

Algorithms HW

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4. Let

$$1 \rightarrow N \rightarrow G \xrightarrow{\pi} H \rightarrow 1$$

be an extension of groups. Show that there is a homomorphism

$$\rho: H \rightarrow \text{Out}(N)$$

sending an element $h \in H$ to the outer automorphism of N given by conjugation by any $\tilde{h} \in G$ such that $\pi(\tilde{h}) = h$. In the particular case that $G = N \rtimes_{\theta} H$ is the semidirect product of H by N via θ , show that ρ is equal to the composition

$$H \xrightarrow{\theta} \text{Aut}(N) \rightarrow \text{Out}(N).$$

Firstly, we will show that ρ is a well defined map $H \rightarrow \text{Out}(N)$. Let $h \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = \pi(\tilde{h}_2) = h$. We have $\rho(\tilde{h}_1) = f := (n \mapsto \tilde{h}_1 n \tilde{h}_1^{-1})$ and $\rho(\tilde{h}_2) = g := (n \mapsto \tilde{h}_2 n \tilde{h}_2^{-1})$. Note that these are indeed automorphisms of N , as in the previous homework we showed that conjugation by a fixed element is an automorphism. If we show that $\rho(\tilde{h}_1)$ and $\rho(\tilde{h}_2)$ lie in the same coset of $\text{Inn}(N)$ then ρ is well-defined. (Note: I believe this map is not well defined as a map $H \rightarrow \text{Aut}(N)$).

Recall that two elements g, h of a group lie in the same coset of a normal subgroup N if $g^{-1}h \in N$. For our automorphisms f, g we have $g^{-1} = (n \mapsto \tilde{h}_2^{-1} n \tilde{h}_2)$. And so we have $(g^{-1} \circ f)(n) = \tilde{h}_2^{-1} \tilde{h}_1 n \tilde{h}_1^{-1} \tilde{h}_2$. Recall that $N \trianglelefteq G$ and so is closed under conjugation by definition. In particular then $\tilde{h}_1 n \tilde{h}_1^{-1} \in N$ and $\tilde{h}_2^{-1}(\tilde{h}_1 n \tilde{h}_1^{-1}) \tilde{h}_2 \in N$ since $\tilde{h}_1, \tilde{h}_2 \in G$. Thus f, g have the same image in $\text{Out}(N)$ and so ρ is well defined with respect to the choice of \tilde{h} .

Next we show that ρ is a group homomorphism. Let $h_1, h_2 \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = h_1$ and $\pi(\tilde{h}_2) = h_2$. Moreover, since π is a group homomorphism we have $\pi(\tilde{h}_1 \tilde{h}_2) = \tilde{h}_1 \tilde{h}_2$. Following a similar, calculation to last week's homework, consider the following

$$\begin{aligned}
\rho(h_1 h_2) &= \gamma_{\tilde{h}_1 \tilde{h}_2} \\
&= (n \mapsto \tilde{h}_1 \tilde{h}_2 n (\tilde{h}_1 \tilde{h}_2)^{-1}) \\
&= (n \mapsto \tilde{h}_1 \tilde{h}_2 n \tilde{h}_2^{-1} \tilde{h}_1^{-1}) \\
&= \gamma_{\tilde{h}_1} \circ \gamma_{\tilde{h}_2} \\
&= \rho(h_1) \rho(h_2).
\end{aligned}$$

Thus, the given ρ is indeed a group homomorphism.

Now suppose $G = N \rtimes_{\theta} H$. We can state more precisely the outer automorphism given by ρ . Let $h \in H$ and then all lifts are of the form $\tilde{h} = (m, h)$ for some $m \in N$. Then, being explicit about the details of the semidirect product, our map $\rho(h) : \iota(N) \rightarrow \iota(N)$ acts as follows

$$\begin{aligned}
\rho_h(n) &= (m, h) \cdot_{\theta} (n, e_H) \cdot_{\theta} (m, h)^{-1} \\
&= (m, h)(n, e_H)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\
&= (m\theta_h(n), h)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\
&= (m\theta_h(n)(\theta_h \circ \theta_{h^{-1}}(m^{-1}), hh^{-1}) \\
&= (m\theta_h(n)m^{-1}, e_H).
\end{aligned}$$

Which induces the automorphism $f = (n \mapsto m\theta_h(n)m^{-1}) : N \rightarrow N$. Note that $(\theta_h \theta_{h^{-1}}) = id_H$ since θ is a group homomorphism $H \rightarrow Aut(N)$.

We show that this is the same as the composition $H \rightarrow Aut(N) \rightarrow Out(N)$. We have $h \mapsto \theta_h \mapsto \overline{\theta_h}$. Notice now that θ_h and f lie in the same coset of $Inn(N)$. In particular

$$\overline{\theta_h} = \overline{\gamma_m \theta_h} = \overline{f}$$

since $\gamma_m = (n \mapsto mn m^{-1})$ is one of the inner automorphisms of N . Hence, in the case where $G = N \rtimes_{\theta} H$ we have ρ and $H \rightarrow Aut(N) \rightarrow Out(N)$ give the same map.

One interpretation of this is that, whilst ρ is a well defined map $H \rightarrow Out(N)$, it is not a well defined map $H \rightarrow Aut(N)$. However, in the case where G is a semidirect product of

N and H via θ , we have a preferred lift $h \mapsto (e_N, h) \in G$, and in fact there is a well defined map $H \rightarrow \text{Aut}(N)$, namely θ , whose projection gives the same map as ρ .

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5. (Aluffi Exercise IV.5.15) Let G be a group of order 28.

- Prove that G contains a subgroup of order 4, and a normal subgroup of order 7. Deduce that G is either a split extension of C_4 by C_7 , or is a split extension of $C_2 \times C_2$ by C_7 .
- Prove that there are only two homomorphisms $C_4 \rightarrow \text{Aut}(C_7)$ and only two homomorphisms $C_2 \times C_2 \rightarrow \text{Aut}(C_7)$, up to changing the choice of generators for C_4 and $C_2 \times C_2$.
- Deduce that there are exactly four groups of order 28, up to isomorphism.

- Sylow's theorem I gives us that there exists a subgroup of order 7 in G , since $|H| = 7^1 \cdot 4$ and $7 \nmid 4$. Alternatively, Cauchy's theorem gives us that there exists an element $g \in G$ with $|g| = 7$, hence we have $|\langle g \rangle| \leq G$. Moreover, Sylow III gives us that there's only a single Sylow 7 group. Consider, if n_7 is the number of Sylow 7 groups in G then Sylow III gives us that $n_7 \equiv 1 \pmod{7}$ and $n_7 | 4$. The only integer solving both these conditions is $n_7 = 1$. Lastly, if we let N denote the Sylow 7 group then Normality is somehow acquired from Sylow II and the fact that N is the only subgroup of order 7, but I don't quite understand how this works. Sylow II gives us that that N is acquired from N by conjugation

I'm also not super sure why there's necessarily a subgroup of order 4

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