Algorithms HW

1

■

2

■

3

■

4. Let
$$1 \to N \to G \xrightarrow{\pi} H \to 1$$
be an extension of groups. Show that there is a homomorphism
$$\rho\colon H \to \mathrm{Out}(N)$$
sending an element $h \in H$ to the outer automorphism of N given by conjugation by any $\tilde{h} \in G$ such that $\pi(\tilde{h}) = h$. In the particular case that $G = N \rtimes_\theta H$ is the semidirect product of H by N via θ, show that ρ is equal to the composition
$$H \xrightarrow{\theta} \mathrm{Aut}(N) \to \mathrm{Out}(N)\,.$$

Firstly, we will show that ρ is a well defined map $H \to Out(N)$. Let $h \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = \pi(\tilde{h}_2) = h$. We have $\rho(\tilde{h}_1) = f := (n \mapsto \tilde{h}_1 n \tilde{h}_1^{-1})$ and $\rho(\tilde{h}_2) = g := (n \mapsto \tilde{h}_2 n \tilde{h}_2^{-1})$. Note that these are indeed automorphisms of N, as in the previous homework we showed that conjugation by a fixed element is an automorphism. If we show that $\rho(\tilde{h}_1)$ and $\rho(\tilde{h}_2)$ lie in the same coset of $Inn(N)$ then ρ is well-defined. (Note: I believe this map is not well defined as a map $H \to Aut(N)$).

Recall that two elements g, h of a group lie in the same coset of a normal subgroup N if $g^{-1}h \in N$. For our automorphisms f, g we have $g^{-1} = (n \mapsto \tilde{h}_2^{-1} n \tilde{h}_2)$. And so we have $(g^{-1} \circ f)(n) = \tilde{h}_2^{-1} \tilde{h}_1 n \tilde{h}_1^{-1} \tilde{h}_2$. Recall that $N \trianglelefteq G$ and so is closed under conjugation by definition. In particular then $\tilde{h}_1 n \tilde{h}_1^{-1} \in N$ and $\tilde{h}_2^{-1}(\tilde{h}_1 n \tilde{h}_1^{-1})\tilde{h}_2 \in N$ since $\tilde{h}_1, \tilde{h}_2 \in G$. Thus f, g have the same image in $Out(N)$ and so ρ is well defined with respect to the choice of $\tilde{h}$.

Next we show that ρ is a group homomorphism. Let $h_1, h_2 \in H$ and $\tilde{h}_1, \tilde{h}_2 \in G$ such that $\pi(\tilde{h}_1) = h_1$ and $\pi(\tilde{h}_2) = h_2$. Moreover, since π is a group homomorphism we have $\pi(\tilde{h}_1 \tilde{h}_2) = \tilde{h}_1 \tilde{h}_2$. Following a similar, calculation to last week's homework, consider the following

$$\begin{aligned}\rho(h_1h_2) &= \gamma_{\tilde{h}_1\tilde{h}_2} \\ &= (n \mapsto \tilde{h}_1\tilde{h}_2 n(\tilde{h}_1\tilde{h}_2)^{-1}) \\ &= (n \mapsto \tilde{h}_1\tilde{h}_2 n\tilde{h}_2^{-1}\tilde{h}_1^{-1}) \\ &= \gamma_{\tilde{h}_1} \circ \gamma_{\tilde{h}_2} \\ &= \rho(h_1)\rho(h_2).\end{aligned}$$

Thus, the given ρ is indeed a group homomorphism.

Now suppose $G = N \rtimes_\theta H$. We can state more precisely the outer automorphism given by ρ. Let $h \in H$ and then all lifts are of the form $\tilde{h} = (m, h)$ for some $m \in N$. Then, being explicit about the details of the semidirect product, our map $\rho(h) : \iota(N) \to \iota(N)$ acts as follows

$$\begin{aligned}\rho_h(n) &= (m,h) \cdot_\theta (n, e_H) \cdot_\theta (m,h)^{-1} \\ &= (m,h)(n,e_H)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\ &= (m\theta_h(n), h)(\theta_{h^{-1}}(m^{-1}), h^{-1}) \\ &= (m\theta_h(n)(\theta_h \circ \theta_{h^{-1}}(m^{-1}), hh^{-1}) \\ &= (m\theta_h(n)m^{-1}, e_H).\end{aligned}$$

Which induces the automorphism $f = (n \mapsto m\theta_h(n)m^{-1}) : N \to N$. Note that $(\theta_h\theta_{h^{-1}}) = id_H$ since θ is a group homomorphism $H \to Aut(N)$.

We show that this is the same as the composition $H \to Aut(N) \to Out(N)$. We have $h \mapsto \theta_h \mapsto \overline{\theta_h}$. Notice now that θ_h and f are lie in the same coset of $Inn(N)$. In particular

$$\overline{\theta_h} = \overline{\gamma_m\theta_h} = \overline{f}$$

since $\gamma_m = (n \mapsto mnm^{-1})$ is one of the inner automorphisms of N. Hence, in the case where $G = N \rtimes_\theta H$ we have ρ and $H \to Aut(N) \to Out(N)$ give the same map.

One interpretation of this is that, whilst ρ is a well defined map $H \to Out(N)$, it is not a well defined map $H \to Aut(N)$. However, in the case where G is a semidirect product of

N and H via θ, we have a preferred lift $h \mapsto (e_N, h) \in G$, and in fact there is a well defined map $H \to Aut(N)$, namely θ, whose projection gives the same map as ρ.

■

4

■

4

∎

4

■

4

■