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Notes from

Representation Theory

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University of Glasgow

Representation Theory

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These are my notes from the SMSTC course *Lie Theory* taught by Prof Christian Korff. These notes were last updated at 13:05 on June 5, 2025.

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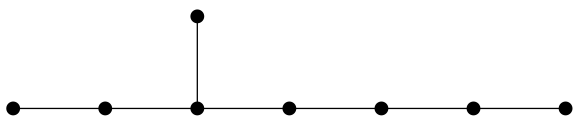

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Lie Algebras

In this section we give a rapid, relatively proof free, tour of the representation theory of Lie algebras. We refer the reader to other sources for details, such as my lecture notes https://github.com/WilloughbySeago/phd-courses-notes/tree/main/lie-theory.

9.1 Lie Algebras

Definition 9.1.1 — Lie Algebra A **Lie algebra**, $\mathfrak{g}$, is a $\Bbbk$-vector space equipped with a linear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ called the **Lie bracket** subject to the following:

- **alternativity**: $[x, x] = 0$ for all $x \in \mathfrak{g}$;
- **Jacobi identity**: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Note that more commonly the definition is given as a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The universal property of the tensor product means that these are equivalent. For fields of characteristic other than 2 the first relation is usually replaced with antisymmetry, $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. With our definition using the tensor product we can pass to the quotient $\Lambda^2\mathfrak{g}$ and we see that $[-, -]$ induces a map $[-, -] \colon \Lambda^2\mathfrak{g} \to \mathfrak{g}$ which trivially is such that $[x, x] = 0$ since $x \otimes x$ maps to zero in $\Lambda^2\mathfrak{g}$.

Definition 9.1.2 Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over the same field, $\Bbbk$. A morphism of Lie algebras, $\varphi \colon \mathfrak{g} \to \mathfrak{g}'$ is a linear map which preserves the Lie bracket, that is

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \tag{9.1.3}$$

where the bracket on the left is that of $\mathfrak{g}$ and on the right it's that of $\mathfrak{g}'$.

Example 9.1.4

- Let A be an associative algebra, then we can make this into a Lie algebra by defining the bracket $[a, b] = ab - ba$. A special case of this is $A = \operatorname{End} V$ for some vector space, V. Then we call the corresponding Lie algebra $\mathfrak{gl}(V)$, or if $\dim V = n$ we call it $\mathfrak{gl}_n$ (note that as vector spaces $\mathfrak{gl}(V)$ is exactly $A = \operatorname{End} V$, the name change just reflects a shifting view point from associative algebras to Lie algebras).

- Any vector space, V, can be made into a Lie algebra by defining $[x, y] = 0$ for all $x, y \in V$. Such a Lie algebra is called **abelian**. The idea is that the commutator vanishing means that multiplication is commutative, an idea that only makes sense if $[-, -]$ really is the commutator, like in the previous example.

Definition 9.1.5 — Lie Subalgebra Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. A Lie subalgebra, $\mathfrak{h}$, is a Lie algebra over $\Bbbk$ equipped with an injective Lie algebra morphism $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

An almost identical definition is that a Lie subalgebra is a subspace, $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{h}$ is a Lie algebra in its own right (with the same bracket as $\mathfrak{g}$). One can then show that this is true so long as the $\mathfrak{h}$ is closed under the Lie bracket. That is, $[\mathfrak{h}, \mathfrak{h}]$ is a subset of $\mathfrak{h}$. Note that in general if U and V are subspaces of $\mathfrak{g}$ then $[U, V]$ is defined to be the span of all $[u, v]$ with $u \in U$ and $v \in V$. Similarly, if $x \in \mathfrak{g}$ then $[x, U]$ is the span of all $[x, y]$ with $y \in \mathfrak{g}$.

The only subtle difference between these two definitions is that the existence of a monomorphism $\mathfrak{h} \hookrightarrow \mathfrak{g}$ only implies that $\mathfrak{h}$ is isomorphic to a subalgebra of $\mathfrak{g}$ with the second definition, but we'll only consider things up to isomorphism most the time so this is really the definition we want.

Example 9.1.6

- Let $\mathfrak{g}$ be any Lie algebra. Any one-dimensional subspace, $\mathfrak{l}$, is an abelian subalgebra, since if $l, l' \in \mathfrak{l}$ then $l = \lambda l'$ for some $\lambda \in \Bbbk$, and so $[l, l'] = [kl', l'] = k[l', l'] = 0$ and $0 \in \mathfrak{l}$.
- The **centre** of a Lie algebra, $\mathfrak{g}$, is the abelian subalgebra

$$\mathfrak{z}(\mathfrak{g}) \coloneqq \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\} \subseteq \mathfrak{g}. \tag{9.1.7}$$

- For V a finite-dimensional vector space of dimension n we know that $\mathfrak{gl}_n = \operatorname{End} V$ is a Lie algebra. Fixing a basis the elements of $\mathfrak{gl}_n$ are just all $n \times n$ matrices with entries in $\Bbbk$. There is a subalgebra, $\mathfrak{sl}_n \subset \mathfrak{gl}_n$, consisting of only the matrices with zero trace. This follows because we have

$$\operatorname{tr}([x, y]) = \operatorname{tr}(xy) - \operatorname{tr}(yx) = 0. \tag{9.1.8}$$

This holds for all $x, y \in \mathfrak{gl}_n$, not just for the traceless case, and so this turns out to be a special case of another construction, called the derived subalgebra, $\mathfrak{g}' \coloneqq [\mathfrak{g}, \mathfrak{g}]$.

Definition 9.1.9 — Ideal Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra, $\mathfrak{i} \subseteq \mathfrak{g}$, is an **ideal** if $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$.

Compare this to the definition of a subalgebra, which only requires that $[\mathfrak{i}, \mathfrak{i}] \subseteq \mathfrak{i}$. Compare this also to the notion of an ideal, I, of a ring, R, which is a subgroup of the additive group such that $IR \subseteq I$.

The idea is that ideals are to Lie algebras as ideals are to rings, or as normal subgroups are to groups. In particular, we have a correspondence between ideals, $\mathfrak{i} \subseteq g$ and Lie algebra morphisms, $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ given by $\mathfrak{i} \leftrightarrow \ker \varphi$ (where the kernel is defined as it is for any linear map). We also have that $\mathfrak{g}/\mathfrak{i}$ is a well defined quotient and a Lie algebra. Note that the quotient of any vector space by a subspace is again a vector space, but it's only a Lie algebra again if we quotient by an ideal. The bracket of this quotient is defined by $[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}$.

Definition 9.1.10 — Derived Subalgebra Let $\mathfrak{g}$ be a Lie algebra, then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the **derived subalgebra**.

Definition 9.1.11 — Solvable Lie Algebra A Lie algebra, $\mathfrak{g}$, is solvable if the series

$$\mathfrak{g} \supseteq \mathfrak{g}' \supseteq \mathfrak{g}'' \supseteq \cdots \tag{9.1.12}$$

terminates.

Definition 9.1.13 — Nilpotent Lie ALgebra A Lie algebra, $\mathfrak{g}$, is solvable if the series

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots \tag{9.1.14}$$

terminates.

The difference between these two is subtle, one nests brackets on both sides, and the other only on the other side. More concretely, the upper triangular matrices form a solvable subalgebra of $\mathfrak{gl}_n$ (in fact, this is a maximal solvable subalgebra, also known as a **Borel subalgebra**), and the *strictly* upper triangular matrices form a (maximal) nilpotent subalgebra of $\mathfrak{gl}_n$.

Definition 9.1.15 The maximal solvable *ideal* of $\mathfrak{g}$ is called its **radical**, $\operatorname{Rad} \mathfrak{g}$.

Definition 9.1.16 A Lie algebra, $\mathfrak{g}$, is **semisimple** if $\operatorname{Rad} \mathfrak{g} = 0$, that is, if $\mathfrak{g}$ has no proper solvable ideals. Similarly, $\mathfrak{g}$ is **simple** if it has no proper ideals (solvable or not).

Definition 9.1.17 — Linear Lie Algebra A **linear Lie algebra** is any Lie algebra which is isomorphic to a Lie subalgebra of some $\mathfrak{gl}(V)$ for V a finite-dimensional vector space.

Ado's theorem tells us that (over a field of characteristic zero) every finite-dimensional Lie algebra is linear.

Theorem 9.1.18 — Ado's Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ admits a faithful representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ for some finite-dimensional vector space, V. Further, one can choose this representation such that the maximal nilpotent ideal, $\mathfrak{n} \subseteq \mathfrak{g}$ acts nilpotently on V.

There are some special linear Lie algebras. Over $\mathbb{C}$ these are

- $\mathfrak{gl}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}\}$ (real dimension $2n^2$)
- $\mathfrak{sl}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) \mid \operatorname{tr} x = 0\}$ (real dimension $2(n^2-1)$);
- $\mathfrak{so}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) \mid x^{\top} + x^{\top} = 0\}$ (real dimension $n(n-1)$);
- $\mathfrak{sp}_{2n} = \{x \in \mathrm{Mat}_{2n}(\mathbb{C}) \mid Jx + x^{\top}J = 0\}$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with $I_n \in \mathrm{Mat}_n(\mathbb{C})$ the identity matrix (real dimension $2\binom{2n+1}{2}$).

Over $\mathbb{R}$ these are

- $\mathfrak{gl}_n = \{x \in \mathrm{Mat}_n(\mathbb{R})\}$ (real dimension n^2);
- $\mathfrak{so}_n = \{x \in \mathrm{Mat}_n(\mathbb{R}) \mid \operatorname{tr} x = 0\}$ (real dimension n^2-1);
- $\mathfrak{u}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) \mid x + x^* = 0\}$ (real dimension n^2);
- $\mathfrak{su}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) \mid x + x^* = 0 \text{ and } \operatorname{tr} x = 0\}$ (real dimension n^2-1);
- $\mathfrak{sp}_{2n} = \{x \in \mathrm{Mat}_n(\mathbb{H}) \mid x + x^* = 0\}$ (real dimension $2n^2+n$).

9.2 Representation Theory of Lie Algebras

Definition 9.2.1 — Representation A **representation**, $\mathfrak{g}$ (over $\Bbbk$), is a $\Bbbk$-vector space, V, equipped with a Lie algebra morphism

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V). \tag{9.2.2}$$

Equivalently, a $\mathfrak{g}$-**module**, V, is a vector space equipped with a (left) Lie algebra action of $\mathfrak{g}$, that is, a map $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \,.\, v$ subject to the following:

- Linearity in the first argument: $(\alpha x + \beta y) \,.\, v = \alpha(x \,.\, v) + \beta(y \,.\, v)$ for all $\alpha, \beta \in \Bbbk$, $x, y \in \mathfrak{g}$ and $v \in V$;
- Linearity in the second argument: $x \,.\, (\alpha v + \beta w) = \alpha(x \,.\, v) + \beta(x \,.\, w)$ for all $\alpha, \beta \in \Bbbk$, $x \in \mathfrak{g}$ and $v, w \in V$;

- Respects the bracket: $[x,y] . v = x . (y . v) - y . (x . v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$.

As with groups and associative algebras the $\mathfrak{g}$-module and representation of $\mathfrak{g}$ carry exactly the same information, and as such which we use is a matter of preference.

Definition 9.2.3 — Adjoint Representation Every Lie algebra, $\mathfrak{g}$, is a $\mathfrak{g}$-module in a canonical way, known as the **adjoint representation**

$$\begin{aligned} \mathrm{ad} \colon \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}_x \end{aligned} \tag{9.2.4}$$

where $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_x(y) = [x,y]$ for all $x, y \in \mathfrak{g}$.

For the adjoint representation to be a representation we need ad to be a Lie algebra morphism. That is, we need to have $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$ for $x, y \in \mathfrak{g}$. It turns out that this is true precisely because the this statement, upon applying both sides of the above to $z \in \mathfrak{g}$, expands to the Jacobi identity:

$$\mathrm{ad}_{[x,y]}(z) = [[x,y],z] \tag{9.2.5}$$

$$[\mathrm{ad}_x, \mathrm{ad}_y](z) = (\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z) = [x,[y,z]] - [y,[x,z]]. \tag{9.2.6}$$

Equality between the two lines above is, after applying the antisymmetry property, exactly the Jacobi identity.

Definition 9.2.7 Given $\mathfrak{g}$-modules V and W we can define

- the **direct sum**, $V \oplus W$, which has the action $x . (v+w) = x . v + x . w$;
- the **tensor product**, $V \otimes W$, which has the action $x . (v \otimes w) = (x . v) \otimes w + v \otimes (x . w)$;
- the **dual representation**, V^*, which has the action $\rho_{V^*}(x) = -\rho_V(x)^*$

all for $x \in \mathfrak{g}$, $v \in V$, and $w \in W$.

9.3 Universal Enveloping Algebra

Definition 9.3.1 — Universal Enveloping Algebra Let $\mathfrak{g}$ be a Lie algebra. An enveloping algebra, (E, i), is an associative unital algebra, E, and an inclusion of vector spaces $i \colon \mathfrak{g} \hookrightarrow E$ such that

$$i([x,y]) = i(x)i(y) - i(y)i(x). \tag{9.3.2}$$

The **universal enveloping algebra** is the[a] enveloping algebra $(U(\mathfrak{g}), \iota)$ such that for any other enveloping algebra, (E, i), there is a unique morphism of associative unital algebras, $\varphi : U(\mathfrak{g}) \to E$ such that $i = \varphi \circ \iota$.

[a]turns out that the universal enveloping algebra both exists, and is unique up to unique isomorphism

The definition is a bit terse, the idea is that $U(\mathfrak{g})$ (dropping ι from the notation) is the smallest associative unital algebra containing $\mathfrak{g}$ in such a way that the bracket of $\mathfrak{g}$ in $U(\mathfrak{g})$ really is just the commutator. For example, the universal enveloping algebra of $\mathfrak{gl}(V)$ is simply $\mathrm{End}(V)$, which is just $\mathfrak{gl}(V)$ but viewed as an associative algebra.

Theorem 9.3.3. The universal enveloping algebra exists. An explicit construction is as follows. Let $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the ideal of the tensor algebra, $T(\mathfrak{g})$, generated by elements of the form

$$[x, y] - x \otimes y + y \otimes x \tag{9.3.4}$$

for $x, y \in \mathfrak{g}$.

The universal property of the universal enveloping algebra can be characterised as the statement that there is an isomorphism

$$\mathrm{Hom}_{\Bbbk\text{-}\mathsf{Lie}}(\mathfrak{g}, L(A)) \cong \mathrm{Hom}_{\Bbbk\text{-}\mathsf{Alg}}(U(\mathfrak{g}), A) \tag{9.3.5}$$

where

- $\Bbbk$-Lie is the category of Lie algebras and Lie algebra homomorphisms;
- $\mathfrak{g}$ is a Lie algebra
- A is an unital associative algebra;
- $L(A)$ is the Lie algebra given by equipping A with the commutator;
- $\Bbbk$-Alg is the category of unital associative algebras and their homomorphisms.

Simply send the Lie algebra homomorphism $\varphi : \mathfrak{g} \to L(A)$ to the associative algebra homomorphism $\tilde{\varphi} : U(\mathfrak{g}) \to A$ defined by $\tilde{\varphi}(x) = \varphi(x)$ for $x \in \mathfrak{g}$ and extended by linearity and the requirement that $\tilde{\varphi}$ preserves multiplication. This works precisely because of the universal property. For the inverse, send $\psi : U(\mathfrak{g}) \to A$ to the restriction $\psi|_{\mathfrak{g}}$.

It turns out that $L : \Bbbk\text{-}\mathsf{Alg} \to \Bbbk\text{-}\mathsf{Lie}$ is a functor, if $f : A \to B$ is a morphism of associative algebras then we can define $L(f) : L(A) \to L(B)$ by defining $L(f)([x, y]) = [f(x), f(y)] = f(x)f(y) - f(y)f(x)$ for $x, y \in A$. That is, we just require that $L(f)$ is a Lie algebra homomorphism. Similarly, $U : \Bbbk\text{-}\mathsf{Lie} \to \Bbbk\text{-}\mathsf{Alg}$ is a functor, if $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras then we can define $U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$ by defining $U(f)(xy) = U(f)(x)U(f)(y)$ for $x, y \in \mathfrak{g}$ and similarly for products of more than two elements, and extended by linearity to all of $U(\mathfrak{g})$. That is, we just require that $U(f)$ respects the multiplication of the associative algebra. Then the above isomorphism happens to be natural, and we thus have that L is right adjoint to U.

The important thing here is that if we take $A = \operatorname{End} V$ then we have

$$\operatorname{Hom}_{\Bbbk\text{-Lie}}(\mathfrak{g}, \mathfrak{gl}(V)) \cong \operatorname{Hom}_{\Bbbk\text{-Alg}}(U(\mathfrak{g}), \operatorname{End} V). \tag{9.3.6}$$

This means that a map $\mathfrak{g} \to \mathfrak{gl}(V)$ carries the same data as a map $U(\mathfrak{g}) \to \operatorname{End} V$. We can identify a map of the first type as a Lie algebra representation of $\mathfrak{g}$, and a map of the second type as a unital associative algebra representation of $U(\mathfrak{g})$. That is, representations of $\mathfrak{g}$ are "the same" as representations of $U(\mathfrak{g})$.

Another way of thinking about this is that $U(\mathfrak{g})$ is to $\mathfrak{g}$ as $\Bbbk G$ is to G for a finite group, G. We can study the representation theory of $\mathfrak{g}$ or G just by studying the representation theory of the universal enveloping algebra or group algebra.

Proposition 9.3.7 The universal enveloping algebra, $U(\mathfrak{g})$, is a Hopf algebra with the comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \tag{9.3.8}$$

counit

$$\varepsilon(x) = 0, \tag{9.3.9}$$

and antipode

$$\chi(x) = -x. \tag{9.3.10}$$

Compare and contrast this to the group algebra, $\Bbbk G$, which is a Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \text{and} \quad \chi(g) = g^{-1}. \tag{9.3.11}$$

These are, in some ways, two opposite ends of the scale for how a Hopf algebra can behave.

Definition 9.3.12 — Filtred Algebra Let A be an associative algebra. We say that A is $\mathbb{Z}_{\geq 0}$-**filtred** if we have a chain of subspaces

$$0 = F_{-1}A \subseteq F_0A \subseteq F_1A \subseteq \cdots \subseteq F_nA \subseteq \cdots \tag{9.3.13}$$

such that $1 \in F_0A$,

$$\bigcup_{n=0}^{\infty} F_nA = A, \tag{9.3.14}$$

and $F_iA \cdot F_jA \subseteq F_{i+j}A$.

Definition 9.3.15 — Degree Filtration If A is an associative algebra generated by $\{x_\alpha\}$ then we can define a filtration on A by declaring all x_α to be of degree 1, and defining $F_nA \coloneqq (F_1A)^n$ to be formed of all terms of degree at most n (note that the degree of $x_\alpha x_{\alpha'}$ is 2, as is the degree of x_α^2, and so on).

Definition 9.3.16 — Associated Graded Algebra Given a filtred algebra, A, we define the **associated graded algebra** to be

$$\mathrm{gr}(A) := \bigoplus_{n=0}^{\infty} F_n(A)/F_{n-1}(A). \tag{9.3.17}$$

For the degree filtration the associated graded algebra is

$$\mathrm{gr}(A) = \bigoplus_{n=0}^{\infty} A_n \tag{9.3.18}$$

where A_n is the span of all words of degree exactly n.

If $\mathfrak{g}$ is a Lie algebra then we can define a degree filtration on $U(\mathfrak{g})$ by setting the degree of any $x \in \mathfrak{g}$ to be 1. Then $F_n U(\mathfrak{g})$ is the image of $\bigoplus_{k=0}^{n} \mathfrak{g}^{\otimes k} \subset T(\mathfrak{g})$ under the quotient map $T(\mathfrak{g}) \twoheadrightarrow T(\mathfrak{g})/I$. Since in $U(\mathfrak{g})$ we have $xy - yx = [x, y]$ for $x \in \mathfrak{g}$ and $y \in U(\mathfrak{g})$ it follows that $[F_i U(\mathfrak{g}), F_j U(\mathfrak{g})] \subseteq F_{i+j-1} U(\mathfrak{g})$. It then follows that when we take $F_n U(\mathfrak{g})/F_{n-1} U(\mathfrak{g})$ in $\mathrm{gr}(U(\mathfrak{g}))$ we are quotenting by (among other things) all commutators of elements of degree less than n. This makes $\mathrm{gr}(U(\mathfrak{g}))$ commutative. This in turn means that there is an epimorphism of associative algebras

$$S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}(U(\mathfrak{g})). \tag{9.3.19}$$

This is a statement that $S(A)$ is universal amongst commutative subalgebras of $T(A)$, i.e., that any such subalgebra can be recognised by taking $S(A)$ and applying some quotient to identify certain terms.

Definition 9.3.20 — PBW Theorem The homomorphism $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is an isomorphism.

Corollary 9.3.21 If $\{x_i\}$ is a basis of $\mathfrak{g}$ we can fix an order on the basis. Then $U(\mathfrak{g})$ is spanned by ordered monomials $\prod_i x_i^{n_i}$ with $n_i \in \mathbb{Z}_{\geq 0}$.

Theorem 9.3.22 — PBW Theorem. The ordered monomials described above are actually linearly independent, and thus form a basis for $U(\mathfrak{g})$.

Example 9.3.23 Consider $\mathfrak{sl}_2(\mathbb{C})$. This is a three-dimensional Lie algebra with generators $\{e, h, f\}$. If we order them so that $e < h < f$ then a basis for $U(\mathfrak{sl}_2(\mathbb{C}))$ is $e^a h^b f^c$ with $a, b, c \in \mathbb{Z}_{\geq 0}$.

9.4 Representation Theory of $\mathfrak{sl}_2(\mathbb{C})$

The representation theory of all finite dimensional semisimple Lie algebras over $\mathbb{C}$ is almost entirely controlled by the representation theory of $\mathfrak{sl}_2$. For this reason we'll now devote some time to the study of $\mathfrak{sl}_2$.

Recall that $\mathfrak{sl}_2$ (working over $\mathbb{C}$) is defined to consist of all traceless 2×2 complex matrices. There is a basis for these given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{9.4.1}$$

One can check that these satisfy the commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad \text{and} \quad [e, f] = h. \tag{9.4.2}$$

We can then abstract the definition of $\mathfrak{sl}_2$ to be $\operatorname{span}_{\mathbb{C}}\{e, h, f\}$ subject to the above commutation relations, without needing an explicit matrix form.

Lemma 9.4.3 Let V be a finite-dimensional representation of $\mathfrak{sl}_2$. Then we have the decomposition

$$V \cong \bigoplus_{\alpha \in \mathbb{C}} V_\alpha \tag{9.4.4}$$

where V_α is the **weight space**, defined to be the eigenspace

$$V_\alpha = \{v \in V \mid h \,.\, v = \alpha v\}. \tag{9.4.5}$$

Proof. It is a fact that finite-dimensional $\mathfrak{sl}_2$-representations are completely reducible. Thus, we may assume without loss of generality that V is irreducible, since if it isn't we can decompose it into a sum of irreducibles and then treat each of these separately.

Let W be the subspace of eigenvectors of h. It is then sufficient to show that $W = V$. To do this we show that W is a subrepresentation, that is, it's closed under h, e, and f. Then irreducibility will imply that $W = V$.

By definition h acts as a scalar on W, so W is closed under h. For e let $v \in W$ be an eigenvector of h, that is $hv = \alpha v$. Then a direct computation gives

$$\begin{aligned} he \,.\, v &= ([h, e] + eh) \,.\, v && (9.4.6)\\ &= (2e + eh) \,.\, v && (9.4.7)\\ &= 2e \,.\, v + eh \,.\, v && (9.4.8)\\ &= 2e \,.\, v + \alpha e \,.\, v && (9.4.9)\\ &= (\alpha + 2)e \,.\, v. && (9.4.10) \end{aligned}$$

Thus, $e \,.\, v$ is again an eigenvector of h, with eigenvalue $\alpha + 2$. Similarly, one can show that $f \,.\, v$ is an eigenvector of h with eigenvalue $\alpha - 2$.

Thus, W is closed under the action of e, h, and f, and thus is a subrepresentation, and so by irreducibility $W = V$. Thus, if V is not irreducible is a direct sum of irreducibles, each of which is an eigenspace of h with some given eigenvalue α. We may as well sum over all possible eigenvalues, $\alpha \in \mathbb{C}$, and simply have $V_\alpha = 0$ for many terms. $\square$

Example 9.4.11 The definition of $\mathfrak{sl}_2$ in terms of 2×2 matrices gives us a natural action of $\mathfrak{sl}_2$ on $\mathbb{C}^2$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. We have $he_1 = e_1$ and $he_2 = -e_2$, so we have two eigenvectors, and the corresponding eigenspaces $V_1 = \mathbb{C}e_1$ and $V_{-1} = \mathbb{C}e_2$. Then we have the following picture:

$$\begin{tikzcd} V_1 \arrow[loop right, "h"] \arrow[d, bend left, "f"] \\ V_{-1} \arrow[loop right, "h"] \arrow[u, bend left, "e"] \end{tikzcd} \tag{9.4.12}$$

The interpretation of this picture is that e and f act to shift the eigenvalue up and down by 2. Note that applying e to e_1 gives $ee_1 = 0$, and likewise, $fe_2 = 0$. Thus, we can add 0 to the top and bottom of this picture:

$$\begin{tikzcd} 0 \arrow[d, bend left, "f"] \\ V_1 \arrow[loop right, "h"] \arrow[d, bend left, "f"] \arrow[u, bend left, "e"] \\ V_{-1} \arrow[loop right, "h"] \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \\ 0 \arrow[u, bend left, "e"] \end{tikzcd} \tag{9.4.13}$$

The picture above actually generalises to any finite dimensional representation, we can always draw a picture like the following:

$$\begin{tikzcd}
0 \arrow[d, bend left, "f"] \\
V_{\alpha+2k} \arrow[u, "e"] \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\
\vdots \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \\
V_{\alpha+2} \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\
V_{\alpha} \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\
V_{\alpha-2} \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\
\vdots \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \\
V_{\alpha-2\ell} \arrow[u, bend left, "e"] \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\
0 \arrow[u, bend left, "e"]
\end{tikzcd} \tag{9.4.14}$$

The fact that we must always eventually get to 0 going either up or down is simply due to the fact that V is finite-dimensional.

Example 9.4.15 Consider the vector space $S^k(\mathbb{C}^2)$. We may identify this with the space of degree k homogenous polynomials (with coefficients in $\mathbb{C}$). For example, for $S^3(\mathbb{C}^2)$ we identify $e_1 \otimes e_1 \otimes e_1$ with x^3, $e_1 \otimes e_1 \otimes e_2 = e_1 \otimes e_2 \otimes e_1 = e_2 \otimes e_1 \otimes e_1$ with x^2y, and so on. Basically, send e_1 to x, e_2 to y, and remember that all tensor products are symmetrised. Note then that we can identify $S(\mathbb{C}^2)$ and $\mathbb{C}[x, y]$ (more generally, $S(\mathbb{C}^m)$ and $\mathbb{C}[x_1, \ldots, x_m]$), an important identification in algebraic geometry.
There is a representation of $\mathfrak{sl}_2$ on $\mathbb{C}[x, y]$ given by

$$e = -y\partial_x, \quad h = -x\partial_x + y\partial_y, \quad \text{and} \quad f = -x\partial_y. \tag{9.4.16}$$

Note that each operator preserves the total degree of any polynomial (so long as it doesn't send it to zero). Thus, we can identify submodules of degree k-polynomials. More generally, the above identification defines an action of $\mathfrak{sl}_2$ on smooth functions $\mathbb{C}^2 \to \mathbb{C}$, of which the $S^k(\mathbb{C}^2)$ are submodules.
Consider $S^k(\mathbb{C}^2)$, which we now identify with the space of degree k poly-

nomials in x and y. A basis for this space consists of vectors

$$v_r = \binom{k}{r} x^r y^{k-r}. \tag{9.4.17}$$

Acting on this with h we have

$$\begin{aligned} hv_r &= (-x\partial_x + y\partial_y)\binom{k}{r} x^r y^{k-r} \\ &= -r\binom{k}{r} x^r y^{k-r} - (k-r)\binom{k}{r} x^r y^{k-r}) = (k-2r)v_r, \end{aligned} \tag{9.4.18}$$

so v_r has h-eigenvalue $\alpha = k - 2r$. We also have

$$ev_r = -y\partial_x \binom{k}{r} x^r y^{k-r} = -r\binom{k}{r} x^{r-1} y^{k-r+1} = (r-k-1)v_{r-1} \tag{9.4.19}$$

and the h-eigenvalue of v_{r-1} is $k - 2(r-1) = k - 2r + 2 = \alpha + 2$. Similarly, we have

$$fv_r = -x\partial_y \binom{k}{r} x^r y^{k-r} = -(k-r)\binom{k}{r} x^{r+1} y^{k-r-1} = -(1+r)v_{r+1} \tag{9.4.20}$$

and the h-eigenvalue of v_{r+1} is $k - 2(r+1) = k - 2r - 2 = \alpha - 2$. Then letting $V_{k-2r} = \mathbb{C}v_r$ we have

$$\begin{tikzcd} V_{k-2r+2} \arrow[loop right, "h\sim k-2r"] \arrow[d, bend left, "f\sim-(1+r)"] \\ V_{k-2r} \arrow[loop right, "h\sim k-2r"] \arrow[u, bend left, "e\sim r-k-1"] \end{tikzcd} \tag{9.4.21}$$

Here $a \sim \lambda$ we mean that a acts by sending the basis vector of one space to the basis vector of the next multiplied by λ.
Let $V(k) = S^k(\mathbb{C}^2)$ be this $\mathfrak{sl}_2$-module. This is an irreducible module. Given any basis vector it lives in one of the V_α, and if we continuously act with e we eventually get v_0. Then v_0 generates this entire module by acting with f and scalar multiplication. Note that $\dim V(k) = k + 1$, since we have the basis $\{v_0, \dots, v_k\}$.

The previous example actually captures all irreducible modules of $\mathfrak{sl}_2$, as the following proves. The argument basically mirrors the argument above without reference to an explicit structure of polynomials.

Proposition 9.4.22 — Classification of Finite Dimensional Irreducible $\mathfrak{sl}_2$-Modules Let V be a $(k+1)$-dimensional $\mathfrak{sl}_2$-module. Then $V \cong V(k)$ with $V(k)$ as defined in Example 9.4.15.

Proof. By the same argument as in the proof of Lemma 9.4.3 we know that the eigenvectors of h span V (which we're assuming is irreducible). Since V is finite-dimensional h has a finite number of eigenvalues, so there must be some h-eigenvector, v_0, for which we have $hv_0 = 0$. Consider $f^k v_0$, as we have a finite-dimensional space, and thus finitely many eigenvectors of h, we must have for some N that $f^N v_0 = 0$, and suppose N is the smallest such value. If we take $B = \{v_0, fv_0, \dots, f^{N-1}v_0\}$ then this is a submodule of V, and thus is all of V. Thus, knowing that V has dimension $k+1$ we know that $N = k+1$. In particular, $f^{N-1}v_0 = f^k v_0$ is the last element of this basis.

For what follows it's useful to absorb some scale factor into the basis, define $v_r = f^r v_0/r!$ for $r = 0, \dots, k$. Then $\{v_r\}$ is a basis of V.

All that remains is to show that the action of e and f on this basis is fully determined. Starting with e we use the fact that $hv_r = (\alpha_0 - 2r)v_r$ where α_0 is the h-eigenvalue of v_0. We then have

$$\begin{aligned}
ev_0 &= 0 && (9.4.23)\\
ev_1 &= efv_0 = [e,f]v_0 + fev_0 = hv_0 + 0 = \alpha_0 v_0 && (9.4.24)\\
ev_2 &= efv_1/2 = [e,f]v_1/2 + fev_1/2 = hv_1/2 + \alpha_0 fv_0/2 && (9.4.25)\\
&= (\alpha_0 - 2)v_1/2 + \alpha_0 v_1/2 = (\alpha_0 - 1)v_1. && (9.4.26)
\end{aligned}$$

We thus make the induction hypothesis that

$$ev_n = (\alpha_0 - n + 1)v_{n-1}. \tag{9.4.27}$$

Assuming the equivalent statement for v_{n-1} holds we then have

$$\begin{aligned}
ev_n &= efv_{n-1}/n = [e,f]v_{n-1}/n + fev_{n-1}/n && (9.4.28)\\
&= hv_{n-1}/n + fev_{n-1}/n && (9.4.29)\\
&= (\alpha_0 - 2n + 2)v_{n-2} + (\alpha_0 - n + 2)fv_{n-2}/n && (9.4.30)\\
&= (\alpha_0 - 2n + 2)v_{n-1}/n + (n-1)(\alpha_0 - n + 2)v_{n-1}/n && (9.4.31)\\
&= (\alpha_0 - n + 1)v_{n-1}. && (9.4.32)
\end{aligned}$$

This shows that the structure of V is entirely determined by α_0, we now show that α_0 is fixed. We know that $fv_k = 0$, and we have

$$\begin{aligned}
efv_k &= [e,f]v_k + fev_k = hv_k + (\alpha_0 - k + 1)fv_{k-1} && (9.4.33)\\
&= (\alpha_0 - 2k)v_k + (\alpha_0 - k + 1)kv_k && (9.4.34)\\
&= (k+1)(\alpha_0 - k)v_{k-1}. && (9.4.35)
\end{aligned}$$

For this to vanish, given that $k+1$, the dimension, is positive (for $k+1 = 0$ clearly all zero dimensional $\mathfrak{sl}_2$-modules are isomorphic), and thus $\alpha_0 = k$ is fixed, and so as soon as we know the dimension of a finite-dimensional irreducible $\mathfrak{sl}_2$-module we know everything about it. $\square$

Definition 9.4.36 — Weight Vectors Let V be an $\mathfrak{sl}_2$-module. We call eigenvectors of h **weight vectors**, and the eigenvalue is called its weight. If v is a weight vector and $ev = 0$ we call v a **highest weight vector**, similarly, if $fv = 0$ we call v a **lowest weight vector**.

The above proposition then says that any finite-dimensional irreducible $\mathfrak{sl}_2$-module is generated by a highest weight vector, v_0.

9.5 Classification of Semisimple Lie Algebras Over $\mathbb{C}$

The steps followed for classifying irreducible finite-dimensional irreducible $\mathfrak{sl}_2$-modules actually generalise remarkably well to classifying not just representations of other Lie algebras, but classifying a whole type of algebra, just by studying the adjoint representations in which these algebras act on themselves.

There were three steps we followed with $\mathfrak{sl}_2$. First, decompose V into eigenspaces of h. Second, use the commutation relations to determine how e and f act on these eigenspaces. Finally, use the irreducibility of the module to show that it is generated by a single highest weight vector.

In order to apply this method to other Lie algebras we'll need to generalise some things. The main one is that instead of just a single operator, h, we end up with a whole subalgebra of operators, $\mathfrak{h}$. Before we get to this we need a few definitions.

Definition 9.5.1 — Semisimple and Nilpotent Elements Let $\mathfrak{g}$ be a Lie algebra. We say that $x \in \mathfrak{g}$ is **semisimple** if ad_x is diagonalisable, and **nilpotent** if ad_x is nilpotent.

For example, in $\mathfrak{sl}_2$ h is semisimple, since in the adjoint representation, with the ordered basis $\{e, h, f\}$, we have

$$\mathrm{ad}_h = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}. \tag{9.5.2}$$

On the other hand, e and f are nilpotent, since in the adjoint representation

$$\mathrm{ad}_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \tag{9.5.3}$$

both of which have vanishing third power.

An abelian subalgebra, $\mathfrak{h} \subseteq \mathfrak{g}$ is called **toral**[1] if it consists of only semisimple elements. For any toral subalgebra we have the following decomposition:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \tag{9.5.4}$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_\alpha \mid \mathrm{ad}_h(x) = [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\}. \tag{9.5.5}$$

[1] This name comes from the fact that if G is a Lie group with Lie algebra $\mathfrak{g}$ then any toral subgroup, H, will have a Lie algebra isomorphic to $\mathfrak{h}$. In turn, a toral subgroup is a Lie subgroup of G which is isomorphic to a torus.

This is simply the weight space decomposition of $\mathfrak{g}$ viewed as an $\mathfrak{h}$-module through (restricted) adjoint action.

One can show that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \tag{9.5.6}$$

In particular, $\mathfrak{g}_0$ is a Lie subalgebra, since $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$, and $\mathfrak{h} \subseteq \mathfrak{g}_0$.

Definition 9.5.7 — Cartan Subalgebra If $\mathfrak{g}$ is a Lie algebra with toral subalgebra, $\mathfrak{h}$, such that, with the notation above, we have $\mathfrak{g}_0 = \mathfrak{h}$ then we call $\mathfrak{h}$ a **Cartan subalgebra** of $\mathfrak{g}$.

Note that while Cartan subalgebras aren't unique they are all conjugate, so we typically speak of *the* Cartan subalgebra, when it exists.

When we have a Cartan subalgebra we can change the decomposition to

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \tag{9.5.8}$$

where $\Delta = \{\alpha \in \mathfrak{h}^* \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\}$ is the subset of $\mathfrak{h}^*$ for which $\alpha \neq 0$ and $\mathfrak{g}_\alpha$ is nontrivial. We call Δ a set of **simple roots**.

For example, for $\mathfrak{sl}_2$ we have the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$. In this case we have $\mathfrak{g}_2 = \mathbb{C}e$ and $\mathfrak{g}_{-2} = \mathbb{C}f$, and we get the decomposition

$$\mathfrak{sl}_2 = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f. \tag{9.5.9}$$

9.5.1 Root Systems

Definition 9.5.10 — Reflection Let E be a Euclidean space with inner product $(-,-)\colon E \otimes E \to \mathbb{R}$. A **reflection** is a linear map $s\colon E \to E$ such that there exists some $v \in E$ such that $s(v) = -v$ and the hyperplane $(\mathbb{R}v)^\perp$ is fixed pointwise by s. Then we call s a reflection along v.

Note that given v the following formula gives a reflection along v:

$$s_v(w) = w - 2\frac{(v,w)}{(v,v)}v. \tag{9.5.11}$$

Definition 9.5.12 — Root System Let E be a real Euclidean space with inner product $(-,-)$. A **root system**, Φ, in E is a finite set of nonzero vectors or **roots** such that

1. $\operatorname{span}_{\mathbb{R}} \Phi = E$;
2. if $\alpha \in \Phi$ then $c\alpha \in \Phi$ only for $c = \pm 1$;
3. $s_\alpha(\Phi) = \Phi$ for $\alpha \in \Phi$;
4. $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$.

Sometimes the second condition isn't required, root systems for which the second condition holds are known as **reduced root systems**.

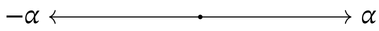


Figure 9.1: The A_1 root system, $\Phi = \{\alpha, -\alpha\}$, with chosen positive roots, $\Pi = \{\alpha\}$, and simple roots, $\Delta = \{\alpha\}$.

Table 9.1: Information on the root systems of rank at most 2. Notice that $\Phi = \Pi \sqcup (-\Pi)$ and in all cases we have chosen our naming of roots such that $\Delta = \{\alpha, \beta\}$. Notice that the positive roots, Π, are always found in the cone between the simple roots.

	Φ	Π	Δ
A_1	$\pm\alpha$	α	α
$\mathrm{A}_1 \oplus \mathrm{A}_1$	$\pm\alpha, \pm\beta$	α, β	α, β
A_2	$\pm\alpha, \pm\beta, \pm(\alpha+\beta)$	$\alpha, \beta, \alpha+\beta$	α, β
B_2	$\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)$	$\alpha, \beta, \alpha+\beta, 2\alpha+\beta$	α, β
G_2	$\pm\alpha, \pm\beta, \alpha+\beta, \pm(2\alpha+\beta), \pm(3\alpha+\beta)$	$\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta$	α, β

The **rank** of the root system is $\dim_{\mathbb{R}} E$.

Definition 9.5.13 — Positive and Simple Roots Given a root system we can make arbitrary choice of a hyperplane containing none of the roots. We then choose one side of this hyperplane, again, arbitrarily, and declare roots in this half to be **positive**. The **simple roots** are the positive roots which cannot be written as a sum, $\alpha + \beta$, of two elements of the positive roots, α and β, alternatively, the simple roots are precisely the subset of the positive roots which generate the positive roots through linear combinations with positive integral coefficients.

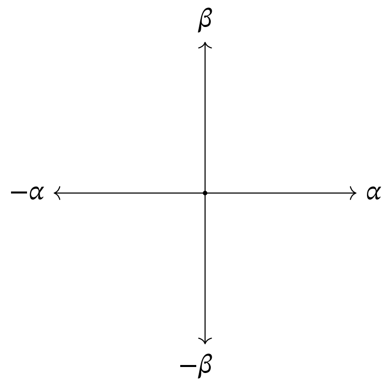
Notation 9.5.14 Notation varies here, but we'll call Φ the set of roots, Π the set of positive roots and Δ the set of simple roots.

It turns out that root systems actually turn up in many different areas of mathematics, but we'll focus on how they're relevant to Lie algebras.

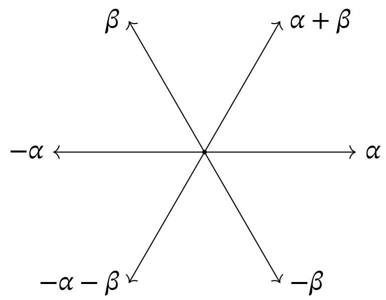
It turns out that, up to scaling, there is only one rank 1 root system. For reasons we'll get into later this root system is known as A_1. This root system is depicted in Figure 9.1. There are also only four rank 2 root systems, known as $\mathrm{A}_1 \oplus \mathrm{A}_1$ (being two orthogonal copies of A_1), A_2, B_2 (or C_2) and G_2. These are depicted in Figure 9.2. Table 9.1 lists the roots, Φ, positive roots, Π, and simple roots, Δ. In all cases we've chosen to label our roots by expressing them in terms of two chosen simple roots, α and β.

9.5.2 Connection to Semisimple Lie Algebras

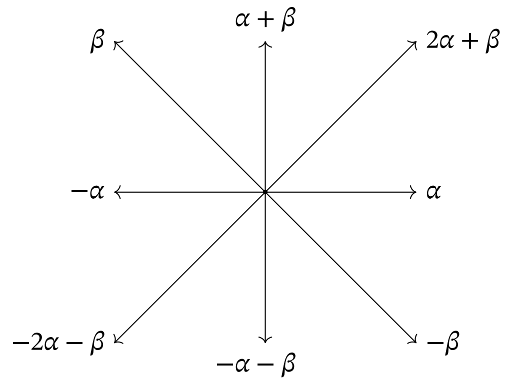
The reason that these root systems, as abstract subsets of some Euclidean space, are relevant is that given a semisimple Lie algebra the set of simple roots, Δ, (that is $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$) is actually the set of simple roots of a corresponding root system.



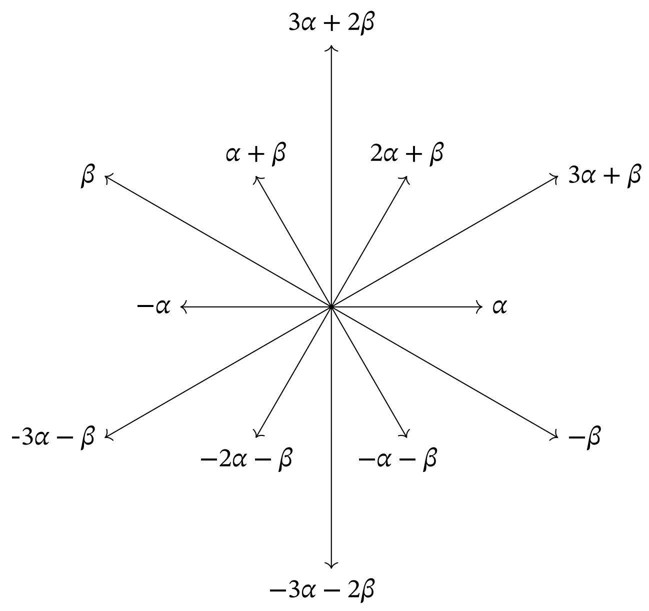
(a) The $A_1 \oplus A_1$ root system.



(b) The A_2 root system.



(c) The B_2 root system.



(d) The G_2 root system.

Figure 9.2: The rank 2 root systems.

Theorem 9.5.15. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{h}$. Let E be a Euclidean space such that the complexification of E is $\mathfrak{h}^*$. Then

- Δ forms a reduced root system in E;
- Eigenspaces are one-dimensional, $\mathfrak{g}_\alpha \cong \mathbb{C}$ for $\alpha \in \Delta$;
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

It turns out that these properties are exactly as is required in order for the following result to hold.

Theorem 9.5.16. There is a bijection between semisimple Lie algebras over $\mathbb{C}$ and reduced root systems.

We've constructed the root system from a semisimple Lie algebra. Since these objects are in bijection we can construct a semisimple Lie algebra in a unique way from a given root system. The process is unfortunately not that insightful, and basically reduces to imposing a bunch of relations on a free Lie algebra according to information encoded in the root system. The nice thing about this result is that it turns out to be much simpler to classify all of the finite-rank root systems.

Definition 9.5.17 — Cartan Matrix A (finite-type) **Cartan matrix** is an $n\times n$ matrix, $A = (a_{ij})_{1\le i,j\le n}$ such that

- $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}_{\le 0}$ for $i \ne j$;
- A is symmetrisable (there exists some diagonal matrix, D, such that DA is a symmetric matrix);
- A is positive (all principle minors of A are positive).

We consider two Cartan matrices to be the same if they are equal up to a simultaneous permutation of the rows and columns. That is, A and B are the same if $a_{i,j} = b_{\sigma(i),\sigma(j)}$ for some $\sigma \in S_n$.

Lemma 9.5.18 Let Φ be a root system with chosen simple roots, $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Define a matrix $A = (a_{ij})_{1\le i,j\le n}$ by

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{9.5.19}$$

This is a Cartan matrix, and is uniquely determined by the root system (up to permutation of the labels of our simple roots). Conversely, given a Cartan matrix one can construct a root system with that Cartan matrix.

The above result means that classifying Cartan matrices classifies root systems, which in turn classifies semisimple Lie algebras.

We're now ready to state the reverse process, for going from a root system or Cartan matrix to the corresponding semisimple Lie algebra.

Proposition 9.5.20 Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix. Let $\mathfrak{g}$ be the Lie algebra generated by $\{e_i, h_i, f_i \mid 1 \leq i \leq n\}$ subject to the relations

- $[h_i, e_j] = a_{ij} e_j$;
- $[h_i, f_j] = -a_{ij} f_j$;
- $[e_i, f_j] = \delta_{ij} h_i$;
- $[h_i, h_j] = 0$;
- $(\mathrm{ad}_{e_i})^{1-a_{ij}} e_j = 0$;
- $(\mathrm{ad}_{f_i})^{1-a_{ij}} f_i = 0$.

Then this is a semisimple Lie algebra over $\mathbb{C}$ and is uniquely determined by A.

The last two relations above are called the **Serre relations**.

Note that in the above $1 - a_{ij}$ is always positive, and $(\mathrm{ad}_{e_i})^k$ means the k-nested bracket with e_i, for example, $(\mathrm{ad}_{e_i})^3(x) = [e_i, [e_i, [e_i, x]]]$.

Example 9.5.21 — $\mathfrak{sl}_2$ Consider $\mathfrak{sl}_2$. We will demonstrate here that $\mathfrak{sl}_2$ is precisely the semisimple Lie algebra corresponding to A_1.
To do so we start with finding the Cartan matrix of A_1. Since $\Phi = \{\pm\alpha\}$ and $\Delta = \{\alpha\}$ this Cartan matrix is just 1×1, with the single entry being

$$a_{11} = \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} = 2. \tag{9.5.22}$$

So, $A = (2)$, of course the diagonal of the Cartan matrix is, by definition, always 2s, so we didn't actually need this calculation.
Then we can take $\mathfrak{g}$ to be the Lie algebra generated by $\{e_1, h_1, f_1\}$ subject to the relations

- $[h_1, e_1] = a_{11} e_1 = 2e_2$;
- $[h_1, f_1] = -a_{11} e_1 = -2f_1$;
- $[e_1, f_1] = \delta_{11} h_1 = h_1$;
- $[h_1, h_1] = 0$.

The last of these is always true, the first three are exactly the relations on $\{e, h, f\}$ which we impose on $\mathfrak{sl}_2$, so $\mathfrak{g} \cong \mathfrak{sl}_2$.
More generally, if we construct a Lie algebra from an arbitrary root system and take the subalgebra generated by e_i, h_i and f_i for fixed i then, since $a_{ii} = 2$ we always get a copy of $\mathfrak{sl}_2$.

Example 9.5.23 — $\mathfrak{sl}_3$ Let's go one dimension up and consider A_2. This root system has $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ and $\Delta = \{\alpha, \beta\}$. Let $\alpha_1 = \alpha$ and $\alpha_2 = \beta$ in what follows. Then the Cartan matrix has diagonals 2. Looking at the root diagram in Figure 9.2b the angle between α and β is $2\pi/3$, and both roots are the same length. Thus, $(\alpha, \beta) = (\alpha_1, \alpha_2) = \cos(2\pi/3) = -1/2$, and thus

$$a_{12} = \frac{2(\alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} = -1, \quad \text{and} \quad a_{21} = \frac{2(\alpha_2, \alpha_1)}{(\alpha_2, \alpha_2)} = 1 \tag{9.5.24}$$

having chosen a normalisation such that $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 1$. The Cartan matrix of A_2 is thus

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{9.5.25}$$

The corresponding semisimple Lie algebra is generated by $\{e_1, e_2, h_1, h_2, f_1, f_2\}$ subject to

- $[h_1, e_1] = 2e_1$, $[h_1, e_2] = -e_2$, $[h_2, e_1] = -e_1$, $[h_2, e_2] = 2e_2$;
- $[h_1, f_1] = -2e_1$, $[h_1, f_2] = f_2$, $[h_2, f_1] = f_1$, $[h_2, f_2] = -2f_2$;
- $[e_1, f_1] = h_1$, $[e_2, f_2] = h_2$, $[e_1, f_2] = [e_2, f_1] = 0$;
- $[h_1, h_2] = 0$;
- $(\mathrm{ad}_{e_1})^{1-a_{12}} e_2 = (\mathrm{ad}_{e_1})^2 e_2 = [e_1, [e_1, e_2]] = 0$, $[e_2, [e_2, e_1]] = 0$;
- $[f_1, [f_1, f_2]] = [f_2, [f_2, f_1]] = 0$.

This algebra is isomorphic to $\mathfrak{sl}_3$.

Example 9.5.26 — $\mathfrak{so}_5$ Consider the root system B_3, which has $\Delta = \{\alpha_1, \alpha_2\}$. Looking at the root diagram, Figure 9.2c, we see that if we choose $\alpha = \alpha_1$ to have length 1 then $\alpha_2 = \beta$ has length $\sqrt{2}$, and the angle between α and β is $3\pi/4$, and $\cos(3\pi/4) = -\sqrt{2}/2$. Thus,

$$a_{12} = \frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = \frac{2\|\alpha_1\|\|\alpha_2\|\cos(3\pi/4)}{\|\alpha_1\|^2} = \frac{2 \cdot 1 \cdot \sqrt{2} \cdot (-\sqrt{2}/2)}{1} = -2,$$
$$a_{21} = \frac{2(\alpha_2, \alpha_1)}{(\alpha_2, \alpha_2)} = \frac{2\|\alpha_2\|\|\alpha_1\|\cos(3\pi/4)}{\|\alpha_2\|^2} = \frac{2 \cdot \sqrt{2} \cdot 1 \cdot (-\sqrt{2}/2)}{(\sqrt{2})^2} = -1.$$

So, the Cartan matrix of B_3 is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}. \tag{9.5.27}$$

Note that this is symmetrisable:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies DA = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}. \tag{9.5.28}$$

The corresponding Lie algebra is generated by $\{e_1, e_2, h_1, h_2, f_1, f_2\}$, subject to the relations that

- $[h_1, e_1] = 2e_1$, $[h_2, e_2] = 2e_2$, $[h_1, e_2] = -2e_2$, $[h_2, e_1] = -e_1$;
- $[h_1, f_1] = -2f_1$, $[h_2, f_2] = -2f_2$, $[h_1, f_2] = 2f_2$, $[h_2, f_1] = f_1$;
- $[e_1, f_1] = h_1$, $[e_2, f_2] = h_2$, $[e_1, f_2] = [e_2, f_1] = 0$;
- $[h_i, h_j] = 0$ for $i, j \in \{1, 2\}$;
- $(\mathrm{ad}_{e_1})^{1-a_{12}} e_2 = (\mathrm{ad}_{e_1})^3 e_2 = [e_1, [e_1, [e_1, e_2]]] = 0$, $[e_2, [e_2, e_1]] = 0$;
- $[f_1, [f_1, [f_1, f_2]]] = [f_2, [f_2, f_1]] = 0$.

This Lie algebra is isomorphic to that of $\mathfrak{so}_5$.

Notice that in all of these examples, and more generally by inspecting the relations defining $\mathfrak{g}$, we always have that $\{e_i, h_i, f_i\}$ (for fixed i) generates a copy of $\mathfrak{sl}_2$. These copies of $\mathfrak{sl}_2$ are such that the e_is and f_js of distinct copies don't "interact" (i.e., they commute). The interaction only occurs when h_is are involved. The h_is themselves form a subalgebra, which is exactly the Cartan subalgebra, which we can see from these relations is always abelian.

9.5.3 Classification of Cartan Matrices

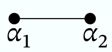
The final part to classifying all finite-dimensional semisimple Lie algebras over $\mathbb{C}$ is to classify all finite-type Cartan matrices. This has been done. The tidiest way to frame this classification is to encode the information of a root system into a labelled graph, and then it turns out that all of the corresponding graphs either fall into one of four families of graphs, or one of five exceptional cases.

First, given an $n \times n$ Cartan matrix, A, or the corresponding root system, (Φ, Π, Δ), we can construct a labelled graph as follows:

- The nodes are the simple roots, $\alpha_i \in \Delta$;
- Draw $a_{ij} a_{ji}$ edges between α_i and α_j ($i \neq j$);
- If α_i is longer than α_j draw an arrow on the edge pointing towards the shorter root.

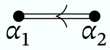
The graph that we get is called the **Dynkin diagram** of the root system/Cartan matrix.

Example 9.5.29 Consider A_2, this has two simple roots, α_1 and α_2. We have $a_{12} a_{21} = (-1)(-11) = 1$, and so the corresponding Dynkin diagram is



(9.5.30)

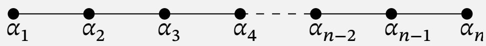
Now consider B_2, this has two simple roots, α_1 and α_2. We have $a_{12}a_{21} = (-2)(-1) = 2$, and α_2 is longer than α_1, so the corresponding Dynkin diagram is



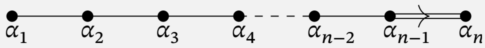
(9.5.31)

This process is invertible, since the Dynkin diagram fully encodes the angles between roots and their relative lengths (well, it encodes which is longer, the actual relative length can then be computed by requiring that the Cartan matrix have integral entries).

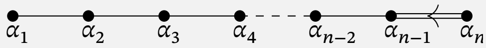
Theorem 9.5.32 — Classification of Root Systems. Every (finite-type) $n \times n$ Cartan matrix and its corresponding root system has a Dynkin diagram which is in one of the following infinite families (all with n vertices),

A_n


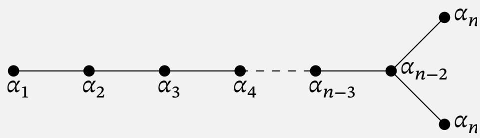
(9.5.33)

B_n


(9.5.34)

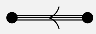
C_n


(9.5.35)

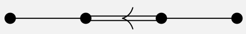
D_n


(9.5.36)

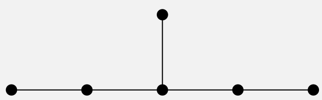
or is one of the following exceptional cases,

G_2


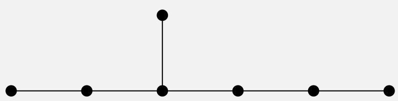
(9.5.37)

F_4


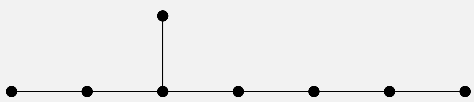
(9.5.38)

E_6


(9.5.39)

E_7


(9.5.40)

E_8


(9.5.41)

There is much more to be said about Dynkin diagrams and the things that they classify, but this is all we have time for here.

9.6 Verma Modules

We can use this classification to say something about the representation theory of semisimple Lie algebras over $\mathbb{C}$. To start with, when $\mathfrak{g}$ is defined from a root system in terms of the generators e_i, h_i, and f_i we can make the following definition.

Definition 9.6.1 — Verma Module Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$, and let $\lambda \in \mathfrak{h}^*$ be a weight. Let $I_\lambda \subseteq U(\mathfrak{g})$ be the left ideal generated by the elements $h - \lambda(h)1$ for $h \in \mathfrak{h}$ and e_i for $i = 1, \dots, r$. The **Verma module**, M_λ, is $U(\mathfrak{g})/I_\lambda$.

The idea of this definition is that M_λ is the largest (with respect to inclusion) highest weight representation with highest weight λ. Recall that by "highest weight representation" we mean that M_λ is generated (as a $U(\mathfrak{g})$-module) by some highest weight vector, v, which is such that $h \,.\, v = \lambda(h)v$ and $e_i \,.\, v = 0$. Thus, M_λ consists of linear combinations of elements of the form $f_{i_1} \cdots f_{i_k} . v$. The only relations imposed amongst these elements are those that are enforced by the commutation relations of the f_is. As a consequence f_i need not act nilpotently, and thus M_λ is infinite dimensional.

Let $\mathfrak{n}_+$ ($\mathfrak{n}_-$) denote the subalgebra of $\mathfrak{g}$ generated by the e_i (f_i). Then one can show that the Verma module, M_λ, is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ in which $h \in \mathfrak{h}$ acts as $h \,.\, v = \lambda(h)v$ and $e \in \mathfrak{n}_+$ acts as $e \,.\, v = 0$ (define $\lambda_+ : \mathfrak{h} \oplus \mathfrak{n}_+ \to \mathbb{C}$ by $\lambda_+(h) = h$ and $\lambda_+(e) = 0$ and then this is the "obvious" one-dimensional representation). We can identify this construction as inducing $\mathbb{C}_\lambda$ up to all of $\mathfrak{g}$, so

$$M_\lambda \cong \operatorname{Ind}_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}^{U(\mathfrak{g})} \mathbb{C}_\lambda. \tag{9.6.2}$$

This makes sense, the Verma module is such that $\mathfrak{h} \oplus \mathfrak{n}_+$ acts by highest weight, which is what $\mathbb{C}_\lambda$ captures, and then $\mathfrak{n}_-$ acts freely imposing only the required commutation relations, which is captured by inducing up to $\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

The Verma module is infinite dimensional, but nevertheless it is still important in the theory of finite dimensional representations of $\mathfrak{g}$.

Proposition 9.6.3 Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and fix a weight $\lambda \in \mathfrak{h}^*$. Let $\mathbb{C}\langle f_1, \dots, f_n\rangle$ be the free algebra generated by the noncommuting symbols $f_1, \dots, f_n$, and let $\tilde{M}_\lambda = \mathbb{C}\langle f_1, \dots, f_n\rangle v$ be the free module generated by v. There exists an action of $\mathfrak{g}$ on $\tilde{M}_\lambda$ such that

$$f_i \,.\left(\prod_k f_{j_k} v\right) = \left(f_i \prod_k f_{jk}\right); \tag{9.6.4}$$

$$h_i \,.\left(\prod_k f_{j_k} v\right) = \left(\lambda(h_i) - \sum_k a_{i,j_k}\right)\left(\prod_k f_{j_k} v\right); \tag{9.6.5}$$

$$e_i \,.\left(\prod_{k=1}^{l} f_{j_k} v\right) = \sum_{k|j_k=i} f_{j_1} \cdots f_{j_{k-1}} h_i f_{j_{k+1}} \cdots f_{j_l} v. \tag{9.6.6}$$

Proof. The $\mathfrak{g}$-module defined here is simply the Verma module, the only difference is that we're not imposing any condition on the f_is in the monomials in $\tilde{M}_\lambda$, whereas in M_λ we impose the Serre relations. □

The **weight lattice** of $\mathfrak{g}$ is $P = \mathbb{Z}\Phi \subset E$, the lattice generated by the roots. For example, for A_1 the weight lattice is just $\mathbb{Z}$, for $\mathrm{A}_1 \oplus \mathrm{A}_1$ it's $\mathbb{Z}^2$, for A_2 it's a hexagonal lattice, and for B_2 it's again a square lattice, $\mathbb{Z}^2$ (but scaled differently to $\mathrm{A}_1 \oplus \mathrm{A}_1$).

Corollary 9.6.7 The Verma module, M_λ, has a weight decomposition. In this weight decomposition the weight lattice is $P = \lambda - \mathbb{Z}\Phi$, and the λ-weight eigenspace of M_λ is one-dimensional, further, all weight subspaces are finite dimensional.

We are now ready to give the result which links M_λ to the finite-dimensional representations.

Proposition 9.6.8 — Universal Property of Verma Modules Let $\mathfrak{g}$ be a semisimple Lie algebra and use notation as above. If V is a $\mathfrak{g}$-module and $v \in V$ is a highest weight vector ($h . v = \lambda(h)v$ for $h \in \mathfrak{h}$ and $e_i . v = 0$) then there exists a unique homomorphism $\varphi \colon M_\lambda \to V$ such that $\eta(v_\lambda) = v$ where $v_\lambda \in M_\lambda$ is the highest weight element of the Verma module M_λ. In particular, if such a nonzero v generates V, that is V is a highest weight representation with weight vector v, then V is a quotient of M_λ.

The above result says that M_λ is universal amongst highest weight representations of $\mathfrak{g}$. Any map into any highest weight representation, V, can be achieved by first mapping into M_λ, then mapping into V in a unique way (using φ).

Proposition 9.6.9 Every highest weight representation has a weight decomposition into finite-dimensional weight subspaces.

So, every highest weight module is a quotient of the Verma module. It turns out that only one of these quotients is irreducible.

Proposition 9.6.10 For every $\lambda \in \mathfrak{h}^*$ the Verma module, M_λ, has a unique simple quotient, L_λ. Further, L_λ arises as a quotient of any highest weight $\mathfrak{g}$-module with highest weight λ.

The idea of the above is that as long as we never include v_λ in any submodule of M_λ we never get all of M_λ, and so we can sum all proper submodules of M_λ, and we know that the result will still be a proper submodule. We can then quotient by this sum, and the result is L_λ, we've quotiented out all submodules which could appear, and thus L_λ is simple.

Corollary 9.6.11 Simple highest weight $\mathfrak{g}$-modules (for $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$) are classified by their highest weight, $\lambda \in \mathfrak{h}^*$, by the bijection $\lambda \mapsto L_\lambda$.

Ten

Braids, Knots, and Hecke Algebras

10.1 The Pure Braid Group

We start with a technical definition, assuming the reader is familiar with the notion of a braid group, if not maybe skip the definition and look at the pictures.

> **Definition 10.1.1 — Pure Braid Group** Let $M_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$, which is a topological space as a subspace of $\mathbb{C}^n$. The **pure braid group**, is the fundamental group, $\mathcal{PB}_n = \pi_1(M_n)$.

A pure braid is then a (homotopy class) continuous function $\beta\colon [0,1] \to M_n$ with $\beta(0) = \beta(1)$, given by $t \mapsto (\beta_1(t), \dots, \beta_n(t))$ where the β_i are continuous functions $[0,1] \to \mathbb{C} \setminus \{z_1, \dots, \widehat{z}_i, \dots, z_n\}$ such that at no $t \in [0,1]$ do we have $\beta_i(t) = \beta_j(t)$.

Let $\mathbb{C}_n$ be the n-punctured complex plane[1]. Then for a pure braid, β, fixing some $t \in [0,1]$ we can view $\beta(t)$ as a choice of n distinct points in $\mathbb{C}$. Further, as t varies these points move around continuously. We can draw the whole path by considering $t \in [0,1]$ as a third dimension, and considering the positions traced by these points as time goes from 0 to 1. By lining up the punctures we can then project this down onto two dimensions, but keeping track of when a path goes over or under another. This gives us the standard picture of a pure braid. For example, Figure 10.1 shows an element of $\mathcal{PB}_4$.

The group operation of $\pi_1(M_n)$ is path concatenation (with rescaling of time so that we still have $t \in [0,1]$). The corresponding operation for pure braids is given by taking $\beta\beta'$ to be given by concatenating the diagram for β below the diagram for β' (reading the braid from the top down we want to do β' first[2]).

[1] We're treating $\mathbb{C}$ as a topological space here, the position of the points doesn't matter, we won't use any algebraic properties of this copy of $\mathbb{C}$

[2] The alternative convention gives us a perfectly well defined group, but to match conventions with the symmetric group we want this order.

10.2 The Braid Group

So far we've restricted our pure braids so that if a strand ends at the same puncture it begins at. The braid group relaxes this condition.

Let M_n be as in Definition 10.1.1. There is an obvious action of S_n on M_n given by permuting elements within a tuple, and this defines an equivalence relation on M_n, in which two tuples are equivalent if they are related by permuting elements. Let M_n/S_n be the quotient of M_n by this equivalence relation.

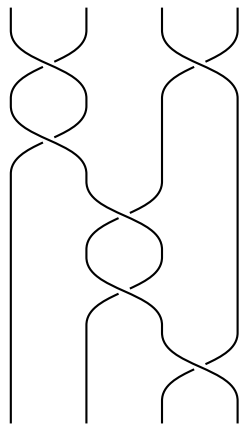


Figure 10.1: An element of the pure braid group, $\mathcal{PB}_4$.

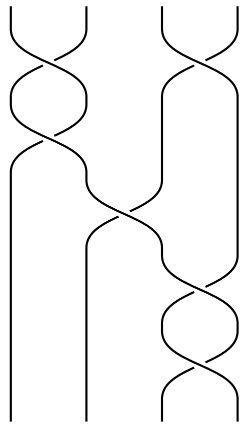


Figure 10.2: An element of the braid group, $\mathcal{B}_n$. Notice that the strands starting at 1, 2, 3 and 4 end at 1, 3, 4, and 2 respectively, defining a permutation $(2\,3\,4)$.

Definition 10.2.1 — Braid Group The **braid group** is $\mathcal{B}_n = \pi_1(M_n/S_n)$.

In terms of the pictures of braids the only difference is that we no longer require that braids start and end at the same point. See Figure 10.2. The group operation is still concatenation. Notice that by tracking where each strand starts and ends we get a permutation, $w \in S_n$. It is always possible to write a braid, $b\colon [0,1] \to M_n/S_n$, as a composite, $b = \beta \circ p$, where β is a pure braid and $w \in S_n$ is a permutation such that $\beta(1) = w^{-1}(b(0))$.

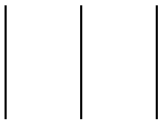
Theorem 10.2.2 — [Artin]. The braid group has the standard presentation

$$\mathcal{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \rangle.$$

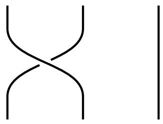
The relationship

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \tag{10.2.3}$$

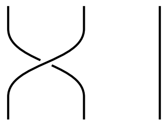
is called the **braid relation**. The identification between this presentation and $\mathcal{B}_n$ is pretty simple. For simplicity we'll just look at the $n = 3$ case, but for other values of n the pictures generalise in the obvious way. First, the identity, e, is simply leaving all strands fixed:

$e =$  . (10.2.4)

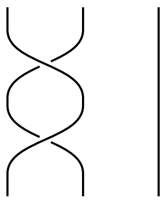
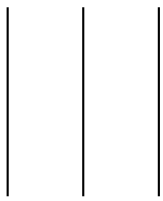
Then σ_1 is the braid

$\sigma_1 =$  , (10.2.5)

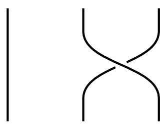
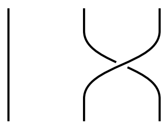
and σ_1^{-1} is given by crossing in the other direction:

$\sigma_1^{-1} =$  . (10.2.6)

This makes sense, since we then have

$\sigma_1\sigma_1^{-1} =$  $=$  $= e.$ (10.2.7)

Similarly, we have

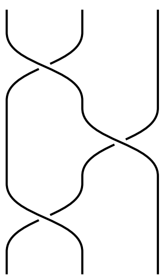
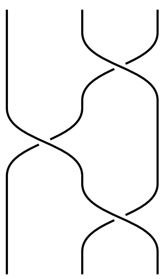
$\sigma_2 =$  and $\sigma_2^{-1} =$  . (10.2.8)

So, σ_i means passing the ith strand over strand $i+1$, and σ_i^{-1} means passing the ith strand *under* strand $i+1$.

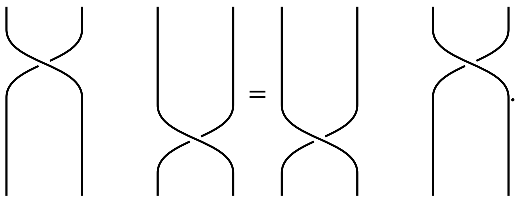
The braid relation in this case tells us that

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \tag{10.2.9}$$

which is just the following picture:

 $=$  . (10.2.10)

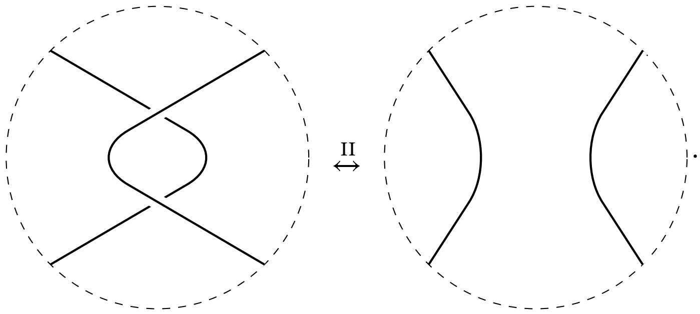
For $n = 3$ we never have $|i - j| > 1$, so let's look at $n = 4$, where this relation simply tells us that "sufficiently separated" swaps commute:



(10.2.11)

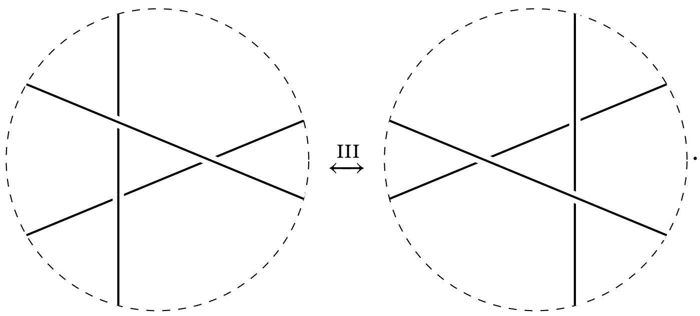
So far, we've just been looking at braids and deciding if they're equal if they intuitively give the same picture after rearranging strands without passing them through each other. This can be made rigorous as follows.

First, we define the **Reidemeister moves** of types II and III. These are "local" operations on braids, in that we can apply them to any portion of the diagram without changing the rest of the diagram. To represent this we use a dashed circle to "zoom in" on just a portion of the diagram: The Reidemeister move of type II is



(10.2.12)

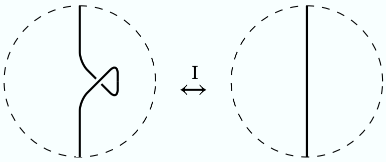
The Reidemeister move of type III is



(10.2.13)

These are simply capturing the fact that we want $\sigma_i^{-1}\sigma_i = e$ and the braid relation.

Remark 10.2.14 The Reidemeister moves first arose in knot theory, in which there is a third Reidemenster move, move number I, which is


(10.2.15)

We don't consider this as strands in braids aren't allowed to loop back up.

Proposition 10.2.16 Two braids are the same if and only if they are related by an isotopy and a sequence of Reidemeister moves of type II and III.

Remark 10.2.17 In physics a braid describes the adiabatic exchange of indistinguishable quasi particles in two dimensions. This is important in, for example, the fractional quantum Hall effect. This idea has applications to quantum computing. Particles whose exchange is governed by the braid group are called *anyons* (cf. bosons and whose exchange is governed by the trivial and antisymmetric representations of the symmetric group).

Remark 10.2.18 From a geometric view point $\mathcal{B}_n$ is the "mapping class group" of the punctured disc with n points. We swap the plane to a disc just because it's nicer to work with compact things, and we're not allowing punctures at infinity anyway.
The mapping class group is defined as follows. Let S be a surface, and $Q \subset S$ a finite set of marked points. Denote by $\mathrm{Homeo}(S, Q)$ the group of homeomorhpisms of S which fix Q as a set and fix the boundary pointwise. That is, $\varphi \in \mathrm{Homeo}(S, Q)$ is such that for every marked point, p, there is some marked point q (not necessarily distinct) such that $\varphi(p) = q$, and for every boundary point, x we have $\varphi(x) = x$. The **mapping class group** of the marked surface (S, Q) is

$$\mathrm{Mod}(S, Q) = \mathrm{Homeo}^+(S, Q)/\,\mathrm{Homeo}_0(S, Q) \tag{10.2.19}$$

where $\mathrm{Homeo}^+(S, Q)$ denotes the collection of orientation preserving homeomorphisms in $\mathrm{Homeo}(S, Q)$, and $\mathrm{Homeo}_0(S, Q)$ denotes the connected component of $\mathrm{Homeo}(S, Q)$ containing the identity (in the compact-open topology).
This group is also sometimes called the modular group, hence the notation $\mathrm{Mod}(S, Q)$. This is because when we take the torus with no marked points the mapping class group ends up being isomorphic to the modular group, $\mathrm{SL}_2(\mathbb{Z})$.
When we say that $\mathcal{B}_n$ is the mapping class group of the disc with n punctured points we mean that if D is this punctured disc and Q is our set of punctures then there is an isomorphism $\mathcal{B}_n \to \mathrm{Mod}(D, Q)$ given by $\sigma_i \mapsto H_i$ where σ_i is a generator of the braid group in the standard presen-

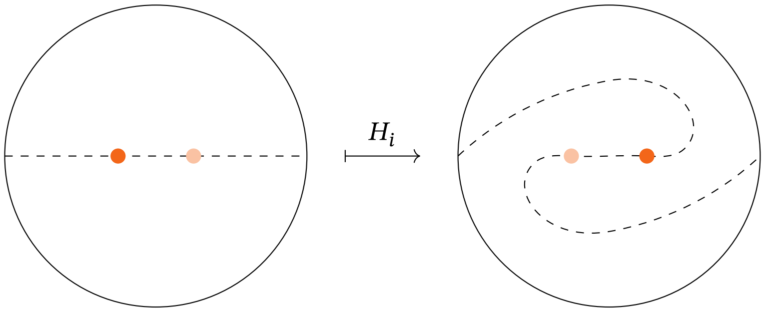


Figure 10.3: Half twist of the disc exchanging i and $i+1$. The dashed line shows some curve and its image under H_i.

tation and H_i is the homeomorphism of the n-punctured disc given by a half twist exchanging the points numbered i and $i+1$. See Figure 10.3.