Algorithms HW

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3. In this problem, we will make good on our promise from class to show that the rank of a free module is well-defined (over a commutative ring!). Let R be a non-zero commutative ring. We are going to show that the R-modules R^n and R^m are non-isomorphic if $n \neq m$.

   - Suppose that $I \trianglelefteq R$ is an ideal of a commutative ring R and let M be an R-module. Let IM be the R-submodule of M spanned by all elements ax for $a \in I$ and $x \in M$. Show that the quotient M/IM has the structure of an R/I-module. When $M = R^n$, show that M/IM is isomorphic to $(R/I)^n$ as an R/I-module.
   - Show that any R-linear isomorphism $R^n \xrightarrow{\sim} R^m$ induces an R/I-linear isomorphism $(R/I)^n \xrightarrow{\sim} (R/I)^m$.
   - By taking I to be a maximal ideal, deduce that R^n and R^m are isomorphic if and only if $n = m$. (Standard linear algebra results may be used without proof.)

- To show that M/IM has the structure of an R/I-module, we must show that M/IM has an underlying abelian group structure with respect to addition, and that there is a well-defined R/I scaler multiplication on M/IM.

  First, we show that M/IM has a well-defined abelian group structure with respect to addition. By definition M has an abelian group structure, and so IM is a normal subgroup of M. It follows that the quotient group M/IM is well-defined and moreover is also abelian. One way to see that the quotient group is abelian is to recall that the quotient map $\pi : M \to M/IM$ is a group homomorphism and so

$$\pi(x) + \pi(y) = \pi(x+y) = \pi(y+x) = \pi(y) + \pi(x),$$

  for all $x, y \in M$.

  Next, we propose an R/I scaler multiplication on M/IM. If $r + I \in R/I$ and $m + IM \in M/IM$. Then I claim that scaler multiplication given by $(r+I)\cdot(m+IM) := rm + IM$ is well defined. Let $r + I = r' + I$ be equivalent elements of R/I. Then we have that there's some $a \in I$ such that $r = r' + a$. Now let $m + IM \in M/IM$ and consider

$$rm + IM = (r'+a)m + IM = r'm + IM + am + IM = r'm + IM + 0 + IM = r'm + IM,$$

since $am \in IM$ by definition. That is, our proposed scaler multiplication maps $m \in M$ to the same class mod IM, regardless of choice of representative of $r + IM$. Hence our multiplication is well defined and we have found an R/I-module structure on M/IM.

- Suppose $\phi : R^n \to R^m$ is an R-linear isomorphism. Since R^n is a free R-module, it follows that ϕ is defined exactly by its action on the basis $\{e_i\}$ where $e_i = (0, \cdots, 1, \cdots 0)$ with the 1 in the ith position.

  Now notice that $\bar{e}_i := \pi(e_i)$ is an R/I-basis for $(R/I)^n$. And so maps out of $(R/I)^n$ can be defined by their action on $\bar{e}_i$. We define a map $\bar{\phi} : (R/I)^n \to (R/I)^m$ by $\bar{\phi}(\bar{e}_i) = (\pi \circ \phi)(e_i)$. This map is R/I linear by construction, we claim that it is also a bijection.

  First we show surjectivity. Suppose $(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m) \in (R/I)^m$. Since $\pi : R \to R/I$ is a surjection, there is an element $(b_1, b_2, \cdots, b_m) \in R^m$ such that $\pi(b_1, b_2, \cdots, b_m) = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m)$. Moreover, ϕ is an isomorphism and so there also exists an element $(a_1, a_2, \cdots, a_n) \in R^n$ such that $\phi(a_1, \cdots, a_n) = (b_1, \cdots, b_m)$. It then follows that $\bar{\phi}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n) = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_m)$. Thus, $\bar{\phi}$ is surjective.

  Now consider $0 \in (R/I)^m$. Tuples of the form $(b_1, \cdots, b_m) \in R^m$ with $b_i \in I$ map to 0 under π. Since ϕ is an isomorphism from R to R, only elements $(a_1, \cdots, a_n) \in R^n$ with $a_i \in I$ satisfy $\phi(a_1, \cdots, a_n) = (b_1, \cdots, b_m)$ with $b_i \in I$. Then, only elements of the form $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n) \in (R/I)^n$ map to $0 \in (R/I)^m$ under $\bar{\phi}$. However, since $a_i \in I$ we have $(\bar{a}_1, \cdots, \bar{a}_n) = 0 \in (R/I)^n$. And so $\bar{\phi}$ is injective.

  We have found a bijective R/I-linear map $\bar{\phi} : (R/I)^n \to (R/I)^m$ and so we have found an induced R/I-linear isomorphism $(R/I)^n \to (R/I)^m$.

- Suppose $n = m$, then $R^n = R^m$ and so $R^n \cong R^m$ as R-modules.

  Now suppose that $R^n \cong R^m$. Since R is a non-zero commutative ring we have (via the axiom of choice (or perhaps Zorn's lemma)) that there exists a maximal ideal $I \trianglelefteq R$. Now by part (b) we have an induced isomorphism on the R/I modules $(R/I)^n \cong (R/I)^m$. However, since I is maximal, it follows that R/I is a field and so $(R/I)^n$ and $(R/I)^m$ are in fact R/I-vector spaces. Vector spaces are characterized by their dimension and so it follows that $(R/I)^n \cong (R/I)^m$ as R/I vector spaces implies $n = m$.

Hence the rank of a free module over a non-zero commutative ring is a well defined notion.

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4. Prove the following identities for tensor products (where M, N, L are arbitrary R-modules):
   - $M \otimes_R N \cong N \otimes_R M$ ("commutativity")
   - $R \otimes_R M \cong M$ ("identity")
   - $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$ ("distributivity")
   - (harder) $M \otimes_R (N \otimes_R L) \cong (M \otimes_R N) \otimes_R L$ ("associativity")

   [Hint: the universal property of tensor products is a handy way of defining R-linear maps out of tensor products]

- Consider the map $M \times N \to N \otimes_R M$ given by $((m, n) \mapsto n \otimes m)$. Notice that this map is R-bilinear on $M \times N$ since $\otimes : M \times N \to M \otimes_R N$ is. Then by the universal property of tensor products we have a unique R-linear map $f : M \otimes_R N \to N \otimes_R M$ such that $f \circ \otimes = (m, n) \mapsto n \otimes m$. The same argument on the R-bilinear map $N \times M \to N \otimes_R M$ given by $(n, m) \mapsto (m \otimes n)$ gives a unique R-linear map $\tilde{f} : N \otimes_R M \to M \otimes_R N$ such that[1] $\tilde{f} \circ \times = (n, m) \mapsto m \otimes n$.

  Notice now that $f \circ \tilde{f} = id_{N \otimes_R M}$ and $\tilde{f} \circ f = id_{M \otimes_R N}$. Indeed $(f \circ \tilde{f})(m \otimes n) = f(n \otimes m) = m \otimes n$. And likewise for the other direction.

  That is, we have found a bijective R-linear map $M \otimes_R N \to N \otimes_R M$ and so in fact $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

- Consider the following map $M \times N \to M$ via $(r, m) \mapsto rm$ given by the module structure on M. Notice that this map is bilinear come back and write the computation out And so by the universal property of tensor products we have a unique R-linear map $f : R \otimes_R M \to M$ such that $r \otimes n \mapsto rm$.

  I claim that this map is bijective and so is an isomorphism of R-modules. First notice that f is surjective. Indeed, if $m \in M$ then $f(1 \otimes m) = 1 \cdot m = m$, so long as R is not the zero ring. If R is the zero ring, then M must be the zero module, and then our desired isomorphism trivially holds.

  Next we show that f is injective. Suppose we have $r \cdot m' = m$ for some $m, m' \in M$

[1] The "$\otimes$" in the following phrase now refers to the R-bilinear map $N \times M \to N \otimes_R M$, whereas earlier it referred to the R-bilinear map $M \times N \to M \otimes_R N$.

and $r \in R$. Then notice

$$r \otimes m' = 1 \cdot r \otimes m' = r \otimes r \cdot m' = 1 \otimes m,$$

And so $rm' = m$ implies $r \otimes m' = 1 \otimes m$. This suffices to show that f is injective because if generally we have $r_1m_1 = r_2m_2$ then by definition $r_1m_1 = m'$ for some $m' \in M$ and then we have $m' = r_2m_2$.

Overall we have a bijective R-linear map $R \otimes_R M \to M$, and so $R \otimes_R M \cong M$.

- First we acquire an R-linear map $M \otimes_R (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$ by leveraging the universal property of tensor products. And then we show that this map is in fact an isomorphism.

  First define a map $h : M \times (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$ by $h(m,(n,l)) = (m \otimes n, m \otimes l)$. Note that in R-Mod finite coproducts are also products, and so the domain of the function h is the R-module $M \oplus N \oplus L$. We claim that this map is R-bilinear. Consider

$$\begin{aligned} h(r_1m_1 + r_2m_2,(n,l)) &= ((r_1m_1 + r_2m_2) \otimes n,(r_1m_1 + r_2m_2) \otimes l) \\ &= (r_1(m_1 \otimes n) + r_2(m_1 \otimes n), r_1(m \otimes l) + r_2(m_2 \otimes n)) \\ &= r_1(m_1 \otimes n, m_1 \otimes l_1) + r_2(m_2 \otimes n, m_2 \otimes l). \end{aligned}$$

  By the definition of the tensor product relations, and of the R-module structure on the direct sum of two R-modules. In other words, we have shown that h is R-linear in the first argument. Now consider the second argument

$$\begin{aligned} h(m,r_1(n_1,l_1) + r_2(n_2,l_2)) &= (m \otimes r_1n_1 + r_2n_2, m \otimes r_1l_1 + r_2l_2) \\ &= (r_1(m \otimes n_1) + r_2(m \otimes n_2), r_1(m \otimes l_1) + r_2(m \otimes l_2)) \\ &= r_1(m \otimes n_1, m \otimes l_1) + r_2(m \otimes n_2, m \otimes l_2), \end{aligned}$$

  using the bilineararity of the tensor product, and by using the definition of the R-module structure on the direct sum of R-modules. That is, we've shown h is R-linear in the second argument also. And so h is an R-bilinear map.

We are now free to use the universal property of tensor products to acquire a new R-linear map $\phi : M \otimes_R (N \oplus L) \to (M \otimes_R N) \oplus (M \otimes_R L)$. where $\phi(m \otimes (n, l)) = (m \otimes n, m \otimes l)$.

Next, we will construct an R-linear map $\psi : (M \otimes_R N) \oplus (M \otimes_R L) \to M \otimes_R (N \otimes L)$ using the universal property of coproducts. We have a commutative diagram

$$\begin{tikzcd}
M \times N \arrow[r, "f_1"] \arrow[d, "\otimes"'] & M \otimes_R (N \oplus L) & M \times L \arrow[l, "f_2"'] \arrow[d, "\otimes"] \\
M \otimes_R N \arrow[r, hook] \arrow[ur, dashed, "\psi_1"] & (M \otimes_R N) \oplus (M \otimes_R L) \arrow[u, dashed, "\psi"] & M \otimes_R L \arrow[l, hook'] \arrow[ul, dashed, "\psi_2"']
\end{tikzcd}$$

where the maps ψ_1, ψ_2 will be built out of the universal property of tensor products, and the desired map ψ will consequently be determined by the universal property of corpoduct in R-mod. First, we must specify bilinear maps $f_1 : M \times N \to M \otimes_R (N \oplus L)$ and $f_2 : M \times L \to M \otimes_R (N \oplus L)$.

I claim that the map $f_1(m, n) = m \otimes (n, 0)$ is bilinear. If n is fixed then for each $r_1, r_2 \in R$ and $m_1, m_2 \in M$ we have $f_1(r_1 m_1 + r_2 m_2, n) = (r_1 m_2 + r_2 m_2) \otimes (n, 0) = r_1(m_1 \otimes (n, 0)) + r_2(m_2 \otimes (n, 0)) = r_1 f_1(m_1, n) + r_2 f_2(m_2, n)$, by the bilinearity of tensor product. I.e. f_1 is R-linear in its first argument. Moreover, if we fix $m \in M$ we have

$$\begin{aligned}
f_1(m, r_1 n_1 + r_2 n_2) &= m \otimes (r_1 n_1 + r_2 n_2, 0) \\
&= m \otimes [r_1(n_1, 0) + r_2(n_2, 0)] \\
&= r_1(m \otimes (n_1, 0)) + r_2(m \otimes (n_2, 0)) \quad = r_1 f_1(m, n_1) + r_2 f_1(m, n_2).
\end{aligned}$$

And so, f_1 is R-linear in its second argument. Hence, $f_1 : M \times N \to M \otimes_R (N \oplus L)$ is R-bilinear. A very similar calculation will show that $f_2 : M \times L \to M \otimes_R (N \oplus L)$ is also R-bilinear.

Thus, by the universal property of tensor products, we have R-linear maps

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6. Suppose that $f\colon M \to M'$ and $g\colon N \to N'$ are homomorphisms of R-modules. Use f and g to define a homomorphism
$$M \otimes_R N \to M' \otimes_R N'.$$
(This homomorphism is usually denoted by $f \otimes g$.)

We define the desired map by using the universal property of tensor products. Consider the following composition

$$M \oplus N \xrightarrow{(f,g)} M' \oplus N' \xrightarrow{\otimes} M' \otimes_R N',$$

via

$$(m,n) \mapsto (f(m), g(n)) \mapsto f(m) \otimes g(n).$$

We claim that this composition $\phi : M \oplus N \to M \otimes_R N$ is R-bilinear. Indeed, first let $n \in N$ be fixed and then consider

$$\begin{aligned}
\phi(r_1 m_1 + r_2 m_2, n) &= (r_1 f(m_1) + r_2 f(m_2)) \otimes g(n) \\
&= (r_1 f(m_1)) \otimes g(n) + (r_2 f(m_2)) \otimes g(n) \\
&= r_1(f(m_1) \otimes g(n)) + r_2(f(m_2) \otimes g(n)) \\
&= r_1 \phi(m_1, n) + r_2 \phi(m_2, n),
\end{aligned}$$

by the relations on the elements of the tensor product. In other words, we have shown that $\phi(-,n)$ is R-linear for each $n \in N$. A very similar calculation will show that $\phi(m,-)$ is R-linear for each $m \in M$. That is, ϕ is R-bilinear.

By the universal property of the tensor product we then have an R-linear map $\psi : M \otimes_R N \to M' \otimes_R N$ given by $m \otimes n \mapsto f(m) \otimes g(n)$, as desired.

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7. Let V be an $\mathbb{R}$-vector space of dimension n. Recall from class that the tensor product $\mathbb{C}\otimes_{\mathbb{R}} V$ can be viewed as a $\mathbb{C}$-vector space in a natural way. In this question, we are going to show that the dimension of $\mathbb{C}\otimes_{\mathbb{R}} V$ (as a $\mathbb{C}$-vector space) is equal to n (the dimension of V as an $\mathbb{R}$-vector space).
   - Suppose that $e_1, \ldots, e_n$ is an $\mathbb{R}$-linear basis of V. Write down n elements of $\mathbb{C}\otimes_{\mathbb{R}} V$ which could plausibly be a $\mathbb{C}$-linear basis.
   - Suppose that $\delta_1, \ldots, \delta_n : V \to \mathbb{R}$ is the dual basis to $e_1, \ldots, e_n$. Write down n $\mathbb{C}$-linear maps $\mathbb{C}\otimes_{\mathbb{R}} V \to \mathbb{C}$ which could plausibly be a $\mathbb{C}$-linear basis of the dual space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} V, \mathbb{C})$. [Hint: $\mathbb{C} = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}$, so the previous part gives you a way of constructing a homomorphism.]
   - Show that the elements you defined in the previous two parts are dual bases of the $\mathbb{C}$-vector space $\mathbb{C}\otimes_{\mathbb{R}} V$, and hence that $\dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} V) = n$.

- I claim that $1\otimes e_1, 1\otimes, e_2, \cdots, 1\otimes e_n \in \mathbb{C}\otimes_{\mathbb{R}} V$ is a basis. First we show that these tensors span $\mathbb{C}\otimes_{\mathbb{R}} V$. First let $\alpha\otimes v \in \mathbb{C}\otimes_{\mathbb{R}} V$ be a pure tensor. Then, since $\{e_i\}$ is a basis for V we have $v = \lambda_1 e_1 + \cdots \lambda_n e_n$ for $\lambda_i \in \mathbb{R}$. Now notice

$$\begin{aligned}
\alpha\lambda_1(1\otimes e_1) + \alpha\lambda_2(1\otimes e_2) + \cdots + \alpha\lambda_n(1\otimes e_n) &= \alpha\left[1\otimes\lambda_1 e_1 + 1\otimes\lambda_2 e_2 + \cdots 1\otimes\lambda_n e_n\right] \\
&= \alpha\otimes(\lambda_1 e_1 + \cdots + \lambda_n e_n) \\
&= \alpha\otimes v.
\end{aligned}$$

  That is, every pure tensor in $\mathbb{C}\otimes_R V$ is a finite $\mathbb{C}$-linear combination of the $1\otimes e_i$. It then follows that every element of $\mathbb{C}\otimes_{\mathbb{R}} V$, which is a finite $\mathbb{C}$-linear combination of pure tensors, is also a finite $\mathbb{C}$-linear combination of the $1\otimes e_i$. Thus, the $1\otimes e_i$ span $\mathbb{C}\otimes_R V$ over $\mathbb{C}$.

  I will show some approximation of the $1\otimes e_i$ being $\mathbb{C}$-linearly independent. Suppose we have $a_1(1\otimes e_1) + \cdots + a_n(1\otimes e_n) = 0$ for some $a_i \in \mathbb{R}$. Then, we have

$$\begin{aligned}
0 &= a_1(1\otimes e_1) + a_2(1\otimes e_2) + \cdots + a_n(1\otimes e_n) \\
&= 1\otimes(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n) = 0 \\
&\implies a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0 \\
&\implies a_i = 0 \text{ for all } i,
\end{aligned}$$

  where the last line follows since the e_i are an $\mathbb{R}$-linear basis for V, and so are $\mathbb{R}$-linearly independent. The second to last line follows since $1 \neq 0 \in \mathbb{C}$. This shows that the elements $1 \otimes e_i$ are $\mathbb{R}$-linearly independent. This argument does not work for $a_i \in \mathbb{C}$, becuase we are unable to "bring the coefficients into the second 'coordinate' of the pure tensors." That is, we do not have a $\mathbb{C}$-vector space structure on V exactly.

However, recalling generally that $M \otimes_{\mathbb{R}} N$ is an $\mathbb{R}$-vector space for any $\mathbb{R}$-vector spaces M, N, we expect to be able to find some basis for $\mathbb{C} \otimes_{\mathbb{R}} V$. The argument above gives that the $\mathbb{R}$-vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ has a basis $1 \otimes e_1$.

- We want to study the $\mathbb{C}$-linear maps $\mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$. We first consider maps which descend from bilinear maps $C \times V \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ induced by $f \in V^*$. First, fix a map $f \in V^\star$. And then we define a map $\phi_f : \mathbb{C} \times V \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ via $\phi_f(\alpha, v) = \alpha \otimes f(v)$. First notice that this map is $\mathbb{R}$-linear in the second coordinate: if we fix $\alpha \in \mathbb{C}$ and let $a_1, a_2 \in \mathbb{R}$ indeed, we have

  $$\phi_f(\alpha, a_1 v_1 + a_2 v_2) = \alpha \otimes f(a_1 v_1 + a_2 v_2) = \alpha \otimes (a_1 f(v_1) + a_2 f(v_2)) = a_1(\alpha \otimes f(v_1)) + a_2(\alpha \otimes f(v_2)),$$

  Since f is an $\mathbb{R}$-linear map and by the bilinearity of the tensor product. A similar calculation shows that ϕ_f is $\mathbb{C}$-linear in the first coordiante.

  Now, since $\delta_1, \cdots, \delta_n$ is a basis for V^* then we can represent $f(v) = a_1\delta_1(v) + a_2\delta_2(v) + \cdots a_n\delta_n(v)$ for $a_i \in \mathbb{R}$. Then we can represent

  $$\phi_f(\alpha, v) = \alpha \otimes (a_1\delta_1(v) + \cdots + a_n\delta_n(v)) = \alpha a_1(1 \otimes \delta_1(v)) + \alpha a_2(1 \otimes \delta_2(v)) + \cdots + \alpha a_n(1 \otimes \delta_n(v)),$$

  Note that the a_i in the computation above depends only on the map $f \in V^*$. This suggests that a plausible basis for $Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$ could be the maps $1 \otimes \delta_i \in Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}}, \mathbb{C})$.

  is there more that we can show?

- See the previous parts to see some partial arguments that the elements given are bases for their corresponding spaces. I will, however, show that $\{1 \otimes e_i\} \subseteq \mathbb{C} \otimes_{\mathbb{R}} V$ and $\{1 \otimes \delta_i\} \subseteq Hom_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$ are dual to each other. Consider

  $$(1 \otimes \delta_i)(1 \otimes e_i) = 1 \otimes \delta_i(e_i) = 1 \otimes 1 = 1 \in \mathbb{C}.$$

Whereas, for $i \neq j$

$$(1 \otimes \delta_i)(1 \otimes e_j) = 1 \otimes \delta_i(e_j) = 1 \otimes 0 = 0 \in \mathbb{C}.$$

This shows that $\{1 \otimes \delta_i\}$ and $\{1 \otimes e_i\}$ are dual and in fact orthonormal.

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