Algorithms HW

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1. Show (without using Cauchy's theorem or any of its consequences) that if $|a| < r < |b|$, then

$$\int_{S_r} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where S_r denotes the circle of radius 1 centered at the origin.

We start by splitting the integrand with partial fractions. Notice that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}.$$

Next we paramaterize $S_r = \gamma(t) = r\exp(it)$ with $|a| < r < |b|$. Now, computing our integral directly gives

$$\int_{S_r} \frac{1}{(z-a)(z-b)} dz = \int_{t=0}^{2\pi} f(\gamma(t))\gamma'(t)dt$$

$$= \int \frac{i}{a-b} \left(\frac{r\exp(it)}{r\exp(it)-a} - \frac{r\exp(it)}{r\exp(it)-b} \right) dt$$

$$= \frac{i}{a-b} \int_{t=0}^{2\pi} \left(\frac{1}{1 - \frac{a}{r}\exp(-it)} - \frac{1}{1 - \frac{b}{r}\exp(-it)} \right) dt$$

if we can show that the integrand is 1 then we are done ∎

2. Show that
$$\int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

(See Stein-Shakarchi Ch.2 Exercise 1 for hints).

Notice that $Re(e^{ix^2}) = \cos(x^2)$ for $x \in \mathbb{R}$. To that end we consider integrating $\int_\gamma f(z)dz$ where $f(z) = e^{-z^2}$ over the following contour. insert contour

Notice that $f'(z)$ is entire with $f'(z) = 2ize^{iz^2}$, and so also has continuous derivative. By Cauchy's Theorem we have that $\int_\gamma f(z)dz = 0$ for all R. The integral we are interested in is $\lim_{R\to\infty} Re \int_A f(z)dz$. To that end, we compute the integral over the other parts of the curve.

Consider the integral along C. We can parameterize this part of the curve as $\gamma(t) = te^{i\pi/4}$ with $t : R \to 0$. Then consider

$$\begin{aligned}
\int_C f(z)dz &= -\int_0^R f(\gamma(t))\gamma'(t)dt \\
&= -\int_0^R e^{it^2(e^{i\pi/4})^2}e^{i\pi/4}dt \\
&= -e^{i\pi/4}\int_0^R e^{-t^2}dt
\end{aligned}$$

Then in the limit $R \to \infty$,

$$\int_C f(z)dz = -e^{i\pi/4}\frac{\sqrt{\pi}}{2}.$$

We integrate along part A of the curve above. We parameterize this part of the curve at $\gamma(t) = t \in [0, R]$. Then we have

$$\begin{aligned}
\lim_{R\to\infty}\int_A f(z)dt &= \lim_{R\to\infty}\int_{t=0}^R f(\gamma(t))\gamma'(t)dt \\
&= \lim_{R\to\infty}\int_{t=0}^R e^{-t^2}dt \\
&= \int_{t=0}^\infty e^{-t^2}dt \\
&= \frac{\sqrt{\pi}}{2},
\end{aligned}$$

as given in the book.

How do we show that the integral over part B is zero?

Notice that f is entire with continuous derivative. By Cauchy's theorem we have

$$-\int_C f(z)dz = \int_A f(z)dz = \sqrt{\pi}/2.$$

Consider the integral along C. We paramaterize this part of the curve as $\gamma(t) = te^{i\pi/4}$ with $t : R \to 0$. Now consider

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\begin{aligned}
-\int_C f(z)dz &= -\int_{t=R}^0 f(\gamma(t))\gamma'(t)dt \\
&= e^{i\pi/4}\int_{t=0}^R e^{(te^{i\pi/4})^2}dt \\
&= e^{i\pi/4}\int_{t=0}^R e^{(\frac{t}{\sqrt{2}}(1+i))^2}dt \\
&= e^{i\pi/4}\int_{t=0}^R e^{-it^2}dt \\
&= e^{i\pi/4}\int_{t=0}^R \cos(-t^2) + i\sin(-t^2)dt \\
&= e^{i\pi/4}\int_{t=0}^R \cos(t^2) - i\sin(-t^2)dt \\
&= \frac{1}{\sqrt{2}}(1+i)\int_{t=0}^R \cos(t^2) - i\sin(-t^2)dt \\
&= \frac{1}{\sqrt{2}}\left[\int_0^R \cos(t^2) + \sin(t^2)dt + i\left(\int_0^R \cos(t^2) - \sin(t^2)dt\right)\right]
\end{aligned}
$$

Now, taking the limit $R \to \infty$ and recalling Cauchy's formula gives

$$\int_0^\infty \cos(t^2) + \sin(t^2)dt + i\left(\int_0^\infty \cos(t^2) - \sin(t^2)dt\right) = \frac{\sqrt{2\pi}}{2}$$

Comparing the real and imaginary part of this equation gives $\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt$ and then gives $\int_0^\infty \cos(t^2)dt = (\sqrt{2\pi})/4$

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