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MPhys Project Report

Computational Group Theory

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One

Mathematical Preliminaries

Physics is full of tensors, they appear in every field, from the moment of inertia tensor in classical mechanics to observables in quantum mechanics, from the electromagnetic field strength in electrodynamics to the curvature tensor in general relativity. As such being able to quickly and efficiently manipulate and perform computations with tensors is of utmost importance to all physicists. Unfortunately the classic physicist definition of a tensor is the famously unhelpful

A tensor is something which transforms like a tensor.

Usually this is then followed by a definition of how a tensor transforms in a given setting, say rotations in classical mechanics or Lorentz transformations in relativity. We will instead start with a more general definition of a tensor, but for this we will require some more mathematics first.

1.1 Groups

Groups capture the idea of a symmetry in a precise and mathematical way. Intuitively a symmetry is something we can do to a system which leaves the system unchanged, or **invariant**, under that symmetry. We can abstract the notion of a symmetry through four requirements. Given some collection of symmetries there must be a way to combine them, do nothing, and undo any of the symmetries, and the final requirement is that the way we use brackets doesn't matter. This leads to the following definition.

Definition 1.1.1 — Group A **group**, (G, \cdot) , is a set, G , and a binary operation, $\cdot : G \times G \rightarrow G$ such that the following axioms are satisfied [1]:

Identity there exists some distinguished element, $1 \in G$, such that for all $x \in G$ we have $1 \cdot x = x \cdot 1 = x$;

Inverse for all $x \in G$ there exists some $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$;

Associativity for all $x, y, z \in G$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We follow the common abuse of terminology and refer to G alone as the group with the operation left implicit. We will also write most group operations as juxtaposition, writing xy for $x \cdot y$.

The prototype for a group is the **symmetric group** on n objects, S_n [2]. This is defined as the set

$$S_n := \{\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \mid \sigma \text{ is a bijection}\} \quad (1.1.2)$$

with function composition as the group operation. We will use cycle notation for elements of S_n , where a cycle sends each element to the next in the list and the last element in the list to the first. For example, $(1\ 3\ 4)$ sends 1 to 3, 3 to 4, and 4 to 1. In this notation the identity is the empty cycle, $()$.

Another important family of groups are various collections of matrices with matrix multiplication as the group operation. In this case the group identity is the identity matrix of the appropriate dimension, \mathbb{I} . Let $\text{Mat}(\mathbb{k}, n)$ denote the set of $n \times n$ matrices with entries in \mathbb{k} . The following are all groups under matrix multiplication [3]:

- **general linear group** $\text{GL}(n, \mathbb{k}) := \{M \in \text{Mat}(\mathbb{k}, n) \mid M \text{ is invertible}\};$
- **special linear group** $\text{SL}(n, \mathbb{k}) := \{M \in \text{GL}(n, \mathbb{k}) \mid \det M = 1\};$
- **orthogonal group** $\text{O}(n) := \{O \in \text{Mat}(\mathbb{R}, n) \mid O^T O = \mathbb{I}\};$
- **special orthogonal group** $\text{SO}(n) := \{O \in \text{O}(n) \mid \det O = 1\};$
- **unitary group** $\text{U}(n) := \{U \in \text{Mat}(\mathbb{C}, n) \mid U^\dagger U = \mathbb{I}\};$
- **special unitary group** $\text{SU}(n) := \{U \in \text{U}(n) \mid \det U = 1\}.$

These matrix groups are actually examples of a Lie group:

Definition 1.1.3 — Lie Group A **Lie group**, G , is a group which is also a manifold in a compatible way [4]. By this we mean that the product map $\cdot : G \times G \rightarrow G$ is analytic.

Most of the time when considering a group we think of the symmetries it represents being applied to some object. This leads to the following definition.

Definition 1.1.4 — Group Action Let G be a group and X a set. A **(left) group action** of G on X is a function $\varphi : G \times X \rightarrow X$ such that [5, p. 713]

Identity for all $x \in X$ we have $\varphi(1, x) = x$,

Compatibility for all $g, h \in G$ and $x \in X$ we have $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ where gh is the product of g and h in G .

We usually write $\varphi(g, x) = g \cdot x$ or even $\varphi(g, x) = gx$. In this case we have $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$, with gh being the product of g and h computed as elements of G .

The symmetric group, S_n , acts on n -tuples, $\langle a_1, \dots, a_n \rangle$, by permuting the elements. For example, $(1\ 3\ 4) \in S_5$ acts on $\langle a_1, a_2, a_3, a_4, a_5 \rangle$ by

$$(1\ 3\ 4) \cdot \langle a_1, a_2, a_3, a_4, a_5 \rangle = \langle a_4, a_2, a_1, a_3, a_5 \rangle. \quad (1.1.5)$$

Note that this action is done by permuting the symbols a_i , rather than by permuting the values of i , which would instead give $\langle a_3, a_2, a_4, a_1, a_5 \rangle$.

Matrix groups, such as $GL(n, \mathbb{k})$, have a natural action on the n -dimensional vector space \mathbb{k}^n by interpreting elements of \mathbb{k}^n as column vectors and then acting on them through matrix multiplication. The action of matrices on vector spaces in this form turns out to be a very useful way of thinking about a group, since matrix multiplication is simple and easy to perform on a computer. This insight leads to the idea of a representation, the subject of the next section.

Definition 1.1.6 — Morphisms Let G and H be groups. A **group homomorphism** is a map $\varphi : G \rightarrow H$ such that $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g, g' \in G$ [5, p. 705]. If φ is invertible then we call it an **isomorphism**. If there is an isomorphism between G and H we say that G and H are **isomorphic**, and denote this $G \cong H$.

Notice that if 1_G and 1_H are the identity elements of G and H respectively then we have $\varphi(1_G) = 1_H$ as well as $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

Definition 1.1.7 — Group Algebra Let G be a group and R a ring. The group ring, $R[G]$, is the set of formal sums

$$\sum_{g \in G} r_g g \quad (1.1.8)$$

where $r_g \in R$ is zero for all but a finite number of elements g . This set of formal sums is a ring with addition defined as

$$\sum_{g \in G} r_g g + \sum_{g \in G} s_g g := \sum_{g \in G} (r_g + s_g) g \quad (1.1.9)$$

and multiplication defined as

$$\left(\sum_{g \in G} r_g g \right) \left(\sum_{h \in G} s_h h \right) := \sum_{g, h \in G} r_g s_h gh \quad (1.1.10)$$

with the product $r_g s_h$ computed in R and the product gh computed in G . If R is a field then $R[G]$ is an associative algebra (a vector space with an associative product), called the **group algebra** [5, p. 740].

The most common case of a group algebra is the group algebra $\mathbb{C}[G]$ for some arbitrary group, G .

1.2 Representations

Definition 1.2.1 — Representation Let G be a group. A **group representation**, (ρ, V) , is a pair consisting of a vector space, V , called the representation space, and a homomorphism $\rho : G \rightarrow GL(V)$ [5, p. 726]. Here $GL(V) := \{T : V \rightarrow V \mid T \text{ is linear}\}$ is the group of automorphisms of V with function composition as the group operation. Fixing some basis for

V we can identify $\text{GL}(V) \cong \text{GL}(\dim V, \mathbb{k})$ if V is a \mathbb{k} -vector space.

Another way of defining a representation is as a group action of G on V given by $g \cdot v = \rho(g)v$ for $g \in G$ and $v \in V$.

It is common to refer to both ρ , V , alone, as well as the pair (ρ, V) as the representation. The simplest example of a representation is the **trivial representation**, $(\rho_{\text{trivial}}, V)$, which acts trivially on V , that is $\rho_{\text{trivial}}(g) = \mathbb{I}$ for all $g \in G$ and so $g \cdot v = v$ for all $g \in G$.

Example 1.2.2 — Permutation Representation Consider the symmetric group on three elements, S_3 . This has a representation on \mathbb{R}^3 given by identifying

$$\rho(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } \rho(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This then acts by permuting the basis vectors $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$, and $\mathbf{e}_3 = (0, 0, 1)^\top$, so we call this the **permutation representation**. The representation of any other group element can be found by writing the element as a product of **transpositions** (two element cycles). For example, $(123) = (12)(23)$ and so

$$\begin{aligned} \rho(123) &= \rho((12)(23)) = \rho(12)\rho(23) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (1.2.3)$$

There are an infinite number of representations of any group, given some representation (ρ, V) we can always consider some larger space $W \supset V$ and define $\rho' : G \rightarrow \text{GL}(W)$ so that $\rho'(g)$ acts on the subspace V as $\rho(g)$. Clearly this representation doesn't really give us any new information. For this reason we define irreducible representations.

Definition 1.2.4 — Irreducible Representation Let G be a group and (ρ, V) a representation of G . We say that (ρ, V) is an **irreducible representation**, or **irrep**, if V has no G -invariant subspaces [6]. That is, there is no $W \subset V$ such that $g \cdot w \in W$ for all $w \in W$.

Definition 1.2.5 — Decomposable Representation Let G be a group and (ρ, V) a representation of G . We say that (ρ, V) is a **decomposable representation** if $\{\rho(g)\}$ can be simultaneously diagonalised [6]. In other words, there exist representations (ρ_i, V_i) such that $\rho = \bigoplus_i \rho_i$ and $V = \bigoplus_i V_i$.

For a finite group, G , and a representation space over \mathbb{R} or \mathbb{C} all indecomposable representations are irreducible [6]. This also the case if G is compact. We will assume that all indecomposable are irreducible and vice versa.

We are now in a position to make the physicist definition of a tensor more precise.

Definition 1.2.6 — Tensor Let G be a group and $\rho : G \rightarrow GL(V)$, and denote by V^* the dual space to V . A **tensor**, $T \in V^p \otimes V^{*q}$, is an object with components $T^{a_1 \dots a_q}_{b_1 \dots b_p}$, which transform under the action of $g \in G$ according to

$$T'^{a_1 \dots a_q}_{b_1 \dots b_p} = \rho(g)^{a_1}_{c_1} \dots \rho(g)^{a_q}_{c_q} \rho(g)^{d_1}_{b_1} \dots \rho(g)^{d_p}_{b_p} T^{c_1 \dots c_q}_{d_1 \dots d_p}$$

where we employ the Einstein summation convention and sum over repeated indices [7, p. 18].

Example 1.2.7 In classical mechanics we consider Cartesian tensors, which transform under the rotation group $SO(3)$. For example, if $R \in SO(3)$ is a rotation matrix then $\mathbf{x} \in \mathbb{R}^3$ is a vector if it has components transforming according to

$$x_i = R_{ij} x_j. \quad (1.2.8)$$

The moment of inertia tensor, I , has components I_{ij} and transforms as

$$I'_{ij} = R_{ik} R_{jl} I_{kl}. \quad (1.2.9)$$

In relativity we consider Lorentz tensors, which transform under the Lorentz group $SO^+(1, 3)$. For example, if $\Lambda \in SO^+(1, 3)$ is a Lorentz transformation then $x \in \mathbb{R}^{1,3}$ is a vector if it has components transforming according to

$$x^\mu = \Lambda^\mu_\nu x^\nu. \quad (1.2.10)$$

The energy-momentum tensor, T , has components $T^{\mu\nu}$ which transform as

$$T'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho\sigma}. \quad (1.2.11)$$

1.3 Birdtracks

A permutation in S_n can be pictured with a **braid diagram**, which tracks n elements swapping through wires connecting inputs and outputs [7, p. 49]. For example, the permutation $(1\ 2\ 4) \in S_5$ can be drawn, including labels normally left implicit,

as

$$(1.3.1)$$

In this notation two permutations can be composed by writing the diagrams *in the opposite order to the product* and joining up the inputs of one to the outputs of the other. For example, the product $(1\ 2\ 4)(3\ 4) = (1\ 2\ 4\ 3)$, viewed in S_5 , can be computed by connecting up the relevant diagrams:

$$(1.3.2)$$

We can then simplify the diagram by assuming that the wires can pass through each other and rearrange them until the diagram is more readable. More formally two diagrams represent the same permutation if they are equivalent up to a four-dimensional spatial isotopy, the fourth dimension allowing us to pass the wires around each other, when in three dimensions they would collide. This results in the following diagram:

$$(1.3.3)$$

A common operation on tensors is to symmetrise or antisymmetrise over a certain set of indices [7, pp. 50–51]. Denoting by $S_{i_1 \dots i_k}$ the symmetriser over the indices i_1, \dots, i_k this symmetriser is given by

$$S_{i_1 \dots i_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \sigma, \quad (1.3.4)$$

where the permutations act on the k indices i_1, \dots, i_k . This formal sum of permutations is an element of the group algebra, $\mathbb{C}[S_k]$. Similarly, the antisymmetriser is defined as

$$A_{i_1 \dots i_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \quad (1.3.5)$$

where sgn is the sign function, defined to be 1 if σ can be decomposed as an even number of transpositions, and -1 otherwise.

For example, $S_{12} = (() + (1\ 2))/2$ and $A_{12} = (() - (1\ 2))/2$. Acting on a two index tensor, T^{ij} , with these gives

$$S_{12} \cdot T^{ij} = T^{(ij)} = \frac{1}{2}(T^{ij} + T^{ji}), \quad (1.3.6)$$

$$A_{12} \cdot T^{ij} = T^{[ij]} = \frac{1}{2}(T^{ij} - T^{ji}). \quad (1.3.7)$$

In the braid notation we write a symmetriser as an empty box into which the wires being symmetrised are fed, and the antisymmetriser as a filled in box, so

$$S_{12} = \text{[Diagram: Two vertical lines connected by a rectangle]} = \text{[Diagram: Two parallel horizontal lines]} + \text{[Diagram: Two crossing horizontal lines]}, \quad (1.3.8)$$

$$A_{12} = \text{[Diagram: Two horizontal lines entering a black square from the left and exiting to the right]} = \text{[Diagram: Two parallel horizontal lines]} - \text{[Diagram: Two horizontal lines crossing each other]} . \quad (1.3.9)$$

Two

Young Tableau

2.1 What are Young Tableau

Definition 2.1.1 — Partition Let $k \in \mathbb{N}$. A **partition** of k is a tuple, $\langle \lambda_1, \dots, \lambda_n \rangle$ such that [5, p. 720]

$$k = \sum_{i=1}^n \lambda_i. \quad (2.1.2)$$

A partition is **ordered** if $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, \dots, n - 1$.

For example, $\langle 1, 5, 3 \rangle$ is a partition of 9, this is not ordered, the equivalent ordered partition is $\langle 5, 3, 1 \rangle$.

Definition 2.1.3 — Young Diagram Given an ordered partition $\langle \lambda_1, \dots, \lambda_n \rangle$ of some $k \in \mathbb{N}$ the **Young diagram** is formed from a row of λ_1 boxes above a row of λ_2 boxes and so on down to a row of λ_n boxes, all aligned to the left [7, p. 87].

For example, the ordered partitions of four are $\langle 4 \rangle$, $\langle 3, 1 \rangle$, $\langle 2, 2 \rangle$, $\langle 2, 1, 1 \rangle$, and $\langle 1, 1, 1, 1 \rangle$. The corresponding Young diagrams are

$$\begin{array}{|c|c|c|c|}, & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, & \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, & \text{and} & \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}. \end{array} \quad (2.1.4)$$

Definition 2.1.5 — Young Tableau Given a Young diagram we form a **Young tableau** by writing numbers in the boxes in such a way that the numbers are increasing from left to right along a row and strictly increasing from top to bottom down a column [7]. A k box **standard Young tableau** is a Young tableau using the numbers 1 through k exactly once.

For example, given the Young diagram associated with the partition $\langle 2, 1, 1 \rangle$ there are three distinct standard tableaux:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}. \quad (2.1.6)$$

Definition 2.1.7 — Hook Number Given a Young diagram, Y , the **hook length** of a box is the number of boxes to the right of that box, plus the number of boxes below that box, plus 1 for the box itself. The **hook number**, $|Y|$, is the product of the hook lengths [7, p. 88].

Example 2.1.8 Consider the Young diagram given by the partition $\langle 3, 2 \rangle$:

$$Y = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}. \quad (2.1.9)$$

The hooks of this diagram are

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \text{ and } \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array}. \quad (2.1.10)$$

Labelling each box with the associated hook length gives

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \hline \end{array} \quad (2.1.11)$$

The hook number is

$$|Y| = 4 \cdot 3 \cdot 2 = 24. \quad (2.1.12)$$

2.2 Young Projectors

Definition 2.2.1 — Young Projector Given a k box Young tableau, Y , with r rows and c columns the **Young projector**, P_Y associated with Y is given by the following procedure [7, p. 91]:

1. Symmetrise over the numbers in each row.
2. Antisymmetrise over the numbers in each column.
3. Normalise by a factor of

$$\alpha_Y := \frac{1}{|Y|} \left(\prod_{i=1}^r |S_i|! \right) \left(\prod_{j=1}^c |A_j|! \right). \quad (2.2.2)$$

Here $|Y|$ is the hook number of the diagram, $|S_i|$ is the length of the i th row and $|A_j|$ is the length of the j th column.

Example 2.2.3 Consider the 5 box Young tableau

$$Y = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}. \quad (2.2.4)$$

Numbering the boxes we get

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}. \quad (2.2.5)$$

We need to symmetrise over 1, 2 and 3, and 4 and 5, using S_{123} and S_{45} , and antisymmetrise over 1 and 4, 2 and 5, and 3 on its own, using A_{14} , A_{25} , and $A_3 = () = 1$.

The normalisation factor is

$$\alpha_Y = \frac{1}{24}(3!2!)(2!2!1!) = 6. \quad (2.2.6)$$

The Young projector is then

$$P_Y = 6 \cdot \begin{array}{c} \text{Diagram showing 6 boxes (3 white, 3 black) connected by lines, representing the Young projector.} \end{array} \quad (2.2.7)$$

2.3 Garnir Relations

The standard Young tableaux of a given shape, or rather the associated Young projectors, form a basis for the Young projectors formed from Young tableaux of that shape. The **Garnir relations** give us a way of taking a projector corresponding to a nonstandard Young tableau and writing it as a sum of projectors corresponding to standard Young tableaux of the same shape. While the relations are on the level of the Young projectors, which are elements of the group algebra, $\mathbb{C}[S_n]$, they are applied on the level of the Young tableaux. So just keep in mind that whenever we write a sum of two Young tableau the sum is actually happening one layer down in the group algebra.

The Garnir relations work by taking a symmetriser, a strip of boxes in a Young tableau. We then apply this to the Young tableau, and there are only so many ways that this can be done without getting zero. We choose this symmetriser such that it involves symmetrising over indices which the projector antisymmetrises, and so the result is zero overall. This allows us to write a Young tableau as a sum of Young tableau which are closer to being standard Young tableau.

We start with a nonstandard k box Young tableau. The Garnir relations are best expressed through an example. We'll start with the $k = 11$ nonstandard Young tableau

$$Y = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 8 & 10 & \\ \hline 4 & 7 & & \\ \hline 9 & 11 & & \\ \hline \end{array}. \quad (2.3.1)$$

Since a Young projector involves symmetrising over a row the order of the numbers in a row is not important, so we can always sort them in increasing order. The numbers in each row are already sorted in increasing order here, but if they weren't then doing so would be the first step. The first step to applying the Garnir relations

is to identify the first space, from top left going left to right top to bottom, where the tableau fails to be standard. In our example this is the 8 which is above a 7. We then highlight from this point to the point directly below it:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 8 & 10 & \\ \hline 4 & 7 & & \\ \hline 9 & 11 & & \\ \hline \end{array} \quad (2.3.2)$$

This gives us a strip of values, $\langle 8, 10, 4, 7 \rangle$. However, we want to think of this strip not by the numbers in the boxes but the numbers of the boxes when labelled 1 to k :

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & & \\ \hline 10 & 11 & & \\ \hline \end{array} . \quad (2.3.3)$$

So the boxes highlighted above are 6, 7, 8, and 9.

The next step is to perform all permutations of this strip which give distinct Young projectors, which means that swaps within a row are all the same. In this case these swaps are

$$(68), (69), (78), (79), \text{ and } (68)(79). \quad (2.3.4)$$

These permutations act on the Young tableau by permuting the values in the boxes with these numbers. This gives us the tableaux

$$\begin{aligned} (68) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 10 & \\ \hline 8 & 7 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (68) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 10 & \\ \hline 4 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (78) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 8 & 4 & \\ \hline 10 & 7 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, \\ (79) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 8 & 7 & \\ \hline 4 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (68)(79) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \hline 8 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}. \end{aligned} \quad (2.3.5)$$

We can again sort all of the rows, giving

$$\begin{aligned} (68) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 10 & \\ \hline 7 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (68) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 10 & \\ \hline 4 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (78) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 8 & \\ \hline 7 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, \\ (79) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 8 & \\ \hline 4 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}, & (68)(79) \cdot Y &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \hline 8 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}. \end{aligned} \quad (2.3.6)$$

The Garnir relations then tell us that

$$\begin{aligned} 0 &= Y + (68) \cdot Y + (68) \cdot Y + (78) \cdot Y + (79) \cdot Y + (68)(79) \cdot Y \quad (2.3.7) \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 8 & 10 & \\ \hline 4 & 7 & & \\ \hline 9 & 11 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 10 & \\ \hline 7 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 10 & \\ \hline 4 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 8 & \\ \hline 7 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 8 & \\ \hline 4 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \hline 8 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}. \end{aligned}$$

Thus, we can write the initial nonstandard Young tableau as

$$Y = -(68) \cdot Y - (68) \cdot Y - (78) \cdot Y - (79) \cdot Y - (68)(79) \cdot Y \quad (2.3.8)$$

$$= -\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 10 & \\ \hline 7 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 10 & \\ \hline 4 & 8 & & \\ \hline 9 & 11 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 8 & \\ \hline 7 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 8 & \\ \hline 4 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \hline 8 & 10 & & \\ \hline 9 & 11 & & \\ \hline \end{array}. \quad (2.3.9)$$

At this point there are two possibilities, if all of the Young tableaux appearing on the right are standard, as is the case here, then we are finished. If one or more is not standard then the first point at which it fails to be standard will have moved right and/or down, meaning it is, in a sense, closer to being standard. We can then apply the Garnir relations to this tableau and the failure point will once again move right and/or down. Since tableaux are finite eventually this process must terminate and we will be left with only standard tableaux.

To summarise, the process of writing a nonstandard tableau in terms of standard tableaux is as follows:

1. Sort the rows of the tableau into ascending order, if the result is a standard tableau stop here.
2. Identify the first point from the top left going left to right and top to bottom where the tableau fails to be standard.
3. Identify a strip going from this point to the box under this point.
4. Apply all permutations to this strip which result in distinct tableaux.
5. The original tableau is then the negative of the sum of these permuted tableaux.
6. If any of the tableaux in this sum are nonstandard recursively apply the Garnir relations. When no nonstandard tableaux remain stop.

Another example, where the Garnir relations are applied recursively, is in [Chapter A](#)

2.4 Irreducible Representations of The Symmetric Group

Given a Young tableau, for example

$$Y = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}, \quad (2.4.1)$$

we can identify a permutation, σ , which acts on the Young tableau of the same shape but with all numbers in order from left to right top to bottom, which we'll call the **ordered tableau** of this shape, by permuting the boxes based on their numbers to produce Y , that is

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} = \sigma \cdot \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}, \quad (2.4.2)$$

In our example $\sigma = (3\ 4)$. In this way we get a one-to-one correspondence between n box Young tableaux and permutations in S_n .

Representations of S_n can be labelled by ordered partitions of n . This is because for a finite group there is a one-to-one correspondence between irreducible representations and conjugacy classes [8], and the conjugacy classes are grouped by cycle type [9], which is simply the length of the cycles, for example $(1\ 2\ 3)(4\ 5)$ has cycle type $\langle 3, 2 \rangle$ in S_5 , or viewed in S_7 we can write it as $(1\ 2\ 3)(4\ 5)(6)(7)$, in which case it has cycle type $\langle 3, 2, 1, 1 \rangle$. So, we have a one-to-one correspondence between partitions and conjugacy classes, a one-to-one correspondence between conjugacy classes and representations, and a one-to-one correspondence between partitions and Young diagrams. Hence, there is a one-to-one correspondence between n box

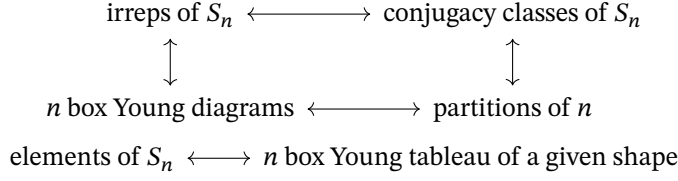


Figure 2.1: One-to-one relations between various concepts relating to Young tableaux and representations of S_n .

Young diagrams and representations of S_n allowing us to label representations of S_n by n box Young diagrams. This is summarised in Figure 2.1

The question now is given some $\sigma \in S_n$ and some n box Young diagram, Y , how do we find the matrix corresponding to σ in the representation labelled by Y ? We'll follow demonstrate the procedure with the example $\sigma = (1\ 3)$ and the representation labelled by the Young diagram

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}. \quad (2.4.3)$$

Start by identifying this Young diagram with the ordered Young tableau of the same shape:

$$Y = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}. \quad (2.4.4)$$

The dimension of the representation is given by the number of standard tableaux of the same shape as Y , in this example there are two:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}. \quad (2.4.5)$$

From each of these we can identify a standard permutation, the permutation giving the standard tableau from the ordered tableau, in this case the standard permutations are $()$ and $(2\ 3)$. We need the Young projector associated with Y :

$$P_Y = \frac{4}{3} \begin{array}{c} \text{diagram with two boxes and two crossings} \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}. \quad (2.4.6)$$

If we act on this with our two standard permutations we get

$$() \cdot P_Y = P_Y = e_1, \quad \text{and} \quad (2\ 3) \cdot P_Y = \begin{array}{c} \text{diagram with two boxes and one crossing} \end{array} = e_2. \quad (2.4.7)$$

We take these two projectors to be the basis vectors in our representation space.

We then act on them with $\sigma = (1\ 3)$, giving

$$\begin{aligned}
 (1\ 3) \cdot \mathbf{e}_1 &= \text{[diagram: crossing of first two wires into a box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} = \text{[diagram: box with Young diagram } \begin{smallmatrix} 3 & 2 \\ 1 \end{smallmatrix} \text{]} = \text{[diagram: sum of two boxes with Young diagrams } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{ and } \begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix} \text{]} \\
 &= -\text{[diagram: box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} - \text{[diagram: crossing of first two wires into a box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} = -\mathbf{e}_1 - \mathbf{e}_2. \quad (2.4.8)
 \end{aligned}$$

Here we act on the inputs to the Young projector, then interpret this as acting on the underlying Young tableau, which we can then decompose into a sum of standard Young tableaux using the Garnir relations, and then we can write each standard tableau as a permutation acting on the Young projector corresponding to the ordered Young tableau. We can do exactly the same with the second basis vector to get

$$(1\ 3) \cdot \mathbf{e}_2 = \text{[diagram: crossing of first two wires into a box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} = \text{[diagram: crossing of first two wires into a box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} \quad (2.4.9)$$

$$= \text{[diagram: crossing of first two wires into a box with Young diagram } \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \text{]} = \mathbf{e}_2. \quad (2.4.10)$$

In the last step we identified that the projector is symmetric in the first two inputs and so we are free to swap the first two input wires, which leaves us with \mathbf{e}_2 . If we hadn't seen that this was possible then we could have applied the permutation to the Young tableau and then decomposed the result with the Garnir relations, and the answer would be the same, it's just slower. We can then use the action on these two basis vectors to give the action on the entire representation space by considering the equation

$$(\mathbf{e}_1 \quad \mathbf{e}_2) \rho(1\ 3) = (-\mathbf{e}_1 - \mathbf{e}_2 \quad \mathbf{e}_2) \implies \rho(1\ 3) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}. \quad (2.4.11)$$

To summarise the process given some $\sigma \in S_n$ and an n box Young diagram labelling a representation we can find the action of σ in this representation as follows:

1. Write down the d standard tableaux of the same shape, these give the standard permutations, τ_i .
2. Act on the ordered Young projector with the standard permutations to get the basis elements, $\mathbf{e}_i = \tau_i \cdot \mathbf{P}_Y$.
3. Act on each basis element with the permutation, $\sigma \cdot \mathbf{e}_i$.
4. Simplify the result using Garnir relations until the result is written in terms of \mathbf{e}_i .

5. Solve the equation

$$(\mathbf{e}_1 \cdots \mathbf{e}_d)\rho(\sigma) = (\sigma \cdot \mathbf{e}_1 \cdots \sigma \cdot \mathbf{e}_d) \quad (2.4.12)$$

for the $d \times d$ representation matrix, $\rho(\sigma)$.

Appendices

A

Garnir Relations Example

A.1 Recursive Application

Consider the Young tableau

$$Y = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & 5 \\ \hline 2 & \\ \hline \end{array}. \quad (\text{A.1.1})$$

The rows are already sorted. The problem strip starts with the 3:

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & 5 \\ \hline 2 & \\ \hline \end{array} \quad (\text{A.1.2})$$

The box numbers are

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}, \quad (\text{A.1.3})$$

so the boxes in the strip are numbers 3, 4, and 5. We need to consider the permutations (3 5) and (4 5), giving

$$(3\ 5) \cdot Y = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}, \quad \text{and} \quad (4\ 5) \cdot Y = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & 2 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.4})$$

Sorting the rows we get the tableaux

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.5})$$

Thus

$$Y = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & 5 \\ \hline 2 & \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.6})$$

The first tableau is standard, but the second isn't, we have a four above a 3. So, we need to recursively apply the Garnir relations.

Starting again with

$$\tilde{Y} = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array} \quad (\text{A.1.7})$$

we identify the problem strip:

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array} \quad (\text{A.1.8})$$

Notice that the problem point has moved right compared to the starting tableau Y . We need to consider the permutations $(2\ 3)$ and $(2\ 4)$. Applying these gives

$$(2\ 3) \cdot \tilde{Y} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & \\ \hline \end{array}, \quad \text{and} \quad (2\ 4) \cdot \tilde{Y} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.9})$$

Sorting the rows gives

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.10})$$

Hence, we have

$$\tilde{Y} = -\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.11})$$

Going back to our original tableau we can express Y as a sum of standard tableaux:

$$Y = -\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array} \quad (\text{A.1.12})$$

$$= -\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} - \left(-\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} \right) \quad (\text{A.1.13})$$

$$= -\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}. \quad (\text{A.1.14})$$

Notice that since the Young projectors, denoted here by the corresponding Young tableaux, are elements of an algebra, $\mathbb{C}[S_5]$, we can do all of the expected algebra with them such as distributing scalars across brackets since $\mathbb{C}[S_5]$ is a vector space.

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