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Notes from

Differential Topology

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SMSTC

Differential Topology

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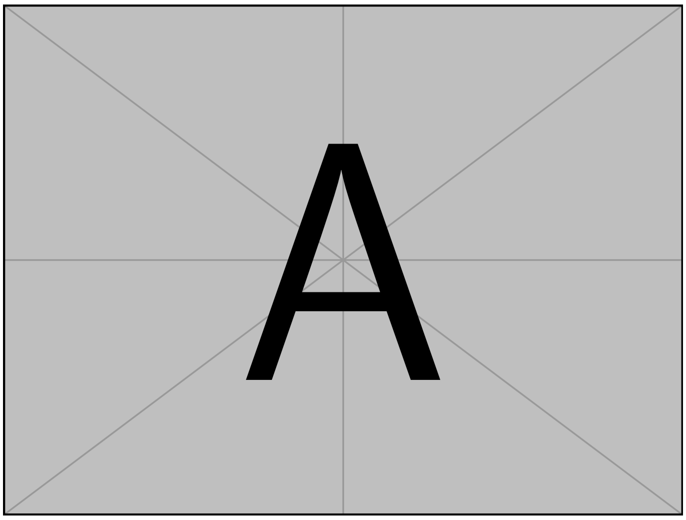
These are my notes from the SMSTC course *Differential Topology* taught by Proff Murad Alim. These notes were last updated at 16:27 on November 30, 2024.



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Smooth Manifolds

1.1 Definitions in $\mathbb{R}^n$

Definition 1.1.1 — Smooth Map Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets[a]. A map $f: U \to V$ is **smooth** is called **smooth** if it is infinitely differentiable. That is, all partial derivatives of all orders[b],

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\cdots+\alpha_k}}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}, \qquad \text{where} \qquad \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k \tag{1.1.2}$$

exist and are continuous.

[a] Here, and throughout, we assume the standard topology on Euclidean space, $\mathbb{R}^m$, so a set, U, is open if and only if every point $x \in U$ there is an open ball centred on x contained entirely within U.

[b] we take $0 \in \mathbb{N}$

Definition 1.1.3 Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets. For a smooth map $f = (f_1, \dots, f_n): U \to V$ and a point $x \in U$ the **derivative**, **differential**, or **pushforward** of f at x is the linear map

$$f_*(x) = \mathrm{d}f_x = \mathrm{d}f(x): \mathbb{R}^m \to \mathbb{R}^n \tag{1.1.4}$$

defined by acting on $\xi \in \mathbb{R}^m$ according to

$$f_*(x)\xi = \mathrm{d}f_x(\xi) = \mathrm{d}(f)(x)\xi = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(x+t\xi) = \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t}. \tag{1.1.5}$$

Fixing coordinates $x = (x^1, \dots, x^m)$ the differential is represented by the Jacobian matrix

$$(\mathrm{d}f(x))_{ij} = \left(\frac{\partial f_i}{\partial x^j}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(x) & \dots & \frac{\partial f_1}{\partial x^m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x^1}(x) & \dots & \frac{\partial f_n}{\partial x^m}(x) \end{pmatrix} \in \mathcal{M}_{n\times m}(\mathbb{R}). \tag{1.1.6}$$

The differential is defined in terms of a derivative of a function $\mathbb{R} \to \mathbb{R}$ (that function is $t \mapsto f(x+t\xi)$), and as such has many of the properties of this derivative, including linearity and the chain rule. That is, if $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$, and $W \subseteq \mathbb{R}^p$ are open and $f: U \to V$ and $g: V \to W$ are smooth then the differential of $g \circ f: U \to W$ at $x \in U$ is the linear map $\mathrm{d}(g \circ f)(x): \mathbb{R}^m \to \mathbb{R}^p$ given by

$$\mathrm{d}(g \circ f)(x) = \mathrm{d}g(f(x)) \circ \mathrm{d}f(x). \tag{1.1.7}$$

The identity map $\mathrm{id}_U: U \to U$, given by $x \mapsto x$, is smooth, its derivatives all vanish. Thus, we have

$$\mathrm{d}(\mathrm{id}_U)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{id}_U(x + t\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x + t\xi) = \xi, \tag{1.1.8}$$

and so $\mathrm{d}(\mathrm{id}_U)(x) = \mathrm{id}_{\mathbb{R}^n}$.

If $f: U \to V$ is a **diffeomorphism** (smooth map with smooth inverse) then for $x \in V$ we can use the above result to get

$$\mathrm{d}(f \circ f^{-1})(x) = \mathrm{d}\mathrm{id}_U(x) = \mathrm{id}_{\mathbb{R}^n}(x) = x, \tag{1.1.9}$$

and we can use the chain rule to get

$$\mathrm{d}(f \circ f^{-1})(x) = \mathrm{d}f(f^{-1}(x)) \circ \mathrm{d}f^{-1}(x). \tag{1.1.10}$$

Thus, we may conclude that

$$\mathrm{d}f(f^{-1}(x)) \circ \mathrm{d}f^{-1}(x) = \mathrm{id}_{\mathbb{R}^n} \tag{1.1.11}$$

which tells us that $\mathrm{d}f(x)$ is invertible, with inverse $\mathrm{d}f(f^{-1}(x))$. This requires that $m = n$.

1.2 Smooth Manifold

A smooth manifold is a space that locally looks like $\mathbb{R}^m$ for some fixed m.

Definition 1.2.1 — Smooth Manifold A **smooth manifold**, $(M, \mathcal{A})$, of dimension $m \in \mathbb{N}$, is a topological space, M, equipped with an open cover, $\{U_\alpha\}_{\alpha \in A}$ for some indexing set A. This open cover must be equipped with a collection of homeomorphisms, $\varphi_\alpha: U_\alpha \to \Omega_\alpha$, where $\Omega_\alpha \subseteq \mathbb{R}^m$ is open. These maps must be such that for any $\alpha, \beta \in A$ the **transition map**

$$\varphi_{\beta\alpha} \coloneqq \varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{1.2.2}$$

is smooth[a]. We call the pair $(U_\alpha, \varphi_\alpha)$ a **coordinate chart**, and the collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called an atlas.

[a] Note that this is a map between open subsets of $\mathbb{R}^m$, so Definition 1.1.1 applies

The requirement that the transition maps are diffeomorphisms is why we call these manifolds *smooth*. If we just require that they by homeomorphisms then we get the notion of a topological manifold, and if we require they're k-times differentiable we get a C^k-manifold. It's also possible to replace $\mathbb{R}$ with $\mathbb{C}$ in all of our previous definitions and consider complex manifolds. This is mostly done when we then require that the transition maps are holomorphic, and we get a holomorphic manifold.

It can be shown that $U \subseteq M$ is open if and only if $\varphi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^m$ for all $\alpha \in A$. Thus, the charts uniquely determine the topology on M and vice versa.

If we have a homeomorphism $\psi : V \to \Omega$ for $V \subseteq M$ and $\Omega \subseteq \mathbb{R}^m$ open then we call (ψ, V) **compatible** with the atlas $\mathcal{A}$ if the transition map $\varphi_\alpha \circ \varphi^{-1} : \psi(V \cap U_\alpha) \to \varphi_\alpha(U \cap U_\alpha)$ is a diffeomorphism for all $\alpha \in A$.

We call the atlas $\mathcal{A}$ **maximal** if it contains every chart that is compatible with all of its members. It can be shown that every atlas is contained in a unique maximal atlas, $\bar{\mathcal{A}}$. Sometimes it's useful to assume that an atlas is maximal, and we can always do this because adding compatible charts to an atlas doesn't change the smooth structure of the manifold. We can extend the notion of compatibility to whole atlases, and we say that two atlases are compatible if their union is an atlas, and in this case both induce the same smooth structure on M, and the maximal atlas containing them is the same.

Example 1.2.3

- $\mathbb{R}^m$ is a manifold of dimension m and it has an atlas given by the single chart $(\mathbb{R}^m, \mathrm{id}_{\mathbb{R}^m})$.
- The sphere, $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$, is a manifold of dimension n, and it is covered by two charts $(S^n \setminus \{(1, 0, \dots, 0)\}, s_+)$ and $(S^n \setminus \{(-1, 0, \dots, 0)\}, s_-)$ where s_+ and s_- are stereographic projection from the points $(\pm 1, 0, \dots, 0)$.
- The m-torus, $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ (with the quotient topology, and $\mathbb{Z}^m$ has the discrete topology, which is also the subset topology as a subset of $\mathbb{R}^m$ with the standard topology) is a manifold. Two vectors $x, y \in \mathbb{R}^m$ project to the same point in $\mathbb{T}^m$ if their difference, $x - y$, is an integer vector (that is, it's in $\mathbb{Z}^m$). Denote the projection by $\pi : \mathbb{R}^m \twoheadrightarrow \mathbb{T}^m$, and this acts on $x \in \mathbb{R}^m$ by $\pi(x) = [x] = x + \mathbb{Z}^m$. A set $U \subseteq \mathbb{T}^m$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^m$ (this is just the definition of the quotient topology). An atlas on $\mathbb{T}^m$ is given by taking the $\mathbb{R}^m$-indexed open cover

$$U_\alpha = \{[x] \mid x \in \mathbb{R}^m \text{ and } |x - \alpha| < 1/2\}, \tag{1.2.4}$$

and the corresponding coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ defined by

$$\varphi_\alpha([x]) = x \tag{1.2.5}$$

for $x \in \mathbb{R}^m$ with $|x - \alpha| < 1/2$. This is well defined since by the construction of the U_α there is only one $x \in \mathbb{R}^m$ satisfying this distance constraint. It can be shown that the transition maps are translation by an integer vector. One way to think of the quotient torus $\mathbb{R}^m/\mathbb{Z}^m$ is as a fixed $1 \times 1 \times \dots \times 1$ cube in $\mathbb{R}^m$ with periodic boundary conditions, which we can view as the cube repeating in all directions forever. Then the transition maps just move us from one cube to the same point in some other cube. Finally, note that the torus may equivalently be defined as $\mathbb{T}^m = S^1 \times \dots \times S^1$ with m copies of the circle, S^1, and the product topology.

- Fixing a smooth function $f\colon U \to \mathbb{R}^n$ from some open subset $U \subseteq \mathbb{R}^m$ the curve $\{(x, f(x)) \in U \times \mathbb{R}^n \mid x \in U\}$ is a smooth manifold.

- $\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathcal{M}_n(\mathbb{R}) \mid \det A \neq 0\}$ is a smooth manifold. One way to show this is to realise that $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$. That is, $\mathrm{GL}(n, \mathbb{R})$ is the preimage of the set $\mathbb{R} \setminus \{0\}$ under the determinant. The determinant is a smooth map $\mathbb{R}^{n^2} \to \mathbb{R}$ (note that $\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ as vector spaces), since the determinant is just a polynomial in the entries in the matrix, which are the components of the vectors in $\mathbb{R}^{n^2}$. The set $\mathbb{R} \setminus \{0\}$ is open. It is known that the preimage of an open set under a smooth map is open. Thus, we may consider $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ as an open subset, and it is known that any open subset of $\mathbb{R}^k$ is a manifold (more generally, an open subset of a manifold is again a manifold).

One important thing about manifolds is that they have a well defined notion of dimension. If some space is locally homeomorphic to $\mathbb{R}^m$ in some area, but locally homeomorphic to $\mathbb{R}^n$ in another and $m \neq n$ then it cannot be a manifold. For example, consider a pair of lines that cross at a point. Each line is a manifold on its own, but their union isn't, since we can't really define dimension in a neighbourhood of the intersection.

1.3 Smooth Maps

Definition 1.3.1 — Smooth Map Let $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A})$ and $(N, \{(V_\beta, \psi_\beta)\}_{\beta \in B})$ be smooth manifolds of dimension m and n respectively. A **smooth map**, $f\colon M \to N$, is a function such that for all charts $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is smooth as a map between the open sets $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$ and $\psi_\beta(V_\beta) \subseteq \mathbb{R}^n$.

The idea here is that we can pass smoothness of a map between manifolds to smoothness of corresponding patches of Euclidean space, where we can use Definition 1.1.1. The important thing here is that we have the commutative diagram

$$\begin{tikzcd} & U_\alpha \arrow[r, "f"] \arrow[d, "\varphi_\alpha"'] & V_\beta \arrow[d, "\psi_\beta"] & \\ \mathbb{R}^m \supseteq & \varphi_\alpha(U_\alpha) \arrow[r, "\psi_\beta \circ f \circ \varphi_\alpha^{-1}"'] & \psi_\beta(V_\beta) & \subseteq \mathbb{R}^n \end{tikzcd} \tag{1.3.2}$$

1.4 Tangent Space

The tangent space to a manifold is most immediate when we view that manifold as a subset of $\mathbb{R}^n$. For example, the tangent space to the circle, S^1, at some point p is just the tangent line to that point. This line is a copy of $\mathbb{R}$, and so comes with lots of structure, and in particular is a vector space. If we go up a dimension we can consider the sphere, S^2, and the tangent space at some point p is the tangent plane at that point. This plane is a copy of $\mathbb{R}^2$, and is, again, a vector space.

The formal definition of the tangent space just formalises this by defining the tangent space to be the space of all tangent curves through p.

Definition 1.4.1 — Tangent Space Let $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A})$ be a smooth manifold. Fix some point $p \in M$. The tangent space of M at p is the quotient space

$$T_pM := \bigsqcup_{\alpha \text{ s.t. } p\in U_\alpha} \mathbb{R}^m/\overset{p}{\sim}. \tag{1.4.2}$$

Here we use the disjoint union, but we could instead use the normal union and explicitly label each point in $\mathbb{R}^m$ with α from the corresponding term in the union:

$$T_pM = \bigcup_{\alpha \text{ s.t. } p\in U_\alpha} (\{\alpha\} \times \mathbb{R}^m)/\overset{p}{\sim}. \tag{1.4.3}$$

Then the corresponding equivalence relation is

$$(\alpha, \xi)\overset{p}{\sim}(\beta, \eta) \iff \mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1})(x)\xi = \eta, \quad x = \varphi_\alpha(p). \tag{1.4.4}$$

That is, two pairs (α, ξ) and (β, η) are equivalent if one is the pushforward of the other along the transition maps. The equivalence class of $(\alpha, \xi) \in A \times \mathbb{R}^m$ with $p \in U_\alpha$ is denoted $[\alpha, \xi]_p$.

This quotient space, T_pM, is a real vector space of dimension m.

Example 1.4.5 Consider the circle, $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, with the charts $U_\pm = S^1 \setminus \{(0, \mp 1)\}$ where

$$\varphi_+(x, y) = \frac{x}{1+y}, \quad \text{and} \quad \varphi_-(x, y) = \frac{x}{1-y}. \tag{1.4.6}$$

We have

$$\varphi_-^{-1}(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right), \quad \text{and} \quad \varphi_+^{-1}(t) = \left(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\right). \tag{1.4.7}$$

Take the point $p = (\sqrt{3}/2, 1/2) \in S^1$. Then we have

$$\varphi_-(p) = \sqrt{3}, \quad \varphi_+(p) = \frac{1}{\sqrt{3}}. \tag{1.4.8}$$

The transition maps are

$$\varphi_+ \circ \varphi_-^{-1}(t) = \frac{1}{t} = \varphi_- \circ \varphi_+^{-1}(t). \tag{1.4.9}$$

Consider the pair $(-, q) \in A \times \mathbb{R}$ where $A = \{+, -\}$ is the index set of the chart and $\mathbb{R}$ is the one-dimensional vector space corresponding to the tangent space to the one-dimensional manifold S^1. What is the equivalence class of $(-, q)$ in the tangent space at p? That is, what is $[-, q]_p$?

To compute this we take the definition of the equivalence relation, that $(-,q) \overset{p}{\sim} (+,\tilde{q})$ if and only if

$$\mathrm{d}(\varphi_+ \circ \varphi_-^{-1})(x)q = \tilde{q} \tag{1.4.10}$$

where $x = \varphi_-(p) = \sqrt{3}$. Then we are looking for some real number, $\tilde{q}$, given by evaluating

$$\mathrm{d}(\varphi_+ \circ \varphi_-^{-1})(\sqrt{3})q. \tag{1.4.11}$$

Using the definition of the differential we have that

$$\mathrm{d}(\varphi_+ \circ \varphi_-^{-1})(x)q = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_+ \circ \varphi_-^{-1}(x+tq) \tag{1.4.12}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{x+tq} \tag{1.4.13}$$

$$= -\left.\frac{q}{(x+tq)^2}\right|_{t=0} \tag{1.4.14}$$

$$= -\frac{q}{x^2} \tag{1.4.15}$$

and so taking $x = \sqrt{3}$ we get

$$\mathrm{d}(\varphi_+ \circ \varphi_-^{-1})(\sqrt{3})q = -\frac{q}{3}. \tag{1.4.16}$$

So, $(-,q) \overset{p}{\sim} (+,-q/3)$, and so

$$[-,q]_p = \{(-,q),(+,-q/3)\}. \tag{1.4.17}$$

Thus, the tangent space to the circle at $p = (\sqrt{3}/2, 1/2)$ is

$$T_{(\sqrt{3}/2,1/2)}S^1 = \{[-,q]_{(\sqrt{3}/2,1/2)} \mid p \in \mathbb{R}\} \tag{1.4.18}$$

$$= \{\{(-,q),(+,-q/3)\} \mid p \in \mathbb{R}\}. \tag{1.4.19}$$

Often we want to consider all of the tangent spaces at once. To do this we construct the tangent bundle.

Definition 1.4.20 — Tangent Bundle The **tangent bundle** of a manifold M is

$$TM := \coprod_{p \in M} T_pM. \tag{1.4.21}$$

The tangent bundle of an m-dimensional smooth manifold is a $2m$-dimensional smooth manifold in a way that we shall define later once we have some more machinery.

1.5 Derivatives

Suppose that we have two smooth manifolds, $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A})$ and $(N, \{(V_\beta, \psi_B)\}_{\beta\in B})$, of dimensions m and n respectively, and a smooth map $f : M \to N$. Can we construct a map

$$T_pM \to T_{f(p)}N? \tag{1.5.1}$$

The answer is yes, and to derive[1] a sensible definition of this map we should look at what maps we really have available to us.

[1] no pun intended

The map we're after is $[\alpha, \xi]_p \mapsto [\beta, \eta]_{f(p)}$, we just need to specify what η is, it should in general depend on ξ, α, β, p, and f. Note that it only makes sense to consider α and β such that $p \in U_\alpha$ and $f(p) \in V_\beta$, and we may further assume that p has a neighbourhood that is contained entirely in V_β, so we'll assume that $f(U_\alpha) \subseteq V_\beta$.

The maps we have involving the manifolds directly may be summarised in the following diagram:

$$\begin{tikzcd} U_\alpha \arrow[r, "f"] \arrow[d, "\varphi_\alpha"'] & V_\beta \arrow[d, "\psi_\beta"] \\ \varphi_\alpha(U_\alpha) \arrow[r, "\psi_\beta\circ f\circ\varphi_\alpha^{-1} = f_{\beta\alpha}"'] & \psi_\beta(V_\beta). \end{tikzcd} \tag{1.5.2}$$

The action of these maps is

$$\begin{tikzcd} p \arrow[r, mapsto] \arrow[d, mapsto] & f(p) \arrow[d, mapsto] \\ \varphi_\alpha(p) \arrow[r, mapsto] & \psi_\beta(f(p)). \end{tikzcd} \tag{1.5.3}$$

The map $f_{\beta\alpha}$ is between an open subset of $\mathbb{R}^m$ and an open subset of $\mathbb{R}^n$. This allows us to consider its differential, $\mathrm{d}f_{\beta\alpha}$, as defined in Definition 1.1.3. That is, we have the map

$$\mathrm{d}f_{\beta\alpha}(x) : \mathbb{R}^m \to \mathbb{R}^n. \tag{1.5.4}$$

So, given some $x \in \mathbb{R}^m$, we can define η to be $\mathrm{d}f_{\beta\alpha}(x)\xi$. The only question then is what x should we use. The only piece of data we haven't yet used is p, so we should take $x = \varphi_\alpha(p)$. This leads to us defining

$$\eta = \mathrm{d}f_{\beta\alpha}(\varphi_\alpha(p))\xi. \tag{1.5.5}$$

Definition 1.5.6 — Derivative Let $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A})$ and $(N, \{(V_\beta, \psi_B)\}_{\beta\in B})$ be smooth manifolds, and let $f : M \to N$ be a smooth map. We define the **derivative**, **differential**, or **pushforward** of f at p as the linear map

$$f_*(p) = \mathrm{d}f_p = \mathrm{d}f(p) : T_pM \to T_{f(p)}N \tag{1.5.7}$$

given by

$$\mathrm{d}f(p)[\alpha,\xi]_p \coloneqq [\beta, \mathrm{d}f_{\beta\alpha}(\varphi_\alpha(p))\xi]_{f(p)} \tag{1.5.8}$$

where $p \in U_\alpha$ and $f(p) \in V_\beta$ and $f_{\beta\alpha} = \psi_\beta \circ f \circ \varphi_\alpha^{-1}$. Note that $\mathrm{d}f_{\beta\alpha}$ is as defined in Definition 1.1.3.

Example 1.5.9 Consider the special case of the derivative when $N = (\mathbb{R}^n, \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\})$ is $\mathbb{R}^n$ with a single chart. This means that the indexing set for charts on N is the singleton, $B = \{\bullet\}$. Then for $q \in N$ we can identify $[\bullet, \eta]_q \in T_qN$ with $\eta \in \mathbb{R}^n$, since there's only one label $\bullet$, giving us a canonical isomorphism $T_qN \cong \mathbb{R}^n$. If $f : M \to N$ is smooth in this case, given $p \in M$ and $[a, \xi]_p \in T_pM$, we have the derivative

$$\mathrm{d}f(p)[a,\xi]_p = [\bullet, \mathrm{d}f_{\beta\alpha}(\varphi_\alpha(p))\xi]_{f(p)}. \tag{1.5.10}$$

Using this canonical isomorphism we identify this with

$$\mathrm{d}f(p)[a,\xi]_p = \mathrm{d}f_{\bullet\alpha}(\varphi_\alpha(p))\chi. \tag{1.5.11}$$

This further simplifies since $f_{\bullet\alpha} = \psi_\bullet \circ f \circ \varphi_\alpha^{-1} = \mathrm{id}_{\mathbb{R}^n} \circ f \circ \varphi_\alpha^{-1} = f \circ \varphi_\alpha^{-1}$, and so, after identifying $T_{f(p)}N$ and $\mathbb{R}^n$, we have

$$\mathrm{d}f(p)[a,\xi]_p = \mathrm{d}(f \circ \varphi_\alpha^{-1})(\varphi_\alpha(p))\xi. \tag{1.5.12}$$

Once f is known this can be calculated using Definition 1.1.3.

Example 1.5.13 We can apply this formula to maps defined on just an open subset of M, rather than all of M. The most important case of this is when the smooth map in consideration is a chart map, $f = \varphi_{\tilde{\alpha}} : U_{\tilde{\alpha}} \to \mathbb{R}^m$. In this case we may still identify N as $(\mathbb{R}^n, \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^m})\})$, and so the results of Example 1.5.9 apply. In particular, after identifying $T_{\varphi_{\tilde{\alpha}}}N$ with $\mathbb{R}^n$ we have

$$\mathrm{d}\varphi_{\tilde{\alpha}}(p)[\alpha,\xi]_p = \mathrm{d}(\varphi_{\tilde{\alpha}} \circ \varphi_\alpha^{-1})(\varphi_\alpha(p))\xi. \tag{1.5.14}$$

This isn't that interesting, unless we take $\alpha = \tilde{\alpha}$, then we have

$$\begin{aligned}
\mathrm{d}\varphi_\alpha(p)[\alpha,\xi]_p &= \mathrm{d}(\varphi_\alpha \circ \varphi_\alpha^{-1})(\varphi_\alpha(p))\xi && (1.5.15)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_\alpha \circ \varphi_\alpha^{-1})(\varphi_\alpha(p) + t\xi) && (1.5.16)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{id}(\varphi_\alpha(p) + t\xi) && (1.5.17)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_\alpha(p) + t\xi) && (1.5.18)\\
&= \xi. && (1.5.19)
\end{aligned}$$

This allows us to identify $[\alpha, \xi]_p \in T_pM$ with $\xi \in \mathbb{R}^m$, giving an isomorphism $T_pM \cong \mathbb{R}^m$. So, $\mathrm{d}\varphi_\alpha$ is the canonical vector space isomorphism determined by α.

Notation 1.5.20 Let $\{e_i\}_{i\in\{1,\dots,m\}}$ be the standard basis for $\mathbb{R}^m$. Then for the tangent vector $[\alpha, e_i]_p \in T_pM$ we write

$$[\alpha, e_i]_p = \frac{\partial}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p. \tag{1.5.21}$$

This is (for now) just notation, we're not thinking of these as derivatives. We'll see why this is a sensible notation shortly.

1.6 Submersions, Immersions, and Embeddings

In this section we state some definitions and results which are standard in the theory of smooth manifolds, but for which we will have little use.

Definition 1.6.1 — Submersion A **submersion** of a smooth manifold, M, in a smooth manifold, N, is a smooth map, $f\colon M \to N$, whose derivative, $\mathrm{d}f$, exists and is a surjective linear map

$$\mathrm{d}f(p)\colon T_pM \to T_{f(p)}N \tag{1.6.2}$$

for all $p \in M$.
Points $p \in M$ at which $\mathrm{d}f(p)$ is a surjective linear map are called **regular points**. So, a submersion is a map for which every $p \in M$ is regular.

Definition 1.6.3 — Immersion An **immersion** of a smooth manifold, M, in a smooth manifold, N, is a smooth map, $f\colon M \to N$, whose derivative, $\mathrm{d}f$, exists and is an injective linear map

$$\mathrm{d}f(p)\colon T_pM \to T_{f(p)}N \tag{1.6.4}$$

for all $p \in M$.
If an immersion is also a topological embedding, meaning that $f(M)$ (with the subspace topology in N) is homeomorphic to M, then we call f an **embedding**.

The notion of an embedding uses the fact that $f(M)$ is a topological space in its own right, and in fact, it's a manifold in its own right. This leads to the notion of a submanifold.

Definition 1.6.5 — Submanifold An **embedded submanifold** of M is a subset, $S \subseteq M$, which is itself a manifold and for which the inclusion $S \hookrightarrow M$ is an embedding.
More explicitly, S is a topological space in the subspace topology such that for every $p \in S$ there exists a neighbourhood, U_α, of $p \in M$ making a chart $(U_\alpha, \varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{R}^m)$ (in some maximal atlas for M) such that $S \cap U_\alpha = \varphi_\alpha^{-1}(\mathbb{R}^k \cap \Omega_\alpha)$ for some fixed $k \in \mathbb{N}$, which is the dimension of the submanifold, S. Note that there is some freedom in how we embed $\mathbb{R}^k$ in $\mathbb{R}^m$ in order to take this intersection, and in general we only require that there is an embedding that makes this work, not that it works for all embeddings.

There is a weaker notion of an immersed submanifold in which the inclusion is merely required to be an immersion.

A classic example of a submanifold is a smooth curve on some surface. This will correspond to the intersection $\mathbb{R} \cap \Omega_\alpha$.

Theorem 1.6.6 — Regular Value Theorem. If $f : M \to N$ is a smooth map of manifolds then $f^{-1}(y)$ (that is, the preimage of some point $y \in N$) is a smooth submanifold of M for every $y \in N$ if y is a regular value.

Two

Vector Fields

2.1 Definitions

Definition 2.1.1 — Vector Field Let M be a smooth manifold. A **vector field** on M is an assignment of a tangent vector, $X(p) \in T_pM$, to each $p \in M$. Thus, a vector field on M is a map $X\colon M \to TM$ such that $X(p) = X_p \in T_pM$ for all $p \in M$.

Example 2.1.2 Consider a chart $(U_\alpha, \varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{R}^m)$ for the manifold M. Let X be a vector field on M. For each chart index, α, there is a corresponding vector field, $X_\alpha : \Omega_\alpha \to \mathbb{R}^m$ (identifying each T_pM with $\mathbb{R}^m$), defined by

$$X_\alpha(\varphi_\alpha(p)) = \mathrm{d}\varphi_\alpha(X(p)). \tag{2.1.3}$$

Note that this is a vector field on Ω_α viewed as a submanifold of $\mathbb{R}^m$.

Definition 2.1.4 — Smooth Vector Field Let M be a manifold and X a vector field on M. Then X is a **smooth vector field** if X_α (as defined in Example 2.1.2) is a smooth map between subsets of Euclidean space. Equivalently, X is smooth if $X\colon M \to TM$ is a smooth map between manifolds. The set of all smooth vector fields on M is denoted $\mathfrak{X}(M)$.

Since $X\colon M \to TM$ takes values in $T_pM \cong \mathbb{R}^m$ we inherit a lot of structure from $\mathbb{R}^m$. In particular, we can define pointwise addition,

$$(X + Y)(p) = X(p) + Y(p), \tag{2.1.5}$$

and scalar multiplication,

$$(\lambda X)(p) = \lambda X(p), \tag{2.1.6}$$

for $X, Y \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$, and the results are still smooth vector fields. Since we also have the zero vector field, defined by $0(p) = 0 \in \mathbb{R}^m$, and negation (take $\lambda = -1$) we can easily see that $\mathfrak{X}(M)$ is a real vector space.

Further, we can define multiplication by a smooth function, $f \in C^\infty(M)$, by

$$(fX)(p) = f(p)X(p), \tag{2.1.7}$$

and the result is again a smooth vector field. This is a bit like scaling, but now we scale by a different amount at each point of the manifold. This, combined with addition of vector fields tells us is that $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. The vector space structure is then a special case where we restrict only to constant functions in $C^\infty(M)$.

Example 2.1.8 Let $\{e_i\}_{i\in\{1,\dots,m\}}$ be the standard basis of $\mathbb{R}^m$. Let M be an m-dimensional smooth manifold with coordinate chart $(U_\alpha, \varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{R}^m)$. We can define a vector field on U_α,

$$\frac{\partial}{\partial x^i} : U_\alpha \to TM, \tag{2.1.9}$$

by taking

$$\frac{\partial}{\partial x^i}(p) = [\alpha, e_i]_p. \tag{2.1.10}$$

This is such that

$$(\mathrm{d}\varphi_\alpha(p))\left(\frac{\partial}{\partial x^i}(p)\right) = e_i \tag{2.1.11}$$

for all $p \in U_\alpha$. Since $\mathrm{d}\varphi_\alpha(p)$ is always an isomorphism $T_pM \to \mathbb{R}^m$ we see that $\{\partial/\partial x^i(p)\}_{i\in\{1,\dots,m\}}$ is a basis for T_pM, since it's mapped to a basis by an isomorphism. We call $\{\partial/\partial x^i\}_{i\in\{1,\dots,m\}}$ a local **frame** on U_α, since it assigns to each point a basis vector.
We write e_i for the constant vector field that assigns to each point in M the constant vector $e_i \in T_pM$. We may then express any constant vector field as $\sum_i X^i e_i$ for some real constants X^i, and any smooth vector field as $\sum_i X^i e_i$ where the X^i are smooth functions $M \to \mathbb{R}$.

2.2 Algebras

Definition 2.2.1 — Algebra An **algebra** over a field, $\mathbb{F}$, is a vector space, A, equipped with a bilinear map $\cdot : A \times A \to A$.

Definition 2.2.2 — Derivation If A is an algebra then a **derivation** is a linear map $D : A \to A$ satisfying the **Leibniz rule** for all $a, b \in A$:

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b). \tag{2.2.3}$$

The classic example of a derivation is, of course, a derivative acting on some algebra of functions, such as $C^\infty(\mathbb{R})$, where the Leibniz rule is exactly the product rule.

Definition 2.2.4 — Lie Algebra A **Lie algebra**, $\mathfrak{g}$, is a vector space over $\mathbb{F}$, equipped with a bilinear map, $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, which is alternating, meaning $[x,x]=0$ for all $x\in\mathfrak{g}$, and the Jacobi identity, that for all $x,y,z\in\mathfrak{g}$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0. \tag{2.2.5}$$

A homomorphism of Lie algebras is a linear map $\varphi\colon \mathfrak{g}\to\mathfrak{h}$ which preserves the bracket, so $\varphi([x,y])=[\varphi(x),\varphi(y)]$.

Let V be a vector space, then $\operatorname{End}V$ is a Lie algebra where we define the bracket by $[\varphi,\psi]=\varphi\circ\psi-\psi\circ\varphi$. Let A be an algebra and write $\operatorname{Der}A$ for the subset of $\operatorname{End}A$ consisting of derivations. Then it can be shown that $\operatorname{Der}A$ is a subalgebra of the Lie algebra $\operatorname{End}A$. To do so one only needs to show that the commutator of two derivations is again a derivation.

Theorem 2.2.6. Write $\operatorname{Der}(C^\infty(M))$ for the Lie algebra of derivations of the algebra of smooth functions. There is an isomorphism of $C^\infty(M)$-modules, $\Phi\colon \mathfrak{X}(M)\to\operatorname{Der}(\mathbb{C}^\infty(M))$, and this equips $\mathfrak{X}(M)$ with a natural Lie algebra structure.

Proof. Let $\Omega\subseteq\mathbb{R}^m$ be open and $p\in\Omega$. Let $X=\sum_i X^i e_i$ be a smooth vector field on Ω expressed in the standard basis. Let $f\colon\Omega\to\mathbb{R}$ be a smooth function. We define another smooth function, $\partial_X f$, on Ω by

$$\begin{aligned}(\partial_X f)(p) &= \mathrm{d}f_p(X_p) && (2.2.7)\\ &= \mathrm{d}f_p\left(\textstyle\sum_i X^i(p)e_i(p)\right) && (2.2.8)\\ &= \sum_{i=1}^m \frac{\partial f}{\partial x^i}(p)X^i(p). && (2.2.9)\end{aligned}$$

This is smooth since it's a product of smooth functions $\partial f/\partial x^i, X^i\colon\Omega\to\mathbb{R}$. Further, the map $f\mapsto\partial_X f$ is a derivation on $C^\infty(\Omega)$, since we have

$$\partial_X(fg)=\mathrm{d}(fg)X=f\,\mathrm{d}gX+g\,\mathrm{d}fX=f\partial_X g+g\partial_X f. \tag{2.2.10}$$

Now taking X to be a vector field on some smooth manifold, M, and $f\in C^\infty(M)$ we can define $\partial_X f(p)=\mathrm{d}f_pX_p$, which we may also write as$\partial_{X_p}f$. The map $\Phi\colon\mathfrak{X}(M)\to\operatorname{Der}(C^\infty(M))$ defined by $X\mapsto\partial_X$ is a morphism of $C^\infty(M)$-modules $\mathfrak{X}(M)\to\operatorname{Der}(C^\infty(M))$.
It can be shown that Φ is in fact an isomorphism. □

Using this result it is standard to identify a smooth vector field, X, with the corresponding derivation[1], $\partial_X\in\operatorname{Der}(C^\infty(M))$. Note that every tangent vector at a

[1] In fact, this is a standard way to define a vector field, and then one proves the above in reverse.

point, X_p, defines a linear map $\partial_{X_p} : C^\infty(M) \to \mathbb{R}$ satisfying

$$\partial_{X_p}(fg) = g(p)\partial_{X_p}(f) + f(p)\partial_{X_p}(g), \tag{2.2.11}$$

which is almost the Leibniz rule, except for the fact that ∂_{X_p} isn't an endomorphism.

Three

Integral Curves and Lie Groups

3.1 Integral Curves

Consider the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ given by

$$X(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{3.1.1}$$

which is interpreted as an element of $T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2$. If we attach $X(x, y)$ to the point (x, y) for some number of points we get an image like Figure 3.1. This is why we call these vector fields.

Notice that every arrow is tangent to a circle centred on the origin of radius $\sqrt{x^2 + y^2}$.

> **Definition 3.1.2** A smooth **path** on a manifold, M, is a smooth map $c : \mathbb{R} \to M$.

What we see here is that the vectors in the vector field X are all tangent to the

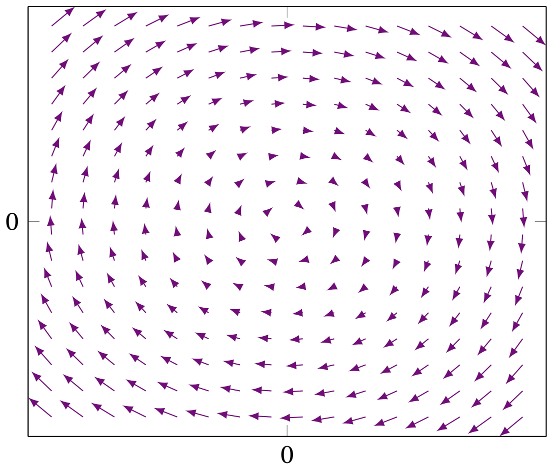

Figure 3.1: The vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ given by $X(x, y) = (-y\ x)^\top$.

smooth curve defined by

$$c(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}. \tag{3.1.3}$$

Proposition 3.1.4 Let M, N, and O be smooth manifolds and let $f\colon M \to N$ and $g\colon N \to O$ be smooth maps. Then for $x \in M$ we have

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x. \tag{3.1.5}$$

If c is a path we can relate the tangent vector at 0, that is $V = \dot{c}(0)$, to the corresponding derivation by

$$\partial_V f = \mathrm{d}f_{c(0)}(V) = \mathrm{d}f_{c(0)}\left(\mathrm{d}c_0\left(\frac{\partial}{\partial t}\right)\right) = \mathrm{d}(f \circ c)_0\left(\frac{\partial}{\partial t}\right) = (f \circ c)'(0). \tag{3.1.6}$$

Definition 3.1.7 — Local Integral Curve Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field on M. A **local integral curve** of X through $p \in M$ is a local path $c\colon (-\varepsilon, \varepsilon) \to M$ (for some $\varepsilon > 0$) such that $c(0) = p$ and $\dot{c}(t) = X_{c(t)}$ for all $t \in (-\varepsilon, \varepsilon)$.

Note that for a smooth curve, $c\colon \mathbb{R} \to M$ with $c(0) = p \in M$ the derivative $\dot{c}(0) \in T_pM$ is defined to be the equivalence class

$$\dot{c}(0) = \left[\alpha, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_\alpha(c(t))\right]_p. \tag{3.1.8}$$

The vector $\dot{c}(0) \in T_pM$ is then the image of the vector $1 \in T_0\mathbb{R} = \mathbb{R}$ under the linear map $\mathrm{d}c_0\colon T_0\mathbb{R} \to T_pM$.

Every $p \in M$ and $v \in T_pM$ has a corresponding smooth curve $c\colon \mathbb{R} \to M$ such that $c(0) = p$ and $\dot{c}(0) = v$. To see this take some coordinate chart $\varphi_\alpha\colon U_\alpha \to \Omega_\alpha$ such that $p \in U_\alpha$, and define $x = \varphi_\alpha(p)$ and $\xi = \mathrm{d}{\varphi_\alpha}_p(v)$. Take some constant $\varepsilon > 0$ such that $x + t\xi \in \Omega_\alpha$ for all $t \in (-\varepsilon, \varepsilon)$. Then we may define c by

$$c(t) = \varphi_\alpha^{-1}\left(x + \frac{\varepsilon t}{\sqrt{\varepsilon^2 + t^2}}\right) \tag{3.1.9}$$

for $t \in \mathbb{R}$ and this has the desired property.

Proposition 3.1.10 Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a smooth vector field. Then the following hold:

1. For every $p \in M$ there exists some $\varepsilon > 0$ such that for every $\delta \leq \varepsilon$ there exists a unique integral curve of X through p defined on $(-\delta, \delta)$.

2. For every $p \in M$ there exists an open neighbourhood, U_p, of p and $\varepsilon > 0$ such that the integral curve of X through every $q \in U_p$ is

defined for $t \in (-\varepsilon, \varepsilon)$.

3. For $p \in M$ let U_p and ε be as in the previous point. If $t < \varepsilon$ we define the map $\varphi_t : U_p \to M$ by $\varphi_t(q) = c_q(t)$ where c_q is the integral curve of X through q. Then we have

$$\varphi_t \circ \varphi_s = \varphi_{s+t} \tag{3.1.11}$$

for all $t, s, s + t \in (-\varepsilon, \varepsilon)$ on whatever open set this composite is defined on.

4. For every $t < \varepsilon$ the local map φ_t is a local diffeomorphism.

We call a family, $\{\varphi_t\}$, of local diffeomorphisms satisfying $\varphi_t \circ \varphi_s = \varphi_{t+s}$ a one-parameter pseudogroup of local diffeomorhpisms of M. It's like a group, but sometimes the composite isn't defined.

3.2 Lie Groups and Lie Algebras

Definition 3.2.1 A group, G, which is also a smooth manifold for which the multiplication map, $\mu : G \times G \to G$, $\mu(g, h) = gh$, and inverse map, $\iota : G \to G$, $\iota(g) = g^{-1}$, are smooth is called a Lie group.

For a Lie group, G, define $L_g, R_g : G \to G$ to be left and right multiplication by g, that is $L_g(h) = gh$ and $R_g(h) = hg$. We call these left and right translation, because we view them as shifting the manifold around by some amount g, the canonical example being $G = \mathbb{R}^n$ with addition as the group operation, in which case these really are translations. Note that L_g and R_g are invertible, with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$ respectively, and they are defined in terms of the smooth multiplication map, so are themselves smooth. Thus L_g and R_g are diffeomorphisms of G.

A vector field, $X \in \mathfrak{X}(G)$, is called **left invariant** (similar definitions hold for right translations) if $\mathrm{d}L_g(X) = X$ for all $g \in G$. That is, if

$$(\mathrm{d}L_g)_{g^{-1}a}(X_{g^{-1}a}) = X_a \tag{3.2.2}$$

for all $a, g \in G$. Note that we're writing $g^{-1}a$ as a generic element of G, since all $b \in G$ can be written in this way, just take $a = gb$. If we take $g = a$ in the above then we find that

$$(\mathrm{d}L_g)_e(X_e) = X_g \tag{3.2.3}$$

and so the value of the vector field at g, that is X_g, is determined entirely from its value at the identity, that is X_e, and the translation L_g. Conversely, this allows us to define a smooth left invariant vector field X by specifying just it's value at e.

We write $\mathfrak{g} = T_eG$ and call this the **Lie algebra** of G. Every vector in T_eG defines a smooth left-invariant vector field on G by the above equation. This correspondence allows us to alternatively identify $\mathfrak{g}$ as the vector space of left-invariant vector fields on G.

Four

The Exterior Algebra

The order in which things were introduced in this section is questionable, and I've decided to take a more standard approach. At a few points this means results which were stated as definitions in the lecture would more properly be understood as theorems here, and I'll attempt to flag these as they arise.

Our goal in this section is to begin the introduction of calculus to manifolds, in particular integrals, as we've already seen some derivatives. It turns out that for a general manifold, M, (as opposed to $\mathbb{R}^m$) functions on M are not the "correct" thing to be integrating. Instead, we need to integrate differential forms

4.1 Differential Forms

4.1.1 Symmetric Group

You know what this is. We write $\varepsilon(\sigma)$ for the sign of the permutation σ.

4.1.2 Forms

Definition 4.1.1 — Form Let V be an m-dimensional real vector space and fix some positive integer k. An alternating k-form is a multilinear map

$$\omega : V^k \to \mathbb{R} \tag{4.1.2}$$

satisfying

$$\omega(v_1, \dots, v_k) = \varepsilon(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \tag{4.1.3}$$

for $S \in S_k$.

In physics parlance an alternating k-form is just an antisymmetric rank k tensor. In a given basis, $\{e_i\}$, for V the components of $\omega : V^k \to \mathbb{R}$ are

$$\omega_{i_1 \cdots i_k} = \omega_I \coloneqq \omega(e_{i_1}, \dots, e_{i_k}). \tag{4.1.4}$$

Here $I = i_1 \cdots i_k$ is **multi-index** notation, it's just shorthand for a bunch of indices.

Lemma 4.1.5 For a fixed k and V the set of all alternating k-forms on V is a vector space, with pointwise addition and scalar multiplication.

We denote this vector space by $\Lambda^k V^*$.

Example 4.1.6

- For $k = 0$ we have $V^0 = \mathbb{R}$ and all maps $\mathbb{R} \to \mathbb{R}$ are trivially alternating (since S_0 is trivial). So 0-forms are just linear functions $\mathbb{R} \to \mathbb{R}$. Such a function is completely determined by where it sends 1, say it sends 1 to λ, then the action on n is $n = n \cdot 1 \mapsto n \cdot \lambda = \lambda n$, so the function is just scaling by λ, and hence $\Lambda^0 V^* \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$.

- For $k = 1$ we have $V^1 = V$ and all maps $V \to \mathbb{R}$ are trivially alternating (since S_1 is trivial). So 1-forms are just linear maps $V \to \mathbb{R}$, which is to say elements of V^*, and so $\Lambda^1 V^* = V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

  For $V = \mathbb{R}^m$ denote by $\mathrm{d}x^i : \mathbb{R}^m \to \mathbb{R}$ the projection onto the ith coordinate in some basis $\{e_i\}$ for $\mathbb{R}^m$. So if $\xi = \xi^1 e_1 + \cdots + \xi^m e_m$ then $\mathrm{d}x^i(\xi) = \xi^i$. The linear maps $\mathrm{d}x^i : \mathbb{R}^m \to \mathbb{R}$ then form a basis for $(\mathbb{R}^m)^* = \Lambda^1(\mathbb{R}^m)^*$.

- For $k = 2$ we finally have a nontrivial S_2, and so not all linear maps $V \to \mathbb{R}$ are alternating k-forms. An alternating 2-form is is a skew-symmetric linear map $\omega : V \times V \to \mathbb{R}$, which is to say that

  $$\omega(u, v) = -\omega(v, u). \tag{4.1.7}$$

  For $V = \mathbb{R}^m$ any such alternating 2-form corresponds to a skew-symmetric matrix, A, which is such that

  $$\omega(u, v) = \langle u, Av \rangle \tag{4.1.8}$$

  where $\langle -, - \rangle$ is the standard inner product on $\mathbb{R}^m$. Further, this identification provides an isomorphism $\Lambda^2(\mathbb{R}^m)^* \cong \mathfrak{so}(m)$ (it just happens that $\mathfrak{so}(m)$ is the space of skew-symmetric matrices, there's not really anything rotational going on here). Simple parameter counting tells us that $\dim \mathfrak{so}(m) = \dim(\Lambda^2(\mathbb{R}^m)^*) = m(m-1)/2$ (this is just the number of above-diagonal elements in an $m \times m$ matrix).

Definition 4.1.9 *This is not the standard definition of* $\mathrm{d}x^I$. *That would be* $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, *instead this is a theorem for us here, which is both wrong and bad.*
Let $\mathcal{I}_{k,m}$ denote the set of tuples $(i_1, \dots, i_k) \in \mathbb{N}^k$ (with $\mathbb{N} = \{1, 2, 3, \dots\}$ (also wrong and bad)) such that $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. Then define the alternating k-form

$$\mathrm{d}x^I : (\mathbb{R}^m)^k \to \mathbb{R} \tag{4.1.10}$$

by

$$\mathrm{d}x^I(v_1, \dots, v_k) = \det \begin{pmatrix} u_1^{i_1} & u_2^{i_1} & \dots & u_k^{i_1} \\ u_1^{i_2} & u_2^{i_2} & \dots & u_k^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{i_k} & u_2^{i_k} & \dots & u_k^{i_k} \end{pmatrix}. \tag{4.1.11}$$

If we allow for $i_j > m$ then we just declare that $\mathrm{d}x^I = 0$, which is enforced by the alternating property of $\mathrm{d}x^I$.

Lemma 4.1.12 Let V be an m-dimensional real vector space. Then

$$\dim \Lambda^k V^* = \binom{m}{k}. \tag{4.1.13}$$

Proof. First note that if $\varphi : V \to \mathbb{R}^m$ is an isomorphism then this induces an isomorphism $\Lambda^k V^* \to \Lambda^k(\mathbb{R}^m)^*$ by $\omega \mapsto \omega \circ \varphi^k$ where φ^k simply applies φ to each input of ω. Thus, it is sufficient to prove the $V = \mathbb{R}^m$ case. In this case we claim that $\{\mathrm{d}x^I\}_{I \in \mathcal{I}_{k,m}}$ forms a basis for $\Lambda^k(\mathbb{R}^m)^*$, and then it is a simple counting exercise to show that

$$|\mathcal{I}_{k,m}| = \binom{m}{k}. \tag{4.1.14}$$

We make three observations:

1. If $\{e_i\}$ is the standard basis of $\mathbb{R}^m$ and $J = (j_1, \dots, j_k) \in \mathcal{I}_{k,m}$ then for $I \in \mathcal{I}_{k,m}$ we have

$$\mathrm{d}x^I(e_J) \coloneqq \mathrm{d}x^I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J, \\ 0 & I \neq J. \end{cases} \tag{4.1.15}$$

   This follows because if $I \neq J$ then the resulting matrix that we take the determinant of will have a zero on the diagonal, and if $I = J$ then the resulting matrix is the identity.

2. For $\omega \in \Lambda^k(\mathbb{R}^m)^*$ we have

$$(\omega = 0) \iff (\omega(e_I) = \omega(e_{i_1}, \dots, e_{i_k}) = \omega_I = 0 \forall I \in \mathcal{I}_{k,m}). \tag{4.1.16}$$

   That is, ω vanishes only if all of its components do in the standard basis.

3. For every $\omega \in \Lambda^k(\mathbb{R}^m)^*$ we have

$$\omega = \sum_{I \in \mathcal{I}_{k,m}} \omega_I \, \mathrm{d}x^I. \tag{4.1.17}$$

   This follows by simply both sides on some set of vectors and using the definitions and observing that they give the same result. Alternatively this is simply the fact that $\mathrm{d}x^I$ is defined via a determinant, which is the only alternating map up to a scale factor, here ω_I, and thus this follows from the first two observations.

The third observation proves that $\mathrm{d}x^I$ span $\Lambda^k(\mathbb{R}^m)^*$, and linear independent of the $\mathrm{d}x^I$ follows from these observations also: if $\sum_I \omega_I \, \mathrm{d}x^I = 0$ then define $\omega = \sum_I \omega_I \, \mathrm{d}x^I$ and $\omega(e_J) = \omega_J = 0$ only if ω_J vanishes in this sum, so $\omega = 0$ only if all of its components vanish. □

4.2 Exterior Product

Definition 4.2.1 — Exterior Product The **exterior product** or **wedge product** is the product $\Lambda^k V^* \times \Lambda^\ell V^* \to \Lambda^{K+\ell} V^*$ defined by

$$(\omega\wedge\eta)(v_1,\dots,v_{k+\ell}) = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} \omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}). \tag{4.2.2}$$

This is not the definition given in lectures, which is in terms of shuffles, to avoid the over-counting fixed by the prefactor of $1/k!\ell!$ at the cost of being much worse in every other way.

Example 4.2.3

- For $k = \ell = 1$ we have

$$(\omega \wedge \eta)(u, v) = \omega(u)\eta(v) - \omega(v)\eta(u). \tag{4.2.4}$$

- For $k = 1$ and $\ell = 2$ taking $S_3 = \{(), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$ in this order we have

$$\begin{aligned}(\omega \wedge \eta)(u, v, w) &= \frac{1}{2!}[\omega(u)\eta(v, w) - \omega(v)\eta(u, w) - \omega(u)\eta(w, v)\\ &\quad - \omega(w)\eta(v, u) + \omega(v)\eta(w, u) + \omega(w)\eta(u, v)]\\ &= \frac{1}{2}[\omega(u)\eta(v, w) + \omega(v)\eta(w, u) + \omega(u)\eta(v, w)\\ &\quad + \omega(w)\eta(u, v) + \omega(v)\eta(w, u) + \omega(w)\eta(u, v)]\\ &= \omega(u)\eta(v, w) + \omega(v)\eta(w, u) + \omega(w)\eta(u, v)\end{aligned}$$

Lemma 4.2.5 The exterior product is associative, distributes over addition, and is graded-commutative (or supercommutative), meaning

$$\omega \wedge \eta = (-1)^{\deg(\omega)\deg(\eta)}\eta \wedge \omega \tag{4.2.6}$$

where $\deg\omega = k$ and $\deg\eta = \ell$ for $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^\ell V^*$.

4.2.1 Better Definitions

The exterior algebra, $\Lambda V^* = \Lambda^* V$, is formally defined to be the quotient of the tensor algebra

$$T(V^*)/I = (\mathbb{R} \oplus V^* \oplus (V^* \otimes V^*) \oplus \cdots)/I \tag{4.2.7}$$

where I is the ideal generated by $\omega \otimes \omega$ for $\omega \in V^*$. This is $\mathbb{N}$-graded and $\Lambda^k V^*$ is just the k-graded part of this. The wedge product is just the tensor product mod I, so $\omega \wedge \eta = \tilde{\omega} \otimes \tilde{\eta} + I$ where $\omega = \tilde{\omega} + I$ and $\eta = \tilde{\eta} + I$ are such that $\tilde{\omega}, \tilde{\eta} \in T(V^*)$.

Theorem 4.2.8 — Determinant Theorem. For $\omega_i \in V^* = \Lambda^1 V^*$ and $v_i \in V$ we have

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \dots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \dots & \alpha_1(v_k) \\ \alpha_2(v_1) & \alpha_2(v_2) & \dots & \alpha_2(v_l) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \alpha_k(v_2) & \dots & \alpha_k(v_k) \end{pmatrix}. \quad (4.2.9)$$

Corollary 4.2.10 $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

Of course, with sensible definitions this corollary is the definition and what we have as the definition is a special case of the determinant theorem.

4.3 Pullback

Definition 4.3.1 — Pullback Let $\varphi\colon V \to W$ be a linear map between real vector spaces V and W. The **pullback** of the alternating k-form $\omega \in \Lambda^k W^*$ along φ is the alternating k-form $\varphi^*\omega \in \Lambda^k V^*$ defined by

$$(\varphi^*\omega)(v_1, \dots, v_k) = \omega(\varphi(v_1), \dots, \varphi(v_k)) \quad (4.3.2)$$

for $v_i \in V$.

Lemma 4.3.3 For fixed $\varphi\colon V \to W$ the map $\Lambda^* W \to \Lambda^* V$ given by $\omega \mapsto \varphi^*\omega$ is linear and preserves the exterior product, that is

$$\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta \quad (4.3.4)$$

for all $\omega, \eta \in \Lambda^* W$.

Lemma 4.3.5 Let $\varphi\colon V \to W$ and $\psi\colon W \to U$ be linear maps between real vector spaces. Then $(\psi \circ \varphi)^*\omega = \varphi^*(\psi^*\omega)$ for all $\omega \in \Lambda^* U$, so $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. Further, if $W = V$ we can take $\varphi = \mathrm{id}_V$ and $\mathrm{id}_V^*\omega = \omega$ so $\mathrm{id}_V^* = \mathrm{id}_{\Lambda^* V}$. In other words, taking exterior algebras and pullbacks defines a contravariant functor $\mathsf{Vect}_{\mathbb{R}}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{R}}$ sending a vector space to $\Lambda^* V$ and sending a linear map, φ, to its pullback, φ^*.

Lemma 4.3.6 If $\varphi\colon V \to V$ is an endomorphism then $\varphi^*\omega = \det(\varphi)\omega$ for all $\omega \in \Lambda^* V$.

Five

Differential Forms

5.1 Differential Forms

When we take the vector space $V = T_pM$ and impose a smoothness condition on the resulting alternating k-forms we get differential forms.

Definition 5.1.1 — Differential Form Let M be a smooth manifold and let $k \in \mathbb{N}$. A **differential k-form**, ω, is a collection of alternating k-forms, $\omega_p \in \Lambda^k T_p^* M$, one for each $p \in M$. That is, we have differential forms

$$\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R} \tag{5.1.2}$$

with k copies of T_pM in the product. The differential form, ω, itself is a function $\omega : M \to \Lambda^k T_p^* M$.
We impose the smoothness condition that for any k smooth vector fields, $X_1, \dots, X_k \in \mathfrak{X}(M)$ the function $p \mapsto \omega_p(X_1(p), \dots, X_k(p))$ must be smooth.
We denote the set of smooth differential k-forms on M by $\Omega^k(M)$. We call k the **degree** of ω and write $k = \deg(\omega)$.

Remark 5.1.3 The set

$$\Lambda^k T^* M = \{(p, \omega) \mid p \in M, \omega \in \Lambda^k T_p^* M\} \tag{5.1.4}$$

is a vector bundle over M, which we'll define in a future section. This requires that $\Lambda^k T^* M$ is a manifold, and indeed it is. The projection $\pi : \Lambda^k T^* M \to M$ given by $(p, \omega) \mapsto p$ is a sooth submersion, and each fibre (to be defined) $\Lambda^k T_p^* M$ is a vector space, with addition and scalar multiplication given by smooth maps. The manifold structure is uniquely determined by the fact that each $\omega \in \Omega^k(M)$ defines a smooth map $M \to \Lambda^k T^* M$ given by $p \mapsto (p, \omega_p)$. The composition of this map with the projection π is the identity on M, and maps with this property are called (smooth) sections of the vector bundle. This allows us to identify $\Omega^k(M)$ with the space of smooth sections of $\Lambda^k T^* M$, sometimes denoted $\Gamma(\Lambda^k T^* M)$.
The space $\Omega^k(M)$ is a vector space, and is generally infinite dimensional (the dimension is finite only if M is a finite set or $k > \dim M$ (in which

case $\Omega^k(M) = 0$)). Note that $\Omega^0(M) = \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}$ is the set of real valued smooth functions on M, also denoted $C^\infty(M)$.

The operators on the wedge product and pullback generalise pointwise to differential forms, as we see in the following definitions.

Definition 5.1.5 — Exterior Product Let M be a smooth manifold and $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ differential forms on M. We define the **exterior product** pointwise by

$$(\omega \wedge \eta)_p \coloneqq \omega_p \wedge \eta_p \tag{5.1.6}$$

where on the right the exterior product is that of alternating forms defined in Definition 4.2.1.

Definition 5.1.7 — Pullback Let M and N be smooth manifolds and let $\omega \in \Omega^k(N)$ be a differential form on N. Let $\varphi\colon M \to N$ be a smooth map between manifolds. The **pullback** of ω is the differential k-form $\varphi^*\omega \in \Omega^k(M)$ on M given by

$$(\varphi^*\omega)_p(v_1, \dots, v_k) \coloneqq \omega_{\varphi(p)}(\mathrm{d}\varphi_p v_1, \dots, \mathrm{d}\varphi_p v_k) \tag{5.1.8}$$

for all $p \in M$ and $v_i \in T_pM$.

The following lemma is immediate from the same properties for the exterior product and pullback of alternating forms, everything just factors through the pointwise operations.

Lemma 5.1.9 The exterior product is associative, distributes over addition, and is graded-commutative.
The pullback is linear, preserves the exterior product, and provides a contravariant functor between vector bundles.

Lemma 5.1.10 Let M and N be smooth manifolds, $\omega \in \Omega^k(N)$ a differential form, and $X_1, \dots, X_k \in \mathfrak{X}(N)$ vector fields. Let $\varphi\colon M \to N$ be a diffeomorphism. Then

$$(\varphi^*\omega)(\varphi^*X_1, \dots, \varphi^*X_k) = \omega(X_1, \dots, X_k) \circ \varphi. \tag{5.1.11}$$

5.2 Differential Forms in Local Coordinates

For explicit calculations (if you're a physicist) it can be useful to work in local coordinates. That is, we take some chart $(U_\alpha, \varphi_\alpha)$, and we restrict our attention to only the part of M covered by U_α. We can then use φ_α to get explicit functions between subsets of $\mathbb{R}^m$ and $\mathbb{R}$ instead of functions $M \to \mathbb{R}$. In this section we'll look at how this lets us express differential forms in local coordinates.

Remark 5.2.1 We have so far assumed, and will continue to assume, that manifolds do not have boundaries. A manifold with boundary can be defined in the same way as a manifold without boundary, just replace $\mathbb{R}^m$ with the upper half space, $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$. Then a boundary point is any point in M which maps to coordinates with $x_m = 0$ in a chart, and all other points are interior points.

Suppose that M is a smooth manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Let $\omega \in \Omega^k(M)$ be a k-form on M. For a chosen chart, $(U_\alpha, \varphi_\alpha)$, we can express ω in local coordinates. To do this we define a differential form on $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$. Denote this form by $\omega_\alpha \in \Omega^k(\varphi_\alpha(U_\alpha))$. We choose this form to be such that it pulls back to the restriction of ω to U_α, that is

$$\omega|_{U_\alpha} = \varphi_\alpha^* \omega_\alpha. \tag{5.2.2}$$

We can determine ω_α uniquely from this. To do so we use the fact that any vector we wish to evaluate ω_α on should arise from a vector in the tangent space T_pM. We turn this into a vector in the tangent space $T_{\varphi_\alpha(p)}U_\alpha$ with the pushforward, $\mathrm{d}\varphi_\alpha(p)$. Then, for $p \in U_\alpha \subseteq M$, $v_i \in T_pM$, $x = \varphi_\alpha(p)$, and $\xi_i = \mathrm{d}\varphi_\alpha(p)v_i$ we have

$$\omega_\alpha(x; \xi_1, \dots, \xi_k) = \omega_p(v_1, \dots, v_k). \tag{5.2.3}$$

The right hand side is known, so this fully determines ω_α.

In more detail, recall that one way of constructing the tangent space was as

$$T_pM = \left(\bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m \right) / \sim, \tag{5.2.4}$$

so elements are of the form $v_i = [\alpha, \xi_i]_p$ for some $\alpha \in A$ and $\xi_i \in \mathbb{R}^m$. Then if $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{R}^m$ we can define

$$f_{\alpha,I} \colon U_\alpha \to \mathbb{R} \tag{5.2.5}$$

for $\alpha \in A$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_{m,k}$ by

$$f_{\alpha,I}(x) = \omega_\alpha(x; e_{i_1}, \dots, e_{i_k}) = \omega_p([\alpha, e_{i_1}]_p, \dots, [\alpha, e_{i_k}]_p) \tag{5.2.6}$$

for $x \in \varphi_\alpha(U_\alpha)$ and $p = \varphi_\alpha^{-1}(x) \in U_\alpha$. Then we can write ω_α as

$$\omega_\alpha = \sum_{I \in \mathcal{I}_{m,k}} f_{\alpha,I} \mathrm{d}x^I. \tag{5.2.7}$$

In order for this local definition to make sense we must have that on the overlap of two charts, $U_\alpha \cap U_\beta$, the two definitions agree. That is, if we define ω_β in the same way but for the chart (U_β, φ_β) then we require that

$$\omega|_{U_\alpha \cap U_\beta} = \varphi_\alpha^* \omega_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = \varphi_\beta^* \omega_\beta|_{\varphi_\beta(U_\alpha \cap U_\beta)}. \tag{5.2.8}$$

Using the fact that the pullback of an inverse is the inverse of the pullback we must have

$$\omega_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = (\varphi_\alpha^{-1})^* (\varphi_\beta^* \omega_\beta|_{\varphi_\beta(U_\alpha \cap U_\beta)}). \tag{5.2.9}$$

Then using the fact that the pullback is contravariant we can write this as

$$\omega_\alpha|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = (\varphi_\beta \circ \varphi_\alpha^{-1})^* \omega_\beta|_{\varphi_\beta(U_\alpha\cap U_\beta)}. \tag{5.2.10}$$

It turns out that this is always true, so local coordinates work as we would like. Further, one can show that if we define $\omega_\alpha \in \Omega^k(\varphi_\alpha(U_\alpha))$ for all charts $(U_\alpha, \varphi_\alpha)$ and this condition holds then there is a unique $\omega \in \Omega^k(M)$ which corresponds to these local coordinate forms, so we can specify a differential form just by specifying its local coordinates for all charts.

5.3 Exterior Derivative and Lie Derivative

5.3.1 Motivation

Consider two smooth functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$. We have the well known operations

$$\operatorname{grad} f = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix}, \quad \operatorname{curl} g = \begin{pmatrix} \partial_2 g_3 - \partial_3 g_2 \\ \partial_3 g_1 - \partial_1 g_3 \\ \partial_1 g_2 - \partial_2 g_1 \end{pmatrix}, \quad \text{and} \quad \operatorname{div} g = \partial_1 g_1 + \partial_2 g_2 + \partial_3 g_3.$$

These operations also satisfy the identities

$$\operatorname{curl} \circ \operatorname{grad} = 0, \qquad \text{and} \qquad \operatorname{div} \circ \operatorname{curl} = 0. \tag{5.3.1}$$

Our goal of this section is to see how all three operations are actually specific cases of the exterior derivative, and these identities are then special cases of the more general identity $\mathrm{d}^2 = 0$. This identity in turn will lead us to define cohomology later.

The key to identifying this is to note that actually we shouldn't think of the curl and gradient as acting on the same objects, rather div acts on $\Omega^1(\mathbb{R}^3)$ and curl acts on $\Omega^2(\mathbb{R}^3)$. What we're doing when we think of them as acting on $g : \mathbb{R}^3 \to \mathbb{R}^3$ is identifying that $\Lambda^1\mathbb{R}^3 \cong \mathbb{R}^3 \cong \Lambda^2\mathbb{R}^3$.

5.3.2 The Exterior Derivative

Let $U \subseteq \mathbb{R}^m$ be open. The exterior derivative on U is a linear operation $\mathrm{d} : \Omega^K(U) \to \Omega^{k+1}(U)$. There are a few ways to define the exterior derivative. The choice mainly depends on how we write the differential form in the first place. We give two definitions here, which are equivalent (although we won't prove this).

Definition 5.3.2 — Exterior Derivative Let $\omega \in \Omega^k(U)$. Then ω defines a smooth map[a]

$$\omega : U \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R}, \tag{5.3.3}$$

$$(x, \xi_1, \dots, \xi_k) \mapsto \omega_x(\xi_1, \dots, \xi_k) \tag{5.3.4}$$

which is such that for any fixed $x \in U$

$$\underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R} \tag{5.3.5}$$

$$(\xi_1, \dots, \xi_k) \mapsto \omega_x(\xi_1, \dots, \xi_k) \tag{5.3.6}$$

is an alternating k-form on $\mathbb{R}^m$. The exterior derivative of ω is the $(k+1)$-form $\mathrm{d}\omega$ defined by

$$\mathrm{d}\omega_x(\xi_1, \dots, \xi_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \omega_{x+t\xi_j}(\xi_1, \dots, \widehat{\xi_j}, \dots \xi_{k+1}) \tag{5.3.7}$$

for all $x \in U$ and $\xi_1, \dots, \xi_{k+1} \in \mathbb{R}^m$ where $\widehat{\xi_j}$ means that we skip the ξ_j evaluation.

[a] The notes use the notation $\omega(x; \xi_1, \dots, \xi_k)$ for this.

Definition 5.3.8 — Exterior Derivative Let $\omega \in \Omega^k(U)$ and define $f_I : U \to \mathbb{R}$ for $I \in \mathcal{I}_{k,m}$ by $f_I(x) = \omega_x(e_{i_1}, \dots, e_{i_k}) = \omega_x(e_I)$ where $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{R}^m$. We then define the exterior derivative of ω to be the $(k+1)$-form $\mathrm{d}\omega$ given by

$$\mathrm{d}\omega = \sum_{I \in \mathcal{I}_{k,m}} \mathrm{d}f_I \wedge \mathrm{d}x^I \tag{5.3.9}$$

where

$$\mathrm{d}f_I = \sum_{i=1}^m \frac{\partial f_I}{\partial x^i} \mathrm{d}x^i. \tag{5.3.10}$$

Note that $\mathrm{d}f_I$ here is just a special case of the exterior derivative when we view f_I as a 0-form.

Lemma 5.3.11 — Properties of the Exterior Derivative

- $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is a linear map;
- for $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$ we have $\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k \omega \wedge \mathrm{d}\tau$, this is called the **super-Leibniz rule** or the **graded-Leibniz rule**;
- $\mathrm{d}^2 = 0$; and
- for $\varphi : U \to V$ a map between open subsets of $\mathbb{R}^m$ we have $\varphi^*(\mathrm{d}\omega) = \mathrm{d}(\varphi^*\omega)$, that is, the exterior derivative commutes with pullbacks.

Proof.

- Linearity follows immediately from the definition, linearity of the partial derivative, and linearity of forms and the wedge product.

- Suppose

$$\omega = \sum_{I\in\mathcal{I}_{k,m}} f_I \,\mathrm{d}x^I, \qquad \text{and} \qquad \tau = \sum_{J\in\mathcal{I}_{\ell,m}} g_J \,\mathrm{d}x^J. \tag{5.3.12}$$

By linearity it is sufficient to prove this for a single term, so we may take

$$\omega = f\,\mathrm{d}x^I, \qquad \text{and} \qquad \tau = g\,\mathrm{d}x^J \tag{5.3.13}$$

for some smooth functions $f, g\colon U \to \mathbb{R}$ and fixed $I \in \mathcal{I}_{k,m}$ and $J \in \mathcal{I}_{\ell,m}$. We then have

$$\begin{aligned}
\mathrm{d}(\omega\wedge\tau) &= \mathrm{d}(f\,\mathrm{d}x^I \wedge g\,\mathrm{d}x^J) &(5.3.14)\\
&= \mathrm{d}(fg\,\mathrm{d}x^I \wedge \mathrm{d}x^J) &(5.3.15)\\
&= \mathrm{d}(fg)\wedge(\mathrm{d}x^I \wedge \mathrm{d}x^J) &(5.3.16)\\
&= ((\mathrm{d}f)g + f\mathrm{d}g)\wedge \mathrm{d}x^I \wedge \mathrm{d}x^J &(5.3.17)\\
&= \mathrm{d}f \wedge \mathrm{d}x^I \wedge g\,\mathrm{d}x^J + \mathrm{d}g\wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J &(5.3.18)\\
&= \mathrm{d}f \wedge \mathrm{d}x^I \wedge g\,\mathrm{d}x^J + (-1)^k f\,\mathrm{d}x^I \wedge \mathrm{d}g \wedge \mathrm{d}x^J &(5.3.19)\\
&= \mathrm{d}\omega\wedge\tau + (-1)^k\omega\wedge\mathrm{d}\tau &(5.3.20)
\end{aligned}$$

where we've used the Leibniz rule for the exterior derivative of 0-forms, which we know holds because it's just the derivative, and the anticommutativity of the wedge product.

- This follows from Schwarz's theorem, which states that partial derivatives commute. The proof is by induction on k, the base case is the 0-form, which is just a smooth function, f. We have

$$\begin{aligned}
\mathrm{d}^2 f &= \mathrm{d}\left(\sum_{i=1}^m \frac{\partial f}{\partial x^i}\mathrm{d}x^i\right) &(5.3.21)\\
&= \sum_{i=1}^m \mathrm{d}\frac{\partial f}{\partial x^i}\wedge \mathrm{d}x^i &(5.3.22)\\
&= \sum_{i=1}^m\sum_{j=1}^m \frac{\partial^2 f}{\partial x^j\,\partial x^i}\wedge \mathrm{d}x^j\wedge\mathrm{d}x^i. &(5.3.23)
\end{aligned}$$

This then vanishes because the derivative is symmetric in i and j while $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is antisymmetric in i and j.

Now, suppose that d^2 vanishes for all $k-1$ forms, and consider a k-form ω. By linearity it is again sufficient to prove this for a single term, so we may take

$$\omega = f\mathrm{d}x^I, \tag{5.3.24}$$

for some smooth function, f, and some fixed $I \in \mathcal{I}_{k,m}$. We then have

$$\mathrm{d}\omega = \mathrm{d}f\wedge\mathrm{d}x^I = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i. \tag{5.3.25}$$

Computing the second exterior derivative and using the graded Leibniz rule, and the fact that f is a zero form and $\mathrm{d}f$ is a 1-form, we have

$$\mathrm{d}^2\omega = \mathrm{d}(\mathrm{d}f \wedge \mathrm{d}x^I) \tag{5.3.26}$$

$$= \mathrm{d}^2 f \wedge \mathrm{d}x^I - \mathrm{d}f \wedge \mathrm{d}^2 x^I. \tag{5.3.27}$$

The first term vanishes because f is a 0-form, so this is the base case. The second term vanishes because we can use the graded Leibniz rule to write

$$\mathrm{d}^2 x^I = \mathrm{d}(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) \tag{5.3.28}$$

$$= \mathrm{d}^2 x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} - \mathrm{d}x^{i_1} \wedge \mathrm{d}^2 x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$
$$+ \cdots + (-1)^{k-1}\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}^2 x^{i_k} \tag{5.3.29}$$

and each $\mathrm{d}^2 x^i$ vanishes as x^i is a 0-form.

- First note that

$$\varphi^*(\mathrm{d}x^i) = \mathrm{d}(x^i \circ \varphi) \tag{5.3.30}$$

by definition of the pullback. Further, this map is $\mathrm{d}\varphi^i$, it projects out the ith component of the result of applying φ to the input. By linearity it is sufficient to show the desired result for a single term, $\omega = f\,\mathrm{d}x^I$. We then have

$$\varphi^*\mathrm{d}\omega = \varphi^*(\mathrm{d}f \wedge \mathrm{d}x^I) \tag{5.3.31}$$

$$= \mathrm{d}(f \circ \varphi) \wedge \mathrm{d}\varphi^I \tag{5.3.32}$$

where we've used the fat that pullbacks distribute over wedge products. This result is exactly the same thing that we get if we take the exterior derivative of

$$\varphi^*\omega = \varphi^*(f\,\mathrm{d}x^I) = ((f \circ \varphi)\mathrm{d}\varphi^I). \tag{5.3.33}$$

□

So far we've defined the exterior derivative only on open subsets of $\mathbb{R}^m$. It generalises to open subsets of a manifold by taking our open subsets of $\mathbb{R}^m$ to be charts for the manifold. Specifically, let $(U_\alpha, \varphi_\alpha)$ be a chart of some manifold, M, and $\omega \in \Omega^k(M)$ a differential form on M. Then we know that their is a corresponding differential form on $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$ such that

$$\omega|_{U_\alpha} = \varphi_\alpha^*\omega_\alpha. \tag{5.3.34}$$

We may define $\mathrm{d}\omega_\alpha$ as above. We can then use the fact that the exterior derivative commutes with the pullback to lift this definition to the manifold. Specifically we define $\mathrm{d}\omega$ to be the unique $(k+1)$-form on M satisfying

$$\mathrm{d}\omega|_{U_\alpha} = \varphi_\alpha^*\mathrm{d}\omega_\alpha = \mathrm{d}\varphi_\alpha^*\omega_\alpha. \tag{5.3.35}$$

To show that this uniquely defines $\mathrm{d}\omega$ we need to show that the definitions agree on the overlap of charts. Specifically, we know that ω_α satisfy

$$\omega_\alpha|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = (\varphi_\beta \circ \varphi_\alpha^{-1})^*\omega_\beta|_{\varphi_\beta(U_\alpha\cap U_\beta)} \tag{5.3.36}$$

and applying the exterior derivative to this we get

$$\mathrm{d}\omega_\alpha|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = \mathrm{d}(\varphi_\beta\circ\varphi_\alpha^{-1})^*\omega_\beta|_{\varphi_\beta(U_\alpha\cap U_\beta)} = (\varphi_\beta\circ\varphi_\alpha^{-1})\mathrm{d}\omega_\beta|_{\varphi_\beta(U_\alpha\cap U_\beta)} \tag{5.3.37}$$

and so $\mathrm{d}\omega_\alpha$ and $\mathrm{d}\omega_\beta$ agree on the overlap, so $\mathrm{d}\omega$ is uniquely defined.

The properties of the last lemma all lift to the manifold.

Lemma 5.3.38 — Exterior Derivatives Properties

- $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is linear;
- $\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^{\deg\omega}\omega \wedge \tau$;
- $\mathrm{d}^2 = 0$;
- $\varphi^*\mathrm{d}\omega = \mathrm{d}\varphi^*\omega$.

5.3.3 Vector Calculus

Now consider the case of $M = \mathbb{R}^3$. Consider a 0-form, that is a smooth function, $f : \mathbb{R}^3 \to \mathbb{R}$. We have

$$\mathrm{d}f = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} = \partial_1 f + \partial_2 f + \partial_3 f. \tag{5.3.39}$$

So, under the isomorphism of vector spaces $\Omega^1(M) \to \mathbb{R}^3$ given on the basis by $\mathrm{d}x^i \mapsto e_i$ this is exactly grad f.

Now consider a 1-form, $\omega = g_1\,\mathrm{d}x^1 + g_2\,\mathrm{d}x^2 + g_3\,\mathrm{d}x^3$, we have

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}g_1 \wedge \mathrm{d}x^1 + \mathrm{d}g_2 \wedge \mathrm{d}x^2 + \mathrm{d}g_3 \wedge \mathrm{d}x^3 && (5.3.40)\\
&= \sum_{i=1}^{3}\left(\frac{\partial g_1}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^1 + \frac{\partial g_2}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^2 + \frac{\partial g_3}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^3\right) && (5.3.41)\\
&= \frac{\partial g_1}{\partial x^1}\mathrm{d}x^1 \wedge \mathrm{d}x^1 + \frac{\partial g_1}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial g_1}{\partial x^3}\mathrm{d}x^3 \wedge \mathrm{d}x^1 && (5.3.42)\\
&+ \frac{\partial g_2}{\partial x^1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial g_2}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^2 + \frac{\partial g_2}{\partial x^3}\mathrm{d}x^3 \wedge \mathrm{d}x^2\\
&+ \frac{\partial g_3}{\partial x^1}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial g_3}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial g_3}{\partial x^3}\mathrm{d}x^3 \wedge \mathrm{d}x^3\\
&= -\frac{\partial g_1}{\partial x^2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \frac{\partial g_1}{\partial x^3}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial g_2}{\partial x^1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 && (5.3.43)\\
&- \frac{\partial g_2}{\partial x^3}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial g_3}{\partial x^1}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial g_3}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^3\\
&= (\partial_2 g_3 - \partial_3 g_2)\mathrm{d}x^2 \wedge \mathrm{d}x^3 + (\partial_3 g_1 - \partial_1 g_3)\mathrm{d}x^1 \wedge \mathrm{d}x^3 && (5.3.44)\\
&+ (\partial_1 g_2 - \partial_2 g_1)\mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned}$$

We see that under the isomorphism of vector spaces $\Omega^2(\mathbb{R}^3) \mapsto \mathbb{R}^3$ given on the basis by $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \mapsto e_3$, $\mathrm{d}x^1 \wedge \mathrm{d}x^3 \mapsto e_2$ and $\mathrm{d}x^2 \wedge \mathrm{d}x^3 \mapsto e_1$ we have exactly curl g where $g = (g_1, g_2, g_3) : \mathbb{R}^3 \to \mathbb{R}^3$.

Finally, consider a 2-form, $\tau = h_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + h_{23}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + h_{13}\,\mathrm{d}x^1 \wedge \mathrm{d}x^3$, we have

$$\mathrm{d}\tau = \mathrm{d}h_{12} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}h_{23} \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathrm{d}h_{13} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \tag{5.3.45}$$

$$= \sum_{i=1}^{3} \left(\frac{\partial h_{12}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial h_{23}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \right. \tag{5.3.46}$$

$$\left. + \frac{\partial h_{13}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \right) \tag{5.3.47}$$

$$= \frac{\partial h_{12}}{\partial x^3} \mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial h_{23}}{\partial x^1} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{5.3.48}$$

$$+ \frac{\partial h_{13}}{\partial x^2} \mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \tag{5.3.49}$$

$$= \left(\frac{\partial h_{12}}{\partial x^3} + \frac{\partial h_{23}}{\partial x^1} - \frac{\partial h_{13}}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{5.3.50}$$

We see that under the isomorphism of vector spaces $\Omega^3(M) \to \mathbb{R}$ given on the basis by $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = 1$ this is exactly div h where $h = (h_{12}, h_{13}, -h_{23}) : \mathbb{R}^3 \to \mathbb{R}^3$.

This suggests that really we should be thinking of grad, div and curl as all being the exterior derivative acting on forms of different degrees. This leads us to the following picture

$$\Omega^0(M) \xrightarrow{\mathrm{d}=\mathrm{grad}} \Omega^1(M) \xrightarrow{\mathrm{d}=\mathrm{curl}} \Omega^2(M) \xrightarrow{\mathrm{d}=\mathrm{div}} \Omega^3(M). \tag{5.3.51}$$

We can extend this picture to

$$0 \xrightarrow{\mathrm{d}} \Omega^0(M) \xrightarrow{\mathrm{d}=\mathrm{grad}} \Omega^1(M) \xrightarrow{\mathrm{d}=\mathrm{curl}} \Omega^2(M) \xrightarrow{\mathrm{d}=\mathrm{div}} \Omega^3(M) \xrightarrow{\mathrm{d}} 0 \tag{5.3.52}$$

by defining $\mathrm{d}0 = 0$ for an "(-1)-form", 0 and noting that the exterior derivative of a 3-form is a 4-form, which must therefore vanish since $\mathbb{R}^3$ is only three-dimensional.

This picture is our first example of a cochain complex. Later we'll generalise this to other manifolds than $\mathbb{R}^3$.

5.3.4 Lie Derivative

The exterior product gives a $(k+1)$-form from a k-form. It is then natural to ask if there is an operation that gives a k-form from a $(k+1)$-form, ω. The answer is yes, but it's not unique. Instead we get one such operation for each vector field, X, it's simply partially evaluating ω_p at X_p. Of course there are other options, such as which slot of ω we evaluate X in, but this is just a choice of sign, so it's natural to take this operation to always act on the first slot.

Definition 5.3.53 — Interior Product/Contraction Let $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$. The **interior product** of X with ω is defined to be the $(k-1)$-form

$\iota(X)\omega = \iota_X\omega$ given by

$$(\iota_X(\omega))_p(v_1, \dots, v_{k-1}) = \omega_p(X_p, v_1, \dots, v_{k-1}) \tag{5.3.54}$$

for $p \in M$ and $v_i \in T_pM$.

One question we might ask now is if there is some way to combine the interior and exterior product to give a notion of a derivative which doesn't change the degree of a form. The answer is yes, there is.

Definition 5.3.55 — Lie Derivative Let $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$. Then if X is complete and $\varphi_t \in \mathrm{Diffeo}(M)$ is the flow associated with X then the **Lie derivative** of ω along X is

$$\mathcal{L}_X\omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^*\omega. \tag{5.3.56}$$

If X is not complete we can still make this definition pointwise when restricting to regions on which the flow is defined.

The idea of this derivative is that we want to compute the usual derivative via the limit definition. If this worked we'd be looking at something of the form

$$\lim_{h\to 0} \frac{\omega_(p+h) - \omega_p}{h}. \tag{5.3.57}$$

There are a few problems with this. The first is that $p + h$ doesn't make sense, we cannot add a point $p \in M$ to some $h \in \mathbb{R}$. Ok, let's fix this by considering two points $p, p' \in M$ which are "close", then we want to consider

$$\lim_{h\to 0} \frac{\omega_p - \omega_{p'}}{h} \tag{5.3.58}$$

where h is the "distance" between the two points, p and p'. There's one further problem, ω_p lives in $\Lambda^k T_p^*M$ and ω_p' lives in $\Lambda^k T_{p'}^*M$. These are different spaces and we cannot subtract vectors in one from vectors in the other. This is where the flow φ_t comes in, this is a diffeomorphism $\varphi_t : M \to M$ which sends p to p' after "time" t, that is $\varphi_t(p) = p'$. Thus, $\varphi_t^* : \Lambda^k T_{p'}^*M \to \Lambda^k T_p^*M$, and thus we can replace ω_p with ω and $\omega_{p'}$ with $\varphi_t^*\omega$, both of which can be taken at $p \in M$ to give an element of $\Lambda^k T_p^*M$, and we can now take their difference. We have replaced h with t now as our parameter that we will take to be small, so we get the limit

$$\lim_{t\to 0} \frac{\omega_p - \varphi_t^*\omega_p}{t}. \tag{5.3.59}$$

This defines a k-form for each $p \in M$, so we can drop the p and take this to be an element of $\Omega^k(M)$. Then this limit is exactly the t derivative in the definition of the Lie derivative.

Our goal was to find a derivative that doesn't change the degree of a form. The Lie derivative is certainly a derivative, but does it change the degree of the form? The answer is no, it doesn't and we can see this through the following result.

Theorem 5.3.60 — Cartan's Magic Formula. For $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$ we have

$$\mathcal{L}_X\omega = \mathrm{d}\iota_X\omega + \iota_X\mathrm{d}\omega. \tag{5.3.61}$$

We see that the first term in Cartan's magic formula decreases the degree then increases it, and the second term does this in reverse. It turns out that the effect of doing both of these in this order is exactly the derivative.

Six

Integration on Manifolds

6.1 Defining Integration

Consider a manifold, M. For integration to make sense M has to be oriented. We won't go into details, and there are many equivalent definitions of oriented manifolds. The one that we'll work with is that the Jacobian determinant of the transition maps is positive. Examples of oriented manifolds include spheres, $\mathbb{R}^n$, S^n, and the cylinder, but not the Möbius loop. An equivalent definition is that there is a nowhere-vanishing volume form $\omega \in \Omega^m(M)$ for $m = \dim M$. Note that nowhere-vanishing means that the map ω_p is never the zero map for any $p \in M$, not that the map ω_p has no zeros, as opposed to being non-zero, which means that $\omega \neq 0$ as elements of $\Omega^m(M)$.

The standard notation for an integral,

$$\int f(x)\,\mathrm{d}x, \tag{6.1.1}$$

informs our guess as to how an integral should be defined on a manifold. The key idea here is that this is a 1-dimensional integral over $\mathbb{R}$ and we're integrating a 1-form, hence the $\mathrm{d}x$. The correct generalisation turns out to be that we should always integrate a top-form, that is an m-form.

We will perform the integration by mapping to $\mathbb{R}^m$ (or $\mathbb{H}^m$ if our manifold has a boundary). We can then integrate normally here. There are a few subtleties to this though. First, in order to get a finite result we should only consider integrals of smooth differential top-forms with compact support. Recall that the support, $\operatorname{supp} f$, of a function, f, (including forms) is the subset of the domain on which the function takes a non-zero value. We write $\Omega^k_{\mathrm{c}}M) \subseteq \Omega^k(M)$ for the set of k-forms on M with compact support.

Suppose that $U \subseteq M$ is an open set region we wish to integrate over, and that $U \subseteq U_\alpha \cap U_\beta$. How should we map ω to be a differential form on $\mathbb{R}^m$ that we can integrate? We could map it under φ_α to ω_α or under φ_β to ω_β, and there's no reason to prefer one over the other. Actually, it turns out that the correct thing to do is some mix of these.

> **Definition 6.1.2 — Partition of Unity** Let M be a manifold and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of M. That is, $U_\alpha \subseteq M$ is open and $\bigcup_{\alpha\in A} U_\alpha = M$. A **partition of unity** subordinate to $\mathcal{U}$ is a collection of smooth maps

$\{\rho_\alpha : U_\alpha \to [0,1]\}_{\alpha\in A}$ such that on any U_α only finitely many of the ρ_α are nonzero and

$$\sum_{\alpha\in A} \rho_\alpha = 1 \tag{6.1.3}$$

and

$$\operatorname{supp} \rho_\alpha \subseteq U_\alpha. \tag{6.1.4}$$

The idea is that a partition of unity gives a weighting to each set in the open cover in a smooth way and we can use this to weight which of ω_α and ω_β we choose on the overlap. We can now define the integral on a manifold.

Definition 6.1.5 — Integrals on a Manifold Let M be a manifold with chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and the corresponding open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. Let $\{\rho_\alpha\}_{\alpha\in A}$ be a partition of unit subordinate to $\mathcal{U}$. Let $\omega \in \Omega^m_{\mathrm{c}}(M)$ be a top-form.
Then, for $U_\alpha \in \mathcal{U}$ we can work in local coordinates to get $\omega_\alpha \in \Omega^m(\varphi_\alpha(U_\alpha))$. This is such that $\omega|_{U_\alpha} = \varphi_\alpha^* \omega_\alpha$. Expanding ω_α in the standard basis we have $\omega_\alpha = g_\alpha \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ where $g_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is a smooth function.
The integral of ω over M is the real number

$$\int_M \omega := \sum_{\alpha\in A} \int_{\varphi_\alpha(U_\alpha)} \rho_\alpha(\varphi_\alpha^{-1}(x)) g_\alpha(x) \, \mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{6.1.6}$$

Remark 6.1.7

- The sum on the right of the definition of the integral is finite because only finitely many of the products $\rho_\alpha \omega$ are nonzero.
- We *do not* write dx on the left, the “d” which is standard in the notation for an integral is “hidden” inside the ω.
- The integral forms a functional

$$\Omega^c_{\mathrm{c}}(M) \to \mathbb{R} \tag{6.1.8}$$

$$\omega \mapsto \int_M \omega. \tag{6.1.9}$$

Linearity follows from the fact that everything making up the definition of the integral is linear.

Proposition 6.1.10 The value of the integral of some compactly-supported top-form, ω, over a manifold, M, is independent of the choice of oriented atlas and partition of unity.

Proof. Suppose that we have a second oriented atlas, $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$, and a corresponding partition of unity $\{\theta_\beta\}_{\beta\in B}$. Then for $\beta \in B$ to evaluate the integral we need to define $\omega_\beta \in \Omega^m(\psi_\beta(V_\beta))$, such that $\omega|_{V_\beta} = \psi_\beta^*\omega_\beta$. Expanding in the basis we have $\omega_\beta = h_\beta \, \mathrm{d}y^1 \wedge \dots \wedge \mathrm{d}y^m$. Then, we have

$$g_\alpha(x) = h_\beta(\psi_\beta \circ \varphi_\alpha^{-1}(x)) \det(\mathrm{d}(\psi_\beta \circ \varphi_\alpha^{-1})(x)) \tag{6.1.11}$$

for all $x \in \varphi_\alpha(U_\alpha \cap V_\beta)$. This determinant is always positive, since we've chosen oriented atlases. We then have

$$\int_M \omega = \sum_{\alpha\in A} \int_{\varphi_\alpha(U_\alpha)} (\rho_\alpha \circ \varphi_\alpha^{-1}) g_\alpha \, \mathrm{d}x^1 \cdots \mathrm{d}x^m \tag{6.1.12}$$

$$= \sum_{\alpha\in A} \sum_{\beta\in B} \int_{\varphi_\alpha(U_\alpha\cap V_\beta)} (\rho_\alpha \circ \varphi_\alpha^{-1})(\theta_\beta \circ \varphi_\alpha^{-1}) g_\alpha \, \mathrm{d}x^1 \cdots \mathrm{d}x^m \tag{6.1.13}$$

$$= \sum_{\alpha\in A} \sum_{\beta\in B} \int_{\psi_\beta(U_\alpha\cap V_\beta)} (\rho_\alpha \circ \psi_\beta^{-1})(\theta_\beta \circ \varphi_\alpha^{-1}) h_b \, \mathrm{d}y^1 \cdots \mathrm{d}y^m \tag{6.1.14}$$

$$= \sum_{\beta\in B} \int_{\psi_\beta(V_\beta)} (\theta_\beta \circ \psi_\beta^{-1}) h_\beta \, \mathrm{d}y^1 \cdots \mathrm{d}y^m. \tag{6.1.15}$$

This is exactly the integral if we'd started with the second atlas. The first second equation follows from the fact that the θ_β form a partition of unity, so actually their sum is just 1, the third equation follows from the expression for g_α in terms of h_β, and the change of variables formula, then the last equation from the fact that ρ_α is a partition of unity. □

6.2 Stokes' Theorem

Theorem 6.2.1 — Stokes. Let M be an oriented manifold of dimension m, potentially with boundary. Let $\omega \in \Omega_c^{m-1}(M)$. Then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{6.2.2}$$

The proof of the above is somewhat long, but essentially comes down to using the version of Stokes' theorem that we all learn in vector calculus with $\mathbb{H}^m$ as our region of integration and then mapping all cases to this one.

Seven

De Rham Cohomology