THE UNIVERSITY *of* EDINBURGH
School of Physics
and Astronomy

Computational Group Theory

MPhys Project Presentation

Willoughby Seago
Supervised by Anthony Kennedy

The University of Edinburgh

Tensor Symmetry

Classifying Symmetries

Young Projectors

Any Questions?

Tensor Symmetry

Overview

Tensors appear everywhere in physics.

Often they have some symmetry we can exploit to simplify calculations.

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This presentation will give an overview of the theory.

Kronecker Delta

Kronecker delta:

$$\delta^{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

This is totally symmetric:

$$\delta^{ij} = \delta^{ji}.$$

Field Strength Tensor

Combine $\mathbf{E}$ and $\mathbf{B}$ fields into one tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \qquad (c = 1)$$

This is totally antisymmetric:

$$F^{\mu\nu} = -F^{\nu\mu}.$$

Levi–Civita Symbol

Levi–Civita:

$$\varepsilon^{ijk} = \begin{cases} 1 & ijk \in \{123, 231, 312\}, \\ -1 & ijk \in \{132, 213, 321\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is totally antisymmetric:

$$\varepsilon^{ijk} = -\varepsilon^{jik} = -\varepsilon^{ikj} = -\varepsilon^{kji}.$$

Riemann Tensor

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The Riemann tensor has the following symmetries:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\ R_{\mu\nu\rho\sigma} &= \quad R_{\rho\sigma\mu\nu}. \end{aligned}$$

This is clearly quite a bit more complicated than our other examples.

Classifying Symmetries

Exploiting Symmetry

We've seen a variety of symmetries:

- Totally symmetric: δ^{ij}
- Totally antisymmetric: $F^{\mu\nu}$, ε^{ijk}
- A mixture: $R_{\mu\nu\rho\sigma}$

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Exploiting these symmetries efficiently can greatly simplify calculations, e.g.

$$\delta^{ij}\varepsilon^{ijk} = 0$$

$$F^{\mu\nu}F_{\mu\nu} = 2(\partial^{\mu}A^{\nu})F_{\mu\nu}, \qquad \text{(1/2 as many terms)}$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0.$$

To exploit these symmetries we need to classify them. This can be done using the symmetric group.

The Symmetric Group

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Definition

The symmetric group, S_n, is the set of all permutations of n objects. Two permutations are combined by performing them one after another. We think of this as multiplying the two permutations.

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Classify tensor symmetries using Young Tableaux.

Diagrams made from numbered boxes, which tell us about the symmetry of an object.

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Represent the symmetry of δ^{ij} with

$$\boxed{i}\boxed{j}\,.$$

Put antisymmetric pairs of indices in the same column

Represent the antisymmetry of $F^{\mu\nu}$ and ε^{ijk} as

$$\begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \end{array}\,, \qquad \text{and} \qquad \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array}\,.$$

Classifying Symmetries of the Riemann Tensor

The Riemann tensor has more complicated symmetries.

- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$ tells us μ and ν are in the same column.
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- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ tells us that μ and ρ, as well as ν and σ, must be on the same row.

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Hence, the symmetries of $R_{\mu\nu\rho\sigma}$ are represented by

$$\begin{array}{|c|c|}\hline \mu & \rho \\ \hline \nu & \sigma \\ \hline \end{array}\,.$$

Young Projectors

Symmetrisers

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Take linear combinations of permutations.

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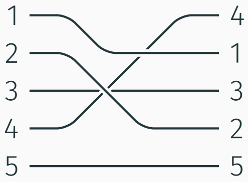
Totally antisymmetric $\to$ antisymmetrise $\to$ alternating sum over permutations:

$$T^{[\mu\nu]} = A_{12} \,.\, T^{\mu\nu} = \frac{1}{2!}(T^{\mu\nu} - T^{\nu\mu}).$$

Diagrammatic Notation

We can represent a permutation as a diagram with wires tracking the elements through the permutation, for example



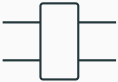
represents the permutation

$$1 \to 2, \quad 2 \to 4, \quad 3 \to 3, \quad 4 \to 1, \quad 5 \to 5$$

or $(1\,2\,4)$ in cycle notation, where each element is sent to the next one in the list, and the list wraps round to the start at the end.

Symmetriser

In this notation symmetrising over indices is written with an empty box:

$S_{12} =$  $= \frac{1}{2}\Big[$  $+$  $\Big] = \frac{1}{2}[() + (1\,2)]$

Symmetriser

In this notation symmetrising over indices is written with an empty box:

$$S_{12} = \frac{1}{2}[() + (1\,2)]$$

Similarly, antisymmetrisation is represented by a filled box:

$$A_{12} = \frac{1}{2}[() - (1\,2)]$$

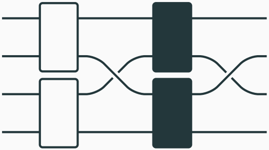
Young Projectors

We can generate tensors with any symmetry using Young projectors, which we get from Young tableaux:

- Symmetrise over rows
- Antisymmetrise over columns

For example:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \propto$$



Expanding the (anti)symmetrisers we have $2^4 = 16$ terms, these get out of hand quickly.

Representation

The permutations in the symmetric group can act on tensors, and this is done through a representation, a collection of matrices

$$\{\rho(\sigma) \mid \sigma \in S_n\}$$

with the same multiplication rules as the symmetric group, so

$$\underbrace{\rho(\sigma)\rho(\tau)}_{\text{product of two matrices, } \rho(\sigma) \text{ and } \rho(\tau)} = \rho(\underbrace{\sigma\tau}_{\text{product of two permutations, } \sigma \text{ and } \tau})$$

These are much easier to work with, but, we have to find them first. This is what most of my code has focused on so far.

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    - $\mathrm{SO}(n) \to$ Rotations $\to$ Classical Mechanics
    - $\mathrm{SO}(1,3) \to$ Lorentz transformations $\to$ Relativity
    - $\mathrm{Sp}(2n) \to$ Transformations preserving Hamilton's equations $\to$ Hamiltonian Dynamics

Any Questions?

What's the Field Strength Tensor?

Suppose we have a scalar potential, φ, and vector potential $\mathbf{A}$.

The electric and magnetic fields are then

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

We can define the four-potential, $A^\mu = (\varphi, \mathbf{A})$ ($c = 1$), then we have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (c = 1)$$

What’s the Riemann Tensor?

One way of defining the Riemann tensor is as follows:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\rho\nu} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\nu}{}_{\nu\rho}$$

where

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\alpha\mu}(\partial_{\nu}g_{\rho\alpha} + \partial_{\rho}g_{\nu\alpha} - \partial_{\alpha}g_{\nu\rho})$$

and $g_{\mu\nu}$ is the metric, with inverse $g^{\mu\nu}$. The metric, and hence $\Gamma^{\mu}{}_{\nu\rho}$ and $R_{\mu\nu\rho\sigma}$, are determined by solving Einstein’s field equations.

Where does $F^{\mu\nu}F_{\mu\nu} = 2(\partial^\mu A^\nu)F_{\mu\nu}$ Come From?

Use

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Expand the first $F^{\mu\nu}$ to get

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)F_{\mu\nu} = (\partial^\mu A^\nu)F_{\mu\nu} - (\partial^\nu A^\mu)F_{\mu\nu}.$$

Rename the indices in the second term, $\mu \leftrightarrow \nu$:

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu)F_{\mu\nu} - (\partial^\mu A^\nu)F_{\nu\mu} = (\partial^\mu A^\nu)(F_{\mu\nu} - F_{\nu\mu}).$$

Use antisymmetry, $F_{\nu\mu} = -F_{\mu\nu}$:

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu)(F^{\mu\nu} - (-F_{\mu\nu})) = 2(\partial^\mu A^\nu)F_{\mu\nu}.$$

What's a Group?

A group is a collection of symmetries, things we can apply to some object without changing it. We can combine two symmetries to get another symmetry, with the rules

- We must be able to do nothing
- We must be able to undo symmetries
- Brackets don't matter (order does)

A set, G, and a binary operation, $G \times G \to G$, such that

- there is an identity element, $1 \in G$, such that $1g = g1 = g$ for all $g \in G$
- there are inverses, $g^{-1} \in G$ for all $g \in G$ such that $gg^{-1} = g^{-1}g = 1$
- the operation is associative, $g(hj) = (gh)j$ for all $g, h, j \in G$

How are you adding permutations?

Given a group, G (such as S_n), we can form the group algebra, $\mathbb{C}[G]$, by defining a vector space with G as a basis, whose vectors are formal sums of group elements, scaled by factors from $\mathbb{C}$. This is, the elements take the form

$$a = a_i g_i \in \mathbb{C}[G] \qquad a_i \in \mathbb{C} \quad \text{and} \quad g_i \in G.$$

Vector addition is defined as one would expect, $a_i g_i + b_i g_i = (a_i + b_i) g_i$, so is scalar multiplication, $z(a_i g_i) = (z a_i) g_i$.

In an *algebra* there is also a notion of multiplying two vectors, in this case it is simply $(a_i g_i)(b_j g_j) = (a_i b_j)(g_i g_j)$, where $a_i b_j$ is complex multiplication and $g_i g_j$ is group multiplication.

What do you mean “acts on”? or What’s a representation?

Given a group, G, a (left) group action on a set S is a function $f_g \colon S \to S$ for ever $g \in G$ such that:

- $f_1 = 1$, the identity element gives the identity function, and
- $f_g(f_h(x)) = f_{gh}(x)$, composition of these functions is group multiplication.

Most groups have an obvious group action, for the symmetric group it is permuting elements of the set.

All groups can act on themselves through group multiplication.

A representation is just a group action on a set which happens to be a vector space.

What is Garnir?

- Standard Young tableaux have their numbers increasing along each row/down each column
- Projectors formed from standard Young tableaux are a basis for Young projectors
- Garnir is a method for decomposing a Young tableaux in terms of standard Young tableaux, before we have to compute the projectors
- It works because there are only so many ways to connect two projectors and not get zero, which happens most of the time since “symmetric $\cdot$ antisymmetric $= 0$”
- Garnir allows us to iteratively make a tableaux more closer to standard, and since tableaux are finite this process terminates and results in standard tableaux

What are $3j$s and $6j$s?

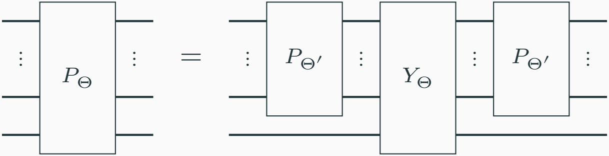
- A tensor can be drawn as a box with legs for each index (arrow direction distinguishes up/down indices)
- Contracting indices $\rightarrow$ joining legs
- Scalar $\rightarrow$ no indices $\rightarrow$ no external legs
- Any scalar can be reduced into a sum of terms formed only from $3j$s and $6j$s, which are the diagrams

- These are just traces of Young projectors

How are Hermitian Young Projectors defined?

- Let Y_Θ be the Young projector corresponding to the n box Young tableau Θ
- Let Θ' be the Young tableau given by removing the nth box from Θ
- Then P_Θ are the Hermitian Young projectors defined recursively by

$$P_\Theta = \begin{cases} Y_\Theta & n \leq 2, \\ (P_{\Theta'} \otimes I) Y_\Theta (P_{\Theta'} \otimes I) & n > 2. \end{cases}$$



What are those other Groups

$\mathrm{SU}(n)$ is the group of unitary transformations of $\mathbb{C}^n$ with determinant one.

$\mathrm{SO}(n)$ is the group of orthogonal transformations of $\mathbb{R}^n$ with determinant one.

$\mathrm{SO}(1,3)$ is the group of proper Lorentz transformations in $1+3$ dimensions.

$\mathrm{Sp}(2n)$ is the group of transformations of $(q_1, \dots, q_n, p_1, \dots, p_n)$ preserving Hamilton's equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Alternatively, transformations of $\mathbb{H}^{2n}$ preserving $\overline{x}_i y_i$ with $x, y \in \mathbb{H}^{2n}$ quaternion "vectors". Since quaternion multiplication is not commutative $\mathbb{H}$ is not a field, but it is a ring, so $\mathbb{H}^{2n}$ is an $\mathbb{H}$-module, not a vector space.