4th Seminar

Exercise

7.1. Given the three data points $(-1,1)$, $(0,0)$, $(1,1)$, determine the interpolating polynomial of degree two:
(*a*) Using the monomial basis
(*b*) Using the Lagrange basis
(*c*) Using the Newton basis
Show that the three representations give the same polynomial.

Solution: Denote $A(t_1, y_1)$, $B(t_2, y_2)$, $C(t_3, y_3)$

(a) Using the monomial basis

$$p(t) = x_1 + x_2 t + x_3 t^2$$

$$\Rightarrow y_1 = p(-1) = x_1 - x_2 + x_3, \quad y_2 = p(0) = x_1 + 0\cdot t + 0 \cdot t^2$$

$$y_3 = p(1) = x_1 + x_2 + x_3$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Denote $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow [A\ \vec{y}] = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 \end{array}\right) \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus the interpolating polynomial $p(t) = t^2$

(b) Using the Lagrange basis

$$p(t) = y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

$$= 1 \cdot \frac{t(t-1)}{(-1)\cdot(-2)} + 0 + 1 \cdot \frac{(t+1)t}{2 \cdot 1}$$

$$= \frac{t(t-1)}{2} + \frac{t(t+1)}{2} = t^2$$

(c) Using the Newton basis

$$\pi_j(t) = \prod_{k=1}^{j-1}(t - t_k) \Rightarrow p(t) = x_1 + x_2(t-t_1) + x_3(t-t_1)(t-t_2)$$

$$\Rightarrow y_1 = x_1, \quad y_2 = x_1 + x_2(t_2 - t_1), \quad y_3 = x_1 + x_2(t_3 - t_1) + x_3(t_3 - t_1)(t_3 - t_2)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3-t_1)(t_3-t_2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{Denote } B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow [B\ \vec{b}] = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 2 & 0 \end{array}\right) \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, interpolating polynomial $p(t) = 1 + (-1)\cdot(t+1) + 1 \cdot (t+1)t = t^2$

7.2. Express the following polynomial in the correct form for evaluation by Horner's method: $p(t) = 5t^3 - 3t^2 + 7t - 2$.

Solution:

Rearrange the polynomial

$$p(t) = -2 + 7t - 3t^2 + 5t^3$$

$$= -2 + t(7 + t(-3 + 5t))$$

7.5. (*a*) Determine the polynomial interpolant to the data

t	1	2	3	4
y	11	29	65	125

using the monomial basis.
(*b*) Determine the Lagrange polynomial interpolant to the same data and show that the resulting polynomial is equivalent to that obtained in part *a*.
(*c*) Compute the Newton polynomial interpolant to the same data using each of the three methods given in the text (triangular matrix, incremental interpolation, and divided differences) and show that each produces the same result as the previous two methods.

Solution:

Denote the given four data points by

$A(t_1, y_1)$, $B(t_2, y_2)$, $C(t_3, y_3)$, $D(t_4, y_4)$

$(t_1 < t_2 < t_3 < t_4)$

(a) Using the monomial basis

$$p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$$

$$\Rightarrow y_1 = p(t_1), \ y_2 = p(t_2), \ y_3 = p(t_3), \ y_4 = p(t_4)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 29 \\ 65 \\ 125 \end{pmatrix}$$

Denote $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 11 \\ 29 \\ 65 \\ 125 \end{pmatrix}$

$$\Rightarrow [A\ \vec{y}] = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 11 \\ 1 & 2 & 4 & 8 & 29 \\ 1 & 3 & 9 & 27 & 65 \\ 1 & 4 & 16 & 64 & 125 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 7 & 18 \\ 0 & 2 & 8 & 26 & 54 \\ 0 & 3 & 15 & 63 & 114 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 7 & 18 \\ 0 & 0 & 2 & 12 & 18 \\ 0 & 0 & 6 & 42 & 60 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 7 & 18 \\ 0 & 0 & 2 & 12 & 18 \\ 0 & 0 & 0 & 6 & 6 \end{array}\right)$$

$\Rightarrow \vec{x} = (5\ 2\ 3\ 1)^T$ Thus the interpolating polynomial $p(t) = 5 + 2t + 3t^2 + t^3$

(b) Using the Lagrange basis

$$p(t) = y_1 \cdot \frac{(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + y_2 \cdot \frac{(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} + y_3 \cdot \frac{(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)}$$

$$+ y_4 \cdot \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}$$

$$= 11 \frac{(t-2)(t-3)(t-4)}{(1-2)(1-3)(1-4)} + 29 \frac{(t-1)(t-3)(t-4)}{(2-1)(2-3)(2-4)} + 65 \frac{(t-1)(t-2)(t-4)}{(3-1)(3-2)(3-4)} + 125 \frac{(t-1)(t-2)(t-3)}{(4-1)(4-2)(4-3)}$$

$$= -\frac{11}{6}(t^3 - 9t^2 + 26t - 24) + \frac{29}{2}(t^3 - 8t^2 + 19t - 12) - \frac{65}{2}(t^3 - 7t^2 + 14t - 8) + \frac{125}{6}(t^3 - 6t^2 + 11t - 6)$$

$$= 5 + 2t + 3t^2 + t^3$$

(c) Using Newton basis (triangular matrix)

$$\pi_j(t) = \prod_{k=1}^{j-1}(t - t_k) \Rightarrow p(t) = x_1 + x_2(t - t_1) + \cdots + x_n(t-t_1)(t-t_2)\cdots(t-t_{n-1})$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & t_2 - t_1 & 0 & 0 \\ 1 & t_3 - t_1 & (t_3-t_1)(t_3-t_2) & 0 \\ 1 & t_4 - t_1 & (t_4-t_1)(t_4-t_2) & (t_4-t_1)(t_4-t_2)(t_4-t_3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 29 \\ 65 \\ 125 \end{pmatrix}$$

$$\Rightarrow \vec{x} = (11\ 18\ 9\ 1)^T \Rightarrow p(t) = 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3)$$

$$= 5 + 2t + 3t^2 + t^3$$

$$x_{j+1} = \frac{y_{j+1} - P_j(t_{j+1})}{\pi_{j+1}(t_{j+1})}$$

(7.5 continued) (Incremental interpolation)

$P_1(t) = y_1 = 11$, $P_2(t) = P_1(t) + x_2\pi_2(t) = P_1(t) + \frac{y_2 - P_1(t_2)}{\pi_2(t_2)}\pi_2(t)$

$$= 11 + \frac{29-11}{2-1}(t-1)$$

$$= 11 + 18(t-1)$$

$P_3(t) = P_2(t) + x_3\pi_3(t) = 11 + 18(t-1) + \frac{y_3 - P_2(t_3)}{\pi_3(t_3)}\pi_3(t)$

$$= 11 + 18(t-1) + \frac{65 - 11 - 18(3-1)}{(3-1)(3-2)}(t-1)(t-2)$$

$$= 11 + 18(t-1) + 9(t-1)(t-2)$$

$P_4(t) = P_3(t) + x_4\pi_4(t) = 11 + 18(t-1) + 9(t-1)(t-2) + \frac{y_4 - P_3(t_4)}{\pi_4(t_4)}\pi_4(t)$

$$= 11 + 18(t-1) + 9(t-1)(t-2) + \frac{125 - 11 - 18(4-1) - 9(4-1)(4-2)}{(4-1)(4-2)(4-3)}(t-1)(t-2)(t-3)$$

$$= 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3)$$

$$= 5 + 2t + 3t^2 + t^3$$

(Divided difference)

$f[t_1] = 11$, $f[t_2] = 29$, $f[t_3] = 65$, $f[t_4] = 125$

$\Rightarrow f[t_1, t_2] = \frac{f[t_2] - f[t_1]}{t_2 - t_1} = 18$, $f[t_2, t_3] = \frac{f[t_3] - f[t_2]}{t_3 - t_2} = 36$

$\Rightarrow f[t_1, t_2, t_3] = \frac{f[t_2, t_3] - f[t_1, t_2]}{t_3 - t_1} = 9$

$f[t_3, t_4] = \frac{f[t_4] - f[t_3]}{t_4 - t_3} = 60$ $\Rightarrow f[t_2, t_3, t_4] = \frac{f[t_3, t_4] - f[t_2, t_3]}{t_4 - t_2} = 12$

$\Rightarrow f[t_1, t_2, t_3, t_4] = \frac{f[t_2, t_3, t_4] - f[t_1, t_2, t_3]}{t_4 - t_1} = 1$

$\Rightarrow P_3(t) = f[t_1]\pi_1(t) + f[t_1, t_2]\pi_2(t) + f[t_1, t_2, t_3]\pi_3(t) + f[t_1, t_2, t_3, t_4]\pi_4(t)$

$$= 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3)$$

$$= 5 + 2t + 3t^2 + t^3$$

7.10. (*a*) For a given set of data points, $t_1, \ldots, t_n$, define the function $\pi(t)$ by

$$\pi(t) = (t - t_1)(t - t_2) \cdots (t - t_n).$$

Show that

$$\pi'(t_j) = (t_j - t_1) \cdots (t_j - t_{j-1})(t_j - t_{j+1}) \cdots (t_j - t_n).$$

(*b*) Use the result of part *a* to show that the jth Lagrange basis function can be expressed as

$$\ell_j(t) = \frac{\pi(t)}{(t - t_j)\,\pi'(t_j)}.$$

proof

(a) From the definition

$$\pi(t) = (t-t_1)(t-t_2) \cdots (t-t_j) \cdots (t-t_n)$$

$$\Rightarrow \pi'(t) = (t-t_1)'(t-t_2) \cdots (t-t_j) \cdots (t-t_n) + (t-t_1)[(t-t_2)(t-t_3) \cdots (t-t_j) \cdots (t-t_n)]'$$

$$= (t-t_1)'(t-t_2) \cdots (t-t_j) \cdots (t-t_n) + (t-t_1)(t-t_2)' \cdots (t-t_j) \cdots (t-t_n) + \cdots$$

$$+ (t-t_1)(t-t_2) \cdots (t-t_j)' \cdots (t-t_n) + \cdots + (t-t_1)(t-t_2) \cdots (t-t_j) \cdots (t-t_n)'$$

$$= (t-t_2) \cdots (t-t_j) \cdots (t-t_n) + (t-t_1) \cdots (t-t_j) \cdots (t-t_n) + (t-t_1)(t-t_2) \cdots (t-t_{j-1})(t-t_{j+1}) \cdots (t-t_n)$$

Thus, $\pi'(t_j) = (t_j-t_2) \cdots 0 \cdots (t_j-t_n) + (t_j-t_1) \cdots 0 \cdots (t_j-t_n) + (t_j-t_1)(t_j-t_2) \cdots (t_j-t_{j-1})(t_j-t_{j+1}) \cdots (t_j-t_n)$

$$= (t_j-t_1) \cdots (t_j-t_{j-1})(t_j-t_{j+1}) \cdots (t_j-t_n)$$

(b) From the definition of Lagrange basis

$$l_j(t) = \frac{\prod_{k=1,\, k\neq j}^{n} (t-t_k)}{\prod_{k=1,\, k\neq j}^{n} (t_j-t_k)} = \frac{(t-t_1)(t-t_2) \cdots (t-t_{j-1})(t-t_{j+1}) \cdots (t-t_n)}{(t_j-t_1)(t_j-t_2) \cdots (t_j-t_{j-1})(t_j-t_{j+1}) \cdots (t_j-t_n)}$$

$$= \frac{\frac{\pi(t)}{t-t_j}}{\pi'(t_j)} = \frac{\pi(t)}{(t-t_j)\pi'(t_j)}$$