Complex Analysis Homework #2

> 1. We define the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ via the power series $\sum_{n \geq 0} \frac{z^n}{n!}$. Show that $\exp(z + w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$.

Following the definition, for $z, w \in \mathcal{C}$ we have the power series

$$exp(z + w) = \sum_{n \geq 0} \frac{(z + w)^n}{n!},$$

which expands to

$$exp(z + w) = \sum_{n \geq 0} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} z^m w^{n-m}.$$

Collect the terms containing z^2. We have

$$n = 2 : \qquad \frac{1}{2!}\binom{2}{2}z^2 + \frac{1}{3!}\binom{3}{2}z^2 w + \frac{1}{4!}\binom{4}{2} + \cdots .$$

Factoring out $z^2/2!$ gives

$$n = 2 : \qquad \frac{z^2}{2!}\sum_{m \geq 2}\binom{m}{2}\frac{2!}{m!}w^{m-2} = \frac{z^2}{2!}\sum_{m \geq 2}\frac{m!}{2!(m-2)!}\frac{2!}{m!}w^{m-2} \quad \text{factorial definition of binom}$$

$$= \frac{z^2}{2!}\sum_{m \geq 2}\frac{1}{(m-2)!}w^{m-2}$$

$$= \frac{z^2}{2!}\sum_{m \geq 0}\frac{1}{m!}w^m \qquad\qquad \text{Re-indexing sum}$$

$$= \frac{z^2}{2!}\cdot exp(w).$$

Repeating this argument for the other z terms of degree n gives

$$exp(z + w) = \sum_{n \geq 0}\frac{z^n}{n!}exp(w) = exp(z)exp(w).$$

$\blacksquare$

2. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is infinitely differentiable on $\mathbb{R}$, but f does not have a convergent power series expansion $\sum_n a_n x^n$ near the origin.

We can show that the given function is C^∞ by induction on n in $f^{(n)}(x)$. We claim that

$$f^{(n)}(x) = \begin{cases} 0 & x \leq 0 \\ p(1/x)e^{-1/x^2} & x > 0 \end{cases},$$

where $p(1/x)$ is some polynomial in $1/x$. We show that $f \in C^1$. The only place we need to check differentiability is at the origin. The limit of the difference quotients approaching the origin from the left is 0. From the right we have

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} = 0,$$

where the final limit can be seen by applying the change of variables $t = 1/h^2$ and then using L'Hopital's rule. It follows that the first derivative exists at 0 and evaluates to 0. Then using the chain rule for all other x we have that

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{x^3}e^{-1/x^2} & x > 0 \end{cases}.$$

For the induction step we can use a similar limit argument to check that the nth derivative exists and is equal to 0 at $x = 0$. Moreover, the product rule will give the claimed form of the nth derivative.

We have shown that $f \in C^\infty$. The discussion above also shows that $f^{(n)}(0) = 0$ for all $n \geq 0$. And so, it follows that the power series expansion about the origin is $f(x) \approx 0$. However, f is not identically zero on any neighbourhood of 0, in other words there does

not exist a neighbourhood about 0 where the power series expansion equals the function. Thus f does not have a convergent power series about 0.

I think the takeaway from this question is to see an example of a smooth function $\mathbb{R} \to \mathbb{R}$ which is not analytic, with the intuition being that it goes to zero at the origin slower than any polynomial. I think Ben alluded to the idea that we'll see that holomorphic functions always have a convergent power series over the domain where it is holomorphic. And so I suppose this same thing cannot happen in the complex case.

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3. (a) For each of the power series $\sum_n nz^n$ and $\sum_n z^n/n^2$, find all values $z \in \mathbb{C}$ for which the series converges.

   (b) Find $z_1, z_2 \in \mathbb{C}$ with $|z_i| = 1$ such that the series $\sum z^n/n$ converges and diverges at z_1, z_2, respectively.

(a) Let $f(z) = \sum_{n \geq 0} nz^n$. Recalling the radius of convergence theorem, we have that $1/R = L = \limsup |a_n|^{1/n}$. Consider the following, we want to evaluate

$$L = \lim_{m \to \infty} \sup_{n \geq m} n^{1/n} = \lim_{m \to \infty} m^{1/m},$$

Using a logarithm transformation and L'Hopital's rule we find that

$$\ln L = \lim_{m \to \infty} \frac{\ln m}{m}$$
$$= \lim_{m \to \infty} \frac{1/m}{1}$$
$$= 0.$$

And so $L = e^0 = 1 = 1/R$. Thus the power series f converges absolutely for all $|z| < R = 1$.

Let purple indicate some string of thoughts that I do not feel completely resolved on. In principle, I think we also need to check for all the terms of $|z| = 1$. For $z \in \mathbb{R}$ and $|z| \geq 1$ this sum definitely diverges Write $z = r\exp(i\theta)$ with $r = 1$. Then our power series becomes $f(z) = \sum_{n \geq 0} n\exp(in\theta)$, the exponential term only adding a phase to each term. I think this series then diverges for all $|z| = 1$.

Now suppose $f(z) = \sum_{n \geq 0} z^n/n^2$. Using similar reasoning we want to evaluate

$$L = \lim_{m \to \infty} \sup_{n \geq m} n^{-2/n} = \lim_{m \to \infty} m^{-2/m}.$$

Using the same ln computation as above, we find that

$$\ln L = \lim_{m \to \infty} \left(\frac{-2}{m} \ln m \right) = 0$$
$$L = 1 \implies R = 1.$$

Hence the power series f converges absolutely for all $|z| < 1$. again, should we check what happens at $R = 1$. Perhaps it depends on choice of z?

(b) Let $f(z) = \sum_{n \geq 1} z^n / n$. Notice that $f(1) = \sum_{n \geq 1} 1/n$, this is the harmonic series and so diverges. On the other hand $f(-1) = \sum_{n \geq 1} (-1)^n / n$ is the alternating harmonic series and converges to $-\ln(2)$ (the minus sign being to do with where I stared indexing the sum above). The value of this series can be found by considering the Taylor expansion of $\ln(1 + x)$.

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4. Find the power series expansion of $\log z$ centered at $z = 1$.

Recall that the power series coefficients of an infinitely differentiable function, at a point $z_0 \in \mathcal{C}$, is given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

There exist neighbourhoods of $z = 1$ which do not include the origin and which do not include the branch cut along the negative part of the real axis. And so $\log z$ is differentiable on some open set about $z = 1$ with derivative $1/n$. It follows then that the coefficients of the power series for $\log z$ about $z = 1$ for $n \geq 1$ are given by

$$a_n = (-1)^{n-1}\frac{(n-1)!}{n!}(\frac{1}{z^n})|_{z=1} = (-1)^{n-1}\frac{1}{n}.$$

And then, recalling that $\log 1 = 0$, our power series expansion about $z = 1$ is

$$\log z = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n}(z-1)^n$$

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5. Find a power series $f(z) = \sum_n a_n z^n$ that converges absolutely on the unit disc, but does not extend to a holomorphic map on any larger domain. (That is, if g is holomorphic on an open set U containing the unit disk Δ and $f(z) = g(z)$ on Δ, then $U = \Delta$.)

Apologies, I didn't make a huge amount of headway into this problem. Two questions prior we found a few power series whose radii of convergence was $R = 1$ (i.e. power series which converge absolutely on the unit disc). However, I was playing a bit instead with the geometric series $f(z) = \sum_{n \geq 0} z^n$. However, I had some trouble understanding how to formualte the constraints of having a function which is equal to the geometric series on the unit disc, and which is holomorphic. I'll probs come chat about this one at office hours.

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6. Find all values $z \in \mathbb{C}$ at which $\exp : \mathbb{C} \to \mathbb{C}$ is conformal.

Recall that a complex function $f : \mathcal{C} \to \mathcal{C}$ is conformal at $z \in \mathcal{C}$ if and only if it is holomorphic at z and $f'(z) \neq 0$. For $f = \exp$ we have that f is entire with $f'(z) = \exp(z)$. Moreover, $f'(a + bi) = \exp(a + bi) = \exp(a)\exp(ib)$. In particular, $|f'(z)| = |\exp(a)| > 0$ for $a \in \mathbb{R}$, the inequality being strict. That is, $f'(z) \neq 0$ for all $z \in \mathcal{C}$ and so f is conformal for all $z \in \mathcal{C}$.

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