Algorithms HW

> 1. We define the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ via the power series $\sum_{n\geq 0} \frac{z^n}{n!}$. Show that $\exp(z+w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$.

Following the definition, for $z, w \in \mathcal{C}$ we have the power series

$$exp(z+w) = \sum_{n\geq 0} \frac{(z+w)^n}{n!},$$

which expands to

$$exp(z+w) = \sum_{n\geq 0} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} z^m w^{n-m}.$$

Collect the terms containing z^2. We have

$$n = 2: \qquad \frac{1}{2!}\binom{2}{2} z^2 + \frac{1}{3!}\binom{3}{2} z^2 w + \frac{1}{4!}\binom{4}{2} + \cdots .$$

Factoring out $z^2/2!$ gives

$$\begin{aligned} n = 2: \qquad \frac{z^2}{2!} \sum_{m\geq 2} \binom{m}{2} \frac{2!}{m!} w^{m-2} &= \frac{z^2}{2!} \sum_{m\geq 2} \frac{m!}{2!(m-2)!} \frac{2!}{m!} w^{m-2} && \text{factorial definition of binom} \\ &= \frac{z^2}{2!} \sum_{m\geq 2} \frac{1}{(m-2)!} w^{m-2} \\ &= \frac{z^2}{2!} \sum_{m\geq 0} \frac{1}{m!} w^{m} && \text{Re-indexing sum} \\ &= \frac{z^2}{2!} \cdot exp(w). \end{aligned}$$

Repeating this argument for the other z terms of degree n gives

$$exp(z+w) = \sum_{n\geq 0} \frac{z^n}{n!} exp(w) = exp(z)exp(w).$$

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2. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is infinitely differentiable on $\mathbb{R}$, but f does not have a convergent power series expansion $\sum_n a_n x^n$ near the origin.

We can show that the given function is C^∞ by induction on n in $f^{(n)}(x)$. We claim that

$$f^(n)(x) = \begin{cases} 0 & x \leq 0 \\ p(1/x)e^{-1/x^2} & x > 0 \end{cases},$$

where $p(1/x)$ is some polynomial in $1/x$. We show that $f \in C^1$. The only place we need to check differentiability is at the origin. The limit of the difference quotients approaching the origin from the left is 0. From the right we have

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} = 0,$$

where the final limit can be seen by applying the change of variables $t = 1/h^2$ and then using L'Hopital's rule. It follows that the first derivative exists at 0 and evaluates to 0. Then using the chain rule for all other x we have that

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{x^3}e^{-1/x^2} & x > 0 \end{cases}.$$

Now suppose that fill in the rest of the argument here

We have shown that $f \in C^\infty$. The discussion above also shows that $f^{(n)}(x)(0) = 0$ for all $n \geq 0$. And so, it follows that the power series expansion about the origin is $f(x) \approx 0$. However, f is not identically zero on any neighbourhood of 0 and so there does not exist a neighbourhood about 0 where the power series expansion equals the function. Thus f does not have a convergent power series about 0.

I think the takeaway from this question is to see a smooth function $\mathbb{R} \to \mathbb{R}$ which is not analytic, with the intuition being that it goes to zero at the origin slower than any polynomial. I think Ben alluded to the idea that we'll see that holomorphic functions always have a convergent power series over the domain where it is holomorphic. And so I suppose this same thing cannot happen in the complex case.

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3. (a) For each of the power series $\sum_n nz^n$ and $\sum_n z^n/n^2$, find all values $z \in \mathbb{C}$ for which the series converges.

   (b) Find $z_1, z_2 \in \mathbb{C}$ with $|z_i| = 1$ such that the series $\sum z^n/n$ converges and diverges at z_1, z_2, respectively.

(a) Let $f(z) = \sum_{n\geq 0} nz^n$. Recalling the radius of convergence theorem, we have that $1/R = L = \limsup |a_n|^{1/n}$. Consider the following, we want to evaluate

$$\lim_{m\to\infty} \sup_{n\geq m} n^{1/n} = \lim_{m\to\infty} m^{1/m},$$

Using a logarithm transformation and L'Hopital's rule we find that

$$\begin{aligned}\ln L &= \lim_{m\to\infty} \frac{\ln m}{m} \\ &= \lim_{m\to\infty} \frac{1/m}{1} \\ &= 0.\end{aligned}$$

And so $L = e^0 = 1 = 1/R$. Thus the power series f converges absolutely for all $|z| < R = 1$.

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