2nd Seminar Problems

Review Questions

2.42. (*a*) What is the inverse of the following matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 \\ 0 & m_2 & 0 & 1 \end{bmatrix}$$

(*b*) How might such a matrix arise in computational practice?

Solution:

Denote the given matrix by A

(a)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 \\ 0 & m_2 & 0 & 1 \end{pmatrix} \Rightarrow B = [A|I] = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & m_2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow E_2 B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_1 & 1 & 0 \\ 0 & -m_2 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & m_2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -m_1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -m_2 & 0 & 1 \end{array}\right)$$

Thus, $A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_1 & 1 & 0 \\ 0 & -m_2 & 0 & 1 \end{pmatrix}$

(b) $O(n^3)$

2.67. Let $\boldsymbol{A}$ be an arbitrary square matrix and c an arbitrary scalar. Which of the following statements must necessarily hold?

(*a*) $\|c\boldsymbol{A}\| = |c| \cdot \|\boldsymbol{A}\|$.

(*b*) $\text{cond}(c\boldsymbol{A}) = |c| \cdot \text{cond}(\boldsymbol{A})$.

Solution:

(a) must necessarily hold

Proof:

$$\|CA\| = \max_{\vec{x} \neq \vec{0}} \frac{\|CA\vec{x}\|}{\|\vec{x}\|} = |C| \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = |C|\|A\|$$

(b) impossibly holds

proof:

$$\text{cond}(CA) = \|CA\| \, \|(CA)^{-1}\| = C\|A\| \cdot \tfrac{1}{C}\|A^{-1}\| = \|A\|\|C^{-1}\| = \text{cond}(A)$$

2.5. What is the inverse of the following matrix?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Solution:

$$[A \; I] = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) = [I \; A^{-1}] \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Exercises

2.4. (*a*) Show that the following matrix is singular.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

(*b*) If $\boldsymbol{b} = [2 \;\; 4 \;\; 6]^T$, how many solutions are there to the system $\boldsymbol{Ax} = \boldsymbol{b}$?

Solution:

(a)

$$\det(A) = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 2 + 1 = 0$$

Thus, matrix A is singular

(b)

The equation $A\vec{x} = \vec{b} \Leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$

$$\Rightarrow [A|\vec{b}] = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 4 \\ 1 & 3 & 2 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 2 \\ x_2 + x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_2 = 2 - x_1 \\ x_3 = x_1 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2 - x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, there are infinite solutions for $\vec{x}$, depending on the value of x_1.

2.10. Let $\boldsymbol{P}$ be any permutation matrix.

(*a*) Prove that $\boldsymbol{P}^{-1} = \boldsymbol{P}^T$.

proof

Let $\vec{p_i}$ denote the ith column of P

Since $\vec{p_i}$ contains only an "one" entry and "zeros" elsewhere

$\Rightarrow \vec{p_i}\,\vec{p_i}^T = 1$

For $i \neq j$, $\vec{p_i}\,\vec{p_j}^T = 0$ since there are no two distinct columns containing 1 in the same row

$\Rightarrow PP^T = I \Rightarrow P^T = P^{-1}$

2.17. Write out the LU factorization of the following matrix (show both the L and U matrices explicitly):

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution:

Denote the given matrix by A

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow M_1A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow M_2M_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$[M_1\ I] = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[M_2\ I] = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \Rightarrow M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow L = M_1^{-1}M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

2.27. Prove that the Sherman-Morrison formula

$$(\boldsymbol{A} - \boldsymbol{u}\boldsymbol{v}^T)^{-1} =$$
$$\boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{u}(1 - \boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{u})^{-1}\boldsymbol{v}^T\boldsymbol{A}^{-1}$$

given in Section 2.4.9 is correct. (*Hint:* Multiply both sides by $\boldsymbol{A} - \boldsymbol{u}\boldsymbol{v}^T$.)

Proof

We can derive that

$$(A - \vec{u}\vec{v}^T)[A^{-1} + A^{-1}\vec{u}(1 - \vec{v}^TA^{-1}\vec{u})^{-1}\vec{v}^TA^{-1}]$$

$$= I + \vec{u}(1 - \vec{v}^TA^{-1}\vec{u})^{-1}\vec{v}^TA^{-1} - \vec{u}\vec{v}^TA^{-1} - \vec{u}\vec{v}^TA^{-1}\vec{u}(1 - \vec{v}^TA^{-1}\vec{u})^{-1}\vec{v}^TA^{-1}$$

$$= I + \vec{u}[(1 - \vec{v}^TA^{-1}\vec{u})^{-1} - 1 - \vec{v}^TA^{-1}\vec{u}(1 - \vec{v}^TA^{-1}\vec{u})^{-1}]\vec{v}^TA^{-1}$$

$$= I + \vec{u} \cdot \frac{1 - (1 - \vec{v}^TA^{-1}\vec{u}) - \vec{v}^TA^{-1}\vec{u}}{1 - \vec{v}^TA^{-1}\vec{u}} \cdot \vec{v}^TA^{-1}$$

$$= I$$

$$\Rightarrow (A - \vec{u}\vec{v}^T)^{-1} = A^{-1} + A^{-1}\vec{u}(1 - \vec{v}^TA^{-1}\vec{u})^{-1}\vec{v}^TA^{-1}$$

2.25. (*a*) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero n-vectors, prove that the $n \times n$ outer product matrix $\boldsymbol{u}\boldsymbol{v}^T$ has rank one.
(*b*) If $\boldsymbol{A}$ is an $n \times n$ matrix such that $\text{rank}(\boldsymbol{A}) = 1$, prove that there exist nonzero n-vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$.

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \vec{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

proof

(a)

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ $\left(\prod_{i=1}^{n} u_i \neq 0,\ \prod_{i=1}^{n} v_i \neq 0\right)$

$$\Rightarrow \vec{u}\vec{v}^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{pmatrix}$$

Since $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ $\Rightarrow$ at least one entry $u_i \neq 0$ and $v_i \neq 0$

Thus, at least one entry $u_iv_j \neq 0$ in $\vec{u}\vec{v}^T$. $\Rightarrow$ We can eliminate all other non-zero entries by row operation $\Rightarrow \text{rank}(\vec{u}\vec{v}^T) = 1$

(b)

Let the matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

Since $\text{rank}(A) = 1 \Rightarrow A$ has only one linearly independent column, assuming it is the 1st column, denoted by $v_1\vec{u}$ $(v_1 \neq 0)$.

For other columns, which are multiples of the 1st row

$\Rightarrow A = \vec{u}\vec{v}^T$