Algorithms HW

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4. Let
$$1 \to N \to G \xrightarrow{\pi} H \to 1$$
be an extension of groups. Show that there is a homomorphism
$$\rho \colon H \to \mathrm{Out}(N)$$
sending an element $h \in H$ to the outer automorphism of N given by conjugation by any $\tilde{h} \in G$ such that $\pi(\tilde{h}) = h$. In the particular case that $G = N \rtimes_\theta H$ is the semidirect product of H by N via θ, show that ρ is equal to the composition
$$H \xrightarrow{\theta} \mathrm{Aut}(N) \to \mathrm{Out}(N)\,.$$

Firstly, we will show that ρ is a well defined map $H \to Out(N)$. Let $h \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = \pi(\tilde{h}_2) = h$. We have $\rho(\tilde{h}_1) = f := (n \mapsto \tilde{h}_1 n \tilde{h}_1^{-1})$ and $\rho(\tilde{h}_2) = g := (n \mapsto \tilde{h}_2 n \tilde{h}_2^{-1})$. Note that these are indeed automorphisms of N, as in the previous homework we showed that conjugation by a fixed element is an automorphism. If we show that $\rho(\tilde{h}_1)$ and $\rho(\tilde{h}_2)$ lie in the same coset of $Inn(N)$ then ρ is well-defined. (Note: I believe this map is not well defined as a map $H \to Aut(N)$).

Recall that two elements g, h of a group lie in the same coset of a normal subgroup N if $g^{-1}h \in N$. For our automorphisms f, g we have $g^{-1} = (n \mapsto \tilde{h}_2^{-1} n \tilde{h}_2)$. And so we have $(g^{-1} \circ f)(n) = \tilde{h}_2^{-1}\tilde{h}_1 n \tilde{h}_1^{-1}\tilde{h}_2$. Recall that $N \trianglelefteq G$ and so is closed under conjugation by definition. In particular then $\tilde{h}_1 n \tilde{h}_1^{-1} \in N$ and $\tilde{h}_2^{-1}(\tilde{h}_1 n \tilde{h}_1^{-1})\tilde{h}_2 \in N$ since $\tilde{h}_1, \tilde{h}_2 \in G$. Thus f, g have the same image in $Out(N)$ and so ρ is well defined with respect to the choice of $\tilde{h}$.

Next we show that ρ is a group homomorphism. Let $h_1, h_2 \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = h_1$ and $\pi(\tilde{h}_2) = h_2$. Moreover, since π is a group homomorphism we have $\pi(\tilde{h}_1\tilde{h}_2) = \tilde{h}_1\tilde{h}_2$. Following a similar, calculation to last week's homework, consider the following

$$\begin{aligned}
\rho(h_1h_2) &= \gamma_{\tilde{h}_1\tilde{h}_2} \\
&= (n \mapsto \tilde{h}_1\tilde{h}_2 n(\tilde{h}_1\tilde{h}_2)^{-1}) \\
&= (n \mapsto \tilde{h}_1\tilde{h}_2 n\tilde{h}_2^{-1}\tilde{h}_1^{-1}) \\
&= \gamma_{\tilde{h}_1} \circ \gamma_{\tilde{h}_2} \\
&= \rho(h_1)\rho(h_2).
\end{aligned}$$

Thus, the given ρ is indeed a group homomorphism.

Now suppose $G = N \rtimes_\theta H$. We can state more precisely the outer automorphism given by ρ. Let $h \in H$ and then all lifts are of the form $\tilde{h} = (m, h)$ for some $m \in N$. Then, being explicit about the details of the semidirect product, our map $\rho(h) : \iota(N) \to \iota(N)$ acts as follows

$$\begin{aligned}
\rho_h(n) &= (m,h) \cdot_\theta (n, e_H) \cdot_\theta (m,h)^{-1} \\
&= (m,h)(n,e_H)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\
&= (m\theta_h(n), h)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\
&= (m\theta_h(n)(\theta_h \circ \theta_{h^{-1}}(m^{-1}), hh^{-1}) \\
&= (m\theta_h(n)m^{-1}, e_H).
\end{aligned}$$

Which induces the automorphism $f = (n \mapsto m\theta_h(n)m^{-1}) : N \to N$. Note that $(\theta_h\theta_{h^{-1}}) = id_H$ since θ is a group homomorphism $H \to Aut(N)$.

We show that this is the same as the composition $H \to Aut(N) \to Out(N)$. We have $h \mapsto \theta_h \mapsto \overline{\theta_h}$. Notice now that θ_h and f are lie in the same coset of $Inn(N)$. In particular

$$\overline{\theta_h} = \overline{\gamma_m\theta_h} = \overline{f}$$

since $\gamma_m = (n \mapsto mnm^{-1})$ is one of the inner automorphisms of N. Hence, in the case where $G = N \rtimes_\theta H$ we have ρ and $H \to Aut(N) \to Out(N)$ give the same map.

One interpretation of this is that, whilst ρ is a well defined map $H \to Out(N)$, it is not a well defined map $H \to Aut(N)$. However, in the case where G is a semidirect product of

N and H via θ, we have a preferred lift $h \mapsto (e_N, h) \in G$, and in fact there is a well defined map $H \to Aut(N)$, namely θ, whose projection gives the same map as ρ.

$\blacksquare$

5. (Aluffi Exercise IV.5.15) Let G be a group of order 28.

- Prove that G contains a subgroup of order 4, and a normal subgroup of order 7. Deduce that G is either a split extension of C_4 by C_7, or is a split extension of $C_2 \times C_2$ by C_7.
- Prove that there are only two homomorphisms $C_4 \to \text{Aut}(C_7)$ and only two homomorphisms $C_2 \times C_2 \to \text{Aut}(C_7)$, up to changing the choice of generators for C_4 and $C_2 \times C_2$.
- Deduce that there are exactly four groups of order 28, up to isomorphism.

- Sylow's theorem I gives us that there exists a subgroup of order 7 in G, since $|H| = 7^1 \cdot 4$ and $7 \not| 4$. Alternatively, Cauchy's theorem gives us that there exists an element $g \in G$ with $|g| = 7$, hence we have $|\langle g \rangle| \leq G$. Moreover, Sylow III gives us that there's only a single Sylow 7 group. Consider, if n_7 is the number of Sylow 7 groups in G then Sylow III gives us that $n_7 \equiv 1 \mod 7$ and $n_7 | 4$. The only integer solving both these conditions is $n_p = 1$.

  Next we argue that N is normal. If $g \in G$ then recall $\gamma_g = (\ell \mapsto g \ell g^{-1}) \in Aut(G)$. Therefore $|\gamma_g(N)| = |N|$. However, there's a unique subgroup of order 7 in G and so the image $\gamma_g(N) = N$ for all $g \in G$. That is, N is closed under conjugation by elements in G and so N is normal by definition. We have shown that G has a normal subgroup of order 7 and in fact we have found that $N \cong C_7$.

  I'm not super sure why there's a subgroup of order 4.

- Recall or perhaps I shall prove that $Aut(N) = Aut(C_7) \cong C_6$. Consider C_4, once we have specified where a generator $\sigma \in C_4$ is mapped to in C_6 then we have determined the homomorphism $C_4 \to C_6$. Since $|\sigma| = 4$ we must have $|\theta(\sigma)| = 4$ or $|\theta(\sigma)| = 2$, for θ non-trivial, since a homomorphism must map an element to an element whose order divides the original order. Notice that there's only a single element of order 2 in C_6. And so there's one trivial map and one non-trivial map $\overline{\theta} : C_4 \to N$. Since $\overline{\theta}(\sigma)$ has order two we can deduce that it is the automorphism which sends each element of C_7 to its inverse. That is $\overline{\theta}(\sigma) = (n \mapsto 7 - n)$. And, of course, the trivial map $\theta_{\text{triv}}(\sigma) = (n \mapsto 0)$ for each $\sigma \in C_4$.

  We use similar reasoning to determine the maps $\theta : C_2 \times C_2 \to Aut(N) \cong C_6$.

One generating set of $C_2 \times C_2$ is $\{(0,1),(1,0)\}$ and again, once we determine where these elements are mapped to by θ we have determine the entire homomorphism $\theta : C_2 \times C_2 \to C_6$. Now each generating element has order two, and so any non-trivial θ maps both the generating elements to the unique element of order 2 in C_6. And so, again, we have one trivial map $\theta_{\text{triv}} : C_2 \times C_2 \to C_6$ and one non-trivial map $\tilde{\theta} : C_2 \times C_2 \to C_6$. The automorphisms $\tilde{\theta}((0,1)) = \tilde{\theta}(1,0)$ are both the same as the one described above — $(n \mapsto 7 - n)$.

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