1 Q2

(MWG, Exercise 13.D.1-variation) Extend the screening model of MWG, Chapter 13.D, to the case in which tasks are productive. Assume that a type θ worker produces $\theta(1+t)$ units of output when her task level is t. As before, $\theta_h > \theta_l > 0$, the probability of θ_h is $\varphi \in (0,1)$, and the worker's cost of task is $c(t,\theta)$, where $c(t,\theta)$ is increasing and convex in t, decreasing in θ, $c(0,\theta) = 0$, and $c_{t\theta}(t,\theta) < 0$ for $t > 0$.
Assume that a subgame perfect Nash equilibrium exists, identify the equilibrium allocations for the low and high-type workers.

Solution. I begin with two claims.

Claim: In equilibrium, expected profits for firms will be zero.

Proof. If expected profits are negative, the firm will simply not continue operation and guarantee themselves zero profit. Now, assume aggregate expected profits, Π, are positive. Since there are many firms, there is at least one firm making at most $\Pi/2$ in profits. Thus, this firm can set wages an amount $\epsilon > 0$ above the equilibrium wage and attract all workers to their firm. If ϵ is close enough to zero, this firm will get arbitrarily close to Π in profits. Thus, a deviation exists when expected profits are positive and thus cannot be an equilibrium. ■

Claim: No pooling equilibrium exists.

Proof. This is seen in Figure 1. Since expected profits must be zero in equilibrium, the equilibrium allocation must lie on the pooling break-even line. However, we can see in the graph that a firm can always make an offer in the shaded region and make a profit. Thus, a profitable deviation always exists and thus a pooling equilibrium does not exist. ■

Using these proven claims, I will argue that the following allocation $\{t_l^*, w_l^*, t_h^*, w_h^*\}$, depicted in 2, is the equilibrium allocation:

- ▷ Expected profits are zero.
- ▷ $w_l^* = (1+t_l^*)\theta_l$ and $w_h^* = (1+t_h^*)\theta_h$.
- ▷ t_l^* is given by the low type worker maximizing their utility.
- ▷ t_h^* is given by the intersection of the low-type worker's indifference curve and the profit line for high type workers.
- ▷ The high-type worker's indifference curve is above the pooling break-even line.

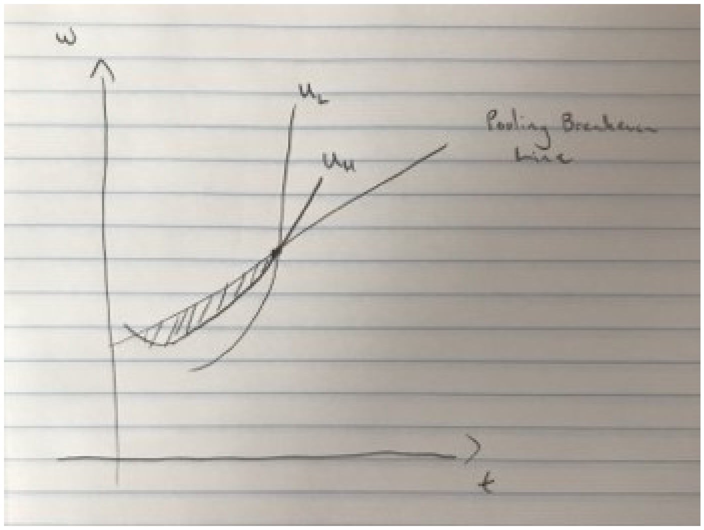

Figure 1: Nonexistence of pooling equilibria

Bullets one and two are the same- I showed above that in equilibrium expected profits must be zero. Since we are conjecturing that this is a separating equilibrium, it must be that workers are paid their productivity, since firms are competitive.

Since the low-type worker is known in this equilibrium, they will be paid somewhere on the line $\theta_l(1+t)$ and thus the low type workers will pick the point that maximizes utility along that line.

In particular, low-type workers will maximize the following utility function

$$\theta_l(1+t_l) - c(t_l, \theta_l).$$

Thus, $t_l^*(\theta_l)$ will be given implicitly by the equation

$$\theta_l = c_t(t_l^*(\theta_l), \theta_l).$$

The task level for the high-worker, $t_h^*(\theta_h)$, will be given by the intersection of the low-type worker's indifference curve $u_l = \{(t,w)|u(t,w|\theta_l) = \theta_l(1+t_l^*(\theta_L)) - c(t_l^*(\theta_l), \theta_l)\}$ and the high type zero profit line $\theta_h(1+t)$. Letting $t_l^* = t_l^*(\theta_l)$ and $t_h^* = t_h^*(\theta_h)$, the equilibrium allocations (t_l^*, w_l^*) and (t_h^*, w_h^*) are depicted in Figure **??** below.

Now that I have described the allocation, I will show why there is no profitable deviation. Here, the fact that the high type's indifference curve is above the pooling break-even line is crucial.

The regions below both indifference curves will never be deviated to since workers prefer the conjectured equilibrium allocation over them. Any region above both indifference curves would lead to allocations above the pooling break-even line and thus firms will make negative expected profits, which we showed above cannot happen. Any offer above the high-type's indifference curve and below the low-type's indifference curve will only be accepted by high-types. In this case, firms make no money on low types and lose money on high-types, and thus firms would have negative expected profits. Therefore, this firms would not deviate to this region.

This leaves the region above the low type's indifference curve and below the high type's indifference curve. However, we know that any offer in this region will only be accepted by low types and is above the profit line for low-type workers. Thus firms can not profitably deviate here either. Thus there are no profitable deviations and this allocation is indeed an equilibrium.

The key assumption is that the pooling break-even line does not lie above the high-type worker's indifference curve. If it did, a profitable pooling contract deviation would indeed exist. Since pooling contracts cannot be an equilibrium, there would be no equilibrium in this model. However, since we are given that an equilibrium exists, the aforementioned allocation is indeed an equilibrium.

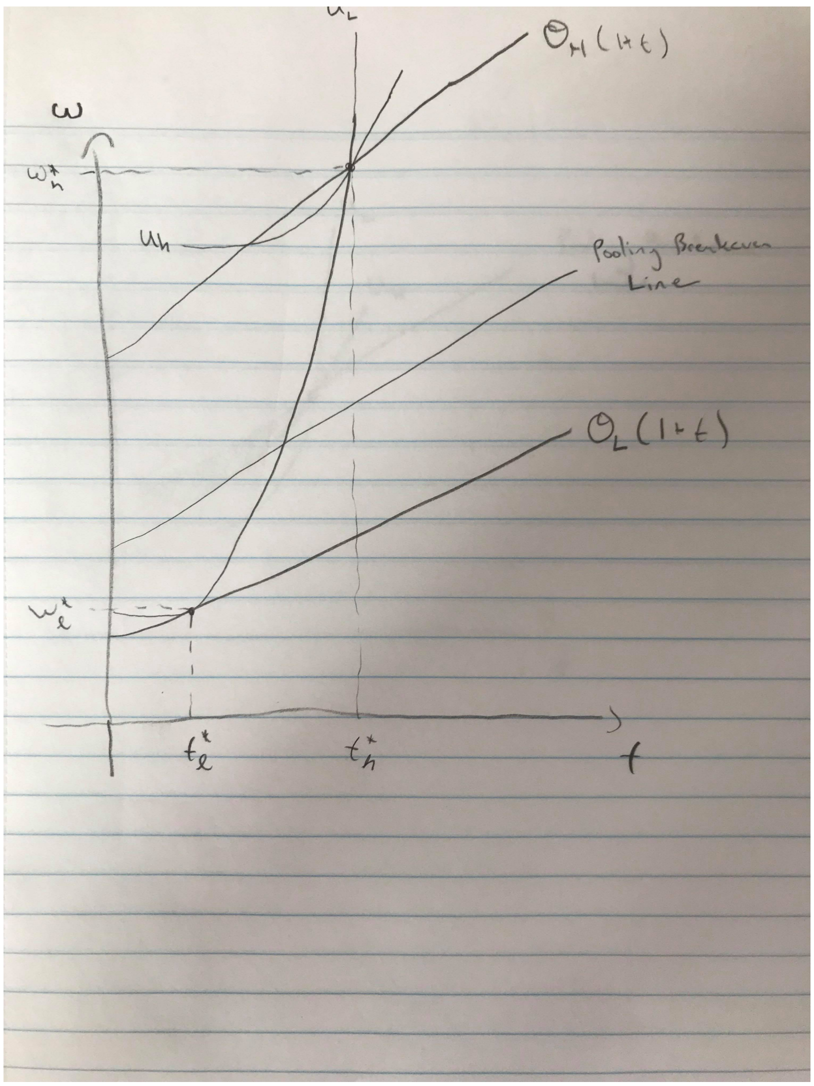

Figure 2: Equilibrium in screening with productive tasks

2 Q4:

Consider a principal-agent model with moral hazard in which the principal is risk neutral and the agent is also risk neutral. There are two efforts, $e_h > e_l,$ with personal cost to the agent of $\psi(e_h) = 10$ and $\psi(e_l) = 0$. The agent's expected utility is $E[w|e] - \psi(e)$.
There are two outcomes. The high-output payoff is $x_2 = 200,$ but the low-outcome payoff represents a loss, $x_1 = -100$. When the agent exerts low effort, each output is equally likely.
When the agent exerts high effort, the probability of high output is $\frac{3}{4}$ (and the probability of low output is $\frac{1}{4}$) . The agent's reservation utility is $\underline{U} = 0$.

Procblem 2.1. Solve for an optimal output-contingent wage contract, $\{w_1^*, w_2^*\}$. Does the worker earn any surplus above his reservation value?

Solution. First note that we proved in 3a that MLRP is satisfied in this setting, so $w_1^* < w_2^*$. The principal's problem is:

$$\begin{aligned}\max_{w_1^*, w_2^*, e\in\{e_l,e_h\}} \; & \mathbb{E}[x - w(x)|e] \\ \text{s.t. } & \mathbb{E}[u(w(x))|e] - \psi(e) \geq \underline{U} = 0 \quad (IR) \\ \text{and } & e \in \arg\max_{\tilde{e}\in\{e_l,e_h\}} \mathbb{E}[u(w(x))|\tilde{e}] - \psi(e) \quad (IC)\end{aligned}$$

Case 1: e_l is optimal.

Then

$$\begin{aligned}\mathbb{E}[x - w(x)|e_l] &= 50 - \frac{1}{2}(w_1 + w_2) \\ (IR) &\leftrightarrow \frac{1}{2}w_1 + \frac{1}{2}w_2 \geq 0 \\ (IC) &\leftrightarrow \frac{1}{2}w_1 + \frac{1}{2}w_2 - 0 \geq \frac{1}{4}w_1 + \frac{3}{4}w_2 - 10\end{aligned}$$

. We ignore (IC) for the moment and note that the principal's problem becomes

$$\begin{aligned}\max_{w_1^*, w_2^*, e\in\{e_l,e_h\}} \; & 50 - \frac{1}{2}(w_1 + w_2) \\ \text{s.t. } & \frac{1}{2}w_1 + \frac{1}{2}w_2 \geq 0 \quad (IR)\end{aligned}$$

Note that at the optimum (IR) will bind, because otherwise the principal can lower at least one of w_1 of W_2, still satisfy the constraint, and strictly increase his payoff. Thus, since (IR) binds:

$$w_1 = -w_2.$$

Plugging this relationship into the principal's objective function, we see that the maximized value is

$$50 - \frac{1}{2}(w_1 + w_2) = 50.$$

We now check that (IC) is satisfied for $w_1 = -w_2$:

$$\frac{1}{2}w_1 + \frac{1}{2}w_2 - 0 = 0 \geq \frac{1}{4}w_1 + \frac{3}{4}w_2 - 10 = -\frac{1}{2}w_1 - 10$$

holds for $w_1 \geq -20$. So in particular, $w_1 = w_2 = 0$ is a solution.

Case 2: e_h is optimal.

Then

$$\begin{aligned}
\mathbb{E}[x - w(x)|e_h] &= 125 - \frac{1}{4}w_1 - \frac{3}{4}w_2 \\
(IR) &\leftrightarrow \frac{1}{4}w_1 + \frac{3}{4}w_2 - 10 \geq 0 \\
(IC) &\leftrightarrow \frac{1}{2}w_1 + \frac{1}{2}w_2 - 0 \leq \frac{1}{4}w_1 + \frac{3}{4}w_2 - 10 \\
&\leftrightarrow -\frac{1}{4}w_1 + \frac{1}{4}w_2 - 10 \geq 0.
\end{aligned}$$

So we write the Lagrangian:

$$L = 125 - \frac{1}{4}w_1 - \frac{3}{4}w_2 + \mu(-\frac{1}{4}w_1 + \frac{1}{4}w_2 - 10) + \lambda(\frac{1}{4}w_1 + \frac{3}{4}w_2 - 10).$$

We now take the FOCs:

$$\begin{aligned}
[w_1]: \quad 1 &= \lambda + \mu\frac{1/4 - 1/2}{1/4} = \lambda - \mu \\
[w_2]: \quad 1 &= \lambda + \mu\frac{3/4 - 1/2}{3/4} = \lambda + \frac{1}{3}\mu.
\end{aligned}$$

Solving these two equations we find

$$\begin{aligned}
\mu &= 0 \\
\lambda &= 1 > 0,
\end{aligned}$$

which means that (IR) binds but (IC) is slack. Using that (IR) binds, we find

$$\frac{1}{4}w_1 + \frac{3}{4}w_2 = 10 \leftrightarrow w_1 = 40 - 3w_2.$$

Plugging this result into the slack (IC) we get:

$$\begin{aligned}
-\frac{1}{4}w_1 + \frac{1}{4}w_2 &> 10 \\
\leftrightarrow -\frac{1}{4}(40 - 3w_2) + \frac{1}{4}w_2 &> 10 \\
\leftrightarrow w_2 &> 20,
\end{aligned}$$

which impilies that

$$w_1 = 40 - 3w_2 < -20.$$

Plugging the relation $w_1 = 40 - 3w_2$ into the principal's objective function, we see that its maximized value is

$$125 - \frac{1}{4}w_1 - \frac{3}{4}w_2 = 125 - \frac{1}{4}(40 - 3w_2) - \frac{3}{4}w_2 = 115.$$

Since $115 > 50$, e_h is optimal for the principal. So the principal will set wages

$$w_1 = 40 - 3w_2, w_2 > 20.$$

In particular, $(w_1^*, w_2^*) = (-20 - 3\epsilon, 20 + \epsilon), \epsilon \to 0$ **is an optimal output-wage contract. Since (IR) binds, the worker gets no suprlus over** $\underline{U} + \psi(e_h) = 10$**.**

Procblem 2.2. Now suppose that the agent is still risk neutral, but legally the principal is not allowed to pay a wage that is negative. [Aside: This is sometimes called the limited-liability assumption. It is also equivalent to a situation in which an agent is infinitely risk averse at $w < 0$, i.e., $u(w) = w$ for $w \geq 0$ and $u(w) = -\infty$ for $w < 0$.] Solve for an optimal output-contingent wage contract, $\{w_1^*, w_2^*\}$. What is the expected cost to the principal of the limited-liability constraint?

Solution. Here we add the following constraints to the principal's problem:

$$w_1 \geq 0, w_2 \geq 0.$$

So the problem solved in part a is a relaxed version of this new program.

Case 1: e_l is optimal.

In Case 1 in part a, we found $w_1 = w_2 = 0 \geq 0$ solved the problem, and so will also solve this more restrictive problem, yielding a maximized objective for the principal of 50 again.

Case 2: e_h is optimal.

In Case 2 in part a we found that for (IR) to bind and (IC) to be satisfied to bind we needed:

$$w_1 = 40 - 3w_2, w_2 > 20,$$

which together imply $w_1 < -20$, which is not allowed in this case. Thus, in order to satisfy (IC) it must be the case that (IR) doesn't bind:

$$\begin{aligned} \frac{1}{4}w_1 + \frac{3}{4}w_2 > 10 \\ \leftrightarrow w_1 > 40 - 3w_2. \end{aligned} \tag{1}$$

Plugging this last inequality into (IC) yields:

$$\begin{aligned}-\frac{1}{4}w_1 + \frac{1}{4}w_2 &> 10 \\ \leftrightarrow w_2 &> 40 + w_1 \\ \leftrightarrow &> 40 + 40 - 3w_2 \\ &= 80 - 3w_2,\end{aligned} \tag{2}$$

which implies $w_2 > 20$, as in part a. Thus, $w_1 > 40 - 3w_2 > -20$, which is satisfied since we require $w_1 > -20$.

Hence

$$(w_1^*, w_2^*) = (0, 40 + \epsilon), \epsilon \to 0$$

satisfies (1) , (2), and $w_1, w_2 \geq 0$. Moreover, this contract is the least-cost contract for the principal subject to (IR) and (IC) e_h and limited liability. Thus, the maximized value of the objective is:

$$125 - \frac{1}{4}w_1 - \frac{3}{4}w_2 = 125 - \frac{1}{4}(0) - \frac{3}{4}(40 + \epsilon) = 95 - \frac{3}{4}\epsilon, \epsilon \to 0.$$

So the expected cost of limited liability to the principal is the difference in the maximized objectives:

$$115 - 95 = 20.$$

3 Q6:

Consider a risk-averse individual with utility function of money $u(\cdot)$ with initial wealth y who faces the risk of having an accident and losing an amount x of her wealth. She has access to a perfectly competitive market of risk-neutral insurers who can offer coverage schedules $b(x)$ (i.e., in the event of loss x, the insurance company pays out $b(x)$) in exchange for an insurance premium, p. Assume that the distribution of x, which depends on accident-prevention effort e, has an atom at $x = 0$:

$$f(x|e) = \begin{cases} 1 - \phi(e) & \text{if } x = 0 \\ \phi(e)g(x) & \text{if } x > 0, \text{ where } g(\cdot) \text{ is a probability density} \end{cases}$$

Assume $\phi'(e) < 0 < \phi''(e)$. Also assume that the individual's (increasing and convex) cost of effort, separable from her utility of money, is $\psi(e)$.

Procblem 3.1. What is the full information insurance contract when e is contractible? Specifically, what is e and $b(x)$.

Solution. Assume the support of x is $X = [0, \infty]$. First, note that because the market of risk-neutral insurers is perfectly competitive, the expected profit of any insurer should be 0. Call this break-even condition the insurer's IR constraint. Also, for the insurer to be operating in this market, it should provide the bundle of a premium and coverage schedules, $(p, b(x))$ so that drivers would actually buy the insurance, i.e. satisfying insurance buyer's individual rationality (IR). Hence, $(p, b(x))$ should satisfy:

$$p = \int_X b(x)f(x|e)dx \qquad \text{(Insurer's IR)}$$

$$e = \arg\max_{\hat{e}} \int_X u(y - x + b(x) - p)f(x|\hat{e})dx - \psi(\hat{e}) \qquad \text{(Buyer's IC)}$$

$$\underline{\text{U}} \leq \int_X u(y - x + b(x) - p)f(x|e)dx - \psi(e) \qquad \text{(Buyer's IR)}$$

where

$$\underline{\text{U}} = \max_{\tilde{e}} \int_X u(y - x)f(x|\tilde{e})dx - \psi(\tilde{e}),$$

which is the reservation utility of an agent from not buying an insurance.

Now, we consider a case where e is observable and thus contractible. The insurer would then want to set the coverage, $b(x)$, as low as possible to induce the driver exert the highest effort while still wanting the insurance. And because e is contractible, e is tied directly with $b(x)$ in the insurer's problem. Given this relationship, we can equivalently set the insurer's problem so as to choose e and $b(x)$ that maximizes the insurance buyer's utility subject to the insurer's break-even (IR) constraint:

$$\max_{e,b(x)} \int_X u(y-x+b(x)-p)f(x|e)dx - \psi(e)$$
$$s.t. \quad p = \int_X b(x)f(x|e)dx \qquad \text{(Insurer's IR)}$$

Using the functional form of $f(x|e)$, the buyer's problem becomes:

$$\max_{e,b(x)} \quad (1-\phi(e))u(y+b(0)-p) + \phi(e)\int_0^\infty u(y-x+b(x)-p)g(x)dx - \psi(e)$$
$$s.t. \quad p = (1-\phi(e))b(0) + \phi(e)\int_0^\infty b(x)g(x)dx \qquad \text{(Insurer's IR)}$$

(Note that $b(0)=0$ since the insurance coverage will be positive only when there is a loss, i.e. $x>0$, but we keep $b(0)$ in the equations above for differentiability at $b(x)$ when $x=0$)

Using λ as the Lagrange multiplier on the insurer's constraint, the FOC with respect to $b(x)$ is given by:

$$[b(x)]: \quad \begin{cases} \phi(e)g(x)[u'(y-x+b(x)-p)-\lambda]=0, & \forall x>0 \\ (1-\phi(e))[u'(y+b(0)-p)-\lambda]=0, & x=0 \end{cases}$$
$$\Rightarrow \lambda = u'(y-x+b(x)-p), \quad \forall x \geq 0$$

Given that λ is constant and that $u''(\cdot)<0$, we must have $b(x)-x = b(x')-x'$ for all $x,x' \in X$ such that $x \neq x'$. Hence, $b(x)$ must be such that $b(x)-x=b(0)$. Since $b(0)=0$, we have $b(x)=x$, which means that the optimal contract consists of full coverage schedules and this guarantees a constant utility of the insurance buyer in any state. Since $b(x)=x$ for every e, the optimal effort, e is given as

$$\begin{aligned} e &= \arg\max_{\tilde{e}}(1-\phi(\tilde{e}))u(y-p) + \phi(\tilde{e})\int_0^\infty u(y-p)g(x)dx - \psi(\tilde{e}) \\ &= \arg\max_{\tilde{e}} u(y-p) - \psi(\tilde{e}) \\ &= 0 \qquad \because \psi(\cdot) < 0. \end{aligned}$$

Hence, p is given as:

$$\begin{aligned} p &= \int_X b(x)f(x|e)dx \\ &= \int_X xf(x|0)dx \\ &= (1-\phi(0)\cdot 0 + \phi(0)\int_0^\infty xg(x)dx \\ &= \phi(0)\int_0^\infty xg(x)dx. \end{aligned}$$

Procblem 3.2. If e is not observable, prove that the optimal contract consists of a premium and a deductible; i.e. $b(x) = x - \delta$. You may assume that a solution exists and the first-order approach is valid.

Solution. Now, since e is not observable, the IC constraint enters into the program in part a. Assuming the solution exists and the first-order approach is valid, we use the first-order condition of IC instead:

$$\begin{aligned} &\max_{e,b(x)} \int_X u(y - x + b(x) - p) f(x|e) dx - \psi(e) \\ s.t. \quad & p = \int_X b(x) f(x|e) dx && \text{(Insurer's IR)} \\ & \psi'(e) = \int_X u(y - x + b(x) - p) f_e(x|e) dx && \text{(IC-FOC)} \end{aligned}$$

Using the functional form of $f(x|e)$, the problem can be rewritten as:

$$\begin{aligned} &\max_{e,b(x)} (1 - \phi(e)) u(y + b(0) - p) + \phi(e) \int_0^\infty u(y - x + b(x) - p) g(x) dx - \psi(e) \\ s.t. \quad & p = (1 - \phi(e)) b(0) + \phi(e) \int_0^\infty b(x) g(x) dx && \text{(Insurer's IR)} \\ & \psi'(e) = \phi'(e) \left[\int_0^\infty u(y - x + b(x) - p) g(x) dx - u(y + b(0) - p) \right] && \text{(IC-FOC)} \end{aligned}$$

Using λ and μ as the Lagrange multiplier on the (IR) and (IC-FOC) constraint, respectively, the corresponding FOCs with respect to $b(x)$ become:

Case 1: $x > 0$

$$\begin{aligned} &\phi(e) g(x) [u'(y - x + b(x) - p) - \lambda] + \mu g(x) \phi'(e) u'(y - x + b(x) - p) = 0 \\ &\Rightarrow \phi(e) [u'(y - x + b(x) - p) - \lambda] + \mu \phi'(e) u'(y - x + b(x) - p) = 0 \\ &\Rightarrow u'(y - x + b(x) - p) = \frac{\lambda \phi(e)}{\phi(e) + \mu \phi'(e)} \\ &\Rightarrow \frac{1}{u'(y - x + b(x) - p)} = \frac{1}{\lambda} + \frac{\mu \phi'(e)}{\lambda \phi(e)} \end{aligned}$$

Case 2: $x = 0$

$$\begin{aligned} &(1 - \phi(e)) [u'(y + b(0) - p) - \lambda] - \mu \phi'(e) u'(y + b(0) - p) = 0 \\ &\Rightarrow [1 - \phi(e) - \mu \phi'(e)] u'(y + b(0) - p) = \lambda (1 - \phi(e)) \\ &\Rightarrow u'(y + b(0) - p) = \frac{\lambda (1 - \phi(e))}{1 - \phi(e) - \mu \phi'(e)} \\ &\Rightarrow \frac{1}{u'(y + b(0) - p)} = \frac{1}{\lambda} - \frac{\mu \phi'(e)}{\lambda (1 - \phi(e))} \end{aligned}$$

Because $\phi'(e) < 0$ and $\phi(e) \in (0, 1)$, we have:

$$
\begin{aligned}
\frac{1}{u'(y + b(0) - p)} > \frac{1}{\lambda} > \frac{1}{u'(y - x + b(x) - p)} & \\
\Rightarrow u'(y + b(0) - p) < u'(y - x + b(x) - p) & \\
\Rightarrow y + b(0) - p > y - x + b(x) - p & \qquad \because u''(\cdot) < 0 \\
\Rightarrow b(0) > b(x) - x & \\
\Rightarrow b(x) < x + b(0) & \\
\Rightarrow b(x) < x & \qquad \because b(0) = 0
\end{aligned}
$$

Hence, we see that $b(x)$ is strictly less than x, meaning that in the event that a loss x occurs, the optimal contract always consists of a premium, p, and a deductible, $\delta = x - b(x) > 0$.