Numerical Methods Midterm Exam

Feb 24, 2022

Name (Last, First):

Student ID:

Instructions: Read through all questions first. Plan your time. (e.g. you might want to do all the conceptual steps first and leave calculations until the end.) Answer neatly and succinctly in the space provided. Show steps you wish to have graded, but not e.g. scribbles for numerical calculations etc. (If needed, use additional blank sheets to first work out your solution, then structure it and transfer it neatly to the exam).

Allowed: 2 single sided sheets 8 1/2 by 11 inches with your own handwritten notes. These are to be handed in with your exam. Non-programmable calculator.

1 4 points

a. For a function $y = f(x)$, show the following

$$\frac{\Delta y/y}{\Delta x/x} \approx \frac{xf'(x)}{f(x)}.$$

$f(x) = \frac{1}{x}$

b. If the function $1/x$ is evaluated with a particular value of x with 2% error in the input, what will be the relative $\boxed{\text{forward error?}}$

c. If the backward error was due to $\boxed{\text{underestimation,}}$ would you increase the output value or decrease to adjust for it?

a.

proof

Using derivative approximation

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \approx f'(x)$$

$$\Rightarrow f(x+\Delta x) - f(x) = \Delta y = f'(x)\Delta x$$

Thus, we can derive that

$$\frac{\Delta y}{y} = \frac{f'(x)\Delta x}{f(x)} \Rightarrow \frac{\Delta y/y}{\Delta x/x} = \frac{f'(x)\Delta x}{f(x)} \cdot \frac{x}{\Delta x} = \frac{xf'(x)}{f(x)}$$

Therefore the statement is proved.

b. Solution.

The conditional number $= \left|\frac{\Delta y/y}{\Delta x/x}\right| \approx \left|\frac{xf'(x)}{f(x)}\right| = \left|\frac{x \cdot (-\frac{1}{x^2})}{\frac{1}{x}}\right| = 1$

$\Rightarrow |\Delta y/y| \approx |\Delta x/x|$

Given that 2% error for value of $x \Rightarrow |\Delta x/x| = 2\%$

$\Rightarrow |\Delta y/y| =$ relative forward error $\approx 2\%$

C. Solution.

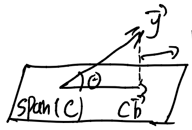
Given that the backward error is due to underestimation

$\Rightarrow \Delta x/x = -0.02$

Since $\frac{\Delta y/y}{\Delta x/x} = \frac{xf'(x)}{f(x)} = -1 \Rightarrow \frac{\Delta y}{y} = 0.02$

since $\Delta y = \hat{y} - y \Rightarrow \frac{\hat{y}-y}{y} = \frac{\hat{y}}{y} - 1 = 0.02 \Rightarrow y = \frac{1}{1.02}\hat{y}$

Therefore, we would decrease the output value to adjust for it.



$\Delta\vec{y} = \|\vec{y} - C\vec{b}\|_2$

$\cos\theta = \dfrac{\|C\vec{b}\|_2}{\|\vec{y}\|_2}$ $\quad \mathrm{cond}(C)\cdot\dfrac{1}{\cos\theta} = \dfrac{\|C\|_2\|C^+\|_2\,\|\vec{y}\|_2}{\|C\vec{b}\|_2}$

2 4 points

Let's consider a $\boxed{\text{least squares problem}}$ $\mathbf{Cb} \cong \mathbf{y}$ where $\mathbf{C}$ has full column rank. For a perturbation to the right hand side vector $\mathbf{y}$, show that the condition number for the least-squares solution is upper bounded by $\mathrm{cond}(\mathbf{C})/\cos\theta$, where θ is the angle between $\mathbf{y}$ and $\mathbf{Cb}$.

proof

Since it is a least squares problem and assume that C is a $m\times n$ matrix

$\Rightarrow m > n$, indicating that $m \neq n$

Thus, this matrix has no inverse but has pesudoinverse $C^+ = (C^TC)^{-1}C^T$

Thus, for the system $C\vec{b} \cong \vec{y}$, we can derive

$$\vec{b} = C^+\vec{y} = (C^TC)^{-1}C^T\vec{y} \quad ①$$

Likewise, for the system $C\Delta\vec{b} \cong \Delta\vec{y}$

$$\Rightarrow \Delta\vec{b} = C^+\Delta\vec{y} = (C^TC)^{-1}C^T\Delta\vec{y} \quad ②$$

From the geometric representation of least squares system, we can derive that

$$\cos\theta = \frac{\|C\vec{b}\|_2}{\|\vec{y}\|_2} \quad ③$$

Additionally, we can know that the condition number of matrix C is

$$\mathrm{cond}(C) = \|C\|_2\|C^+\|_2 \quad ④$$

Thus, we can derive the condition number of the solution of the linear squares system

$$\mathrm{cond}(\vec{b}) = \frac{\|\Delta\vec{b}\|_2/\|\vec{b}\|_2}{\|\Delta\vec{y}\|_2/\|\vec{y}\|_2} = \frac{\|\Delta\vec{b}\|_2\,\|\vec{y}\|_2}{\|\Delta\vec{y}\|_2\,\|\vec{b}\|_2}$$

$$= \frac{\|C^+\Delta\vec{y}\|_2\,\|\vec{y}\|_2}{\|\Delta\vec{y}\|_2\,\|\vec{b}\|_2}$$

$$\leq \frac{\|C^+\|_2\|\Delta\vec{y}\|_2\,\|\vec{y}\|_2}{\|\Delta\vec{y}\|_2\cdot\|\vec{b}\|_2}$$

$$= \frac{\|C^+\|_2\,\|\vec{y}\|_2}{\|\vec{b}\|_2}$$

$$= \frac{\|C\|_2\|C^+\|_2\,\|\vec{y}\|_2}{\|C\|_2\,\|\vec{b}\|_2}$$

$$\leq \frac{\|C\|_2\|C^+\|_2\,\|\vec{y}\|_2}{\|C\vec{b}\|_2}$$

$$= \frac{\mathrm{cond}(C)}{\cos\theta}$$

Thus, we can prove $\mathrm{cond}(\vec{b}) \leq \dfrac{\mathrm{cond}(C)}{\cos\theta}$

3 4 points

a. Given the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

calculate $\mathbf{AC}$, $\mathbf{BC}$, $(\mathbf{AB})^{-1}$, and $(\mathbf{CAB})^{-1}$.

b. Given the following matrix and vector, solve $\mathbf{Dy} = \mathbf{b}$ using forward-substitution:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = [1, 0, 0, 0]^\top .$$

Solution.

a. Notice the given matrices, we will find that A, B and C are elementary elimination matrices of D.

$\Rightarrow B = M_1,\ A = M_2,\ C = M_3$

$$\Rightarrow AC = M_2 M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$BC = M_1 M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 0 & -1 & 1 \end{pmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1} = M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

$$(CAB)^{-1} = B^{-1}(CA)^{-1} = B^{-1}A^{-1}C^{-1} = M_1^{-1} M_2^{-1} M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 \end{pmatrix}$$

b. The constructed system should be

$$D\vec{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{b}$$

Using forward substitution

$y_1 = 1$

$y_1 + y_2 = 0 \Rightarrow y_2 = -1$

$-3y_1 + 2y_2 + y_3 = 0 \Rightarrow y_3 = 3y_1 - 2y_2 = 5$

$2y_1 - y_2 + y_3 + y_4 = 0 \Rightarrow y_4 = -2y_1 + y_2 - y_3 = -8$

$\Rightarrow \vec{y} = (1\ -1\ 5\ -8)^T$

4 4 points

Given the three data points (-1,1), (0,0), (1,1), determine the interpolating polynomial of degree two: using the monomial basis and the Newton basis, and show that they give the same polynomial.

Solution.

Case 1: Using monomial basis

Denote the given 3 data points

$A(t_1, y_1)$, $B(t_2, y_2)$, $C(t_3, y_3)$

$\Rightarrow p(t) = x_1 + x_2 t + x_3 t^2$

$\Rightarrow y_1 = p(t_1)$, $y_2 = p(t_2)$, $y_3 = p(t_3)$

Thus, the system should be

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Denote $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow (A \; \vec{y}) = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \end{cases} \Rightarrow \vec{x} = (0\ 0\ 1)^T \Rightarrow p(t) = t^2$$

Case 2: Using the Newton basis

$$\pi_j(t) = \prod_{k=1}^{j-1} (t - t_k)$$

$\Rightarrow p(t) = x_1 + x_2(t - t_1) + x_3(t - t_1)(t - t_2)$

$\Rightarrow y_1 = x_1$, $y_2 = x_1 + x_2(t_2 - t_1)$, $y_3 = x_1 + x_2 \cdot (t_3 - t_1) + x_3(t_3 - t_1)(t_3 - t_2)$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

The system should be

$$A\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{y}$$

$\Rightarrow \vec{x} = (1\ -1\ 1)^T \Rightarrow p(t) = 1 - 1(t+1) + 1(t-1) \cdot t$

$= 1 - t - 1 + t^2 - t$

$= t^2$