Algorithms HW

1. Consider the following functions (morphisms in Set):

$$f\colon \{1,2,3\} \to \{1,2,3,4\} \qquad f(x) = \begin{cases} x & \text{if } x \le 2 \\ 4 & \text{if } x = 3 \end{cases}$$

$$g\colon \{1,2,3,4\} \to \{1,2,3\} \qquad g(x) = \begin{cases} 1 & \text{if } x = 1 \\ x-1 & \text{if } x \ge 2 \end{cases}$$

$$h\colon \{1,2,3\} \to \{1,2\} \qquad h(x) = \begin{cases} 1 & \text{if } x = 1 \\ x-1 & \text{if } x \ge 2 \end{cases}$$

$$k\colon \{1,2\} \to \{1,2,3\} \qquad k(x) = \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}$$

Show that the square

$$\begin{tikzcd} \{1,2,3\} \arrow[r, "f"] \arrow[d, "h"] & \{1,2,3,4\} \arrow[d, "g"] \\ \{1,2\} \arrow[r, "k"] & \{1,2,3\} \end{tikzcd}$$

commutes.

To show that the given square commuts, we must show that $(gf)(x) = (kh)(x)$ for all $x \in \{1,2,3\}$. Consider the image of $1 \in \{1,2,3\}$:

$$gf(1) = g(1) = 1 = k(1) = kh(1).$$

Now, consider the image of $2 \in \{1,2,3\}$:

$$gf(2) = g(2) = 1 = k(1) = kh(2).$$

Finally, here's the image of 3:

$$gf(3) = g(4) = 3 = k(2) = kh(3).$$

Hence, the given square commutes by definition. Note that we did not need to check whether $\operatorname{im} g = \operatorname{im} k$ (and in fact these images in $\{1,2,3\}$ are not equal.) ∎

2. We work in the category $\mathsf{Mod}_{\mathbb{R}}$ of real vector spaces. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the $\mathbb{R}$-linear map given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Let $g\colon \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \to \mathbb{R}$

be the $\mathbb{R}$-linear map induced by the $\mathbb{R}$-bilinear map

$$\beta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \qquad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz\,.$$

For which $\mathbb{R}$-linear maps $h\colon \mathbb{R} \to \mathbb{R}$ does the square

$$\begin{tikzcd} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "f\otimes f"] & \mathbb{R} \arrow[d, "h"] \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

commute?

Suppose we have $(w,x) \otimes (y,z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Following This element clockwise around the diagram we have that $(h \circ g)((w,x) \otimes (y,z)) = h(wy + xz)$ and following this element counter-clockwise around the diagram we have $(g \circ f \otimes f)((w,x) \otimes (y,z)) = g((w - x, w + x) \otimes (y - z, y + z)) = (w - x)(y - z) + (w + x)(y + z)$. That is, any $\mathbb{R}$-linear map $h : \mathbb{R} \to \mathbb{R}$ must satisfy

$$h(wy + xz) = (w - x)(y - z) + (w + x)(y + z)$$

for all $w, x, y, z \in \mathbb{R}$ is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since h is an $\mathbb{R}$-linear map we have that

$$h(wy + xz) = h(1)(wy + xz).$$

Moreover, since $\mathbb{R}$ is a rank 1 free module over $\mathbb{R}$, we have that any $\mathbb{R}$-linear map $\mathbb{R} \to \mathbb{R}$ is determined by where it sends the basis $\{1\}$. Given the expression above we have that any such map h satisfies

$$\begin{aligned} h(1) &= \frac{(w-x)(y-z)+(w+x)(y+z)}{wy+xz} \\ &= \frac{wy-wz-xy+xz+wy+wz+xy+xz}{wy+xz} \\ &= \frac{2(wy+xz)}{wy+xz} \\ &= 2. \end{aligned}$$

That is, there is a single map $h : \mathbb{R} \to \mathbb{R}$ which makes the above diagram commute — namely the one which sends the basis $1 \mapsto 2$, i.e $h(x) = 2x$.

I'm curious if there's any geometric significance to this thing that we've just shown

$\blacksquare$

3. Consider the following diagram in an arbitrary category $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[r, "g"] \arrow[d, "h'"] & Z \arrow[d, "h''"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

Show that if the left and right squares in this diagram commute, then so does the outer rectangle. That is, if the two squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "h'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h'"] & Z \arrow[d, "h''"] \\ Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

both commute, then so does the square

$$\begin{tikzcd} X \arrow[r, "gf"] \arrow[d, "h"] & Z \arrow[d, "h''"] \\ X' \arrow[r, "g'f'"] & Z' \end{tikzcd}$$

We need to show that $h''gf = g'f'h$. Suppose $x \in X$ If $\mathcal{C}$ is not "sets with extra structure" can we still reason about functions by considering their actions on elements in their domain?

Consider the right-handed commuting square. Let $f(x) \in Y$. Since this second square commutes, we have $h''gf = g'h'f$. Moreover, since the left-handed square commutes, we have $h'f = f'h$. Substituting this relation into our first equation gives us

$$h''gf = g'h'f = g'f'h,$$

as desired. ■

4. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in a category $\mathcal{C}$. Let $gf\colon X \to Z$ be their composition.
   - Show that if f and g are both monomorphisms, then gf is a monomorphism.
   - Show that if f and g are both epimorphisms, then gf is an epimorphism.
   - Show that if gf is a monomorphism, then f is a monomorphism. Must g be a monomorphism?
   - Show that if gf is an epimorphism, then g is an epimorphism. Must f be an epimorphism?

For this question, recall that the universal properties of monic maps and of epic maps. Let $\mathcal{C}$ be a category and let $X, Y \in \mathcal{C}$, then a morphism $f : X \to Y$ is called monic if for all $Z \in \mathcal{C}$ and all $g, h : Z \to X$ we have $fg = fh$ implies $g = h$. Likewise, $f : X \to Y$ is called epic if for all $Z \in \mathcal{C}$ and all $g, h : Y \to Z$ we have $gf = hf$ implies $g = h$.

- Suppose f, g are monic and now consider gf. Let $W \in \mathcal{C}$ and suppose $h, k : W \to X$ such that $gfh = gfk$. Now, since g is a monomorphism and since function composition in $\mathcal{C}$ is associative, we have $g(fh) = gfh = gfk = g(fk)$ implies $fh = fk$. Now, since f is a monomorphism we have $h = k$. In other words, we have shown that $gfh = gfk$ implies $h = k$ for all morphisms $h, k : Z \to X$. That is, g, f monic imply that gf is monic.

- Now suppose f, g are epic and let $Z \in \mathcal{C}$ with $h, k : Z \to W$ such that $hgf = kgf$. Since f is epic we have that $(hg)f = hgf = kgf = (kg)f$ implies $hg = kg$. Moreover, g epic implies that $h = k$. That is, we have $hgf = kgf$ implies $h = k$ and so gf is epic.

- Suppose $gf : X \to Z$ is a monomorphism. Let $W \in \mathcal{C}$ and $h, k : W \to X$ such that $fh = fk$. We have that im $(fh) =$ im $(fk) \in Y$ and so, since $g = g$ we have that $gfh = gfk$. Now, since gf is monic we have that $h = k$. That is $fh = fk$ implies $h = k$, i.e. f is monic by definition. It is not necessary that g be monic.

- Now suppose gf is epic. Let $W \in \mathcal{C}$ with $h, k : Z \to W$ such that $hg = kg$. We have that $hgf = kgf$ as maps $X \to W$. But now, since gf is epic, we have that $h = k$. Thus g is epic by definition. It was not necessary that f be epic.

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5. Fix a group G. The category Set_G of *G-sets* is defined as follows:

- The objects of Set_G are sets X with an action of G.
- The morphisms $f: X \to Y$ in Set_G are functions $X \to Y$ which satisfy
$$f(\sigma x) = \sigma f(x)$$
for all $\sigma \in G$ and $x \in X$. (Such functions are called *G-equivariant*.)
- Composition in Set_G is given by composition of functions.
- The identity element $1_X : X \to X$ is the identity function.

Prove carefully that Set_G is a category (check the axioms). Prove that finite products exist in Set_G.

Recall the axioms of a category. Given the objects and morphisms of $\mathbf{Set}_G$ we need to verify (1) : that we have a well-defined composition rule, i.e., that the given composition gives us a $\mathbf{Set}_G$ morphism, (2) given our composition rule, that the given identity morphism satisfies $g1_X = 1_X g = g$ for all $g \in Hom_{\mathbf{Set}_G}(X, X)$ for all $X \in \mathbf{Set}_G$, and (3) that the given composition rule is associative.

Firstly, the by theory of group actions, sets with group actions and G-equivariant functions are a well defined collection of objects and morphisms between those objects.

(1) *Composition:* Let $X, Y, Z \in \mathbf{Set}_G$ and let $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$. Note that we have a well defined function composition gf from the category $\mathbf{Set}$. Now we must verify that gf is also G-equivariant. Since f, g are G-equivariant we have, for $\sigma \in G$ and $x \in X$,

$$gf(\sigma x) = g(\sigma f(x)) = \sigma gf(x).$$

And so, $gf \in Hom_{\mathbf{Set}}(X, Z)$ is G-equivariant by definition and gf is indeed a morphism in $Hom_{\mathbf{Set}_G}(X, Z)$.

(2) Identity: Let $X \in \mathbf{Set}_G$ and let $1_X : X \to X$ be the identity function on X as a set. Since 1_X is the identity function for X as a set we already have $g1_X = 1_X g = g$ for all functions $g \in Hom_{\mathbf{Set}}(X, X)$. And so 1_X, if it is G-equivariant, already satisfies $g1_X = 1_X g = g$ for all $g \in Hom_{\mathbf{Set}_G}(X, X)$. Let $\sigma \in G$ and consider

$$1_X(\sigma x) = \sigma x = \sigma f(x),$$

by definition of the action of 1_X as a function. And so, indeed, 1_X is G-equivariant and so is a morphism in $Hom_{\mathbf{Set}_X}(X, X)$, thus every $X \in \mathbf{Set}_X$ has an identity morphism.

(3) Associativity of function composition: Note that composition of functions is associative since **Set** is a category. It follows immediately that G-equivariant function composition is associative, since the G-equivariant functions from $X \to Y$ are a "sub-class" of the class of functions $X \to Y$.

Now we show that finite products exists in $\mathbf{Set}_G$. We claim that binary products exist in $\mathbf{Set}_G$ (and so it will follow that finite products exist in $\mathbf{Set}_G$ by iterating the construction for binary products). Recall that the universal property for binary products is the pullback of the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$. I claim that, given $X, Y \in \mathbf{Set}_G$, the cartesian product $X \times Y$ with the usual projections π_X, π_Y satisfy the universal property for $\mathbf{Set}_G$. To verify this claim we have two things to show: (1): that the cartesian product has some G-action for which π_X and π_Y are G-equivariant (i.e. that $X \times Y$ is indeed an object in $\mathbf{Set}_G$ and $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ are indeed morphisms in $\mathbf{Set}_G$). And (2) that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of products

(1): We show that $X \times Y$ has a G-action which makes π_X, π_Y into G-equivariant functions. Recall that given $X, Y \in \mathbf{Set}_G$ we can construct the set $X \times Y := \{(x, y) : x \in X \quad y \in Y\}$. Define a G-action on $X \times Y$ by $\sigma(x, y) = (\sigma x, \sigma y)$ for $\sigma \in G$ where $\sigma x, \sigma y$ are given by the G-action structure on X, Y. We verify that this is indeed a G-action on $X \times Y$. Note that if $1 \in G$ is the identity then we have $1(x, y) = (1x, 1y) = (x, y)$ for all $(x, y) \in X \times Y$. Moreover, if $\sigma, \delta \in G$ we have $\sigma(\delta(x, y)) = \sigma(\delta x, \delta y) = (\sigma\delta x, \sigma\delta y) = (\sigma\delta)(x, y)$. That is, our proposed action is indeed a G-action on $X \times Y$ and so the set $X \times Y$ is also an object in $\mathbf{Set}_G$.

Now we verify that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are G-equivariant. Let $(x, y) \in X \times Y$ and $\sigma \in G$ and consider

$$pi_X(\sigma(x, y)) = \pi_X((\sigma x, \sigma y)) = \sigma x = \sigma \pi_X(x, y).$$

That is, indeed, π_X is G-equivariant. Extremely similar reasoning shows that π_Y is G-equivariant. Hence π_X, π_Y are indeed morphisms in $\mathbf{Set}_G$.

(2): Finally, we need to show that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of binary products. But recall that $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property of binary products in **Set**. That is, for any set Z with functions $f_X : Z \to X$ and $f_Y : Z \to Y$ we have a unique function $f : Z \to X \times Y$ such that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dashed, "f"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Moreover, we know that

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