Problem Set

1. Consider a random variable with a cumulative distribution function of $F(x) = \frac{x^2}{16}$ for $0 \leqslant x \leqslant 4$.

(a) Find the variance of this random variable;

(b) Find the standard variation of this random variable;

(c) Find the upper and lower quartiles of this random variable and the interquartile range.

2. Suppose that two continuous random variables X and Y have a joint probability density function

$$f(x, y) = \begin{cases} ae^{-(3x+y)}, & x > 0, \ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the joint cumulative distribution function $F(x, y)$ of X and Y;

(b) Find $P(Y \leqslant X)$.

3. Suppose that two continuous random variables X and Y have a joint probability density function

$$f(x, y) = \begin{cases} a, & x^2 \leqslant y \leqslant x \\ 0, & \text{elsewhere} \end{cases}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

4. Suppose we have a marginal probability density function

$$f_X(x) = \begin{cases} \frac{x}{8}, & 0 < x < 4 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability density function of the random variable Y for $Y = 3X + 6$.

5. Suppose we have

$$\Omega_0(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\Omega_{k+1}(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \Omega_k(t)\,dt,\ k \in \mathbb{N}$$

(a) Find $\Omega_1(x)$ and $\Omega_2(x)$;

(b) If random variable X has a probability density function $\Omega_1(x)$, find the expectation of X.

(c) Consider again the random variable X has a probability density function $\Omega_2(x)$, find the expectation of X.

6. Use integration by parts to show that if X has an exponential distribution with parameter λ, then

(a) $E(x) = \frac{1}{\lambda}$, (b) $E(X^2) = \frac{2}{\lambda^2}$

7. Suppose that you are waiting for a friend to call you and that the time you wait in minutes has an exponential distribution with parameter $\lambda = 0.1$.

(a) What is the expectation of your waiting time?

(b) What is the probability that you will wait longer than 10 minutes?

(c) What is the probability that you will wait less than 5 minutes?

(d) Suppose that after 5 minutes you are still waiting for the call. What is the distribution of your *additional* waiting time? In this case, what is the probability that your total waiting time is longer than 15 minutes?

(e) Suppose now that the time you wait in minutes for the call has a $U(0, 20)$ distribution. What is the expectation of your waiting time? If after 5 minutes you are still waiting for the call. what is the distribution of your additional waiting time?

8. A researcher plants 12 seeds whose germination times in days are independent exponential distributions with $\lambda = 0.31$.
(a) What is the probability that a given seed germinates within five days?
(b) What are the expectation and variance of the number of seeds germinating within five days?
(c) What is the probability that no more than nine seeds have germinated within five days?

9. A double exponential distribution, often called a **Laplace distribution**, has a probability density function

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x-\theta|}$$

for $x \in \mathbb{R}$, depending on two parameters λ and θ. Sketch the probability density function and cumulative distribution function of this distribution. What is the expectation of the distribution? If $\lambda = 3$ and $\theta = 2$, calculate:

(a) $P(X \leqslant 0)$, (b) $P(X \geqslant 1)$

10. A multiple-choice quiz consists of ten questions, each with five possible answers of which only one is correct. A student passes the quiz if seven or more correct answers are obtained. What is the probability that a student who guesses blindly at all of the questions will pass the quiz? What is the probability of passing the quiz if, on each question, a student can eliminate three incorrect answers and then guesses between the remaining two?

11. A flu virus hits a company employing 180 people. Independent of the other employees, there is a probability of $p = 0.35$ that each person needs to take sick leave. What are the expectation and variance of the proportion of the workforce who need to take sick leave? In general, what value of the sick rate p produces the *largest* variance for this proportion?

12. Consider the two independent binomial random variables $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$. If $Y = X_1 + X_2$, explain why $Y \sim B(n_1 + n_2, p)$.

13. A company receives 60% of its orders over the Internet. Within a collection of 18 independently placed orders, what is the probability that
(a) between eight and ten of the orders are received over the Internet?
(b) no more than four of the orders are received over the Internet?

14. Suppose that $E(X_1) = \mu$, $\mathrm{Var}(X_1) = 10$, $E(X_2) = \mu$, and $\mathrm{Var}(X_2) = 15$,

and consider the point estimates

$$\hat{\mu}_1 = \frac{X_1}{2} + \frac{X_2}{2},\ \hat{\mu}_2 = \frac{X_1}{4} + \frac{3X_2}{4},\ \hat{\mu}_3 = \frac{X_1}{6} + \frac{X_2}{3} + 9$$

(a) Calculate the bias of each point estimate. ls any one of them unbiased?
(b) Calculate the variance of each point estimate. Which one has the smallest variance?
(c) Calculate the mean square error of each point estimate. Which point estimate has the smallest mean square error when $\mu = 8$?

15. Suppose that $E(X_1) = \mu$, $\mathrm{Var}(X_1) = 7$, $E(X_2) = \mu$, $\mathrm{Var}(X_2) = 13$,

$E(X_3) = \mu$ and $\mathrm{Var}(X_3) = 20$, and consider the point estimates

$$\hat{\mu}_1 = \frac{X_1}{3} + \frac{X_2}{3} + \frac{X_3}{3},\ \hat{\mu}_2 = \frac{X_1}{4} + \frac{X_2}{3} + \frac{X_3}{5},\ \hat{\mu}_3 = \frac{X_1}{6} + \frac{X_2}{3} + \frac{X_4}{4} + 2$$

(a) Calculate the bias of each point estimate. ls any one of them unbiased?
(b) Calculate the variance of each point estimate. Which one has the smallest variance?
(c) Calculate the mean square error of each point estimate. Which point estimate has the smallest mean square error when $\mu = 3$?

16. Suppose that $E(X_1) = \mu$, $\mathrm{Var}(X_1) = 4$, $E(X_2) = \mu$, and $\mathrm{Var}(X_2) = 6$

(a) What is the variance of $\hat{\mu}_1 = \frac{X_1}{2} + \frac{X_2}{2}$

(b) What value of p minimizes the variance of $\hat{\mu}_2 = pX_1 + (1 - p)X_2$

(c) What is the relative efficiency of $\hat{\mu}_1$ to the point estimate with the smallest variance that you have found?

17. Repeat problem 16 with $\mathrm{Var}(X_1)=1$ and $\mathrm{Var}(X_2)=7$ with the remaining given relations unchanged.

18. Suppose that the random variable X has a probability density function

$$f(x)=2x$$

for $0\leqslant x\leqslant 1$. Find the probability density function and the expectation of the random variable Y in the following cases.

(a) $Y=X^3$ (b) $Y=\sqrt{X}$ (c) $Y=\dfrac{1}{1+X}$ (d) $Y=2^X$

19. The radius of a soap bubble has a probability density function

$$f(r)=A[1-(r-1)^2]$$

for $0\leqslant r\leqslant 2$

(a) What is the value of A?

(b) What is the probability density function of the *volume* of the soap bubble?

(c) What is the expected value of the volume of the soap bubble?

20. A rod of length L is bent until it snaps in two. The point of breakage X, as measured from one end of the rod, has a probability density function

$$f(x)=Ax(L-x)$$

for $0\leqslant x\leqslant L$

(a) What is the value of A?

(b) What is the probability density function of the *difference* in the lengths of the two pieces of the rod?

(c) What is the expected difference in the lengths of the two pieces of the rod?

21. If $\$x$ is invested in mutual fund A, the annual return has an expectation of $\$0.1x$ and a standard deviation of $\$0.02x$. If $\$x$ is invested in mutual fund B, the annual return has an expectation of $\$0.1x$ and a standard deviation of $\$0.03x$. Suppose that the returns on the two funds are independent of each other and that I have \$1000 to invest.

(a) What are the expectation and variance of my annual return if I invest all my money in fund A?

(b) What are the expectation and variance of my annual return if I invest all my money in fund B?

(c) What are the expectation and variance of my total annual return if I invest half of my money in fund A and half in fund B?

(d) Suppose I invest $\$x$ in fund A and the rest of my money in fund B. What value of x minimizes the variance of my total annual return?

22. If $A \subset B$ and $B \neq \varnothing$, is $P(A)$ larger or smaller than $P(A|B)$? Provide some intuitive reasoning for your answer.

23. A ball is chosen at random from a bag containing 150 balls that are either red or blue and either dull or shiny. There are 36 red shiny balls and 54 blue balls.
What is the probability of the chosen ball being shiny conditional on it being red?
What is the probability of the chosen ball being dull conditional on it being red?

24. A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.85 that it is satisfactory. There is a probability of 0.77 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?