

Algorithms HW

1. Consider the following functions (morphisms in **Set**):

$$\begin{aligned}
 f: \{1, 2, 3\} &\rightarrow \{1, 2, 3, 4\} & f(x) &= \begin{cases} x & \text{if } x \leq 2 \\ 4 & \text{if } x = 3 \end{cases} \\
 g: \{1, 2, 3, 4\} &\rightarrow \{1, 2, 3\} & g(x) &= \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases} \\
 h: \{1, 2, 3\} &\rightarrow \{1, 2\} & h(x) &= \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases} \\
 k: \{1, 2\} &\rightarrow \{1, 2, 3\} & k(x) &= \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}
 \end{aligned}$$

Show that the square

$$\begin{array}{ccc}
 \{1, 2, 3\} & \xrightarrow{f} & \{1, 2, 3, 4\} \\
 \downarrow h & & \downarrow g \\
 \{1, 2\} & \xrightarrow{k} & \{1, 2, 3\}
 \end{array}$$

commutes.

To show that the given square commutes, we must show that $(gf)(x) = (kh)(x)$ for all $x \in \{1, 2, 3\}$. Consider the image of $1 \in \{1, 2, 3\}$:

$$gf(1) = g(1) = 1 = k(1) = kh(1).$$

Now, consider the image of $2 \in \{1, 2, 3\}$:

$$gf(2) = g(2) = 1 = k(1) = kh(2).$$

Finally, here's the image of 3:

$$gf(3) = g(4) = 3 = k(2) = kh(3).$$

Hence, the given square commutes by definition. Note that we did not need to check whether $\text{im } g = \text{im } k$ (and in fact these images in $\{1, 2, 3\}$ are not equal.) ■

2. We work in the category $\text{Mod}_{\mathbb{R}}$ of real vector spaces. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the \mathbb{R} -linear map given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Let $g: \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \rightarrow \mathbb{R}$

be the \mathbb{R} -linear map induced by the \mathbb{R} -bilinear map

$$\beta: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz.$$

For which \mathbb{R} -linear maps $h: \mathbb{R} \rightarrow \mathbb{R}$ does the square

$$\begin{array}{ccc} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \\ \downarrow f \otimes f & & \downarrow h \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \end{array}$$

commute?

Suppose we have $(w, x) \otimes (y, z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Following this element clockwise around the diagram we have that $(h \circ g)((w, x) \otimes (y, z)) = h(wy + xz)$ and following this element counter-clockwise around the diagram we have $(g \circ f \otimes f)((w, x) \otimes (y, z)) = g((w - x, w + x) \otimes (y - z, y + z)) = (w - x)(y - z) + (w + x)(y + z)$. That is, any \mathbb{R} -linear map $h: \mathbb{R} \rightarrow \mathbb{R}$ must satisfy

$$h(wy + xz) = (w - x)(y - z) + (w + x)(y + z)$$

for all $w, x, y, z \in \mathbb{R}$ is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since h is an \mathbb{R} -linear map we have that

$$h(wy + xz) = h(1)(wy + xz).$$

Moreover, since \mathbb{R} is a rank 1 free module over \mathbb{R} , we have that any \mathbb{R} -linear map $\mathbb{R} \rightarrow \mathbb{R}$ is determined by where it sends the basis $\{1\}$. Given the expression above we have that

any such map h satisfies

$$\begin{aligned} h(1) &= \frac{(w-x)(y-z) + (w+x)(y+z)}{wy+xz} \\ &= \frac{wy - wz - xy + xz + wy + wz + xy + xz}{wy + xz} \\ &= \frac{2(wy + xz)}{wy + xz} \\ &= 2. \end{aligned}$$

That is, there is a single map $h : \mathbb{R} \rightarrow \mathbb{R}$ which makes the above diagram commute — namely the one which sends the basis $1 \mapsto 2$, i.e $h(x) = 2x$.

I'm curious if there's any geometric significance to this thing that we've just shown

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3. Consider the following diagram in an arbitrary category \mathcal{C} :

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow h & & \downarrow h' & & \downarrow h'' \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

Show that if the left and right squares in this diagram commute, then so does the outer rectangle. That is, if the two squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow h' \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow h' & & \downarrow h'' \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

both commute, then so does the square

$$\begin{array}{ccc} X & \xrightarrow{gf} & Z \\ \downarrow h & & \downarrow h'' \\ X' & \xrightarrow{g'f'} & Z' \end{array}$$

We need to show that $h''gf = g'f'h$. Suppose $x \in X$ If \mathcal{C} is not “sets with extra structure” can we still reason about functions by considering their actions on elements in their domain?

Consider the right-handed commuting square. Let $f(x) \in Y$. Since this second square commutes, we have $h''gf = g'h'f$. Moreover, since the left-handed square commutes, we have $h'f = f'h$. Substituting this relation into our first equation gives us

$$h''gf = g'h'f = g'f'h,$$

as desired. ■

4. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms in a category \mathcal{C} . Let $gf: X \rightarrow Z$ be their composition.

- Show that if f and g are both monomorphisms, then gf is a monomorphism.
- Show that if f and g are both epimorphisms, then gf is an epimorphism.
- Show that if gf is a monomorphism, then f is a monomorphism. Must g be a monomorphism?
- Show that if gf is an epimorphism, then g is an epimorphism. Must f be an epimorphism?

For this question, recall that the universal properties of monic maps and of epic maps. Let \mathcal{C} be a category and let $X, Y \in \mathcal{C}$, then a morphism $f: X \rightarrow Y$ is called monic if for all $Z \in \mathcal{C}$ and all $g, h: Z \rightarrow X$ we have $fg = fh$ implies $g = h$. Likewise, $f: X \rightarrow Y$ is called epic if for all $Z \in \mathcal{C}$ and all $g, h: Y \rightarrow Z$ we have $gf = hf$ implies $g = h$.

- Suppose f, g are monic and now consider gf . Let $W \in \mathcal{C}$ and suppose $h, k: W \rightarrow X$ such that $gfh = gfk$. Now, since g is a monomorphism and since function composition in \mathcal{C} is associative, we have $g(fh) = gfh = gfk = g(fk)$ implies $fh = fk$. Now, since f is a monomorphism we have $h = k$. In other words, we have shown that $gfh = gfk$ implies $h = k$ for all morphisms $h, k: Z \rightarrow X$. That is, g, f monic imply that gf is monic.
- Now suppose f, g are epic and let $Z \in \mathcal{C}$ with $h, k: Z \rightarrow W$ such that $hgf = kgf$. Since f is epic we have that $(hg)f = hgf = kgf = (kg)f$ implies $hg = kg$. Moreover, g epic implies that $h = k$. That is, we have $hgf = kgf$ implies $h = k$ and so gf is epic.
- Suppose $gf: X \rightarrow Z$ is a monomorphism. Let $W \in \mathcal{C}$ and $h, k: W \rightarrow X$ such that $fh = fk$. We have that $\text{im}(fh) = \text{im}(fk) \in Y$ and so, since $g = g$ we have that $gfh = gfk$. Now, since gf is monic we have that $h = k$. That is $fh = fk$ implies $h = k$, i.e. f is monic by definition. It is not necessary that g be monic.
- Now suppose gf is epic. Let $W \in \mathcal{C}$ with $h, k: Z \rightarrow W$ such that $hg = kg$. We have that $hgf = kgf$ as maps $X \rightarrow W$. But now, since gf is epic, we have that $h = k$. Thus g is epic by definition. It was not necessary that f be epic.



5. Fix a group G . The category \mathbf{Set}_G of G -sets is defined as follows:

- The objects of \mathbf{Set}_G are sets X with an action of G .
- The morphisms $f: X \rightarrow Y$ in \mathbf{Set}_G are functions $X \rightarrow Y$ which satisfy

$$f(\sigma x) = \sigma f(x)$$

for all $\sigma \in G$ and $x \in X$. (Such functions are called G -equivariant.)

- Composition in \mathbf{Set}_G is given by composition of functions.
- The identity element $1_X: X \rightarrow X$ is the identity function.

Prove carefully that \mathbf{Set}_G is a category (check the axioms). Prove that finite products exist in \mathbf{Set}_G .

Recall the axioms of a category. Given the objects and morphisms of \mathbf{Set}_G we need to verify (1) : that we have a well-defined composition rule, i.e., that the given composition gives us a \mathbf{Set}_G morphism, (2) given our composition rule, that the given identity morphism satisfies $g1_X = 1_X g = g$ for all $g \in \text{Hom}_{\mathbf{Set}_G}(X, X)$ for all $X \in \mathbf{Set}_G$, and (3) that the given composition rule is associative.

Firstly, the by theory of group actions, sets with group actions and G -equivariant functions are a well defined collection of objects and morphisms between those objects.

(1) *Composition:* Let $X, Y, Z \in \mathbf{Set}_G$ and let $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, Z)$. Note that we have a well defined function composition gf from the category \mathbf{Set} . Now we must verify that gf is also G -equivariant. Since f, g are G -equivariant we have, for $\sigma \in G$ and $x \in X$,

$$gf(\sigma x) = g(\sigma f(x)) = \sigma gf(x).$$

And so, $gf \in \text{Hom}_{\mathbf{Set}}(X, Z)$ is G -equivariant by definition and gf is indeed a morphism in $\text{Hom}_{\mathbf{Set}_G}(X, Z)$.

(2) *Identity:* Let $X \in \mathbf{Set}_G$ and let $1_X: X \rightarrow X$ be the identity function on X as a set. Since 1_X is the identity function for X as a set we already have $g1_X = 1_X g = g$ for all functions $g \in \text{Hom}_{\mathbf{Set}}(X, X)$. And so 1_X , if it is G -equivariant, already satisfies $g1_X = 1_X g = g$ for all $g \in \text{Hom}_{\mathbf{Set}_G}(X, X)$. Let $\sigma \in G$ and consider

$$1_X(\sigma x) = \sigma x = \sigma f(x),$$

by definition of the action of 1_X as a function. And so, indeed, 1_X is G -equivariant and so is a morphism in $\text{Hom}_{\mathbf{Set}_X}(X, X)$, thus every $X \in \mathbf{Set}_X$ has an identity morphism.

(3) *Associativity of function composition:* Note that composition of functions is associative since **Set** is a category. It follows immediately that G -equivariant function composition is associative, since the G -equivariant functions from $X \rightarrow Y$ are a “sub-class” of the class of functions $X \rightarrow Y$.

Now we show that finite products exists in \mathbf{Set}_G . ■

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