Problem 1

Solution:

Since X has an exponential distribution, we have

$f(x) = \lambda e^{-\lambda x} \ (\lambda > 0, x > 0)$

(a) Thus, we can obtain

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x} dx$$

Integrate by parts, we have

$$\int x e^{-\lambda x} dx = -\frac{1}{\lambda} x e^{-\lambda x} + \int \frac{1}{\lambda} e^{-\lambda x} dx$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x} dx$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda}\left(-\frac{1}{\lambda} e^{-\lambda x} + C\right)$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C_1$$

(where $C_1 = \frac{C}{\lambda}$)

$$\Rightarrow \int_0^\infty x e^{-\lambda x} dx = \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x}\right]_0^\infty$$

$$= \lim_{t \to \infty} \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x}\right]_0^t$$

$$= \frac{1}{\lambda^2}$$

Thus, $E(X) = \lambda \int_0^\infty x e^{-\lambda x} dx = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$

(b) Thus, we can obtain

$$E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x} dx$$

Integrate by parts, we have

$$\int x^2 e^{-\lambda x} dx = -\frac{1}{\lambda} x^2 e^{-\lambda x} + \int 2x \cdot \frac{1}{\lambda} e^{-\lambda x} dx$$

$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda} \int x e^{-\lambda x} dx$$

$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda}\left(-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C\right)$$

$$= -\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} + C_1$$

(where $C_1 = \frac{2}{\lambda} C$)

$$\Rightarrow \int_0^\infty x^2 e^{-\lambda x} dx = \left[-\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x}\right]_0^\infty$$

$$= \lim_{t \to \infty} \left[-\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x}\right]_0^t$$

$$= \frac{2}{\lambda^3}$$

Thus, $E(X^2) = \lambda \int_0^\infty x^2 e^{-\lambda x} dx = \frac{2}{\lambda^3}$

Problem 2

Solution:

Denote the time for waiting by X.

(a)

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.1} = 10 \text{ mins}$$

(b)

Since $\lambda = 0.1$, we can obtain the probability density function

$f(x) = 0.1 e^{-0.1x}$

Thus, $P(X \le 10) = \int_0^{10} 0.1 e^{-0.1x} dx = 0.1 \int_0^{10} e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^{10}$

$$= 1 - \frac{1}{e}$$

$\Rightarrow P(X > 10) = 1 - P(X \le 10) = \frac{1}{e} \approx 0.3679$

(c)

Likewise, we can obtain that

$P(X \le 5) = \int_0^5 0.1 e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^5 = 1 - \frac{1}{\sqrt{e}} \approx 0.3935$

(d)

Without anything changed, the additional time still has the exponential distribution.

For the probability of waiting longer than 15 minutes,

$P(X \le 15) = \int_0^{15} 0.1 e^{-0.1x} dx = \left[-e^{-0.1x}\right]_0^{15} = 1 - e^{-\frac{3}{2}}$

Thus, $P(X > 15) = 1 - P(X \le 15) = e^{-\frac{3}{2}} \approx 0.2231$

(e)

Since the waiting time has a $U(0, 20)$, we can obtain that

$f(x) = \frac{1}{20}$

$\Rightarrow E(X) = \int_0^{20} x \cdot \frac{1}{20} dx = \frac{1}{40}\left[x^2\right]_0^{20} = 10 \text{ mins}$

Likewise, after waiting 5 minutes, the additional time has a $U(0, 15)$ distribution

Problem 3

Denote the germination times by Y.

Since Y has an exponential distribution, we can obtain the probability density function

$f(y) = 0.31e^{-0.31y} \quad (y>0)$

(a)

$\Rightarrow P(Y\le 5) = \int_0^5 0.31e^{-0.31y}dy = 0.31\int_0^5 e^{-0.31y}dy$

$= [-e^{-0.31y}]_0^5$

$= 1-e^{-1.55} \approx 0.7878$

(b) Since the germination time is independent each other, the number of seeds that germinate has a binomial distribution $X \sim B(12, 0.7878)$

Thus, $E(X) = 12\times 0.7878 = 9.4536$

$Var(X) = 12\times 0.7878\times(1-0.7878) \approx 2.0061$

(c)

Since we know that the number of seeds that germinate has a binomial distribution, we can obtain

$P(X=10) = C_{12}^{10}\times 0.7878^{10}\times 0.2122^2 \approx 0.2736$

$P(X=11) = C_{12}^{11}\times 0.7878^{11}\times 0.2122 \approx 0.1847$

$P(X=12) = C_{12}^{12}\times 0.7878^{12}\times 0.2122^0 \approx 0.0571$

$\Rightarrow P(X\le 9) = 1-[P(X=10)+P(X=11)+P(X=12)]$

$= 0.4846$

Problem 4

Solution:

Denote the cumulative distribution by $F(X)$

Case 1: when $X-\theta<0 \Leftrightarrow X<\theta$

$\Rightarrow F(X) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x}\frac{1}{2}\lambda e^{-\lambda(\theta-y)}dy = \frac{\lambda}{2e^{\lambda\theta}}\int_{-\infty}^{x} e^{\lambda y}dy$

$= \frac{1}{2e^{\lambda\theta}}[e^{\lambda y}]_{-\infty}^{x}$

$= \frac{1}{2}e^{\lambda(x-\theta)}$

Determine the range of x first before integrating when finding CDF.

Case 2: when $X-\theta>0 \Leftrightarrow X>\theta$

$\Rightarrow F(X) = \frac{1}{2}+\int_{\theta}^{x} f(y)dy = \frac{1}{2}+\int_{\theta}^{x}\frac{1}{2}\lambda e^{-\lambda(y-\theta)}dy = \frac{1}{2}+\frac{\lambda e^{\lambda\theta}}{2}\int_{\theta}^{x} e^{-\lambda y}dy$

when $X=\theta$

$= \frac{1}{2}-\frac{e^{\lambda\theta}}{2}[e^{-\lambda y}]_{\theta}^{x}$

$= 1-\frac{1}{2}e^{\lambda(\theta-x)}$

When $\lambda=3$ and $\theta=2$

(a) $P(X\le 0) = \frac{1}{2}e^{3\times(-2)} = \frac{1}{2e^6} \approx 0.0012$

(b) $P(X\ge 1) = 1-F(1) = 1-\frac{1}{2}e^{3\times(-1)} = 1-\frac{1}{2e^3} \approx 0.9751$