Derivative Pricing - Lecture 2

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1 Preliminaries

Download Anaconda v5.0.1 at `https://www.anaconda.com/download/` or `https://repo.continuum.io/pkgs/` and run the installation.

Open Spider and create a new file. At the beginning of the file, please include the following lines :

```
import numpy as np
import pylab as pl
from random import gauss, uniform, randint
from math import exp, sqrt, log
import matplotlib.pyplot as plt
import scipy from scipy.stats import norm
```

2 The Black-Scholes model

1) Implement the Black-Scholes formula for a call option. As a reminder, the formula is :

$$Call^{BS}(t, S_t; T, K; r, q, \sigma) = S_t \exp(-q(T-t))\mathcal{N}(d_1) - K\exp(-r(T-t))\mathcal{N}(d_0) \tag{1}$$

where :

$$\begin{cases} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln(\frac{S_t}{K}) + (r - q + \frac{\sigma^2}{2})(T-t)\right) \\ d_0 &= d_1 - \sigma\sqrt{T-t} \end{cases} \tag{2}$$

2) Implement the Δ and Γ functions of the call option which are the 1st and 2nd derivative of the price with respect to the underlying asset value.

3) Implement the $Vega$ of the call option. The $Vega$ is defined as the sensitivity of the price with respect to the volatility parameter of the Black-Scholes model.

3 The Heston Model

In the Heston model, the volatility is not considered constant as in the Black-Scholes model but is a stochastic process. The joint dynamics of the equity share and its variance, under the "historical probability" $\mathbb{P}$ is given by :

$$\left\{ \begin{array}{ll} dS_t & = (\mu - q)S_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t & = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t^V \end{array} \right. \tag{3}$$

where B_t^S and B_t^V are standard brownian motions under $\mathbb{P}$, correlated such as $\langle B^S, B^V \rangle_t = \rho t$. The parameters μ, q, κ, θ, σ are (possibly non-negative) real numbers.

Remark : under the "risk-neutral probability" $\mathbb{Q}$, only the μ parameter is changed (to be equal to the risk free rate r).

1) Explicit the Euler scheme of the above SDE. What can be said about it ?

2) Implement a function generating equity paths using the Heston model (based on Euler scheme). The function prototype might be :

```
def generatePathsWithHeston(dates,S0,V0,mu,q,sigma,corr,kappa,theta,numberOfPaths)
```

where *dates* is the list of dates on which is performed the Euler scheme with *dates[0] = 0*.

3) Implement a function performing the **pricing of a European Call option** under the Heston model by Monte-Carlo methods and returning the estimation of both the **price and the standard error**. The function prototype could be :

```
def HestonCallPriceByMC(t,St,Vt,T,K,r,q,corr,kappa,theta,numberOfPaths, numberOfSteps)
```

Numerical application : $t = 0$, $S_t = 100$, $V_t = 0.04$, $T = 1$, $K = 100$, $r = 0.0$, $q = 0.0$, $\rho = -0.75$, $\kappa = 0.2$, $\theta = 0.04$, $numberOfPaths = 10'000$, $numberOfSteps = 1'000$. The timestep within the Euler scheme is set to $\frac{T-t}{numberOfSteps}$.

4) Upgrade your Monte-Carlo pricer using the antithetic variable variance reduction technique.

5) Implement a control variate approach. A good candidate for the control variate is :

$$\exp(-rT)(X_T - K)^+ - Call^{BS}\left(0, X_0; T, K; r, q, \sqrt{V_0}\right) \tag{4}$$

where X_t is the process defined as :

$$\left\{ \begin{array}{ll} dX_t & = (r - q)X_t dt + \sqrt{V_0} X_t dB_t^S \\ X_0 & = S_0 \end{array} \right. \tag{5}$$

4 Barrier Option : Up-and-In Put

The payoff of an *Up-and-In Put* is given by :

$$(K - S_T)^+.\mathbb{I}_{\max_{t\in[0,T]}(S_t)>B} \tag{6}$$

At the maturity of the option T, the buyer has access to the payoff (of Put type) only if the maximum of the equity value observed during the life of the option is higher than the "barrier" level B.

1) Compute the price of the Up-and-In Put option using a Monte-Carlo method. The function prototype should be :

```
def UpInPutBSByMC(t,St,T,K,B,r,q,sigma,numberOfPaths,numberOfSteps)
```

2) Compare the price obtained with the following formula :

$$\left(\frac{S_0}{B}\right)^{\alpha-2} Put^{BS}(0, S0; T, \frac{S_0^2 K}{B^2}; r, q, \sigma) \tag{7}$$

where $\alpha = 1 - 2\frac{r-q}{\sigma^2}$.

3) Same under the Heston model.

4) Compare the sensitivities Δ, Γ and $Vega$ between the 2 models