Design of IIR digital filters

$$H(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_mz^{-m}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}$$

$$H(e^{j\omega})=\frac{N(e^{j\omega})}{D(e^{j\omega})}=\frac{b^Te_b(\omega)}{1+a^Te_a(\omega)}$$

$$a=[a_1\ a_2\ \cdots\ a_n]^T$$

$$b=[b_0\ b_1\ \cdots\ b_m]^T$$

$$e_a(\omega)=[e^{-j\omega}\ e^{-j2\omega}\ \cdots\ e^{-jn\omega}]^T$$

$$e_b(\omega)=[1\ e^{-j\omega}\ e^{-j2\omega}\ \cdots\ e^{-jm\omega}]^T$$

Desired frequency response : $H_d(\omega)$

Objective error function : $J(a,b)=\frac{1}{2}\int_R W(\omega)\left|H_d(\omega)-H(e^{j\omega})\right|^2 d\omega$ R : designed bands

$$=\frac{1}{2}\int_R W(\omega)\left|H_d(\omega)-\frac{b^Te_b(\omega)}{1+a^Te_a(\omega)}\right|^2 d\omega$$

$$=\frac{1}{2}\int_R \frac{W(\omega)}{|1+a^Te_a(\omega)|^2}\left|H_d(\omega)(1+a^Te_a(\omega))-b^Te_b(\omega)\right|^2 d\omega$$

$$=\frac{1}{2}\int_R \frac{W(\omega)}{|1+a^Te_a(\omega)|^2}\left|H_d(\omega)+H_d(\omega)e_a^T(\omega)a-e_b^T(\omega)b\right|^2 d\omega$$

$$=\frac{1}{2}\int_R \frac{W(\omega)}{|1+a^Te_a(\omega)|^2}\left|H_d(\omega)+\begin{bmatrix}H_d(\omega)e_a(\omega)\\-e_b(\omega)\end{bmatrix}^T\begin{bmatrix}a\\b\end{bmatrix}\right|^2 d\omega$$

$$=\frac{1}{2}\int_R \hat{W}(\omega)\left|H_d(\omega)+e^T(\omega)X\right|^2 d\omega$$

$$\hat{W}(\omega)=\frac{W(\omega)}{|1+a^Te_a(\omega)|^2}\qquad e(\omega)=\begin{bmatrix}H_d(\omega)e_a(\omega)\\-e_b(\omega)\end{bmatrix}\qquad X=\begin{bmatrix}a\\b\end{bmatrix}$$

$$J(X)=\frac{1}{2}\int_R \hat{W}(\omega)\left(H_d(\omega)+e^T(\omega)X\right)^*\left(H_d(\omega)+e^T(\omega)X\right)d\omega$$

$$=\frac{1}{2}\int_R \hat{W}(\omega)\left(H_d^*(\omega)+e^H(\omega)X\right)\left(H_d(\omega)+e^T(\omega)X\right)d\omega$$

$$=\frac{1}{2}\int_R \hat{W}(\omega)\left(H_d^*(\omega)H_d(\omega)+H_d^*(\omega)e^T(\omega)X+H_d(\omega)e^H(\omega)X+X^Te(\omega)e^H(\omega)X\right)d\omega$$

$$=s+r^TX+\frac{1}{2}X^TQX$$

$$s = \frac{1}{2}\int_R \hat{W}(\omega)\, H_d^*(\omega)\, H_d(\omega)\, d\omega = \frac{1}{2}\int_R \hat{W}\,|H_d(\omega)|^2\, d\omega$$

$$r = \int_R \hat{W}(\omega)\, \mathrm{Re}\{ H_d^*(\omega)\, e(\omega) \}\, d\omega$$

$$Q = \int_R \hat{W}(\omega)\, e(\omega)\, e^H(\omega)\, d\omega$$

Stability constraint:

$$\mathrm{Re}\left[1 + a^T e_a(\omega) \right] > 0 \qquad \omega \in [0, \pi]$$

$$1 + a_1\cos(\omega) + a_2\cos(2\omega) + \cdots + a_n\cos(n\omega) \geq \delta$$

$$a_1\cos(\omega) + a_2\cos(2\omega) + \cdots + a_n\cos(n\omega) \geq \delta - 1$$

$$-a_1\cos(\omega) - a_2\cos(2\omega) - \cdots - a_n\cos(n\omega) \leq 1 - \delta$$

$$\begin{bmatrix} -\cos(\omega) & -\cos(2\omega) & \cdots & -\cos(n\omega) \end{bmatrix} a \leq 1 - \delta$$

$$\begin{bmatrix} -\cos(\omega_0) & -\cos(2\omega_0) & \cdots & -\cos(n\omega_0) & 0 & 0 & \cdots & 0 \\ -\cos(\omega_1) & -\cos(2\omega_1) & \cdots & -\cos(n\omega_1) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \vdots \\ -\cos(\omega_M) & -\cos(2\omega_M) & \cdots & -\cos(n\omega_M) & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} 1-\delta \\ 1-\delta \\ \vdots \\ 1-\delta \end{bmatrix} = d \qquad \omega_i = \frac{i\pi}{M} \quad i = 0, \cdots, M$$

$$Bx \leq d$$

Problem:

minimize $J(x) = s + r^T x + \frac{1}{2} x^T Q x$

subject to $Bx \leq d$

Matlab tool:

quadratic programming:

$$\min_x \; 0.5\, x^T Q x + r^T x \quad \text{subject to} \quad Bx \leq d$$

language: $x = \mathrm{quadprog}(Q, r, B, d)$

Iterative optimization:

initial condition: $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

k-th iteration:

minimize $J_k(x_k) = s_k + r_k^T x_k + \frac{1}{2} x_k^T Q_k x_k$

subject to $Bx_k \leq d$

terminal criterion:

$$\delta_x = \frac{\| x_k - x_{k-1} \|}{\| x_k \|} \leq \epsilon$$

Example: $M = N = 14$

$$H_d(\omega) = \begin{cases} 0 & \omega \le 0.475\pi \\ e^{-j12\omega} & \omega \ge 0.525\pi \end{cases}$$

Example: $M = 30$ $N = 15$

$$H_d(\omega) = \begin{cases} e^{-j20\omega} & \omega \le 0.5\pi \\ 0 & \omega \ge 0.55\pi \end{cases}$$