Algorithms HW

1. (Lang Exercise I.6, Aluffi Exercise II.4.8) Let G be a group, and let $g \in G$ be an element. Let $\gamma_g : G \to G$ be the function given by $h \mapsto ghg^{-1}$. Show that:

   - γ_g is an automorphism of G;
   - the function $G \to \mathrm{Aut}(G)$ given by $g \mapsto \gamma_g$ is a homomorphism;
   - the image of the homomorphism $G \to \mathrm{Aut}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

   (The image is the group $\mathrm{Inn}(G)$ of *inner automorphisms* of G, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is the *outer automorphism group* of G.)

1. We show that γ_g is a bijective homomorphism, for some fixed $g \in G$. Let $k, \ell \in G$ then we have

$$\gamma_g(k \cdot \ell) = g \cdot (k \cdot \ell) \cdot g^{-1} = g \cdot k \cdot e \cdot \ell \cdot g^{-1} = (g \cdot k \cdot g^{-1}) \cdot (g \cdot \ell \cdot g^{-1}) = \gamma_g(k) \cdot \gamma_g(\ell),$$

since group products are associative and by definition of the identity element. Hence γ_g is a homomorphism for all $g \in G$.

Now suppose $\gamma_g(h) = e$ for some $h \in G$ we have

$$\begin{aligned} \gamma_g(h) &= e \\ ghg^{-1} &= e \\ (g^{-1}g)h(g^{-1}g) &= g^{-1}eg \\ h &= g^{-1}eg \\ h &= e. \end{aligned}$$

Thus, $\gamma_g(h)$ is injective. Now let $k \in G$ and notice that $\gamma_g(g^{-1}kg) = g \cdot g^{-1}kg^{-1}g = k$. Moreover, $g^{-1}kg \in G$ since G is closed under its group operation. That is, γ_g is surjective for all $g \in G$. Hence, we have shown that γ_g is an automorphism of G.

2. Let $g, h \in G$. And let $f : G \to Aut(G)$ be the map $f(g) = \gamma_g$.

Consider the action of γ_{gh} on some group element k. We have

$$\begin{aligned}\gamma_{gh}(k) &= (gh)k(gh)^{-1} \\ &= (gh)k(h^{-1}g^{-1}) \\ &= g(hkh^{-1})g^{-1} \\ &= (\gamma_g \circ \gamma_h)(k),\end{aligned}$$

holds for all $k \in G$. That is, we have shown $f(g \cdot h) = f(g) \circ f(h)$, where $\cdot$ denotes the product in G and $\circ$ denotes function composition — the group operation in $Aut(G)$. Hence, f is a homomorphism

3. We show directly that $\operatorname{im} f$ is closed under conjugation by homomorphism in $Aut(G)$. Let $h \in Aut(G)$ and $\gamma_g \in \operatorname{im} f$. There then exists an inverse homomorphism h^{-1} and consider the action of

$$h \circ \gamma_g \circ h^{-1}.$$

This is an automorphism since the composition of group homomorphisms is again a group homomorphism check this.

Let $k \in G$ and consider

$$\begin{aligned}(h \circ \gamma_g \circ h^{-1})(k) &= h(g \cdot h^{-1}(k) \cdot g^{-1}) \\ &= h(g) \cdot k \cdot h(g^{-1}), \qquad \text{since } h \text{ is a homomorphism}\end{aligned}$$

Moreover, $h(g) = g' \in G$ since h is an automorphism of G. That is, we have shown $(h \circ \gamma_g \circ g^{-1}) = f(g') \in \operatorname{im} f$. And so, $\operatorname{im} f$ is a normal sunbgroup of $Aut(G)$ by definition.

■

2. What is the size of the symmetry group of the cube? Explain how you got your answer.

■

3. Determine the conjugacy classes in the alternating group A_6. For each conjugacy class, you should give its size and say what its elements are.

We search for the conjugacy classes of S_n whose elements are even permutations. Note that this is a well-defined notion, since if $\sigma, \tau \in S_n$ are even permutations then $\sigma\tau\sigma^{-1}$ has a transposition decomposition whose length is a product of three even numbers, and is thus even.

Recall that the conjugacy classes are exactly given by the type of a permutation and that the number of valid conjugacy classes correspond to the number of partitions of n. And so, we will enumerate the partitions of 6 and then acquire the conjugacy classes of A_6 by choosing the classes which correspond to even partitions. The partitions of 6 are:

$$[\mathbf{1,1,1,1,1,1,1}] \qquad [2,2,2] \qquad [\mathbf{2,2,1,1}] \qquad [2,1,1,1,1] \qquad [\mathbf{3,3}] \qquad [3,2,1],$$
$$[\mathbf{3,1,1,1}] \qquad [\mathbf{4,2}] \qquad [4,1,1] \qquad [\mathbf{5,1}] \qquad [6].$$

The bolded types are those which correspond to even partitions and so are the conjugacy classes of A_6. Recall that these are the types whose number of entries have the same parity as 6 (i.e. these are the types with an even number of rows). Suppose $\sigma \in S_n$ has type $[a_1, \cdots, a_k]$ then the parity of σ is $(a_1 - 1) + \cdots (a_k - 1)$, since each a_i denotes the length of a cycle which composes σ. Now notice $(a_1 - 1) + \cdots (a_k - 1) = \sum_i a_i - \sum_{i=1}^{k}(-1) = 6 - 2\ell = 6 - 2\ell$ is even. And so indeed the chosen permutations give the conjugacy classes of A_6.

However, we have a bit more counting to do. Recall that a conjugacy $[\sigma] \subseteq S_n$ splits into two conjugacy classes in A_n exactly when the type of σ consists of distinct odd numbers, and otherwise it splits into a single class in A_n. In our case we have $[3,3]$ and $[5,1]$ split into two clases in A_n. Hence, overall we have $1 + 1 + 2 + 1 + 1 + 2 = 8$ conjugacy classes in A_6.

Next we determine the sizes of each conjugacy class in A_6. Note that the classes not of type $[3,3]$ and $[5,1]$ have the same size as the corresponding classes in S_n. The classes of type $[3,3]$ and $[5,1]$ split into two classes of equal sizes in A_6. Recall that the class type gives the sizes of the cycles in cycle decomposition of $\sigma \in [\sigma]$. And so, we can determine the size of each class by counting each distinct way of writing a permutation with the given types. For example, $[2,2,1,1]$ corresponds to $\sigma = (a_1, a_2)(b_1 b_2)(c_1)(d_1)$ where $a_i, b_i, c_i, d_i \in [n]$. There are 6! ways to populate these numbers, but then we have equivalent permutations given by cycling the elements in (a_1, a_2) and (b_1, b_2) and another equivalence given by interchanging the cycles, then a final equivalence given by interchanging the two trivial cycles. We do not need to consider any equivalence given by interchanging the positions of the 2-cycles and the trivial cycles, since this was included in our enumeration of the partitions of 6. And so the number of elements in the class of type $[2,2,1,1]$ is given by $\frac{6!=720}{2\cdot 2\cdot 2\cdot 2} = \frac{720}{16}$.

A similar kind of counting gives us the following data. In the following $|[t_i]|$ means the number of elements in the conjugacy class whose type is given by $[t_i]$.

$$|[1,1,1,1,1,1,1]| = 1 \qquad |[2,2,1,1]| = \frac{720}{16} \qquad |[3,3]| = \frac{720}{3\cdot 3\cdot 2}\cdot\frac{1}{2} = \frac{720}{36}$$
$$|[3,1,1,1]| = \frac{720}{3\cdot 3!} = \frac{720}{18} \qquad |[4,2]| = \frac{720}{4\cdot 2} = \frac{720}{8} \qquad |[5,1]| = \frac{720}{5}\cdot\frac{1}{2} = \frac{720}{10}$$

Here the classes with type $[3,3]$ and $[5,1]$ in S_n split into two distinct equal sized classes in A_6 and so we have denoted the size of each split class in the data above. Then we can write the class formuala

$$1 + 45 + 2(20) + 40 + 90 + 2(72) = 360 = |A_6|$$

Showing that we have counted the size of our conjugacy classes correctly.

Lastly, we write the elements of our classes. First consider the classes which do not split in A_6. These classes have the same elements in A_6 as they do in S_6. The type of the class tells us the cycle decomposition of its elements. For example the class whose type is $[2,2,1,1]$ contains even permutations whose cycle decomposition is $\sigma = (a_1, a_2)(b_1, b_2)(c_1)(d_1)$ for $a_i, b_i, c_i, d_i \in [n]$ and distinct. Since permutation type is preserved by conjugation, this argument is well defined for a given conjugacy class. The same reasoning applies to the classes whose type is $[1,1,1,1,1,1]$, $[2,2,1,1]$, $[3,1,1,1]$, or $[4,2]$.

The classes in S_6 whose type is $[3,3]$ or $[5,1]$ split into two distinct equal size classes in A_6. how do we figure out which elements belong to which class?

■

4. Let G be a group and $H \leq G$ a subgroup of index 2. Show that H is a normal subgroup of G.

■

5. (Lang Exercise I.15) Let G be a finite group acting transitively on a finite set X, with $\#X \geq 2$. Show that there exists an element $g \in G$ which acts on X without fixed points (i.e. $g \cdot x \neq x$ for all $x \in X$).

■

6. (Lang Exercise I.35) Show that there are exactly two non-abelian groups of order 8, up to isomorphism.

■

7. (Goursat's Lemma, Lang Exercise I.5) Let G_1 and G_2 be groups, and let H be a subgroup of $G_1 \times G_2$ such that the two projections $p_1 \colon H \to G_1$ and $p_2 \colon H \to G_2$ are surjective. Let N_1 be the kernel of p_2, and let N_2 be the kernel of p_1. We can view N_1 and N_2 as subgroups of G_1 and G_2.

   - Show that N_1 is normal in G_1 and N_2 is normal in G_2.
   - Prove that the image of H in $(G_1/N_1) \times (G_2/N_2)$ is the graph of an isomorphism $G_1/N_1 \cong G_2/N_2$.

■