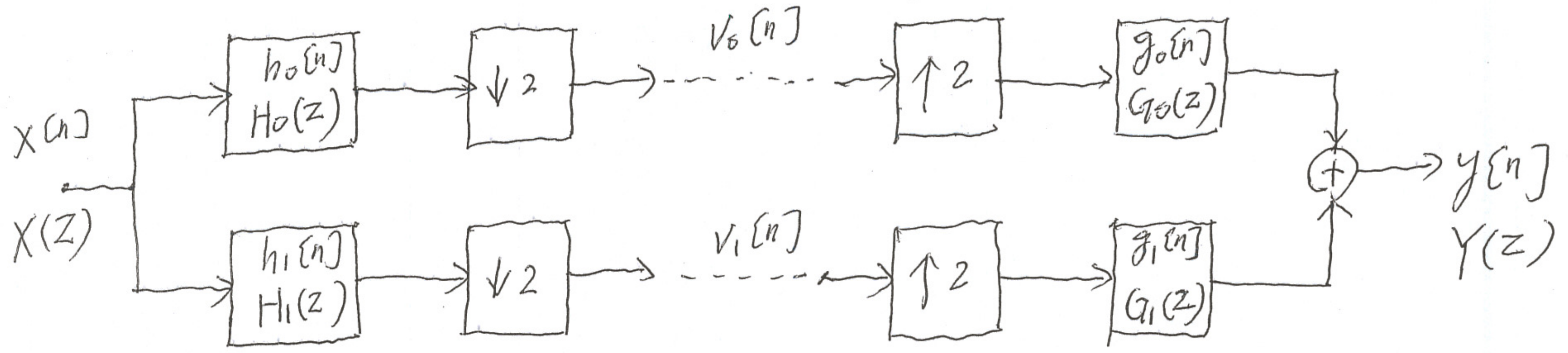
# Design of two-channel filter banks




$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

let $h_1[n] = (-1)^n h_0[n] \longleftrightarrow H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$

$g_0[n] = 2h_0[n] \longleftrightarrow G_0(e^{j\omega}) = 2H_0(e^{j\omega})$

$g_1[n] = -2h_1[n] \longleftrightarrow G_1(e^{j\omega}) = -2H_0(e^{j(\omega-\pi)})$

$$\Rightarrow Y(e^{j\omega}) = \left[H_0^2(e^{j\omega}) - H_0^2(e^{j(\omega-\pi)})\right]X(e^{j\omega})$$

$$H_0(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-jn\omega} = e^{-j\frac{N-1}{2}\omega}\hat{H}_0(\omega)$$

↑ Type I or Type II linear phase lowpass filter?

$$Y(e^{j\omega}) = e^{-j(N-1)\omega}\left[\hat{H}_0^2(\omega) - (-1)^{N-1}\hat{H}_0^2(\omega-\pi)\right]X(e^{j\omega})$$

$\Rightarrow N$ : even

$\Rightarrow H_0(Z)$ : Type II linear phase lowpass filter.

$$\hat{H}_0(\omega) = \sum_{n=1}^{N/2} a(n)\cos\left[\left(n-\frac{1}{2}\right)\omega\right] = A^T C(\omega)$$

$$A = \left[a(1)\ a(2)\ a(3) \cdots a\left(\frac{N+1}{2}\right)\right]^T$$

$$C(\omega) = \left[\cos\frac{\omega}{2}\ \cos\frac{3\omega}{2} \cdots \cos\frac{N}{2}\omega\right]^T$$

$$a(n) = 2h\left(\frac{N}{2}-n\right) \quad n = 1, 2, \cdots, \frac{N}{2}$$

objective error function:

$$\hat{J}(A) = \int_0^{\pi}\left[1 - \hat{H}_0^2(\omega) - \hat{H}_0^2(\omega-\pi)\right]^2 d\omega + \alpha\int_{\omega_s}^{\pi}\hat{H}_0^2(\omega)\,d\omega \leftarrow \text{nonlinear problem.}$$

modified objective error function:

$$J_k(A) = \int_0^{\pi}\left[1 - \hat{H}_{0,k-1}(\omega)\hat{H}_{0,k}(\omega) - \hat{H}_{0,k-1}(\omega-\pi)\hat{H}_{0,k}(\omega-\pi)\right]^2 d\omega + \alpha\int_{\omega_s}^{\pi}\hat{H}_{0,k}^2(\omega)\,d\omega$$

$$\hat{H}_{0,k}(\omega) = A_k^T C_k(\omega) = C_k^T(\omega) A_k$$

$$J_k(A) = \int_0^{\pi} \left[1 - \hat{H}_{0,k-1}(\omega) A_k^T C_k(\omega) - \hat{H}_{0,k-1}(\omega-\pi) A_k^T C_k(\omega-\pi)\right]^2 d\omega + \alpha \int_{\omega_s}^{\pi} A_k^T C_k(\omega) C_k^T(\omega) A_k \, d\omega$$

$$= \pi + A_k^T \left[\int_0^{\pi} -2\left(\hat{H}_{0,k-1}(\omega) C_k(\omega) + \hat{H}_{0,k-1}(\omega-\pi) C_k(\omega-\pi)\right) d\omega\right]$$

$$+ A_k^T \left[\int_0^{\pi} \left(\hat{H}_{0,k-1}(\omega) C_k(\omega) + \hat{H}_{0,k-1}(\omega-\pi) C_k(\omega-\pi)\right)(\cdot)^T d\omega\right] A_k + A_k^T \left[\alpha \int_{\omega_s}^{\pi} C_k C_k^T d\omega\right] A_k$$

$$= \pi + A_k^T P + A_k^T Q_1 A_k + A_k^T Q_2 A_k$$

$$P = -2 \int_0^{\pi} \left(\hat{H}_{0,k-1}(\omega) C_k(\omega) + \hat{H}_{0,k-1}(\omega-\pi) C_k(\omega-\pi)\right) d\omega$$

$$Q_1 = \int_0^{\pi} \left(\hat{H}_{0,k-1}(\omega) C_k(\omega) + \hat{H}_{0,k-1}(\omega-\pi) C_k(\omega-\pi)\right)(\cdot)^T d\omega$$

$$Q_2 = \int_{\omega_s}^{\pi} C_k(\omega) C_k^T(\omega) d\omega$$

design procedures:

initialization: design of a Type II lowpass filter

k-th iteration: minimize $J_k(A) \Rightarrow A_k = -\frac{1}{2}(Q_1 + Q_2)^{-1} P \Rightarrow A_k \leftarrow \beta A_k + (1-\beta) A_{k-1}$

termination: $\delta_k = \frac{\| A_k - A_{k-1} \|}{\| A_k \|} < \epsilon$

**design of a Type II lowpass filter**

$$e(A) = \int_0^{\omega_p} \left(1 - A^T C(\omega)\right)^2 d\omega + \int_{\omega_s}^{\pi} \left(A^T C(\omega)\right)^2 d\omega$$

$$= s + A^T P_0 + A^T Q_p A + A^T Q_s A$$

$$\Rightarrow A = -\frac{1}{2}(Q_p + Q_s)^{-1} P_0$$

$$s = \int_0^{\omega_p} 1 \, d\omega$$

$$P_0 = -2 \int_0^{\omega_p} C(\omega) d\omega$$

$$Q_p = \int_0^{\omega_p} C(\omega) C^T(\omega) d\omega$$

$$Q_s = \int_{\omega_s}^{\pi} C(\omega) C^T(\omega) d\omega$$