Problem 1

Solution:

Since X has an exponential distribution, we have

$f(x) = \lambda e^{-\lambda x} \ (\lambda > 0, x > 0)$

(a) Thus we can obtain

$$E(X) = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$
$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

Integrate by parts, we have

$$\int x e^{-\lambda x} dx = -\frac{1}{\lambda} x e^{-\lambda x} + \int \frac{1}{\lambda} e^{-\lambda x} dx$$
$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x} dx$$
$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda}\left(-\frac{1}{\lambda} e^{-\lambda x} + C\right)$$
$$= -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C_1$$
$$\left(\text{where } C_1 = \frac{C}{\lambda}\right)$$

$$\Rightarrow \int_0^{\infty} x e^{-\lambda x} dx = \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x}\right]_0^{\infty}$$
$$= \lim_{t \to \infty} \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x}\right]_0^t$$
$$= \frac{1}{\lambda^2}$$

Thus. $E(X) = \lambda \int_0^{\infty} x e^{-\lambda x} dx = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$

(b) Thus, we can obtain

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$
$$= \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

Integrate by parts, we have

$$\int x^2 e^{-\lambda x} dx = -\frac{1}{\lambda} x^2 e^{-\lambda x} + \int 2x \cdot \frac{1}{\lambda} e^{-\lambda x} dx$$
$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda} \int x e^{-\lambda x} dx$$
$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda}\left(-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C\right)$$
$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} + C_1$$
$$\left(\text{where } C_1 = \frac{2}{\lambda} C\right)$$

$$\Rightarrow \int_0^{\infty} x^2 e^{-\lambda x} dx = \left[-\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x}\right]_0^{\infty}$$
$$= \lim_{t \to \infty} \left[-\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x}\right]_0^t$$
$$= \frac{2}{\lambda^3}$$

Thus, $E(X^2) = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx = \frac{2}{\lambda^3}$

Problem 2

Solution:

Denote the time for waiting by X.

(a)

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.1} = 10 \text{ mins}$$

(b)

Since $\lambda = 0.1$, we can obtain the probability density function

$f(x) = 0.1 e^{-0.1x}$

Thus, $P(X \le 10) = \int_0^{10} 0.1 e^{-0.1x} dx = 0.1 \int_0^{10} e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^{10}$
$$= 1 - \frac{1}{e}$$

$\Rightarrow P(X > 10) = 1 - P(X \le 10) = \frac{1}{e} \approx 0.3679$

(c)

Likewise, we can obtain that

$P(X \le 5) = \int_0^5 0.1 e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^5 = 1 - \frac{1}{\sqrt{e}} \approx 0.3935$

(d)

Without anything changed, the additional time still has the exponential distribution.

For the probability of waiting longer than 15 minutes.

$P(X \le 15) = \int_0^{15} 0.1 e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^{15} = 1 - e^{-\frac{3}{2}}$

Thus. $P(X > 15) = 1 - P(X \le 15) = e^{-\frac{3}{2}} \approx 0.2231$

(e)

Since the waiting time has a $U(0,20)$, we can obtain that

$f(x) = \frac{1}{20}$

$\Rightarrow E(X) = \int_0^{20} x \cdot \frac{1}{20} dx = \frac{1}{40}\left[x^2\right]_0^{20} = 10 \text{ mins}$

Likewise, after waiting 5 minutes, the additional time has a $U(0,15)$ distribution

Problem 3

Denote the germination times by Y.

Since Y has an exponential distribution, we can obtain the probability density function

$$f(y) = 0.31e^{-0.31y} \ (y>0)$$

(a)

$$\Rightarrow P(Y \le 5) = \int_0^5 0.31e^{-0.31y}dy = 0.31\int_0^5 e^{-0.31y}dy$$

$$= \left[-e^{-0.31y}\right]_0^5$$

$$= 1-e^{-1.55} \approx 0.7878$$

(b) Since the germination time is independent each other, the number of seeds that germinate has a binomial distribution $X \sim B(12, 0.7878)$

Thus, $E(X) = 12 \times 0.7878 = 9.4536$

$Var(X) = 12 \times 0.7878 \times (1-0.7878) \approx 2.0061$

(c)

Since we know that the number of seeds that germinate has a binomial distribution, we can obtain

$$P(X=10) = C_{12}^{10} \times 0.7878^{10} \times 0.2122^2 \approx 0.2736$$

$$P(X=11) = C_{12}^{11} \times 0.7878^{11} \times 0.2122 \approx 0.1847$$

$$P(X=12) = C_{12}^{12} \times 0.7878^{12} \times 0.2122^0 \approx 0.0571$$

$$\Rightarrow P(X \le 9) = 1-[P(X=10)+P(X=11)+P(X=12)]$$

$$= 0.4846$$

Problem 4

Solution:

Denote the cumulative distribution by $F(X)$

Case 1: When $x-\theta<0 \Leftrightarrow x<\theta$

$$\Rightarrow F(x) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x} \frac{1}{2}\lambda e^{-\lambda(\theta-y)}dy = \frac{\lambda}{2e^{\lambda\theta}}\int_{-\infty}^{x} e^{\lambda y}dy$$

$$= \frac{1}{2e^{\lambda\theta}}\left[e^{\lambda y}\right]_{-\infty}^{x}$$

$$= \frac{1}{2}e^{\lambda(x-\theta)}$$

Determine the range of x first before integrating when finding CDF.

Case 2: when $x-\theta>0 \Leftrightarrow x>\theta$

$$\Rightarrow F(x) = \underbrace{\frac{1}{2}}_{\text{when } x=\theta}+\int_{\theta}^{x} f(y)dy = \frac{1}{2}+\int_{\theta}^{x}\frac{1}{2}\lambda e^{-\lambda(y-\theta)}dy = \frac{1}{2}+\frac{\lambda e^{\lambda\theta}}{2}\int_{\theta}^{x} e^{-\lambda y}dy$$

$$= \frac{1}{2}-\frac{e^{\lambda\theta}}{2}\left[e^{-\lambda y}\right]_{\theta}^{x}$$

$$= 1-\frac{1}{2}e^{\lambda(\theta-x)}$$

When $\lambda=3$ and $\theta=2$

(a) $P(X \le 0) = \frac{1}{2}e^{3\times(-2)} = \frac{1}{2e^6} \approx 0.0012$

(b) $P(X \ge 1) = 1-F(1) = 1-\frac{1}{2}e^{3\times(-1)} = 1-\frac{1}{2e^3} \approx 0.9751$

Problem 5

Solution:

Case 1: When a student guesses blindly at all of the questions, the number of correct answers has a binomial distribution $X \sim B(10, 0.2)$

To ensure the student can pass the quiz,

$$P(X \ge 7) = P(X=7)+P(X=8)+P(X=9)+P(X=10)$$

$$= C_{10}^{7} \times 0.2^7 \times 0.8^3 + C_{10}^{8} \times 0.2^8 \times 0.8^2 + C_{10}^{9} \times 0.2^9 \times 0.8 + C_{10}^{10} \times 0.2^{10} \times 0.8^0$$

$$\approx 0.0009$$

Case 2: When a student can only guess between two possible answer, the number of correct answers has a binomial distribution $X \sim B(10, 0.5)$

$$\Rightarrow P(X \ge 7) = P(X=7)+P(X=8)+P(X=9)+P(X=10)$$

$$= C_{10}^{7} \times 0.5^7 \times 0.5^3 + C_{10}^{8} \times 0.5^8 \times 0.5^2 + C_{10}^{9} \times 0.5^9 \times 0.5 + C_{10}^{10} \times 0.5^{10} \times 0.5^0$$

$$= 0.1719$$

Problem 6

Solution:

Denote the number of people taking sick leave by X, and the proportion is $Y=\frac{X}{180}$

Since the employees are independent each other, X has a binomial distribution $X \sim B(180, 0.35)$

Thus, we can obtain the expectation and variance of X, respectively

$$E(Y)=E\left(\frac{X}{180}\right)=\frac{1}{180}E(X)=\frac{1}{180}np=\frac{1}{180}\times 180\times 0.35=0.35$$

$$Var(Y)=Var\left(\frac{X}{180}\right)=\frac{1}{180^2}Var(X)=\frac{1}{180^2}np(1-p)=\frac{1}{180^2}\times 180\times 0.35\times 0.65 \approx 0.00126$$

Generally, we have no idea of the specific value of p, thus X still has a binomial distribution $X \sim B(180, p)$

Thus, the variance of Y is

$$Var(Y)=Var\left(\frac{X}{180}\right)=\frac{1}{180^2}Var(X)=\frac{1}{180^2}180p(1-p)$$

$$=\frac{1}{180}p(1-p)=-\frac{1}{180}\left(p-\frac{1}{2}\right)^2+\frac{1}{720}$$

Hence, when $p=0.5$, it produces the largest variance for this proportion, which is approximately 0.00139.

Problem 7

Solution:

For distribution $X_1 \sim B(n_1, p)$, it can be considered as the number of successes within n_1 trials.

Analogically, for distribution $X_2 \sim B(n_2, p)$, it can be consider as the number of successes within n_2 trials.

Hence, for $Y=X_1+X_2$, which is the sum of X_1 and X_2, it can be considered as the number of successes within (n_1+n_2) trials with the unchanged successful probability p.

Finally, we can conclude that $Y \sim B(n_1+n_2, p)$

Problem 8

Solution:

Denote the number of the orders received from the Internet by X.

Since the placed orders are independent each other, X has a binomial distribution $X \sim B(18, 0.6)$

(a) $P(8 \le X \le 10)=P(X=8)+P(X=9)+P(X=10)$

$$=C_{18}^{8}\times 0.6^8\times 0.4^{10}+C_{18}^{9}\times 0.6^9\times 0.4^{10}+C_{18}^{10}\times 0.6^{10}\times 0.4^{8}$$

$$=0.0771+0.1284+0.1734$$

$$=0.3789$$

(b) $P(X\le 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$$=C_{18}^{0}\times 0.6^0\times 0.4^{18}+C_{18}^{1}\times 0.6^1\times 0.4^{17}+C_{18}^{2}\times 0.6^2\times 0.4^{16}$$

$$+C_{18}^{3}\times 0.6^3\times 0.4^{15}+C_{18}^{4}\times 0.6^4+0.4^{14}$$

$$\approx 0.0013$$

Problem 9

Solution:

(a)

For $\hat{\mu}_1$, $E(\hat{\mu}_1) = E(\frac{X_1+X_2}{2}) = \frac{1}{2}[E(X_1)+E(X_2)] = \mu$

$\Rightarrow$ bias $= E(\hat{\mu}_1) - \mu = 0$

Thus, point estimate $\hat{\mu}_1$ is unbiased.

For $\hat{\mu}_2$, $E(\hat{\mu}_2) = E(\frac{X_1}{4} + \frac{3X_2}{4}) = \frac{1}{4}E(X_1) + \frac{3}{4}E(X_2)$

$= \mu$

$\Rightarrow$ bias $= E(\hat{\mu}_2) - \mu = 0$

Thus, point estimate $\hat{\mu}_2$ is unbiased.

For $\hat{\mu}_3$, $E(\hat{\mu}_3) = E(\frac{X_1}{6} + \frac{X_2}{3} + 9) = \frac{1}{6}E(X_1) + \frac{1}{3}E(X_2) + 9$

$= \frac{\mu}{6} + \frac{\mu}{3} + 9$

$= \frac{\mu}{2} + 9$

$\Rightarrow$ bias $= E(\hat{\mu}_3) - \mu = 9 - \frac{\mu}{2}$

(b)

For $\hat{\mu}_1$, $Var(\hat{\mu}_1) = Var(\frac{X_1+X_2}{2}) = \frac{1}{4}[Var(X_1)+Var(X_2)] = \frac{25}{4} = 6.25$

For $\hat{\mu}_2$, $Var(\hat{\mu}_2) = Var(\frac{X_1}{4} + \frac{3X_2}{4}) = \frac{1}{16}Var(X_1) + \frac{9}{16}Var(X_2)$

$= \frac{10}{16} + \frac{9}{16} \times 15 = \frac{145}{16} = 9.0625$

For $\hat{\mu}_3$, $Var(\hat{\mu}_3) = Var(\frac{X_1}{6} + \frac{X_2}{3} + 9) = \frac{1}{36}Var(X_1) + \frac{1}{9}Var(X_2)$

$= \frac{10}{36} + \frac{15}{9}$

≈ 1.9444

Thus, $\hat{\mu}_3$ has the smallest variance

(c)

Since $MSE(\hat{\theta}) = Var(\hat{\theta}) + bias^2$

$\Rightarrow$ For $\hat{\mu}_1$, $MSE(\hat{\mu}_1) = Var(\hat{\mu}_1) = 6.25$

For $\hat{\mu}_2$, $MSE(\hat{\mu}_2) = Var(\hat{\mu}_2) = 9.0625$

For $\hat{\mu}_3$, $MSE(\hat{\mu}_3) = Var(\hat{\mu}_3) + (9 - \frac{\mu}{2})^2$

$= 1.9444 + (9 - \frac{\mu}{2})^2$

when $\mu = 8$, $MSE(\hat{\mu}_3) = 1.9444 + 25 = 26.9444$

Problem 10

Solution:

(a)

For $\hat{\mu}_1$, $E(\hat{\mu}_1) = E(\frac{X_1+X_2+X_3}{3}) = \frac{1}{3}[E(X_1)+E(X_2)+E(X_3)] = \mu$

$\Rightarrow$ bias $= E(\hat{\mu}_1) - \mu = 0$

This, this point estimate is unbiased.

For $\hat{\mu}_3$, $E(\hat{\mu}_3) = E(\frac{X_1}{4} + \frac{X_2}{3} + \frac{X_3}{5}) = \frac{1}{4}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{5}E(X_3)$

$= \frac{\mu}{4} + \frac{\mu}{3} + \frac{\mu}{5}$

$\approx 0.783\mu$

$\Rightarrow$ bias $= E(\hat{\mu}_2) - \mu = 0.783\mu - \mu = -0.217\mu$

For $\hat{\mu}_3$, $E(\hat{\mu}_3) = E(\frac{X_1}{6} + \frac{X_2}{3} + \frac{X_3}{4} + 2) = \frac{1}{6}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{4}E(X_3) + 2$

$= \frac{\mu}{6} + \frac{\mu}{3} + \frac{\mu}{4} + 2$

$= \frac{3}{4}\mu + 2$

$\Rightarrow$ bias $= E(\hat{\mu}_3) - \mu = 2 - \frac{\mu}{4}$

(b)

For $\hat{\mu}_1$, $Var(\hat{\mu}_1) = Var(\frac{X_1+X_2+X_3}{3})$

$= \frac{1}{9}[Var(X_1)+Var(X_2)+Var(X_3)]$

$= \frac{1}{9} \times (7+13+20)$

≈ 4.444

For $\hat{\mu}_2$, $Var(\hat{\mu}_2) = Var(\frac{X_1}{4} + \frac{X_2}{3} + \frac{X_3}{5}) = \frac{1}{16}Var(X_1) + \frac{1}{9}Var(X_2) + \frac{1}{25}Var(X_3)$

$= \frac{7}{16} + \frac{13}{9} + \frac{20}{25}$

≈ 2.682

For $\hat{\mu}_3$, $Var(\hat{\mu}_3) = Var(\frac{X_1}{6} + \frac{X_2}{3} + \frac{X_3}{4} + 2) = \frac{1}{36}Var(X_1) + \frac{1}{9}Var(X_2) + \frac{1}{16}Var(X_3)$

$= \frac{7}{36} + \frac{13}{9} + \frac{20}{16}$

≈ 2.889

Thus, $\hat{\mu}_2$ has the smallest version

(c)

$MSE(\hat{\theta}) = Var(\hat{\theta}) + bias^2$

$\Rightarrow$ For $\hat{\mu}_1$, $MSE(\hat{\mu}_1) = Var(\hat{\mu}_1) \approx 4.4444$

For $\hat{\mu}_2$, $MSE(\hat{\mu}_2) = Var(\hat{\mu}_2) + bias^2 \approx 3.104$

For $\hat{\mu}_3$, $MSE(\hat{\mu}_3) = Var(\hat{\mu}_3) + bias^2 \approx 4.452$

Problem 11

Solution:

(a)

For $\hat{\mu}_1$, $Var(\hat{\mu}_1) = Var(\frac{X_1+X_2}{2}) = \frac{1}{4}[Var(X_1)+Var(X_2)]$

$= \frac{1}{4}\times(4+6)$

$= 2.5$

(b)

For $\hat{\mu}_2$, $Var(\hat{\mu}_2) = Var[pX_1+(1-p)X_2]$

$= p^2 Var(X_1)+(1-p)^2 Var(X_2)$

$= 4p^2+6(1-p)^2$

$= 10p^2-12p+6$

$= 10(p^2-\frac{6}{5}p)+6$

$= 10(p-\frac{3}{5})^2+\frac{12}{5}$

Thus, when $p=0.6$, $Var_{min}(\hat{\mu}_2) = 2.4$.

(c)

$$Efficiency = \frac{Var_{min}(\hat{\mu}_2)}{Var(\hat{\mu}_1)} = \frac{2.4}{2.5} = 0.96$$

Problem 12

omitted

Problem 13

Solution:

Since $f(x)=2x$

$\Rightarrow F_X(x) = x^2 \quad (0\le x\le 1)$

(a)

When $Y=X^3$, to obtain the CDF

$\Rightarrow P(Y\le y) = P(X^3\le y) = P(X\le y^{\frac{1}{3}})$

$= F_X(y^{\frac{1}{3}}) = y^{\frac{2}{3}}$

Thus, the PDF of Y is

$f(y) = \frac{d}{dy}F_X(y^{\frac{1}{3}}) = \frac{2}{3}y^{-\frac{1}{3}}$

$\Rightarrow E(Y) = \int_0^1 y\cdot\frac{2}{3}y^{-\frac{1}{3}}dy = \int_0^1 \frac{2}{3}y^{\frac{2}{3}}dy$

$= \frac{2}{3}\int_0^1 y^{\frac{2}{3}}dy$

$= \frac{2}{5}[y^{\frac{5}{3}}]_0^1$

$= \frac{2}{5}$

(b)

When $Y=\sqrt{X}$, obtain the CDF

$\Rightarrow P(Y\le y) = P(\sqrt{X}\le y) = P(X\le y^3) = y^6 \quad (0\le y\le 1)$

Thus, the PDF is

$f(y) = \frac{d}{dy}F_X(y^3) = 6y^5$

$\Rightarrow E(y) = \int_0^1 y\cdot 6y^5 dy = \int_0^1 6y^6 dy = 6\int_0^1 y^6 dy = \frac{6}{7} \approx 0.857$

(c)

When $Y=\frac{1}{1+X}$, obtain the CDF

$\Rightarrow P(Y\le y) = P(\frac{1}{1+X}\le y) = P(X\ge\frac{1-y}{y}) = 1-P(X\le\frac{1-y}{y}) = -\frac{1}{y^2}+\frac{2}{y}$

Thus, the PDF is

$f(y) = \frac{d}{dy}P(Y\le y) = \frac{1}{y}+2\ln y \quad (\frac{1}{2}\le y\le 1)$

$\Rightarrow E(y) = \int_{\frac{1}{2}}^1 y(\frac{1}{y}+2\ln y)dy = \int_{\frac{1}{2}}^1 (1+2y\ln y)dy$

$= [y+y^2\ln y-\frac{y^2}{2}]_{\frac{1}{2}}^1$

$= \frac{1}{8}+\frac{\ln 2}{4} \approx 0.298$

Problem 14

Solution:

(a)

Since the sum of the probability produced by the PDF is equal to 1

$\Rightarrow \int_0^2 f(r)dr = \int_0^2 A[1-(r-1)^2]dr$

$= A\int_0^2 [1-(r-1)^2]dr$

$= A[r-\frac{(r-1)^3}{3}]_0^2$

$= \frac{4}{3}A = 1$

$\Rightarrow A = \frac{3}{4} = 0.75$

(b)

We have obtained the PDF of r is

$f(r) = \frac{3}{4}[1-(r-1)^2]$

Thus, its PDF is

$P(R\le r) = \int f(r)dr = \frac{3}{4}\int[1-(r-1)^2]dr = \frac{3}{4}[r-\frac{(r-1)^3}{3}]$

$= F_R(r)$

Denote the volume of soap bubble by V.

Since the shape of the bubble can be approximately considered as a sphere

$\Rightarrow V = \frac{4}{3}\pi R^3$

Assume that there exists a quantity v in the distribution function of V.

$\Rightarrow P(V\le v) = P(\frac{4}{3}\pi R^3\le v) = P(R\le(\frac{3v}{4\pi})^{\frac{1}{3}})$

$= F_R[(\frac{3v}{4\pi})^{\frac{1}{3}}]$

$= \frac{3}{4}\{(\frac{3v}{4\pi})^{\frac{1}{3}} - \frac{[(\frac{3v}{4\pi})^{\frac{1}{3}}-1]^3}{3}\}$

Thus, the PDF of the volume is

$f(v) = \frac{d}{dv}P(V\le v) = \frac{3}{4}\frac{d}{dv}\{(\frac{3v}{4\pi})^{\frac{1}{3}} - \frac{[(\frac{3v}{4\pi})^{\frac{1}{3}}-1]^3}{3}\}$

$= \frac{1}{4}(\frac{3}{4\pi})^{\frac{1}{3}}v^{-\frac{2}{3}}\{1-[(\frac{3v}{4\pi})^{\frac{1}{3}}-1]^2\}$

$= \frac{1}{2}(\frac{3}{4\pi})^{\frac{2}{3}}v^{-\frac{1}{3}} - \frac{3}{16\pi} \quad (0\le v\le\frac{32\pi}{3})$

(c)

From the definition of expectation,

$\Rightarrow E(V) = \int_0^{\frac{32\pi}{3}} vf(v)dv$

$= \int_0^{\frac{32\pi}{3}}[\frac{1}{2}(\frac{3}{4\pi})^{\frac{2}{3}}v^{\frac{2}{3}} - \frac{3}{16\pi}v]dv$

$= [\frac{3}{10}(\frac{3}{4\pi})^{\frac{2}{3}}v^{\frac{5}{3}} - \frac{3}{32\pi}v^2]_0^{\frac{32\pi}{3}}$

$= \frac{32}{15}\pi$

Problem 15

Solution:

(a)

Since the sum of the probability produced by the PDF is equal to 1.

$\Rightarrow \int_0^L f(x)dx = \int_0^L Ax(L-x)\,dx = A\int_0^L x(L-x)dx$

$= A\left[\frac{1}{2}Lx^2 - \frac{1}{3}x^3\right]_0^L = 1$

$\Rightarrow A = \frac{6}{L^3}$

(b)

We have obtained that

$f(x) = \frac{6}{L^3}x(L-x)\ (0 \le x \le L)$

Thus, the PDF of x is

$F_X(x) = P(X \le x) = \int \frac{6}{L^3}x(L-x)dx$

$= -\frac{2}{L^3}x^3 + \frac{3}{L^2}x^2\ (0 \le x \le L)$

From the description, we can obtain that the difference $= |L-2x|$, denoted by D.

$\Rightarrow D = |L-2x|$

Assume that there exists a d in the distribution function of D.

To find the CDF of d, we need to assume that $D \le d$ holds

$\Rightarrow |L-2x| \le d \Rightarrow \frac{L-d}{2} \le x \le \frac{L+d}{2}$

Thus, the CDF of d is

$P(D \le d) = P\left(\frac{L-d}{2} \le x \le \frac{L+d}{2}\right) = F_X\left(\frac{L+d}{2}\right) - F_X\left(\frac{L-d}{2}\right)$

$= -\frac{2}{L^3}\left(\frac{L+d}{2}\right)^3 + \frac{3}{L^2}\left(\frac{L+d}{2}\right)^2 + \frac{2}{L^3}\left(\frac{L-d}{2}\right)^3 - \frac{3}{L^2}\left(\frac{L-d}{2}\right)^2$

$= \underline{-\frac{1}{4L^3}d^3} + \frac{3}{4L}d + \frac{1}{2} \underline{-\frac{1}{4L^3}d^3} + \frac{3}{4L}d - \frac{1}{2}$

$= -\frac{1}{2L^3}d^3 + \frac{3}{2L}d\ (0 \le d \le L)$

Thus, the PDF of d is

$f(d) = (P(D \le d))'$ (with respect to d)

$= \left(-\frac{1}{2L^3}d^3 + \frac{3}{2L}d\right)'$

$= -\frac{3}{2L^3}d^2 + \frac{3}{2L}\ (0 \le d \le L)$

(c)

From the definition of expectation, we can obtain that

$E(D) = \int_0^L d f(d)\,dd = \int_0^L \left(-\frac{3}{2L^3}d^3 + \frac{3}{2L}d\right)dd$

$= \left[-\frac{3}{8L^3}d^4 + \frac{3}{4L}d^2\right]_0^L$

$= \frac{3L}{8}$

Problem 16

Solution:

(a)

The annual return has the expectation of \$100, and variance of 400 for fund A.

(b)

The annual return has the expectation of \$100, and variance of 900 for fund B.

(c)

The total return has the expectation of \$100, and variance of 325 for half of money in fund A, and half of money in fund B.

(d)

From the description, we can obtain the variance of the money.

$Var = (0.02x)^2 + [0.03(1000-x)]^2$

$= 0.0004x^2 + 0.0009(x-1000)^2$

$= 0.0013x^2 - 1.8x + 900$

when $x = -\frac{-1.8}{0.0026} = \frac{1.8}{0.0026} \approx 692$,

Var has the minimum value of 276.9.

Problem 17

Solution:

When $A \subset B$ and $B \neq \phi$,

we can obtain that

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

Since $0 \leq P(A) \leq 1$, $0 < P(B) \leq 1$

we can also rewrite $P(A) = \frac{P(A)}{1}$

Thus we can conclude that $P(A|B) \geqslant P(A)$,

which indicates that $P(A)$ is smaller than $P(A|B)$

Problem 18

Solution:

Denote the event that the ball is red by A,

the ball is shiny by B, and the ball is dull by C.

(a)

Before deriving the conditional probability,

we need $P(A) = \frac{150-54}{150} = \frac{96}{150} = 0.64$

Thus, we can obtain that

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{36}{150}}{0.64} = 0.375$$

(b)

We can obtain that

$$P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{96-36}{150}}{0.64} = 0.625$$

Problem 19

Solution:

Denote the event that a car repair is on time by A, and

the car repair is satisfactory by B.

From the description, we can obtain that

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = 0.85$$

$$\Rightarrow P(A \cap B) = P(B|A) \cdot P(A) = 0.85 \times 0.77 = 0.6545$$