Homework Assignment 2

Due: Friday, September 27, 2024, 11:59 p.m. Mountain time

Total marks: 20

Policies:

For all multiple-choice questions, note that multiple correct answers may exist. However, selecting an incorrect option will cancel out a correct one. For example, if you select two answers, one correct and one incorrect, you will receive zero points for that question. Similarly, if the number of incorrect answers selected exceeds the correct ones, your score for that question will be zero. Please note that it is not possible to receive negative marks. **You must select all the correct options to get full marks for the question.**

While the syllabus initially indicated the need to submit a paragraph explaining the use of AI or other resources in your assignments, this requirement no longer applies as we are now utilizing eClass quizzes instead of handwritten submissions. Therefore, you are **not** required to submit any explanation regarding the tools or resources (such as online tools or AI) used in completing this quiz.

This PDF version of the questions has been provided for your convenience should you wish to print them and work offline.

Only answers submitted through the eClass quiz system will be graded. Please do not submit a written copy of your responses.

Question 1. [1 MARK]

Suppose you flip three coins. Suppose the first coin is represented by random variable $X_1 \in \{0,1\}$, the second coin by $X_2 \in \{0,1\}$, and the third coin by $X_3 \in \{0,1\}$. Which of the following is the outcome space of the random variable $X = (X_1, X_2, X_3)$?

a. $\{0,1\}^3$

b. $\{1,2,3\}$

c. $\{(x_1, x_2, x_3) \mid x_1 \in \{0,1\}, x_2 \in \{0,1\}, x_3 \in \{0,1\}\}$

d. $\{0,1\}$

Solution:

The correct answers are:

- **a.** $\{0,1\}^3$
- **c.** $\{(x_1, x_2, x_3) \mid x_1 \in \{0,1\}, x_2 \in \{0,1\}, x_3 \in \{0,1\}\}$

Explanation:

The outcome space of the random variable $X = (X_1, X_2, X_3)$ consists of all possible combinations of the coin flips, where each X_i can be either 0 or 1.

a. $\{0,1\}^3$: Correct. This notation represents the Cartesian product of $\{0,1\}$ taken three times, resulting in all 3-tuples where each component is 0 or 1.

b. $\{1, 2, 3\}$: Incorrect. This set does not represent the possible outcomes of the coin flips.

c. $\{(x_1, x_2, x_3) \mid x_1 \in \{0, 1\}, x_2 \in \{0, 1\}, x_3 \in \{0, 1\}\}$: Correct. This explicitly defines the set of all 3-tuples where each x_i is either 0 or 1.

d. $\{0, 1\}$: Incorrect. This set includes only two outcomes and does not account for all combinations of three coin flips.

Both options **a** and **c** correctly represent the outcome space of X.

Question 2. [1 MARK]

Suppose you roll a fair twenty-sided die. The outcome space is $\mathcal{X} = \{1, 2, 3, \ldots, 20\}$. Which of the following is an event?

a. $\{0, 1, 2\}$

b. $\{x \in \mathcal{X} \mid x > 10\}$

c. 12

d. $\{12\}$

Solution:

The correct answers are:

- **b.** $\{x \in \mathcal{X} \mid x > 10\}$
- **d.** $\{12\}$

Explanation:
An *event* is any subset of the outcome space $\mathcal{X}$.

a. $\{0, 1, 2\}$: Incorrect. This set includes 0, which is not in $\mathcal{X}$, so it is not a subset of $\mathcal{X}$ and therefore not an event.

b. $\{x \in \mathcal{X} \mid x > 10\}$: Correct. This set includes all outcomes greater than 10 within $\mathcal{X}$, specifically $\{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. It is a subset of $\mathcal{X}$ and thus an event.

c. 12: Incorrect. This is an element of $\mathcal{X}$, not a subset. An event must be a subset of $\mathcal{X}$.

d. $\{12\}$: Correct. This is the singleton set containing the element 12, which is a subset of $\mathcal{X}$ and therefore an event.

Question 3. [1 MARK]

Which of the following is an event from the outcome space $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{Y} = \mathbb{R}$?

a. $\mathcal{X} \times \mathcal{Y}$

b. A function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

c. $((1, 2), 3)$

d. $\{((x_1, x_2), y) \in \mathcal{X} \times \mathcal{Y} \mid y \geq 300\}$

Solution:

Answers: a. $\mathcal{X} \times \mathcal{Y}$ and **d.** $\{((x_1, x_2), y) \in \mathcal{X} \times \mathcal{Y} \mid y \geq 300\}$
Explanation:
An event is any subset of the outcome space. Given the outcome space $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{Y} = \mathbb{R}$, let's analyze each option:

1. **Option a:** $\mathcal{X} \times \mathcal{Y}$ **Correct.** This represents the entire outcome space itself, which is a valid event (the certain event).

2. **Option b:** A function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ **Incorrect.** This describes a mapping between spaces, not an event within the outcome space.

3. **Option c:** $((1, 2), 3)$ **Incorrect.** While $((1, 2), 3)$ is an element (outcome) of $\mathcal{X} \times \mathcal{Y}$, it is not an event. An event is a set of outcomes, not a single outcome unless specified as a singleton set.

4. **Option d:** $\{((x_1, x_2), y) \in \mathcal{X} \times \mathcal{Y} \mid y \geq 300\}$ **Correct.** This is a subset of the outcome space where the third component y meets the condition $y \geq 300$. It is a valid event.

Question 4. [1 MARK]

Which of the following is an event from the outcome space $(\mathcal{X} \times \mathcal{Y})^n$, where $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{Y} = \mathbb{R}$?

a. $(\mathcal{X} \times \mathcal{Y})^n$

b. $(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n))$ where $x_{i,1}, x_{i,2}, y_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$

c. $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X} \times \mathcal{Y})^n \mid y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$

d. $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X} \times \mathcal{Y})^n \mid x_{i,1} \geq 3 \text{ and } y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$

Solution:

Answers: a. $(\mathcal{X} \times \mathcal{Y})^n$, **c.** $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X} \times \mathcal{Y})^n \mid y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$, and **d.** $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X} \times \mathcal{Y})^n \mid x_{i,1} \geq 3 \text{ and } y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$
Explanation:
An event is a subset of the outcome space $(\mathcal{X} \times \mathcal{Y})^n$. Let's evaluate each option:

1. **Option a:** $(\mathcal{X} \times \mathcal{Y})^n$ **Correct.** This represents the entire outcome space, which is a valid event (the certain event).

2. **Option b:** $(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n))$ where $x_{i,1}, x_{i,2}, y_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$ **Incorrect.** This describes a single ordered n-tuple (a specific outcome), not an event. An event should be a set of such outcomes.

3. **Option c:** $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X} \times \mathcal{Y})^n \mid y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$ **Correct.** This is a subset of $(\mathcal{X} \times \mathcal{Y})^n$ where all y_i satisfy $y_i \geq 300$. It is a valid event.

4. **Option d:** $\{(((x_{1,1}, x_{1,2}), y_1), \ldots, ((x_{n,1}, x_{n,2}), y_n)) \in (\mathcal{X}\times\mathcal{Y})^n \mid x_{i,1} \geq 3 \text{ and } y_i \geq 300 \text{ for all } i \in \{1, \ldots, n\}\}$ **Correct.** This subset imposes conditions on both $x_{i,1}$ and y_i for all i, making it a valid event.

Question 5. [1 MARK]

Suppose you have a random variable Y representing the house prices in a city. You know Y is distributed according to the normal distribution with mean $100,000$ and standard deviation $10,000$. Is the following True or False? Y is a continuous random variable.

Solution:

Answer: True.
Explanation:
A random variable is *continuous* if it can take on any value within a certain interval or range, and its probability distribution is described by a continuous probability density function (pdf). The normal distribution is a continuous distribution defined over the entire real line $\mathbb{R}$. Since Y follows a normal distribution with mean $100,000$ and standard deviation $10,000$, it can take any real value, and its probabilities are determined by a continuous pdf. Therefore, Y is a continuous random variable.

Question 6. [1 MARK]

Suppose you have a random variable X representing the age of houses in a city. You know X is distributed according to the continuous uniform distribution with outcome space $\mathcal{X} = [10, 20]$. Let p be the pdf of X. Is the following True or False? $p(17)$ is the probability that $X = 17$.

Solution:

Answer: False.
Explanation:
For a continuous random variable X, the probability that X takes on any specific value x is zero, i.e., $\mathbb{P}(X = x) = 0$. The probability density function $p(x)$ does not give the probability at a specific point but rather represents the density of the distribution at that point. To find the probability that X falls within an interval $[a, b]$, we integrate the pdf over that interval:

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)\, dx$$

Therefore, $p(17)$ is not the probability that $X = 17$; it is the value of the pdf at $x = 17$.

Question 7. [1 MARK]

For a continuous random variable $X \in \mathcal{X}$, we know that $\mathbb{P}(X = x) = 0$ for all $x \in \mathcal{X}$. How is it possible that the pdf $p(x) \neq 0$ for all $x \in \mathcal{X}$?

a. Because $p(x)$ is not the probability of x. Instead, it is just a function that we can integrate over to get a probability.

b. Because the pdf measures the probability mass at each point.

c. Because the pdf is always zero for continuous variables.

d. Because the random variable is discrete.

Solution:

The correct answer is:

- **a.** Because $p(x)$ is not the probability of x. Instead, it is just a function that we can integrate over to get a probability.

Explanation:
For continuous random variables, the probability of any specific value x is zero, i.e., $\mathbb{P}(X = x) = 0$. The probability density function $p(x)$ represents the density of the distribution at point x and is used to calculate probabilities over intervals through integration:

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)\, dx$$

Therefore, even though $\mathbb{P}(X = x) = 0$, the pdf $p(x)$ can be non-zero for all $x \in \mathcal{X}$ because it is not the probability at x, but a function that describes how probability is distributed over the range of X.
Option **b** is incorrect because the pdf does not measure probability mass at a point. Option **c** is incorrect because the pdf is not always zero for continuous variables. Option **d** is irrelevant because X is a continuous random variable, not discrete.

Question 8. [1 MARK]

Suppose that $Y \in \mathbb{R}$ is distributed according to Laplace(10, 2). Is the following True or False? The probability distribution of Y is $\mathbb{P}(Y = y) = \frac{1}{4} \exp\left(-\frac{|y-10|}{2}\right)$ for $y \in \mathbb{R}$.

Solution:

Answer: False.
Explanation:
For a continuous random variable Y following a Laplace distribution with mean $\mu = 10$ and scale parameter $b = 2$, the *probability density function* (pdf) is:

$$p(y) = \frac{1}{2b} \exp\left(-\frac{|y-\mu|}{b}\right) = \frac{1}{4} \exp\left(-\frac{|y-10|}{2}\right)$$

However, the probability that Y takes on any specific value y is zero, i.e., $\mathbb{P}(Y = y) = 0$ for all $y \in \mathbb{R}$.
Therefore, the expression $\mathbb{P}(Y = y) = \frac{1}{4} \exp\left(-\frac{|y-10|}{2}\right)$ is incorrect. It should be stated as the probability *density* function $p(y)$, not the probability mass function $\mathbb{P}(Y = y)$.

Question 9. [1 MARK]

Suppose you have two discrete random variables $X \in \{0, 1\}$ and $Y \in \{1, 2, 3\}$. The joint probability mass function (pmf) of X and Y is given by the following values:

$$p(0,1) = \frac{1}{24}, \quad p(0,2) = \frac{1}{12}, \quad p(0,3) = \frac{1}{3},$$
$$p(1,1) = \frac{1}{6}, \quad p(1,2) = \frac{3}{24}, \quad p(1,3) = \frac{3}{12}.$$

Which of the following is the marginal pmf of X?

a. $p_X(0) = \frac{11}{24}, \quad p_X(1) = \frac{13}{24}$

b. $p_X(0) = \frac{1}{2}, \quad p_X(1) = \frac{1}{2}$

c. $p_X(0) = \frac{7}{12}, \quad p_X(1) = \frac{5}{12}$

d. $p_X(0) = \frac{5}{12}, \quad p_X(1) = \frac{7}{12}$

Solution:

Answer:

- **a.** $p_X(0) = \frac{11}{24}, \quad p_X(1) = \frac{13}{24}$

Explanation:
To find the marginal pmf $p_X(x)$, we sum the joint pmf over all possible values of Y for each X:
For $x = 0$:

$$\begin{aligned} p_X(0) &= p(0,1) + p(0,2) + p(0,3) \\ &= \frac{1}{24} + \frac{1}{12} + \frac{1}{3} \\ &= \frac{1}{24} + \frac{2}{24} + \frac{8}{24} \quad \text{(common denominator of 24)} \\ &= \frac{11}{24} \end{aligned}$$

For $x = 1$:

$$\begin{aligned} p_X(1) &= p(1,1) + p(1,2) + p(1,3) \\ &= \frac{1}{6} + \frac{3}{24} + \frac{3}{12} \\ &= \frac{4}{24} + \frac{3}{24} + \frac{6}{24} \quad \text{(common denominator of 24)} \\ &= \frac{13}{24} \end{aligned}$$

Therefore, the marginal pmf of X is:

$$p_X(0) = \frac{11}{24}, \quad p_X(1) = \frac{13}{24}$$

This corresponds to option **a**.

Question 10. [1 MARK]

Using the same random variables X and Y from the previous question, which of the following is the conditional pmf $p_{Y|X}(y|x)$?

a.

$$p_{Y|X}(1|0) = \frac{1}{11}, \quad p_{Y|X}(2|0) = \frac{2}{11}, \quad p_{Y|X}(3|0) = \frac{8}{11}$$
$$p_{Y|X}(1|1) = \frac{4}{13}, \quad p_{Y|X}(2|1) = \frac{3}{13}, \quad p_{Y|X}(3|1) = \frac{6}{13}$$

b.

$$p_{Y|X}(1|0) = \frac{1}{3}, \quad p_{Y|X}(2|0) = \frac{1}{3}, \quad p_{Y|X}(3|0) = \frac{1}{3}$$

$$p_{Y|X}(1|1) = \frac{1}{3}, \quad p_{Y|X}(2|1) = \frac{1}{3}, \quad p_{Y|X}(3|1) = \frac{1}{3}$$

c.

$$p_{Y|X}(1|0) = \frac{1}{2}, \quad p_{Y|X}(2|0) = \frac{1}{4}, \quad p_{Y|X}(3|0) = \frac{1}{4}$$

$$p_{Y|X}(1|1) = \frac{1}{4}, \quad p_{Y|X}(2|1) = \frac{1}{4}, \quad p_{Y|X}(3|1) = \frac{1}{2}$$

d.

$$p_{Y|X}(1|0) = \frac{1}{6}, \quad p_{Y|X}(2|0) = \frac{1}{3}, \quad p_{Y|X}(3|0) = \frac{1}{2}$$

$$p_{Y|X}(1|1) = \frac{2}{7}, \quad p_{Y|X}(2|1) = \frac{1}{7}, \quad p_{Y|X}(3|1) = \frac{4}{7}$$

Solution:

Answer:

- **a.** The conditional pmf given in option **a**.

Explanation:
The conditional pmf $p_{Y|X}(y|x)$ is calculated using:

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)}$$

From the previous solution, we have:

$$p_X(0) = \frac{11}{24}, \quad p_X(1) = \frac{13}{24}$$

For $x = 0$:

$$p_{Y|X}(1|0) = \frac{p(0, 1)}{p_X(0)} = \frac{\frac{1}{24}}{\frac{11}{24}} = \frac{1}{11}$$

$$p_{Y|X}(2|0) = \frac{p(0, 2)}{p_X(0)} = \frac{\frac{1}{12}}{\frac{11}{24}} = \frac{\frac{2}{24}}{\frac{11}{24}} = \frac{2}{11}$$

$$p_{Y|X}(3|0) = \frac{p(0, 3)}{p_X(0)} = \frac{\frac{1}{3}}{\frac{11}{24}} = \frac{\frac{8}{24}}{\frac{11}{24}} = \frac{8}{11}$$

For $x = 1$:

$$p_{Y|X}(1|1) = \frac{p(1,1)}{p_X(1)} = \frac{\frac{1}{6}}{\frac{13}{24}} = \frac{\frac{4}{24}}{\frac{13}{24}} = \frac{4}{13}$$

$$p_{Y|X}(2|1) = \frac{p(1,2)}{p_X(1)} = \frac{\frac{3}{24}}{\frac{13}{24}} = \frac{3}{13}$$

$$p_{Y|X}(3|1) = \frac{p(1,3)}{p_X(1)} = \frac{\frac{3}{12}}{\frac{13}{24}} = \frac{\frac{6}{24}}{\frac{13}{24}} = \frac{6}{13}$$

Therefore, the conditional pmf matches the one provided in option **a**.

Question 11. [1 MARK]

Based on the previous two questions, determine whether the following statement is True or False: The random variables X and Y are independent.

Solution:

Answer: False.
Explanation:
Two random variables X and Y are independent if and only if:

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad \text{for all } x, y.$$

First, compute the marginal pmf of Y:
For $y = 1$:

$$\begin{aligned} p_Y(1) &= p(0,1) + p(1,1) \\ &= \frac{1}{24} + \frac{1}{6} \\ &= \frac{1}{24} + \frac{4}{24} = \frac{5}{24} \end{aligned}$$

For $y = 2$:

$$\begin{aligned} p_Y(2) &= p(0,2) + p(1,2) \\ &= \frac{1}{12} + \frac{3}{24} \\ &= \frac{2}{24} + \frac{3}{24} = \frac{5}{24} \end{aligned}$$

For $y = 3$:

$$\begin{aligned} p_Y(3) &= p(0,3) + p(1,3) \\ &= \frac{1}{3} + \frac{3}{12} \\ &= \frac{8}{24} + \frac{6}{24} = \frac{14}{24} = \frac{7}{12} \end{aligned}$$

Now, check if $p(x,y) = p_X(x) \cdot p_Y(y)$ for all x and y:

Example for $x = 0, y = 1$:

$$p_X(0) \cdot p_Y(1) = \frac{11}{24} \cdot \frac{5}{24} = \frac{55}{576}$$
$$p(0,1) = \frac{1}{24} = \frac{24}{576}$$

Since $\frac{55}{576} \neq \frac{24}{576}$, they are not equal.
Example for $x = 1, y = 3$:

$$p_X(1) \cdot p_Y(3) = \frac{13}{24} \cdot \frac{7}{12} = \frac{91}{288}$$
$$p(1,3) = \frac{3}{12} = \frac{6}{24} = \frac{72}{288}$$

Again, $\frac{91}{288} \neq \frac{72}{288}$.
Since there exists an x, y such that $p(x,y) \neq p_X(x) \cdot p_Y(y)$, the random variables X and Y are **not independent**.
Alternatively, the fact that the conditional pmf $p_{Y|X}(y|x)$ is not equal to $p_Y(y)$ confirms that X and Y are not independent.

Question 12. [1 MARK]

Let $X = (X_1, \ldots, X_n) \in \{0,1\}^n$ be a random variable representing the outcome of n coin flips. Let $\mathbb{P}_X$ be the distribution of X. Let $\mathbb{P}_{X_i}$ be the marginal distribution of X_i for each $i \in \{1, \ldots, n\}$. Assume $X_1, \ldots, X_n$ are independent and identically distributed with $\mathbb{P} = \text{Bernoulli}(0.7)$, meaning each flip results in heads (1) with probability 0.7 and tails (0) with probability 0.3. Let

$$\mathcal{E} = \{(x_1, \ldots, x_n) \mid x_n = 1 \text{ and } x_i = 0 \text{ for all } i \in \{1, \ldots, n-1\}\}$$

be the event that you get tails for the first $n-1$ flips and then heads on flip n. What is $\mathbb{P}_X(\mathcal{E})$ if $n = 4$?
(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: $\mathbb{P}_X(\mathcal{E}) = 0.0189$
Explanation:
Since the $\mathbb{P} = \text{Bernoulli}(0.7)$ the pmf p is $p(1) = 0.7$ and $p(0) = 0.3$. Since each coin flip is independent and identically distributed, the joint distribution $\mathbb{P}_X$ can be written as the product of the marginal distribution $\mathbb{P}$:

$$\begin{aligned}
\mathbb{P}_X(\mathcal{E}) &= \mathbb{P}_X(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1) \\
&= \mathbb{P}(X_1 = 0) \cdot \mathbb{P}(X_2 = 0) \cdot \mathbb{P}(X_3 = 0) \cdot \mathbb{P}(X_4 = 1) \\
&= p(X_1 = 0) \cdot p(X_2 = 0) \cdot p(X_3 = 0) \cdot p(X_4 = 1) \\
&= 0.3 \times 0.3 \times 0.3 \times 0.7 \\
&= 0.0189
\end{aligned}$$

Question 13. [1 MARK]

Let $X \in \mathcal{X}$ be a random variable. Let $g : \mathcal{X} \to \mathcal{F}$, where

$$\mathcal{F} = \{f \mid f : \mathcal{X} \to \mathbb{R} \text{ and } f(x) = xw \text{ where } w \in \mathbb{R}\}.$$

Is $g(X)$ a random variable and what is its outcome space?

a. Yes, $\mathcal{F}$

b. Yes, $\mathbb{R}$

c. No, $\mathcal{F}$

d. No, $\mathbb{R}$

Solution:

Answer: a. Yes, $\mathcal{F}$
Explanation:
$g(X)$ is a function of a random variable, and it maps elements from $\mathcal{X}$ to $\mathcal{F}$, where $\mathcal{F}$ consists of functions from $\mathcal{X}$ to $\mathbb{R}$ of the form $f(x) = xw$, with $w \in \mathbb{R}$. Since the output of $g(X)$ is a function in $\mathcal{F}$, $g(X)$ is indeed a random variable, and its outcome space is $\mathcal{F}$.

Question 14. [1 MARK]

If $X_1, \ldots, X_n$ are random variables with the same distribution $\mathcal{N}(\mu, \sigma^2)$, what is the expected value of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

a. μ

b. σ^2

c. $\frac{\mu}{n}$

d. $n\mu$

Solution:

Answer: a. μ
Explanation:
The expected value of the sample mean $\bar{X}$ is:

$$\mathbb{E}[\bar{X}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n} \cdot n\mu = \mu.$$

Question 15. [1 MARK]

Let X be a random variable with outcome space $\mathcal{X} = \{a, b, c\}$ and pmf $p(a) = 0.1, \quad p(b) = 0.2, \quad p(c) = 0.7$. The function $f(x)$ is given by:

$$f(a) = 10, \quad f(b) = 5, \quad f(c) = \frac{10}{7}.$$

What is $\mathbb{E}[f(X)]$?
(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: 3
Explanation:
Compute the expected value:

$$\begin{aligned}\mathbb{E}[f(X)] &= f(a)\cdot p(a) + f(b)\cdot p(b) + f(c)\cdot p(c)\\ &= 10 \times 0.1 + 5 \times 0.2 + \frac{10}{7} \times 0.7\\ &= 1 + 1 + \left(\frac{10 \times 0.7}{7}\right)\\ &= 1 + 1 + \left(\frac{7}{7}\right)\\ &= 1 + 1 + 1 = 3.\end{aligned}$$

Question 16. [1 MARK]

Alberta Hospital occasionally has electrical problems. It can take some time to find the problem, though it is always found in no more than 10 hours. The amount of time is variable; for example, one time it might take 0.3 hours, and another time it might take 5.7 hours. The time (in hours) necessary to find and fix an electrical problem at Alberta Hospital is a random variable X, whose density is given by the following uniform distribution:

$$p(x) = \frac{1}{10} \quad \text{if } 0 \leq x \leq 10,$$

$$p(x) = 0 \quad \text{otherwise.}$$

Such electrical problems can be costly for the Hospital, more so the longer it takes to fix it. The cost of an electrical breakdown of duration x is $C(x) = x^3$. What is the expected cost of an electrical breakdown?
(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: $\mathbb{E}[C(X)] = 250$
Explanation:
We are asked to compute the expected cost:

$$\mathbb{E}[C(X)] = \mathbb{E}[X^3] = \int_0^{10} x^3 \cdot p(x)\, dx$$

Given that X is uniformly distributed over $[0, 10]$, the density function is:

$$p(x) = \frac{1}{10} \quad \text{for } 0 \leq x \leq 10$$

Compute the expected value:

$$
\begin{aligned}
\mathbb{E}[X^3] &= \int_0^{10} x^3 \cdot \frac{1}{10}\,dx \\
&= \frac{1}{10}\int_0^{10} x^3\,dx \\
&= \frac{1}{10}\left[\frac{x^4}{4}\right]_0^{10} \\
&= \frac{1}{10}\left(\frac{10^4}{4} - 0\right) \\
&= \frac{1}{10}\left(\frac{10000}{4}\right) \\
&= \frac{1}{10} \times 2500 \\
&= 250
\end{aligned}
$$

Question 17. [1 MARK]

Let N be a random variable that takes values based on the roll of a biased four-sided die. The pmf of N is:

$$p_N(n) = \frac{n}{10} \quad \text{for } n \in \{1, 2, 3, 4\},$$

$$p_N(n) = 0 \quad \text{otherwise.}$$

What is $\mathbb{E}[N]$?

(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: $\mathbb{E}[N] = 3$

Explanation:

First, list the probabilities:

$$
\begin{aligned}
p_N(1) &= \frac{1}{10} \\
p_N(2) &= \frac{2}{10} \\
p_N(3) &= \frac{3}{10} \\
p_N(4) &= \frac{4}{10}
\end{aligned}
$$

Compute the expected value:

$$\begin{aligned}
\mathbb{E}[N] &= \sum_{n=1}^{4} n \cdot p_N(n) \\
&= (1) \cdot \frac{1}{10} + (2) \cdot \frac{2}{10} + (3) \cdot \frac{3}{10} + (4) \cdot \frac{4}{10} \\
&= \frac{1}{10} + \frac{4}{10} + \frac{9}{10} + \frac{16}{10} \\
&= \frac{1+4+9+16}{10} \\
&= \frac{30}{10} \\
&= 3
\end{aligned}$$

Question 18. [1 MARK]

Let everything be as defined in the previous question. What is $\mathrm{Var}[N]$?
(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: $\mathrm{Var}[N] = 1$
Explanation:
First, compute $\mathbb{E}[N^2]$:

$$\begin{aligned}
\mathbb{E}[N^2] &= \sum_{n=1}^{4} n^2 \cdot p_N(n) \\
&= (1^2) \cdot \frac{1}{10} + (2^2) \cdot \frac{2}{10} + (3^2) \cdot \frac{3}{10} + (4^2) \cdot \frac{4}{10} \\
&= \frac{1}{10} + \frac{8}{10} + \frac{27}{10} + \frac{64}{10} \\
&= \frac{1+8+27+64}{10} \\
&= \frac{100}{10} \\
&= 10
\end{aligned}$$

Now compute the variance:

$$\mathrm{Var}[N] = \mathbb{E}[N^2] - (\mathbb{E}[N])^2 = 10 - (3)^2 = 10 - 9 = 1$$

Question 19. [1 MARK]

Let everything be as defined in the previous two questions. Given the random variable N, a biased coin is flipped with the random variable X representing the outcome of the coin flip. If $X = 1$ then the coin shows heads and if $X = 0$ the coin shows tails. Assume the conditional pmf of N given $X = 1$ is:

$$p_{N|X}(1|1) = \frac{1}{7}, \quad p_{N|X}(2|1) = \frac{3}{14}, \quad p_{N|X}(3|1) = \frac{2}{7}, \quad p_{N|X}(4|1) = \frac{5}{14}.$$

What is the conditional expectation $\mathbb{E}[N|X=1]$ (i.e., the expectation with respect to this conditional pmf)?
(Do not write your answer as a fraction. Instead, express it as a decimal number, rounded to two decimal places if necessary. For example, write 0.33 instead of 1/3.)

Solution:

Answer: $\mathbb{E}[N|X=1] = \dfrac{20}{7}$
Explanation:
The conditional expectation $\mathbb{E}[N|X=1]$ is computed as:

$$\mathbb{E}[N|X=1] = \sum_{n=1}^{4} n \cdot p_{N|X}(n|1)$$

Substituting the values of $p_{N|X}(n|1)$:

$$\begin{aligned}
\mathbb{E}[N|X=1] &= (1) \cdot \frac{1}{7} + (2) \cdot \frac{3}{14} + (3) \cdot \frac{2}{7} + (4) \cdot \frac{5}{14} \\
&= \frac{1}{7} + \frac{6}{14} + \frac{6}{7} + \frac{20}{14} \\
&= \frac{1}{7} + \frac{3}{7} + \frac{6}{7} + \frac{10}{7} \\
&= \frac{1+3+6+10}{7} \\
&= \frac{20}{7}
\end{aligned}$$

Question 20. [1 MARK]

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $\mathcal{N}(\mu, \sigma^2)$. What is $\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$ in terms of μ and σ^2?

a. σ^2

b. $n\sigma^2$

c. $\dfrac{\sigma^2}{n}$

d. $n^2\sigma^2$

Solution:

Answer: **b.** $n\sigma^2$
Explanation:
We are tasked with finding the variance of the sum of n independent and identically distributed (i.i.d.) normal random variables $X_1, X_2, \ldots, X_n$, each with mean μ and variance σ^2.

1. **Understanding Variance of Independent Variables:**

   For independent random variables, the variance of their sum is equal to the sum of their variances. Mathematically, for independent X and Y:

   $$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

   This property extends to any finite number of independent random variables.

2. **Applying the Property to n Variables:**

   Given that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with $\text{Var}(X_i) = \sigma^2$ for all $i \in \{1, 2, \ldots, n\}$, the variance of their sum is:

   $$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$