Homework Assignment 1

Due: Friday, September 20, 2024, 11:59 p.m. Mountain time
Total marks: 25

Policies:

For all multiple-choice questions, note that multiple correct answers may exist. However, selecting an incorrect option will cancel out a correct one. For example, if you select two answers, one correct and one incorrect, you will receive zero points for that question. Similarly, if the number of incorrect answers selected exceeds the correct ones, your score for that question will be zero. Please note that it is not possible to receive negative marks.

While the syllabus initially indicated the need to submit a paragraph explaining the use of AI or other resources in your assignments, this requirement no longer applies as we are now utilizing eClass quizzes instead of handwritten submissions. Therefore, you are **not** required to submit any explanation regarding the tools or resources (such as online tools or AI) used in completing this quiz.

This PDF version of the questions has been provided for your convenience should you wish to print them and work offline.

Only answers submitted through the eClass quiz system will be graded. Please do not submit a written copy of your responses.

Question 1. [1 MARK]

Is the following True or False. $(\mathbb{R}^2) \times \mathbb{R}$ is a set.

Solution:

True. A Cartesian product is always a set. It is defined as $(\mathbb{R}^2) \times \mathbb{R} = \{((x_1, x_2), y) | x_1, x_2, y \in \mathbb{R}\}$, which is a set of tuples where each tuple has an element from $\mathbb{R}^2$ and an element from $\mathbb{R}$.

Question 2. [1 MARK]

Is the following True or False. $(((1, 2), 3), ((1/2, 1/3), 1/4))$ is a tuple.

Solution:

True. Since parentheses are used at the outermost level, this is a tuple. It doesn't matter what the elements of the tuple are; as long as they are enclosed in parentheses, it is a tuple. It is also clear that it is not an open interval, since an open interval would have exactly two elements and would be written as (a, b) where $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a < b$.

Question 3. [1 MARK]

Is the following True or False. A tuple can have duplicates.

Solution:

True. Tuples are ordered collections of elements, and they can contain duplicates.

Question 4. [1 MARK]

Which of the following is an element of $\mathbb{N}^4$?

a. $(1, 2, 3, 4)$

b. $(1, 2, 3)$

c. $(1, 2, 3, 4, 5)$

d. $(1, 2, 3, \pi)$

Solution:

The correct answer is:

- **a.** $(1, 2, 3, 4)$

Explanation:
$\mathbb{N}^4$ represents the set of all 4-tuples where each element is a natural number ($\mathbb{N}$).

a. $(1, 2, 3, 4)$: Correct. This is a 4-tuple of natural numbers.

b. $(1, 2, 3)$: Incorrect. This is a 3-tuple, not a 4-tuple.

c. $(1, 2, 3, 4, 5)$: Incorrect. This is a 5-tuple, not a 4-tuple.

d. $(1, 2, 3, \pi)$: Incorrect. π is not a natural number.

Question 5. [1 MARK]

Which of the following is an element of $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} = \mathbb{R}^3$ and $\mathcal{Y} = \mathbb{R}$?

a. $(1, 2, 3)$

b. $((1, 2, 3), (1, 2, 3))$

c. $((1, 2, 3), 4)$

d. $(1, (2, 3, 4))$

Solution:

The correct answer is:

- **c.** $((1, 2, 3), 4)$

Explanation:
An element of $\mathcal{X} \times \mathcal{Y}$ is an ordered pair $(\mathbf{x}, y)$ where $\mathbf{x} \in \mathbb{R}^3$ and $y \in \mathbb{R}$.

a. $(1, 2, 3)$: Incorrect. This is a single 3-tuple in $\mathbb{R}^3$, not an ordered pair.

b. $((1, 2, 3), (1, 2, 3))$: Incorrect. Both elements are in $\mathbb{R}^3$; the second should be in $\mathbb{R}$.

c. $((1, 2, 3), 4)$: Correct. The first element is in $\mathbb{R}^3$ and the second is in $\mathbb{R}$.

d. $(1, (2, 3, 4))$: Incorrect. The first element is in $\mathbb{R}$, but it should be in $\mathbb{R}^3$.

Question 6. [1 MARK]

Let $\mathcal{D} = ((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n))$ where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$. Then, how would you write the set of all possible $\mathcal{D}$?

a. $((\mathbb{R}^d) \times \{0, 1\})^n$

b. $(\mathcal{X} \times \mathcal{Y})^n$ where $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{0, 1\}$

c. $\{(z_1, \ldots, z_n) \mid z_i \in \mathbb{R}^d \times \mathbb{R},\ i \in \{1, \ldots, n\}\}$

d. $\{((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)) \mid \mathbf{x}_i \in \mathbb{R},\ y_i \in \{0, 1\},\ i \in \{1, \ldots, n\}\}$

Solution:

The correct answers are:

- **a.** $((\mathbb{R}^d) \times \{0, 1\})^n$
- **b.** $(\mathcal{X} \times \mathcal{Y})^n$ where $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{0, 1\}$

Explanation:
We need to represent all sequences $\mathcal{D}$ of n pairs $(\mathbf{x}_i, y_i)$ with $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{0, 1\}$.

a. $((\mathbb{R}^d) \times \{0, 1\})^n$: Correct. This set includes all n-tuples of pairs $(\mathbf{x}_i, y_i)$ where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{0, 1\}$.

b. $(\mathcal{X} \times \mathcal{Y})^n$ where $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{0, 1\}$: Correct. This is equivalent to option a., using variable names.

c. $\{(z_1, \ldots, z_n) \mid z_i \in \mathbb{R}^d \times \mathbb{R},\ i \in \{1, \ldots, n\}\}$: Incorrect. Here, y_i can be any real number, but it should be restricted to $\{0, 1\}$.

d. $\{((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)) \mid \mathbf{x}_i \in \mathbb{R},\ y_i \in \{0, 1\},\ i \in \{1, \ldots, n\}\}$: Incorrect. This specifies $\mathbf{x}_i \in \mathbb{R}$ instead of $\mathbb{R}^d$.

Question 7. [1 MARK]

Suppose you wanted to keep information of houses being sold. You decide to use two features to represent each house and to keep track of the price (an element of $[0, \infty)$) it was sold at. The first feature was the number of rooms (a natural number), the second feature was age (an element of $[0, \infty)$). How would you write the set of all possible houses that are represented in this way? Elements of this set should look like $((x_1, x_2), y)$ where x_1 represents the number of rooms, x_2 represents the age, and y represents the price.

a. $(\mathbb{N} \times [0, \infty)) \times [0, \infty)$

b. $\mathbb{N} \times [0, \infty) \times [0, \infty)$

c. $(\mathbb{N} \times [0, \infty)) \times \mathbb{N}$

d. $\mathbb{R}^2 \times [0, \infty)$

Solution:

The correct answers are:

- **a.** $(\mathbb{N} \times [0, \infty)) \times [0, \infty)$

Explanation:
Each house is represented as $((x_1, x_2), y)$ where:

- $x_1 \in \mathbb{N}$ (number of rooms)
- $x_2 \in [0, \infty)$ (age)
- $y \in [0, \infty)$ (price)

a. $(\mathbb{N} \times [0, \infty)) \times [0, \infty)$: Correct. It matches the required structure.

b. $\mathbb{N} \times [0, \infty) \times [0, \infty)$: Incorrect. An element of this set would look like (x_1, x_2, y), not $((x_1, x_2), y)$.

c. $(\mathbb{N} \times [0, \infty)) \times \mathbb{N}$: Incorrect. The price y should be in $[0, \infty)$, not $\mathbb{N}$.

d. $\mathbb{R}^2 \times [0, \infty)$: Incorrect. The number of rooms x_1 should be in $\mathbb{N}$, not $\mathbb{R}$.

Question 8. [1 MARK]

Let $f : (\mathbb{R}^3) \times (\mathbb{R}^3) \to \mathcal{Y}$ be such that $f(\mathbf{x}, \mathbf{w}) = \mathbf{x}^\top \mathbf{w}$, where $\mathbf{x}, \mathbf{w}$ are vectors. If $\mathbf{x} = (1, 4, 2)^\top$ and $\mathbf{w} = (1, 2, 3)^\top$, then what is $f(\mathbf{x}, \mathbf{w})$?

Solution:

The correct answer is:

- 15

Explanation:
To find $f(\mathbf{x}, \mathbf{w})$, we compute the dot product of $\mathbf{x}$ and $\mathbf{w}$:

$$f(\mathbf{x}, \mathbf{w}) = \mathbf{x}^\top \mathbf{w} = (1)(1) + (4)(2) + (2)(3) = 1 + 8 + 6 = 15$$

Therefore, $f(\mathbf{x}) = 15$.

Question 9. [1 MARK]

Let $f : \mathbb{R} \to \mathcal{Y}$ be such that $f(x) = 4 + x^2$. What is the range of f, and what is a valid codomain $\mathcal{Y}$?

a. The range of f is $[4, \infty)$, and a valid codomain is $\mathbb{R}$.

b. The range of f is $[0, \infty)$, and a valid codomain is $[0, \infty)$.

c. The range of f is $[4, \infty)$, and a valid codomain is $[4, \infty)$.

d. The range of f is $\mathbb{R}$, and a valid codomain is $\mathbb{R}$.

Solution:

The correct answers are:

- **a.** The range of f is $[4, \infty)$, and a valid codomain is $\mathbb{R}$.
- **c.** The range of f is $[4, \infty)$, and a valid codomain is $[4, \infty)$.

Explanation:
$f(x) = 4 + x^2$ is a quadratic function that achieves a minimum value of 4 when $x = 0$ and increases without bound as $|x|$ increases. Therefore, the range is $[4, \infty)$.

a. Correct. $\mathbb{R}$ includes the range $[4, \infty)$.

b. Incorrect. The range starts at 4, not 0.

c. Correct. Setting the codomain to $[4, \infty)$ precisely matches the range.

d. Incorrect. While $\mathbb{R}$ is a valid codomain, the range is not all of $\mathbb{R}$; it is restricted to $[4, \infty)$.

Question 10. [1 MARK]

A polynomial of degree 3 or less is a function that looks like $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ are considered to be fixed constants (i.e., they are not variables of the function f). Which of the following is a set of all functions that have domain $\mathbb{R}$ and are polynomials of degree 3 or less?

a. $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \text{ and } a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

b. $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \text{ and } a_0, a_1, a_2, a_3 \in \mathbb{N}\}$

c. $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + a_2x^2 \text{ and } a_0, a_1, a_2 \in \mathbb{R}\}$

d. $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \text{ and } a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

Solution:

The correct answer is:

- **a.** $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + a_2x^2 + a_3x^3,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

Explanation:

a. Correct. This set precisely defines all real polynomials of degree up to 3.

b. Incorrect. This set has coefficients as only natural numbers, whereas the definition restricts coefficients to $\mathbb{R}$.

c. Incorrect. This set only includes polynomials of degree up to 2.

d. Incorrect. This set includes polynomials of degree up to 4, which exceeds the specified degree.

Question 11. [1 MARK]

Which of the following is a function

$$\mathcal{A} : (\mathbb{R} \times \mathbb{R})^n \to \{f \mid f : \mathbb{R} \to \mathbb{R} \text{ and } f(x) = xw,\ w \in \mathbb{R}\}$$

where $n = 2$?

a. $\mathcal{A}((a, b), (c, d)) = f(x) = ax + b$

b. $\mathcal{A}((a, b), (c, d)) = f(x) = cx$

c. $\mathcal{A}((a, b), (c, d)) = f(x) = ax + d$

d. $\mathcal{A}((a, b), (c, d)) = f(x) = \frac{ab+cd}{a^2+b^2}x$

Solution:

The correct answers are:

- **b.** $\mathcal{A}((a, b), (c, d)) = f(x) = cx$

Explanation:
The function $\mathcal{A}$ should map input tuples to functions of the form $f(x) = xw$, where $w \in \mathbb{R}$.

a. Incorrect. $f(x) = ax + b$ includes a constant term b, which does not fit the required form $f(x) = xw$.

b. Correct. $f(x) = cx$ matches the form with $w = c$.

c. Incorrect. $f(x) = ax + d$ includes a constant term d, which does not fit the required form.

d. Incorrect. $f(x) = \frac{ab+cd}{a^2+b^2}x$ is undefined when $a = 0$ and $b = 0$.

Question 12. [1 MARK]

Let $f(x, w) = xw$ where $x, w \in \mathbb{R}$. What is the partial derivative of $f(x, w)$ with respect to w?

a. x

b. w

c. $x + w$

d. 1

Solution:

The correct answer is:

- **a.** x

Explanation:
The function $f(x, w) = xw$ is a linear function in w. To find the partial derivative of f with respect to w, we treat x as a constant:

$$\frac{\partial f}{\partial w} = \frac{\partial}{\partial w}(xw) = x$$

Therefore, the partial derivative is x.

a. x: Correct. The partial derivative of xw with respect to w is x.

b. w: Incorrect. This would be the partial derivative if the function were $\frac{1}{2}w^2$.

c. $x + w$: Incorrect. This suggests adding x and w, which is not applicable here.

d. 1: Incorrect. This would be the partial derivative if the function were w.

Question 13. [1 MARK]

Let $\ell(\hat{y}, y) = (\hat{y} - y)^2$ where $\hat{y}, y \in \mathbb{R}$. What is the partial derivative of $\ell(\hat{y}, y)$ with respect to $\hat{y}$?

a. $2(\hat{y} - y)$

b. $(\hat{y} - y)$

c. $2\hat{y}$

d. $-2y$

Solution:

The correct answer is:

- **a.** $2(\hat{y} - y)$

Explanation:
Given $\ell(\hat{y}, y) = (\hat{y} - y)^2$, the partial derivative with respect to $\hat{y}$ is:

$$\frac{\partial \ell}{\partial \hat{y}} = 2(\hat{y} - y)$$

a. $2(\hat{y} - y)$: Correct.

b. $(\hat{y} - y)$: Incorrect. Missing the factor of 2 from the power rule.

c. $2\hat{y}$: Incorrect. This would be the partial derivative if y were zero.

d. $-2y$: Incorrect. This ignores the $\hat{y}$ term.

Question 14. [1 MARK]

Let $\ell(\hat{y}, y) = (\hat{y} - y)^2$ where $\hat{y}, y \in \mathbb{R}$. Let $f(x, w) = xw$ where $x, w \in \mathbb{R}$. What is the partial derivative of $\ell(f(x, w), y)$ with respect to w?

a. $2x(xw - y)$

b. $2(xw - y)$

c. x

d. $2xw$

Solution:

The correct answer is:

- **a.** $2x(xw - y)$

Explanation:
First, express $\ell(f(x, w), y)$:

$$\ell(f(x, w), y) = (xw - y)^2$$

To find the partial derivative with respect to w:

$$\frac{\partial}{\partial w}\ell(f(x, w), y) = 2(xw - y) \cdot x = 2x(xw - y)$$

a. $2x(xw - y)$: Correct. Applies the chain rule correctly.

b. $2(xw - y)$: Incorrect. Missing the multiplication by x.

c. x: Incorrect. This is the partial derivative of $f(x, w)$ with respect to w, not of ℓ.

d. $2xw$: Incorrect. This ignores the y term and the product rule.

Question 15. [1 MARK]

If $\mathcal{X} = \{1, 2, 3\}$, what is $\frac{1}{3}\sum_{x\in\mathcal{X}} x$?

Solution:

The correct answer is:

- 2

Explanation:
First, compute the sum:

$$\sum_{x\in\mathcal{X}} x = 1 + 2 + 3 = 6$$

Then,

$$\frac{1}{3}\sum_{x\in\mathcal{X}} x = \frac{6}{3} = 2$$

Question 16. [1 MARK]

If $\mathcal{X} = (x_1, \ldots, x_n)$, how would you write the sum over all $x \in \mathcal{X}$?

a. $\sum_{i=1}^{n} x_i$

b. $\prod_{x\in\mathcal{X}} x$

c. $\int_{\mathcal{X}} x\,dx$

d. $\sum_{x\in\mathcal{X}} x$

Solution:

The correct answers are:

- **a.** $\sum_{i=1}^{n} x_i$
- **d.** $\sum_{x\in\mathcal{X}} x$

Explanation:
When $\mathcal{X} = \{x_1, \ldots, x_n\}$, the sum over all $x \in \mathcal{X}$ is written as:

$$x_1 + x_2 + \cdots + x_n = \sum_{x\in\mathcal{X}} x = \sum_{i=1}^{n} x_i$$

a. $\sum_{i=1}^{n} x_i$: Correct.

b. $\prod_{x\in\mathcal{X}} x$: Incorrect. This represents the product, not the sum.

c. $\int_{\mathcal{X}} x\,dx$: Incorrect. This represents an integral, which is not applicable for finite sums.

d. $\sum_{x\in\mathcal{X}} x$: Correct.

Question 17. [1 MARK]

Let $\mathcal{X} = (x_1, \ldots, x_n)$. Let $f(x, w) = xw$ where $x, w \in \mathbb{R}$. How would you write the sum of $f(x, w)$ over all $x \in \mathcal{X}$?

a. $\sum_{i=1}^{n} x_i w$

b. $w \sum_{i=1}^{n} x_i$

c. $\sum_{x=1}^{n} w$

d. $w \prod_{i=1}^{n} x_i$

Solution:

The correct answers are:

- **a.** $\sum_{i=1}^{n} x_i w$
- **b.** $w \sum_{i=1}^{n} x_i$

Explanation:
The sum of $f(x, w) = xw$ over all $x \in \mathcal{X}$ is:

$$\sum_{x \in \mathcal{X}} f(x, w) = \sum_{i=1}^{n} x_i w = w \sum_{i=1}^{n} x_i$$

Thus, both expressions $\sum_{i=1}^{n} x_i w$ and $w \sum_{i=1}^{n} x_i$ are correct.

a. $\sum_{i=1}^{n} x_i w$: Correct. Direct substitution of $f(x_i, w)$.

b. $w \sum_{i=1}^{n} x_i$: Correct. Factor w out of the sum.

c. $\sum_{x=1}^{n} w$: Incorrect. This sums w n times, ignoring x_i.

d. $w \prod_{i=1}^{n} x_i$: Incorrect. This represents w multiplied by the product of all x_i, not the sum.

Question 18. [1 MARK]

Let $\mathcal{D} = ((x_1, y_1), \ldots, (x_n, y_n))$. Let $\ell(\hat{y}, y) = (\hat{y} - y)^2$ where $\hat{y}, y \in \mathbb{R}$. Let $f(x, w) = xw$ where $x, w \in \mathbb{R}$. How would you write the sum of $\ell(f(x, w), y)$ over all $(x, y) \in \mathcal{D}$?

a. $\sum_{i=1}^{n} (\hat{y}_i - y_i)^2$

b. $\sum_{i=1}^{n} (x_i w - y_i)^2$

c. $\sum_{i=1}^{n} x_i w y_i$

d. $\prod_{i=1}^{n} (x_i w - y_i)^2$

Solution:

The correct answer is:

- **b.** $\sum_{i=1}^{n} (x_i w - y_i)^2$

Explanation:
Given $\mathcal{D} = ((x_1, y_1), \ldots, (x_n, y_n))$, the sum is:

$$\sum_{(x,y) \in \mathcal{D}} \ell(f(x, w), y) = \sum_{i=1}^{n} (\underbrace{x_i w}_{f(x_i, w)} - y_i)^2 = \sum_{i=1}^{n} (x_i w - y_i)^2$$

a. $\sum_{i=1}^{n} (\hat{y}_i - y_i)^2$: Incorrect. $\hat{y}_i$ is not defined in this context.

b. $\sum_{i=1}^{n} (x_i w - y_i)^2$: Correct. Direct substitution of $f(x_i, w)$.

c. $\sum_{i=1}^{n} x_i w y_i$: Incorrect. This represents a different operation, not the squared loss.

d. $\prod_{i=1}^{n} (x_i w - y_i)^2$: Incorrect. This represents the product of squared losses, not the sum.

Question 19. [1 MARK]

Let $L(z_1, \ldots, z_n) = \frac{1}{n} \sum_{i=1}^{n} z_i$ where $z_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. Let $\ell(\hat{y}, y) = (\hat{y} - y)^2$ where $\hat{y}, y \in \mathbb{R}$. Let $f(x, w) = xw$ where $x, w \in \mathbb{R}$. What is the partial derivative of $L(\ell(f(x_1, w), y_1), \ldots, \ell(f(x_n, w), y_n))$ with respect to w?

a. $\frac{2}{n}\sum_{i=1}^{n} x_i(x_i w - y_i)$

b. $\sum_{i=1}^{n} x_i(x_i w - y_i)$

c. $\frac{1}{n}\sum_{i=1}^{n} 2(x_i w - y_i)$

d. $2\sum_{i=1}^{n} x_i w$

Solution:

The correct answer is:

- **a.** $\frac{2}{n}\sum_{i=1}^{n} x_i(x_i w - y_i)$

Explanation:
First, express $L(\ell(f(x_1, w), y_1), \ldots, \ell(f(x_n, w), y_n))$:

$$L = \frac{1}{n}\sum_{i=1}^{n} \ell(f(x_i, w), y_i) = \frac{1}{n}\sum_{i=1}^{n}(x_i w - y_i)^2$$

To find the partial derivative with respect to w:

$$\frac{\partial L}{\partial w} = \frac{1}{n}\sum_{i=1}^{n} 2(x_i w - y_i) \cdot x_i = \frac{2}{n}\sum_{i=1}^{n} x_i(x_i w - y_i)$$

a. $\frac{2}{n}\sum_{i=1}^{n} x_i(x_i w - y_i)$: Correct. Applies the chain rule and factor of $\frac{1}{n}$.

b. $\sum_{i=1}^{n} x_i(x_i w - y_i)$: Incorrect. Missing the factor $\frac{2}{n}$.

c. $\frac{1}{n}\sum_{i=1}^{n} 2(x_i w - y_i)$: Incorrect. Missing multiplication by x_i.

d. $2\sum_{i=1}^{n} x_i w$: Incorrect. This does not account for the y_i terms.

Question 20. [1 MARK]

Let $f(x, y) = xy$ where $x, y \in \mathbb{R}$. What is

$$\sum_{y \in \mathcal{Y}} f(x, y)$$

where $\mathcal{Y} = \{1, 2, 3\}$?

a. $6x$

b. 6

c. $6y$

d. x

Solution:

The correct answer is:

- **a.** $6x$

Explanation:
We are asked to compute the sum:

$$\sum_{y \in \mathcal{Y}} f(x,y) = \sum_{y \in \{1,2,3\}} xy$$

Substituting the values of y from the set $\mathcal{Y}$:

$$\sum_{y \in \{1,2,3\}} xy = x \times 1 + x \times 2 + x \times 3 = x + 2x + 3x = 6x$$

Thus, the sum is $6x$, which corresponds to option **a**.

a. $6x$: **Correct**. As shown above, the sum evaluates to $6x$.

b. 6: **Incorrect**. This would be the sum if $x = 1$, but x is a variable and not necessarily equal to 1.

c. $6y$: **Incorrect**. The sum results in $6x$, not $6y$. y is a variable within the summation and does not factor out.

d. x: **Incorrect**. This would be the case if the sum had only one term, but there are three terms being summed.

Question 21. [1 MARK]

Let $f(x,y) = xy$ where $x, y \in \mathbb{R}$. What is

$$\int_{\mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y)\, dx = \int_{\mathcal{X}} \left(\sum_{y \in \mathcal{Y}} f(x,y) \right) dx$$

where $\mathcal{Y} = \{1,2,3\}$ and $\mathcal{X} = [0,5]$?

Solution:

The correct answer is:

- **a.** 75

Explanation:
First, compute the sum inside the integral:

$$\sum_{y \in \mathcal{Y}} f(x,y) = \sum_{y \in \{1,2,3\}} xy = x \times 1 + x \times 2 + x \times 3 = x + 2x + 3x = 6x$$

Now, substitute this back into the integral:

$$\int_{\mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y)\, dx = \int_0^5 6x\, dx$$

Evaluate the integral:

$$\int_0^5 6x\,dx = 6 \times \int_0^5 x\,dx = 6 \times \left[\frac{x^2}{2}\right]_0^5 = 6 \times \left(\frac{25}{2} - 0\right) = 6 \times 12.5 = 75$$

Thus, the integral evaluates to 75, which corresponds to option **a**.

Question 22. [1 MARK]

Let $f(x, y) = xy$ where $x, y \in \mathbb{R}$. What is

$$\int_{\mathcal{Y}} f(x, y)\,dy$$

where $\mathcal{Y} = [1, 3]$?

a. $4x$

b. $6x$

c. $2x$

d. x

Solution:

The correct answer is:

- **a.** $4x$

Explanation:
Compute the integral:

$$\int_{\mathcal{Y}} f(x, y)\,dy = \int_1^3 xy\,dy$$

Since x is treated as a constant with respect to y, factor it out of the integral:

$$\int_1^3 xy\,dy = x \int_1^3 y\,dy$$

Evaluate the integral of y:

$$\int_1^3 y\,dy = \left[\frac{y^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

Multiply by x:

$$x \times 4 = 4x$$

Thus, the integral evaluates to $4x$, which corresponds to option **a**.

a. $4x$: **Correct**. As calculated above.

b. $6x$: **Incorrect**. Overestimation of the integral's value.

c. $2x$: **Incorrect**. Underestimation of the integral's value.

d. x: **Incorrect**. Incorrect calculation of the integral.

Question 23. [1 MARK]

Let $f(x, y) = xy$ where $x, y \in \mathbb{R}$. What is

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y)\, dy\, dx = \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} f(x, y)\, dy \right) dx$$

where $\mathcal{Y} = [1, 3]$ and $\mathcal{X} = [0, 5]$?

Solution:

The correct answer is:

- **a.** 50

Explanation:
First, compute the inner integral:

$$\int_{\mathcal{Y}} f(x, y)\, dy = \int_1^3 xy\, dy = x \int_1^3 y\, dy = x \left[\frac{y^2}{2} \right]_1^3 = x \left(\frac{9}{2} - \frac{1}{2} \right) = x \times 4 = 4x$$

Now, substitute this back into the outer integral:

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y)\, dy\, dx = \int_0^5 4x\, dx$$

Evaluate the outer integral:

$$\int_0^5 4x\, dx = 4 \times \int_0^5 x\, dx = 4 \times \left[\frac{x^2}{2} \right]_0^5 = 4 \times \left(\frac{25}{2} - 0 \right) = 4 \times 12.5 = 50$$

Thus, the double integral evaluates to 50, which corresponds to option **a**.

Question 24. [1 MARK]

Let X be a discrete random variable uniformly distributed with outcome space $\mathcal{X} = \{3, 5, 7, 9\}$. The probability mass function (pmf) of X is given by $p(x) = \frac{1}{4}$ for each $x \in \mathcal{X}$. What is the probability of the event that X is either 5 or 9?

Solution:

The correct answer is:

- **b.** $\frac{1}{2}$

Explanation:
Since X is uniformly distributed over $\{3, 5, 7, 9\}$, each outcome has an equal probability of $\frac{1}{4}$. The event X is either 5 or 9 is represented as the set $\{5, 9\}$. Therefore,

$$\mathbb{P}(X \in \{5, 9\}) = p(5) + p(9) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Thus, the probability is $\frac{1}{2}$, which corresponds to option **b**.

Question 25. [1 MARK]

Let Y be a continuous random variable uniformly distributed with outcome space $\mathcal{Y} = [2, 10]$. The probability density function (pdf) of Y is given by $p(y) = \frac{1}{8}$ for $y \in [2, 10]$. What is the probability of the event that Y lies between 4 and 7?

Solution:

The correct answer is:

- **b.** $\frac{3}{8}$

Explanation:
For a continuous uniform distribution over $[a, b]$, the probability that Y lies between c and d (where $a \leq c < d \leq b$) is calculated as:

$$\mathbb{P}(c \leq Y \leq d) = \int_c^d p(y)\, dy = p(y) \times (d - c)$$

Given:

$$\begin{aligned} a &= 2 \\ b &= 10 \\ p(y) &= \frac{1}{8} \\ c &= 4 \\ d &= 7 \end{aligned}$$

Calculating the probability:

$$\mathbb{P}(4 \leq Y \leq 7) = \frac{1}{8} \times (7 - 4) = \frac{1}{8} \times 3 = \frac{3}{8}$$

Thus, the correct probability is $\frac{3}{8}$, which corresponds to option **b**.