

Algorithms HW

1. Consider the following functions (morphisms in **Set**):

$$f: \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\} \quad f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 4 & \text{if } x = 3 \end{cases}$$

$$g: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3\} \quad g(x) = \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$h: \{1, 2, 3\} \rightarrow \{1, 2\} \quad h(x) = \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$k: \{1, 2\} \rightarrow \{1, 2, 3\} \quad k(x) = \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}$$

Show that the square

$$\begin{array}{ccc} \{1, 2, 3\} & \xrightarrow{f} & \{1, 2, 3, 4\} \\ \downarrow h & & \downarrow g \\ \{1, 2\} & \xrightarrow{k} & \{1, 2, 3\} \end{array}$$

commutes.

To show that the given square commutes, we must show that $(gf)(x) = (kh)(x)$ for all $x \in \{1, 2, 3\}$. Consider the image of $1 \in \{1, 2, 3\}$:

$$gf(1) = g(1) = 1 = k(1) = kh(1).$$

Now, consider the image of $2 \in \{1, 2, 3\}$:

$$gf(2) = g(2) = 1 = k(1) = kh(2).$$

Finally, here's the image of 3:

$$gf(3) = g(4) = 3 = k(2) = kh(3).$$

Hence, the given square commutes by definition. Note that we did not need to check whether $\text{im } g = \text{im } k$ (and in fact these images in $\{1, 2, 3\}$ are not equal.) ■

2. We work in the category $\text{Mod}_{\mathbb{R}}$ of real vector spaces. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the \mathbb{R} -linear map given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Let $g: \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \rightarrow \mathbb{R}$

be the \mathbb{R} -linear map induced by the \mathbb{R} -bilinear map

$$\beta: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz.$$

For which \mathbb{R} -linear maps $h: \mathbb{R} \rightarrow \mathbb{R}$ does the square

$$\begin{array}{ccc} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \\ \downarrow f \otimes f & & \downarrow h \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R} \end{array}$$

commute?

Suppose we have $(w, x) \otimes (y, z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Following This element clockwise around the diagram we have that $(h \circ g)((w, x) \otimes (y, z)) = h(wy + xz)$ and following this element counter-clockwise around the diagram we have $(g \circ f \otimes f)((w, x) \otimes (y, z)) = g((w - x, w + x) \otimes (y - z, y + z)) = (w - x)(y - z) + (w + x)(y + z)$. That is, any \mathbb{R} -linear map $h: \mathbb{R} \rightarrow \mathbb{R}$ must satisfy

$$h(wy + xz) = (w - x)(y - z) + (w + x)(y + z)$$

for all $w, x, y, z \in \mathbb{R}$ is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since h is an \mathbb{R} -linear map we have that

$$h(wy + xz) = h(1)(wy + xz).$$

Moreover, since \mathbb{R} is a rank 1 free module over \mathbb{R} , we have that any \mathbb{R} -linear map $\mathbb{R} \rightarrow \mathbb{R}$ is determined by where it sends the basis $\{1\}$. Given the expression above we have that

any such map h satisfies

$$\begin{aligned}h(1) &= \frac{(w-x)(y-z) + (w+x)(y+z)}{wy+xz} \\&= \frac{wy-wz-xy+xz+wy+wz+xy+xz}{wy+xz} \\&= \frac{2(wy+xz)}{wy+xz} \\&= 2.\end{aligned}$$

That is, there is a single map $h : \mathbb{R} \rightarrow \mathbb{R}$ which makes the above diagram commute — namely the one which sends the basis $1 \mapsto 2$, i.e $h(x) = 2x$.

I'm curious if there's any geometric significance to this thing that we've just shown

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3. Consider the following diagram in an arbitrary category \mathcal{C} :

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow h & & \downarrow h' & & \downarrow h'' \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

Show that if the left and right squares in this diagram commute, then so does the outer rectangle. That is, if the two squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow h' \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow h' & & \downarrow h'' \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

both commute, then so does the square

$$\begin{array}{ccc} X & \xrightarrow{gf} & Z \\ \downarrow h & & \downarrow h'' \\ X' & \xrightarrow{g'f'} & Z' \end{array}$$

We need to show that $h''gf = g'f'h$. Suppose $x \in X$. If \mathcal{C} is not “sets with extra structure” can we still reason about functions by considering their actions on elements in their domain?

Consider the right-handed commuting square. Let $f(x) \in Y$. Since this second square commutes, we have $h''gf = g'h'f$. Moreover, since the left-handed square commutes, we have $h'f = f'h$. Substituting this relation into our first equation gives us

$$h''gf = g'h'f = g'f'h,$$

as desired. ■

4. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms in a category \mathcal{C} . Let $gf : X \rightarrow Z$ be their composition.

- Show that if f and g are both monomorphisms, then gf is a monomorphism.
- Show that if f and g are both epimorphisms, then gf is an epimorphism.
- Show that if gf is a monomorphism, then f is a monomorphism. Must g be a monomorphism?
- Show that if gf is an epimorphism, then g is an epimorphism. Must f be an epimorphism?

For this question, recall that the universal properties of monic maps and of epic maps. Let \mathcal{C} be a category and let $X, Y \in \mathcal{C}$, then a morphism $f : X \rightarrow Y$ is called monic if for all $Z \in \mathcal{C}$ and all $g, h : Z \rightarrow X$ we have $fg = fh$ implies $g = h$. Likewise, $f : X \rightarrow Y$ is called epic if for all $Z \in \mathcal{C}$ and all $g, h : Y \rightarrow Z$ we have $gf = hf$ implies $g = h$.

- Suppose f, g are monic and now consider gf . Let $W \in \mathcal{C}$ and suppose $h, k : W \rightarrow X$ such that $gfh = gfk$. Now, since g is a monomorphism and since function composition in \mathcal{C} is associative, we have $g(fh) = gfh = gfk = g(fk)$ implies $fh = fk$. Now, since f is a monomorphism we have $h = k$. In other words, we have shown that $gfh = gfk$ implies $h = k$ for all morphisms $h, k : W \rightarrow X$. That is, g, f monic imply that gf is monic.
- Now suppose f, g are epic and let $Z \in \mathcal{C}$ with $h, k : Z \rightarrow W$ such that $hgf = kgf$. Since f is epic we have that $(hg)f = hgf = kgf = (kg)f$ implies $hg = kg$. Moreover, g epic implies that $h = k$. That is, we have $hgf = kgf$ implies $h = k$ and so gf is epic.
- Suppose $gf : X \rightarrow Z$ is a monomorphism. Let $W \in \mathcal{C}$ and $h, k : W \rightarrow X$ such that $fh = fk$. We have that $\text{im}(fh) = \text{im}(fk) \in Y$ and so, since $g = g$ we have that $gfh = gfk$. Now, since gf is monic we have that $h = k$. That is $fh = fk$ implies $h = k$, i.e. f is monic by definition. It is not necessary that g be monic.
- Now suppose gf is epic. Let $W \in \mathcal{C}$ with $h, k : Z \rightarrow W$ such that $hg = kg$. We have that $hgf = kgf$ as maps $X \rightarrow W$. But now, since gf is epic, we have that $h = k$. Thus g is epic by definition. It was not necessary that f be epic.

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5. Fix a group G . The category \mathbf{Set}_G of G -sets is defined as follows:

- The objects of \mathbf{Set}_G are sets X with an action of G .
- The morphisms $f: X \rightarrow Y$ in \mathbf{Set}_G are functions $X \rightarrow Y$ which satisfy

$$f(\sigma x) = \sigma f(x)$$

for all $\sigma \in G$ and $x \in X$. (Such functions are called G -equivariant.)

- Composition in \mathbf{Set}_G is given by composition of functions.
- The identity element $1_X: X \rightarrow X$ is the identity function.

Prove carefully that \mathbf{Set}_G is a category (check the axioms). Prove that finite products exist in \mathbf{Set}_G .

Recall the axioms of a category. Given the objects and morphisms of \mathbf{Set}_G we need to verify (1) : that we have a well-defined composition rule, i.e., that the given composition gives us a \mathbf{Set}_G -morphism, (2) given our composition rule, that the given identity morphism satisfies $g1_X = 1_Xg = g$ for all $g \in \text{Hom}_{\mathbf{Set}_G}(X, X)$ for all $X \in \mathbf{Set}_G$, and (3) that the given composition rule is associative.

Firstly, the by theory of group actions, sets with group actions and G -equivariant functions are a well defined collection of objects and morphisms between those objects.

(1) *Composition*: Let $X, Y, Z \in \mathbf{Set}_G$ and let $f \in \text{Hom}_{\mathbf{Set}_G}(X, Y)$ and $g \in \text{Hom}_{\mathbf{Set}_G}(Y, Z)$. Note that we have a well defined function composition gf from the category \mathbf{Set} . Now we must verify that gf is also G -equivariant. Since f, g are G -equivariant we have, for $\sigma \in G$ and $x \in X$,

$$gf(\sigma x) = g(\sigma f(x)) = \sigma gf(x).$$

And so, $gf \in \text{Hom}_{\mathbf{Set}_G}(X, Z)$ is G -equivariant by definition and gf is indeed a morphism in $\text{Hom}_{\mathbf{Set}_G}(X, Z)$.

(2) *Identity*: Let $X \in \mathbf{Set}_G$ and let $1_X: X \rightarrow X$ be the identity function on X as a set. Since 1_X is the identity function for X we already have $g1_X = 1_Xg = g$ for all functions $g \in \text{Hom}_{\mathbf{Set}}(X, X)$. And so 1_X , if it is G -equivariant, satisfies the axiom for identity G -Set morphism. Let $\sigma \in G$ and consider

$$1_X(\sigma x) = \sigma x = \sigma f(x),$$

by definition of the action of 1_X as a function. And so, indeed, 1_X is G -equivariant and so is a morphism in $\text{Hom}_{\mathbf{Set}_X}(X, X)$, thus every $X \in \mathbf{Set}_X$ has an identity morphism.

(3) *Associativity of function composition*: Note that composition of functions is associative since \mathbf{Set} is a category. It follows immediately that G -equivariant function composition is associative, since the G -equivariant functions from $X \rightarrow Y$ are a “sub-class” of the class of functions $X \rightarrow Y$.

Now we show that finite products exist in \mathbf{Set}_G . We claim that binary products exist in \mathbf{Set}_G (and so it will follow that finite products exist in \mathbf{Set}_G by iterating the construction for binary products). Recall that the universal property for binary products is the pullback of the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$. I claim that, given $X, Y \in \mathbf{Set}_G$, the cartesian product $X \times Y$ with the usual projections π_X, π_Y satisfy the universal property for \mathbf{Set}_G . To verify this claim we have two things to show: (1) that the cartesian product has some G -action for which π_X and π_Y are G -equivariant (i.e. that $X \times Y$ is indeed an object in \mathbf{Set}_G and $\pi_X : X \times Y \rightarrow X, \pi_Y : X \times Y \rightarrow Y$ are indeed morphisms in \mathbf{Set}_G). And (2), that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of products

(1): We show that $X \times Y$ has a G -action which makes π_X, π_Y into G -equivariant functions. Recall that given $X, Y \in \mathbf{Set}_G$ we can construct the set $X \times Y := \{(x, y) : x \in X, y \in Y\}$. Define a G -action on $X \times Y$ by $\sigma(x, y) = (\sigma x, \sigma y)$ for $\sigma \in G$ and where $\sigma x, \sigma y$ are given by the G -action structure on X, Y . We verify that this is indeed a G -action on $X \times Y$. Note that if $1 \in G$ is the identity element of G then we have $1(x, y) = (1x, 1y) = (x, y)$ for all $(x, y) \in X \times Y$. Moreover, if $\sigma, \delta \in G$ we have $\sigma(\delta(x, y)) = \sigma(\delta x, \delta y) = (\sigma \delta x, \sigma \delta y) = (\sigma \delta)(x, y)$. That is, our proposed action is indeed a G -action on $X \times Y$ and so the set $X \times Y$ is also an object in \mathbf{Set}_G .

Now we verify that the projections $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are G -equivariant. Let $(x, y) \in X \times Y$ and $\sigma \in G$ and consider

$$\pi_X(\sigma(x, y)) = \pi_X((\sigma x, \sigma y)) = \sigma x = \sigma \pi_X(x, y).$$

That is, indeed, π_X is G -equivariant. Extremely similar reasoning shows that π_Y is G -equivariant. Hence π_X, π_Y are indeed morphisms in \mathbf{Set}_G .

(2): Finally, we need to show that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of binary products. Recall that $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property of binary products in **Set**. That is, for any set Z with functions $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ we have a unique function $f : Z \rightarrow X \times Y$ such that the following diagram commutes

$$\begin{array}{ccccc} & & Z & & \\ & f_X \swarrow & \downarrow f & \searrow f_Y & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Moreover, we know that f necessarily has the form $f(z) = (f_X(z), f_Y(z))$ for all $z \in Z$. If we verify that f is G -equivariant, i.e. that it's a valid morphism $Z \rightarrow X \times Y$ in **Set_G**, then it will follow that $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property in **Set_G**. So, suppose now that $Z \in \mathbf{Set}_G$ is a set with a G action, and that $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ are G -equivariant functions. Indeed, let $\sigma \in G$, and consider

$$f(\sigma z) = (f_X(\sigma z), f_Y(\sigma z)) = (\sigma f_X(z), \sigma f_Y(z)) = \sigma(f_X(z), f_Y(z)) = \sigma f(z),$$

by definition of f , by G -equivariance of f_X, f_Y , and by definition of the G -action on $X \times Y$. That is, $f : Z \rightarrow X \times Y$ is a valid morphism in **Set_G** and it follows that f is the unique morphism which makes the following diagram commute, now as a diagram in **Set_G**

$$\begin{array}{ccccc} & & Z & & \\ & f_X \swarrow & \downarrow f & \searrow f_Y & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Hence, $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property of binary product in **Set_G**. It follows then that finite products exist in **Set_G**.

Do empty products count as a finite product? I think very similar reasoning with products over a set would do the trick. Or, if we really want, we can show that **Set_G** has an initial object, and we know that we have unary products.

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6. Prove that each of the following is a functor.

$$F: \mathbf{Ring} \rightarrow \mathbf{Set} \quad F(R) = R$$

$$G: \mathbf{Ring} \rightarrow \mathbf{Gp} \quad G(R) = R^\times$$

$$H: \mathbf{Ring} \rightarrow \mathbf{Ring} \quad H(R) = R^2$$

$$K: \mathbf{Ring} \rightarrow \mathbf{Set} \quad K(R) = \{\text{pairs } (x, y) \in R^2 \text{ such that } y^2 = x^3 - x\}$$

In each case, make sure to specify how your functor acts on morphisms, i.e. if $f: R \rightarrow S$ is a ring homomorphism, you will need to specify the function $F(f): F(R) \rightarrow F(S)$, the group homomorphism $G(f): G(R) \rightarrow G(S)$, etc.

still need to proofread this one

We note for reference the definition of a functor here. A functor \mathcal{F} from categories $\mathcal{C} \rightarrow \mathcal{D}$ is an association of objects from \mathcal{C} to objects in \mathcal{D} such that $\text{Hom}_{\mathcal{C}}(X, Y)$ has a corresponding association $\text{Hom}_{\mathcal{D}}(\mathcal{F}X, \mathcal{F}Y)$ for all $X, Y \in \mathcal{C}$. Furthermore, this association of morphisms map identity morphisms to identity morphisms, $\mathcal{F}1_X = 1_{\mathcal{F}X}$ for all $X \in \mathcal{C}$, and should respect composition, $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ for all $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$.

- We are given an association of **Ring** \rightarrow **Set** by associating a ring to its underlying set. We specify that F acts on morphisms by taking a ring homomorphism $R \rightarrow S$ to its underlying function on the sets $R \rightarrow S$.

Note that the identity morphism on a given ring then is associated to the identity function on the underlying set, and so F takes identity morphisms to identity morphism. Moreover, recall that composition of ring homomorphisms was defined as the composition of the underlying functions (and then we verified that this was still a ring homomorphism), but then it follows by definition that $F(g \circ f) \in \text{Hom}_{\mathbf{Set}}(FR, FT)$ is mapped to $Fg \circ Ff$, for all $F, T \in \mathbf{Ring}$.

- We are G which takes a ring R to its set of units R^\times , with a group structure given by

R -multiplication. Note that this is a well-defined functor, since if $g \in R^\times$ then it has a two-sided inverse by definition, and so R^\times is closed under inverses. Moreover, R must contain a multiplicative identity 1 which is also in R^\times (it is its own two-sided inverse). The given group operation is associative since ring multiplication is associative.

If $f : R \rightarrow S$ is a ring homomorphism we associate $Gf : GR \rightarrow GS$ to be the restriction to R^\times . We verify that this gives a well defined group homomorphism $GR \rightarrow GS$. First note that if $g \in R^\times$ with two-sided inverse $h \in R^\times$ then we have $1 = f(1) = f(gh) = f(g)f(h) = f(h)f(g)$. That is the image $f(g)$ has two-sided inverse $f(h)$ and so $f(R^\times) \subseteq S^\times$. Moreover, informally, f respects the group operation on GR because f , by definition, respects the ring multiplication on R . And so the association $(f : R \rightarrow S) \mapsto (Gf : GR \rightarrow GS)$ is a valid association of morphisms in **Ring** to morphisms in **Gp**.

Now we verify that that G sends identity morphisms to identity morphisms and that it respects composition. If 1_R is the identity morphism on a ring R , then $G1_R$ is the identity morphism on GR since it is just the restriction onto a subset of R . Moreover, if $g \circ f : R \rightarrow T$ is a composition then the association $G(g \circ f)$ is the restriction of $g \circ f$ to R^\times . Moreover, unpacking the definitions as functions on sets will show that $(g \circ f)|_{R^\times} = g|_{R^\times} \circ f|_{R^\times}$. And so G respects composition of morphisms. Thus, G is a functor.

- We have a functor from **Ring** \rightarrow **Ring** by mapping $R \rightarrow R^2$. We propose the following association of ring homomorphisms. If we have a ring homomorphism $f : R \rightarrow S$ then associate $Hf : R^2 \rightarrow S^2$ by $Hf(r_1, r_2) := (fr_1, fr_2)$. This definition is certainly a well defined map from R^2 to S^2 .

Now, we verify that such an association satisfies the axioms of functor. Suppose we have the identity morphism on a ring 1_R and consider, for $r_1, r_2 \in R$,

$$H1_R(r_1, r_2) = (1_R r_1, 1_R r_2) = (r_1, r_2).$$

And so indeed, $H1_R$ is the identity morphism on R^2 . Now suppose we have a com-

position $g \circ f : R \rightarrow T$. Consider the following

$$H(g \circ f)(r_1, r_2) = ((g \circ f)r_1, (g \circ f)r_2) = Hg(fr_1, fr_2) = (Hg)(Hf)(r_1, r_2).$$

That is, H also respects ring homomorphism composition. Thus, the given H is indeed a functor.

- We are given a functor **Ring** \rightarrow **Set**. We propose the following association of ring morphisms to set functions.

Suppose $f : R \rightarrow S$ is a ring homomorphism and then notice, if $x, y \in R$ satisfy $y^2 = x^3 - x$ then we have

$$f(y)^2 - f(x)^3 - f(x) = f(y^2 - x^3 - x) = f(0) = 0,$$

That is $u := f(x), v := f(y) \in S$ then satisfy $v^2 = u^3 - u$. This discussion then shows that the association $Kf : KR \rightarrow KS$ defined by $Kf(x, y) = (fx, fy)$ is a well defined set function $KR \rightarrow KS$.

We verify that this definition satisfies the axioms of functor. Essentially the same discussion showing that F, H are functors applies to show that K sends identity ring morphisms to the identity function $KR \rightarrow KR$. Moreover, informally, the same computation showing that H respects composition also shows that K respects composition. And so K is indeed a functor from **Ring** \rightarrow **Set**.

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