Algorithms HW

1. Consider the following functions (morphisms in **Set**):

$$f\colon \{1,2,3\} \to \{1,2,3,4\} \qquad f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 4 & \text{if } x = 3 \end{cases}$$

$$g\colon \{1,2,3,4\} \to \{1,2,3\} \qquad g(x) = \begin{cases} 1 & \text{if } x = 1 \\ x-1 & \text{if } x \geq 2 \end{cases}$$

$$h\colon \{1,2,3\} \to \{1,2\} \qquad h(x) = \begin{cases} 1 & \text{if } x = 1 \\ x-1 & \text{if } x \geq 2 \end{cases}$$

$$k\colon \{1,2\} \to \{1,2,3\} \qquad k(x) = \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}$$

Show that the square

$$\begin{tikzcd} \{1,2,3\} \arrow[r, "f"] \arrow[d, "h"] & \{1,2,3,4\} \arrow[d, "g"] \\ \{1,2\} \arrow[r, "k"] & \{1,2,3\} \end{tikzcd}$$

commutes.

To show that the given square commuts, we must show that $(gf)(x) = (kh)(x)$ for all $x \in \{1,2,3\}$. Consider the image of $1 \in \{1,2,3\}$:

$$gf(1) = g(1) = 1 = k(1) = kh(1).$$

Now, consider the image of $2 \in \{1,2,3\}$:

$$gf(2) = g(2) = 1 = k(1) = kh(2).$$

Finally, here's the image of 3:

$$gf(3) = g(4) = 3 = k(2) = kh(3).$$

Hence, the given square commutes by definition. Note that we did not need to check whether $\operatorname{im} g = \operatorname{im} k$ (and in fact these images in $\{1,2,3\}$ are not equal.) ∎

2. We work in the category $\mathsf{Mod}_{\mathbb{R}}$ of real vector spaces. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the $\mathbb{R}$-linear map given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Let $g\colon \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \to \mathbb{R}$

   be the $\mathbb{R}$-linear map induced by the $\mathbb{R}$-bilinear map

$$\beta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \qquad \beta\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = wy + xz\,.$$

   For which $\mathbb{R}$-linear maps $h\colon \mathbb{R} \to \mathbb{R}$ does the square

$$\begin{tikzcd} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "f \otimes f"] & \mathbb{R} \arrow[d, "h"] \\ \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

   commute?

Suppose we have $(w,x) \otimes (y,z) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. Following This element clockwise around the diagram we have that $(h \circ g)((w,x) \otimes (y,z)) = h(wy + xz)$ and following this element counter-clockwise around the diagram we have $(g \circ f \otimes f)((w,x) \otimes (y,z)) = g((w - x, w + x) \otimes (y - z, y + z)) = (w - x)(y - z) + (w + x)(y + z)$. That is, any $\mathbb{R}$-linear map $h : \mathbb{R} \to \mathbb{R}$ must satisfy

$$h(wy + xz) = (w - x)(y - z) + (w + x)(y + z)$$

for all $w, x, y, z \in \mathbb{R}$ is this last statement true, since our inputs are tensor products and so there's some relation between these symbols, right?

Since h is an $\mathbb{R}$-linear map we have that

$$h(wy + xz) = h(1)(wy + xz).$$

Moreover, since $\mathbb{R}$ is a rank 1 free module over $\mathbb{R}$, we have that any $\mathbb{R}$-linear map $\mathbb{R} \to \mathbb{R}$ is determined by where it sends the basis $\{1\}$. Given the expression above we have that any such map h satisfies

$$\begin{aligned} h(1) &= \frac{(w-x)(y-z)+(w+x)(y+z)}{wy+xz} \\ &= \frac{wy-wz-xy+xz+wy+wz+xy+xz}{wy+xz} \\ &= \frac{2(wy+xz)}{wy+xz} \\ &= 2. \end{aligned}$$

That is, there is a single map $h : \mathbb{R} \to \mathbb{R}$ which makes the above diagram commute — namely the one which sends the basis $1 \mapsto 2$, i.e $h(x) = 2x$.

I'm curious if there's any geometric significance to this thing that we've just shown

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3. Consider the following diagram in an arbitrary category $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[r, "g"] \arrow[d, "h'"] & Z \arrow[d, "h''"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

Show that if the left and right squares in this diagram commute, then so does the outer rectangle. That is, if the two squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "h'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h'"] & Z \arrow[d, "h''"] \\ Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

both commute, then so does the square

$$\begin{tikzcd} X \arrow[r, "gf"] \arrow[d, "h"] & Z \arrow[d, "h''"] \\ X' \arrow[r, "g'f'"] & Z' \end{tikzcd}$$

We need to show that $h''gf = g'f'h$. Suppose $x \in X$ If $\mathcal{C}$ is not "sets with extra structure" can we still reason about functions by considering their actions on elements in their domain?

Consider the right-handed commuting square. Let $f(x) \in Y$. Since this second square commutes, we have $h''gf = g'h'f$. Moreover, since the left-handed square commutes, we have $h'f = f'h$. Substituting this relation into our first equation gives us

$$h''gf = g'h'f = g'f'h,$$

as desired. ■

4. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in a category $\mathcal{C}$. Let $gf\colon X \to Z$ be their composition.

   - Show that if f and g are both monomorphisms, then gf is a monomorphism.
   - Show that if f and g are both epimorphisms, then gf is an epimorphism.
   - Show that if gf is a monomorphism, then f is a monomorphism. Must g be a monomorphism?

- Show that if gf is an epimorphism, then g is an epimorphism. Must f be an epimorphism?

For this question, recall that the universal properties of monic maps and of epic maps. Let $\mathcal{C}$ be a category and let $X, Y \in \mathcal{C}$, then a morphism $f : X \to Y$ is called monic if for all $Z \in \mathcal{C}$ and all $g, h : Z \to X$ we have $fg = fh$ implies $g = h$. Likewise, $f : X \to Y$ is called epic if for all $Z \in \mathcal{C}$ and all $g, h : Y \to Z$ we have $gf = hf$ implies $g = h$.

- Suppose f, g are monic and now consider gf. Let $W \in \mathcal{C}$ and suppose $h, k : W \to X$ such that $gfh = gfk$. Now, since g is a monomorphism and since function composition in $\mathcal{C}$ is associative, we have $g(fh) = gfh = gfk = g(fk)$ implies $fh = fk$. Now, since f is a monomorphism we have $h = k$. In other words, we have shown that $gfh = gfk$ implies $h = k$ for all morphisms $h, k : Z \to X$. That is, g, f monic imply that gf is monic.

- Now suppose f, g are epic and let $Z \in \mathcal{C}$ with $h, k : Z \to W$ such that $hgf = kgf$. Since f is epic we have that $(hg)f = hgf = kgf = (kg)f$ implies $hg = kg$. Moreover, g epic implies that $h = k$. That is, we have $hgf = kgf$ implies $h = k$ and so gf is epic.

- Suppose $gf : X \to Z$ is a monomorphism. Let $W \in \mathcal{C}$ and $h, k : W \to X$ such that $fh = fk$. We have that im $(fh) =$ im $(fk) \in Y$ and so, since $g = g$ we have that $gfh = gfk$. Now, since gf is monic we have that $h = k$. That is $fh = fk$ implies $h = k$, i.e. f is monic by definition. It is not necessary that g be monic.

- Now suppose gf is epic. Let $W \in \mathcal{C}$ with $h, k : Z \to W$ such that $hg = kg$. We have that $hgf = kgf$ as maps $X \to W$. But now, since gf is epic, we have that $h = k$. Thus g is epic by definition. It was not necessary that f be epic.

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5. Fix a group G. The category Set_G of *G-sets* is defined as follows:

- The objects of Set_G are sets X with an action of G.
- The morphisms $f\colon X \to Y$ in Set_G are functions $X \to Y$ which satisfy
$$f(\sigma x) = \sigma f(x)$$
for all $\sigma \in G$ and $x \in X$. (Such functions are called *G-equivariant.*)
- Composition in Set_G is given by composition of functions.
- The identity element $1_X : X \to X$ is the identity function.

Prove carefully that Set_G is a category (check the axioms). Prove that finite products exist in Set_G.

Recall the axioms of a category. Given the objects and morphisms of $\mathbf{Set}_G$ we need to verify (1) : that we have a well-defined composition rule, i.e., that the given composition gives us a $\mathbf{Set}_G$-morphism, (2) given our composition rule, that the given identity morphism satisfies $g1_X = 1_X g = g$ for all $g \in Hom_{\mathbf{Set}_G}(X, X)$ for all $X \in \mathbf{Set}_G$, and (3) that the given composition rule is associative.

Firstly, the by theory of group actions, sets with group actions and G-equivariant functions are a well defined collection of objects and morphisms between those objects.

(1) *Composition:* Let $X, Y, Z \in \mathbf{Set}_G$ and let $f \in Hom_{\mathbf{Set}_G}(X, Y)$ and $g \in Hom_{\mathbf{Set}_G}(Y, Z)$. Note that we have a well defined function composition gf from the category **Set**. Now we must verify that gf is also G-equivariant. Since f, g are G-equivariant we have, for $\sigma \in G$ and $x \in X$,

$$gf(\sigma x) = g(\sigma f(x)) = \sigma gf(x).$$

And so, $gf \in Hom_{\mathbf{Set}}(X, Z)$ is G-equivariant by definition and gf is indeed a morphism in $Hom_{\mathbf{Set}_G}(X, Z)$.

(2) Identity: Let $X \in \mathbf{Set}_G$ and let $1_X : X \to X$ be the identity function on X as a set. Since 1_X is the identity function for X we already have $g1_X = 1_X g = g$ for all functions $g \in Hom_{\mathbf{Set}}(X, X)$. And so 1_X, if it is G-equivariant, satisfies the axiom for identity G-Set morphism. Let $\sigma \in G$ and consider

$$1_X(\sigma x) = \sigma x = \sigma f(x),$$

by definition of the action of 1_X as a function. And so, indeed, 1_X is G-equivariant and so is a morphism in $Hom_{\mathbf{Set}_X}(X, X)$, thus every $X \in \mathbf{Set}_X$ has an identity morphism.

(3) Associativity of function composition: Note that composition of functions is associative since **Set** is a category. It follows immediately that G-equivariant function composition is associative, since the G-equivariant functions from $X \to Y$ are a "sub-class" of the class of functions $X \to Y$.

Now we show that finite products exists in $\mathbf{Set}_G$. We claim that binary products exist in $\mathbf{Set}_G$ (and so it will follow that finite products exist in $\mathbf{Set}_G$ by iterating the construction for binary products). Recall that the universal property for binary products is the pullback of the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$. I claim that, given $X, Y \in \mathbf{Set}_G$, the cartesian product $X \times Y$ with the usual projections π_X, π_Y satisfy the universal property for $\mathbf{Set}_G$. To verify this claim we have two things to show: (1) that the cartesian product has some G-action for which π_X and π_Y are G-equivariant (i.e. that $X \times Y$ is indeed an object in $\mathbf{Set}_G$ and $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ are indeed morphisms in $\mathbf{Set}_G$). And (2), that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of products

(1): We show that $X \times Y$ has a G-action which makes π_X, π_Y into G-equivariant functions. Recall that given $X, Y \in \mathbf{Set}_G$ we can construct the set $X \times Y := \{(x, y) : x \in X \quad y \in Y\}$. Define a G-action on $X \times Y$ by $\sigma(x, y) = (\sigma x, \sigma y)$ for $\sigma \in G$ and where $\sigma x, \sigma y$ are given by the G-action structure on X, Y. We verify that this is indeed a G-action on $X \times Y$. Note that if $1 \in G$ is the identity element of G then we have $1(x, y) = (1x, 1y) = (x, y)$ for all $(x, y) \in X \times Y$. Moreover, if $\sigma, \delta \in G$ we have $\sigma(\delta(x, y)) = \sigma(\delta x, \delta y) = (\sigma\delta x, \sigma\delta y) = (\sigma\delta)(x, y)$. That is, our proposed action is indeed a G-action on $X \times Y$ and so the set $X \times Y$ is also an object in $\mathbf{Set}_G$.

Now we verify that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are G-equivariant. Let $(x, y) \in X \times Y$ and $\sigma \in G$ and consider

$$\pi_X(\sigma(x, y)) = \pi_X((\sigma x, \sigma y)) = \sigma x = \sigma \pi_X(x, y).$$

That is, indeed, π_X is G-equivariant. Extremely similar reasoning shows that π_Y is G-equivariant. Hence π_X, π_Y are indeed morphisms in $\mathbf{Set}_G$.

(2): Finally, we need to show that $(X \times Y, \pi_X, \pi_Y)$ satisfy the universal property of binary products. Recall that $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property of binary products in **Set**. That is, for any set Z with functions $f_X : Z \to X$ and $f_Y : Z \to Y$ we have a unique function $f : Z \to X \times Y$ such that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dashed, "f"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Moreover, we know that f necessarily has the form $f(z) = (f_X(z), f_Y(z))$ for all $z \in Z$. If we verify that f is G-equivariant, i.e. that it's a valid morphism $Z \to X \times Y$ in $\mathbf{Set}_G$, then it will follow that $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property in $\mathbf{Set}_G$ So, suppose now that $Z \in \mathbf{Set}_G$ is a set with a G action, and that $f_X : Z \to X$ and $f_Y : Z \to Y$ are G-equivariant functions. Indeed, let $\sigma \in G$, and consider

$$f(\sigma z) = (f_X(\sigma z), f_Y(\sigma z)) = (\sigma f_X(z), \sigma f_Y(z)) = \sigma(f_X(z), f_Y(z)) = \sigma f(z),$$

by definition of f, by G-equivariance of f_X, f_Y, and by definition of the G-action on $X \times Y$. That is, $f : Z \to X \times Y$ is a valid morphism in $\mathbf{Set}_G$ and it follows that f is the unique morphism which makes the following diagram commute, now as a diagram in $\mathbf{Set}_G$

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dashed, "f"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Hence, $(X \times Y, \pi_X, \pi_Y)$ satisfies the universal property of binary product in $\mathbf{Set}_G$. It follows then that finite products exist in $\mathbf{Set}_G$.

Do empty products count as a finite product? I think very similar reasoning with products over a set would do the trick. Or, if we really want, we can show that $\mathbf{Set}_G$ has an initial object, and we know that we have unary products.

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6. Prove that each of the following is a functor.

$$\begin{aligned} F\colon \ \mathsf{Ring} \to \mathsf{Set} \qquad & F(R) = R \\ G\colon \ \mathsf{Ring} \to \mathsf{Gp} \qquad & G(R) = R^\times \\ H\colon \ \mathsf{Ring} \to \mathsf{Ring} \qquad & H(R) = R^2 \\ K\colon \ \mathsf{Ring} \to \mathsf{Set} \qquad & K(R) = \{\text{pairs } (x,y) \in R^2 \text{ such that } y^2 = x^3 - x\} \end{aligned}$$

In each case, make sure to specify how your functor acts on morphisms, i.e. if $f\colon R \to S$ is a ring homomorphism, you will need to specify the function $F(f)\colon F(R) \to F(S)$, the group homomorphism $G(f)\colon G(R) \to G(S)$, etc.

still need to proofread this one

We note for reference the definition of a functor here. A functor $\mathcal{F}$ from categories $C \to D$ is an association of objects from C to objects in D such that $Hom_C(X,Y)$ has a corresponding association $Hom_D(\mathcal{F}X, \mathcal{F}Y)$ for all $X, Y \in C$. Furthermore, this association of morphisms map identity morphisms to identity morphisms, $\mathcal{F}1_X = 1_{\mathcal{F}X}$ for all $X \in C$, and should respect composition, $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ for all $f, g \in Hom_C(X,Y)$ for all $X, Y \in C$.

- We are given an association of **Ring** $\to$ **Set** by associating a ring to its underlying set. We specify that F acts on morphisms by taking a ring homomorphism $R \to S$ to its underlying function on the sets $R \to S$.

  Note that the identity morphism on a given ring then is associated to the identity function on the underlying set, and so F takes identity morphisms to identity morphism. Moreover, recall that composition of ring homomorphisms was defined as the composition of the underlying functions (and then we verified that this was still a ring homomorphism), but then it follows by definition that $F(g \circ f) \in Hom_{Set}(FR, FT)$ is mapped to $Fg \circ Ff$, for all $F, T \in$ **Ring**.

- We are G which takes a ring R to its set of units $R^\times$, with a group structure given by

R-multiplication. Note that this is a well-defined functor, since if $g \in R^\times$ then it has a two-sided inverse by definition, and so $R^\times$ is closed under inverses. Moreover, R must contain a multiplicative identity 1 which is also in $R^\times$ (it is its own two-sided inverse). The given group operation is associative since ring multiplication is associative.

If $f : R \to S$ is a ring homomorphism we associate $Gf : GR \to GS$ to be the restriction to $R^\times$. We verify that this gives a well defined group homomorphism $GR \to GS$. First note that if $g \in R^\times$ with two-sided inverse $h \in R^\times$ then we have $1 = f(1) = f(gh) = f(g)f(h) = f(h)f(g)$. That is the image $f(g)$ has two-sided inverse $f(h)$ and so $f(R^\times) \subseteq S^\times$. Moreover, informally, f respects the group operation on GR because f, by definition, respects the ring multiplication on R. And so the association $(f : R \to S) \mapsto (Gf : GR \to GS)$ is a valid association of morphisms in **Ring** to morphisms in **Gp**.

Now we verify that that G sends identity morphisms to identity morphisms and that it respects composition. If 1_R is the identity morphism on a ring R, then $G1_R$ is the identity morphism on GR since it is just the restriction onto a subset of R. Moreover, if $g \circ f : R \to T$ is a composition then the association $G(g \circ f)$ is the restriction of $g \circ f$ to $R^\times$. Moreover, unpacking the definitions as functions on sets will show that $(g \circ f)|_{R^\times} = g|_{R^\times} \circ f|_{R^\times}$. And so G respects composition of morphisms. Thus, G is a functor.

- We have a functor from **Ring** $\to$ **Ring** by mapping $R \to R^2$. We propose the following association of ring homomorphisms. If we have a ring homomorphism $f : R \to S$ then associate $Hf : R^2 \to S^2$ by $Hf(r_1, r_2) := (fr_1, f_2)$. This definition is certainly a well defined map from R^2 to S^2.

  Now, we verify that such an association satisfies the axioms of functor. Suppose we have the identity morphism on a ring 1_R and consider, for $r_1, r_2 \in R$,

  $$H1_R(r_1, r_2) = (1_R r_1, 1_R r_2) = (r_1, r_2).$$

  And so indeed, $H1_R$ is the identity morphism on R^2. Now suppose we have a composition $g \circ f : R \to T$. Consider the following

$$H(g \circ f)(r_1, r_2) = ((g \circ f)r_1, (g \circ f)r_2) = Hg(fr_1, fr_2) = (Hg)(Hf)(r_1, r_2).$$

That is, H also respects ring homomorphism composition. Thus, the given H is indeed a functor.

- We are given a functor **Ring** $\to$ **Set**. We propose the following association of ring morphisms to set functions.

  Suppose $f : R \to S$ is a ring homomorphism and then notice, if $x, y \in R$ satisfy $y^2 = x^3 - x$ then we have

  $$f(y)^2 - f(x)^3 - f(x) = f(y^2 - x^3 - x) = f(0) = 0,$$

  That is $u := f(x), v := f(y) \in S$ then satisfy $v^2 = u^3 - u$. This discussion then shows that the association $Kf : KR \to KS$ defined by $Kf(x, y) = (fx, fy)$ is a well defined set function $KR \to KS$.

  We verify that this definition satisifies the axioms of functor. Essentially the same discussion showing that F, H are functors applies to show that K sends identity ring morphisms to the identity function $KR \to KR$. Moreover, informally, the same computation showing that H respects composition also shows that K respects composition. And so K is indeed a functor from **Ring** $\to$ **Set**.

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7. Prove that each of the functors in the previous question is representable when viewed as a functor from Ring to Set. [Hint to get you started: for F, it will be representable by the polynomial ring $\mathbb{Z}[x]$. To prove this, you need to write down a bijection $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], R) \cong R$ for each ring R and check that your bijection is natural in R.]

this is to be read. For reference, a functor $K : \mathcal{C} \to \mathbf{Set}$ is *representable* if there exists an object $M \in \mathcal{C}$ such that the functor $Hom_{\mathcal{C}}(M, -)$ is naturally isomorphic to K.

- We show that $Hom_{\mathbf{Ring}}(\mathbb{Z}[x], -)$ is naturally isomorphic to F. First we note that maps $f : \mathbb{Z}[x] \to R$ are determined exactly by the image of x (since $1 \in \mathbb{Z}[x]$ must map to $1 \in R$ and the rest of the images are given by using the ring homomorphism properties of f).

  Next we construct a natural transformation $Hom(\mathbb{Z}[x], -) \to F$. The data of a natural transformation between these two functors is, an association from each $R \in \mathbf{Ring}$ to a morphism in $\mathbf{Set}$ $f_R : Hom(\mathbb{Z}[x], R) \to FR$. The above discussion gives an "obvious" function $Hom(\mathbb{Z}[x], R) \xrightarrow{\sim} R$ in $\mathbf{Set}$, namely evaluating the map at x, $f \mapsto f(x)$. Moreover, this function is in fact a bijection because it has inverse

$$r \mapsto f(\ell) := \begin{cases} 1 & \ell = 1 \\ r & \ell = x \\ \text{“extend using homomorphism property”} & \text{otherwise} \end{cases}$$

  Let $g \in Hom_{\mathbf{Ring}}(R, S)$. We are left to show that the above association $R \mapsto f_R$ makes the following diagram commute in $\mathbf{Set}$

$$\begin{tikzcd} Hom(\mathbb{Z}[x], R) \arrow[r, "f_R"] \arrow[d, "g'"'] & R \arrow[d, "g"] \\ Hom(\mathbb{Z}[x], S) \arrow[r, "f_S"] & S \end{tikzcd}$$

  where g' is given by the action of the functor $Hom(\mathbb{Z}[x], -)$ on $g : R \to S$. Recalling that $f \in Hom(\mathbb{Z}[x], R)$ outputs to elements in R we can see that $g'(f) = g \circ f$ is a map $\mathbb{Z}[x] \to S$.

We verify that this diagram commutes. Let $f \in Hom(\mathbb{Z}[x], R)$. Following the top route gives the element

$$f \mapsto f(x) \mapsto g(f(x)) = (g \circ f)(x).$$

On the other hand, following the bottom route gives

$$f \mapsto g \circ f \mapsto (g \circ f)(x).$$

And so this diagram commutes.

Thus we have found a natural transformation $Hom(\mathbb{Z}[x], -) \to F$ with each morphism f_R a bijection. Thus we in fact have F is naturally isomorphic to $Hom(\mathbb{Z}[x], -)$ and so F is a representable functor, represented by $\mathbb{Z}[x]$.

- I will go a bit faster through this argument since many of the details follow similar logic to the above part.

  I claim that G : **Ring** $\to$ **Set** is representable functor, represented by the ring $\mathbb{Z}[x, x^{-1}]$. We must show that there exists a natural isomorphism $Hom_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], -) \to G$. First note that the maps $f \in Hom_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}])$ are still determined by their action on $x \in \mathbb{Z}[x, x^{-1}]$. But now, since $1 = f(1) = f(xx^{-1}) = f(x)f(x^{-1})$ any ring homomorphism $\mathbb{Z}[x, x^{-1}] \to R$ must send x to a unit in R. That is, the maps $f \in Hom_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], R)$ are exactly determined by the image on x, with the restriction that the image of x must be a unit in R.

  Now we construct a natural transformation $Hom_{\mathbf{Ring}}(\mathbb{Z}[x, x^{-1}], -) \to G$. Given $R \in$ **Ring** we assign the function $f_R : Hom(\mathbb{Z}[x, x^{-1}], R) \to R^{\times}$ given by evaluating the input map at x, $f_R(f) := f(x)$. We note that under this definition f_R is a well defined map into $R^{\times}$ becuase, as we noted above, the image of x under a ring homomorphism $\mathbb{Z}[x, x^{-1}]$ must be a unit. Next, notice that each f_R is actually a bijection since it has inverse $r \in R^{\times}$ maps to the homomorphism where $f(x) = r$, using similar reasoning to the discussion for F above.

  Lastly, we need to show that with these definitions, and given a ring morphism

$g : R \to S$, the following diagram commutes

$$\begin{tikzcd} Hom(\mathbb{Z}[x,x^{-1}],R) \arrow[r, "f_R"] \arrow[d, "g'"'] & R \arrow[d, "g"] \\ Hom(\mathbb{Z}[x,x^{-1}],S) \arrow[r, "f_S"] & S \end{tikzcd}$$

Following the top side we have

$$f \mapsto f(x) \mapsto (g \circ f)(x),$$

meanwhile following the bottom side gives

$$f \mapsto g \circ f \mapsto (g \circ f)(x).$$

And so this diagram commutes for any ring homomorphism. We have found a natural isomorphism from $Hom_{\mathbf{Ring}}(\mathbb{Z}[x,x^{-1}],-) \to G$ and so G is a representable functor, represented by the ring $\mathbb{Z}[x,x^{-1}]$.

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