Theory of Income III
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Note:

These lecture notes are based on Professor Shimer's lectures in Theory of Income III, Spring Quarter 2018-19.

Contents

1 Canonical Business Cycle Model

1.1 Canonical Model and Balanced Growth Path

We will set up a basic neoclassical growth model and think about conditions needed to obtain a balanced growth path.

1.1.1 Setup

The planner's problem is given by

$$\begin{aligned}\max_{\{C_t,H_t,K_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty}\beta^t u\left(C_t,H_t\right)\\ \text{s.t. } & K_{t+1}+C_t=F_t\left(K_t,H_t\right)\\ & K_0 \text{ given}\end{aligned}$$

where H_t represents the hours worked. Depreciation is embedded in F_t. Also assume nice properties about the utility function and the choice set.

Question. *Should I solve this recursively?*

Solution. I could write down the value function recursively, but in many cases there is no need.

1.1.2 Solving the Planner's Problem

Solving the problem through a Lagrangian:

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\left\{u\left(C_t,H_t\right)+\lambda_t\left[F_t\left(K_t,H_t\right)-K_{t+1}-C_t\right]\right\}$$

▷ Taking the first-order conditions:

$$\begin{aligned}\left[C_t\right]&: u_C\left(C_t,H_t\right)=\lambda_t\\ \left[H_t\right]&: -u_H\left(C_t,H_t\right)=\lambda_t F_{H,t}\left(K_t,H_t\right)\\ \left[K_{t+1}\right]&: \beta^t\lambda_t=\beta^{t+1}\lambda_{t+1}F_{K,t+1}\left(K_{t+1},H_{t+1}\right)\end{aligned}$$

▷ First, we obtain:

$$\underbrace{-\frac{U_H\left(C_t,H_t\right)}{U_C\left(C_t,H_t\right)}}_{=MRS}=\underbrace{F_{H,t}\left(K_t,H_t\right)}_{=MPL}$$

which is referred to as a **static** or **intra-temporal** first-order condition.

▷ Second, we obtain:

$$[EE]: \beta^t U_C\left(C_t,H_t\right)=\beta^{t+1}U_C\left(C_{t+1},H_{t+1}\right)F_{K,t+1}\left(K_{t+1},H_{t+1}\right)$$

which is referred to as an **inter-temporal** condition.

▷ The **resource constraint** is given as

$$K_{t+1} + C_t = F_t\left(K_t, H_t\right) \equiv \tilde{F}_t\left(K_t, H_t\right) + \left(1-\delta\right) K_t$$

In the big picture, we have three time-series to solve for, and three equations for every period t: the intra-temporal, the inter-temporal, and the resource constraint. Using these three equations, we can solve for the paths. Note that transversality condition is necessary, but we are not going to write it out explicitly at this moment. This would ensure that we don't choose an exploding sequence of capital stock.

1.2 Calibration

There are certain parts of the economy that we understand really well (such as the long-run growth paths) and we want the model to match these facts where we have a lot of data. We want the model to be consistent with business cycle as well as balanced growth facts.

1.2.1 Kaldor Facts & Their Validity

Kaldor emphasized that there are variables that seem to grow over time and some that seem to be stationary. We think about the aggregate consumption, output, and capital stock to be trending upward for 100+ years in the entire post-war period and across different economies. Here are some stylized facts:

1. C_t and K_t grow at similar rate g. $Y_t \equiv \tilde{F}\left(K_t, H_t\right) \equiv F\left(K_t, H_t\right) - \left(1-\delta\right) K_t$ also grows at rate g.
2. H_t doesn't grow. But this is subject to dispute.
   - ▷ If you look at periods where hours were measured accurately, the total number of hours worked went down a little bit in the early post-war period, came up for a while (in the 70s until the 2000s) and has the trend has fallen downward again. Behind the scenes, for example in the 1970s, there was a huge increase in the labor supply of women as well. The fact also breaks down if you break it down into different education levels or other countries.
   - ▷ The idea of retirement or vacation didn't exist back in the days, so it is true that people are working in smaller fraction of their life times. We will later ponder the question of how to modify the assumptions to match the new stylized trends about H_t.
3. $w_t \equiv MRS_t = MPL_t$ also grows at rate g.
   - ▷ If you haven't studied standard utility functions, there is some tension in here where there is a trend growth in wages but people are not working more. Of course, the standard answer in macro models is that there are offsetting income and substitution effects (as some microeconomists recognized) and income and substitution effects work in different directions. With a particular class of preferences, we will be able to get the income and substitution effects to cancel each other, which we refer to as the **balanced growth class**.
   - ▷ If we want people to work less as wage increases, we want the income effect to be a little stronger than the substitute effect.

Now the key is to identify which conditions are necessary and sufficient to generate these facts.

1.2.2 Assumptions

We will make the following assumptions.

1. Production: $F_t(K_t, H_t) = \hat{F}\left(K_t, (1+g)^t H_t\right)$ and CRS.

   ▷ If we have Cobb-Douglas, it doesn't really matter where we stick the growth term since it can be separated; if we don't, then it matters where we stick in the growth term.

2. Utility:

$$u(C, H) = \frac{\left(Ce^{-v(H)}\right)^{1-\sigma} - 1}{1-\sigma}, \qquad \lim_{\sigma \to 1} u(C, H) = \log C - v(H)$$

   where v is convex.

   ▷ If we make σ slightly larger than one, you will have income effect dominating the substitution effect.

   ▷ If we used instead the following utility form:

$$\frac{C^{1-\sigma} - 1}{1-\sigma} - v(H), \quad \sigma \neq 1$$

   it will not give us balanced growth.

We can show that assumptions 1 and 2 are sufficient to explain the Kaldor facts.

1.2.3 Finding BGP

New notation:

$$c_t \equiv C_t (1+g)^{-t}, \quad k_t = K_t (1+g)^{-t}, \quad y_t = Y_t (1+g)^{-t}$$

▷ This is useful since we can look for constant c_t, k_t and y_t from our model.

We can rewrite the three constraints with $c_t = \bar{c}$, $k_t = \bar{k}$, and $H_t = \bar{H}$,

▷ From the intra-temporal constraint:

$$\bar{c} v'\left(\bar{H}\right) = \hat{F}_H\left(\bar{k}, \bar{H}\right)$$

▷ From the inter-temporal constraint:

$$(1+g)^\sigma = \beta \hat{F}_K\left(\bar{k}, \bar{H}\right)$$

▷ From the resource constraint and the properties of F :

$$(1+g)\bar{k} + \bar{c} = \hat{F}\left(\bar{k}, \bar{H}\right)$$

1.2.4 Specialization

We will further specialize the production and utility functions:

1. Production: Cobb-Douglas with explicit depreciation term:

$$\hat{F}(k, H_t) \equiv Ak^\alpha H^{1-\alpha} + (1-\delta) k$$

   or since $F_t(K_t, H_t) = \hat{F}\left(K_t, (1+g)^t H_t\right)$:

$$F_t(K, H) = AK^\alpha \left((1+g)^t H\right)^{1-\alpha} + (1-\delta) K$$

- ▷ Cobb-Douglas immediately implies that labor share of income is constant (fraction of output going to labor). This is another one of Kaldor facts that is increasingly being questionable (new data says that it's decreasing). To see this:

$$w_t = (1-\alpha) A k^{\alpha} H^{-\alpha} (1+g)^t$$

which yields:

$$\frac{w_t H_t}{Y_t} = \frac{w_t H_t}{A K_t^{\alpha} \left\{(1+g)^t H_t\right\}^{1-\alpha}} = 1-\alpha$$

- ▷ Note that Cobb Douglas yields constant labor share; CRS yields constant labor share on a balanced growth path.

2. Other parameters: $\alpha, \beta, g, \sigma, \delta, v(H)$

1.2.5 Parameter Calibration

We need to obtain parameter values for

$$\alpha, \beta, g, \delta$$

and the function

$$v(H).$$

The idea is to use the long-run facts to calibrate the parameters. This ensures that when we use the calibrated model to study real business cycles (which is a short-run analysis), the model provides long-run implications that are consistent with the aforementioned long-run stylised facts.

1. Relative risk aversion ($\sigma = 1$): We won't be able to tell this from balanced growth, so set $\sigma = 1$.

2. Balanced path growth rate ($g = 0.02$): output and capital have been growing at around 2% per year on average.

3. Capital share of output ($\alpha = 0.4$): The long-run average capital share of output in the US has been around 0.4. A decade ago, it would have been around 0.33.

4. Depreciation rate ($\delta = 0.06$): To get this number, recall the law of motion for capital:

$$K_{t+1} = (1-\delta) K_t + X_t,$$

where X_t is investment. Dividing through by K_t and evaluating this on the balanced growth path,

$$1 + g = \frac{K_{t+1}}{K_t} = (1-\delta) + \frac{X_t}{K_t}$$

We already know g, so if we can obtain a value of X_t/K_t (ratio of investment to capital), then we can back out δ. In the US, long-run average capital to output ratio is

$$\frac{K}{Y} = \begin{cases} 3.2 & \text{annual} \\ 12.8 & \text{quarterly} \end{cases}, \quad \frac{X}{Y} = 0.26$$

Note that K is a stock while Y is a flow variable. This means that conversion between annual and quarterly involves multiplying by 4 (K is the "same" whether we consider annual or quarterly, while quarterly output c. 1/4 of annual output). In contrast, because investment and output are both flow variables, the value of X/Y is the same whether we consider annual or quarterly time periods. Using this, we obtain that

$$\delta = \frac{X}{Y}\frac{Y}{K} - g = \begin{cases} 0.06 & \text{annual} \\ 0.015 & \text{quarterly} \end{cases}$$

5. Discount factor ($\beta = 0.958872$): According to the $[EE]$ with $\sigma = 1$, we obtain:

$$\begin{aligned} 1 + g &= \beta \left(\alpha \left(\frac{\bar{k}}{\bar{H}} \right)^{-(1-\alpha)} + 1 - \delta \right) \\ &= \beta \left(\alpha \frac{\bar{k}^{\alpha} \bar{H}^{1-\alpha}}{\bar{k}} + 1 - \delta \right) = \beta \left(\alpha \frac{Y}{K} + 1 - \delta \right) \\ \Rightarrow \beta &= \frac{1+g}{\alpha \frac{Y}{K} + 1 - \delta} \end{aligned}$$

Substituting the values we obtained already, we get

$$\beta = \begin{cases} 0.958872 & \text{annual} \\ 0.989234 & \text{quarterly} \end{cases}$$

6. Disutility of working (v): Note that this pins down the average hours worked. Furthermore, on the balanced growth path, we can only observe the value of $v'(H)$ evaluated at $H = \bar{H}$, and so it is not possible to obtain the function v from long-run data.

   ▷ Approach #1: Impose a one-parameter functional form for $v(H)$ so that knowing the value of the function at a particular point is sufficient to obtain the function in full.

   * For example, Prescott imposes that

   $$v(H) = -\gamma \log(1 - H),$$

   where γ describes the disutility of working.

   ▷ Approach #2: Allow more flexibility in the functional form.

   * We will see later when we consider log-linearised equilibrium conditions that what matters is v' and v''. This motivates us to use the following functional form:

   $$v(H) = \gamma \frac{\varepsilon}{1+\varepsilon} H^{\frac{1+\varepsilon}{\varepsilon}}$$

   where ε is the Frisch elasticity of labour supply (i.e. percentage change in labour supply due to a one percentage change in income keeping the marginal utility of wealth fixed). To calibrate the value of ε, we need to look beyond the data on the balanced growth path.

   * The estimates of ε are wide ranging (anywhere from zero to "∞") but (Shimer says that) it is reasonable to use an estimate of around 1. The problem with estimating ε arises from the fact that, here, we are assuming a representative agent who has much larger margin of labour supply adjustment than an individual (e.g. extensive vs intensive margin). In any case, once we have an estimate of ε, we can back out the value of γ using the intra-temporal condition.

   * Suppose we assume the Frisch elasticity of labour supply, ε, to be one; i.e. $\varepsilon = 1$. As it turns out, γ, the disutility from working, only affects the level of the stationary point but not the dynamics. We therefore use γ to normalise the level/unit of $\bar{H}$. Since data suggests average weekly hours to be around 23 hours, given $\varepsilon = 1$, this implies that $\gamma = 0.00153$.

Question. *How do you measure stock of capital?*

Solution. We use the depreciation rate to back out the stock of capital using the perpetual inventory method. Different types of capital get different depreciation rates; here there is only one type of capital.

1.3 Transition Dynamics

We now consider how the economy behaves off of the balanced growth path; i.e. its transition dynamics.

1.3.1 Characterization of the Equilibrium

Recall the following equations that characterized the equilibrium:

$$
\begin{aligned}
c_t v'(H_t) &= (1-\alpha) k_t^{\alpha} H_t^{-\alpha} \\
\frac{1+g}{c_t} &= \beta \frac{\alpha k_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}} \\
k_{t+1}(1+g) &= k_t^{\alpha} H_t^{1-\alpha} + (1-\delta) k_t - c_t
\end{aligned}
$$

Question. *How would you argue that the solution is unique?*

Solution. The argument is the following:

- ▷ In the second equation, the $\bar{c}$ will cancel each other, so all there is left is a function homogenous with degree zero in k_{t+1} and H_{t+1}. Here you can pin down a capital-to-hours ratio.
- ▷ In the third equation, you can pin down the capital-to-consumption ratio.
- ▷ In the first equation, as long as v' is monotone (for example, v is monotone and convex) then this will uniquely pin down H_t and hence the balanced growth path.

There is one exception – if you have zero capital to start with, you are stuck there. Mathematically, this does not show up since you have $0/0$ which does not apply to the argument about ratio. Furthermore, the resource constraint tells you that you should set consumption equal to zero, which is not well-defined.

We also have a transversality condition:

$$
\lim_{T\to\infty} \beta^T F_{K,T}(K_T, H_T) U_C(C_T, H_T) K_T = 0.
$$

- ▷ Intuition: If the horizon is finite, to the extent capital has some value, agents would want to run down the capital stock in the last period. For the social planner's problem to have a solution in the infinite horizon, the same must be true, which is why we require the limit condition as above.

1.3.2 Case #1: Inelastic Labor Supply

Mathematically, we can have inelastic labor supply by imposing the following condition:

$$
v(H) = \begin{cases} 0 & \text{if } H \leq 1 \\ \infty & \text{if } H > 1 \end{cases} \Rightarrow H_t = 1
$$

i.e. the disutility of work is infinite if $H > 1$.

▷ This naturally implies that the intratemporal condition:

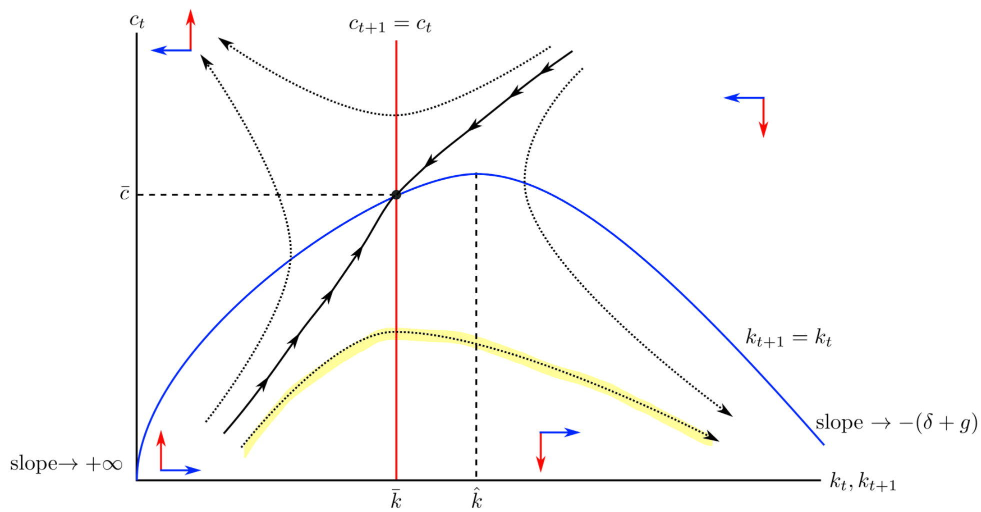
$$c_t v'(H_t) = (1-\alpha) k_t^\alpha H_t^{-\alpha}$$

drops out of the system since v is not differentiable.

▷ Thus we are left with the following two equations:

$$\begin{aligned}\frac{c_{t+1}}{c_t}(1+g) &= \beta\left(\alpha k_{t+1}^{\alpha-1} H^{1-\alpha} + 1 - \delta\right), \\ k_{t+1} - k_t &= \frac{k_t^\alpha H^{1-\alpha} + (1-\delta) k_t - c_t}{1+g} - k_t \\ &= \frac{k_t^\alpha H^{1-\alpha} - (\delta+g) k_t - c_t}{1+g}.\end{aligned}$$

We can represent the dynamics in the following phase diagram.



1. Locus of $k_{t+1} = k_t$:

$$c_t = k_t^\alpha H^{1-\alpha} - (\delta+g) k_t$$

2. Locus of $c_{t+1} = c_t$:

$$k_{t+1}^{\alpha-1} = \frac{1}{\alpha H^{1-\alpha}}\left(\frac{1+g}{\beta} - (1-\delta)\right)$$

3. Observations from the phase diagram:

   ▷ As $k \to 0$, the slope is positive infinity since

$$\begin{aligned}\lim_{k\to 0} \frac{\partial}{\partial k}\left[k^\alpha H^{1-\alpha} - (\delta+g) k\right] &= \lim_{k\to 0} \alpha k^{\alpha-1} H^{1-\alpha} - (\delta+g) \\ &= \lim_{k\to 0} \alpha \left(\frac{H}{k}\right)^{1-\alpha} - (\delta+g) = \infty\end{aligned}$$

▷ As $k \to \infty$, the slope is $-(\delta + g)$:

$$\lim_{k\to\infty} \frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right] = \lim_{k\to\infty} \alpha\left(\frac{H}{k}\right)^{1-\alpha} - (\delta+g) = -(\delta+g)\,.$$

▷ The intersection is attained at the positive-sloped portion of the hump.

$$\begin{aligned}\frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right]\Big|_{k=\bar{k}} &= \alpha k^{\alpha-1}H^{1-\alpha} - (\delta+g)\big|_{k_t=\bar{k}} \\ &= \alpha\frac{1}{\alpha H^{1-\alpha}}\left(\frac{1+g}{\beta} - (1-\delta)\right)H^{1-\alpha} - (\delta+g) \\ &= \frac{1+g}{\beta} - (1+g) > 0\end{aligned}$$

▷ Maximum value $\hat{k}$ can be computed:

$$\begin{aligned}0 &= \frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right] \\ \Rightarrow \alpha\hat{k}^{\alpha-1}H^{1-\alpha} &= \delta+g \\ \Rightarrow \hat{k} &= \left(\frac{\delta+g}{\alpha}\right)^{\frac{1}{\alpha-1}} H.\end{aligned}$$

4. Intuition for $\bar{k} < \hat{k}$: We want the interest rate to be positive.

5. Highlighted transition dynamics: The agent accumulates a lot of capital and consumes little. The transversality condition ensures that the agent doesn't suddnely consume a whole lot ("have a party") with the accumulated capital.

6. Saddle Path: These transition dynamics mean that there is a one-dimensional set of points, called the saddle path, that converges to the balanced growth path $(\bar{k}, \bar{c})$.

How can we solve this model numerically? The following is the outline of a **forward shooting algorithm:**

▷ We fix the initial capital to be some value k_0. The idea is to guess the value of c_0 and iterate using the difference equations.

▷ If we guessed the correct value of c_0 (i.e. (k_0, c_0) is on the saddle path), then we know that $(k_t, c_t) \to (\bar{k}, \bar{c})$.

▷ If we guessed too high, then we know that $c_t \not\to 0$ and $k_t \to 0$; if we guessed too low, then $c_t \to 0$ but $k_t \not\to 0$.

Alternatively, we can go backwards in time. This is known as reverse shooting algorithm.

▷ Shooting algorithm is not very efficient (see Problem Set 1). What we can do is to reverse.

▷ Suppose we set the "initial" point to be $(\bar{k}, \bar{c} - \varepsilon)$, where ε is some small positive number. If we use the difference equations backwards, then this gives us the path that leads to the point $(\bar{k}, \bar{c} - \varepsilon)$.

▷ Specifically, given c_{t+1} and k_{t+1}, we can solve for c_t and k_t using the two equations:

$$\begin{aligned}\frac{1+g}{c_t} &= \beta\frac{\alpha k_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}} \\ k_{t+1}(1+g) &= k_t^{\alpha}H_t^{1-\alpha} + (1-\delta)\,k_t - c_t\end{aligned}$$

▷ We can do this until we reach some desired level of initial capital (e.g. $k_0 = \bar{k}/2$), and read off the value of c_t at this point. This will not give us an exact value but by choosing ε small, we can be very close to the correct initial point on the saddle path.

1.3.3 Case #2: Elastic Labor Supply

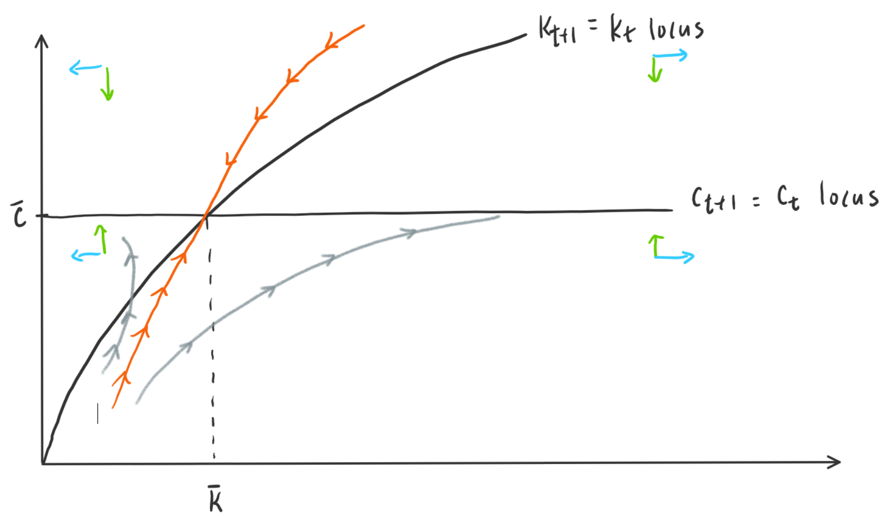
We will let $v(H) = \gamma H$, which yields ∞ Frisch elasticity of labor supply and $v'(H) = \gamma$.

Remark 1.1. When you're talking about elasticity, you are always fixing other variables. Economists like to use the Frisch Elasticity of Labor Supply for dynamic models, which holds the marginal utility of wealth constant.

Once again, we start with the system of equations:

$$c_t v'(H_t) = (1-\alpha) k_t^{\alpha} H_t^{-\alpha}$$
$$\frac{1+g}{c_t} = \beta \frac{\alpha k_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}}$$
$$k_{t+1}(1+g) = k_t^{\alpha} H_t^{1-\alpha} + (1-\delta) k_t - c_t$$

which we can solve analogously. Take the first equation to derive an expression of k_t/H_t as a function of γ and plug into the next two equations. We can once again represent the dynamics in the following phase diagram (the algebra is skipped for brevity):



▷ After getting the saddle path, we can plug it into the equations to get the dynamics of H_t as well.

Remark 1.2. For poor countries, people will have income effect dominating and the hours of work will decrease.

1.4 Log-Linearization

We can log linearise the system of equations (the intratemporal condition, the Euler equation, and the feasibility condition) to study the dynamics around the balanced growth path. In doing so, we also assume that there are no shocks in the future.

1.4.1 Log-linearizing the Model

Start by assuming
$$v\left(H\right)=\frac{\gamma\epsilon}{1+\epsilon}H^{\frac{1+\epsilon}{\epsilon}},\epsilon>0$$
where ϵ is the Frisch elasticity of labor supply. As $\epsilon\to 0$, we have $v\left(H\right)\to\gamma H$. Note that this formulation yields $v'\left(H\right)=\gamma H^{1/\epsilon}$. I will also denote $\phi_t\left(c_t,H_t,k_t\right)$

- ▷ We will linearize around the balanced growth path. The reason is that we spend a lot of time here.
- ▷ In this type of models, the log-linearization works pretty well even globally. Essentially, we are looking for slope of the saddle path around the steady state, since we spend the most time around the steady state.

Ideally, we want to write $\phi_{t+1}=A\phi_t$ but we can't since we have a static equation (MRS = MPL) which only has period-t variables. Thus, we will eliminate the static equation by writing:
$$\psi_t=\left(c_t,k_t\right),\quad \psi_{t+1}=B\psi_t$$
We will also use the following calibration from the characterization of the steady-state:
$$\begin{aligned} g&=0.005\\ \beta&=0.989\\ \alpha&=0.4\\ \delta&=0.04\end{aligned}$$
and the following parameters that were not attained from the steady-state:
$$\begin{aligned}\epsilon&=1\\ \gamma&=0.0.00152\end{aligned}$$
which yields $H^*=23$. Ultimately, we have the following system of equations after eliminating H_t:
$$\begin{aligned}\hat{k}_t&=\log k_t-\log k^*\\ \hat{c}_t&=\log c_t-\log c^*\end{aligned}$$
which yields:
$$\begin{bmatrix}\hat{k}_{t+1}\\ \hat{c}_{t+1}\end{bmatrix}=\begin{bmatrix}1.024 & -0.088\\ -0.013 & 0.989\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix}$$

- ▷ The eigenvalues of the matrix are 0.968 and 1.044, each of which is associated with an eigenvector e_1,e_2 which yields the following general solution:
$$\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix}=\mu_1\lambda_1^te_1+\mu_2\lambda_2^te_2$$
- ▷ Since $\lambda_2>1$, it must be that $\mu_2=0$. Furthermore, it must be that $\mu_1e_1\neq\mathbf{0}$. These will pin down what c_0 and k_0 has to be.
- ▷ Imposing this constraint, we have
$$\hat{c}_t=0.632\hat{k}_t\Leftrightarrow e_1=\begin{bmatrix}0.632\\ 1\end{bmatrix}$$

Plugging into the static equation, we have

$$\hat{h}_t = 0.268\hat{k}_t - 0.714\hat{c}_t = -0.160\hat{k}_t$$

It also turns out that we have

$$\begin{aligned}\hat{k}_{t+1} &= \lambda_1 \hat{k}_t = 0.968\hat{k}_t \\ \hat{y}_t &= 0.301\hat{k}_t \\ \hat{x}_t &= -0.653\hat{k}_t\end{aligned}$$

where $\hat{x}_t$ is investment.

- ▷ Note that when capital falls, output does not fall as much since (1) diminishing marginal product of capital and (2) people work harder.
- ▷ We obtained the equation for investment through:

$$x_t = k_{t+1} - (1-\delta)\, k_t$$

1.4.2 Log linearization versus Linearization

There is no rule that says how one should choose between log-linearisation or linearisation. Of course, log-linearisation will not be appropriate if we expect the variables to be negative (e.g. the real interest rate, the inflation rate). However, in general, the approximation tends to be more accurate with log-linearisation if variables are positive. Another advantage of log-linearisation is that we need not worry about the units of the variable as we can interpret in terms of percentage changes with log linearisation.

1.5 Real Business Cycles

Now we will add additional shocks to the neoclassical model to generate real business cycles.

1.5.1 Setup

We use the following notation consistent with Sargent's:

- ▷ t: period
- ▷ s_t: state at time t
- ▷ $\mathbf{s}^t \equiv \{s_0, ..., s_t\}$: history
- ▷ $\Pi_t\left(\mathbf{s}^t\right)$: time-0 probability of history s^t
- ▷ $\mathbf{s}^{t+1} = \left\{\mathbf{s}^t, s_{t+1}\right\}$: successor history of s^t, denoted as $\mathbf{s}^{t+1} \succ \mathbf{s}^t$
- ▷ $\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)/\Pi_t\left(\mathbf{s}^t\right)$: condtional probability of $s_{t+1}|\mathbf{s}^t$

Note that it is standard to impose a Markovian assumption but we will step away from this now to deliver more general results

1.5.2 Planner's Problem

The planner now solves the similar problem in terms of *expected* utility. The objective is now modified to be

$$\max \sum_{t=0}^{\infty} \beta^t \mathbb{E}\left[u\left(C_t, H_t\right)\right]$$

This implies that planner at time 0 can plan out the entire time-series of consumption and leisure. Writing out the expectation and adding the constraints:

$$\begin{aligned}
\max_{\{C_t(\mathbf{s}^t), H_t(\mathbf{s}^t), K_{t+1}(\mathbf{s}^t)\}} & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right)\left\{\log C_t\left(\mathbf{s}^t\right) - v\left(H_t\left(\mathbf{s}^t\right)\right)\right\} \\
\text{such that } & K_{t+1}\left(\mathbf{s}^t\right) = K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(Z_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + (1-\delta) K_t\left(\mathbf{s}^{t-1}\right) - C_t\left(\mathbf{s}^t\right), \forall t, \mathbf{s}^t \\
& K_0\left(\mathbf{s}^{-1}\right) \text{ given}
\end{aligned}$$

▷ Note that previously we had $Z_t\left(\mathbf{s}^t\right) \equiv \left(1+g_z\right)^t$ i.e. exogenous growth.

To solve this, you can write down a Bellman equation. Without a Markov assumption, however, the state variables will be an entire history and it does not lend naturally to a Bellman approach. The alternative approach is to use the Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right)\left\{\log C_t\left(\mathbf{s}^t\right) - v\left(H_t\left(\mathbf{s}^t\right)\right)\right\} \\
& + \sum_{t=0}^{\infty} \lambda_t\left(\mathbf{s}^t\right) \beta^t\left[K_{t+1}\left(\mathbf{s}^t\right) - K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(Z_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} - (1-\delta) K_t\left(\mathbf{s}^{t-1}\right) + C_t\left(\mathbf{s}^t\right)\right]
\end{aligned}$$

▷ **Intra-temporal condition:**

$$C_t\left(\mathbf{s}^t\right) v'\left(H_t\left(\mathbf{s}^t\right)\right) = (1-\alpha) Z_t\left(\mathbf{s}^t\right)^{1-\alpha} K_t\left(\mathbf{s}^{t-1}\right)^{\alpha} H_t\left(\mathbf{s}^t\right)^{-\alpha}$$

which is analogous to:

$$\underbrace{-\frac{U_H\left(C_t, H_t\right)}{U_C\left(C_t, H_t\right)}}_{=MRS} = \underbrace{F_{H,t}\left(K_t, H_t\right)}_{=MPL}$$

▷ **Inter-temporal condition:**

$$\frac{1}{C_t\left(\mathbf{s}^t\right)} = \beta \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)} \frac{\alpha K_{t+1}\left(\mathbf{s}^t\right)^{\alpha-1}\left(Z_{t+1}\left(\mathbf{s}^{t+1}\right) H_{t+1}\left(\mathbf{s}^{t+1}\right)\right)^{1-\alpha} + 1 - \alpha}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}$$

which is analogous to:

$$[EE]: \beta^t U_C\left(C_t, H_t\right) = \beta^{t+1} \mathbb{E}_t\left[U_C\left(C_{t+1}, H_{t+1}\right) F_{K,t+1}\left(K_{t+1}, H_{t+1}\right)\right]$$

▷ **Resource condition:**

$$K_{t+1}\left(\mathbf{s}^t\right) = K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(Z_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + (1-\delta) K_t\left(\mathbf{s}^{t-1}\right) - C_t\left(\mathbf{s}^t\right)$$

1.5.3 Assumptions on Shocks

As opposed to the deterministic case, we now look for sequences that depend not only on t but also on $\mathbf{s}^t$. Previously, we could just use the shooting algorithm to find the sequence but now we can't since you'd have to do that for each $\mathbf{s}^t$. And this is where log-linearization proves to be very useful, and in order to do so we need to make some assumptions about the shock. Here are a few options:

- ▷ $Z_t\left(\mathbf{s}^t\right) = Z_0\left(1+g\right)^t \mathbf{s}_t$ combined with $\pi\left(\mathbf{s}_{t+1}|\mathbf{s}_t\right)$ – deterministic trend
    - * This says that in the long-run, the deterministic component drives the growth and we have temporary fluctations around this deterministic trend.
    - * The shock will wear off –
- ▷ $Z_t\left(\mathbf{s}^t\right) = Z_{t-1}\left(\mathbf{s}^{t-1}\right)\mathbf{s}_t$ combined with $\pi\left(\mathbf{s}_{t+1}|\mathbf{s}_t\right)$ – stochastic trend
    - * Cumulative level of productivity is growing stochastically – log of productivity is a random walk. We don't have any exogenous growth here.

Each assumption requires a different way of solving the model, but they will deliver almost identical behaviors for consumption and savings. People generally use the deterministic trend, so we will do the same here.

1.5.4 Equilibrium under Deterministic Trend Assumption

Define the following normalized variables:

$$c_t\left(\mathbf{s}^t\right) \equiv \left(1+g\right)^{-t} C_t\left(\mathbf{s}^t\right)$$
$$k_t\left(\mathbf{s}^{t-1}\right) \equiv \left(1+g\right)^{-t} K_t\left(\mathbf{s}^{t-1}\right)$$

and assume:

$$Z_t\left(\mathbf{s}^t\right) = Z_0\left(1+g\right)^t \mathbf{s}_t = \left(1+g\right)^t \mathbf{s}_t$$

and re-write the three equations in terms of these normalized variables:

- ▷ Intra-temporal condition:

$$c_t\left(\mathbf{s}^t\right) v'\left(H_t\left(\mathbf{s}^t\right)\right) = \left(1-\alpha\right)\mathbf{s}_t^{1-\alpha} k_t\left(\mathbf{s}^{t-1}\right)^{\alpha} H\left(\mathbf{s}^t\right)^{-\alpha}$$

- ▷ Inter-temporal condition:

$$\frac{1+g}{c_t\left(\mathbf{s}^t\right)} = \beta\mathbb{E}_t\left[\frac{\alpha k_{t+1}\left(\mathbf{s}^t\right)^{\alpha-1}\left(\mathbf{s}_{t+1}H_{t+1}\left(\mathbf{s}^{t+1}\right)\right)^{1-\alpha} + 1 - \delta}{c_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]$$

- ▷ Resource constraint:

$$\left(1+g\right)k_{t+1}\left(\mathbf{s}^t\right) = k_t\left(\mathbf{s}^t\right)^{\alpha}\left(\mathbf{s}_t H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + \left(1-\delta\right)k_t\left(\mathbf{s}^{t-1}\right) - c_t\left(\mathbf{s}^t\right)$$

1.5.5 Log-Linearization

To get started, we will shut down shocks but not shut down $\mathbf{s}_t$. Thus, suppose that $\mathbf{s}_t \to 1$ via: $\log \mathbf{s}_{t+1} = \rho \log \mathbf{s}_t$ with $\mathbf{s}_0$ given and $\rho = 0.95$. This will get rid of the expectation operator in the inter-temporal condition. Log-linearzing, we ultimately have

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} = \begin{bmatrix} 1.024 & -0.088 & 0.064 \\ -0.013 & 0.989 & 0.025 \\ 0 & 0 & 0.95 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{s}_t \end{bmatrix}$$

▷ Note that the 2×2 block is unchanged from before:

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} 1.024 & -0.088 \\ -0.013 & 0.989 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}$$

This is because when we do a first-order approximation, there is no interaction term and introducing new stuff cannot affect the coefficients from before.

The eigenvalues are now $0.968, 1.044, 0.95$ – once again, 0.968 and 1.044 are from before – and we can use this to get the eigenvectors:

$$e_1 = \begin{bmatrix} 0.845 \\ 0.534 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} -0.828 \\ 0.465 \\ 0.314 \end{bmatrix}$$

which yields the following constraint on the initial condition that does not diverge:

$$\begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \\ \hat{s}_0 \end{bmatrix} = \mu_1 \begin{bmatrix} 0.845 \\ 0.534 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} -0.828 \\ 0.465 \\ 0.314 \end{bmatrix}$$

which gives us three equations and three variables to get: $\hat{c}_0, \mu_1, \mu_2$.

▷ Three stable eigenvalues: model is undetermined (too may unknowns)

▷ One stable eigenvalue: model is overdetermined (and generically you probably forgot a part of the model)

Thus we ultimately get

$$\hat{c}_t = 0.632\hat{k}_t + 0.186\hat{s}_t$$

and once again, in the log linear approximation, we are not changing anything about how $\hat{c}_t$ responds to $\hat{k}_t$. We can also get an expression for $\hat{h}_t$:

$$\hat{H}_t = -0.160\hat{k}_t + 0.273\hat{s}_t$$

and an expression for $\hat{k}_{t+1}$ and $\hat{x}_t$:

$$\begin{aligned} \hat{k}_{t+1} &= 0.968\hat{k}_t + 0.048\hat{s}_t \\ \hat{x}_t &= -0.653\hat{k}_t + 2.474\hat{s}_t \end{aligned}$$

Interpretation* Suppose we start at period 0 with capital stock on trend $\left(\hat{k}_0 = 0\right)$ but productivity is 1% below trend $(\hat{s}_0 = -1)$. What is the impact going to be? From the equations, we can read off the immediate impact this period:

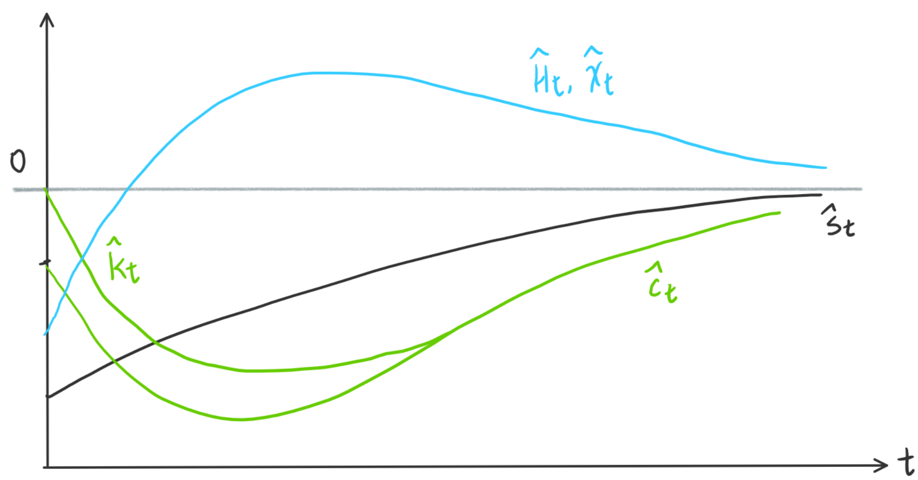
▷ The lack of productivity means we will consume relatively less and work relatively less.

▷ This is consistent with our observation that if a country starts with little capital, labor supply will be high. But if you start with little productivity, it is kind of like having a lot of capital, so they don't want to work that much. They also anticipate productivity will rise in the future, which induces them to take leisure today and work later.

▷ If hours worked is low and productivity is low, then output today will be low, consistent with low investment and low consumption.

▷ Investment is low because these guys have permanent income and smooth consumption. They don't want to drop consumption as much as by drop in output, so you want to borrow against the future by reducing investment (capital stock) which allows you to have a smaller reduction in consumption today. This shows up in next period's capital stock.

Impulse Responses* We can draw the impulse responses:



1.5.6 Getting to Real Business Cycles

We make one modification:

$$\log \mathbf{s}_{t+1} = \rho \log \mathbf{s}_t + \upsilon_{t+1}, \quad \mathbb{E}_t [\upsilon_{t+1}] = 0$$

This leaves the log-linearization the same as before because the mean of υs are equal to zero. Thus we have

$$\begin{aligned}
\hat{c}_t &= 0.632\hat{k}_t + 0.186\hat{s}_t \\
\hat{h}_t &= -0.160\hat{k}_t + 0.273\hat{s}_t \\
\hat{k}_{t+1} &= 0.968\hat{k}_t + 0.048\hat{s}_t \\
\hat{x}_t &= -0.653\hat{k}_t + 2.474\hat{s}_t \\
\hat{s}_{t+1} &= 0.95\hat{s}_t + \underline{\nu_{t+1}}
\end{aligned}$$

This is the limitation of log-linearization.

▷ The recent literature has focused on uncertainty shocks, which you can model as increasing the variance of υ_{t+1}. In the log-linearization case, this will have no effect.

▷ Of course, if you do a second-order approximation, it will show up.

The graph above would be thus consistent with the conditional expectations at each point in time.

1.6 Connecting Model with Data

Now that we have generated model, we will think about how we can properly compare the model with data.

1.6.1 Connecting Model with Data

Using the log-linearization, you can compute the ergodic joint distribution analytically. Moreover, you can compute the joint distribution of $\hat{c}_{t+7}$ and $\hat{k}_t$ i.e. in different weeks. There are, however, some cautions.

▷ If you do this, you are not comparing data to the model in the right way.

  * We take the data series from BEA and BLS and so on. For these time-series, we start worrying that it doesn't look like the real world, so we may tweak the data: de-trend (to get rid of low frequency series), seasonal adjustments (booms in summer, Christmas). This is mostly because adding these elements to the model is too cumbersome.
  * How do we de-trend or adjust? For de-trending, one example is HP filter; for seasonal adjustments, there's a census 12x algorithm that is partly black box.

▷ Thus, we want to treat model the same way we treat the data. In this class, we will do it in terms of detrending.

So the steps are the following:

1. Get initial conditions (either through some burn-in or drawing from the joint distribution)
2. Construct simulated data series from the model and do what you did to the real data.
3. You can compute moments and correlations using the model-generated data and the real data.

1.6.2 Monte Carlo simulation

We can also compare comovements implied by the model against the data using Monte Carlo simulation. Suppose we have 50 years worth of quarterly data (i.e. 200 observations). Then, we would proceed as follows.

1. Specify the distribution of the shocks, υ_{t+1}. Take 200 draws of υ_{t+1}.
2. Feed the shocks into the model and compute the variables of interest for 200 periods.
3. Detrend the data (more on this below).
4. Calculate the variance-covariance matrix.
5. Repeat a "bunch" of times.

1.6.3 Thoughts on Detrending

The aggregate variables in the data generally has both trend and cyclical part. The idea of detrending is to isolate the cyclical part (i.e. the business cycle) from the trend part.

▷ If you're comfortable using a bandpass filter, there's no issue.

▷ The HP filter has nice properties:

  * *It behaves as a linear operator.* If you give me a time-series (a vector of outputs), I will take it and multiply it by a fixed matrix which will give a trend. This fixed matrix is used for detrending the model and the actual output. Consequently, this is also quick.

* *Linearity*. Trend in consumption + trend in investment = trend in output, so you have variables that add up in a natural way.
* *It will always remove a linear trend.* So the 2% growth rate will not affect the detrended data. It will also remove low-frequency stuff like increases in the labor supply.

1.6.4 HP Filter

Let y_t denote a time series data of a variable. We want to decompose this into two parts: the trend, g_t, and the cyclical part, d_t. We suppose that

$$y_t = g_t + d_t, \forall t.$$

The Hodrick-Prescott filtering solves the following problem.

$$\min_{\{g_t, d_t\}_{t=1}^T} \left\{ \sum_{t=1}^{T} (d_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \right\}.$$

▷ The first component of the objective function penalises deviations from the trend. To interpret the second, note that $(g_{t+1} - g_t)$ and $(g_t - g_{t-1})$ are slopes of the trend.

▷ The second part of the objective function penalises changes in slope of the trend, and λ is a parameter that specifies the relative weights between the two "losses". Note that

$$\lambda = 0 \Rightarrow \begin{cases} d_t = 0 \\ g_t = y_t \end{cases},$$

$$\lambda \to \infty \Rightarrow \tau_t = \text{a linear trend.}$$

The customary values of λ are:

Data frequency	Yearly	Quarterly	Monthly
λ	100	1,600	14,400

If we think that variables are growing exponentially over time, then we would detrend the log of the series. If a variable grows linear, there is no need. (We can only detrend the log of the series if the variable is always nonnegative (e.g. we cannot take logs of inflation or the interest rate!).

1.7 Wedge Accounting

We will introduce something that looks like taxes but actually represents model misspecification.

1.7.1 Motivation

The table below gives some statistics on logged output (y), consumption (c), investment (x) and hours worked (h). Since housing can be thought of as a capital good (i.e. it provides services to the owner over time), it is common to include house purchases in investment. Sometimes, other durable goods (e.g. cars) are also included in investment for the same reason. Hours worked does not include vacation.

Table 1: Historical US data vs model output.

	US data				Model output				
	y	c	x	h	y	c	x	h	z
Standard deviation[1]	1	0.563	3.684	1.017	1	0.293	0.181	0.378	1.284
Autocorrelation	0.992	0.935	0.912	0.917	0.911	0.933	0.909	0.909	0.910
Cross correlation (log)									
y	1	0.827	0.864	0.849	1	0.876	0.992	0.982	0.997
c	.	1	0.756	0.759	.	1	0.807	0.771	0.837
x	.	.	1	0.699	.	.	1	0.998	0.999
h	.	.	.	1	.	.	.	1	0.994
z	n.a.				.	.	.	.	1

Comparing the data with the output, we can see that:

- ▷ consumption, investment and hours worked are all too "smooth" in the model;
- ▷ cross-correlations tend to be too high in the model.

With respect to hours worked, recall that we had set the Frisch elasticity of labour, ε, to be one. We can introduce more volatility in the hours worked in the model by increasing this parameter; however, for reasonable values (up to around 4), we cannot reproduce the volatility observed in the real data (see Problem Set 3). To fix other aspects, we may wish extend the RBC model by adding government and/or export/imports into the model.

Alternatively, we could look at the three optimality conditions to see which of them are violated in the data. This is what is referred to as *wedge accounting*.

1.7.2 Modified Economy

The basic modifications, following Chari-Kehoe-McGrattan, we introduce the following notation:

- ▷ $\tau_{h,t}\left(\mathbf{s}^t\right)$: tax on labour income
- ▷ $\tau_{x,t}\left(\mathbf{s}^t\right)$: tax on investment
- ▷ $z_t\left(\mathbf{s}^t\right)$: technology shock
- ▷ $G_t\left(\mathbf{s}^t\right)$: (exogenous) government purchases + net exports

Adding these to the economy and solving for the first-order conditions:

- ▷ Intratemporal condition (modified due to **labor wedge**):

$$\gamma C_t\left(s\right)^t H_t\left(\mathbf{s}^t\right)^{\frac{1}{\varepsilon}} = \left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)\left(1-\alpha\right)\frac{Y_t\left(\mathbf{s}^t\right)}{H_t\left(\mathbf{s}^t\right)}$$

  and we call $1-\tau_{h,t}$ the labor wedge.

- ▷ Intertemporal condition (modified due to **investment wedge**)

$$\frac{1+\tau_{x,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)} = \beta\mathbb{E}_t\left[\frac{\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+\left(1-\delta\right)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]$$

  and we call $1/\left(1+\tau_{x,t}\right)$ the investment wedge.

[1] Normalised by the standard deviation of y.

▷ Resource constraint (modified due to **efficiency wedge**):

$$\left(z_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}=\frac{Y_t\left(\mathbf{s}^t\right)}{\left(K_t\left(\mathbf{s}^{t-1}\right)\right)^{\alpha}\left(H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}}$$

and we call z_t the efficiency wedge.

▷ Output accounting:

$$Y_t\left(\mathbf{s}^t\right)=C_t\left(\mathbf{s}^t\right)+X_t\left(\mathbf{s}^t\right)+G_t\left(\mathbf{s}^t\right)$$

▷ Law of motion for capital:

$$K_{t+1}\left(\mathbf{s}^{t+1}\right)=(1-\delta)\,K_t\left(\mathbf{s}^t\right)+X_t\left(\mathbf{s}^t\right)$$

1.7.3 Which wedges matter?

One simple interpretation of the wedges is that they represent taxes, TFP and government spending that exists in real life.

▷ Chari, Kehoe and McGrattan (2007) interprets the wedges to be misspecification of the model.

▷ Specifically, they back out the time series for each wedge.

* Since the time series may contain interaction among the wedges, they run a VAR to back out the "pure" effect of each wedge.
* They then plug this back into the model, one wedge at at a time, and see which ones matter the most in replicating the observed data (note that with all four wedges "in action", the model is able to fully replicate the data by construction).

1.7.4 Future Directions

Nominal rigidites are about "markups" in the economy.

▷ If there are no markups, they will equate wage to the marginal productivity. But if it has market power, it will want some markup over the wage.

▷ Thus we will introduce something that drives cyclicality in the markup (price / marginal product). It turns out that price stickiness is a potential source of this "markup" and this will create a labor wedge.

2 Price Rigidity

2.1 Monopolistic Competition

This will provide a wedge between w_t and MPL_t. There is a representative household, but it will now consume a lot of goods (a continuum of different goods). Each of these goods is produced by different monopolistic producer of that good, and each producer chooses which price to charge, recognizing that it faces some demand for the good. Firms will produce output using only labor here with CRS technology. The novel part will be pricing decision of the firms, and each firm will decide given the demand curve they face.

Why is this useful? We need to move away from perfect competition to talk about price rigidity. It's also useful for talking about other stuff like innovation, since the reward for innovation is the possibility of a monopoly.

2.1.1 Setup

Household's Problem The household solves the following problem:

$$\begin{aligned}\max_{\{C_t, c_{j,t}, H_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t U(C_t, H_t) \\ s.t. \quad & A_0 = \sum_{t=0}^{\infty} Q_0^t \left(\int_0^1 p_{j,t} c_{j,t} dj - W_t H_t \right) \\ & A_0, \left\{ Q_0^t, W_t, p_{j,t} \right\}_{t=0}^{\infty} \text{ given,}\end{aligned}$$

where C_t is the aggregate consumption index:

$$C_t := \left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}$$

where $\eta > 1$ is the elasticity of substitution between goods and $c_{j,t}$ is household's consumption of goods produced by firm j in period t.

- C_t is concave – the household likes to have a variety of goods, and the marginal utility of consumption is infinite if the consumption at zero.
- H_t is the hours worked by the household
- Q_0^t gives the period-0 worth of a dollar in period t
- W_t is the nominal wage
- $p_{j,t}$ is the nominal price of good j at time t

Firm's Problem The firm solves the following problem: (recall that there is still no capital)

$$\begin{aligned}V_{o,j} := \max_{\{p_{j,t} y_{j,t} c_{j,t} h_{j,t}\}} \quad & \sum_{t=0}^{\infty} Q_0^t (p_{j,t} y_{j,t} - W_t h_{j,t}) \\ s.t. \quad & y_{j,t} = z_t h_{j,t} \\ & y_{j,t} = c_{j,t} = \Phi_t(p_{j,t}) \qquad \text{[Market Clearing]} \\ & \{W_t, P_t, C_t\} \text{ given.}\end{aligned}$$

Market Clearing We have the following conditions

$$
\begin{aligned}
H_t &= \int_0^1 h_{j,t} dj \\
A_0 &= \int_0^1 V_{o,j} dj
\end{aligned}
$$

2.1.2 Characterizing Equilibrium

Demand Function Write down the Lagrangian for the Household's problem:

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U \left(\left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}, H_t \right) + +\lambda \left(A_0 - \sum_{t=0}^{\infty} Q_0^t \left(\int_0^1 p_{j,t} c_{j,t} dj - W_t H_t \right) \right)
$$

Taking the **first-order condition** with respect to $c_{j,t}$:

$$
\begin{aligned}
[c_{j,t}] : \beta^t U_C (C_t, H_t) \left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}} dj \right)^{\frac{1}{\eta-1}} c_{j,t}^{-\frac{1}{\eta}} &= \lambda Q_0^t p_{j,t} \\
\Rightarrow \beta^t U_C (C_t, H_t) C_t^{\frac{1}{\eta}} c_{j,t}^{-\frac{1}{\eta}} &= \lambda Q_0^t p_{j,t}
\end{aligned}
$$

and $c_{0,t}$:

$$
[c_{0,t}] : \beta^t U_C (C_t, H_t) C_t^{\frac{1}{\eta}} c_{0,t}^{-\frac{1}{\eta}} = \lambda Q_0^t p_{0,t}
$$

▷ Combining these two equations:

$$
\frac{c_{j,t}^{-\frac{1}{\eta}}}{c_{0,t}^{-\frac{1}{\eta}}} = \frac{p_{j,t}}{p_{0,t}} \Rightarrow c_{j,t} = \left(\frac{p_{j,t}}{p_{0,t}} c_{0,t}^{-\frac{1}{\eta}} \right)^{-\eta} = c_{0,t} \left(\frac{p_{0,t}}{p_{j,t}} \right)^{\eta}
$$

▷ Substituting into C_t:

$$
\begin{aligned}
C_t &= \left(\int_0^1 \left(p_{j,t}^{-\eta} p_{0,t}^{\eta} c_{0,t} \right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} \\
&= \left(\int_0^1 p_{j,t}^{-(\eta-1)} p_{0,t}^{\eta-1} c_{0,t}^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} \\
\Rightarrow C_t &= c_{0,t} p_{0,t}^{\eta} \left(\int_0^1 p_{j,t}^{1-\eta} dj \right)^{\frac{\eta}{\eta-1}}
\end{aligned}
$$

▷ The expenditure $\int p_{j,t} c_{j,t} dj$ can also be computed:

$$
\int_0^1 p_{j,t} c_{j,t} dj = c_{0,t} p_{0,t}^{\eta} \int_0^1 p_{j,t}^{1-\eta} dj.
$$

We can also combine expression to eliminate $c_{0,t}$ to get the demand function:

$$
c_{j,t} = C_t \left(\frac{P_t}{p_{j,t}} \right)^{\eta} \equiv \Phi_t (p_{j,t}) \qquad \forall j \in [0, 1]
$$

Remark 2.1. We are using the continuum of firms assumption implicitly since each firm takes as given P_t and C_t (aggregate price index and aggregate demand) to decide $p_{j,t}$. If there are two firms in the economy, this assumption would be an egregious one.

Price Index Define the ideal price index as

$$P_t := \left(\int_0^1 p_{j,t}^{1-\eta} dj\right)^{\frac{1}{1-\eta}}$$

▷ This summarizes the price vector nicely:

$$\int_0^1 p_{j,t} c_{j,t} dj = C_t P_t^{\eta} P_t^{1-\eta} = C_t P_t.$$

▷ This relationship holds since:

$$C_t = c_{0,t} p_{0,t}^{\eta} P_t^{-\eta} \Leftrightarrow C_t P_t^{\eta} = c_{0,t} p_{0,t}^{\eta}$$

and

$$\int_0^1 p_{j,t} c_{j,t} dj = c_{0,t} p_{0,t}^{\eta} P_t^{1-\eta}.$$

Household's Optimization Rearranging the **first-order condition** with respect to $c_{j,t}$:

$$\beta^t U_C(C_t, H_t) C_t^{\eta} = \lambda Q_0^t \left(c_{j,t} p_{j,t}^{\eta}\right)^{\frac{1}{\eta}}$$

and substituting in the expression for $c_{j,t}$:

$$\beta^t U_C(C_t, H_t) C_t^{\frac{1}{\eta}} = \lambda Q_0^t \left(C_t P_t^{\eta}\right)^{\frac{1}{\eta}}$$

which yields:

$$[C_t] : \beta^t U_C(C_t, H_t) = \lambda Q_0^t P_t$$
$$[C_{t+1}] : \beta^{t+1} U_C(C_{t+1}, H_{t+1}) = \lambda Q_t^{t+1} P_{t+1}$$

Taking the **first-order conditio**n with respect to H_t:

$$[H_t] : -\beta^t U_H(C_t, H_t) = \lambda Q_0^t W_t$$

Combining $[C_t]$ and $[H_t]$:

$$\frac{-U_H(C_t, H_t)}{U_C(C_t, H_t)} = \frac{W_t}{P_t}.$$

▷ The condition equates the marginal rate of substitution between consumption and leisure (i.e. the left-hand side) with the real wage (i.e. the right-hand side).

Combining $[C_t]$ and $[C_{t+1}]$:

$$U_C(C_t, H_t) = \beta \frac{Q_0^t P_t}{Q_0^{t+1} P_t^{t+1}} U_C(C_{t+1}, H_{t+1})$$

Demand curve:

$$c_{j,t} = C_t \left(\frac{P_t}{p_{j,t}}\right)^{\eta} \equiv \Phi_t(p_{j,t}) \qquad \forall j \in [0,1]$$

Firm's Optimization Rewrite the firm's problem as:

$$\begin{aligned}\max_{\{p_{j,t}y_{j,t}c_{j,t}h_{j,t}\}} \quad & p_{j,t}y_{j,t} - W_t h_{j,t} \\ s.t. \quad & y_{j,t} = z_t h_{j,t} \\ & y_{j,t} = c_{j,t} = \frac{C_t P_t^\eta}{p_{j,t}^\eta} \\ & \{W_t, P_t, C_t\} \text{ given.}\end{aligned}$$

Using the constraints, we can write the objective function in terms of $p_{j,t}$'s (and C_t, P_t) only:

$$\begin{aligned}\max_{p_{j,t},y_{j,t}} p_{j,t}y_{j,t} - W_t\frac{y_{j,t}}{z_t} &= \max_{p_{j,t},y_{j,t}} p_{j,t}\frac{C_tP_t^\eta}{p_{j,t}^\eta} - \frac{W_t}{z_t}\frac{C_tP_t^\eta}{p_{j,t}^\eta} \\ &= \max_{p_{j,t}} C_tP_t^\eta\left(p_{j,t}^{1-\eta} - \frac{W_t}{z_t}p_{j,t}^{-\eta}\right)\end{aligned}$$

Since $\eta > 1$, first-order conditions are necessary and sufficient:

$$[p_{j,t}] : 0 = (1-\eta)\, p_{j,t}^{-\eta} + \eta\frac{W_t}{z_t}p_{j,t}^{-\eta-1}$$

which yields

$$p_{j,t} = \frac{\eta}{\eta-1}\frac{W_t}{z_t}$$

- ▷ Note that W_t/z_t is the marginal cost of production and $\eta/(\eta-1)$ is the mark-up over the marginal cost. We see that as $\eta \to \infty$, $p_{j,t} \to W_t/z_t$ so that we get the perfect competition outcome. In contrast as $\eta \downarrow 1$, we see that the firms would charge arbitrarily high prices.

- ▷ Since the right-hand side of oes not depend on j, we realise that all firms charge the same price. This implies that

$$P_t = \left(\int_0^1 p_{j,t}^{1-\eta}dj\right)^{\frac{1}{1-\eta}} = p_{j,t}$$

  and so

$$P_t := \frac{\eta}{\eta-1}\frac{W_t}{z_t},\ \forall j \in [0,1]\,.$$

- ▷ This allows us to write the real wage as

$$\frac{W_t}{P_t} = z_t\frac{\eta-1}{\eta} < z_t.$$

  Thus, we see that there is a wedge, $(\eta-1)/\eta$, between the real wage and the marginal product of labour.

Market Clearing Labour market clearing condition is:

$$H_t = \int_0^1 h_{j,t}dj.$$

Using the production function and the consumer's demand function, we can write

$$H_t = \int_0^1 \frac{C_tP_t^\eta}{p_{j,t}^\eta z_t}dj = \int_0^1 \frac{C_tP_t^\eta}{P_t^\eta z_t}dj = \frac{C_t}{z_t}.$$

Rearranging this gives the aggregate production in the economy:

$$C_t = z_t H_t.$$

FF problem

To summarise, we have obtained the following optimality conditions in aggregate variables:

$$\frac{W_t}{P_t} = z_t \frac{\eta - 1}{\eta},$$
$$P_t = p_{j,y}, \forall j \in [0,1].$$

Goods market clearing

$$C_t = z_t H_t.$$

2.1.3 Equilibrium

An equilibrium here is a sequence of allocations

$$\left\{C_t, H_t, \{c_{t,j}, h_{j,t}, y_{j,t}\}_{j \in [0,1]}\right\}_{t=0}^{\infty}$$

and a sequence of prices

$$\left\{P_t, Q_0^t, W_t, \{p_{t,j}\}_{j \in \{0,1\}}\right\}$$

and initial conditions A_0 such that: (i) household solves its maximization problem; (ii) firms solves its maximization problem in all periods; and (iii) markets clear:

$$H_t = \int_0^1 h_{j,t} dj, \ \forall t = 1, \ldots, \infty,$$
$$y_{j,t} = c_{j,t}, \ \forall j \in [0,1], t = 1, \ldots, \infty,$$
$$A_0 = \sum_{t=0}^{\infty} Q_0^t \int_0^1 v_t^j dj.$$

2.1.4 Balanced Growth Preferences

Let us assume that

$$U(C, H) := \frac{\left(Ce^{-v(H)}\right)^{1-\sigma}}{1-\sigma}$$

Using the second HH equation and the first and third FF equation, we can write

$$z_t H_t v'(H_t) = C_t v'(H_t) = \frac{W_t}{P_t} = z_t \frac{\eta - 1}{\eta},$$

which simplifies to

$$H_t v'(H_t) = \frac{\eta - 1}{\eta} \Rightarrow H \text{ constant}$$

- ▷ In the social planner's problem, the left-hand side would equal one (which coincides with the $\eta \to \infty$ case) so we can interpret $(\eta - 1)/\eta$ as a labour wedge.
- ▷ In this setup, however, since η is a fixed parameter, this labour wage does not affect cyclicality. For η (the elasticity of demand) or equivalently $(\eta - 1)/\eta$ (mark up over the marginal cost charges by firms) to affect business cycles, they must vary cyclically. One rationale is that, in times of recession, due to some firms exiting, market power of remaining firms may increase (and the opposite in times of booms).

2.1.5 Comments

1. There is so much homogeneity here – there are no idiosyncratic shocks $(z_{j,t})$ or individual productivity shocks. In real life, there is a dispersion in how prices move along with inflation. In this model, we assume that there is a monetary policy that keeps relative prices the same.

2. What is the cost of changing these prices? Why are prices sticky?

   ▷ They exist in the reality (including one by Klenow) – you can imagine, before the internet, that there was a physical cost of putting prices on objects or changing the restaurant menus. People think of them as real costs of adjusting.

   ▷ It's difficult to find this cost to be high – print newspapers can change the prices but they don't.

   ▷ It's possible that the optimal price does not change that much and the cost of choosing a new price may outweigh the benefits.

3. We have multiple models. We will start with an economy where there is a physical cost of adjusting the prices.

2.2 Incorporating Productivity Shocks

Increase in productivity will lead to decrease in employment.

2.2.1 Setup

Household's Problem The household solves the following problem:

$$\begin{aligned} \max_{\{C_t(\mathbf{s}^t), c_{j,t}(\mathbf{s}^t), H_t(\mathbf{s}^t)\}_{t=0}^\infty} \quad & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right) U\left(C_t, H_t\right) \\ s.t. \quad & A_0 = \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} Q_0^t\left(\mathbf{s}^t\right) \left(\int_0^1 p_{j,t}\left(\mathbf{s}^t\right) c_{j,t}\left(\mathbf{s}^t\right) dj - W_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right) \right) \\ & A_0, \left\{Q_0^t, W_t, p_{j,t}\right\}_{t=0}^\infty \text{ given,} \end{aligned}$$

where C_t is the aggregate consumption index:

$$C_t\left(\mathbf{s}^t\right) := \left(\int_0^1 \left(c_{j,t}\left(\mathbf{s}^t\right)\right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}$$

and Q_0^t is a claim to a dollar in period t if history $\mathbf{s}^t$ has been realized.

Firm's Problem The firm solves the following problem: (recall that there is still no capital)

$$\begin{aligned} & V_{j,0}\left(p_{j,-1}\right) := \\ & \max_{\{p_{j,t} y_{j,t} c_{j,t} h_{j,t}\}} \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} Q_0^t\left(\mathbf{s}^t\right) \left(p_{j,t}\left(\mathbf{s}^t\right) y_{j,t}\left(\mathbf{s}^t\right) - W_t\left(\mathbf{s}^t\right) h_{j,t}\left(\mathbf{s}^t\right) - \underbrace{W_t\left(\mathbf{s}^t\right) \bar{H} \mathbb{I}\left(p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right)}_{\text{cost of changing price}} \right) \\ & \quad s.t. \quad y_{j,t}\left(\mathbf{s}^t\right) = z_t\left(\mathbf{s}^t\right) h_{j,t}\left(\mathbf{s}^t\right) \qquad \text{[Production Technology]} \\ & \qquad\quad\;\; y_{j,t}\left(\mathbf{s}^t\right) = c_{j,t} = \Phi_t\left(p_{j,t}, \mathbf{s}^t\right) \qquad \text{[Market Clearing]} \\ & \qquad\quad\;\; \{W_t, P_t, C_t\} \text{ given.} \end{aligned}$$

▷ Since $p_{j,t}$ is a price to commit to, it's possible that p falls below the marginal cost if the adjustment cost is too high.

Market Clearing We have the following conditions

$$H_t = \int_0^1 h_{j,t}\left(\mathbf{s}^t\right) + \bar{H}\mathbb{I}\left(p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right) dj$$

Flexible Price We will assume that $\bar{H} = 0$.

2.2.2 Characterizing the Equilibrium

Household's Problem The conditions are:

▷ Combining $[C_t]$ and $[H_t]$:

$$\frac{-U_H\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right)}{U_C\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right)} = \frac{W_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)}.$$

 * The condition equates the marginal rate of substitution between consumption and leisure (i.e. the left-hand side) with the real wage (i.e. the right-hand side).

▷ Combining $[C_t]$ and $[C_{t+1}]$:

$$\Pi_t\left(\mathbf{s}^t\right)\frac{U_C\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right)}{P_t\left(\mathbf{s}^t\right)Q_0^t\left(\mathbf{s}^t\right)} = \beta\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\frac{U_C\left(C_{t+1}\left(\mathbf{s}^{t+1}\right), H_{t+1}\left(\mathbf{s}^{t+1}\right)\right)}{P_{t+1}\left(\mathbf{s}^{t+1}\right)Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)} \quad \forall t, \mathbf{s}^{t+1} \succ \mathbf{s}^t$$

▷ Demand curve:

$$c_{j,t}\left(\mathbf{s}^t\right) = C_t\left(\mathbf{s}^t\right)\left(\frac{P_t\left(\mathbf{s}^t\right)}{p_{j,t}\left(\mathbf{s}^t\right)}\right)^{\eta} \equiv \Phi_t\left(p_{j,t}, \mathbf{s}^t\right) \qquad \forall j \in [0,1]$$

Firm's Problem The conditions are:

▷ Productivity is strictly higher than wage due to market power:

$$\frac{W_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)} = z_t\left(\mathbf{s}^t\right)\frac{\eta-1}{\eta}$$
$$P_t\left(\mathbf{s}^t\right) = p_{j,t}\left(\mathbf{s}^t\right), \forall j \in [0,1].$$

▷ Labour market clearing

$$C_t\left(\mathbf{s}^t\right) = z_t\left(\mathbf{s}^t\right)H_t\left(\mathbf{s}^t\right).$$

2.2.3 Balanced Growth Preferences

Similarly as before, we obtain:

$$C_t\left(\mathbf{s}^t\right)v'\left(H_t\left(\mathbf{s}^t\right)\right) = \frac{W_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)} = z_t\left(\mathbf{s}^t\right)\frac{\eta-1}{\eta}$$

which gives us:

$$H_t\left(\mathbf{s}^t\right)v'\left(H_t\left(\mathbf{s}^t\right)\right) = \frac{\eta-1}{\eta} \Rightarrow H \text{ constant}$$

2.2.4 Nominal Interest Rates

Going back to the Euler equation:

$$\Pi_t\left(\mathbf{s}^t\right)\frac{U_C\left(C_t\left(\mathbf{s}^t\right),H_t\left(\mathbf{s}^t\right)\right)}{P_t\left(\mathbf{s}^t\right)Q_0^t\left(\mathbf{s}^t\right)}=\beta\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\frac{U_C\left(C_{t+1}\left(\mathbf{s}^{t+1}\right),H_{t+1}\left(\mathbf{s}^{t+1}\right)\right)}{P_{t+1}\left(\mathbf{s}^{t+1}\right)Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}\quad\forall t,\mathbf{s}^{t+1}\succ\mathbf{s}^t$$

To talk about the nominal interest rate, it's good think about a safe nominal asset that pays 1 in the future. We can construct the security that costs the price of all securities associated with all possible future histories.

▷ Mathematically, the **(gross) nominal interest rate** amounts to

$$i_t\left(\mathbf{s}^t\right)\equiv\frac{Q_0^t\left(\mathbf{s}^t\right)}{\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}$$

▷ Plugging this into the Euler equation, we can express the nominal interest rate s a function of expected inflation and other stuff:

$$i_t\left(\mathbf{s}^t\right)=\left[\beta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}\frac{U_{C,t+1}\left(\mathbf{s}^{t+1}\right)}{U_{C,t}\left(\mathbf{s}^t\right)}\frac{P_t\left(\mathbf{s}^t\right)}{P_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]^{-1}$$

What determines $P_t\left(\mathbf{s}^t\right)$ and $P_{t+1}\left(\mathbf{s}^{t+1}\right)$? As of now, we currently we have too many degrees of freedom. So how can we pin down the nominal interest rate?

1. Given prices, we can compute the nominal interest rate.
2. Alternatively, we can ask: which interest rates are consistent with the prices being stable i.e. $P_t=P_{t+1}$.
3. If $i_t\left(\mathbf{s}^t\right)$ is set at the level consistent with $P_t\left(\mathbf{s}^t\right)=P_{t+1}\left(\mathbf{s}^t\right),\forall t$ and $P_{j,-1}=P_{-1},\forall j$, then there is no change if $\bar{H}>0$.
   ▷ $P_{j,-1}=P_{-1},\forall j$ is important – if we have initial price dispersion among firms, this may not hold.

2.2.5 Comments

▷ In a model with flexible prices, the productivity shock does not affect the hours worked.

▷ In the sticky price model, we are going to get out that change in z_t is not going to affect C_t but lead to a proportional decline in H_t.

▷ The nature of the model in which all firms are hit with the same productivity shock.

  * You can imagine idiosyncratic shocks, so different firms will want to have different prices in time.
  * Therefore, there will not be a monetary policy where there are no firms who want to change their prices. This is more realistic.
  * So this is the kind of an artificial feature of this economy that it's possible to run a monetary policy that keeps the prices stable.

▷ To introduce money, stick it into the utility function and try changing the money supply on how it affects the households.

2.3 Caplin & Leahy (1991) Model

There are firms with fixed adjustment costs to price. The equilibrium had a couple of features:

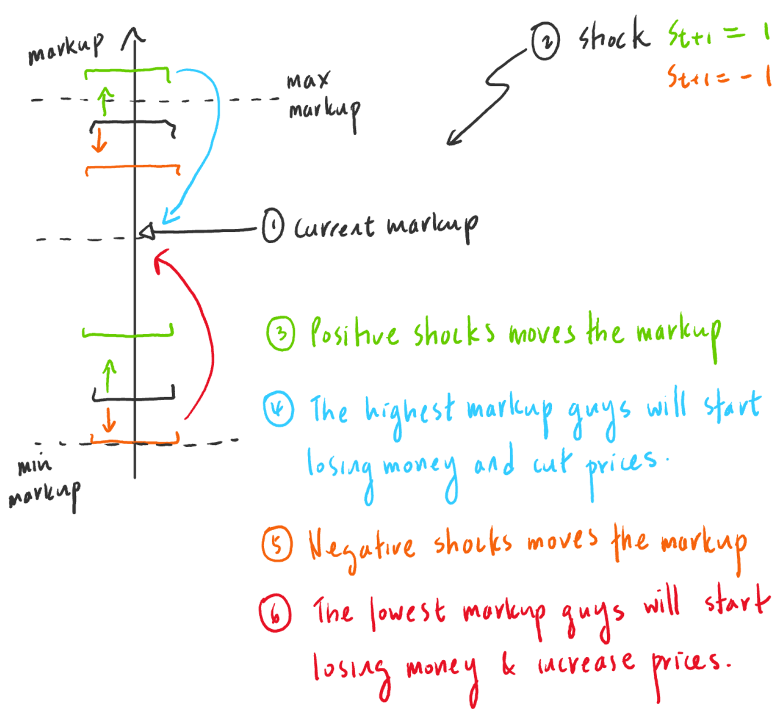
- ▷ There is a "target markup," and there is a fixed number of steps that firms could have away from the "target markup."
- ▷ We were looking for an equilibrium where the nominal wage is constant. Note that there can be different monetary policies which could lead to prices being constant as well.

2.3.1 Assumptions

We introduce the following assumptions:

1. $U(C, H) = \log C - \gamma H$
2. Productivity is a random walk:
$$z_t\left(\mathbf{s}^{t+1}\right) = \left(1 + \Delta \cdot \mathbf{s}_{t+1}\right) z_t\left(\mathbf{s}^t\right)$$
where $\mathbf{s}_{t+1}$ is an i.i.d. random variable that takes on values in $\{-1, 0, 1\}$.
3. There exists a monetary policy i_t such that $W_t\left(\mathbf{s}^t\right) = W$
    - ▷ This is in contrast with the previous case where we set the prices constant through the monetary policy.
4. $\bar{H} > 0$

What are sticky prices doing? You won't change the price unless it's too far away from your desired price. As a consequence, you will have a distribution of markups (as opposed to the same markup for all firms).



2.3.2 Setup

Define the thresholds: y, n (note that y, n are endogenous).

▷ The behavior of the firms in this economy is such that

$$e^{y-n\Delta}\frac{W}{z_t\left(\mathbf{s}^t\right)} < p_{j,t-1}\left(\mathbf{s}^{t-1}\right) < e^{y+n\Delta}\frac{W}{z_t\left(\mathbf{s}^t\right)} \Rightarrow p_{j,t-1}\left(\mathbf{s}^{t-1}\right) = p_{j,t}\left(\mathbf{s}^t\right)$$

Otherwise, we will have

$$p_{j,t}\left(\mathbf{s}^{t-1}\right) = e^{y}\frac{W}{z_t\left(\mathbf{s}^t\right)}$$

i.e. if your price is not too low or not too high, it will keep the price constant.

* y is how much markup you have on average.
* n is an integer that dictates how many of these Δ steps we can go from y before you start adjusting your price.
* The max markup and the min markup are therefore expressed as

$$e^{y-n\Delta}, \qquad e^{y+n\Delta}$$

since the markup is defined to be price divided by marginal cost.

▷ Distribution of firms is a discrete uniform in log markup space.

▷ Lowest log markup

$$x_t\left(\mathbf{s}^t\right) \in \{y-(n-1)\Delta, \cdots, y\}$$

yields the distribution of markups uniform on $\left\{x_t\left(\mathbf{s}^t\right), x_t\left(\mathbf{s}^t\right)+\Delta, \cdots, x_t\left(\mathbf{s}^t\right)+(n-1)\Delta\right\}$

* For example, if $x_t = y$ and we have positive productivity then the $y+(n-1)\Delta$ becomes y and he will now want to adjust prices.
* Solving the firm's problem yields the new markup to be equal to y.

2.3.3 Characterizing the Equilibrium

We are looking for a particular equilibrium in which there's a return point (y), a set of symmetric endpoints $(y \pm (n-1)\Delta)$, and where the firms are a uniform discrete distribution on a set of points. We also want the nominal wage to be fixed in this equilibrium. There could be other equilibria, but for tractability, we will focus on an equilibrium of this kind.

Household's Problem Solving the houseshold's problem implies:

▷ **Intratemporal** condition: Since $U(C,H) = \log C - \gamma H$:

$$\frac{-U_H\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right)}{U_C\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right)} = \frac{W}{P_t\left(\mathbf{s}^t\right)}.$$
$$\Leftrightarrow \gamma C_t\left(\mathbf{s}^t\right) = \frac{W}{P_t\left(\mathbf{s}^t\right)}$$

which implies:

* Nominal expenditure P_tC_t is constant since $P_tC_t = W/\gamma$.

* When firms do not change their prices, real consumption stays constant.

▷ **Intertemporal** condition:

$$\frac{1}{C_t\left(\mathbf{s}^t\right)} = \beta\left(\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^{t+1}\right)}\right)\left(\frac{Q_0^t\left(\mathbf{s}^t\right)P_0^t\left(\mathbf{s}^t\right)}{Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)P_0^{t+1}\left(\mathbf{s}^{t+1}\right)}\right)\frac{1}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}$$

Combining with the intratemporal condition that $P_tC_t = P_{t+1}C_{t+1} = W/\gamma$, we have

$$1 = \beta\left(\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^{t+1}\right)}\right)\left(\frac{Q_0^t\left(\mathbf{s}^t\right)}{Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}\right)$$

Using these results, we obtain the following conclusions:

▷ **Monetary Policy**: The nominal interest rate

$$i_t\left(\mathbf{s}^t\right) \equiv \frac{Q_0^t\left(\mathbf{s}^t\right)}{\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t} Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}$$

Plugging into the EE:

$$\begin{aligned} i_t\left(\mathbf{s}^t\right) &= \left[\beta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}\frac{U_{C,t+1}\left(\mathbf{s}^{t+1}\right)}{U_{C,t}\left(\mathbf{s}^t\right)}\frac{P_t\left(\mathbf{s}^t\right)}{P_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]^{-1} \\ &= \left[\beta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}\right]^{-1} \qquad (\because)\ P_tC_t = W/\gamma \\ &= \frac{1}{\beta}\left[\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}\right]^{-1} = \frac{1}{\beta} \end{aligned}$$

i.e. the nominal interest rate policy consistent with fixed nominal wage is the one in which $i_t = 1/\beta$.

* If you want constant real consumption, you have $r_t\left(\mathbf{s}^t\right) = 1/\beta$ i.e. the real interest rate should equal $1/\beta$.

▷ **Demand Curve:** This can be obtained via

$$c_{j,t}\left(\mathbf{s}^t\right) = C_t\left(\mathbf{s}^t\right)\left(\frac{P_t\left(\mathbf{s}^t\right)}{p_{j,t}\left(\mathbf{s}^t\right)}\right)^{\eta} \equiv \Phi_t\left(p_{j,t},\mathbf{s}^t\right) \qquad \forall j\in[0,1]$$

* Given a sequence of aggregate shocks, we can find the prices that individual firms charge (given we know its initial conditions).
* In other models, the individual histories matter, but in this case only the aggregate shock histories matter.

Price Index Recall that the ideal price index was defined to be:

$$P_t := \left(\int_0^1 p_{j,t}^{1-\eta}dj\right)^{\frac{1}{1-\eta}}$$

Given our assumptions about the distribution of the firms and x which locates the lowest markup in the economy:

$$= \left(\frac{1}{n} \sum_{i=0}^{n-1} \left\{ e^{x+i\Delta} \frac{W}{z_t\left(\mathbf{s}^t\right)} \right\}^{1-\eta} \right)^{\frac{1}{1-\eta}}$$

Factoring out the terms that do not depend on i:

$$\underline{P_t\left(\mathbf{s}^t\right) = e^{x_t\left(\mathbf{s}^t\right)} \frac{W}{z_t\left(\mathbf{s}^t\right)} \left(\frac{1}{n} \sum_{i=0}^{n-1} e^{i\Delta(1-\eta)} \right)^{\frac{1}{1-\eta}}}$$

▷ The price level depends on the productivity shock and $\mathbf{s}^t$.

▷ Interpretation:

  * If we are in the intermediate region: e^x/z is the interesting part. e^x moves proportionally to the productivity shock so it will cancel the increase in z_t. Therefore, P_t does move.
  * If we are at the top and hit with positive productivity shock: then x doesn't move and z increases, so we get the price level moving downward and consumption increasing.
  * If we are at the bottom and hit with negative productivity shock: then x doesn't move and z decreases, so we get the price level moving upward and consumption decreasing.

We could derive this price index alternately from the following:

$$C_t\left(\mathbf{s}^t\right) = \left(\int_0^1 c_{j,t}\left(\mathbf{s}^t\right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}$$

Inefficiency from Price Dispersion Labor market clearing implies

$$\begin{aligned} \int_0^1 h_{j,t}\left(\mathbf{s}^t\right) dj &= \frac{1}{z_t\left(\mathbf{s}^t\right)} \int_0^1 c_{j,t}\left(\mathbf{s}^t\right) dj \\ &= \frac{C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta}}{z_t\left(\mathbf{s}^t\right)} \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{-\eta} dj \\ &= \frac{C_t\left(\mathbf{s}^t\right)}{z_t\left(\mathbf{s}^t\right)} \underbrace{\left\{ \left(\frac{1}{n} \sum_{i=0}^{n-1} e^{i\Delta(1-\eta)} \right)^{\frac{1}{1-\eta}} \right\}^{\eta} \left(\frac{1}{n} \sum_{i=0}^{n-1} e^{-i\Delta\eta} \right)}_{[A]} \end{aligned}$$

▷ If $n = 1$, then each of the bracket terms is 1.

▷ If $n > 1$, then we can use Jensen's inequality to prove that $[A]$ is strictly greater than 1. This number tells you the inefficiency caused by the price dispersion in this economy.

The social planner who doesn't care about price rigidity will want to produce equal amount of each good. Price rigidity gives different prices for different goods, which leads to different demand, which leads to different labor usage.

Inefficiency from Price Dispersion Note that C_t/z_t looks like $e^{-x_t(\mathbf{s}^t)}$, so we have

$$\int_0^1 h_{j,t}\left(\mathbf{s}^t\right) dj = \underbrace{\frac{C_t\left(\mathbf{s}^t\right)}{z_t\left(\mathbf{s}^t\right)}}_{=e^{-x_t(\mathbf{s}^t)}} \left\{ \left(\frac{1}{n} \sum_{i=0}^{n-1} e^{i\Delta(1-\eta)} \right)^{\frac{1}{1-\eta}} \right\}^{\eta} \left(\frac{1}{n} \sum_{i=0}^{n-1} e^{-i\Delta\eta} \right)$$

▷ If higher lower bound of markup (x_t): $\Rightarrow$ we will employ less in this economy.

▷ In the intermediate region, aggregate consumption doesn't change but you employ less workers. This is welfare-improving, since people don't like to work.

▷ If we are at the top and hit with productivity shock, it leads to a proportionate change in consumption and therefore we don't get any further changes in hours worked. The amount of production labor used is inversely proportional to e^x.

Firm's Problem The firm's problem is set up as:

$$\sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} Q_0^t\left(\mathbf{s}^t\right) \left(p_{j,t}\left(\mathbf{s}^t\right) c_{j,t}\left(\mathbf{s}^t\right) - \frac{W c_{j,t}\left(\mathbf{s}^t\right)}{z_t\left(\mathbf{s}^t\right)} - W\bar{H}\mathbb{I}\left\{p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right\} \right)$$

▷ Using the inter-temporal condition of the household,

$$1 = \beta \left(\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^{t+1}\right)} \right) \left(\frac{Q_0^t\left(\mathbf{s}^t\right)}{Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)} \right)$$

we can rewrite:

$$\sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \underline{\beta^t \Pi_t\left(\mathbf{s}^t\right)} \left(p_{j,t}\left(\mathbf{s}^t\right) c_{j,t}\left(\mathbf{s}^t\right) - \frac{W c_{j,t}\left(\mathbf{s}^t\right)}{z_t\left(\mathbf{s}^t\right)} - W\bar{H}\mathbb{I}\left\{p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right\} \right)$$

▷ Using the expression for $c_{j,t}\left(\mathbf{s}^t\right)$, we can further rewrite:

$$\sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right) \left(C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta} \left(p_{j,t}\left(\mathbf{s}^t\right)^{1-\eta} - \frac{W p_{j,t}\left(\mathbf{s}^t\right)^{-\eta}}{z_t\left(\mathbf{s}^t\right)} \right) - W\bar{H}\mathbb{I}\left\{p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right\} \right)$$

  * This is probably wrong. As z increases, we will get proportionally higher consumption and lower P.
  * So $C_t P_t^{\eta}$ will become smaller and smaller over time, and the adjustment cost term is getting dominant, so we need an additional term that also scales down the adjustment cost term.

▷ Prof. Shimer conjectures that adding this term solves the issue:

$$\sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right) \left(C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta} \left(p_{j,t}\left(\mathbf{s}^t\right)^{1-\eta} - \frac{W p_{j,t}\left(\mathbf{s}^t\right)^{-\eta}}{z_t\left(\mathbf{s}^t\right)} \right) - W\bar{H}\mathbb{I}\left\{p_{j,t}\left(\mathbf{s}^t\right) \neq p_{j,t-1}\left(\mathbf{s}^{t-1}\right)\right\} z_t\left(\mathbf{s}^t\right)^{1-\eta} \right)$$

Intuition

▷ If we are at the top of the shelf, $x_t = y$ and the price index is inversely proportional to z. But the stochastic process for C and P going forward – the relative growth of consumption and prices – only depends on the fact that the elevator is at the top. It does not depend on current level of consumption and productivity and prices.

▷ If we are at the middle of the shelf, the positive productivity shock has no impact on C and P. If we are at the top, it increases C and decreases P. The increase is the same for all cases in which we are at the top of the shelf.

This makes the problem difficul to solve fully. Thus a recursive formulation might be good to think about.

2.3.4 Recursive Formulation

We will define the state by x, the lowest markup in the economy. You can write out the problem as a function of your markup and the lowest markup in the economy which dictates the stochastic process in this economy.

2.3.5 Comments

▷ This economy is hard to work with, just as in any menu costs model.

▷ If the initial distribution of the firms is messed up, it will be really difficult to work with.

  * Suppose $n = 2$ and 2/3 of the firms charge a high price and 1/3 of the firms charge a low price. When they adjust, 2/3 of the firms change their price by Δ, it has a disproportionate effect on the aggregate price level.

▷ When firms change their prices, they do it not because of aggregate productivity shock but due to idiosyncratic shock (in reality, that is). There are all sorts of reasons why some firms are cutting prices and some firms are raising prices.

▷ It also depends on industries – for example, higher education leading to higher labor costs, leading to higher prices. Getting this out of the model with only an aggregate TFP shock is a futile task.

▷ Markup for the firm is very similar to the firm that has to decide periodically to build a new plant or scrap a new plant, where there is a fixed cost in adjusting its capital stock.

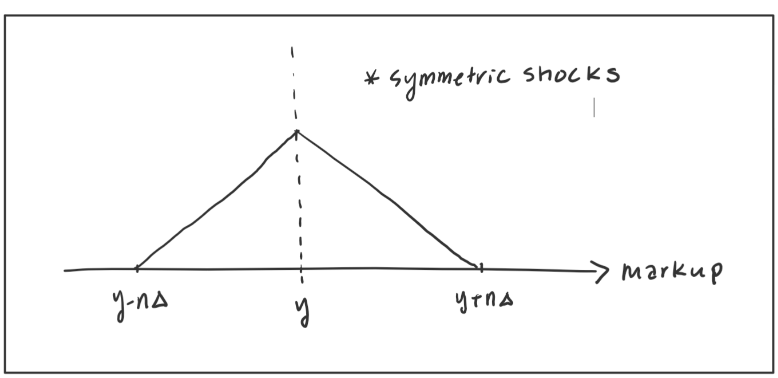
2.4 Golosov & Lucas Model (2007)

The paper by Golosov and Lucas is more complicated but conceptually it's easier to understand. The main modification is going to be replacing $z_t\left(\mathbf{s}^t\right)$ with a $z_{j,t}\left(\mathbf{s}^t\right)$. It is also in continuous time. Every point in time, some firms are hit by a positive productivity shock and some are hit by a negative productivity shock. This allows us to start from a degenerate markup distribution to fan out over time due to the idiosyncratic shocks – firms with positive productivity shocks show increase in markup and firms with negative productivity shocks show decrease in markup, until they decide to adjust their prices. This naturally gives rise to an ergodic distribution of markups.

Most of the mass will be in the center of the distribution (y) and decreasing mass towards the outskirts, i.e. a triangular distribution.

2.4.1 Ergodic Distribution of Markups

In the paper, the model generates an ergodic distribution of markups:



2.4.2 Monetary Policy Shock

They introduce a one-time unanticipated change to the nominal interest rate.

▷ They find that it does have real effects in the model, but the effects are incredibly short-lived (only for half a quarter).

 * If you have an aggregate shock, it will move around the markup distribution (for shocks to technology or money supply).
 * Take technology shocks, for example. It will move the whole triangular distribution to the right (positive shocks), forcing a bunch of firms on the right to adjust to the middle. Since the idiosyncratic shocks are orders of magnitude larger than aggregate shocks, it will make firms vulnerable to $y + n\Delta$ threshold adjust much more quickly than they would have with aggregate shocks. Thus we will return to the ergodic distribution relatively quickly.
 * The relative size of idiosyncratic shocks is microfounded in economic data.

▷ They also find that relative to Calvo model, the monetary policy shocks are much less persistent.

2.4.3 Sources of Cost of Price Adjustment

Another point they make is that menu costs are a better way of thinking about price adjustment than time-dependent pricing.

▷ We can think of decision to adjust prices in two ways:

 * One way is to think that there is a physical menu cost to adjusting prices (some amount of labor you have to use to change prices) and a deliberate decision on when to change the prices. In making these decisions, you are looking at the state of the world and making profit-maximizing decision each period in time. If you thought that the cost of adjusting prices is relabelling the prices, then this is a reasonable way of thinking about this.
 * Another source of the cost might be actually sitting down and figuring out which prices to charge. This requires information on competitor prices, a forecast of what future prices of P and C might look like.
 * Therefore, you have teams of people at places like Walmart where they think about what prices you should be charged for different goods. This will get you, at least most of the times, a periodic series of decisions to think about what the price should be. Thus, the cost here is not the physical labor of replacing costs but the cost of thinking about these prices.

▷ The alternate interpretation implies that you may change your price periodically.

 * One version is called Taylor pricing – John Taylor argued that there will be an annual meeting and they will decide on what the prices will be for the next year.
 * He talked about this in the context of labor, which is reasonable. A lot of workers have annual performance reviews and have adjustments.
 * For other things, it's probably less regular than that. Taylor pricing turns out to be a difficult problem to deal with, since you have a distribution of prices that depends on when it's going to be reset. So the clever assumption of Calvo is to replace this deterministic pricing cycle with the assumption that you have a probability $1/12$ of changing your price next month. This parameter for how frequently you can change is called the Calvo Parameter, which will be a measure of how flexibly you can change the prices.

▷ There are some arguments that menu costs (physical labor associated) are the only right way to think about this, but Prof. Shimer thinks that the alternate interpretation is also as valid. People meet regularly to change their prices. It is a little bit of assumption (in the Calvo model) since the probability of price adjusting does depend on how stale the price is.

2.5 Calvo Model (1983)

The key assumption is that with probability $1-\theta_j$, firm j can change its price in period t, i.i.d. across $j, t, \mathbf{s}^t$. We can show how we reduce the number of state variables.

2.5.1 Setup

The firm chooses p to maximize

$$\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t} \theta^{t'-t} Q_0^{t'}\left(\mathbf{s}^{t'}\right)\left(p c_{t'}\left(p ; \mathbf{s}^{t'}\right)-W_{t'}\left(\mathbf{s}^{t'}\right) \frac{c_{t'}\left(p ; \mathbf{s}^t\right)}{z_{t'}\left(\mathbf{s}^{t'}\right)}\right)$$

where $c_{t'}\left(p ; \mathbf{s}^{t'}\right)=C_t\left(\mathbf{s}^t\right)\left(P_t\left(\mathbf{s}^t\right) / p\right)^\eta$.

▷ The demand function is static, which means that today's price does not affect future period optimization. If the demand function was dynamic (i.e. depends on customer base) then we will not have the same setup.

▷ Thus, we have $\theta^{t'-t}$ term which allows us to only consider future profits that depends on the price we set today. In the full optimization problem, there are some continuation value terms, but since they are not relevant to the firm's choice, it does not matter.

2.5.2 Solving for Equilibrium

The first-order condition for p:

$$\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t} \theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\left((1-\eta) p^{-\eta}+\eta \frac{W_{t'}\left(\mathbf{s}^{t'}\right)}{z_{t'}\left(\mathbf{s}^{t'}\right)} p^{-\eta-1}\right)=0$$

▷ Dividing each side by $p^{-\eta-1}$:

$$\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t} \theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\left((1-\eta) p+\eta \frac{W_{t'}\left(\mathbf{s}^{t'}\right)}{z_{t'}\left(\mathbf{s}^{t'}\right)}\right)=0$$

which yields:

$$p=\left(\frac{\eta}{\eta-1}\right) \frac{\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\left(\frac{W_{t'}\left(\mathbf{s}^{t'}\right)}{z_{t'}\left(\mathbf{s}^{t'}\right)}\right)\right\}}{\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\right\}} \equiv P_t^*\left(\mathbf{s}^t\right)$$

which is the profit-maximizing price for the firm.

▷ All firms adjusting their price at time t is $P_t^*\left(\mathbf{s}^t\right)$. This is forward-looking.

* You weight the future marginal cost of production – $W_{t'}/z_{t'}$ – by some weight, and multiply it by the markup.

2.5.3 Price Index

The ideal price index

$$P_t\left(\mathbf{s}^t\right)^{1-\eta} \equiv \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{1-\eta} dj = \theta P_{t-1}\left(\mathbf{s}^{t-1}\right)^{1-\eta} + (1-\theta) P_t^*\left(\mathbf{s}^t\right)^{1-\eta}$$

i.e. the weighted average of the price index of the previous period and the ideal price in the current period.

▷ This is quite intuitive.

2.5.4 Market Clearing

The labor market clearing condition:

$$H_t\left(\mathbf{s}^t\right) = \int_0^1 h_{j,t}\left(\mathbf{s}^t\right) dj = \frac{C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta}}{z_t\left(\mathbf{s}^t\right)} \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{-\eta} dj$$

We also define a "Production Price Index":

$$\tilde{P}_t\left(\mathbf{s}^t\right) \equiv \left(\int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{-\eta} dj\right)^{-\frac{1}{\eta}}$$

which satisfies the following recursion:

$$\tilde{P}_t\left(\mathbf{s}^t\right)^{-\eta} \equiv \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{-\eta} dj = \theta \tilde{P}_{t-1}\left(\mathbf{s}^{t-1}\right)^{-\eta} + (1-\theta) P_t^*\left(\mathbf{s}^t\right)^{-\eta}$$

This implies that

$$C_t\left(\mathbf{s}^t\right) = z_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right) \left(\frac{\tilde{P}_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)}\right)^{\eta}$$

▷ If $\tilde{P}_t > P_t$, then we are getting more consumption than is produced, but using Jensen's inequality, we can show that $\tilde{P}_t\left(\mathbf{s}^t\right) \leq P_t\left(\mathbf{s}^t\right)$.

▷ This is just a statement about if prices of different good are different, then we will allocate different amounts of consumption to different goods. The maximum utility is to consume equal amounts of goods.

▷ Sticky prices lead to different prices, which lead to $\tilde{P}_t < P_t$ i.e. strict inequality.

2.5.5 Combining Equations

We obtain the following system of equations (underdetermined):

$$
\begin{aligned}
[HH1]: & C_t\left(\mathbf{s}^t\right) v'\left(H_t\left(\mathbf{s}^t\right)\right)=\frac{W_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)} \\
[HH2]: & \frac{1}{C_t\left(\mathbf{s}^t\right)}=\beta \frac{\Pi_0^{t+1}\left(\mathbf{s}^{t+1}\right) Q_0^t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)}{\Pi_0^t\left(\mathbf{s}^t\right) Q_0^{t+1}\left(\mathbf{s}^t\right) P_{t+1}\left(\mathbf{s}^{t+1}\right) C_{t+1}\left(\mathbf{s}^{t+1}\right)} \\
[3]: & C_t\left(\mathbf{s}^t\right)=z_t\left(\mathbf{s}^t\right) H_t\left(\mathbf{s}^t\right)\left(\frac{\tilde{P}_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)}\right)^\eta \\
[4]: & P_t\left(\mathbf{s}^t\right)^{1-\eta} \equiv \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{1-\eta} dj=\theta P_{t-1}\left(\mathbf{s}^{t-1}\right)^{1-\eta}+(1-\theta) P_t^*\left(\mathbf{s}^t\right)^{1-\eta} \\
[5]: & \tilde{P}_t\left(\mathbf{s}^t\right)^{-\eta} \equiv \int_0^1 p_{j,t}\left(\mathbf{s}^t\right)^{-\eta} dj=\theta \tilde{P}_{t-1}\left(\mathbf{s}^{t-1}\right)^{-\eta}+(1-\theta) P_t^*\left(\mathbf{s}^t\right)^{-\eta} \\
[6]: & P_t^*\left(\mathbf{s}^t\right)=\left(\frac{\eta}{\eta-1}\right) \frac{\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\left(\frac{W_{t'}\left(\mathbf{s}^{t'}\right)}{z_{t'}\left(\mathbf{s}^{t'}\right)}\right)\right\}}{\sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'} Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta\right\}}
\end{aligned}
$$

with seven unknowns: three price indices, wage (W), consumption (C), aggregate hours (H) and intertemporal prices (Q). This means we are short one equation.

▷ We can back out the **nominal interest rate** through

$$
i_t\left(\mathbf{s}^t\right)=\frac{Q_0^t\left(\mathbf{s}^t\right)}{\sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}
$$

So we can look for a specific path of nominal interest rate that can get us the seventh equation.

We want to express [6] as a recursive form.

2.5.6 Obtaining a Recursive Formulation

We want to obtain an expression of the following form:

$$
P_t^*\left(\mathbf{s}^t\right)=\frac{\eta}{\eta-1} \frac{m_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)}{d_t\left(\mathbf{s}^t\right)}
$$

where $m_t\left(\mathbf{s}^t\right)$ is the real marginal cost, appropriately discounted while prices are sticky:

$$
\begin{aligned}
m_t\left(\mathbf{s}^t\right) & \equiv \sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'-t} \frac{Q_0^{t'}\left(\mathbf{s}^{t'}\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^{\eta+1}}{Q_0^t\left(\mathbf{s}^t\right) C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta+1}} \frac{W_{t'}\left(\mathbf{s}^{t'}\right)}{z_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)}\right\} \\
d_t\left(\mathbf{s}^t\right) & \equiv \sum_{t'=t}^{\infty} \sum_{\mathbf{s}^{t'} \succ \mathbf{s}^t}\left\{\theta^{t'-t} \frac{Q_0^{t'}\left(\mathbf{s}^t\right) C_{t'}\left(\mathbf{s}^{t'}\right) P_{t'}\left(\mathbf{s}^{t'}\right)^\eta}{Q_0^t\left(\mathbf{s}^t\right) C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^\eta}\right\}
\end{aligned}
$$

To get us into the recursive form, write;

$$
\begin{aligned}
m_t\left(\mathbf{s}^t\right) & =\frac{W_t\left(\mathbf{s}^t\right)}{z_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)}+\theta \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \frac{Q_0^{t+1}\left(\mathbf{s}^{t+1}\right) C_{t+1}\left(\mathbf{s}^{t+1}\right) P_{t+1}\left(\mathbf{s}^{t+1}\right)^{\eta+1}}{Q_0^t\left(\mathbf{s}^t\right) C_t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^{\eta+1}} m_{t+1}\left(\mathbf{s}^{t+1}\right) \\
d_t\left(\mathbf{s}^t\right) & =1+\theta \sum_{\mathbf{s}^{t+!} \succ \mathbf{s}^t} \frac{Q_0^{t+1}\left(\mathbf{s}^{t+1}\right) C_{t+1}\left(\mathbf{s}^{t+1}\right) P_{t+1}\left(\mathbf{s}^{t+1}\right)^\eta}{Q_0^t\left(\mathbf{s}^t\right) C_0^t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)^\eta} d_{t+1}\left(\mathbf{s}^{t+1}\right)
\end{aligned}
$$

2.5.7 Adding Inflation

Define inflation i.e. the gross increase in the price level:

$$\pi_t\left(\mathbf{s}^t\right) = \frac{P_t\left(\mathbf{s}^t\right)}{P_{t+1}\left(\mathbf{s}^{t+1}\right)}$$

and define a measure of productive inefficiency:

$$x_t\left(\mathbf{s}^t\right) = \frac{\tilde{P}_t\left(\mathbf{s}^t\right)}{P_t\left(\mathbf{s}^t\right)}$$

This will allow us to simplify the system.

2.5.8 Conclusion

In the end (after much algebra):

$$[1] : \left\{x_{t+1}\left(\mathbf{s}^{t+1}\right)\pi_{t+1}\left(\mathbf{s}^{t+1}\right)\right\}^{-\eta} = \theta\left\{x_t\left(\mathbf{s}^t\right)\right\}^{-\eta} + (1-\theta)\left(\frac{\left\{\pi_{t+1}\left(\mathbf{s}^{t+1}\right)\right\}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{\eta}{\eta-1}}$$

$$[2] : m_t\left(\mathbf{s}^t\right)\pi_t\left(\mathbf{s}^t\right)\left(\frac{\pi_{t+1}\left(\mathbf{s}^{t+1}\right)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}$$

$$= \frac{\eta-1}{\eta} + \beta\theta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_0^{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_0^t\left(\mathbf{s}^t\right)}m_{t+1}\left(\mathbf{s}^{t+1}\right)\pi_{t+1}\left(\mathbf{s}^{t+1}\right)\left(\frac{\pi_{t+1}\left(\mathbf{s}^{t+1}\right)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}$$

$$[3] : m_t\left(\mathbf{s}^t\right) - H_t\left(\mathbf{s}^t\right)\left\{x_t\left(\mathbf{s}^t\right)\right\}^{\eta}v'\left(H_t\left(\mathbf{s}^t\right)\right) = \theta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\beta\left[\frac{\Pi_0^{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_0^t\left(\mathbf{s}^t\right)}\right]\left\{\pi_{t+1}\left(\mathbf{s}^{t+1}\right)\right\}^{\eta}m_{t+1}\left(\mathbf{s}^{t+1}\right)$$

- ▷ Four variables: x, π, m, and H (instead of seven equations as before)
- ▷ We are still short one equation since we don't have any monetary policy. We can pick a path for π and back out all the other variables.
- ▷ $\pi = 1$ gives us $x = 1$; other values of π gives us $x < 1$. x is equivalently a measure of price dispersion.

2.5.9 Reducing State Variables

Initially, we had two state variables: P_t and $\tilde{P}_t$.

- ▷ We have a continuum of firms that may have a stagnant price coming from the past. So you may think that we have to keep the track of a distribution of prices.

We further reduced the state variables to just one (x_t) thanks to homogeneity.

- ▷ If I double all the firms' prices, the equilibrium will be just doubling the wage and each of the price indices. Nothing else changes. So we can normalize everything using inflation, and we only need x to be $\tilde{P}/P$.

We can get rid of this state variable through log-linearization (see problem set 4)

- ▷ We obtain the distortion in the price indices follows $\hat{x}_{t+1} = \theta\hat{x}_t$ i.e. the distortion converges monotonically to its steady-state value.

- ▷ If you log-linearized around other steady states, you wouldn't get a nice form like this.
- ▷ If there are no shocks, all firms will converge to the same price.
- ▷ The small deviations are all second-order; the first-order tells you that you would want to equate h_j across the different goods. The difference in prices only has a second-order cost (effect) in output, so if you log-linearize, the distortion disppears.

2.5.10 Three Equation Model

At $\hat{x}_t = 0$, we have the following equation [1]:

$$[1] : \hat{\pi}_t = \kappa \hat{H}_t + \beta \mathbb{E}_t \left[\hat{\pi}_{t+1}\right]$$

where

$$\kappa \equiv \left(\frac{(1-\theta)(1-\theta\beta)}{\theta}\right)\left(1+\frac{1}{\epsilon}\right) > 0, \quad \epsilon \equiv \frac{v'\left(\bar{H}\right)}{\bar{H}v''\left(\bar{H}\right)} > 0$$

- ▷ Inflation deviation today is equal to labor deviation and expected inflation deviation
- ▷ **Equation** [1] **is called New Keynesian Phillips Curve (NKPC)**. Phillips originally wrote about static inflation and had output gap. Here we have something different – When firms are thinking about adjusting prices, they are thinking about (1) statically (given marginal cost, what's the price to set today, and higher labor will induce higher prices and higher labor) and (2) dynamically (they are forward looking and if they expect higher inflation.

Furthermore, we can derive a second important equation.

- ▷ Starting with the household EE:[2]

$$\frac{1}{C_t\left(\mathbf{s}^t\right)} = \beta \frac{\Pi_0^{t+1}\left(\mathbf{s}^{t+1}\right) Q_0^t\left(\mathbf{s}^t\right) P_t\left(\mathbf{s}^t\right)}{\Pi_0^t\left(\mathbf{s}^t\right) Q_0^{t+1}\left(\mathbf{s}^t\right) P_{t+1}\left(\mathbf{s}^{t+1}\right) C_{t+1}\left(\mathbf{s}^{t+1}\right)}$$

  and recalling the definition of inflation

$$i_t\left(\mathbf{s}^t\right) = \frac{Q_0^t\left(\mathbf{s}^t\right)}{\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t} Q_0^{t+1}\left(\mathbf{s}^{t+1}\right)}$$

- ▷ Then we can combine these two equations to obtain:

$$i_t\left(\mathbf{s}^t\right) = \frac{1}{\beta \mathbb{E}_t\left[\frac{P_t}{P_{t-1}}\frac{C_t}{C_{t+1}}\right]}$$

- ▷ Furthermore, since $C_t = z_t H_t x_t^{\eta}$, we can plug this into the above expression and solve for zero inflation steady state $\hat{x}_t = \hat{x}_{t+1} = 0$ and $H_t = H_{t+1} = \bar{H}$:

$$[2] : \hat{H}_t = \mathbb{E}_t\left[\hat{z}_{t+1}\right] - \hat{z}_t - \hat{i}_t + \mathbb{E}_t\left[\hat{\pi}_{t+1}\right] + \mathbb{E}_t\left[\hat{H}_{t+1}\right]$$

- ▷ **Equation** [2] **is called the Dynamic IS Curve**. This is not the same IS curve we saw in undergrad micro class.

[2]Note that we don't need the sum on the RHS. Certainly we can have it, but we don't need it because we are assuming complete markets.

Equation [1] and [2], conditional on path of interest rates and exogenous shocks, should determine the path of inflation and hours worked, relative to the zero inflation steady state values. We need one more equation to tell us what $\hat{i}_t$ is.

▷ We add an extra equation:

$$[3] : \hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_H \hat{H}_t + \nu_t$$

  * **Equation** [3] **is known as the Taylor Rule.**
  * Government looks at current inflation, employments, and shocks (ex. Trump tweeting) to determine the nominal interest rate.

Note on the Taylor Rule The original paper by Taylor was an empirical description of monetary policy in the United States.

▷ It was originally a regression of nominal interest rates on inflation or employment. This served as a useful summary of the Fed policy through these parameters.

▷ He also noticed that these two parameters were not constant over time – in particular, until around 1970s, ϕ_π was positive and less than 1, but it shifted when Volcker came in and ϕ_π became greater than 1. The magnitude of this parameter plays a big role in the dynamics.

Equation [3] became a description, rather than a prescription of the Fed policy model. Through the introduction of these three-equation model, the Taylor rule became prescriptive.

▷ The last piece of the Taylor rule is that $\phi_\pi > 1$. (Taylor has 1.5 on his business card).

What is ν_t? It can represent the error in the forecast. In the model, it should be a surprise to the participants in the model.

▷ People call this the monetary policy shock. One thing you can look at is the futures market for the federal funds rate (ver active market) and around the time of the announcement of monetary policy, there are changes in the fed funds futures market. These changes are viewed as surprise information coming out of the Fed.

  * For example, Jay Powell eats a bad shrimp the night before and cannot voice his opinion during the FOMC meeting.

▷ An alternative explanation is the following. What's happening is that Fed has a large group of forecasters, and maybe one thing happens is the Fed understand the economy better than other people, so setting nominal interest rates is somewhat forward looking. In this case, ν_t is not a shock per say but something that's left out of the model.

▷ Most people think that $\text{Var}[\nu_t]$ is small. They also think that it will be useful for understand the economy.

2.5.11 Solving the Three-Equation Model

Recall that we had

$$\begin{aligned}
[1] &: \hat{\pi}_t = \kappa \hat{H}_t + \beta \mathbb{E}_t[\hat{\pi}_{t+1}] \\
[2] &: \hat{H}_t = \mathbb{E}_t[\hat{z}_{t+1}] - \hat{z}_t - \hat{i}_t + \mathbb{E}_t[\hat{\pi}_{t+1}] + \mathbb{E}_t\left[\hat{H}_{t+1}\right] \\
[3] &: \hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_H \hat{H}_t + \nu_t
\end{aligned}$$

Plugging [3] into [2]:

$$(1 + \phi_H)\hat{H}_t = \mathbb{E}_t[\hat{z}_{t+1}] - \hat{z}_t - \phi_\pi \hat{\pi}_t + \mathbb{E}_t[\hat{\pi}_{t+1}] + \mathbb{E}_t\left[\hat{H}_{t+1}\right] - \nu_t$$

No shocks Let us shutdown the exogenous shocks in the model and set $\hat{z}_t = \upsilon_t = 0$ for all t.

- ▷ What are the state variables in this problem? None. The only state variable was x_t and we set $\hat{x}_t = 0$.
  - * Now suppose we solved the system of equations and found that both eigenvalues lay inside the unit circle. What does this mean?
    - Every initial condition would converge to the steady state.
  - * What if one is stable and one is not?
    - Then there is a one-dimensional subset which would converge to the steady state, and the rest would blow up. This implies that there is a large set of initial conditions such that the system would converge to the steady state (and zero inflation and so on). All the dynamics are well described by the log-linearization.
  - * What if both lie outside the unit circle?
    - Then there is only a unique set of initial values that would keep us towards the steady state of this model. This is referred to as local determinacy.

Therefore, we want both eigenvalues to be greater than 1. This is equivalent to having the following condition:

$$(\phi_\pi - 1)\,\kappa + \phi_H\,(1-\beta) > 0$$

- ▷ If $\phi_\pi > 1$ and $\phi_H > 0$, then this condition is automatically satisfied, which means that there is a unique set of initial values that wouldn't blow up the system.
- ▷ We want to talk about transversality to rule out explosive solutions to the model, but there is none here.
- ▷ Taylor argues that when US inflation was high, $\phi_\Pi \approx 0.5$ and in the Volcker period, inflation was brought down with $\phi_\Pi \approx 1.5$ – we went from a period where a lot of things could happen to a subset of things that could happen.
- ▷ $\phi_\pi < 1$ implies that 1% in inflation represents a cut in the real interest rate (less than 1% in nominal interest rate).

With monetary policy shocks, υ_t We assume that $\phi_\Pi > 1$ and that

$$\mathbb{E}_t\left[\upsilon_{t+1}\right] = \rho_\upsilon \upsilon_t,\ \hat{z}_t = 0,\ \forall t.$$

Then we get the following solution:

$$\hat{\pi}_t = -\frac{\kappa\nu_t}{\beta\left(\lambda_1-\rho_\nu\right)\left(\lambda_2-\rho_\nu\right)}, \qquad \hat{H}_t = -\frac{\left(1-\beta\rho_\nu\right)\nu_t}{\beta\left(\lambda_1-\rho_\nu\right)\left(\lambda_2-\rho_\nu\right)}$$

where $\lambda_2 > \lambda_1 > 1$ solves

$$\beta\lambda^2 - \lambda\left(1+\beta\left(1+\phi_H\right)+\kappa\right) + \left(1+\phi_H+\kappa\phi_\pi\right) = 0$$

- ▷ To see where we got this equation, recall our system of equations after plugging [3] into [2] :

$$\begin{aligned}\left(1+\phi_H\right)\hat{H}_t &= \mathbb{E}_t\left[\hat{z}_{t+1}\right] - \hat{z}_t - \phi_\pi\hat{\pi}_t + \mathbb{E}_t\left[\hat{\pi}_{t+1}\right] + \mathbb{E}_t\left[\hat{H}_{t+1}\right] - \nu_t \\ \hat{\pi}_t &= \kappa\hat{H}_t + \beta\mathbb{E}_t\left[\hat{\pi}_{t+1}\right]\end{aligned}$$

Then we can write the system as:

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \beta & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{H}_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & \kappa \\ -\phi_\Pi & -(1+\phi_H) \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{H}_t \end{bmatrix} \\
&\equiv \mathbf{G}^*_{t+1}\hat{\mathbf{x}}_{t+1} + \mathbf{G}^*_t \hat{\mathbf{x}}_t \\
\Rightarrow \hat{\mathbf{x}}_{t+1} &= -\left(\mathbf{G}^*_{t+1}\right)^{-1} \mathbf{G}^*_t \hat{\mathbf{x}}_t \\
&\equiv \mathbf{M}\hat{\mathbf{x}}_t \\
\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{H}_{t+1} \end{bmatrix} &= \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\beta} + \phi_\Pi & 1 + \frac{\kappa}{\beta} + \phi_H \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{H}_t \end{bmatrix}.
\end{aligned}
$$

▷ The characteristic polynomial is given as

$$
\begin{aligned}
0 &= \left(\frac{1}{\beta} - \lambda\right)\left(1 + \frac{\kappa}{\beta} + \phi_H - \lambda\right) - \left(-\frac{\kappa}{\beta}\right)\left(-\frac{1}{\beta} + \phi_\Pi\right) \\
&= (1 - \beta\lambda)\left(1 + \frac{\kappa}{\beta} + \phi_H - \lambda\right) + (\kappa)\left(\phi_\Pi - \frac{1}{\beta}\right) \\
&= 1 + \frac{\kappa}{\beta} + \phi_H - \lambda - \beta\lambda - \kappa\lambda - \beta\lambda\phi_H + \beta\lambda^2 + \kappa\phi_\Pi - \frac{\kappa}{\beta} \\
&= \beta\lambda^2 - (1 + \beta(1+\phi_H) + \kappa)\lambda + (1 + \phi_H + \kappa\phi_\Pi).
\end{aligned}
$$

which yields the desired equation.

Using guess-and-verify, we can solve for the actual expressions:

$$
\begin{aligned}
\hat{H}_t &= \frac{1 - \beta\rho_\upsilon}{\kappa(\rho_\upsilon - \phi_\Pi) - (1 - \beta\rho_\upsilon)(1 + \phi_H - \rho_\upsilon)}\upsilon_t \\
\hat{\Pi}_t &= \frac{\kappa}{\kappa(\rho_\upsilon - \phi_\Pi) - (1 - \beta\rho_\upsilon)(1 + \phi_H - \rho_\upsilon)}\upsilon_t,
\end{aligned}
$$

where both coefficients are negative.

▷ This means that following a positive monetary policy shock, both employment and inflation falls.

▷ That $\hat{H}_t$ is affected also implies that monetary policy has a real effect on the economy (which directly affects $\hat{Y}_t$).

▷ Why does monetary policy have a real effect? Suppose that we think that after some period T, there will be no monetary shocks; i.e. $\upsilon_t = 0$ for all $t > T$. We think of solving this model backwards. The impact of monetary shock, υ_t affects nominal interest and it ends up affecting real interest rate. High value of υ_t (contractionary monetary policy), raises nominal interest and raises real interest. After period T, we must be in the flexible price case and so $\hat{H}_t = 0$ and $H = H^*$.

▷ So when each consumer sees that interest rate will be higher from today to tomorrow, the consumption Euler equation tells us that consumers would want steeply upward consumption profile. But since the end point cannot move, in order to achieve this, consumers must cut back consumption today and "save". Here, since there is no capital, saving is only possible if consumers demand less today. This means that less labour is required and consumers feel poor. As hours worked falls, wages decreases until the intertemporal condition is satisfied.

▷ Note that if ρ_υ small, then high υ_t implies high $\hat{\imath}_{t+1}$; but if ρ_υ close to one, then high υ_t implies low $\hat{\imath}_{t+1}$.

3 Unemployment

Consider the following setup:

▷ w: constant payoff until terminal date

▷ Terminal date t is exponentially distributed with parameter χ

  * The probability of termination prior to t is $F(t) = 1 - e^{-\chi t}$ i.e. the cdf of the exponential distribution.
  * This implies that the hazard rate is constant:

$$h(t) \equiv \frac{F'(t)}{1 - F(t)} = \chi$$

▷ r: discount rate

▷ $\bar{V}$: termination payoff

Define V to be the expected present value of facing this environment:

$$V = \int_0^\infty e^{-rt} e^{-\chi t} w dt + \int_0^\infty e^{-rt} \chi e^{-\chi t} \bar{V} dt$$

▷ $F'(t) = \chi e^{-\chi t}$ is the probability that you are terminated at date t

▷ Explicitly solving the integral since t only appears in the exponent:

$$V = \frac{w + \chi \bar{V}}{r + \chi}$$

which yields:

$$\underline{rV = w + \chi\left(\bar{V} - V\right)}$$

▷ V is the present value of the asset, and rV is the average payoff to the asset. The flow payoff is a sum of two terms:

  * w is the "dividend" from this asset
  * At a constant rate χ, you get a capital gain of change from $V \to \bar{V}$.

3.1 Pissarides (2000)

Pissarides (2000) introduces two major simplifications when compared to real busienss cycle models.

1. Workers are risk-neutral (thereby getting around consumption/savings problem)

2. There is no capital in this model; output is produced using only labor.

On top of this, we add a notion of unemployment.

3.1.1 Setup

We work in continuous time. Consider the following notation:

- ▷ u: number of those unemployed
- ▷ v: number of vacancies
- ▷ $m(u, v)$: matching function, a reduced-form representation of how firms find workers and how workers find jobs; CRS.
- ▷ $\theta = v/u$: measure of market tightness – The situation where $\theta > 1 \Leftrightarrow v > u$ is preferred by the worker whereas $\theta < 1 \Leftrightarrow v < u$ is preferred by the firm; i.e. if market is tight for one side of the market, then it is relaxed for the other side of the market.

Using this notation, we can denote the **rate at which the unemployed finds a job** is

$$\frac{m(v, u)}{u} = m\left(1, \frac{v}{u}\right) = m(1, \theta) := f(\theta).$$

where we used the property that m is CRS. f is increasing in θ.

- ▷ This shows an important deviation from the neoclassical models—the rate at which a worker finds a job depends on how many other workers are seeking to find a job (through θ).
- ▷ That is, there is a congestion externality in the model.

Analogously, we can define the **rate at which vacancies are filled** is

$$\frac{m(v, u)}{v} = m\left(\frac{1}{\theta}, 1\right) := \mu(\theta)$$

where μ is decreasing in θ since $\mu(\theta) = f(\theta)/\theta$.

- ▷ There is a congestion externality on this side of the market too—the rate at which a firm fills a vacancy depends on how many other firms are seeking to fill positions.

To rule our corner solutions, we assume:

$$\begin{aligned} f(0) = 0, &\quad \lim_{\theta \to \infty} \mu(\theta) = 0 \\ \mu(0) = 0, &\quad \lim_{\theta \to \infty} f(\theta) = 0 \end{aligned}$$

Matching Function The matching function gives the rate (flow) of new matches per unit of time (in discrete time, m would be the number of job matches). If there are no frictions (e.g. geographical, skill constraints) in the economy, then

$$m(u, v) = \min\{u, v\}.$$

However, we assume instead there are frictions and impose the following properties on the matching function:

- ▷ m is increasing
- ▷ m exhibits constant returns to scale (CRS) – The CRS assumption means that, with respect to any one factor (i.e. u or v), there is diminishing returns. That is, if you double the number of job vacancies holding fixed the number of unemployed workers, then job matches less than double; i.e. the rate of matches falls. This reflects the fact that it is harder for each firm to hire workers when more firms are trying to recruit workers.

3.1.2 Value Functions

Workers We define a value function for each state.

- ▷ Workers have two "states": employed and unemployed. Denote them V^e and V^u respectively.
- ▷ If the worker is **unemployed**, and, at rate $f(\theta)$, he is matched with a job and earns some $w = w^*$, where w^* denotes the equilibrium wage rate. The value function for an unemployed worker is therefore

  $$rV^u = 0 + f(\theta)\left(V^e(w^*) - V^u\right)$$

  The term $V^e(w^*) - V^u$ is the net benefit from changing the state from being unemployed to employed. $-V^u$ represents the opportunity cost of becoming employed.
- ▷ If, instead, the worker is **employed** at wage rate w, then he receives this wage, w while suffering disutility of work γ and at rate χ, he loses the job. The "gain" from losing a job is given by $V^u - V^e(w)$. Thus, the value function for an employed worker is

  $$rV^e(w) = w - \gamma + \chi\left(V^u - V^e(w)\right)$$

  where $-V^e(w)$ represents the opportunity cost of becoming unemployed.

Firms Given the assumption that firms have CRS technology using only labor, we can analyse the firm's decision to hire job-by-job. Every worker will produce z units of output, and we assume $z > \gamma$. There is cost c for each job vacancy.

- ▷ There are two "states" for the firms: with one vacancy (v) or with the vacancy filled (f). Let V^v and V^f be the value function in each state respectively.
- ▷ Unlike workers, firms do not have an opportunity cost of hiring. This is because firms have no upper bound on how many workers to hire given the assumption of constant returns to scale/free entry.
- ▷ If the **firm has the vacancy filled** for job j, the profit is $z - w$. At rate χ, the worker is fired (remember, firing happens for exogenous reasons in the model), in which case the firm's value is reduced by $V^j(w)$. Thus, the value function of a firm with no vacancy is given by:

  $$[1] : rV^j(w) = z - w - \chi V^j(w)$$

  $$[2] : rV^j(w) = z - w - \chi\left[V^v - V^j(w)\right]$$
- ▷ If the **firm has vacancy**, then it does not earn z but must pay a cost of c to post a vacancy. At rate $\mu(\theta)$, the firm finds a worker, which gives an additional value of $V^j(w^*)$. Hence, the value function of a firm with vacancy is

  $$rV^v = -c + \mu(\theta)\left[V^j(w^*) - V^v\right]$$
- ▷ We assume *free entry of vacancies*. This means that, if $V^v > 0$, then firms will continue to post vacancies. In equilibrium, it must be that

  $$V^v = 0$$

  which implies that

  $$c = \mu(\theta) V^j(w)$$
- ▷ Note that we can aggregate these at the firm level to, for example, compare them against the stock market wealth.

In sum, we have the following five equations:

$$\begin{aligned}
&[1] : rV^u = 0 + f(\theta)(V^e(w^*) - V^u) \\
&[2] : rV^e(w) = w - \gamma + \chi(V^u - V^e(w)) \\
&[3] : rV^j(w) = z - w - \chi V(w) \\
&[4] : rV^v = -c + \mu(\theta)\left[V^j(w^*) - V^v\right] \\
&[5] : V^v = 0
\end{aligned}$$

but six unknowns. Thus, we need an additional equation.

3.1.3 Pinning down w^*

To make the problem nontrivial, we assume that $z > \gamma$. For any wage level $w \in (z, \gamma)$, the worker is willing to work, and the firm is willing to hire; i.e. the value functions above cannot pin down the wage level. This is a consequence of having search costs in the labour market so that the labour market is no longer competitive.

▷ In short, both workers and firms have some bargaining power—for firms, if workers quit, then it may take several periods and incurs cost in filing the vacancy, and similarly, for workers, if they are fired, it may take several periods and loss in income to find a new job.

▷ Another way to think about this is the firms have a range of wages that they are willing to offer, and workers also have a range of wages they are willing to offer. The intersection contains multiple values, so the standard argument cannot pin down the value of w^*.

Nash Bargaining To pin down the equilibrium wage, w^*, we adopt the asymmetric Nash bargaining solution

$$w^* \in \arg\max_w \left(V^e(w) - V^u\right)^{\phi}\left(V^j(w) - 0\right)^{1-\phi}$$

which introduces an additional parameter ϕ.

▷ $V^e(w) - V^u$ is the net surplus for the worker from working (relative to his threat point or outside option);

▷ $V^j(w) - 0$ is the firm's net surplus from hiring a worker (relative to its threat point of zero profits);

▷ $\phi \in (0, 1)$ is a parameter that relates to the worker's relative bargaining power over firms.

  * $0 < \phi < 1$ implies that both terms must be positive i.e. agreement is better than the have the threat materialize.
  * If $\phi = 1/2$, then firms an workers have equal bargaining power.
  * If $\phi \to 1$, then the solution would involve setting $V^e(w) - V^u$ as high as possible while setting $V^j(w)$ small (non-negativity of $V^j(w)$).

Before solving for w^*, we can re-write the Nash bargaining problem as

$$\begin{aligned}
&\arg\max_w \left(\frac{w - \gamma - rV^u}{r + x}\right)^{\phi}\left(\frac{z - w}{r + x}\right)^{1-\phi} \\
\equiv &\arg\max_w \phi \ln(w - \gamma - rV^u) + (1 - \phi)\ln(z - w) - \ln(r + x).
\end{aligned}$$

since

$$\begin{aligned} rV^e(w) &= w - \gamma + \chi(V^u - V^e(w)) \\ \Rightarrow rV^e(w) - rV^u &= w - \gamma + \chi(V^u - V^e(w)) - rV^u \\ \Rightarrow V^e(w) - V^u &= \frac{w - \gamma - rV^u}{r + \chi}. \end{aligned}$$

and

$$\begin{aligned} rV^j(w) &= z - w - xV^j(w) \\ \Rightarrow V^j(w) &= \frac{z - w}{w + x} \end{aligned}$$

The first-order condition implies

$$w^* = \phi z + (1 - \phi)(\gamma + rV^u)$$

Set of Equations Now, we have six equations:

$$\begin{aligned} &[1] : rV^u = 0 + f(\theta)(V^e(w^*) - V^u) \\ &[2] : rV^e(w) = w - \gamma + \chi(V^u - V^e(w)) \\ &[3] : rV^j(w) = z - w - \chi V^j(w) \\ &[4] : rV^v = -c + \mu(\theta)\left[V^j(w^*) - V^v\right] \\ &[5] : V^v = 0 \\ &[6] : w^* = \phi z + (1 - \phi)(\gamma + rV^u) \end{aligned}$$

We solve for θ.

▷ First, plugging in [6] into [2] and [3] yields:

$$[A] : \frac{V^e(w^*) - V^u}{\phi} = \frac{V^j(w^*)}{1 - \phi} = V^e(w^*) + V^j(w^*) - V^u \equiv V^S$$

▷ Second, using $[1]$, $[2]$ and $[3]$ yields:

$$rV^S = z - \gamma - \chi V^S - f(\theta)(V^e(w^*) - V^u)$$

and using the fact that $V^e(w^*) - V^u = \phi V^S$ from $[A]$, we have**firm has vacancy**

$$[B] : rV^S = z - \gamma - \chi V^S - \phi f(\theta) V^S$$

▷ Third, using $[4]$, $[5]$ and $[A]$, we have

$$[C] : c = \mu(\theta)(1 - \phi) V^S$$

▷ Fourth, using $[B]$ and $[C]$ yields:

$$\frac{r + \chi}{\mu(\theta)} = (1 - \phi)\frac{z - \gamma}{c} - \phi\frac{f(\theta)}{\mu(\theta)}$$

$$\Rightarrow \underline{\frac{r + \chi}{\mu(\theta)} = (1 - \phi)\frac{z - \gamma}{c} - \phi\theta}$$

* LHS is an increasing function of θ, and RHS is a decreasing function of θ, and this yields a unique θ.
* If we don't use the Inada assumptions, we can get a negative value of vacancies, which is not interesting.
* You can also do comparative statics here and look at the effect of ϕ. Increasing ϕ increases w^*, but you're reducing the vacancy/employment ratio. There are these two forces fighting against each other. So what's optimal is an intermediate bargaining power.

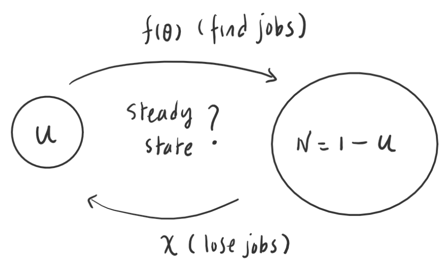
Now that we have the value of θ, we can get other variables of interest. Specifically, plugging this into the wage yields:

$$\underline{w^* = \phi (z + c\theta) + (1 - \phi) \gamma}$$

▷ If we didn't have $c\theta$, then it looks pretty intuitive. $c\theta$ is recognizing that if you lose a worker, you lose more than just z and future output as well – you have to post a vacancy and hire a worker. It turns out that $c\theta$ is exactly how you represent this additional cost.

3.1.4 Unemployment

Consider the following diagram:



▷ We can characterize the steady state as

$$uf(\theta) = \chi (1 - u) \Rightarrow u = \frac{\chi}{\chi + f(\theta)} \equiv u^*$$

What should the dynamics look like outside the steady state? In principle, we will have to go through the similar algebra as before. It turns out, fortUunately, that the algebra is exactly the same, so we can introduce a new state variable proxying for unemployment. Then it turns out that the above result also holds for outside the equilibrium as well.

What does this mean? We can now generalize the logic in the figure above to have a time-varying unemployment rate that represents the discrepancy between these two flows:

$$\dot{u}(t) = \chi (1 - u(t)) - f(\theta) u(t)$$

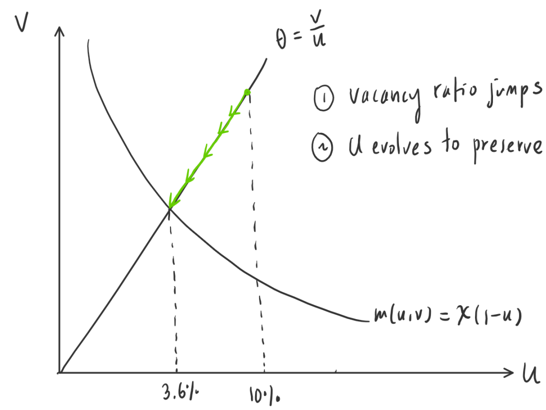
where we assume θ is constant (this is an assertion):

$$u(t) = u^* + (u(0) - u^*) e^{-(f(\theta) + \chi)t}$$

3.1.5 Thanos Shock

Using the equation above, we can consider the following dynamics. A Thanos shock is an initial condition that is below the steady-state.

▷ Since $uf(\theta) \equiv m(u, v)$, we can draw the following graph:



▷ Note that that in this model, the convergence happens quite quickly. The half-time of convergence is given as

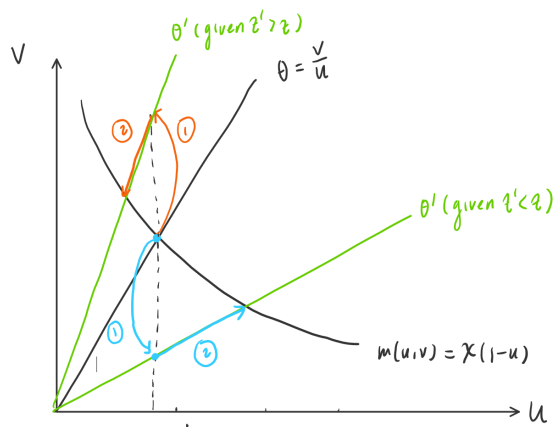
$$e^{-(f(\theta)+\chi)t} = \frac{1}{2} \Rightarrow t = \frac{\log 2}{f(\theta) + \chi}$$

The denominator is around $1/3$ and $\log 2$ is about 0.3010, so it takes about a month from 10% to go to 7.5% (the midpoint).

This model is not well suited for thinking about χ, but it is well-suited for thinking about $f(\theta)$.

3.1.6 MIT Shock

Consider an one-time unanticipated increase (orange) / decrease (blue) in labour productivity, z (i.e. MIT shock). We have the following dynamics:



▷ Comparative statics with respect to z:

* If we raise z, from the following equation:

$$\frac{r+\chi}{\mu(\theta)} = (1-\phi)\frac{z-\gamma}{c} - \phi\theta$$

you're increasing the RHS of the equation, so you need something to reduce the RHS or increase the LHS.

  * Since LHS is an increasing function in θ and RHS is a decreasing in function in θ, this means you will get to a higher θ. Intuitively, productivity increase implies that you need to restore the free entry condition, so it must be more difficult to obtain the workers and restore equilibrium.

- ▷ Furthermore from the wage equation:
$$w^* = \phi(z + c\theta) + (1 - \phi)\gamma$$
since z and θ both increase, w^* also increases, thus driving up the equilibrium wage.

  * Direct force (z): "I realize that any potential employer will produce more output with me, so the need to hire me has gone up so I get a fraction of the new rent."
  * Indirect force (θ): This is the threat point of the worker. It's easy for workers to find jobs, so when I go to my employer to bargain my wage, I have a viable threat point in bargaining. We will note this as a "change in threat point" as opposed to "change in bargaining power."

- ▷ Persistent productivity shocks make it seem like unemployment is also persistent.

One thing you can do is to ask which percentage change in productivity is necessary to get a doubling of θ. In recessions, θ decreases by a factor of 4, and it turns out that you need an immediate productivity shock in order to get your desired value.

- ▷ If z and γ are close to each other, then small changes in z will lead to large changes in θ. The reason is that if $z - \gamma$ is .05, then 1% change in z is a 20% change in $z - \gamma$, which has a big impact on the θ. This paramter γ becomes a very important parameter.γ previously didn't have a big impact on business cycles, but now it does.

- ▷ Suppose we are hit continuously by MIT shocks and they are shifting θ around (up and down), then we will remain close to $\dot{u} = 0$. We are unlikely to see large movements along the $\dot{u} = 0$ locus because the adjustment mechanism is fast. We therefore would not expect to see prolonged period in which movement in u and v in the same direction.

3.1.7 Data

Unemployment data comes from household surveys.

3.2 Social Planner's Problem

It's worth thinking about what the social planner would do in this setting.

3.2.1 Setup

The SP solves

$$\begin{aligned}
\max_{\{v(t),u(t)\}} \int_0^\infty & e^{-rt}\left((1 - u(t))(z - \gamma) - cv(t)\right) dt \\
\text{s.t. } \dot{u}(t) &= \chi(1 - u(t)) - m(u(t), v(t)) \\
u(0) &= u_0
\end{aligned}$$

so the SP has the same technology. Since agents are risk neutral, we take their utility function to be linear. $u(t)\gamma$ is the "output" from the unemployed, $(1 - u(t))z$ is the output from the employed. $v(t)c$ is the cost firms pay to fill vacancies. We will solve this using Bellman equation.

3.2.2 Bellman Formulation

The bellman equation can be constructed as:

$$rW(u) = \max_{v\in[0,1]} \left\{u\gamma + (1-u)z - vc + W'(u)(x(1-u) - m(u,v))\right\}$$

▷ To review the construction of Bellman equations in continuous-time, suppose we had the following model:

$$V(a) = \max_{a,c} \int_0^\infty e^{-\rho t} u(c)\, dt$$
$$\text{s.t. } \dot{a} = ra + w - c$$

which yields

$$\underline{\rho V(a) = \max_{a,c} u(c) + (ra + w - c) V'(a)}$$

whereas its discrete counterpart is the following program:

$$\max_{a,c} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } a_{t+1} = (1+r) a_t + w_t - c_t$$

which yields:

$$\underline{V(a) = \max_{a,c} u(c) + \beta V((1+r)a + w - c)}$$

Solving the Bellman equation yields by taking the FOC with respect to v:

$$[FOC]: -c - W'(u) m_v(u,v) = 0$$
$$[EC]: rW'(u) = -(z-\gamma) + W'(u)(-x - m_u(u,v))$$

Combining them yields:

$$\underline{\frac{r+\chi}{m_v(u,v)} = \frac{z-\gamma}{c} - \frac{m_u(u,v)}{m_v(u,v)}}$$

Since the partial derivatives of m are homogeneous of degree zero, we have:

$$\underline{\frac{r+\chi}{m_v(1,\theta)} = \frac{z-\gamma}{c} - \frac{m_u(1,\theta)}{m_v(1,\theta)}}$$

so that social planner's problem amounts to choosing θ (given u).

3.2.3 Example: Cobb-Douglas Matching Function

Assume the following functional form for the matching function:

$$m(u,v) \coloneqq \bar{m} u^\eta v^{1-\eta},\ \eta \in (0,1)$$

We are thus assuming a Cobb-Douglas matching function. Given the functional form,

$$m_u(u,v) = \eta \bar{m} u^{\eta-1} v^{1-\eta} = \eta \bar{m} \left(\frac{v}{u}\right)^{1-\eta} = \eta \bar{m} \theta^{1-\eta}$$
$$m_v(u,v) = (1-\eta) \bar{m} u^\eta v^{-\eta} = (1-\eta) \bar{m} \left(\frac{v}{u}\right)^{-\eta} = (1-\eta) \bar{m} \theta^{-\eta}$$

Then we can write the equilibrium condition as

$$\begin{aligned}\frac{r+\chi}{(1-\eta)\,\bar{m}\theta^{-\eta}} &= \frac{z-\gamma}{c} - \frac{\eta\bar{m}\theta^{1-\eta}}{(1-\eta)\,\bar{m}\theta^{-\eta}} \\ &= \frac{z-\gamma}{c} - \frac{\eta}{1-\eta}\theta.\end{aligned}$$

Note that

$$\mu(\theta) = \frac{m(u,v)}{v} = \frac{\bar{m}u^{\eta}v^{1-\eta}}{v} = \bar{m}\theta^{-\eta}.$$

So, in fact, we have

$$\frac{r+\chi}{\mu(\theta)} = (1-\eta)\frac{z-\gamma}{c} - \eta\theta,$$

which is exactly what we had before with $\eta = \phi$. In other words, the equilibrium solves the planner's problem if and only if $\eta = \phi$. This is referred to as the *(Mortensen-) Hosios condition*.

3.2.4 Effect of Bargaining Power

Assume as before the following functional form for the matching function:

$$m(u,v) := \bar{m}u^{\eta}v^{1-\eta}$$

Consider what happens if the bargaining power of the worker, ϕ, increases. Then more of the net surplus accrues to the worker from the bargaining so that w^* increases. Consequently, firms will reduce vacancies so that the probability of filling the vacancy increases to restore the zero profit condition, and the equilibrium θ^* increases. Observe that ϕ does not play a role in the planner's problem since the planner has access to lump-sum transfers. Mortensen (1982, AER) provides some intuition for the result. Let us consider the two extremes, $\eta = 1$ and $\eta = 0$ to see what the efficient outcome would be.

- ▷ If $\eta = 0$, then $m(u,v) = \bar{m}v$ so that the matching function depends only on the number of vacancies and not on the number of unemployed.
  - * We can think of this situation as one in which the employers are making all the effort to fill vacancies—unemployed workers simply wait for the "phone call" from firms offering them jobs. So the firm can create a lot of vacancies and obtain all the surplus.
- ▷ If $\eta = 1$, then $m(u,v) = \bar{m}u$. Now, it is the worker that makes all the effort in the matching process.
  - * This means that there will be vacancies and the rate at which vacancies are filled $\mu(\theta)$ is infinite. Analogously as before, we require all of the net surplus from matching to go to the workers.
- ▷ For intermediate values of η, elasticity tells you the importance of each's role in the matching process.

Now we can compare outcomes in the CE against outcomes in SP when $\eta \neq \phi$.

- ▷ For example, suppose $\phi > 0 = \eta$. Then the workers do have some bargaining power, so the firms will not be creating as many jobs as it was in the SP's case.

3.3 Incorporating search friction into NCG

As mentioned before when we discussed the dynamics, whether the model fits the data well depends upon the ratio z/γ being close to one or not. The problem was that the linearity in the model meant that there was no distinction between marginal and average effects. Our goal here is to introduce search friction into an RBC model (which is non-linear) and to study whether this extension means that the model fits the data better. We first solve the planner's problem. (If we were to decentralise this, then we would need to specify the worker's bargaining power, ϕ.)

3.3.1 Setup

Households choose consumption and whether to work or not; i.e. the work decision is binary so that there is no intensive decision regarding how much to work.[3] We assume that the disutility from not working is zero, while γ is the disutility of work conditional on working. This means that the total disutility of work in the economy is given by γ times the number of individuals working, N_t. Workers lose jobs with constant probability χ.

Firms (representative) choose capital and labour. In particular, past hiring decisions of the firm affects the availability of labour that can be used in production, L_t, in period t. In this way, firms *invest* in hiring labour, and doing so requires some labour, which we denote as V_t (we can think of this as firms having an HR/recruitment department).

(Aggregate) matching technology The matching technology, $m(u_t, V_t)$, depends on the number of unemployed, u_t, as well as the amount of labour in the HR department, V_t. So V_t plays the role of vacancies in the Pissarides model. As before, we assume that m is increasing in both elements and exhibits constant returns to scale.

Planner's problem We assume that the planner has the same utility as the household and 2 laws of motions (one for capital and one for labour). The planner maximises household utility subject to these law of motions, as well as the aggregate feasibility constraint for goods and labour. The planner solves the following maximisation problem with respect to $\{K_{t+1}, N_{t+1}, C_t, V_t, L_t, u_{t+1}\}$:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right)\left(\log C_t\left(\mathbf{s}^t\right) - \gamma N_t\left(\mathbf{s}^{t-1}\right)\right) \\
s.t. \quad & [1] : K_{t+1}\left(\mathbf{s}^t\right) = K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right) L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + (1-\delta) K_t\left(\mathbf{s}^{t-1}\right) - C_t\left(\mathbf{s}^t\right) \\
& [2] : N_{t+1}\left(\mathbf{s}^t\right) = (1-\chi) N_t\left(\mathbf{s}^{t-1}\right) + m\left(u_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right) \\
& [3] : u_t\left(\mathbf{s}^{t-1}\right) + N_t\left(\mathbf{s}^{t-1}\right) = 1, \\
& [4] : V_t\left(\mathbf{s}^t\right) + L_t\left(\mathbf{s}^t\right) = N_t\left(\mathbf{s}^{t-1}\right) \\
& [5] : K_0\left(\mathbf{s}^{-1}\right) = K_0,\ N_0\left(\mathbf{s}^{-1}\right) = N_0.
\end{aligned}
$$

V_t can be interpreted as "recruiting labor" and L_t can be interpreted as "production labor." Since the planner keeps track of the Ns, it's $N_t\left(\mathbf{s}^{t-1}\right)$ not $N_t\left(\mathbf{s}^t\right)$. Furthermore, note:

- ▷ $\gamma N_t\left(\mathbf{s}^t\right)$ represents the fact that decision to work involves only the extensive margin;
- ▷ [1] is the law of motion for capital, which is the standard one except that we now denote labour (used in production of goods) as L_t;
- ▷ [2] is the law of motion for labour. The number of total workers in the next period is the remaining workers from the previous period (since worker lose job with probability χ) plus the number of job matches given by the matching function $m(u_t, V_t)$;
- ▷ [3] just says that there is a unit measure of individuals, and each individual is either employed or unemployed;
- ▷ [4] says that total number of workers are determined one period before the decision as to whether workers work on producing output, L_t, or in the HR department, V_t.

[3]The data suggests about about $1/3$ of the variation in aggregate hours worked is due to the intensive margin, and $2/3$ is due to the extensive margin.

3.3.2 Characterizing Equilibrium

We can express the social planner's problem as a Lagrangian:

$$\begin{aligned}\mathcal{L}=\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)&\left[\log C_t\left(\mathbf{s}^t\right)-\gamma N_t\left(\mathbf{s}^{t-1}\right)\right.\\
&+\lambda_t\left(\mathbf{s}^t\right)\Big(K_{t+1}\left(\mathbf{s}^t\right)-K_t\left(\mathbf{s}^{t-1}\right)^\alpha\left(z_t\left(\mathbf{s}^t\right)\left(N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)\right)\right)^{1-\alpha}\\
&\qquad-(1-\delta)K_t\left(\mathbf{s}^{t-1}\right)+C_t\left(\mathbf{s}^t\right)\Big)\\
&\left.+\mu_t\left(\mathbf{s}^t\right)\left(N_{t+1}\left(\mathbf{s}^t\right)-(1-\chi)N_t\left(\mathbf{s}^{t-1}\right)-m\left(1-N_t\left(\mathbf{s}^{t-1}\right),V_t\left(\mathbf{s}^t\right)\right)\right)\right].\end{aligned}$$

The first-order conditions are:

$$\begin{aligned}
\left[C_t\left(\mathbf{s}^t\right)\right]:&\quad 0=\beta^t\Pi_t\left(\mathbf{s}^t\right)\left[\frac{1}{C_t\left(\mathbf{s}^t\right)}-\lambda_t\left(\mathbf{s}^t\right)\right],\\
\left[V_t\left(\mathbf{s}^t\right)\right]:&\quad 0=\lambda_t\left(\mathbf{s}^t\right)(1-\alpha)K_t\left(\mathbf{s}^{t-1}\right)^\alpha z_t\left(\mathbf{s}^t\right)^{1-\alpha}\left(N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)\right)^{-\alpha}\\
&\qquad-\mu_t\left(\mathbf{s}^t\right)m_v\left(1-N_t\left(\mathbf{s}^{t-1}\right),V_t\left(\mathbf{s}^t\right)\right),\\
\left[K_{t+1}\left(\mathbf{s}^t\right)\right]:&\quad 0=\beta^t\Pi_t\left(\mathbf{s}^t\right)\lambda_t\left(\mathbf{s}^t\right)-\beta^{t+1}\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\lambda_t\left(\mathbf{s}^{t+1}\right)\times\\
&\qquad\left[\alpha K_{t+1}\left(\mathbf{s}^t\right)^{\alpha-1}\left(z_{t+1}\left(\mathbf{s}^{t+1}\right)\left(N_{t+1}\left(\mathbf{s}^t\right)-V_{t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)^{1-\alpha}-(1-\delta)\right],\\
\left[N_{t+1}\left(\mathbf{s}^t\right)\right]:&\quad 0=\beta^{t+1}\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\gamma\\
&\qquad-\beta^{t+1}\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\lambda_t\left(\mathbf{s}^{t+1}\right)(1-\alpha)K_{t+1}\left(\mathbf{s}^t\right)^\alpha z_{t+1}\left(\mathbf{s}^{t+1}\right)^{1-\alpha}\left(N_{t+1}\left(\mathbf{s}^t\right)-V_{t+1}\left(\mathbf{s}^{t+1}\right)\right)^{-\alpha}\\
&\qquad+\beta^t\Pi_t\left(\mathbf{s}^t\right)\mu_t\left(\mathbf{s}^t\right)\\
&\qquad-\beta^{t+1}\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\mu_{t+1}\left(\mathbf{s}^{t+1}\right)\left[(1-\chi)-m_u\left(1-N_{t+1}\left(\mathbf{s}^t\right),V_{t+1}\left(\mathbf{s}^t\right)\right)\right].
\end{aligned}$$

Combining these equations, we get the following Euler equations:

$$[1]:\frac{1}{C_t\left(\mathbf{s}^t\right)}=\beta\sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}\frac{F_{K,t+1}\left(\mathbf{s}^{t+1}\right)+1-\delta}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}=\beta\mathbb{E}_t\left[\frac{F_{K,t+1}\left(\mathbf{s}^{t+1}\right)+1-\delta}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]$$

$$[2]:\frac{F_{\ell,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)}=\beta m_{v,t}\left(\mathbf{s}^t\right)\mathbb{E}_t\left[\frac{F_{\ell,t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\left(1+\underbrace{\frac{1-\chi-m_{u,t+1}\left(\mathbf{s}^{t+1}\right)}{m_{v,t+1}\left(\mathbf{s}^{t+1}\right)}}_{(*)}\right)-\gamma\right]$$

▷ [1] is the standard consumption Euler equation.

  * This tells us that, on the margin, consuming today (the left-hand side) should give the same benefit as foregoing consumption today and investing today. The consumption Euler equation tells us the optimal trade off between consuming today and investing in capital today.

▷ [2] is the "labor" Euler equation.

  * This tells us the optimal trade off between working for production today and working in recruitment/HR.

* The left-hand side gives the marginal utility from working in production today (working in production today gives additional output of $F_{\ell,t}$ which is worth $F_{\ell,t}/C_t$ in utilities).
* If we instead work in recruiting, then this increases the number of matches by $m_{v,t}$. The return from this is the additional production worker we get the next period worth $F_{\ell,t+1}/C_{t+1}$ in utils. The $(*)$ term plays the same role as $1-\delta$ in the consumption Euler equation. It reflects the fact that the additional workers are likely to be around in $t+2$ which allows the planner to move more workers into production in $t+1$. Of course, having additional workers mean that there is an associated disutility of work γ.

3.4 Decentralization

We want to decentralize the planner's problem which would allow us to study equilibrium wage determination.

3.4.1 Assumption of Complete Markets

We assume in the RBC model with search frictions that individuals can work or not work. We can think of this as individuals experiencing idiosyncratic shock. However, the optimal consumption was constant across all individuals—in effect, the social planner provided insurance for individuals so as to ensure that consumption is the same whether an individual is working or not. This means that, in order for our decentralisation to replicate the planner's solution, we must insure agents against shocks.

▷ One way is to introduce securities that allow individuals to insure against idiosyncratic shocks. But this requires, in principle, Arrow-Debreu securities for every individuals for every possible history; i.e. many many securities. We would rather keep the problem simple.

The way we proceed is to assume that each individuals belongs to a household, and that the household provides insurance for its members.

▷ As Shimer puts it, each household is lead by a "grandpa" that tells members what to do and what to consume.

▷ Within each household, some members could be working or not working, but the insurance means that all members consume the same amount. We assume that there is a measure one of households and each household has a measure one of members. There are no incentive problems in here.

3.4.2 Competitive equilibrium

An equilibrium is defined to be the sequences of allocations

$$\left\{C_t\left(\mathbf{s}^t\right), L_t\left(\mathbf{s}^t\right), V_t\left(\mathbf{s}^t\right), N_{t+1}\left(\mathbf{s}^t\right), K_{t+1}\left(\mathbf{s}^t\right)\right\}$$

and prices $\left\{w_t\left(\mathbf{s}^t\right), q_0^t\left(\mathbf{s}^t\right)\right\}$, and $\theta_t\left(\mathbf{s}^t\right)$ (which is linked to V_t and N_t through the market clearing) such that:

1. Households choose $\left\{C_t\left(\mathbf{s}^t\right), N_{t+1}\left(\mathbf{s}^t\right)\right\}$ to maximize:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)\left[\log C_t\left(\mathbf{s}^t\right)-\gamma N_t\left(\mathbf{s}^{t-1}\right)\right] \\
\text{s.t.} \quad & a_0=\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}q_0^t\left(\mathbf{s}^t\right)\left[C_t\left(\mathbf{s}^t\right)-w_t\left(\mathbf{s}^t\right)N_t\left(\mathbf{s}^t\right)\right] \\
& N_{t+1}\left(\mathbf{s}^t\right)=(1-\chi)N_t\left(\mathbf{s}^t\right)+f\left(\theta_t\left(\mathbf{s}^t\right)\right)\left(1-N_t\left(\mathbf{s}^{t-1}\right)\right)
\end{aligned}$$

given $\left\{W_t\left(\mathbf{s}^t\right), q_0^t\left(\mathbf{s}^t\right), \theta_t\left(\mathbf{s}^t\right)\right\}$ and $N_0\equiv N_0\left(\mathbf{s}^{-1}\right)$ and a_0.

- ▷ The fact that this is a single budget constraint comes from our complete markets assumption.
- ▷ Employed people are worse off than unemployed people since they consume the same amount, but employment gives the disutility of work. This is a feature of complete markets economy with unemployment.
- ▷ N_t is determined exogenously, which is in contrast with the frictionless model where individual chooses employment. This was the case in the Pissarides model – there's nothing the household can do to affect unemployment. Alternatively, you could take N_t as given and choose C_t.
- ▷ If the wage is too low, the "grandpa" may choose not to send some people to work, in which case the constraint changes to:

$$N_{t+1}\left(\mathbf{s}^t\right) \leq (1-\chi) N_t\left(\mathbf{s}^t\right) + f\left(\theta_t\left(\mathbf{s}^t\right)\right)\left(1 - N_t\left(\mathbf{s}^{t-1}\right)\right)$$

where the wedge is an endogenous choice variable representing the number of people that "grandpa" does not send to work.

2. Firms choose $\left\{L_t\left(\mathbf{s}^t\right), V_t\left(\mathbf{s}^t\right), N_{t+1}\left(\mathbf{s}^t\right), K_{t+1}\left(\mathbf{s}^t\right)\right\}$ to maximize:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t} q_0^t\left(\mathbf{s}^t\right)\left\{K_t\left(\mathbf{s}^t\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + (1-\delta)K_t\left(\mathbf{s}^{t-1}\right) - K_{t+1}\left(\mathbf{s}^t\right) - w_t\left(\mathbf{s}^t\right)N_t\left(\mathbf{s}^{t-1}\right)\right\} \\
\text{s.t.} \quad & N_{t+1}\left(\mathbf{s}^t\right) = (1-\chi)N_t\left(\mathbf{s}^t\right) + \mu\left(\theta_t\left(\mathbf{s}^t\right)\right)V_t\left(\mathbf{s}^t\right) \\
& N_t\left(\mathbf{s}^{t-1}\right) = V_t\left(\mathbf{s}^t\right) + L_t\left(\mathbf{s}^t\right)
\end{aligned}$$

given $\left\{W_t\left(\mathbf{s}^t\right), q_0^t\left(\mathbf{s}^t\right), \theta_t\left(\mathbf{s}^t\right)\right\}$ and $N_0 \equiv N_0\left(\mathbf{s}^{-1}\right), K_0 \equiv K_0\left(\mathbf{s}^{-1}\right)$.

- ▷ If $L_t = N_t$, then this would be identical to the generic model.
- ▷ Firm can choose how much to put into recruiting, $V_t\left(\mathbf{s}^t\right)$.

3. Goods market clearing:

$$K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} = C_t\left(\mathbf{s}^t\right) + K_{t+1}\left(\mathbf{s}^t\right) - (1-\delta)K_t\left(\mathbf{s}^{t-1}\right)$$

4. Internal consistency of labor market variables:

$$\theta_t\left(\mathbf{s}^t\right) = \frac{V_t\left(\mathbf{s}^t\right)}{1 - N_t\left(\mathbf{s}^{t-1}\right)}$$

- ▷ It is important that firms are "small" in the labor market, thereby taking θ as given. In the aggregate, it must satisfy the above condition. So we don't want to plug this into the firm's optimization problem.

5. Given $\left\{K_t\left(\mathbf{s}^{t-1}\right), L_t\left(\mathbf{s}^t\right), \theta_t\left(\mathbf{s}^t\right), C_t\left(\mathbf{s}^t\right)\right\}$, $\left\{w_t\left(\mathbf{s}^t\right)\right\}$ solves the Nash bargaining problem

$$\begin{aligned}
w_t\left(\mathbf{s}^t\right) &= \phi\overbrace{(1-\alpha)\left(K_t\left(\mathbf{s}^{t-1}\right)\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}\left(L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}}^{MPL}\left(1+\theta_t\left(\mathbf{s}^t\right)\right) \\
&\quad + (1-\phi)\underbrace{\gamma C_t\left(\mathbf{s}^t\right)}_{MRS}, \ \forall t, \mathbf{s}^t. \\
&= \phi F_{\ell,t}\left(1+\theta_t\left(\mathbf{s}^t\right)\right) + (1-\phi)\gamma C_t
\end{aligned}$$

- ▷ Note that this requires some derivation to get this expression. $\left(1+\theta_t\left(\mathbf{s}^t\right)\right)$ is recognizing that the firm's problem is dynamic – if a worker leaves, it needs to replace the worker in the future as well.

▷ Note that we will have

$$MRS < w_t < MPL$$

Remark 3.1. (Ownership of Capital) In the standard NCG, letting firms own capital made it a dynamic problem, whereas letting households own the capital made both problems static. So in this respect, it was easier to just have a rental market and make both problem static. In this setup, the firm's problem is already dynamic in the absence of this concern, so it's totally fine to stick capital ownership into the firm's problem.

Remark 3.2. (Departure from Classic NCG)

1. We no longer have the static version of MRS equaling the wage. The wage is always larger than MRS – the household always wants to have as many people as it can.
2. We no longer have the saimple link between MPL equaling wage, since $L_t \neq N_t$. There is still some link because if you anticipated that the wage is going to be higher than MPL in all future periods, there's no reason to forego output today and engage in recruiting. In such case, recruiting is not as useful. If the wage is a lot less than MPL and if you anticipate that in the future, that incentivizes the firm to engage in recruiting, leading to a high V_t which leads to high employment, thereby lowering the MPL. So there is some link, but it is not as straightforward as before.

These results imply that there will a lot of equilibria for different wage paths, and we need the Nash bargaining constraint to pin down the unique equilibrium.

3.4.3 Characterizing the Equilibrium

By taking the first-order conditions, we get the following equations. Note that we assumed Nash bargaining in its derivation.

$$[1] : \frac{1}{C_t\left(\mathbf{s}^t\right)} = \beta\mathbb{E}_t\left[\frac{F_{K,t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right], \quad F_{K,t+1} \equiv \alpha K_{t+1}^{\alpha-1}\left(z_{t+1}L_{t+1}\right)^{1-\alpha} + (1-\delta)$$

$$[2] : \frac{F_{\ell,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)} = \beta\mu\left(\theta_t\left(\mathbf{s}^t\right)\right)\left(\frac{F_{\ell,t+1}\left(\mathbf{s}^{t+1}\right)\left(1+\frac{1-\chi}{\mu\left(\theta_{t+1}\left(\mathbf{s}^{t+1}\right)\right)}\right) - w_{t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right), \quad F_{\ell,t+1} = (1-\alpha)K_t^{\alpha}z_t$$

▷ Interpretation of $[1]$:

* In the planner's problem, we can reduce consumption by one today, thereby increasing output tomorrow by the MPL . It must be the case that the additonal consumption next period, evaluated at the marginal utility of consumption and discounted, should be equal.
* In the decentralized equilibrium, households tell firms how much they shoudl engage in investment, and the firm could return one extra unit of dividend (LHS, thereby increasing consumption), or the firm could make the investment and raise output / pay out dividend in the subsequent period. To evaluate this, the firm uses q_0^t.

▷ Interpretation of $[2]$:

* In the planner's problem, we had

$$[2^*] : \frac{F_{\ell,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)} = \beta m_{v,t}\mathbb{E}_t\left[\frac{F_{\ell,t+1}\left(\mathbf{s}^{t+1}\right)\left(1+\frac{1-\chi-m_{u,t+1}}{m_{v,t+1}}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)} - \gamma\right]$$

This is also an intertemporal choice that the planner i making. This choice is between allocating marginal worker between production and recruiting. If you put a worker in production, he produces $F_{\ell,t}$ evaluated at marginal utility. If you put him in recruiting, it increases future number of workers by $m_{v,t}$ and raise output in the next period. The additional workers could be put into production next period $(F_{\ell,t+1})$ which we evaluate at marginal utility, and they also suffer the disutility of work γ. The

$$\frac{1-\chi-m_{u,t+1}}{m_{v,t+1}}$$

term is analogous to why $(1-\delta)$ is in the standard EE. Once a new guy is employed, then I can save on recruiting at $t+2$ by the above mount. The workers that I hire this period that are around $(1-\chi)$ save me some matching efficiency and also changes the number of unemployed workers in the future period. So the term represents the additional workers that are saved in the period afterwards.

* In the decentralized equilibrium, this is also similar. It has many of the same terms with **two key differences.**

$$[2]:\frac{F_{\ell,t}\left(\mathbf{s}^{t}\right)}{C_{t}\left(\mathbf{s}^{t}\right)}=\beta\mu\left(\theta_{t}\left(\mathbf{s}^{t}\right)\right)\left(\frac{F_{\ell,t+1}\left(\mathbf{s}^{t+1}\right)\left(1+\frac{1-\chi}{\mu\left(\theta_{t+1}\left(\mathbf{s}^{t+1}\right)\right)}\right)-w_{t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right),\quad F_{\ell,t+1}=(1-\alpha)K_{t}^{\alpha}z_{t}$$

- **First**, from the planner what matters is the derivative of the matching function. In equilibrium, however, what matters is the average product of the matching function, i.e. matching divided by vacancy. This is because in equilibrium people take given θ_t in solving their problem and decide on the number of people to put into recruiting. So from the firm's perspective, each firm produces $\mu(\theta)$ and not the derivative.
- **Second**, firm does not care about the disutility that workers suffer; they only care about the wage that the workers will get. Households do own the firms, but they don't think back to the fact that people working will suffer disutility. So wage is appearing instead of γ.
- If the firm puts an additional guy into production, you get profit and pay back to the household. If you put one guy in recruiting, you get additional output next period but you have to pay additional wage to the workers in this period, and you value this at the marginal utility of consumption next period.

Here, the previous results apply that if $m(u,v)=\bar{m}u^{\eta}v^{1-\eta}$ and $\eta=\phi$, then the equilibrium allocation solves the planner's problem.

3.4.4 Thoughts on Calibration

If we want this model to be matching facts about labor share of income, long-run growth rates, capital-output ratio, investment-output ratio, and so on, there need to be some parameter restrictions that need to be satisfied. We can show that if z_t grows at a constant rate, then the model has a nice balanced growth path – this is true for both SP and CE problem.

▷ In such path, K_t, C_t, w_t grow at the same rate, but employment variables are all constant. In the Pissarides model, if you had a trend growth in productivity, you need something exogenous to offset it to avoid having unemployment trending down to zero. Here, it happens naturally here due to the income and substitution effect.

▷ Features of the BGP can be used to calibrate the standard parameters of the model. Through the usual way, we can calibrate the discount rate, the depreciation rate.

▷ α is not quite standard since it's not competitive anymore, but it turns out to be quite close. Recall that α represents the capital share of income. Unlike in the standard NCG/RBC model, there are now two types of labour: workers can be involved in production or recruitment. This means that α, the capital share of income, is based on L_t and not on N_t as before. Since the proportion of those working in recruitment is small (0.4%), α does not differ much from before.

- ▷ To calibrate χ, note that the job separation rate χ in the US, the average rate is around 3.4% (on a monthly basis): $\chi = 0.034$.

If we suppose that $\log z_{t+1} = 0.98 \log z_t + \nu_{t+1}$, we obtain

$$\lambda_1 =\sim 0.999,\ \lambda_2 = 0.98,\ \lambda_3 =\sim 0.27$$

- ▷ $\lambda_3 \approx 0.3$ represents N_t so we reach steady-state really quickly. This is exactly the same forces used in the Pissarides.
- ▷ $\lambda_1 \approx 0.999$ is associated with capital.

3.4.5 Effect of Incorporating Search Frictions

What change as we incorporate search friction?

- ▷ In the frictionless employment world, the volatility of employment was 81.5% of standard deviation of output; incorporating fictions reduces the volatility to 16%.
- ▷ In response to a positive productivity shock, you would instanteously raise employment before; now you have to shift get people off production line, shift to recruiting, and recruiting technology gets unattractive. So at the end of the day you adjust employment less.

This is a disappointing result. We've dampened the fluctuations in employment, the opposite of what we wanted to do. Suppose wages change deterministically; they don't satisfy Nash bargaining in general, and in response to productivity shock they grow together.

- ▷ In this setting, the decentralized equilibrium is no longer efficient.
- ▷ We still have positive co-movements between these variables, and employment volatility is around 98.2%, which gives us some hope. Now a boom is a period when there is a burst of recruiting activity to take advantage of the fact that productivity increases and wages have not gone up in response to that. To equilibrate the market, you have to make it very hard to find unemployed people and keep employment going up.
- ▷ We have volatility of productivity shock that is 78% of volatility of output there, whereas in the model of nash bargaining, we had volatility 150% of output.

3.4.6 Note on Labor Wedge

Recall that we had

$$\begin{aligned} w_t\left(\mathbf{s}^t\right) &= \phi \overbrace{(1-\alpha)\left(K_t\left(\mathbf{s}^{t-1}\right)\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}\left(L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}}^{MPL}\left(1+\theta_t\left(\mathbf{s}^t\right)\right) \\ &\quad + (1-\phi)\underbrace{\gamma C_t\left(\mathbf{s}^t\right)}_{MRS},\ \forall t, \mathbf{s}^t. \\ &= \phi F_{\ell,t}\left(1+\theta_t\left(\mathbf{s}^t\right)\right) + (1-\phi)\gamma C_t \end{aligned}$$

which results in $MRS < w_t < MPL$.

- ▷ MPL gets pulled down to the wage level.
- ▷ We could consider an alternative where $w_t = (1+g)^t w_0$. Then we don't have this endogenous increase in wage coming from θ_t and C_t from the previous expression. Also, if I stick this wage form into the model, we can do a decomposition and show that MPL is almost constant, since firms will adjust the hiring to a point that MPL is slightly above the wage.

Why are we interested in this? We want to see what the path of wage look like.

- ▷ One component decomposes the labor wedge between w and MRS vs. the w and MPL. The paper concludes that the majority of the wedge is between w_t and MRS.

4 Investment

We will first explore incorporating convex adjustment costs and then extend it to non-convex adjustment costs. This extension is motivated by the empirical evidence that the adjustment costs are not actually convex. Instead, micro-evidence suggests two kinds of adjustment costs: (1) fixed-cost of investing and (2) wedge between the purchase price and the sell price of capital.

- ▷ Fixed-cost of investing is simialr to fixed cost of price changes that we saw in the New-Keynesian models.
- ▷ If you were to go and buy a car and re-sell it, you will lose money. The same thing is true for more complicated machines. If you are a manufacturer of aircrafts and you buy a wind tunnel to test your aircraft, it turns out that you will have to sell this at a substantial loss. This would introduce a wedge between the buy price and the sell price of capital.

4.1 Convex Adjustment Cost

We introduce a model that is useful for understanding the aggregates but not so much at the individual level. One of the empirical motivations is that investments are much more slow-moving than what RBC would predict. In the data, the value of capital (firm) moves a lot and it is predictive of investment, and the way we generate it in the model is through convex adjustment costs. If the firm's value is relatively higher than the capital stock, firms will invest a lot.

4.1.1 Firm problem

Consider the value of a firm:

$$
\begin{aligned}
& V_0\left(\mathbf{s}^0, k_0\right) \\
:= & \max_{\{k_{t+1}(\mathbf{s}^t), h_t(\mathbf{s}^t)\}_{t,\mathbf{s}^t}} \sum_{t=0}^{\infty} \beta^t \sum_{\mathbf{s}^t} q_0^t\left(\mathbf{s}^t\right)\left[f\left(k_t\left(\mathbf{s}^{t-1}\right), z_t\left(\mathbf{s}^t\right) h_t\left(\mathbf{s}^t, k_0\right)\right) - w_t\left(\mathbf{s}^t\right) h_t\left(\mathbf{s}^t\right) - \underbrace{c\left(\frac{k_{t+1}\left(\mathbf{s}^t\right)}{k_t\left(\mathbf{s}^{t-1}\right)}\right) k_t\left(\mathbf{s}^{t-1}, k_0\right)}_{\text{cost of adjustmnet}}\right]
\end{aligned}
$$

where the adjustment cost function c is convex.

- ▷ $V_0\left(\mathbf{s}^0, k_0\right)$ is the value of the firm (expected present discounted value of future profits);
- ▷ k_0 is the initial level of capital;
- ▷ f is an increasing, concave, constant returns to scale (CRS) production function;
- ▷ c represents the adjustment cost that is increasing, convex, constant returns to scale function (continuously differentiable). The CRS assumption implies that the adjustment cost is proportional to different level of k_0.
- ▷ If $k_{t+1} = k_t$, then adjustment cost if equal to $c(1)\, k_t$. We can think of this as depreciation.

Claim 4.1. The value function is linear in k_0.

Proof. One thing that the firm with twice as many capital stock as another firm will choose twice as much labor (h_t) and capital (k_{t+1}). This would immediately give us that the value function is linear under the assumption that the production function is CRS.

1. For any $\lambda > 0$, $V_0(\lambda k_0) \geq \lambda V_0(k_0)$.

   Proof. Denote $k^*_{t+1}(\mathbf{s}^t; k_0)$ as our optimal policy and take $k'_0 = \lambda k_0$. First, the policy $k_{t+1}(\mathbf{s}^t; \lambda k_0) \equiv \lambda k^*_{t+1}(\mathbf{s}^t, k_0)$ is still feasible. Second, the cost of adjustment are the same since

$$\frac{k^*_{t+1}(\mathbf{s}^t; k_0)}{k^*_t(\mathbf{s}^{t-1}; k_0)} = \frac{k^*_{t+1}(\mathbf{s}^t; \lambda k_0)}{k^*_t(\mathbf{s}^{t-1}; \lambda k_0)}$$

   and the production is also the same since it's CRS. Therefore, we have $V_0(\lambda k_0) \geq \lambda V_0(k_0)$. ■

2. For any $\lambda > 0$, $V_0(k_0) \geq \frac{1}{\lambda} V_0(\lambda k_0)$.

   Proof. This follows from a symmetric argument as before. ■

Combining these two equations, we arrive at $v_0(\lambda k_0) = \lambda v_0(k_0)$. Note that this argument is not unique to the model with convex adjustment costs. We can also find a linear optimal policy. ■

Now since the value function is linear,

$$v_0(\mathbf{s}^0, k) = v_0(\mathbf{s}^0) \equiv \underbrace{\frac{V_0(\mathbf{s}^0, k)}{k}}_{\text{Tobin's Q}} = \frac{\partial V_0(\mathbf{s}^0, k)}{\partial k}$$

Taking the first-order conditions:

$$\begin{aligned}
[h_t] &: f_{h,t}(\mathbf{s}^t) = w_t(\mathbf{s}^t) \\
[k_0] &: v_0(\mathbf{s}^0) = f_{k,0}(\mathbf{s}^0) - c\left(\frac{k_1}{k_0}\right) + c'\left(\frac{k_1}{k_0}\right)\frac{k_1}{k_0}
\end{aligned}$$

Define $\gamma_t(\mathbf{s}^t) = k_{t+1}(\mathbf{s}^t) / k_t(\mathbf{s}^{t-1})$ i.e. the growth rate of capital, to rewrite:

$$\begin{aligned}
[h_t] &: f_{h,t}(\mathbf{s}^t) = w_t(\mathbf{s}^t) \\
[k_0] &: v_0(\mathbf{s}^0) = f_{k,0}(\mathbf{s}^0) \underbrace{- c(\gamma_0(\mathbf{s}^0)) + c'(\gamma_0(\mathbf{s}^{-1}))\gamma_0(\mathbf{s}^0)}_{[1]}
\end{aligned}$$

- ▷ $f_{h,t}$ is homogenous of degree zero, so $[h_t]$ tells us that k_t/h_t depends on w_t and z_t.
- ▷ Note that differentiating $[k_0]$ with respect to γ yields a positive number, so [1] is increasing in γ_0. Therefore, growth rate of capital is increasing in Tobin's Q minus the marginal productivity of capital. $(v_0(\mathbf{s}^0) - f_{k,0}(\mathbf{s}^0))$. Note that there is nothing special about period zero, so the same argument applies to any generic time t.
- ▷ The $-f_{k,t}(\mathbf{s}^t)$ appears because firms are forward-looking. Firms will make decisions that affect future profitability. When we compute the marginal value to the firm, it's affected by two things: (1) If I have more capital today, it raises output today; (2) It increases the value of the firm going forward. The difference is therefore the value to the firm after taking out the profits of having more capital today. It's saying that the investment of the firm is increasing in this forward-looking aspect of the marginal value of capital for this firm.
- ▷ Another way to do this is to write out the expression for the value of the firm *after* removing the current value net of the labor cost. In other words, we can consider the value of the firm ex-dividend.
- ▷ For short time-periods, $v_t(\mathbf{s}^t)$ will be orders of magnitude larger. It also says that this will be the only thing that predicts investment decision.

4.1.2 Interpretation

The results show that the expected value of the firm v_0^* is sufficient to determine the optimal investment rate for the firm. In other words, investment is only a function of average value of a unit of capital—called the Tobin's q:

$$\frac{v(k_0)}{k_0} := \text{Tobin's average } q,$$
$$v'(k_0) := \text{Tobin's marginal } q.$$

Of course, investment decision is decided on the marginal (rather than average). However, as we can see here, under our set of assumptions (perfect competition and constant returns to scale production and investment cost functions), $v(k_0)$ is linear such that average and marginal Tobin's q are equal. Since $v(k_0)$ is linear, as it is under our set of assumptions (perfect competition and constant returns to scale production and investment cost functions), the two are equal. This helps us empirically since average q is easier to measure than marginal q.

▷ If we did not have constant returns to scale, then there will be a wedge between average and marginal q. Hayashi (1982) constructs this wedge that also includes the effect of taxes. He then regresses aggregate investment rate on this tax-adjusted measure of marginal q and finds that it explains around 46% of the variation (R^2).

However, this represented the high point of q-theory. Subsequent work has found that q-theory does not perform well on micro data although one might expect it to work better (since micro data suffers less from the effect of aggregating over heterogenous firms). They find that other things explains investment behaviour, despite Tobin's q being a sufficient statistic theoretically.

▷ For example, current cash flow of the firms, even conditional on tax adjusted Tobin's q. (This did not appear in our model since we assumed perfect access to capital markets.)

One possibility of failure is that we are not measuring q accurately.

▷ This might be due to the fact that what we observe in the data is the average q and so we may be misspecifying how we adjust the average q to obtain estimates of marginal q.

▷ Another possibility is that we may have the whole adjustment cost specification wrong. The nature of adjustment cost often seems non-convex and exhibits irreversibility of investment (e.g. a wedge between a buy and sell price of capital). It turns out that if we can calculate marginal q in models with non-convex costs, marginal q is still a sufficient statistic for investment. But the relationship between marginal q and investment becomes highly non-linear—we can obtain inaction regions—which would be difficult to pick up in regressions.

4.2 Non-Convex Adjustment Cost

The literature has focused on more natural representations of adjustment costs. We will cover the following representations:

1. **Wedge between purchase and sale price of capital**

   In one period, you buy a truck and you sell a truck the next day. You will make a negative profit on this transaction. Sources of the wedge may include adverse selection and search friction. An interesting research question may be looking into the sources of this wedge.

2. **Fixed cost of adjusting the capital**

   One thing you can think about is a firm running a factory and wants to move to a larger factory. So it builds a new building – adjusting the size of your building is too expensive each period. In an environment like that, the firm will mostly stay in its building. It bears a lot of resemblance to the menu cost model.

4.2.1 Wedge between purchase and sale price of capital

Consider a firm whose revenue is determined by capital and the technology shock:

$$k_t\left(\mathbf{s}^t\right)^\alpha z_t\left(\mathbf{s}^t\right)^{1-\alpha}$$

When it buys capital, it costs

$$-p_b \max\left\{k_t\left(\mathbf{s}^t\right)-k_{t-1}\left(\mathbf{s}^{t-1}\right),0\right\}$$

and when it sells capital, it earns

$$p_s \max\left\{k_{t-1}\left(\mathbf{s}^{t-1}\right)-k_t\left(\mathbf{s}^t\right),0\right\}$$

where $p_b \geq p_s$.

Assumptions

- ▷ Firm discounts profits at rate $\beta < 1$
- ▷ Productivity $z_{t+1}\left(\mathbf{s}^{t+1}\right)/z_t\left(\mathbf{s}^t\right)$ is i.i.d.

Using the assumptions above, we can show that the value function is homogeneous of degree one. This is because the profit functions are homogeneous of degree one, and the productivity process keeps the probability distribution also homogeneous of degree one. Using the argument we used earlier that the value function is linear, we have the result that

$$V\left(k_{t-1},z_t\right) \text{ is homogeneous of degree 1in } (k,z)$$

Furthermore, define

$$v\left(\frac{k}{z}\right) \equiv \frac{V(k,z)}{z} = V\left(\frac{k}{z},1\right)$$

and let $x \equiv k/z$.

Recursive Formulation We can write the value function in a recursive form:

$$V(k,z) = \max_{k'}\left\{\left(k'\right)^\alpha z^{1-\alpha} - p_b\max\left\{k'-k,0\right\} + p_s\max\left\{k-k',0\right\} + \beta\mathbb{E}\left[v\left(k',z'\right)|z\right]\right\}$$

Dividing each side by z:

$$\begin{aligned} v(x) &= \max_{x'}\left\{\left(x'\right)^\alpha - p_b\max\left\{x'-x,0\right\} + p_s\max\left\{x-x',0\right\} + \beta\mathbb{E}\left[\frac{z'}{z}v\left(\frac{k'}{z'},1\right)|z\right]\right\} \\ &= \max_{x'}\left\{\left(x'\right)^\alpha - p_b\max\left\{x'-x,0\right\} + p_s\max\left\{x-x',0\right\} + \beta\mathbb{E}\left[\gamma v\left(\frac{x'}{\gamma}\right)|x\right]\right\} \end{aligned}$$

where $x' = k'/z$.

- ▷ Without adjustment costs, we can set $p_b = p_s = p$, in which case

$$v(x) = \max_{x'}\left(\left(x'\right)^\alpha - p\left(x'-x\right) + \beta\mathbb{E}\left[\gamma v\left(\frac{x'}{\gamma}\right)|x\right]\right)$$

Differentiating, we get:

$$\begin{aligned} [x] &: v'(x) = p \\ \left[x'\right] &: \alpha\left(x'\right)^{\alpha-1} - p + \beta\mathbb{E}\left[v'\left(\frac{x'}{\gamma}\right)\right] = 0 \\ &\Rightarrow \alpha\left(x'\right)^{\alpha-1} = p(1-\beta) \end{aligned}$$

since

$$v'\left(\frac{x'}{\gamma}\right) = v'\left(\frac{k'/z}{\gamma}\right) = v'\left(\frac{k'}{z\gamma}\right) = v'\left(\frac{k'}{z'}\right) = v'(x) = p$$

Specifying the Productivity Process We assume $\gamma \in \left\{e^{-\Delta}, 1, e^{\Delta}\right\}$ with probability $\pi, 1-2\pi$, and π. In this case, we have

$$\mathbb{E}\left[\log \gamma\right] = 0, \quad \operatorname{Var}\left[\log \gamma\right] = 2\Delta^2 \pi$$

Note that the probability specification implies that there is no drift. We also assume in addition that

$$1 > \beta \mathbb{E}\left[\gamma\right] = \beta\left[\pi e^{-\Delta} + (1-2\pi) + \pi e^{\Delta}\right]$$

Conjecturing Optimal Policy We *conjecture* that the optimal policy consists of two thresholds $\underline{x}$ and $\bar{x}$:

$$x' = \begin{cases} \underline{x} & x < \underline{x} \\ x & \bar{x} \in [\underline{x}, \bar{x}] \\ \bar{x} & x > \bar{x} \end{cases}$$

with $\bar{x} > \underline{x}$. We consider each case in detail.

1. $x \in [\underline{x}, \bar{x}]$

   In this case, the value function becomes:

   $$v(x) = x^{\alpha} + \beta\left[\pi e^{-\Delta} v\left(x e^{\Delta}\right) + (1-2\pi) v(x) + \pi e^{\Delta} v\left(x e^{-\Delta}\right)\right]$$

   which looks like a second-order difference equation.

   ▷ To solve this, we can define a new variable y as

   $$y = \frac{\log x}{\Delta}, \quad w(y) = v(x)$$

   which would make the above look like a standard second-order difference equation of the following form:

   $$w(y) = e^{\alpha \Delta y} + \beta\left[\pi e^{-\Delta} w(y+1) + (1-2\pi) w(y) + \pi e^{\Delta} w(y-1)\right]$$

   ▷ Solving for the homogeneous part, we have

   $$w(y) = c_1 \lambda_1^y + c_2 \lambda_2^y$$

   where λ_1, λ_2 solve

   $$\lambda = \beta\left[\pi e^{\Delta} + (1-2\pi)\lambda + \pi e^{-\Delta}\lambda^2\right]$$

   ▷ Solving for the non-homogeneous part, it is enough to propose a functional form that satisfies the non-homogeneous part. We will use

   $$w(y) = k e^{\alpha \Delta y}$$

   which yields:

   $$v(x) = \frac{x^{\alpha}}{1 - \beta\left(\pi e^{\Delta(1-\alpha)} + (1-2\pi) + \pi e^{-\Delta(1-\alpha)}\right)} + c_1 x^{\frac{\log \lambda_1}{\Delta}} + c_2 x^{\frac{\log \lambda_2}{\Delta}}$$

   It remains to pin down $\underline{x}, \bar{x}, c_1, c_2$.

- ▷ Consider an arbitrary $\underline{x} < \bar{x}$. Then we can write based on our conjecture (using intuition!):

$$v\left(\underline{x}e^{-\Delta}\right) = -p_b\left(\underline{x} - \underline{x}e^{-\Delta}\right) + v\left(\underline{x}\right)$$

Expanding at $\Delta = 0$:

$$v\left(\underline{x}\right) - \underline{x}v'\left(\underline{x}\right)\left(\Delta - 0\right) = -p_b\underline{x} + v\left(\underline{x}\right) + p_b\underline{x} - p_b\underline{x}\left(\Delta - 0\right)$$

Rearranging:

$$-\underline{x}v'\left(\underline{x}\right)\Delta = -p_b\underline{x}\Delta$$

which yields:

$$\underline{v'\left(\underline{x}\right) = p_b}$$

Similarly, we get a similar expression:

$$\underline{v'\left(\bar{x}\right) = p_s}$$

These conditions are called **smoothing pasting.**

- ▷ Thus, we can write $c_1\left(\underline{x}, \bar{x}\right)$ and $c_2\left(\underline{x}, \bar{x}\right)$. Note that if $\left(\underline{x}, \bar{x}\right)$ maximizes c_1, they also maximize c_2. Thus, we can substitute $\underline{x}$ and $\bar{x}$ and maximizing either c_1 or c_2 eventually amounts to

$$\underline{v''\left(\underline{x}\right) = v''\left(\bar{x}\right) = 0}$$

which is called **super contact.**

2. $x < \underline{x} \Rightarrow x' = x$

Note that we don't go the optimal point as in the fixed-cost case. If you have a fixed-cost model, then the fixed cost is a sunk cost and the firm will move to the optimal level. Here the firm will not since we don't have a fixed cost here.

Here's a brute force recipe for a more general problem that we solved.

- ▷ Consider $v\left(x\right) = f\left(x\right) + c_1x^{\mu_1} + c_2x^{\mu_2}$ where

$$v'\left(\underline{x}\right) = p_b, \quad v'\left(\bar{x}\right) = p_s$$

and $c_1 = c_1\left(\underline{x}, \bar{x}\right), c_2 = c_2\left(\underline{x}, \bar{x}\right)$.

- ▷ Consider $v\left(x; \underline{x}, \bar{x}\right) = f\left(x\right) + c_1\left(\underline{x}, \bar{x}\right)x^{\mu_1} + c_2\left(\underline{x}, \bar{x}\right)x^{\mu_2}$ where

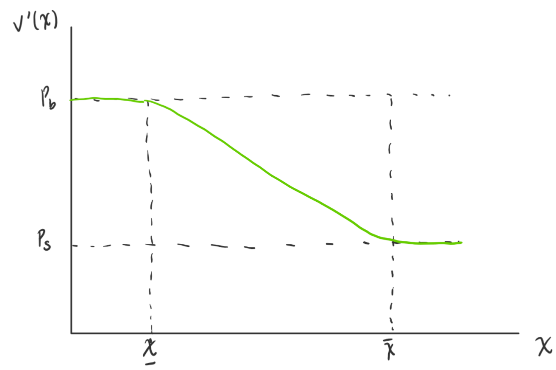
$$v_1\left(\underline{x}; \underline{x}, \bar{x}\right) = p_b, \quad v_1\left(\bar{x}; \underline{x}, \bar{x}\right) = p_s$$

The super contact condition can be derived as

$$v_{11}\left(\underline{x}; \underline{x}, \bar{x}\right) + \frac{\partial c_1\left(\underline{x}, \bar{x}\right)}{\partial \underline{x}}\left(\cdot\right) + \frac{\partial c_2\left(\underline{x}, \bar{x}\right)}{\partial \underline{x}}\left(\cdot\right) = 0$$

where the last two terms are equal to zero due to optimality.

A graphical depiction:



- $\triangleright$ According to the figure, between $\underline{x}$ and $\bar{x}$, the firm will not trade capital. This is consistent.
- $\triangleright$ Tobin's q is a perfect predictor of investment.
- $\triangleright$ Running a linear regression will be extremely misleading.

Ergodic Distribution via Kolmogorov Forward Equation Assume for simplicity that $\bar{x} - \underline{x} = n\Delta$. (In the menu cost model, we assumed that the firms want to adjust to the halfway point between the minimum and maximum prices). Furthermore, for any $x' \in [\underline{x} + \Delta, \bar{x} - \Delta]$, define the probability distribution of xs at time t **after** the investment has been made:

$$\phi_t\left(x'\right) = \pi\phi_{t-1}\left(x'e^{-\Delta}\right) + (1-2\pi)\,\phi_{t-1}\left(x'\right) + \pi\phi_{t-1}\left(x'e^{\Delta}\right)$$

Note that with positive productivity shock, x declines. At the boundaries:

$$\phi_t\left(\bar{x}\right) = (1-\pi)\,\phi_{t-1}\left(\bar{x}\right) + \pi\phi_{t-1}\left(\bar{x}e^{-\Delta}\right)$$
$$\phi_t\left(\underline{x}\right) = (1-\pi)\,\phi_{t-1}\left(\underline{x}\right) + \pi\phi_{t-1}\left(\underline{x}e^{\Delta}\right)$$

The ergodic distribution can be found by setting:

$$\phi_t\left(x'\right) = \phi\left(x'\right), \forall t$$

- $\triangleright$ Plugging into the equation:

$$\phi\left(x'\right) = \frac{\phi\left(x'e^{-\Delta}\right) + \phi\left(x'e^{\Delta}\right)}{2}$$
$$\phi\left(\bar{x}\right) = \phi\left(\bar{x}e^{-\Delta}\right)$$
$$\phi\left(\underline{x}\right) = \phi\left(\underline{x}e^{\Delta}\right)$$

  and since ϕ is a probability distribution, we have

$$\sum_{j=0}^{n} \phi\left(\underline{x}e^{j\Delta}\right) = 1$$

- $\triangleright$ We can see that the solutions are homogeneous of degree one.
- $\triangleright$ Plugging in $x' = \bar{x}e^{-\Delta}$, we have

$$\phi\left(\bar{x}\right) = \phi\left(\bar{x}e^{-\Delta}\right) = \frac{\phi\left(\bar{x}e^{-2\Delta}\right) + \phi\left(\bar{x}\right)}{2} \Rightarrow \phi\left(\bar{x}e^{-2\Delta}\right) = \phi\left(\bar{x}\right)$$

  By induction, we have a uniform distribution of $\log\left(k/z\right)$. The key assumption is that there is no drift in this process.

This will deliver serially correlated investment, which doesn't seem much different from the model with convex adjustment costs.

Incorporating into RBC Model The biggest issue is that the state space is too large, since you have to keep track of the distribution of the firms.

- ▷ By increasing π, we can increase the volatility of the shocks. Then firms will delay purchase of the capital, so the boundaries will widen (new values). Following this, there will be a period where investments and disinvestments are really low.

4.2.2 Fixed Cost of Adjustments

What changes in this new setup?

- ▷ Anytime you want to adjust, you need to pay a cost λ times your productivity z. This implies that you don't want to make small adjustments to your capital.
- ▷ The optimal policy will now depend on three numbers: $\underline{x}, \bar{x}, x^*$ where x^* is a quantity that the firm adjusts to.

The Bellmans will look similar to what we did here. What changes is what c_1 and c_2 are and how they relate to the three thresholds. So we now have five numbers to figure out. How can we find this?

- ▷ If we are at $\underline{x}$, it's like being at x^* except for the fact that you have to buy capital to get up to x^* and pay the adjustment cost. This is called "value matching conditions" that look similar to the smooth pasting conditions.
- ▷ Another condition is from optimality, the optimal choice of the thresholds. Here they are first-derivative conditions and also called smooth pasting conditions. At the points when you are adjusting, the marginal value of capital is one. If you took a firm that was below the adjustment threshold, it was buying capital;
- ▷ We will NOT get a uniform distribution. We will get a tent.

See problem set questions for more details.

4.3 Financing Constraints

There is a lot of empirical evidence regarding the nature of financing constraints and the imperfect nature of capital markets. There is a huge industry in finance that moves capital to the most productive usages. One thing that papers like Brunnermeier and Sannikov / Kiyotaki and Moore emphasize that the wealth of people who are in the position to make investments will matter for the aggregate behavior of the economy.

4.3.1 Motivation

Hubbar (JEL 1998) finds that cash flows matter for firms' investment behaviour. In the model we considered so far, we implicitly assumed that firms have perfect access to capital markets. However, for cash flows to matter, we would need some kind of violations of the Modigliani-Miller theorem.

One concrete example is the price of earthquake insurance following earthquakes. The consequences of earthquakes are well-known. Earthquakes wipes out capital, and the price of earthquakes goes up. This is because people who are not good at providing earthquake insurance had to provide insurance since traditional players had their capital wiped out.

4.3.2 Bernanke & Gertler (AER 1989)

Costly state verification: Existence of moral hazard in financing projects creates a wedge (if a Kickstarter project fails, could it be because the developer decided to buy a new kitchen with the money from the funders?)

- ▷ In case the project fails, the investor would have to undertake costly verification of the "effort" exerted by the firm.
- ▷ The problem is the same in case the funding comes from banks—the cost of verification (and perhaps also for monitoring) would be part of the interest rate. These imply that projects financed by external funding would have to have a higher expected return than if the firm had cash on hand.

4.3.3 Kiyotaki & Moore (JPE 1997); Brunnermeier & Sannikov (AER 2014)

The amount that the banks would lend to firms depends on the amount of collateral that the firm can post. Requiring firms to post collateral incentives the firm to exert appropriate effort (think Myerson's model of moral hazard).

- ▷ We can also think about endogenising the value of collateral.
- ▷ For example, if collateral value collapses, then firms would not be able to borrow and the collateral may be used to finance less efficient uses. Self-fulfilling equilibrium in Kiyotaki Moore (1997), and large responses to small shocks in Brunnermeier Sannikov (2014).

5 Bubbles

We gives ourselves a working definition of a bubble:

Definition 5.1. A bubble is a situation in which the price of an asset exceeds the discounted present value of its future dividends.

This implies that all fiat money are bubbles since it pays no dividends. In NCG, there can be no bubbles. Bitcoin is also a timely example.

5.1 Failure of the First Welfare Theorem (FWT)

We will find that rational bubbles and failure of the First Welfare Theorem are closely related. We first study the failure of FWT in an OLG-like setting. However, to emphasis the fact that the failure of the FWT is unrelated to time, we work with a static model.

5.1.1 Setup

We start with an endowment economy. Suppose there are infinitely many individuals indexed by $i \in \mathbb{Z}$ (i can be positive/negative integers) and infinitely many goods indexed by $j \in \mathbb{Z}$. We assume that each individual has the same measure.

- ▷ Endowments: Let e_{ij} denote individual i's endowment of good j. We assume that individual i has endowments of just good $j = i$ and $j = i + 1$; i.e.

$$\begin{aligned} e_{ii} &\coloneqq e_1 > 0, \\ e_{ii+1} &\coloneqq e_2 > 0, \\ e_{ij} &\coloneqq 0, \ \forall j \neq i, i+1. \end{aligned}$$

- ▷ Utility: Let c_{ij} denote individual i's consumption of good j. We assume that i's utility is given by

$$\log c_{ii} + \log c_{ii+1}$$

  so that i only values goods $j = i$ and $j = i + 1$. Let q_i^{i+1} denote the price of good $i + 1$ in terms of good i.

- ▷ Prices: q_i^{i+1} is the amount of good i needed to purchase one unit of good $i + 1$
- ▷ Budget constraint: $c_{i,i} + q_i^{i+1} c_{i,i+1} = e_{i,i} + q_i^{i+1} e_{i,i+1} = e_1 + q_i^{i+1} e_2$

The setup here is similar to OLG but differs in that there is no concept of "initial period" and i (which corresponds to t in the OLG) extends is not one directional. Moreover, any individual can trade with another individual (in OLG, you cannot trade with a "dead" generation).

5.1.2 Characterizing the Equilibrium

Definition 5.2. *(Competitive equilibrium)* A competitive equilibrium is a nonnegative sequence $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ and $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$ such that

1. utility is maximised; i.e. for all $i \in \mathbb{Z}$, $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ solves

$$\begin{aligned} \max_{\{c_{ii}, c_{ii+1}\}} \quad & \log c_{ii} + \log c_{ii+1} \\ s.t. \quad & c_{ii} + q_i^{i+1} c_{ii+1} \leq e_{ii} + q_i^{i+1} e_{ii+1} = e_1 + q_i^{i+1} e_2 \end{aligned}$$

2. goods market clears; i.e.
$$c_{ii} + c_{i-1i} = e_{ii} + e_{i-1i} = e_1 + e_2, \ \forall i \in \mathbb{Z}.$$

Let's characterize the equilibrium.

▷ Let λ_i denote the Lagrange multiplier on the budget constraint in the individual's utility maximisation problem.

▷ The first-order conditions are
$$\frac{1}{c_{ii}} = \lambda_i, \ \frac{1}{c_{ii+1}} = q_i^{i+1}\lambda_i \Rightarrow c_{ii} = q_i^{i+1} c_{ii+1}$$

▷ Substituting into the budget constraint:
$$c_{i,i} = \frac{e_1 + q_i^{i+1} e_2}{2}, \ c_{i,i+1} = \frac{e_1 + q_i^{i+1} e_2}{2q_i^{i+1}}.$$

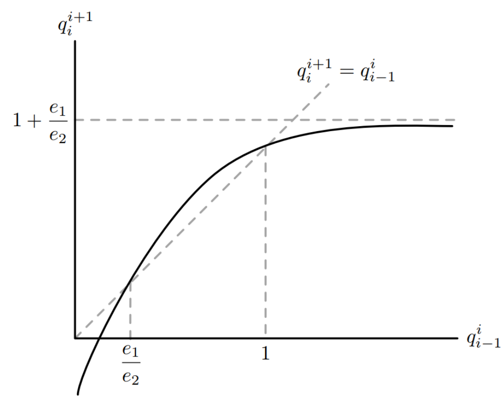
▷ Imposing goods market clearing:
$$\begin{aligned} e_1 + e_2 &= c_{ii} + c_{i-1,i} \\ &= \frac{e_1 + q_i^{i+1} e_2}{2} + \frac{e_1 + q_{i-1}^{i} e_2}{2q_{i-1}^{i}} \\ \Leftrightarrow q_i^{i+1} e_2 &= 2\left(e_1 + e_2\right) - e_1 - \frac{e_1 + q_{i-1}^{i} e_2}{q_{i-1}^{i}} \\ &= e_1 - \frac{e_1}{q_{i-1}^{i}} + 2e_2 - e_2 \\ \Leftrightarrow q_i^{i+1} &= 1 + \frac{e_1}{e_2}\frac{q_{i-1}^{i} - 1}{q_{i-1}^{i}}. \end{aligned}$$

Therefore, any $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$ that satisfies this sequence is associated with an equilibrium. Now we show that this yields multiple equilibria.

Multiple Equilibria We will show that the difference equation has only two fixed points given by
$$q_{i-1}^{i} = 1, \frac{e_1}{e_2}.$$

▷ To see that these are the only two fixed points, note that the RHS
$$f\left(q_{i-1}^{i}\right) \equiv 1 + \frac{e_1}{e_2}\frac{q_{i-1}^{i} - 1}{q_{i-1}^{i}}$$
is strictly increasing and concave in q_{i-1}^{i}.

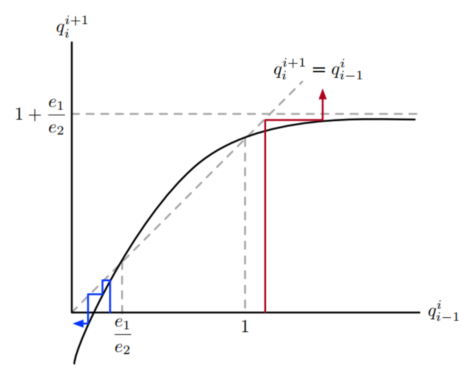
▷ Drawing the figure:



- ▷ It is clear from the figure that there can be at most two fixed points, and the knife edge case is when $e_1 = e_2$.

This tells us that any $q_0^1 \in [e_1/e_2, 1]$ is consistent with equilibrium.

- ▷ Assuming $e_1/e_2 < 1$, then if $q_0^1 < e_1/e_2$, then $q_i^{i+1} < 0$ for some $i \geq 0$. Symmetrically, if $q_0^1 > 1$, then $q_i^{i+1} > 1 + e_1/e_2$ for some $i \leq 0$.
- ▷ To see this, first note that for any $q_{i-1}^i < e_1/e_2$, if we solve the difference equation forward, we will eventually find that for some $i' > i$, $q_{i'-1}^{i'} < 0$ (blue arrow in the figure below). Similarly, for any $q_{i-1}^i > 1$, there exists $i'' > i'$ such that $q_{i''-1}^{i''}$ does not exist (red arrow in the figure below).



Let's focus on the two special equilibria: $q_i^{i+1} = 1$ for all i and $q_i^{i+1} = e_1/e_2$ for all i.

- ▷ If $q_i^{i+1} = 1, \forall i$, then

$$c_{i,i} = \frac{e_1 + e_2}{2}, \quad c_{i,i+1} = \frac{e_1 + e_2}{2}$$

- ▷ If $q_i^{i+1} = e_1/e_2$, then we have a "no-trade" equilibria where

$$c_{i,i} = e_1, \quad c_{i,i+1} = e_2$$

Ranking the Equilibria Now, for any $q_{i-1}^i \in [e_1/e_2, 1]$, we converge to either of the two fixed points so that the difference equation is satisfied for such sequence. We now claim that the equilibria are Pareto rankable. In particular,

- ▷ The best equilibrium is $q_{i-1}^i = q_i^{i+1} = 1$.

▷ the worst equilibrium is $q^i_{i-1} = q^{i+1}_i = e_1/e_2$;

The argument is through revealed preference: since we know that individuals are utility maximising, and consuming the endowment is always possible for any relative prices, for individuals to pick another bundle, it must be that the new bundle gives him a higher utilities (this is a revealed preference argument).

▷ To be more rigorous, consider the derivative of the utility with respect to q^{i+1}_i when the individual is consuming the optimal bundle.

$$\begin{aligned}\frac{\partial}{\partial q^{i+1}_i}\left(\log\frac{e_1+q^{i+1}_i e_2}{2}+\log\frac{e_1+q^{i+1}_i e_2}{2q^{i+1}_i}\right) &= \frac{\partial}{\partial q^{i+1}_i}\left(\log\frac{e_1+q^{i+1}_i e_2}{2}+\log\frac{\frac{e_1}{q^{i+1}_i}+e_2}{2}\right)\\ &= \frac{e_2/2}{\frac{e_1+q^{i+1}_i e_2}{2}}+\frac{-\frac{e_1}{2\left(q^{i+1}_i\right)^2}}{\frac{e_1+q^{i+1}_i e_2}{2q^{i+1}_i}}\\ &= \frac{1}{e_1+q^{i+1}_i e_2}\left(e_2-\frac{e_1}{q^{i+1}_i}\right).\end{aligned}$$

▷ The derivative is smallest when q^{i+1}_i is lowest, which is e_1/e_2, the value of the derivative when $q^{i+1}_i = e_1/e_2$ is

$$\frac{1}{e_1+q^{i+1}_i e_2}\left(e_2-\frac{e_1}{e_1/e_2}\right)=0.$$

▷ Hence, we realise that the higher the q^{i+1}_i, the higher is the individual's utility from consuming the optimum bundle (since the derivative is strictly positive when $q^{i+1}_i > e_1/e_2$). This tells us that that $q^i_{i-1} = q^{i+1}_i = 1$ is the best equilibrium (in any equilibrium sequence of prices, the relative prices cannot be greater than one).

So we conclude that not all competitive equilibria are Pareto optimal; i.e. the First Welfare Theorem fails.

5.1.3 Failure of the FWT

There was a step in the proof for the FWT where we summed all the individual's budget constraints. In this case, if $e_1 > e_2$, then q^{1000}_0 for example will be explosive and the conditions for the FWT will break. Another critical part that contributes to the breakdown of FWT is that there are infinitely many agents.

5.2 Bubbles in an OLG model

The previous model was to illustrate the failure of the FWT. Now we introduce dynamics and introduce an asset to see how a bubble can emerge in this economy. Specifically, this asset will be intrinsically worthless i.e. money which pays zero dividend.

5.2.1 Setup

For each cohort born at $t \in \{0, 1, \dots\}$, suppose the "young cohort" solves the following problem:

$$\begin{aligned}\max_{c_{1,t}c_{2,t}} \quad & \log c_{1,t} + \log c_{2,t+1}\\ s.t. \quad & c_{1,t} + q^{t+1}_t c_{2,t+1} \le e_1 + q^{t+1}_t e_2.\end{aligned}$$

We assume a unit measure of initial old (at period 0) whose preference is simply

$$\max_{c_{2,0}} \quad \log c_{2,0}$$
$$s.t. \quad c_{2,0} \leq e_2.$$

Definition 5.3. Competitive equilibrium is $\left\{c_{1,t}, c_{2,t}, q_t^{t+1}\right\}_{t=0}^{\infty}$ such that every cohort maximises their utility and resource constraint holds:

$$c_{1,t} + c_{2,t} = e_1 + e_2, \ \forall t.$$

5.2.2 Characterizing the Equilibrium

Observe that for those born at $t = 0, 1, \ldots$, the problem is the same as before so that optimality and market clearing require

$$q_i^{i+1} = 1 + \frac{e_1}{e_2} \frac{q_{i-1}^i - 1}{q_{i-1}^i}, \ \forall t \geq 1.$$

1. For period $t = 0$, we know that the initial old will consume all his endowment so that $c_{2,0} = e_2$.
2. By market clearing, i.e. $c_{1,0} + c_{2,0} = e_1 + e_2$, it follows that
$$c_{1,0} = e_1.$$
3. For this to be optimal for the cohort born in period 0,
$$c_{1,0} = \frac{e_1 + q_0^1 e_2}{2} = e_1 \Rightarrow q_0^1 = \frac{e_1}{e_2}.$$
4. This gives us a boundary condition for the difference equation. But since e_1/e_2 is a fixed point of the difference equation, it follows that
$$q_t^{t+1} = \frac{e_1}{e_2}, \ \forall t \geq 0.$$
5. This, in turn, implies that
$$c_{1,t} = e_1, \ c_{2,t} = e_2, \ \forall t \geq 0,$$
which gives us the (unique) competitive equilibrium.

5.2.3 Examining Pareto Efficiency

Once again, we want to see if the FWT breaks down.

- ▷ If $e_1 < e_2$, then the standard FWT proof goes through and the CE is PO.
- ▷ If $e_1 = e_2$, the proof breaks down but the CE is PO.
- ▷ If $e_1 > e_2$, the competitive equilibrium is not Pareto optimal— we can make everyone at least as well off by changing consumption bundles to
$$c'_{1,t} = c'_{2,t} = \frac{e_1 + e_2}{2}.$$
since:

1. This is a feasible allocation.
2. Since the utility function is concave, if $e_1 \neq e_2$,

$$\log c_{1,t} + \log c_{2,t} < \log c'_{1,t} + \log c'_{2,t}, \ \forall t \geq 0.$$

3. For the initial old,

$$e_1 > e_2 \Leftrightarrow e_2 = \log c_{2,0} < \log c'_{2,0} = \frac{e_1 + e_2}{2}.$$

Note that if $e_1 > e_2$, in the competitive equilibrium we have:

$$q_t^{t+1} = \frac{e_1}{e_2} > 1.$$

Here, we think of q_t^{t+1} as the price of goods in the next period in terms of goods this period; i.e.

$$q_t^{t+1} = \frac{1}{1+r},$$

where r is the interest rate. Hence, we realise that this equilibrium is equivalent to saying that

$$\frac{1}{1+r} > 1 \Leftrightarrow r < 0.$$

We interpret this to mean that the interest rate is less than the growth rate of the economy, which is zero in this case. Moreover, note that the value of endowment blows up in this case:

$$\sum_{t=0}^{\infty} \left(\frac{e_1}{e_2}\right)^t (e_1 + e_2) = \infty.$$

5.2.4 Bubbles

We now "give" to the initial old a piece of paper that lasts forever (and no more is produced). Let p_t denote the price of the paper. We will show that there exists a competitive equilibrium in which $p_t = 0$ for all t but, if $e_1/e_2 > 1$, then there exists many equilibria in which $p_t \neq 0$ for all t. Punchline: bubbles can exist in competitive equilibria.

Definition 5.4. Competitive equilibrium is $\{c_{1,t}, c_{2,t}, a_t, p_t\}_{t=0}^{\infty}$ such that:

1. For those born in $t = 0, 1, \ldots,$

$$\begin{aligned} \max_{\{c_{1,t}, c_{2,t}, a_t\}} \quad & \log c_{1,t} + \log c_{2,t+1} \\ s.t. \quad & c_{1,t} + p_t a_t \leq e_1, \\ & c_{2,t+1} \leq e_2 + p_{t+1} a_t; \end{aligned}$$

2. For the initial old:

$$\begin{aligned} \max_{\{c_{2,0}\}} \quad & \log c_{2,0} \\ s.t. \quad & c_{2,0} \leq e_2 + p_0; \end{aligned}$$

3. Resource constraints hold:

$$a_t = 1,$$
$$c_{1,t} + c_{2,t} = e_1 + e_2,$$

for all t.

No bubble equilibrium We want to find a no-bubble equilibrium i.e. an equilibrium where $p_t = 0$. So we would have $p_t = 0$, $c_{1,t} = e_1$ and $c_{2,t} = e_2$ for all t.

- ▷ To see this, consider the case in which the individual faces $p_{t+1} = 0$.
- ▷ Since purchasing asset in period t does not yield any benefit in period $t+1$ (since $p_{t+1} = 0$), the optimal consumption profile is for the individual to consume their endowment in each period.
- ▷ Moreover, in order for the markets to clear, i.e. $a_t = 1$, we must have $p_t = 0$ since, if $p_{t+1} = 0$, the demand for asset is is zero for any positive p_t, and arbitrary if p_t is zero. This tells us that $p_{t+1} = 0$ implies $p_t = 0$, which, in turn, implies that $p_t = 0$ for all t.

Equilibrium with a bubble Now suppose $p_t > 0$ for all t.

- ▷ The budget constraint becomes:
$$c_{it} + \frac{p_t}{p_{t+1}} c_{2,t+1} = e_1 + \frac{p_t}{p_{t+1}} e_2$$
- ▷ If we let $p_t/p_{t+1} = q_t^{t+1}$, then the problem is isomorphic to the OLG model we considered previously. Hence, optimality and market clearing requires
$$\begin{aligned}
\frac{p_t}{p_{t+1}} &= 1 + \frac{e_1}{e_2} \frac{\frac{p_{t-1}}{p_t} - 1}{\frac{p_{t-1}}{p_t}}, \ \forall t \geq 1 \\
\Leftrightarrow \frac{p_t}{p_{t+1}} &= 1 + \frac{e_1}{e_2} \frac{p_{t-1} - p_t}{p_{t-1}} \\
&= \frac{e_2 p_{t-1} + e_1 \left(p_{t-1} - p_t\right)}{e_2 p_{t-1}}, \ \forall t \geq 1 \\
\Leftrightarrow p_{t+1} &= \frac{e_2 p_{t-1} p_t}{\left(e_2 + e_1\right) p_{t-1} - e_1 p_t}, \ \forall t \geq 1.
\end{aligned}$$

  and for the initial old:
$$c_{2,0} = e_2 + p_0$$
- ▷ Market clearing also requires that
$$c_{1,0} = e_1 + e_2 - e_2 - p_0 = e_1 - p_0.$$
  For this to be optimal for cohort zero, it must be that
$$\begin{aligned}
c_{1,0} = \frac{e_1 + \frac{p_0}{p_1} e_2}{2} &= e_1 - p_0 \\
\Leftrightarrow e_1 + \frac{p_0}{p_1} e_2 &= 2e_1 - 2p_0 \\
\Leftrightarrow p_1 &= \frac{e_2 p_0}{e_1 - 2p_0}.
\end{aligned}$$
  This gives us the boundary condition for the (second-order) difference equation. For any given p_0, the expression above gives us p_1, and we can use
$$\Leftrightarrow p_{t+1} = \frac{e_2 p_{t-1} p_t}{\left(e_2 + e_1\right) p_{t-1} - e_1 p_t}, \ \forall t \geq 1.$$
  to obtain a sequence of prices $p_3, p_4, \ldots$.

However, not all sequence generated in this way is a competitive equilibrium. We have to ensure that

$$0 \leq p_t < e_1, \ \forall t.$$

- ▷ The second inequality ensures that the young, who has an endowment of e_1, is able to purchase one unit of the asset in every period (if the price is larger than e_1, then the asset market would not clear.) The following proposition gives the condition for the existence of a bubble equilibrium.

Proposition 5.1. *There exists a bubble equilibrium if and only if $e_1 > e_2$. In other words, the unique competitive equilibrium if $e_1 < e_2$ is the no bubble equilibrium in which $p_t = 0$ for all t.*

Proof. Admitted. ■

Assuming $e_1 > e_2$, an example of a sequence of prices that form a bubble equilibrium is

$$p_t = \frac{e_1 - e_2}{2}, \ \forall t.$$

This implies that

$$\frac{p_t}{p_{t+1}} = 1 \Rightarrow c_{1,t} = c_{2,t} = \frac{e_1 + e_2}{2}, \ \forall t.$$

There is a continuum of bubble equilibrium for all $p_0 \in (0, (e_1 - e_2)/2)$ that are Pareto rankable. For such initial price p_0, p_t is a decreasing sequence that converges to zero. The Pareto dominant bubble equilibrium is the example given above.

- ▷ If the price is too high, the young people will not be able to afford it. In fact, once you get out the sequence of prices, it will be an increasing sequence.
- ▷ If the price is too low, we cannot have negative prices – the old initial cohort will not sell the asset.

What happens if $e_1 < e_2$?

- ▷ Recall that if $e_1 > e_2$, then absent a monetary asset we have an autarky equilibrium that is Pareto inefficient. Including a monetary asset, we achieved a Pareto superior equilibrium.
- ▷ If $e_1 < e_2$, we can show that the autarky equilibrium is actually a Pareto-optimal allocation. Moreover, you can prove using the similar logic as before that any equilibrium must have zero price on the intrinsically worthless asset. The idea is to construct a sequence of $\{q_t\}$s.

5.3 Extension involving Asset with Dividends

We start with a model where $e_1 > e_2$ i.e. a Pareto-inefficient equilibrium, and including a worthless asset allowed us to go to a Pareto-dominating equilibrium. Now we consider introducing an asset that pays ϵ in every period and asking if there are equilibria in which the worthless asset has a positive price. The answer to this question is that the equilibria collapses to an equilibrium when we don't have a bubble. Some people see this troublesome in this literature which represents fragility in this equilibria.

5.3.1 Preview of the Logic

When $e_1 > e_2$ and we don't have the valuable asset, the nature of these equilibria are characterized by $\{q\}$s. If the monetary asset is worthless, $q = e_1/e_2$ and if it's valuable, $q = 1$. Recall that q is $1/(1+r)$. If you have an asset that pays out D, we are not discounting the future at all, so the price of the asset is infinite. Then people would not be able to afford the asset. If $q = e_1/e_2$ is worse – the real interest rate is negative, so there is no value on the monetary asset. The present value of the valuable asset is infinite. So neither of these will make sense once we have an asset that pays positive dividends in the future.

In fact, the nature of Pareto inefficiency that arises in a decentralized equilibrium is that the PV of the endowment stream was infinite. So we always had the case that the intermporal prices were such that when we discount the e_1 and e_2 using $\{q\}$, it was ill-defined. Now when we have the dividend stream D (which is constant), the PV of these streams will also be infinite. Thus, the breakdown in the FWT told us that it will come from the discounted sum. We try doing this with a valuable asset, and we know this cannot be an equilibrium. In fact, we need discounted sum to be less than e_1 for the young people to be able to afford this, which means that the FWT will go through.

5.3.2 Setup

We will make a modification to the OLG setup explored before. Individuals born at period t wants to maximize:

$$\begin{aligned} \max_{c_{1,t}, c_{2,t+1}, a_t, m_t} & \log c_{1t} + \log c_{2,t+1} \\ \text{s.t. } & c_{1,t} + p_{a,t} a_t + p_{m,t} m_t = e_1 \\ & c_{2,t+1} = e_2 + (p_{a,t+1} + d)\, a_t + p_{m,t+1} m_t \end{aligned}$$

where $p_{a,t}$ is the price of the "valuable" asset and $p_{m,t}$ is the price of the "money." Without the a_t terms, this is the same problem as before. For the initial old cohort, we have

$$c_{2,0} = e_2 + (p_{a,0} + d)\, 1 + p_{m,0} 1$$

where 1 is the endowments of the asset. Finally, market clearing requires that

$$\begin{aligned} c_{1,t} + c_{2,t} &= e_1 + e_2 + d \\ a_t &= 1, \forall t \\ m_t &= 1, \forall t \end{aligned}$$

Therefore, an equilibrium will be $\{c_{1,t}, c_{2,t}, p_{a,t}, p_{m,t}, a_t, m_t\}$ that solves the above problem.

5.3.3 Characterizing the Equilibria

We will show that any equilibrium has $p_{m,t} = 0$ and $p_{a,t} = p_{a,0} > 0$ and $c_{1,t} = c_{1,0}, c_{2,t} = c_{2,0}, \forall t$.

▷ **Non-monetary Equilibrium**: We can set $p_{m,t} = 0$ and solve for the equilibrium.

In this case, we can then proceed as before. Unlike before, there's nothing wonky happening at $e_1 = e_2$ or $e_1 = e_2 + d$. We will have a unique equilibrium, and it will always be PO. The idea is that $p_{a,t}$ will have to be the discounted sum of the dividend streams and bounded by e_1.

▷ **Monetary Equilibrium:**

* From the young generation, a_t and m_t are perfect substitutes and they will save them using these two technologies. The only way we have an equilibrium in which the $p_{m,t} > 0$ is if $p_{m,t+1}/p_{m,t} = (p_{a,t+1} + d)/p_{a,t}$. When $p_{m,t} = 0$ obviously we cannot use this logic.

5.3.4 Modification via Long-lived Asset

Given that the bubble equilibria are fragile, should we be concerned? We can introduce a dividend that is vanishing asymptotically i.e. smaller every period. In that case, the geometric decline will be enough to offset the fact that we are discounting with a negative real interest rate. As long as d is declining geometrically, we can still get a bubble.

Alternatively, we can recognize that there is a long-lived asset.

- ▷ We can introduce a permanently growing population that is growing at rate n_t and endow each initial old people with one tree. And whenever someone is born, they will be endowed with a tree just enough to make per capita holdings stationary.
- ▷ The violation will come from
$$\sum q_0^{t+1}\left(e_1\left(1+n\right)^t+e_2\left(1+n\right)^{t-1}\right)=\infty$$

The idea is that asset today becomes less valuable in 100 years, and we cannot trade a new asset in 100 years from today. Thus the problem looks like

$$\begin{aligned}\max \quad & \log c_1+\log c_2\\ \text{s.t. } & e_1+d\frac{n}{1+n}=c_1+p_{m,t}m_t+p_{a,t}\left(a_t-\frac{n}{1+n}\right)\\ & e_2+p_{m,t+1}m_t+\left(p_{a,t+1}+d\right)a_t=c_2\end{aligned}$$

Then the equilibrium with $p_{m,t}=0,\forall t$ is Pareto inefficient; the equilibrium with $p_{m,t}=\left(1+n\right)^t p_{m,0}>0$ is Pareto efficient.

5.4 Bubbles in Incomplete Markets

We will find that in this case also, each individual can have a present value of endowment of infinity and the competitive equilibrium can be inefficient, and a bubble can lead to a Pareto dominant allocation.

Instead of infinite types of people, we will have “odd” and “even” types of people. An odd (even) type receives high (low) endowment in odd periods and low (high) endowments in even periods. Let the two possible possible values of endowment be e_1 and e_2, where
$$\beta e_1>e_2.$$

The two types have a common discount rate factor of $\beta\in(0,1)$. $e_1>e_2$ is not enough; we need $\beta e_1>e_2$ so that the even type to smooth consumption instead of consuming their endowments in every period. **Crucially**, we assume that neither types can go into debt. This means that both types must consume their endowment.

- ▷ Each type solves the problem:

$$\begin{aligned}\max_{\{c_t^i\}} \quad & \sum_{t=0}^{\infty}\beta^t\log c_t^i\\ s.t. \quad & c_t^i+q_t^{t+i}a_{t+1}^i=e_t^i+a_t^i\\ & a_t^i\geq 0,\end{aligned}$$

where the second constraint reflects the fact that individuals cannot go into debt.

▷ **Case 1: Assets in zero Supply** – Market clearing condition for the asset market is

$$a_t^{\text{odd}} + a_t^{\text{even}} = 0,$$

which implies that there is zero net supply of assets in the economy. From here it immediately follows that

$$\begin{aligned} a_t^i &= 0, \ \forall t \\ e_t^i &= c_t^i. \end{aligned}$$

How is it possible that both types hold zero assets? One type has high endowment today and low endowment tomorrow, and q_t^{t+1} will move to ensure they save zero; the other type has low endowment today and is constrained. We will also see that $\beta e_1 > e_2$. implies that the equilibrium is Pareto inefficient.

▷ **Case 2: Assets in positive supply** – We can also look for an equilibrium with $a_t^{\text{odd}} + a_t^{\text{even}} = \bar{a} > 0$.

  * In this case, the high endowment guy purchases assets and the low endowment guy sells the assets.

What's wrong with bubbles? The problem is that we are worse off when the bubble bursts. It could also be a problem even if it does not burst.

▷ Suppose there is a new asset with bubble at $t = 7$. The old cohort needs to trade it with the richer children.

▷ In the production economy of Diamond, the bubble attached to houses leads to an inefficient, over-production of houses where we need to build houses first.

Bewley-Aiyagari Model Note that there is also an extension involving Aiyagari's model where an intrinsically worthless asset in positive supply has a positive price. The setting is one of stochastic endowments. Just consuming the endowments (no consumption smoothing) is always feasible, and introducing intrinsically worthless asset may improve welfare by a revealed preference argument.

5.5 Monetary Search a la Trejos-Wright (JPE, 1995)

The idea is that we will have people who would like to consume (and they are able to produce). The important thing is that there is specialization in this economy. The way we model specialization is that you can't consume the good you're able to produce; instead, you need to meet somebody who wants to consume the good you produce. Moreover, we will have "double coincidence of wants" never happening. We will model this using search friction – periodically, you meet somebody and the interesting meetings will be the one with a single coincidence of wants. Moreover, we assume the preferences are "fleeting" i.e. people go their separate ways after the rendezvous.

5.5.1 Kiyotaki & Wright vs. Trejos & Wright

In Kiyotaki & Wright, there are three types of individuals who consume i and produce $i + 1$. There is an absence of double coincidence and both goods and money are indivisible. In Trejos & Wright, on the other hand, goods are divisible while money is not, and we operate in a decentralized market where there exist gains from trade.

5.5.2 Preview of Intuition

What can we do for this person? Each period, some fraction of the people are endowed with the paper money. If you have something I want, I can give you the paper money and you will produce a good that I'd like to consume. Thus, we will construct an equilibrium where people develop Pareto-improving trades thanks to this intrinsically worthless piece of paper. We will also have a non-monetary equilibrium. We will also assume everything is non-storable (Kiyotaki-Wright do have a storable asset).

Another important assumption here that makes our life easier is that even though money is easy to carry around, people only have the capacity to carry around one piece of the paper (instead of multiple papers). This is to prevent something from happening when two people with money meet and one guy gives the other guy some money. If two people don't have any money, obviously there will not be any trade. Therefore, the interesting case is when somebody who has a unit of money meets someone who does not have any unit of money.

5.5.3 Setup

We have a unit measure of individuals and measures $m < 1$ of indivisible money holdings.

- ▷ Utility over consumption $u(q)$ and cost is $c(q)$ with $u(0) = c(0) = 0$ and $u'(0) > c'(0) = 0$ where u is concave and c is convex.
- ▷ We also assume that $\exists q^*$ such that $u'(q^*) = c'(q^*)$

An agent randomly meets someone in a single coincidence meeting at rate $2\alpha > 0$ where α is the parameter of the Poisson process, and the discount rate $\rho > 0$. There are two types of people: one with money and one without money.

5.5.4 Bellman Equations

We will look for an equilibrium where q units of goods are traded "whenever possible" – a single coincidence meeting. The equilibrium is defined as $\{q, V_0, V_1\}$ such that

1. Given q, V_0 and V_1 satisfy the Bellman equations described below;
2. Given V_0, V_1, q satisfies the Nash bargaining solution.

We characterize the Bellman equations where V_0 is the value of not having any money while V_1 is the value of having money.

- ▷ Individuals without money:
$$[1] : \rho V_0 = \alpha m \left[-c(q) + V_1 - V_0\right]$$
  where
  - * αm is the probability of meeting person who holds money and wants to buy the good I produce
  - * I produce at the cost $c(q)$ and become someone who holds money (V_1) instead of someone who does not (V_0). So $-c(q) + V_1 - V_0$ is the gain from switching.
- ▷ Individuals with money:
$$[2] : \rho V_1 = \alpha(1-m)\left[u(q) + V_0 - V_1\right]$$
- ▷ We are bargaining over quantity (q) that will be produced, and q is measured in utility units.

* Therefore, the solution can be obtained from

$$\max_{q} \left(-c\left(q\right)+V_1-V_0\right)^{\phi}\left(u\left(q\right)+V_0-V_1\right)^{1-\phi}$$

Taking logs and maximizing yields:

$$[3]:\frac{c'\left(q\right)}{-c\left(q\right)+V_1-V_0}=\frac{u'\left(q\right)}{u\left(q\right)+V_0-V_1}$$

We have three unknowns: $V_0, V_{1,}$ and q.

▷ In this system, one solution will be such that

$$V_1=V_0=q=0$$

i.e. the non-monetary equilibrium to the economy.

* In this equilibrium, money is being treated as a worthless asset. Models like this will always have an equilibria in which money is worthless since everyone expects it to be worthless.

▷ Another solution will be such that $q^* > q > 0$. In this solution, people are willing to produce some good in return for money, since they anticipate they will get something in return for money.

* It's less than q^* which maximizes the static gain from trade, but here we have a dynamic problem.
* If I produce for you, I pay $c\left(q\right)$ today and in sometime later I will get $u\left(q\right)$. Therefore, this is an economy where an intrinsically worthless asset has a positive value.

5.5.5 Characterizing the Equilibrium

Denote $q_b\left(\Delta\right)$ as offer by the buyer and $q_s\left(\Delta\right)$ as offer by the seller when delay is Δ.

▷ Seller's perspective:

$$V_1-c\left(q_b\left(\Delta\right)\right)=e^{-\rho\Delta}\left[V_1-c\left(q_s\left(\Delta\right)\right)\right]$$
$$V_1-c\left(q_s\left(\Delta\right)\right)=e^{-\rho\Delta}\left[V_1-c\left(q_b\left(\Delta\right)\right)\right]$$

i.e. the value that the seller gets by accepting offer today (LHS) is equal to the value of waiting Δ and making an offer himself (RHS).

▷ Buyer's perspective: Buyer loses money $\left(V_1\to V_0\right)$ but gains utility, so we have:

$$V_0+u\left(q_s\left(\Delta\right)\right)=e^{-\rho\Delta}\left[V_0+u\left(q_b\left(\Delta\right)\right)\right]$$

If we know V_0, V_1, we can then solve this equation. Then we have

$$\lim_{\Delta\to 0} q_s\left(\Delta\right)=\lim_{\Delta\to 0} q_b\left(\Delta\right)=q$$

that satisfies

$$\frac{V_1-c\left(q\right)}{V_0-u\left(q\right)}=\frac{c'\left(q\right)}{u'\left(q\right)}\Leftrightarrow q\in\max\left(V_0+u\left(q\right)\right)\left(V_1-c\left(q\right)\right)$$

which kinda looks like "in the hut" Nash bargaining. The difference here is that the threat point here is zero. An alternative Nash bargaining solution – "out of the hut" would be

$$q\in\max\left(V_0+u\left(q\right)-V_1\right)\left(V_1-c\left(q\right)-V_0\right)$$

where V_1 is the threat point. We can solve model under either assumption, but the results are a little nicer with the "out of the hut" solution.

▷ The "in the hut" solution would be :

$$\frac{c'(q)}{u'(q)} = \frac{(p+\alpha m)(\alpha(1-m)(u(q)-c(q))-pc(q))}{(p+\alpha(1-m))(\alpha m(u(q)-c(q))+pu(q))}$$

▷ If you take the limit of the RHS as $q \to 0$, it goes to zero, and at q^*, the LHS is equal to 1, so there will be a value of $q > 0$ that solves this.

▷ If we do the "out of the hut" bargaining, then the RHS above may not be less than 1.

5.5.6 Comparison vs. Cash-in-Advance

The key difference here is that in this model we are assuming that the non-monetary equilibria can arise. People like Wright and Wallace are emphatic that every model should account for the non-monetary equilibria. Cash-in-advance is ruling out the possibility of non-monetary equilibria. There is also an intermediate model by Lagos and Wright that look at bridging the gap between cash-in-advance and money search.

5.6 Heterogeneous Beliefs (Harrison-Kreps)

We now consider bubbles from a different perspective—with heterogeneous beliefs.

5.6.1 No-Trade Theorem (Milgrom & Stokey, 1982)

Imagine that we have this asset – bitcoin – and you have privileged information regarding the future value of bitcoin. I am aware of what the structure of the economy is. The only reason someone wants to buy bitcoin from me is because this other person has more information than me; in such case, since we are both rational agents, I should interpret that as meaning the value of bitcoin is equal to more than what I think.

Formalizing this idea, the **no-trade theorem states** that (1) if markets are in a state of efficient equilibrium, (2) if there are no noise traders or other non-rational interferences with prices, and (3) if the structure by which traders or potential traders acquire information is itself common knowledge, then even though some traders may possess private information, none of them will be in a position to profit from it. The assumptions are deliberately unrealistic, but the theorem may nonetheless be pertinent to debates over inside information.

In this setup, we will assume that the investors are incredibly stubborn. People are not Bayesian actors, and they just have different beliefs.

5.6.2 Preview of Results

Suppose that you hold an asset that you think is no good. You might believe that if you hold it for long enough, some "sucker" would come by and purchase the asset from you.

▷ To model this, we can think of individuals as having signals about the future dividends of the assets in question.

▷ If you believe that there are others that have a signal that is wrong in a way that makes them pay more for the asset. In this case, you might be willing to pay more for the asset than what you believe is its worth expecting it to sell the asset to the one with the incorrect signal.

Milgrom & Stokey (1982) shows that, in a rational expectations equilibrium, these types of trades motivated by differences in information cannot exist—the intuition is that one is able to infer (enough about) the other's signal from the fact that he is willing to trade at that price.

We consider a model with heterogeneous expectations model by Harrison and Kreps—we can think of such individual as very stubborn Bayesians who are unwilling to update their priors. In this environment, Harrison and Kreps show that we can have assets that trade, not just above the value that an individual thinks it is worth, but also above what everyone in the economy think it is worth. Such prices come about because individuals believe that they will be able to sell the asset onto other individuals with "stupid" beliefs in the future, and that person believes the same with respect to (the same or) another individual.

5.6.3 Setup

We assume each individual $i \in I$ is risk neutral, discounts the future at rate β and has a belief about the transition probabilities for the future dividends d' conditional on the current dividends d as as $\pi^i\left(d'|d\right)$. Suppose the truth probabilities $\pi\left(d'|d\right)$ is the truth.

We assume that individual's holding of assets has to be nonnegative (without the nonnegative constraints on the asset holding, there will be infinite supply of the asset). In this case, the post dividend price of the asset is given by

$$p\left(d\right) = \max_{i \in I}\ \beta \sum_{d'} \pi^i\left(d'|d\right)\left(d' + p\left(d'\right)\right).$$

That is, the price of the asset is determined by the individual with the highest valuation of the next period's dividends based on their beliefs.

- ▷ If $p > p\left(d\right)$, then everyone believes that holding it today makes him/her worse off.
- ▷ If $p < p\left(d\right)$, then some type i can have unbounded profits.

In contrast, each individual i's valuation of the asset (assuming that i holds the asset forever) is

$$q^i\left(d\right) = \beta \sum_{d'} \pi^i\left(d'|d\right)\left(d' + q^i\left(d'\right)\right).$$

The highest valuation of the asset among the I individual is

$$q\left(d\right) = \max_{i \in I}\ q^i\left(d\right).$$

Definition 5.5. We say that there is a *speculative behaviour* or a *bubble* if and only if

$$p\left(d\right) > q\left(d\right) = \max_i q^i\left(d\right)$$

That is, there is a speculative behaviour if each individual believes that it is not worth holding the asset at that price, but someone is willing to purchase at that price. In such situations, it must be that the one willing to purchase believes that he will be willing to sell it for a higher price at a latter point. We can also think of the equation as saying that the price of the asset is greater than the buy-and-hold value/price of the asset.

Now suppose there are two types of investors A and B, and two dividend levels, High (H) and Low (L).

- ▷ A thinks that dividends payments are persistent while B thinks that dividends payments are i.i.d over time.

- ▷ If the current dividend is high, A will put a high valuation on the asset since they think high dividend payments will persist. In contrast, if current dividend is low, A thinks that it will stay low for a long time.
- ▷ But A believes there are "suckers" (B) who believe that the states are transitory (so they think dividends will recover to H quickly).
- ▷ Then, A will be willing to pay more for the asset today (since it has an option value) when dividends are high, because they anticipate that they can sell it at to B if the dividends are low. In contrast, B are willing to buy when dividends are low, because they believe that shocks are transitory so that dividends will recover to H quickly. When it does recover, B think that they will be able to sell it to the "suckers" A who think that high dividends will persist.

5.6.4 Characterizing the Equilibria

It will be the case that there will always be a bubble. Consider a two-state case where A is the high dividend state and B is the low dividend state.

- ▷ Suppose $d_0 < d_1$ and $\rho^A > \rho^B$ with

$$\Pi^i = \begin{bmatrix} \rho^i & 1-\rho^i \\ 1-\rho^i & \rho^i \end{bmatrix}$$

  i.e. A thinks that dividends payments are more persistent than B thinks.

- ▷ Under this setup, each individual i's valuation of the asset j

$$\begin{aligned} q^i(d_j) &= \beta\left(\rho^i\left[d_j + q^i(d_j)\right] + \left(1-\rho^i\right)\left[d_{1-j} + q^i(d_{1-j})\right]\right) \\ q^i(d_{1-j}) &= \beta\left(\rho^i\left[d_{1-j} + q^i(d_{1-j})\right] + \left(1-\rho^i\right)\left[d_j + q^i(d_j)\right]\right) \end{aligned}$$

- ▷ Solving this we can show that

$$q^A(d_1) > q^B(d_1), \quad q^A(d_0) < q^B(d_0)$$

  so the price is more volatile for A.

Given this setup, you can conjecture that there is no bubble. Then it must be the case that

$$p(d_1) = \max\left\{q^A(d_1), q^B(d_1)\right\}$$

In the d_1 state, the person A will want to pay more for the good asset since I can dump it to stupid guys (who think that bad states are transitory) when the bad state actually hits. So the A guys buy in the good state and the B guys buy in the bad state. Therefore, this mechanism leads to

$$q^A(d_0) < q^B(d_0) < p(d_0)$$

We will now show that A buys when $d = d_1$ and B buys when $d = d_0$.

- ▷ Suppose that this is true. Then:

$$\begin{aligned} p(d_0) &= \beta\left[\rho^B(d_0 + p(d_0)) + \left(1-\rho^B\right)(d_1 + p(d_1))\right] > \max\left\{q^A(d_0), q^B(d_0)\right\} = q^B(d_0) \\ p(d_1) &= \beta\left[\rho^A(d_1 + p(d_1)) + \left(1-\rho^A\right)(d_0 + p(d_0))\right] > \max\left\{q^A(d_1), q^B(d_1)\right\} = q^A(d_1) \end{aligned}$$

  and thus we have a bubble.

- ▷ Each type of investor "falsely" believes that the state is persistent and wants to pay more.