

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx, \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx. \tag{6.20}$$

The ideas can be extended to more general situations in an analogous way, if other functions are required in the separation of variables (such examples will be seen later). The series (infinite sum) in (6.19) is approximated by the *partial sum*

$$s_N : [-L, L] \to \mathbb{R}, \quad s_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N}\left[a_k\cos\left(\frac{k\pi x}{L}\right) + b_k\sin\left(\frac{k\pi x}{L}\right)\right]. \tag{6.21}$$

One important question is related to ***convergence***: letting $N \to \infty$, will s_N converge and will f be the limit? In which sense? To answer this, we use the definitions below. We mention that the convergence of the series is equivalent to the convergence of the

partial sum, namely we say that the series converges if the limit $\lim_{N\to\infty} s_N$ exists. The series converges to f if the limit before exists and it is equal to f.
Below we use notions from the measure theory, like *almost everywhere (a.e.)*, the *essential supremum*, and the L^∞ and L^2 norms

$$\|f\|_\infty = \operatorname{ess\,sup}|f(x)|, \quad \|f\|_{L^2(-L,L)} = \Big(\int_{-L}^{L} f^2(x)dx\Big)^{\frac{1}{2}}, \tag{6.22}$$

where the integral should be understood in Lebesgue sense.

Definition 5 *The Fourier series* ***converges pointwise*** *to f if for all $x \in [-L, L]$ one has $\lim\limits_{N\to\infty} s_N(x) = f(x)$. If the equality of the limit only holds for a.e. $x \in [-L, L]$ (thus not neccesarily for all x) then the convergence is a.e..*

Definition 6 *The Fourier series* ***converges uniformly*** *to f on $[-L, L]$ if one has $\lim\limits_{N\to\infty} \|s_N - f\|_\infty = 0$.*

Definition 7 *The Fourier series* ***converges in L^2 sense*** *to f on $[-L, L]$ if one has $\lim\limits_{N\to\infty} \|s_N - f\|_{L^2(-L,L)} = 0$.*

Clearly, the uniform convergence implies the pointwise convergence, which implies the convergence a.e.. Also, the uniform convergence implies the L^2 convergence. The converse results do not hold.
Recalling that *piecewise continuous* on $[-L, L]$ means that the interval can be partitioned in sub-intervals, in which the function is continuous, we introduce the following spaces of functions

$$\begin{aligned} &PC[-L, L] = \{f : [-L, L] \to \mathbb{R}, f \text{ is piecewise continuous and 2L - periodic}\},\\ &PC^1[-L, L] = \{f \in PC[-L, L], f' \in PC[-L, L]\}. \end{aligned} \tag{6.23}$$

In the second space f is piecewise continuously differentiable, and the derivative f' exists has left and right limits everywhere. We have the following convergence results.

Theorem 5 *Let $f \in L^2(-L, L)$, then the Fourier series converges in L^2-sense to f and one has*

$$\|f\|^2_{L^2(-L,L)} = \frac{L}{2}a_0^2 + L\sum_{k=1}^{\infty}\big(a_k^2 + b_k^2\big).$$

The equality above is called ***Parseval's equality***.

Theorem 6 *Let $f \in PC^1[-L, L]$, then the Fourier series converges pointwise and for all $x \in [-L, L]$ one has*

$$\lim_{N\to\infty} s_N(x) = \frac{1}{2}\big(f(x-0) + f(x+0)\big).$$

Remark 14 *In the above, $f(x \pm 0)$ stand for the left and the right limits of f in x. At $x = L$, since f is assumed $2L$ - periodic, instead of $f(L+0)$ we take $f(-L+0)$, and $f(L-0)$ for $f(-L-0)$. If f is continuous then the series converges pointwise to f.*

Theorem 7 *Let $f \in PC^1[-L, L] \cap C[-L, L]$ be s.t. $f(-L) = f(L)$. Then the Fourier series converges uniformly to f.*

Remark 15 *For functions f like in Theorem 7, the Fourier series can be differentiated term by term, and $\{s'_N, N \in \mathbb{N}\}$ are the partial sums of the Fourier series of f'.*

Theorem 8 *Let $f \in PC[-L, L]$. Then $S_N(x) = \int_{-L}^{x} s_N(z)dz$ is the partial sum of the Fourier series of the primitive of f defined by $F(x) = \int_{-L}^{x} f(z)dz$.*

6.2.2 Convergence proof for the separation of variables

We consider the parabolic problem discussed first as an example for the separation of variables,

$$(P) \left\{ \begin{array}{rclll} \partial_t u & = & D\partial_{xx}u, & & \text{for } x \in (0, L), t > 0 \\ u(0,t) = 0, & & u(L,t) = 0, & & \text{for } t > 0, \\ u(x, 0) & = & u_0(x), & & \text{for } x \in (0, L). \end{array} \right. \tag{6.24}$$

The initial condition u_0 and the diffusion $D > 0$ are given (as, of course, $L > 0$). We further assume that $u_0 \in L^2(0, L)$.
By applying the separation of variables we obtain the functions

$$u_k(x,t) = e^{-D\left(\frac{k\pi}{L}\right)^2 t} \sin\left(\frac{k\pi x}{L}\right), \quad k \in \mathbb{N}_0 \tag{6.25}$$

satisfying the equation and the boundary conditions as in Problem P, stated in (6.24)).
With the Fourier coefficients

$$b_k = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{k\pi x}{L}\right)dx, \quad k \in \mathbb{N}_0, \tag{6.26}$$

we claim that $u(x,t) = \sum_{k=1}^{\infty} b_k u_k(x,t)$ is the solution of Problem P. To prove this claim, we use the coefficients b_k from above and the partial sums

$$s_N(x,t) = \sum_{k=1}^{N} b_k u_k(x,t), \quad N \in \mathbb{N}_0. \tag{6.27}$$

Observe that only the sinus functions are involved in this construction, while the results in the provious section are stated for the general Fourier series. However, since the boundary conditions here are of Dirichlet type (and homogeneous), one can extend the problem straightforwardly to $[-L, L]$ in an odd way (see Remark 12). More precisely,

if u solves Problem P on $[0, L]$, then the extension $\tilde{u}$ of u satisfying $\tilde{u}(x,t) = -u(-x,t)$ for $x \in [-L, 0]$ will also solve the same problem, but now on $[-L, L]$, and for the odd extension of the initial condition to $[-L, L]$. Since $\tilde{u}$ is odd, its Fourier coefficients corresponding to the cosinus functions will be 0, therefore only the sinus functions will appear. For $\tilde{u}$ one can apply the results for Fourier series to obtain all convergence results in terms of sinus series. On the other hand, these results are equivalent to the ones for u, but on the interval $[0, L]$.
The main result in this section is

Theorem 9 *Let $u_0 \in L^2(0, L)$. Then the function $u(x,t) = \sum_{k=1}^{\infty} b_k u_k(x,t)$ with the coefficients b_k in* (6.26) *is a solution to Problem P. More precisely, u solves the equation* $(6.24)_1$ *for all $t > 0$ and $x \in (0, L)$, and satisfies the homogeneous Dirichlet boundary conditions in* $(6.24)_2$. *Also, it satisfies the initial condition in the L^2 sense,*

$$\lim_{t \searrow 0} \|u(\cdot, t) - u_0\|_{L^2(0,L)} = 0.$$

Finally, u is smooth for all $t > 0$, namely $u \in C^\infty([0, L] \times (0, \infty))$.

Note: Here $t \searrow 0$ means that t approaches 0 from above (the right limit) and $u(\cdot, t)$ is viewed as a function in x.

Proof. We observe that, since $u_0 \in L^2(0, L)$, by Theorem 5 one has

$$\frac{L}{2} \sum_{k=1}^{\infty} b_k^2 = \|u_0\|_{L^2(0,L)}^2 < \infty.$$

Therefore the series $\sum_{k=1}^{\infty} b_k^2$ is convergent, implying that

$$\lim_{k \to \infty} |b_k| = 0, \quad \text{and} \quad \lim_{k \to \infty} \sum_{p=k}^{\infty} b_p^2 = 0. \tag{6.28}$$

We use these limits to prove the result. This is done in two steps: in the first we prove the uniform convergence of s_N, and in the second the properties of the limit u
Step 1. Given any $M > 0$, form (6.28) it follows that a $k_M \in \mathbb{N}_0$ exists s.t. $|b_k| < M$ for all $k \geq k_M$. Taking M large enough, one gets $|b_k| \leq M$ for all $k \in \mathbb{N}_0$. This implies that, for all $x \in [0, L]$ and $t > 0$,

$$|b_k u_k(x,t)| \leq M e^{-D\left(\frac{k\pi}{L}\right)^2 t}.$$

Let now $\delta > 0$ be arbitrary small, then

$$|b_k u_k(x,t)| \leq M e^{-D\left(\frac{k\pi}{L}\right)^2 \delta}$$

for all $x \in [0, L]$ and $t \geq \delta$. Applying now the Weierstraß M-test, since the series $\sum_{k=1}^{\infty} M e^{-D\left(\frac{k\pi}{L}\right)^2 \delta}$ is convergent, it implies the uniform convergence of the partial sums s_N, $N \in \mathbb{N}_0$, namely

$$s_N \to u \quad \text{uniformly in } [0, L] \times [\delta, \infty). \tag{6.29}$$

Step 2. This step requires proving three things: the smoothness of u, that u satisfies the equation and the boundary conditions, and also the initial condition in the L^2 sense.

Step 2A. Note that $s_N \in C^\infty([0, L] \times [\delta, \infty))$ for all $N \in \mathbb{N}_0$ and $\delta > 0$. The uniform convergence argument in *Step 1* can be extended to any (mixed) derivative of the partial sums, namely

$$\frac{\partial^{k,p}}{\partial x^k \partial t^p} s_N \to v^{k,p} \quad \text{uniformly in } [0, L] \times [\delta, \infty), \qquad \text{for all } k, p \in \mathbb{N}.$$

Moreover, all (uniform) limit functions $v^{k,p}$ above are continuous and, since all convergences are uniform for any k and p, one obtains that $\partial_{x^k t^p} u = v^{k,p}$ (see Remark 15) and, in fact, $u \in C^\infty([0, L] \times [\delta, \infty))$. Since $\delta > 0$ is arbitrary, it follows that $u \in C^\infty([0, L] \times (0, \infty))$.

Step 2B. Clearly, $s_N(0, t) = s_N(L, t) = 0$, and the same holds for u due to the convergence of the function series. Further, for every $x \in (0, L)$ and $t > 0$ one can differentiate the series term by term to obtain

$$\partial_t u - D\partial_{xx} u = (\partial_t - D\partial_{xx})(\lim_{N\to\infty} s_N) = \lim_{N\to\infty} \big[(\partial_t - D\partial_{xx}) s_N\big] = 0.$$

Step 2C. It only remains to give the proof for the initial condition, namely

$$\lim_{t \searrow 0} \|u(\cdot, t) - u_0\|^2_{L^2(0,L)} = 0$$

(the square in the norm makes no difference). Observe that, by Theorem 5,

$$\begin{aligned}
\|u(\cdot, t) - u_0\|^2_{L^2(0,L)} &= \int_0^L |u(x, t) - u_0(x)|^2 dx \\
&= \int_0^L \sum_{k=1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1\right]^2 \sin^2\big(\frac{k\pi x}{L}\big) dx \\
&= \frac{L}{2} \sum_{k=1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1\right]^2.
\end{aligned} \tag{6.30}$$

We prove that the series on the right in (6.30) converges to 0 as $t \searrow 0$. To this aim we use the elementary inequalities (please, check!)

$$0 \leq 1 - e^{-z} \leq z, \qquad \text{for all } z \geq 0. \tag{6.31}$$

Since $u_0 \in L^2$, from (6.28) it follows that, for any $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}_0$ s.t. $\sum_{k=N_\varepsilon+1}^{\infty} b_k^2 < \frac{\varepsilon}{L}$. Since $0 < 1 - e^{-D\left(\frac{k\pi}{L}\right)^2 t} < 1$ for all $t > 0$, this gives

$$\frac{L}{2} \sum_{k=N_\varepsilon+1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 < \frac{\varepsilon}{2}. \tag{6.32}$$

To deal with the remaining terms in (6.30) we observe that, by (6.31) one has

$$\left[1 - e^{-D\left(\frac{k\pi}{L}\right)^2 t} \right]^2 \leq \left[D\left(\frac{k\pi}{L}\right)^2 t \right]^2.$$

For $k \leq N_\varepsilon$ this gives

$$\frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 \leq D^2 \left(\frac{N_\varepsilon \pi}{L}\right)^4 t^2 \ \frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \leq D^2 \left(\frac{N_\varepsilon \pi}{L}\right)^4 t^2 \|u_0\|^2_{L^2(0,L)},$$

where we have used, again, that u_0 is an L^2 function. Finally, with $t_\varepsilon = \frac{1}{D\|u_0\|_{L^2(0,L)}} \left(\frac{L}{N_\varepsilon \pi}\right)^2 \sqrt{\frac{\varepsilon}{2}}$, for any $t \in (0, t_\varepsilon)$ one gets

$$\frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 < \frac{\varepsilon}{2}. \tag{6.33}$$

From (6.32) and (6.33) it follows that, for any $\varepsilon > 0$, a $t_\varepsilon > 0$ exists s.t.

$$\|u(\cdot, t) - u_0\|^2_{L^2(0,L)} < \varepsilon$$

for all $t \in (0, t_\varepsilon)$. This completes the proof of the theorem. $\square$

Remark 16 *Since $u_0 \in L^2(0, L)$, the initial condition is satisfied in L^2 sense. If u_0 is better, e.g. $C^k[0, L]$ and is also the restriction of an odd, $2L$ periodic function, then one can prove that u has the same type of smoothness up to $t = 0$, and that $u(\cdot, t)$ converges to u_0 pointwise as $t \searrow 0$.*

We conclude this section with the result presenting the behaviour of the solution as $t \to \infty$.

Proposition 8 *Let u solve Problem P stated in (6.24). For any $x \in [0, L]$ one has*

$$\lim_{t \to \infty} u(x, t) = 0.$$

Proof. For any $N \in \mathbb{N}_0$ one has $|s_N(x, t)| \leq \sum_{k=1}^{N} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 t}$ for all $x \in [0, L]$ and $t > 0$. Similarly, for all $x \in [0, L]$ and $t > 0$

$$|u(x, t)| \leq \sum_{k=1}^{\infty} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 t}.$$

Take now a $\tau > 0$ fixed (arbitrary) and rewrite the above as

$$|u(x,t)| \leq \sum_{k=1}^{\infty} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} e^{-D\left(\frac{k\pi}{L}\right)^2 (t-\tau)}.$$

We recall that the coefficients b_k are bounded, see Step 1 in the proof of Theorem 9. Using the fact that the exponential function grows more rapid than the quadratic one, from (6.28) one obtains the existence of a $k^* \in \mathbb{N}_0$ s.t.

$$|b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} \leq \frac{M}{k^2},$$

for all $k \geq k^*$. Using now the convergence of the p-series with $p = 2$ and that k^* is fixed, finite, it follows that the series below is convergent, namely that

$$R := \sum_{k=1}^{N} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} < \infty.$$

Finally, for all $t > \tau$ one has obviously $e^{-D\left(\frac{k\pi}{L}\right)^2 (t-\tau)} < e^{-D\left(\frac{\pi}{L}\right)^2 (t-\tau)}$ for all $k \in \mathbb{N}_0$, and therefore

$$|u(x,t)| \leq R\, e^{-D\left(\frac{\pi}{L}\right)^2 (t-\tau)}.$$

The result follows immediately, by letting $t \to \infty$. $\square$

Exercise set 6 *"Parabolic problems, separation of variables"*

1. ***Separation of variables, Dirichlet boundary conditions***
   The separation of variables can be applied to more general, linear and homogeneous problems. In this sense, an example is given below:

   (a) *With given $D > 0$ and $k \in \mathbb{R}$, consider the diffusion/heat problem with reaction*

   $$(P) \begin{cases} \dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2} - ku & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = 0,\ u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x(1-x), & \text{for } 0 < x < 1. \end{cases}$$

   i. *Determine the problems for the separated variables X satisfying the boundary conditions and subsequently find the family of solutions for u; disregard here the initial condition.*
   ii. *Determine the solution u satisfying the given initial condition.*

iii. *Alternatively, you can reduce the equation to the standard heat equation by applying an appropriate transformation for u (see also Part A, Problem 3 from the Instruction in Week 2). Compare the results provided by the two methods.*

iv. *Show that problem (P) cannot have two different solutions.*

2. ***Separation of variables, other boundary conditions***
As discussed in the lecture, the boundary conditions are determining the eigenpairs (λ, X). Two examples are considered below:

(a) *With given $D > 0$ and $k \in \mathbb{R}$, consider the diffusion/heat problem with reaction*

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - 1 & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = 0,\ \dfrac{\partial u}{\partial x}u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x(1-x), & \text{for } 0 < x < 1. \end{cases}$$

Note that the boundary conditions are of Neumann type and that an inhomogeneous term appears on the right of the equation.

i. *Reduce Problem (P) to the standard heat equation by using the transform $v(x,t) = u(x,t) + f(x,t)$ with an appropriate function f (see also Part A, Problem 5 from the Instruction in Week 2). What are the initial and the boundary conditions?*

ii. *Determine the solution v and subsequently u satisfying the given initial condition.*

(b) *Consider the parabolic (heat) problem ($D > 0$ being a given constant)*

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = 1,\ u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x - 1 + \cos\left(\frac{\pi}{2}x\right) & \text{for } 0 < x < 1. \end{cases}$$

Note that the boundary conditions are not homogeneous.

i. *Reduce Problem (P) to a problem involving homogeneous boundary conditions in $x = 0$ and $x = 1$. Use the transform $v(x,t) = u(x,t) + f(x,t)$ with an appropriate function f.*

ii. *Determine the problems for the separated variables X corresponding to the transformed problem (including the homogeneous boundary conditions) and subsequently find the family of solutions for v. Disregard here the initial condition.*

iii. *Determine the solution v of the transformed problem and further determine u.*

Chapter 7

Elliptic equations (week 6-7)

Until now we have mainly studied the diffusion equation, as a representative example of a parabolic problem. We now focus on the Poisson/Laplace equation, which is an elliptic equation. We study first some qualitative aspects valid for such equations defined in a general, bounded domain, and then discuss some techniques to determine the solution effectively.

7.1 General, bounded domains in $\mathbb{R}^d$ ($d \in \mathbb{N}_0$)

As in the first part of Week 4, we assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a **bounded domain**. Its boundary is denoted $\partial\Omega$, where ***boundary conditions*** will be imposed. For simplicity, we only work here with Dirichlet and Neumann type boundary conditions, but other types may also be considered. Therefore, $\partial\Omega$ consists of two disjoints parts, Γ_D and Γ_N, so $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_N = \Phi$. Observe that both Γ_D and Γ_N may consist of several subsets, and may even be void. We consider the *Poisson* problem

$$\begin{cases} -\Delta u(\vec{x}) &= f, & \text{for } \vec{x} \in \Omega, \\ u(\vec{x}) &= u_D(\vec{x}), & \text{for } \vec{x} \in \Gamma_D, \\ \vec{\nu} \cdot \nabla u(\vec{x}) &= q_N(\vec{x}), & \text{for } \vec{x} \in \Gamma_N, \end{cases} \tag{7.1}$$

where the functions $f : \Omega \to \mathbb{R}$, $u_D : \Gamma_D \to \mathbb{R}$, $q_N : \Gamma_N \to \mathbb{R}$ are given.

7.1.1 Harmonic functions

Before stating the results for the problem in (7.1), we give the following

Definition 8 *A function $f : \Omega \to \mathbb{R}$ with the property that $\Delta u(\vec{x}) = 0$ for all $\vec{x} \in \Omega$ is called **harmonic**. If one has $\Delta u(\vec{x}) \leq 0$ for all $\vec{x} \in \Omega$, the function is called **super-harmonic**. Analogously, if $\Delta u(\vec{x}) \geq 0$ for all $\vec{x} \in \Omega$, the function is called **sub-harmonic**.*

To understand why in the latter cases, the function is called super/sub-harmonic, observe that, in one spatial dimension, a harmonic function is affine (a first order polynomial). If $u''(x) \geq 0$ for all x, then the function is convex, so, between any two points x_1, x_2, its graph lies under the chord connecting the points $(x_k, u(x_k))$ $(k = 1, 2)$, which justifies the name sub-harmonic. Moreover, in this case the maximal value of u for all arguments between the two points will be found in one of these two points. In particular, if the sub-harmonic function u is defined on an interval $[a, b]$, then it attains its maximum in one of the boundary points of the interval. As will be seen below, this property remains the same in the multi-dimensional case. Also, a similar property holds for the case of super-harmonic functions, which, in one spatial dimension, are concave, and attain their minimum value at the boundary.

7.1.2 Compatibility conditions

Observe that the problem in (7.1) involves two types of boundary conditions, Dirichlet and Neumann. Let us assume first that $\Gamma_D = \Phi$, so only Neumann type of boundary conditions are involved (thus $\Gamma_N = \partial\Omega$). Then, if u is a solution, integrating $(7.1)_1$ over the entire Ω and using the Gauß Divergence Theorem (see Problem B6, Exercises week 1, or Proposition 9.1.3 [2]) and the boundary condition, one obtains

$$\int_\Omega f(\vec{x})d\vec{x} = -\int_\Omega \Delta u(\vec{x})d\vec{x} = -\int_{\partial\Omega} \partial_{\vec{\nu}} u(\vec{x})d\sigma = -\int_{\partial\Omega} q_N(\vec{x})d\sigma,$$

where $\vec{\nu}$ is the unit outward normal to Ω, and $\partial_{\vec{nu}}$ denotes the directional derivative in the direction of $\vec{nu}$. This implies that, a ***necessary condition*** for the existence of a solution to (7.1) is that the source term f and the boundary data q_N are chosen in such a way that they satisfy

$$\int_\Omega f(\vec{x})d\vec{x} + \int_{\partial\Omega} q_N(\vec{x})d\sigma = 0. \tag{7.2}$$

This is called ***compatibility condition*** for the problem in (7.1). Observe that this compatibility condition is only necessary if Γ_D is void, and it is strictly related to the equation and the boundary condition in (7.1).

To understand the importance of this compatibility condition, we mention that, if this is not satisfied, the problem in (7.1) cannot have a solution. On the other hand, for the case $\Gamma_D = \Phi$, if f and q_N do satisfy the compatibility condition, since only derivatives of u are appearing in (7.1) and thus not the function u itself, if a solution u has been found, then one does have infinitely many solutions, since, or any $C \in \mathbb{R}$, $u_C = u + C$ will also be a solution, so uniqueness will not hold anymore, as any solution can be translated by any constant. Clearly, this will change if either Γ_D is non-void (more details being provided below), or the equation involves the function u itself (again, under certain assumptions, e.g. $-\Delta u + u = f$).

7.1.3 Comparison principle, uniqueness of a solution

We can continue now with the some qualitative aspects related to (7.1). We follow Chapter 9.1 in [2] and prove first a comparison principle that is similar to the result in Theorem 4, stated for the diffusion equation, and then conclude the uniqueness and boundedness of a solution.

We assume that Ω is a **bounded, connected domain** and that Γ_D is not void. The former means that Ω cannot be written as the union of two non-empty, separated subdomains Ω_1 and Ω_2 (i.e. Ω_1 and Ω_2 are open and $\overline{\Omega_1} \cap \Omega_2 = \Omega_1 \cap \overline{\Omega_2} = \Phi$, where $\overline{X}$ stands for the closure of X).

We assume that, if existing, the solutions are sufficiently smooth everywhere, i.e. twice continuously differentiable w.r.t. the spatial variables and, if needed, up to the boundary of Ω (e.g. that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$). We have the following

Theorem 10 (Comparison principle) *Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}_0$) be bounded and connected, and $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \Phi$ and $\Gamma_D \neq \Phi$. Let $u^{(k)}$ ($k = 1, 2$) be solutions to the problem*

$$\left\{ \begin{array}{rcll} -\Delta u^{(k)}(\vec{x}) & = & f^{(k)}(\vec{x}), & \textit{for } \vec{x} \in \Omega, \\ u^{(k)}(\vec{x}) & = & u_D^{(k)}(\vec{x}), & \textit{for } \vec{x} \in \Gamma_D, \\ \vec{\nu} \cdot \nabla u^{(k)}(\vec{x}) & = & q_N^{(k)}(\vec{x}), & \textit{for } \vec{x} \in \Gamma_N. \end{array} \right. \tag{7.3}$$

Further, assume that the functions $f^{(k)}$ and the boundary conditions are ordered, namely

$$f^{(1)}(\vec{x}) \leq f^{(2)}(\vec{x}), \quad u_D^{(1)}(\vec{x}) \leq u_D^{(2)}(\vec{x}) \quad \textit{and} \quad q_N^{(1)}(\vec{x}) \leq q_N^{(2)}(\vec{x}),$$

for all $\vec{x} \in \Omega$, respectively $(\vec{x}) \in \Gamma_D$, or $(\vec{x}) \in \Gamma_N$. Then, the solutions are ordered everywhere,

$$u^{(1)}(\vec{x}) \leq u^{(2)}(\vec{x}) \quad \textit{for all } x \in \Omega.$$

Proof. We follow the ideas appearing in the proof of Therem 4, and show that for the difference $u = u^{(1)} - u^{(2)}$ one has $u(\vec{x}) \leq 0$ for all $\vec{x} \in \Omega$. We use again the functions defined in (5.23), namely

$$j, J : \mathbb{R} \to \mathbb{R}, \quad j(z) = \left\{ \begin{array}{ll} 0, & \text{if } z < 0, \\ z, & \text{if } z \geq 0, \end{array} \right. \qquad \text{and } J(z) = \left\{ \begin{array}{ll} 0, & \text{if } z < 0, \\ \frac{1}{2}z^2, & \text{if } z \geq 0, \end{array} \right.$$

and show that $J(u(\vec{x})) = 0$ for all $x \in \Omega$.

Clearly, for u one has

$$\Delta u(\vec{x}) = f^{(2)}(\vec{x}) - f^{(1)}(\vec{x}) \geq 0 \tag{7.4}$$

for all $\vec{x} \in \Omega$, and $u \leq 0$ on Γ_D respectively $\vec{\nu} \cdot \nabla u \leq 0$ on Γ_N, which is due to the assumptions on the boundary data.

Multiplying (7.4) by $j(u(\vec{x}))$ and integrating over Ω gives

$$0 \leq \int_\Omega j(u(\vec{x}, t))\Big(f^{(2)}(\vec{x}) - f^{(1)}(\vec{x})\Big) d\vec{x} = \int_\Omega j(u(\vec{x}))\Delta u(\vec{x}) d\vec{x}. \tag{7.5}$$

For the last integral, we use (a corollary of) the Gauß Divergence Theorem to obtain

$$\int_\Omega j(u(\vec{x}))\Delta u(\vec{x})d\vec{x} = \int_{\partial\Omega} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma - \int_\Omega \nabla j(u(\vec{x})) \cdot \nabla u(\vec{x})d\vec{x}. \tag{7.6}$$

For the boundary integral in the above it holds that

$$\int_{\partial\Omega} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma = \int_{\Gamma_D} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma + \int_{\Gamma_N} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma.$$

Using the boundary conditions for u, since $u(\vec{x}) \leq 0$ on Γ_D, one has $j(u(\vec{x})) = 0$ there and therefore the first boundary integral on the right vanishes. For the second integral, although the sign of u is unknown on Γ_N, one still has that $j(u(\vec{x})) \geq 0$ (by the definition of j) and, due to the boundary conditions on Γ_N, one obtains that $\partial_{\vec{\nu}} u(\vec{x}) = (q_N^{(1)} - q_N^{(2)})(\vec{x}) \leq 0$. Therefore, on Γ_N one has $j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x}) \leq 0$, and the last integral in the above is non-positive. In this way, we obtain that

$$\int_{\partial\Omega} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma \leq 0. \tag{7.7}$$

Using the above in (7.5), one ends up with

$$0 \geq \int_{\Gamma_N} j(u(\vec{x}))\partial_{\vec{\nu}} u(\vec{x})d\sigma \geq \int_\Omega \nabla j(u(\vec{x})) \cdot \nabla u(\vec{x})d\vec{x}.$$

By the chain rule, $\nabla j(u(\vec{x})) = j'(u(\vec{x})\nabla u(\vec{x})$ for a.e. value of u, i.e. whenever $u \neq 0$. Further, $j'(u) \geq 0$ (as before, j' should be interpreted in a *weak sense*), so, from the above, one gets

$$0 \leq \int_\Omega j'(u(\vec{x})\|\nabla u(\vec{x})\|^2 d\vec{x} \leq 0 \tag{7.8}$$

(where $\|\nabla u\|$ denotes the norm of the vector ∇u,

$$\|\nabla u(\vec{x})\| = \left[\sum_{k=1}^{d} |\partial_{x_k} u(\vec{x})|^2\right]^{\frac{1}{2}}.$$

By the Vanishing Lemma, this implies that, for any $\vec{x} \in \Omega$, either $j'(u(\vec{x})) = 0$, or $\nabla u(\vec{x}) = \vec{0}$.

We now rule out the possibility to have $u > 0$. In this sense, we assume that a $\vec{x}_0 \in \Omega$ exists s.t. $u(\vec{x}_0) > 0$. Since u is assumed continuous, a neighbourhood $\omega \subset \Omega$ of $\vec{x}_0$ exists s.t. $u(\vec{x}) > 0$ for all $\vec{x} \in \Omega$. In this case, $j(u(\vec{x})) = u(\vec{x})$, and $j'(u(\vec{x})) = 1$ for all $\vec{x} \in \omega$, which means that $\nabla u(\vec{x}) = \vec{0}$ there. In other words, u is constant in ω.

Let now N be the largest subset of Ω s.t. $\vec{x}_0 \in N$ and $u(\vec{x}) = u(\vec{x}_0)$ for all $\vec{x} \in N$ (i.e. we extend ω to the largest possible neighbourhood of $\vec{x}_0$ where u remains constant, namely $u \equiv u(\vec{x}_0) > 0$ there). We consider now its boundary ∂N, and study its intersection with Ω. Assume now that this intersection is non-void, i.e. an $\vec{x}^* \in \Omega \cap \partial N$ exists. Being on ∂N, it can be approached by a sequence of points lying in the interior of

N, where the function u is constant. Due to the continuity of u, one therefore gets that $u(\vec{x}^*) = u(\vec{x}_0)$. However, $\vec{x}^* \in \Omega$, which is an open set, so it means that, in fact, a neighbourhood γ of $\vec{x}^*$ exists, completely inside Ω, where u is, again, strictly positive. By the reasoning used above, the function will be constant in γ. Further, since $\vec{x}^* \in \partial N$, it follows that γ is not completely inside N, so $\gamma \cup N$ is a larger subset of Ω than N, still containing $\vec{x}_0$, and where $u \equiv u(\vec{x}_0)$. This contradicts the assumption that N is maximal.
From the above, it follows that $\partial N \cap \Omega = \Phi$. Since Ω is connected and $N \subseteq \overline{\Omega}$, it follows that, in fact, $\partial N = \partial\Omega$. Now, since Γ_D is non-void, by the argument used above, it follows that, for any $\vec{x} \in \Gamma_D$ one has $u(\vec{x}) = u(\vec{x}_0) > 0$. However, at Γ_D the boundary conditions are assumed ordered in such a way that $u(\vec{x}) \leq 0$, which leads to a contradiction.
In this way, we have ruled out the existence of an $\vec{x}_0 \in \Omega$ where $u(\vec{x}_0) > 0$ and the proof is complete. □

Remark 17 *The assumptions on Ω and Γ_D are essential. The boundedness of Ω will be justified later, through Example 5. Without having Ω_D non-void, the last argument leading to the contradiction when assuming $u(\vec{x}_0) > 0$ cannot be used. Moreover, we have seen that, in this case, multiple solutions are possible. Finally, assume that Ω is not connected, e.g. let $\Omega = \Omega_1 \cup \Omega_2$ with open, bounded sets $\Omega_{1,2}$ s.t. $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \Phi$. Then one may have $\Gamma_D = \partial\Omega_1$ and $\Gamma_N = \partial\Omega_2$, and, once a solution u is found, one can add any constant to the restriction of u to Ω_2 to obtain a new solution. By choosing the constants properly, one ends up with solutions where the ordering is violated.*

An immediate consequence of Theorem 10 is the uniqueness of a solution. We have

Corollary 5 *Assume that $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}_0$) is bounded and connected, and that $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \Phi$ and $\Gamma_D \neq \Phi$. Then, the problem in* (7.1) *has at most one (smooth) solution.*

Task: The proof can be done by following the ideas in the proof of Corollary 4. Work out the details.

7.1.4 Boundedness, maximum principle

We use here the comparison principle stated in Theorem 10 to obtain the boundedness of the solution u of (7.1). In particular, we show that, for the Laplace equation (i.e. $f \equiv 0$, which means that the solution is harmonic), the extreme values are attained at the boundary. This is a "maximum principle" for elliptic equations.

Lemma 3 (Maximum principle) *$\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}_0$) is bounded and connected, and that $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \Phi$ and $\Gamma_D \neq \Phi$. Further, let $f : \Omega \to \mathbb{R}$, $u_D : \Gamma_D \to \mathbb{R}$*

and $q_N : \Gamma_N \to \mathbb{R}$ be given, and assume that $f(\vec{x}) \leq 0$ for all $\vec{x} \in \Omega$, $q_N \leq 0$ for all $x \in \Gamma_N$, and that $u_D \leq M$. Finally, let u be a solution to the problem in (7.1). Then,

$$u(\vec{x}) \leq M \text{ for all } \vec{x} \in \Omega.$$

Proof. We observe that the constant function $u_M \equiv M$ is a solution to the equation in $(7.1)_1$, when 0 replaces f on the right, and satisfies the boundary conditions $u_M = M$ on Γ_D and $\partial_{\vec{\nu}} u_M = 0$ on Γ_N. By the assumptions on f and the boundary data, one has $f \leq 0$ in Ω, $u_D \leq M$ on Γ_D, and $q_N \leq 0$ on Γ_N. Applying Theorem 10 with $u^{(1)} = u$ and $u^{(2)} = u_M$ gives that $u(\vec{x}) \leq u_M = M$ for all $\vec{x} \in \Omega$. □

Observe that the function u in Lemma 3 is sub-harmonic. As in the one-dimensional case, this function attains its maximum at the boundary of Ω. A similar result can be proved for super-harmonic functions, namely that they attain their minimum at the boundary. Moreover, for harmonic functions the extreme values are attained at the boundary.

Task: Work out the details for the statements above.

Example 5 *We conclude this part with an example that supports the boundedness requirement for Ω. If this is not fulfilled, even for connected domains and with Dirichlet boundary conditions, an elliptic equation may have multiple solutions. A simple example in this sense is*

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{for } (x,y) \in \Omega = (0,\infty) \times (0,\infty), \\ u(x,0) & = & 0, & \text{for } x > 0, \\ u(0,y) & = & 0, & \text{for } y > 0. \end{array} \right. \tag{7.9}$$

Obviously, $u \equiv 0$ is a solution to (7.9). A simple calculation shows that also $u(x,y) = xy$ is a solution. Both solutions are defined on the closure of Ω, but only the first is bounded.

7.2 Elliptic equations, separation of variables

In this part we use examples to discuss some additional aspects related to the separation of variables for elliptic problems. As already seen for parabolic problems, the basic idea is to seek solutions in a ***separated form***. We restrict ourselves to the case of two spatial dimensions, but the ideas can be extended in an absolutely analogous manner to multiple dimensions. As for the parabolic case, we assume that u can be written as

$$u(x,y) = X(x)Y(y). \tag{7.10}$$

Observe that this only works for Cartesian domains, namely $(x,y) \in \Omega = \Omega_x \times \Omega_y$, where Ω_x and Ω_y are intervals (why?). We use the form stated in (7.10) to determine X and Y, and, actually write u as a series of functions $u_k = X_k Y_k$, $k \in \mathbb{N}$.

Later we consider domains with radial symmetry. We use a similar approach, but rewrite the problem in polar coordinates.
In any case, we will use "as much as possible" homogeneous boundary conditions. Later we make the statement "as much as possible" clear.

7.2.1 Elliptic equation in a strip

We consider an elliptic problem defined in an infinite strip $(0, \infty) \times (0, 1)$,

$$\left\{ \begin{array}{rcll} \Delta u & = & \partial_x u & \text{for } (x, y) \in (0, \infty) \times (0, 1), \\ u(x, 0) & = & 0, \;\; \partial_y u(x, 1) = 1 & \text{for } x > 0, \\ \partial_x u(0, y) & = & \sin\left(\frac{\pi}{2} y\right) - \sin\left(\frac{3\pi}{2} y\right) + y & \text{for } y \in (0, 1). \end{array} \right. \tag{7.11}$$

A similar problem is presented in discussed in Section 9.2 in [2], but there the strip is vertical, and the equation is a bit simpler (no term involving a first-order derivative is appearing). Please, study that example too. We will see below that here it is easier consider first the y-direction, namely to find Y in the separated form (7.10). Therefore we homogenise the conditions for $y = 1$ by considering e.g.

$$v(x, y) = u(x, y) - y,$$

which satisfies the problem in (7.11) but with the homogeneous condition $\partial_y v(x, 1) = 0$ for all $x > 0$:

$$\left\{ \begin{array}{rcll} \Delta v & = & \partial_x v & \text{for } (x, y) \in (0, \infty) \times (0, 1), \\ v(x, 0) & = & 0, \;\; \partial_y v(x, 1) = 0 & \text{for } x > 0, \\ \partial_x v(0, y) & = & \sin\left(\frac{\pi}{2} y\right) - \sin\left(\frac{3\pi}{2} y\right) & \text{for } y \in (0, 1). \end{array} \right. \tag{7.12}$$

Now we seek v in a separated form, $v(x, y) = X(x)Y(y)$ (see (7.10)). Using this in the equation of (7.11) gives $X''Y + XY'' = X'Y$, leading to (see also (6.3))

$$\frac{X'' - X'}{X} = -\frac{Y''}{Y}, \text{ for } x > 0, y \in (0, 1). \tag{7.13}$$

As before, we conclude that the terms in the left and right of the equality equal a constant, $\beta \in \mathbb{R}$, and seek β in such a way that non-trivial solutions are obtained.
From (7.13) and using the boundary conditions at $y = 0$ and $y = 1$ one ends up with

$$\left\{ \begin{array}{ll} Y'' + \beta Y = 0, & \text{for } y \in (0, 1), \\ Y(0) = 0, & Y'(1) = 0. \end{array} \right. \tag{7.14}$$

As argued for parabolic equations, to obtain nontrivial solutions one needs to take $\beta > 0$. In this case one obtains the solution

$$Y(y) = a \cos\left(\sqrt{\beta} y\right) + b \sin\left(\sqrt{\beta} y\right), \text{ for } a, b \in \mathbb{R}.$$

From $Y(0) = 0$ one obtains $a = 0$. Next, $Y'(1) = 0$ gives the pairs $\{(\beta_k, Y_k), k \in \mathbb{N}\}$

$$\beta_k = \Big[\frac{(2k+1)\pi}{2}\Big]^2, \quad Y_k(y) = \sin\Big(\frac{(2k+1)\pi}{2}y\Big), \qquad . \tag{7.15}$$

We use the β_k above to find the corresponding X_k. From (7.13) one obtains ($k \in \mathbb{N}$)

$$X_k'' - X_k' - \beta_k X_k = 0, \text{ for } x > 0.$$

The solutions to the characteristic equation $r^2 - r - \beta_k = 0$ are

$$\delta_k = \frac{1}{2}\big(1 - \sqrt{1+4\beta_k}\big), \gamma_k = \frac{1}{2}\big(1 + \sqrt{1+4\beta_k}\big), \tag{7.16}$$

providing the solutions ($k \in \mathbb{N}$)

$$X_k(x) = c_k e^{\delta_k x} + d_k e^{\gamma_k x}, \qquad \text{for } x > 0. \tag{7.17}$$

Here $c_k, d_k \in \mathbb{R}$ are arbitrary and will be determined in such a way that they lead to a solution that satisfies the condition at $x = 0$.
At this point we observe that (7.15) and (7.17) provide two families of functions ($k \in \mathbb{N}$)

$$w_k(x,y) = e^{\delta_k x} \sin\big(\beta_k y\big), \text{ and } z_k(x,y) = e^{\gamma_k x} \sin\big(\beta_k y\big) \tag{7.18}$$

satisfying all the equation in (7.12) and the boundary conditions for $y = 0$ and $y = 1$. We seek now the function v as the series

$$v(x,y) = \sum_{k \in \mathbb{N}} \big\{c_k w_k(x,y) + d_k z_k(x,y)\big\}. \tag{7.19}$$

In other words we identify the coefficients c_k, d_k so that the v also satisfies the remaining boundary condition, for $x = 0$.
From (7.18) and (7.19) one obtains

$$\partial_x v(0,y) = \sum_{k \in \mathbb{N}} \big\{(c_k \delta_k + d_k \gamma_k) \sin\big(\beta_k y\big)\big\}$$

The boundary condition for $x = 0$ immediately implies

$$c_0\delta_0 + d_0\gamma_0 = 1, c_1\delta_1 + d_1\gamma_1 = -1, \text{ and } c_k\delta_k + d_k\gamma_k = 0 \text{ for } k > 1. \tag{7.20}$$

Observe that one can have multiple solutions. For example, one can take $c_k = d_k = 0$ for $k > 1$, $d_0 = d_1 = 0$ and $c_0 = \frac{1}{\delta 0} = -\frac{2}{\sqrt{\pi^2+1}-1}$, $c_1 = -\frac{1}{\delta 1} = \frac{2}{\sqrt{9\pi^2+1}-1}$, yielding

$$\begin{aligned} v(x,y) = & -\frac{2}{\sqrt{\pi^2+1}-1} e^{-\frac{\sqrt{\pi^2+1}-1}{2}x} \sin\Big(\frac{\pi}{2}y\Big) \\ & + \frac{2}{\sqrt{9\pi^2+1}-1} e^{-\frac{\sqrt{9\pi^2+1}-1}{2}x} \sin\Big(\frac{3\pi}{2}y\Big), \end{aligned} \tag{7.21}$$

and finally $u(x,y) = v(x,y) + y$.
However, one can find another solution, e.g. by taking $d_0 = 1$, $c_0 = \frac{1-\gamma_0}{\delta_0}$, and the other coefficients as above. Also, the coefficients c_k, d_k can be chosen differently for $k > 1$ as long as they satisfy $c_k\delta_k = -d_k\gamma_k$. One may be puzzled, is this in contradiction with the uniqueness result, as following from Lemma 9.1.1 (see Corollary 9.1.2)? The answer is no, as these results only apply for bounded domains, whereas the strip considered here is unbounded in the x direction (see also Example 5).
Clearly, to identify a solution uniquely, one needs an additional criterion. In doing so we observe that δ_k, γ_k introduced in (7.16) have different signs, $\delta_k < 0 < \gamma_k$ for all $k \in \mathbb{N}$. Consequently, the function w_k defined in (7.18) is bounded, whereas z_k not. This suggests the following selection criterion:

Boundedness: Out of all possible solutions to (7.11), find the solution u that is bounded for all $(x,y) \in (0,\infty) \times (0,1)$.

Clearly, the boundendess of u is equivalent to the one for v. Now, since $\gamma_k > 0$ for all k, the functions z_k are not bounded. To find bounded solutions, one needs to take $d_k = 0$ in (7.19). Together with the equalities for c_k, d_k emerging from the condition at $x = 0$ one identifies then c_k uniquely. In the present case, this is the choice made for the solution found in (7.21).
We mention that the choice $d_k = 0$ for all $k > 0$ is only determined by the boundedness criterion. In fact, any choice of the d_k coefficients would provide a solution to (7.11), so there are infinitely many solutions possible.

7.2.2 Elliptic equation in a rectangle

We consider now the situation of a bounded domain, a rectangle. In this case, homogenising the boundary conditions should be done in a selective manner. More precisely, one is focusing alternatively on either the x-direction, or the y- direction.
To make the ideas more specific, we consider the problem

$$\left\{\begin{array}{rcll} \Delta u &=& 0 & \text{for } (x,y) \in (0,L) \times (0,H), \\ u(x,0) &=& f_1(x),\ \ \partial_y u(x,H) = f_2(x) & \text{for } x \in (0,L), \\ u(0,y) &=& f_3(y),\ \ u(L,y) = 0 & \text{for } y \in (0,H), \end{array}\right. \tag{7.22}$$

where f_1, f_2 and f_3 are given functions.
We use the linearity of the problem and decompose the solution into

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y).$$

The functions u_1, u_2 and u_3 are all solutions to the elliptic equation (i.e. $\Delta u_j = 0$ for $j = 1,2,3$) and satisfy homogeneous conditions excepting on one part of the boundary. More precisely, for u_1 we assume

$$u_1(x,0) = f_1(x), \partial_y u_1(x,H) = 0 \text{ for } x \in (0,L), u_1(0,y) = u_1(L,y) = 0 \text{ for } y \in (0,H).$$

Similarly, for u_2 and u_3 we assume

$$u_2(x,0) = 0, \partial_y u_2(x,H) = f_2(x) \text{ for } x \in (0,L), u_2(0,y) = u_2(L,y) = 0 \text{ for } y \in (0,H),$$

respectively

$$u_3(x,0) = \partial_y u_3(x,H) = 0 \text{ for } x \in (0,L), u_3(0,y) = f_3(y), u_3(L,y) = 0 \text{ for } y \in (0,H),$$

(why did we not consider a fourth function u_4?)
The remaining reduces to finding the functions u_j, $j = 1,2,3$. This can be done by separation of variables, and accounting for the specific boundary conditions. For each k, the separated equations become

$$\frac{X''}{X} = -\frac{Y''}{Y},$$

implying that both sides of the equality are equal to a constant β
For $j = 1$ and $j = 2$ the boundary conditions at $x = 0$ and $x = L$ are homogeneous. Therefore in these cases we find β s.t. the problems

$$\begin{cases} X'' - \beta X = 0, & \text{for } x \in (0,L), \\ X(0) = 0, & X(L) = 0. \end{cases}$$

have non-trivial solutions. As already seen, we end up with $\beta_k = -\left(\frac{k\pi}{L}\right)^2$ and $X_k(x) = \sin\left(\frac{k\pi x}{L}\right)$, $k \in \mathbb{N}_0$. Next we find Y_k as solutions to

$$Y_k'' + \beta_k Y_k = 0, \text{ for } y \in (0,H),$$

namely

$$Y_k(y) = a_k e^{-\frac{k\pi y}{L}} + b_k e^{\frac{k\pi y}{L}}.$$

Finally, we determine $a_k, b_k \in \mathbb{R}$ s.t.

$$u_1(x,y) = \sum_{k\in\mathbb{N}_0} \left\{ a_k e^{-\frac{k\pi y}{L}} + b_k e^{\frac{k\pi y}{L}} \right\} \sin\left(\frac{k\pi x}{L}\right)$$

satisfies the boundary conditions for $y = 0$ and $y = H$, and in a similar fashion u_2.
Finally, we mention that, for determining u_3, the procedure is absolutely analogous, the only difference being that, in this case, one finds first the Y- functions and then X. We get

$$u_3(x,y) = \sum_{k=0}^{\infty} \left\{ a_k e^{-\frac{(2k+1)\pi x}{2H}} + b_k e^{\frac{(2k+1)\pi x}{2H}} \right\} \sin\left(\frac{(2k+1)\pi y}{2H}\right),$$

with $a_k, b_k \in \mathbb{R}$ determined s.t. $u_3(0,y) = f_3(y)$, $u_3(L,y) = 0$ for $y \in (0,H)$.

Remark 18 *We note that the only difference between u_1 and u_2 is in the boundary conditions for $y = 0$ and $y = H$. The series used for determining u_1 and u_2 being the same, one can find directly a function $\tilde{u}$ that satisfies both boundary conditions at $y = 0$ and $y = H$ simultaneously. In other words, we seek a_k, b_k s.t.*

$$\tilde{u}(x,0) = f_1(x), \partial_y \tilde{u}(x,H) = f_2(x) \textit{ for } x \in (0,L).$$

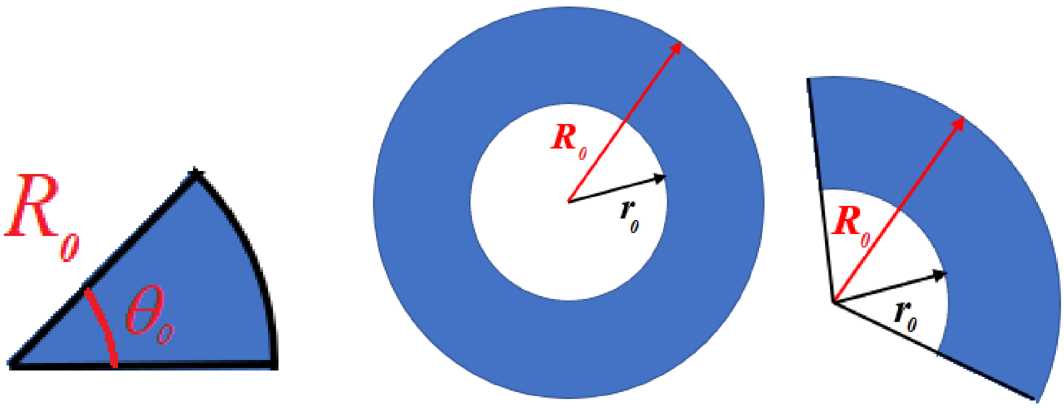


Figure 7.1: A circle/disk sector of radius R_0 and spanning an angle θ_0 (left), an annulus between two concentric circles of radii r_0 and R_0 (middle), and an annulus sector between two circle sectors of radii r_0 and R_0 and spanning the same angle θ_0 (right).

7.2.3 Elliptic equation in domains featuring radial symmetry

This part is based on Section 9.3 of [2]. We consider elliptic equations in two-dimensional domains featuring some radial symmetry: a disk, a circular/disk sector, an annulus, or an annuls sector (see Figure 7.1). In what follows, we assume that $\Omega \subset \mathbb{R}^2$ is either the entire disk of a given radius $R_0 \in (0, \infty)$

$$\Omega = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < R_0^2\}, \tag{7.23}$$

or an annulus between two concentric circles of radii r_0 and R_0 (with $0 < r_0 < R_0 < \infty$),

$$\Omega = \{(x, y) \in \mathbb{R}^2, r_0^2 < x^2 + y^2 < R_0^2\}, \tag{7.24}$$

or a sector of a disk or of an annulus.
As seen before, to apply the separation of variables, one needs that the domains can be written as a (Cartesian) product of intervals. Although the domains above are not satisfying this condition, one can consider working in polar coordinates,

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta, \tag{7.25}$$

with $r \geq 0$ and $\theta \in [0, 2\pi)$. In this way, the above-mentioned domains in (7.23) and (7.24) can be written as product-type sets,

$$\Omega_{\mathcal{P}} = [0, R_0) \times [0, 2\pi), \text{ or } \Omega_{\mathcal{P}} = (r_0, R_0) \times [0, 2\pi). \tag{7.26}$$

The sector of a disk or of an annulus spanning over an angle θ_0 is given by

$$\Omega_{\mathcal{P}} = [0, R_0) \times (0, \theta_0), \text{ or } \Omega_{\mathcal{P}} = (r_0, R_0) \times (0, \theta_0). \tag{7.27}$$

Using the transformation in (7.25), the unknown function u becomes

$$u_{\mathcal{P}}(r,\theta) = u(r\cos\theta, r\sin\theta). \tag{7.28}$$

By an abuse of notation, we give up the subscript $\mathcal{P}$ and use u and Ω to denote the unknown, respectively the domain either Cartesian or polar coordinates.
We observe that, in polar coordinates, the domain $\Omega_{\mathcal{P}}$ can be written as a Cartesian product of intervals (for the radius r and the angle θ). This opens the possibility to write the solution in a separated form. However, the separated form is not in terms of Cartesian variables (like in (7.10)), but of polar ones

$$u(r,\theta) = R(r)\Theta(\theta). \tag{7.29}$$

Clearly, when using polar coordinates, one needs to rewrite the operators (the partial derivatives) in terms of the new variables. More precisely, the Laplacian

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

becomes

$$\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{7.30}$$

To illustrate how the separation of variables works in this case, we now consider the following.

Example 6 *We solve the Laplace equation in the disk Ω introduced in* (7.23)*,*

$$\Delta u(x,y) = 0, \ \textit{for } (x,y) \in \Omega,$$

with the Dirichlet boundary condition $u = f$ on $\partial\Omega$. In polar coordinates, Ω is given in (7.26)*, and the problem becomes*

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & \textit{for } (r,\theta) \in (0,R_0)\times[0,2\pi), \\ u(R_0,\theta) = f(\theta) & \textit{for } \theta \in [0,2\pi). \end{cases} \tag{7.31}$$

The boundary data f is already written in polar coordinates.

Before constructing a solution, we observe that we have not defined the equation in the origin, where $r = 0$ gives a singularity. However, this is strictly due to the reformulation in polar coordinates, as, in Cartesian ones, no singularities appear. We explain this later, in Section 7.2.4. On the other hand, since the domain is a complete disk, we expect that the solution u is periodic w.r.t. the angle θ. This is why we only consider $\theta \in [0,2\pi)$, but, in fact, we have u defined for all $\theta \in \mathbb{R}$, satisfying the periodicity condition

$$u(r,\theta) = u(r,\theta+2\pi), \quad \text{for all } r \in (0,R_0) \text{ and } \theta \in \mathbb{R}. \tag{7.32}$$

Assuming that u has the form in (7.29), the partial derivatives in r or θ simply become derivatives of R of Θ, and the equation becomes

$$\frac{1}{r}\left(rR'(r)\right)'\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0,$$

for all $r \in (0, R_0)$ and $\theta \in [0, 2\pi)$. As we seek non-trivial solutions, the above becomes

$$\frac{r\left(rR'(r)\right)'}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \beta \in \mathbb{R}. \tag{7.33}$$

When setting the two ratios in the above to a constant, we have used the same argument that led to (6.4). Continuing in the same spirit, we observe that we have, in fact, two different equations (one in R and the other in Θ) and try to find non-trivial solutions for these.

Recalling the periodicity of u, we first focus on Θ, which has to be periodic as well. In fact, Θ solves

$$\Theta''(\theta) + \beta\Theta(\theta) = 0, \tag{7.34}$$

for all $\theta \in \mathbb{R}$, and satisfying $\Theta(\theta) = \Theta(\theta + 2\pi)$ for all $\theta \in \mathbb{R}$.

Using the periodicity condition, one can show that non-trivial solutions θ can only be obtained if $\beta \geq 0$. Moreover these solutions are obtained for $\beta_k = k^2$, with $k \in \mathbb{N}$, yielding

$$\Theta_k(\theta) = a_k \cos(k\theta) + b_k \sin(k\theta), \qquad \text{where } a_k, b_k \in \mathbb{R}. \tag{7.35}$$

Clearly, $\Theta_0(\theta) = a_0$ and b_0 plays no role here.

Task: Please, verify the statements above!

We now find the R-functions that correspond to $\beta_k = k^2$, as identified above. From (7.33) one gets

$$r^2 R_k''(r) + rR_k'(r) - k^2 R_k(r) = 0, \qquad \text{for } r \in (0, R_0). \tag{7.36}$$

Observe that this is a linear equation with a particular structure: the terms involving derivatives of order 0, 1, or 2 for R are multiplied by r raised to the same power. Such equations are called ***Euler equations***. In this case, the solutions can be found as powers of r. More precisely, we seek the appropriate powers $\alpha \in \mathbb{R}$ s.t. r^α is a solution to (7.36). Since this equation is of second order and linear, if two different values of α are found, one can consider linear combinations of these power functions r^α to generate all possible solutions.

More precisely, with $k \in \mathbb{N}_0$ (the case $k = 0$ will be considered separately), assuming that r^α is a solution to (7.36) one gets

$$r^2\alpha(\alpha - 1)r^{\alpha-2} + r\alpha r^{\alpha-1} - k^2 r^\alpha = 0,$$

implying

$$(\alpha^2 - k^2)r^\alpha = 0$$

for $r \in (0, R_0)$. This immediately gives the two powers, $\alpha_{1,2} = \pm k$ and therefore

$$R_k(r) = c_k r^k + d_k r^{-k}, \text{ for } k \in \mathbb{N}_0 \text{ and with } c_k, d_k \in \mathbb{R}. \tag{7.37}$$

The case $k = 0$ gives $(rR_0'(r))' = 0$, leading to

$$R_0(r) = c_0 \ln r + d_0, \text{ with } c_0, d_0 \in \mathbb{R}. \tag{7.38}$$

From (7.35) – (7.38) one obtains the following families of solutions

$$u_k(r,\theta) = r^k \cos(k\theta), v_k(r,\theta) = r^{-k}\cos(k\theta), w_k(r,\theta) = r^k \sin(k\theta), z_k(r,\theta) = r^{-k}) \sin(k\theta), \tag{7.39}$$

for $k \in \mathbb{N}_0$. Further, for $k = 0$ one has

$$u_0(r,\theta) = 1, v_0(r,\theta) = \ln r. \tag{7.40}$$

Each of the functions u_k, v_k, w_k, z_k above solve the equation in (7.31) and is periodic w.r.t. θ. Therefore, one can find the solution u as a series of functions

$$u(r,\theta) = \frac{a_0}{2} + \frac{c_0}{2}\ln r + \sum_{k=1}^{\infty} \{a_k u_k + c_k v_k + b_k w_k + d_k z_k\} \tag{7.41}$$

where $a_k, b_k, c_k, d_k \in \mathbb{R}$ are to be determined in such a wa that the remaining boundary condition (at $r = R_0$) is satisfied (the division by 2 in the first terms was taken to simplify the writing later). We have

$$u(R_0, \theta) = f(\theta), \text{ for } \theta \in [0, 2\pi),$$

so, the coefficients $a_k, b_k, c_k, d_k \in \mathbb{R}$ are given by the Fourier coeffcinets of f. More precisely, we get

$$f(\theta) = \frac{a_0}{2} + \frac{c_0}{2}\ln(R_0) + \sum_{k=1}^{\infty}\left(a_k R_0^k + c_k R_0^{-k}\right)\cos(k\theta) + \sum_{k=1}^{\infty}\left(b_k R_0^k + d_k R_0^{-k}\right)\sin(k\theta),$$

giving (see also (6.20))

$$a_0 + c_0 \ln(R_0) \quad = \quad \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta, \tag{7.42}$$

for $k = 0$ and, for $k \in \mathbb{N}_0$,

$$\left(a_k R_0^k + \frac{1}{R_0^k}c_k\right) \quad = \quad \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)d\theta, \tag{7.43}$$

$$\left(b_k R_0^k + \frac{1}{R_0^k}d_k\right) \quad = \quad \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)d\theta. \tag{7.44}$$

Observe that, for each $k \in \mathbb{N}$, two unknown coefficients need to be determined. This would not be a problem if a second boundary condition would be available, as would

happen e.g. if the problem is formulated in an annulus. Then, a second boundary condition would be specified for $r = r_0$, so, for each k one gets similar conditions like in (7.42), (7.43). This would allow detemining all coefficients a_k, b_k, c_k and d_k uniquely. However, since we work on a disk, we have no second boundary condition for the problem in (7.31).
To determine the coefficients we observe that, as $r \searrow 0$, the solution in (7.41) develops a singularity (it becomes unbounded and, actually, it is not defined at $r = 0$) unless the terms involving positive powers of r, or $\ln r$ are vanishing. On the other hand, one expects that the solution u remains bounded, since it solves an elliptic equation in a bounded domain (see also Section 7.2.4 below). Therefore, on has to take $c_k = d_k = 0$ for all k in (7.41), yielding

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta) d\theta, \tag{7.45}$$

respectively, for $k \in \mathbb{N}_0$,

$$a_k = \frac{1}{\pi R_0^k} \int_0^{2\pi} f(\theta) \cos(k\theta) d\theta, \quad \text{and} \quad b_k = \frac{1}{\pi R_0^k} \int_0^{2\pi} f(\theta) \sin(k\theta) d\theta. \tag{7.46}$$

Remark 19 *In* (7.31) *(or* (7.56)*) we have considered a Dirichlet boundary condition, namely $u = f$ on $\partial\Omega$. Neumann conditions can be considered too, namely $\vec{\nu} \cdot \nabla u = g$. In polar coordinates, this becomes*

$$\partial_r u(R_0, \theta) = g(\theta), \tag{7.47}$$

and the rest follows as above. Observe that , due to the equivalence between the problems in Cartesian, respectively polar coordinates, the discussion in Section 7.1.2 concerning the compatibility condition, respectively the (non-)uniqueness remain valid. For (7.31) *but with the boundary condition in* (7.47)*, since there is no source on the right of the equation, the compatibility condition reads*

$$\int_0^{2\pi} g(\theta) d\theta = 0.$$

Example 7 *The domain in Example 6 was bounded. We now consider the same equation, but in the complement of the closed disk, thus in an unbounded domain. Given f, we solve the Laplace equation outside the disk Ω introduced in* (7.23)*,*

$$\left\{ \begin{array}{rcll} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} & = & 0 & \text{for } (r,\theta) \in (R_0, \infty) \times [0, 2\pi), \\ u(R_0, \theta) & = & f(\theta) & \text{for } \theta \in [0, 2\pi). \end{array} \right. \tag{7.48}$$

All the steps carried out for Example 6 remain the same until setting the coefficients c_k, d_k to 0. This was justified by the possible singularity of the solution in the origin. Here, the origin does not belong to the domain, but we work in an unbounded domain.

As seen in Section 7.2.1, in this case uniqueness does not hold anymore. On the other hand, like in the case of a strip, only one of the solutions is bounded, namely when taking $c_0 = 0$, and further $a_k = b_k = 0$ for $k \in \mathbb{N}_0$. This gives a_0 as in (7.45), but

$$c_k = \frac{R_0^k}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)d\theta, \qquad \textit{and} \qquad d_k = \frac{R_0^k}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)d\theta. \tag{7.49}$$

Example 8 *Until now, the domains were involving complete disks, which means that $\theta \in [0, 2\pi)$ and, by periodicity, $\theta \in \mathbb{R}$. For given $\theta_0 \in (0, 2\pi)$, we now consider a disk sector and solve*

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} & = & 0 & \textit{for } (r,\theta) \in (0, R_0) \times (0, \theta_0), \\ u(R_0, \theta) & = & f(\theta), & \textit{for } \theta \in [0, \theta_0], \\ u(r, 0) = u(r, \theta_0) & = & 0, & \textit{for } r \in [0, R_0]. \end{cases} \tag{7.50}$$

Again, one can follow the ideas above and seek the solution u in the separated form stated in (7.29). Also, the equations in (7.33) remain, leading to (7.34). However, this equation only holds for $\theta \in (0, \theta_0)$, while at the endpoint of the interval one has

$$\Theta(0) = \Theta(\theta_0) = 0.$$

This follows from the boundary conditions along the boundaries $(0, R_0) \times \{0\}$, respectively $(0, R_0) \times \{\theta_0\}$, which are homogeneous. Then, as before, nontrivial solutions can only be obtained if $\beta > 0$. Specifically, with $k \in \mathbb{N}_0$ one gets

$$\beta_k = \left(\frac{k\pi}{\theta_0}\right)^2, \qquad \text{and} \qquad \Theta_k(\theta) = \sin\left(\frac{k\pi\theta}{\theta_0}\right). \tag{7.51}$$

With this, the equation for R in (7.36) becomes

$$r^2 R_k''(r) + rR_k'(r) - \left(\frac{k\pi}{\theta_0}\right)^2 R_k(r) = 0, \qquad \text{for } r \in (0, R_0), \tag{7.52}$$

giving $\alpha_{1,2} = \pm\frac{k\pi}{\theta_0}$, and the solutions

$$R_k(r) = b_k r^{\frac{k\pi}{\theta_0}} + d_k r^{-\frac{k\pi}{\theta_0}}, \text{ with } b_k, d_k \in \mathbb{R}. \tag{7.53}$$

As before, we set $d_k = 0$ to avoid the singularity in $r = 0$, and write

$$u(r,\theta) = \sum_{k=1}^{\infty} b_k r^{\frac{k\pi}{\theta_0}} \sin\left(\frac{k\pi\theta}{\theta_0}\right). \tag{7.54}$$

To find the coefficients b_k, we use the boundary condition at $r = R_0$ and obtain

$$b_k = \frac{2}{\pi R_0^{\frac{k\pi}{\theta_0}}}\int_0^{\theta_0} f(\theta)\sin\left(\frac{k\pi\theta}{\theta_0}\right)d\theta. \tag{7.55}$$

Remark 20 *As in Example 7, the approach for the unbounded sector*

$$\Omega = (R_0, \infty) \times (0, \theta_0)$$

remains unchanged. Also, when dealing with a sector annulus, one has sufficient boundary conditions to determine all coefficients. Finally, the homogeneous Dirichlet boundary conditions along any of the "lateral" sides $\theta = 0$, or $\theta = \theta_0$ is replaced by a homogeneous Neumann one, this becomes $\partial_\theta u = 0$ there, leading to the condition $\Theta'(0) = 0$, or $\Theta'(\theta_0) = 0$.

7.2.4 Some final remarks for the case of domains with radial symmetry

Refering to (7.31), the equation in polar coordinates has a singularity at the origin, where $r = 0$. There the equation cannot be stated. However, this only appears due to the switch to polar coordinates, as the origin $(0, 0)$ in Cartesian coordinates expands to a segment $\{0\} \times (0, \phi_0)$ in polar coordinates. This makes it impossible to interpret the equation at the origin, whereas in Cartesian coordinates the origin has no particular meaning for the equation: the equation simply holds there as well.
Recalling the uniqueness result stated in Theorem 10, the problem in (7.31) is equivalent to

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < R_0^2\}, \\ u(x, y) = f(x, y) & \text{on } \partial\Omega = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = R_0^2\}, \end{cases} \tag{7.56}$$

where we use the same notation for u and f when written in the different coordinate systems. In other words, u is a solution to (7.31) iff u is a solution to (7.56). This holds at least for smooth solutions (i.e. $u \in C^2(\Omega) \cap C(\overline{\Omega})$) and can be extended to weaker concepts of solutions, requiring less smoothness. Moreover, this result extends to more classes of elliptic equations, like the one in (7.1).
Therefore, all results that can be obtained for the problem posed in Cartesian coordinates transfer to the problem in polar coordinates and its solution. In particular, one has the following:

1. *Uniqeness*: If uniqueness is obtained for the problem in Cartesian coordinates, it will hold for the translated problem into polar coordinates. Alternatively, if a problem is given in polar coordinates, to prove the uniqueness of a solution one can reformulate it into Cartesian coordinates and prove the uniqueness for this variant.

2. *Boundedness/comparison principle* Similarly, to prove the boundedness of a solution for the problem formulated in polar coordinates, or to prove a comparison principle, one may find it convenient to formulate first the problem in Cartesian

coordinates. The results obtained in this form are transferred to the problem in polar coordinates. For example, on page 202 of [2] a "physical argument" is used to justify the choice $B = 0$. This is nothing but claiming that the solution must be bounded, which is not so obvious for the problem formulated in polar coordinates. However, if one goes back to the Cartesian coordinates, then the boundedness is guaranteed by Lemma 3 (Lemma 9.1.1 in [2]) and its consequences (the maximum principle). Therefore the choice $B = 0$ is justified whenever the origin is included in the domain Ω if the equation should hold there, as otherwise the solution would have a singularity in the origin. This singularity would be then artificial, as in Cartesian coordinates the solution remains bounded.

3. *Radially symmetric solution*: The solution in polar coordinates is sought as $u(r, \varphi) = R(r)\Theta(\theta)$. Assuming now that the domain Ω is the entire disc (meaning that the solution is periodic w.r.t. θ) and the boundary data f does not depend on θ (in fact, it is constant), then one may seek first solutions that are radially symmetric, namely $u(r, \theta) = R(r)$ (thus not depending on θ). This can simplify the solution strategy dramatically.

   A similar situation can appear if a disk sector or an annulus sector are considered (see (7.27)), but the boundary conditions along the sector boundaries $\theta = 0$ and $\theta = \theta_0$ are of homogeneous Neumann type, $\partial_\theta u(r, 0) = \partial_\theta u(r, \theta_0) = 0$, and the ones along the boundary $r = R_0$ and, if applicable, $r = r_0$, are θ-independent (e.g. $u(R_0, \theta) = u_{R_0}$ for all $\theta \in (0, \theta_0)$, where $u_{R_0} \in \mathbb{R}$ is given). Also in this case one seeks $u = u(r)$, thus only depending on r and not on θ.

   In such situations, since the θ-derivatives are 0, the Laplacian reduces to

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Exercise set 7 *"Elliptic equations, bounded domains"*

1. ***Elliptic problems, bounded domains***
   In the lecture we have proven the comparison principle for the Poisson problem

$$(\mathcal{P})\begin{cases} -\Delta u(x) &= f(x), \qquad x \in \Omega, \\ \partial_n u(x) &= q(x), \qquad x \in \Gamma_N, \\ \\ u(x) &= u_D(x) \qquad x \in \Gamma_D, \end{cases}$$

   where $\Omega \subset \mathbb{R}^d$ is bounded and connected domain with the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ s.t. $\Gamma_D \neq \Phi$ and $\Gamma_D \cap \Gamma_N = \Phi$. The functions $f : \Omega \to \mathbb{R}$, $u_D : \Gamma_D \to \mathbb{R}$ and $q : \Gamma_N \to \mathbb{R}$ are given. We consider smooth solutions, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Use the comparison principle to prove the following:

   (a) Problem $\mathcal{P}$ has at most one solution.

(b) *If $f(x) \geq 0$ for all $x \in \Omega$, $u_D(x) \geq 0$ for all $x \in \Gamma_D$ and $q(x) \geq 0$ for all $x \in \Gamma_N$, then one has $u(x) \geq 0$ for all $x \in \bar{\Omega}$.*

(c) *Let $f \equiv 0$ (i.e. u is harmonic) and $q \equiv 0$ if the boundary Γ_N is not void. Further, assume that there exist $m, M \in \mathbb{R}$ s.t. for all $x \in \Gamma_D$ it holds $m \leq u(x) \leq M$. Then the inequalities hold for all $x \in \Omega$. In other words, if u is harmonic and either the Neumann boundary conditions are homogeneous, or only dirichlet type of boundary conditions are imposed, then u attains its extreme values on the boundary.*

2. ***Separation of variables, Laplace equation: cartesian coordinates***

(a) *Let $\Omega = (0, L) \times (0, \infty)$ and consider the elliptic problem*

$$(P) \begin{cases} \Delta u = u, & \text{for } (x,y) \in \Omega \\ \dfrac{\partial u}{\partial y}(x,0) = \sin\left(\frac{\pi x}{L}\right), & \text{for } x \in (0,L) \\ u(0,y) = 0, \quad u(L,y) = 0, & \text{for } y \in (0,\infty). \end{cases}$$

Find the bounded solution u of Problem P. Give also another solution, which may not be bounded.

(b) *Consider the Laplace problem in a strip $\mathcal{S} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < \infty\}$,*

$$(P) \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{for } (x,y) \in \mathcal{S}, \\ \frac{\partial u}{\partial x}(0,y) = 1, \ \dfrac{\partial u}{\partial x}(1,y) = 1 & \text{voor } 0 < y < \infty, \\ u(x,0) = f(x) & \text{voor } 0 < x < 1. \end{cases}$$

Note that the boundary conditions are inhomogeneous.

i. *Transform Problem (P) to a problem with homogeneous boundary conditions along $x = 0$ and $x = 1$.*

ii. *Find the bounded solution v of the transformed problem and subsequently find u (of course, he coefficients depend on f, which is not given explicitly).*

iii. *Give another solution to the probleem (P).*

(c) *Let $\Omega = (0, L) \times (0, H)$ and consider the elliptic problem*

$$(P) \begin{cases} \Delta u = u, & \text{for } (x,y) \in \Omega \\ \dfrac{\partial u}{\partial x}(0,y) = 0, \quad \dfrac{\partial u}{\partial x}(L,y) = f(y), & \text{for } y \in (0,H) \\ u(x,0) = 0, \quad u(x,H) = g(x), & \text{for } x \in (0,L). \end{cases}$$

Let further $f(y) = \sin\left(\frac{2\pi y}{H}\right)$ and $g(x) = \cos\left(\frac{\pi x}{L}\right)$. We seek u solving Problem P.

i. *First, find the solution u_f of the problem P_f obtained from P by replacing g by 0 (i.e. considering homogeneous Dirichlet boundary conditions on the boundary $\{(x,H), x \in (0,L)\}$) and leaving f unchanged. Apply the separation of variables and identify first the factor Y (why is this more convenient?).*

ii. *Second, find the solution u_g of the problem P_g obtained from P by replacing now f by 0 and leaving g unchanged. Apply the again the separation of variables.*

iii. *Exploit the linearity of Problem P to find its solution, based on the steps above.*

(d) *Supplementary: Problem 1 on p. 203 in the book by C.J. van Duijn and M.J. de Neef.*

3. ***Separation of variables, Laplace equation: polar coordinates***

(a) *Consider the Laplace problem in the disk sector $\Omega = \{(x,y), x = r\cos\theta, y = r\sin\theta, 0 \le r < 1, 0 < \theta < \frac{\pi}{2}\}$:*

$$(P)\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & \text{for } (x,y) \in \Omega \\ \dfrac{\partial u}{\partial \theta}(r,0) = \dfrac{\partial u}{\partial \theta}(r,\dfrac{\pi}{2}) = 0, & \text{for } r \in (0,1) \\ u(1,\theta) = \frac{1}{2} + \cos(2\theta), & \text{for } \theta \in \left(0, \frac{\pi}{2}\right). \end{cases}$$

Note that u is written now directly in polar coordinates, $u = u(r,\theta)$.

i. *Explain why one has to seek a bounded solution u in spite of the singularity appearing in the differential operator at $r = 0$. Think at the properties of the solution to the Laplace problem posed in the original, cartesian coordinates.*

ii. *Find the solution u to Problem P by separation of variables.*

(b) *Consider the Poisson problem in the unit disc $\Omega = \{(x,y), x = r\cos\theta, y = r\sin\theta, 0 \le r < 1, 0 \le \theta < 2\pi\}$:*

$$(P)\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 1, & \text{for } 0 < r < 1 \in \Omega \\ u(1,\theta) = 1, & \text{for } \theta \in \left[0, 2\pi\right). \end{cases}$$

i. *Show that this problem admits at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Think at the properties of the Poisson problem posed in the original, cartesian coordinates.*

ii. *Find the solution u to Problem P in the form $u = u(r)$. Why finding such a solution u is meaningful? Think at the boundary conditions.*

iii. *Consider now the same problem in the complementary domain,* $\Omega^C = \{(x, y), x = r\cos\theta, y = r\sin\theta, r > 1, 0 \leq \theta < 2\pi\}$. *Find again a bounded,* φ*-independent solution to the problem*

$$(P^C)\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 1, & \text{for } (x, y) \in \Omega^C \\ u(1, \theta) = 1, & \text{for } \theta \in \big[0, 2\pi\big). \end{cases}$$

(c) *Find the bounded solution to the problem below, defined in the exterior of the disk woth radius 2*

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & \text{for } r > 2 \text{ and } \theta \in [0, \infty) \\ u(2, \varphi) = 2\cos^2\theta, & \text{for } \theta \in \big[0, 2\pi\big). \end{cases}$$

Chapter 8

Hyperbolic equations (week 8)

In Section 2.1.1 we studied linear, first order hyperbolic equations, which we have solved by the method of characteristics (MOC). Here we consider second order problems, also of hyperbolic type. We start with bounded domains, where we derive energy estimates for the solution. Then we apply the separation of variables to solve such equations defined on one-dimensional intervals. Finally,we construct solutions in unbounded intervals.

8.1 Energy estimates

We consider first the problem in a bounded domain $\Omega \subset \mathbb{R}^d$,

$$\left\{ \begin{array}{rcll} \dfrac{\partial^2 u}{\partial t^2} &=& c^2 \Delta u & \text{for } \vec{x} \in \Omega \text{ and } t > 0, \\ u(\vec{x}, t) &=& 0 & \text{for } \vec{x} \in \partial\Omega \text{ and } t > 0, \\ u(\vec{x}, 0) &=& u_0(\vec{x}) & \text{for } \vec{x} \in \Omega, \\ \dfrac{\partial u}{\partial t}(\vec{x}, 0) &=& u_1(\vec{x}) & \text{for } \vec{x} \in \Omega, \end{array} \right. \tag{8.1}$$

where the constant $c > 0$ and the initial conditions u_0, u_1 are given. Before deriving an energy estimate we observe that there are two differences when compared to the parabolic and elliptic problems:

1. All derivatives are of second order, like in the case of the Laplace/Poisson equation. However, if all terms of the equation are brought to one side, on sees that the derivatives in t have a different sign than the ones in the spatial variables. Therefore time and space are treated differently.

2. Compared to the diffusion/heat equation, where only one time derivative appears, here we have two time derivatives. Therefore here we also take two initial conditions (the last two equalities in (8.1)), compared to one in the parabolic case. Recalling the physical motivation, which is the vibration of a membrane, or the oscillation of a spring-mass system, the need to impose two initial conditions is

explained as follows. The oscillations can be initiated in two ways: either by an initial displacement from the equilibrium position (the first condition), or by giving an initial impuls (the second condition). Of course, a one can have the two initiators simultaneously.

We now derive energy estimates for the solution to the problem in (8.1), for which we assume sufficient regularity. For the case of parabolic equations, such estimates were obtained after multiplying the equation with the solution, and integrating the result over the domain Ω (see Section 5.1). Unfortunately, this strategy does not lead to useful estimates, as the terms resulting after integration by parts do not have the same sign.
Task: Work out the details.
We therefore adopt a different strategy: we multiply the equation by $\partial - tu$ and integrate again the result over Ω. This gives

$$\int_\Omega \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} d\vec{x} = c^2 \int_\Omega \frac{\partial u}{\partial t}\Delta u d\vec{x} \tag{8.2}$$

The integral on the left can be rewritten as (see also (5.9))

$$\int_\Omega \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} d\vec{x} = \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 d\vec{x} = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 d\vec{x}. \tag{8.3}$$

For the integral on the right, like in (5.10) one gets

$$\int_\Omega \frac{\partial u}{\partial t}\Delta u d\vec{x} = \int_{\partial\Omega} \frac{\partial u}{\partial t}\, \partial_{\vec{\nu}} u d\sigma - \int_\Omega \nabla \frac{\partial u}{\partial t} \cdot \nabla u d\vec{x}. \tag{8.4}$$

The boundary condition in (8.1) states that $u(\vec{x}, t) = 0$ at $\partial\Omega$ for all $t > 0$. Therefore, one also gets that $\partial_t u(\vec{x}, t) = 0$ at $\partial\Omega$ for all $t > 0$, so the boundary integral in the above vanishes. Note that a similar result would be obtained if homogeneous Neumann boundary conditions would be imposed at the boundary (of parts of it), instead of homogeneous Dirichlet ones. For the last integral in (8.4), one uses the identity

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 = \frac{1}{2}\frac{\partial}{\partial t}(\nabla u \cdot \nabla u) = \nabla u \cdot \nabla \frac{\partial u}{\partial t}$$

to obtain

$$\int_\Omega \nabla \frac{\partial u}{\partial t} \cdot \nabla u d\vec{x} = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 d\vec{x}. \tag{8.5}$$

Using (8.3)–(8.5), (8.2) gives

$$\frac{d}{dt}\int_\Omega \left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right] d\vec{x} = 0. \tag{8.6}$$

Defining the *energy* as

$$E(t) = \int_\Omega \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] dx, \tag{8.7}$$

(8.6) means nothing but

$$\frac{d}{dt} E(t) = 0.$$

With this, we actually have proved

Proposition 9 *Let u be a sufficiently smooth solution to* (8.1). *Then the energy E defined in* (8.7) *remains constant in time.*

A straightforward consequence of this proposition is the uniqueness result in the following

Corollary 6 *The problem in* (8.1) *can have at most one solution.*

Clearly, if other terms appear in the first equation of (8.1), one needs to use further strategies to estimate the energy, and this may not remain constant. For example, if the equation is replaced by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + f,$$

where $f : \Omega \times [0, T] \to \mathbb{R}$ is s.t. a constant $C_f > 0$ exists s.t. $\int_\Omega f^2(\vec{x}, t) d\vec{x} \leq C_f$. Repeating the steps above, one gets

$$\frac{d}{dt} \int_\Omega \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] d\vec{x} = 2c^2 \int_{\partial\Omega} \frac{\partial u}{\partial t} (\vec{\nu} \cdot \nabla u) d\sigma + 2 \int_\Omega f \frac{\partial u}{\partial t} d\vec{x}. \tag{8.8}$$

Using the Cauchy-Schwarz inequality (5.15) and the Young inequality (5.17), the last term on the right can be estimated as

$$2 \int_\Omega f \frac{\partial u}{\partial t} dx \leq 2 \|f(t)\| \|\partial_t u(t)\| \leq \|f(t)\|^2 + \|\partial_t u(t)\|^2.$$

In the above $\| \cdot \|$ stands for the $L^2(\Omega)$ norm, and by $f(t)$ we mean the function in $\vec{x}$ obtained for a fixed t.
With this, using the definition of the energy in (8.7) and the boundary conditions stated in (8.1), from (8.8) one gets

$$E'(t) \leq \|f(t)\|^2 + \|\partial_t u(t)\|^2 \leq \|f(t)\|^2 + E(t).$$

Now one can use Gronwall's inequality to prove that the energy has a bounded growth in time. This growth is, however, exponential. We leave the details as a homework!
We end this part by mentioning that the uniqueness result stated in Corollary 6 also holds if an additional term f is added in the equation, as considered above. The proof is absolutely identical.

8.2 The wave equation: separation of variables

Separation of variables can be applied for hyperbolic equations too. As for elliptic or parabolic problems, one needs that the domain Ω is a Cartesian product (in either Cartesian, or polar coordinates), and the equation has a suitable form (linear, allowing a separation of the different variables).
To show how the method works, with $L > 0$ we consider the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } x \in (0, L) \text{ and } t > 0, \\ u(0,t) = u(L,t) = 0 & \text{for } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in (0,L), \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in (0,L), \end{cases} \tag{8.9}$$

for given $c > 0$ and $u_0, u_1 \in C(0, L)$.
Note that the boundary conditions in (8.9) are homogeneous. As for parabolic problems, if this is not the case, one has to homogenise them before applying the method of separation of variables. Now we seek u in separated form, $u(x,t) = X(x)T(t)$, and identify the functions X and T s.t. their product form a solution to the equation $(8.9)_1$, and satisfies the boundary conditions $(8.9)_2$. After a straightforward calculation, assuming that X and T are non-trivial, the former gives

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \beta \tag{8.10}$$

for some $\beta \in \mathbb{R}$, while from the latter one gets $X(0) = X(L) = 0$.
In view of the above, we solve first the equation in X (see also (6.6)),

$$\begin{cases} X''(x) - \beta X(x) = 0, & \text{for } x \in (0,L), \\ X(0) = X(L) = 0. \end{cases} \tag{8.11}$$

As in Section 6.1, the nontrivial solutions are obtained for $\beta_k = -\left(\frac{k\pi}{L}\right)^2$, giving

$$X_k(x) = \sin\left(\frac{k\pi x}{L}\right), \tag{8.12}$$

for $x \in (0, L)$ and with $k \in \mathbb{N}_0$.
From (8.10), the factor T in the expression of u solves

$$T_k''(t) - c^2 \beta_k T_k(t) = 0,$$

for $t > 0$ and with $k \in \mathbb{N}_0$. This gives the family of solutions

$$T_k(t) = a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right), \tag{8.13}$$

with $a_k, b_k \in \mathbb{R}$. We therefore obtained two families of functions, $\{u_k, k \in \mathbb{N}_0\}$ and $\{v_k, k \in \mathbb{N}_0\}$, where

$$u_k(x,t) = \cos\left(\frac{ck\pi t}{L}\right)\sin\left(\frac{k\pi x}{L}\right) \quad \text{and} \quad v_k(x,t) = \sin\left(\frac{ck\pi t}{L}\right)\sin\left(\frac{k\pi x}{L}\right). \tag{8.14}$$

Each member of the two families satisfies the equation $(8.9)_1$ and the boundary conditions $(8.9)_2$, but not necessarily the initial conditions $(8.9)_3$ and $(8.9)_4$. We seek u as the series

$$u(x,t) = \sum_{k=1}^{\infty} a_k u_k(x,t) + \sum_{k=1}^{\infty} b_k v_k(x,t),$$

and identify the coefficients a_k, b_k so that u satisfies the initial conditions. More precisely, $u(x,0) = u_0(x)$ gives

$$u_0(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right),$$

while $\partial_t u(x,0) = u_1(x)$ leads to

$$u_1(x) = \partial_t u(x,0) = \sum_{k=1}^{\infty} b_k \frac{ck\pi}{L} \sin\left(\frac{k\pi x}{L}\right).$$

Hence, the coefficients a_k, b_k $(k \in \mathbb{N}_0)$ are obtained from the Fourier coefficients of u_0 and u_1, namely (see also Section 6.2.1)

$$a_k = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{k\pi x}{L}\right)dx, \quad \text{and} \quad b_k = \frac{2}{ck\pi}\int_0^L u_1(x)\sin\left(\frac{k\pi x}{L}\right)dx. \tag{8.15}$$

Remark 21 *The above is only one example where the separation of variables is applied for solving a linear hyperbolic equation. For other situations, one may obtain different* sin *or* cos *functions, but the ideas remain the same. Also, note that, unlike the case of elliptic equations in a strip, here the solution is uniquely determined because we have two initial conditions.*

8.3 The method of D'Alembert (unbounded domains)

We now consider the problem in $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{8.16}$$

where $c > 0$ and u_0, u_1 are given. Since the domain is unbounded on both sides, no boundary conditions can be prescribed.

This problem is solved in [2] by means of a change of variables (the method of D'Alembert). Here we give an alternative approach, following the ideas presented in Chapter 2. We start by observing that the equation can be rewritten as

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u\right]=0 \qquad \text{for } x\in\mathbb{R} \text{ and } t>0. \tag{8.17}$$

This suggests using the auxiliary function $v(x,t) = \partial_t u - c\partial_x u$, which satisfies the equation

$$\partial_t v + c\partial_x v = 0,$$

for $x \in \mathbb{R}$ and $t > 0$. Applying the method of characteristics, we obtain that v satisfies

$$v(x,t) = v(x-ct,0)$$

(see also Section 2.1.1). Using the initial conditions and the definition of v we get

$$v(x,t) = \partial_t u(x-ct,0) - c\partial_x u(x-ct,0) = u_1(x-ct) - cu_0'(x-ct) \tag{8.18}$$

(note that u_0 is a function in one variable, therefore the we speak about u_0' and not about a partial derivative of u_0).
Having found v we now draw our attention to u, which satisfies

$$\partial_t u(x,t) - c\partial_x u(x,t) = v(x,t)$$

for all $x \in \mathbb{R}$ and $t > 0$. For the above, the characteristics are defined by $x'(t) = -c$, yielding $x(t) = x_0 - ct$.
Let now x_0 be fixed. Along the characteristic $x(t) = x_0 - ct$, for the function $w : [0,\infty) \to \mathbb{R}$, $w(t) = u(x_0 - ct, t)$ one has

$$w'(t) = -c\partial_x u(x_0-ct,t) + \partial_t u(x_0-ct,t) = v(x_0-ct,t).$$

From this and using (8.18), after integration one obtains straightforwardly

$$\begin{aligned} u(x_0-ct,t)=w(t) &= w(0)+\int_0^t v(x_0-c\tau,\tau)d\tau \\ &= w(0)+\int_0^t \big(u_1(x_0-c\tau-c\tau)-cu_0'(x_0-c\tau-c\tau)\big)d\tau \\ &= w(0)+\int_0^t \big(u_1(x_0-2c\tau)-cu_0'(x_0-2c\tau)\big)d\tau. \end{aligned}$$

Since $w(0) = u(x_0, 0)$, after applying the substitution $y = x_0 - 2c\tau$, the above leads to

$$\begin{aligned} u(x_0-ct,t) &= u_0(x_0)-\frac{1}{2c}\int_{x_0}^{x_0-2ct}\big(u_1(y)-cu_0'(y)\big)dy \\ &= u_0(x_0)+\frac{1}{2}\big[u_0(x_0-2ct)-u_0(x_0)\big]+\frac{1}{2c}\int_{x_0-2ct}^{x_0}u_1(y)dy \\ &= \frac{1}{2}\big[u_0(x_0)+u_0(x_0-2ct)\big]+\frac{1}{2c}\int_{x_0-2ct}^{x_0}u_1(y)dy. \end{aligned}$$

Finally, with $x = x_0 - ct$ gives

$$u(x,t) = \frac{1}{2}\big[u_0(x+ct) + u_0(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy.$$

Remark 22 *If the initial impulse is 0 ($u_1 \equiv 0$) one has the solution*

$$u(x,t) = \frac{1}{2}\big[u_0(x+ct) + u_0(x-ct)\big].$$

In other words, the initial profile is divided by two, and these two parts travel with the velocity c to the left and to the right. A straightforward calculations gives

$$\int_{\mathbb{R}} u(x,t)dx = \frac{1}{2}\left\{\int_{\mathbb{R}} u_0(x-ct)dx + \int_{\mathbb{R}} u_0(x+ct)dx\right\} = \int_{\mathbb{R}} u_0(y)dy,$$

which means that the initial displacement is conserved.

Exercise set 8 *"Hyperbolic problems, second order wave equation"*

1. ***Energy estimates***
   Consider the problem (8.1), but with homogeneous Neumann boundary conditions $\vec{n}\cdot\nabla u = 0$ on $\partial\Omega$ and for all $t \geq 0$. Prove that the total impulse is conserved,

   $$\int_\Omega \frac{\partial u}{\partial t}(\vec{x},t)d\vec{x} = \int_\Omega u_1(\vec{x})d\vec{x}.$$

   Then prove that in the absence of an initial impulse, the total displacement in Ω is conserved.
   Hint: In general no explicit solution is available, but you may integrate the equation in (8.1) to deduce that the total impulse remains 0, and that the total displacement is constant in time.

2. ***The wave equation: separation of variables***

   (a) *Let $c, \alpha > 0$ be two positive constants and $f : [0,L] \to \mathbb{R}$ a given function in $C^1(([0,L])$ satisfying $f(0) = f(L) = 0$. Consider the problem*

   $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} - \alpha^2 u & \text{in } \{(x,t) : 0 < x < L, t > 0\}, \\[2ex] u(0,t) = 0, \quad u(L,t) = 0 & \text{for } t > 0, \\[2ex] u(x,0) = 0, \quad \dfrac{\partial u}{\partial t}(x,0) = f(x) & \text{for } 0 < x < L. \end{cases}$$

   Determine the solution u.

(b) *Let $c > 0$ be a positive constant and consider the problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{in } \{(x,t) : 0 < x < 1, t > 0\}, \\ u(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(1,t) = 0, & \text{for } t > 0, \\ u(x,0) = 2\sin\left(\frac{\pi}{2}x\right), \quad \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{for } 0 < x < 1. \end{cases}$$

Find the solution to the given problem.

(c) *Let $c > 0$ be a positive constant and consider the problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{in } \{(x,t) : 0 < x < 1, t > 0\}, \\ \dfrac{\partial u}{\partial x}(0,t) = 1, \quad u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{for } 0 < x < 1. \end{cases}$$

the initial displacement $f(x) = x - 1$ whereas the initial velocity g and its derivative g' are piecewise continuous. Prove that regardless of the initial velocity, the displacement u at $t = \frac{2}{c}$ is equal to the initial f.

3. ***The wave equation in unbounded domains***

(a) *We use a different approach to obtain the solution to the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0,$$

with given $c > 0$ and initial data

$$u(x,0) = u_0(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x \in \mathbb{R}.$$

More precisely, we consider a change of variable based on the characteristics

$$\zeta = x + ct, \qquad \eta = x - ct.$$

i. *With U being the solution to the wave equation, let $v : \mathbb{R}^2 \to \mathbb{R}$, $v(\zeta, \eta) = u(x,t)$. Show that v satisfies the equation*

$$4c^2 \frac{\partial^2}{\partial \zeta \partial \eta} v(\zeta, \eta) = 0, \qquad \text{for all } \zeta, \eta \in \mathbb{R}.$$

ii. *Show that there exist $f, g : \mathbb{R} \to \mathbb{R}$ s.t. $v(\zeta, \eta) = f(\zeta) + g(\eta)$ for all $\zeta, \eta \in \mathbb{R}$.*

iii. *Determine f and g explicitly, depending on the initial data. Then determine the solution u.*

(b) *With given $c > 0$, find the solution to the problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} &= c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } x \in \mathbb{R}, t > 0, \\ u(x,0) &= \begin{cases} 1 - |x|, & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases} \quad , \quad \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

4. ***Analysis:***
Prove the following:

Lemma 4 *Let $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable and assume that the following limits exist:*

$$\lim_{x\to\infty} f(x) = a, \quad \text{and} \quad \lim_{x\to\infty} f'(x) = b, \quad \text{with } a, b \in \mathbb{R}.$$

Prove that $b = 0$.

5. ***Qualitative methods for ordinary differential equations (remember...)***

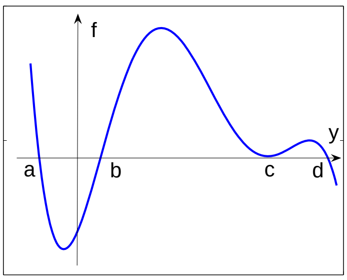
(a) *Sketch the phase line for the ODE*

$$\frac{du}{dt} = 3u^3 - 12u^2.$$

Identify the equilibrium points and their type. Then sketch the solutions satisfying $u(0) = -1$, $u(0) = 0$, $u(0) = 3$, respectively $u(1) = 3$.

(b) *Draw the phase line of the autonomous equation $\frac{dy}{dt} = f(y)$, where the graph of f is sketched in the left plot of Figure 8.1.*

(c) *The right plot in Figure 8.1 displays the phase line of an autonomous equation $\frac{dy}{dt} = f(y)$. Sketch the graph of the function f.*



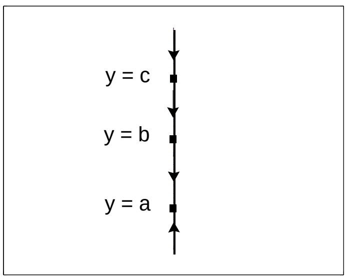


Figure 8.1: Left: Graph of f (Problem C2); right: phase line (Problem C3).

(d) *Suppose that f is smooth and that the autonomous equation $\frac{dy}{dt} = f(y)$ has an equilibrium point at $y = y^*$. Investigate its type in each of the following three cases:*

i. $f'(y^*) = 0,\ f''(y^*) = 0,\ f'''(y^*) > 0;$

ii. $f'(y^*) = 0,\ f''(y^*) = 0,\ f'''(y^*) < 0;$

iii. $f'(y^*) = 0,\ f''(y^*) > 0.$

Chapter 9

Nonlinear evolution equations (week 9/10)

Until now we have only considered linear equations, for which we have presented, next to qualitative aspects, different ways to construct solutions. Here we address nonlinear equations, for which we construct, again, special types of solutions: travelling waves, respectively similarity solutions.

9.1 Some background

A well-known nonlinear equations is *the Burgers equation*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu \frac{\partial^2 u}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0. \tag{9.1}$$

This is an important equation as it can be seen as the one-dimensional counterpart of the Navier-Stokes model, still featuring several of the aspects that makes the latter a challenging mathematical issue. Moreover, when $\nu = 0$, the resulting is a nonlinear hyperbolic equation (the inviscid Burgers equation) which, in case of smooth solutions, can be seen as prototypical for any nonlinear equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \text{ for } x \in \mathbb{R} \text{ and } t > 0, \tag{9.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is any twice continuously differentiable function. To see the connection between the two equations, one rewrites $\partial_x f(u) = f'(u)\partial_x u$ and multiplies the resulting (9.2) by $f''(u)$ to obtain

$$f''(u)\partial_t u + f''(u)f'(u)\partial_x u = 0.$$

Letting now $v = f'(u)$, the latter equation reduces to

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = 0,$$

which is nothing but (9.1) for $\nu = 0$. Observe that this transformation *is only possible* if the solution u is sufficiently smooth, at least continuously differentiable.
However, this smoothness cannot be guaranteed. One can see it immediately if considering (9.5) with a non-smooth initial condition. The solution will not become smoother, as it happened for parabolic problems. Moreover, in the nonlinear case the solution may become non-smooth even when starting with a smooth initial data. To see this, recall that the solution to first order hyperbolic problems like (9.5) can be solved by the method of characteristics. For (9.2) the characteristics are obtained as solutions to

$$x'(t) = f'(u(x(t), t)), \text{ for } t > 0. \tag{9.3}$$

For the inviscid Burgers equation one has $x'(t) = u(x(t), t)$. Clearly, in the $x - -t$ plane the characteristics for larger values of u are more inclined to the x-axis than those for smaller values of u. Other said, larger values of u will travel faster than smaller ones, and the former can catch up the latter. At points where something like this occurs, the solution u will have different left- and right limits, so it becomes discontinuous. In other words, smooth solutions do not exist any more.
This pleads for a more general concept of solutions, called *weak solutions*, in which derivatives are interpreted in a weak sense by using smooth test functions. More precisely, u will be called a solution if one has

$$\int_0^\infty \int_{\mathbb{R}} u(x,t)\partial_t\varphi(x,t)dxdt + \int_0^\infty \int_{\mathbb{R}} f(u(x,t))\partial_x\varphi(x,t)dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)dx = 0$$

for all *test functions* $\varphi \in C^\infty(\mathbb{R} \times [0,\infty))$ having a compact support in $\mathbb{R} \times [0,\infty)$. Without entering into details, we only mention that this new concept does not require that u is smooth, it allows even solutions that are discontinuous, so it solves the "existence" problem. On the other hand, one is facing the "uniqueness" problem. More precisely, there can be more than one "weak solution" to the same equation, and the question then is how to choose one of them as being physically relevant?
One option is to select the solution as the limit

$$u = \lim_{\nu \searrow 0} u^\nu,$$

where u^ν is the solution to the parabolic equation

$$\frac{\partial u^\nu}{\partial t} + \frac{\partial}{\partial x} f(u^\nu) = \nu \frac{\partial^2 u^\nu}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0, \tag{9.4}$$

(the so-called *viscous regularisation* of the hyperbolic equation obtained for $\nu = 0$). This choice is commonly called *entropy solution* and is motivated by physics. More precisely, the hyperbolic equation is a simplified mathematical model for e.g. flow, in which diffusive effects are neglected. However, although these effects are disregarded, to make the correct choice of a solution one has to recall what was neglected when the model was simplified!

9.2 Travelling waves for the nonlinear parabolic equation (9.4)

We have already discussed the simplest form of the wave equation

$$\partial_t u + a\partial_x u = 0, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{9.5}$$

and with a given initial condition $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}$. Here $a \in \mathbb{R}$ is given. The solution to (9.5) is a *wave* travelling with the *speed* a, $u(x, t) = u_0(x - at)$, see Chapter **??**. This situation was extended in Chapter 8 the case of a second order hyperbolic equation, where two waves travelling in opposite direction were encountered.

One question appearing naturally is whether such solutions with a constant profile, and travelling with a given velocity, are also possible for (9.4) (of course, for suitable initial conditions). In other words, the sought solutions will have the form

$$u^\nu(x, t) = v(x - ct), \tag{9.6}$$

for a function v and a *wave speed* $c \in \mathbb{R}$ that need to be determined? Such solutions are called *travelling waves.*

Clearly, the answer to the question above also depends on the expected behaviour as x, respectively η are approaching $\pm\infty$. From now on we seek solutions satisfying

$$\lim_{x\to-\infty} u^\nu(x, t) = u_\ell, \text{ and } \lim_{x\to\infty} u^\nu(x, t) = u_r, \tag{9.7}$$

for all $t > 0$, and where $u_\ell, u_r \in \mathbb{R}$ are assumed given. Consequently, one has

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \text{ and } \lim_{\eta\to\infty} v(\eta) = u_r. \tag{9.8}$$

With this we formulate the following

Definition 9 *(Travelling wave)*
Given the left and right states $u_\ell, u_r \in \mathbb{R}$, a travelling wave (TW) solution to (9.4) is a solution $u^\nu : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ to (9.4) for which a travelling wave speed $c \in \mathbb{R}$ and a function $v : \mathbb{R} \to \mathbb{R}$ exist s.t.

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \lim_{\eta\to\infty} v(\eta) = u_r, \text{ and } u^\nu(x, t) = v(x - ct),$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

Observe that, given $u_\ell, u_r \in \mathbb{R}$, two unknowns have to be determined when speaking of TW solutions: the function v itself, and the wave speed c. For the ease of writing we use $\eta = x - ct$ as new variable. Observe that assuming that u^ν has the form stated in Definition 9 (or in (9.6)) implicitly relates the originally independent variables x and t and reduces them to only one. We therefore adopt a strategy that has been

used before: to assume a certain relation between x and t, and to reduce the partial differential equation to an ordinary one.
We discuss below the existence of a TW solution and how to obtain the function v and the TW velocity c. For obvious reasons we restrict to the case $u_\ell \neq u_r$, the case $u_\ell = u_r$ being trivial. First of all, with η as above one uses the chain rule to obtain

$$\frac{\partial}{\partial t} = -c\frac{d}{d\eta}, \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{d}{d\eta}. \tag{9.9}$$

With this, since u^ν is a solution to (9.4), v solves the equation

$$-cv' + \big(f(v)\big)' = \nu v'', \text{ for all } \eta \in \mathbb{R}.$$

In the above, $\frac{d}{d\eta}$ is replaced by the more simple notation $'$. Integrating the above, one obtains that v solves the first order differential equation

$$A - cv + f(v) = \nu v', \text{ for all } \eta \in \mathbb{R}, \tag{9.10}$$

where $A \in \mathbb{R}$ is an arbitrary constant. As mentioned before, $c \in \mathbb{R}$ is an unknown too. We now use the stated behaviour for $\eta \to \pm\infty$ to determine the constant A and the TW velocity c. More precisely, we assume the existence of a v solving (9.10) and satisfying (9.8), and use the latter to determine A and c. We observe that (9.10) reduces to

$$v'(\eta) = g(v(\eta)), \text{ for all } \eta \in \mathbb{R} \text{ and with } g(v) = \frac{1}{\nu}\big(f(v) - cv + A\big). \tag{9.11}$$

The function f is assumed continuous, therefore the same holds for g. Since one has $\lim\limits_{\eta\to\infty} v(\eta) = u_r$ and $\lim\limits_{\eta\to-\infty} v(\eta) = u_\ell$, one obtains that

$$\lim_{\eta\to\infty} g(v(\eta)) = g(u_r), \text{ and } \lim_{\eta\to-\infty} g(v(\eta)) = g(u_\ell).$$

By (9.11), this immediately shows that v' has limits as $\eta \to \pm\infty$ as well,

$$\lim_{\eta\to\infty} v'(\eta) = g(u_r), \text{ and } \lim_{\eta\to-\infty} v'(\eta) = g(u_\ell).$$

We use now an elementary result, of which proof is left as homework:

Proposition 10 *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and assume that the following limits exist:*

$$\lim_{\eta\to\infty} h(\eta) = a, \quad \text{and} \quad \lim_{\eta\to\infty} h'(\eta) = b, \quad \text{with } a, b \in \mathbb{R}.$$

Then one has $b = 0$.

Clearly, the same result can be stated for $\eta \to -\infty$. With this, we observe that Proposition 10 can be applied to v to obtain that

$$g(u_\ell) = g(u_r) = 0.$$

From (9.11) one gets

$$f(u_r) - cu_r + A = 0, \text{ and } f(u_\ell) - cu_\ell + A = 0.$$

This is a linear system of two equations with two unknowns, giving

$$c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}, \text{ and } A = cu_\ell - f(u_\ell) = cu_r - f(u_r).$$

The latter equality follows from the specific form of c. We have therefore proved the following

Proposition 11 *(Necessary condition for the existence of TW solutions)*
Assume that (9.4) *has a TW solution v connecting the given left and right states u_ℓ and u_r ($u_\ell, u_r \in \mathbb{R}$). Then, the corresponding TW velocity is*

$$c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}, \tag{9.12}$$

and v is a solution to

$$v' = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), \text{ for all } \eta \in \mathbb{R}. \tag{9.13}$$

Observe that, by (9.12), the term on the right in (9.13) can be replaced by $\frac{1}{\nu}\big(f(v) - f(u_r) - c(v - u_r)\big)$.
At this point we note that (9.12) agrees with a general property of TW solutions to an equation of the form

$$\partial_t(\alpha(u)) + \partial_x(\beta(u)) = \partial_x(\gamma(u)\partial_x u), \tag{9.14}$$

where α, β, γ are C^1 functions s.t. α is strictly increasing and γ non-negative. One has

Proposition 12 *Assume that* (9.14) *admits TW solutions connecting two given states $u_\ell, u_r \in \mathbb{R}$ s.t. $u_\ell \neq u_r$. Then, the TW velocity is*

$$c = \frac{\beta(u_\ell) - \beta(u_r)}{\alpha(u_\ell) - \alpha(u_r)}.$$

Task: Give a proof of this general property.

Remark 23 *The TW velocity in Proposition 12 does not depend on the diffusion function γ.*

Returning to the TW solutions to (9.4), Proposition 11 only gives necessary conditions for the existence of TW solutions, but not sufficient ones. The existence depends on the properties of f. Moreover, the uniqueness is not true. To see this, observe that, if v is a TW solution and hence solves

$$\begin{cases} v' = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), & \text{for all } \eta \in \mathbb{R}, \\ \lim\limits_{\eta\to-\infty} v(\eta) = u_\ell, \quad \text{and } \lim\limits_{\eta\to\infty} v(\eta) = u_r, \end{cases} \tag{9.15}$$

then, for any $\zeta \in \mathbb{R}$, so does the ζ-translation of v, $v_\zeta(\eta) = v(\eta - \zeta)$. Therefore, to fix the ideas, we will use a *normalisation* of the TW by requiring that

$$v(0) = v_0 \tag{9.16}$$

for some value v_0 between u_ℓ and u_r. E.g., one may choose $v_0 = (u_\ell + u_r)/2$, but, in some situations, other choices may be more convenient. This shows that, actually, one can reformulate the question of existence to the following

Question. Le $v_0 = (u_\ell + u_r)/2$ (or any value between the two states) be given and v be the solution to the initial value problem

$$\begin{cases} v' &= \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), \qquad \text{for all } \eta \in \mathbb{R}, \\ v(0) &= v_0. \end{cases} \tag{9.17}$$

Does v satisfy $\lim_{\eta \to -\infty} v(\eta) = u_\ell, \quad$ and $\lim_{\eta \to \infty} v(\eta) = u_r$?

Note that the existence and uniqueness of a solution v to (9.17) is guaranteed by the standard theory for ordinary differential equations (why?). It only remains to investigate the behaviour of v as $\eta \to \pm\infty$, which can be done by phase line arguments. We start by observing that (9.13) is an ordinary differential equation $v' = g(v)$, and that the two states u_ℓ and u_r are equilibrium solutions to it, $g(u_\ell) = g(u_r) = 0$. Since g is a C^1 function and because at $\eta = 0$ the solution lies between u_ℓ and u_r, it follows the same for $v(\eta)$ for all $\eta \in \mathbb{R}$. Also, v should approach u_ℓ, respectively u_r as η approaches $-\infty$, respectively ∞. This rules out the possibility that g has another zero between u_ℓ and u_r. To see this, assume that an u^* between the two states exists s.t. $g(u^*) = 0$. Then u^* is an equilibrium for (9.13), and then v could never cross it to approach either u_ℓ at $-\infty$, or u_r at ∞. This implies that no TW solutions connecting the given states will exist.

Therefore one concludes that g has constant sign for all arguments between u_ℓ and u_r. Essentially, the existence of a TW solution reduces to the analysis of the (sign of the) function g, in connection with the ordering of u_ℓ and u_r. We have the following cases.

Case A. If $u_\ell > u_r$, then v decays from u_ℓ to u_r. Therefore, there exists an $\eta_0 \in \mathbb{R}$ s.t. $v(\eta_0) \in (u_r, u_\ell)$ and $v'(\eta_0) < 0$. This implies that $g(v(\eta_0)) < 0$, and therefore $g(v) < 0$ for all $v \in (u_r, u_\ell)$. If a $v \in (u_r, u_\ell)$ exists s.t. $g(v) \geq 0$ then no TW connecting u_ℓ to u_r exist.

On the other hand, if $u_\ell > u_r$ and g are s.t. $g(v) < 0$ for all $v \in (u_r, u_\ell)$, then by standard ordinary differential equation arguments one obtains that the solution to (9.17) has the desired behaviour, implying the existence of a TW solution.

Case B. If $u_\ell < u_r$, the analysis is similar. If a $v \in (u_\ell, u_r)$ exists s.t. $g(v) \leq 0$ then no TW connecting u_ℓ to u_r exist. Otherwise, if $u_\ell < u_r$ and g are s.t. $g(v) > 0$ for all $v \in (u_\ell, u_r)$, then a TW solution does exist.

The discussion above can be summarised in the following

Lemma 5 *Let $u_\ell, u_r \in \mathbb{R}$ be given s.t. $u_\ell \neq u_r$ and assume $f \in C^1(\mathbb{R})$. Then* (9.4) *does admit TW solutions in the sense of Definition 9 if and only if the function $g : \mathbb{R} \to \mathbb{R}$ defined by*

$$g(v) = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big) \quad \textit{with} \quad c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}$$

has a constant sign for all v between the two states, and this sign is the same as the sign of $u_r - u_\ell$. The TW is unique up to a translation, namely up to a normalisation as in (9.16).

Remark 24 *For the Burgers equation* (9.1) *one has $f(u) = \frac{1}{2}u^2$. In this case, a direct calculation shows gives $c = \frac{1}{2}(u_\ell + u_r)$ and $g(v) = \frac{1}{2\nu}(v - u_\ell)(v - u_r)$ (please, check!). Clearly, for any v between u_ℓ and u_r one has $g(v) < 0$, showing that TW solutions are possible if and only if $u_\ell > u_r$.*

Remark 25 *A similar conclusion can be drawn if the function f in* (9.4) *is convex. To see this, we note that the function $g(v) = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big)$ will be convex too. Since $g(u_\ell) = g(u_r) = 0$, the convexity of g implies that $g(v) < 0$ for all arguments v between u_ℓ and u_r (please, give a proof). Therefore, for a convex f TW solutions exist if and only if $u_\ell > u_r$ (again the case $u_\ell = u_r$ is not considered since it is trivial). Similarly if f is concave, then TW solutions exist if and only if $u_\ell < u_r$.*
We finally remark that whether f is convex or concave is not essential, the sign of g. The convexity/concavity of f is only relevant for proving that g is negative/positive. If this can be proved by employing arguments, then the result is still valid.

The arguments above show that the TW is a monotone function and the range is either (u_ℓ, u_r) or (u_r, u_ℓ) (depending on the ordering of the states). To fix the ideas assume that $u_r < u_\ell$ and that $f \in C^1$ is s.t. $g(v) < 0$ for all $v \in (u_r, u_\ell)$, the other case being analogous. Also, assume that the TW is normalised s.t. $v(0) = \frac{u_\ell + u_r}{2}$.
We assume further that the states are approached only asymptotically, i.e.

$$\begin{aligned} &\lim_{\eta\to\infty} v(\eta) = u_r, \quad \lim_{\eta\to-\infty} v(\eta) = u_\ell, \quad \text{and} \\ &u_r < v(\eta) < u_\ell \quad \text{for all } \eta \in \mathbb{R}. \end{aligned} \tag{9.18}$$

The discussion on when this is fulfilled is postponed, we only mention that, alternatively, one may have e.g. that $v(\eta) > u_r$ for all $\eta < \eta_r$ and $v(\eta) = u_r$ for all $\eta \geq \eta_r$.
Under the assumptions in (9.18), the TW $v : \mathbb{R} \to (u_r, u_\ell)$ is a bijection, so the inverse function $\eta : (u_r, u_\ell) \to \mathbb{R}$, $\eta = \eta(v)$ is well defined. This is a decreasing function satisfying the equation

$$\eta'(v) = \frac{1}{v'(\eta(v))} = \frac{1}{g(v)},$$

where we have used the rule for differentiating the inverse function, and the equation for v. Moreover, $\eta\big(\frac{u_\ell + u_r}{2}\big) = 0$. From this, by integration one obtains

$$\eta = \int_{\frac{u_\ell + u_r}{2}}^{v} \frac{1}{g(z)} dz,$$

with g as in Lemma 5. In other words, we have determined the inverse of the (normalised) TW. This can be interpreted as the implicit definition of the TW, also called as the ***integral representation*** of the TW,

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{1}{g(z)} dz, \tag{9.19}$$

for all $\eta \in \mathbb{R}$. If (9.19) can be solved in terms of $v(\eta)$, one obtains the TW explicitly. However, compared with the integral representation, this last step is not always possible.

As discussed in Remark 24, when considering the Burgers equation (9.1) one has $f(u) = \frac{1}{2}u^2$ and TW solutions are possible if and only if $u_\ell > u_r$. These solutions travel with the velocity $c = \frac{1}{2}(u_\ell + u_r)$. When considering the normalisation $v(0) = \frac{u_\ell+u_r}{2}$, the TW are solutions to

$$\begin{cases} v' &= \frac{1}{2\nu}(v-u_\ell)(v-u_r), \qquad \text{for all } \eta \in \mathbb{R}, \\ v(0) &= \frac{u_\ell+u_r}{2}. \end{cases} \tag{9.20}$$

From (9.19), the integral representation of the TW is

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{2\nu}{(z-u_\ell)(z-u_r)} dz. \tag{9.21}$$

In this case, the TW can be obtained explicitly. Using partial fractions,

$$\frac{1}{(z-u_\ell)(z-u_r)} = \frac{1}{u_\ell - u_r}\left[\frac{1}{z-u_\ell} - \frac{1}{z-u_r}\right],$$

one obtains

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{2\nu}{u_\ell - u_r}\left[\frac{1}{z-u_\ell} - \frac{1}{z-u_r}\right] dz.$$

Since $v(\eta) \in (u_r, u_\ell)$, this gives

$$\eta = \frac{2\nu}{u_\ell - u_r} \ln \left|\frac{z-u_\ell}{z-u_r}\right| \Bigg|_{z=\frac{u_\ell+u_r}{2}}^{v(\eta)} = \frac{2\nu}{u_\ell - u_r} \ln \frac{u_\ell - v(\eta)}{v(\eta) - u_r}.$$

From this, it follows that

$$v(\eta) = u_r + (u_\ell - u_r)\left[1 + e^{\frac{u_\ell - u_r}{2\nu}\eta}\right]^{-1},$$

yielding the TW solution

$$u(x,t) = u_r + (u_\ell - u_r)\left[1 + e^{\frac{u_\ell - u_r}{2\nu}\left(x - \frac{u_\ell + u_r}{2}t\right)}\right]^{-1}. \tag{9.22}$$

Observe that this solution satisfies all required properties, including that $u(x,t) = \frac{u_\ell+u_r}{2}$ whenever $x = \frac{u_\ell+u_r}{2}t$ (which corresponds to $\eta = 0$).
Recall that in the introduction of this paragraph we discussed the viscous Burgers equation (9.1), in which $\nu > 0$, as a regularisation of the inviscid Burgers equation obtained for $\nu = 0$. For the latter, as a particular case of (9.2), we saw that it may admit more than one weak solution, and we mentioned the limit $u = \lim\limits_{\nu\searrow 0} u^\nu$ as the physically relevant solution, with u^ν solving (9.1). Having now the TW solution computed explicitly, one can pass $\nu \to 0$ in (9.22) to obtain the limit

$$u(x,t) = \begin{cases} u_\ell, & \text{if } x < \frac{u_\ell+u_r}{2}t, \\ u_r, & \text{if } x > \frac{u_\ell+u_r}{2}t. \end{cases} \tag{9.23}$$

Observe that this is a discontinuous solution to the inviscid Burgers equation, called a *shock*. Based on the arguments above, such a solution can be obtained (for $f(u) = \frac{1}{2}u^2$) if and only if $u_\ell > u_r$, and will be called *admissible*. This strategy is being adopted for more general situations, in which TW solutions and, more general, solutions to regularised hyperbolic equations, are used to decide whether a shock solution is admissible or not.
We finally come back to the assumption made in (9.18), namely that the TW only reaches the left and the right states asymptotically. Using the integral representation in (9.19), it follows that the (improper) integral on the right should diverge if v approaches either u_r or u_ℓ,

$$\lim_{v\searrow u_r} \int_{\frac{u_\ell+u_r}{2}}^{v} \frac{1}{g(z)}dz = \infty, \quad \text{and} \quad \lim_{v\nearrow u_\ell} \int_{\frac{u_\ell+u_r}{2}}^{v} \frac{1}{g(z)}dz = -\infty. \tag{9.24}$$

Recalling the comparison theorem for integrals and the convergence of p-integrals, it follows that g should have the following asymptotic behaviour

$$\lim_{v\searrow u_r} \frac{g(v)}{(v-u_r)^p} = a, \quad \text{and} \quad \lim_{v\nearrow u_\ell} \frac{g(v)}{(u_\ell - v)^q} = b,$$

for some exponents $p, q \geq 1$ and where $a, b \in \mathbb{R}_0$. E.g. for the Burgers equation one has $p = q = 1$ and $a = b = -\frac{u_\ell - u_r}{2\nu}$.
However, one may have the case when the asymptotic behaviour of g is different, implying that at least one of the integrals in (9.24) is convergent. To fix the ideas, assume that a (finite!) $\eta_+ \in \mathbb{R}$ exists s.t.

$$\lim_{v\searrow u_r} \int_{\frac{u_\ell+u_r}{2}}^{v} \frac{1}{g(z)}dz = \eta_+,$$

whereas the limit $v \nearrow u_\ell$ is still $-\infty$. Then for the function η in (9.19) one has $\eta(u_r) = \eta_+$, and its range becomes $(-\infty, \eta_+]$. In other words, we have a bijection $\eta : [u_r, u_\ell) \to (-\infty, \eta_+]$, having an inverse $v : (-\infty, \eta_+] \to [u_r, u_\ell)$. To obtain the TW solution, one needs to extend this function by a constant to the right of η_+, namely $v(\eta) = u_r$ for $\eta \geq \eta_+$. Such an example will be discussed later.

9.3 Degenerate Burgers equation

With $\nu > 0$ and $m > 1$ we consider the degenerate Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu \frac{\partial^2(u^m)}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0. \tag{9.25}$$

Given two states $u_\ell, u_r \geq 0$, we seek travelling wave (TW) solutions to (9.25) connecting them.

Remark 26 *To understand the name "degenerate" we refer to the porous medium equation, where the same nonlinear diffusion term is encountered. In particular, the diffusion is vanishing if $u = 0$. We discuss such aspects in Section 9.4 below.*

Problem: Show that travelling waves are possible if $u_\ell \geq u_r$, but not if $u_\ell < u_r$. Then, show that TW solutions exist whenever $u_\ell \geq u_r \geq 0$, and find the TW solution for the case $m = 2$, $u_\ell = 1$ and $u_r = 0$.

Solution. TW are solutions in the form

$$u(x,t) = v(\eta), \text{ where } \eta = x - ct.$$

Here $c \in \mathbb{R}$ is a constant TW velocity to be determined. Assuming that such solutions exist, and using the differentiation rules in which $'$ stands for the derivative w.r.t. η

$$\frac{\partial}{\partial t} = -c()' \text{ and } \frac{\partial}{\partial x} = ()',$$

if u is a TW solution to (9.25) then v must satisfy the equation

$$-cv'(\eta) + \frac{1}{2}(v^2(\eta))' = \nu(v^m(\eta))'', \text{ for all } \eta \in \mathbb{R}.$$

Integrating w.r.t η one gets the ordinary differential equation

$$(v^m(\eta))' = \frac{1}{\nu}\left[A - cv(\eta) + \frac{1}{2}v^2(\eta)\right], \text{ for all } \eta \in \mathbb{R}, \tag{9.26}$$

where the constant $A \in \mathbb{R}$ is yet unspecified. To determine it explicitly, as well as to find the TW velocity $c \in \mathbb{R}$ we use the behaviour of v at $\pm\infty$.
One has $\lim_{\eta\to\infty} v(\eta) = u_r$, therefore, by (9.26) the following limits exist:

$$\lim_{\eta\to\infty} v(\eta)^m = u_r^m, \text{ and } \lim_{\eta\to\infty}(v^m(\eta))' = A - cu_r + \frac{1}{2}u_r^2.$$

Using now Lemma 1 in the Instruction 8 for the function $f = v^m$ one obtains that $A = A - cu_r + \frac{1}{2}u_r^2$. Therefore, for all $\eta \in \mathbb{R}$, v solves the TW equation

$$(v^m(\eta))' = g(v(\eta)), \text{ where } g(v) = \frac{1}{\nu}\left[\frac{1}{2}(v^2 - u_r^2) - c(v - u_r)\right]. \tag{9.27}$$

Furthermore, letting $\eta \to -\infty$ and using the argument above leads to

$$0 = cu_r - \frac{1}{2}u_r^2 - cu_\ell + \frac{1}{2}u_\ell^2,$$

which gives

$$c = \frac{1}{2}\frac{u_\ell^2 - u_r^2}{u_\ell - u_r} = \frac{u_\ell + u_r}{2}. \tag{9.28}$$

Returning to the TW solution of (9.25), we observe that for the function g in (9.27) one has $g(u_\ell) = g(u_r) = 0$, and has constant sign between the two states. In other words, u_ℓ and u_r are two equilibrium solutions for the equation in (9.27). Since no equilibrium point exists between the two states, a TW *may* exist. It is a solution v to (9.27) s.t.

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \text{ and } \lim_{\eta\to\infty} v(\eta) = u_r. \tag{9.29}$$

Non-existence. Assume that $u_r > u_\ell \geq 0$. Form (9.29) it follows that v, hence v^m is increasing at least in an interval of $\mathbb{R}$, and this for some value $v \in (u_\ell, u_r)$. Therefore an $\eta \in \mathbb{R}$ exists s.t. $v(\eta) \in (u_\ell, u_r)$ and $(v^m(\eta))' > 0$, implying that $g(v(\eta)) > 0$. However, this is a contradiction since $g(v) < 0$ for all $v \in (u_\ell, u_r)$. this shows that no TW exist whenever $u_\ell < u_r$.
Existence. The case $u_\ell = u_r$ is trivial, then the equilibrium solution is a TW. We consider separately the cases $u_\ell > u_r > 0$ and $u_\ell > u_r = 0$.
The case $u_\ell > u_r > 0$. Since $u_r \leq v(\eta) \leq u_\ell$ for all $\eta \in \mathbb{R}$, it follows that v is bounded away from 0 everywhere. We rewrite (9.27) as

$$v'(\eta) = g(v(\eta)), \text{ with } g(v) = \frac{1}{2m\nu}\frac{(v-u_r)(v-u_\ell)}{v^{m-1}}. \tag{9.30}$$

Clearly, $g \in C^1[u_r, u_\ell]$, and therefore it is Lipschitz continuous in this closed interval. For any initial condition $v_0 \in (u_r, u_\ell)$, the equation in (9.30) has a unique solution v satisfying $v(0) = v_0$, at least locally. In fact, the existence is global since $g \in C^1$, and the two equilibria u_r and u_ℓ are lower and upper bounds for v. Moreover, $g(v) < 0$ for all $v \in (u_r, u_\ell)$, so v has the behaviour in (9.29). This means that (9.25) admits a TW solution connecting u_ℓ to u_r.
Integral representation. The TW can be given in the *integral form.* To fix the ideas, we choose $v_0 = \frac{1}{2}(u_r + u_\ell)$, but any other choice $v_0 \in (u_r, u_\ell)$ is possible. Observe that $v : \mathbb{R} \to (u_r, u_\ell)$ is a bijection, so it make sense to consider the function $\eta : (u_r, u_\ell) \to \mathbb{R}$. Since $\eta'(v) = \frac{1}{v'(\eta(v))}$ from (9.30) one obtains

$$\eta'(v) = \frac{1}{g(v)} \text{ for all } v \in (u_r, u_\ell).$$

Integrating the above and using the initial condition $v(0) = \frac{1}{2}(u_r + u_\ell)$ one obtains

$$\eta(v) = \int_{\frac{1}{2}(u_r+u_\ell)}^{v} \frac{1}{g(z)} dz. \tag{9.31}$$

Again, observe that for all $v \in (u_r, u_\ell)$ the integrand in the above is strictly negative and therefore η is well defined. If h is s.t. the integral in (9.31) can be computed explicitly, one gets the TW in implicit form, as $\eta = H(v)$, with H a primitive of the function $1/g$ s.t. $H\big(\frac{1}{2}(u_r + u_\ell)\big) = 0$. If, further, the inverse of H (say $G = H^{-1}$) can be found explicitly, then v can be obtained as $v = G(\eta)$, and with this the TW solution $u(x,t) = v(x - ct)$ with c given in (9.28).

Remark 27 *Observe that if $v \nearrow u_\ell$, g behaves asymptotically like $(v - u_\ell)$. Since $g < 0$ in the given interval, the integrand $1/g$ approaches $-\infty$ as $v \nearrow 0$. Moreover, due to the asymptotic behaviour of g in the left neighbourhood of u_ℓ it follows that $1/g \notin L^1((v^*, u_\ell)$ for any $v^* \in (u_r, u_\ell)$, in the sense that any primitive H of $1/g$ blows down to $-\infty$ as $v \nearrow u_\ell$. From (9.31) it follows that $\eta \to -\infty$ when $v \nearrow u_\ell$.*
Analogously, $\eta \to \infty$ if $v \searrow u_r$. This confirms that the integral representation of the TW has the expected behaviour, approaching u_ℓ, u_r if η goes to $-\infty$, respectively ∞.

The case $u_\ell > u_r = 0$. For simplicity we only consider $u_\ell = 1$. If $u_\ell \neq 1$, the situation can be reduced to this case by rescaling the TW, namely by working with $w(\eta) = \frac{1}{u_\ell} v(\eta)$.
We start by observing that v needs not to be a C^1 function, but v^m. This is important if $m > 1$, as will be seen below. At this point, from (9.27) it follws that v satisfies

$$2\nu(v^m(\eta))' = v(v(\eta) - u_\ell), \text{ for all } \eta \in \mathbb{R}. \tag{9.32}$$

Since $m > 1$, this means that for any η one either has $v(\eta) = 0$, or it holds that

$$\frac{2\nu m}{m-1}(v^{m-1}(\eta))' = v(\eta) - 1,$$

and $0 < v(\eta) < 1$.
As before, v is monotone decreasing, and we fix v by taking $v(0) = \frac{1}{2}$. This gives the integral representation of the TW

$$\eta = 2\nu m \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz. \tag{9.33}$$

Since $g(z) = 2\nu m(z^{m-2})/(z-1)$ has a constant sign for any argument z between 0 and 1, η is a bijection mapping (0, 1) to its range. It is unclear whether its range is the entire $\mathbb{R}$ or not. Without making the proof rigorous (I invite you to do it!) and in the spirit of Remark 27, we observe the following:

- As $z \nearrow 1$, $1/g(z)$ behaves like $1/(z-1)$. This means that

  $$\lim_{v \nearrow 1} \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz = -\infty,$$

  showing that $\lim_{v \nearrow 1} \eta = -\infty$. This also implies that the TW v only reaches the left state 1 asymptotically.

- For $z \searrow 0$, $1/g(z)$ behaves like $-z^{m-2}$. If $m \geq 2$, this is continuous up to 0, and therefore the integral in (9.33) has a finite limit when $v \searrow 0$. More precisely, an $\eta_0 > 0$ exists (why strictly positive?) s.t.

$$\lim_{v \searrow 0} \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz = \eta_0.$$

  The situation for $m \in (1, 2)$ is quite similar. Here the reciprocal of h is sigular in 0, but this singularity is integrable since $m - 2 > -1$ and therefore the primitive of z^{m-2} is defined in 0. This means that also in this case an $\eta_0 > 0$ exists as the limit of the integral if $v \searrow 0$.

We have therefore shown that the integral representation in (9.33) defines the strictly decreasing function $\eta : [0, 1) \to (-\infty, \eta_0]$, which can be inverted to find the TW v defined only on the interval $(-\infty, \eta_0]$. Observe that this TW reaches the right state $u_r = 0$ at a finite η_0.
To extend the TW for arguments $\eta > \eta_0$ we recall that v constructed above was using only one possibility offered by the TW equation (9.32). The other possibility was simply $v = 0$. This means that the TW solution on the entire $\mathbb{R}$ is the extension of the solution above by 0 for arguments $\eta \geq \eta_0$.

Remark 28 *Clearly, the right limit in η_0 of the TW v' is 0. From the discussion below, for $\eta < \eta_0$ but close, $\eta_0 - \eta$ has the same order as $v(\eta)^{m-1}$. This means that $v(\eta)$ behaves like $(\eta_0 - \eta)^{\frac{1}{m-1}}$. For $m \in (1, 2)$, this power is greater than 1, which means that actually even $v'(\eta)$ approaches 0 when $\eta \nearrow \eta_0$ and implying that the solution constructed above is C^1 everywhere on $\mathbb{R}$. For $m = 2$ the left limit of v' is finite but not 0, so the solution v has a kink at η_0. For $m > 2$ the left limit is $-\infty$. However, for all $m > 1$, v^m is C^1 everywhere, and the equation (9.32) is satisfied by the TW everwhere in $\mathbb{R}$.*

For the case $m = 2$ the TW can be found explicitly from (9.33), which becomes

$$\eta = 4\nu \int_{\frac{1}{2}}^{v} \frac{1}{z-1} dz.$$

This gives for all $v \in (0, 1)$

$$\eta = 4\nu\big(\ln|v - 1| - \ln(1/2)\big),$$

and therefore

$$v = 1 - \frac{1}{2} e^{\frac{\eta}{4\nu}}.$$

In other words we found the TW v explicitly, but this only holds if $v \in (0, 1)$. Whereas the expression on the right in the above is less than 1 for all values of η, the value 0 is achieved for $\eta_0 = 4\nu \ln 2$. With this we found the TW

$$v(\eta) = \begin{cases} 1 - \frac{1}{2} e^{\frac{\eta}{4\nu}}, & \text{if } \eta \leq \eta_0, \\ 0, & \text{if } \eta > \eta_0, \end{cases} \tag{9.34}$$

where η_0 is given above. Clearly, one has

$$\lim_{\eta \nearrow \eta_0} v'(\eta) = -\frac{1}{4\nu} < 0, \text{ and } \lim_{\eta \nearrow \eta_0} (v^2)'(\eta) = 0.$$

This means that v' has finite left and right limits in η_0, while v^2 is C^1 (see Remark 28). Finally, with the TW velocity $c = \frac{1}{2}$ as given in (9.28), the TW solution u is

$$u(x,t) = \begin{cases} 1 - \frac{1}{2} e^{\frac{1}{4\nu}(x - \frac{1}{2}t)}, & \text{if } x \leq \frac{1}{2}t + 4\nu \ln 2, \\ 0, & \text{if } x > \frac{1}{2}t + 4\nu \ln 2. \end{cases} \tag{9.35}$$

9.4 The porous medium equation

The equation studied in Section 9.3 is called *degenerate*, as the diffusion may vanish for certain values of the solution (there, if $u = 0$), and the equation may change its type. We present here a prominent example, the ***porous medium equation***.

9.4.1 Modelling background

Porous media are encountered practically everywhere: in geological formations (rocks, subsurface), different types of soils (clay, sand), civil engineering (concrete, bricks), technology (filters, complex materials), biosystems (tissues, bones), to name a few. The porous medium equation is a mathematical model for gas flow in a porous medium.

A porous medium is a mixture of solid parts (the porous skeleton) and void spaces (the pores). To understand this, imagine a recipient filled by sand grains. Clearly, the typical diametre of a sand grain is much smaller than the size of the recipient. When considering a small volume (commonly called a ***representative elementary volume, REV***) inside the recipient, one observes that it contains a mixture of grains and void spaces. The ratio of the void volume inside the REV, and the total volume of the REV, is called ***porosity*** Φ ($[-]$, thus a dimensionless quantity), The porosity is one of the quantities characterising the structure of the porous medium. another quantity is the ***permeability*** K $[m^2]$, which is expressing the surface available to flow in a section through an REV. In fact, the permeability expresses how easy a fluid/gas can flow through a specific location in a porous medium. Note that the section through the REV depends on the normal to the plane in which the section is considered, so K is usually a tensor. For simplicity, here we consider that the permeability is the same in all directions, so that K reduces to a scalar quantity. Also, both Φ and K are assumed to be the same everywhere in the recipient, so that these are medium-dependent constants that can be determined experimentally. We assume them known here.

To express the gas flow through a porous medium in terms of mathematical equations, we let ρ $\left[\frac{g}{m^3}\right]$ denote the ***density of the gas***, and $\vec{u}$ $\left[\frac{m}{s}\right]$ the gas ***velocity***. Note that both ρ and $\vec{u}$ are unknown. The first equation of the mathematical model is

$$\partial_t(\Phi\rho) + \nabla \cdot \Big(\rho\vec{u}\Big) = 0, \tag{9.36}$$

which expresses the ***conservation of mass***, in the absence of any source term (the term on the right being 0).
The second equation is the ***Darcy law***,

$$\vec{u} = -\frac{1}{\mu} K \nabla p, \tag{9.37}$$

where $\mu \left[\frac{kg}{ms}\right]$ is the ***dynamic viscosity*** of the gas, and $p \left[\frac{kg}{ms^2}\right]$ is the ***pressure*** inside the gas.
Whereas the gas viscosity can be determined (so, it is assumed known), the pressure is the third unknown quantity in the model. The model is then completed by the assumption that the gas is ideal and isentropic. In this case, there exists a constant $\gamma \geq 1$ s.t.

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{9.38}$$

where p_0 and ρ_0 are gas-specific constants (with the dimension of a pressure, respectively density).
With this, we have completed the system of three equations and three unknowns, ρ, $\vec{u}$ and p. Using (9.38) into (9.37), and the resulting into (9.36), one ends up with the ***porous medium equation***

$$\Phi \partial_t \rho - \frac{K}{\mu} \frac{p_0}{\rho_0^{\gamma}} \nabla \cdot (\rho \nabla \rho^{\gamma}) = 0, \tag{9.39}$$

where we have used the fact that Φ, K, μ, p_0 and ρ_0 are all constant.
Now, (9.39) is a mathematical model containing only one unknown, ρ. This equation is holds in all points of the domain occupied by the medium (the recipient) and for all times. The model will be completed by boundary and initial conditions.
Observe that the model is dimensional, in the sense that all quantities involved have dimensions. To analyse the model, and, in particular, to identify whether a quantity or a constant is small or large, one reduces the model to a dimensionless form. To do so, one chooses characteristic and reduces all quantities to a dimensionless form. For example, a charactersitic time T_R $[s]$ can be the time of the observation, and the dimensionless time $\tilde{t}$ is defined as $\tilde{t} = \frac{t}{T_R}$. By the chain rule, the time derivative becomes $\frac{\partial}{\partial t} = \frac{1}{T_R} \frac{\partial}{\partial \tilde{t}}$.
Analogously, one uses the characteristic length L_R (hence, the spatial derivatives are multiplied by $\frac{1}{L_R}$), and the characteristic density ρ_R to define the dimensionless unknown density $\tilde{\rho} = \frac{\rho}{\rho_R}$. Using this into (9.39) gives

$$\frac{\rho_R \Phi}{T_R} \partial_{\tilde{t}} \tilde{\rho} - \frac{K}{\mu} \frac{p_0 \rho_R^{\gamma+1}}{\rho_0^{\gamma} L_R^2} \tilde{\nabla} \cdot \left(\tilde{\rho} \tilde{\nabla} \tilde{\rho}^{\gamma}\right) = 0,$$

with $\tilde{\nabla}$ denoting the derivatives w.r.t. the dimensionless spatial variables. Assuming that the characteristic quantities are chosen s.t.

$$(\gamma + 1)\Phi\mu L_R^2 \rho_0^\gamma = \gamma p_0 T_R K \rho_R^\gamma,$$

a straightforward manipulation gives the porous medium equation in dimensionless form

$$\partial_{\tilde{t}}\tilde{\rho} = \tilde{\Delta}\tilde{\rho}^{\gamma+1}, \tag{9.40}$$

with $\tilde{\Delta}$ being the Laplace operator in terms of dimensionless spatial variables.

9.4.2 A fundamental solution for the porous medium equation

Using a similarity transformation, we construct here a solution for the porous medium equation (9.40) in one spatial dimension. This solution has the same properties as the fundamental solution for the diffusion equation, which was derived in Section 4.1.
More precisely, with given $m > 1$ we consider the nonlinear parabolic problem defined on the entire real axis

$$\begin{cases} \partial_t u(x,t) &= \partial_{xx} u^m(x,t), \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) &= \delta(x), \qquad \text{for } x \in \mathbb{R}, \end{cases} \tag{9.41}$$

where δ is the Dirac δ distribution discussed in Section 4.2.1.
Before constructing a solution, we make some observations. First of all, compared to (9.40), the unknown is now u, and $m = \gamma + 1$. Having the modelled process in mind, u is similar to the gas density, so the initial mass of the gas in the "domain" $\mathbb{R}$ is 1. Also one expects that $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. One can show that, if the initial condition is assumed positive, and with appropriate boundary conditions, the solution u remains positive everywhere and for all times.
Next, the term "degenerate" is motivated by the following observation. The equation $(9.41)_1$ seems to be (nonlinear) parabolic. However, one can write the term on the right as $\Delta u^m = \nabla \cdot \Big(m u^{m-1} \nabla u \Big)$. In other words, the diffusion is now a nonlinear function in terms of the unknown u, namely $D(u) = m u^{m-1}$. Therefore, whenever $u = 0$, one has $D(0) = 0$, so the diffusion vanishes and $(9.41)_1$ degenerates into an ordinary differential equation. As will be seen below, the solution will not have the same properties as the fundamental solution for the diffusion equation. In particular, in this case the solution is loses its smoothness. Also, it may have a compact support, which evolves in time in an apriori unknown manner, and with a finite speed of propagation. The boundary of this support is a ***free boundary***
Recalling Definition 4, we determine a fundamental solution u to (9.41). This solution will satisfy the following properties:

$$\int_{\mathbb{R}} u(x,t)\, dx = 1, \qquad \text{for all } t > 0; \tag{9.42}$$

$$\lim_{t \searrow 0} u(x,t) = 0, \qquad \text{for all } x \neq 0. \tag{9.43}$$

However, this solution cannot be used to construct solutions to the porous medium equation for arbitrary initial conditions, as the problem is non-linear and, in general, the convolution does not commute with the nonlinearity.
For $\alpha, \beta \in \mathbb{R}$ properly chosen, we seek $v : \mathbb{R} \to [0, \infty)$ s.t.

$$u(x,t) = t^{-\alpha} v(\eta), \qquad \text{where} \quad \eta = xt^{-\beta}, \tag{9.44}$$

is a solution for (9.41), with the properties above. Observe that, since $x \in \mathbb{R}$ and $t > 0$, ne gets $\eta \in \mathbb{R}$.
Now, from (9.42), one gets

$$1 = \int_{\mathbb{R}} u(x,t)\ dx = t^{-\alpha} \int_{\mathbb{R}} v(xt^{-\beta})\ dx = t^{\beta-\alpha} \int_{\mathbb{R}} v(\eta)\ d\eta,$$

implying $\alpha = \beta$. Next, to check whether u in the given form is satisfies $(9.41)_1$, we us the chain and the product rules to express the time derivative as

$$\partial_t u(x,t) = -\alpha t^{-(\alpha+1)} v(\eta) + t^{-\alpha} v'(\eta)(-\beta) x t^{-(\beta+1)} = -\alpha t^{-\alpha+1} \big(v(\eta) + \eta v'(\eta)\big),$$

since $\alpha = \beta$. This gives

$$\partial_t u(x,t) = -\alpha t^{-\alpha+1} \big(\eta v(\eta)\big)', \tag{9.45}$$

For the spatial derivatives, one has

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x} = t^{-\alpha} \frac{\partial}{\partial x}.$$

Therefore, by (9.44),

$$\partial_{xx} u^m(x,t) = t^{-(m+2)\alpha} \big(v^m(\eta)\big)''. \tag{9.46}$$

Since u solves $(9.41)_1$, one gets

$$-\alpha t^{-(\alpha+1)} \big(\eta v(\eta)\big)' = t^{-(m+2)\alpha} \big(v(\eta)^m\big)'', \qquad \text{for all } \eta \in \mathbb{R}.$$

If $\alpha = \frac{1}{m+1}$, the time variable t can be eliminated in the above. In this case, one gets $u(x,t) = t^{-\frac{1}{m+1}} v(\eta)$, whith $\eta = xt^{-\frac{1}{m+1}}$ and v solving

$$-\frac{1}{m+1} \big(\eta v(\eta)\big)' = \big(v^m(\eta)\big)'', \qquad \text{for all } \eta \in \mathbb{R}$$

and, after integration in η,

$$\big(v^m(\eta)\big)' + \frac{1}{m+1} \eta v(\eta) = C, \qquad \text{for all } \eta \in \mathbb{R}, \tag{9.47}$$

for some $C \in \mathbb{R}$.
We now observe that, for $x > 0$, one has $\eta \to \infty$ if $t \searrow 0$. Analogously, for $x < 0$, $\eta \to -\infty$ if $t \searrow 0$. Observe that, if $x \neq 0$, from (9.41) one has

$$u(x,t) = \frac{1}{x} \eta v(\eta).$$

By (9.43), one concludes that

$$\lim_{\eta\to\pm\infty} \eta\, v(\eta) = 0. \tag{9.48}$$

Clearly, this implies that also $v(\eta)$ has the limit 0 as $\eta \to \pm\infty$.
We rewrite now (9.47) as $(v^m(\eta))' = C - \frac{1}{m+1}\eta\, v(\eta)$ and use the above to conclude that the following limits exist

$$\lim_{\eta\to\pm\infty} v^m(\eta) = 0, \qquad \text{and} \qquad \lim_{\eta\to\pm\infty} \Big(v^m(\eta)\Big)' = C.$$

By Proposition 10, this means that $C = 0$. Therefore, v solves

$$\big(v^m(\eta)\big)' + \frac{1}{m+1}\eta\, v(\eta) = 0, \qquad \text{for all } \eta \in \mathbb{R}, \tag{9.49}$$

with the condition $\lim\limits_{\eta\to\pm\infty} v(\eta) = 0$. Observe that (9.49) implies that, for any $\eta \in \mathbb{R}$, one either has $v(\eta) = 0$, or

$$m v^{m-2}(\eta) v'(\eta) + \frac{1}{m+1}\eta = 0$$

whenever $v(\eta) > 0$. In the latter case, one gets (recall that $m > 1$)

$$\frac{m}{m-1}\big(v^{m-1}(\eta)\big)'(\eta) + \frac{1}{m+1}\eta = 0.$$

Therefore, a constant $A > 0$ exists s.t.

$$v(\eta) = \left[A - \frac{m-1}{2m(m+1)}\eta^2\right]^{\frac{1}{m-1}}, \tag{9.50}$$

for some $A \in \mathbb{R}$. Recalling that the expression on (9.50) is obtained only for positive values of v, it follows that $A > 0$, and that the solution v is given by

$$v(\eta) = \begin{cases} 0, & \text{if } |\eta| \geq \sqrt{\frac{2m(m+1)}{m-1}A}, \\ \left[A - \frac{m-1}{2m(m+1)}\eta^2\right]^{\frac{1}{m-1}}, & \text{if } |\eta| < \sqrt{\frac{2m(m+1)}{m-1}A}. \end{cases} \tag{9.51}$$

In this way, we obtain the solution

$$u(x,t) = \begin{cases} 0, & \text{if } |x| \geq t^{\frac{1}{m+1}}\sqrt{\frac{2m(m+1)}{m-1}A}, \\ \frac{1}{t^{\frac{1}{m+1}}}\left[A - \frac{m-1}{2m(m+1)}\frac{x^2}{t^{\frac{2}{m+1}}}\right]^{\frac{1}{m-1}}, & \text{if } |x| < t^{\frac{1}{m+1}}\sqrt{\frac{2m(m+1)}{m-1}A}. \end{cases} \tag{9.52}$$

The constant A in the above is taken in such a way that u satisfies (9.42). The exact value requires tedious calculations, so we omit the details here.
The solution given in (9.52) is called the ***Barenblatt-Pattle solution***. It has some remarkable features, out of which two are mentioned below.

1. Free boundaries;
2. Lack of smoothness.

Both are due to the degenerate character of the equation. More precisely, one can encounter them for solutions to degenerate parabolic equations, but not for the solution to the diffusion equation, or, more general, a linear parabolic equation, or even a nonlinear, but nondegenerate one.

Free boundaries.

With respect to the first aspect, we note that the solution u in (9.52) vanishes whenever $|x| \geq t^{\frac{1}{m+1}}\sqrt{\frac{2m(m+1)}{m-1}A}$. With $\gamma = \sqrt{\frac{2m(m+1)}{m-1}A}$, its (closed) support is

$$\mathrm{supp}(u) = \cup_{t\geq 0}\big[-\gamma t^{\frac{1}{m+1}}, \gamma t^{\frac{1}{m+1}}\big].$$

This is in contrast with the fundamental solution of the diffusion equation (see Section 4.1), which was strictly positive for all $t > 0$. The region where $u > 0$ is separated from the ones where $u \equiv 0$ by the two curves

$$\Gamma_{\pm} = \{(x,t) \in \mathbb{R} \times [0,\infty), x = \pm\gamma t^{\frac{1}{m+1}}\}.$$

As these curves depend on the solution u, these are not known apriori. In other words, to determine them, one has to solve first the equation. Such curves, separating regions that can be identified through remarkable properties of the solution, are called ***free boundaries***. Observe that here these curves are the boundaries of the space-time domain where $u > 0$. In general, free boundaries are determined by an equation relating the normal velocity (here, $(x'(t), 1)$) to the solution u. Free boundaries are a characteristic feature of degenerate parabolic equations.

Lack of smoothness.

Another aspect that distinguishes the fundamental solution for the porous medium equation from the one for the diffusion equation is related to the smoothness. Whereas the fundamental solution in Section 4.1 is C^∞ for all $t > 0$, the Barenblatt solution does not have this feature. In fact, if $m \geq 2$, u is not differentiable precisely along the free boundaries above, where the left and the right limits of $\partial_x u$ are different. Moreover, one of these limits are unbounded for $m > 2$. More details are discussed in the 9^{th} set of exercises.

Exercise set 9 *"Nonlinear problems"*

1. ***Travelling waves***

   (a) *With given $\nu > 0$, $Q > 0$ and $p > 0$, consider de nonlinear parabolic equation*

$$\frac{\partial(u^p)}{\partial t} + \frac{\partial(u^{2p})}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0.$$

Let $u_\ell, u_r \in [0,\infty)$ be given left- and right states. We seek travelling wave solutions (TW) satisfying

$$u(-\infty,t) = u_\ell, \quad u(\infty,t) = u_r, \qquad \text{for } t > 0.$$

i. *Assuming the existence of TW solutions, determine the TW speed c and the equation satisfied by the TW solution.*
ii. *Prove that no TW solutions exists if $u_\ell < u_r$.*
iii. *Let $u_r = 0$ and $u_\ell > 0$. Determine the integral representation of the TW solution. To fix the TW, you may take a convenient value of the TW at $\eta = 0$.*
iv. *Detemine the TW for $p = 1$, $u_\ell = 1$ and $u_r = 0$.*

(b) *The system below appears in the modeling of reactive porous media flows:*

$$\begin{cases} \partial_t(u+v) + Q\partial_x u &= 0, \\ \partial_t v + k\big(v - \varphi(u)\big) &= 0, \end{cases}$$

where $t > 0$ and $x \in \mathbb{R}$. Here Q and k are strictly positive constants, while $\varphi : [0,\infty) \to [0,\infty)$ is a $C^\infty((0,\infty)) \cap C([0,\infty))$ function satisfying $\varphi(0) = 0$, and $\varphi' > 0$, $\varphi'' < 0$ in $(0,\infty)$. We assume the following behavior at $\pm\infty$:

$$(P) \begin{cases} u(t,-\infty) = u_l > 0, & u(t,+\infty) = 0, \\ v(t,-\infty) = v_l > 0, & v(t,+\infty) = 0, \end{cases}$$

i. *Determine a necessary condition on u_l and v_l for the existence of traveling wave solutions to (P). Then show that this condition is also sufficient.*
ii. *Compute the traveling waves for $\varphi(u) = u^p$ with $0 < p < 1$.*

(c) *With given $Q > 0$ and $b : \mathbb{R} \to \mathbb{R}$ a C^2 function that is also increasing and convex, consider the nonlinear equation*

$$\frac{\partial b(u)}{\partial t} + Q\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0.$$

Given $u_r, u_\ell \in [0,\infty)$, we seek travelling wave solutions satisfying

$$u(-\infty,t) = u_\ell, \quad u(\infty,t) = u_r \quad \text{for all } t > 0.$$

i. *Determine the travelling wave equation and, assuming the existence of a travelling wave, its speed.*
ii. *Prove that no travelling waves exist if $u_\ell > u_r$ (recall that b is convex).*
iii. *Find the travelling wave for $b(u) = u^2$, $u_\ell = 0$ en $u_r = 1$. You may normalise the travelling wave by choosing $v(0) = \frac{1}{2}$.*

(d) *Let $f : \mathbb{R} \to \mathbb{R}$ be a concave, C^2 function and $D : \mathbb{R} \to \mathbb{R}$ increasing and C^1. Consider the nonlinear equation*

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 D(u)}{\partial x^2}, \quad x \in \mathbb{R}, t > 0.$$

Given $u_r, u_\ell \in \mathbb{R}$, we seek travelling wave solutions satisfying

$$u(-\infty, t) = u_\ell, \quad u(\infty, t) = u_r \quad \textit{for all } t > 0.$$

i. *Determine the travelling wave equation and, assuming the existence of a travelling wave, its speed.*

ii. *Prove that no travelling waves exist if $u_\ell > u_r$ (recall that f is concave and D is increasing).*

(e) *Give a proof for Proposition 12.*

2. ***The porous medium equation in 1D: smoothness***
We have detemined an explicit solution for the porous medium equation

$$\partial_t u = \partial_{xx}([u]_+)^m, \quad \textit{for } x \in \mathbb{R}, t > 0.$$

Unlike solutions to the heat equation (the case $m = 1$) is this solution strictly positive only in a bounded interval, and not C^∞.

(a) *Although this solution has the similarity form $u(x,t) = t^{-\frac{1}{\alpha}} f(\eta)$ (with $\eta = xt^{-\frac{1}{\beta}}$), it cannot be used to determine a solution of the porous medium equation for a general initial condition, as in the case of the diffusion equation. Explain why this not possible in the present case.*

(b) *For which $m > 1$ is this solution continuously differentiable?*

(c) *Observe that the term on the right does not require that u is continuously differentiable, but u^m. This rises the following question: for which power $\lambda \geq 1$ is u^λ continuously differentiable?*

3. ***The porous medium equation in 2D (supplementary)***
The approach by A can be extended to find a solution in multiple space dimensions. Here we consider the two-dimensional case,

$$\partial_t u = \Delta([u]_+)^m, \quad \textit{for } x = (x_1, x_2) \in \mathbb{R}^2, t > 0.$$

We assume that an even function $f : \mathbb{R} \to \mathbb{R}$ exists s.t. the solution u can be written as

$$u(x,t) = t^{-\alpha} f(\eta) \textit{ with } \eta = rt^{-\beta}) \textit{ and } r = (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

(a) *Prove that the mass balance is achieved only if $\alpha = 2\beta$.*

(b) Prove the following:

$$
\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= -t^{-(\alpha+1)}\big[\alpha f(\eta)+\beta\eta f'(\eta)\big],\\
\frac{\partial}{\partial x_k}u^m(x,t) &= t^{-(m\alpha+\beta)}\frac{x_k}{\|x\|}\big(f^m(\eta)\big)',\\
\frac{\partial^2}{\partial x_k^2}u^m(x,t) &= t^{-(m\alpha+\beta)}\big[(f^m(\eta))"t^{-\beta}\frac{x_k^2}{\|x\|^2}+(f^m(\eta))'\big(\tfrac{1}{\|x\|}-\frac{x_k^2}{\|x\|^3}\big)\big].
\end{aligned}
$$

(c) Use the above to prove that $\alpha=\frac{1}{m}$, $\beta=\frac{1}{2m}$ *and that* f *solves*

$$\big[\eta(f^m(\eta))'\big]'+\frac{1}{2m}(\eta^2 f(\eta))'=0, \qquad \textit{for all } \eta>0.$$

(d) Prove that $\lim_{\eta\to\infty}\eta^2 f(\eta)=0$.

(e) Prove that f *satisfies*

$$\eta(f^m(\eta))'+\frac{1}{2m}\eta^2 f(\eta)=0, \qquad \textit{for all } \eta>0.$$

(f) Determine f *en consequently* u*. You may leave the constant appearing in the solution open.*

(g) For which values of m *and* λ *are* u*, resp.* u^λ *continuously differentiable?*

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