

## Algorithms HW

1. Consider the following functions (morphisms in **Set**):

$$f: \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\} \quad f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 4 & \text{if } x = 3 \end{cases}$$

$$g: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3\} \quad g(x) = \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$h: \{1, 2, 3\} \rightarrow \{1, 2\} \quad h(x) = \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$k: \{1, 2\} \rightarrow \{1, 2, 3\} \quad k(x) = \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}$$

Show that the square

$$\begin{array}{ccc} \{1, 2, 3\} & \xrightarrow{f} & \{1, 2, 3, 4\} \\ \downarrow h & & \downarrow g \\ \{1, 2\} & \xrightarrow{k} & \{1, 2, 3\} \end{array}$$

commutes.

To show that the given square commutes, we must show that  $(gf)(x) = (kh)(x)$  for all  $x \in \{1, 2, 3\}$ . Consider the image of  $1 \in \{1, 2, 3\}$ :

$$gf(1) = g(1) = 1 = k(1) = kh(1).$$

Now, consider the image of  $2 \in \{1, 2, 3\}$ :

$$gf(2) = g(2) = 1 = k(1) = kh(2).$$

Finally, here's the image of 3:

$$gf(3) = g(4) = 3 = k(2) = kh(3).$$

Hence, the given square commutes by definition. Note that we did not need to check whether  $\text{im } g = \text{im } k$  (and in fact these images in  $\{1, 2, 3\}$  are not equal.) ■

2. We work in the category  $\text{Mod}_{\mathbb{R}}$  of real vector spaces. Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the  $\mathbb{R}$ -linear map given by the matrix  $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ . Let  $g: \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \rightarrow \mathbb{R}$

be the  $\mathbb{R}$ -linear map induced by the  $\mathbb{R}$ -bilinear map

$$\beta: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz.$$

For which  $\mathbb{R}$ -linear maps  $h: \mathbb{R} \rightarrow \mathbb{R}$  does the square

$$\begin{array}{ccc} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \\ \downarrow f \otimes f & & \downarrow h \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \end{array}$$

commute?

Suppose we have  $(w, x) \otimes (y, z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ . Following This element clockwise around the diagram we have that  $(h \circ g)((w, x) \otimes (y, z)) = h(wy + xz)$  and following this element counter-clockwise around the diagram we have  $(g \circ f \otimes f)((w, x) \otimes (y, z)) = g((w - x, w + x) \otimes (y - z, y + z)) = (w - x)(y - z) + (w + x)(y + z)$ . That is, any  $\mathbb{R}$ -linear map  $h: \mathbb{R} \rightarrow \mathbb{R}$  must satisfy

$$h(wy + xz) = (w - x)(y - z) + (w + x)(y + z)$$

for all  $w, x, y, z \in \mathbb{R}$  is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since  $h$  is an  $\mathbb{R}$ -linear map we have that

$$h(wy + xz) = h(1)(wy + xz).$$

Moreover, since  $\mathbb{R}$  is a rank 1 free module over  $\mathbb{R}$ , we have that any  $\mathbb{R}$ -linear map  $\mathbb{R} \rightarrow \mathbb{R}$  is determined by where it sends the basis  $\{1\}$ . Given the expression above we have that

any such map  $h$  satisfies

$$\begin{aligned}h(1) &= \frac{(w-x)(y-z) + (w+x)(y+z)}{wy+xz} \\&= \frac{wy-wz-xy+xz+wy+wz+xy+xz}{wy+xz} \\&= \frac{2(wy+xz)}{wy+xz} \\&= 2.\end{aligned}$$

That is, there is a single map  $h : \mathbb{R} \rightarrow \mathbb{R}$  which makes the above diagram commute — namely the one which sends the basis  $1 \mapsto 2$ , i.e  $h(x) = 2x$ .

I'm curious if there's any geometric significance to this thing that we've just shown

■

3. Consider the following diagram in an arbitrary category  $\mathcal{C}$ :

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow h & & \downarrow h' & & \downarrow h'' \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

Show that if the left and right squares in this diagram commute, then so does the outer rectangle. That is, if the two squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow h' \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow h' & & \downarrow h'' \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

both commute, then so does the square

$$\begin{array}{ccc} X & \xrightarrow{gf} & Z \\ \downarrow h & & \downarrow h'' \\ X' & \xrightarrow{g'f'} & Z' \end{array}$$

We need to show that  $h''gf = g'f'h$ . Suppose  $x \in X$ . If  $\mathcal{C}$  is not “sets with extra structure” can we still reason about functions by considering their actions on elements in their domain?

Consider the right-handed commuting square. Let  $f(x) \in Y$ . Since this second square commutes, we have  $h''gf = g'h'f$ . Moreover, since the left-handed square commutes, we have  $h'f = f'h$ . Substituting this relation into our first equation gives us

$$h''gf = g'h'f = g'f'h,$$

as desired. ■

4. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms in a category  $\mathcal{C}$ . Let  $gf : X \rightarrow Z$  be their composition.

- Show that if  $f$  and  $g$  are both monomorphisms, then  $gf$  is a monomorphism.
- Show that if  $f$  and  $g$  are both epimorphisms, then  $gf$  is an epimorphism.
- Show that if  $gf$  is a monomorphism, then  $f$  is a monomorphism. Must  $g$  be a monomorphism?
- Show that if  $gf$  is an epimorphism, then  $g$  is an epimorphism. Must  $f$  be an epimorphism?

For this question, recall that the universal properties of monic maps and of epic maps. Let  $\mathcal{C}$  be a category and let  $X, Y \in \mathcal{C}$ , then a morphism  $f : X \rightarrow Y$  is called monic if for all  $Z \in \mathcal{C}$  and all  $g, h : Z \rightarrow X$  we have  $fg = fh$  implies  $g = h$ . Likewise,  $f : X \rightarrow Y$  is called epic if for all  $Z \in \mathcal{C}$  and all  $g, h : Y \rightarrow Z$  we have  $gf = hf$  implies  $g = h$ .

- Suppose  $f, g$  are monic and now consider  $gf$ . Let  $W \in \mathcal{C}$  and suppose  $h, k : W \rightarrow X$  such that  $gfh = gfk$ . Now, since  $g$  is a monomorphism and since function composition in  $\mathcal{C}$  is associative, we have  $g(fh) = gfh = gfk = g(fk)$  implies  $fh = fk$ . Now, since  $f$  is a monomorphism we have  $h = k$ . In other words, we have shown that  $gfh = gfk$  implies  $h = k$  for all morphisms  $h, k : W \rightarrow X$ . That is,  $g, f$  monic imply that  $gf$  is monic.
- Now suppose  $f, g$  are epic and let  $Z \in \mathcal{C}$  with  $h, k : Z \rightarrow W$  such that  $hgf = kgf$ . Since  $f$  is epic we have that  $(hg)f = hgf = kgf = (kg)f$  implies  $hg = kg$ . Moreover,  $g$  epic implies that  $h = k$ . That is, we have  $hgf = kgf$  implies  $h = k$  and so  $gf$  is epic.
- Suppose  $gf : X \rightarrow Z$  is a monomorphism. Let  $W \in \mathcal{C}$  and  $h, k : W \rightarrow X$  such that  $fh = fk$ . We have that  $\text{im}(fh) = \text{im}(fk) \in Y$  and so, since  $g = g$  we have that  $gfh = gfk$ . Now, since  $gf$  is monic we have that  $h = k$ . That is  $fh = fk$  implies  $h = k$ , i.e.  $f$  is monic by definition. It is not necessary that  $g$  be monic.
- Now suppose  $gf$  is epic. Let  $W \in \mathcal{C}$  with  $h, k : Z \rightarrow W$  such that  $hg = kg$ . We have that  $hgf = kgf$  as maps  $X \rightarrow W$ . But now, since  $gf$  is epic, we have that  $h = k$ . Thus  $g$  is epic by definition. It was not necessary that  $f$  be epic.

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5. Fix a group  $G$ . The category  $\mathbf{Set}_G$  of  $G$ -sets is defined as follows:

- The objects of  $\mathbf{Set}_G$  are sets  $X$  with an action of  $G$ .
- The morphisms  $f: X \rightarrow Y$  in  $\mathbf{Set}_G$  are functions  $X \rightarrow Y$  which satisfy

$$f(\sigma x) = \sigma f(x)$$

for all  $\sigma \in G$  and  $x \in X$ . (Such functions are called  $G$ -equivariant.)

- Composition in  $\mathbf{Set}_G$  is given by composition of functions.
- The identity element  $1_X: X \rightarrow X$  is the identity function.

Prove carefully that  $\mathbf{Set}_G$  is a category (check the axioms). Prove that finite products exist in  $\mathbf{Set}_G$ .

Recall the axioms of a category. Given the objects and morphisms of  $\mathbf{Set}_G$  we need to verify (1) : that we have a well-defined composition rule, i.e., that the given composition gives us a  $\mathbf{Set}_G$ -morphism, (2) given our composition rule, that the given identity morphism satisfies  $g1_X = 1_Xg = g$  for all  $g \in \text{Hom}_{\mathbf{Set}_G}(X, X)$  for all  $X \in \mathbf{Set}_G$ , and (3) that the given composition rule is associative.

Firstly, the by theory of group actions, sets with group actions and  $G$ -equivariant functions are a well defined collection of objects and morphisms between those objects.

(1) *Composition*: Let  $X, Y, Z \in \mathbf{Set}_G$  and let  $f \in \text{Hom}_{\mathbf{Set}_G}(X, Y)$  and  $g \in \text{Hom}_{\mathbf{Set}_G}(Y, Z)$ . Note that we have a well defined function composition  $gf$  from the category  $\mathbf{Set}$ . Now we must verify that  $gf$  is also  $G$ -equivariant. Since  $f, g$  are  $G$ -equivariant we have, for  $\sigma \in G$  and  $x \in X$ ,

$$gf(\sigma x) = g(\sigma f(x)) = \sigma gf(x).$$

And so,  $gf \in \text{Hom}_{\mathbf{Set}_G}(X, Z)$  is  $G$ -equivariant by definition and  $gf$  is indeed a morphism in  $\text{Hom}_{\mathbf{Set}_G}(X, Z)$ .

(2) *Identity*: Let  $X \in \mathbf{Set}_G$  and let  $1_X: X \rightarrow X$  be the identity function on  $X$  as a set. Since  $1_X$  is the identity function for  $X$  we already have  $g1_X = 1_Xg = g$  for all functions  $g \in \text{Hom}_{\mathbf{Set}}(X, X)$ . And so  $1_X$ , if it is  $G$ -equivariant, satisfies the axiom for identity  $G$ -Set morphism. Let  $\sigma \in G$  and consider

$$1_X(\sigma x) = \sigma x = \sigma f(x),$$

by definition of the action of  $1_X$  as a function. And so, indeed,  $1_X$  is  $G$ -equivariant and so is a morphism in  $\text{Hom}_{\mathbf{Set}_X}(X, X)$ , thus every  $X \in \mathbf{Set}_X$  has an identity morphism.

(3) *Associativity of function composition*: Note that composition of functions is associative since  $\mathbf{Set}$  is a category. It follows immediately that  $G$ -equivariant function composition is associative, since the  $G$ -equivariant functions from  $X \rightarrow Y$  are a “sub-class” of the class of functions  $X \rightarrow Y$ .

Now we show that finite products exist in  $\mathbf{Set}_G$ . We claim that binary products exist in  $\mathbf{Set}_G$  (and so it will follow that finite products exist in  $\mathbf{Set}_G$  by iterating the construction for binary products). Recall that the universal property for binary products is the pullback of the diagram  $\cdot \leftarrow \cdot \rightarrow \cdot$ . I claim that, given  $X, Y \in \mathbf{Set}_G$ , the cartesian product  $X \times Y$  with the usual projections  $\pi_X, \pi_Y$  satisfy the universal property for  $\mathbf{Set}_G$ . To verify this claim we have two things to show: (1) that the cartesian product has some  $G$ -action for which  $\pi_X$  and  $\pi_Y$  are  $G$ -equivariant (i.e. that  $X \times Y$  is indeed an object in  $\mathbf{Set}_G$  and  $\pi_X : X \times Y \rightarrow X, \pi_Y : X \times Y \rightarrow Y$  are indeed morphisms in  $\mathbf{Set}_G$ ). And (2), that  $(X \times Y, \pi_X, \pi_Y)$  satisfy the universal property of products

(1): We show that  $X \times Y$  has a  $G$ -action which makes  $\pi_X, \pi_Y$  into  $G$ -equivariant functions. Recall that given  $X, Y \in \mathbf{Set}_G$  we can construct the set  $X \times Y := \{(x, y) : x \in X, y \in Y\}$ . Define a  $G$ -action on  $X \times Y$  by  $\sigma(x, y) = (\sigma x, \sigma y)$  for  $\sigma \in G$  and where  $\sigma x, \sigma y$  are given by the  $G$ -action structure on  $X, Y$ . We verify that this is indeed a  $G$ -action on  $X \times Y$ . Note that if  $1 \in G$  is the identity element of  $G$  then we have  $1(x, y) = (1x, 1y) = (x, y)$  for all  $(x, y) \in X \times Y$ . Moreover, if  $\sigma, \delta \in G$  we have  $\sigma(\delta(x, y)) = \sigma(\delta x, \delta y) = (\sigma \delta x, \sigma \delta y) = (\sigma \delta)(x, y)$ . That is, our proposed action is indeed a  $G$ -action on  $X \times Y$  and so the set  $X \times Y$  is also an object in  $\mathbf{Set}_G$ .

Now we verify that the projections  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  are  $G$ -equivariant. Let  $(x, y) \in X \times Y$  and  $\sigma \in G$  and consider

$$\pi_X(\sigma(x, y)) = \pi_X((\sigma x, \sigma y)) = \sigma x = \sigma \pi_X(x, y).$$

That is, indeed,  $\pi_X$  is  $G$ -equivariant. Extremely similar reasoning shows that  $\pi_Y$  is  $G$ -equivariant. Hence  $\pi_X, \pi_Y$  are indeed morphisms in  $\mathbf{Set}_G$ .



(2): Finally, we need to show that  $(X \times Y, \pi_X, \pi_Y)$  satisfy the universal property of binary products. Recall that  $(X \times Y, \pi_X, \pi_Y)$  satisfies the universal property of binary products in **Set**. That is, for any set  $Z$  with functions  $f_X : Z \rightarrow X$  and  $f_Y : Z \rightarrow Y$  we have a unique function  $f : Z \rightarrow X \times Y$  such that the following diagram commutes

$$\begin{array}{ccccc} & & Z & & \\ & f_X \swarrow & \downarrow f & \searrow f_Y & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Moreover, we know that  $f$  necessarily has the form  $f(z) = (f_X(z), f_Y(z))$  for all  $z \in Z$ . If we verify that  $f$  is  $G$ -equivariant, i.e. that it's a valid morphism  $Z \rightarrow X \times Y$  in **Set<sub>G</sub>**, then it will follow that  $(X \times Y, \pi_X, \pi_Y)$  satisfies the universal property in **Set<sub>G</sub>**. So, suppose now that  $Z \in \mathbf{Set}_G$  is a set with a  $G$  action, and that  $f_X : Z \rightarrow X$  and  $f_Y : Z \rightarrow Y$  are  $G$ -equivariant functions. Indeed, let  $\sigma \in G$ , and consider

$$f(\sigma z) = (f_X(\sigma z), f_Y(\sigma z)) = (\sigma f_X(z), \sigma f_Y(z)) = \sigma(f_X(z), f_Y(z)) = \sigma f(z),$$

by definition of  $f$ , by  $G$ -equivariance of  $f_X, f_Y$ , and by definition of the  $G$ -action on  $X \times Y$ . That is,  $f : Z \rightarrow X \times Y$  is a valid morphism in **Set<sub>G</sub>** and it follows that  $f$  is the unique morphism which makes the following diagram commute, now as a diagram in **Set<sub>G</sub>**

$$\begin{array}{ccccc} & & Z & & \\ & f_X \swarrow & \downarrow f & \searrow f_Y & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Hence,  $(X \times Y, \pi_X, \pi_Y)$  satisfies the universal property of binary product in **Set<sub>G</sub>**. It follows then that finite products exist in **Set<sub>G</sub>**.

Do empty products count as a finite product? I think very similar reasoning with products over a set would do the trick. Or, if we really want, we can show that **Set<sub>G</sub>** has an initial object, and we know that we have unary products.

■

6. Prove that each of the following is a functor.

$$F: \mathbf{Ring} \rightarrow \mathbf{Set} \quad F(R) = R$$

$$G: \mathbf{Ring} \rightarrow \mathbf{Gp} \quad G(R) = R^\times$$

$$H: \mathbf{Ring} \rightarrow \mathbf{Ring} \quad H(R) = R^2$$

$$K: \mathbf{Ring} \rightarrow \mathbf{Set} \quad K(R) = \{\text{pairs } (x, y) \in R^2 \text{ such that } y^2 = x^3 - x\}$$

In each case, make sure to specify how your functor acts on morphisms, i.e. if  $f: R \rightarrow S$  is a ring homomorphism, you will need to specify the function  $F(f): F(R) \rightarrow F(S)$ , the group homomorphism  $G(f): G(R) \rightarrow G(S)$ , etc.

still need to proofread this one

We note for reference the definition of a functor here. A functor  $\mathcal{F}$  from categories  $\mathcal{C} \rightarrow \mathcal{D}$  is an association of objects from  $\mathcal{C}$  to objects in  $\mathcal{D}$  such that  $\text{Hom}_{\mathcal{C}}(X, Y)$  has a corresponding association  $\text{Hom}_{\mathcal{D}}(\mathcal{F}X, \mathcal{F}Y)$  for all  $X, Y \in \mathcal{C}$ . Furthermore, this association of morphisms map identity morphisms to identity morphisms,  $\mathcal{F}1_X = 1_{\mathcal{F}X}$  for all  $X \in \mathcal{C}$ , and should respect composition,  $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$  for all  $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$  for all  $X, Y \in \mathcal{C}$ .

- We are given an association of **Ring**  $\rightarrow$  **Set** by associating a ring to its underlying set. We specify that  $F$  acts on morphisms by taking a ring homomorphism  $R \rightarrow S$  to its underlying function on the sets  $R \rightarrow S$ .

Note that the identity morphism on a given ring then is associated to the identity function on the underlying set, and so  $F$  takes identity morphisms to identity morphism. Moreover, recall that composition of ring homomorphisms was defined as the composition of the underlying functions (and then we verified that this was still a ring homomorphism), but then it follows by definition that  $F(g \circ f) \in \text{Hom}_{\mathbf{Set}}(FR, FT)$  is mapped to  $Fg \circ Ff$ , for all  $F, T \in \mathbf{Ring}$ .

- We are  $G$  which takes a ring  $R$  to its set of units  $R^\times$ , with a group structure given by

$R$ -multiplication. Note that this is a well-defined functor, since if  $g \in R^\times$  then it has a two-sided inverse by definition, and so  $R^\times$  is closed under inverses. Moreover,  $R$  must contain a multiplicative identity 1 which is also in  $R^\times$  (it is its own two-sided inverse). The given group operation is associative since ring multiplication is associative.

If  $f : R \rightarrow S$  is a ring homomorphism we associate  $Gf : GR \rightarrow GS$  to be the restriction to  $R^\times$ . We verify that this gives a well defined group homomorphism  $GR \rightarrow GS$ . First note that if  $g \in R^\times$  with two-sided inverse  $h \in R^\times$  then we have  $1 = f(1) = f(gh) = f(g)f(h) = f(h)f(g)$ . That is the image  $f(g)$  has two-sided inverse  $f(h)$  and so  $f(R^\times) \subseteq S^\times$ . Moreover, informally,  $f$  respects the group operation on  $GR$  because  $f$ , by definition, respects the ring multiplication on  $R$ . And so the association  $(f : R \rightarrow S) \mapsto (Gf : GR \rightarrow GS)$  is a valid association of morphisms in **Ring** to morphisms in **Gp**.

Now we verify that that  $G$  sends identity morphisms to identity morphisms and that it respects composition. If  $1_R$  is the identity morphism on a ring  $R$ , then  $G1_R$  is the identity morphism on  $GR$  since it is just the restriction onto a subset of  $R$ . Moreover, if  $g \circ f : R \rightarrow T$  is a composition then the association  $G(g \circ f)$  is the restriction of  $g \circ f$  to  $R^\times$ . Moreover, unpacking the definitions as functions on sets will show that  $(g \circ f)|_{R^\times} = g|_{R^\times} \circ f|_{R^\times}$ . And so  $G$  respects composition of morphisms. Thus,  $G$  is a functor.

- We have a functor from **Ring**  $\rightarrow$  **Ring** by mapping  $R \rightarrow R^2$ . We propose the following association of ring homomorphisms. If we have a ring homomorphism  $f : R \rightarrow S$  then associate  $Hf : R^2 \rightarrow S^2$  by  $Hf(r_1, r_2) := (fr_1, fr_2)$ . This definition is certainly a well defined map from  $R^2$  to  $S^2$ .

Now, we verify that such an association satisfies the axioms of functor. Suppose we have the identity morphism on a ring  $1_R$  and consider, for  $r_1, r_2 \in R$ ,

$$H1_R(r_1, r_2) = (1_R r_1, 1_R r_2) = (r_1, r_2).$$

And so indeed,  $H1_R$  is the identity morphism on  $R^2$ . Now suppose we have a com-

position  $g \circ f : R \rightarrow T$ . Consider the following

$$H(g \circ f)(r_1, r_2) = ((g \circ f)r_1, (g \circ f)r_2) = Hg(fr_1, fr_2) = (Hg)(Hf)(r_1, r_2).$$

That is,  $H$  also respects ring homomorphism composition. Thus, the given  $H$  is indeed a functor.

- We are given a functor **Ring**  $\rightarrow$  **Set**. We propose the following association of ring morphisms to set functions.

Suppose  $f : R \rightarrow S$  is a ring homomorphism and then notice, if  $x, y \in R$  satisfy  $y^2 = x^3 - x$  then we have

$$f(y)^2 - f(x)^3 - f(x) = f(y^2 - x^3 - x) = f(0) = 0,$$

That is  $u := f(x), v := f(y) \in S$  then satisfy  $v^2 = u^3 - u$ . This discussion then shows that the association  $Kf : KR \rightarrow KS$  defined by  $Kf(x, y) = (fx, fy)$  is a well defined set function  $KR \rightarrow KS$ .

We verify that this definition satisfies the axioms of functor. Essentially the same discussion showing that  $F, H$  are functors applies to show that  $K$  sends identity ring morphisms to the identity function  $KR \rightarrow KR$ . Moreover, informally, the same computation showing that  $H$  respects composition also shows that  $K$  respects composition. And so  $K$  is indeed a functor from **Ring**  $\rightarrow$  **Set**.

■

7. Prove that each of the functors in the previous question is representable when viewed as a functor from **Ring** to **Set**. [Hint to get you started: for  $F$ , it will be representable by the polynomial ring  $\mathbb{Z}[x]$ . To prove this, you need to write down a bijection  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \cong R$  for each ring  $R$  and check that your bijection is natural in  $R$ .]

**this is to be read.** For reference, a functor  $K : \mathcal{C} \rightarrow \mathbf{Set}$  is *representable* if there exists an object  $M \in \mathcal{C}$  such that the functor  $\text{Hom}_{\mathcal{C}}(M, -)$  is naturally isomorphic to  $K$ .

- We show that  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], -)$  is naturally isomorphic to  $F$ . First we note that maps  $f : \mathbb{Z}[x] \rightarrow R$  are determined exactly by the image of  $x$  (since  $1 \in \mathbb{Z}[x]$  must map to  $1 \in R$  and the rest of the images are given by using the ring homomorphism properties of  $f$ ).

Next we construct a natural transformation  $\text{Hom}(\mathbb{Z}[x], -) \rightarrow F$ . The data of a natural transformation between these two functors is, an association from each  $R \in \mathbf{Ring}$  to a morphism in **Set**  $f_R : \text{Hom}(\mathbb{Z}[x], R) \rightarrow FR$ . The above discussion gives an “obvious” function  $\text{Hom}(\mathbb{Z}[x], R) \xrightarrow{\sim} R$  in **Set**, namely evaluating the map at  $x$ ,  $f \mapsto f(x)$ . Moreover, this function is in fact a bijection because it has inverse

$$r \mapsto f(\ell) := \begin{cases} 1 & \ell = 1 \\ r & \ell = x \\ \text{“extend using homomorphism property”} & \text{otherwise} \end{cases}$$

Let  $g \in \text{Hom}_{\mathbf{Ring}}(R, S)$ . We are left to show that the above association  $R \mapsto f_R$  makes the following diagram commute in **Set**

$$\begin{array}{ccc} \text{Hom}(\mathbb{Z}[x], R) & \xrightarrow{f_R} & R \\ g' \downarrow & & \downarrow g \\ \text{Hom}(\mathbb{Z}[x], S) & \xrightarrow{f_S} & S \end{array}$$

where  $g'$  is given by the action of the functor  $\text{Hom}(\mathbb{Z}[x], -)$  on  $g : R \rightarrow S$ . Recalling that  $f \in \text{Hom}(\mathbb{Z}[x], R)$  outputs to elements in  $R$  we can see that  $g'(f) = g \circ f$  is a map  $\mathbb{Z}[x] \rightarrow S$ .

We verify that this diagram commutes. Let  $f \in \text{Hom}(\mathbb{Z}[x], R)$ . Following the top route gives the element

$$f \mapsto f(x) \mapsto g(f(x)) = (g \circ f)(x).$$

On the other hand, following the bottom route gives

$$f \mapsto g \circ f \mapsto (g \circ f)(x).$$

And so this diagram commutes.

Thus we have found a natural transformation  $\text{Hom}(\mathbb{Z}[x], -) \rightarrow F$  with each morphism  $f_R$  a bijection. Thus we in fact have  $F$  is naturally isomorphic to  $\text{Hom}(\mathbb{Z}[x], -)$  and so  $F$  is a representable functor, represented by  $\mathbb{Z}[x]$ .

- I will go a bit faster through this argument since many of the details follow similar logic to the above part.

I claim that  $G : \mathbf{Ring} \rightarrow \mathbf{Set}$  is representable functor, represented by the ring  $\mathbb{Z}[x, x^{-1}]$ . We must show that there exists a natural isomorphism  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], -) \rightarrow G$ . First note that the maps  $f \in \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}])$  are still determined by their action on  $x \in \mathbb{Z}[x, x^{-1}]$ . But now, since  $1 = f(1) = f(xx^{-1}) = f(x)f(x^{-1})$  any ring homomorphism  $\mathbb{Z}[x, x^{-1}] \rightarrow R$  must send  $x$  to a unit in  $R$ . That is, the maps  $f \in \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], R)$  are exactly determined by the image on  $x$ , with the restriction that the image of  $x$  must be a unit in  $R$ .

Now we construct a natural transformation  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], -) \rightarrow G$ . Given  $R \in \mathbf{Ring}$  we assign the function  $f_R : \text{Hom}(\mathbb{Z}[x, x^{-1}], R) \rightarrow R^\times$  given by evaluating the input map at  $x$ ,  $f_R(f) := f(x)$ . We note that under this definition  $f_R$  is a well defined map into  $R^\times$  because, as we noted above, the image of  $x$  under a ring homomorphism  $\mathbb{Z}[x, x^{-1}]$  must be a unit. Next, notice that each  $f_R$  is actually a bijection since it has inverse  $r \in R^\times$  maps to the homomorphism where  $f(x) = r$ , using similar reasoning to the discussion for  $F$  above.

Lastly, we need to show that with these definitions, and given a ring morphism

$g : R \rightarrow S$ , the following diagram commutes

$$\begin{array}{ccc} \text{Hom}(\mathbb{Z}[x, x^{-1}], R) & \xrightarrow{f_R} & R \\ g' \downarrow & & \downarrow g \\ \text{Hom}(\mathbb{Z}[x, x^{-1}], S) & \xrightarrow{f_S} & S \end{array}$$

Following the top side we have

$$f \mapsto f(x) \mapsto (g \circ f)(x),$$

meanwhile following the bottom side gives

$$f \mapsto g \circ f \mapsto (g \circ f)(x).$$

And so this diagram commutes for any ring homomorphism. We have found a natural isomorphism from  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], -) \rightarrow G$  and so  $G$  is a representable functor, represented by the ring  $\mathbb{Z}[x, x^{-1}]$ .

- We claim that  $H$  can be represented by the ring  $\mathbb{Z}[x, y]$ . **Notes to self: maps out of this ring are determined exactly by the images of  $x, y$ . similar reasoning to showing  $\mathbb{Z}[x]$  represented  $F$  then ensues.**
- We claim that  $K$  is can be represented by the ring  $\tilde{Z} = \mathbb{Z}[x, y] / ((y^2 - x^3 - x^2))$ . To prove this claim we need to find a natural bijection of sets  $\text{Hom}(\tilde{Z}, R) \xrightarrow{\sim} K(R)$  for each  $R \in \mathbf{Ring}$ .

**we definitely want to read over this paragraph again.** Let us first describe the maps  $\mathbb{Z}[x, y]/I \rightarrow R$  for some fixed ring  $R$ . Given two elements  $r_1, r_2 \in R$  there is a unique ring homomorphism  $f : \mathbb{Z}[x, y] \rightarrow R$  with  $f(x) = r_1$  and  $f(y) = r_2$ . If  $f$  further satisfies  $f(I) = 0$  then, by the universal property of quotients, there exist a unique ring morphism  $\bar{f} : \tilde{Z} \rightarrow R$  making the following diagram commute

$$\begin{array}{ccc} \mathbb{Z}[x, y] & \xrightarrow{f} & R \\ \pi \downarrow & \nearrow \bar{f} & \\ \mathbb{Z}[x, y]/I & & \end{array}$$

That is,  $\tilde{f} : \tilde{Z} \rightarrow R$  satisfying  $\tilde{f}(\tilde{x}) = r_1$  and  $\tilde{f}(\tilde{y}) = r_2$ . Conversely, a map in  $\tilde{f} \in \mathbb{Z}[x, y]/I$  is determined exactly by homomorphisms  $f : \mathbb{Z}[x, y] \rightarrow R$  satisfying  $f(I) = 0$ . Suppose we have a ring morphism  $\tilde{g} : \mathbb{Z}[x, y]/I \rightarrow R$  then we have a ring homomorphism  $(\pi \circ g) : \mathbb{Z}[x, y] \rightarrow R$  satisfying  $(\pi \circ g)(I) = 0$ . The universal property of quotients then recovering the map  $g$ .  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y]/I, R) \xrightarrow{\sim} \{f \in \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y], R) : f(I) = 0\}$  And so a homomorphism  $\mathbb{Z}[x, y]/I \rightarrow R$  is determined by the images of  $x, y$ .

Now, still with a fixed ring  $R$ , consider the function  $\text{Hom}_{\mathbf{Ring}}(\tilde{Z}, R) \rightarrow K(R)$  defined by  $f \mapsto (f(x), f(y))$ . We verify that this is a well-defined function to  $K(R)$ . Since,  $f$  is a ring homomorphism, we have

$$f(y)^3 - f(x)^3 - f(x)^2 = f(y^3 - x^3 - x^2) = 0$$

By definition of  $f$ . And so indeed the pair  $(f(x), f(y))$  is an element in  $K(R)$ . Moreover, this function is in fact a bijection. Given  $r_1, r_2 \in K(R)$  we have a unique function  $f : \mathbb{Z}[x, y] \rightarrow R$  determined by  $f(x) = r_1$  and  $f(y) = r_2$ . Furthermore, by definition of  $r_1, r_2$  we further have

$$f(y^2 - x^3 + x) = r_2^2 - r_1^3 + r_1 = 0.$$

That is,  $f(I) = 0$  and so  $f$  has a unique extension to the quotient  $\mathbb{Z}[x, y]/I \rightarrow R$ . That is, the components  $\alpha_R$  are isomorphisms for all  $R \in \mathbf{Ring}$ .

Lastly, given a ring homomorphism  $f : R \rightarrow S$ , we need to verify that our definition for the components  $\alpha_R$  makes the following diagram commute

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y]/I, R) & \xrightarrow{\alpha_R} & K(R) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y]/I, S) & \xrightarrow{\alpha_S} & K(S) \end{array}$$

Before, we can check commutivity of this diagram, we should be clear about the vertical maps. Given  $h : \mathbb{Z}[x, y]/I \rightarrow R$  the image of  $f$  under the functor  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y]/I, -)$  sends  $h$  to  $f \circ h$ . Moreover, given  $(r_1, r_2) \in KR$ , the image of  $f$  under the functor  $K$



sends  $(r_1, r_2)$  to  $(f(r_1), f(r_2))$ . See the solution to the previous question to see that this is the correct image under  $K$ .

Now, for some fixed  $R, S \in \mathbf{Ring}$  and  $f : R \rightarrow S$ , given  $h \in \text{Hom}(\mathbb{Z}[x, y]/I, R)$  following the top path gives

$$h \mapsto (h(x), h(y)) \rightarrow ((f \circ h)(x), (f \circ h)(y)),$$

whereas, following the bottom path gives

$$h \mapsto f \circ h \mapsto ((f \circ h)(x), (f \circ h)(y)).$$

And so we have found a natural bijection  $\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x, y]/I, R) \xrightarrow{\sim} K(R)$  for each  $R \in \mathbf{Ring}$ . Hence the functor  $K$  is represented by the ring  $\mathbb{Z}[x, y]/(y^2 - x^3 + x)$ .

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