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Eigenfilter Design of 1-D and 2-D IIR Digital All-Pass Filters

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***Abstract*—In this paper, the eigenfilter approach is extended to design 1-D and 2-D IIR digital all-pass filters. Instead of the integration of the filter magnitude error, the integration of phase error is minimized such that the design problem can be characterized by an eigenformulation. The stability is obtained by constraining the phase specification.**

I. Introduction

This paper presents a new method for designing 1-D and 2-D IIR digital all-pass filters. They are usually used to equalize the nonlinear phase systems as well as to compensate the phase distortions produced by the communication channel. Moreover, they can introduce desired amounts of phase response into a system.

There are several papers concerning 1-D IIR all-pass filter designs with their respective features [1]–[6]. The most attractive scheme of these is that of a phase approximation algorithm using Chebyshev criterion for polynomial transfer functions proposed by Z. Jing [6]. His algorithm exploits a scheme of iteratively linearizing the nonlinear constraints in a nonlinear programming, and converges theoretically.

In this paper, we develop another efficient scheme for designing 1-D IIR all-pass filters, which can be easily extended to design 2-D IIR all-pass filters. Our method is based on minimizing a quadratic measure of the phase error and the computation of an eigenvector of an appropriate real, symmetric, and positive-definite matrix. Several examples are presented and compared with those of the existing methods, which demonstrate the design capability and the efficiency of this approach.

II. Problem Formulation for 1-D IIR All-Pass Filter Design

The transfer function of a 1-D IIR all-pass filter of order N has the characteristic representation as below

$$H(z) = \frac{\sum_{n=0}^{N} a(N-n)z^{-n}}{\sum_{n=0}^{N} a(n)z^{-n}} = \frac{z^{-N}\sum_{n=0}^{N} a(n)z^{n}}{\sum_{n=0}^{N} a(n)z^{-n}}$$

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$$= \frac{z^{-N}R(z^{-1})}{R(z)} \tag{1}$$

where

$$R(z) = \sum_{n=0}^{N} a(n)z^{-n}. \tag{2}$$

Hence

$$P_H(\omega) = -N\omega - 2P_R(\omega) \tag{3}$$

where $P_H(\omega) = \arg[H(e^{j\omega})]$ and $P_R(\omega) = \arg[R(e^{j\omega})]$, so the phase design of $H(z)$ is equivalent to the phase design of $R(z)$, and we can only concentrate on the design of the denominator phase for designing IIR all-pass filters. Obviously

$$P_R(\omega) = -\frac{N\omega + P_H(\omega)}{2}. \tag{4}$$

For stabilizing IIR digital all-pass filters, we directly apply a necessary and sufficient condition for the group delay $\tau(\omega)$ of $H(z)$ as below [6], [7]

$$\int_0^{\pi} \tau(\omega)\,d\omega = N\pi \tag{5}$$

where

$$\tau(\omega) = -\frac{d}{d\omega}P_H(\omega). \tag{6}$$

Moreover, an all-pass filter is also stable if it has a phase approximating error less than π at $\omega = \pi$ [6].

From (2)

$$\frac{-\sum_{n=0}^{N} a(n)\sin(n\omega)}{\sum_{n=0}^{N} a(n)\cos(n\omega)} = \tan(P_R(\omega)) = \frac{\sin(P_R(\omega))}{\cos(P_R(\omega))} \tag{7}$$

so

$$\sin(P_R(\omega))\sum_{n=0}^{N} a(n)\cos(n\omega) + \cos(P_R(\omega))\sum_{n=0}^{N} a(n)\sin(n\omega) = 0. \tag{8}$$

Let

$$\mathbf{A} = [a(0) \quad a(1) \quad \cdots \quad a(N)]^t \tag{9}$$

and

$$\mathbf{C}(\omega) = [c(0) \quad c(1) \quad \cdots \quad c(N)]^t \tag{10}$$

where

$$c(i) = \sin(P_R(\omega))\cos(i\omega) + \cos(P_R(\omega))\sin(i\omega), \quad i = 0, 1, \cdots, N \tag{11}$$

then (8) can be rewritten as an inner product form as

$$\mathbf{A}^t\mathbf{C}(\omega) = 0. \tag{12}$$

However, it is impossible for finite N to realize $R(z)$ such that the actual phase $P_R(\omega)$ of $R(z)$ is exactly equal to the desired phase $P_{R_d}(\omega)$ of $R(z)$, and the problem we face is to replace $P_R(\omega)$ in (11)

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by $P_{R_d}(\omega)$ and find a set of filter coefficients to minimize $\mathbf{A}^t\hat{\mathbf{C}}(\omega)$ where

$$\hat{\mathbf{C}}(\omega) = [\hat{c}(0) \quad \hat{c}(1) \quad \cdots \quad \hat{c}(N)]^t \tag{13}$$

in which

$$\hat{c}(i) = \sin(P_{R_d}(\omega))\cos(i\omega) + \cos(P_{R_d}(\omega))\sin(i\omega), \quad i = 0, 1, \cdots, N. \tag{14}$$

In this paper, we use the least-squares approach to solve this problem with the phase error function as

$$\begin{aligned} E &= \int_0^\pi W(\omega)[\mathbf{A}^t\hat{\mathbf{C}}(\omega)]^2\,d\omega \\ &= \mathbf{A}^t\left[\int_0^\pi W(\omega)\hat{\mathbf{C}}(\omega)\hat{\mathbf{C}}^t(\omega)\,d\omega\right]\mathbf{A} \\ &= \mathbf{A}^t\mathbf{Q}\mathbf{A} \end{aligned} \tag{15}$$

where $W(\omega)$ is a weighting function and $\mathbf{Q}$ is a real, symmetric, and positive-definite matrix. By Rayleigh's principle [8], the smallest E is just the minimum eigenvalue, and the eigenvector corresponding to this minimum eigenvalue is the desired coefficient vector. Note that the elements of $\mathbf{Q}$ are given by

$$\begin{aligned} q(n,m) = \int_0^\pi & W(\omega)[\sin(P_{R_d}(\omega))\cos(n\omega) \\ &+ \cos(P_{R_d}(\omega))\sin(n\omega)][\sin(P_{R_d}(\omega))\cos(m\omega) \\ &+ \cos(P_{R_d}(\omega))\sin(m\omega)]\,d\omega \qquad 0 \le n, m \le N \end{aligned} \tag{16}$$

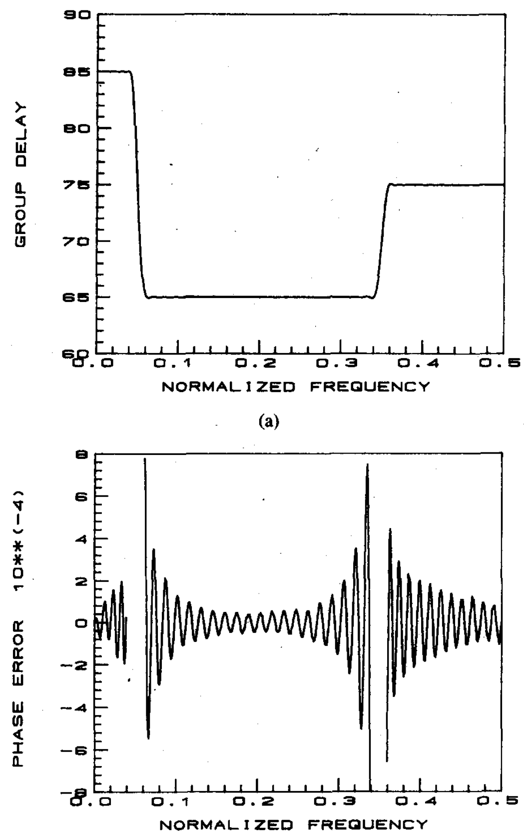
Example 1: In this example, we want to design a seventieth-order piecewise constant delay all-pass filter with the desired phase

$$P_{H_d}(\omega) = \begin{cases} -85\omega, & 0 \le \omega < 0.1\pi \\ -65\omega - 2\pi, & 0.1\pi \le \omega < 0.7\pi \\ -75\omega + 5\pi, & 0.7\pi \le \omega < \pi. \end{cases} \tag{17}$$

For a good phase approximation, the transition bands are needed. The group delay and its phase error are shown in Fig. 1 with two transition bands $(0.08\pi, 0.12\pi), (0.685\pi, 0.715\pi)$ and

$$W(\omega) = \begin{cases} 90, & 0 \le \omega \le 0.08\pi \\ 1, & 0.12\pi \le \omega \le 0.685\pi \\ 30, & 0.715\pi \le \omega \le \pi. \end{cases} \tag{18}$$

From Fig. 1(b), we find that the phase error is smaller than the peak error 2.7×10^{-4} rad in [6] over the piecewise constant delay region except the narrow-band region near the band edges. The stability is guaranteed by the phase specification (17) and the actual phase error in $\omega = \pi$, which is less than π.



(a)

(b)

Fig. 1. Example 1: A seventieth-order piecewise constant delay all-pass filter design: (a) Group delay response; (b) phase error.

III. Design of 2-D IIR All-Pass Digital Filters

For a 2-D IIR all-pass filter, its transfer function can be characterized by

$$\begin{aligned} H(z_1, z_2) &= \frac{\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2} a(N_1 - n_1, N_2 - n_2) z_1^{-n_1} z_2^{-n_2}}{\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2} a(n_1, n_2) z_1^{-n_1} z_2^{-n_2}} \\ &= \frac{z_1^{-N_1} z_2^{-N_2} R(z_1^{-1}, z_2^{-1})}{R(z_1, z_2)} \end{aligned} \tag{19}$$

where

$$R(z_1, z_2) = \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2} a(n_1, n_2) z_1^{-n_1} z_2^{-n_2}. \tag{20}$$

Like the 1-D case

$$\begin{aligned} &\arg[R(e^{j\omega_1}, e^{j\omega_2})] \\ &\quad = \frac{-N_1\omega_1 - N_2\omega_2 - \arg[H(e^{j\omega_1}, e^{j\omega_2})]}{2} \\ &\quad = P_R(\omega_1, \omega_2) \end{aligned} \tag{21}$$

so the phase design of $H(z_1, z_2)$ is equivalent to the phase design of $R(z_1, z_2)$.

For stabilizing a 2-D IIR all-pass filter, there exists a similar necessary and sufficient condition like 1-D case, i.e.

$$\int_0^\pi \tau_1(\omega_1, \omega_2)\,d\omega_1 = N_1\pi, \qquad -\pi \le \omega_2 \le \pi \tag{22}$$

and

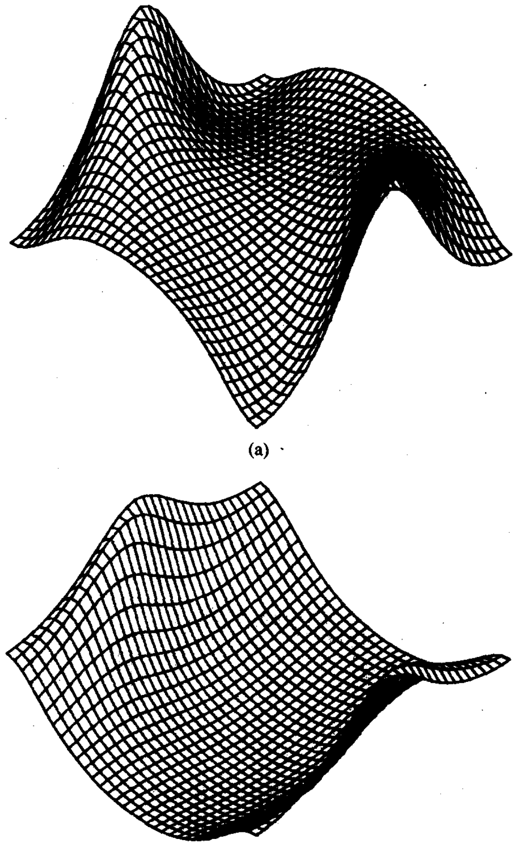
$$\int_0^\pi \tau_2(\omega_1, \omega_2)\,d\omega_2 = N_2\pi, \qquad -\pi \le \omega_1 \le \pi \tag{23}$$

where

$$\tau_1(\omega_1, \omega_2) = -\frac{d}{d\omega_1}\arg[H(e^{j\omega_1}, e^{j\omega_2})] \tag{24}$$

and

$$\tau_2(\omega_1, \omega_2) = -\frac{d}{d\omega_2}\arg[H(e^{j\omega_1}, e^{j\omega_2})]. \tag{25}$$



(a)

(b)

Fig. 2. Example 2: An $N_1 = N_2 = 10$ 2-D all-pass filter design: (a) Group delay corresponding to ω_1-axis; (b) group delay corresponding to ω_2-axis.

The design problem is also to find a set of filter coefficients to minimize

$$E = \mathbf{A}^t \left[\int_0^\pi \int_0^\pi W(\omega_1, \omega_2) \hat{\mathbf{C}}(\omega_1, \omega_2) \hat{\mathbf{C}}^t(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \right] \mathbf{A} = \mathbf{A}^t \mathbf{Q} \mathbf{A} \tag{26}$$

where

$$\mathbf{A} = [a(0,0) \quad a(0,1) \cdots a(0,N_2) \quad a(1,0) \quad a(1,1) \cdots a(1,N_2) \cdots a(N_1,0) \quad a(N_1,1) \cdots a(N_1,N_2)]^t \tag{27}$$

$$\hat{\mathbf{C}}(\omega_1,\omega_2) = [\hat{c}(0,0) \quad \hat{c}(0,1) \cdots \hat{c}(0,N_2) \quad \hat{c}(1,0) \quad \hat{c}(1,1) \cdots \hat{c}(1,N_2) \cdots \hat{c}(N_1,0) \quad \hat{c}(N_1,1) \cdots \hat{c}(N_1,N_2)]^t. \tag{28}$$

$W(\omega_1, \omega_2)$ is a weighting function and $\mathbf{Q}$ is also a real, symmetric, and positive-definite matrix, in which

$$\hat{c}(i,j) = \sin(P_{R_d}(\omega_1,\omega_2)) \cos(i\omega_1 + j\omega_2) + \cos(P_{R_d}(\omega_1,\omega_2)) \sin(i\omega_1 + j\omega_2), \quad 0 \le i \le N_1, 0 \le j \le N_2 \tag{29}$$

and $P_{R_d}(\omega_1, \omega_2)$ is the desired phase of $R(z_1, z_2)$. By the same steps stated in Section II, the desired coefficient vector is easy to get.

Example 2: In this example, an $N_1 = N_2 = 10$ 2-D all-pass filter with the desired phase

$$P_{H_d}(\omega_1,\omega_2) = -10\omega_1 - 10\omega_2 + 20 \tan^{-1} \left[\frac{a \sin(\omega_1) + b \sin(\omega_2) + c \sin(\omega_1 + \omega_2)}{1 + a \cos(\omega_1) + b \cos(\omega_2) + c \cos(\omega_1 + \omega_2)} \right] \tag{30}$$

is designed where $a = -0.048\,544, b = 0.342\,476$, and $c = 0.086\,698$, which are adopted from [11]. When $W(\omega_1, \omega_2) = 1$ is used, Fig. 2(a) shows the group delay corresponding to ω_1-axis, in which the maximum delay error is $0.054\,950\,7$, and the group delay corresponding to ω_2-axis is presented in Fig. 2(b), in which the maximum delay error is $0.111\,169\,8$.

IV. Conclusions

An efficient method for designing 1-D and 2-D IIR digital all-pass filters has been introduced in this paper. The algorithm is based on minimizing a quadratic measure of the phase error using the eigenfilter approach [9], [10]. A few examples are presented and compared with those of the existing methods, which demonstrate the design capability and the efficiency of this approach.

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