Algorithms HW

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2. We work in the category $\mathsf{Mod}_{\mathbb{R}}$ of real vector spaces. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the $\mathbb{R}$-linear map given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Let $g\colon \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \to \mathbb{R}$

be the $\mathbb{R}$-linear map induced by the $\mathbb{R}$-bilinear map

$$\beta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \qquad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz\,.$$

For which $\mathbb{R}$-linear maps $h\colon \mathbb{R} \to \mathbb{R}$ does the square

$$\begin{tikzcd} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "f\otimes f"] & \mathbb{R} \arrow[d, "h"] \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

commute?

Suppose we have $(w,x) \otimes (y,z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Following This element clockwise around the diagram we have that $(h \circ g)((w,x) \otimes (y,z)) = h(wy+xz)$ and following this element counter-clockwise around the diagram we have $(g \circ f \otimes f)((w,x) \otimes (y,z)) = g((w-x, w+x) \otimes (y-z, y+z)) = (w-x)(y-z) + (w+x)(y+z)$. That is, any $\mathbb{R}$-linear map $h : \mathbb{R} \to \mathbb{R}$ must satisfy

$$h(wy+xz) = (w-x)(y-z) + (w+x)(y+z)$$

for all $w,x,y,z \in \mathbb{R}$ is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since h is an $\mathbb{R}$-linear map we have that

$$h(wy+xz) = h(1)(wy+xz).$$

Moreover, since $\mathbb{R}$ is a rank 1 free module over $\mathbb{R}$, we have that any $\mathbb{R}$-linear map $\mathbb{R} \to \mathbb{R}$ is determined by where it sends the basis $\{1\}$. Given the expression above we have that any such map h satisfies

$$\begin{aligned}
h(1) &= \frac{(w-x)(y-z)+(w+x)(y+z)}{wy+xz} \\
&= \frac{wy-wz-xy+xz+wy+wz+xy+xz}{wy+xz} \\
&= \frac{2(wy+xz)}{wy+xz} \\
&= 2.
\end{aligned}$$

That is, there is a single map $h : \mathbb{R} \to \mathbb{R}$ which makes the above diagram commute — namely the one which sends the basis $1 \mapsto 2$, i.e $h(x) = 2x$.

I'm curious if there's any geometric significant to this thing that we've just shown

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