Willoughby Seago

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Computational Group Theory

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Supervised by Tony Kennedy

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Chapters

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Mathematical Preliminaries

Physics is full of tensors, they appear in every field, from the moment of inertia tensor in classical mechanics to observables in quantum mechanics, from the electromagnetic field strength in electrodynamics to the curvature tensor in general relativity. As such being able to quickly and efficiently manipulate and perform computations with tensors is of utmost importance to all physicists. Unfortunately the classic physicist definition of a tensor is the famously unhelpful

> A tensor is something which transforms like a tensor.

Usually this is then followed by a definition of how a tensor transforms in a given setting, say rotations in classical mechanics or Lorentz transformations in relativity. We will instead start with a more general definition of a tensor, but for this we will require some more mathematics first.

1.1 Groups

Groups capture the idea of a symmetry in a precise and mathematical way. Intuitively a symmetry is something we can do to a system which leaves the system unchanged, or invariant, under that symmetry. We can abstract the notion of a symmetry through four requirements. Given some collection of symmetries there must be a way to combine them, do nothing, and undo any of the symmetries, and the final requirement is that the way we use brackets doesn't matter. This leads to the following definition.

Definition 1.1.1 — Group A **group**, $(G, \cdot)$, is a set, G, and a binary operation, $\cdot\colon G \times G \to G$ such that the following axioms are satisfied:

Identity there exists some distinguished element, $1 \in G$, such that for all $x \in G$ we have $1 \cdot x = x \cdot 1 = x$;

Inverse for all $x \in G$ there exists some $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$;

Associativity for all $x, y, z \in G$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We follow the common abuse of terminology and refer to G alone as the group with the operation left implicit. We will also write most group operations as juxtaposition, writing xy for $x \cdot y$.

The prototype for a group is the **symmetric group** on n objects, S_n. This is defined as the set

$$S_n \coloneqq \{\sigma : \{1, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ is a bijection}\} \tag{1.1.2}$$

with function composition as the group operation. We will use cycle notation for elements of S_n, where a cycle sends each element to the next in the list and the last element in the list to the first. For example, $(1\,3\,4)$ sends 1 to 3, 3 to 4, and 4 to 1.

Another important collection of groups are various collections of matrices with matrix multiplication as the group operation. In this case the group identity is the identity matrix of the appropriate dimension, $\mathbb{I}$. Let $\mathrm{Mat}(\mathbb{k}, n)$ denote the set of $n \times n$ matrices with entries in $\mathbb{k}$. The following are all groups under matrix multiplication:

- **general linear group** $\mathrm{GL}(n, \mathbb{k}) \coloneqq \{M \in \mathrm{Mat}(\mathbb{k}, n) \mid M \text{ is invertible}\}$;
- **special linear group** $\mathrm{SL}(n, \mathbb{k}) \coloneqq \{M \in \mathrm{GL}(n, \mathbb{k}) \mid \det M = 1\}$;
- **orthogonal group** $\mathrm{O}(n) \coloneqq \{O \in \mathrm{Mat}(\mathbb{R}, n) \mid O^{\top} O = \mathbb{I}\}$;
- **special orthogonal group** $\mathrm{SO}(n) \coloneqq \{O \in \mathrm{O}(n) \mid \det O = 1\}$;
- **unitary group** $\mathrm{U}(n) \coloneqq \{U \in \mathrm{Mat}(\mathbb{C}, n) \mid U^{\dagger} U = \mathbb{I}\}$;
- **special unitary group** $\mathrm{SU}(n) \coloneqq \{U \in \mathrm{U}(n) \mid \det U = 1\}$.

Most of the time when considering a group we think of the symmetries it represents being applied to some object. This leads to the following definition.

Definition 1.1.3 — Group Action Let G be a group and X a set. A (left) **group action** of G on X is a function $\varphi : G \times X \to X$ such that

Identity for all $x \in X$ we have $\varphi(1, x) = x$,

Compatibility for all $g, h \in G$ and $x \in X$ we have $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ where gh is the product of g and h in G.

We usually write $\varphi(g, x) = g \,.\, x$ or even $\varphi(g, x) = gx$. In this case we have $1 \,.\, x = x$ and $g \,.\, (h \,.\, x) = (gh) \,.\, x$, with gh being the product of g and h computed as elements of G.

The symmetric group, S_n, acts on n-tuples, $\langle a_1, \dots, a_n \rangle$, by permuting the elements. For example, $(1\,3\,4) \in S_5$ acts on $\langle a_1, a_2, a_3, a_4, a_5 \rangle$ by

$$(1\,3\,4) \,.\, \langle a_1, a_2, a_3, a_4, a_5 \rangle = \langle a_4, a_2, a_1, a_3, a_5 \rangle. \tag{1.1.4}$$

Note that this action is done by permuting the symbols a_i, rather than by permuting the values of i, which would instead give $\langle a_3, a_2, a_4, a_1, a_5 \rangle$.

Matrix groups, such as $\mathrm{GL}(n, \mathbb{k})$, have a natural action on the n-dimensional vector space $\mathbb{k}^n$ by interpreting elements of $\mathbb{k}^n$ as column vectors and then acting on them through matrix multiplication. The action of matrices on vector spaces in this form turns out to be a very useful way of thinking about a group, since matrix multiplication is simple and easy to perform on a computer. This insight leads to the idea of a representation, the subject of the next section.

Definition 1.1.5 — Morphisms Let G and H be groups. A **group homomorphism** is a map $\varphi\colon G \to H$ such that $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g, g' \in G$. If φ is invertible then we call it an **isomorphism**. If there is an isomorphism between G and H we say that G and H are **isomorphic**, and denote this $G \cong H$.

Notice that if 1_G and 1_H are the identity elements of G and H respectively then we have $\varphi(1_G) = 1_H$ as well as $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

Definition 1.1.6 — Group Algebra Let G be a group and R a ring. The group ring, $R[G]$, is the set of formal sums

$$\sum_{g\in G} r_g g \tag{1.1.7}$$

where $r_g \in R$ is zero for all but a finite number of elements g. This set of formal sums is a ring with addition defined as

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g := \sum_{g\in G} (r_g + s_g) g \tag{1.1.8}$$

and multiplication defined as

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right) := \sum_{g,h\in G}^{g} r_g s_h g h \tag{1.1.9}$$

where within each the operations occur either within R or within G. If R is a field then $R[G]$ is an associative algebra (a vector space with an associative product).

The most common case of a group algebra is the group algebra $\mathbb{C}[G]$ for some arbitrary group, G.

1.2 Representations

Definition 1.2.1 — Representation Let G be a group. A **group representation**, (ρ, V), is a pair consisting of a vector space, V, called the representation space, and a homomorphism $\rho\colon G \to \mathrm{GL}(V)$. Here $\mathrm{GL}(V) := \{T\colon V \to V \mid T \text{ is linear}\}$ is the group of automorphisms of V with function composition as the group operation. Fixing some basis for V we can identify $\mathrm{GL}(V) \cong \mathrm{GL}(\dim V, \Bbbk)$ if V is a $\Bbbk$-vector space.

Another way of defining a representation is as a group action of G on V given by $g\,.\,v = \rho(g)v$ for $g \in G$ and $v \in V$.

It is common to refer to both ρ, V, alone, as well as the pair (ρ, V) as the representation. The simplest example of a representation is the **trivial representation**, $(\rho_{\text{trivial}}, V)$, which acts trivially on V, that is $\rho_{\text{trivial}}(g) = \mathbb{1}$ for all $g \in G$ and so $g\,.\,v = v$ for all $g \in G$.

Example 1.2.2 — Permutation Representation Consider the symmetric group on three elements, S_3. This has a representation on $\mathbb{R}^3$ given by identifying

$$\rho(1\,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \rho(1\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } \rho(2\,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This then acts by permuting the basis vectors $\boldsymbol{e_1} = (1, 0, 0)^\top$, $\boldsymbol{e_2} = (0, 1, 0)^\top$, and $\boldsymbol{e_3} = (0, 0, 1)^\top$, so we call this the **permutation representation**. The representation of any other group element can be found by writing the element as a product of **transpositions** (two element cycles). For example, $(1\,2\,3) = (1\,2)(2\,3)$ and so

$$\begin{aligned}\rho(1\,2\,3) &= \rho((1\,2)(2\,3)) = \rho(1\,2)\,\rho(2\,3) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{1.2.3}$$

There are an infinite number of representations of any group, given some representation (ρ, V) we can always consider some larger space $W \supset V$ and define $\rho' : G \to \mathrm{GL}(W)$ so that $\rho'(g)$ acts on the subspace V as $\rho(g)$. Clearly this representation doesn't really give us any new information. For this reason we define irreducible representations.

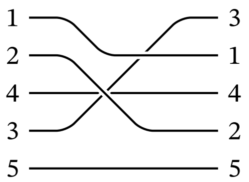
Definition 1.2.4 — Irreducible Representation Let G be a group and (ρ, V) a representation of G. We say that (ρ, V) is an **irreducible representation**, or irrep, if V has no G-invariant subspaces. That is, there is no $W \subset V$ such that $g \,.\, w \in W$ for all $w \in W$.

Definition 1.2.5 — Decomposable Representation Let G be a group and (ρ, V) a representation of G. We say that (ρ, V) is a **decomposable representation** if $\{\rho(g)\}$ can be simultaneously diagonalised. In other words, there exist representations (ρ_i, V_i) such that $\rho = \bigoplus_i \rho_i$ and $V = \bigoplus_i V_i$.

For a finite group, G, and a representation space over $\mathbb{R}$ or $\mathbb{C}$ all indecomposable representations are irreducible. This also the case if G is compact. We will assume that all indecomposable are irreducible and vice versa.

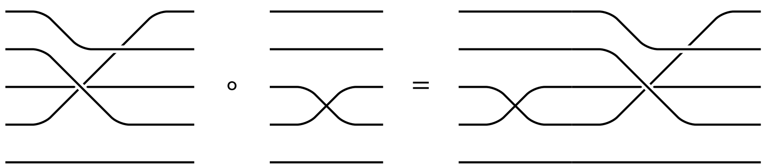
1.3 Birdtracks

A permutation in S_n can pictured with a **braid diagram**, which tracks n elements swapping through wires connecting inputs and outputs. For example, the permutation $(1\,2\,4) \in S_5$ can be drawn, including labels normally left implicit, as



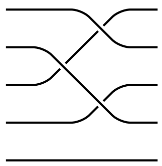
. (1.3.1)

In this notation two permutations can be composed by writing the diagrams *in the opposite order to the product* and joining up the inputs of one to the outputs of the other. For example, the product $(1\,2\,4)(3\,4) = (1\,2\,4\,3)$, viewed in S_5, can be computed by connecting up the relevant diagrams:



. (1.3.2)

We can then simplify the diagram by assuming that the wires can pass through each other and rearrange them until the diagram is more readable. More formally two diagrams represent the same permutation if they are equivalent up to a four-dimensional spatial isotopy, the fourth dimension allowing us to pass the wires around each other, when in three dimensions they would collide. This results in the following diagram:



. (1.3.3)

A common operation on tensors is to symmetrise or antisymmetrise over a certain set of indices. Denoting by $S_{i_1 \dots i_k}$ the symmetriser over the indices $i_1, \dots, i_k$ this symmetriser is given by

$$S_{i_1 \dots i_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \sigma, \tag{1.3.4}$$

where the permutations act on the k indices $i_1, \dots, i_k$. This formal sum of permutations is an element of the group algebra, $\mathbb{C}[S_k]$. Similarly, the antisymmetriser is defined as

$$A_{i_1 \dots i_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma \tag{1.3.5}$$

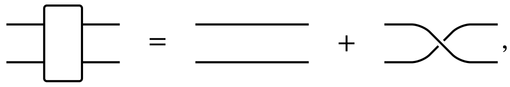
where sgn is the sign function, defined to be 1 if σ can be decomposed as an even number of transpositions, and -1 otherwise.

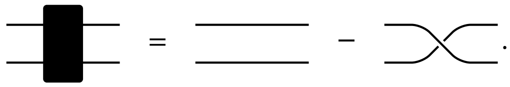
For example, $S_{12} = (() + (1\,2))/2$ and $A_{12} = (() + (1\,2))/2$. Acting on a two index tensor, T^{ij}, with these gives

$$S_{12} \,.\, T^{ij} = T^{(ij)} = \frac{1}{2}(T^{ij} + T^{ji}), \tag{1.3.6}$$

$$A_{12} \,.\, T^{ij} = T^{[ij]} = \frac{1}{2}(T^{ij} - T^{ji}). \tag{1.3.7}$$

In the braid notation we write a symmetriser as an empty box into which the wires being symmetrised are fed, and the antisymmetriser as a filled in box, so

$S_{12} =$  (1.3.8)

$A_{12} =$  (1.3.9)

1.4 Representations of the Symmetric Group

Two

Young Tableau

2.1 What are Young Tableau

Definition 2.1.1 — Partition Let $k \in \mathbb{N}$. A **partition** of k is a tuple, $\langle \lambda_1, \dots, \lambda_n \rangle$ such that

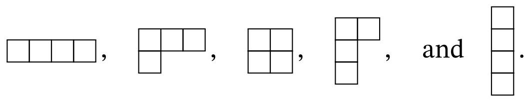
$$k = \sum_{i=1}^{n} \lambda_i. \tag{2.1.2}$$

A partition is **ordered** if $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, \dots, n-1$.

For example, $\langle 1, 5, 3 \rangle$ is a partition of 9, this is not ordered, the equivalent ordered partition is $\langle 5, 3, 1 \rangle$.

Definition 2.1.3 — Young Tableau Given an ordered partition $\langle \lambda_1, \dots, \lambda_n \rangle$ of some $k \in \mathbb{N}$ the **Young diagram** is formed from a row of λ_1 boxes above a row of λ_2 boxes and so on down to a row of λ_n boxes, all aligned to the left.

For example, the ordered partitions of four are $\langle 4 \rangle$, $\langle 3, 1 \rangle$, $\langle 2, 2 \rangle$, $\langle 2, 1, 1 \rangle$, and $\langle 1, 1, 1, 1 \rangle$. The corresponding Young diagrams are



(2.1.4)

2.2 Young Projectors

2.2.1 Garnir Relations

2.2.2 Orthogonal Projectors

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