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Notes from

Algebraic Geometry

October 6th, 2025

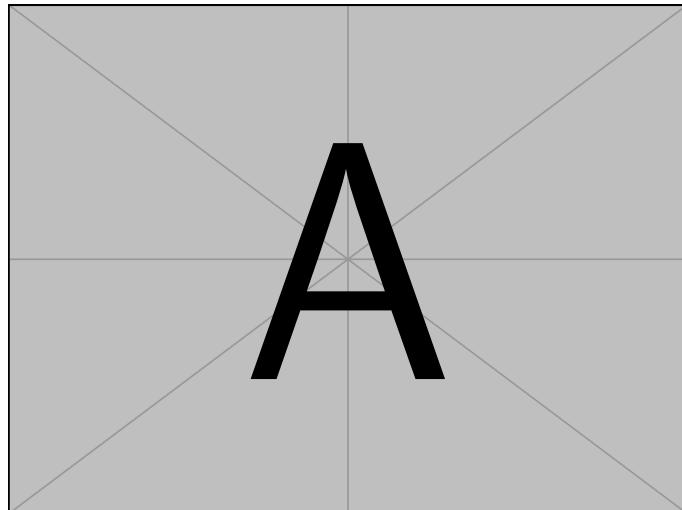
UNIVERSITY OF GLASGOW

Algebraic Geometry

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October 6th, 2025

These are my notes from the SMSTC course *Algebraic Geometry* taught by Dr Giulia Gugliatti and Prof Ivan Cheltsov. The lectures, and hence these notes, follow the *Algebraic Geometry* notes of Andreas Gathmann. These notes were last updated at 17:12 on November 17, 2025.



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One

Introduction

1.1 Conventions and Notation

Throughout the notes the ground field, K , will always be assumed to be *algebraically closed*, up to the point where we introduce schemes. Taking $K = \mathbb{C}$ is usually reasonable.

All rings, R , are assumed to be *commutative* with *unity*. That J is an ideal of R will be denoted $J \trianglelefteq R$. The ideal generated by a subset, $S \subseteq R$, is denoted $\langle S \rangle$.

We write $K[x_1, \dots, x_n]$ for the ring of polynomials with coefficients in K in the variables x_1, \dots, x_n . We write $f(a)$ to mean the evaluation of an element of this ring at the point $a = (a_1, \dots, a_n) \in K^n$, and where no confusion may arise we'll usually call this point $x = (x_1, \dots, x_n)$.

The natural numbers, \mathbb{N} , are assumed to contain 0.

1.2 Motivation

This section contains various motivating examples of algebro-geometric thinking, in varying levels of precision. Since the goal is to motivate some precision may be lacking.

1.2.1 Systems of Polynomial Equations

When we first learned algebra in high school it was to study the zeros of polynomials. Later we learned linear algebra, which it can be argued is the study of the zeros of systems of linear equations. Algebraic geometry combines these two fundamental fields into the study of zeros of systems of polynomials.

Given $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ the basic object of study of algebraic geometry is the **affine variety**

$$X = \{x \in K^n \mid f_i(x) = 0 \text{ for } i = 1, \dots, m\}. \quad (1.2.1)$$

What questions can we ask about this set? Just as a single complex polynomial, $f \in \mathbb{C}[x]$, cannot be solved exactly for $\deg f > 4$ we cannot possibly hope to explicitly list the points in X . Instead we reason about the geometric structure of the solutions. We will ask geometric questions about X , which we then aim to answer by an algebraic study of the f_i .

In the following sections we will give several examples of the sorts of geometric objects which can arise. We will focus on the existence of connections to other areas of mathematics.

1.2.2 Riemann Surfaces

¹Note that this is a “curve” since it’s complex dimension is 1 (we’ll define dimension of affine varieties later, for now just use your intuition for the dimension of a manifold). Of course, in our pictures this single complex dimension is drawn as two real dimensions.

Fix some positive integer, n . We can define a curve¹

$$c_n = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x - 1)(x - 2)(x - 3) \cdots (x - 2n)\} \subseteq \mathbb{C}^2. \quad (1.2.2)$$

We can view the defining equation as defining the quantity y . Since we have $y^2 = \dots$ to find the value of y we have to take a square root. What we get depends on the value of x . For most cases, specifically $x \neq 1, 2, \dots, 2n$, we have

$$y = \pm\sqrt{(x - 1)(x - 2) \cdots (x - 2n)}. \quad (1.2.3)$$

For $x = 1, 2, \dots, 2n$ we have

$$y = 0. \quad (1.2.4)$$

Consider what values y can take. For $x \neq 1, \dots, 2n$ we have two copies of \mathbb{C} , one for $+\sqrt{(x - 1) \cdots (x - 2n)}$ and one for $-\sqrt{(x - 1) \cdots (x - 2n)}$. For $x = 1, \dots, 2n$ we only have one possible value, 0. The picture this suggests is two copies of \mathbb{C} identified at the points $1, \dots, 2n$.

However, this isn’t quite right. We know that $z \in \mathbb{C}^\times$ doesn’t have a distinguished choice of \sqrt{z} . Upon passing once around the origin they are exchanged. For example, if we take the path $x = re^{i\theta}$, with $r \geq 0$ fixed and $\theta \in [0, 2\pi]$ then $\sqrt{x} = \sqrt{re^{i\theta/2}}$. Then at $\theta = 0$ we get \sqrt{r} and at $\theta = 2\pi$ we get $-\sqrt{r}$. The result is that as we go around the points $x = 1, \dots, 2n$ we move from one copy of \mathbb{C} to the other.

Fortunately, we know how to deal with this, we take branch cuts between zeros. Take both copies of \mathbb{C} , and perform branch cuts along alternate intervals, $[1, 2], [3, 4], \dots, [2n - 1, 2n]$. For $n = 3$ this produces Figure 1.1a. Now glue these along the cuts, which gives the picture Figure 1.1b. Finally, because it makes things nicer, add two points at infinity, one for each copy of \mathbb{C} , compactifying everything to get the picture Figure 1.1c. We see that this leaves us with a Riemann surface of genus $g = n - 1$. This relates algebraic geometry to the theory of Riemann surfaces.

We can change our curve to

$$\{(x, y) \in \mathbb{C}^2 \mid y^2 = (x - 1)^2(x - 2)(x - 3) \cdots (x - 2n)\} \subseteq \mathbb{C}^2. \quad (1.2.5)$$

Then the same analysis can be applied, except that we have a singular point at the repeated root, $x = 1$. This relates algebraic geometry to singularity theory.

1.2.3 Lines on Spaces

Consider the surface

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 + x_1^3 + x_2^3 + x_3^3 - (1 + x_1 + x_2 + x_3)\} \subseteq \mathbb{R}^3. \quad (1.2.6)$$

This is called the **Clebsch surface**. It’s plotted in ???. This is a cubic surface because it’s defined by a single cubic equation. It’s possible to draw straight lines on this surface. One can ask how many such straight lines exist. The answer over \mathbb{C} , surprisingly, is always 27, at least under some mild conditions. The Clebsch surface has the nice property that all of these lines are real. Cubic surfaces are actually a weird middle ground, between the infinite families of lines on a quadratic surface, and the general absence of lines on surfaces defined by any higher degree equation.

The question of how many geometric objects of a certain type exist is one of enumerative geometry, which makes heavy use of algebraic geometry.

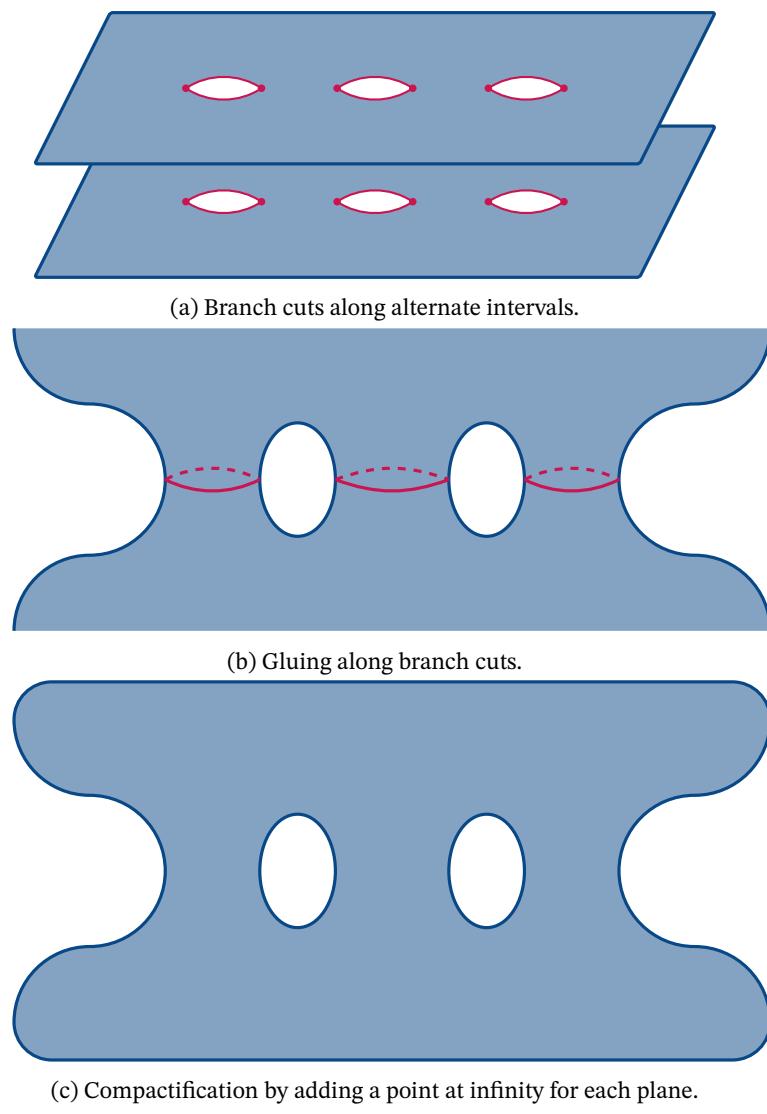


Figure 1.1: Producing a Riemann surface from a curve

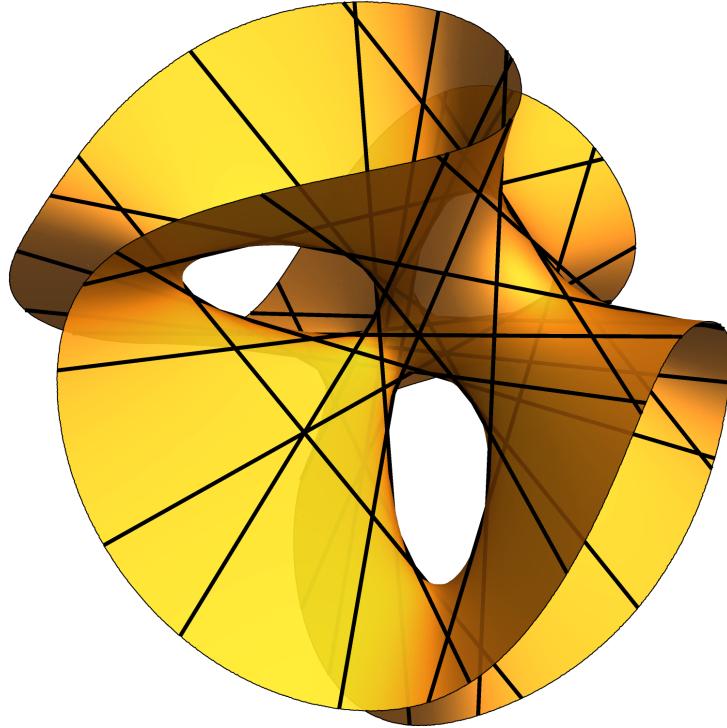


Figure 1.2: The Clebsch surface, as well as the 27 lines which lie on it.

1.2.4 String Theory

Strings, world sheets, those are surfaces, physicists should care about algebraic geometry.

1.2.5 Curves in Space

Consider the following curve

$$X = \{(x_1, x_2, x_3) = (t^3, t^4, t^5) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^3. \quad (1.2.7)$$

This is a parametric definition of this surface. We can equally define it explicitly as

$$X = \{(x_1, x_2, x_3) \mid x_1^3 = x_2 x_3, x_2^2 = x_1 x_3, x_3^2 = x_1^2 x_2\} \subseteq \mathbb{C}^3. \quad (1.2.8)$$

This is a surface, so it's two (complex) dimensional. However, we need all three of these equations to define it, if we remove any of them we don't get the same surface. This is very different to the world of linear algebra, where we'd have linear defining relations. There any codimension d subspace can be defined by d (linear) equations. Here X is one-dimensional, so it has codimension 2, but we need three equations to define it.

The general problem of taking an affine variety, X , defined as the vanishing set of some polynomials, and determining its dimension is actually very hard. We can use Gröbner bases to do this, but the algebra is pretty unwieldy, and we're forced to use computers to solve it most of the time. A Gröbner basis is a certain generating

set of the ideal generated by the polynomials defining the affine variety. Actually, even defining dimension for an arbitrary affine variety is not that straight forward, but for now the intuition from manifolds and vector spaces should be enough.

1.2.6 Different Fields

Over \mathbb{R} or \mathbb{C} we can use real or complex analytic methods to study the zeros of polynomials, and hence affine varieties.

Over \mathbb{Q} or finite fields we can use number theoretic techniques to study the zeros of polynomials, and hence affine varieties.

For example, Fermat's last theorem can be stated as the study of the affine variety

$$X = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 \mid x_1^n + x_2^n = x_3^n\}, \quad (1.2.9)$$

where the question we ask is if this has any non-trivial points.

Two

Affine Varieties

2.1 Affine Varieties

Definition 2.1.1 — Affine Space **Affine n -space** over K is the set

$$\mathbb{A}^n = \mathbb{A}_K^n := \{(a_1, \dots, a_n) \mid a_i \in K \forall i = 1, \dots, n\}. \quad (2.1.2)$$

Note that as sets $\mathbb{A}^n = K^n$. However, we write \mathbb{A}^n when we wish to forget the additional algebraic structure of K^n , specifically the vector space and ring, that is, we want to forget about the ability to scale, add and multiply elements.

For the time being we will take \mathbb{A}^n as our ambient space. Then a polynomial, $f \in K[x_1, \dots, x_n]$, defines a **polynomial function**

$$\mathbb{A}^n \rightarrow K \quad (2.1.3)$$

$$a \mapsto f(a). \quad (2.1.4)$$

We'll usually call this function f as well.

Definition 2.1.5 — Affine Variety Let $S \subseteq K[x_1, \dots, x_n]$ be some set of polynomials. The **zero locus** or **vanishing set** of S , denoted $V(S)$, is all points of \mathbb{A}^n on which the polynomial functions defined by polynomials in S vanish. That is,

$$V(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \forall f \in S\} \subseteq \mathbb{A}^n \quad (2.1.6)$$

Any subset of \mathbb{A}^n of this form is called an **affine variety**.



Note that some authors require that affine varieties have the additional property of being irreducible. These authors would then call all sets like $V(S)$ **affine algebraic sets**.

Notation 2.1.7 If $S = \{f_1, \dots, f_n\}$ is a finite set we write

$$V(S) = V(\{f_1, \dots, f_n\}) = V(f_1, \dots, f_n). \quad (2.1.8)$$

There are some properties we can immediately prove about affine varieties.

Lemma 2.1.9 — Reversal of Inclusion If $S_1 \subseteq S_2 \subseteq K[x_1, \dots, x_n]$ then $V(S_2) \subseteq V(S_1)$.

Proof. Suppose $x \in V(S_2)$. Then $f(x) = 0$ for all $f \in S_2$, and so certainly $f(x) = 0$ for $f \in S_1 \subseteq S_2$, and thus $x \in V(S_1)$. \square

Lemma 2.1.10 — Union If $S_1, S_2 \subseteq K[x_1, \dots, x_n]$ then $V(S_1) \cup V(S_2) = V(S_1S_2)$ where

$$S_1S_2 = \{fg \mid f \in S_1, g \in S_2\}. \quad (2.1.11)$$

Proof. We start by showing that $V(S_1) \cup V(S_2) \subseteq V(S_1S_2)$. Suppose that $x \in V(S_1) \cup V(S_2)$. Then $x \in V(S_1)$, so $f(x) = 0$ for all $f \in S_1$, and $x \in V(S_2)$, so $g(x) = 0$ for all $g \in S_2$. Thus, for $f \in S_1$ and $g \in S_2$ we have $(fg)(x) = f(x)g(x) = 0 \cdot 0 = 0$, so $x \in V(S_1S_2)$.

We now show that $V(S_1S_2) \subseteq V(S_1) \cup V(S_2)$. We do so by supposing that $x \notin V(S_1) \cup V(S_2)$. Then there exist polynomials, $f \in S_1$ and $g \in S_2$, for which $f(x) \neq 0$ and $g(x) \neq 0$. Thus, $(fg)(x) = f(x)g(x) \neq 0$ (since we work in a field, so have no nonzero zero divisors). Thus, $x \notin V(S_1S_2)$ since $fg \in S_1S_2$. By the contrapositive then we have that if $x \in V(S_1S_2)$ then $x \in V(S_1) \cup V(S_2)$. \square

Lemma 2.1.12 — Intersection Let J be an index set, and $\{S_j\}_{j \in J}$ an indexed family of subsets of $K[x_1, \dots, x_n]$. Then

$$\bigcap_{j \in J} V(S_j) = V\left(\bigcup_{j \in J} S_j\right). \quad (2.1.13)$$

Proof. Suppose $x \in \bigcap_{j \in J} V(S_j)$. Then $x \in V(S_j)$ for all $j \in J$. Thus, $f(x) = 0$ for all $f \in S_j$ for all $j \in J$. Thus, $x \in V\left(\bigcup_{j \in J} S_j\right)$.

Conversely, suppose $x \in V\left(\bigcup_{j \in J} S_j\right)$. Then $f(x) = 0$ for all $f \in \bigcup_{j \in J} S_j$, which means $f(x) = 0$ for all $f \in S_j$ for any $j \in J$, and therefore $x \in \bigcap_{j \in J} V(S_j)$. \square

We can also give some examples of simple affine varieties.

Example 2.1.14 — Affine Varieties

1. Affine n -space is itself an affine variety. Specifically, $\mathbb{A}^n = V(0)$, since the zero polynomial vanishes.
2. The empty set is an affine variety. Specifically, $\emptyset = V(1)$, since the constant polynomial at 1 vanishes nowhere.

3. Any linear subspace of $K^n = \mathbb{A}^n$ is an affine variety since a linear subspace is defined by the vanishing of linear equations.
4. If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine varieties then $X \times Y$ is too when viewed as a subspace of \mathbb{A}^{m+n} . The defining equations of $X \times Y$ are those of X and Y where we view those of X as a function of x_1, \dots, x_m and those of Y as a function of x_{m+1}, \dots, x_{m+n} .

Remark 2.1.15 The above results say that \emptyset and \mathbb{A}^n are both affine varieties, and that affine varieties are closed under finite union and arbitrary intersections. This is very close to the definition of a topology on \mathbb{A}^n in terms of open sets, \emptyset and X should be open, and the topology should be closed under finite intersections and arbitrary unions. Notice how unions and intersections exchange roles. Instead what we have is actually the requirements to define a topology on \mathbb{A}^n via the *closed* sets. We'll do exactly this in [Chapter 3](#).

Example 2.1.16 — Affine 1-Space The only affine varieties in \mathbb{A}^1 are \mathbb{A}^1 , \emptyset , and all finite sets. Any finite set, $\{\alpha_1, \dots, \alpha_n\}$, is the vanishing set of $(x - \alpha_1) \cdots (x - \alpha_n)$. To show that infinite sets cannot be affine varieties here (other than \mathbb{A}^1) suppose $X = V(S)$ is infinite for some $S \subseteq K[x]$. Fix some $f \in S$. Then $\{f\} \subseteq S$, so by [Lemma 2.1.9](#) $V(S) \subseteq V(f)$, and so $x \in V(f)$ for all $x \in X$, which means that $f(x) = 0$ for all $x \in X$, and so f has infinitely many roots, which is not possible for a polynomial.

If $f, g \in K[x_1, \dots, x_n]$ vanish on $X \subseteq \mathbb{A}^n$ then so do $f + g$ and fh for any $h \in K[x_1, \dots, x_n]$. Thus, the set, S , defining an affine variety, $X = V(S)$, is certainly not unique. We can always add $f + g$ and fh . From this we see that $V(S) = V(\langle S \rangle)$ where $\langle S \rangle \trianglelefteq K[x_1, \dots, x_n]$ is the ideal generated by S . This means that any affine variety can be expressed as the vanishing set of some ideal of a polynomial ring.

Hilbert's basis theorem ([Theorem A.2.9](#) and [Corollary A.2.10](#)) along with a standard characterisation of noetherian rings ([Lemma A.2.5](#)) tells us that all ideals of $K[x_1, \dots, x_n]$ are finitely generated. Given an affine variety, $X = V(S)$, we can then take $X = V(\langle S \rangle)$, and then we can find some finite generating set for this ideal, S' . Then $X = V(S')$. Thus, every affine variety is the zero locus of a finite set of polynomials.

Definition 2.1.17 — Radical Let R be a ring with ideal J . The **radical** of J is

$$\sqrt{J} = \{f \in R \mid f^k \in J \text{ for some } k \in \mathbb{N}\}. \quad (2.1.18)$$

We say J is **radical** if $J = \sqrt{J}$.

Lemma 2.1.19 Let $J \trianglelefteq R$. Then $J \subseteq \sqrt{J}$.

Proof. Suppose that $f \in J$, then $f^1 \in J$, and so $f \in \sqrt{J}$. \square

We can now state some results which are the analogues of [Lemmas 2.1.9](#) to [2.1.12](#) when we work with zero loci of ideals.

Lemma 2.1.20 Let $J \trianglelefteq K[x_1, \dots, x_n]$. Then $V(\sqrt{J}) = V(J)$.

Proof. First, [Lemma 2.1.19](#) gives us $J \subseteq \sqrt{J}$. Thus, by [Lemma 2.1.9](#) we have that $V(\sqrt{J}) \subseteq V(J)$.

Now suppose that $x \in V(J)$ and $f \in \sqrt{J}$. Then $f^k \in J$, so $f^k(x) = 0$, and since we're in a field with no nonzero zero divisors we must have that $f(x) = 0$, and so $x \in V(\sqrt{J})$. \square

This result, combined with our earlier analysis, means that every affine variety is the zero locus of a radical ideal.

Lemma 2.1.21 — Union If $J_1, J_2 \trianglelefteq K[x_1, \dots, x_n]$ then $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$.

Proof. That $V(J_1) \cup V(J_2) = V(J_1 J_2)$ is [Lemma 2.1.10](#). It remains to show that $V(J_1 J_2) = V(J_1 \cap J_2)$. Note that it is not generally true that $J_1 J_2 = J_1 \cap J_2$. However, it is true that $\sqrt{J_1 J_2} = \sqrt{J_1} \cap \sqrt{J_2}$ ([Lemma A.1.4](#)), and the result follows from this. \square

Lemma 2.1.22 — Intersection If $J_1, J_2 \trianglelefteq K[x_1, \dots, x_n]$ then $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Proof. From [Lemma 2.1.12](#) we have that $V(J_1) \cap V(J_2) = V(J_1 \cup J_2)$. We also have that $\langle J_1 \cup J_2 \rangle = J_1 + J_2$, so $V(J_1 \cup J_2) = V(\langle J_1 \cup J_2 \rangle) = V(J_1 + J_2)$. \square

Remark 2.1.23 With these results we have set up a pairing between geometric objects and algebraic objects. Specifically, we've defined a map

$$V: \{\text{algebraic objects}\} \rightarrow \{\text{geometric objects}\} \quad (2.1.24)$$

$$\text{ideal} \mapsto \text{affine variety}. \quad (2.1.25)$$

Studying the map going in the opposite direction will be the focus of the next section.

2.2 Ideal of an Affine Variety

Definition 2.2.1 — Ideal Let X be a subset of \mathbb{A}^n . The **ideal** of X is

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in X\}. \quad (2.2.2)$$

This is indeed an ideal, if $f, g \in I(X)$ then $f(x) = g(x) = 0$ for all $x \in X$ and $f(x) + g(x) = 0$, so $f + g \in I(X)$, and $-f(x) = 0$ so $-f \in I(X)$, and if $h \in K[x_1, \dots, x_n]$ then $f(x)h(x) = 0h(x) = 0$ so $fh \in I(X)$.

Lemma 2.2.3 — Reversal of Inclusion Suppose $X_1 \subseteq X_2 \subseteq \mathbb{A}^n$. Then $I(X_2) \subseteq I(X_1)$.

Proof. Suppose that $f \in I(X_2)$, that is, $f(x) = 0$ for all $x \in X_2$. Then $f(x) = 0$ for all $x \in X_1 \subseteq X_2$, and so $f \in I(X_1)$. \square

Lemma 2.2.4 — Ideal is Radical If $X \subseteq \mathbb{A}^n$ then $I(X)$ is radical.

Proof. Suppose $f \in \sqrt{I(X)}$. Then $f^k \in I(X)$ for some $k \in \mathbb{N}$. Then $f^k(x) = 0$ for all $x \in X$, and since we're in a field $f(x) = 0$ for all $x \in X$, and thus $f \in I(X)$, and hence $\sqrt{I(X)} \subseteq I(X)$. We also have $I(X) \subseteq \sqrt{I(X)}$ by Lemma 2.1.19. Thus, $I(X) = \sqrt{I(X)}$. \square

Remark 2.2.5 This gives us the other side of the pairing between algebraic objects and geometric objects:

$$I : \{\text{subsets of } \mathbb{A}^n\} \rightarrow \{\text{radical ideals of } K[x_1, \dots, x_n]\}. \quad (2.2.6)$$

These aren't quite inverses, since in this direction we only produce radical ideals. However, as we've seen radical ideals are good enough if we're applying V . The following important theorem tells us that these maps, while not quite inverses, are essentially inverses, so long as we're happy to only deal with radical ideals, which we can do by liberally taking radicals.

Theorem 2.2.7 — Hilbert's Nullstellensatz.

1. For any affine variety, $X \subseteq \mathbb{A}^n$, we have $V(I(X)) = X$.
2. For any ideal, $J \trianglelefteq K[x_1, \dots, x_n]$, we have $I(V(J)) = \sqrt{J}$.

Proof. We first prove that $X \subseteq V(I(X))$. If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$, and thus $x \in V(I(X))$.

Next, we prove that $\sqrt{J} \subseteq I(V(J))$. If $f \in \sqrt{J}$ then $f^k \in J$ for some $k \in \mathbb{N}$. Thus, $f^k(x) = 0$ for all $x \in V(J)$, and so $f(x) = 0$ for all $x \in V(J)$, and so

$f \in I(V(J))$.

Third, we prove that $V(I(X)) \subseteq X$. Since X is an affine variety we know that there is some ideal, $J \trianglelefteq K[x_1, \dots, x_n]$, for which $X = V(J)$. Then $\sqrt{J} \subseteq I(V(J))$ by the previous step, and $J \subseteq \sqrt{J}$, so $J \subseteq I(V(J))$. Taking the zero locus, which reverses the inclusion ([Lemma 2.1.9](#)), we have $V(I(V(J))) \subseteq V(J)$. Since $X = V(J)$ this is then exactly $V(I(X)) \subseteq X$, and so combined with the first step we have that $V(I(X)) = X$.

The only hard step of the proof is showing that $I(V(J)) \subseteq \sqrt{J}$. This requires some pretty heavy commutative algebra, so we'll skip it. It is this step of the proof which requires that K is algebraically closed. \square

Remark 2.2.8 Nullstellensatz means “theorem of the zeroes”.

Example 2.2.9 Consider a nonzero ideal, $J \trianglelefteq K[x]$. Since $K[x]$ is a PID we have that $J = \langle f \rangle$ for some $f \in K[x]$. Over an algebraically closed field we can always write f as

$$f(x) = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \quad (2.2.10)$$

for some $a_i \in K$ and $k_i, r \in \mathbb{N}$. Note that $J = \langle f \rangle$ consists of all polynomials vanishing at a_i with order at least k_i . We therefore have $V(J) = V(f) = \{a_1, \dots, a_n\} \subseteq \mathbb{A}^1$. This affine variety captures the zeros of f , but loses information about their multiplicities.

Hilbert's Nullstellensatz ([Theorem 2.2.7](#)) tells us that $I(V(J)) = \sqrt{J}$, and in this case we have

$$\sqrt{J} = \langle (x - a_1) \cdots (x - a_r) \rangle, \quad (2.2.11)$$

consisting of all polynomials vanishing at a_i with *any* order. So, \sqrt{J} too contains the information of the zeros of f while losing the information on their multiplicities. In this way the algebraic object, \sqrt{J} , and the geometric object, $V(J)$, contain exactly the same information.

Example 2.2.12 — Not Algebraically Closed Note that the fact K is algebraically closed is essential. In this example we'll consider the field \mathbb{R} , which is not algebraically closed. The ideal $\langle x^2 + 1 \rangle \trianglelefteq \mathbb{R}[x]$ is prime, and hence radical ([Lemma A.1.5](#)). However, $V(x^2 + 1) = \emptyset \neq \sqrt{\langle x^2 + 1 \rangle}$. Thus, Hilbert's Nullstellensatz doesn't hold as $I(V(x^2 + 1)) = I(\emptyset) = \mathbb{R}[x]$, when the Nullstellensatz would have $I(V(\langle x^2 + 1 \rangle)) = \sqrt{\langle x^2 + 1 \rangle} = \langle x^2 + 1 \rangle$, which is a proper ideal.

Example 2.2.13 Consider the ideal $J = \langle x - a_1, \dots, x - a_n \rangle \trianglelefteq K[x_1, \dots, x_n]$ for some $a_i \in K$. This is a maximal ideal since $K[x_1, \dots, x_n]/J \cong K$ (setting $x_i = a_i$). Hence, it is also prime, and so radical (Lemma A.1.5). The vanishing set of this ideal is $V(J) = \{a\}$ for $a = (a_1, \dots, a_n) \in \mathbb{A}^n$. Then by Hilbert's Nullstellensatz (Theorem 2.2.7) we have

$$I(\{a\}) = I(V(J)) = \sqrt{J} = J = \langle x_1 - a_1, \dots, x_n - a_n \rangle. \quad (2.2.14)$$

This lets us identify points in \mathbb{A}^n with minimal non-empty affine varieties. By the inclusion-reversing pairings of the Nullstellensatz points in \mathbb{A}^n are in one-to-one correspondence with maximal ideals in $K[x_1, \dots, x_n]$. This gives us another pairing of algebraic and geometric objects,

$$\{\text{maximal ideals of } K[x_1, \dots, x_n]\} \xleftrightarrow{1:1} \{\text{points in } \mathbb{A}^n\}. \quad (2.2.15)$$

This also shows that maximal ideals of the form of J above are actually the only maximal ideals of $K[x_1, \dots, x_n]$, a fact which can be proven purely algebraically, but this proof passes through geometry.

We can now prove a couple of results about how I interacts with unions and intersections. These are analogous to the results Lemmas 2.1.10 and 2.1.12 for V .

Lemma 2.2.16 Let X_1 and X_2 be affine varieties in \mathbb{A}^n . Then $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

Proof. Suppose $f \in I(X_1 \cup X_2)$. Then f vanishes on any point of X_1 or X_2 , and thus $f \in I(X_1)$ and $f \in I(X_2)$, so $f \in I(X_1) \cap I(X_2)$.

Conversely, suppose $f \in I(X_1) \cap I(X_2)$. Then f vanishes on X_1 and X_2 , and so it vanishes on $X_1 \cup X_2$, and hence $f \in I(X_1 \cup X_2)$. \square

Corollary 2.2.17 The intersection of two radical ideals of $K[x_1, \dots, x_n]$ is again radical.

Proof. If J_1 and J_2 are radical ideals then there exist affine varieties, X_1 and X_2 , such that $J_1 = I(X_1)$ and $J_2 = I(X_2)$. Then $J_1 \cap J_2 = I(X_1 \cup X_2)$, which is radical since the ideal of any affine variety is radical. \square

Note that it's possible to prove this corollary purely algebraically as well.

Lemma 2.2.18 Let X_1 and X_2 be affine varieties in \mathbb{A}^n . Then $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. By Hilbert's Nullstellensatz (Theorem 2.2.7) we have that $X_1 = V(I(X_1))$ and $X_2 = V(I(X_2))$. Thus, we have

$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))). \quad (2.2.19)$$

Then, by [Lemma 2.1.12](#) we have $V(J_1) \cap V(J_2) = V(J_1 + J_2)$, and so

$$I(X_1 \cap X_2) = I(V(I(X_1) + I(X_2))). \quad (2.2.20)$$

Then by the Nullstellensatz again we have $I(V(J)) = \sqrt{J}$, and so

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}. \quad \square$$

Remark 2.2.21 It is not, in general, true that the sum of two radical ideals is radical. This shouldn't be surprising, the algebraic explanation is that exponentiating a sum doesn't behave particularly simply, we need the binomial theorem. This is why we have to take the radical in the lemma above.

There is also a geometric explanation for this, in addition to the algebraic one. Consider the affine varieties $X_1, X_2 \subseteq \mathbb{A}_{\mathbb{C}}^2$ with $I(X_1) = \langle x_2 - x_1^2 \rangle$ and $I(X_2) = \langle x_2 \rangle$. The real points of these varieties are shown in [Figure 2.1](#). These correspond to $y = x^2$ and $y = 0$, although we're only really able to visualise these for $x, y \in \mathbb{R}$.

The intersection of these two varieties is $X_1 \cap X_2 = \{(0, 0)\}$. Thus, $I(X_1 \cap X_2) = I((0, 0)) = \langle x_1, x_2 \rangle$. Here we've used the identification of points of $\mathbb{A}_{\mathbb{C}}^2$ with maximal ideals of $\mathbb{C}[x_1, x_2]$ from [Example 2.2.13](#).

We have that

$$I(X_1) + I(X_2) = \langle x_2 - x_1^2 \rangle + \langle x_2 \rangle = \langle x_2 - x_1^2, x_2 \rangle = \langle x_1^2, x_2 \rangle. \quad (2.2.22)$$

This is not a radical ideal, we have

$$\sqrt{\langle x_1^2, x_2 \rangle} = \langle x_1, x_2 \rangle. \quad (2.2.23)$$

Which we expect from [Lemma 2.2.18](#).

The geometric interpretation is then as follows. The varieties X_1 and X_2 are tangent at their intersection point. Thus, in a linear approximation their defining equations, $x_2 = x_1^2$ and $x_2 = 0$, are the same, and both pick out the x_1 axis. This means we can imagine that the intersection, $X_1 \cap X_2$, actually extends a small distance from the origin, an infinitesimal amount in the x_1 direction. But, in this extended region x_1 doesn't vanish, and so it doesn't lie in $I(X_1) + I(X_2)$.

There are various ways to deal with this problem. One is to keep track of the multiplicities of curve intersections. The algebraic-geometry approach is to define schemes. These enlarge our class of geometric objects to include “objects extending by infinitesimally small amounts in some direction”. Then the result that we get mirroring that of Hilbert's Nullstellensatz ([Theorem 2.2.7](#)) is that affine schemes are in one-to-one correspondence with *arbitrary* ideals of $K[x_1, \dots, x_n]$. Then the intersection of X_1 and X_2 is replaced with the scheme corresponding to the non-radical ideal $\langle x_1, x_2 \rangle$.

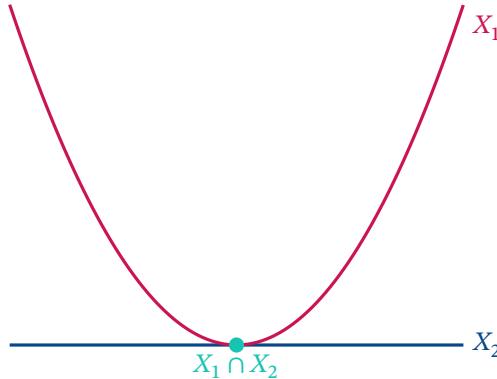


Figure 2.1: The two varieties used to demonstrate why the sum of radical ideals is not necessarily radical.

If $J \trianglelefteq K[x_1, \dots, x_n]$ is proper then J has a zero, that is $V(J)$ is non-empty. Otherwise, we'd have that $\sqrt{J} = I(V(J)) = I(\emptyset) = K[x_1, \dots, x_n]$, which means $1 \in \sqrt{J}$ and so $1 \in J$ meaning $J = K[x_1, \dots, x_n]$, violating the assumption that J is proper.

Proposition 2.2.24 — Weak Nullstellensatz If J is a proper ideal of $K[x_1, \dots, x_n]$ then $V(J)$ is non-empty.

Remark 2.2.25 Historically the weak nullstellensatz was proven first. This result is the reason for the name, “theorem of the zeros”. Despite the “weak” in the name of this result the weak Nullstellensatz is actually equivalent to the full Nullstellensatz. There's a trick, known as Rabinowitz's trick, which allows one to reduce the full Nullstellensatz in n variables to the weak Nullstellensatz in $n + 1$ variables.

2.3 Polynomial Functions

Definition 2.3.1 A **polynomial function** on \mathbb{A}^n is any function $\mathbb{A}^n \rightarrow K$ determined by $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$.

Note that such functions form a ring.

An immediate consequence of the Nullstellensatz is that polynomials and polynomial functions on \mathbb{A}^n agree. That is, two polynomials in $K[x_1, \dots, x_n]$ are equal if and only if the polynomial functions they determine on \mathbb{A}^n are equal.

If $f, g \in K[x_1, \dots, x_n]$ determine the same polynomial function on \mathbb{A}^n then $f(x) = g(x)$ for all $x \in \mathbb{A}^n$ by definition of equality of functions. Then $(f - g)(x) = 0$ for all $x \in \mathbb{A}^n$. Then by the Nullstellensatz we have

$$f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle \quad (2.3.2)$$

and thus $f - g = 0$ in $K[x_1, \dots, x_n]$, which means $f = g$ as polynomials.

The trickiest thing here is distinguishing between a polynomial and the polynomial function it determines. The solution to this is to use the work above to mostly ignore the distinction. We identify $K[x_1, \dots, x_n]$ with the ring of polynomial functions on \mathbb{A}^n .

We can just as well define polynomial functions on any subset of \mathbb{A}^n , and the most useful subsets to define them on are affine varieties. Note that this subsumes the above definition by considering \mathbb{A}^n as an affine variety.

Definition 2.3.3 Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then a **polynomial function** on X is any function $X \rightarrow K$ determined by $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$.

The ring of all polynomial functions on X is called the **coordinate ring**, denoted $A(X)$.

Notation 2.3.4 A common alternative notation for the coordinate ring of X is $K[X]$, not to be confused with the polynomial ring in a single variable, $K[x]$, or say the group algebra or K -span of X .

Lemma 2.3.5 Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the coordinate ring is given by

$$A(X) \cong K[x_1, \dots, x_n]/I(X). \quad (2.3.6)$$

Proof. The isomorphism simply identifies the equivalence class of a polynomial, $[f]$, with the corresponding function $x \mapsto f(x)$, which is clearly a ring homomorphism. We need only show that this is independent of choice of representative. To do so suppose that $f, g \in [f]$. That is $f - g \in I(X)$. Then $f(x) - g(x) = 0$ for all $x \in X$, and thus $f(x) = g(x)$, so f and g determine the same polynomial function on X . \square

We will identify $A(X)$ and $K[x_1, \dots, x_n]/I(X)$ from now on.

The idea here is that as far as X is concerned two polynomials are the same if they are equal for all $x \in X$. Whether these polynomials differ outside of X is not a question relevant when we're studying X . Thus, the difference of these two polynomials should vanish on X , which is exactly what it means for the difference of these two polynomials to be in $I(X)$.

Note that as well as being a ring $A(X)$ is actually a vector space, and the multiplication of two polynomial functions is K -bilinear. This means $A(X)$ is actually a K -algebra. Despite this, the name coordinate *ring* remains.

Example 2.3.7 Consider the affine variety $X = V(y - x^2) \subseteq \mathbb{A}^2$. Then $A(X) = K[x, y]/I(X)$. We have that $K[x, y]/\langle y - x^2 \rangle \cong K[x, x^2] = K[x]$, which is an integral domain. Thus, $\langle y - x^2 \rangle$ is prime, and so by Lemma A.1.5 we have $\langle y - x^2 \rangle = \sqrt{\langle y - x^2 \rangle}$. Thus, $I(X) = \sqrt{\langle y - x^2 \rangle} = \langle y - x^2 \rangle$ and so $A(X) = K[x, y]/I(X) \cong K[x]$.

Note that we almost always only identify coordinate rings up to isomorphism.

2.4 Affine Subvarieties

We will now repeat much of our previous work to define *relative* versions of many concepts. These replace the ambient space, \mathbb{A}^n , with some other affine variety, $Y \subseteq \mathbb{A}^n$, and then make the equivalent definitions for $X \subseteq Y$ given by the vanishing set of some polynomials.

Definition 2.4.1 Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety. For a subset, $S \subseteq A(Y)$, we define its **relative zero locus** to be

$$V_Y(S) = \{x \in Y \mid f(x) = 0 \forall f \in S\} \subseteq Y. \quad (2.4.2)$$

Subsets of this form are called **affine subvarieties** of Y .

Notation 2.4.3 When no confusion is likely to occur we drop the subscript Y and just write $V(S)$. This is usually fine due to the following point.

Note that affine subvarieties of Y are exactly the affine varieties (subsets of \mathbb{A}^n) which are also subsets of Y . In the definition we're just restricting the polynomial functions determined on \mathbb{A}^n to polynomial functions defined on Y before restricting further to X . This doesn't actually change anything¹.

¹This is an important part of the definition of a sheaf, which we'll see later

Definition 2.4.4 Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety. For a subset, $X \subseteq Y$, we define the **relative ideal** of X in Y to be

$$I_Y(X) = \{f \in A(Y) \mid f(x) = 0 \forall x \in X\} \trianglelefteq A(Y). \quad (2.4.5)$$

Notation 2.4.6 When no confusion is likely to occur we drop the subscript Y and just write $I(X)$.

Lemma 2.4.7 Let $X \subseteq Y \subseteq \mathbb{A}^n$ be affine varieties. Then

$$A(X) \cong A(Y)/I_Y(X). \quad (2.4.8)$$

Proof. The isomorphism identifies an equivalence class, $[f]$, of polynomial functions on Y with the polynomial function on X defined by $x \mapsto f(x)$. This is clearly an isomorphism. It is independent of the choice of representatives because if $f, g \in [f]$ then $f - g \in I_Y(X)$, which means $f(x) - g(x) = 0$ on X , which means $f(x) = g(x)$ for $x \in X$ and therefore f and g both determine the same polynomial function on X . \square

There are many relative results we can now state, but won't prove. First, all of the properties of V and I with respect to inclusions, unions, and intersections still hold for the relative versions. That is, we get analogous relative results for Lemmas 2.1.9 to 2.1.12, 2.2.3, 2.2.16 and 2.2.18.

Theorem 2.4.9 — Relative Nullstellensatz. Let $X \subseteq Y \subseteq \mathbb{A}^n$ be affine varieties. Then we have $V_Y(I_Y(X)) = X$. Let $J \trianglelefteq A(Y)$, then $I_Y(V_Y(J)) = \sqrt{J}$.

This gives us a bijection

$$\{\text{affine subvarieties of } Y\} \xleftrightarrow{1:1} \{\text{radical ideals of } A(Y)\}. \quad (2.4.10)$$

Three

Zariski Topology

In this section we see that there is a natural topology on any affine variety, given by declaring all affine subvarieties to be closed.

3.1 Topological Preliminaries

A topology can be defined by specifying open sets. It is also possible to define a topology by specifying closed sets (complements of open sets). This gives an equivalent definition of a topology, which is what we will work with.

Lemma 3.1.1 Let X be a set. We can declare a **topology** on X by declaring a collection of closed sets so long as

1. the empty set and X are closed;
2. arbitrary intersections of closed sets are closed;
3. finite unions of closed sets are closed.

Notice that the standard definition of a topology has arbitrary unions/finite intersections of open sets. These get swapped because taking complements turns unions into intersections and vice versa by De Morgan's laws.

Lemma 3.1.2 If Y is a topological space and $X \subseteq Y$ is a set then the **subspace topology** on X is given by declaring the closed sets of X to be those sets, $A \subseteq X$, of the form $A = C \cap Y$ for $C \subseteq Y$ closed in the topology of Y .

Lemma 3.1.3 A function, $f : X \rightarrow Y$, between topological spaces is **continuous** if the preimage of a closed set is closed.

3.2 Zariski Topology

Definition 3.2.1 — Zariski Topology Let X be an affine variety. The **Zariski topology** on X is given by declaring the closed sets to be the affine subvarieties of X .

That is, the closed subsets are exactly those of the form $V_X(S) = V(S)$ where $S \subseteq A(X)$.

Unless stated otherwise all topological notions for an affine variety will be considered with respect to the Zariski topology. Likewise, any topological notions for a subset of an affine variety will be considered with respect to the subspace topology of the affine variety (which is itself considered with respect to the Zariski topology).

Lemma 3.2.2 The Zariski topology is really a topology.

Proof. Let X be an affine variety. Since $X = V(I(X))$ and $\emptyset = V(1)$ we have that X and \emptyset are closed. A collection of closed subsets is a collection of affine subvarieties. This is closed under arbitrary intersection by the relative version of Lemma 2.1.12, and is closed under finite unions by the relative version of Lemma 2.1.10. \square

Notice that if we have affine varieties, $X \subseteq Y$, then there are *a priori* two topologies we could consider on X :

1. The Zariski topology;
2. The subspace topology.

However, these are actually exactly the same. To see this note that the affine subvarieties of X (that is, the closed sets of X in the Zariski topology) are precisely the affine subvarieties of Y which are a subset of X , that is, they're of the form $Z \cap Y$ where $Z \subseteq Y$ is closed, but that's precisely the closed sets of the subspace topology.

To showcase some of the slightly unusual features of the Zariski topology we'll consider $\mathbb{A}_{\mathbb{C}}^1$ and compare things to the standard topology on \mathbb{C} .

Example 3.2.3 Consider the unit ball,

$$B = \{x \in \mathbb{A}_{\mathbb{C}}^1 \mid |x| \leq 1\}. \quad (3.2.4)$$

Viewing this as a subset of \mathbb{C} in the standard topology it is clearly closed. Viewing it as a subset of $\mathbb{A}_{\mathbb{C}}^1$ in the Zariski topology it is not closed, since it is an infinite set and the only affine varieties of \mathbb{A}^1 are \mathbb{A}^1 and finite sets (Example 2.1.16).

This example informs our intuition for closed sets in the Zariski topology. Specifically, closed sets are, in a sense, “small”. Meaning that open sets are “big”. Now, in dimensions greater than 1 we can have infinite closed sets, so we have to be a bit careful about the meaning of “small”, but it's a reasonable intuition to have.

Note that any Zariski closed subset of $\mathbb{A}_{\mathbb{C}}^n$ is also closed in the standard topology of \mathbb{C}^n . This is because given $X = V(f_1, \dots, f_n)$ a Zariski-closed subset we have that $X = (f_1, \dots, f_n)^{-1}(0)$, where we're considering a function $(f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}$ and $\{0\} \subseteq \mathbb{C}$ is closed in the standard topology and polynomials are clearly continuous (with respect to the standard topology), so X is the preimage of a closed set under a continuous map and so is closed in the standard topology also.

Only very few closed subsets in the standard topology are also closed in the Zariski topology. The Zariski topology is coarser than the standard topology.

Example 3.2.5 Let $f : \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be any injective map. Then if $X \subseteq \mathbb{A}_{\mathbb{C}}^1$ is finite (i.e., Zariski-closed) then $f^{-1}(X)$ is also finite, and hence Zariski-closed. We also have that $f^{-1}(\emptyset) = \emptyset$ and any injective polynomial from $\mathbb{C} \rightarrow \mathbb{C}$ necessarily has domain $f^{-1}(\mathbb{A}_{\mathbb{C}}^1) = \mathbb{A}_{\mathbb{C}}^1$. Thus, the preimage of any Zariski-closed subset is again Zariski-closed, and so f is always continuous.

Example 3.2.6 — Product Topology Given topological spaces, X and Y , their product, $X \times Y$, can be equipped with a topology by declaring open subsets to be those of the form $\bigcup_{i \in I} U_i \times V_i$ where $U_i \subseteq X$ and $V_i \subseteq Y$ are families of open subsets in their respective topologies.

The standard topology on \mathbb{C}^n is precisely the product topology induced by the standard topology on each copy of \mathbb{C} . This is not so for the Zariski topology.

Let $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ and $Y \subseteq \mathbb{A}_{\mathbb{C}}^m$ be affine varieties. Then we have seen that $X \times Y \subseteq \mathbb{A}_{\mathbb{C}}^{n+m}$ is an affine variety (Example 2.1.14). However, the Zariski topology on $X \times Y$ does not coincide with the product topology on $X \times Y$ induced by the Zariski topology on X and Y .

To see this note that $V(x - y) = \{(a, a) \mid a \in K\} \subseteq \mathbb{A}_{\mathbb{C}}^2$ is closed in the Zariski topology of $\mathbb{A}_{\mathbb{C}}^2$, but it is not closed in the product topology, since the only way to write it as a union of products is

$$\bigcup_{a \in K} \{a\} \times \{a\}, \tag{3.2.7}$$

but $\{a\}$ is not open in the Zariski-topology (its complement is an infinite subset of $\mathbb{A}_{\mathbb{C}}^1$).

Note that the diagonal, $\Delta = \{(a, a) \mid a \in X\}$, is a closed subset of X if and only if X is Hausdorff (Lemma B.1.1). This shows that the Zariski topology is not Hausdorff, at least when we're working over an infinite field.

These examples show that the notion of continuous functions and products of spaces aren't that useful when it comes to the Zariski topology. In the next section we'll define some much more useful properties.

3.3 Irreducible Spaces

Consider the affine variety $X = V(x_1 x_2) \subseteq \mathbb{A}^2$. This consists of all points $(x_1, x_2) \in \mathbb{A}^2$ where $x_1 = 0$ or $x_2 = 0$. We can see this by just considering the solutions to

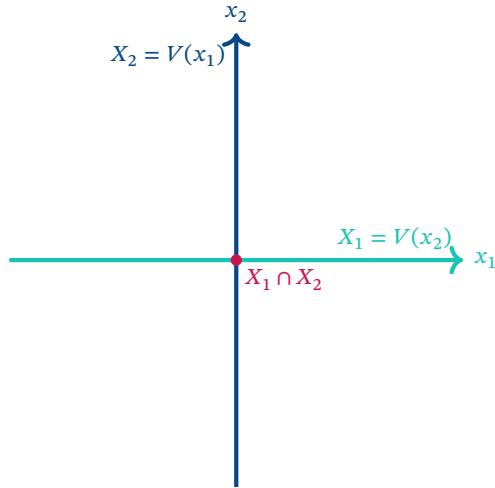


Figure 3.1: The real points of the affine variety $X = V(x_1x_2)$, which breaks into two components, $X = X_1 \cup X_2 = V(x_2) \cup V(x_1)$. Note the nonempty intersection $X_1 \cap X_2 = \{(0,0)\}$.

$x_1x_2 = 0$, or by noticing that $V(x_1x_2) = V(x_1) \cup V(x_2)$ by Lemma 2.1.10. When we take $K = \mathbb{C}$ we can plot the real points of this affine variety, they're simply the coordinate axes, $X_1 = V(x_2)$ and $X_2 = V(x_1)$ (see Figure 3.1). Note the exchange of indices, the x_i coordinate axis is where all other coordinates vanish. We see that $X = X_1 \cup X_2$, giving us a way to decompose X into two “smaller” affine varieties. Notice also that $(0,0) \in X_1$ and $(0,0) \in X_2$, so these are not disjoint affine varieties. This leads us to make the following definitions.

Definition 3.3.1 — Connected Space A topological space, X , is **disconnected** if there exist closed proper (i.e., nonempty) subsets, $X_1, X_2 \subsetneq X$, such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Otherwise X is called **connected**.

Definition 3.3.2 — Irreducible Space A topological space, X , is **reducible** if there exist closed proper (i.e., nonempty) subsets, $X_1, X_2 \subsetneq X$, such that $X = X_1 \cup X_2$. Otherwise X is called **irreducible**.

Note that the only difference between these is that for a space to be disconnected it needs to split into non-overlapping sets, whereas to be reducible the sets can be overlapping. In particular, irreducibility implies connectedness (if it doesn't split as a union it definitely doesn't split as a union of disjoint sets).

We can see that $X = V(x_1x_2)$ is reducible, since $X = X_1 \cap X_2 = V(x_2) \cap V(x_1)$ (remembering that $V(x_1)$ and $V(x_2)$ are closed in the Zariski topology). However, X is not disconnected, since if it split into two Zariski-closed disjoint subsets then these would also be closed in the standard topology, and we can see from the picture that this space is not disconnected.

Example 3.3.3 Note that reducibility depends on the topology. For example, the complex plane, \mathbb{C} , is reducible in the standard topology because we can write it as

$$\mathbb{C} = \{z \in \mathbb{C} \mid |z| \leq 1\} \cup \{z \in \mathbb{C} \mid |z| \geq 1\}. \quad (3.3.4)$$

However, in the Zariski topology any such decomposition would require at least one of the sets to be infinite, and the only infinite affine subvariety of $\mathbb{A}_{\mathbb{C}}^1$ is $\mathbb{A}_{\mathbb{C}}^1$ itself, so there is no way to write $\mathbb{A}_{\mathbb{C}}^1$ as a union of *proper* Zariski-closed subsets.

Example 3.3.5

1. Consider a single point, $p \in \mathbb{A}^n$. The set $\{p\} = V(x - p)$ is an affine variety. This is clearly irreducible, if $\{p\} = X_1 \cup X_2$ then one of X_1 or X_2 must be $\{p\}$, so these aren't proper subsets.
2. The emptyset is reducible, since we cannot write it as a union of nonempty sets.
3. Let $X = \{p_1, \dots, p_m\} \subseteq \mathbb{A}^n$ be any finite set with $m \geq 2$. This is an affine variety, $X = V((x - p_1) \cdots (x - p_m))$. We can always write $X = \{p_1, \dots, p_{m-1}\} \cup \{p_m\} = V((x - p_1) \cdots (x - p_{m-1})) \cup V(x - p_m)$, showing that X is reducible.

Combining these three, we see that a finite affine variety is irreducible if and only if it contains exactly one point.

Connectedness and reducibility, as stated above, are topological properties. It turns out that there are alternative algebraic characterisation of these properties for the Zariski topology.

Proposition 3.3.6 Let X be a disconnected affine variety such that $X = X_1 \cup X_2$ with $X_1, X_2 \subsetneq X$ closed subsets. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof. In $A(X)$ by [Lemma 2.2.16](#) we have that

$$I(X_1) \cap I(X_2) = I(X_1 \cup X_2) = I(X) = \langle 0 \rangle. \quad (3.3.7)$$

We also have $X_1 \cap X_2 = \emptyset$, and so by [Lemma 2.2.18](#)

$$\sqrt{I(X_1) + I(X_2)} = I(X_1 \cap X_2) = I(\emptyset) = A(X). \quad (3.3.8)$$

Since $1^k = 1$ it must be that $1 \in I(X_1) + I(X_2)$, and thus $I(X_1) + I(X_2) = A(X)$. Then by the Chinese remainder theorem ([Lemma A.1.6](#)) we have an isomorphism

$$A(X) \cong \frac{A(X)}{I(X_1)} \times \frac{A(X)}{I(X_2)} = A(X_1) \times A(X_2). \quad (3.3.9)$$

Proposition 3.3.10 Let X be a nonempty affine variety. Then X is irreducible if and only if $A(X)$ is an integral domain.

Proof. Since X is nonempty $A(X)$ is not the zero ring, which is required to be an integral domain. We will prove that X is reducible if and only if $A(X)$ is not an integral domain.

Suppose that $A(X)$ is not an integral domain. That is, there exist nonzero $f_1, f_2 \in A(X)$ with $f_1 f_2 = 0$. Then $X_1 = V(f_1)$ and $X_2 = V(f_2)$ are closed subsets of X and since f_i are nonzero $X_i \subsetneq X$. By Lemma 2.1.10, $X_1 \cup X_2 = V(f_1) \cup V(f_2) = V(f_1 f_2) = V(0) = X$, and so X is reducible.

Suppose instead that X is reducible, so $X = X_1 \cup X_2$ for some closed proper subsets, $X_1, X_2 \subsetneq X$. By the relative Nullstellensatz (Theorem 2.4.9) we know that $I(X_i) \neq \{0\}$, since under the bijection between affine subvarieties and radical ideals of $A(X)$ the ideal $\{0\}$ corresponds to X itself. Thus, there exists nonzero $f_i \in I(X_i)$. Then $f_1 f_2$ vanishes on $X_1 \cup X_2 = X$ since f_1 vanishes on X_1 and f_2 vanishes on X_2 . Thus, $f_1 f_2 = 0$ in $A(X)$, and so $A(X)$ is not an integral domain. \square

Example 3.3.11 Affine space, \mathbb{A}^n , is irreducible (and hence connected) since its coordinate ring, $A(\mathbb{A}^n) = K[x_1, \dots, x_n]$, is an integral domain.

More generally, any affine variety given as the vanishing set of some linear polynomials is irreducible, since its coordinate ring is again isomorphic to a polynomial ring over a field, which is an integral domain.

Remark 3.3.12 Note that $A(X) = K[x_1, \dots, x_n]/I(X)$ being an integral domain means that $I(X)$ is a prime ideal. This gives us yet another bijection between algebraic and geometric objects:

$$\{\text{nonempty irreducible affine subvarieties of } X\} \xleftrightarrow{1:1} \{\text{prime ideals of } A(X)\}.$$

In other words,

$$\{\text{nonempty irreducible affine subvarieties of } X\} \xleftrightarrow{1:1} \text{Spec } A(X).$$

3.4 Noetherian Spaces

In this section we'll see that any affine variety can always be written as a finite union of irreducible spaces. We'll actually show that this is true for a much broader class of spaces. These are called Noetherian spaces, having a very similar definition to that of a Noetherian ring.

Definition 3.4.1 — Noetherian Space A topological space, X , is **Noetherian** if every *descending* chain of closed subsets,

$$X \supseteq X_1 \supseteq X_2 \supseteq \dots, \quad (3.4.2)$$

stabilises. That is, for sufficiently large i we have $X_{i+1} = X_i$.

Note that the corresponding definition for Noetherian *rings* has an ascending chain of ideals, $I_1 \subseteq I_2 \subseteq \dots$. We can reformulate the definition of Noetherian spaces in terms of ascending chains of *open* subsets. We can also view this as the reversal of inclusions under $X \mapsto I(X)$.

Lemma 3.4.3 Any affine variety with the Zariski topology is a Noetherian topological space.

Proof. Suppose X is an affine variety admitting an infinite descending chain, $X_1 \supsetneq X_2 \supsetneq \dots$. Then by the relative Nullstellensatz (Theorem 2.4.9) this gives rise to an infinite ascending chain, $I(X_1) \subsetneq I(X_2) \subsetneq \dots$, in $A(X)$, but $A(X)$ is always Noetherian as it is a quotient of the polynomial ring, which is Noetherian (Theorem A.2.9 and Corollary A.2.10) and a quotient of a Noetherian ring is always Noetherian (Lemma A.2.8). \square

Lemma 3.4.4 Any subspace of a Noetherian space is Noetherian.

Proof. Let X be a Noetherian topological space and Y a subspace of X . Consider a descending chain, $Y_1 \supseteq Y_2 \supseteq \dots$, of closed subsets of Y . By definition of the subspace topology each of these closed subsets is of the form $Y_i = Y \cap X_i$ with $X_i \subseteq X$ some closed subset of X . Thus, we have the chain $Y \cap X_1 \supseteq Y \cap X_2 \supsetneq \dots$ in X . This gives rise to a descending chain $X_1 \supseteq X_1 \cap X_2 \supseteq X_1 \cap X_2 \cap X_3 \supseteq \dots$. This must stabilise, so $X_1 \cap \dots \cap X_{n+1} = X_1 \cap \dots \cap X_n$ for sufficiently large n . This then implies that $Y \cap X_{n+1} = Y \cap X_n$ for sufficiently large n , and thus $Y_{n+1} = Y_n$ for sufficiently large n , so our original chain stabilises and Y is Noetherian. \square

Corollary 3.4.5 Any subspace of an affine variety is a Noetherian topological space.

Proposition 3.4.6 — Irreducible Decomposition Any Noetherian space, X , decomposes as a finite union, $X = X_1 \cup \dots \cup X_r$, of nonempty irreducible closed subsets. Further, if $X_i \not\subseteq X_j$ for $i \neq j$ then the X_i are unique up to order. We call the X_i the **irreducible components** of X .

Proof. If $X = \emptyset$ then X is such a union with $r = 0$, so suppose that $X \neq \emptyset$. Suppose that X doesn't admit such a decomposition. This means that X is

reducible, else it decomposes as itself with $r = 1$. This means $X = X_1 \cup X'_1$ for some closed proper subsets $X_1, X'_1 \subsetneq X$. If both of these sets admit such a decomposition then so would X , so it must be that at least one of them doesn't, say X'_1 . Then by the same logic X'_1 is reducible, so $X'_1 = X_2 \cup X'_2$ for some closed proper subsets $X_2, X'_2 \subsetneq X'_1$. Again, one of these must not admit a decomposition, so is reducible. Repeating like this we define an infinite chain $X \supseteq X_1 \supseteq X_2 \supseteq \dots$. This contradicts the assumption that X is Noetherian, proving existence.

Suppose now that we have two such decompositions for X ,

$$X = X_1 \cup \dots \cup X_r = X'_1 \cup \dots \cup X'_s. \quad (3.4.7)$$

For any $i \in \{1, \dots, r\}$ we have $X_i \subseteq \bigcup_j X'_j$, and so $X_i = \bigcup_j (X_i \cap X'_j)$. By assumption X_i is irreducible, so we must have that all but one of these terms is empty, and so $X_i = X_i \cap X'_j$ for some j , meaning $X_i \subseteq X'_j$ for some j . Similarly, we have that $X'_j \subseteq X_k$ for some k . Thus, we have $X_i \subseteq X'_j \subseteq X_k$, and by assumption this is only possible for $i = k$, which then implies that $X_i = X'_j$. Thus, every set on the left appears on the right. The same logic can be applied to show that every set on the right appears on the left. Thus the two decompositions are the same, up to the order of terms. \square

Remark 3.4.8 One can compute the irreducible decomposition of an affine variety from the corresponding primary decomposition (Definition A.3.2) of its ideal, which always exists (Lemma A.3.3). Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $I(X) = Q_1 \cap \dots \cap Q_r$, with Q_i primary ideals of $K[x_1, \dots, x_n]$ be the primary decomposition of $I(X)$. Then by Hilbert's Nullstellensatz (Theorem 2.2.7) and Lemma 2.1.21 we have

$$\begin{aligned} X = V(I(X)) &= V(Q_1 \cap \dots \cap Q_r) = V(Q_1) \cup \dots \cup V(Q_r) \\ &= V(\sqrt{Q_1}) \cup \dots \cup V(\sqrt{Q_r}) \end{aligned} \quad (3.4.9)$$

and since $\sqrt{Q_i}$ is prime the $V(\sqrt{Q_i})$ are irreducible by Remark 3.3.12. Keeping only the minimal prime ideals, corresponding to maximal affine subvarieties, we obtain the irreducible decomposition of X .

This gives us the following bijection:

$$\{\text{irreducible components of } X\} \xleftrightarrow{1:1} \{\text{minimal prime ideals of } A(X)\}. \quad (3.4.10)$$

We have previously remarked that open sets are “big” in the Zariski topology. For example, in \mathbb{A}^1 the open sets are precisely the cofinite sets. We see this particularly when we consider irreducible affine varieties.

Let X be an irreducible topological space, and let $U, U' \subsetneq X$ be open and nonempty. Then $U \cap U'$ is never empty. Suppose that $U \cap U' = \emptyset$, then taking the complement of this we have $X \setminus (U \cap U') = (X \setminus U) \cup (X \setminus U') = X \setminus \emptyset = X$, and since U and U' are open their complements are closed, so this contradicts X

being irreducible. Intuitively, any two open sets are (edge cases aside) always so large that they overlap, no matter how we choose them.

Further, the closure of U , the smallest closed subset containing U , denoted \overline{U} , is all of X . That is, U is **dense** in X . To see this suppose that $Y \subseteq X$ is a closed subset containing U . Then $X = Y \cup (X \setminus U)$, and since X is irreducible and $X \setminus U \neq X$ it must be that $Y = X$, and in particular this is true when $Y = \overline{U}$. Intuitively, this means that if U is open then while it may not contain all of X it contains something “close” to any given point of X .

Four

Dimension

We have an intuitive notion of dimension, from our knowledge of vector spaces or manifolds. The dimension is the number of degrees of freedom, it's the number of pieces of information we need to specify to pick out a particular point. We know from manifolds that precisely how this information specifies a point only works in a neighbourhood of the point.

Here we'll define the dimension in terms of topological properties. Then we'll show that it aligns with a notion of dimension for the corresponding coordinate rings.

4.1 Dimension of a Topological Space

The key idea here is that if X is irreducible then any closed proper subset aught to have a smaller dimension than X . If we want this to hold then we have to look at all chains of inclusions of closed subsets, and define the dimension to be large enough that all of the subsets can have lower dimension, remembering that of course we want the dimension to be a natural number if finite.

Definition 4.1.1 — Dimension Let X be a nonempty topological space. The **dimension** of X , $\dim X$, is the supremum over all $n \in \mathbb{N}$ such that there exists a chain,

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n, \quad (4.1.2)$$

of length n consisting of irreducible closed subsets, $Y_i \subseteq X$. If the supremum doesn't exist the dimension is ∞ .

The idea here is that we can take Y_i to have dimension i , so that X having dimension n still leaves room to fit all of these smaller spaces.

Definition 4.1.3 — Codimension Let X be a nonempty topological space, and let Y be a nonempty irreducible closed subset of X . The **codimension** of Y in X , $\text{codim}_X Y$, is the supremum over all $n \in \mathbb{N}$ such that there exists a chain,

$$Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \quad (4.1.4)$$

of length n consisting of irreducible closed subsets of X containing Y . If

the supremum doesn't exist the codimension is ∞ .

Similar to the notion of dimension, the codimension of Y_i being $i + \dim Y$ lets us fit all of these spaces between Y and X , and so we should have $\dim X = n + \dim Y$, or $n = \dim X - \dim Y$, which intuitively is what the codimension should be.

Notation 4.1.5 For the dimension of a vector space, V , over K we write $\dim_K V$, leaving $\dim X$ without a subscript for the topological dimension.

Example 4.1.6 The affine space, \mathbb{A}^1 , has dimension 1, since the maximal chains of nonempty irreducible closed subsets of \mathbb{A}^1 are just $\{p\} \subsetneq \mathbb{A}^1$ for $p \in \mathbb{A}^1$, which all have length 1. Similarly, $\text{codim}_{\mathbb{A}^1}\{p\} = 1$.

Remark 4.1.7 We typically think of being Noetherian as a finiteness condition. However, it is not strong enough to imply finite dimension. For example, consider $X = \mathbb{N}$ equipped with the topology in which the closed subsets are \emptyset , X , and $Y_n = \{0, \dots, n\}$ for $n \in \mathbb{N}$. Then X is Noetherian, but has chains $Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n$ of nonempty irreducible closed subsets of arbitrary length, and thus the supremum of their lengths is ∞ .

Fortunately, for affine varieties this infinite-dimension problem cannot occur. To see this we need an algebraic notion of dimension.

4.2 Dimension of a Ring

We now give an algebraic notion of the dimension of a ring. This definition is constructed precisely so that it corresponds to the notion of dimension in the previous section.

Definition 4.2.1 — Krull Dimension Let R be a ring. The **Krull dimension**, $\text{Kdim } R$, of R is the supremum over all $n \in \mathbb{N}$ such that there exists a chain,

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \tag{4.2.2}$$

of length n consisting of prime ideals, $\mathfrak{p}_i \trianglelefteq R$.

Similarly, we can give an algebraic definition of codimension. Note that since we've moved to the algebraic side we're looking at chains ending with \mathfrak{p} , whereas on the topology side we looked at chains starting with Y .

Definition 4.2.3 — Height Let R be a ring with prime ideal \mathfrak{p} . The **height** of \mathfrak{p} , also known as the **codimension**, $\text{codim}_R \mathfrak{p}$, is the supremum over all n such that there exists a chain,

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \subsetneq \mathfrak{p} \tag{4.2.4}$$

of length n consisting of prime ideals contained in \mathfrak{p} .

We can now show that these two notions of (co)dimension actually agree for affine varieties.

Lemma 4.2.5 Let X be a nonempty affine variety. Then $\dim X = \text{Kdim } A(X)$.

Further, if Y is a nonempty irreducible affine subvariety of X then $\text{codim}_X Y = \text{codim}_{A(X)} I(Y)$.

Proof. Every chain of nonempty irreducible closed subsets of X corresponds to a chain of prime ideals of $A(X)$. Thus, the corresponding notions of dimension are equivalent.

We know that $I(Y)$ is a prime ideal of $A(X)$. If we require that the chain in X starts with Y this is equivalent to requiring that the corresponding chain in $A(X)$ ends with $I(Y)$, since I reverses inclusions (Lemma 2.2.3). Thus, the corresponding notions of codimension are equivalent. \square

Note that a finitely generated algebra over a field always has finite dimension. Hence $A(X)$ has finite dimension, since it's a quotient of $K[x_1, \dots, x_n]$ which is the free commutative algebra over K generated by x_1, \dots, x_n . Thus, X has finite dimension. Note that $\text{Kdim } K[x_1, \dots, x_n] = n$ [1, Prop 11.9(a)].

Proposition 4.2.6 Let X and Y be nonempty irreducible affine varieties.

1. $\dim(X \times Y) = \dim X + \dim Y$ (where $X \times Y$ is equipped with the Zariski topology, not the product topology).
2. As a special case of the above, $\dim \mathbb{A}^n = n$.
3. If $Y \subseteq X$ then $\dim X = \dim Y + \text{codim}_X Y$.
4. As a special case of the above $\text{codim}_X \{p\} = \dim X$ for all $p \in X$.
5. If $f \in A(X)$ is nonzero then every irreducible component of $V(f)$ has codimension 1 in X , and hence dimension $\dim X - 1$.

Proof. Item 1 follows from the equivalent statement for the ideals, which in turn follows from [1, Ex 11.33].

Item 3 holds because all maximal chains of prime ideals in $A(X)$ have the same length (which is not the case in an arbitrary ring) [1, Crl 11.12]. Thus, any maximal chain containing the prime ideal $I(Y)$ has length $\dim X$.

Item 5 follows from the equivalent algebraic statement, which is known as Krull's principal ideal theorem [1, Prop 11.15]. \square

Example 4.2.7 Consider the affine variety $X = V(x_2 - x_1^2) \subseteq \mathbb{A}_{\mathbb{C}}^2$. The real points of this are shown in Figure 4.1.

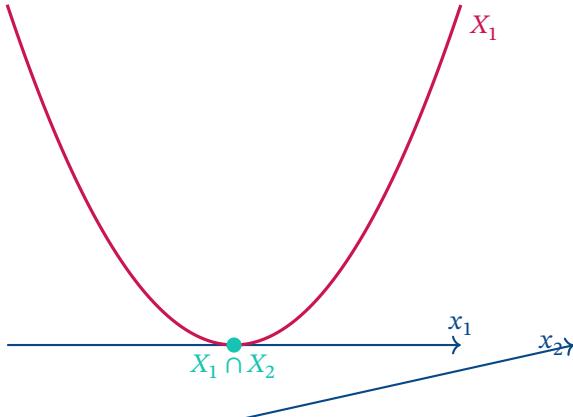


Figure 4.1: The real points of the affine variety $V(x_2 - x_1^2)$.

This is irreducible, since its coordinate ring is $A(X) = \mathbb{C}[x_1, x_2]/\langle x_2 - x_1^2 \rangle \cong \mathbb{C}[x_1, x_1^2] = \mathbb{C}[x_1]$.

The dimension of this variety is, as expected, 1, since it is the zero locus of a single nonzero polynomial in $\mathbb{A}_{\mathbb{C}}^2$ and $\dim \mathbb{A}_{\mathbb{C}}^2 = 2$ so $\dim X = \dim \mathbb{A}^2 - 1$.

Note that irreducibility is required for the above statements to be true. However, if we relax this condition then we can still say something provided we know the irreducible decomposition of the affine variety, $X = X_1 \cup \dots \cup X_r$. The dimension of X is simply the largest dimension of any of its irreducible components,

$$\dim X = \max\{\dim X_1, \dots, \dim X_r\}. \quad (4.2.8)$$

We must have that $\dim X$ is at least the greatest dimension of an irreducible component, since any chain in an irreducible component is also a chain in X . We must have that $\dim X$ is bounded above by the largest dimension of one of its irreducible components because if $Y_0 \subsetneq \dots \subsetneq Y_n$ is a chain of nonempty irreducible affine subvarieties of X then $Y_n = (Y_n \cap X_1) \cup \dots \cup (Y_n \cap X_r)$ is a union of closed subsets, and since Y_n is irreducible it must be that these are empty except for one term, so $Y_n = Y_n \cap X_i$, but then this chain is also a chain in X_i , meaning that $\dim X_i \geq n$.

This result allows us to mostly focus on irreducible spaces when we're considering dimension.

Another result that can allow us to determine the dimension of X is that, even if X is reducible,

$$\dim X = \sup\{\text{codim}_X\{a\} \mid a \in X\}. \quad (4.2.9)$$

If $\text{codim}_X\{a\} \geq n$ there must be some chain, $\{a\} \subseteq Y_0 \subsetneq \dots \subsetneq Y_n$ of irreducible affine subvarieties of X , which shows $\dim X \geq n$. If $\dim X \geq n$ then there is a chain $Y_0 \subsetneq \dots \subsetneq Y_n$ of nonempty affine subvarieties of X , and then for any $a \in Y_0$ this chain shows that $\text{codim}_X\{a\} \geq n$.

Example 4.2.10 Consider the affine variety $X = V(x_1x_3, x_2x_3) \subseteq \mathbb{A}^3$. Looking at the equations we see that a point in X must either have $x_3 = 0$ or both $x_1 = 0$ and $x_2 = 0$. That is, $X = V(x_3) \cup V(x_1, x_2)$. Both of these are irreducible, since their defining equations are linear. The first, $V(x_3)$, consists of all points $(x_1, x_2, 0)$, so it's the (x_1, x_2) -plane. The second, $V(x_1, x_2)$, consists of all points $(0, 0, x_3)$, so it's the x_3 -axis. This is shown in Figure 4.2a. We have that $\dim V(x_3) = 1$, because it's defined by a single nonzero equation. Alternatively, the largest chain we can make inside $V(x_3)$ is $X_0 \subsetneq X_1$ where X_0 is a point on the x_3 -axis and $X_1 = V(x_3)$ is the whole x_3 -axis. This is shown in Figure 4.2b.

We also have $\dim V(x_1, x_2) = 2$ since the largest chain we can make in this space is $Y_0 \subsetneq Y_1 \subsetneq Y_2$ where Y_0 is a point in the (x_1, x_2) -plane, Y_1 is a line in the (x_1, x_2) plane containing Y_0 , and $Y_2 = V(x_1, x_2)$ is the entire (x_1, x_2) -plane. This is shown in Figure 4.2c.

Of course, these are exactly the dimensions we would expect a line and a plane to have. By the remark above we have that $\dim X = \max\{1, 2\} = 2$. Intuitively, $\dim X$ is the maximum number of degrees of freedom needed to specify a point anywhere on the variety, even if some parts of the variety, the line in this case, don't require that amount of information.

Continuing with this example note that X_0 , a point on the line, has codimension 1, and Y_0 , a point on the plane, has codimension 2. This demonstrates how we can think of the codimension of a point as being the local dimension of the affine variety. Then Proposition 4.2.6 Item 3 is a statement that the local dimension of an *irreducible* affine variety is the same everywhere.

Often it's useful to restrict to cases where all irreducible components have the same dimension. For this we introduce the following terminology.

Definition 4.2.11 — Pure Dimension A Noetherian topological space, X , is of **pure dimension** n if every irreducible component of X has dimension n .

There are common shorthands for some special pure (co)dimensions.

Definition 4.2.12 An affine variety is

1. a **curve** if it has pure dimension 1;
2. a **surface** if it has pure dimension 2;
3. a **hypersurface** in a pure-dimensional affine variety Y if it is an affine subvariety of Y of pure dimension $\dim Y - 1$.



These terms are not used consistently. Some authors require that a curve is irreducible, and others allow a curve to refer to any affine variety whose irreducible components have dimension at most 1. Our choice is somewhere between these, we do not require that a curve is irreducible, but we do require that its irreducible components have dimension *exactly* 1.

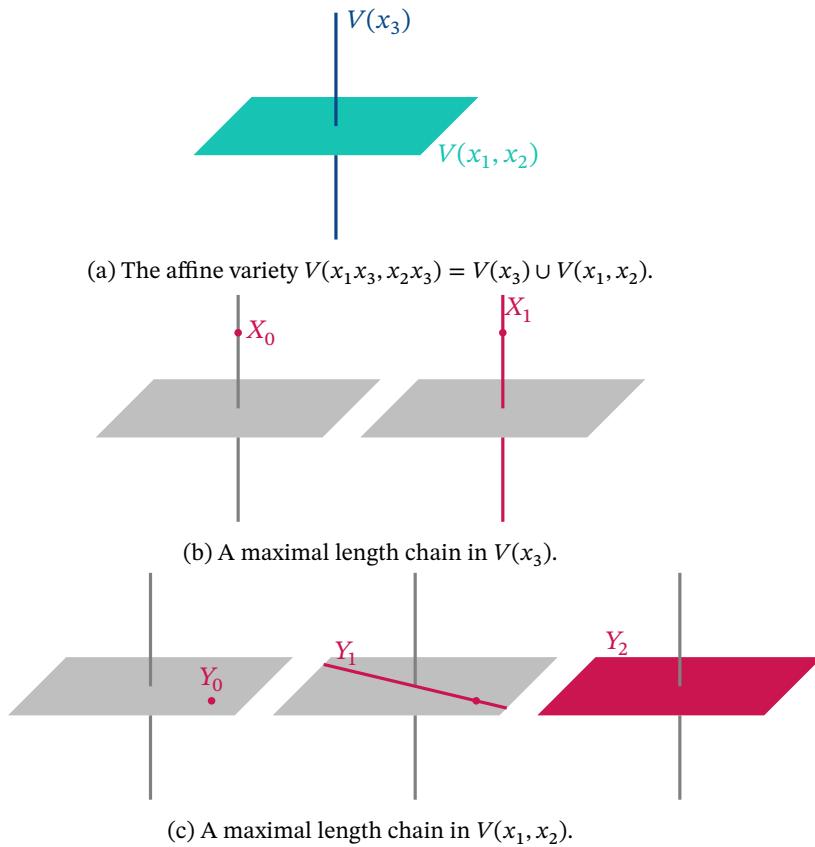


Figure 4.2: The affine variety $V(x_1x_3, x_2x_3)$ with its irreducible decomposition, and maximal chains in each irreducible component.

Notice that [Proposition 4.2.6 Item 5](#) is a statement that if $f \in A(X)$ is nonzero then $V(f)$ is a hypersurface.

This suggests that we should ask if *every* hypersurface is of this form. It turns out that the answer actually depends on some algebraic properties of $A(X)$. Specifically, whether $A(X)$ is a UFD. This is because of the following result.

Proposition 4.2.13 Let R be a Noetherian integral domain. Then the following are equivalent:

1. R is a UFD.
2. Every prime ideal of codimension 1 in R is principal.

Proof. We first show that [Item 2](#) implies [Item 1](#). Since R is Noetherian we can decompose any nonzero nonunit, $f \in R$, as a product of irreducible elements. If this were not the case then f could not itself be irreducible, and so $f = f_1f'_1$ and one of these must also not be decomposable, say f'_1 , so

$f'_1 = f_2 f'_2$. Repeating this gives a chain $\langle f \rangle \subsetneq \langle f'_1 \rangle \subsetneq \langle f'_2 \rangle \subsetneq \dots$, contradicting the fact that R is Noetherian.

To prove that R is a UFD it is sufficient to show that every irreducible element, $f \in R$, is prime. To do so choose a minimal prime ideal, \mathfrak{p} , containing f . Then by Krull's principal ideal theorem ([Proposition 4.2.6 Item 5](#)) we have that $\text{codim } \mathfrak{p} = 1$, and so \mathfrak{p} is principal by assumption. That is, $\mathfrak{p} = \langle g \rangle$ for some prime element, $g \in R$. However, g divides f and f is irreducible, so up to units $f = g$, and thus f is also prime.

We now show that [Item 1](#) implies [Item 2](#). Let \mathfrak{p} be a prime ideal of codimension 1 in R . Then we can choose a nonzero element, $f \in \mathfrak{p}$. Since $\mathfrak{p} \neq \langle 1 \rangle$ we know that f is not a unit. Since R is a UFD we can uniquely (up to order) write $f = f_1 \cdots f_k$ for $f_i \in R$ prime. Then since \mathfrak{p} is prime it must be that at least one of the f_i is in \mathfrak{p} . Since the codimension of \mathfrak{p} is 1 this requires that $\mathfrak{p} = \langle f_i \rangle$, and thus \mathfrak{p} is principal. \square

Let X be an irreducible hypersurface in \mathbb{A}^n . Then $I(X) \trianglelefteq K[x_1, \dots, x_n]$ is a prime ideal of codimension 1. Since $K[x_1, \dots, x_n]$ is a UFD it follows that $I(X) = \langle f \rangle$ for some $f \in K[x_1, \dots, x_n]$.

If X is a hypersurface, but not necessarily irreducible, then we can apply the same argument to each irreducible component of $X = X_1 \cup \dots \cup X_r$, showing that $I(X_i) = \langle f_i \rangle$ for some $f_i \in K[x_1, \dots, x_n]$. Then $I(X) = \langle f \rangle$ with $f = f_1 \cdots f_k$, which is again principal.

Definition 4.2.14 Let X be an affine hypersurface in \mathbb{A}^n with ideal $I(X) = \langle f \rangle$. Then the **degree** of X is the degree of f as a polynomial.

This degree is well defined because of the uniqueness of factorisation. For example, over \mathbb{C} , up to units and reordering, $f(x) = (x - a_1) \cdots (x - a_k)$ with a_i the roots of f , and then f has degree k . This lets us talk of linear, quadric, or cubic hypersurfaces, and so on.

It is generally a hard problem to find out if $A(X)$ is a UFD for a given affine variety. The following example is just one case in which $A(X)$ is not principal.

Example 4.2.15 Let $R = K[x_1, x_2, x_3, x_4]/\langle x_1 x_4 - x_2 x_3 \rangle$. This is a three dimensional integral domain. The elements x_1, x_2, x_3 and x_4 are irreducible, but not prime. Thus, R is not a UFD. Both $x_1 x_4$ and $x_2 x_3$ are decompositions of the same element of R into irreducible elements, and they don't agree up to units. The ideal $\langle x_1, x_2 \rangle$ is prime and of codimension 1 in R , but it is not principal.

Thus, by [Proposition 4.2.13](#) the plane, $V(x_1, x_2)$, is a hypersurface in $X = V(x_1 x_4 - x_2 x_3)$, but the ideal of $V(x_1, x_2)$ cannot be generated by a single element of $A(X)$.

Five

The Sheaf of Regular Functions

Now that we've defined affine varieties, and given them some structure, we're ready to look at maps between them. As usual, we'll look for maps preserving the relevant structure. This is actually a fairly hard question, requiring us to define something called a sheaf. For this section we'll only consider the simplest example, we'll look for morphisms from an affine variety, X , to the ground field, which we view as an affine variety, $K = \mathbb{A}^1$. When we say "function" (on X) we will mostly mean functions $X \rightarrow K$.

5.1 Regular Functions

Much of our thinking will be inspired by the theory of manifolds. So keep in mind the idea of smooth functions from a manifold to \mathbb{R} . These are just normal functions, but with an extra local condition of smoothness. Local meaning that smoothness can be checked on a neighbourhood of a point, and we get that a function is smooth everywhere precisely when it is smooth on any open neighbourhood of all points.

Unlike the manifold case we don't have a notion of smoothness, at least not yet. Instead our local condition will be that our functions should look like rational functions in a neighbourhood of any given point. They need not have this structure globally, by which we mean that the polynomials forming our rational function need not be the same on different neighbourhoods.

Of course, the assumption that we're working with polynomials is valid only if we're working with a subvariety of \mathbb{A}^n , if we want to work relative to some other variety then we have to talk of quotients of polynomial functions in $A(X)$.

Definition 5.1.1 — Regular Function Let X be an affine variety, and let $U \subseteq X$ be open. A **regular function** on U is a map, $\varphi : U \rightarrow K$, with the following property: for all $a \in U$ there are polynomial functions, $f, g \in A(X)$, with $g(x) \neq 0$ and

$$\varphi(x) = \frac{f(x)}{g(x)} \tag{5.1.2}$$

for all $x \in U_a$ where $U_a \subseteq U$ is an open neighbourhood of a .
We write $\mathcal{O}_X(U)$ for the set of all regular functions on U .

Note that $\mathcal{O}_X(U)$ is a K -algebra under pointwise operations. However, the common terminology is to refer to $\mathcal{O}_X(U)$ as the *ring* of regular functions on U , ignoring the additional vector space structure.

Notation 5.1.3 For the condition that $\varphi(x) = f(x)/g(x)$ for all $x \in U_a$ we usually say that $\varphi = f/g$ on U_a , although this is really an abuse of notation. In this case f/g is pointwise division of functions, not say, some element of a ring localised at g , although sometimes it can be interpreted as such.

Example 5.1.4 Consider the three dimensional affine variety $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$. An open subset of this is given by

$$U = X \setminus V(x_2, x_4) = \{(x_1, x_2, x_3, x_4) \in X \mid \text{one of } x_2 \text{ and } x_3 \text{ is nonzero}\}. \quad (5.1.5)$$

We can define a function on U by

$$\varphi : U \rightarrow K \quad (5.1.6)$$

$$(x_1, x_2, x_3, x_4) \mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0; \\ \frac{x_3}{x_4} & x_4 \neq 0. \end{cases} \quad (5.1.7)$$

Notice that since $x_1x_4 - x_2x_3 = 0$ on X when both fractions are defined we always have that $x_1/x_2 = x_3/x_4$. This means that this is a well-defined function. Clearly this function is locally a quotient of polynomials. In a neighbourhood of $x_2 = 0$ it's given by x_3/x_4 , and in a neighbourhood of $x_4 = 0$ it's given by x_1/x_2 . However, neither of these two ratios can be used to define the function on all of U , since one is undefined at $(0, 0, 0, 1)$ and the other at $(0, 1, 0, 0)$.

Algebraically we're using the fact that, as mentioned in [Example 4.2.15](#), $A(X)$ is not a UFD.

While regular functions are not (in general) polynomials it still makes sense to consider their zero loci, and we can in fact see that regular functions are sufficiently close to being polynomial for several facts to carry over to zero loci of regular functions.

Definition 5.1.8 Let X be an affine variety and $U \subseteq X$ open. The zero locus of a regular function, $\varphi \in \mathcal{O}_X(U)$, is

$$V(\varphi) = \{x \in U \mid \varphi(x) = 0\}. \quad (5.1.9)$$

Lemma 5.1.10 The zero locus of a regular function is closed.

Proof. Let X be an affine variety, $U \subseteq X$ an open subset, and $\varphi \in \mathcal{O}_X(U)$ a regular function. Then any $a \in U$ admits an open neighbourhood, $U_a \subseteq U$, on which $\varphi = f_a/g_a$ for some $f_a, g_a \in A(X)$ with g_a nonzero on U_a . Then we have that

$$U_a \setminus V(\varphi) = \{x \in U_a \mid \varphi(x) \neq 0\} = U_a \setminus V(g_a), \quad (5.1.11)$$

since φ is only nonzero on U_a when g_a is nonzero. Thus, $U_a \setminus V(\varphi)$ is the complement of an open set by a closed one, so is open, and hence $U_a \setminus V(\varphi)$ is open in X . Then the union over a of all such sets is also open, and this union is just $U \setminus U(\varphi)$. Thus, $V(\varphi)$ is closed in U . \square

A consequence of this result is the following. If we have nonempty open subsets, $U \subseteq V \subseteq X$, with X an irreducible affine variety then for regular functions $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ if φ_1 and φ_2 agree on U they must agree on all of V . The reason for this is that the locus $V(\varphi_1 - \varphi_2)$, where the two functions agree, contains U , and thus is closed in V . Thus it also contains the closure, \overline{U} , of U in V , but $\overline{V} = X$ by our earlier remark that open sets in the Zariski topology are dense (end of [Section 3.4](#)). Thus, V is irreducible, and then again this means that the closure of U in V is V , and so it must be that $\varphi_1 - \varphi_2$ actually vanishes on V , so $\varphi_1 = \varphi_2$ on all of V .

This statement is true because open subsets in the Zariski topology are so large, so their overlap is always substantial. For example, over \mathbb{C} it is also true that the closure of U in V is all of V , and thus $\varphi_1 = \varphi_2$ on V follows from the fact that $\varphi_1|_U = \varphi_2|_U$ and the φ_i are continuous. In fact, this statement also holds if we replace regular functions in the standard topology with holomorphic functions. Two holomorphic functions on a connected open subset, $V \subseteq \mathbb{C}^n$, must be the same function if they agree on a smaller nonempty subset $U \subseteq V$. This is known as the **identity theorem** for holomorphic functions, so we might call our result the **identity theorem** for regular functions. Much of the power of complex analysis comes from the fact that U can be very small, such as being the boundary of V . This makes the result much more surprising than the same result in the Zariski topology, where U will always be large. This is just one of many results that has both a complex-analytic version and an algebro-geometric version. Another such result which we'll see soon is the existence of removable singularities ([Example 5.1.26](#)).

Next we'll compute what the K -algebra $\mathcal{O}_X(U)$ is in some cases. An important case is when U is the complement of the zero locus of a single polynomial function. It turns out in this case that it's always possible to define regular functions globally as a ratio of polynomial functions, and that the denominator is always just some power of the original polynomial function.

Definition 5.1.12 — Distinguished Open Subset Let X be an affine variety and $f \in A(X)$ a polynomial function on X . We call

$$D(f) = X \setminus V(f) = \{x \in X \mid f(x) \neq 0\} \quad (5.1.13)$$

the **distinguished open subset** of f in X .

The nice thing about the distinguished open subsets is that they behave nicely with respect to intersections and unions. For any $f, g \in A(X)$ we have that

$$D(f) \cap D(g) = D(fg) \quad (5.1.14)$$

since $x \in D(f) \cap D(g)$ if and only if $f(x) \neq 0$ and $g(x) \neq 0$ if and only if $f(x)g(x) \neq 0$ if and only if $x \in D(fg)$. In particular, this means that finite intersections of distinguished open sets are distinguished open sets.

Further, all open sets, $U \subseteq X$, arise in this way. By definition to be open U must be the complement of some closed subset, so $U = X \setminus V(f_1, \dots, f_k)$ for some $f_i \in A(X)$. Then by Lemma 2.1.12 and De Morgan's laws

$$U = X \setminus V(f_1, \dots, f_k) = X \setminus (V(f_1) \cap \dots \cap V(f_k)) = D(f_1) \cup \dots \cup D(f_k). \quad (5.1.15)$$

This means that, in a sense, the distinguished open subsets are the “smallest” open subsets of X . In a more precise sense, the distinguished open subsets form a basis of the Zariski topology.

For this reason it's particularly important to understand the regular functions on distinguished open subsets, which we do with the following result.

Proposition 5.1.16 Let X be an affine variety, and let $f \in A(X)$. Then

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N} \right\}. \quad (5.1.17)$$

In particular, on a distinguished open subset a regular function is always globally the quotient of two polynomial functions.

Proof. The inclusion of the right hand side in the left is clear, every function of the form f/g^n with $g \in A(X)$ is a regular function on $D(f)$.

For the opposite inclusion let $\varphi : D(f) \rightarrow K$ be a regular function. Then for any $a \in D(f)$ there exists some local representation, $\varphi = g_a/f_a$ for some $f_a, g_a \in A(X)$, valid on an open neighbourhood, $U_a \subseteq U$, of a . We can always then find within U_a a distinguished open subset, $D(h_a)$, for some $h_a \in A(X)$. We can then write $g_a/f_a = g_a h_a/f_a h_a$ on $D(h_a)$, since by definition h_a is nonzero on $D(h_a)$. Then both the numerator and denominator of this representation of φ vanish on $V(h_a)$, which is the complement of $D(h_a)$. This means that the denominator vanishes on $V(h_a)$ and not on $D(h_a)$, meaning the denominator has exactly the same zero locus as h_a . This allows us to assume that the denominator is h_a .

As a consequence we have that in $A(X)$ $g_a f_b = g_b f_a$ for all $a, b \in D(f)$, since these two functions must agree on $D(f_a) \cap D(f_b)$ as on this region both are valid representations of φ , so must be equal, and outside this region both vanish, so are again equal.

We have that $D(f) = \bigcup_{a \in D(f)} D(f_a)$, and taking the complement

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(\{f_a \mid a \in D(f)\}). \quad (5.1.18)$$

Then by the relative Nullstellensatz (Theorem 2.4.9) we have

$$f \in I(V(f)) = I(V(\{f_a \mid a \in D(f)\})) = \sqrt{\langle f_a \mid a \in D(f) \rangle}. \quad (5.1.19)$$

Thus, $f^n = \sum_a k_a f_a$ for some $n \in \mathbb{N}$ and $k_a \in A(X)$, and k_a is nonzero for only finitely many $a \in D(f)$. Then if we set $g = \sum_a k_a g_a$, with the same coefficients as f^n , we have that $\varphi = g/f^n$ on $D(f)$ since for all $b \in D(f)$ we have $\varphi = g_b/f_b$ and

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b f^n, \quad (5.1.20)$$

having used that $g_a f_b = g_b f_a$, which shows that these functions agree on $D(f_b)$, and since these open subsets cover $D(f)$ these functions agree on all of $D(f)$. \square

Corollary 5.1.21 Let X be an affine variety. Then $\mathcal{O}_X(X) = A(X)$.

Proof. Setting $f = 1$ in Proposition 5.1.16 we have that $D(1) = X \setminus V(1) = X \setminus \emptyset = X$, and thus

$$\mathcal{O}_X(X) = \mathcal{O}_X(D(1)) = \left\{ \frac{g}{1^n} \mid g \in A(X), n \in \mathbb{N} \right\} = A(X). \quad (5.1.22)$$

This result is not true without the Nullstellensatz. For example, over the non-algebraically closed field \mathbb{R} we get everywhere defined functions which are not polynomial functions, such as $1/(x^2 + 1)$.

This result is really one of commutative algebra. We've worked with polynomial functions here where there's an existing notion of division, but it's possible to restate things in terms of localisations.

Corollary 5.1.23 Let X be an affine variety and $f \in A(X)$. Then $\mathcal{O}_X(D(f))$ is isomorphic as a K -algebra to the localisation, $A(X)_f$, of the coordinate ring at the multiplicatively closed subset $\{f^n \mid n \in \mathbb{N}\}$.

Proof. Consider the obvious K -algebra homomorphism

$$A(X)_f \rightarrow \mathcal{O}_X(D(f)) \quad (5.1.24)$$

$$\frac{g}{f^n} \mapsto \frac{g}{f^n}. \quad (5.1.25)$$

This is interpreting a formal fraction in $A(X)_f$ as an honest-to-god quotient of polynomials on $D(f)$. We have to show that this is well-defined. If $g/f^n = g'/f^m$ in $A(X)_f$ then by definition there exists $k \in \mathbb{N}$ such that $f^k(gf^m - g'f^n) = 0$ in $A(X)$. Then, since we work over a field, $g^m = g'f^n$, and so $g/f^n = g'/f^m$ as ratios of polynomials.

The homomorphism is surjective since by Proposition 5.1.16 all elements of $\mathcal{O}_X(D(f))$ are of the form g/f^n . It is injective since $g/f^n = 0$ as a function

on $D(f)$ if and only if $g = 0$ on $D(f)$, and thus $fg = 0$ on all of X and so $f(g \cdot 1 - 0 \cdot f^n) = 0$ in $A(X)$, and thus $g/f^n = 0/1$ as formal fractions in $A(X)_f$, and $0/1$ is exactly zero in $A(X)_f$, so this map has trivial kernel. \square

Example 5.1.26 Consider $U = \mathbb{A}^2 \setminus \{0\}$. This is an open subset since $\{0\} = V(x_1, x_2)$ is closed. We claim that

$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = K[x_1, x_2]. \quad (5.1.27)$$

This then means that $\mathcal{O}_X(U) = \mathcal{O}_X(X)$, that is, every regular function on U can be extended to X . This is another example of a result which is similar in the algebraic-geometry and complex-analysis settings.

To see why this is true let $\varphi \in \mathcal{O}_X(U)$. Then φ is regular on $D(x_1) = (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$ and $D(x_2) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$. Thus, $\varphi = f/x_1^m$ on $D(x_1)$ and $\varphi = g/x_2^n$ for some $f, g \in A(\mathbb{A}^2 \setminus \{0\}) = K[x_1, x_2]$ and $m, n \in \mathbb{N}$. We can choose m and n such that f/x_1^m and g/x_2^n are reduced.

On the intersection, $D(x_1) \cap D(x_2)$, both representations are valid, and so $f x_2^n = g x_1^m$. The locus $V(f x_2^n - g x_1^m)$, upon which this equation holds, is closed, and so $f x_2^n = g x_1^m$ on $\overline{D(x_1) \cap D(x_2)} = \mathbb{A}^2$. Thus, $f x_2^n = g x_1^m$ in the polynomial ring, $A(\mathbb{A}^2) = K[x_1, x_2]$.

If $m > 0$ then x_1 must divide $f x_2^n$, and since $K[x_1, x_2]$ is a unique factorisation domain this means that x_1 must divide f , but this contradicts our assumption that f/x_1^m is reduced. So $m = 0$. Then $\varphi = f$ is a polynomial, and so $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = K[x_1, x_2]$ as claimed.

5.2 Sheaves

In this section we'll develop some machinery to help us work with function-like objects with some local property. This puts a lot of the learning work up front, once we've learned the things covered in this section a lot of things just work out in later sections.

The key idea is that we want to combine all of the rings $\mathcal{O}_X(U)$ into some object as U varies over open sets. This should of course be subject to constraints that mean like the elements of $\mathcal{O}_X(U)$ are sufficiently “function like”, although they need not be functions in the abstract definition.

Definition 5.2.1 — Presheaf A **presheaf**, \mathcal{F} , of rings on a topological space, X , consists of the following data

1. for every open set, $U \subseteq X$, a ring, $\mathcal{F}(U)$;
2. for every inclusion of open sets, $U \subseteq V \subseteq X$, a ring homomorphism, $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, called the **restriction map**.

This is subject to the following conditions

1. $\mathcal{F}(\emptyset) = 0$;

2. $\rho_{U,U} = \text{id}_{\mathcal{F}(U)}$;
3. for any inclusions, $U \subseteq V \subseteq W$, of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ are called **sections** of \mathcal{F} over U . We often write $\rho_{V,U}(\varphi) = \varphi|_U$.

The idea is that sections, elements of $\mathcal{F}(U)$, are like functions on U , and the restriction maps are like restriction of a function. See the following example.

Example 5.2.2 Let X be a topological space. Consider the sheaf, \mathcal{F} , of functions on X . That is, for open sets $U \subseteq X$ we set $\mathcal{F}(U)$ to be the ring of functions, $U \rightarrow K$. Then $\rho_{V,U}$ is restricting a function from the domain V to the domain U .

We have that $\mathcal{F}(\emptyset)$ consists of the single function $\emptyset \rightarrow \mathbb{R}$. Thus, $\mathcal{F}(\emptyset)$ must be the zero ring. The fact that $\rho_{U,U} = \text{id}_{\mathcal{F}(U)}$ simply means that restricting a function to its domain doesn't do anything. The composition of restriction maps also makes sense: $\rho_{V,U} \circ \rho_{W,V}$ first restricts from W to V then from V to U , which is clearly the same as $\rho_{W,U}$, which restricts directly from W to U .

The word “section” comes from fibre bundles, $\pi : E \rightarrow B$, where a local section is a map $\sigma : U \rightarrow E$ (for $U \subseteq B$ open) such that $\pi(\sigma(x)) = x$ for all $x \in U$. The set of all local sections forms an important example of a presheaf (in fact a sheaf).

Remark 5.2.3 We have defined above a presheaf of rings. We can replace rings with many other categories, such as sets, K -algebras, abelian groups, or modules. The only changes that we have to make to the definition are

1. $\mathcal{F}(U)$ should be an object in the relevant category;
2. $\rho_{V,U}$ should be a morphism in the relevant category;
3. $\mathcal{F}(\emptyset)$ should be the terminal object of the relevant category (that is the object which is both initial and terminal).

Remark 5.2.4 — Presheaf is a Functor Let U be the poset category of open sets of X . That is, objects are open sets, $U \subseteq X$, and there is a unique morphism $i_{U,V} : U \rightarrow V$ if $U \subseteq V$. Consider a functor $F : U^{\text{op}} \rightarrow \text{Ring}$. This assigns to each open subset, $U \subseteq X$, a ring, $F(U)$. To each pair of open subsets, $U \subseteq V \subseteq X$, it assigns a map $i_{V,U} = F(i_{U,V}) : F(V) \rightarrow F(U)$. The fact that this is a contravariant functor means that

1. $F(\emptyset) = 0$, since a contravariant functor sends the initial object to the terminal objects.
2. for $U \subseteq V \subseteq W \subseteq X$ open we have unique maps $i_{U,V} : U \rightarrow V$ and

$i_{V,W} : V \rightarrow W$, which we can compose to get a map $i_{V,W} \circ i_{U,V} : U \rightarrow W$, which by uniqueness of maps in \mathbf{U} must just be $i_{U,W} : U \rightarrow W$. Applying F we have $F(i_{V,W} \circ i_{U,V}) = F(i_{V,W}) \circ F(i_{U,V}) = r_{W,V} \circ r_{V,U}$. Applying F to $i_{U,W}$ we get $F(i_{U,W}) = r_{W,U}$. Thus, $r_{W,V} \circ r_{V,U} = r_{W,U}$.

3. for $U \subseteq X$ we have the (unique) identity arrow, $\text{id}_U : U \rightarrow U$, and applying F to this we get $F(\text{id}_U) = \text{id}_{F(U)}$.

We see that we can define a presheaf, \mathcal{F} , by taking $\mathcal{F}(U) = F(U)$ and $r_{V,W} = r_{V,W}$. Conversely, any sheaf defines such a functor. So a presheaf is nothing but a functor $\mathbf{U}^{\text{op}} \rightarrow \text{Ring}$ (or replace Ring with some other category).

Example 5.2.5 — Preheaf of Functions Let X be a topological space. The sheaf of functions, \mathcal{F} , on X has $\mathcal{F}(U) = \{f : U \rightarrow \mathbb{C}\}$, which is a ring under pointwise operations. There's a unique function $\emptyset \rightarrow \mathbb{C}$, and so $\mathcal{F}(\emptyset) = 0$ is the zero ring. The restriction maps are just the normal restriction of functions, which then work out to have all the desired properties.

Example 5.2.6 — Constant Preheaf Let X be a topological space, and let S be a set. The constant presheaf, $\underline{S}_{\text{pr}}$, consists of all constant functions. That is, $\underline{S}_{\text{pr}}(U)$ consists of functions $U \rightarrow S$ which are constant. The restriction maps are the usual restriction of functions. Since there is a unique function $\emptyset \rightarrow S$, which is vacuously constant, we have that $\underline{S}_{\text{pr}}(\emptyset) = 0$, and as discussed restriction works out.

The notion of a presheaf captures a lot of what it means to be a function, even if the sections aren't actually functions. The notion of a sheaf captures a lot of what it means to be a function with some additional local property, such as smoothness, or more importantly for us, regularity.

Definition 5.2.7 — Sheaf A **sheaf** of rings is a presheaf, \mathcal{F} , satisfying the **gluing property**: if $U \subseteq X$ is an open set with an open cover, $\{U_i\}_{i \in I}$, and there are $\varphi_i \in \mathcal{F}(U_i)$ such that $\varphi|_{U_i \cap U_j} = \varphi|_{U_i \cap U_j}$ then there exists some unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Remark 5.2.8 Sometimes the gluing property is split up into two parts, uniqueness and existence:

1. locality/identity: if $\varphi, \psi \in \mathcal{F}(U)$ are such that $\varphi|_{U_i} = \psi|_{U_i}$ for all i then $\varphi = \psi$;
2. gluability: if $\varphi_i \in \mathcal{F}(U_i)$ are such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ then there exists $\varphi \in \mathcal{F}(U)$ such that $\varphi_i = \varphi|_{U_i}$ for all $i \in I$.

As we've stated it our glubability axiom has "there exists some unique φ ", and this takes care of the locality property also.

The idea here is that we can take the φ_i to be functions defined on covering sets, and then as long as they agree on the overlap of these sets we can glue them together into a single function defined on the whole set. For this to be possible we usually require some nice property of our functions. For example, they may need to be continuous, smooth, or holomorphic, depending on whether we want the resulting function to be continuous, smooth, or holomorphic, so depending on whether we're doing topology, differential geometry, or complex analysis. Functions on their own rarely glue together nicely.

Example 5.2.9 Let X be an affine variety. Then the **sheaf of regular functions** is \mathcal{O}_X where $\mathcal{O}_X(U)$ is the ring of regular functions on the open set $U \subseteq X$.

Example 5.2.10 — Locally Constant Sheaf Let X be a topological space, and let S be a set. The locally constant sheaf (of sets), \underline{S} , consists of all locally constant functions. That is, $\underline{S}(U)$ consists of functions $U \rightarrow S$ where for $p \in U$ there exists a neighbourhood of p , $U_p \subseteq U$, on which the function is constant. The restriction maps are the usual restriction of functions. Since there is a unique function $\emptyset \rightarrow S$ which is vacuously locally constant we have that $\underline{S}(\emptyset) = 0$. We have the gluing property. Let $\{U_i\}_{i \in I}$ be an open cover of U and $\varphi_i \in \underline{S}(U_i)$ with $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. We claim that these glue uniquely to give $\varphi \in \underline{S}(U)$ defined by $\varphi(p) = \varphi_i(p)$ if $p \in U_i$. We need only to show that this is well-defined, that is, if $p \in U_i \cap U_j$ then $\varphi_i(p) = \varphi_j(p)$. This is true since $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$.

Note that we have to take *locally constant* functions, not just constant ones, if we want a sheaf, as opposed to a presheaf. The reason for this is if we take $U = U_1 \sqcup U_2$, then we may define $\varphi_1(p) = s$ for $p \in U_1$ and $\varphi_2(p) = s'$ for $p \in U_2$ with $s \neq s'$. Then these don't glue to define a function which is constant on $U = U_1 \sqcup U_2$. However, if our space is connected then we cannot have this scenario arise and we do get a sheaf of constant functions.

Definition 5.2.11 — Restriction Let \mathcal{F} be a presheaf on X . Then we define the restriction of \mathcal{F} to an open set, $U \subseteq X$, to be the presheaf $\mathcal{F}|_U$, on U , where $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all open sets $V \subseteq U$, and the restriction maps are those of \mathcal{F} .

If \mathcal{F} is a sheaf so is $\mathcal{F}|_U$.

Definition 5.2.12 — Stalks Let \mathcal{F} be a presheaf on X . The **stalk** of \mathcal{F} at $a \in X$ is

$$\mathcal{F}_a = \{(U, \varphi) \mid U \subseteq X \text{ open}, a \in U, \varphi \in \mathcal{F}(U)\}/\sim \quad (5.2.13)$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists some open subset, $V \subseteq U \cap U'$, on which $\varphi|_V = \varphi'|_V$.

We call elements of \mathcal{F}_a **germs** of \mathcal{F} at a .

The stalk, \mathcal{F}_a , inherits the structure of a ring from the $\mathcal{F}(U)$ for U a neighbourhood of a . For example, $(U, \varphi) + (V, \psi) = (U \cap V, \varphi + \psi)$. Of course, one needs to show that this is well-defined.

There is a morphism $\mathcal{F}(U) \rightarrow \mathcal{F}_a$, given by $\varphi \mapsto (U, \varphi)$.

The idea here is that $\mathcal{F}(U)$ contains information about the sheaf in a neighbourhood and \mathcal{F}_a contains information about the sheaf at a point. That is, we think of $\mathcal{F}(U)$ as consisting of functions defined on an open set, U , and we think of \mathcal{F}_a as consisting of functions defined on an arbitrarily small open neighbourhood of a . This can be made more precise by defining the stalk to be a directed limit:

$$\mathcal{F}_a = \varinjlim_{U \ni a} \mathcal{F}(U). \quad (5.2.14)$$

Here we have a directed system of functions on open neighbourhoods of a , joined by restrictions, and part of the definition of a directed limit is that we consider two elements the same if they eventually are the same in this limit, that is, if after sufficient restriction they are the same. Intuitively, we take some nested neighbourhoods of a and if we can keep getting smaller and smaller, “zooming in” on a , until the two functions agree, we consider them to be the same function.

Despite stalks seemingly having less information it is often sufficient to look at stalks to determine whether a given property holds for a sheaf.

Note that when we have a sheaf with a name like \mathcal{O}_X we typically denote the stalk at a as $\mathcal{O}_{X,a}$, rather than, say $(\mathcal{O}_X)_a$, \mathcal{O}_{Xa} , or some other clunky notation.

Example 5.2.15 Let S be the locally constant sheaf. Then $S_a = S$. We identify a germ, (U, φ) , with $\varphi(a) \in S$, and there is one such germ for each value of S . Then two germs, (U, φ) and (U', φ') are only the same if they agree on some open subset of $U \cap U'$ containing a , but because φ and φ' are locally constant this just means that $\varphi(a) = \varphi'(a)$, so this identification provides a bijection between S_a and S . This is all taking place with sheaves of sets, so we have an isomorphism, $S_a \cong S$ for all $a \in X$, as claimed.

Consider the sheaf of differentiable functions on the real line (with the standard topology). A germ at $a \in \mathbb{R}$ allows us to compute the value of the corresponding function at a , and the value of the derivative of the function at a . It does not allow us to compute the value of the function anywhere else.

Consider the sheaf of holomorphic functions on \mathbb{C} (with the standard topology). Then since the value of such functions is fully determined by the value on some smaller open set the germs of these functions contain enough information to compute the value of the function at other points (as long as they’re part of the same connected open set as the point at which we take the germ).

Given an affine variety, X , there is a well-defined evaluation map, $\mathcal{O}_{X,a} \rightarrow K$, $(U, \varphi) \mapsto \varphi(a)$. This tells us the value of φ at a , but not at any other point.

It is possible to describe regular functions algebraically in terms of a localisation of the coordinate ring. This is, in fact, the reason why we call localisations localisations.

Lemma 5.2.16 Let a be a point in the affine variety X . Then the stalk, $\mathcal{O}_{X,a}$, is isomorphic as a K -algebra to the localisation $A(X)_{I(a)}$ at the maximal ideal $I(a) \trianglelefteq A(X)$. That is,

$$\mathcal{O}_{X,a} \cong \left\{ \frac{f}{g} \mid f, g \in A(X), g(a) \neq 0 \right\}. \quad (5.2.17)$$

This means that $\mathcal{O}_{X,a}$ is a local ring (in the sense of [Definition A.1.8](#)), having the unique maximal ideal

$$I_a := \{(U, \varphi) \in \mathcal{O}_{X,a} \mid \varphi(a) = 0\} \cong \left\{ \frac{f}{g} \mid f, g \in A(X), f(a) = 0, g(a) \neq 0 \right\}. \quad (5.2.18)$$

We call this the **local ring** of X at a .

Proof. Consider the map

$$\begin{aligned} F: A(X)_{I(a)} &\rightarrow \mathcal{O}_{X,a} \\ \frac{f}{g} &\mapsto \left(D(g), \frac{f}{g} \right). \end{aligned} \quad (5.2.19)$$

This sends a formal fraction, f/g , to the corresponding quotient of polynomial functions on the open set $D(g)$ on which the denominator doesn't vanish. Notice that $a \in D(g)$ since $g(a) \neq 0$ by definition. This is a K -algebra homomorphism by construction of the operations on $\mathcal{O}_{X,a}$. This map is well-defined, if $f/g = f'/g'$ in $A(X)_{I(a)}$ then $h(fg' - f'g) = 0$ for some $h \in A(X) \setminus I(a)$. Thus, the functions f/g and f'/g' agree on the open neighbourhood $D(h) \cap D(g) \cap D(g')$ of a , and so determine the same element of the stalk.

Further, F is surjective since by definition any regular function in a sufficiently small neighbourhood of a is representable as a fraction, f/g , with $f \in A(X)$ and $g \in A(X) \setminus I(a)$.

This map is injective. Suppose f/g is mapped to the zero element. That is, f/g is zero on an open neighbourhood of a . We can always shrink this open neighbourhood to be $D(h)$ for some $h \in A(X)$, since the distinguished open sets form a basis for the Zariski topology. Further, since $h(a) \neq 0$ (as we have $a \in D(h)$ by assumption) we know that $h \notin I(a)$, so $h \in A(X) \setminus I(a)$. Thus, we have $h(f \cdot 1 - 0 \cdot g) = 0$ on all of X , and thus this function is zero in $A(X)$. Therefore $g/f = 0/1$ in $A(X)_{I(a)}$, so the map has trivial kernel. \square

Six

Morphisms

We have looked at regular functions on some affine variety X . These are maps from some open subset of X to the field, K . Thinking of K as itself being the affine variety \mathbb{A}^1 we will consider these regular functions to be our first example of morphisms between affine varieties. Our goal in this chapter is to extend this definition to define morphisms between any two affine varieties.

It turns out that the correct way to define morphisms, $X \rightarrow Y$, is actually to look at regular functions out of both X and Y . This gives rise to two sheaves, and morphisms between sheaves give us what we want, although of course we still have to define morphisms between sheaves.

6.1 Ringed Spaces

In order to make these definitions it's important that the data of the regular functions is attached to our affine variety, as later we'll want to be able to define an affine variety abstractly in such a way that changing the regular functions also changes the structure of the affine variety. To this end we make the following definition pairing up the required data.

Definition 6.1.1 — Ringed Spaces A **ringed space**, (X, \mathcal{O}_X) , is a pair consisting of a topological space, X , and a sheaf, \mathcal{O}_X , on X . We call \mathcal{O}_X the **structure sheaf** of the ringed space.

Notation 6.1.2 We will typically just write X for a ringed space, in which case the structure sheaf is always denoted \mathcal{O}_X .

We will always consider an affine variety, X , as a ringed space, (X, \mathcal{O}_X) where the structure sheaf, \mathcal{O}_X , is the sheaf of regular functions. Likewise, any open subset, U , of a ringed space, X (including the case where X is an affine variety) will be considered as a ringed space with the structure sheaf given by restriction, $\mathcal{O}_U = \mathcal{O}_X|_U$ ([Definition 5.2.11](#)).

The reason that this is important for defining morphisms is we've attached extra information to our affine varieties. This means that our definition of a morphism, $f : X \rightarrow Y$, should preserve this extra information. In fact, we can define the notion of a morphism of ringed spaces more generally, and then morphisms of affine varieties are just a special case of this.

The correct notion of preserving the structure of the sheaves of regular functions is that given a regular function $\varphi : U \rightarrow K$ on an open subset $U \subseteq Y$ the composite, $\varphi \circ f : f^{-1}(U) \rightarrow K$ should again be a regular function. Note that this requires that $f^{-1}(U)$ is an open subset of X , so at the very least f must be continuous.

There is a problem with this requirement if we're aiming for full generality. While the composite $\varphi \circ f$ is defined for regular functions, φ , if we take an arbitrary ringed space then we cannot always assume that the elements of $\mathcal{O}_X(U)$ are functions, that is, composition isn't necessarily defined. There are two ways around this problem:

1. Lots of technicalities.
2. Ignore it.

We'll pick the later.

From now on until we specify otherwise we will assume that all sheaves of rings are actually sheaves of K -valued functions.

Note that this also makes all sheaves sheaves of K -algebras, since we can always define pointwise scalar multiplication. This allows us to make the following definitions.

6.2 Morphisms

Definition 6.2.1 — Pullback Let $f : X \rightarrow Y$ be a map between ringed spaces. For any map, $\varphi : U \rightarrow K$, from the open subset $U \subseteq Y$ we define the **pullback** of φ by f , $f^*\varphi : f^{-1}(U) \rightarrow K$ by precomposition, $f^*\varphi = \varphi \circ f$.

Definition 6.2.2 — Morphism of Ringed Spaces Let $f : X \rightarrow Y$ be a map of ringed spaces. This is a **morphism of ringed spaces** if

1. f is continuous; and
2. for all open subsets $U \subseteq Y$ and for all $\varphi \in \mathcal{O}_Y(U)$ we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. That is, pulling back by f defines a K -algebra homomorphism

$$\begin{aligned} f^* : \mathcal{O}_Y(U) &\rightarrow \mathcal{O}_X(f^{-1}(U)) \\ \varphi &\mapsto f^*\varphi. \end{aligned} \tag{6.2.3}$$

As usual, we say that a morphism of ringed spaces is an isomorphism of ringed spaces if it has a two sided inverse which is also a morphism of ringed spaces.

Definition 6.2.4 — Morphism of Affine Varieties A morphism of (open subsets of) affine varieties is exactly a morphism of the corresponding ringed spaces.

Remark 6.2.5 Here are some of the technicalities we avoid by assuming all sheaves are sheaves of K -valued functions. Let \mathcal{F} and \mathcal{G} be sheaves on X . A **morphism of sheaves**, $\varphi: \mathcal{F} \rightarrow \mathcal{G}$, is a collection of morphisms, $\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for $U \subseteq X$ an open subset. These must be compatible with restriction, that is, for $U \subseteq V \subseteq X$ open the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(U) \\ \rho_{V,U}^{\mathcal{F}} \downarrow & & \downarrow \rho_{V,U}^{\mathcal{G}} \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array} \quad (6.2.6)$$

must commute, where $\rho_{V,U}^{\mathcal{F}}$ and $\rho_{V,U}^{\mathcal{G}}$ are the restriction maps of \mathcal{F} and \mathcal{G} respectively.

Composition of morphisms of sheaves is given by $(\varphi \circ \varphi')_U = \varphi_U \circ \varphi'_U$. With this definition sheaves of objects from some category, C , on a topological space, X , form a category themselves, often denoted C_X or $\text{Sh}_X(C)$. Note that if we interpret our sheaves as functors then $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is precisely a natural transformation.

Let $f: X \rightarrow Y$ be a continuous map of topological spaces. The **direct image functor**, $f_*: C_X \rightarrow C_Y$ sends a sheaf, \mathcal{F} , on X to the sheaf, $f_*\mathcal{F}$, on Y defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. If $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves then $f_*\varphi: f_*\mathcal{F}(U) \rightarrow f_*\mathcal{G}(U)$ is a morphism of sheaves given by the family of morphisms $\mathcal{F}(f^{-1}(U)) \rightarrow \mathcal{G}(f^{-1}(U))$ for $U \subseteq Y$ open given by $\psi \mapsto \varphi_{f^{-1}(U)}(\psi)$.

A **morphism of ringed spaces**, $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, is a pair, (f, φ) where $f: X \rightarrow Y$ is a continuous map between the underlying topological spaces and $\varphi: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves.

Unpacking this definition a little, a morphism of ringed spaces is a continuous function, $f: X \rightarrow Y$, and a family of morphisms $\varphi_U: \mathcal{O}_Y \rightarrow \mathcal{O}_X(f^{-1}(U))$ for all open sets $U \subseteq Y$ such that if $U \subseteq V \subseteq Y$ are open then

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\varphi_V} & \mathcal{O}_X(f^{-1}(V)) \\ \rho_{V,U}^{\mathcal{O}_Y} \downarrow & & \downarrow \rho_{V,U}^{f_*\mathcal{O}_X} \\ \mathcal{O}_Y(U) & \xrightarrow{\varphi_U} & \mathcal{O}_X(f^{-1}(U)) \end{array} \quad (6.2.7)$$

commutes.

Note that if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are morphisms of ringed spaces then so is $g \circ f: X \rightarrow Z$. This follows immediately from the definition since $g \circ f$ is also continuous and for $\varphi \in \mathcal{O}_Z(U)$ we have $(g \circ f)^*\varphi = \varphi \circ g \circ f = f^*(g^*\varphi) = f^*(g^*\varphi)$

and since $f^* : \mathcal{O}_Y(g^{-1}(U)) \rightarrow \mathcal{O}_X(f^{-1}(g^{-1}(U)))$ and $g^* : \mathcal{O}_Z(U) \rightarrow \mathcal{O}_Y(g^{-1}(U))$ are both K -algebra homomorphisms so is $(g \circ f)^* = f^* \circ g^*$.

The identity map of ringed spaces is $\text{id}_X : X \rightarrow X$ and $\text{id}_X^* : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(\text{id}_X^{-1}(U)) = \mathcal{O}_X(U)$ is defined by $\varphi \mapsto \text{id}_X^*\varphi = \varphi \circ \text{id}_X = \varphi$.

Associativity of composition follows from associativity of composition of continuous functions and ring homomorphisms. Thus, ringed spaces form a category.

Restrictions of morphisms are also morphisms. That is, if $f : X \rightarrow Y$ is a morphism of ringed spaces and we have open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$ then the restricted map $\tilde{f} : U \rightarrow V$ given by $\tilde{f}(x) = f(x)$ is again a morphism of ringed spaces. This again essentially follows from the definition and the fact that sheaves are set up to work with restriction.

There is also a gluing property similar to that of sheaves.

Lemma 6.2.8 — Gluing Property of Morphisms Let $f : X \rightarrow Y$ be a map of ringed spaces, and let $\{U_i\}_{i \in I}$ be an open cover of X such that all restrictions, $f|_{U_i} : U_i \rightarrow Y$, are morphisms of ringed spaces. Then f is a morphism of ringed spaces.

Proof. We need to check two things. We start with continuity. Let $V \subseteq Y$ be open. Then

$$f^{-1}(V) = \bigcap_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} (f|_{U_i})^{-1}(V). \quad (6.2.9)$$

Since the restrictions are all continuous and V is open we know that $(f|_{U_i})^{-1}(V)$ is open in U_i and thus their union is open in X .

The second thing we have to show is that f maps pullback sections of \mathcal{O}_Y to sections of \mathcal{O}_X . Again, let $V \subseteq Y$ be open, and let $\varphi \in \mathcal{O}_Y(V)$. Then

$$(f^*\varphi)|_{U_i \cap f^{-1}(V)} = (f|_{U_i \cap f^{-1}(V)})^* \varphi \in \mathcal{O}_X(U_i \cap f^{-1}(V)) \quad (6.2.10)$$

since $f|_{U_i}$ and therefore also $f|_{U_i \cap f^{-1}(V)}$ are morphisms. Then by the gluing property for sheaves this means that $f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$. \square

We can apply this result to morphisms between (open subsets of) affine varieties.

Proposition 6.2.11 Let U be an open subset of an affine variety, X , and let $Y \subseteq \mathbb{A}^n$ be an affine variety. Then a morphism, $f : U \rightarrow Y$, is a map of the form

$$\begin{aligned} f &= (\varphi_1, \dots, \varphi_n) : U \rightarrow Y \\ x &\mapsto (\varphi_1(x), \dots, \varphi_n(x)) \end{aligned} \quad (6.2.12)$$

where $\varphi_i \in \mathcal{O}_X(U)$.

Proof. Suppose $f : U \rightarrow Y$ is a morphism of affine varieties. Then for $i = 1, \dots, n$ we can define the i th coordinate function, $y_i : Y \rightarrow K$ to simply be projecting out the i th coordinate, $y_i(a_1, \dots, a_n) = a_i$. This is a regular

function on Y , and thus we have that $\varphi_i := f^*y_i \in \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U)$ by the properties of a morphism. We then also have that $f^*y_i = y_i \circ f$, which is just the i th component function of f . Thus, we have $f = (\varphi_1, \dots, \varphi_n)$. Conversely, suppose that $f = (\varphi_1, \dots, \varphi_n)$ with $\varphi_i \in \mathcal{O}_X(U)$ and $f(U) \subseteq Y$. First, we show that f is continuous. Let Z be any closed subset of Y . Then $Z = V(g_1, \dots, g_m)$ for some $g_i \in A(Y)$ and

$$f^{-1}(Z) = \{x \in U \mid g_i(\varphi_1(x), \dots, \varphi_n(x)) = 0 \forall i = 1, \dots, m\}. \quad (6.2.13)$$

The functions $x \mapsto g_i(\varphi_1(x), \dots, \varphi_n(x))$ are regular, since the φ_i are regular and the g_i are polynomials, and evaluating a polynomial at a quotient of polynomials again gives a quotient of polynomials locally. Thus, $f^{-1}(Z)$ is closed in U since the zero loci of regular functions are closed ([Lemma 5.1.10](#)). Thus, f is continuous.

Now suppose that $\psi \in \mathcal{O}_Y(W)$ is a regular function on an open subset, $W \subseteq Y$. Then we have

$$\begin{aligned} f^*\psi &= \psi \circ f : f^{-1}(W) \rightarrow K \\ x &\mapsto \psi(\varphi_1(x), \dots, \varphi_n(x)), \end{aligned} \quad (6.2.14)$$

which is again regular, since replacing the variables in a quotient of polynomials by other quotients of polynomials results again in a quotient of polynomials. Thus, $f^*\psi \in \mathcal{O}_X(f^{-1}(W))$, and so f is indeed a morphism. \square

Corollary 6.2.15 Regular functions from $U \rightarrow \mathbb{A}^1$ are precisely the regular functions in $\mathcal{O}_X(U)$.

These results are a sanity check on our definitions. A morphism mapping to an affine subvariety of \mathbb{A}^n is simply an n -tuple of regular functions whose image lies in said affine subvariety. Further, we see that regular functions are really just the special case of $n = 1$.

We can take $U = X$. Then in this case we can interpret morphisms entirely algebraically through the following corollary.

Corollary 6.2.16 Let X and Y be affine varieties. Then there is a bijection

$$\{\text{morphisms } X \rightarrow Y\} \xleftrightarrow{1:1} \{\text{K-algebra homomorphisms } A(Y) \rightarrow A(X)\}. \quad (6.2.17)$$

Proof. By definition, any morphism, $f : X \rightarrow Y$, determines a K -algebra homomorphism, $f^* : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$, and since $\mathcal{O}_Y(Y) = A(Y)$ and $\mathcal{O}_X(X) = A(X)$ this gives us one direction of the bijection.

Conversely, let $g : A(Y) \rightarrow A(X)$ be a K -algebra homomorphism. Take $Y \subseteq \mathbb{A}^n$ and let y_1, \dots, y_n be the coordinate functions of \mathbb{A}^n . Then we can define $\varphi_i := g(y_i) \in A(X) = \mathcal{O}_X(X)$. Setting $f = (\varphi_1, \dots, \varphi_n) : X \rightarrow \mathbb{A}^n$ we

then have that for any $h \in K[y_1, \dots, y_n]$

$$(f^*h)(x) = h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = g(h)(x) \quad (6.2.18)$$

for all $x \in X$. The last equality holds because both sides of the equation are K -algebra homomorphisms in h and on the generators y_i they both give $\varphi_i(x)$, so they must be the same K -algebra homomorphisms.

This shows that $h(f(x)) = 0$ for all $h \in I(Y)$, since these are the polynomials which map to zero in $A(Y)$, and thus g will vanish on these polynomials. Then the image of f is in $V(I(Y)) = Y$, so we have a map $f : X \rightarrow Y$ as needed. Its coordinate functions are regular, since they are just φ_i , and so by [Proposition 6.2.11](#) f is a morphism. The calculation above shows that $f^* = g$, and so this is indeed inverse to the first map stated. \square

Corollary 6.2.19 Under the above bijection isomorphisms of affine varieties correspond to isomorphisms of K -algebras.

Proof. This follows immediately upon noting that for $f : X \rightarrow Y$ and $g : Y \rightarrow X$ we have

$$(f \circ g)^* = g^* \circ f^* \quad \text{and} \quad (g \circ f)^* = f^* \circ g^* \quad (6.2.20)$$

and so if f and g are inverses then we have $\text{id}_Y^* = \text{id}_{A(Y)} = g^* \circ f^*$ so f^* has a left inverse and $\text{id}_X^* = \text{id}_{A(X)} = f^* \circ g^*$ so f^* has a right inverse. The converse also holds, if we assume that $A(Y) \rightarrow A(X)$ is an isomorphism it must arise as the pullback of some invertible morphism of affine varieties. \square

Remark 6.2.21 We can state this all a bit more formally with the language of category theory. There is a contravariant functor, A , from the category of affine varieties to the full subcategory of K -algebras which can arise as the coordinate rings of affine varieties^{a,b}. On objects this map is $X \mapsto A(X)$, and on morphisms it is $f \mapsto f^*$. The correspondence of isomorphisms in both categories is due to the fact that this functor both preserves and reflects isomorphisms.

^aThis is, as best I can tell, the full subcategory of finitely generated K -algebras with no nilpotent elements.

^bOne way to get around having to take this subcategory is to work with schemes, then we will have a corresponding functor between the category of affine schemes and the category of commutative unital rings. In fact, this functor is an equivalence, and can be used abstractly to define schemes.

Note that an isomorphism of affine varieties is *not* a bijective morphism, just as for topological spaces where the inverse of a continuous map need not be continuous. This is demonstrated in the following example.

Example 6.2.22 Let $X = V(x_1^2 - x_2^3) \subseteq \mathbb{A}^2$. Consider the map

$$\begin{aligned} f: \mathbb{A}^1 &\rightarrow X \\ t &\mapsto (t^3, t^2). \end{aligned} \tag{6.2.23}$$

Note that $(t^3, t^2) \in X$ since $(t^3)^2 - (t^2)^3 = 0$. Thus, f is a morphism of affine varieties since its component functions, $t \mapsto t^3$ and $t \mapsto t^2$, are regular (even polynomial) functions.

The corresponding K -algebra homomorphism is $f^*: A(X) \rightarrow A(\mathbb{A}^1)$. We have $A(X) = K[x_1, x_2]/\langle x_1^2 - x_2^3 \rangle$ and $A(\mathbb{A}^1) = K[t]$. Then $f^*(\overline{x_1}) = t^3$ and $f^*(\overline{x_2}) = t^2$, which we get by composing f with the coordinate functions on \mathbb{A}^2 .

The function f is a bijection, with inverse

$$\begin{aligned} f^{-1}: X &\rightarrow \mathbb{A}^1 \\ (x_1, x_2) &\mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0, \\ 0 & x_2 = 0. \end{cases} \end{aligned} \tag{6.2.24}$$

However, f is not an isomorphism, that is, f^{-1} is not a morphism, since this would require that f^* is an isomorphism, which it isn't as we can readily check that the linear polynomial, t , is not in the image of f^* .

6.3 Products

Recall that when we defined the product of affine varieties we didn't equip it with the product topology, instead we equipped it with the subspace topology viewing it as a subspace of $\mathbb{A}^m \times \mathbb{A}^n$ with the Zariski topology. This is the "correct" topology for the product because it means that the product is indeed an affine variety and satisfies the universal property of products. We just couldn't say this until now because the universal property of products of affine varieties involves morphisms between affine varieties.

Proposition 6.3.1 Let X and Y be affine varieties. Let $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ be projections onto the corresponding factors. Then $X \times Y$ equipped with these projections satisfies the universal property of a product.

Proof. First note that π_X and π_Y are indeed morphisms of affine varieties since they are of the form $(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n}) \mapsto (a_1, \dots, a_m)$ and $(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n}) \mapsto (a_{m+1}, \dots, a_{m+n})$, which clearly have regular component functions (being identities).

The universal property of products states that for any other affine variety, Z , equipped with morphisms $f_X: Z \rightarrow X$ and $f_Y: Z \rightarrow Y$ there exists a unique map $f: Z \rightarrow X \times Y$ such that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$. Given such a Z with f_X and f_Y we can simply take $f(z) = (f_X(z), f_Y(z))$. This

is then a morphism of affine varieties since its component functions are regular since f_X and f_Y are morphisms. By construction we have $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$. Thus, $X \times Y$ is really the product in the category of affine varieties. \square

Under the correspondence between affine varieties and their coordinate rings the universal property of the product becomes the universal property of the coproduct of K -algebras. The coproduct of K -algebras is just the tensor product over K . Thus, the coordinate ring of $A(X \times Y)$ is precisely $A(X) \otimes_K A(Y)$.

Appendices

A

Commutative Algebra

Here we collect some results from commutative algebra which we'll make use of in the course. This won't be very well organised, and is more for reference than actual reading. The conditions to be included here are pretty much "I had to look it up" or "I had to think about it for more than 10 seconds" while writing these notes, or "I thought it was worth recapping".

A.1 Ideals

Definition A.1.1 — Prime Ideal A proper ideal, $\mathfrak{p} \trianglelefteq R$, is **prime** if whenever $ab \in \mathfrak{p}$ for $a, b \in R$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Equivalently, \mathfrak{p} is prime if R/\mathfrak{p} is an integral domain.

Definition A.1.2 — Maximal Ideal A proper ideal, $\mathfrak{m} \trianglelefteq R$, is **maximal** if whenever there is another ideal, $I \trianglelefteq R$, with $\mathfrak{m} \subseteq I$ then either $I = \mathfrak{m}$ or $I = R$. Equivalently, \mathfrak{m} is maximal if R/\mathfrak{m} is a field.

Lemma A.1.3 Let R be a ring with ideals I and J . Then $IJ \subseteq I \cap J$.

Proof. If $a \in I$ and $b \in J$ then $ab \in I$ and $ab \in J$ by definition of an ideal. Then $ab \in I \cap J$. \square

Lemma A.1.4 Let R be a ring with ideals I and J . Then $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Proof. We prove a circle of inclusions. We start with $\sqrt{IJ} \subseteq \sqrt{I \cap J}$, which follows from Lemma A.1.3.
If $a \in \sqrt{I \cap J}$ then $a^k \in I \cap J$ for some $k \in \mathbb{N}$. Thus, $a^k \in I$ and $a^k \in J$. Hence, $a \in \sqrt{I} \cap \sqrt{J}$.
If $a \in \sqrt{I} \cap \sqrt{J}$ then $a^k \in I$ and $a^\ell \in J$ for some $k, \ell \in \mathbb{N}$. Then $a^k a^\ell =$

$a^{k+\ell} \in IJ$, and so $a \in \sqrt{IJ}$.

□

Lemma A.1.5 Every prime ideal is radical.

Proof. Let \mathfrak{p} be a prime ideal of a ring, R . Consider $\sqrt{\mathfrak{p}}$. If $a \in \sqrt{\mathfrak{p}}$ then there exists some $k \in \mathbb{N}$ such that $a^k \in \mathfrak{p}$. Suppose that k is minimal in making this true. If $k = 1$ then $a \in \mathfrak{p}$. If $k > 1$ then by the definition of a prime ideal have $x \cdot x^{k-1} \in \mathfrak{p}$ implying $x \in \mathfrak{p}$ or $x^{k-1} \in \mathfrak{p}$. However, the later cannot be the case because k was assumed minimal. Therefore, $x \in \mathfrak{p}$, and since $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$ (Lemma 2.1.19) it must be that $\mathfrak{p} = \sqrt{\mathfrak{p}}$. □

Lemma A.1.6 — Chinese Remainder Theorem Let R be a ring with ideals I and J . Consider the ring homomorphism

$$\varphi: R \rightarrow \frac{R}{I} \times \frac{R}{J} \quad (\text{A.1.7})$$

which sends r to $(r \bmod I, r \bmod J)$. Then

1. φ is injective if and only if $I \cap J = \langle 0 \rangle$;
2. φ is surjective if and only if $I + J = R$.

Proof. Notice that $\ker \varphi$ consists of all $r \in R$ such that $(r \bmod I, r \bmod J) = (0, 0)$, which means precisely that $r \in I$ and $r \in J$, so $r \in I \cap J$. Thus, if $I \cap J = \langle 0 \rangle$ then $\ker \varphi = \langle 0 \rangle$, so φ is injective. The same logic in reverse proves the converse.

Suppose that φ is surjective. Then $(1, 0)$ is in the image of φ , so there exists some element $r \in R$ with $(r \bmod I, r \bmod J) = (1, 0)$, which means $r \bmod J = 0$ so $r \in J$. Then $1 - r \bmod I = 1 - 1 = 0$, so $1 - r \in I$ and $r \in J$, so $1 = (1 - r) + r \in I + J$ so $I + J = R$.

Now suppose that $I + J = R$. Then there exist $a \in I$ and $b \in J$ such that $a + b = 1$, so $b = 1 - a$, meaning $b \bmod I = 1 - a \bmod I = 1 - 1 = 0$ and $a + b \bmod I = a \bmod I = 1$. Thus, $(1, 0)$ is in the image of φ . Similarly, $(0, 1)$ is in the image of φ . Since $(1, 0)$ and $(0, 1)$ generate $R/I \times R/J$ we are done. □

Definition A.1.8 — Local Ring A **local ring** is a ring with a unique maximal ideal.

A.2 Noetherian Rings

Definition A.2.1 — Noetherian Ring A ring, R , is **Noetherian** if it satisfies the ascending chain condition. That is, if every chain of ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \tag{A.2.2}$$

terminates, so $I_{n+1} = I_n$ for sufficiently large n .

Note that all fields are Noetherian, and so is \mathbb{Z} .

Definition A.2.3 — Noetherian Module Let R be a ring, and let M be an R -module. We say that M is **Noetherian** if it satisfies the ascending chain condition. That is, if every chain of submodules,

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \tag{A.2.4}$$

terminates, so $M_{n+1} = M_n$ for sufficiently large n .

Note that R is Noetherian as a ring exactly when R is Noetherian as an R -module.

Lemma A.2.5 Let R be a ring. The following are equivalent:

1. R is a Noetherian.
2. Every ideal of R is finitely generated.

Lemma A.2.6 Let R be a ring and M an R -module. The following are equivalent:

1. M is Noetherian.
2. Every submodule of M is finitely generated.
3. Every nonempty family of submodules of M has a maximal element.

Lemma A.2.7 Let M be a Noetherian R -module. Then any submodule of M is also Noetherian.

Proof. Let N be a submodule of M . Then any ascending chain, $N_1 \subseteq N_2 \subseteq \cdots$, in N is also an ascending chain in M , and thus stabilises. Thus N is also Noetherian. \square

Lemma A.2.8 Let M be a Noetherian R -module. Then any quotient of M is also Noetherian.

Proof. Let N be a submodule of M and consider the quotient M/N with the quotient map $\pi : M \rightarrow M/N$. Let $P_1 \subseteq P_2 \subseteq \dots$ be an ascending chain in M/N . Then setting $M_i = \pi^{-1}(P_i)$ defines an ascending chain, $M_1 \subseteq M_2 \subseteq \dots$, in M , which must stabilise. Thus, for sufficiently large n we have $M_{n+1} = M_n$. Since π is surjective this implies that $P_{n+1} = \pi(M_{n+1}) = \pi(M_n) = P_n$, and thus our original chain in M/N stabilises. \square

A.2.1 Hilbert's Basis Theorem

Theorem A.2.9 — Hilbert's Basis Theorem. If R is a Noetherian ring then $R[x]$ is also Noetherian.

Corollary A.2.10 If R is a Noetherian ring then $R[x_1, \dots, x_n]$ is Noetherian.

A.3 Primary Ideals

Definition A.3.1 — Primary Ideal Let R be a ring. An ideal, $Q \trianglelefteq R$, with $Q \neq R$ is called **primary** if for all $a, b \in R$ with $ab \in Q$ we have $a \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$.

Note that this definition is equivalent to $b \in \sqrt{Q}$.

Every prime ideal is primary, with $n = 1$. Conversely, the definition of a primary ideal is exactly such that \sqrt{Q} is a prime ideal whenever Q is a primary ideal.

The idea to keep in mind, which is true for PIDs, is that primary ideals are powers of prime ideals. For example, (p^n) is a primary ideal of \mathbb{Z} for p prime (so (p) is a prime ideal).

Definition A.3.2 — Primary Decomposition Let R be a ring and $I \trianglelefteq R$ an ideal. Then a **primary decomposition** of I is a finite set of primary ideals, Q_1, \dots, Q_r , such that $I = Q_1 \cap \dots \cap Q_r$.

Lemma A.3.3 In a Noetherian ring every ideal admits a primary decomposition.

A.4 Localisation

Definition A.4.1 — Multiplicatively Closed Set Let R be a commutative ring. A set, S , is called **multiplicatively closed** if S is closed under multiplication and contains 1.

Definition A.4.2 — Localisation Let R be a ring, and let S be a multiplicatively closed subset of R . The **localisation** of R at S is the ring,

$$S^{-1}R = (R \times S)/\sim \quad (\text{A.4.3})$$

where $(r, s) \sim (r', s')$ if there exists some $t \in S$ such that

$$t(sr' - s'r) = 0. \quad (\text{A.4.4})$$

The operations in this ring are

$$(r, s) + (r', s') = (rs' + r's, ss'), \quad (\text{A.4.5})$$

$$(r, s)(r', s') = (rr', ss'). \quad (\text{A.4.6})$$

The idea is that (r, s) is really the fraction, $\frac{r}{s}$. This is where the definition of the operations comes from. The key example being $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$, in which case $S^{-1}R = \mathbb{Q}$. The equivalence relation is then enforcing that things like $1/2$ and $2/4$ are considered the same. The t is needed to deal with the case where there are zero divisors. The multiplicative identity of $S^{-1}R$ is $1/1$, and the additive identity is $0/s$ for any $s \in S$ (they're all equivalent). There's an embedding, $R \hookrightarrow S^{-1}R$, $r \mapsto r/1$.

Note that if R is an integral domain then $S^{-1}R$ is its field of fractions.

If $0 \in S$ then $S^{-1}R = 0$.

The localisation of S in R has the universal property that the map $i : R \hookrightarrow S^{-1}R$ is universal in that if $f : R \rightarrow T$ is a ring homomorphism sending every element of S to a unit of T then there exists a unique ring homomorphism, $g : S^{-1}R \rightarrow T$, such that $f = g \circ i$. That is,

$$\begin{array}{ccc} R & \xrightarrow{i} & S^{-1}R \\ & \searrow f & \downarrow \exists! g \\ & & T. \end{array} \quad (\text{A.4.7})$$

Notation A.4.8 Let R be a commutative ring and $x \in R$. Then we write R_x for $S^{-1}R$ with $S = \{1, x, x^2, \dots\}$.

Let \mathfrak{p} be a prime ideal of R . Then we write $R_{\mathfrak{p}}$ for $S^{-1}R$ with $S = R \setminus \mathfrak{p}$.

This notation can be a bit confusing. For R_x we've forced x to be invertible. For $R_{\mathfrak{p}}$ we've forced everything *not* in \mathfrak{p} to be invertible.

B

Topology

Here we collect some results from topology which we'll make use of in the course. This won't be very well organised, and is more for reference than actual reading. The conditions to be included here are pretty much "I had to look it up" or "I had to think about it for more than 10 seconds" while writing these notes, or "I thought it was worth recapping".

B.1 Results

Lemma B.1.1 Let X be a topological space. Then X is Hausdorff if and only if

$$\Delta = \{(x, x) \mid x \in X\} \tag{B.1.2}$$

is a closed subset of $X \times X$ with the product topology.

Proof. Suppose that Δ is closed. Let $x, y \in X$ be distinct points. We look for open sets, $U, V \subseteq X$, such that $x \in U, y \in V$ and $U \cap V = \emptyset$. To do so consider the point $p = (x, y) \in X \times X$. Since $x \neq y$ we know that $p \notin \Delta$, and so $p \in (X \times X) \setminus \Delta$, which is an open set by assumption. The product topology on $X \times X$ is generated by sets of the form $U \times V$ with open sets, $U, V \subseteq X$. Specifically, this means that there must be some choice of such a U and V with $p \in U \times V \subseteq (X \times X) \setminus \Delta$. Since $p \in U \times V$ we know that $x \in U$ and $y \in V$. If U and V are not disjoint we can always take open neighbourhoods, $U' \subseteq U$ and $V' \subseteq V$, containing x and y respectively, which are disjoint, which then separate x and y showing that X is Hausdorff.

Suppose instead that X is Hausdorff. We will show that $(X \times X) \setminus \Delta$ is open. To do so we show that any $p \in (X \times X) \setminus \Delta$ has an open neighbourhood disjoint from Δ . We know that $p \in X \times X$, so $p = (x, y)$ for some $x, y \in X$, and since $p \notin \Delta$ we know that $x \neq y$. Then since X is Hausdorff there exist disjoint open sets, $U, V \subseteq X$, such that $x \in U$ and $y \in V$ and then $p = (x, y) \in U \times V$, and so $U \times V$ is an open neighbourhood of P , and since U and V are disjoint $U \times V \cap \Delta = \emptyset$. \square

Bibliography

- [1] Andreas Gathmann. *Commutative Algebra*. 2014.