Price Theory, III | **Professor Lars Stole**
Economics 33003 | **Spring 2019**

Assignment 7
(Due Friday, May 31, prior to the start of the Review session)

Problem 1 (JR Exercise 9.4) There are n bidders participating in a first-price auction. Each bidder's value is independently drawn from $[0, 1]$ according to the distribution function F, having continuous and strictly positive density f. If a bidder's value is θ_i and he wins the object with a bid of $b_i < \theta_i$, then his von Neumann-Morgenstern utility is $(\theta_i - b_i)^{\frac{1}{\alpha}}$, where $\alpha \geq 1$ is fixed and common to all bidders. Consequently, the bidders are risk averse when $\alpha > 1$ and risk neutral when $\alpha = 1$. Given the risk-aversion parameter α, let $\bar{b}_\alpha(\theta)$ denote the (symmetric) equilibrium bid of a bidder when his value is θ. The following parts will guide you toward finding $\bar{b}_\alpha(\theta)$ and uncovering some of its implications.

(a). Let $U(\hat{\theta}|\theta)$ denote a bidder's expected utility from bidding $\bar{b}_\alpha(\hat{\theta})$, given that all other bidders employ $\bar{b}_\alpha(\cdot)$. Show that

$$U(\hat{\theta}|\theta) = F(\hat{\theta})^{n-1}(\theta - \bar{b}_\alpha(\hat{\theta}))^{\frac{1}{\alpha}}.$$

Why must $U(\hat{\theta}|\theta)$ be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$?

(b). Use part (a) to argue that

$$U(\hat{\theta}|\theta)^\alpha = F(\hat{\theta})^{\alpha(n-1)}(\theta - \bar{b}(\hat{\theta})).$$

must be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$.

(c). Use part (b) to argue that a first-price auction with the n risk-averse bidders above whose values are each independently distributed according to $F(\theta)$, is equivalent to a first-price auction with n risk-neutral bidders whose values are each independently distributed according to $F(\theta)^\alpha$. Use the solution for the risk-neutral case to conclude that

$$\bar{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(n-1)} dx.$$

(d). Prove that $\bar{b}_\alpha(\theta)$ is strictly increasing in α. Does this make sense? Conclude that as bidders become more risk averse, the seller's revenue from a first-price auction increases.

(e). Use part (d) and the revenue equivalence result for the standard auctions in the risk-neutral case to argue that when bidders are risk averse as above, a first-price auction raises more revenue for the seller than a second-price auction. Hence, these two standard auctions no longer generate the same revenue when bidders are risk averse.

(f) What happens to the seller's revenue as the bidders become infinitely-risk averse (i.e., as $\alpha \to \infty$)?

Problem 2 (Auctioning procurement contracts.) (This is closely related to Problem 1 from last week's Problem Set 6. You may want to consult that question and solution before tackling this one.)

A monopsony buyer is interested in purchasing a large quantity of output from one of n possible suppliers. Each supplier i has a constant marginal cost of production equal to c_i which is private information to the supplier and is uniformly distributed on $[1, 2]$. Supplier i's payoff from producing $q \in [0, Q]$ units of output for a transfer of t dollars is

$$t - c_i q.$$

Each supplier's outside option is 0. The buyer's payoff from purchasing q units of output at a total price of t dollars is

$$vq - \frac{1}{2}q^2 - t,$$

where we assume $v \geq 3$.

The buyer's objective is to design an optimal direct-revelation mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, where each component is a mapping from cost reports $c = (c_1, \ldots, c_n)$ to probabilities of selecting firm i, output for the selected firm, and transfers to each supplier, respectively, in order to maximize

$$E_c\left[\sum_{i=1}^{n} \phi_i(c)\left(vq_i(c) - \frac{1}{2}q_i(c)^2 - c_i q_i(c)\right) - U_i(c_i)\right].$$

(a). State the two conditions that any incentive compatible mechanism must satisfy. [Hint: the monotonicity condition will now involve both ϕ_i and q_i.]

(b). Using your conditions in (a), find an expression of $E_c[U_i(c_i)]$ that is entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

(c). Using your result in (b), substitute into the buyer's objective function to obtain a maximization program that is expressed entirely in terms $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$

(d). Find the optimal $\phi_i(c)$ and $q_i(c)$ components of the optimal mechanism. Make whatever regularity assumptions you use to this end explicit.

Problem 3 (Mineral-rights auctions). Consider a common-value auction with n bidders. Each bidder privately learns a signal, θ_i, independently distributed according to $F(\theta_i)$ on $[0, 1]$ (with positive density, $f(\theta_i)$). For now, notice we are allowing distributions to differ across bidders and the signals are independently distributed. Each bidder i has a common value of the good given by

$$v(\theta) = \frac{1}{n}\sum_{j=1} \theta_j.$$

The seller's value for the good is $\theta_0 = 0$, and does not depend upon the signals of the bidders.

We first want to compute the optimal auction using Myerson's framework.

(a). Write down the two conditions which characterize an incentive-compatible direct mechanism, $\{\phi_i(\cdot), t_i(\cdot)\}_i$.

(b). Using the conditions in (a), find an expression for $E_\theta[U_i(\theta_i)]$.

(c). Using (b), write the seller's objective function expressed only in terms of $\phi_i(\cdot)$ and $U(0)$.

(d). Using (c), find the optimal $\phi_i(\cdot)$ which maximizes the seller's expected profit and find an expression determining the reservation type for each i. Make any regularity assumption(s) that you use explicit.

(e). Now further assume that the signal distributions are symmetric across bidders and uniform on $[0,1]$. Suppose that the seller uses a first-price auction without reserve price to sell the good. Find the symmetric equilibrium bid function $\overline{b}(\cdot)$. [Hint: it is linear.]

(f). Is the auction in (e) optimal? Why or why not?

(g). Show that for the equilibrium in (e), if $n > 4$, the bid function declines as the number of bidders increases. Explain.

Problem 4 (Auctions with adverse selection.) The economics department is trying to procure teaching services from one of n potential external lecturers. Candidate i has an outside opportunity of $\theta_i \in [0,1]$ with distribution $F(\cdot)$. This opportunity is private information and can be thought of as the candidate's type. The department gets teaching value $v(\theta_i)$ from a lecturer with type θ_i; the function $v(\cdot)$ is increasing and differentiable.

Consider a direct revelation mechanism consisting of an allocation function $\phi_i(\theta_1, \ldots, \theta_n) \in [0,1]$ for each lecturer i, and a transfer function $t_i(\theta_1, \ldots, \theta_n)$ which is the payment to each lecturer i. i's utility from reporting $\hat{\theta}_i$ when her true type is θ_i is

$$U_i(\hat{\theta}|\theta) = E_{\theta_{-i}}\left[t_i(\hat{\theta}_i, \theta_{-i}) - \phi_i(\hat{\theta}_i, \theta_{-i})\theta_i\right] = \overline{t}_i(\theta_i) - \overline{\phi}_i(\theta_i)\theta_i.$$

[Note that the lecturer utility is decreasing in θ_i and the single-crossing term is negative.] The department's objective is to maximize

$$\Pi = E_\theta\left[\sum_{i=1}^{n} \phi_i(\theta)v(\theta_i) - t_i(\theta)\right].$$

(a) Characterize incentive compatibility in terms of an integral equation for the agent's utility, $U_i(\theta_i)$ and a monotonicity constraint.

(b). Using (a), what is the department's profit expressed in terms of $\phi_i(\cdot)$ and $U_i(1)$?

For the rest of the question, assume that $F(\theta_i)/f(\theta_i)$ is increasing in θ_i.

(c). If $v'(\theta_i) \le 1$, what is the department's optimal hiring policy, $\{\phi_i(\cdot)\}_i$?

(d). Suppose instead that $v'(\theta_i) > 2$ and $E[v(\theta_i)] \ge 1$. What is the department's optimal hiring policy? [Hint: if an unrelaxed program violates the required monotonicity at every point of θ_i, then the constrained-optimal solution must be constant.]

Problem 5 (Auctions with endogenous entry.) Consider an IPV model of auctions with a large number of bidders, each of whom must pay k in order to learn their type before they can participate in the auction. The timing is as follows.

1. Seller offers an auction mechanism and offers invitations to bid to some subset of the population of bidders; each bidder sees how many bidders in total, n, are invited to the auction.

2. bidders with invitations to attend the auction decide whether or not to spend k to learn their type and participate in the auction; those that decide to do so learn their type θ_i which is distributed i.i.d. according to $F(\cdot)$ on $[\underline{\theta}, \overline{\theta}]$. The seller's cost of the good is θ_0.

3. bidders report their types into the mechanism and the winner and transfers are determined accordingly.

(a). Characterize incentive compatibility for those who decide to participate in the mechanism. As usual, you should have a monotonicity condition and an integral condition for the bidder's indirect utility function, $U_i(\theta)$.

(b). What is the relevant individual rationality constraint for the participating bidders that ensures all the invited bidders are willing to spend k to learn their type? Note that the bidders decide whether or not to participate *before* they learn their type, so this will involve the bidder's expected utility, $E[U_i(\theta)]$. In other words, the seller must satisfy ex ante IR constraints and not interim IR constraints for the bidders.

(c). Suppose that it is optimal for the seller to invite n bidders to the auction. Using (a) and (b), write the seller's program in terms of $\phi_i(\cdot)$ and $U_i(\underline{\theta})$.

(d). What is the optimal allocation for the seller? What is the optimal reserve type, θ^*? How does your answer differ from the standard optimal auction of Myerson? [Hint: to solve for the optimal auction, use a Lagrange multiplier, λ, for the participation constraint in (b), incorporate it into the seller's objective function, and prove that at the optimum the multiplier is $\lambda = 1$.]

(e). Using you answer from (d), show that the value function of the seller (as a function of n) is simply

$$\Pi(n) = E\left[\max\{\theta_1, \ldots, \theta_n, \theta_0\} - nk\right].$$

Conclude that the seller will want to invite the socially optimal number of bidders to the auction when there is a cost to entry.

Problem 6 (Reserve prices in IPV auctions.) Consider an IPV setting with $n = 2$ and θ_i uniformly distributed on $[0, 1]$ for both bidders. Suppose that $\theta_0 = 0$ (the seller's opportunity cost of the object is zero).

(a). In the Myerson optimal auction, what is the allocation function $\phi_i(\theta_i, \theta_{-i})$ and what is the reserve type, θ^* (i.e., what is the highest type buyer θ_i such that $\overline{\phi}_i(\theta_i) = 0$). In describing the optimal allocation function, you may ignore situations with zero probability (i.e., ignore ties).

(b). Show that a first-price auction with an appropriately chosen reserve price is also optimal. Argue that the optimal reserve price is set at $r^* = \theta^*$.

(c). Compute the equilibrium bidding function in (b) given the optimal reserve. [Hint: it is *not* linear. Try using the envelope theorem to find $\overline{b}(\cdot)$.]

(d). In order to implement the optimal auction allocation in a standard all-pay auction (i.e., highest bidder wins but everyone pays their bids), what is the optimal reserve price, r^*, that must be set. [Hint: for the all-pay auction, the answer is not the same as in (b). The equilibrium bid function will be of the form $\overline{b}(\theta) = 0$ for all $\theta < \theta^*$, a jump at $b(\theta^*) = r^*$, and $\overline{b}(\theta) > r^*$ for all $\theta > \theta^*$. Try using the envelope theorem.]

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Answers to Assignment 7

1 (a). In a first-price auction, the bidder with the highest bid wins and pays the winning bid. If the bidder does not win, then his utility is zero. The probability that bidder i who bids $\bar{b}_\alpha(\hat{\theta})$ wins is the probability that all other bidders bid less than $\bar{b}_\alpha(\hat{\theta})$. Let us assume for now that $\bar{b}_\alpha(\hat{\theta})$ is strictly increasing. Hence,

$$\bar{b}_\alpha\left(\hat{\theta}\right) > \bar{b}_\alpha\left(\theta_j\right),\ \forall j \Leftrightarrow \hat{\theta} > \theta_j,\ \forall j.$$

Given that values are drawn independently, the probability of winning with bid $\bar{b}_\alpha(\hat{\theta})$ is

$$\left[F\left(\hat{\theta}\right)\right]^{n-1},$$

where we implicitly assume that all other bidders bid according to $\bar{b}_\alpha$. In case the bidder wins, he must pay $\bar{b}_\alpha(\hat{\theta})$ so that his expected utility is given by

$$U\left(\hat{\theta}|\theta\right) = \left[F\left(\hat{\theta}\right)\right]^{n-1}\left(\theta - \bar{b}_\alpha\left(\hat{\theta}\right)\right)^{\frac{1}{\alpha}}.$$

We want to argue that the bidder maximises utility by acting truthfully, i.e. that bidding $\hat{b}_\alpha(\theta)$ when his value is θ is optimal. This is equivalent to requiring that $U(\hat{\theta}|\theta)$ is maximised in $\hat{\theta}$ when $\hat{\theta} = \theta$; i.e.

$$\theta = \arg\max_{\hat{\theta}}\ U\left(\hat{\theta}|\theta\right).$$

(b). From Part (a),

$$\left[U\left(\hat{\theta}|\theta\right)\right]^\alpha = \left[F\left(\hat{\theta}\right)\right]^{\alpha(n-1)}\left(\theta - \bar{b}_\alpha\left(\hat{\theta}\right)\right).$$

Since $\alpha \geq 1$ is a strictly increasing transformation of $U(\hat{\theta}|\theta)$, the point of maximum does not change; i.e. $[U(\hat{\theta}|\theta)]^\alpha$ is maximised in $\hat{\theta}$ at the same point as $U(\hat{\theta}|\theta)$, which is when $\hat{\theta} = \theta$; i.e.

$$\theta = \arg\max_{\hat{\theta}}\ U\left(\hat{\theta}|\theta\right) = \arg\max_{\hat{\theta}}\ \left[U\left(\hat{\theta}|\theta\right)\right]^\alpha,\ \forall\alpha \geq 1.$$

(c). In a first-price auction with n *risk-neutral* bidders whose values are each independently distributed according to $G(\theta) = F(\theta)^\alpha$, the expected utility from bidding $\bar{b}(\hat{\theta})$ when value is θ (assuming symmetric bidding function) is

$$\begin{aligned}\tilde{U}\left(\hat{\theta}|\theta\right) &= \left[G\left(\hat{\theta}\right)\right]^{(N-1)}\left(\theta - \bar{b}\left(\hat{\theta}\right)\right)\\ &= \left[F\left(\hat{\theta}\right)\right]^{\alpha(N-1)}\left(\theta - \bar{b}\left(\hat{\theta}\right)\right)\\ &= \left[U\left(\hat{\theta}|\theta\right)\right]^\alpha.\end{aligned}$$

From parts (a) and (b), we know that $[U(\hat{\theta}|\theta)]^\alpha$ and $U(\hat{\theta}|\theta)$ are maximised when $\hat{\theta} = \theta$. Using the symmetric equilibrium bidding function from the risk-neutral case when values are each independently distributed according to $G(\theta) = F(\theta)^{\alpha}$ gives us that

$$\begin{aligned}
\bar{b}(\theta) &= \theta - \int_0^{\theta} \left(\frac{G(x)}{G(\theta)}\right)^{N-1} \mathrm{d}x \\
&\equiv \theta - \int_0^{\theta} \left(\frac{[F(x)]^{\alpha}}{[F(\theta)]^{\alpha}}\right)^{N-1} \mathrm{d}x \\
&= \theta - \int_0^{\theta} \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x \equiv \bar{b}_{\alpha}(\theta).
\end{aligned}$$

(d). Consider

$$\begin{aligned}
\frac{\partial \bar{b}_{\alpha}(\theta)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(\theta - \int_0^{\theta} \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x\right) \\
&= -\frac{\partial}{\partial \alpha}\left(\int_0^{\theta} \exp\left[\alpha(N-1)\ln\left(\frac{F(x)}{F(\theta)}\right)\right] \mathrm{d}x\right) \\
&= -\int_0^{\theta} (N-1)\ln\left(\frac{F(x)}{F(\theta)}\right)\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x.
\end{aligned}$$

Note that

$$\begin{aligned}
F(x) \leq F(v), \ \forall 0 \leq x \leq \theta \\
\Rightarrow \ln\left(\frac{F(x)}{F(\theta)}\right) \leq 0, \ \forall 0 \leq x \leq \theta.
\end{aligned}$$

Thus, so long as $\theta > 0$,

$$\frac{\partial \bar{b}_{\alpha}(\theta)}{\partial \alpha} > 0.$$

his tells us that bidders bid more as they become more risk averse reflecting the fact that they are more willing to pay to avoid uncertainty of not winning the object. Since the seller's revenue is the winning bid, it follows that the seller's revenue from a first-price auction increases as agents become more risk averse.

(e). We know that when agents are risk neutral, a first-price auction raises the same revenue for the seller as a second-price auction. In part (d), we argued that a first-price auction with risk-averse agents raises more revenue than a first-price auction with risk neutral agents. So it follows that a first-price auction with risk-averse agents raises more revenue than a second-price auction with risk-neutral agents. Now, to compare the seller's revenue from a first-price auction with risk-averse agents and a second-price auction with risk-averse agents, consider what the players would do in the latter case.

In a second-price auction, the bidder with the highest bid wins and pays an amount equal to the second highest bid. The (weakly) dominant strategy with risk-neutral agents was that the bidders will bid their true value. In fact, even if the agents are risk averse, bidding v remains a dominant strategy. To see this, suppose the bidder i bids $\hat{\theta}_i < \theta_i$.

- If $\hat{\theta}_i < \theta_i < \max\{\theta_j\}_{j\neq i}$, then the bidder does not win and obtains zero utility;
- If $\hat{\theta}_i < \max\{\theta_j\}_{j\neq i} < \theta_i$, then reporting θ_i gives the bidder utility of

$$\left(\theta_i - \max\{\theta_j\}_{j\neq i}\right)^{1/\alpha} > 0$$

and reporting $\hat{\theta}_i$ gives zero utility.

- If $\max\{\theta_j\}_{j\neq i} < \hat{\theta}_i < \theta_i$, then the bidder wins whether he reports $\hat{\theta}_i$ or θ_i and the utility is given by
$$\left(\theta_i - \max\{\theta_j\}_{j\neq i}\right)^{1/\alpha}.$$

Hence, bidding θ_i is not dominated by bidding $\hat{\theta}_i < \theta_i$. Analogous argument tells us that bidding θ_i is also not dominated by bidding $\hat{\theta}_i > \theta_i$. It follows then that the revenue for the seller from a second-price auction with risk-averse agents are the same as the revenue from a second-price auction with risk-neutral agents.

Putting all of these together, we realise that the seller's revenue from a first-price auction with risk-averse agents must be greater than his revenue from a second-price auction with risk-averse agents.

(f). The seller's revenue is the winning bid. Let $\theta = \max\{\theta_i\}$, then his revenue is

$$\overline{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x.$$

Taking limit as $\alpha \to \infty$ while recalling that $0 \leq F(x)/F(\theta) \leq 1$ for all $x \in [0, \theta]$,

$$\lim_{\alpha\to\infty} \overline{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\lim_{\alpha\to\infty}\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)}\right) \mathrm{d}x = \theta.$$

So the seller's revenue is the highest value among the bidders and the seller is able to extract the entire surplus from the bidders.

2 (a). A supplier's expected utility given report $\hat{c}_i$ and true type c_i is

$$U_i(\hat{c}_i|c_i) = \overline{t}_i(\hat{c}_i) - E_{c_{-i}}\left[\phi_i(\hat{c}_i, c_{-i})q_i(\hat{c}_i, c_{-i})\right] c_i.$$

For notational ease, define $\overline{\Phi}_i(\hat{c}_i) \equiv E_{c_{-i}}\left[\phi_i(\hat{c}_i, c_{-i})q_i(\hat{c}_i, c_{-i})\right]$ to be the expected value (over c_{-i}) of the product ϕ_i and q_i for a seller i who reports $\hat{c}_i$. Incentive compatibility requires $U_i(c_i) = U_i(c_i|c_i) \geq U_i(\hat{c}_i|c_i)$, which is equivalent to

$$U_i(c_i) - U_i(\hat{c}_i) \geq \overline{\Phi}_i(\hat{c}_i)(\hat{c}_i - c_i).$$

Reversing the roles of $\hat{c}_i$ and c_i, we have

$$\overline{\Phi}_i(c_i)(\hat{c}_i - c_i) \geq U_i(c_i) - U_i(\hat{c}_i) \geq \overline{\Phi}_i(\hat{c}_i)(\hat{c}_i - c_i).$$

Ignoring the middle part,
$$(\overline{\Phi}_i(c_i) - \overline{\Phi}_i(\hat{c}_i))(\hat{c}_i - c_i) \geq 0.$$

Thus, the expected product of $\phi_i(c)q_i(c)$, denoted by $\overline{\Phi}_i(c_i)$, must be nonincreasing in c_i. This is the relevant monotonicity condition. Because $\overline{\Phi}_i$ is bounded, the envelope theorem immediately tells us that $U_i(c_i)$ is absolutely continuous and

$$U_i(c_i) = U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s)ds.$$

This is the relevant integral condition. [Aside: I have not proven sufficiency here, but you should do that. It proceeds exactly as in the monopoly-screening notes.]

(b). We want an expression for $E[U_i(c_i)]$. Using our integral condition,

$$E[U_i(c_i)] = U_i(2) + \int_1^2 \left(\int_{c_i}^2 \overline{\Phi}_i(s)ds \right) f(c_i)dc_i.$$

Integrating by parts,

$$E[U_i(c_i)] = U_i(2) + \left(\int_{c_i}^2 \overline{\Phi}_i(s)ds \right) F(c_i) \Big|_{c_i=1}^2 + \int_1^2 c_i \overline{\Phi}_i(c_i) F(c_i) dc_i = U_i(2) + E\left[\overline{\Phi}_i(c_i) \frac{F(c_i)}{f(c_i)} \right].$$

Undoing the expectations of $\overline{\Phi}_i$, we have

$$E_c[U_i(c)] = U_i(2) + E_c\left[\phi_i(c) q_i(c) \frac{F(c_i)}{f(c_i)} \right].$$

(c). The buyer's objective is to maximize

$$E_c\left[\sum_{i=1}^n \phi_i(c) \left(vq_i(c) - \frac{1}{2} q_i(c)^2 - c_i q_i(c) \right) - U_i(c) \right],$$

subject to IC and IR. Substituting out the transfer functions by $t_i(c) \equiv \phi_i(c)q_i(c)c_i + U_i(c)$ and using our result in (b), we can restate the program entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$:

$$E_c\left[\sum_{i=1}^n \phi_i(c) \left(vq_i(c) - \frac{1}{2} q_i(c)^2 - \left(c_i + \frac{F(c_i)}{f(c_i)} \right) q_i(c) \right) - U_i(2) \right].$$

The seller maximizes this subject to IR (which is now the requirement that $U_i(2) \geq 0$ for all i), and $E_{c_{-i}}[\phi_i(c)\overline{q}_i(c)]$ is nonincreasing in c_i.

(d). Define

$$J_i(c_i) = c_i + \frac{F(c_i)}{f(c_i)}.$$

Because $F(c_i)/f(c_i) = (c_i - 1)$, this is increasing in c_i. We can rewrite the objective as

$$E_c\left[\sum_{i=1}^n \phi_i(c) \left(vq_i(c) - \frac{1}{2} q_i(c)^2 - J_i(c_i) q_i(c) \right) - U_i(2) \right].$$

Ignoring the monotonicity condition, pointwise optimization yields the following.

1. $U_i(2) = 0$ for all i

2. For any seller who is selected ($\phi_i(c) > 0$), the optimal $q_i(c)$ solves

$$q_i(c) = \arg \max_{q \in [0,Q]} vq_i(c) - \frac{1}{2} q_i(c)^2 - J_i(c_i) q_i(c).$$

The first-order condition (using the fact that c_i is uniform on $[1, 2]$) provides

$$q_i(c) = v - c_i - (c_i - 1) = v - 2c_i + 1.$$

Notice that $q_i(c)$ is a function of only c_i and it is decreasing in c_i. Furthermore, given $c_i \leq 2$ and $v \geq 3$, we know that $q_i(c_i) \geq 0$ and the lower bound constraint on q does not bind. Finally, we can now compute

$$V_i^*(c_i) = \max_{q \in [0,Q]} \; vq_i(c) - \frac{1}{2}q_i(c)^2 - J_i(c_i)q_i(c) = \frac{1}{2}(v + 1 - 2c_i) \geq 0.$$

3. Because $V_i^*(c_i) \geq 0$ for all c_i, it will always be optimal to award the supply contract to some supplier. Hence, $\sum_i \phi_i(c) = 1$ for all $c \in [1,2]^n$. The optimal pointwise solution is to award it to the highest $V_i^*(c_i)$.

$$\phi_i(c) = \begin{cases} 1 & \text{if } V_i^*(c_i) > \max_{j \neq i} V_j^*(c_j) \\ 0 & \text{if } V_i^*(c_i) < \max_{j \neq i} V_j^*(c_j) \\ \frac{1}{k} & \text{if } V_i^*(c_i) = \max_{j \neq i} V_j^*(c_j) \text{ and there exist } k-1 \text{ firms with } V_j^*(c_j) = V_i^*(c_i). \end{cases}$$

Lastly, we need to check monotonicity. $q_i(c_i)$ is decreasing in c_i. Because ϕ_i is weakly increasing in $V_i^*(c_i)$ and $V_i^*(c_i)$ is strictly decreasing in c_i, $\phi_i(c)$ is weakly decreasing in c_i. Together, the product $\phi_i(c)q_i(c)$ must also be weakly decreasing in c_i. Hence, the solution to the unconstrained program satisfies the monotonicity constraints.

Aside: This problem is very similar to the results of Laffont-Tirole (*JPE*, 1987); see also Laffont-Tirole (1993, ch. 7). The same conclusions emerge. First, there is a separation property where the selected firm produces at the level of output that it would have in absence of competition. Second, there is a rent reduction because of the underlying competitive auction. It is worth noticing that the expected rent that the selected firm receives under the mechanism is the same as the expected rent if there were no competition but costs were distributed according to F truncated on $[\underline{c}, c^{(2)}]$, where $c^{(2)}$ is the second lowest cost firm. This also explains the separation property: such a "truncation" does not change the choice of q_i for the selected firm because $\frac{F(c)}{f(c)}$ is invariant to *upward* truncations. That is,

$$\frac{F(c|c \leq c^{(2)})}{f(c|c \leq c^{(2)})} = \frac{F(c)}{F(c^{(2)})}\frac{F(c^{(2)})}{f(c)} = \frac{F(c)}{f(c)}.$$

3 (a). Write bidder i's expected utility in the auction when θ_i is the true type and i reports $\hat{\theta}_i$:

$$\begin{aligned} U_i(\hat{\theta}_i|\theta_i) &= E_{\theta_{-i}}\left[\phi_i(\hat{\theta}_i, \theta_{-i})\left(\frac{1}{n}\theta_i + \frac{1}{n}\sum_{j \neq i}\theta_j\right)\right] - \bar{t}_i(\hat{\theta}_i) \\ &= \frac{\overline{\phi}_i(\hat{\theta}_i)}{n}\theta_i + E_{\theta_{-i}}\left[\frac{\phi_i(\hat{\theta}_i, \theta_{-i})}{n}\sum_{j \neq i}\theta_j\right] - \bar{t}_i(\hat{\theta}_i) \\ &= \frac{\overline{\phi}_i(\hat{\theta}_i)}{n}\theta_i - \overline{T}_i(\hat{\theta}_i), \end{aligned}$$

where for notational simplicity, I've defined

$$T_i(\hat{\theta}_i) \equiv \bar{t}_i(\hat{\theta}_i) - E_{\theta_{-i}}\left[\frac{\phi_i(\hat{\theta}_i, \theta_{-i})}{n}\sum_{j \neq i}\theta_j\right],$$

The key feature is that $T_i(\hat{\theta}_i)$ is independent of the true θ_i.

We are now in the familiar format of Myerson's optimal auctions model, but where it is "as if" the agent's type is θ_i/n. Hence, it is immediate to apply our result for incentive compatibility in auctions. We conclude that a DRM mechanism is incentive compatible if and only if

$$\overline{\phi}_i(\theta_i) \text{ is nondecreasing,}$$

$$U_i(\theta_i) = U_i(0) + \frac{1}{n}\int_0^{\theta_i} \overline{\phi}_i(s)ds.$$

(b). As in the optimal auction model, we have immediately

$$E[U_i(\theta_i)] = U_i(0) + \frac{1}{n}E\left[\phi_i(\theta)\frac{1-F(\theta)}{f(\theta)}\right].$$

(c). We can fold the result from (b) into the seller's objective function to obtain

$$E\left[\sum_{i=1}^{n}\phi_i(\theta)\left(\sum_{j=1}^{n}\theta_j - \theta_0\right) + \theta_0 - U_i(0) - \phi_i(\theta)\frac{1-F(\theta_i)}{nf(\theta_i)}\right]$$

or more compactly

$$E\left[\sum_{i=1}^{n}\phi_i(\theta)\left(\sum_{j=1}^{n}\theta_j - \frac{1-F(\theta_i)}{nf(\theta_i)} - \theta_0\right) - U_i(0)\right].$$

(d). We ignore the monotonicity condition and solve the pointwise problem. Define

$$V_i(\theta) = \left(\sum_{j=1}^{n}\theta_j\right) - \frac{1-F(\theta_i)}{nf(\theta_i)}.$$

Assume that $V_i(\theta_i, \theta_{-i})$ is increasing in θ_i. This will be true if F satisfies the MHRC for each i. Note that in this case, for any vector of types, $V_i(\theta) > V_j(\theta)$ if and only if $\theta_i > \theta_j$. The following is immediate:

1. $U_i(0) = 0$ for all i.
2. The optimal allocation is

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \max_{j\neq i}\theta_j \text{ and } V_i(\theta) \geq 0 \\ 0 & \text{if } \theta_i < \max_{j\neq i}\theta_j \text{ or } V_i(\theta) < 0 \\ \frac{1}{k} & \text{if } \theta_i = \max_{j\neq i}\theta_j \text{ and there exist } k-1 \text{ firms with } \theta_i = \theta_j, \text{ and } V_i(\theta) \geq 0. \end{cases}$$

(e). To find the symmetric bid function, we are going to use the envelope theorem. [You could have also done this by introducing the inverse bid function and looking at bidder i's first-order conditions. In this case, the hint that the function is linear would be particularly useful.] Suppose that $\overline{b}(\cdot)$ is strictly increasing and differentiable. In equilibrium, this implies that the highest type wins the object. Thus,

$$U_i(\theta_i) = \text{Prob}[\max_{j \neq i} \theta_j \leq \theta_i] E_{\theta_{-i}} \left[\left(\sum_{j=1}^{n} \theta_j \right) - \overline{b}(\theta_i) \mid \max_{j \neq i} \theta_j \leq \theta_i \right].$$

We can simplify this with some computations.

$$U_i(\theta_i) = F(\theta_i)^{n-1} \left(\frac{1}{n}\theta_i + \frac{n-1}{n} \int_0^{\theta_i} \frac{sf(s)}{F(\theta_i)} ds \right) - F(\theta_i)^{n-1}\overline{b}(\theta_i).$$

Using the fact that $F(\theta) = \theta$, we have

$$U_i(\theta_i) = \theta_i^n \left(\frac{n+1}{2n} \right) - \theta_i^{n-1}\overline{b}(\theta_i).$$

The envelope theorem tells us that

$$U_i(\theta_i) = U_i(0) + \frac{1}{n} \int_0^{\theta_i} F(s)^{n-1} ds = U_i(0) + \frac{1}{n^2}\theta_i^n.$$

Because the lowest type bidder in the auction, $\theta_i = 0$, can only win if $\max_{j \neq i} \theta_j = 0$, in which case $v(0, \ldots, 0) = 0$, we know $\overline{b}(0) = 0$ and $U_i(0) = 0$. Collecting terms

$$\theta_i^n \left(\frac{n+1}{2n} \right) - \theta_i^{n-1}\overline{b}(\theta_i) = \frac{1}{n^2}\theta_i^n,$$

or

$$\overline{b}(\theta_i) = \theta_i \left(\frac{n+1}{2n} - \frac{1}{n^2} \right) = \theta_i \left(\frac{(n+2)(n-1)}{2n^2} \right).$$

This function is clearly increasing and differentiable as assumed. What remains is for us to verify that bidder i will follow this strategy given that all other bidders are using it. First, we note that bidder i would never bid in excess of $\overline{b}(1)$, because she would win with probability one using a lower bid of $\overline{b}(1)$. Because $\overline{b}$ is strictly increasing, we can think of bidder i as choose a report $\hat{\theta}_i$ to generate any bid in $[0, \overline{b}(1)]$. Her problem is

$$\max_{\hat{\theta}_i} U_i(\hat{\theta}_i | \theta_i),$$

where

$$U_i(\hat{\theta}_i|\theta_i) = \hat{\theta}_i^{n-1} \left(\frac{1}{n}\theta_i + \frac{n-1}{2n}\hat{\theta}_i \right) - \hat{\theta}_i^{n-1} \left(\frac{(n+2)(n-1)}{2n^2} \right) \hat{\theta}_i,$$

or after simplifying

$$U_i(\hat{\theta}_i|\theta_i) = \frac{1}{n}\hat{\theta}_i^{n-1}\theta_i - \frac{n-1}{n^2}\hat{\theta}_i^n.$$

The first-order condition for $\hat{\theta}_i$ is

$$\hat{\theta}^{n-2}\theta_i \frac{n-1}{n} - \hat{\theta}_i^{n-1} \frac{n-1}{n},$$

which has a unique root at $\hat{\theta}_i = \theta_i$. The second derivative is negative at $\theta_i = \hat{\theta}_i$, indicating the program is strictly quasi-concave and truth telling (i.e., following the strategy $\overline{b}(\cdot)$) is optimal for player i.

(f). The first-price auction awards the good to the highest type θ_i; the seller never keeps the object. The optimal auction awards the good to the highest type θ_i, providing that $\frac{1}{n}\sum_j \theta_j \geq \frac{1-F(\theta)}{nf(\theta)}$. Rewriting the latter condition, the good is awarded only if there is some θ_i such that

$$\sum_{j=1}^{n} \theta_j \geq (1-\theta_i).$$

If $\theta_j \in (0, \frac{1}{2n})$ for all j, for example, then we have

$$\sum_{j=1}^{n} \theta_j \leq \frac{1}{2} \leq (1-2\frac{1}{2n}) \leq (1-\theta_i),$$

and thus the good would be retained by the seller. Hence, the first-price auction is not optimal, though it could be made optimal with the correct reservation price.

(g). You could differentiate the bidding function in (e) with respect to n (which would ignore integer issues), but to be careful let's look at the first difference:

$$\Delta(n) \equiv \left(\frac{(n+2)(n-1)}{2n^2}\right) - \left(\frac{(n+1)(n-2)}{2(n-1)^2}\right) = \frac{n(5-n)-2}{2(n-1)^2 n^2}.$$

It is immediate that the slope of the bidding function falls once n exceeds 4 (i.e., once $n \geq 5$). The reason for this is the winner's curse. As more bidders participate, winning the auction provides worse news about the value of the object to the winner.

4 (Thanks to Simon Board for this question.) Note that although θ_i is appearing in the principal's utility and so there is an element of adverse selection, there is no reason why we cannot use our optimal-auctions toolbox to solve the problem.

(a). Following standard arguments, the direct-revelation mechanism is IC if and only if $\overline{\phi}_i(\theta_i)$ is nondecreasing and the lecturer's utility can be written as

$$U_i(\theta_i) = U_i(1) + \int_{\theta_i}^{1} \overline{\phi}_i(s)ds.$$

(b). Following standard arguments, the result in (a) allows us to express $E[U_i(\theta_i)]$ as

$$E[U_i(\theta)] = U_i(1) + E\left[\phi_i(\theta)\frac{F(\theta)}{f(\theta)}\right].$$

Using $t_i(\theta) = \phi_i(\theta)\theta_i + U_i(\theta)$, the department's objective function can be written as

$$E\left[\sum_{i=1}^{n} \phi_i(\theta)\left(v(\theta_i) - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}\right) - U_i(1)\right].$$

(c). Define

$$V(\theta_i) = v(\theta_i) - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}.$$

Using the fact that $F(\theta) = \theta$, we have

$$V(\theta_i) = v(\theta_i) - 2\theta_i.$$

By assumption, $v'(\theta_i) \leq 1$, so $V(\theta_i)$ is decreasing in θ_i.

We solve the program pointwise, ignoring the monotonicity constraint. The solution (ignoring probability-zero events) is

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } V_i(\theta_i) > \max_{j \neq i} V_j(\theta_j) \text{ and } V_i(\theta_i) \geq 0 \\ 0 & \text{if } V_i(\theta_i) < \max_{j \neq i} V_j(\theta_j) \text{ or } V_i(\theta_i) < 0. \end{cases}$$

Because $V(\theta_i)$ is decreasing in θ_i, the ϕ_i solution above is nonincreasing as required, and thus the relaxed program solves the unrelaxed program.

(d). The pointwise solution to the relaxed program is the same as in (c). Unfortunately, when $v'(\theta_i) > 2$, the function $V(\theta_i)$ is now strictly increasing. Thus, $\overline{\phi}_i(\theta_i)$ is strictly increasing for all θ_i. This violates the monotonicity condition everywhere. You can apply the hint from the problem and conclude that the optimal allocation rule is constant. Hence, it is optimal to choose a lecturer randomly among the n candidates and pay the highest price $t_i(\theta_i) = 1$ (required for incentive compatibility). Intuitively, the department cares so much more about quality relative to costs that it gives up trying to reduce expenditures by screening among the lecturers.

5 (This question is loosely based on Levin and Smith, "Equilibrium in Auctions with Entry," *AER*, 1994.) Before we get started, note that the seller is designing the auction before the bidders learn their types. So while the seller needs to somehow cover the bidding cost k for each bidder, the seller should be able to get the first best by somehow selling an option to consume the good efficiently at the auction. Thus, before doing any math, you may guess the solution. Bidders earn no rents: bidders' expected payoffs after learning their type are exactly equal to the cost of learning, k. The seller takes everything. Hence, it is efficient to allocate the good to the highest bidder and to sell the good whenever $\max_i \theta_i \geq \theta_0 = \theta^*$. In terms of asking bidders to show up, the seller should invite the socially optimal number of bidders (because the seller obtains the social surplus). We will find all of these conclusions once we apply the optimal auctions framework.

(a). This is standard stuff from our lectures on auctions. An direct-revelation auction mechanism is IC if and only if $\overline{\phi}_i(c_i)$ is nondecreasing and

$$U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} \overline{\phi}_i(s)ds.$$

(b). It must be that the expected utility of participating in the mechanism, prior to learning θ_i weakly exceeds k: $E_\theta[U_i(\theta)] \geq k$. Using our result in (a) and integrating by parts, we can restate this requirement as

$$E[U_i(\theta_i)] = U_i(\underline{\theta}) + E\left[\overline{\phi}_i(\theta_i)\frac{1 - F(\theta_i)}{f(\theta_i)}\right] \geq k,$$

or equivalently

$$E[U_i(\theta)] = U_i(\underline{\theta}) + E\left[\phi_i(\theta)\frac{1 - F(\theta_i)}{f(\theta_i)}\right] \geq k.$$

(c). Fixing n and substituting in for $E[U_i(\theta)]$, the seller's objective is to maximize

$$E\left[\theta_0 + \sum_{i=1}^{n} \phi_i(\theta)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \theta_0\right) - U_i(\underline{\theta})\right],$$

subject to $\overline{\phi}_i(\theta_i)$ nondecreasing and

$$U_i(\underline{\theta}) + E\left[\phi_i(\theta)\frac{1-F(\theta_i)}{f(\theta_i)}\right] \geq k.$$

(d). Let λ be the Lagrange multiplier on the ex ante IR constraint. We can write the seller's Lagrangian as

$$\mathcal{L} = E\left[\theta_0 + \sum_{i=1}^{n} \phi_i(\theta)\left(\theta_i - \frac{1-F(\theta_i)-\theta_0}{f(\theta_i)}\right) - U_i(\underline{\theta}) + \lambda\left(\phi_i(\theta)\frac{1-F(\theta_i)}{f(\theta_i)} + U_i(\underline{\theta}) - k\right)\right],$$

or alternatively,

$$\mathcal{L} = E\left[\theta_0 + \sum_{i=1}^{n} \phi_i(\theta)\left(\theta_i - (1-\lambda)\frac{1-F(\theta_i)}{f(\theta_i)} - \theta_0\right) - (1-\lambda)U_i(\underline{\theta}) - \lambda k\right].$$

Consider the first derivative of this function $\mathcal{L}$ with respect to $U_i(\underline{\theta}_i)$. It is $\lambda - 1$. If this were a positive number, then $U_i(\underline{\theta}_i)$ would be set to infinity, but this cannot be optimal for the seller. Similarly, if it were negative, then $U_i(\underline{\theta}_i)$ would be set to negative infinity which cannot happen because there would be a violation of the ex ante IR constraint. (Note that there is no interim IR constraint which would require that $U_i(\underline{\theta}_i) \geq 0$.) Thus, if a solution to the program exists, we must have $\lambda = 1$.

Fixing $\lambda = 1$ reduces the seller's pointwise program to maximizing

$$\mathcal{L} = E\left[\theta_0 - nk + \sum_{i=1}^{n} \phi_i(\theta)\left(\theta_i - \theta_0\right)\right].$$

Alternatively, we can define $\phi_0(\theta) \equiv 1 - \sum_{i=1}^{n} \phi_i(\theta)$ and reduce the seller's program to choosing ϕ_i to maximize

$$E\left[-nk + \sum_{i=0}^{n} \phi_i(\theta)\theta_i\right].$$

In this case, the optimal solution is easy – award the good to the highest type agent, possibly the seller. $\theta^* = \theta_0$. Unlike the optimal auction in Myerson (1918), in this setting the seller never inefficiently keeps the good. This is because the auction design takes place before the bidders learn their types. The ex ante IR constraint allows the seller to extract all of the bidder's information rents up front.

Aside: How would this auction work in practice? Here is one way. Suppose that the auction is set up as a second-price auction with reserve price/type $r^* = \theta^* = \theta_0$. Because it is a second-price auction, there is an expected surplus. Denote the ex ante surplus in a second-price auction when there are n total bidders as $E[U^{2d(n)}(\theta_i)]$. Now charge each invited bidder the auction fee $E[U^{2d(n)}(\theta_i)] - k$. This makes all invited bidders willing to pay k and participate in the auction.

(e). It is immediate from (d) that the seller's expected payoff for a given n is

$$\Pi(n) = E\left[\max\{\theta_1, \ldots, \theta_n, \theta_0\} - nk\right].$$

Hence, the optimal number of bidders will be invited to learn their valuation at cost k. The auction is design so that the expected surplus is exactly equal to k for the invited bidders.

6 (a). Each bidder's virtual type is $J(\theta_i) = \theta - (1-\theta) = 2\theta - 1$. The optimal allocation should award the object to the bidder with the highest virtual valuation, providing that the valuation exceeds $\theta_0 = 0$. This implies a threshold type of $J(\theta^*) = 0$ or $\theta^* = \frac{1}{2}$. Thus,

$$\phi_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \text{if } \theta_i > \theta_{-i} \text{ and } \theta_i \geq \frac{1}{2} \\ 0 & \text{if } \theta_i < \max\{\theta_{-i}, \frac{1}{2}\} \end{cases}$$

(b). The first-price auction with reserve will also be optimal if it implements the same ϕ_i's and $U_i(0) = 0$.

In the first-price auction, for bidders who submit bids above the reserve price, the highest bidder wins. Because the distributions are symmetric, this corresponds to the highest virtual type winning, as in the optimal auction in (a).

We next show that a reserve price of $r^* = \theta^*$ implements the same threshold type, θ^*. To see this, note that any bidder with value $\theta_i < r^*$ will not bid and will earn $U(\theta_i) = 0$. Any bidder with type $\theta_i > r^*$ will find it optimal to bid and will choose a bid in $[r^*, \theta_i)$. Hence, types $\theta_i > r^*$ will win with some probability in the first-price auction. It follows that by setting $r^* = \theta^*$, that only $\theta \geq \theta^*$ will bid and have any probability of winning.

Because $U_i(\theta_i) = 0$ for all $\theta_i \leq \theta^*$, the utility of the lowest type is $U_i(0) = 0$, just as in the optimal auction. Because these utilities are the same and the first-price auction with reserve implements the same ϕ_i's, revenue-equivalence holds and the first-price auction is also optimal.

(c). The answer is not a truncation of the equilibrium bid function in the no-reserve case. Indeed, it is no longer a linear function of type. Here's one way to find it.

We know in equilibrium,

$$U_i(\theta) = F(\theta_i)(\theta_i - \overline{b}(\theta_i)) \text{ for } \theta_i \geq \theta^*.$$

Furthermore, by our incentive-compatibility theorem, we know

$$U_i(\theta) = \int_0^{\theta_i} \overline{\phi}_i(s)ds = \int_{\theta^*}^{\theta_i} F(s)ds \text{ for } \theta_i \geq \theta^*.$$

so, for $\theta_i \geq \theta^*$, (and using $F(\theta) = \theta$ and $\theta^* = 1/2$), we have

$$\theta_i(\theta_i - \overline{b}(\theta_i)) = \frac{\theta_i^2}{2} - \frac{1}{8}.$$

Solving this for $\overline{b}(\theta_i)$ yields

$$\overline{b}(\theta_i) = \frac{1}{8\theta_i} + \frac{\theta_i}{2}.$$

We can see immediately that $\overline{b}(1/2) = 1/2$ and it is strictly increasing on $[\frac{1}{2}, 1]$, but the relationship is nonlinear.

(d). We need to find a reserve price r^* such that all types $\theta \leq \theta^*$ choose $\overline{b}(\theta) = 0$ and all types $\theta > \theta^*$ choose $\overline{b}(\theta) \geq r^*$ and win with positive probability.

In equilibrium, for all $\theta_i > \theta^*$, the probability of winning the auction is $F(\theta_i)$ (i.e., the probability the other bidder's type is below θ_i). Thus,

$$U_i(\theta_i) = F(\theta_i)\theta_i - \bar{b}(\theta_i).$$

The envelope theorem implies this must equal

$$U_i(\theta_i) = \int_0^{\theta_i} \overline{\phi}_i(s)ds.$$

Because $\overline{\phi}_i(\theta_i) = F(\theta_i)$ if $\theta_i \geq \theta^*$ and 0 otherwise,

$$U_i(\theta_i) = \int_{\theta^*}^{\theta_i} F(s)ds \quad \text{for } \theta_i \geq \theta^*.$$

Thus, we have (using $F(\theta) = \theta$ and $\theta^* = 1/2$) for all $\theta_i \geq \theta^*$,

$$\theta_i^2 - \bar{b}(\theta_i) = \frac{\theta_i^2}{2} - \frac{1}{8}.$$

Hence, for all $\theta_i \geq \theta^*$,

$$\bar{b}(\theta_i) = \frac{\theta_i^2}{2} + \frac{1}{8}.$$

For $\theta_i = \theta^*$, we need that $\bar{b}(\theta^*) = r^*$. Thus,

$$\bar{b}(\theta^*) = \frac{(\theta^*)^2}{2} + \frac{1}{8} = \frac{1}{4}.$$

Hence,

$$r^* = \frac{1}{4}.$$

At this reserve, a bidder of type θ^* is just willing to bid r^* and will win with positive probability. For $\theta_i < r^*$, the bidder chooses not to participate (or bids zero). Hence, are candidate bidding strategy for the all-pay auction with reserve $r^* = \frac{1}{4}$ is

$$\bar{b}(\theta_i) = \begin{cases} \frac{1}{2}\theta_i^2 + \frac{1}{8} & \text{if } \theta_i \geq \theta^* \\ 0 & \text{if } \theta_i < \theta^* \end{cases}$$

What remains to be shown is that this bidding function is in fact an equilibrium in the all-pay auction with a reserve of $r^* = \frac{1}{4}$.

If bidder j chooses to play the equilibrium strategy, then $b_j \in \{0\} \cup [\frac{1}{4}, \frac{5}{8}]$. Given this, bidder i would never choose to bid $b_i \in (0, \frac{1}{4})$ because he cannot win with such a bid. Bidder i would also never choose a bid $b_i > \frac{5}{8}$ because he would win with probability 1 with a lower bid of $\frac{5}{8}$. Hence, the only optimal bids for bidder i correspond to the bid of some type $\hat{\theta}_i$ who follows the equilibrium strategy. We can thus reduce bidder i's strategy to choosing a report to maximize

$$U_i(\hat{\theta}_i|\theta_i) = F(\hat{\theta}_i)\theta_i - \bar{b}(\hat{\theta}_i).$$

Substituting for $\bar{b}(\cdot)$ and $F(\theta) = \theta$ yields

$$U_i(\hat{\theta}_i|\theta_i) = \begin{cases} 0 & \text{if } \hat{\theta}_i < \theta^* = \frac{1}{2} \\ \hat{\theta}_i\theta_i - \frac{1}{2}\hat{\theta}_i^2 - \frac{1}{8} & \text{if } \hat{\theta}_i \geq \theta^* = \frac{1}{2} \end{cases}$$

Conditional on reporting $\hat{\theta}_i \geq \frac{1}{2}$, bidder i's objective in strictly concave and achieves a maximum at $\hat{\theta}_i = \theta_i$. The only remaining part to determine is whether the bidder with type below $\frac{1}{2}$ would want to pretend to be above (or vice versa). If $\theta_i < \frac{1}{2}$, bidding the truth yields 0. Reporting any other type below $\frac{1}{2}$ yields the same outcome. If bidder i chooses some report $\hat{\theta}_i \geq \frac{1}{2}$, then the best report is $\hat{\theta}_i = \frac{1}{2}$. At this report, bidder i's payoff is

$$\frac{1}{2}\theta_i - \frac{1}{4} < 0.$$

Now suppose that $\theta_i \geq \frac{1}{2}$. Telling the truth yields

$$U_i(\theta_i|\theta_i) = \frac{1}{2}\theta_i^2 - \frac{1}{8} \geq 0.$$

If this bidder chooses to report $\hat{\theta}_i < \frac{1}{2}$, instead, the payoff is zero. Hence, it is optimal to follow the strategy in $\overline{b}(\cdot)$ if the rival follows it as well.