

### Problem 1

Solution:

Since  $X$  has an exponential distribution, we have

$$f(x) = \lambda e^{-\lambda x} \quad (\lambda > 0, x > 0)$$

(a) Thus, we can obtain

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x} dx$$

Integrate by parts, we have

$$\begin{aligned} \int x e^{-\lambda x} dx &= -\frac{1}{\lambda} x e^{-\lambda x} + \int \frac{1}{\lambda} e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \left( -\frac{1}{\lambda} e^{-\lambda x} + C \right) \\ &= -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C, \\ &\quad (\text{where } C_1 = \frac{C}{\lambda}) \end{aligned}$$

$$\Rightarrow \int_0^\infty x e^{-\lambda x} dx = \left[ -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \left[ -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^x \\ &= \frac{1}{\lambda^2} \end{aligned}$$

$$\text{Thus, } E(X) = \lambda \int_0^\infty x e^{-\lambda x} dx = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

(b) Thus, we can obtain

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty x^2 e^{-\lambda x} dx \end{aligned}$$

Integrate by parts, we have

$$\begin{aligned} \int x^2 e^{-\lambda x} dx &= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \int 2x \cdot \frac{1}{\lambda} e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda} \int x e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} x^2 e^{-\lambda x} + \frac{2}{\lambda} \left( -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C \right) \\ &= -\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} + C, \\ &\quad (\text{where } C_1 = \frac{2}{\lambda} C) \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^\infty x^2 e^{-\lambda x} dx &= \left[ -\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} \right]_0^\infty \\ &= \lim_{x \rightarrow \infty} \left[ -\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} \right]_0^x \\ &= \frac{2}{\lambda^3} \end{aligned}$$

$$\text{Thus, } E(X^2) = \lambda \int_0^\infty x^2 e^{-\lambda x} dx = \frac{2}{\lambda^3}$$

### Problem 2

Solution:

Denote the time for waiting by  $X$ .

(a)

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.1} = 10 \text{ mins}$$

(b)

Since  $\lambda = 0.1$ , we can obtain the probability density function

$$f(x) = 0.1 e^{-0.1x}$$

$$\begin{aligned} \text{Thus, } P(X \leq 10) &= \int_0^{10} 0.1 e^{-0.1x} dx = 0.1 \int_0^{10} e^{-0.1x} dx = [-e^{-0.1x}]_0^{10} \\ &= 1 - \frac{1}{e} \end{aligned}$$

$$\Rightarrow P(X > 10) = 1 - P(X \leq 10) = \frac{1}{e} \approx 0.3679$$

(c)

Likewise, we can obtain that

$$P(X \leq 5) = \int_0^5 0.1 e^{-0.1x} dx = [-e^{-0.1x}]_0^5 = 1 - \frac{1}{e^5} \approx 0.3935$$

(d)

Without anything changed, the additional time still has the exponential distribution.

For the probability of waiting longer than 15 minutes,

$$P(X \geq 15) = \int_0^{15} 0.1 e^{-0.1x} dx = [-e^{-0.1x}]_0^{15} = 1 - e^{-\frac{3}{2}}$$

$$\text{Thus, } P(X > 15) = 1 - P(X \leq 15) = e^{-\frac{3}{2}} \approx 0.2231$$

(e)

Since the waiting time has a  $U(10, 20)$ , we can obtain that

$$f(x) = \frac{1}{20}$$

$$\Rightarrow E(X) = \int_0^{20} x \cdot \frac{1}{20} dx = \frac{1}{20} [x^2]_0^{20} = 10 \text{ mins}$$

Likewise, after waiting 5 minutes, the additional time has a  $U(10, 15)$  distribution

### Problem 3

Denote the germination times by  $Y$ .

Since  $Y$  has an exponential distribution,  
we can obtain the probability density function

$$f(y) = 0.31e^{-0.31y} \quad (y > 0)$$

(a)

$$\begin{aligned} \Rightarrow P(Y \leq 5) &= \int_0^5 0.31e^{-0.31y} dy = 0.31 \int_0^5 e^{-0.31y} dy \\ &= [ -e^{-0.31y} ]_0^5 \\ &= 1 - e^{-1.55} \approx 0.7878 \end{aligned}$$

(b) Since the germination time is independent each other, the number of seeds that germinate has a binomial distribution  $X \sim B(12, 0.7878)$

$$\text{Thus, } E(X) = 12 \times 0.7878 = 9.4536$$

$$\text{Var}(X) = 12 \times 0.7878 \times (1 - 0.7878) \approx 2.0061$$

(c)

Since we know that the number of seeds that germinate has a binomial distribution, we can obtain

$$P(X=10) = C_{12}^{10} \times 0.7878^{10} \times 0.2122^2 \approx 0.2736$$

$$P(X=11) = C_{12}^{11} \times 0.7878^{11} \times 0.2122^1 \approx 0.1847$$

$$P(X=12) = C_{12}^{12} \times 0.7878^{12} \times 0.2122^0 \approx 0.0571$$

$$\begin{aligned} \Rightarrow P(X \leq 9) &= 1 - [P(X=10) + P(X=11) + P(X=12)] \\ &= 0.4846 \end{aligned}$$

### Problem 4

Solution:

Denote the cumulative distribution by  $F(x)$

Case 1: When  $x=0 \Leftrightarrow x < 0$

$$\begin{aligned} \Rightarrow F(x) &= \int_{-\infty}^x f(y) dy = \int_{-\infty}^x \frac{1}{2} \lambda e^{-\lambda(y-\theta)} dy = \frac{\lambda}{2e^{\lambda\theta}} \int_{-\infty}^x e^{\lambda y} dy \\ &= \frac{1}{2e^{\lambda\theta}} [e^{\lambda y}]_{-\infty}^x \\ &= \frac{1}{2} e^{\lambda(x-\theta)} \end{aligned}$$

Determine the range of  $x$  first  
before integrating when finding CDF.

Case 2: When  $x-\theta > 0 \Leftrightarrow x > \theta$

$$\begin{aligned} \Rightarrow F(x) &= \frac{1}{2} + \int_0^x f(y) dy = \frac{1}{2} + \int_0^x \frac{1}{2} \lambda e^{-\lambda(y-\theta)} dy = \frac{1}{2} + \frac{\lambda e^{\lambda\theta}}{2} \int_0^x e^{-\lambda y} dy \\ &\quad \text{when } x=\theta \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} - \frac{e^{\lambda\theta}}{2} [e^{-\lambda y}]_0^x \\ &= 1 - \frac{1}{2} e^{\lambda(\theta-x)} \end{aligned}$$

When  $\lambda=3$  and  $\theta=2$

$$(a) P(X \leq 0) = \frac{1}{2} e^{3x+6} = \frac{1}{2e^6} \approx 0.0012$$

$$(b) P(X \geq 1) = 1 - F(1) = 1 - \frac{1}{2} e^{3x+1} = 1 - \frac{1}{2e^3} \approx 0.9751$$

### Problem 5

Solution:

Case 1: When a student guesses blindly at all of the questions, the number of correct answers has a binomial distribution  $X \sim B(10, 0.2)$

To ensure the student can pass the quiz,

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$\begin{aligned} &= C_{10}^7 \times 0.2^7 \times 0.8^3 + C_{10}^8 \times 0.2^8 \times 0.8^2 + C_{10}^9 \times 0.2^9 \times 0.8 + \\ &\quad C_{10}^{10} \times 0.2^{10} \times 0.8^0 \end{aligned}$$

$$\approx 0.0609$$

Case 2: When a student can only guess between two possible answer, the number of correct answers has a binomial distribution  $X \sim B(10, 0.5)$

$$\Rightarrow P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$\begin{aligned} &= C_{10}^7 \times 0.5^7 \times 0.5^3 + C_{10}^8 \times 0.5^8 \times 0.5^2 + C_{10}^9 \times 0.5^9 \times 0.5 + C_{10}^{10} \times 0.5^{10} \end{aligned}$$

$$= 0.1719$$

0.50

### problem 6

Solution:

Denote the number of people taking sick leave by  $X$ , and the proportion is  $\gamma = \frac{X}{180}$

Since the employees are independent each other,  $X$  has a binomial distribution  $X \sim B(180, 0.35)$

Thus, we can obtain the expectation and variance of  $X$ , respectively

$$E(Y) = E\left(\frac{X}{180}\right) = \frac{1}{180} E(X) = \frac{1}{180} np = \frac{1}{180} \times 180 \times 0.35 = 0.35$$

$$\text{Var}(Y) = \text{Var}\left(\frac{X}{180}\right) = \frac{1}{180^2} \text{Var}(X) = \frac{1}{180^2} np(1-p) = \frac{1}{180^2} \times 180 \times 0.35 \times 0.65 \approx 0.00126$$

Generally, we have no idea of the specific value of  $p$ , thus  $X$  still has a binomial distribution  $X \sim B(180, p)$

Thus, the variance of  $Y$  is

$$\begin{aligned} \text{Var}(Y) &= \text{Var}\left(\frac{X}{180}\right) = \frac{1}{180^2} \text{Var}(X) = \frac{1}{180^2} 180p(1-p) \\ &= \frac{1}{180} p(1-p) = -\frac{1}{180} (p - \frac{1}{2})^2 + \frac{1}{720} \end{aligned}$$

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Hence, when  $p=0.5$ , it produces the largest variance for this proportion, which is approximately 0.00139,

### problem 7

Solution:

For distribution  $X_1 \sim B(n_1, p)$ , it can be considered as the number of successes within  $n_1$  trials.

Analogically, for distribution  $X_2 \sim B(n_2, p)$ , it can be considered as the number of successes within  $n_2$  trials.

Hence, for  $Y = X_1 + X_2$ , which is the sum of  $X_1$  and  $X_2$ , it can be considered as the number of successes within  $(n_1+n_2)$  trials with the unchanged successful probability  $p$ .

Finally, we can conclude that  $Y \sim B(n_1+n_2, p)$

### problem 8

Solution:

Denote the number of the orders received from the Internet by  $X$ .

Since the placed orders are independent each other,  $X$  has a binomial distribution  $X \sim B(18, 0.6)$

$$\begin{aligned} (a) P(8 \leq X \leq 10) &= P(X=8) + P(X=9) + P(X=10) \\ &= C_{18}^8 \times 0.6^8 \times 0.4^{10} + C_{18}^9 \times 0.6^9 \times 0.4^{10} + C_{18}^{10} \times 0.6^{10} \times 0.4^{10} \\ &= 0.071 + 0.1284 + 0.1734 \\ &= 0.3729 \end{aligned}$$

$$(b) P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$\begin{aligned} &= C_{18}^0 \times 0.6^0 \times 0.4^{18} + C_{18}^1 \times 0.6^1 \times 0.4^{17} + C_{18}^2 \times 0.6^2 \times 0.4^{16} \\ &\quad + C_{18}^3 \times 0.6^3 \times 0.4^{15} + C_{18}^4 \times 0.6^4 \times 0.4^{14} \\ &\approx 0.0013 \end{aligned}$$