Algorithms HW

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4. Prove the following identities for tensor products (where M, N, L are arbitrary R-modules):

   - $M \otimes_R N \cong N \otimes_R M$ ("commutativity")
   - $R \otimes_R M \cong M$ ("identity")
   - $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$ ("distributivity")
   - (harder) $M \otimes_R (N \otimes_R L) \cong (M \otimes_R N) \otimes_R L$ ("associativity")

   [Hint: the universal property of tensor products is a handy way of defining R-linear maps out of tensor products]

- Consider the map $M \times N \to N \otimes_R M$ given by $((m,n) \mapsto n \otimes m)$. Notice that this map is R-bilinear on $M \times N$ since $\otimes : M \times N \to M \otimes_R N$ is. Then by the universal property of tensor products we have a unique R-linear map $f : M \otimes_R N \to N \otimes_R M$ such that $f \circ \otimes = (m,n) \mapsto n \otimes m$. The same argument on the R-bilinear map $N \times M \to N \otimes_R M$ given by $(n,m) \mapsto (m \otimes n)$ gives a unique R-linear map $\tilde{f} : N \otimes_R M \to M \otimes_R N$ such that[1] $\tilde{f} \circ \times = (n,m) \mapsto m \otimes n$.

  Notice now that $f \circ \tilde{f} = id_{N \otimes_R M}$ and $\tilde{f} \circ f = id_{M \otimes_R N}$. Indeed $(f \circ \tilde{f})(m \otimes n) = f(n \otimes m) = m \otimes n$. And likewise for the other direction.

  That is, we have found a bijective R-linear map $M \otimes_R N \to N \otimes_R M$ and so in fact $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

- Consider the following map $M \times N \to M$ via $(r,m) \mapsto rm$ given by the module structure on M. Notice that this map is bilinear come back and write the computation out And so by the universal property of tensor products we have a unique R-linear map $f : R \otimes_R M \to M$ such that $r \otimes n \mapsto rm$.

  I claim that this map is bijective and so is an isomorphism of R-modules. First notice that f is surjective. Indeed, if $m \in M$ then $f(1 \otimes m) = 1 \cdot m = m$, so long as R is not the zero ring. If R is the zero ring, then M must be the zero module, and then our desired isomorphism trivially holds.

  Next we show that f is injective. Suppose we have $r \cdot m' = m$ for some $m, m' \in M$

[1] The "$\otimes$" in the following phrase now refers to the R-bilinear map $N \times M \to N \otimes_R M$, whereas earlier it referred to the R-bilinear map $M \times N \to M \otimes_R N$.

and $r \in R$. Then notice

$$r \otimes m' = 1 \cdot r \otimes m' = r \otimes r \cdot m' = 1 \otimes m,$$

And so $rm' = m$ implies $r \otimes m' = 1 \otimes m$. This suffices to show that f is injective because if generally we have $r_1 m_1 = r_2 m_2$ then by definition $r_1 m_1 = m'$ for some $m' \in M$ and then we have $m' = r_2 m_2$.

Overall we have a bijective R-linear map $R \otimes_R M \to M$, and so $R \otimes_R M \cong M$.

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