Math6310 Algebra Homework #1

1. (Lang Exercise I.6, Aluffi Exercise II.4.8) Let G be a group, and let $g \in G$ be an element. Let $\gamma_g : G \to G$ be the function given by $h \mapsto ghg^{-1}$. Show that:

   - γ_g is an automorphism of G;
   - the function $G \to \mathrm{Aut}(G)$ given by $g \mapsto \gamma_g$ is a homomorphism;
   - the image of the homomorphism $G \to \mathrm{Aut}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

   (The image is the group $\mathrm{Inn}(G)$ of *inner automorphisms* of G, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is the *outer automorphism group* of G.)

1. We show that γ_g is a bijective homomorphism, for some fixed $g \in G$. Let $k, \ell \in G$ then we have

$$\gamma_g(k \cdot \ell) = g \cdot (k \cdot \ell) \cdot g^{-1} = g \cdot k \cdot e \cdot \ell \cdot g^{-1} = (g \cdot k \cdot g^{-1}) \cdot (g \cdot \ell \cdot g^{-1}) = \gamma_g(k) \cdot \gamma_g(\ell),$$

since group products are associative, and by definition of the identity element. Hence γ_g is a homomorphism for all $g \in G$.

Now suppose $\gamma_g(h) = e$ for some $h \in G$ we have

$$\begin{aligned} \gamma_g(h) &= e \\ ghg^{-1} &= e \\ (g^{-1}g)h(g^{-1}g) &= g^{-1}eg \\ h &= g^{-1}eg \\ h &= e. \end{aligned}$$

Thus, $\gamma_g(h)$ is injective. Now let $k \in G$ and notice that $\gamma_g(g^{-1}kg) = g \cdot g^{-1}kg^{-1}g = k$. Moreover, $g^{-1}kg \in G$ since G is closed under its group operation. That is, γ_g is surjective for all $g \in G$. Hence, we have shown that γ_g is an automorphism of G.

2. Let $g, h \in G$. And let $f : G \to Aut(G)$ be the map $f(g) = \gamma_g$.

Consider the action of γ_{gh} on some group element k. We have

$$\begin{aligned}
\gamma_{gh}(k) &= (gh)k(gh)^{-1} \\
&= (gh)k(h^{-1}g^{-1}) \\
&= g(hkh^{-1})g^{-1} \\
&= (\gamma_g \circ \gamma_h)(k),
\end{aligned}$$

holds for all $k \in G$. That is, we have shown $f(g \cdot h) = f(g) \circ f(h)$, where $\cdot$ denotes the product in G and $\circ$ denotes function composition — the group operation in $Aut(G)$. Hence, f is a homomorphism.

3. We show directly that im f is closed under conjugation by homomorphism in $Aut(G)$. Let $h \in Aut(G)$ and $\gamma_g \in \text{im } f$. There then exists an inverse homomorphism h^{-1}, consider the action of

$$h \circ \gamma_g \circ h^{-1}.$$

This is an automorphism since the composition of group homomorphisms is again a group homomorphism.

Let $k \in G$ and consider

$$\begin{aligned}
(h \circ \gamma_g \circ h^{-1})(k) &= h(g \cdot h^{-1}(k) \cdot g^{-1}) \\
&= h(g) \cdot k \cdot h(g^{-1}), && \text{since } h \text{ is a homomorphism}
\end{aligned}$$

Moreover, $h(g) = g' \in G$ since h is an automorphism of G. That is, we have shown $(h \circ \gamma_g \circ g^{-1}) = f(g') \in \text{im } f$. And so, im f is a normal subgroup of $Aut(G)$ by definition.

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2. What is the size of the symmetry group of the cube? Explain how you got your answer.

Big Apologies ■

> 3. Determine the conjugacy classes in the alternating group A_6. For each conjugacy class, you should give its size and say what its elements are.

We search for the conjugacy classes of S_n whose elements are even permutations. Note that this is a well-defined notion, since if $\sigma, \tau \in S_n$ are even permutations then $\sigma\tau\sigma^{-1}$ has a transposition decomposition whose length is a product of three even numbers, and is thus even.

Recall that the conjugacy classes are exactly given by the type of a permutation and that the number of valid conjugacy classes correspond to the number of partitions of n. And so, we will enumerate the partitions of 6 and then acquire the conjugacy classes of A_6 by choosing the classes which correspond to even partitions. The partitions of 6 are:

$$[\mathbf{1,1,1,1,1,1,1}] \qquad [2,2,2] \qquad [\mathbf{2,2,1,1}] \qquad [2,1,1,1,1] \qquad [\mathbf{3,3}] \qquad [3,2,1],$$
$$[\mathbf{3,1,1,1}] \qquad [\mathbf{4,2}] \qquad [4,1,1] \qquad [\mathbf{5,1}] \qquad [6].$$

The bolded types are those which correspond to even partitions, and so correspond to the conjugacy classes of A_6. Recall that these are the types whose number of entries have the same parity as 6 (i.e. these are the types with an even number of rows). Suppose $\sigma \in S_n$ has type $[a_1, \cdots, a_k]$ then the parity of σ is $(a_1 - 1) + \cdots (a_k - 1)$, since each a_i denotes the length of a cycle which composes σ. Now notice $(a_1 - 1) + \cdots (a_k - 1) = \sum_i a_i - \sum_{i=1}^{k}(-1) = 6 - 2\ell = 6 - 2\ell$ is even. And so indeed the chosen permutations give the conjugacy classes of A_6.

However, we have a bit more counting to do. Recall that a conjugacy $[\sigma] \subseteq S_n$ splits into two conjugacy classes in A_n exactly when the type of σ consists of distinct odd numbers, and otherwise it splits into a single class in A_n. In our case we have $[3,3]$ and $[5,1]$ split into two clases in A_n. Hence, overall we have $1+1+2+1+1+2 = 8$ conjugacy classes in A_6.

Next we determine the sizes of each conjugacy class in A_6. Note that the classes not of type $[3,3]$ and $[5,1]$ have the same size as the corresponding classes in S_n. The classes of type $[3,3]$ and $[5,1]$ split into two classes of equal sizes in A_6. Recall that the class type gives the sizes of the cycles in cycle decomposition of $\sigma \in [\sigma]$. And so, we can determine the size of each class by counting each distinct way of writing a permutation with the given types. For example, $[2, 2, 1, 1]$ corresponds to $\sigma = (a_1, a_2)(b_1 b_2)(c_1)(d_1)$ where $a_i, b_i, c_i, d_i \in [n]$. There are 6! ways to populate these numbers, but then we have equivalent permutations given by cycling the elements in (a_1, a_2) and (b_1, b_2), another equivalence given by interchanging the cycles, and a final equivalence given by interchanging the two trivial cycles. We do not need to consider any equivalence given by interchanging the positions of the 2-cycles and the trivial cycles, since this was included in our enumeration of the partitions of 6, by definition. And so the number of elements in the class of type $[2, 2, 1, 1]$ is given by $\frac{6!=720}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{720}{16}$.

A similar kind of counting gives us the following data. In the following $|[t_i]|$ means the number of elements in the conjugacy class whose type is given by $[t_i]$.

$$|[1,1,1,1,1,1,1]| = 1 \qquad |[2,2,1,1]| = \frac{720}{16} \qquad |[3,3]| = \frac{720}{3 \cdot 3 \cdot 2} \cdot \frac{1}{2} = \frac{720}{36}$$
$$|[3,1,1,1]| = \frac{720}{3 \cdot 3!} = \frac{720}{18} \qquad |[4,2]| = \frac{720}{4 \cdot 2} = \frac{720}{8} \qquad |[5,1]| = \frac{720}{5} \cdot \frac{1}{2} = \frac{720}{10}$$

Here the classes with type $[3, 3]$ and $[5, 1]$ in S_n split into two distinct equal sized classes in A_6 and so we have denoted the size of each split class in the data above. Then we can write the class formuala

$$1 + 45 + 2(20) + 40 + 90 + 2(72) = 360 = |A_6|$$

Showing that we have counted the size of our conjugacy classes correctly.

Lastly, we write the elements of our classes. First consider the classes which do not split in A_6. These classes have the same elements in A_6 as they do in S_6. The type of the class tells us the cycle decomposition of its elements. For example the class whose type is $[2, 2, 1, 1]$ contains even permutations whose cycle decomposition is $\sigma = (a_1, a_2)(b_1, b_2)(c_1)(d_1)$ for $a_i, b_i, c_i, d_i \in [n]$ and distinct. Since permutation type is preserved by conjugation, this argument is well defined for a given conjugacy class. The same reasoning applies to the classes whose type is $[1, 1, 1, 1, 1, 1]$, $[2, 2, 1, 1]$, $[3, 1, 1, 1]$, or $[4, 2]$.

The classes in S_6 whose type is $[3, 3]$ or $[5, 1]$ split into two distinct equal size classes in A_6. The two split classes must end up containing all the permutations whose type is $[3, 3]$ or $[5, 1]$ in S_6. Those elements have a similar form to what's argued above.

However, for these split classes, there must be representatives in S_6 which are acquired under conjugation by an odd cycle, hence giving us two classes in A_6. For $[3, 3]$ I could argue one class in A_6 contains $\sigma = (1, 2, 3)(4, 5, 6)$, and then the other class has a representative given by conjugating σ with some odd permutation, and then all other representatives are given by conjugating those elements with even permutations. But this seems less concrete than I would like.

$\blacksquare$

4. Let G be a group and $H \leq G$ a subgroup of index 2. Show that H is a normal subgroup of G.

Recall that a subgroup $H \leq G$ is normal if the set of its left cosets are equal to the set of its right cosets, by definition of normality. That is if $\{gH : g \in G\} = \{Hg : g \in H\}$.

Recall that $[G : H] = 2$ means that H has exactly two left/right cosets[1]. Since $e \in G$, $e \cdot H = H$, and $H \cdot e = H$, it must be that H is one of the left cosets of H and also one of the right cosets of H. Also recall that the left/right cosets of a subgroup partition G as a set. It follows then that the non-H left coset is $G \setminus H$, and the non-H right coset is also $G \setminus H$.

That is, we have shown that the left cosets of H are equal to the right cosets of H and so H must be normal.

Lang proves this using a lot of machinary of the orbits of group actions and the kernel of group actions. That all seems more technical than what I've done here. So I'm a bit worried that I've missed something. ■

[1] Recall that there's a bijection between the left cosets of a subgroup and the right cosets of a subgroup, and so, this is a well-defined quantity.

5. (Lang Exercise I.15) Let G be a finite group acting transitively on a finite set X, with $\#X \geq 2$. Show that there exists an element $g \in G$ which acts on X without fixed points (i.e. $g \cdot x \neq x$ for all $x \in X$).

Note that the same exercise is given in Aluffi IV.1.18, I will be using the hint given in that version of this problem. The hint gives the following information: (1) The action of G on X is isomorphic to G acting on G/H where $H = \text{Stab}(x)$ for some $x \in X$. Here, G acts on G/H by left multiplication. (2) If $H \lneq G$ for finite G then G is not the union of conjugates of H. If time, come back and prove at least the second one.

First, we show that, since G acts transitively on X, if $x \in X$ then $\text{Stab}(x)$ is conjugate to $\text{Stab}(y)$ for all $y \in X$. Suppose g stabilizes x, that is $gx = x$. Since G acts transitively on X we have $y = \overline{g}x$ for some $\overline{g} \in G$. Consider the following

$$\begin{aligned} gx &= x \\ g(\overline{g}y) &= \overline{g}y \\ (\overline{g}^{-1}g\overline{g}) &= y. \end{aligned}$$

That is, a conjugate of g stabilizes y for each $g \in \text{Stab}(x)$. In other words, $g \cdot \text{Stab}(x) \cdot g^{-1} \subseteq \text{Stab}(y)$ for some $g \in G$. A similar calculation gives the reverse inclusion.

Now suppose that, for contradiction, every $g \in G$ fixes some $x \in X$. Then $G \subseteq \bigcup_{x \in X} \text{Stab}(x)$, and so, $G = \bigcup_{x \in X} \text{Stab}(x)$. Let $H = \text{Stab}(x')$ for some $x' \in X$. Notice that $H \lneq G$ because $|X| \geq 2$ and because G acts transitively on X. If H were not proper then we would have $gx' = x'$ for all $g \in G$, however, we know there exists some $x'' \neq x' \in X$ and then no $g \in G$ satisfies $gx' = x''$, a contradiction. From the preceding paragraph, we can rewrite $\text{Stab}(x) = g_x H g_x^{-1}$ for some $g_x \in G$. Then we have

$$G = \bigcup_{x \in X} \text{Stab}(x) = \bigcup_{x \in X} g_x H g_x^{-1},$$

a contradiction of fact (2) above.

$\blacksquare$

6. (Lang Exercise I.35) Show that there are exactly two non-abelian groups of order 8, up to isomorphism.

Big apologies ■

7. (Goursat's Lemma, Lang Exercise I.5) Let G_1 and G_2 be groups, and let H be a subgroup of $G_1 \times G_2$ such that the two projections $p_1 : H \to G_1$ and $p_2 : H \to G_2$ are surjective. Let N_1 be the kernel of p_2, and let N_2 be the kernel of p_1. We can view N_1 and N_2 as subgroups of G_1 and G_2.

   - Show that N_1 is normal in G_1 and N_2 is normal in G_2.
   - Prove that the image of H in $(G_1/N_1) \times (G_2/N_2)$ is the graph of an isomorphism $G_1/N_1 \cong G_2/N_2$.

- First, we can view $N_i \leq G_i$ as say $p_1(N_1 = \ker p_2 = \{(x, e_{G_2}) \in H\}) \leq G_1$. This is indeed a subgroup of G_1 becuase if $x, y \in p_1(N_1)$ then this means $(x, e), (y, e) \in N_2$ and so, since H is a subgroup, $(x +_{G_1} y, e) \in N_1$ hence $x +_{G_1} y \in p(N_1)$. And a similar argument holds for inverses and the identity. If we get time, write out the arg to show that it contains identity and inverses. Put another way, $p_1(N_1) \leq G_1$ since $N_1 \leq H$. The same reasoning shows that we can view $N_2 \leq G_2$.

  Now we show that $N_1 \trianglelefteq G_1$. Let $x \in N_1 \leq G_1$ and let $g \in G_1$. Since $p_1 : H \to G_1$ is surjective there exists $(g, y) \in H$ and $(g, y)^{-1} = (g^{-1}, y^{-1}) \in H$ since H is a subgroup. Lastly $(x, e) \in H$ by definition of $\ker p_2$. Now consider

$$(g, y) \cdot_{(G_1 \times G_2)} (x, e) \cdot_{(G_1 \times G_2)} (g^{-1}, y^{-1}) = (gxg^{-1}, e) \in H$$

  since H is closed under $\cdot_{G_1 \times G_2}$. But then we have shown $gxg^{-1} \in N_1 \leq G_1$. Thus, N_1 is closed under conjugation and is normal in G_1. A very similar argument holds to show that $N_2 \trianglelefteq G_2$.

- Let $\overline{H}$ be the image of H in $G_1/N_1 \times G_2/N_2$. We want to show that $(\overline{x}, \overline{y}) \in \overline{H}$ associates elements $\overline{x} \in G_1/N_1$ to $\overline{y} \in G_2/N_2$, as a function, in a bijective manner, and as a group homomorphism.

  To be clear, $\overline{H} = \overline{H}_1 \times \overline{H}_2$ where $\overline{H}_i = \operatorname{im}(H \twoheadrightarrow^{p_i} G_i \twoheadrightarrow^{\pi_i} G_i/N_i)$. First we show that $\overline{H}$ defines a function. That is, we need to show that there is exactly one element of the form $(\overline{x}, -) \in \overline{H}$ for each $\overline{x} \in G_1/N_1$. Notice that if $(x, y_1), (x, y_2) \in H$ then we have $y_1 - y_2 \in N_2$ and so $\overline{y}_1 = \overline{y}_2 \in G_2/N_2$. That is, any elements of G_2 which are associated with x in H end up in the same class in G_2/N_2. And likewise for any $x' \in G_1$ with $\overline{x'} = \overline{x}$. It then follows that there is at most one element of the form

$(\overline{x}, -) \in \overline{H}$ for each $\overline{x} \in G_1/N_1$. Moreover, $H \twoheadrightarrow G_1 \twoheadrightarrow G_1/N_1$ is surjective since it is the composition of surjective maps. It then follows that for all $\overline{x} \in G_1/N_1$ there is some element $(\overline{x}, -) \in \overline{H}$. Thus $\overline{H}$ defines a function $f : G_1/N_1 \to G_2/N_2$.

Next we show that f is bijective. First notice that $H \twoheadrightarrow G_2 \twoheadrightarrow G_2/N_2$ again is surjective. Thus for each $\overline{y} \in G_2/N_2$ we have some $(-, \overline{y}) \in \overline{H}$. That is, f is surjective. Now notice that if $(x_1, y), (x_2, y) \in H$ then we have $x_1 - x_2 \in N_1$. Hence, again, all elements which are associated to y in H end up in the same class in G_1/N_1. And likewise for $x' \in G_1$ which associate to some $y' \in G_2$ such that $\overline{y'} = \overline{y}$. That is, there is at most one element of the form $(-, \overline{y}) \in \overline{H}$. That is, f is injective.

Lastly, f is a group homomorphism because $\overline{H}$ is a subgroup of $G_1/N_1 \times G_2/N_2$; this follows since H is a subgroup of $G_1 \times G_2$. There's some unpacking and deatil checking to do here, but I currently believe this follows from unpacking all the definitions of the objects.

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