Willoughby Seago

Notes from

Algebraic Geometry

October 6th, 2025

UNIVERSITY OF GLASGOW

Algebraic Geometry

Willoughby Seago

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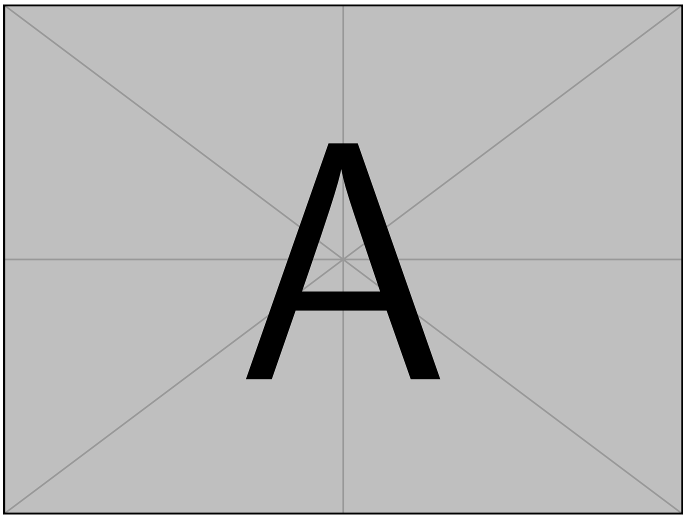
These are my notes from the SMSTC course *Algebraic Geometry* taught by Dr Giulia Gugiatti and Prof Ivan Cheltsov. The lectures, and hence these notes, follow the *Algebraic Geometry* notes of Andreas Gathmann. These notes were last updated at 22:20 on October 19, 2025.



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Introduction

1.1 Conventions and Notation

Throughout the notes the ground field, K, will always be assumed to be *algebraically closed*, up to the point where we introduce schemes. Taking $K = \mathbb{C}$ is usually reasonable.

All rings, R, are assumed to be *commutative* with *unity*. That J is an ideal of R will be denoted $J \trianglelefteq R$. The ideal generated by a subset, $S \subseteq R$, is denoted $\langle S \rangle$.

We write $K[x_1, \dots, x_n]$ for the ring of polynomials with coefficients in K in the variables $x_1, \dots, x_n$. We write $f(a)$ to mean the evaluation of an element of this ring at the point $a = (a_1, \dots, a_n) \in K^n$, and where no confusion may arise we'll usually call this point $x = (x_1, \dots, x_n)$.

1.2 Motivation

This section contains various motivating examples of algebro-geometric thinking, in varying levels of precision. Since the goal is to motivate some precision may be lacking.

1.2.1 Systems of Polynomial Equations

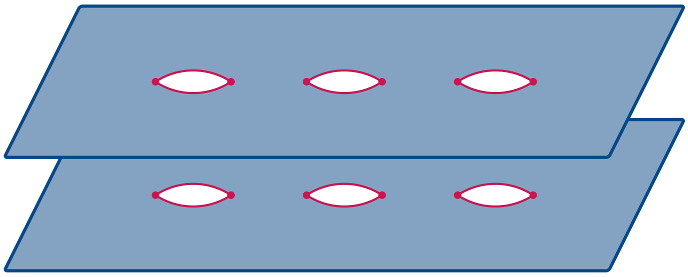
When we first learned algebra in high school it was to study the zeros of polynomials. Later we learned linear algebra, which it can be argued is the study of the zeros of systems of linear equations. Algebraic geometry combines these two fundamental fields into the study of zeros of systems of polynomials.

Given $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ the basic object of study of algebraic geometry is the **affine variety**

$$X = \{x \in K^n \mid f_i(x) = 0 \text{ for } i = 1, \dots, m\}. \tag{1.2.1}$$

What questions can we ask about this set? Just as a single complex polynomial, $f \in \mathbb{C}[x]$, cannot be solved exactly for $\deg f > 4$ we cannot possibly hope to explicitly list the points in X. Instead we reason about the geometric structure of the solutions. We will ask geometric questions about X, which we then aim to answer by an algebraic study of the f_i.

In the following sections we will give several examples of the sorts of geometric objects which can arise. We will focus on the existence of connections to other areas of mathematics.



(a) Branch cuts along alternate intervals.

Figure 1.1: Producing a Riemann surface from a curve

1.2.2 Riemann Surfaces

Fix some positive integer, n. We can define a curve[1]

$$c_n = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)\cdots(x-2n)\} \subseteq \mathbb{C}^2. \tag{1.2.2}$$

[1]Note that this is a "curve" since it's complex dimension is 1 (we'll define dimension of affine varieties later, for now just use your intuition for the dimension of a manifold). Of course, in our pictures this single complex dimension is drawn as two real dimensions.

We can view the defining equation as defining the quantity y. Since we have $y^2 = \dots$ to find the value of y we have to take a square root. What we get depends on the value of x. For most cases, specifically $x \neq 1, 2, \dots, 2n$, we have

$$y = \pm\sqrt{(x-1)(x-2)\cdots(x-2n)}. \tag{1.2.3}$$

For $x = 1, 2, \dots, 2n$ we have

$$y = 0. \tag{1.2.4}$$

Consider what values y can take. For $x \neq 1, \dots, 2n$ we have two copies of $\mathbb{C}$, one for $+\sqrt{(x-1)\cdots(x-2n)}$ and one for $-\sqrt{(x-1)\cdots(x-2n)}$. For $x = 1, \dots, 2n$ we only have one possible value, 0. The picture this suggests is two copies of $\mathbb{C}$ identified at the points $1, \dots, 2n$.

However, this isn't quite right. We know that $z \in \mathbb{C}^\times$ doesn't have a distinguished choice of $\sqrt{z}$. Upon passing once around the origin they are exchanged. For example, if we take the path $x = re^{i\theta}$, with $r \geq 0$ fixed and $\theta \in [0, 2\pi]$ then $\sqrt{x} = \sqrt{r}e^{i\theta/2}$. Then at $\theta = 0$ we get $\sqrt{r}$ and at $\theta = 2\pi$ we get $-\sqrt{r}$. The result is that as we go around the points $x = 1, \dots, 2n$ we move from one copy of $\mathbb{C}$ to the other.

Fortunately, we know how to deal with this, we take branch cuts between zeros. Take both copies of $\mathbb{C}$, and perform branch cuts along the intervals $[1, 2], [3, 4], \dots, [2n-1, 2n]$. For $n = 3$ this produces Figure 1.1a. Now glue these along the cuts, which gives the picture Finally, because it makes things nicer, add two points at infinity, one for each copy of $\mathbb{C}$, compactifying everything to get the picture We see that this leaves us with a Riemann surface of genus $g = n - 1$. This relates algebraic geometry to the theory of Riemann surfaces.

We can change our curve to

$$\{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)^2(x-2)(x-3)\cdots(x-2n)\} \subseteq \mathbb{C}^2. \tag{1.2.5}$$

Then the same analysis can be applied, except that we have a singular point at the repeated root, $x = 1$. This relates algebraic geometry to singularity theory.

1.2.3 Lines on Spaces

Consider the surface

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 + x_1^3 + x_2^3 + x_3^3 - (1 + x_1 + x_2 + x_3)\} \subseteq \mathbb{R}^3. \quad (1.2.6)$$

This is called the **Clebsch surface**. This is a cubic surface because it's defined by a single cubic equation. It's possible to draw straight lines on this surface. One can ask how many such straight lines exist. The answer, surprisingly, is always 27, at least under some mild conditions. Cubic surfaces are actually a weird middle ground, between the infinite families of lines on a quadratic surface, and the general absence of lines on surfaces defined by any higher degree equation.

The question of how many geometric objects of a certain type exist is one of enumerative geometry, which makes heavy use of algebraic geometry.

1.2.4 String Theory

Strings, world sheets, those are surfaces, physicists should care about algebraic geometry.

1.2.5 Curves in Space

Consider the following curve

$$X = \{(x_1, x_2, x_3) = (t^3, t^4, t^5) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^3. \quad (1.2.7)$$

This is a parametric definition of this surface. We can equally define it explicitly as

$$X = \{(x_1, x_2, x_3) \mid x_1^3 = x_2 x_3, x_2^2 = x_1 x_3, x_3^2 = x_1^2 x_2\} \subseteq \mathbb{C}^3. \quad (1.2.8)$$

This is a surface, so it's two (complex) dimensional. However, we need all three of these equations to define it, if we remove any of them we don't get the same surface. This is very different to the world of linear algebra, where we'd have linear defining relations. There any codimension d subspace can be defined by d (linear) equations. Here X is one-dimensional, so it has codimension 2, but we need three equations to define it.

The general problem of taking an affine variety, X, defined as the vanishing set of some polynomials, and determining its dimension is actually very hard. We can use Gröbner bases to do this, but the algebra is pretty unwieldy, and we're forced to use computers to solve it most of the time. A Gröbner basis is a certain generating set of the ideal generated by the polynomials defining the affine variety. Actually, even defining dimension for an arbitrary affine variety is not that straight forward, but for now the intuition from manifolds and vector spaces should be enough.

1.2.6 Different Fields

Over $\mathbb{R}$ or $\mathbb{C}$ we can use real or complex analytic methods to study the zeros of polynomials, and hence affine varieties.

Over $\mathbb{Q}$ or finite fields we can use number theoretic techniques to study the zeros of polynomials, and hence affine varieties.

For example, Fermat's last theorem can be stated as the study of the affine variety

$$X = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 \mid x_1^n + x_2^n = x_3^n\}, \quad (1.2.9)$$

where the question we ask is if this has any non-trivial points.

Two

Affine Varieties

2.1 Affine Varieties

Definition 2.1.1 — Affine Space **Affine $\mathbb{n}$-space** over K is the set

$$\mathbb{A}^n = \mathbb{A}^n_K := \{(a_1, \dots, a_n) \mid a_i \in K \forall i = 1, \dots, n\}. \tag{2.1.2}$$

Note that as sets $\mathbb{A}^n = K^n$. However, we write $\mathbb{A}^n$ when we wish to forget the additional algebraic structure of K^n, specifically the vector space and ring, that is, we want to forget about the ability to scale, add and multiply elements.

For the time being we will take $\mathbb{A}^n$ as our ambient space. Then a polynomial, $f \in K[x_1, \dots, x_n]$, defines a **polynomial function**

$$\mathbb{A}^n \to K \tag{2.1.3}$$
$$a \mapsto f(a). \tag{2.1.4}$$

We'll usually call this function f as well.

Definition 2.1.5 — Affine Variety Let $S \subseteq K[x_1, \dots, x_n]$ be some set of polynomials. The **zero locus** or **vanishing set** of S, denoted $V(S)$, is all points of $\mathbb{A}^n$ on which the polynomial functions defined by polynomials in S vanish. That is,

$$V(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \forall f \in S\} \subseteq \mathbb{A}^n \tag{2.1.6}$$

Any subset of $\mathbb{A}^n$ of this form is called an **affine variety**.

! Note that some authors require that affine varieties have the additional property of being irreducible. These authors would then call all sets like $V(S)$ **affine algebraic sets**.

Notation 2.1.7 If $S = \{f_1, \dots, f_n\}$ is a finite set we write

$$V(S) = V(\{f_1, \dots, f_n\}) = V(f_1, \dots, f_n). \tag{2.1.8}$$

There are some properties we can immediately prove about affine varieties.

Lemma 2.1.9 — Reversal of Inclusion If $S_1 \subseteq S_2 \subseteq K[x_1, \dots, x_n]$ then $V(S_2) \subseteq V(S_1)$.

Proof. Suppose $x \in V(S_2)$. Then $f(x) = 0$ for all $f \in S_2$, and so certainly $f(x) = 0$ for $f \in S_1 \subseteq S_2$, and thus $x \in V(S_1)$. □

Lemma 2.1.10 — Union If $S_1, S_2 \subseteq K[x_1, \dots, x_n]$ then $V(S_1) \cup V(S_2) = V(S_1 S_2)$ where

$$S_1 S_2 = \{fg \mid f \in S_1, g \in S_2\}. \tag{2.1.11}$$

Proof. We start by showing that $V(S_1) \cup V(S_2) \subseteq V(S_1 S_2)$. Suppose that $x \in V(S_1) \cup V(S_2)$. Then $x \in V(S_1)$, so $f(x) = 0$ for all $f \in S_1$, and $x \in V(S_2)$, so $g(x) = 0$ for all $g \in S_2$. Thus, for $f \in S_1$ and $g \in S_2$ we have $(fg)(x) = f(x)g(x) = 0 \cdot 0 = 0$, so $x \in V(S_1 S_2)$.
We now show that $V(S_1 S_2) \subseteq V(S_1) \cup V(S_2)$. We do so by supposing that $x \notin V(S_1) \cup V(S_2)$. Then there exist polynomials, $f \in S_1$ and $g \in S_2$, for which $f(x) \neq 0$ and $g(x) \neq 0$. Thus, $(fg)(x) = f(x)g(x) \neq 0$ (since we work in a field, so have no nonzero zero divisors). Thus, $x \notin V(S_1 S_2)$ since $fg \in S_1 S_2$. By the contrapositive then we have that if $x \in V(S_1 S_2)$ then $x \in V(S_1) \cup V(S_2)$. □

Lemma 2.1.12 — Intersection Let J be an index set, and $\{S_j\}_{j \in J}$ an indexed family of subsets of $K[x_1, \dots, x_n]$. Then

$$\bigcap_{j \in J} V(S_j) = V\left(\bigcup_{j \in J} S_j\right). \tag{2.1.13}$$

Proof. Suppose $x \in \bigcap_{j \in J} V(S_j)$. Then $x \in V(S_j)$ for all $j \in J$. Thus, $f(x) = 0$ for all $f \in S_j$ for all $j \in J$. Thus, $x \in V\left(\bigcup_{j \in J} S_j\right)$.
Conversely, suppose $x \in V\left(\bigcup_{j \in J} S_j\right)$. Then $f(x) = 0$ for all $f \in \bigcup_{j \in J} S_j$, which means $f(x) = 0$ for all $f \in S_j$ for any $j \in J$, and therefore $x \in \bigcap_{j \in J} V(S_j)$. □

We can also give some examples of simple affine varieties.

Example 2.1.14 — Affine Varieties

1. Affine n-space is itself an affine variety. Specifically, $\mathbb{A}^n = V(0)$, since the zero polynomial vanishes.
2. The empty set is an affine variety. Specifically, $\varnothing = V(1)$, since the constant polynomial at 1 vanishes nowhere.

3. Any linear subspace of $K^n = \mathbb{A}^n$ is an affine variety since a linear subspace is defined by the vanishing of linear equations.

4. If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine varieties then $X \times Y$ is too when viewed as a subspace of $\mathbb{A}^{m+n}$. The defining equations of $X \times Y$ are those of X and Y where we view those of X as a function of $x_1, \dots, x_m$ and those of Y as a function of $x_{m+1}, \dots, x_{m+n}$.

Remark 2.1.15 The above results say that $\varnothing$ and $\mathbb{A}^n$ are both affine varieties, and that affine varieties are closed under finite union and arbitrary intersections. This is very close to the definition of a topology on $\mathbb{A}^n$ in terms of open sets, $\varnothing$ and X should be open, and the topology should be closed under finite intersections and arbitrary unions. Notice how unions and intersections exchange roles. Instead what we have is actually the requirements to define a topology on $\mathbb{A}^n$ via the *closed* sets. We'll do exactly this in Chapter 3.

Example 2.1.16 — Affine 1-Space The only affine varieties in $\mathbb{A}^1$ are $\mathbb{A}^1$, $\varnothing$, and all finite sets. Any finite set, $\{\alpha_1, \dots, \alpha_n\}$, is the vanishing set of $(x - \alpha_1) \cdots (x - \alpha_n)$. To show that infinite sets cannot be affine varieties here (other than $\mathbb{A}^1$) suppose $X = V(S)$ is infinite for some $S \subseteq K[x]$. Fix some $f \in S$. Then $\{f\} \subseteq S$, so by Lemma 2.1.9 $V(S) \subseteq V(f)$, and so $x \in V(f)$ for all $x \in X$, which means that $f(x) = 0$ for all $x \in X$, and so f has infinitely many roots, which is not possible for a polynomial.

If $f, g \in K[x_1, \dots, x_n]$ vanish on $X \subseteq \mathbb{A}^n$ then so do $f + g$ and fh for any $h \in K[x_1, \dots, x_n]$. Thus, the set, S, defining an affine variety, $X = V(S)$, is certainly not unique. We can always add $f + g$ and fh. From this we see that $V(S) = V(\langle S \rangle)$ where $\langle S \rangle \trianglelefteq K[x_1, \dots, x_n]$ is the ideal generated by S. This means that any affine variety can be expressed as the vanishing set of some ideal of a polynomial ring.

Hilbert's basis theorem (Theorem A.2.4 and Corollary A.2.5) along with a standard characterisation of noetherian rings (Lemma A.2.3) tells us that all ideals of $K[x_1, \dots, x_n]$ are finitely generated. Given an affine variety, $X = V(S)$, we can then take $X = V(\langle S \rangle)$, and then we can find some finite generating set for this ideal, S'. Then $X = V(S')$. Thus, every affine variety is the zero locus of a finite set of polynomials.

Definition 2.1.17 — Radical Let R be a ring with ideal J. The **radical** of J is

$$\sqrt{J} = \{f \in R \mid f^k \in J \text{ for some } k \in \mathbb{N}\}. \tag{2.1.18}$$

We say J is **radical** if $J = \sqrt{J}$.

Lemma 2.1.19 Let $J \trianglelefteq R$. Then $J \subseteq \sqrt{J}$.

Proof. Suppose that $f \in J$, then $f^1 \in J$, and so $f \in \sqrt{J}$. □

We can now state some results which are the analogues of Lemmas 2.1.9 to 2.1.12 when we work with zero loci of ideals.

Lemma 2.1.20 Let $J \trianglelefteq K[x_1, \dots, x_n]$. Then $V(\sqrt{J}) = V(J)$.

Proof. First, Lemma 2.1.19 gives us $J \subseteq \sqrt{J}$. Thus, by Lemma 2.1.9 we have that $V(\sqrt{J}) \subseteq V(J)$.
Now suppose that $x \in V(J)$ and $f \in \sqrt{J}$. Then $f^k \in J$, so $f^k(x) = 0$, and since we're in a field with no nonzero zero divisors we must have that $f(x) = 0$, and so $x \in V(\sqrt{J})$. □

This result, combined with our earlier analysis, means that every affine variety is the zero locus of a radical ideal.

Lemma 2.1.21 — Union If $J_1, J_2 \trianglelefteq K[x_1, \dots, x_n]$ then $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$.

Proof. That $V(J_1) \cup V(J_2) = V(J_1 J_2)$ is Lemma 2.1.10. It remains to show that $V(J_1 J_2) = V(J_1 \cap J_2)$. Note that it is not generally true that $J_1 J_2 \overset{!}{=} J_1 \cap J_2$. However, it is true that $\sqrt{J_1 J_2} = \sqrt{J_1 \cap J_2}$ (Lemma A.1.4), and the result follows from this. □

Lemma 2.1.22 — Intersection If $J_1, J_2 \trianglelefteq K[x_1, \dots, x_n]$ then $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Proof. From Lemma 2.1.12 we have that $V(J_1) \cap V(J_2) = V(J_1 \cup J_2)$. We also have that $\langle J_1 \cup J_2 \rangle = J_1 + J_2$, so $V(J_1 \cup J_2) = V(\langle J_1 \cup J_2 \rangle) = V(J_1 + J_2)$. □

Remark 2.1.23 With these results we have set up a pairing between geometric objects and algebraic objects. Specifically, we've defined a map

$$V\colon \{\text{algebraic objects}\} \to \{\text{geometric objects}\} \tag{2.1.24}$$
$$\text{ideal} \mapsto \text{affine variety}. \tag{2.1.25}$$

Studying the map going in the opposite direction will be the focus of the next section.

2.2 Ideal of an Affine Variety

Definition 2.2.1 — Ideal Let X be a subset of $\mathbb{A}^n$. The **ideal** of X is

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in X\}. \tag{2.2.2}$$

This is indeed an ideal, if $f, g \in I(X)$ then $f(x) = g(x) = 0$ for all $x \in X$ and $f(x) + g(x) = 0$, so $f + g \in I(X)$, and $-f(x) = 0$ so $-f \in I(X)$, and if $h \in K[x_1, \dots, x_n]$ then $f(x)h(x) = 0h(x) = 0$ so $fh \in I(X)$.

Lemma 2.2.3 — Reversal of Inclusion Suppose $X_1 \subseteq X_2 \subseteq \mathbb{A}^n$. Then $I(X_2) \subseteq I(X_1)$.

Proof. Suppose that $f \in I(X_2)$, that is, $f(x) = 0$ for all $x \in X_2$. Then $f(x) = 0$ for all $x \in X_1 \subseteq X_2$, and so $f \in I(X_1)$. □

Lemma 2.2.4 — Ideal is Radical If $X \subseteq \mathbb{A}^n$ then $I(X)$ is radical.

Proof. Suppose $f \in \sqrt{I(X)}$. Then $f^k \in I(X)$ for some $k \in \mathbb{N}$. Then $f^k(x) = 0$ for all $x \in X$, and since we're in a field $f(x) = 0$ for all $x \in X$, and thus $f \in I(X)$, and hence $\sqrt{I(X)} \subseteq I(X)$. We also have $I(X) \subseteq \sqrt{I(X)}$ by Lemma 2.1.19. Thus, $I(X) = \sqrt{I(X)}$. □

Remark 2.2.5 This gives us the other side of the pairing between algebraic objects and geometric objects:

$$I\colon \{\text{subsets of } \mathbb{A}^n\} \to \{\text{radical ideals of } K[x_1, \dots, x_n]\}. \tag{2.2.6}$$

These aren't quite inverses, since in this direction we only produce radical ideals. However, as we've seen radical ideals are good enough if we're applying V. The following important theorem tells us that these maps, while not quite inverses, are essentially inverses, so long as we're happy to only deal with radical ideals, which we can do by liberally taking radicals.

Theorem 2.2.7 — Hilbert's Nullstellensatz.

1. For any affine variety, $X \subseteq \mathbb{A}^n$, we have $V(II(X)) = X$.
2. For any ideal, $J \trianglelefteq K[x_1, \dots, x_n]$, we have $I(V(J)) = \sqrt{J}$.

Proof. We first prove that $X \subseteq V(I(X))$. If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$, and thus $x \in V(I(X))$.
Next, we prove that $\sqrt{J} \subseteq I(V(J))$. If $f \in \sqrt{J}$ then $f^k \in J$ for some $k \in \mathbb{N}$. Thus, $f^k(x) = 0$ for all $x \in V(J)$, and so $f(x) = 0$ for all $x \in V(J)$, and so

$f \in I(V(J))$.
Third, we prove that $V(I(X)) \subseteq X$. Since X is an affine variety we know that there is some ideal, $J \trianglelefteq K[x_1, \dots, x_n]$, for which $X = V(J)$. Then $\sqrt{J} \subseteq I(V(J))$ by the previous step, and $J \subseteq \sqrt{J}$, so $J \subseteq I(V(J))$. Taking the zero locus, which reverses the inclusion (Lemma 2.1.9), we have $V(I(V(J))) \subseteq V(J)$. Since $X = V(J)$ this is then exactly $V(I(X)) \subseteq X$, and so combined with the first step we have that $V(I(X)) = X$.
The only hard step of the proof is showing that $I(V(J)) \subseteq \sqrt{J}$. This requires some pretty heavy commutative algebra, so we'll skip it. It is this step of the proof which requires that K is algebraically closed. □

Remark 2.2.8 Nullstellensatz means "theorem of the zeroes".

Example 2.2.9 Consider a nonzero ideal, $J \trianglelefteq K[x]$. Since $K[x]$ is a PID we have that $J = \langle f \rangle$ for some $f \in K[x]$. Over an algebraically closed field we can always write f as

$$f(x) = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \tag{2.2.10}$$

for some $a_i \in K$ and $k_i, r \in \mathbb{N}$. Note that $J = \langle f \rangle$ the consists of all polynomials vanishing at a_i with order at least k_i. We therefore have $V(J) = V(f) = \{a_1, \dots, a_n\} \subseteq \mathbb{A}^1$. This affine variety captures the zeros of f, but loses information about their multiplicities.
Hilbert's Nullstellensatz (Theorem 2.2.7) tells us that $I(V(J)) = \sqrt{J}$, and in this case we have

$$\sqrt{J} = \langle (x - a_1) \cdots (x - a_r) \rangle, \tag{2.2.11}$$

consisting of all polynomials vanishing at a_i with *any* order. So, $\sqrt{J}$ too contains the information of the zeros of f while losing the information on their multiplicities. In this way the algebraic object, $\sqrt{J}$, and the geometric object, $V(J)$, contain exactly the same information.

Example 2.2.12 — Not Algebraically Closed Note that the fact K is algebraically closed is essential. In this example we'll consider the field $\mathbb{R}$, which is not algebraically closed. The ideal $\langle x^2 + 1 \rangle \trianglelefteq \mathbb{R}[x]$ is prime, and hence radical (Lemma A.1.5). However, $V(x^2+1) = \varnothing \neq \sqrt{\langle x^2 + 1}$. Thus, Hilbert's Nullstellensatz doesn't hold as $I(V(x^2+1)) = I(\varnothing) = \mathbb{R}[x]$, when the Nullstellensatz would have $I(V(\langle x^2+1 \rangle)) \overset{!}{=} \sqrt{\langle x^2 + 1 \rangle} = \langle x^2+1 \rangle$, which is a proper ideal.

Example 2.2.13 Consider the ideal $J = \langle x - a_1, \dots, x - a_n \rangle \trianglelefteq K[x_1, \dots, x_n]$ for some $a_i \in K$. This is a maximal ideal since $K[x_1, \dots, x_n]/J \cong K$ (setting $x_i = a_i$). Hence, it is also prime, and so radical (Lemma A.1.5). The vanishing set of this ideal is $V(J) = \{a\}$ for $a = (a_1, \dots, a_n) \in \mathbb{A}^n$. Then by Hilbert's Nullstellensatz (Theorem 2.2.7) we have

$$I(\{a\}) = I(V(J)) = \sqrt{J} = J = \langle x_1 - a_1, \dots, x_n - a_n \rangle. \tag{2.2.14}$$

This lets us identify points in $\mathbb{A}^n$ with minimal non-empty affine varieties. By the inclusion-reversing pairings of the Nullstellensatz points in $\mathbb{A}^n$ are in one-to-one correspondence with maximal ideals in $K[x_1, \dots, x_n]$. This gives us another pairing of algebraic and geometric objects,

$$\{\text{maximal ideals of } K[x_1, \dots, x_n]\} \overset{1:1}{\longleftrightarrow} \{\text{points in } \mathbb{A}^n\}. \tag{2.2.15}$$

This also shows that maximal ideals of the form of J above are actually the only maximal ideals of $K[x_1, \dots, x_n]$, a fact which can be proven purely algebraically, but this proof passes through geometry.

We can now prove a couple of results about how I interacts with unions and intersections. These are analogous to the results Lemmas 2.1.10 and 2.1.12 for V.

Lemma 2.2.16 Let X_1 and X_2 be affine varieties in $\mathbb{A}^n$. Then $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

Proof. Suppose $f \in I(X_1 \cup X_2)$. Then f vanishes on any point of X_1 or X_2, and thus $f \in I(X_1)$ and $f \in I(X_2)$, so $f \in I(X_1) \cap I(X_2)$.
Conversely, suppose $f \in I(X_1) \cap I(X_2)$. Then f vanishes on X_1 and X_2, and so it vanishes on $X_1 \cup X_2$, and hence $f \in I(X_1 \cup X_2)$. □

Corollary 2.2.17 The intersection of two radical ideals of $K[x_1, \dots, x_n]$ is again radical.

Proof. If J_1 and J_2 are radical ideals then there exist affine varieties, X_1 and X_2, such that $J_1 = I(X_1)$ and $J_2 = I(X_2)$. Then $J_1 \cap J_2 = I(X_1 \cup X_2)$, which is radical since the ideal of any affine variety is radical. □

Note that it's possible to prove this corollary purely algebraically as well.

Lemma 2.2.18 Let X_1 and X_2 be affine varieties in $\mathbb{A}^n$. Then $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. By Hilbert's Nullstellensatz (Theorem 2.2.7) we have that $X_1 = V(I(X_1))$ and $X_2 = V(I(X_2))$. Thus, we have

$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))). \tag{2.2.19}$$

Then, by Lemma 2.1.12 we have $V(J_1) \cap V(J_2) = V(J_1 + J_2)$, and so

$$I(X_1 \cap X_2) = I(V(I(X_1) + I(X_2))). \tag{2.2.20}$$

Then by the Nullstellensatz again we have $I(V(J)) = \sqrt{J}$, and so

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}. \qquad \square$$

Remark 2.2.21 It is not, in general, true that the sum of two radical ideals is radical. This shouldn't be surprising, the algebraic explanation is that exponentiating a sum doesn't behave particularly simply, we need the binomial theorem. This is why we have to take the radical in the lemma above.

There is also a geometric explanation for this, in addition to the algebraic one. Consider the affine varieties $X_1, X_2 \subseteq \mathbb{A}^2_{\mathbb{C}}$ with $I(X_1) = \langle x_2 - x_1^2 \rangle$ and $I(X_2) = \langle x_2 \rangle$. The real points of these varieties are shown in Figure 2.1. These correspond to $y = x^2$ and $y = 0$, although we're only really able to visualise these for $x, y \in \mathbb{R}$.

The intersection of these two varieties is $X_1 \cap X_2 = \{(0,0)\}$. Thus, $I(X_1 \cap X_2) = I((0,0)) = \langle x_1, x_2 \rangle$. Here we've used the identification of points of $\mathbb{A}^2_{\mathbb{C}}$ with maximal ideals of $\mathbb{C}[x_1, x_2]$ from Example 2.2.13.

We have that

$$I(X_1) + I(X_2) = \langle x_2 - x_1^2 \rangle + \langle x_2 \rangle = \langle x_2 - x_1^2, x_2 \rangle = \langle x_1^2, x_2 \rangle. \tag{2.2.22}$$

This is not a radical ideal, we have

$$\sqrt{\langle x_1^2, x_2 \rangle} = \langle x_1, x_2 \rangle. \tag{2.2.23}$$

Which we expect from Lemma 2.2.18.

The geometric interpretation is then as follows. The varieties X_1 and X_2 are tangent at their intersection point. Thus, in a linear approximation their defining equations, $x_2 = x_1^2$ and $x_2 = 0$, are the same, and both pick out the x_1 axis. This means we can imagine that the intersection, $X_1 \cap X_2$, actually extends a small distance from the origin, an infinitesimal amount in the x_1 direction. But, in this extended region x_1 doesn't vanish, and so it doesn't lie in $I(X_1) + I(X_2)$.

There are various ways to deal with this problem. One is to keep track of the multiplicities of curve intersections. The algebraic-geometry approach is to define schemes. These enlarge our class of geometric objects to include "objects extending by infinitesimally small amounts in some direction". Then the result that we get mirroring that of Hilbert's Nullstellensatz (Theorem 2.2.7) is that affine schemes are in one-to-one correspondence with *arbitrary* ideals of $K[x_1, \dots, x_n]$. Then the intersection of X_1 and X_2 is replaced with the scheme corresponding to the non-radical ideal $\langle x_1, x_2^2 \rangle$.

If $J \trianglelefteq K[x_1, \dots, x_n]$ is proper then J has a zero, that is $V(J)$ is non-empty. Otherwise, we'd have that $\sqrt{J} = I(V(J)) = I(\varnothing) = K[x_1, \dots, x_n]$, which means $1 \in \sqrt{J}$

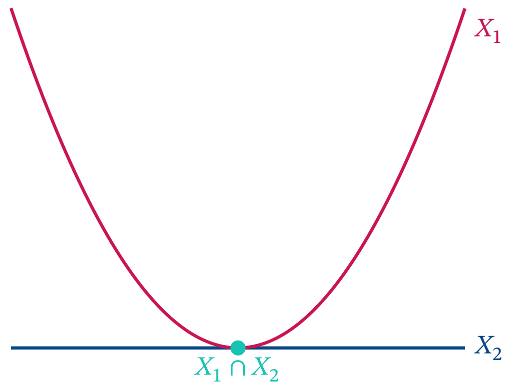


Figure 2.1: The two varieties used to demonstrate why the sum of radical ideals is not necessarily radical.

and so $1 \in J$ meaning $J = K[x_1, \dots, x_n]$, violating the assumption that J is proper.

Proposition 2.2.24 — Weak Nullstellensatz If J is a proper ideal of $K[x_1, \dots, x_n]$ then $V(J)$ is non-empty.

Remark 2.2.25 Historically the weak nullstellensatz was proven first. This result is the reason for the name, "theorem of the zeros". Despite the "weak" in the name of this result the weak Nullstellensatz is actually equivalent to the full Nullstellensatz. There's a trick, known as Rabinowitsch's trick, which allows one to reduce the full Nullstellensatz in n variables to the weak Nullstellensatz in $n + 1$ variables.

2.3 Polynomial Functions

Definition 2.3.1 A **polynomial function** on $\mathbb{A}^n$ is any function $\mathbb{A}^n \to K$ determined by $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$.

Note that such functions form a ring.

An immediate consequence of the Nullstellensatz is that polynomials and polynomial functions on $\mathbb{A}^n$ agree. That is, two polynomials in $K[x_1, \dots, x_n]$ are equal if and only if the polynomial functions they determine on $\mathbb{A}^n$ are equal.

If $f, g \in K[x_1, \dots, x_n]$ determine the same polynomial function on $\mathbb{A}^n$ then $f(x) = g(x)$ for all $x \in \mathbb{A}^n$ by definition of equality of functions. Then $(f - g)(x) = 0$ for all $x \in \mathbb{A}^n$. Then by the Nullstellensatz we have

$$f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle \tag{2.3.2}$$

and thus $f - g = 0$ in $K[x_1, \dots, x_n]$, which means $f = g$ as polynomials.

The trickiest thing here is distinguishing between a polynomial and the polynomial function it determines. The solution to this is to use the work above to mostly ignore the distinction. We identify $K[x_1, \dots, x_n]$ with the ring of polynomial functions on $\mathbb{A}^n$.

We can just as well define polynomial functions on any subset of $\mathbb{A}^n$, and the most useful subsets to define them on are affine varieties. Note that this subsumes the above definition by considering $\mathbb{A}^n$ as an affine variety.

Definition 2.3.3 Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then a **polynomial function** on X is any function $X \to K$ determined by $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$.
The ring of all polynomial functions on X is called the **coordinate ring**, denoted $A(X)$.

Notation 2.3.4 A common alternative notation for the coordinate ring of X is $K[X]$, not to be confused with the polynomial ring in a single variable, $K[x]$, or say the group algebra or K-span of X.

Lemma 2.3.5 Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the coordinate ring is given by

$$A(X) \cong K[x_1, \dots, x_n]/I(X). \tag{2.3.6}$$

Proof. The isomorphism simply identifies the equivalence class of a polynomial, $[f]$, with the corresponding function $x \mapsto f(x)$, which is clearly a ring homomorphism. We need only show that this is independent of choice of representative. To do so suppose that $f, g \in [f]$. That is $f - g \in I(X)$. Then $f(x) - g(x) = 0$ for all $x \in X$, and thus $f(x) = g(x)$, so f and g determine the same polynomial function on X. □

We will identify $A(X)$ and $K[x_1, \dots, x_n]/I(X)$ from now on.

The idea here is that as far as X is concerned two polynomials are the same if they are equal for all $x \in X$. Whether these polynomials differ outside of X is not a question relevant when we're studying X. Thus, the difference of these two polynomials should vanish on X, which is exactly what it means for the difference of these two polynomials to be in $I(X)$.

Note that as well as being a ring $A(X)$ is actually a vector space, and the multiplication of two polynomial functions is K-bilinear. This means $A(X)$ is actually a K-algebra. Despite this, the name coordinate *ring* remains.

Example 2.3.7 Consider the affine variety $X = V(y - x^2) \subseteq \mathbb{A}^2$. Then $A(X) = K[x, y]/I(X)$. We have that $K[x, y]/\langle y - x^2\rangle \cong K[x, x^2] = K[x]$, which is an integral domain. Thus, $\langle y - x^2\rangle$ is prime, and so by Lemma A.1.5 we have $\langle y - x^2\rangle = \sqrt{\langle y - x^2\rangle}$. Thus, $I(X) = \sqrt{\langle y - x^2\rangle} = \langle y - x^2\rangle$ and so $A(X) = K[x, y]/I(X) \cong K[x]$.

Note that we almost always only identify coordinate rings up to isomorphism.

2.4 Affine Subvarieties

We will now repeat much of our previous work to define *relative* versions of many concepts. These replace the ambient space, $\mathbb{A}^n$, with some other affine variety, $Y \subseteq \mathbb{A}^n$, and then make the equivalent definitions for $X \subseteq Y$ given by the vanishing set of some polynomials.

Definition 2.4.1 Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety. For a subset, $S \subseteq A(Y)$, we define it's **relative zero locus** to be

$$V_Y(S) = \{x \in Y \mid f(x) = 0 \forall f \in S\} \subseteq Y. \tag{2.4.2}$$

Subsets of this form are called **affine subvarieties** of Y.

Notation 2.4.3 When no confusion is likely to occur we drop the subscript Y and just write $V(S)$. This is usually fine due to the following point.

Note that affine subvarieties of Y are exactly the affine varieties (subsets of $\mathbb{A}^n$) which are also subsets of Y. In the definition we're just restricting the polynomial functions determined on $\mathbb{A}^n$ to polynomial functions defined on Y before restricting further to X. This doesn't actually change anything[1].

[1]This is an important part of the definition of a sheaf, which we'll see later

Definition 2.4.4 Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety. For a subset, $X \subseteq Y$, we define the **relative ideal** of X in Y to be

$$I_Y(X) = \{f \in A(Y) \mid f(x) = 0 \forall x \in X\} \trianglelefteq A(Y). \tag{2.4.5}$$

Notation 2.4.6 When no confusion is likely to occur we drop the subscript Y and just write $I(X)$.

Lemma 2.4.7 Let $X \subseteq Y \subseteq \mathbb{A}^n$ be affine varieties. Then

$$A(X) \cong A(Y)/I_Y(X). \tag{2.4.8}$$

Proof. The isomorphism identifies an equivalence class, $[f]$, of polynomial functions on Y with the polynomial function on X defined by $x \mapsto f(x)$. This is clearly an isomorphism. It is independent of the choice of representatives because if $f, g \in [f]$ then $f - g \in I_Y(X)$, which means $f(x) - g(x) = 0$ on X, which means $f(x) = g(x)$ for $x \in X$ and therefore f and g both determine the same polynomial function on X. □

There are many relative results we can now state, but won't prove. First, all of the properties of V and Y with respect to inclusions, unions, and intersections still hold for the relative versions. That is, we get analogous relative results for Lemmas 2.1.9 to 2.1.12, 2.2.3, 2.2.16 and 2.2.18.

Theorem 2.4.9 — Relative Nullstellensatz. Let $X \subseteq Y \subseteq \mathbb{A}^n$ be affine varieties. Then we have $V_Y(I_Y(X)) = X$. Let $J \trianglelefteq A(Y)$, then $I_Y(V_Y(J)) = \sqrt{J}$.

This gives us a bijection

$$\{\text{affine subvarieties of } Y\} \xleftrightarrow{1:1} \{\text{radical ideals of } A(Y)\}. \tag{2.4.10}$$

Three

Zariski Topology

In this section we see that there is a natural topology on any affine variety, given by declaring all affine subvarieties to be closed.

3.1 Topological Preliminaries

A topology can be defined by specifying open sets. It is also possible to define a topology by specifying closed sets (complements of open sets). This gives an equivalent definition of a topology, which is what we will work with.

Lemma 3.1.1 Let X be a set. We can declare a **topology** on X by declaring a collection of closed sets so long as

1. the empty set and X are closed;
2. arbitrary intersections of closed sets are closed;
3. finite unions of closed sets are closed.

Notice that the standard definition of a topology has arbitrary unions/finite intersections of open sets. These get swapped because taking complements turns unions into intersections and vice versa by De Morgan's laws.

Lemma 3.1.2 If Y is a topological space and $X \subseteq Y$ is a set then the **subspace topology** on X is given by declaring the closed sets of X to be those sets, $A \subseteq X$, of the form $A = C \cap Y$ for $C \subseteq Y$ closed in the topology of Y.

Lemma 3.1.3 A function, $f : X \to Y$, between topological spaces is **continuous** if the preimage of a closed set is closed.

3.2 Zariski Topology

Definition 3.2.1 — Zariski Topology Let X be an affine variety. The **Zariski topology** on X is given by declaring the closed sets to be the affine subvarieties of X.

That is, the closed subsets are exactly those of the form $V_X(S) = V(S)$ where $S \subseteq A(X)$.

Unless stated otherwise all topological notions for an affine variety will be considered with respect to the Zariski topology. Likewise, any topological notions for a subset of an affine variety will be considered with respect to the subspace topology of the affine variety (which is itself considered with respect to the Zariski topology).

Lemma 3.2.2 The Zariski topology is really a topology.

Proof. Let X be an affine variety. Since $X = V(I(X))$ and $\varnothing = V(1)$ we have that X and $\varnothing$ are closed. A collection of closed subsets is a collection of affine subvarieties. This is closed under arbitrary intersection by the relative version of Lemma 2.1.12, and is closed under finite unions by the relative version of Lemma 2.1.10. □

Notice that if we have affine varieties, $X \subseteq Y$, then there are *a priori* two topologies we could consider on X:

1. The Zariski topology;
2. The subspace topology.

However, these are actually exactly the same. To see this note that the affine subvarieties of X (that is, the closed sets of X in the Zariski topology) are precisely the affine subvarieties of Y which are a subset of X, that is, they're of the form $Z \cap Y$ where $Z \subseteq Y$ is closed, but that's precisely the closed sets of the subspace topology.

To showcase some of the slightly unusual features of the Zariski topology we'll consider $\mathbb{A}^1_{\mathbb{C}}$ and compare things to the standard topology on $\mathbb{C}$.

Example 3.2.3 Consider the unit ball,

$$B = \{x \in \mathbb{A}^1_{\mathbb{C}} \mid |x| \leq 1\}. \tag{3.2.4}$$

Viewing this as a subset of $\mathbb{C}$ in the standard topology it is clearly closed. Viewing it as a subset of $\mathbb{A}^1_{\mathbb{C}}$ in the Zariski topology it is not closed, since it is an infinite set and the only affine varieties of $\mathbb{A}^1$ are $\mathbb{A}^1$ and finite sets (Example 2.1.16).

This example informs our intuition for closed sets in the Zariski topology. Specifically, closed sets are, in a sense, "small". Meaning that open sets are "big". Now, in dimensions greater than 1 we can have infinite closed sets, so we have to be a bit careful about the meaning of "small", but it's a reasonable intuition to have.

Note that any Zariski closed subset of $\mathbb{A}^n_{\mathbb{C}}$ is also closed in the standard topology of $\mathbb{C}^n$. This is because given $X = V(f_1, \dots, f_n)$ a Zariski-closed subset we have that $X = (f_1, \dots, f_n)^{-1}(0)$, where we're considering a function $(f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}$ and $\{0\} \subseteq \mathbb{C}$ is closed in the standard topology and polynomials are clearly continuous (with respect to the standard topology), so X is the preimage of a closed set under a continuous map and so is closed in the standard topology also.

Only very few closed subsets in the standard topology are also closed in the Zariski topology. The Zariski topology is coarser than the standard topology.

Example 3.2.5 Let $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ be any injective map. Then if $X \subseteq \mathbb{A}^1_{\mathbb{C}}$ is finite (i.e., Zariski-closed) then $f^{-1}(X)$ is also finite, and hence Zariski-closed. We also have that $f^{-1}(\varnothing) = \varnothing$ and any injective polynomial from $\mathbb{C} \to \mathbb{C}$ necessarily has domain $f^{-1}(\mathbb{A}^1_{\mathbb{C}}) = \mathbb{A}^1_{\mathbb{C}}$. Thus, the preimage of any Zariski-closed subset is again Zariski-closed, and so f is always continuous.

Example 3.2.6 — Product Topology Given topological spaces, X and Y, their product, $X \times Y$, can be equipped with a topology by declaring open subsets to be those of the form $\bigcup_{i\in I} U_i \times V_i$ where $U_i \subseteq X$ and $V_i \subseteq Y$ are families of open subsets in their respective topologies.
The standard topology on $\mathbb{C}^n$ is precisely the product topology induced by the standard topology on each copy of $\mathbb{C}$. This is not so for the Zariski topology.
Let $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ and $Y \subseteq \mathbb{A}^m_{\mathbb{C}}$ be affine varieties. Then we have seen that $X \times Y \subseteq \mathbb{A}^{n+m}_{\mathbb{C}}$ is an affine variety (Example 2.1.14). However, the Zarisiki topology on $X \times Y$ does not coincide with the product topology on $X \times Y$ induced by the Zariski topology on X and Y.
To see this note that $V(x - y) = \{(a, a) \mid a \in K\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ is closed in the Zariski topology of $\mathbb{A}^2_{\mathbb{C}}$, but it is not closed in the product topology, since the only way to write it as a union of products is

$$\bigcup_{a\in K} \{a\} \times \{a\}, \tag{3.2.7}$$

but $\{a\}$ is not open in the Zariski-topology (its complement is an infinite subset of $\mathbb{A}^1_{\mathbb{C}}$).

Note that the diagonal, $\Delta = \{(a, a) \mid a \in X\}$, is a closed subset of X if and only if X is Hausdorff. This shows that the Zariski topology is not Hausdorff, at least when we're working over an infinite field.

Appendices

A

Commutative Algebra

Here we collect some results from commutative algebra which we'll make use of in the course. This won't be very well organised, and is more for reference than actual reading. The conditions to be included here are pretty much "I had to look it up" or "I had to think about it for more than 10 seconds" while writing these notes, or "I thought it was worth recapping".

A.1 Ideals

Definition A.1.1 — Prime Ideal A proper ideal, $\mathfrak{p} \trianglelefteq R$, is **prime** if whenever $ab \in \mathfrak{p}$ for $a, b \in R$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
Equivalently, $\mathfrak{p}$ is prime if $R/\mathfrak{p}$ is an integral domain.

Definition A.1.2 — Maximal Ideal A proper ideal, $\mathfrak{m} \trianglelefteq R$, is **maximal** if whenever there is another ideal, $I \trianglelefteq R$, with $\mathfrak{m} \subseteq I$ then either $I = \mathfrak{m}$ or $I = R$.
Equivalently, $\mathfrak{m}$ is maximal if $R/\mathfrak{m}$ is a field.

Lemma A.1.3 Let R be a ring with ideals I and J. Then $IJ \subseteq I \cap J$.

Proof. If $a \in I$ and $b \in J$ then $ab \in I$ and $ab \in J$ by definition of an ideal. Then $ab \in I \cap J$. □

Lemma A.1.4 Let R be a ring with ideals I and J. Then $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Proof. We prove a circle of inclusions. We start with $\sqrt{IJ} \subseteq \sqrt{I \cap J}$, which follows from Lemma A.1.3.
If $a \in \sqrt{I \cap J}$ then $a^k \in I \cap J$ for some $k \in \mathbb{N}$. Thus, $a^k \in I$ and $a^k \in J$. Hence, $a \in \sqrt{I} \cap \sqrt{J}$.
If $a \in \sqrt{I} \cap \sqrt{J}$ then $a^k \in I$ and $a^\ell \in J$ for some $k, \ell \in \mathbb{N}$. Then $a^k a^\ell = a^{k+\ell} \in IJ$, and so $a \in \sqrt{IJ}$. □

Lemma A.1.5 Every prime ideal is radical.

Proof. Let $\mathfrak{p}$ be a prime ideal of a ring, R. Consider $\sqrt{\mathfrak{p}}$. If $a \in \sqrt{\mathfrak{p}}$ then there exists some $k \in \mathbb{N}$ such that $a^k \in \mathfrak{p}$. Suppose that k is minimal in making this true. If $k = 1$ then $a \in \mathfrak{p}$. If $k > 1$ then by the definition of a prime ideal have $x \cdot x^{k-1} \in \mathfrak{p}$ implying $x \in \mathfrak{p}$ or $x^{k-1} \in \mathfrak{p}$. However, the later cannot be the case because k was assumed minimal. Therefore, $x \in \mathfrak{p}$, and since $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$ (Lemma 2.1.19) it must be that $\mathfrak{p} = \sqrt{\mathfrak{p}}$. □

A.2 Noetherian Rings

Definition A.2.1 — Noetherian A ring, R, is noetherian if it satisfies the ascending chain condition. That is, if every chain of ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \tag{A.2.2}$$

terminates, so $I_{n+1} = I_n$ for sufficiently large n.

Note that all fields are noetherian, and so is $\mathbb{Z}$.

Lemma A.2.3 Let R be a ring. The following are equivalent:

1. R is a noetherian.
2. Every ideal of R is finitely generated.

A.2.1 Hilbert's Basis Theorem

Theorem A.2.4 — Hilbert's Basis Theorem. If R is a noetherian ring then $R[x]$ is also Noetherian.

Corollary A.2.5 If R is a noetherian ring then $R[x_1, \ldots, x_n]$ is noetherian.