Algorithms HW

1. (Lang Exercise I.6, Aluffi Exercise II.4.8) Let G be a group, and let $g \in G$ be an element. Let $\gamma_g : G \to G$ be the function given by $h \mapsto ghg^{-1}$. Show that:

   - γ_g is an automorphism of G;
   - the function $G \to \mathrm{Aut}(G)$ given by $g \mapsto \gamma_g$ is a homomorphism;
   - the image of the homomorphism $G \to \mathrm{Aut}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

   (The image is the group $\mathrm{Inn}(G)$ of *inner automorphisms* of G, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the *outer automorphism group* of G.)

1. We show that γ_g is a bijective homomorphism, for some fixed $g \in G$. Let $k, \ell \in G$ then we have

   $$\gamma_g(k \cdot \ell) = g \cdot (k \cdot \ell) \cdot g^{-1} = g \cdot k \cdot e \cdot \ell \cdot g^{-1} = (g \cdot k \cdot g^{-1}) \cdot (g \cdot \ell \cdot g^{-1}) = \gamma_g(k) \cdot \gamma_g(\ell),$$

   since group products are associative and by definition of the identity element. Hence γ_g is a homomorphism for all $g \in G$.

   Now suppose $\gamma_g(h) = e$ for some $h \in G$ we have

   $$\begin{aligned} \gamma_g(h) &= e \\ ghg^{-1} &= e \\ (g^{-1}g)h(g^{-1}g) &= g^{-1}eg \\ h &= g^{-1}eg \\ h &= e. \end{aligned}$$

   Thus, $\gamma_g(h)$ is injective. Now let $k \in G$ and notice that $\gamma_g(g^{-1}kg) = g \cdot g^{-1}kg^{-1}g = k$. Moreover, $g^{-1}kg \in G$ since G is closed under its group operation. That is, γ_g is surjective for all $g \in G$. Hence, we have shown that γ_g is an automorphism of G.

2. Let $g, h \in G$. And let $f : G \to Aut(G)$ be the map $f(g) = \gamma_g$.

Consider the action of γ_{gh} on some group element k. We have

$$\begin{aligned}\gamma_{gh}(k) &= (gh)k(gh)^{-1} \\ &= (gh)k(h^{-1}g^{-1}) \\ &= g(hkh^{-1})g^{-1} \\ &= (\gamma_g \circ \gamma_h)(k),\end{aligned}$$

holds for all $k \in G$. That is, we have shown $f(g \cdot h) = f(g) \circ f(h)$, where $\cdot$ denotes the product in G and $\circ$ denotes function composition — the group operation in $Aut(G)$. Hence, f is a homomorphism

3. We show directly that im f is closed under conjugation by homomorphism in $Aut(G)$. Let $h \in Aut(G)$ and $\gamma_g \in$ im f. There then exists an inverse homomorphism h^{-1} and consider the action of

$$h \circ \gamma_g \circ h^{-1}.$$

This is an automorphism since the composition of group homomorphisms is again a group homomorphism check this.

Let $k \in G$ and consider

$$\begin{aligned}(h \circ \gamma_g \circ h^{-1})(k) &= h(g \cdot h^{-1}(k) \cdot g^{-1}) \\ &= h(g) \cdot k \cdot h(g^{-1}), \qquad \text{since } h \text{ is a homomorphism}\end{aligned}$$

Moreover, $h(g) = g' \in G$ since h is an automorphism of G. That is, we have shown $(h \circ \gamma_g \circ g^{-1}) = f(g') \in$ im f. And so, im f is a normal sunbgroup of $Aut(G)$ by definition.

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