Project 1
Design of Linear-Phase FIR Digital Filters

I. Frequency Responses of Linear-Phase FIR Filters

Let $\{h(n)\}$ be a causal finite duration sequence defined over the interval $0 \le n \le N-1$. The z transform of $\{h(n)\}$ is

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} = h(0) + h(1)z^{-1} + \cdots + h(N-1)z^{-(N-1)}. \tag{1}$$

The Fourier transform of $\{h(n)\}$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-jn\omega}. \tag{2}$$

If $\{h(n)\}$ is symmetric or antisymmetric about its central point, Eq.(2) can be rewritten as

$$H(e^{j\omega}) = e^{-j\frac{N-1}{2}\omega}e^{j\frac{L\pi}{2}}\hat{H}(\omega) \tag{3}$$

where $\hat{H}(\omega)$ is a real-valued amplitude response and

$$L = \begin{cases} 1, & \text{if } \{h(n)\} \text{ is antisymmetric,} \\ 0, & \text{if } \{h(n)\} \text{ is symmetric.} \end{cases} \tag{4}$$

According to the symmetric properties of the impulse response $\{h(n)\}$ and the filter length (even or odd), there are four cases to be considered.

Case 1: Symmetric impulse response and odd length: $L = 0$ and

$$\hat{H}(\omega) = \sum_{k=0}^{\frac{N-1}{2}} a(k)cos(k\omega) \tag{5}$$

where

$$a(k) = \begin{cases} h(\frac{N-1}{2}), & k = 0, \\ 2h(\frac{N-1}{2} - k), & k = 1, 2, \cdots, \frac{N-1}{2}. \end{cases} \tag{6}$$

Case 2: Symmetric impulse response and even length: $L = 0$ and

$$\hat{H}(\omega) = \sum_{k=1}^{\frac{N}{2}} a(k)cos((k - \frac{1}{2})\omega) \tag{7}$$

where

$$a(k) = 2h(\frac{N}{2} - k), \quad k = 1, 2, \cdots, \frac{N}{2}. \tag{8}$$

Case 3: Antisymmetric impulse response and odd length: $L = 1$ and

$$\hat{H}(\omega) = \sum_{k=1}^{\frac{N-1}{2}} a(k)sin(k\omega) \tag{9}$$

where

$$a(k) = 2h(\frac{N-1}{2} - k), \quad k = 1, 2, \cdots, \frac{N-1}{2}. \tag{10}$$

Case 4: Antisymmetric impulse response and even length: $L = 1$ and

$$\hat{H}(\omega) = \sum_{k=1}^{\frac{N}{2}} a(k)sin((k - \frac{1}{2})\omega) \tag{11}$$

where

$$a(k) = 2h(\frac{N}{2} - k), \quad k = 1, 2, \cdots, \frac{N}{2}. \tag{12}$$

II. Proof of Eqs.(5) and (6)

Because $\{h(n)\}$ is symmetric and N is odd,

$$\begin{aligned} H(e^{j\omega}) &= h(0) + h(1)e^{-j\omega} + \cdots + h(\frac{N-1}{2})e^{-j\frac{N-1}{2}\omega} + \cdots + \\ &\quad h(N-2)e^{-j(N-2)\omega} + h(N-1)e^{-j(N-1)\omega} \\ &= e^{-j\frac{N-1}{2}\omega}\{h(\frac{N-1}{2}) + [h(\frac{N-1}{2} - 1)e^{j\omega} + h(\frac{N-1}{2} + 1)e^{-j\omega}] + \\ &\quad [h(\frac{N-1}{2} - 2)e^{j2\omega} + h(\frac{N-1}{2} + 2)e^{-j2\omega}] + \cdots + \\ &\quad [h(0)e^{j\frac{N-1}{2}\omega} + h(N-1)e^{-j\frac{N-1}{2}\omega}]\} \qquad (13) \\ &= e^{-j\frac{N-1}{2}\omega}\{h(\frac{N-1}{2}) + 2h(\frac{N-1}{2} - 1)cos(\omega) + \\ &\quad 2h(\frac{N-1}{2} - 2)cos(2\omega) + \cdots + 2h(0)cos(\frac{N-1}{2}\omega)\}. \end{aligned}$$

Comparing (3) and (13), we can obtain (5) and (6) easily.

Exercise 1.1: Derive (7)~(12).

III. Design of Case 1 Low-Pass Filters

Suppose the given desired amplitude response is $D(\omega)$, for designing a **Case 1** low-pass filter with passband $[0, \omega_p]$, stopband $[\omega_s, \pi]$, the objective error function can be defined by

$$e = \int_R W(\omega)[D(\omega) - \hat{H}(\omega)]^2 d\omega \tag{14}$$

where $R = [0, \omega_p] \cup [\omega_s, \pi]$ and $W(\omega)$ is the weighting function. The amplitude response $\hat{H}(\omega)$ can be represented in inner product form:

$$\hat{H}(\omega) = \mathbf{A}^t\mathbf{C}(\omega) = \mathbf{C}^t(\omega)\mathbf{A} \tag{15}$$

where t denotes the transpose operation,

$$\mathbf{A} = [a(0) \quad a(1) \quad a(2) \quad \cdots \quad a(\frac{N-1}{2})]^t \tag{16}$$

and

$$\mathbf{C}(\omega) = [1 \quad cos(\omega) \quad cos(2\omega) \quad \cdots \quad cos(\frac{N-1}{2}\omega)]^t. \tag{17}$$

Subsitituting (15) into (14),

$$\begin{aligned} e &= \int_R W(\omega)[D^2(\omega) - 2D(\omega)\mathbf{C}^t(\omega)\mathbf{A} + \mathbf{A}^t\mathbf{C}(\omega)\mathbf{C}^t(\omega)\mathbf{A}]d\omega \\ &= s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} \end{aligned} \tag{18}$$

where

$$s = \int_R W(\omega)D^2(\omega)d\omega, \tag{19}$$

$$\mathbf{P} = -2\int_R W(\omega)D(\omega)\mathbf{C}(\omega)d\omega \tag{20}$$

and

$$\mathbf{Q} = \int_R W(\omega)\mathbf{C}(\omega)\mathbf{C}^t(\omega)d\omega. \tag{21}$$

Notice that, assuming the desired response and the weighting function are given by

$$D(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p, \\ 0, & \omega_s \le \omega \le \pi \end{cases} \tag{22}$$

and

$$W(\omega) = \begin{cases} W_p, & 0 \le \omega \le \omega_p, \\ W_s, & \omega_s \le \omega \le \pi, \end{cases} \tag{23}$$

respectively, the elements of $\mathbf{P} = [p(i),\ 0 \le i \le \frac{N-1}{2}]$ are

$$p(i) = -2\int_0^{\omega_p} W_p cos(i\omega) d\omega = \begin{cases} -2W_p\omega_p, & i = 0, \\ -2W_p\frac{sin(i\omega_p)}{i}, & i \neq 0. \end{cases} \tag{24}$$

Exercise 1.2: Derive the closed forms of the elements of $\mathbf{Q} = [q(i,l),\ 0 \le i, l \le \frac{N-1}{2}]$.

The desired coefficient vector $\mathbf{A}$ can be obtained by setting $\frac{\partial e}{\partial \mathbf{A}}$ to zero, i.e.

$$\frac{\partial e}{\partial \mathbf{A}} = \mathbf{P} + 2\mathbf{Q}\mathbf{A} = 0, \tag{25}$$

which leads to

$$\mathbf{A} = -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{P}. \tag{26}$$

Once $\mathbf{A}$ is obtained, the filter coefficients $\{h(n)\}$ can be computed by (6).

Exercise 1.3: Use MATLAB software package to design a **Class 1** low-pass filter with $N = 31$, $\omega_p = 0.4\pi$, $\omega_s = 0.5\pi$, $W_p = W_s = 1$. (a) Show the impulse response sequence $\{h(n)\}$. (b) Show the amplitude response. (c) Show the amplitude response in dB scale.

Exercise 1.4: Repeat **Exercise 1.3**, but $W_p = 1$, $W_s = 10$. Compare the obtained results with those of **Exercise 1.3**.

Exercise 1.5: Design of a **Case 1** band-pass filter with length N,

$$D(\omega) = \begin{cases} 0, & 0 \le \omega \le \omega_{s_1}, \\ 1, & \omega_{s_1} < \omega_{p_1} \le \omega \le \omega_{p_2}, \\ 0, & \omega_{p_2} < \omega_{s_2} \le \omega \le \pi \end{cases} \tag{27}$$

and

$$W(\omega) = \begin{cases} W_{s_1}, & 0 \le \omega \le \omega_{s_1}, \\ W_p, & \omega_{s_1} < \omega_{p_1} \le \omega \le \omega_{p_2}, \\ W_{s_2}, & \omega_{p_2} < \omega_{s_2} \le \omega \le \pi. \end{cases} \quad (28).$$

(a) Derive the closed forms of the elements of **P** and **Q**. (b) Use MATLAB software package to design a **Case 1** band-pass filter with $N = 43$, $\omega_{s_1} = 0.2\pi$, $\omega_{p_1} = 0.26\pi$, $\omega_{p_2} = 0.56\pi$, $\omega_{s_2} = 0.64\pi$, $W_{s_1} = W_{s_2} = 5$ and $W_p = 1$. (c) Show the impulse response sequence $\{h(n)\}$. (d) Show the amplitude response. (e) Show the amplitude response in dB scale.

IV. Appendix. Proof of $\frac{\partial \mathbf{P}^t\mathbf{A}}{\partial \mathbf{A}} = \mathbf{P}$

Let g be a scalar-valued function, the derivative of g with respective to the vector **A** is defined by

$$\frac{\partial g}{\partial \mathbf{A}} = [\frac{\partial g}{\partial a(0)} \quad \frac{\partial g}{\partial a(1)} \quad \frac{\partial g}{\partial a(2)} \quad \cdots \quad \frac{\partial g}{\partial a(\frac{N-1}{2})}]^t. \tag{29}$$

Suppose

$$\mathbf{P} = [p(0) \quad p(1) \quad p(2) \quad \cdots \quad p(\frac{N-1}{2})]^t, \tag{30}$$

then

$$\mathbf{P}^t\mathbf{A} = \sum_{i=0}^{\frac{N-1}{2}} p(i)a(i). \tag{31}$$

Because

$$\frac{\partial \mathbf{P}^t\mathbf{A}}{\partial a(i)} = p(i), \quad 0 \le i \le \frac{N-1}{2}, \tag{32}$$

so

$$\frac{\partial \mathbf{P}^t\mathbf{A}}{\partial \mathbf{A}} = [p(0) \quad p(1) \quad p(2) \quad \cdots \quad p(\frac{N-1}{2})]^t = \mathbf{P}. \tag{33}$$

Exercise 1.6: Show that $\frac{\partial \mathbf{A}^t\mathbf{Q}\mathbf{A}}{\partial \mathbf{A}} = 2\mathbf{Q}\mathbf{A}$.

V. Design of Hilbert Transformers and Differentiators

For a zero-phase Hilbert transformer design, the desired frequency response is generally given by

$$D_H(\omega) = \begin{cases} -j, & \omega_{p_1} \le \omega \le \omega_{p_2}, \\ j, & -\omega_{p_2} \le \omega \le -\omega_{p_1} \end{cases} \tag{34}$$

where ω_{p_1} and ω_{p_2} are cutoff frequencies. Hence, **Case 3** and **Case 4** can be applied for the design of Hilbert transformers. Notice that ω_{p_2} can be set to π for **Case 4** design while $\omega_{p_2} < \pi$ for **Case 3** design. The former is generally called by "wide-band Hilbert transformer".

For **Case 3** design, the amplitude response $\hat{H}(\omega)$ can be represented by

$$\hat{H}(\omega) = \mathbf{A}^t\mathbf{S}(\omega) = \mathbf{S}^t(\omega)\mathbf{A} \tag{35}$$

where

$$\mathbf{A} = [a(1) \quad a(2) \quad \cdots \quad a(\frac{N-1}{2})]^t \tag{36}$$

and

$$\mathbf{S}(\omega) = [sin(\omega) \quad sin(2\omega) \quad \cdots \quad sin(\frac{N-1}{2}\omega)]^t. \tag{37}$$

By (14) and (30), the objective error function can also be formulated into

$$e = s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} \tag{38}$$

where

$$s = \int_R W(\omega)D^2(\omega)d\omega, \tag{39}$$

$$\mathbf{P} = -2\int_R W(\omega)D(\omega)\mathbf{S}(\omega)d\omega \tag{40}$$

and

$$\mathbf{Q} = \int_R W(\omega)\mathbf{S}(\omega)\mathbf{S}^t(\omega)d\omega, \tag{41}$$

in which $R = \{\omega,\ \omega_{p_1} \le \omega \le \omega_{p_2}\}$ and $D(\omega) = -1$ for $\omega \in R$. Once $\mathbf{P}$ and $\mathbf{Q}$ are found, the coefficient vector $\mathbf{A}$ can be computed by (26), and the filter coefficients can be obtained by (10).

Exercise 1.7: Design of a Hilbert transformer with $W(\omega) = 1$. (a) Derive the closed-forms of $\mathbf{P}$ and $\mathbf{Q}$. (b) Use MATLAB software package to design an $N = 31$ Hilbert transformer with $\omega_{p_1} = 0.08\pi$ and $\omega_{p_2} = 0.92\pi$. (c) Show the amplitude response.

Exercise 1.8: Following the design procedures stated above, design a **Case 4** Hilbert transformer. (a) Formulate the error function in matrix form likes (38)~(41). (b) Derive the closed-forms of $\mathbf{P}$ and $\mathbf{Q}$ assumming $W(\omega) = 1$ for $\omega \in R$. (c) Use MATLAB software package to design an $N = 32$ Hilbert transformer with $\omega_{p_1} = 0.08\pi$ and $\omega_{p_2} = \pi$. (d) Show the amplitude response.

The desired frequency response of a zero-phase differentiator is given by

$$D_D(\omega) = j\omega, \quad -\omega_p \le \omega \le \omega_p, \tag{42}$$

so only **Case 3** and **Case 4** design can be used. Also, ω_p can be set to π for **Case 4** design, which is generally called by "full-band differentiator". The design procedures are similar to those of Hilbert transformer design.

Exercise 1.9: Following the design procedures of a Hilbert transformer, design a **Case 3** differentiator. (a) Formulate the error function in matrix form likes (38)~(41). (b) Derive the closed-forms of $\mathbf{P}$ and $\mathbf{Q}$ assumming $W(\omega) = 1$ for $\omega \in R$. (c) Use MATLAB software package to design an $N = 31$ differentiator with $\omega_p = 0.9\pi$. (d) Show the amplitude response.

Project 2
Design of Two-Dimensional Linear-Phase Quadrantally Symmetric FIR Digital Filters

I. Frequency Responses of Two-Dimensional Linear-Phase Quadrantally Symmetric FIR Filters

The transfer function of a two-dimensional (2-D) FIR digital filter with impulse response sequence $h(n_1, n_2)$, $n_1 = 0, 1, \cdots, N_1 - 1$, $n_2 = 0, 1, \cdots, N_2 - 1$ can be characterized as

$$H(z_1, z_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{1}$$

and the corresponding frequency response is

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) e^{-jn_1\omega_1} e^{-jn_2\omega_2}. \tag{2}$$

According to the symmetric properties of the impulse response $h(n_1, n_2)$ and the filter lengths $(N_1,\ N_2)$, there are sixteen types of linear-phase quadrantally symmetric 2-D FIR filters and (2) can be further rewritten as

$$H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j\frac{N_1-1}{2}\omega_1} e^{-j\frac{N_2-1}{2}\omega_2} e^{j\frac{M\pi}{2}} \hat{H}(\omega_1, \omega_2) \tag{3}$$

where $\hat{H}(\omega_1, \omega_2)$ is a real-valued amplitude function and

$$M = \begin{cases} 0, & \textbf{Type I}, \\ 1, & \textbf{Type II} \text{ and } \textbf{Type III}, \\ 2, & \textbf{Type IV}. \end{cases} \tag{4}$$

For example, if $h(n_1, n_2)$ is a **Type I** 2-D sequence and N_1, N_2 are odd integers, then

$$\hat{H}(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1-1}{2}} \sum_{n_2=0}^{\frac{N_2-1}{2}} a(n_1, n_2) cos(n_1\omega_1) cos(n_2\omega_2) \tag{5}$$

where

$$a(n_1,n_2) = \begin{cases} h(\frac{N_1-1}{2}, \frac{N_2-1}{2}), & n_1 = n_2 = 0, \\ 2h(\frac{N_1-1}{2}, \frac{N_2-1}{2} - n_2), & n_1 = 0,\ n_2 = 1,2,\cdots,\frac{N_2-1}{2}, \\ 2h(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2}), & n_1 = 1,2,\cdots,\frac{N_1-1}{2},\ n_2 = 0, \\ 4h(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2} - n_2), & n_1 = 1,2,\cdots,\frac{N_1-1}{2},\ n_2 = 1,2,\cdots,\frac{N_2-1}{2}. \end{cases} \tag{6}$$

The corresponding types of $\hat{H}(\omega_1,\omega_2)$ for each type filter and the symmetry relations of the impulse response samples are tabulated in Table I. Also, the relationships between the coefficients $a(n_1,n_2)$ and $h(n_1,n_2)$ are listed in Table II in which

$$L_i = \begin{cases} \frac{N_i-1}{2}, & N_i \text{ odd},\ i = 1,2, \\ \frac{N_i}{2}, & N_i \text{ even},\ i = 1,2. \end{cases} \tag{7}$$

Exercise 2.1: Derive (4), (5) and (6) for **Type I-1** 2-D filters.

II. Design of Type I-1 2-D Filters

Suppose the given desired amplitude response is $D(\omega_1,\omega_2)$, the objective error function for designing **Type I-1** 2-D filters can be defined by

$$e = \int\int_R W(\omega_1,\omega_2)[D(\omega_1,\omega_2) - \hat{H}(\omega_1,\omega_2)]^2 d\omega_1 d\omega_2 \tag{8}$$

where R represents the designed bands including passbands and stopbands, and $W(\omega_1,\omega_2)$ is the weighting function. The amplitude response $\hat{H}(\omega_1,\omega_2)$ can be represented in vector product form:

$$\hat{H}(\omega_1,\omega_2) = \mathbf{A}^t\mathbf{C}(\omega_1,\omega_2) = \mathbf{C}^t(\omega_1,\omega_2)\mathbf{A} \tag{9}$$

where t denotes the transpose operation, $\mathbf{A}$ is the coefficient vector defined by

$$\mathbf{A} = [\mathbf{A}_0^t \quad \mathbf{A}_1^t \quad \mathbf{A}_2^t \quad \cdots \quad \mathbf{A}_{L_1}^t]^t \tag{10}$$

and $\mathbf{C}(\omega_1,\omega_2)$ is the kernel vector given by

$$\mathbf{C}(\omega_1,\omega_2) = [\mathbf{C}_0^t(\omega_1,\omega_2) \quad \mathbf{C}_1^t(\omega_1,\omega_2) \quad \mathbf{C}_2^t(\omega_1,\omega_2) \quad \cdots \quad \mathbf{C}_{L_1}^t(\omega_1,\omega_2)]^t \tag{11}$$

in which

$$\mathbf{A}_i = [a(i,0) \quad a(i,1) \quad a(i,2) \quad \cdots \quad a(i,L_2)]^t, \quad 0 \le i \le L_1 \tag{12}$$

and

$$\mathbf{C}_i(\omega) = [cos(i\omega_1) \quad cos(i\omega_1)cos(\omega_2) \quad cos(i\omega_1)cos(2\omega_2) \quad \cdots \quad cos(i\omega_1)cos(L_2\omega_2)]^t, \quad 0 \le i \le L_1. \tag{13}$$

Subsitituting (9) into (8), we can obtain

$$e = s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} \tag{14}$$

where

$$s = \int\int_R W(\omega_1,\omega_2)D^2(\omega_1,\omega_2)d\omega_1 d\omega_2, \tag{15}$$

$$\mathbf{P} = -2\int\int_R W(\omega_1,\omega_2)D(\omega_1,\omega_2)\mathbf{C}(\omega_1,\omega_2)d\omega_1 d\omega_2 \tag{16}$$

and

$$\mathbf{Q} = \int\int_R W(\omega_1,\omega_2)\mathbf{C}(\omega_1,\omega_2)\mathbf{C}^t(\omega_1,\omega_2)d\omega_1 d\omega_2. \tag{17}$$

The desired coefficient vector $\mathbf{A}$ can be obtained by setting $\frac{\partial e}{\partial \mathbf{A}}$ to zero as

$$\mathbf{A} = -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{P}. \tag{18}$$

Once $\mathbf{A}$ is obtained, the filter coefficients $h(n_1,n_2)$ can be computed by (6).

Exercise 2.2: Use MATLAB software package to design a 15×15 **Type I-1** 2-D circular low-pass filter with $W(\omega_1,\omega_2) = 1$ for $(\omega_1,\omega_2) \in R$ and

$$D(\omega_1,\omega_2) = \begin{cases} 1, & \sqrt{\omega_1^2+\omega_2^2} \le 0.3\pi, \\ 0, & \sqrt{\omega_1^2+\omega_2^2} \ge 0.54\pi. \end{cases} \tag{19}$$

Notice that the elements of $\mathbf{P}$ and $\mathbf{Q}$ can be obtained by numerical methods. Plot the amplitude response.

Table I: $\hat{H}(\omega_1,\omega_2)$ of 2-D sequences with length $N_1 \times N_2$. ($L_i = \frac{N_i-1}{2}$ for odd N_i and $L_i = \frac{N_i}{2}$ for even N_i, $i = 1, 2$. S:symmetry, A:anti-symmetry)

Type and symmetry of impulse response along n_1- and n_2- directions	Sub-type	N_1, N_2	$\hat{H}(\omega_1,\omega_2)$
I	1	N_1:odd, N_2:odd	$\sum_{n_1=0}^{L_1}\sum_{n_2=0}^{L_2} a(n_1,n_2)cos(n_1\omega_1)cos(n_2\omega_2)$
(S,S)	2	N_1:odd, N_2:even	$\sum_{n_1=0}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos(n_1\omega_1)cos((n_2-\frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=0}^{L_2} a(n_1,n_2)cos((n_1-\frac{1}{2})\omega_1)cos(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos((n_1-\frac{1}{2})\omega_1)cos((n_2-\frac{1}{2})\omega_2)$
II	1	N_1:odd, N_2:odd	$\sum_{n_1=0}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos(n_1\omega_1)sin(n_2\omega_2)$
(S,A)	2	N_1:odd, N_2:even	$\sum_{n_1=0}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos(n_1\omega_1)sin((n_2-\frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos((n_1-\frac{1}{2})\omega_1)sin(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)cos((n_1-\frac{1}{2})\omega_1)sin((n_2-\frac{1}{2})\omega_2)$
III	1	N_1:odd, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=0}^{L_2} a(n_1,n_2)sin(n_1\omega_1)cos(n_2\omega_2)$
(A,S)	2	N_1:odd, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin(n_1\omega_1)cos((n_2-\frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=0}^{L_2} a(n_1,n_2)sin((n_1-\frac{1}{2})\omega_1)cos(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin((n_1-\frac{1}{2})\omega_1)cos((n_2-\frac{1}{2})\omega_2)$
IV	1	N_1:odd, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin(n_1\omega_1)sin(n_2\omega_2)$
(A,A)	2	N_1:odd, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin(n_1\omega_1)sin((n_2-\frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin((n_1-\frac{1}{2})\omega_1)sin(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2} a(n_1,n_2)sin((n_1-\frac{1}{2})\omega_1)sin((n_2-\frac{1}{2})\omega_2)$

Table II: Relationship between $a(n_1, n_2)$ in $\hat{H}(\omega_1, \omega_2)$ and $h(n_1, n_2)$ in $H(\omega_1, \omega_2)$.

Type	Relationships between $a(n_1, n_2)$ and $h(n_1, n_2)$
I-1	$a(0,0) = h(L_1, L_2)$ $a(0, n_2) = 2h(L_1, L_2 - n_2) \quad n_2 = 1, \cdots, L_2$ $a(n_1, 0) = 2h(L_1 - n_1, L_2) \quad n_1 = 1, \cdots, L_1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
I-2	$a(0, n_2) = 2h(L_1, L_2 - n_2) \quad n_2 = 1, \cdots, L_2$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
I-3	$a(n_1, 0) = 2h(L_1 - n_1, L_2) \quad n_1 = 1, \cdots, L_1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
I-4	$a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
II-1	$h(n_1, L_2) = 0 \quad n_1 = 0, \cdots, N_1 - 1$ $a(0, n_2) = 2h(L_1, L_2 - n_2) \quad n_2 = 1, \cdots, L_2$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
II-2	$a(0, n_2) = 2h(L_1, L_2 - n_2) \quad n_2 = 1, \cdots, L_2$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
II-3	$h(n_1, L_2) = 0 \quad n_1 = 0, \cdots, N_1 - 1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
II-4	$a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
III-1	$h(L_1, n_2) = 0 \quad n_2 = 0, \cdots, N_2 - 1$ $a(n_1, 0) = 2h(L_1 - n_1, L_2) \quad n_1 = 1, \cdots, L_1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
III-2	$h(L_1, n_2) = 0 \quad n_2 = 0, \cdots, N_2 - 1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
III-3	$a(n_1, 0) = 2h(L_1 - n_1, L_2) \quad n_1 = 1, \cdots, L_1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
III-4	$a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
IV-1	$h(L_1, n_2) = 0 \quad n_2 = 0, \cdots, N_2 - 1$ $h(n_1, L_2) = 0 \quad n_1 = 0, \cdots, N_1 - 1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
IV-2	$h(L_1, n_2) = 0 \quad n_2 = 0, \cdots, N_2 - 1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
IV-3	$h(n_1, L_2) = 0 \quad n_1 = 0, \cdots, N_1 - 1$ $a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$
IV-4	$a(n_1, n_2) = 4h(L_1 - n_1, L_2 - n_2) \quad n_1 = 1, \cdots, L_1 \quad n_2 = 1, \cdots, L_2$

Project 3
Applications of the Lagrange Multiplier Approach to the Design of Special FIR Digital Filters

I. Design of Maximally Flat FIR Digital Filters

From the description in Chapter 1, the amplitude response of a **Case 1** FIR filter can be represented in vector product form as follows:

$$\hat{H}(\omega) = \mathbf{A}^t\mathbf{C}(\omega) = \mathbf{C}^t(\omega)\mathbf{A} \tag{1}$$

where the definitions of $\mathbf{A}$ and $\mathbf{C}(\omega)$ can be found in Chapter 1. After some mathematical manipulations, the objective least squares error function for approximating the desired response $D(\omega)$ by $\hat{H}(\omega)$ can be represented by

$$e = s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} \tag{2}$$

where s, $\mathbf{P}$ and $\mathbf{Q}$ are all defined in Chapter 1, too.

In some applications, certain constraints should be incorporated in the design procedures as well as the desired response $D(\omega)$ is approximated. For example, for designing maximally flat filters, the desired filter passband performance is attained by imposing magnitude or derivtaive constraints F at a discrete set of points $\omega_0,\ \omega_1,\ \cdots,\ \omega_J$ in the passband, i.e.

$$F_{mj}[\hat{H}(\omega_j)] = g_{mj}, \quad 0 \le m \le M \text{ and } 0 \le j \le J \tag{3}$$

where M denotes the order of constraints at a particular frequency ω_j. Eq.(3) can be rewritten as

$$\mathbf{B}^t\mathbf{A} = \mathbf{G} \tag{4}$$

where

$$\begin{aligned}\mathbf{B} = [F_{00}[\mathbf{C}(\omega)] \ \cdots \ F_{M0}[\mathbf{C}(\omega)] \ \ F_{01}[\mathbf{C}(\omega)] \ \cdots \ F_{M1}[\mathbf{C}(\omega)] \ \cdots \\ F_{0J}[\mathbf{C}(\omega)] \ \cdots \ F_{MJ}[\mathbf{C}(\omega)]]\end{aligned} \tag{5}$$

and

$$\mathrm{G} = [g_{00} \quad \cdots \quad g_{M0} \quad g_{01} \quad \cdots \quad g_{M1} \quad \cdots \quad g_{0J} \quad \cdots \quad g_{MJ}]^t. \tag{6}$$

Hence the design of the desired FIR filters with magnitude or derivative constraints can be formulated as a quadratic programming problem:

$$\begin{aligned} &\text{Minimize} \quad e = s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} \\ &\text{subjectto} \quad \mathbf{B}^t\mathbf{A} = \mathrm{G}. \end{aligned} \tag{7}$$

Let the Lagrange multiplier vector be

$$\lambda = [\lambda_{00} \quad \cdots \quad \lambda_{M0} \quad \lambda_{01} \quad \cdots \quad \lambda_{M1} \quad \cdots \quad \lambda_{0J} \quad \cdots \quad \lambda_{MJ}]^t, \tag{8}$$

then the Lagrangian function is

$$\wedge(\mathbf{A}, \lambda) = s + \mathbf{P}^t\mathbf{A} + \mathbf{A}^t\mathbf{Q}\mathbf{A} - \lambda^t(\mathbf{B}^t\mathbf{A} - \mathrm{G}). \tag{9}$$

The necessary and sufficient conditions for optimality are

$$\nabla_{\mathbf{A}} \wedge = 0 \tag{10}$$

and

$$\nabla_{\lambda} \wedge = 0, \tag{11}$$

which lead to

$$\mathbf{P} + 2\mathbf{Q}\mathbf{A} - \mathbf{B}\lambda = 0 \tag{12}$$

and

$$\mathbf{B}^t\mathbf{A} - \mathrm{G} = 0, \tag{13}$$

respectively. Pre-multiplying (12) by $\mathbf{B}^t\mathbf{Q}^{-1}$, and by (13), we can obtain

$$\mathbf{B}^t\mathbf{Q}^{-1}\mathbf{P} + 2\mathrm{G} = \mathbf{B}^t\mathbf{Q}^{-1}\mathbf{B}\lambda, \tag{14}$$

hence

$$\lambda = (\mathbf{B}^t\mathbf{Q}^{-1}\mathbf{B})^{-1}(2\mathrm{G} + \mathbf{B}^t\mathbf{Q}^{-1}\mathbf{P}). \tag{15}$$

Subsitituting (15) into (12), the closed-form solution of $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{B}(\mathbf{B}^t\mathbf{Q}^{-1}\mathbf{B})^{-1}\mathrm{G} + \frac{1}{2}\mathbf{Q}^{-1}[\mathbf{B}(\mathbf{B}^t\mathbf{Q}^{-1}\mathbf{B})^{-1}\mathbf{B}^t\mathbf{Q}^{-1} - \mathbf{I}]\mathbf{P} \tag{16}$$

where $\mathbf{I}$ is an $\frac{N+1}{2} \times \frac{N+1}{2}$ identity matrix.

Exercise 3.1: (a) Design a **Case 1** band-pass filter with length 31,

$$D(\omega) = \begin{cases} 0, & 0 \leq \omega \leq 0.18\pi, \\ 1, & 0.26\pi \leq \omega \leq 0.56\pi, \\ 0, & 0.66\pi \leq \omega \leq \pi, \end{cases} \tag{17}$$

and

$$W(\omega) = \begin{cases} 1, & 0 \leq \omega \leq 0.18\pi, \\ 5, & 0.26\pi \leq \omega \leq 0.56\pi, \\ 1, & 0.66\pi \leq \omega \leq \pi. \end{cases} \tag{18}$$

Show its amplitude response. (b) Imposing third-order derivative constraint over $\omega_0 = 0.41\pi$ for the design in (a). Notice that $\mathbf{B}$ and $\mathbf{G}$ is given by

$$\mathbf{B} = [\mathrm{C}(\omega_0) \quad \frac{d\mathrm{C}(\omega)}{d\omega}|_{\omega=\omega_0} \quad \frac{d^2\mathrm{C}(\omega)}{d\omega^2}|_{\omega=\omega_0} \quad \frac{d^3\mathrm{C}(\omega)}{d\omega^3}|_{\omega=\omega_0}] \tag{19}$$

and

$$\mathrm{G} = [1 \quad 0 \quad 0 \quad 0]^t. \tag{20}$$

Show the amplitude response. (c) Show the passband details of (a) and (b) and compare them.

II. Design of FIR Nyquist Filters

Basically, the design of linear-phase Nyquist filters is similar to that of **Case 1** low-pass filters except the following conditions should be satisfied:

(1) Suppose ω_p and ω_s are passband and stopband edges and M is the intersymbol duration, then

$$\omega_p + \omega_s = \frac{2\pi}{M}. \tag{21}$$

(2) Some time-domain coefficient constraints:

$$h(\frac{N-1}{2}) = \frac{1}{M} \tag{22}$$

and

$$h(n) = 0 \quad \text{for } ((n - \frac{N-1}{2}))_M = 0 \text{ and } n \neq \frac{N-1}{2} \tag{23}$$

where $((x))_M$ denotes modular operation with Mod M.

By the relationships between $a(n)$ and $h(n)$, and by (22) and (23),

$$a(0) = \frac{1}{M} \tag{24}$$

and

$$a(n) = 0 \quad \text{for } ((n))_M = 0 \text{ and } n \neq 0. \tag{25}$$

It is noted that (24) and (25) can be represented in matrix form by

$$\mathbf{B}^t \mathbf{A} = \mathbf{G} \tag{26}$$

where, for example,

$$\mathbf{B}^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{27}$$

and

$$\mathbf{G} = [\frac{1}{2} \ \ 0 \ \ 0 \ \ 0 \ \ \cdots]^t \tag{28}$$

for half-band ($M = 2$) Nyquist filter design. So, the Lagrange multiplier approach can also be applied to design Nyquist filters.

Exercise 3.2: Design a half-band Nyquist filter with $N = 31$, $\omega_p = (\frac{0.9}{4})2\pi$, $\omega_s = (\frac{1.1}{4})2\pi$, $W_p = 1$ and $W_s = 2$. (a) Show the amplitude response in dB scale. (c) Show the impulse response.

Exercise 3.3: Design an FIR Nyquist filter with $M = 4$, $N = 39$, $\omega_p = (\frac{0.85}{8})2\pi$, $\omega_s = (\frac{1.15}{8})2\pi$, $W_p = 1$ and $W_s = 4$. (a) Show the amplitude response in dB scale. (c) Show the impulse response.