Design of variable fractional-order (VFO) FIR differ-integrators

desired frequency response:

$$H_d(\omega,p) = e^{-jI\omega}(j\omega)^p \qquad p_1 \le p \le p_2 \quad \omega_1 \le |\omega| \le \omega_2$$

$$= e^{-jI\omega}|\omega|^p\left[\cos\left(\frac{p\pi}{2}\right) + j\,\mathrm{sgn}(\omega)\sin\left(\frac{p\pi}{2}\right)\right]$$

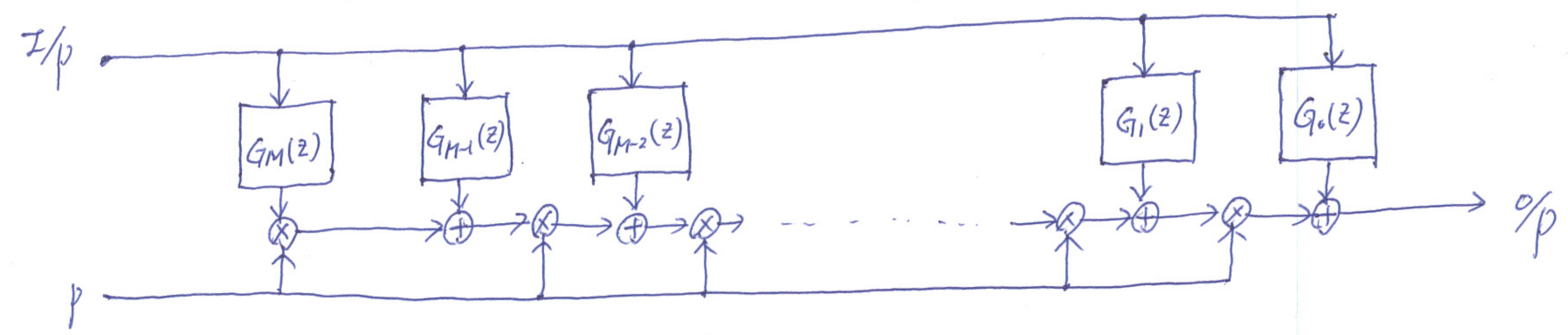
variable FIR digital filter:

$$H(z,p) = \sum_{n=0}^{N} h_n(p) z^{-n}$$

$$h_n(p) = \sum_{m=0}^{M} h(n,m) p^m$$

$$H(z,p) = \sum_{n=0}^{N}\sum_{m=0}^{M} h(n,m) p^m z^{-n} = \sum_{m=0}^{M}\left(\sum_{n=0}^{N} h(n,m) z^{-n}\right) p^m = \sum_{m=0}^{M} G_m(z) p^m$$

$$G_m(z) = \sum_{n=0}^{N} h(n,m) z^{-n}$$



$$h(n,m) = h_e(n,m) + h_o(n,m)$$

$$h_e\left(\frac{N}{2}+n, m\right) = \frac{1}{2}\left[h\left(\frac{N}{2}+n,m\right) + h\left(\frac{N}{2}-n,m\right)\right] \qquad -\frac{N}{2} \le n \le \frac{N}{2} \quad 0 \le m \le M$$

N: even

$$h_o\left(\frac{N}{2}+n, m\right) = \frac{1}{2}\left[h\left(\frac{N}{2}+n,m\right) - h\left(\frac{N}{2}-n,m\right)\right] \qquad -\frac{N}{2} \le n \le \frac{N}{2} \quad 0 \le m \le M$$

$$H(e^{j\omega},p) = e^{-j\frac{N}{2}\omega}\left[\sum_{n=0}^{N/2}\sum_{m=0}^{M} a(n,m)p^m\cos(n\omega) + j\sum_{n=1}^{N/2}\sum_{m=0}^{M} b(n,m)p^m\sin(n\omega)\right]$$

$$= e^{-j\frac{N}{2}\omega}\left[a^T c(\omega,p) + j\, b^T s(\omega,p)\right]$$

$$a(n,m) = \begin{cases} h_e\left(\frac{N}{2},m\right) & n=0 \quad 0 \le m \le M \\ 2h_e\left(\frac{N}{2}-n,m\right) & 1 \le n \le \frac{N}{2} \quad 0 \le m \le M \end{cases}$$

$$b(n,m) = 2h_o\left(\frac{N}{2}-n,m\right) \qquad 1 \le n \le \frac{N}{2},\ 0 \le m \le M$$

$$a = \left[\ a(n,m) \quad 0 \le n \le \frac{N}{2},\ 0 \le m \le M\ \right]^T$$

$$b = \left[\ b(n,m) \quad 1 \le n \le \frac{N}{2},\ 0 \le m \le M\ \right]^T$$

$$c(\omega,p) = \left[\ p^m\cos(n\omega) \quad 0 \le n \le \frac{N}{2},\ 0 \le m \le M\ \right]^T$$

$$s(\omega,p) = \left[\ p^m\sin(n\omega) \quad 1 \le n \le \frac{N}{2} \quad 0 \le m \le M\ \right]^T$$

let $I = \frac{N}{2}$

objective error function:

$$e(a,b) = \int_{p_1}^{p_2}\int_{\omega_1}^{\omega_2} \left| H_d(\omega,p) - H(e^{j\omega},p) \right|^2 d\omega\, dp$$

$$= \int_{p_1}^{p_2}\int_{\omega_1}^{\omega_2} \left| \omega^p \cos\left(\frac{p\pi}{2}\right) + j\,\omega^p \sin\left(\frac{p\pi}{2}\right) - a^T c(\omega,p) - j\, b^T s(\omega,p) \right|^2 d\omega\, dp$$

$$= \int_{p_1}^{p_2}\int_{\omega_1}^{\omega_2} \left[\omega^p \cos\left(\frac{p\pi}{2}\right) - a^T c(\omega,p) \right]^2 d\omega\, dp + \int_{p_1}^{p_2}\int_{\omega_1}^{\omega_2} \left[\omega^p \sin\left(\frac{p\pi}{2}\right) - b^T s(\omega,p) \right]^2 d\omega\, dp$$

$$= e(a) + e(b)$$

$$e(a) = S_a + r_a^T a + a^T Q_a a$$

$$e(b) = S_b + r_b^T b + b^T Q_b b$$

$$S_a = \iint \left(\omega^p \cos\left(\frac{p\pi}{2}\right)\right)^2 d\omega\, dp \qquad S_b = \iint \left(\omega^p \sin\left(\frac{p\pi}{2}\right)\right)^2 d\omega\, dp$$

$$r_a = -2\iint \omega^p \cos\left(\frac{p\pi}{2}\right) c(\omega,p)\, d\omega\, dp \qquad r_b = -2\iint \omega^p \sin\left(\frac{p\pi}{2}\right) s(\omega,p)\, d\omega\, dp$$

$$Q_a = \iint c(\omega,p)\, c^T(\omega,p)\, d\omega\, dp \qquad Q_b = \iint s(\omega,p)\, s^T(\omega,p)\, d\omega\, dp$$

$$\frac{\partial e(a,b)}{\partial a} = \frac{\partial e(a)}{\partial a} = r_a + 2Q_a a = 0 \Rightarrow a = -\frac{1}{2} Q_a^{-1} r_a$$

$$\frac{\partial e(a,b)}{\partial b} = \frac{\partial e(b)}{\partial b} = r_b + 2Q_b b = 0 \Rightarrow b = -\frac{1}{2} Q_b^{-1} r_b$$

Example: $N=40$ $M=5$ $\omega_1 = 0.05\pi$ $\omega_2 = 0.95\pi$ $p_1 = -0.5$ $p_2 = 0.5$

Example: $N=30$ $M=6$ $\omega_1 = 0$ $\omega_2 = 0.9\pi$ $p_1 = 1$ $p_2 = 2$

Example: $N=60$ $M=6$ $\omega_1 = 0.05\pi$ $\omega_2 = 0.9\pi$ $p_1 = -1.5$ $p_2 = -0.5$