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Notes from

Conformal Field Theory and Vertex Operator Algebras

September 26th, 2024

SMSTC

Conformal Field Theory and Vertex Operator Algebras

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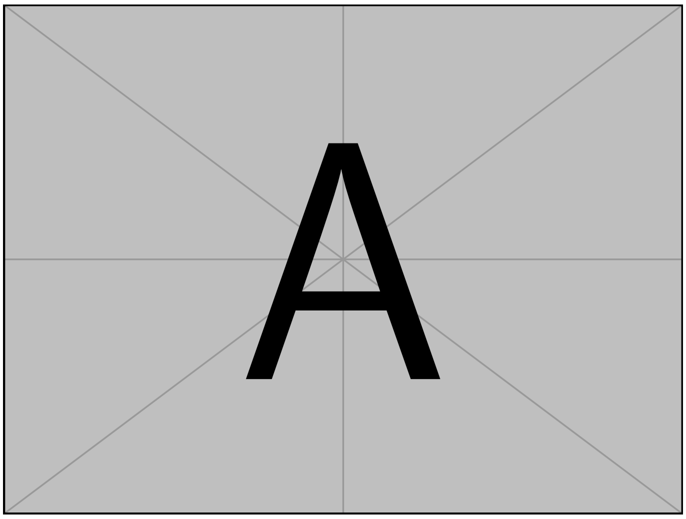
These are my notes from the SMSTC course *Conformal Field Theories and Vertex Operator Algebras* taught by Dr Anatoly Konechny. These notes were last updated at 14:39 on October 8, 2024.



Chapters

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Conformal Geometry

1.1 Local Conformal Maps

Intuitively, we want conformal maps to preserve angles, but not necessarily distances. To do so consider how the angle between two vectors in, say, $\mathbb{R}^3$ is computed,

$$\cos(\theta_{\boldsymbol{u},\boldsymbol{v}}) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}. \tag{1.1.1}$$

We see here that if each vector were made longer by some positive constant, ρ, then we have

$$\cos(\theta_{\rho\boldsymbol{u},\rho\boldsymbol{v}}) \frac{\rho\boldsymbol{u} \cdot \rho\boldsymbol{v}}{\|\rho\boldsymbol{u}\|\|\rho\boldsymbol{v}\|} = \cos(\theta_{\boldsymbol{u},\boldsymbol{v}}). \tag{1.1.2}$$

We can think of scaling all of the vectors here by ρ as the same as scaling the metric by $1/\rho$. It's $1/\rho$ because if we make the units of a measurement smaller then the number we measure gets bigger (hence, *contravariant* vectors).

The following is really just fancy differential-geometry-speak for this rescaling of the metric, and we also allow the scaling of the metric to depend on position.

Definition 1.1.3 — Local Conformal Map Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be n-dimensional Riemannian manifolds. Let $U_1 \subseteq \mathcal{M}_1$ and $U_2 \subseteq \mathcal{M}_2$ be open subsets. A (local) **conformal transformation** is a smooth, injective map, $\varphi : U_1 \to U_2$, satisfying the pullback condition

$$\varphi^* g_2 = \Lambda g_1 \tag{1.1.4}$$

for some function $\Lambda : U_1 \to \mathbb{R}_{>0}$.
A conformal map defined on all of $\mathcal{M}_1$ is a global conformal map.

Remark 1.1.5 It is possible to relax the conditions on φ, and require only that it is differentiable. However, requiring smoothness and injectivity is common when it comes to applications, so we make it a basic requirement. It's also common to further restrict to orientation-preserving maps, but we won't do that just yet.

We can express the pullback condition, $\varphi^* g_2 = \Lambda g_1$, in local coordinates. Let $x = (x^1, \dots, x^n)$ be coordinates covering U_1, and $y = (y^1, \dots, y^n)$ coordinates covering U_2. Then φ is fully specified by the functions φ^i which are defined such that $y^i = \varphi^i(x)$. The metrics, g_i, may be specified by their components, $(g_i)_{jk} : U_i \to \mathbb{R}$. In these coordinates the pullback condition becomes

$$\sum_{k,l} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j} (g_2)_{kl}(\varphi(x)) = \Lambda(x)(g_1)_{ij}(x). \tag{1.1.6}$$

Taking determinants of either side of this equation we have

$$\det\left(\frac{\partial \varphi^k}{\partial x^i}\right) \det(g_2) \det\left(\frac{\partial \varphi^l}{\partial x^j}\right) = \Lambda^n \det(g_1). \tag{1.1.7}$$

Now, $\Lambda^n \neq 0$ and $\det(g_i) \neq 0$, so it follows that $\det(\partial\varphi^k/\partial x^i) \neq 0$, meaning that the matrix with components $\partial\varphi^k/\partial x^i$ is invertible. This means that a conformal map is always **locally invertible**. That is, for any $p \in U_1$ we have a neighbourhood $V_1 \subseteq U_1$ with $p \in V_1$ such that φ restricted to V_1 is a bijection.

Note that it's possible to be locally invertible but not fully invertible. There may be a point in $U_1 \setminus V_1$ which maps to the same point as a point in V_1, so the function will not be injective.

1.1.1 Conformal Maps Preserve Angles

Consider two vectors $u, v \in T_p\mathcal{M}_1$ and some $p \in \mathcal{M}_1$. Taking some open neighbourhood of p, $U_1 \subseteq \mathcal{M}_1$, we can also take coordinates $x = (x^1, \dots, x^n)$ covering U_1. This gives a basis $\{\partial/\partial x^i|_p\}$ for $T_p\mathcal{M}_1$. The angle between these vectors is given, as in $\mathbb{R}^3$, by

$$\cos(\theta_{u,v}) = \frac{(u,v)}{\|u\|\|v\|} = \frac{u^i (g_1)_{ij} v^j}{\sqrt{u^l (g_1)_{lk} u^k v^p (g_1)_{pq} v^q}}. \tag{1.1.8}$$

Here we've started to employ the Einstein summation convention, and we shall do so from now on. Let $\varphi : U_1 \to U_2$ be a conformal transformation and suppose that U_2 is covered by coordinates $y = (y^1, \dots, y^n)$. Consider the pushforward

$$\mathrm{d}\varphi_p : T_p\mathcal{M}_1 \to T_{\varphi(p)}\mathcal{M}_2. \tag{1.1.9}$$

Under this the vectors u and v map to the vectors

$$\tilde{u} = \mathrm{d}\varphi_p(u), \qquad \text{and} \qquad \tilde{v} = \mathrm{d}\varphi_p(v), \tag{1.1.10}$$

which have coordinates

$$\tilde{u}^i = \frac{\partial \varphi^i}{\partial x^j} u^j, \qquad \text{and} \qquad \tilde{v}^i = \frac{\partial \varphi^i}{\partial x^j} v^j. \tag{1.1.11}$$

We can now calculate the angle between these vectors as follows:

$$\begin{aligned}
\cos(\theta_{\tilde{u},\tilde{v}}) &= \frac{(\tilde{u},\tilde{v})}{\|u\|\|v\|} && (1.1.12)\\
&= \frac{\tilde{u}^a (g_2)_{ab} \tilde{v}^b}{\sqrt{\tilde{u}^c (g_2)_{cd} \tilde{u}^d \tilde{v}^e (g_2)_{ef} \tilde{v}^f}} && (1.1.13)\\
&= \frac{\frac{\partial \varphi^a}{\partial x^i} u^i (g_2)_{ab} \frac{\partial \varphi^b}{\partial x^j} u^j}{\sqrt{\frac{\partial \varphi^c}{\partial x^l} u^l (g_2)_{cd} \frac{\partial \varphi^d}{\partial x^k} u^k \frac{\partial \varphi^e}{\partial x^p} v^p (g_2)_{ef} \frac{\partial \varphi^f}{\partial x^q}}} && (1.1.14)\\
&= \frac{u^i \Lambda (g_1)_{ij} v^j}{\sqrt{u^l \Lambda (g_1)_{lk} u^k v^p \Lambda (g_1)_{pq} v^q}} && (1.1.15)\\
&= \frac{u^i (g_1)_{ij} v^j}{\sqrt{u^l (g_1)_{lk} u^k v^p (g_1)_{pq} v^q}} && (1.1.16)\\
&= \cos(\theta_{u,v}) && (1.1.17)
\end{aligned}$$

where we've used the pullback condition

$$\frac{\partial \varphi^a}{\partial x^i} (g_2)_{ab} \frac{\partial \varphi^b}{\partial x^j} = \Lambda (g_1)_{ij}. \qquad (1.1.18)$$

This shows that conformal transformations really do preserve angles as we were looking for.

Appendices

A

Differential Geometry

A.1 Tangent Space

Let $\mathcal{M}$ be a d-dimensional manifold. The tangent space at $p \in M$ is a d-dimensional vector space $T_p\mathcal{M}$. One definition of this is the vector space of derivations at p, where a derivation is a linear map $D\colon C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying

$$D(fg) = D(f)g(x) + f(x)D(g). \tag{A.1.1}$$

Clearly derivatives are derivations, this is just the product rule, and in fact given a coordinate chart (U, x) with $p \in U$ and $x = (x^1, \dots, x^d)$ we have a basis for $T_p\mathcal{M}$ given by

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^d}\bigg|_p \right\}. \tag{A.1.2}$$

Once we have tangent spaces it makes sense to consider the collection of all tangent vectors at any point $p \in \mathcal{M}$. This gives us the tangent bundle

$$TM = \bigsqcup_{p\in\mathcal{M}} T_pM. \tag{A.1.3}$$

This is a bundle since we have the natural projection $\pi\colon TM \twoheadrightarrow \mathcal{M}$ sending a tangent vector $v \in T_pM$ to the point $p \in M$.

A.2 Pushforward and Pullback

Let $\varphi\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds. The **pushforward**, also called the **differential**, of φ at $p \in \mathcal{M}$ is the linear map

$$\mathrm{d}\varphi_p\colon T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N} \tag{A.2.1}$$

defined to act on a derivation, $X\colon C^\infty(\mathcal{M}) \to \mathbb{R}$, by sending it to the derivation $\mathrm{d}\varphi_p(X)\colon C^\infty(\mathcal{N}) \to \mathbb{R}$ defined to act on $f \in C^\infty(\mathcal{N})$ by

$$\mathrm{d}\varphi_p(X)(f) = X(f \circ \varphi). \tag{A.2.2}$$

That is, $\mathrm{d}\varphi_p$ is nothing but precomposition with φ followed by evaluation.

Fix charts (U, x) and V, y for neighbourhoods of $p \in \mathcal{M}$ and $\varphi(p) \in \mathcal{N}$. s Then $T_p\mathcal{M}$ and $T_{\varphi(p)}\mathcal{N}$ have bases $\{\partial/\partial x^i|_p\}$ and $\{\partial/\partial y^i|_{\varphi(p)}\}$. In these bases $\mathrm{d}\varphi_p$ may be expressed as a matrix

$$(\mathrm{d}\varphi_p)^i{}_j = \frac{\partial \varphi^i}{\partial x^j} \tag{A.2.3}$$

where φ^j is such that $y^j = \varphi(x^j)$.

The **pullback** of φ is the map $\varphi^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$ defined by $(\varphi^* f)(x) = f(\varphi(x))$. That is, φ^* is precomposition with φ. We can also define the pullback of a k-form, ω, as

$$(\varphi^*\omega)_p(X_1, \dots, X_k) = \omega_{\varphi(p)}(\mathrm{d}\varphi_p(X_1), \dots, \mathrm{d}\varphi_p(X_k)). \tag{A.2.4}$$

This will be particularly important for a 2-form, g, where we have

$$(\varphi^* g)_p(X_1, X_2) = g_{\varphi(p)}(\mathrm{d}\varphi_p(X_1), \mathrm{d}\varphi_p(X_2)). \tag{A.2.5}$$

A.3 Riemannian Manifolds

A **Riemannian manifold**, $(\mathcal{M}, g)$, is a manifold, $\mathcal{M}$, equipped with a **Riemannian metric**, g, which assigns to each tangent space, $T_p\mathcal{M}$, a positive-definite inner product

$$g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}, \tag{A.3.1}$$

such that the component functions, $g_{ij} : U \to \mathbb{R}$, are smooth on any chart (U, x). These components are defined for a basis $\{e_i\}$ of $T_p\mathcal{M}$ by

$$g_{ij} = g_p(e_i, e_j). \tag{A.3.2}$$

These are such that

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \tag{A.3.3}$$

where $\mathrm{d}x^i$ is the dual basis to $\{e_i\}$, defined by $\mathrm{d}x^i(e_j) = \delta^i{}_j$.