exercises 4.1 & 4.2 omitted

4.3

(a) $T(n)=T(n-1)+n,\ T(n)=O(n^2)$

Solution:

Guess $T(n)\le cn^2$ for $n\ge n_0$ ($\exists c>0$ and c is a constant)

Inducive hypothesis: $T(n-1)\le c(n-1)^2$

Substitution:

$T(n)=T(n-1)+n$

$\le c(n-1)^2+n$

$=c(n^2-2n+1)+n$

$=cn^2+(1-2c)n+c$

$\le cn^2$ (if $1-2c\le 0$

$\Rightarrow c\ge\frac{1}{2}$)

Thus, $T(n)=O(n^2)$

(b) $T(n)=T(\frac{n}{2})+\Theta(1),\ T(n)=O(\log_2 n)$

Solution:

Rewrite $T(n)$ as

$T(n)=T(\frac{n}{2})+c_0\ (n>1)$

Omit the case when $n=1$

Guess: $T(n)\le d\log_2 n$ for $n\ge n_0$

($\exists d>0$ and d is a constant)

Inducive hypothesis: $T(\frac{n}{2})\le d\log_2(\frac{n}{2})$

Substitution:

$T(n)=T(\frac{n}{2})+c_0$

$\le d\log_2(\frac{n}{2})+c_0$

$=d(\log_2 n-1)+c_0$

$=d\log_2 n+c_0-d$

$\le d\log_2 n$ (if $c_0-d\le 0$

$\Rightarrow d\ge c_0$)

Thus, $T(n)=O(\log_2 n)$

(c) $T(n)=2T(\frac{n}{3})+\Theta(n),\ T(n)=\Theta(n)$

Solution:

Rewrite $T(n)$ as

$T(n)=2T(\frac{n}{3})+c_0n\ (n>1)$

(Omit the case when $n=1$)

Firstly, prove $T(n)=O(n)$

Guess: $T(n)\le d_1n$ for $n\ge n_0$ ($\exists d_1>0$ and d_1 is a constant)

Inducive hypothesis: $T(\frac{n}{3})\le d_1\cdot\frac{n}{3}$

Substitution:

$T(n)=2T(\frac{n}{3})+c_0n$

$\le 2d_1\cdot\frac{n}{3}+c_0n$

$=(\frac{2d_1}{3}+c_0)n$

$\le d_1n$ (if $\frac{2d_1}{3}+c_0\le d_1$

$\Rightarrow d_1\ge 3c_0$)

Thus, $T(n)=O(n)$

Then, prove $T(n)=\Omega(n)$

Guess: $T(n)\ge d_2n$ for $n\ge n_0$ ($\exists d_2>0$ and d_2 is a constant)

Inducive hypothesis: $T(n)\ge d_2\cdot\frac{n}{3}$

Substitution:

$T(n)=2T(\frac{n}{3})+c_0n$

$\ge 2d_2\cdot\frac{n}{3}+c_0n$

$=(\frac{2d_2}{3}+c_0)n$

$\ge d_2n$ (if $\frac{2d_2}{3}+c_0\ge d_2$)

$\Rightarrow d_2\le 3c_0$

Thus, $T(n)=\Omega(n)$

Therefore, $T(n)=\Theta(n)$.

4.4

(a) $T(n)=T(\frac{n}{2})+n^3$

Solution:

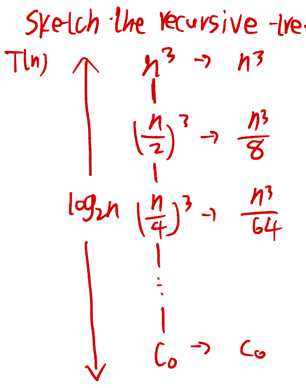
Rewrite $T(n)$ as

$$T(n)=\begin{cases} c_0 & n=1 \\ T(\frac{n}{2})+n^3 & n>1\end{cases}$$

Determine the height of the recursive tree

$\frac{n}{2^h}=1\Rightarrow h=\log_2 n$

Sketch the recursive tree



The total cost of the tree is

$T(n)=\sum_{i=0}^{\log_2 n-1}(\frac{1}{8})^i n^3+c_0$

$=n^3\sum_{i=0}^{\log_2 n-1}(\frac{1}{8})^i+c_0$

$=\frac{8}{7}n^3(1-\frac{1}{n^3})+c_0$

$=\frac{8}{7}n^3+c_0-\frac{7}{8}$

$=O(n^3)$

(b) $T(n)=3T(n-1)+1$

Solution:

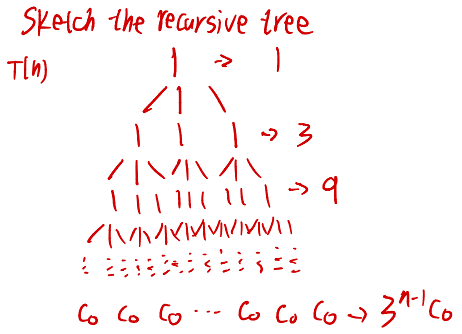
Rewrite $T(n)$ as

$$T(n)=\begin{cases} c_0 & n=1 \\ 3T(n-1)+1 & n>1\end{cases}$$

Determine the height of the recursive tree

$n-h=1\Rightarrow h=n-1$

Sketch the recursive tree



The total cost of the tree is

$T(n)=\sum_{i=0}^{n-2}3^i+3^{n-1}c_0$

$=\frac{1}{2}(3^{n-1}-1)+3^{n-1}c_0$

$=\frac{c_0-\frac{1}{2}}{3}3^n-\frac{1}{2}$

$=O(3^n)$

4.5

(a) $T(n)=2T(\frac{n}{4})+1$

Solution:

we have $a=2,\ b=4$

Thus, $n^{\log_b a}=n^{\log_4 2}=n^{\frac{1}{2}}=\Theta(n^{\frac{1}{2}})$

Since $f(n)=1=n^0=O(n^{\frac{1}{2}-\varepsilon})$ ($\exists\varepsilon>0$ and ε is a constant)

From what we obtained above,

we can know $f(n)=O(n^{\frac{1}{2}})$ is impossible for $\varepsilon>0$,

but we can find a $\varepsilon=\frac{1}{2}$ such that $f(n)=O(1)$.

From analysis above, only case 1 of master theorem satisfies $f(n)$.

$\Rightarrow T(n)=\Theta(n^{\log_b a})=\Theta(\sqrt{n})$

(b) $T(n)=2T(\frac{n}{4})+\sqrt{n}$

Solution:

we have $a=2,\ b=4$

Thus, $n^{\log_b a}=n^{\log_4 2}=\sqrt{n}=\Theta(\sqrt{n})$

Since we can find a constant $k=0$

such that $f(n)=\sqrt{n}=\Theta(n^{\log_b a}\log_2^k n)$.

only case 2 of master theorem satisfies.

$\Rightarrow T(n)=\Theta(n^{\log_b a}\log_2^{k+1}n)=\Theta(\sqrt{n}\log_2 n)$

(c) $T(n)=2T(\frac{n}{4})+n^2$

Solution:

we have $a=2,\ b=4$

Thus, $n^{\log_b a}=n^{\log_4 2}=\sqrt{n}=\Theta(\sqrt{n})$

Since $f(n)=n^2=\Omega(n^{\frac{1}{2}+\varepsilon})$ ($\exists\varepsilon>0$ and ε is a constant)

($\varepsilon\in(0,\frac{3}{2}]$), we can find $\varepsilon\in(0,\frac{3}{2}]$ such that $f(n)=\Omega(n^{\frac{1}{2}+\varepsilon})$.

Only case 3 of master theorem.

Thus, complete regularity test.

Assume $2f(\frac{n}{4})\le cf(n)$ ($\exists c<1$ when n is sufficiently large) holds

$\Rightarrow 2(\frac{n}{4})^2\le cn^2\Rightarrow c\ge\frac{2(\frac{n}{4})^2}{n^2}=\frac{1}{8}$

Thus we can find $c\in[\frac{1}{8},1)$ that satisfies regularity test.

$\Rightarrow T(n)=\Theta(f(n))=\Theta(n^2)$