CS220/SE220 Design and Analysis of Algorithms

School of Computer Science and Engineering
Macau University of Science and Technology
Assignment 1
Due Day: 7 March 2024 (Thursday)

1. Use the **substitution method** to show that each of the following recurrences defined on the reals has the **asymptotic solution** specified. **You have to show both the upper (O) and lower (Ω) bounds of $T(n)$.**
   a. $T(n) = 2T(n/3) + \Theta(n)$ has solution $T(n) = \Theta(n)$.
   b. $T(n) = 4T(n/2) + \Theta(n)$ has solution $T(n) = \Theta(n^2)$.

2. For each of the following recurrences, sketch its **recursion tree**, and guess a good **asymptotic upper bound** on its solution. **You DO NOT need to use the substitution method to verify your answer.**
   a. $T(n) = T(n/3) + \Theta(n^3)$.
   b. $T(n) = 4T(n/3) + \Theta(n)$.

3. Use the **master method** to give **tight asymptotic bounds** for the following recurrences:
   a. $T(n) = 2T(n/4) + \sqrt{n}\,\lg^2 n$.
   b. $T(n) = 2T(n/2) + n^3$.

problem 1

a. proof

upper bound

Guess: $T(n) \le dn$ ($d > 0$, d is constant)

By definition of Θ-notation

$\exists c_1, c_2$ are positive constants, such that

$c_1 g(n) \le f(n) \le c_2 g(n)$ for $n \ge n_0$

To prove upper bound, write

$T(n) \le 2T(\frac{n}{3}) + c_2 n$

Assume $T(\frac{n}{3}) \le d \cdot \frac{n}{3}$ holds

Substitution:

$\Rightarrow T(n) \le 2T(\frac{n}{3}) + c_2 n$

$\le 2d \cdot \frac{n}{3} + c_2 n$

$= (\frac{2d}{3} + c_2) n$

$\le dn$ if $\frac{2d}{3} + c_2 \le d$

$\Rightarrow d \ge 3c_2$

Thus, $T(n) = O(n)$

Lower bound

Guess: $T(n) \ge dn$ ($d > 0$, d is constant)

To prove lower bound, write

$T(n) \ge 2T(\frac{n}{3}) + c_1 n$

Assume $T(\frac{n}{3}) \ge d \cdot \frac{n}{3}$ holds

Substitution:

$\Rightarrow T(n) \ge 2T(\frac{n}{3}) + c_1 n$

$\ge 2d \cdot \frac{n}{3} + c_1 n$

$= (\frac{2d}{3} + c_1) n$

$\ge dn$ if $\frac{2d}{3} + c_1 \ge d$

$\Rightarrow d \le 3c_1$

Thus, $T(n) = \Omega(n)$

Therefore, $T(n) = \Theta(n)$

b. proof

Upper bound

Guess: $T(n) \le dn^2$ ($d>0$, d is constant)

To prove upper bound, write

$T(n) \le 4T(\frac{n}{2}) + c_2 n$ ($c_2 > 0$)

Assume $T(\frac{n}{2}) \le d(\frac{n}{2})^2$ holds

Substitution:

$$
\begin{aligned}
T(n) &\le 4T(\tfrac{n}{2}) + c_2 n \\
&\le 4d(\tfrac{n}{2})^2 + c_2 n \\
&= dn^2 + c_2 n
\end{aligned}
$$

Obviously, we connot prove it from what we obtained above.

Thus, change the guess into

$T(n) \le dn^2 - n$

Assume $T(\frac{n}{2}) \le d(\frac{n}{2})^2 - \frac{n}{2}$

Substitution:

$$
\begin{aligned}
T(n) &\le 4T(\tfrac{n}{2}) + c_2 n \\
&\le 4[d(\tfrac{n}{2})^2 - \tfrac{n}{2}] + c_2 n \\
&= dn^2 + (c_2 - 2)n \\
&\le dn^2 \quad \text{if } c_2 - 2 \le 0 \\
&\quad\quad \Rightarrow c_2 \le 2
\end{aligned}
$$

Thus, $T(n) = O(n^2)$

Lower bound

Guess: $T(n) \ge dn^2$ ($d>0$, d is constant)

To prove lower bound, write

$T(n) \ge 4T(\frac{n}{2}) + c_1 n$ ($c_1 > 0$)

Assume $T(\frac{n}{2}) \ge d(\frac{n}{2})^2$ holds

Substitution:

$$
\begin{aligned}
T(n) &\ge 4T(\tfrac{n}{2}) + c_1 n \\
&\ge 4d(\tfrac{n}{2})^2 + c_1 n \\
&= dn^2 + c_1 n \\
&\ge dn^2
\end{aligned}
$$

Problem 2

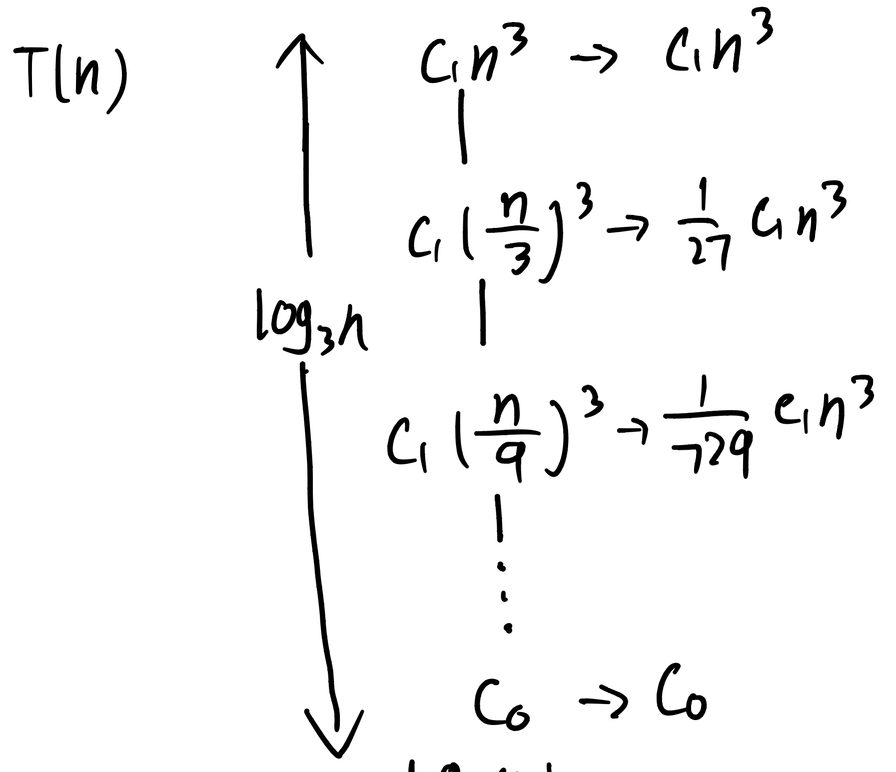
a. Solution:

For $T(n)=T(\frac{n}{3})+\Theta(n^3)$

Write $T(n)$ as

$$T(n)=\begin{cases} c_0 & n=1 \\ T(\frac{n}{3})+c_1n^3 & n>1 \end{cases}$$

Sketch the recursion tree



Thus, $T(n)=\sum_{i=0}^{\log_3 n-1}(\frac{1}{27})^i c_1n^3+c_0$

$$=c_1n^3\sum_{i=0}^{\log_3 n-1}(\frac{1}{27})^i+c_0$$

$$=\frac{27}{26}c_1n^3(1-\frac{1}{n^3})+c_0$$

$$=\frac{27}{26}c_1n^3+c_0-\frac{27}{26}c_1$$

$$=O(n^3)$$

b. Solution:

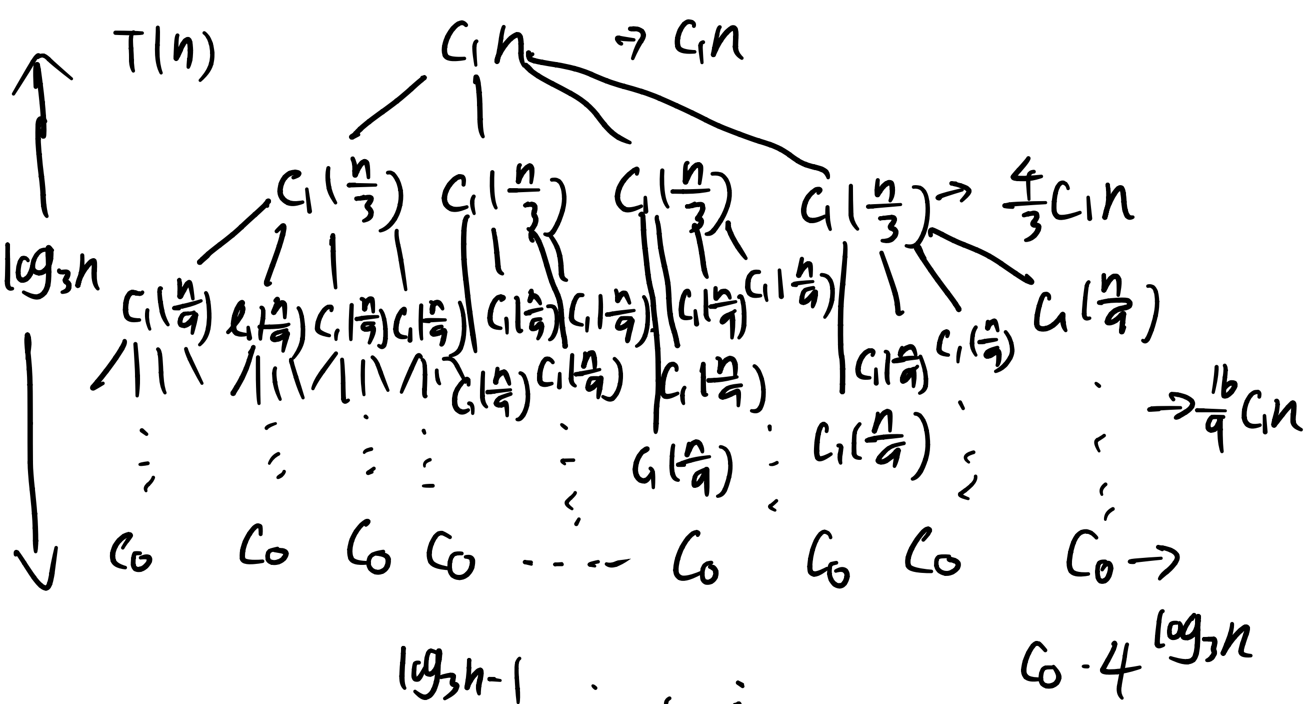
For $T(n)=4T(\frac{n}{3})+\Theta(n)$

write $T(n)$ as

$$T(n)=\begin{cases} c_0 & n=1 \\ 4T(\frac{n}{3})+c_1n & n>1 \end{cases}$$

$\frac{4}{9}c_1n\cdot 4$

Sketch the recursion tree $\quad 4^{\log_3 n}$



Thus, $T(n)=\sum_{i=0}^{\log_3 n-1}4^i(\frac{1}{3})^i c_1n+c_0\cdot 4^{\log_3 n}$

$$=c_1n\sum_{i=0}^{\log_3 n-1}(\frac{4}{3})^i+c_0\cdot 4^{\log_3 n}$$

$$=3c_1n(\frac{2^{\log_3 n^2}}{n}-1)+c_0\cdot 4^{\log_3 n}$$

$$=3c_1n(n^{2\log_3 n-1}-1)+c_0\cdot n^{2\log_3 2}$$

$$=3c_1n^{2\log_3 2}+c_0n^{2\log_3 2}-3c_1n$$

$$=(3c_1+c_0)n^{\log_3 4}-3c_1n$$

$$=O(n^{\log_3 4})$$

Problem 3

a. Solution:

For $T(n) = 2T(\frac{n}{4}) + \sqrt{n}(\log_2 n)^2$,

we have $a=2$ and $b=4$

$\Rightarrow n^{\log_b a} = n^{\log_4 2} = \sqrt{n} = \Theta(\sqrt{n})$

Since $f(n) = \sqrt{n}(\log_2 n)^2 = \Theta(n^{\log_b a} \log_2^2 n)$,

which corresponds with case 2 in textbook

$\Rightarrow T(n) = \Theta(\sqrt{n} \log_2^3 n)$

b. Solution:

For $T(n) = 2T(\frac{n}{2}) + n^3$,

we have $a=2$ and $b=2$

$\Rightarrow n^{\log_b a} = n^{\log_2 2} = n = \Theta(n)$

Since $f(n) = n^3 = \Omega(n^{1+\varepsilon}) = \Omega(n^3)$,

we can find $\varepsilon = 2$ such that $f(n) = \Omega(n^{1+\varepsilon})$

Assume $2f(\frac{n}{2}) \le cf(n)$ ($c<1$ and c is constant)

holds

$\Rightarrow 2(\frac{n}{2})^3 \le cn^3$

$\Rightarrow c \geq \frac{2(\frac{n}{2})^3}{n^3} = \frac{1}{4}$

Therefore, $\exists\, c \in [\frac{1}{4}, 1)$ such that $2f(\frac{n}{2}) \le cf(n)$

$\Rightarrow T(n) = \Theta(n^3)$