

Algorithms HW

1. Show (without using Cauchy's theorem or any of its consequences) that if $|a| < r < |b|$, then

$$\int_{S_r} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where S_r denotes the circle of radius 1 centered at the origin.

We start by splitting the integrand with partial fractions. Notice that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}.$$

Next we parameterize $S_r = \gamma(t) = r \exp(it)$ with $|a| < r < |b|$. Now, computing our integral directly gives

$$\begin{aligned} \int_{S_r} \frac{1}{(z-a)(z-b)} dz &= \int_{t=0}^{2\pi} f(\gamma(t)) \gamma'(t) dt \\ &= \int \frac{i}{a-b} \left(\frac{r \exp(it)}{r \exp(it) - a} - \frac{r \exp(it)}{r \exp(it) - b} \right) dt \\ &= \frac{i}{a-b} \int_{t=0}^{2\pi} \left(\frac{1}{1 - \frac{a}{r} \exp(-it)} - \frac{1}{1 - \frac{b}{r} \exp(-it)} \right) dt \end{aligned}$$

if we can show that the integrand is 1 then we are done ■

2. Show that

$$\int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

(See Stein-Shakarchi Ch.2 Exercise 1 for hints).

Notice that $\operatorname{Re}(e^{ix^2}) = \cos(x^2)$ for $x \in \mathbb{R}$. To that end we consider integrating $\int_\gamma f(z) dz$ where $f(z) = e^{iz^2}$ over the following contour. **insert contour**

Notice that $f'(z)$ is entire with $f'(z) = 2ize^{iz^2}$, and so also has continuous derivative. By Cauchy's Theorem we have that $\int_\gamma f(z) dz = 0$ for all R . The integral we are interested in is $\lim_{R \rightarrow \infty} \operatorname{Re} \int_A f(z) dz$. To that end, we compute the integral over the other parts of the curve.

Consider the integral along C . We can parameterize this part of the curve as $\gamma(t) = te^{i\pi/4}$ with $t : R \rightarrow 0$. Then consider

$$\begin{aligned} \int_C f(z) dz &= - \int_0^R f(\gamma(t)) \gamma'(t) dt \\ &= - \int_0^R e^{it^2(e^{i\pi/4})^2} e^{i\pi/4} dt \\ &= -e^{i\pi/4} \int_0^R e^{-t^2} dt \end{aligned}$$

Then in the limit $R \rightarrow \infty$,

$$\int_C f(z) dz = -e^{i\pi/4} \frac{\sqrt{\pi}}{2}.$$

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