1 Q2

Consider the following moral hazard problem. The worker can choose one of two (unobservable) effort levels, $e \in \{1, 2\}$ If the worker exerts effort e and receives wage w, her payoff is $\sqrt{w} - e$. The outside option of the worker is zero. The owner of the firm is risk neutral. There are two possible realizations of the output, $x_1 = 1$ and $x_2 = 16$.

Problem 1.1. Suppose that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	α	0
$x_2 = 16$	$1 - \alpha$	1

where $\alpha \in [0, 1]$. Characterize the set of those α's for which the owner of the firm can achieve the same payoff as if the effort of the worker was observable and contractible.

Solution. I will solve the firm's problem when effort is observable and when it is unobservable. In the unobservable case, I will calculate the payoffs for each effort level and then compare payoffs. I will then relate how this compares to the observable effort outcome.

▷ **Effort is Observable**

The firm solves

$$\max_{w_1,w_2,e} (1 - w_1)f(1|e) + (16 - w_2)f(16|e)$$
$$\text{s.t. } \sqrt{w_1}f(1|e) + \sqrt{w_2}f(16|e) - e \geq 0$$

Letting λ be the Lagrangian multiplier on the above constraint, we get the FOC for a wage w_i is

$$-f(x_i|e) + \frac{1}{2}w_i^{-\frac{1}{2}}\lambda f(x_i|e) = 0$$

which implies

$$w_i = \frac{\lambda^2}{4}$$

and thus $w_1 = w_2 = w$ is constant across outcomes. From the budget constraint we get

$$w(e) = e^2$$

and thus if low effort is optimal, the wage will be $w(1) = 1$ and if high effort is optimal the wage will be $w(2) = 4$. Low effort will result in a payoff of

$$(1 - 1^2)\alpha + (16 - 1^2)(1 - \alpha) = 15(1 - \alpha)$$

and high effort will result in the payoff

$$(1 - 2^2)0 + (16 - 2^2)1 = 12$$

and thus if $\alpha \leq \frac{1}{5}$, the firm will choose low effort and provide a wage of $w = 1$ and if $\alpha > \frac{1}{5}$ the firm will choose high effort and provide a wage of $w = 4$.

▷ **Effort is Unobservable**

* Low effort is optimal.

  In this case, since this is the lowest effort we do not have to worry about incentive compatibility and thus the solution is the same as in the observable case. Thus, for $e = 1$, the wage $w = 1$ and the payoff is $15(1 - \alpha)$.

* High effort is optimal.

  In this case, the owner maximizes

$$\begin{aligned} &\max_{w_1, w_2} (16 - w_2) \\ &\quad \text{s.t. } \sqrt{w_2} - 2 \geq 0 \\ &\quad \sqrt{w_2} - 2 \geq \sqrt{w_1}\alpha + \sqrt{w_2}(1 - \alpha) - 1 \end{aligned}$$

  Since w_1 only enters in the IC constraint, we get set it equal to zero in order to have the ability to set w_2 lower. Thus, the IC constraint becomes

$$\sqrt{w_2}\alpha \geq 1 \implies w_2 \geq \frac{1}{\alpha^2}.$$

  If $\alpha \geq \frac{1}{2}$, then the two constraints imply $w_2 = 4$ and we get that the high effort will result in the same payoff for the firm. So for $\alpha \in [\frac{1}{2}, 1]$, the payoffs are the same.

However, if $\alpha < \frac{1}{2}$, we have $w_2 = \frac{1}{\alpha^2}$ solves the high effort problem for the firm. Here, the $\bar{\alpha}$ that equates pay offs is

$$(16 - \frac{1}{\bar{\alpha}^2}) = 15(1 - \bar{\alpha}) \implies 15\bar{\alpha}^3 + \bar{\alpha}^2 - 1 = 0 \implies \bar{\alpha} \approx 0.3845.$$

Thus, we get that for $\alpha \in [0, \bar{\alpha}]$, the low effort is optimal and thus the payoff is $15(1 - \alpha)$. Since $\bar{\alpha} > \frac{1}{5}$, we have the payoffs are the same when $\alpha \in [0, \frac{1}{5}]$.

In all, the payoffs will be the same as in the observable case when $\alpha \leq \frac{1}{5}$ and $\alpha \geq \frac{1}{2}$.

Problem 1.2. Suppose now that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	1	β
$x_2 = 16$	0	$1 - \beta$

where $\beta \in (0, 1)$. Again, characterize the set of those β's for which the owner of the firm can achieve the same payoff as if the effort of the worker was observable and contractible.

Solution. I will solve the firm's problem when effort is observable and when it is unobservable. In the unobservable case, I will calculate the payoffs for each effort level and then compare payoffs. I will then relate how this compares to the observable effort outcome.

▷ **Effort is Observable**

The firm solves

$$\max_{w_1, w_2, e} (1 - w_1) f(1|e) + (16 - w_2) f(16|e)$$
$$\text{s.t. } \sqrt{w_1} f(1|e) + \sqrt{w_2} f(16|e) - e \geq 0$$

Letting λ be the Lagrangian multiplier on the above constraint, we get the FOC for a wage w_i is

$$-f(x_i|e) + \frac{1}{2} w_i^{-\frac{1}{2}} \lambda f(x_i|e) = 0$$

which implies

$$w_i = \frac{\lambda^2}{4}$$

and thus $w_1 = w_2 = w$ is constant across outcomes. From the budget constraint we get

$$w(e) = e^2$$

and thus if low effort is optimal, the wage will be $w(1) = 1$ and if high effort is optimal the wage will be $w(2) = 4$. Low effort will result in a payoff of

$$(1 - 1) = 0$$

and high effort will result in the payoff

$$(1 - 4)\beta + (16 - 4)(1 - \beta) = 12 - 15\beta$$

and thus if $\beta >= \frac{4}{5}$, the firm will choose low effort and provide a wage of $w = 1$ and if $\beta < \frac{4}{5}$ the firm will choose high effort and provide a wage of $w = 4$.

▷ **Effort is Unobservable**

* Low effort is optimal.

  In this case, since this is the lowest effort we do not have to worry about incentive compatibility and thus the solution is the same as in the observable case. Thus, for $e = 1$, the wage $w = 1$ and the payoff is 0.

* High effort is optimal.

  In this case, the owner solves

$$\begin{aligned} \max_{w_1, w_2} &(1 - w_1)\beta + (16 - w_2)(1 - \beta) \\ \text{s.t. } &\sqrt{w_1}\beta + \sqrt{w_2}(1 - \beta) - 2 \geq 0 \\ &\sqrt{w_1}\beta + \sqrt{w_2}(1 - \beta) - 2 \geq \sqrt{w_1} - 1 \end{aligned}$$

  The first order conditions are

$$-\beta + \beta\frac{1}{2}w_1^{-\frac{1}{2}}\lambda - (1 - \beta)\frac{1}{2}w_1^{-\frac{1}{2}}\mu = 0.$$

  and

$$-(1-\beta)+(1-\beta)\frac{1}{2}w_2^{-\frac{1}{2}}\lambda+(1-\beta)\frac{1}{2}w_2^{-\frac{1}{2}}\mu=0.$$

which imply

$$w_1^{\frac{1}{2}}=\frac{\lambda-\frac{1-\beta}{\beta}\mu}{2}$$

and

$$w_2^{\frac{1}{2}}=\frac{\mu+\lambda}{2}.$$

Using the binding IR constraint we have

$$\frac{\lambda-\frac{1-\beta}{\beta}\mu}{2}\beta+\frac{\mu+\lambda}{2}(1-\beta)=2 \implies \lambda=4.$$

and the IC constraint tells us

$$\frac{\lambda}{2}-2=\frac{\lambda-\frac{1-\beta}{\beta}\mu}{2}-1 \implies \mu=\frac{2\beta}{1-\beta}$$

and so the optimal wages are

$$w_1=\frac{(4-2)^2}{4}=1$$

and

$$w_2=\frac{(4+\frac{2\beta}{1-\beta})^2}{4}$$

The payoffs are equalized at

$$\begin{aligned}
&(16-\frac{(4+\frac{2\bar{\beta}}{1-\bar{\beta}})^2}{4})=0\\
&64=(4+\frac{2\bar{\beta}}{1-\bar{\beta}})^2\\
&4=\frac{2\bar{\beta}}{1-\bar{\beta}}\\
&4-4\bar{\beta}=2\bar{\beta}\\
&\bar{\beta}=\frac{2}{3}
\end{aligned}$$

Thus, in the unobservable case, $\beta >= \frac{2}{3}$ implies low effort is optimal and $\beta < \frac{2}{3}$ implies high effort is optimal.

Therefore, when $\beta >= \frac{4}{5}$, low effort is optimal in both the unobservable and observable case and so payoffs will be the same. When $\beta < \frac{2}{3}$, high effort is optimal in both cases. The $\tilde{\beta}$ where the payoffs would be equal is

$$\begin{aligned}
&-3\tilde{\beta} + (16 - \frac{(4 + \frac{2\tilde{\beta}}{1-\tilde{\beta}})^2}{4})(1-\tilde{\beta}) = 12 - 15\tilde{\beta} \\
&\implies (16 - \frac{16 + \frac{16\tilde{\beta}}{1-\tilde{\beta}} + \frac{4\tilde{\beta}^2}{(1-\tilde{\beta})^2}}{4})(1-\tilde{\beta}) = 12 - 12\tilde{\beta} \\
&\implies (12 - 4\frac{\tilde{\beta}}{1-\tilde{\beta}} - \frac{\tilde{\beta}^2}{(1-\tilde{\beta})^2})(1-\tilde{\beta}) = 12 - 12\tilde{\beta} \\
&\implies 4\tilde{\beta} + \frac{\tilde{\beta}^2}{(1-\tilde{\beta})} = 0 \\
&\implies \frac{\tilde{\beta}^2}{(1-\tilde{\beta})} = -4\tilde{\beta} \\
&\implies \tilde{\beta}^2 = -4\tilde{\beta} + 4\tilde{\beta}^2 \\
&\implies 3\tilde{\beta}^2 = 4\tilde{\beta} \\
&\implies \tilde{\beta} = \frac{4}{3}.
\end{aligned}$$

However, $\beta \in (0,1)$ and so the payoffs are never equal in the high effort case.

In all, when $\beta \in [\frac{4}{5}, 1)$, the payoffs will be the same in either case.

Problem 1.3. Suppose that the stochastic relationship between effort and output is described by the probability matrix given in part (b) and $\beta = \frac{1}{4}$ Characterize the optimal contract for the owner of the firm.

Solution. If $\beta = \frac{1}{4}$, then $e = 2$ is the optimal effort level and the optimal contract is

$$w_1 = 1$$

and

$$w_2 = \frac{\left(4 + \frac{\frac{1}{2}}{\frac{3}{4}}\right)^2}{4} = \frac{49}{9}$$

Problem 1.4. Suppose that the production technology is the same as in part (c). Assume that the owner of the firm can invest in a monitoring technology which perfectly reveals the effort of the worker. The outcome of this monitoring is contractible. The owner has to make this (observable) investment decision prior to offering a contract and it costs p. For what values of p should the owner invest in monitoring?

Solution. The payoff for $\beta = \frac{1}{4}$ when effort is observable is

$$12 - 15\beta = 12 - \frac{15}{4} = \frac{33}{4} = \frac{99}{12}$$

and the payoff when effort is unobservable is

$$(16 - \frac{49}{9})(1 - \frac{1}{4}) = \frac{95}{9}\frac{3}{4} = \frac{95}{12}$$

and so the max price $\bar{p}$ that an owner would pay would be

$$\frac{95}{12} = \frac{99}{12} - \bar{p} \implies \bar{p} = \frac{1}{3}$$

and so any $p \in [0, \frac{1}{3}]$, the owner should invest in the monitoring technology.

2 Q4

Consider the modified linear managerial-incentive-scheme problem, where the manager's effort, e, affects current profits, $x_1 = e + \epsilon_1$ and future profits $x_2 = e + \epsilon_2$, where ϵ_i are idd with normal distribution $N(0, \sigma_\epsilon^2)$. Note that the single effort equally impacts the short run (period 1) and the long run (period 2). The manager retires at the end of the first period, and the manager's compensation cannot be based on x_2. However, the company can issue stock that she must hold for one year after retirement. The price of the stock one year after retirement is $p = x_1 + x_2 + \eta$ where η is normally distributed, $N\left(0, \sigma_\eta^2\right)$ and η is independently distributed from ϵ_t. The firm maximizes the expectation of $x_1 + x_2 - w - sp$ where s are the shares of stocks given to the manager. There is no time discounting and the manager only cares about the total value of compensation once the stock is sold one year after retirement. The manager's utility is CARA with risk parameter, r, and her monetary cost of effort is $\frac{1}{2}e^2$. Her outside option is $\underline{\text{U}}$. Assume for reasons given in Holmstrom and Milgrom (1987) that the optimal contract is linear. Derive the optimal compensation contract for period 1 output

$$w(x_1) = \alpha x_1 + \beta$$

and derive the optimal amount of stock, s, to give to the manager. Explain the differences in the optimal α and s. In particular, what happens if the stock market price is a perfect aggregator of $x_1 + x_2$ with $\sigma_\eta^2 = 0$.

Note: because I found the question ambiguous. I solved the question through two methods. The first is when the principal can offer either a wage or equity contract. The second is when the principal offers both equity and a wage.

Either Wage or Equity Contract

To solve for the optimal wage contract, we will proceed in two steps. First, we will characterize the certainty equivalent of a contract of the form $w(x_1) = \alpha x_1 + \beta$. We will find α that solves the agent's IC constraint for a given effort level by taking the derivative of the certainty equivalent with respect to α. Then, we will plug in that α into the principal's decision problem. The certainty equivalent of the contract is given by

$$\alpha e + \beta - \frac{1}{2}e^2 - \frac{r}{2}\alpha^2\sigma_\epsilon^2$$

this has an associated first-order condition of

$$\alpha - e = 0$$
$$\alpha = e$$

so we can solve the principal's problem

$$\max_{\alpha,\mu} 2e - \frac{1}{2}e^2 - \frac{r}{2}\alpha^2\sigma_\epsilon^2$$

s.t. $\alpha = e$. This yields that

$$\alpha^* = \left(1 + r\sigma_\epsilon^2\right)^{-1}$$

So the principal will set this $\alpha^\star$ in the optimal contract. And the principal will set $\beta^\star$ high enough such that the IR constraint binds. In particular, $\beta^\star$ will be set s.t.

$$\mathbb{E}[u(\alpha^\star \times x_1 + \beta^\star - \frac{1}{2}(e^\star)^2)] \geq \underline{\mathrm{U}}$$

Now, in the same way, we can consider the certainty equivalent of the stock. This expression is given by

$$s \times 2e - \frac{1}{2}e^2 - rs^2\sigma_\epsilon^2 - s \times \frac{r}{2}\sigma_\eta^2$$

this has an associated first-order condition of

$$\begin{aligned} 2s - e &= 0 \\ s &= \frac{1}{2}e \end{aligned}$$

so we can now solve the principal's problem

$$\max_{s,e} 2e - \frac{1}{2}e^2 - rs^2\sigma_\epsilon^2 - s \times \frac{r}{2}\sigma_\epsilon^2$$

subject to $s = \frac{1}{2}e$. We can then write

$$\max_{s,e} 4s - 2s^2 - rs^2\sigma_\epsilon^2 - s\frac{r}{2}\sigma_\eta^2$$

which yields the foc

$$\begin{aligned} 4 - 4s - 4rs\sigma_\epsilon^2 - \frac{r}{2}\sigma_\epsilon^2 &= 0 \\ 4 - \frac{r}{2}\sigma_\epsilon^2 &= 4s + 4rs\sigma_\epsilon^2 \\ 4 - \frac{r}{2}\sigma_\epsilon^2 &= s\left(4 + 4r\sigma_\epsilon^2\right) \\ \frac{4 - \frac{r}{2}\sigma_\eta^2}{4 + 4r\sigma_\epsilon^2} &= s^\star \end{aligned}$$

Notice that the $s^\star$ term has an additional term $\frac{r}{2}\sigma_\eta^2$ that serves to compensate the risk averse agent for the additional, normally distributed, η term in the value of equity. Notice further that if $\sigma_\eta = 0$ then the $\alpha^\star$ we calculated above and $s^\star$ are identical. Now we need to consider two cases:

1. The IR constraint is slack or holds with equality – in either case we have found the optimal contract. However, the two contracts are not, in general, precisely the same as in the wage contract case the principal will set nonzero β s.t. the IR constraint binds. So, if $s^\star = \alpha^\star$ unless the IR constraint holds with equality the principal will set $\beta^\star < 0$ and the two contracts are not the same.

2. The IR constraint is violated – if the IR constraint is violated then the principal will have to increase the shares awarded to the agent until the IR constraint holds with equality. In particular, under the new optimal contract the shares awarded will have to satisfy

$$4s - 2s^2 - rs^2\sigma_\epsilon^2 - s\frac{r}{2}\sigma_\eta^2 = \underline{\text{U}}$$

denote this value of shares as $\tilde{s}$. Notice that it must be that $\tilde{s} > s^\star$ – which follows because IR was violated under the original $s^\star$. Therefore the level of effort exerted by the agent must be higher under the new contract than the original.

Wage and Equity Contract

The Principal's problem is given by

$$\begin{aligned} \max_{\{e_1,e_2,\alpha,\beta,s\}} & \mathbb{E}\left[x_1 + x_2 - w - sp\right] \\ & \Leftrightarrow 2e\left(1-s\right) - \alpha e - \beta \\ & \Leftrightarrow e\left(2\left(1-s\right)-\alpha\right) - \beta \end{aligned}$$

subject to the IR and IC constraints

$$\begin{aligned} \mathbb{E}\left[u\left(w\left(x\right) - \frac{1}{2}e^2\right)\right] \geq \underline{\text{U}} \quad \text{(IR)} \\ e \in \max_{\tilde{e}} \mathbb{E}\left[u\left(w\left(x\right) - \frac{1}{2}\tilde{e}^2\right)\right] \quad \text{(IC)} \end{aligned}$$

we then have that the certainty equivalent of the agent is given by

$$\alpha e + \beta + 2se - \frac{1}{2}e^2 - \frac{r}{2}\left(\alpha^2\sigma_\epsilon^2 + 2s^2\sigma_\epsilon^2 + s^2\epsilon_\eta^2\right)$$

this yields the following first-order condition with respect to e

$$\begin{aligned} \alpha + 2s - e = 0 \\ e = 2s + \alpha \end{aligned}$$

now, for the moment ignoring the IR constraint, we can plug in $e = 2s + e$ into the principal's optimization problem. This yields

$$\max_{\{\alpha,s,\beta\}} \left\{\left(2s+\alpha\right)\left(2\left(1-s\right)-\alpha\right) - \beta\right\}$$

plugging in for β s.t. the IR constraint holds we have that

$$\max_{\alpha,s} \left(2s+\alpha\right)\left(2\left(1-s\right)-\alpha\right) - \frac{r}{2}\left(\alpha^2\sigma_\epsilon^2 + 2s^2\sigma_\epsilon^2 + s^2\sigma_\eta^2\right)$$

which yields the following FOC with respect to α:

$$\begin{aligned} 2(1-s)-\alpha-(2s+\alpha)-r\left(\alpha\sigma_\epsilon^2\right)&=0\\ \alpha\left(2+r\sigma_\epsilon^2\right)&=2-4s \end{aligned}$$

and the following FOC with respect to s:

$$\begin{aligned} 2(2-2s-\alpha)-2(2s+\alpha)-r\left(2s\sigma_\epsilon^2+s\sigma_\eta^2\right)&=0\\ \alpha&=s\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}\right) \end{aligned}$$

Combining our two expressions yields that

$$\alpha=\frac{2}{2+r\sigma_\epsilon^2}\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}\right)\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}+\frac{4}{2+r\sigma_\epsilon^2}\right)^{-1}$$

and

$$\begin{aligned} s&=\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}\right)^{-1}\frac{2}{2+r\sigma_\epsilon^2}\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}\right)\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}+\frac{4}{2+r\sigma_\epsilon^2}\right)^{-1}\\ &=\frac{2}{2+r\sigma_\epsilon^2}\left(1+\frac{\sigma_\eta^2}{2\sigma_\epsilon^2}+\frac{4}{2+r\sigma_\epsilon^2}\right) \end{aligned}$$

specializing to the case where $\sigma_\eta=0$ we have that

$$s=\alpha=\frac{2}{2+\sigma_\epsilon^2}\left(1+\frac{4}{2+r\sigma_\epsilon^2}\right)^{-1}$$

so in the case where $\sigma_\eta\neq 0$ the principal puts higher weight on compensation through α – i.e. the wage. This is because when the agent's compensation is through the wage the principal doesn't have to compensate the agent for the additional equity risk summarized by σ_η^2. When $\sigma_\eta^2=0$ there is no additional risk and so the linear part of the wage and the (linear) equity component of the contract are identical.

3 Q6

Amend the two-effort-level model with a risk-neutral principal as follows: suppose now that effort has distinct effects on revenues, R, and costs, C, where $x = R - C$. Let $f_R(R, a)$ and $f_C(C, a)$ denote the density functions of R and C conditional on a, and assume that, conditional on a, R and C are independently distributed. Assume $R \in [R_0, R_1]$, $C \in [C_0, C_1]$ and that for all a, $f_R(R, a) > 0$ for all $R \in [R_0, R_1]$ and $f_C(C, a) > 0$ for all $C \in [C_0, C_1]$.
The two effort choices are now $\{a_R, a_C\}$ where a_R is an effort choice that devotes more time to revenue enhancement and less to cost reduction, and the opposite is true for a_C. In particular, assume that $F_R(R, a_R) < F_R(R, a_C)$ for all $R \in (R_0, R_1)$ and that $F_C(C, a_C) > F_C(C, a_R)$ for all $C \in (C_0, C_1)$. Moreover, assume that the monotone likelihood ratio property holds for each of these variables in the following form: $\frac{f_R(R, a_R)}{f_R(R, a_C)}$ is increasing in R, and $\frac{f_C(C, a_R)}{f_C(C, a_C)}$ is increasing in C. Finally, the agent preferes revenue enhancement over cost reduction: that is, $\psi(a_C) > \psi(a_R)$.

Problem 3.1. Suppose that the owner wants to implement effort choice a_C and that both R and C are observable. Derive the first-order condition for the optimal compensation scheme $w(R, C)$. How does it depend on R and C?

Solution. The owner's problem is now

$$\min_{,w(R,C)} \int_R \int_C w(R, C) f_R(R, a_C) f_C(C, a_C) \, dC dR$$

subject to

$$\begin{aligned} \int_R \int_C u(w(R, C)) f_R(R, a_C) f_C(C \mid a_C) \, dC dR - \psi(e) \geq 0 & \quad \text{(IR)} \\ \int_R \int_C w(R, C) f_R(R \mid a_C) f_C(C \mid a_C) \, dC dR - \psi(a_C) \geq & \quad \text{(IC)} \\ \int_R \int_C w(R, C) f_R(R \mid a_R) f_C(C \mid a_R) \, dC dR - \psi(a_R) & \end{aligned}$$

we can then write the lagrangian as

$$\begin{aligned} L = & - \int_R \int_C w(R, C) f_R(R, a_C) f_C(C, a_C) \, dC dR \\ & + \gamma \left(\int_R \int_C v(w(R, C)) f_R(R, a_C) f_C(C \mid a_C) \, dC dR - \psi(a_C) \right) \\ & + \mu \left(\begin{array}{c} \int_R \int_C v(w(R, C)) f_R(R \mid a_C) f_C(C \mid a_C) \, dC dR - \psi(a_C) \\ - \int_R \int_C v(w(R, C)) f_R(R \mid a_R) f_C(C \mid a_R) \, dC dR + \psi(a_R) \end{array} \right) \end{aligned}$$

We can then take a first-order condition and calculate

$$\left\{\frac{\partial L}{\partial w\left(R,C\right)}\right\}: \qquad \begin{aligned} 0 = &- f_R\left(R,a_C\right) f_C\left(C,a_C\right) + \gamma v'\left(w\left(R,C\right)\right) f_R\left(R,a_C\right) f_C\left(C \mid a_C\right) \\ &+ \mu \left(\begin{array}{c} v'\left(w\left(R,C\right)\right) f_R\left(R \mid a_C\right) f_C\left(C \mid a_C\right) \\ -v'\left(w\left(R,C\right)\right) f_R\left(R \mid a_R\right) f_C\left(C \mid a_R\right) \end{array} \right) \end{aligned}$$

Solving this yields that

$$\frac{1}{v'\left(w\left(R,C\right)\right)} = \gamma + \mu \left[1 - \frac{f_R\left(R \mid a_R\right) \times f_C\left(C \mid a_R\right)}{f_R\left(R \mid a_C\right) \times f_C\left(C \mid a_C\right)}\right]$$

Then we have that as R increases, $\frac{f_R(R|e_R)}{f_R(R|e_C)}$ increases, and because $v\left(\cdot\right)$ is concave we have that $w\left(R,C\right)$ decreases. Similarly, as C increases $\frac{f_C(C|a_R)}{f_C(C|e_C)}$ increases and because $v\left(\cdot\right)$ concave implies that $w\left(R,C\right)$ will decrease.

Problem 3.2. How would your answer to (a) change if the agent could always unobservably reduce the revenues of the firm (in a way that is of no direct benefit to him)?

Solution. If the agent can unobservably reduce the revenues of the firm then the principal cannot use R as a variable in $w\left(R,C\right)$. I.e. it must be that $w\left(R,C\right) = w\left(C\right)$. If $\frac{\partial w}{\partial R} > 0$ then the agent will not choose a_C as his wage will increase in R – thus the agent will choose a_R. If $\frac{\partial w}{\partial R} < 0$ then the manager will reduce the revenue of the firm until this derivative is nonnegative – thus this cannot be an optimal contract. Accordingly, an optimal contract that incentivizes a_C will have $\frac{\partial w}{\partial R} = 0$ for all values of R and $\frac{\partial w}{\partial C} < 0$.

Problem 3.3. What if, in addition, costs are now unobservable by a court (so that compensation can be made contingent only on revenues)?

Solution. If costs are unobservable by a court then only a_R is implementable. To see this, recall that we showed in part (B) that an optimal compensation scheme in order to incentivze a_C will be $w\left(C\right)$ with $\frac{\partial w}{\partial C} < 0$ and $\frac{\partial w}{\partial R} = 0$ for all values of R. But if we cannot contract on C then this contract is not possible. Thus we will have to have a contract with $\frac{\partial w}{\partial R} > 0$ or $\frac{\partial w}{\partial R} < 0$ for some value of R. But again recall that if $\frac{\partial w}{\partial R} > 0$ then the contract will not incentivze the agent to choose a_C and if $\frac{\partial w}{\partial R} < 0$ the agent will reduce revenue until the derivative is nonnegative and so the contract is not optimal. Notice further that if $\frac{dW}{dR} = 0$ for all values of R, i.e. there is a fixed wage, then the agent will just pick a_R as it is less costly. Thus it is only possible to implement a_R.