

Algorithms HW

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2. Suppose f is an entire function such that for any each $z_0 \in \mathbb{C}$, at least one coefficient in the power series expansion

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is zero. Prove that f is a polynomial.

If at some point $z_0 \in \mathbb{C}$ we have some power series coefficient $c_n = 0$ then, recalling that the formula for the power series coefficients, we have

$$f^{(n)}(z_0) = 0,$$

for some n . Let us cover \mathbb{C} with sets where the n th derivative is zero. Let $Z_n := \{z_0 \in \mathbb{C} : f^{(n)} = 0\}$ be the set where the n th derivative of f is zero. Then, the statement in the question can be rewritten as

$$\mathbb{C} = \bigcup_{n=0}^{\infty} Z_n.$$

However, there are only countably many such sets Z_n yet \mathbb{C} is an uncountable set. This means there is at least one \tilde{n} where $Z_{\tilde{n}}$ is uncountable.

We claim that $Z_{\tilde{n}}$ then has a limit point. Consider tiling \mathbb{C} with closed unit squares. One of these squares must contain uncountably many points of $Z_{\tilde{n}}$, otherwise we would have shown that $Z_{\tilde{n}}$ is countable. That is, we have some square, a closed and bounded subset of \mathbb{C} , which contains uncountably many points of $Z_{\tilde{n}}$. It follows by Bolzano-Weierstrass that $Z_{\tilde{n}}$ has an accumulation point in that square. But in particular, $Z_{\tilde{n}}$ has an accumulation point.

Now recall that $Z_{\tilde{n}}$ are the points where $f^{(\tilde{n})}$ vanish. We have found a sequence of points in \mathbb{C} converging to some limit point in \mathbb{C} where $f^{(\tilde{n})} = 0$. It follows then by the identity theorem for holomorphic functions that $f^{(\tilde{n})}$ is identically zero. Moreover, it then follows that $f^{(\tilde{n}+k)}$ is also identically zero for all $k \in \mathbb{Z}_+$. In other words we have that the power series coefficients $c_n = 0$ for all $n \geq \tilde{n}$.

Now we can write f as a polynomial by repeatedly writing down the antiderivatives for $f^{(\tilde{n})}$. We can determine the polynomial coefficients by inspecting the power series at

$z = 0$.

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3. Let $U \subset \mathbb{C}$ be open and connected. Let $f_1, \dots, f_n : U \rightarrow \mathbb{C}$ be holomorphic functions. Suppose $\sum_i |f_i(z)|$ is constant on U . Show that each f_i is constant on U .

We have that $\sum_i |f_i(z)| = M$ for some constant $M \in \mathbb{C}$. In particular, we must have that each $|f_i(z)| \leq M$ for all $z \in U$; as a sum of non-negative real numbers, if any $|f_i(z)|$ was unbounded on U then the sum $\sum_i |f_i(z)|$ would be unbounded on U .

That is, for each of the holomorphic functions f_i we have $|f_i(z)|$ is bounded on U . But then by the maximum modulus principle we have that each f_i is constant on U .

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4. Let $u : \Delta \rightarrow \mathbb{R}$ be a real-valued harmonic function on the unit disk. Show that there exists a holomorphic function $f : \Delta \rightarrow \mathbb{C}$ with $\operatorname{Re}(f) = u$. To what extent is such an f unique?

We want to construct $f(x + iy) = u(x, y) + iv(x, y)$ which is holomorphic on Δ . Recall that this means we need both u, v to be differentiable as multivariable real functions. Moreover, we need u, v to satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Recall from multivariable calculus that we can take single-variable integrals to partially recover a function from its derivative by taking indefinite integrals

$$\begin{aligned} \int \left(\frac{\partial v}{\partial x} \right) dx &= v(x, y) + C(x) \\ \int \left(\frac{\partial v}{\partial y} \right) dy &= v(x, y) + D(y), \end{aligned}$$

for any differentiable functions $C, D : \mathbb{R} \rightarrow \mathbb{R}$. **What are the bounds in this set up?**

Here, we are given $u(x, y)$ and so we can partially determine v by computing

$$\begin{aligned} v(x, y) + C(x) &= \int \left(-\frac{\partial u}{\partial y} \right) dx \\ v(x, y) + D(y) &= \int \left(\frac{\partial u}{\partial x} \right) dy. \end{aligned}$$

However, this gives us a system of two equations with three unknowns. And so we can only ever determine v up to either a function of x or a function of y .

Note that this process of integrating our partial derivatives to reconstruct v works (locally) exactly when $\frac{\partial}{\partial x}(\partial v / \partial y) = \frac{\partial}{\partial y}(\partial v / \partial x)$ (this is tantamount to “partials commuting”). If v is a twice differentiable real function then its partials must commute. The harmonic

condition on u satisfies this condition in our case; consider

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right) \quad \text{Cauchy-Riemann}$$

$$= -\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2}$$

Harmonic condition on u

$$= \frac{\partial}{\partial x} \frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial x} \frac{\partial v}{\partial y}$$

Cauchy-Riemann

$$= \frac{\partial^2 v}{\partial x \partial y}.$$

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