

## Complex Analysis Homework #1

1. Let  $v, w$  be distinct points in  $\mathbb{C}$ , and  $\rho > 0$ . Show that the locus of points  $z \in \mathbb{C}$  that satisfy

$$|z - v| = \rho|z - w|$$

is a circle or a line.

Let us consider some special cases. Let us first take one of our fixed points to be 0, say  $v = 0$ . Let us also study the case where  $\rho = 1$  to begin. Our defining equation becomes  $|z| = |z - w|$ , notice that  $z$  satisfies this system if and only if it satisfies  $|z|^2 = |z - w|^2$ . Let us write  $z = a + bi$  and  $w = c + di$  then

$$|z| = |z - w| \iff c^2 + d^2 - 2ac - 2bd = 0.$$

Now let us consider some cases for  $w$ . If  $w$  is real and non-zero then  $w = c$  and we can solve for  $a = \frac{1}{2}c$ .  $b$  is free and so the locus of points which solve this case is  $z = \frac{1}{2}c + bi$  for any fixed  $c \in \mathbb{R}$  and for all  $b \in \mathbb{R}$ . This is exactly a vertical line of points. In particular, it is the vertical line of points through the origin, translated by  $\frac{1}{2}c$ .

Performing an extremely similar calculation, if we consider the case where  $w$  is purely imaginary,  $w = di$  for some  $d \in \mathbb{R}$ , then our locus of points will be  $z = a + \frac{1}{2}d$  for some for all  $a \in \mathbb{R}$ . These are horizontal lines. In particular, these are the horizontal line of points through the origin, translated by  $\frac{1}{2}d$

In the case where  $w = c + di$  with  $c, d$  non-zero. We are now looking for points which have the same distance from the origin after being translated by  $-(c + di)$ . If  $c = d = 1$  then our locus of points is a diagonal line. In particular, it is the diagonal line passing through the origin and  $-(1 + i)$ , translated<sup>1</sup> by  $\frac{1}{2}(1 + i)$ . Geometrically, the  $\rho = 1$  case is the locus of lines which are equidistant from 0 and  $1 + i$ . Likewise, following a similar argument, for general  $c, d$  we have that the locus of points is some diagonal line.

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<sup>1</sup>Another way of saying this is that it's the line defined by the normal vector  $[1, 1]$  and then translated by the vector  $\frac{1}{2}[1, 1]$ .

The entire discussion above holds for  $v \neq 0$  but still  $\rho = 1$ . Except now the defining horizontal/vertical/diagonal lines are those relative to  $v$ , rather than relative to the origin.

Now we consider the  $\rho \neq 1$  case. It turns out we can show that  $\rho \neq 1$  gives a circle by analyzing the conditions directly. Suppose that we have  $\rho \neq 1$  and  $v, w \in \mathbb{C}$  arbitrary. There exists an isometry which translates  $v$  to the origin, and which rotates  $w$  to be on the real axis. And so without loss of generality we can take  $v = 0$  and  $w \in \mathbb{R}$ . I will analyze the case where  $w = 1$  to simplify the arithmetic, but the argument for other real  $w$  is similar to the following. With these transformations our condition becomes

$$|z|^2 = \rho |z - 1|^2$$

If we let  $z = a + bi$  and expand the definitions, we have

$$\begin{aligned} a^2(1 - \rho) + b^2(1 - \rho) &= -2\rho a + \rho \\ \left(a^2 + \frac{2\rho}{1 - \rho}a\right) + b^2 &= \frac{\rho}{1 - \rho} \quad \rho \neq 1 \\ \left(a + \frac{\rho}{1 - \rho}\right)^2 + b^2 &= \frac{\rho}{1 - \rho} + \frac{\rho^2}{(1 - \rho)^2} \quad \text{Completing the square} \\ \left(a + \frac{\rho}{1 - \rho}\right)^2 + b^2 &= \frac{\rho}{(1 - \rho)^2}. \end{aligned}$$

This is manifestly a circle of radius  $(\sqrt{\rho})/(1 - \rho)$ . Note, I think I may have missed an extra factor of  $\rho$  in my starting condition.

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2. Show that it's impossible to define a *total ordering* on  $\mathbb{C}$ , i.e. a relation  $\succ$  between complex numbers so that:
- For any  $z, w \in \mathbb{C}$ , exactly one of the following holds:  $z \succ w$ ,  $w \succ z$ , or  $z = w$ .
  - For all  $z_1, z_2, w \in \mathbb{C}$  the relation  $z_1 \succ z_2$  implies  $z_1 + w \succ z_2 + w$ .
  - For all  $z_1, z_2, w \in \mathbb{C}$  with  $w \succ 0$ , the relation  $z_1 \succ z_2$  implies  $z_1w \succ z_2w$ .

We show that there is no total ordering on  $\mathbb{C}$  by considering the elements  $1, i, 0 \in \mathbb{C}$ . Suppose there is a total order on  $\mathbb{C}$  denoted  $\prec$ . Since  $1, i, 0$  are distinct, we have four possible cases to consider:  $(1 \prec 0, i \prec 0)$ ,  $(1 \prec 0, i \succ 0)$ ,  $(1 \succ 0, i \prec 0)$ , or  $(1 \succ 0, i \succ 0)$ . We show that we arrive at a contradiction in all cases.

$(1 \succ 0, i \succ 0)$  : Since  $i \succ 0$  we have the following

$$\begin{aligned} 1 \succ 0 &\implies i \succ 0 && \text{Property (c)} \\ &\implies -1 \succ 0 && \text{Property (c)} \\ &\implies 0 \succ 1 && \text{Property (b),} \end{aligned}$$

a contradiction.

$(1 \succ 0, i \prec 0)$  : Since  $i \prec 0$  we have that  $0 \prec -i$  by property (b). Then, by property (c) and then (b), we have the following

$$-i \succ 0 \implies -1 \succ 0 \implies 0 \succ 1,$$

a contradiction.

$(1 \prec 0, i \succ 0)$  : Using similar reasoning, with  $i \succ 0$ , applying property (c) a sufficient number of times with  $w = i$  implies  $1 \succ 0$ . A contradiction.

$(1 \prec 0, i \prec 0)$  : Using similar reasoning,  $i \prec 0$  implies  $-i \succ 0$ . Then applying property (c) a sufficient number of times with  $w = -i$  gives  $1 \succ 0$ . A contradiction. ■

3. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region  $r > 0$  and  $-\pi < \theta < \pi$ .

Suppose we have a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $f(z = x + iy) = u(x, y) + iv(x, y)$  where  $x, y \in \mathbb{R}$ . Then recall the Cauchy-Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Recalling the polar form of a complex number  $z = re^{i\theta} = r \cos(\theta) + i \sin(\theta)$  for  $r > 0$ . We can read the variable transformation  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ . Being pedantic about our notation (we will abuse at the end), let us write

$$\tilde{u}(r, \theta) := u(r \cos(\theta), r \sin(\theta)) \quad \tilde{v}(r, \theta) := v(r \cos(\theta), r \sin(\theta)).$$

Now, recalling the partial derivative chain rule for multivariable functions we have

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial r} &= \frac{\partial u}{\partial x} \cos(\theta) + \frac{\partial u}{\partial y} \sin(\theta) \\ \frac{\partial \tilde{u}}{\partial \theta} &= \frac{\partial u}{\partial x} (-r \sin(\theta)) + \frac{\partial u}{\partial y} r \cos(\theta) \\ \frac{\partial \tilde{v}}{\partial r} &= \frac{\partial v}{\partial x} \cos(\theta) + \frac{\partial v}{\partial y} r \cos(\theta) \\ \frac{\partial \tilde{v}}{\partial \theta} &= \frac{\partial v}{\partial x} (-r \sin(\theta)) + \frac{\partial v}{\partial y} r \cos(\theta). \end{aligned}$$

Now, using our original Cauchy-Riemann equations, we can derive relations between

these equations. Consider, for example

$$\begin{aligned} \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} &= \frac{\partial v}{\partial x}(-\sin \theta) + \frac{\partial v}{\partial y}(\cos \theta) && \text{Using the formulae above,} \\ &= \frac{\partial u}{\partial y}(\sin \theta) + \frac{\partial u}{\partial x}(\cos \theta) && \text{Using Cauchy-Riemann,} \\ &= \frac{\partial \tilde{u}}{\partial \tilde{r}}. \end{aligned}$$

Using very similar reasoning will also give us that  $(1/r)(\partial \tilde{u}/\partial \theta) = -(\partial \tilde{v}/\partial r)$ . And now, we abuse notation by setting  $u(r, \theta) = \tilde{u}(r, \theta)$  and  $v(r, \theta) = \tilde{v}(r, \theta)$ .

Recall that our function  $f(z) = u(x, y) + iv(x, y)$  is holomorphic if and only if  $f$  is real differentiable (implying all the relevant partial derivatives exist) and  $u, v$  satisfy the Cauchy-Riemann equations. We verify that  $\log z : \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic where it's defined, by checking if it satisfies the Cauchy-Riemann equations.

For  $\log z$  we have  $u(r, \theta) = \log r$  and  $v(r, \theta) = \theta$ . Then notice we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} = \frac{1}{r} \cdot 1 = \frac{1}{r} \frac{\partial v}{\partial r} \\ \frac{1}{r} \frac{\partial u}{\partial \theta} &= 0 = -\frac{\partial v}{\partial r}, \end{aligned}$$

And so, indeed, the complex logarithm function is holomorphic where it is defined. ■

4. Consider the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(x+iy) = \sqrt{|x||y|}$  for all  $x, y \in \mathbb{R}$ . Show that  $f$  satisfies the Cauchy-Riemann equations at 0, but  $f$  is not holomorphic at 0.

Writing  $f(x+iy) = u(x,y) + iv(x,y)$  we can read that  $u(x,y) = \sqrt{|x||y|}$  and  $v(x,y) = 0$ .

First we need to verify that the partial derivatives for  $u, v$  exist at  $(0,0)$ . Since  $v = 0$  both its partial derivatives exist and evaluate to zero everywhere. Now consider  $u$ . Recalling the definition for partial derivative, consider

$$u_x(0,0) := \lim_{h \rightarrow 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h||0|} - \sqrt{0}}{h} = 0,$$

And so the partial derivative  $u_x$  exists at  $(0,0)$  and evaluates to 0. A similar calculation will show that the partial derivative  $u_y$  exists and evaluates to 0 at 0.

Now we can check if  $u, v$  satisfy the Cauchy-Riemann equations at  $(0,0)$ . Recall that the Cauchy-Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Since all relevant partial derivatives evaluate to 0 at  $z = 0$ , the Cauchy-Riemann equations are satisfied at  $z = 0$ . The fact that the Cauchy-Riemann equations are satisfied shows that the difference quotients of our function along the real and imaginary axes about  $(0,0)$  agree.

However, we show that this function is not holomorphic by constructing two sequences of difference quotients approaching  $z = 0$  which disagree.

Consider a sequence along the real line which approaches  $z = 0$  (such as  $\{1/n\}_{n=1}^{\infty}$ ). Given this sequence of complex numbers we have the difference quotients are all 0:

$$\frac{f(z_n) - f(0)}{z} = \frac{\sqrt{|1/n||0|} - 0}{1/n} = 0.$$

And so the limit of the difference quotients, with respect to this sequence, exists and evaluates to 0.

Now, consider a sequence of complex numbers approaching 0 via the ray which passes through 0 and  $1+i$ . One such sequence is  $\{1/n + i(1/n)\}$ . Our difference quotients are

now

$$\frac{f(z_n) - f(0)}{z_n} = \frac{\sqrt{|1/n| |1/n|}}{1 + i(1/n)} = \frac{1}{1 + i} = \frac{1}{2}(1 - i),$$

for all  $n$ . It follows that the limit of the difference quotients with respect to this sequence exists and evaluates to  $0.5(1 - i)$ .

We have then shown that the limit

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z}$$

does not exist, since it is not well-defined with respect to choice of limit approach. And so,  $f$  is not differentiable at  $0 \in \mathbb{C}$  by definition.

I think this argument shows that, whilst all partial derivatives for  $u, v$  exist at  $(x, y) = (0, 0)$  and the Cauchy-Riemann equations are satisfied, it must be that the given function is not real-differentiable at  $z = 0$ .

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5. Let  $U \subset \mathbb{C}$  open, and  $f : U \rightarrow \mathbb{C}$  holomorphic. Show that if  $\operatorname{Re}(f)$  is constant, then  $f$  is constant.

Let  $f : U \rightarrow \mathbb{C}$  be given by  $f(x + iy) = u(x, y) + iv(x, y)$  with  $u(x, y) = a$  for some  $a \in \mathbb{R}$ . Then both partial derivatives of  $u$  vanish everywhere on  $U$ , i.e.  $\partial u / \partial x = 0$  and  $\partial u / \partial y = 0$ . Since  $f$  is holomorphic on  $U$  we have that  $f$  satisfies the Cauchy-Riemann equations on  $U$ . And so we have

$$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0,$$

everywhere on  $U$ , in other words  $\nabla v = 0$ . Since  $f$  is holomorphic on  $U$  we have that  $v(x, y)$  is differentiable as a real function everywhere on  $U$ . It then follows that  $v(x, y) = b$  for some  $b \in \mathbb{R}$  as a real function on  $U$  (a result from the real-analysis of multivariate functions<sup>2</sup>). Then, by definition,  $f = u + iv$  is a constant function. ■

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<sup>2</sup>I thiiiiink we might also need that  $U$  is connected, but my multivariable real analysis is a bit rusty.