Algebra Homework $\#4$

1. Let $R = \mathbb{Z}[x]$ be the polynomial ring over $\mathbb{Z}$, and let $I \trianglelefteq R$ be the ideal $(2, x)$. We will think of I as an R-module. Show that:

   - Any subset of I of size ≤ 1 does not span I.
   - Any subset of I of size ≥ 2 is not R-linearly independent.

   Deduce that I is not a free R-module.

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2. Let R be a commutative ring. Show that the following two conditions on an R-module M are equivalent:

   (a) every submodule of M is finitely generated;

   (b) M satisfies the ascending chain condition for submodules: if $N_1 \leq N_2 \leq N_3 \leq \dots$ is an increasing chain of submodules of M, then there exists some j such that $N_i = N_j$ for all $i \geq j$.

   (A module M satisfying these conditions is said to be *Noetherian*. The ring R is Noetherian as a ring if and only if it is Noetherian as an R-module. We saw in class that finitely generated modules over Noetherian rings are Noetherian.)

First we show that (a) implies (b). First notice that if every submodule of M is finitely generated then in particular M is finitely generated. note: at first I read (a) incorrectly as "M is finitely generated" and so I wrote this part of the question under that assumption. And so, let $\{e_i\}$ be a finite, minimal, generating set for M with $\#\{e_i\} = n$. And let $N_1 \leq N_2 \leq N_3 \leq \cdots$ be an ascending chain of M submodules.

Let us construct a new chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ where A_i is some minimal generating set of N_i. This is well defined since, as submodules of a finitely generated module, each N_i is also finitely generated. We claim that the chain $A_1 \subseteq A_2 \subseteq \cdots$ stabilizes at some k. Notice that the finite set $\{e_i\}$ is maximal amoungst the set of A_i. And so the chain of A_i's must stabilize either at $\{e_i\}$ or at some subset $B \subsetneq \{e_i\}$. Let k be the index where stabilization occurs for the A_i.

In the first case where $A_k = \{e_i\}$ it follows then that $N_k = M$, since A_k was defined to be a minimal generating set for N_k. It then follows that $N_i = N_k = M$ for all $i > k$. In the second case our chain of A_i's stabilize with $A_k = B$. In this case we claim that our chain of N_i stabilizes at N_k also. Let $m \in N_i$ for some $i > k$. Since A_i is a generating set for N_i we have that m is some finite R-linear combination over A_i. However, $A_k = A_i$ and so the same finite R-linear combination is also contained in N_k. In other words $m \in N_k$. Hence $N_i \leq N_k$ and so $N_i = N_k$ for all $i > k$.

Now we show that (b) implies (a). Assume that M satisfies the ascending chain condition for modules and let $L \leq M$ be a submodule. Suppose, for contradiction, that L is not finitely generated. That is, L has a minimal[1] generating set $\{\ell_i\}$ with $\#\{\ell_i\} = \infty$. Suppose first that $\{\ell_i\}$ is countable.

We construct an asending chain $N_i := \{\sum_{k=1}^{i} r_i \ell_i : r_i \in R\}$, submodules generated by the first i generators of L. We claim that $N_1 \leq N_2 \leq \cdots$ is an ascending chain which does not stabilize. Consider N_i for some i. Note that since $\{\ell_i\}$ is minimal, then there's some $k > i$ such that we cannot write ℓ_k as a finite R-linear combination of elements in $\{\ell_j\}_{j=1}^{i}$. That is, there exists some N_k which contains an element ℓ_k which cannot be written as a finite linear combiantion of elements in N_i. That is, we have found a $k > i$ such that $N_k \neq N_k$. Then our chain $N_1 \leq N_2 \leq \cdots$ does not stabilize. However, this contradicts the submodule ascending chain condition on M. Thus, it must be that L is finitely generated.

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[1] Am I generally just allowed to ask for a minimal generating set?

3. In this problem, we will make good on our promise from class to show that the rank of a free module is well-defined (over a commutative ring!). Let R be a non-zero commutative ring. We are going to show that the R-modules R^n and R^m are non-isomorphic if $n \neq m$.

   - Suppose that $I \trianglelefteq R$ is an ideal of a commutative ring R and let M be an R-module. Let IM be the R-submodule of M spanned by all elements ax for $a \in I$ and $x \in M$. Show that the quotient M/IM has the structure of an R/I-module. When $M = R^n$, show that M/IM is isomorphic to $(R/I)^n$ as an R/I-module.
   - Show that any R-linear isomorphism $R^n \xrightarrow{\sim} R^m$ induces an R/I-linear isomorphism $(R/I)^n \xrightarrow{\sim} (R/I)^m$.
   - By taking I to be a maximal ideal, deduce that R^n and R^m are isomorphic if and only if $n = m$. (Standard linear algebra results may be used without proof.)

- To show that M/IM has the structure of an R/I-module, we must show that M/IM has an underlying abelian group structure with respect to addition, and that there is a well-defined R/I scaler multiplication on M/IM.

  First, we show that M/IM has a well-defined abelian group structure with respect to addition. By definition M has an abelian group structure, and so IM is a normal subgroup of M. It follows that the quotient group M/IM is well-defined and moreover is also abelian. One way to see that the quotient group is abelian is to recall that the quotient map $\pi : M \to M/IM$ is a group homomorphism and so

  $$\pi(x) + \pi(y) = \pi(x + y) = \pi(y + x) = \pi(y) + \pi(x),$$

  for all $x, y \in M$.

  Next, we propose an R/I scaler multiplication on M/IM. If $r + I \in R/I$ and $m + IM \in M/IM$. Then I claim that scaler multiplication given by $(r + I) \cdot (m + IM) := rm + IM$ is well defined. Let $r + I = r' + I$ be equivalent elements of R/I. Then we have that there's some $a \in I$ such that $r = r' + a$. Now let $m + IM \in M/IM$ and consider

  $$rm + IM = (r' + a)m + IM = r'm + IM + am + IM = r'm + IM + 0 + IM = r'm + IM,$$

since $am \in IM$ by definition. That is, our proposed scaler multiplication maps $m \in M$ to the same class mod IM, regardless of choice of representative of $r + IM$. Hence our multiplication is well defined and we have found an R/I-module structure on M/IM.

- Suppose $\phi : R^n \to R^m$ is an R-linear isomorphism. Since R^n is a free R-module, it follows that ϕ is defined exactly by its action on the basis $\{e_i\}$ where $e_i = (0, \cdots, 1, \cdots 0)$ with the 1 in the ith position.

  Now notice that $\bar{e}_i := \pi(e_i)$ is an R/I-basis for $(R/I)^n$. And so maps out of $(R/I)^n$ can be defined by their action on $\bar{e}_i$. We define a map $\bar{\phi} : (R/I)^n \to (R/I)^m$ by $\bar{\phi}(\bar{e}_i) = (\pi \circ \phi)(e_i)$. This map is R/I linear by construction, we claim that it is also a bijection.

  First we show surjectivity. Suppose $(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m) \in (R/I)^m$. Since $\pi : R \to R/I$ is a surjection, there is an element $(b_1, b_2, \cdots, b_m) \in R^m$ such that $\pi(b_1, b_2, \cdots, b_m) = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m)$. Moreover, ϕ is an isomorphism and so there also exists an element $(a_1, a_2, \cdots, a_n) \in R^n$ such that $\phi(a_1, \cdots, a_n) = (b_1, \cdots, b_m)$. It then follows that $\bar{\phi}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n) = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m)$. Thus, $\bar{\phi}$ is surjective.

  Now consider $0 \in (R/I)^m$. Tuples of the form $(b_1, \cdots, b_m) \in R^m$ with $b_i \in I$ map to 0 under π. Since ϕ is an isomorphism from R to R, only elements $(a_1, \cdots, a_n) \in R^n$ with $a_i \in I$ satisfy $\phi(a_1, \cdots, a_n) = (b_1, \cdots, b_m)$ with $b_i \in I$. Then, only elements of the form $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n) \in (R/I)^n$ map to $0 \in (R/I)^m$ under $\bar{\phi}$. However, since $a_i \in I$ we have $(\bar{a}_1, \cdots, \bar{a}_n) = 0 \in (R/I)^n$. And so $\bar{\phi}$ is injective.

  We have found a bijective R/I-linear map $\bar{\phi} : (R/I)^n \to (R/I)^m$ and so we have found an induced R/I-linear isomorphism $(R/I)^n \to (R/I)^m$.

- Suppose $n = m$, then $R^n = R^m$ and so $R^n \cong R^m$ as R-modules.

  Now suppose that $R^n \cong R^m$. Since R is a non-zero commutative ring we have (via the axiom of choice (or perhaps Zorn's lemma)) that there exists a maximal ideal $I \trianglelefteq R$. Now by part (b) we have an induced isomorphism on the R/I modules $(R/I)^n \cong (R/I)^m$. However, since I is maximal, it follows that R/I is a field and so $(R/I)^n$ and $(R/I)^m$ are in fact R/I-vector spaces. Vector spaces are characterized by their dimension and so it follows that $(R/I)^n \cong (R/I)^m$ as R/I vector spaces implies $n = m$.

Hence the rank of a free module over a non-zero commutative ring is a well defined notion.

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4. Prove the following identities for tensor products (where M, N, L are arbitrary R-modules):
   - $M \otimes_R N \cong N \otimes_R M$ ("commutativity")
   - $R \otimes_R M \cong M$ ("identity")
   - $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$ ("distributivity")
   - (harder) $M \otimes_R (N \otimes_R L) \cong (M \otimes_R N) \otimes_R L$ ("associativity")

   [Hint: the universal property of tensor products is a handy way of defining R-linear maps out of tensor products]

- Consider the map $M \times N \to N \otimes_R M$ given by $((m, n) \mapsto n \otimes m)$. Notice that this map is R-bilinear on $M \times N$ since $\otimes : M \times N \to M \otimes_R N$ is. Then by the universal property of tensor products we have a unique R-linear map $f : M \otimes_R N \to N \otimes_R M$ such that $f \circ \otimes = (m, n) \mapsto n \otimes m$. The same argument on the R-bilinear map $N \times M \to N \otimes_R M$ given by $(n, m) \mapsto (m \otimes n)$ gives a unique R-linear map $\tilde{f} : N \otimes_R M \to M \otimes_R N$ such that[2] $\tilde{f} \circ \times = (n, m) \mapsto m \otimes n$.

  Notice now that $f \circ \tilde{f} = id_{N \otimes_R M}$ and $\tilde{f} \circ f = id_{M \otimes_R N}$. Indeed $(f \circ \tilde{f})(m \otimes n) = f(n \otimes m) = m \otimes n$. And likewise for the other direction.

  That is, we have found a bijective R-linear map $M \otimes_R N \to N \otimes_R M$ and so in fact $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

- Consider the following map $M \times N \to M$ via $(r, m) \mapsto rm$ given by the module structure on M. Notice that this map is bilinear come back and write the computation out And so by the universal property of tensor products we have a unique R-linear map $f : R \otimes_R M \to M$ such that $r \otimes n \mapsto rm$.

  I claim that this map is bijective and so is an isomorphism of R-modules. First notice that f is surjective. Indeed, if $m \in M$ then $f(1 \otimes m) = 1 \cdot m = m$, so long as R is not the zero ring. If R is the zero ring, then M must be the zero module, and then our desired isomorphism trivially holds.

  Next we show that f is injective. Suppose we have $r \cdot m' = m$ for some $m, m' \in M$

[2] The "$\otimes$" in the following phrase now refers to the R-bilinear map $N \times M \to N \otimes_R M$, whereas earlier it referred to the R-bilinear map $M \times N \to M \otimes_R N$.

and $r \in R$. Then notice

$$r \otimes m' = 1 \cdot r \otimes m' = r \otimes r \cdot m' = 1 \otimes m,$$

And so $rm' = m$ implies $r \otimes m' = 1 \otimes m$. This suffices to show that f is injective because if generally we have $r_1 m_1 = r_2 m_2$ then by definition $r_1 m_1 = m'$ for some $m' \in M$ and then we have $m' = r_2 m_2$.

Overall we have a bijective R-linear map $R \otimes_R M \to M$, and so $R \otimes_R M \cong M$.

- First we acquire an R-linear map $M \otimes_R (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$ by leveraging the universal property of tensor products. And then we show that this map is in fact an isomorphism.

  First define a map $h : M \times (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$ by $h(m,(n,l)) = (m \otimes n, m \otimes l)$. Note that in R-Mod finite coproducts are also products, and so the domain of the function h is the R-module $M \oplus N \oplus L$. We claim that this map is R-bilinear. Consider

$$\begin{aligned} h(r_1 m_1 + r_2 m_2, (n,l)) &= ((r_1 m_1 + r_2 m_2) \otimes n, (r_1 m_1 + r_2 m_2) \otimes l) \\ &= (r_1(m_1 \otimes n) + r_2(m_1 \otimes n), r_1(m \otimes l) + r_2(m_2 \otimes n)) \\ &= r_1(m_1 \otimes n, m_1 \otimes l_1) + r_2(m_2 \otimes n, m_2 \otimes l). \end{aligned}$$

  By the definition of the tensor product relations, and of the R-module structure on the direct sum of two R-modules. In other words, we have shown that h is R-linear in the first argument. Now consider the second argument

$$\begin{aligned} h(m, r_1(n_1,l_1) + r_2(n_2,l_2)) &= (m \otimes r_1 n_1 + r_2 n_2, m \otimes r_1 l_1 + r_2 l_2) \\ &= (r_1(m \otimes n_1) + r_2(m \otimes n_2), r_1(m \otimes l_1) + r_2(m \otimes l_2)) \\ &= r_1(m \otimes n_1, m \otimes l_1) + r_2(m \otimes n_2, m \otimes l_2), \end{aligned}$$

  using the bilineararity of the tensor product, and by using the definition of the R-module structure on the direct sum of R-modules. That is, we've shown h is R-linear in the second argument also. And so h is an R-bilinear map.

We are now free to use the universal property of tensor products to acquire a new R-linear map $\phi : M \otimes_R (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$. where $\phi(m \otimes (n,l)) = (m \otimes n, m \otimes l)$.

Next, we will construct an R-linear map $\psi : (M \otimes_R N) \oplus (M \otimes_R L) \to M \otimes_R (N \otimes L)$ using the universal property of coproducts. We have a commutative diagram

$$\begin{tikzcd}
M \times N \arrow[rr, "f_1"] \arrow[d, "\otimes"'] & & M \otimes_R (N \oplus L) & & M \times L \arrow[ll, "f_2"'] \arrow[d, "\otimes"] \\
M \otimes_R N \arrow[rru, dashed, "\psi_1"] \arrow[rr, hook] & & (M \otimes_R N) \oplus (M \otimes_R L) \arrow[u, dashed, "\psi"] & & M \otimes_R L \arrow[ll, hook'] \arrow[llu, dashed, "\psi_2"']
\end{tikzcd}$$

where the maps ψ_1, ψ_2 will be built out of the universal property of tensor products, and the desired map ψ will consequently be determined by the universal property of corpoduct in R-mod. First, we must specify bilinear maps $f_1 : M \times N \to M \otimes_R (N \oplus L)$ and $f_2 : M \times L \to M \otimes_R (N \oplus L)$.

I claim that the map $f_1(m,n) = m \otimes (n,0)$ is bilinear. If n is fixed then for each $r_1, r_2 \in R$ and $m_1, m_2 \in M$ we have $f_1(r_1m_1 + r_2m_2, n) = (r_1m_2 + r_2m_2) \otimes (n,0) = r_1(m_1 \otimes (n,0)) + r_2(m_2 \otimes (n,0)) = r_1f_1(m_1,n) + r_2f_2(m_2,n)$, by the bilinearity of tensor product. I.e. f_1 is R-linear in its first argument. Moreover, if we fix $m \in M$ we have

$$\begin{aligned}
f_1(m, r_1n_1 + r_2n_2) &= m \otimes (r_1n_1 + r_2n_2, 0) \\
&= m \otimes [r_1(n_1,0) + r_2(n_2,0)] \\
&= r_1(m \otimes (n_1,0)) + r_2(m \otimes (n_2,0)) \quad = r_1f_1(m,n_1) + r_2f_1(m,n_2).
\end{aligned}$$

And so, f_1 is R-linear in its second argument. Hence, $f_1 : M \times N \to M \otimes_R (N \oplus L)$ is R-bilinear. A very similar calculation will show that $f_2 : M \times L \to M \otimes_R (N \oplus L)$ is also R-bilinear.

Then, by the universal property of tensor products, we have R-linear maps $\psi_1 : M \otimes_R N \to M \otimes_R (N \oplus L)$ and $\psi_2 : M \otimes \otimes_R L \to M \otimes_R (N \oplus L)$ with $\psi_1(m \otimes n) = m \otimes (n,0)$ and $\psi_2(m \otimes l) = m \otimes (0,l)$. Then by the universal property of coproducts, we have an R-linear map $\psi : (M \otimes_R N) \oplus (M \otimes_R L) \to M \otimes_R (N \oplus L)$

where $\psi(m_1 \otimes n, m_2 \otimes l) = m_1 \otimes (n, 0) + m_2 \otimes (0, l)$. We claim that ψ is an $\mathbb{R}$ linear map which is inverse to ϕ. If we show that it will then follow that ϕ is in fact an isomorphism $M \otimes_R (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$.

First let $m \otimes (n, l) \in M \otimes_R (N \oplus L)$ and consider

$$\begin{aligned}(\psi \circ \phi)(m \otimes (n, l)) &= \phi(m \otimes n, m \otimes l) \\ &= m \otimes (n, 0) + m \otimes (0, l) \\ &= m \otimes [(n, 0) + (0, l)] \\ &= m \otimes (n, l).\end{aligned}$$

Now let $(m_1 \otimes n, m_2 \otimes l) \in (M \otimes_R N) \oplus (M \otimes_R L)$ and consider

$$\begin{aligned}(\phi \circ \psi)(m_1 \otimes n, m_2 \otimes l) &= \phi(m_1 \otimes (n, 0) + m_2 \otimes (0, l)) \\ &= \phi(m_1 \otimes (n, 0)) + \phi(m_2 \otimes (0, l)) \\ &= (m_1 \otimes n, m_1 \otimes 0) + (m_2 \otimes 0, m_2 \otimes l) \\ &= (m_1 \otimes n, 0) + (0, m_2 \otimes l) \\ &= (m_1 \otimes n, m_2 \otimes l).\end{aligned}$$

And so, ϕ, ψ are inverse R-linear homomorphisms and $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$.

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5. Compute the following tensor products:
   (a) $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$
   (b) $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^5$
   (c) $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z})$
   (d) $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$
   (e) $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$

(a)

(b) b

(c) We show that $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z}) \cong 0$. First notice that $gcd(5,7) = 1$ and so by Bezout's identity there exists integers x, y such that $5x + 7y = 1$.

Moreover, every element of $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z})$ can be written as a pure tensor since our factors are cyclic, and so, let $[p] \otimes [q] \in (\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z})$. Recalling modular arithmetic rules and distributivity of the tensor product we have

$$\begin{aligned}
[p] \otimes [q] &= [1] \otimes [pq] \\
&= [5x + 7y] \otimes [pq] \\
&= ([5x] + [7y]) \otimes [pq] \\
&= [5x] \otimes [pq] + [7y] \otimes [pq] \\
&= [5x] \otimes [pq] + [y] \otimes [7pq] \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

Thus, we have shown that every elements in $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z})$ is actually the zero tensor. In other words $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z}) = 0$.

(d) We show that $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. First we show that every element in $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ can is either the 0 element or $[1] \otimes [1]$. Since $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ has cyclic factors, we can write every element of $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ as a pure tensor.

Let $[p] \otimes [q] \in (\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$. There exist integers $x, y \in \mathbb{Z}$ such that $6x +$

$8y = gcd(6,8) = 2$. We have two cases either $[p] = [2k]$ or $[p] = [2k+1]$ for $k = 0,1,2$. Consider the first case, we have

$$\begin{aligned}[2k] \otimes [q] &= 2([k] \otimes [q]) \\ &= (6x+8y)([k] \otimes [q]) \\ &= ([6xk] \otimes [q]) + ([k] \otimes [8yq]) \\ &= 0.\end{aligned}$$

Now in the second case, using similar reasoning, we have

$$[2k+1] \otimes [q] = ([2k] \otimes [q]) + ([1] \otimes [q]) = [1] \otimes [q].$$

Now, we have two additional cases, either $[q] = [2\ell]$ or $[q] = [2\ell+1]$ for $\ell = 0,1,2,3$. Again, using similar reasoning, if $[q] = [2k]$ then we have $[1] \otimes [q] = [1] \otimes [2k] = 0$. And in the second case we have

$$\begin{aligned}[1] \otimes [2k+1] &= [1] \otimes [2k] + [1] \otimes [1] \\ &= 0 + [1] \otimes [1] \\ &= [1] \otimes [1].\end{aligned}$$

Thus we have shown that every element in $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ is either the 0 tensor or $[1] \otimes [1]$.

Now, we claim that $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. To show so this we construct a bijective R-linear map $(\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. I claim that $\phi : (\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ via $\phi(0) = 0$ and $\phi([1] \otimes [1]) = [1]$ is such an isomorphism. Okay, to show this for real, we need to show three things, that this map is: R-linear, well-defined with respect to the tensor product realtions, and that this map is bijective. The other way to do this is using the universal property of tensor products, but we cannot write elements $(p,q) \in (\mathbb{Z}/6\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ as 0 or $(1,1)$, like we can for elements in the tensor product. So, I'm not sure this is any easier in this case.

(e) note: pretty sure this is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. write this up

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6. Suppose that $f\colon M \to M'$ and $g\colon N \to N'$ are homomorphisms of R-modules. Use f and g to define a homomorphism
$$M \otimes_R N \to M' \otimes_R N'\,.$$
(This homomorphism is usually denoted by $f \otimes g$.)

We define the desired map by using the universal property of tensor products. Consider the following composition

$$M \oplus N \xrightarrow{(f,g)} M' \oplus N' \xrightarrow{\otimes} M' \otimes_R N',$$

via

$$(m,n) \mapsto (f(m), g(n)) \mapsto f(m) \otimes g(n).$$

We claim that this composition $\phi : M \oplus N \to M \otimes_R N$ is R-bilinear. Indeed, first let $n \in N$ be fixed and then consider

$$\begin{aligned}
\phi(r_1 m_1 + r_2 m_2, n) &= (r_1 f(m_1) + r_2 f(m_2)) \otimes g(n) \\
&= (r_1 f(m_1)) \otimes g(n) + (r_2 f(m_2)) \otimes g(n) \\
&= r_1(f(m_1) \otimes g(n)) + r_2(f(m_2) \otimes g(n)) \\
&= r_1 \phi(m_1, n) + r_2 \phi(m_2, n),
\end{aligned}$$

by the relations on the elements of the tensor product. In other words, we have shown that $\phi(-, n)$ is R-linear for each $n \in N$. A very similar calculation will show that $\phi(m, -)$ is R-linear for each $m \in M$. That is, ϕ is R-bilinear.

By the universal property of the tensor product we then have an R-linear map $\psi : M \otimes_R N \to M' \otimes_R N$ given by $m \otimes n \mapsto f(m) \otimes g(n)$, as desired.

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7. Let V be an $\mathbb{R}$-vector space of dimension n. Recall from class that the tensor product $\mathbb{C} \otimes_{\mathbb{R}} V$ can be viewed as a $\mathbb{C}$-vector space in a natural way. In this question, we are going to show that the dimension of $\mathbb{C} \otimes_{\mathbb{R}} V$ (as a $\mathbb{C}$-vector space) is equal to n (the dimension of V as an $\mathbb{R}$-vector space).

   - Suppose that $e_1, \ldots, e_n$ is an $\mathbb{R}$-linear basis of V. Write down n elements of $\mathbb{C} \otimes_{\mathbb{R}} V$ which could plausibly be a $\mathbb{C}$-linear basis.
   - Suppose that $\delta_1, \ldots, \delta_n : V \to \mathbb{R}$ is the dual basis to $e_1, \ldots, e_n$. Write down n $\mathbb{C}$-linear maps $\mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C}$ which could plausibly be a $\mathbb{C}$-linear basis of the dual space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$. [Hint: $\mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$, so the previous part gives you a way of constructing a homomorphism.]
   - Show that the elements you defined in the previous two parts are dual bases of the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathbb{R}} V$, and hence that $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V) = n$.

- I claim that $1 \otimes e_1, 1\otimes, e_2, \cdots, 1 \otimes e_n \in \mathbb{C} \otimes_{\mathbb{R}} V$ is a basis. First we show that these tensors span $\mathbb{C} \otimes_{\mathbb{R}} V$. First let $\alpha \otimes v \in \mathbb{C} \otimes_{\mathbb{R}} V$ be a pure tensor. Then, since $\{e_i\}$ is a basis for V we have $v = \lambda_1 e_1 + \cdots \lambda_n e_n$ for $\lambda_i \in \mathbb{R}$. Now notice

$$\begin{aligned} \alpha\lambda_1(1 \otimes e_1) + \alpha\lambda_2(1 \otimes e_2) + \cdots + \alpha\lambda_n(1 \otimes e_n) &= \alpha\left[1 \otimes \lambda_1 e_1 + 1 \otimes \lambda_2 e_2 + \cdots 1 \otimes \lambda_n e_n\right] \\ &= \alpha \otimes (\lambda_1 e_1 + \cdots + \lambda_n e_n) \\ &= \alpha \otimes v. \end{aligned}$$

  That is, every pure tensor in $\mathbb{C} \otimes_R V$ is a finite $\mathbb{C}$-linear combination of the $1 \otimes e_i$. It then follows that every element of $\mathbb{C} \otimes_{\mathbb{R}} V$, which is a finite $\mathbb{C}$-linear combination of pure tensors, is also a finite $\mathbb{C}$-linear combination of the $1 \otimes e_i$. Thus, the $1 \otimes e_i$ span $\mathbb{C} \otimes_R V$ over $\mathbb{C}$.

  Later we will show that the $\{1 \otimes e_i\}$ are also $\mathbb{C}$-linearly independent.

- We want to study the $\mathbb{C}$-linear maps $\mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$. We first consider maps which descend from bilinear maps $C \times V \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ induced by $f \in V^*$. First, fix a map $f \in V^\star$. And then we define a map $\phi_f : \mathbb{C} \times V \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ via $\phi_f(\alpha, v) = \alpha \otimes f(v)$. First notice that this map is $\mathbb{R}$-linear in the second coordinate: if we fix $\alpha \in \mathbb{C}$ and let $a_1, a_2 \in \mathbb{R}$ indeed, we have

$$\phi_f(\alpha, a_1 v_1 + a_2 v_2) = \alpha \otimes f(a_1 v_1 + a_2 v_2) = \alpha \otimes (a_1 f(v_1) + a_2 f(v_2)) = a_1(\alpha \otimes f(v_1)) + a_2(\alpha \otimes f(v_2)),$$

  Since f is an $\mathbb{R}$-linear map and by the bilinearity of the tensor product. A similar calculation shows that ϕ_f is $\mathbb{C}$-linear in the first coordiante.

Now, since $\delta_1, \cdots, \delta_n$ is a basis for V^* then we can represent $f(v) = a_1\delta_1(v) + a_2\delta_2(v) + \cdots a_n\delta_n(v)$ for $a_i \in \mathbb{R}$. Then we can represent

$$\phi_f(\alpha, v) = \alpha \otimes (a_1\delta_1(v) + \cdots + a_n\delta_n(v)) = \alpha a_1(1 \otimes \delta_1(v)) + \alpha a_2(1 \otimes \delta_2(v)) + \cdots + \alpha a_n(1 \otimes \delta_n(v)),$$

Note that the a_i in the computation above depends only on the map $f \in V^*$. This suggests that a plausible basis for $Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$ could be the maps $1 \otimes \delta_i \in Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}}, \mathbb{C})$.

Generally though, however, we need to show that an arbitrary $\mathbb{C}$-linear function $\mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C}$ can be written as a finite linear combination of the $\{1 \otimes \delta_i\}$.

- See the previous parts to see some partial arguments that the elements given are R-linear bases for their corresponding spaces. I need to show more to show that they are in fact $\mathbb{C}$-linear bases for their corresponding spaces.

  I will, however, show that $\{1 \otimes e_i\} \subseteq \mathbb{C} \otimes_{\mathbb{R}} V$ and $\{1 \otimes \delta_i\} \subseteq Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$ are dual to each other. Consider

$$(1 \otimes \delta_i)(1 \otimes e_i) = 1 \otimes \delta_i(e_i) = 1 \otimes 1 = 1 \in \mathbb{C}.$$

  Whereas, for $i \neq j$

$$(1 \otimes \delta_i)(1 \otimes e_j) = 1 \otimes \delta_i(e_j) = 1 \otimes 0 = 0 \in \mathbb{C}.$$

  This shows that $\{1 \otimes \delta_i\}$ and $\{1 \otimes e_i\}$ are dual and in fact orthonormal.

  Earlier we showed that $\{1 \otimes e_i\}$ spanned $\mathbb{C} \otimes_{\mathbb{R}} V$. Now, we show that $\{1 \otimes e_i\}$ are $\mathbb{C}$-linearly independent. Suppose we have $\lambda_1(1 \otimes e_1) + \lambda_2(1 \otimes e_2) + \cdots + \lambda_n(1 \otimes e_n) = 0$ for $\lambda_i \in \mathbb{C}$. Applying the $\mathbb{C}$-linear homomorphism $1 \otimes \delta_1$ gives

$$\begin{aligned} \lambda_1(1 \otimes \delta_1)(1 \otimes e_1) + \lambda_2(1 \otimes \delta_2)(1 \otimes e_2) + \cdots + \lambda_n(1 \otimes \delta_n)(1 \otimes e_n) &= (1 \otimes \delta_n)(0) \\ \lambda_1(1) + 0 + \cdots + 0 &= 0 \\ \lambda_1 &= 0. \end{aligned}$$

  Likewise, applying $1 \otimes \delta_i$ for each i will then give $\lambda_i = 0$ for each i. Thus, $\{1 \otimes e_i\}$ is $\mathbb{C}$-linearly independent by definition.

Similarly, earlier we showed that the functions $\{1 \otimes \delta_i\}$ spanned the $\mathbb{C}$-module $Hom(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R})$. Now we show that they are $\mathbb{C}$-linearly independent. Suppose $\lambda_1(1 \otimes \delta_i) + \lambda_2(1 \otimes \delta_2) + \cdots \lambda_n(1 \otimes \delta_n) = 0$ sum to the zero function. Recall that these functions act by $(1 \otimes \delta_i)(\alpha \otimes v) = \alpha \otimes \delta_i(v) \mapsto \alpha\delta_i(v) \in \mathbb{C}$. So now plugging in the element $1 \otimes e_1$ into both sides gives

$$\begin{aligned}\lambda_1(1 \otimes \delta_i)(1 \otimes e_1) + \lambda_2(1 \otimes \delta_2)(1 \otimes e_2) + \cdots \lambda_n(1 \otimes \delta_n)(1 \otimes e_n) &= 0(1 \otimes e_1) \\ \lambda_1(1 \cdot 1) + \lambda_2(1 \cdot 0) + \cdots \lambda_n(1 \cdot 0) &= 0 \\ \lambda_1 &= 0.\end{aligned}$$

Likewise, plugging in the elements $1 \otimes e_i$ will give that $\lambda_i = 0$ for each i. In other words, the $\{1 \otimes \delta_i\}$ are $\mathbb{C}$ linearly independent.

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8. *Let M be an abelian group. Show that the set $\operatorname{End}(M)$ of homomorphisms $f: M \to M$ has the structure of a ring (not necessarily commutative), where addition is pointwise addition of homomorphisms, and multiplication is composition of homomorphisms. Show moreover that giving M the structure of a left R-module is equivalent to specifying a ring homomorphism $\phi: R \to \operatorname{End}(M)$.

(This parallels the way that giving a set X a left action of a group G is equivalent to specifying a group homomorphism $\phi: G \to \operatorname{Sym}(X)$.)

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