Algorithms HW

1. Show (without using Cauchy's theorem or any of its consequences) that if $|a| < r < |b|$, then

$$\int_{S_r} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where S_r denotes the circle of radius 1 centered at the origin.

We start by splitting the integrand with partial fractions. Notice that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}.$$

Next we paramaterize $S_r = \gamma(t) = r\exp(it)$ with $|a| < r < |b|$. Now, computing our integral directly gives

$$\begin{aligned}
\int_{S_r} \frac{1}{(z-a)(z-b)} dz &= \int_{t=0}^{2\pi} f(\gamma(t))\gamma'(t) dt \\
&= \int \frac{i}{a-b} \left(\frac{r\exp(it)}{r\exp(it) - a} - \frac{r\exp(it)}{r\exp(it) - b} \right) dt \\
&= \frac{i}{a-b} \int_{t=0}^{2\pi} \left(\frac{1}{1 - \frac{a}{r}\exp(-it)} - \frac{1}{1 - \frac{b}{r}\exp(-it)} \right) dt
\end{aligned}$$

if we can show that the integrand is 1 then we are done ∎

2. Show that
$$\int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$
(See Stein-Shakarchi Ch.2 Exercise 1 for hints).

Notice that $Re(e^{ix^2}) = \cos(x^2)$ for $x \in \mathbb{R}$. To that end we consider integrating $\int_\gamma f(z)dz$ where $f(z) = e^{-z^2}$ over the following contour. insert contour

We integrate along part A of the curve above. We parameterize this part of the curve at $\gamma(t) = t \in [0, R]$. Then we have

$$\begin{aligned}
\lim_{R\to\infty} \int_A f(z)dt &= \lim_{R\to\infty} \int_{t=0}^{R} f(\gamma(t))\gamma'(t)dt \\
&= \lim_{R\to\infty} \int_{t=0}^{R} e^{-t^2}dt \\
&= \int_{t=0}^{\infty} e^{-t^2}dt \\
&= \frac{\sqrt{\pi}}{2},
\end{aligned}$$

as given in the book.

How do we show that the integral over part B is zero?

Notice that f is entire with continuous derivative. By Cauchy's theorem we have

$$-\int_C f(z)dz = \int_A f(z)dz = \sqrt{\pi}/2.$$

Consider the integral along C. We paramaterize this part of the curve as $\gamma(t) = te^{i\pi/4}$

with $t : R \to 0$. Now consider

$$
\begin{aligned}
-\int_C f(z)dz &= -\int_{t=R}^{0} f(\gamma(t))\gamma'(t)dt \\
&= e^{i\pi/4} \int_{t=0}^{R} e^{(te^{i\pi/4})^2} dt \\
&= e^{i\pi/4} \int_{t=0}^{R} e^{(\frac{t}{\sqrt{2}}(1+i))^2} dt \\
&= e^{i\pi/4} \int_{t=0}^{R} e^{-it^2} dt \\
&= e^{i\pi/4} \int_{t=0}^{R} \cos(-t^2) + i\sin(-t^2)dt \\
&= e^{i\pi/4} \int_{t=0}^{R} \cos(t^2) - i\sin(-t^2)dt \\
&= \frac{1}{\sqrt{2}}(1+i) \int_{t=0}^{R} \cos(t^2) - i\sin(-t^2)dt \\
&= \frac{1}{\sqrt{2}} \left[\int_0^R \cos(t^2) + \sin(t^2)dt + i\left(\int_0^R \cos(t^2) - \sin(t^2)dt \right) \right]
\end{aligned}
$$

Now, taking the limit $R \to \infty$ and recalling Cauchy's formula gives

$$
\int_0^\infty \cos(t^2) + \sin(t^2)dt + i\left(\int_0^\infty \cos(t^2) - \sin(t^2)dt \right) = \frac{\sqrt{2\pi}}{2}
$$

Comparing the real and imaginary part of this equation gives $\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt$ and then gives $\int_0^\infty \cos(t^2)dt = (\sqrt{2\pi})/4$

■

3. Suppose $U \subset \mathbb{C}$ open, and let $f_1, f_2, \ldots : U \to \mathbb{C}$ be continuous functions. Let $\gamma \subset U$ be a piecewise smooth curve, and suppose the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on γ to some function f. Show that the series can be integrated term by term:

$$\int_\gamma f(z)dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)dz.$$

Let us write $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy$. Consider $\sum_{n=1}^{\infty} f_n(z) = \sum_{n=1}^{\infty} u_n(x,y) + iv(x,y)$ converges uniformly to $f(z)$ means that $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge uniformly to u and v respectively.

Now recall that we can decompose our integral into linear combination of integrals over multivariable real functions, and consider the following

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy) \\
&= \int_\gamma \sum_{n\geq 1} u_n dx - \int_\gamma \sum_{n\geq 1} v_n dy + i\int_\gamma \sum_{n\geq 1} v_n dx + i\int_\gamma \sum_{n\geq 1} u_n dy \\
&= \sum_{n\geq 1} \int_\gamma u_n dx - \sum_{n\geq 1} \int_\gamma v_n dy + i\sum_{n\geq 1} \int_\gamma v_n dx + i\sum_{n\geq 1} \int_\gamma u_n dy \\
&= \sum_{n\geq 1} \left(\int_\gamma (u_n dx - v_n dy) + i\int_\gamma (v_n dx + udy) \right) \\
&= \sum_{n\geq 1} \int_\gamma f_n(z)dz,
\end{aligned}$$

where the third equality holds since each of the relevent series of real functions converge uniformally, as discussed above[1]. ■

[1]There may be some other measure theoretic conditions on the curve, perhaps that the curve is finite measure.

4. Let $U \subset \mathbb{C}$ be open, and let R be a rectangle, such that R and its interior are contained in U. Let z_0 be a point in the interior of U. Suppose that $f : U - z_0 \to \mathbb{C}$ is holomorphic and *bounded*. Show that

$$\int_R f(z)dz = 0.$$

In this solution, I will let $\bar{R}$ denote the rectangle and its interior. We split up our solution into cases: $z_0 \in U \setminus \bar{R}$, $z_0 \in R$, $z_0 \in \text{Int } R$. When $z_0 \in U \setminus \bar{R}$ we can find an open set $V \subseteq U$ such that $R \subseteq V$ but $z_0 \notin V$. Since $V \subseteq U \setminus z_0$ we have that f is holomorphic on V. Then, by Goursat's lemma we have that $\int_R f(z)dz = 0$. We cannot have the case where $z_0 \in R$ since for $\int_R f(z)dz$ to be well-defined, we need $f(z)$ to be defined for all $z \in R$. But this is not the case if we have $z_0 \in R$.

Lastly, consider $z_0 \in \text{Int } R$. Suppose $D(z_0, \varepsilon)$ denotes the closed ball about z_0 with radius ε. Then let us define a new region $R' = \text{Int } R \setminus D(z_0, \varepsilon)$. Notice that R' is an open set in $\mathbb{C}$ where f is holomorphic and bounded. Moreover, $\partial R'$ has two components $\partial R' = R \sqcup S_{\varepsilon,z_0}$. Orient R counter clockwise and orient S_{ε,z_0} clockwise. Then there's a version of Cauchy's Theorem (that's a bit more general than the specific one we proved in class, but which Ben used in another proof perhaps I should discuss this at the end) which gives us that

$$\int_R f(z)dz = \int_{S_{\varepsilon,z_0}} f(z)dz.$$

Now consider the integral around the small circle centered at z_0. We show this integral is zero by bounding it. Since f is bounded we have that $|f(z)| \leq M$ for some $M \in \mathbb{C}$ for all $z \in \mathbb{C}$. Now, consider

$$\begin{aligned}
\int_{S_{\varepsilon,z_0}} f(z)dz &\leq \sup_{z \in S_{\varepsilon,z_0}} |f(z)| \cdot 2\pi\varepsilon \\
&= \leq M \cdot 2\pi\varepsilon && \text{Since } f \text{ is bounded,} \\
&\to 0 && \text{as } \varepsilon \to 0.
\end{aligned}$$

This then gives the desired result $\int_R f(z)dz = 0$.

Perhaps come back and discuss the general version of Cauchy's theorem.

■

5. The Weierstrass approximation theorem states that every continuous function on the closed interval $[0, 1]$ can be uniformly approximated arbitrarily well by polynomials. Can every continuous function on the closed unit disk in $\mathbb{C}$ be approximated arbitrarily well by polynomials in z?

Consider the function $f(z) = \bar{z}$ on the unit disc Δ. We saw some time ago that f is not a holomorphic function since it does not satisfy the Cauchy-Riemann equations. Indeed, writing $f(x + iy) = x - iy$ we see that $u(x, y) = x$ and $v(x, y) = -y$ and then neither $\partial u/\partial x = \partial v/\partial y$ nor $\partial u/\partial y = -\partial v/\partial x$ are satisfied for any $z \in \mathbb{C}$. In particular, there is no point in $\mathbb{C}$ where f is holomorphic.

However, f is continuous on Δ (in fact, on all of $\mathbb{C}$). Consider the following limit for some $z_0 \in \mathbb{C}$

$$\begin{aligned}
\lim_{z \to z_0} f(z) &= \lim_{x+iy \to x_0+iy_0} (x - iy) \\
&= \lim_{x \to x_0} x - i \lim_{y \to y_0} y && \text{using linearity of limits,} \\
&= x_0 - iy_0 \\
&= f(z_0).
\end{aligned}$$

And so $f(z) = \bar{z}$ is continuous by definition.

I claim now that f cannot be approximated arbitrarily well by polynomials on Δ. A function $g : \mathbb{C} \to \mathbb{C}$ being approximated arbitrarily well on Δ by polynomials means (I think) that there is some neighbourhood about each point $z_0 \in U \subseteq \Delta$ such that f has a convergent power series. (Perhaps there are other interpretations, I will discuss this more at the end) Now, for any $z_0 \in \mathbb{C}$, if f had a convergent power series about z_0 then we would have found an open neighbourhood where f is holomorphic. This is a contradiction since, as argued above, f does not satisfy the Cauchy-Riemann equations for any point in $\mathbb{C}$.

We have shown in class that if a function is holomorphic on some domain then it is analytic on that same domain. And conversely, any power series is holomorphic on its domain of convergence. However, there still exist complex functions which are continuous but not holomorphic.

I will note, perhaps another way of interpreting "uniformally approximated arbitrarily well by polynomials" might mean something more like: for a given point z_0 there's a sequence of polynomials p_n such that $|f(z_0) - p_n(z_0)| \to 0$ as $n \to \infty$ where p_n can be defined just on some neighbourhood of z_0. This might be the same thing as f having a convergent power series at the point z_0, but it's not entirely clear to me.

$\blacksquare$