Algorithms HW

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2. Suppose f is an entire function such that for any each $z_0 \in \mathbb{C}$, at least one coefficient in the power series expansion

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is zero. Prove that f is a polynomial.

If at some point $z_0 \in \mathbb{C}$ we have some power series coefficient $c_n = 0$ then, recalling that the formula for the power series coefficients, we have

$$f^{(n)}(z_0) = 0,$$

for some n. Let us cover $\mathbb{C}$ with sets where the nth derivative is zero. Let $Z_n := \{z_0 \in \mathbb{C} : f^{(n)} = 0\}$ be the set where the nth derivative of f is zero. Then, the statement in the question can be rewritten as

$$\mathbb{C} = \bigcup_{n=0}^{\infty} Z_n.$$

However, there are only countably many such sets Z_n yet $\mathbb{C}$ is an uncountable set. This means there is at least one $\tilde{n}$ where $Z_{\tilde{n}}$ is uncountable.

We claim that $Z_{\tilde{n}}$ then has a limit point. Consider tiling $\mathbb{C}$ with closed unit squares. One of these squares must contain uncountably many points of $Z_{\tilde{n}}$, otherwise we would have shown that $Z_{\tilde{n}}$ is countable. That is, we have some square, a closed and bounded subset of $\mathbb{C}$, which contains uncountably many points of $Z_{\tilde{n}}$. It follows by Bolzano-Weierstrass that $Z_{\tilde{n}}$ has an accumulation point in that square. But in particular, $Z_{\tilde{n}}$ has an accumulation point.

Now recall that $Z_{\tilde{n}}$ are the points where $f^{(\tilde{n})}$ vanish. We have found a sequence of points in $\mathbb{C}$ converging to some limit point in $\mathbb{C}$ where $f^{(\tilde{n})} = 0$. It follows then by the identity theorem for holomorphic functions that $f^{(\tilde{n})}$ is identically zero. Moreover, it then follows that $f^{(\tilde{n}+k)}$ is also identically zero for all $k \in \mathbb{Z}_+$. In other words we have that the power series coefficients $c_n = 0$ for all $n \geq \tilde{n}$.

Now we can write f as a polynomial by repeatedly writing down the antiderivatives for $f^{(\tilde{n})}$. We can determine the polynomial coefficients by inspecting the power series at

$z = 0$.

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3. Let $U \subset \mathbb{C}$ be open and connected. Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be holomorphic functions. Suppose $\sum_i |f_i(z)|$ is constant on U. Show that each f_i is constant on U.

Hmmm this might be bollocks, in particular, $\sum_i |f_i(z)|$ may not be holomorphic

In this situation the Cauchy integral formulae give us the following

$$\begin{aligned}\frac{d}{dz}(\sum_i |f_i(z)|) &= \frac{1}{2\pi i}\int_\gamma \frac{\sum_i |f_i(z)|}{(z-\xi)^2}d\xi \\ &= \frac{M}{2\pi i}\int_\gamma \frac{1}{(z-\xi)^2}d\xi\end{aligned}$$

Suppose that γ is a circle of radius R centered at z. Then $1/(z-\xi)^2$ has an antiderivative on the punctured domain bounded by γ. By the fundamental theorem of calculus the above integral is then 0. Then we have.

$$\frac{d}{dz}(\sum_i |f_i(z)|) = 0.$$

By linearity of the derivative, this is only true when $\frac{d}{dz}|f_i(z)| = 0$ for each i. And then this implies that each f_i is constant on U. ∎

4. Let $u : \Delta \to \mathbb{R}$ be a real-valued harmonic function on the unit disk. Show that there exists a holomorphic function $f : \Delta \to \mathbb{C}$ with $\mathrm{Re}(f) = u$. To what extent is such an f unique?

We want to construct $f(x+iy) = u(x,y) + iv(x,y)$ which is holomorphic on Δ. Recall that this means we need both u, v to be differentiable as multivariable real functions. Moreover, we need u, v to satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Recall from multivariable calculus that we can take single-variable integrals to partially recover a function from its derivatve by taking indefinite integrals

$$\int \left(\frac{\partial v}{\partial x}\right) dx = v(x,y) + C(x)$$
$$\int \left(\frac{\partial v}{\partial y}\right) dy = v(x,y) + D(y),$$

for any differentiable functions $C, D : \mathbb{R} \to \mathbb{R}$. What are the bounds in this set up?

Here, we are given $u(x,y)$ and so we can paritally determine v by computing

$$v(x,y) + C(x) = \int \left(-\frac{\partial u}{\partial y}\right) dx$$
$$v(x,y) + D(y) = \int \left(\frac{\partial u}{\partial x}\right) dy.$$

However, this gives us a system of two equations with three unknowns. And so we can only ever determine v up to either a function of x or a function of y.

Note that this process of integrating our partial derivatives to reconstruct v works (locally) exactly when $\frac{\partial}{\partial x}(\partial v/\partial y) = \frac{\partial}{\partial y}(\partial v/\partial x)$ (this is tantamount to "partials commuting"). If v is a twice differentiable real function then its partials must commute. The harmonic condition on u satisfies this condition in our case; consider

$$\begin{aligned}
\frac{\partial^2 v}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) && \text{Cauchy-Riemann} \\
&= -\frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial^2 u}{\partial x^2} && \text{Harmonic condition on } u \\
&= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} && \text{Cauchy-Riemann} \\
&= \frac{\partial^2 v}{\partial x \partial y}.
\end{aligned}$$

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