• Decimation-in-frequency FFT algorithms

$$X[k]=\sum_{n=0}^{N-1} x[n]\,W_N^{nk} \qquad k=0,1,\cdots,N-1 \qquad W_N=e^{-j\frac{2\pi}{N}}$$

→ $X[k]$ 之偶數點： $X[2r]=\sum_{n=0}^{N-1} x[n]\,W_N^{n(2r)} \qquad r=0,1,\cdots,\frac{N}{2}-1$

$$X[2r]=\underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[n]\,W_N^{2nr}}_{\text{前半段}}+\underbrace{\sum_{n=N/2}^{N-1} x[n]\,W_N^{2nr}}_{\text{後半段}}$$

後半段：

$$\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_N^{2r[n+\frac{N}{2}]}=\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_N^{2nr}$$

$$W_N^{2r(n+\frac{N}{2})}=W_N^{2rn}\,W_N^{rN}=W_N^{2rn}\,\underbrace{e^{-j\frac{2\pi}{N}rN}}_{1}=W_N^{2rn}$$

$$=\sum_{n=0}^{\frac{N}{2}-1} x[n]\,W_{N/2}^{nr}+\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_{N/2}^{nr}$$

$\because W_N^2=e^{-j\frac{2\pi}{N}2}=e^{-j\frac{2\pi}{N/2}}=W_{N/2}$

$$=\underbrace{\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x[n+\tfrac{N}{2}]\right)W_{N/2}^{nr}}_{\frac{N}{2}\text{-point DFT of } x[n]+x[n+\frac{N}{2}],\ n=0,1,\cdots,\frac{N}{2}}$$

$\Rightarrow$ $X[k]$ 之偶數點可以從 $x[n]+x[n+\frac{N}{2}]$ $n=0,1,\cdots,\frac{N}{2}$ 之 $\frac{N}{2}$-point DFT 獲得

→ $X[k]$ 之奇數點： $X[2r+1]=\sum_{n=0}^{N-1} x[n]\,W_N^{n(2r+1)}$

$$X[2r+1]=\underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[n]\,W_N^{n(2r+1)}}_{\text{前半段}}+\underbrace{\sum_{n=N/2}^{N-1} x[n]\,W_N^{n(2r+1)}}_{\text{後半段}}$$

後半段：

$$\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_N^{(n+\frac{N}{2})(2r+1)}=W_N^{\frac{N}{2}(2r+1)}\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_N^{n(2r+1)}=-\sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\,W_N^{n(2r+1)}$$

$$W_N^{\frac{N}{2}(2r+1)}=e^{-j\frac{2\pi}{N}\frac{N}{2}(2r+1)}=(-1)^{2r+1}=-1$$

decimation-in-frequency

$$X[2r+1]=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]-x[n+\tfrac{N}{2}]\right)W_N^{n(2r+1)}$$
$$=\sum_{n=0}^{\frac{N}{2}-1}\left[\left(x[n]-x[n+\tfrac{N}{2}]\right)W_N^{n}\right]W_{N/2}^{nr}$$

乜 $\frac{N}{2}$-point DFT of $\left[x[n]-x[n+\frac{N}{2}]\right]W_N^n$, $n=0,1,\cdots,\frac{N}{2}-1$

$\Rightarrow$ $X[k]$之奇數點可以從 $\left(x[n]-x[n+\frac{N}{2}]\right)W_N^n$, $n=0,1,\cdots,\frac{N}{2}-1$之 $\frac{N}{2}$-point DFT獲得

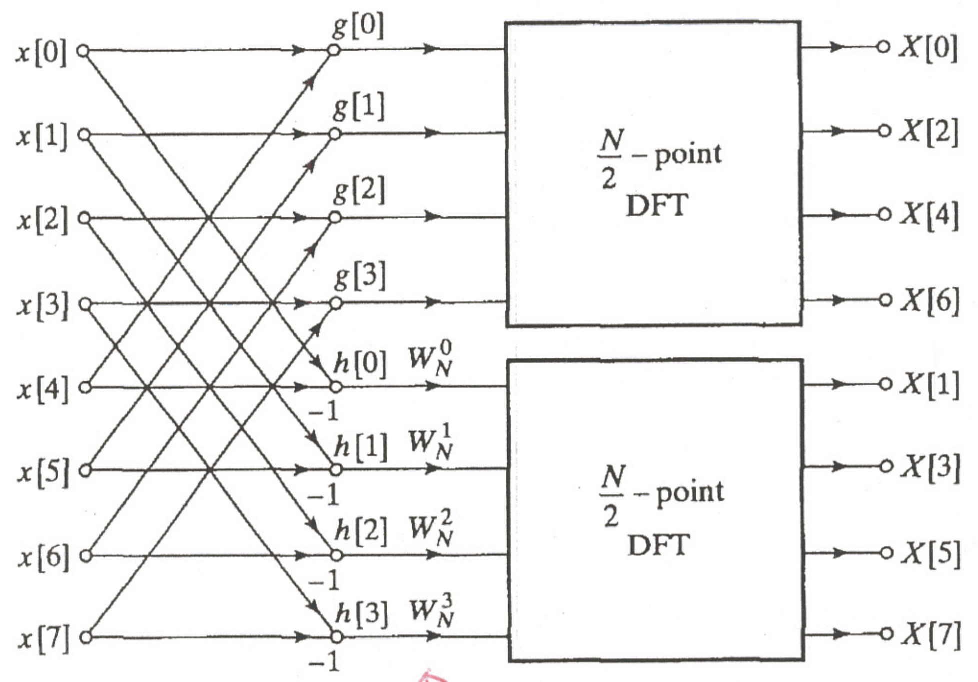


Figure 9.17 Flow graph of decimation-in-frequency decomposition of an N-point DFT computation into two $(N/2)$-point DFT computations $(N = 8)$.

以此類推, $\frac{N}{2}$-point 之DFT 亦可由上下並行之 $\frac{N}{4}$-point DFT 組成, 以 $N=8$ 為例, 方塊圖如下所示:

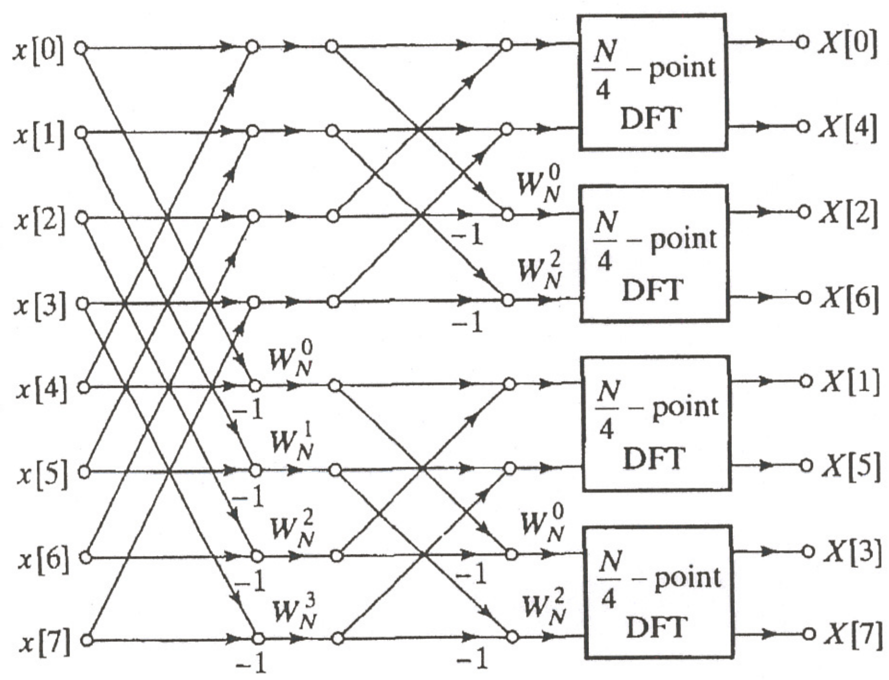
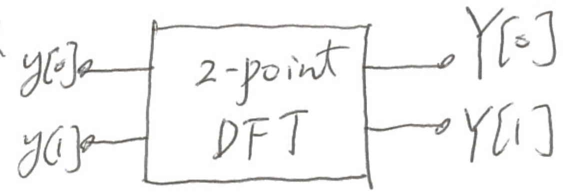


Figure 9.18 Flow graph of decimation-in-frequency decomposition of an 8-point DFT into four 2-point DFT computations.

當 $N=2^m$ 時, 經過 $m-1$ 次上述之步驟, 最後剩 $\frac{m}{2}$ 個 2-point DFT 有待完成。

2-point DFT：以

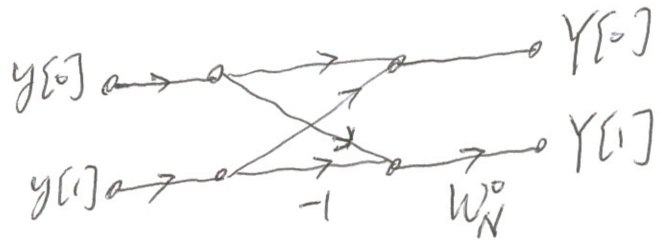


decimation in frequency

$$Y[0]=\sum_{n=0}^{1} y[n]\,W_2^{n\cdot 0} = y[0]+y[1]$$

$$Y[1]=\sum_{n=0}^{1} y[n]\,W_2^{n\cdot 1} = y[0]+y[1]\,W_2$$
$$= y[0]-y[1]\,W_N^{0}$$

$$W_2 = e^{-j\frac{2\pi}{2}} = -1 = -1\cdot e^{-j\frac{2\pi}{N}\cdot 0}$$
$$= -1\cdot W_N^{0}$$



故 decimation-in-frequency 演算法所得之最後架構(以 $N=8$ 為例)如下：

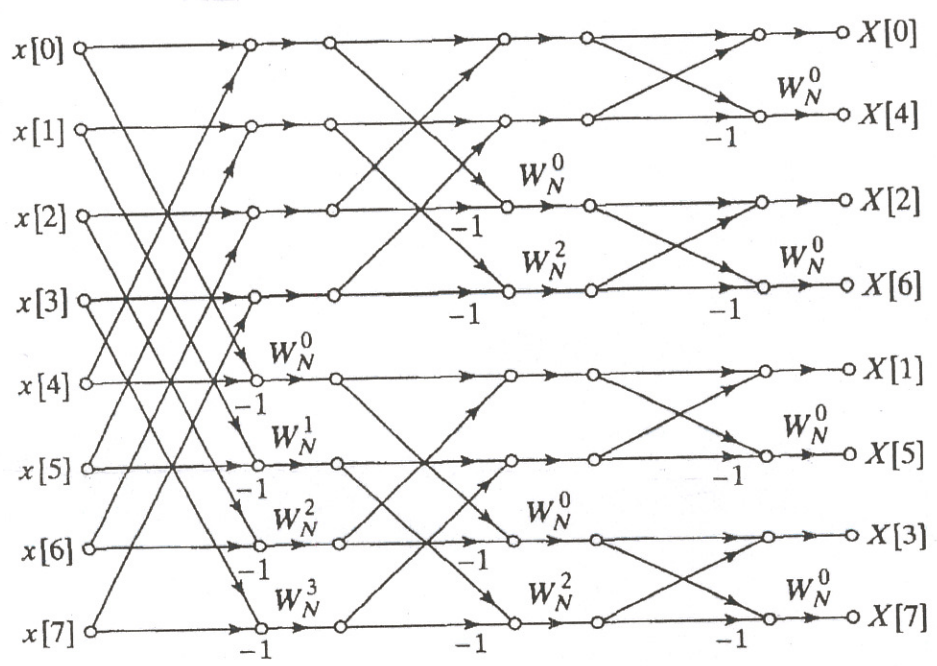


Figure 9.20 Flow graph of complete decimation-in-frequency decomposition of an 8-point DFT computation.

從上圖可知此架構由左至右可分 $\log_2 N$ 段，每段須 $\frac{N}{2}$ 個乘法器及 N 個加法器，因此此架構共須 $\frac{N}{2}\log_2 N$ 個乘法器及 $N\log_2 N$ 個加法器，與 decimation-in-time 演算法所得之結果相同。

和傳統 DFT 比較，舉例結果如下：(以乘法器為例)

m ($N=2^m$)	4	6	8	10	12
DFT N^2	256	4096	65536	1048576	16777216
FFT $\frac{N}{2}\log_2 N$	32	192	1024	5120	24576
$\frac{\frac{N}{2}\log_2 N}{N^2}$	0.125	0.0469	0.0156	0.0049	0.0015

上述架構之變形：

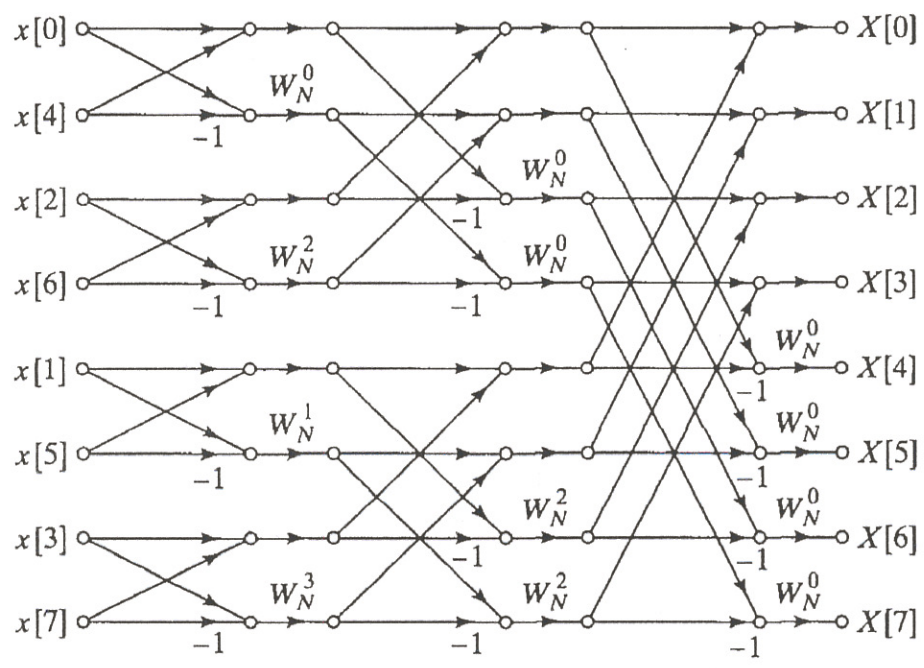


Figure 9.22 Flow graph of a decimation-in-frequency DFT algorithm obtained from Figure 9.20. Input in bit-reversed order and output in normal order. (Transpose of Figure 9.14.)

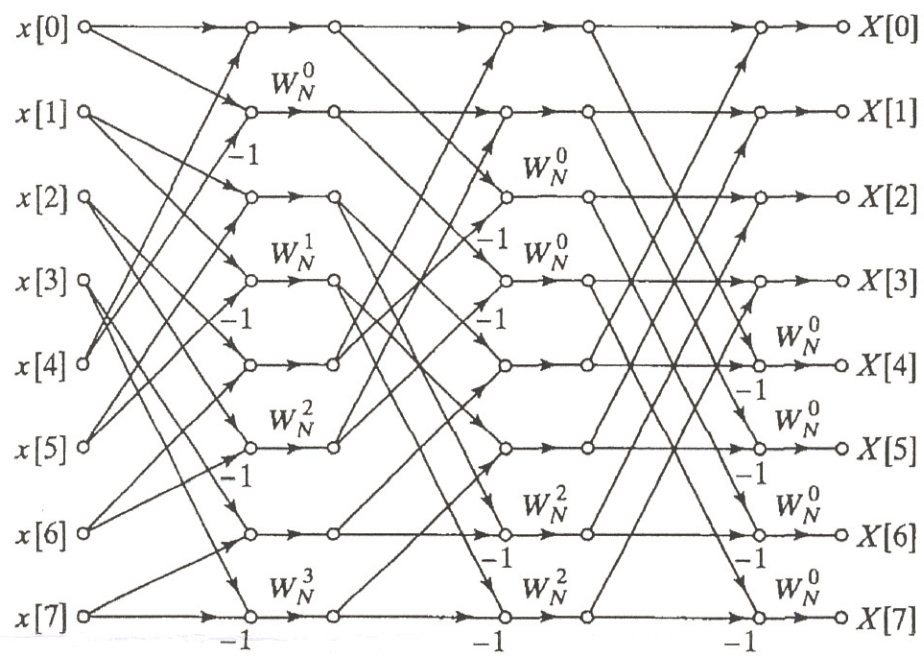


Figure 9.23 Rearrangement of Figure 9.20 with both input and output in normal order. (Transpose of Figure 9.15.)

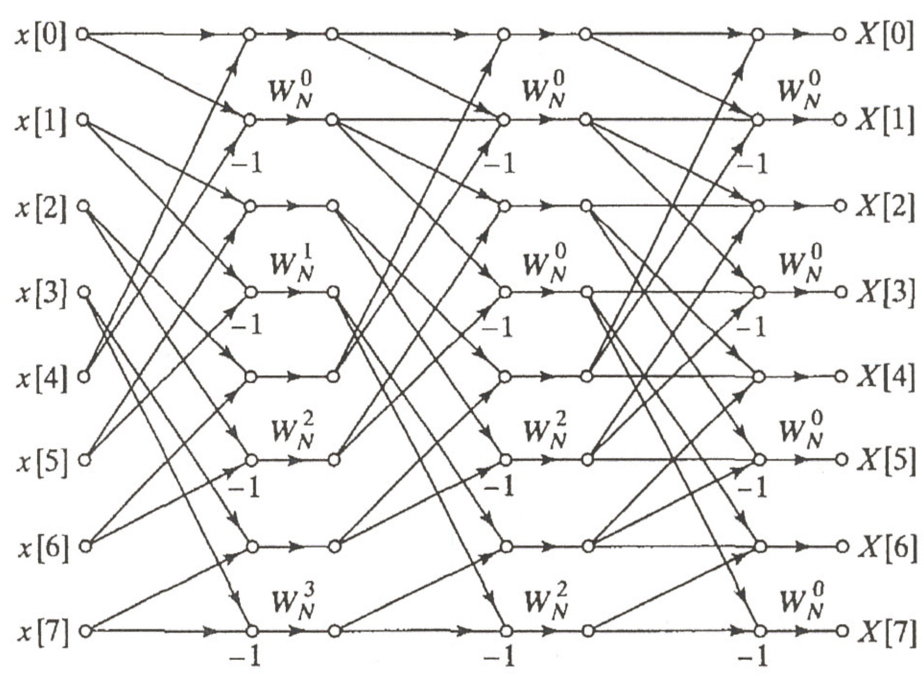


Figure 9.24 Rearrangement of Figure 9.20 having the same geometry *for each stage, thereby permitting* sequential data accessing and storage. (Transpose of Figure 9.16.)