

Complex Analysis Homework #1

1. Let v, w be distinct points in \mathbb{C} , and $\rho > 0$. Show that the locus of points $z \in \mathbb{C}$ that satisfy

$$|z - v| = \rho|z - w|$$

is a circle or a line.

Let us consider some special cases. Let us first take one of our fixed points to be 0, say $v = 0$. Let us also study the case where $\rho = 1$ to begin. Our defining equation becomes $|z| = |z - w|$, notice that z satisfies this system if and only if it satisfies $|z|^2 = |z - w|^2$. Let us write $z = a + bi$ and $w = c + di$ then

$$|z| = |z - w| \iff c^2 + d^2 - 2ac - 2bd = 0.$$

Now let us consider some cases for w . If w is real and non-zero then $w = c$ and we can solve for $a = \frac{1}{2}c$. b is free and so the locus of points which solve this case is $z = \frac{1}{2}c + bi$ for any fixed $c \in \mathbb{R}$ and for all $b \in \mathbb{R}$. This is exactly a vertical line of points. In particular, it is the vertical line of points through the origin, translated by $\frac{1}{2}c$.

Performing an extremely similar calculation, if we consider the case where w is purely imaginary, $w = di$ for some $d \in \mathbb{R}$, then our locus of points will be $z = a + \frac{1}{2}d$ for some for all $a \in \mathbb{R}$. These are horizontal lines. In particular, these are the horizontal line of points through the origin, translated by $\frac{1}{2}d$.

In the case where $w = c + di$ with c, d non-zero. We are now looking for points which have the same distance from the origin after being translated by $-(c + di)$. If $c = d = 1$ then our locus of points is a diagonal line. In particular, it is the diagonal line passing through the origin and $-(1 + i)$, translated¹ by $\frac{1}{2}(1 + i)$. Geometrically, the $\rho = 1$ case is the locus of lines which are equidistant from 0 and $1 + i$. Likewise, following a similar argument, for general c, d we have that the locus of points is some diagonal line.

¹Another way of saying this is that it's the line defined by the normal vector $[1, 1]$ and then translated by the vector $\frac{1}{2}[1, 1]$.

The entire discussion above holds for $v \neq 0$ but still $\rho = 1$, Except now the defining horizontal/vertical/diagonal lines are those relative to v , rather than relative to the origin.

Now we consider the $\rho \neq 1$ case. It turns out we can show that $\rho \neq 1$ gives a circle by analyzing the conditions directly. Suppose that we have $\rho \neq 1$ and $v, w \in \mathbb{C}$ arbitrary. There exists an isometry which translates v to the origin, and which rotates w to be on the real axis. And so without loss of generality we can take $v = 0$ and $w \in \mathbb{R}$. I will analyze the case where $w = 1$ to simplify the arithmetic, but the argument for other real w is similar to the following. With these transformations our condition becomes

$$|z|^2 = \rho |z - 1|^2$$

If we let $z = a + bi$ and expand the definitions, we have

$$\begin{aligned} a^2(1 - \rho) + b^2(1 - \rho) &= -2\rho a + \rho \\ \left(a^2 + \frac{2\rho}{1 - \rho}a\right) + b^2 &= \frac{\rho}{1 - \rho} && \rho \neq 1 \\ \left(a + \frac{\rho}{1 - \rho}\right)^2 + b^2 &= \frac{\rho}{1 - \rho} + \frac{\rho^2}{(1 - \rho)^2} && \text{Completing the square} \\ \left(a + \frac{\rho}{1 - \rho}\right)^2 + b^2 &= \frac{\rho}{(1 - \rho)^2}. \end{aligned}$$

This is manifestly a circle of radius $(\sqrt{\rho})/(1 - \rho)$. **Note, I think I may have missed an extra factor of ρ in my starting condition.**

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2. Show that it's impossible to define a *total ordering* on \mathbb{C} , i.e. a relation \succ between complex numbers so that:

- (a) For any $z, w \in \mathbb{C}$, exactly one of the following holds: $z \succ w$, $w \succ z$, or $z = w$.
- (b) For all $z_1, z_2, w \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + w \succ z_2 + w$.
- (c) For all $z_1, z_2, w \in \mathbb{C}$ with $w \succ 0$, the relation $z_1 \succ z_2$ implies $z_1 w \succ z_2 w$.

We show that there is no total ordering on \mathbb{C} by considering the elements $1, i, 0 \in \mathbb{C}$. Suppose there is a total order on \mathbb{C} denoted \prec . Since $1, i, 0$ are distinct, we have four possible cases to consider: $(1 \prec 0, i \prec 0)$, $(1 \prec 0, i \succ 0)$, $(1 \succ 0, i \prec 0)$, or $(1 \succ 0, i \succ 0)$. We show that we arrive at a contradiction in all cases.

$(1 \succ 0, i \succ 0)$: Since $i \succ 0$ we have the following

$$\begin{array}{ll}
 1 \succ 0 \implies i \succ 0 & \text{Property (c)} \\
 \implies -1 \succ 0 & \text{Property (c)} \\
 \implies 0 \succ 1 & \text{Property (b),}
 \end{array}$$

a contradiction.

$(1 \succ 0, i \prec 0)$: Since $i \prec 0$ we have that $0 \prec -i$ by property (b). Then, by property (c) and then (b), we have the following

$$-i \succ 0 \implies -1 \succ 0 \implies 0 \succ 1,$$

a contradiction.

$(1 \prec 0, i \succ 0)$: Using similar reasoning, with $i \succ 0$, applying property (c) a sufficient number of times with $w = i$ implies $1 \succ 0$. A contradiction.

$(1 \prec 0, i \prec 0)$: Using similar reasoning, $i \prec 0$ implies $-i \succ 0$. Then applying property (c) a sufficient number of times with $w = -i$ gives $1 \succ 0$. A contradiction. ■

3. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where} \quad z = re^{i\theta} \quad \text{with} \quad -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Suppose we have a function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z = x + iy) = u(x, y) + iv(x, y)$ where $x, y \in \mathbb{R}$. Then recall the Cauchy-Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Recalling the polar form of a complex number $z = re^{i\theta} = r \cos(\theta) + i \sin(\theta)$ for $r > 0$. We can read the variable transformation $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Being pedantic about our notation (we will abuse at the end), let us write

$$\tilde{u}(r, \theta) := u(r \cos(\theta), r \sin(\theta)) \quad \tilde{v}(r, \theta) := v(r \cos(\theta), r \sin(\theta)).$$

Now, recalling the partial derivative chain rule for multivariable functions we have

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial r} &= \frac{\partial u}{\partial x} \cos(\theta) + \frac{\partial u}{\partial y} \sin(\theta) \\ \frac{\partial \tilde{u}}{\partial \theta} &= \frac{\partial u}{\partial x} (-r \sin(\theta)) + \frac{\partial u}{\partial y} r \cos(\theta) \\ \frac{\partial \tilde{v}}{\partial r} &= \frac{\partial v}{\partial x} \cos(\theta) + \frac{\partial v}{\partial y} r \cos(\theta) \\ \frac{\partial \tilde{v}}{\partial \theta} &= \frac{\partial v}{\partial x} (-r \sin(\theta)) + \frac{\partial v}{\partial y} r \cos(\theta). \end{aligned}$$

Now, using our original Cauchy-Riemann equations, we can derive relations between

these equations. Consider, for example

$$\begin{aligned}\frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} &= \frac{\partial v}{\partial x}(-\sin \theta) + \frac{\partial v}{\partial y}(\cos \theta) && \text{Using the formulae above,} \\ &= \frac{\partial u}{\partial y}(\sin \theta) + \frac{\partial u}{\partial x}(\cos \theta) && \text{Using Cauchy-Riemann,} \\ &= \frac{\partial \tilde{u}}{\partial \tilde{r}}.\end{aligned}$$

Using very similar reasoning will also give us that $(1/r)(\partial \tilde{u}/\partial \theta) = -(\partial \tilde{v}/\partial r)$. And now, we abuse notation by setting $u(r, \theta) = \tilde{u}(r, \theta)$ and $v(r, \theta) = \tilde{v}(r, \theta)$.

Recall that our function $f(z) = u(x, y) + iv(x, y)$ is holomorphic if and only if f is real differentiable (implying all the relevant partial derivatives exist) and u, v satisfy the Cauchy-Riemann equations. We verify that $\log z : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic where it's defined, by checking if it satisfies the Cauchy-Riemann equations.

For $\log z$ we have $u(r, \theta) = \log r$ and $v(r, \theta) = \theta$. Then notice we have

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r} = \frac{1}{r} \cdot 1 = \frac{1}{r} \frac{\partial v}{\partial r} \\ \frac{1}{r} \frac{\partial u}{\partial \theta} &= 0 = -\frac{\partial v}{\partial r},\end{aligned}$$

And so, indeed, the complex logarithm function is holomorphic where it is defined. ■

4. Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(x + iy) = \sqrt{|x||y|}$ for all $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy-Riemann equations at 0, but f is not holomorphic at 0.

Writing $f(x + iy) = u(x, y) + iv(x, y)$ we can read that $u(x, y) = \sqrt{|x||y|}$ and $v(x, y) = 0$.

First we need to verify that the partial derivatives for u, v exist at $(0, 0)$. Since $v = 0$ both its partial derivatives exist and evaluate to zero everywhere. Now consider u . Recalling the definition for partial derivative, consider

$$u_x(0, 0) := \lim_{h \rightarrow 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h||0|} - \sqrt{0}}{h} = 0,$$

And so the partial derivative u_x exists at $(0, 0)$ and evaluates to 0. A similar calculation will show that the partial derivative u_y exists and evaluates to 0 at 0.

Now we can check if u, v satisfy the Cauchy-Riemann equations at $(0, 0)$. Recall that the Cauchy-Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Since all relevant partial derivatives evaluate to 0 at $z = 0$, the Cauchy-Riemann equations are satisfied at $z = 0$. The fact that the Cauchy-Riemann equations are satisfied shows that the difference quotients of our function along the real and imaginary axes about $(0, 0)$ agree.

However, we show that this function is not holomorphic by constructing two sequences of difference quotients approaching $z = 0$ which disagree.

Consider a sequence along the real line which approaches $z = 0$ (such as $\{1/n\}_{n=1}^{\infty}$). Given this sequence of complex numbers we have the difference quotients are all 0:

$$\frac{f(z_n) - f(0)}{z_n} = \frac{\sqrt{|1/n||0|} - 0}{1/n} = 0.$$

And so the limit of the difference quotients, with respect to this sequence, exists and evaluates to 0.

Now, consider a sequence of complex numbers approaching 0 via the ray which passes through 0 and $1 + i$. One such sequence is $\{1/n + i(1/n)\}$. Our difference quotients are

now

$$\frac{f(z_n) - f(0)}{z_n} = \frac{\sqrt{|1/n||1/n|}}{1 + i(1/n)} = \frac{1}{1+i} = \frac{1}{2}(1-i),$$

for all n . It follows that the limit of the difference quotients with respect to this sequence exists and evaluates to $0.5(1-i)$.

We have then shown that the limit

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z}$$

does not exist, since it is not well-defined with respect to choice of limit approach. And so, f is not differentiable at $0 \in \mathbb{C}$ by definition.

I think this argument shows that, whilst all partial derivatives for u, v exist at $(x, y) = (0, 0)$ and the Cauchy-Riemann equations are satisfied, It must be that the given function is not real-differentiable at $z = 0$.

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5. Let $U \subset \mathbb{C}$ open, and $f : U \rightarrow \mathbb{C}$ holomorphic. Show that if $\operatorname{Re}(f)$ is constant, then f is constant.

Let $f : U \rightarrow \mathbb{C}$ be given by $f(x + iy) = u(x, y) + iv(x, y)$ with $u(x, y) = a$ for some $a \in \mathbb{R}$. Then both partial derivatives of u vanish everywhere on U , i.e. $\partial u / \partial x = 0$ and $\partial u / \partial y = 0$. Since f is holomorphic on U we have that f satisfies the Cauchy-Riemann equations on U . And so we have

$$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0,$$

everywhere on U , in other words $\nabla v = 0$. Since f is holomorphic on U we have that $v(x, y)$ is differentiable as a real function everywhere on U . It then follows that $v(x, y) = b$ for some $b \in \mathbb{R}$ as a real function on U (a result from the real-analysis of multivariate functions²). Then, by definition, $f = u + iv$ is a constant function. ■

²I thiiiiink we might also need that U is connected, but my multivariable real analysis is a bit rusty.