Derivative Pricing

Introduction

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Objectives

- Present the concept of "Derivatives" for different type of underlying assets (equity, interest rate, *etc.*)
- Introduce to models and (numerical) methods used in pricing, such as:
  - Closed Formed formulas
  - Monte-Carlo simulations
  - Partial Differential Equations (PDE)
- Perform concrete applications on financial products (implementations to be done during class)

Bibliography

Some useful references:

- J. Hull, "Options, Futures and Other Derivatives", *Pearson*
- M. Joshi, "The Concepts and Practice of Mathematical Finance", *Cambridge*
- S. Shreve, "Stochastic Calculus for Finance", *Springer*
- D. Brigo & F. Mercurio, "Interest Rate Models - Theory and Practice", *Springer*
- R. Cont & P. Tankov, "Financial Modelling With Jump Processes", *Chapman & Hall*
- R. Portait & P. Poncet, "Finance de Marche", *Dalloz*

What is a derivative ?

- A **derivative** is a contract whose value explicitly depends from the performance of an **underlying asset**.
- The **underlying asset** might be:
  - an equity share (Google, BNP-PARIBAS,...)
  - an index (S&P500, Eurostoxx50, CAC40,...)
  - an interest rate (3-month Libor,...)
  - a commodity (gold, electricity,...) ...
- Derivative products might be traded either on a **Trading Exchange** (organized market) or **Over-the-Counter (OTC)** (no intermediary between parties).
- Among others : forward contracts, future contracts, calls, puts, swaps, swaptions, caps, floors, range accrual,...

How to price a derivative ?

Brief introduction

In modern pricing theory, the **price** of a derivative is computed as the **expected present value of future cashflows under the risk-neutral probability**.

- Future cash-flows $\rightarrow$ depend on the underlying asset future evolution
- Present Value $\rightarrow$ future cash-flows to be discounted using the risk-free rate
- Expected $\rightarrow$ the mathematical expectation
- Risk-Neutral probability $\rightarrow$ under this probability, all assets (including the underlying asset) earn the risk-free rate on average

Generally, a specific **model is chosen** to provide the probability distribution of the underlying asset.

How to price a derivative ?

Example

Let's consider a **call option** on a given underlying equity share whose value at time t is denoted S_t. The call option **strike** is K and its **maturity** is T.

The buyer of this option will therefore receive a single cash-flow at time T equals to:

$$\max(S_T - K, 0)$$

To price this derivative, we can use the **Black-Scholes** model.

How to price a derivative ?

Example

In the Black-Scholes model:

- we assume there exists a risk-free asset which earns a fixed interest rate r.
- The dynamics of S_t under the "risk neutral" probability is given by :

$$dS_t = rS_t dt + \sigma S_t dW_t$$

  where W_t is a standard Brownian Motion. We note $\mathcal{F}_t$, the σ-algebra generated by the past trajectory of S from 0 to t.

At any time t, the option price P_t is provided by the formula:

$$P_t = \mathbb{E}\left[e^{-r(T-t)} \max(S_T - K, 0) \middle| \mathcal{F}_t \right]$$

In this case, the formula above can be mathematically derived. But in more general cases, this would necessitate to use numerical methods (Monte-Carlo, PDE, Trees,...)

Closed Form Formula

Call Option in Black-Scholes model

In the Black-Scholes model, the price of the call is given by:

$$P_t = S_t \mathcal{N}(d_1) - K \exp^{-r(T-t)} \mathcal{N}(d_0)$$

with :

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left(\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2).(T-t) \right)$$

and:

$$d_0 = d_1 - \sigma\sqrt{T-t}$$

Monte-Carlo methods

Concept

Main idea: Use the Law of Large Numbers to numerically compute expectations.

If $X_1, ..., X_N$ are i.i.d. draws from the law of X then:

$$\frac{1}{N}\sum_{i=1}^{N} X_i \to \mathbb{E}[X] \text{ } a.s. \text{ and in } L^2$$

The error can be estimated thanks to the Central Limit Theorem:

$$\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}[X] \to^{L} \mathcal{N}\left(0, \frac{\mathbb{V}[X]}{N}\right)$$

Monte-Carlo methods

Call in Black-Scholes model

In this situation, assuming the underlying equity S follows Black-Scholes SDE, we want to estimate :

$$P_t = e^{-r(T-t)}\mathbb{E}\left[\max(S_T - K, 0) \middle| \mathcal{F}_t\right]$$

Applying the MC method consists in simulating i.i.d. draws in the law of S_T conditionally to $\mathcal{F}_t$ to get a numerical approximation of the expectation above.

As a reminder, the law of S_T conditionally to $\mathcal{F}_t$ is given by:

$$S_T = S_t.e^{(r-\frac{1}{2}\sigma^2).(T-t)+\sigma(W_T - W_t)}$$

Generally, the pricing problem to solve will have the form following form :

$$P_t = \mathbb{E}\left[e^{-\int_t^T r_s ds} F\left((X_s)_{0\le s\le T}\right) \middle| \mathcal{F}_t \right]$$

with $(X_t)_{t\ge 0}$ an **Ito process**, possibly of several dimension, including the short rate process r_t.

The difficulty then lies on how to generate $(X_s)_{0\le s\le T}$ conditionally to $\mathcal{F}_t$. In case no exact solution for the SDE exists, apply a Euler scheme (or any other discretization scheme).

Euler scheme for SDE

Concept

If $(X_s)_{0\leq s\leq T}$ is an Ito process with the following SDE:

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t$$

It is possible to construct a process X^N that will approximate the distribution of the real solution by setting :

- $X^N_{t_0} = X_{t_0}$
- for all $0 < i < N$:

$$X^N_{t_{i+1}} - X^N_{t_i} = \alpha(t_i, X^N_{t_i})(t_{i+1} - t_i) + \beta(t_i, X^N_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- (if necessary) perform linear interpolation for dates t in between all the t_i

Partial Differential Equation methods

Concepts - 1

Example : the Black-Scholes PDE

$$\frac{\partial u(t,x)}{\partial t} + rx\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u(t,x)}{\partial x^2} = ru(t,x)$$

This PDE can be numerically solved using the Euler approach on a finite time-space grid. One possible discretization is:

$$\frac{u(t_{i+1},x_j) - u(t_i,x_j)}{\Delta t} + rx_j \frac{u(t_{i+1},x_{j+1}) - u(t_{i+1},x_{j-1})}{2.\Delta x}$$
$$+\frac{1}{2}\sigma^2 x_j^2 \frac{u(t_{i+1},x_{j+1}) + u(t_{i+1},x_{j-1}) - 2u(t_{i+1},x_j)}{\Delta x^2} = ru(t_i,x_j)$$

where $t_i = i * \Delta t$ and $x_j = x_0 + j * \Delta x$ with $\Delta t = T/N$ and $\Delta x = \frac{x_M - x_0}{M}$

Partial Differential Equation methods

Concepts - 2

From the previous equation, terms can be grouped as :

$$a_j * u(t_{i+1}, x_{j+1}) + b_j * u(t_{i+1}, x_j) + c_j * u(t_{i+1}, x_{j-1}) = u(t_i, x_j)$$

which can be computed thanks to matrix calculation (matrix inversion in this case).

Note: in practice, i) the PDE is simplified when possible (in this case, performing the change of variable $x \to e^y$) and ii) there exist different possible discretization schemes (e.g. Implicit or Explicit Euler scheme, etc.)

Setting the parameters (r and σ) of the Black-Scholes model

In the model, r is the risk free return obtained on a 'deposit account'. Historically, r was the rate of return of an instrument considered as risk free on the interbank market (eonia, libor,...) with the same maturity as for the derivative. This is still the case when no collateral is in place to cover for possible default of one of the parties involved. Though, in case a collateral mechanism is in place, the most recent pricing approach involves both the interbank rates and the rate at which the collateral is remunerated.

For the volatility parameter σ, the 'natural' approach is to estimate the historical volatility observed on the underlying asset, also called the **realized volatility**. Though, in practice i) the volatility is not constant over time and ii) for call/put European options, this parameter depends of the strike and maturity. This is known as the **implied volatility smile and term structure**.

Notion of implied volatility

On the market, the call option are quoted based on their **implied volatility**. The implied volatility is the parameter to plug in the Black-Scholes formula to obtain the price of the option.

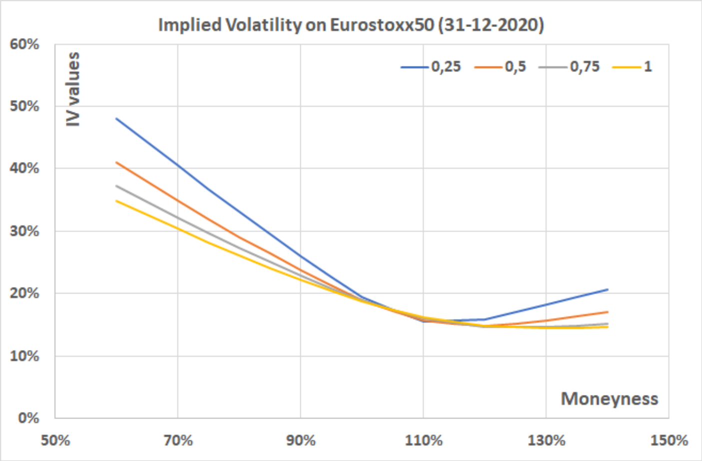
For a given call of maturity T and strike K, the implied volatility $\sigma(T, K)$ is such that:

$$Call^{BS}(0, S_0; r(T), q, \sigma(T, K); T, K) = Call^{Market}(T, K)$$

Note: r(T) is the risk free interest rate of the same maturity as the call option.

Illustration of implied volatility

The following graphics provides with the implied volatility observed for call options on the Eurostoxx50 index:



Black-Scholes with dividends

The Black-Scholes model can be improved to embed dividend payment. Noting the continuous dividend rate q, the equity share SDE under the risk-neutral probability becomes :

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t$$

The price formula (in its expectation for) for a call option, at any time t, is still provided by:

$$P_t = \mathbb{E}\left[e^{-r(T-t)} \max(S_T - K, 0) \middle| \mathcal{F}_t \right]$$

where $\mathcal{F}_t$ represents the knowledge of the past prices of S over the dates $[0, t]$.

Black-Scholes with dividends

Though, the closed form formula needs to be adjusted :

$$P_t = S_t e^{-q(T-t)} \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_0)$$

with :

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left(\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{1}{2}\sigma^2).(T-t) \right)$$

and:

$$d_0 = d_1 - \sigma\sqrt{T-t}$$

Sensitivities of Call Option under BS framework

Delta and Gamma

The *Delta* is the first order sensitivity of the option price with respect to the underlying asset price S_t:

$$\Delta(t, S_t) = \frac{\partial Call^{BS}}{\partial S_t} = e^{-q(T-t)}\mathcal{N}(d_1)$$

with $\mathcal{N}$ is the cumulative density function of a $N(0, 1)$.

The *Gamma* is the second order sensitivity of the option price with respect to the underlying asset price S_t:

$$\Gamma(t, S_t) = \frac{\partial^2 Call^{BS}}{\partial S_t^2} = \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}}\phi(d_1)$$

with ϕ is the likelihood of a $N(0, 1)$.

Sensitivities of Call Option under BS framework

Theta, Vega

The *Theta* is the first order sensitivity of the option price with respect to the time t:

$$\theta(t, S_t) = \frac{\partial Call^{BS}}{\partial t} = -\frac{S\sigma e^{-q(T-t)}\phi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\mathcal{N}(d_0)$$

The *Vega* is the first order sensitivity of the option price with respect to the volatility σ:

$$Vega(t, S_t) = \frac{\partial Call^{BS}}{\partial \sigma} = e^{-q(T-t)}S\sqrt{T-t}\phi(d_1)$$

How is the option price evolution related to sensitivities ?

Provided S_t is an Ito process, the variation of the price is obtained applying Ito's lemma :

$$dCall^{BS}(t, S_t) = \theta_t dt + \Delta_t dS_t + \frac{1}{2}\Gamma_t d\langle S \rangle_t$$

And as (due to Black-Scholes PDE):

$$\theta_t = rCall^{BS} - rS_t\Delta_t - \frac{1}{2}\Gamma_t \sigma^2 S_t^2$$

We obtain :

$$dCall^{BS}(t, S_t) - rCall^{BS} dt = \Delta_t (dS_t - rS_t dt) + \frac{1}{2}\Gamma_t S_t^2 \left(\frac{d\langle S \rangle_t}{S_t^2} - \sigma^2 dt \right)$$

The Heston Model

The Heston model is a stochastic volatility model which provides with the joint evolution of the equity share price S_t and its instantaneous variance V_t. Under the risk neutral probability, the joint dynamics is:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t &= \kappa \left(\theta - V_t\right) dt + \sigma \sqrt{V_t} dB_t^V \end{aligned}$$

where r is the interest rate, κ is the mean reversion speed, θ is the long term variance and σ is the volatility of the variance process V. The two brownian motion are possibly correlated such as $d\langle B^S, B^V\rangle_t = \rho dt$.

Heston Euler Scheme

Given the SDE of the Heston model (under RN probability), the Euler scheme is :

$$S_{t_{i+1}} = S_{t_i} + rS_{t_i}\delta t_i + \sqrt{|V_{t_i}|}S_{t_i}\delta B^S_{t_i}$$

$$V_{t_{i+1}} = V_{t_i} + \kappa\left(\theta - V_{t_i}\right)\delta t_i + \sigma\sqrt{|V_{t_i}|}\delta B^V_{t_i}$$

with $\delta t_i = t_{i+1} - t_i$, $\delta B^S_{t_i} = B^S_{t_{i+1}} - B^S_{t_i}$ and $\delta B^V_{t_i} = B^V_{t_{i+1}} - B^V_{t_i}$.

Note: the absolute value objective is to avoid simulations where V is negative. Other approximation are possible, for instance $\max(V, 0)$.

Generation of correlated Brownian Motion

If U and V are two independent Brownian Motion, then the couple of processes defined as:

$$\begin{aligned} B_t^1 &= U_t \\ B_t^2 &= \rho U_t + \sqrt{1-\rho^2} V_t \end{aligned}$$

is a bi-dimensional brownian motion with correlation ρ.

Note: one can remark that *i)* B^1 and B^2 are both dimension one BM and *ii)* $\langle B^1, B^2 \rangle_t = \rho t$.

Antithetic Variables

The idea of antithetic variables is to group simulation by pairs (X^+, X^-) where both X^+ and X^- follow the same distribution as X but are negatively correlated together. Then :

$$\mathbb{E}\left[\frac{X^+ + X^-}{2}\right] = \mathbb{E}\left[X^+\right] = \mathbb{E}\left[X^-\right] = \mathbb{E}\left[X\right]$$

$$\begin{aligned}
\mathbb{V}\left[\tfrac{X^+ + X^-}{2}\right] \quad &= \mathbb{V}\left[\tfrac{X^+ + X^-}{2}\right] = \frac{\mathbb{V}\left[X^+\right] + \mathbb{V}\left[X^-\right] + 2Cov\left[X^+, X^-\right]}{4} \\
&= \frac{\mathbb{V}\left[X^+\right]}{2} + \frac{Cov\left[X^+, X^-\right]}{2} = \frac{\mathbb{V}[X]}{2} + \frac{\rho\left[X^+, X^-\right]\mathbb{V}[X]}{2} \\
&= \mathbb{V}\left[X\right] \tfrac{1+\rho}{2}
\end{aligned}$$

Antithetic Variables

The initial vector $(X_1, ..., X_N)$ of i.i.d draws is replaced with $N/2$ i.i.d pairs $((X_1^+, X_1^-), ..., ((X_{N/2}^+, X_{N/2}^-)))$ and the monte-carlo estimate becomes:

$$\hat{\mu}_{anti} = \frac{2}{N} \sum_{i=1}^{N/2} \frac{X_i^+ + X_i^-}{2}$$

$$\begin{aligned}\mathbb{V}\left[\hat{\mu}_{anti}\right] &= \mathbb{V}\left[\frac{2}{N} \sum_{i=1}^{N/2} \frac{X_i^+ + X_i^-}{2}\right] = \frac{2}{N}\mathbb{V}\left[\frac{X_i^+ + X_i^-}{2}\right] \\ &= \frac{\mathbb{V}\left[X^+\right]}{N} + \frac{Cov\left[X^+, X^-\right]}{N} = \mathbb{V}\left[X\right]\frac{1+\rho}{N}\end{aligned}$$

Note: the variance is lower than the standard MC estimate only if $\rho < 0$.

Antithetic Variables

Call in Black-Scholes

For a Call option (maturity T and strike K, priced at time $t = 0$), in the Black-Scholes model, the principles of the antithetic variable approach can be applied by setting :

$$S_T^+ = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right)$$

$$S_T^- = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T - \sigma W_T\right)$$

The Monte-Carlo estimate is then build to compute the below expectation:

$$\mathbb{E}\left[\frac{X^+ + X^-}{2}\right]$$

with $X^+ = \max(S_T^+ - K, 0)$ and $X^- = \max(S_T^- - K, 0)$.

Note: the random variable X^+ and X^- are indeed negatively correlated by construction.

Control variate

Find a real number α and a random variable Y such as $\mathbb{E}[Y] = 0$ and $\mathbb{V}[X + \alpha Y] < \mathbb{V}(X)$.

$$\hat{\mu}_\alpha = \frac{1}{N} \sum_{i=1}^{N} X^i + \alpha Y^i$$

$$\mathbb{V}\left[\hat{\mu}_\alpha\right] = \frac{\mathbb{V}\left[X\right]}{N} + 2\alpha \frac{Cov\left[X, Y\right]}{N} + \alpha^2 \frac{\mathbb{V}\left[Y\right]}{N}$$

Note : it is possible to minimize the variance of the estimator by selecting the adequate value of α.

Control variate

The minimum variance of $\hat{\mu}_\alpha$ is reached for α^* defined below:

$$\alpha^* = -\frac{Cov\left[X, Y\right]}{\mathbb{V}\left[Y\right]}$$

For this value, the variance is equal to:

$$\mathbb{V}\left[\hat{\mu}_{\alpha^*}\right] = \frac{\mathbb{V}\left[X\right]}{N}\left(1 - \rho(X, Y)^2\right)$$

with $\rho(X, Y)$ is the correlation between X and Y. The higher the correlation, the better the overall variance reduction.

Barrier options - 1

Barrier options are derivative products for which the payoff is paid depending on the occurrence of the underlying asset prices reaches (or not) a specific level (called the barrier) during the lifetime of the option. There is mainly 4 types :

- **Up-and-In**: payoff is paid if the underlying asset increases above a given 'upside' threshold
- **Up-and-Out**: payoff is paid if the underlying asset never reaches a given 'upside' threshold
- **Down-and-In**: payoff is paid if the underlying asset decreases below a given 'downside' threshold
- **Down-and-Out**: payoff is paid if the underlying asset always stays above a given 'downside' threshold

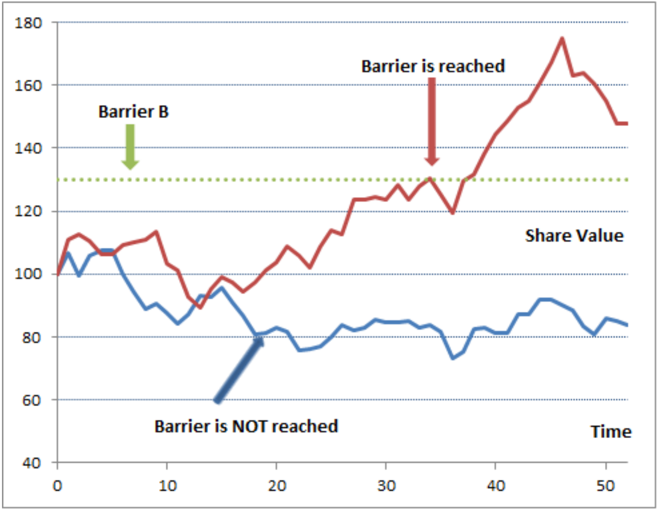
Barrier options - 2

If T is the maturity of the barrier option and B the barrier level, it means that the payoff is paid if the below condition is realized:

- **Up-and-In**: $\max(S_t; 0 \leq t \leq T) > B$
- **Up-and-Out**: $\max(S_t; 0 \leq t \leq T) < B$
- **Down-and-In**: $\min(S_t; 0 \leq t \leq T) < B$
- **Down-and-Out**: $\min(S_t; 0 \leq t \leq T) > B$

Barrier options - 3

Illustration : Up-and-In type



Barrier options - 4

Example : Up-and-In Put option

If we note T the maturity of the Up-and-In Put option and K the strike, the price at time t of an Up-and-In Put option (settled at time 0) is :

$$UIPut(t, S_t) = e^{-r(T-t)}\mathbb{E}\left[\max(K - S_T; 0).\mathbb{1}_{\{\max(S_t; 0 \leq t \leq T) > B\}} \middle| \mathcal{F}_t\right]$$

In particular, at the settlement date, the price is :

$$UIPut(0, S_0) = e^{-rT}\mathbb{E}\left[\max(K - S_T; 0).\mathbb{1}_{\{\max(S_t; 0 \leq t \leq T) > B\}}\right]$$

Note: In the Black-Scholes model, there are closed form formula for simple call and put barrier options.

Model calibration

Example with the Heston model - 1

In the Heston model, there is several parameters to fix, mostly the one for the variance process. As a reminder, the dynamic (under RN probability) is provided by :

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t &= \kappa \left(\theta - V_t \right) dt + \sigma \sqrt{V_t} dB_t^V \end{aligned}$$

where r is the interest rate and is directly observed in the market. On the contrary, the initial value of the variance process V_0, the mean reversion speed κ, the long term variance θ, the volatility of the variance process σ and ρ, the correlation between brownian motion are not observed and need to be calibrated.

Model calibration

Example with the Heston model - 2

In practice, observed prices of call (or put) options for different maturity and strike are used to perform the calibration of the parameters Θ, so that **the prices provided by the model are as close as possible to observed prices**.

After selecting different maturities T_i and strikes K_j, this corresponds to minimize a predefined loss function, for instance:

$$L(\Theta) = \sum_{i,j} \left(Call^{Heston}(T_i, K_j; \Theta) - Call^{Market}(T_i, K_j) \right)^2$$

using absolute value, or weighting the precision desired on each call is also possible.

Model calibration

Notion of implied volatility

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Note: r(T) is the interest rate of the same maturity as the call option.

Model calibration

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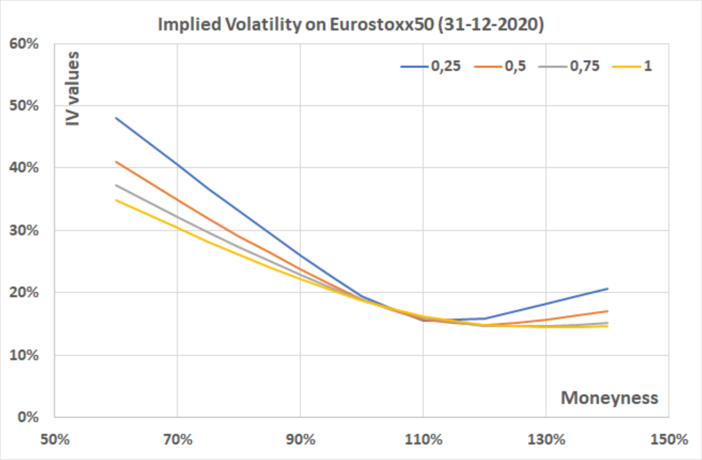

Notes/Comments - 1

Notes/Comments - 2

Notes/Comments - 3

Notes/Comments - 4