OLG Perpetual Youth

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Demographics

- Continuous time. Agent die period of length dt with prob. $p\ dt$.
- Note that $p \in (0, \infty)$. Expected lifetime $1/p$.
- If measure p is born at date s, fraction $p\ e^{-p(t-s)}$ survive at t.
  - Let $N(s,t)$ be the size of the cohort, so $N(s, t + \Delta) = N(s,t)(1 - p\Delta)$ so
  - $dN(s,t)/dt = -N(s,t)p$ and boundary $N(s,s) = p$: initial cohort size.
  - Then solution is $N(s,t) = p\ e^{-p(t-s)}$.
- Agents that die replaced by newborns.
- Adding all cohort alive at time t : $\int_{-\infty}^{t} N(s,t)ds = \int_{-\infty}^{t} p\ e^{-p(t-s)} ds = 1$.

Insurance, Annuities

- All agents have same time expectancy, no young and old.
- Unrealistic, but tractable. Effects of currently alive vs. those yet not born.
- Agents can insure against debt, at fair prices.
- Let r be the net risk-ess interest rate.
- Invest v at t, gets $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t+\Delta$, and zero if dead.
- Insurance company selling annuities breaks even.
- Continuous time (as $\Delta \downarrow 0$) : $v\frac{1+\Delta r}{1-p\Delta} = v + v(r+p)\Delta + o(\Delta)$

Preferences

- Expected discounted utility. Utility only while alive.
- Agent born at t maximizes: $\mathbb{E}\left[\int_t^\infty u(c(z))\, e^{-\theta(z-t)} dz\right]$
- Discount rate $\theta \in (0, \infty)$, so 1 util at time $t + \Delta$ is worth $\frac{1}{1+\Delta\theta}$ at t.
- Expectation only with respect to realization of death.
- Time remaining alive exponentially distributed:

$$\mathbb{E}\left[\int_t^\infty u(c(z))\, e^{-\theta(z-t)} dz\right] = \int_t^\infty u(c(z))\, e^{-(p+\theta)(z-t)} dz$$

Household Problem

- Specialize $u(\cdot)$ to log utility.
- $\max \mathbb{E}\left[\int_t^\infty u\left(c\left(z\right)\right) e^{-\theta(z-t)}\, dz\right] = \int_t^\infty \log\left(c\left(z\right)\right) e^{-(\rho+\theta)(z-t)}\, dz$
- subject to $\int_t^\infty [c(z) - y(z)]\, R(t,z)\, dz = v(t)$ where
  - $R(t,z)$: price of a good in z in terms of goods in t,
  - $v(t)$: non-human (financial) wealth at time t,
  - $y(z)$: labor income at time z.

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  - $R(t,z)$: price of a good in z in terms of goods in t,
  - $v(t)$: non-human (financial) wealth at time t,
  - $y(z)$: labor income at time z.
- define human wealth as $h(t) = \int_t^\infty y(z)R(t,z)\, dz$
- Solution $c(t) = (\theta + p)\left(v\left(t\right) + h\left(t\right)\right)$.

Preferences, log case

- Recall that $max_{c_1, c_2, \ldots, c_n} \; a_1 \log(c_1) + a_1 \log(c_2) + \ldots + a_n \log(c_n)$

  subject to $R_1 \; c_1 + R_1 \; c_2 + \ldots + R_n \; c_n = h + v$.

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  Optimal demand gives: $c_1 = a_1\ [h+v]$.

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- Since weights add up to one $\int_t^\infty \log(c(z))\, e^{-(p+\theta)(z-t)} dz$ and

  $R(t, t) = 1$ by definition, so $c(t) = (\theta + p)\,(v(t) + h(t))$,

  notice that “weights add up to one": $\int_t^\infty e^{-(\theta+p)(z-t)} dz = 1/(\theta + p)$.

Deriving ODEs for HH problem

- Given Intertemporal prices $R(t,z)$ define interest rates $r(\mu)+p$ as :

$$R(t,z) = \exp\left(- \int_t^z \left[r\left(\mu\right) + p \right] d\mu \right) .$$

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- Budget Constraint $\int_t^\infty [c(z)-y(z)]\, R(t,z)\, dz = v(t)$ equivalent to:

$$\frac{dv(z)}{dz} = \left[r(z)+p\right] v(z) + y(z) - c(z) \text{ and } \lim_{z\to\infty} R(t,z)v(z) = 0 .$$

- Definition of human wealth $h(t) = \int_t^\infty y(z)R(t,z)\, dz$ equivalent to:

$$\frac{dh(z)}{dz} = \left[r(z)+p\right] h(z) - y(z) \text{ and } \lim_{z\to\infty} R(t,z)h(z) = 0 .$$

- Differentiate f.o.c. for $c(z)$: $\frac{\exp(-(\theta+p)(z-t))}{c(z)} = \lambda \exp\left(-\int_t^z \left[r(\mu)+p\right] d\mu\right)$

Obtain: $\frac{dc(z)}{dz} = \left[r(z)-\theta\right] c(z)$ (where λ was multiplier in BC).

Deriving ODEs for HH problem, algebra I

- Given Intertemporal prices $R(t,z)$ define interest rates $r(\mu)+p$ as :

$$R(t,z) = \exp\left(-\int_t^z \left[r\left(\mu\right)+p\right] d\mu\right)$$

- Consider discrete time case with Δ the length of time period.

- There are $(z-t)/\Delta$ discrete time periods between t and z, and let $p=0$:

$$R(t,z)^{-1} = (1+r_{t+1}\Delta)(1+r_{t+2}\Delta)\cdots(1+r_z\Delta) = \Pi_{i=1}^{(t-z)/\Delta}(1+r_{t+i}\Delta)$$

$$-\log\left(R(t,z)\right) = \sum_{i=1}^{(t-z)/\Delta} \log(1+r_{t+i}\Delta) = \sum_{i=1}^{(t-z)/\Delta}\left[r_{t+i}+\frac{o(\Delta)}{\Delta}\right]\Delta$$

- $\lim_{\Delta\downarrow 0} \log\left(R(t,z)\right) = -\int_t^z r\left(\mu\right) d\mu$.

- In general with $p>0$: $\lim_{\Delta\downarrow 0} \log\left(R(t,z)\right) = -\int_t^z \left[r\left(\mu\right)+p\right] d\mu$.

Deriving ODEs for HH problem, algebra II

- Let $v(t) = \int_t^T [c(z) - y(z)]\, R(t,z)\, dz + R(t,T) v(T)$
- Use expression for $R(t,z)$ in terms of $r + p$:
- $v(t) = \int_t^T [c(z) - y(z)]\, e^{-\int_t^z (r(\mu)+p) d\mu} dz + e^{-\int_t^T (r(\mu)+p) d\mu} v(T)$
- Differentiate previous expression with respect to t:

$$\frac{dv(t)}{dt} = -[c(t) - y(t)] + [r(t) + p] \int_t^T [c(z) - y(z)]\, e^{-\int_t^z (r(\mu)+p) d\mu} dz$$
$$+ [r(t) + p]\, e^{-\int_t^T (r(\mu)+p) d\mu} v(T) \ .$$

- Replace the expression for $v(t)$: $\underbrace{\frac{dv(t)}{dt}}_{\text{capital gain}} = \underbrace{y(t) - c(t)}_{\text{- dividend}} + \underbrace{[r(t) + p]\, v(t)}_{\text{current income}}$.

Boundary of ODEs for HH problem, interpretation

- For ODE (sequential BC) to be equivalent to Present Value BC:

  $\lim_{T\to\infty} v(T)e^{-\int_t^T (r(\mu)+p)d\mu} = 0$.

- If this is strictly positive, the agent did not maximize its utility. (Why?)

- If this is strictly negative, the agent the plan was not budget feasible.

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- If this is strictly negative, the agent the plan was not budget feasible.

- For $h(t) = \lim_{T\to\infty} \int_t^T y(z)R(t,z)\, dz$ to converge requires:

  $\lim_{T\to\infty}\ y(T)\ e^{-\int_t^T (r(\mu)+p)d\mu} = 0$.

- Interpretation: if this limit does not converge, the agent human wealth is unbounded.

- If human wealth is unbounded, the problem has no solution.

Consumption/Savings Behaviour of Households given Human and Non-Human Wealth

optimal decision rule

- Individuals: $c(s,t) = (p+\theta)\,(h(s,t) + v(s,t))$ or

  $\frac{dc(s,t)}{dt} = [r(t) - \theta]\, c(s,t)$ BC + and boundary condition.

- Define aggregate: $C(t) \equiv \int_{-\infty}^{t} N(s,t)c(s,t)ds$.

- Use $c(s,t)$ above and definitions of $H(t)$ and $V(t)$ as aggregates:

$$
\begin{aligned}
H(t) &\equiv \int_{-\infty}^{t} N(s,t)h(s,t)ds\,, \quad \text{human wealth} \\
V(t) &\equiv \int_{-\infty}^{t} N(s,t)v(s,t)ds\,. \quad \text{non-human wealth}
\end{aligned}
$$

- Aggregate: $C(t) = (p+\theta)\,(H(t) + V(t))$

Exponential model of labor services

- Model endowment of labor services as decreasing with age at rate α.
- Labor income = labor services $\times$ wage.
- $Y(t)$: aggregate labor income of agents alive at time t.
- $y(s,t)$: labor income of agent born at s alive at t.
- $y(s,t) = a\, Y(t)\, e^{-\alpha(t-s)}$ for some constant $a > 0$.
- $Y(t) = \int_{-\infty}^{t} N(s,t)\, y(s,t) ds = \int_{-\infty}^{t} p\, e^{-p(t-s)}\, y(s,t) ds$
  $= \int_{-\infty}^{t} p\, e^{-p(t-s)}\, a\, Y(t)\, e^{-\alpha(t-s)} ds \implies a = \frac{p+\alpha}{p}$.

Individual and Aggregate Human Wealth

- Using definitions $h(t,s) = \int_t^{\infty} y(s,z)R(t,z)\, dz$,

  intertemporal price: $R(t,z) = \exp\left(-\int_t^z \left[r\left(\mu\right) + p\right] d\mu\right)$, and

  exponentially declining labor services: $y(s,t) = a\; Y(t)\; e^{-\alpha(t-s)}$:

- define Aggregate Human Wealth $H(t) = \int_{-\infty}^{t} h(s,t)N(s,t)ds$.

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- define Aggregate Human Wealth $H(t) = \int_{-\infty}^{t} h(s,t) N(s,t) ds$.

- Aggregate Human Wealth satisfies: $\frac{dH(t)}{dt} = \left[r(t) + p + \alpha\right] H(t) - Y(t)$,

  and $\lim_{z \to} H(z)\; \exp\left(-\int_t^z \left[r(\mu) + \alpha + p\right] d\mu\right) = 0$.

- $H(t)$: discounted present value of those currently alive:

  discounted for death (p) and for decline on labor services (α).

Individual and Aggregate Human Wealth, algebra

- $h(t,s) = \int_t^{\infty} a\, Y(z)\, e^{-\alpha(z-s)}\, R(t,z)\, dz$

  $= a\, \left[\int_t^{\infty}\; Y(z)\, e^{-\alpha(z-t)}\, R(t,z)\, dz\right]\; e^{-\alpha(t-s)}$.

- $H(t) = \int_{-\infty}^{t} h(s,t)\, p\, e^{-p(t-s)} ds$

  $= \int_{-\infty}^{t}\; a\, p\left[\int_t^{\infty}\; Y(z)\, e^{-\alpha(z-t)}\, R(t,z)\, dz\right]\; e^{-\alpha(t-s)}\; e^{-p(t-s)} ds$.

- Using the characterization of $R(t,z)$ and a:

  $H(t) = \int_t^{\infty} Y(z)\; \exp\left\{-\int_t^{z} (\alpha + p + r(\mu))\, d\mu\right\} dz$.

- This present value relationship implies the ODE (diff. .w.r.t. t):

  $dH(t) = [\alpha + r(t) + p]\, H(t) - Y(t)$ plus boundary.

Individual and Aggregate Non-Human Wealth

- Define $V(t) = \int_{-\infty}^{t} v(s,t) N(s,t) ds = \int_{-\infty}^{t} v(s,t)\; p\; e^{-p(t-s)} ds$.
- Differentiate $V(t)$ w.r.t. to t to obtain:

$$\frac{dV(t)}{dt} = p\; v(t,t) - p\; V(t) + \int_{-\infty}^{t} \left[\frac{dv(s,t)}{dt} \right] \; p\; e^{-p(t-s)} ds,$$

- $v(t,t) = 0$ since it is the wealth at birth (OLG with no altruism!),
- $\frac{dv(s,t)}{dt} = [r(t) + p]\, v(s,t) + y(t) - c(t)$, i.e. budget constraint,
- Thus $\frac{dV(t)}{dt} = r(t)\; V(t) + Y(t) - C(t)$.
- Aggregate wealth accumulates at r, individual at $r + p$.
- Difference, $p\; V(t)$ is the transfer from annuities.

insurance does not add to aggregate wealth.

Summary Aggregate Behavior

- $C(t) = (p + \theta)(H(t) + V(t))$
- $\frac{dV(t)}{dt} = r(t)\ V(t) + Y(t) - C(t)$
- $\frac{dH(t)}{dt} = [r(t) + p + \alpha]\, H(t) - Y(t)$
- and boundary condition as $t \to \infty$.

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- $\frac{dH(t)}{dt} = [r(t)+p+\alpha]\, H(t) - Y(t)$
- and boundary condition as $t \to \infty$.
- Differentiating $C(t)$ and replacing $\frac{dH(t)}{dt}, \frac{dV(t)}{dt}$:

$$\frac{dC(t)}{dt} = [r(t)+\alpha-\theta]\, C(t) - (p+\alpha)(p+\theta)\, V(t)\ ,$$
$$\frac{dV(t)}{dt} = r(t)\ V(t) + Y(t) - C(t)\ ,$$

and boundary condition as $t \to \infty$.

Summary Aggregate Behavior, algebra

- $C(t) = (p+\theta)(H(t)+V(t))$
- $\dot{V}(r) = r(t)\ V(t) + Y(t) - C(t)$
- $\dot{H}(t) = [r(t)+p+\alpha]\,H(t) - Y(t)$
- Differentiating C, replacing $\dot{H}$, $\dot{V}$, and $C = (\theta+p)\,(H+V)$:

$$
\begin{aligned}
\dot{C} &= (p+\theta)r\,[H+V] - (p+\theta)C + (p+\alpha)(p+\theta)H\,, \\
&= [r-p-\theta]\,C + (p+\alpha)(p+\theta)H\,, \\
&\quad \text{adding and subtracting } (p+\alpha)(p+\theta)V \\
&= [r-p-\theta]\,C + (p+\alpha)(p+\theta)\,(H+V) - (p+\alpha)(p+\theta)V\,, \\
&= [r-p-\theta]\,C + (p+\alpha)C - (p+\alpha)(p+\theta)V\,, \\
&= [r+\alpha-\theta]\,C - (p+\alpha)(p+\theta)V\,.
\end{aligned}
$$

Summary Aggregate Behavior, details on boundaries

- For the agent problem to have finite wealth we require:
- Y(t) $\lim_{T\to\infty} e^{-\int_t^T [r(z)+\alpha+p]dz} = 0$.
- If this is violated agents have unbounded human wealth.
- If this is violated agents can borrow an unbounded amount.
- Note that for $\alpha + p > 0$, we can have $r < 0$.

General Equilibrium: 3 cases

- Interest rates on pure endowment:

  $Y(t) = C(t)$ and $V(t) = 0$.

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- Add capital accumulation, study capital accumulation:

  $F(K) = \mathbb{F}(K, 1) - \delta K$, $\mathbb{F}(\cdot, \cdot)$ CRTS neoclassical,

  $Y(t) = F(K(t)) - F'(K(t)) K(t)$, $r(t) = F'(K(t))$,

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  $V(t) = K(t)$.

- Add government debt:

  $V(t) = B(t) + K(t)$.

Pure Endowment

- $Y(t) = C(t) = Y$.
- $V(t) = 0$, since some agents borrow and others lend:

  $V(t) = \int_{-\infty}^{t} N(s,t)\; v(s,t)\; ds = 0$.
- $v(s,t)$ value of cumulated savings (net assets) of cohort born at s at t.
- $0 = \frac{dC(t)}{dt} = [r(t) + \alpha - \theta]\, C(t) - (p+\alpha)(p+\theta)\, V(t)$ implies
- Equilibrium interest rate $r(t) = \theta - \alpha$.
- Individuals: $\frac{dc(s,t)}{dt} = [r(t) - \theta]\, c(s,t) = -\alpha\; c(s,t)$.
- Thus equilibrium is autarky! $c(s,t) = y(s,t)$ and $v(s,t) = 0$.

Pure Endowment, cont.

- What did we assume for the distribution of wealth among those agents alive at time $t = 0$?
- What is the equilibrium interest rate for $\alpha = 0$? How are the individual consumptions?
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Pure Endowment, cont.

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- What is the equilibrium interest rate for $p = 0$? How are the individual consumptions?
- Consider the class of allocations with $c(s, t) = Y$. Are these Pareto superior to the equilibrium?
- Question: can $r(t) = \theta$ be an equilibrium?

Capital Accumulation, Technology

- Neoclassical Production Function $\mathbb{F}(K, L)$.
- Time t Feasibility: $I(t) + C(t) = \mathbb{F}\left(K(t), L(t)\right)$.
- Discrete time with length Δ gives: $K_{t+\Delta} = \Delta\ I_t\ +\ K_t\left(1 - \Delta\delta\right)$.
- Limit of discrete time as $\Delta \downarrow 0$: $\frac{dK(t)}{dt} = I(t) - \delta K(t)$.
- Replacing Law of Motion of capital into time t Feasibility:
- $\frac{dK(t)}{dt} + C(t) = \mathbb{F}\left(K(t), L(t)\right) - \delta K(t)$.
- We also replace that labor supply is inelastic at $L = 1$:
- Define $F(K) \equiv \mathbb{F}\left(K, 1\right) - \delta K$ and assume no depreciation:

$$\frac{dK(t)}{dt} = F\left(K(t)\right) - C(t) \ .$$

Capital Accumulation, Equilibrium payment to factors

- Aggregate Labor Income $Y(t) = w(t)\, L = F\,(K(t)) - F'\,(K(t))\, K(t)$ for $L = 1$.
- Interest Rate = Marginal Productivity capital: $r(t) = F'\,(K(t))$.
- Discrete time : $\nu_{t+\Delta} = r_t + \delta$ and $\mathbb{F}_k(K_{t+\Delta}, 1) = \nu_{t+\Delta}$.

  where ν_t is the rental rate of capital at t .
- Limit of discrete time as $\Delta \downarrow 0$:

  $\mathbb{F}_k(K(t), 1) = \nu(t) = r(t) + \delta$ or $F'\,(K(t)) = r(t)$.

Capital Accumulation, Equilibrium

- $v(s,t)$ value of cumulated savings (net assets) of cohort born at s at t :

  $V(t) = \int_{-\infty}^{t} N(s,t)\ v(s,t)\ ds = K(t)$.

- Interest payments plus labor income = Output:

  $r(t)\ K(t) + Y(t) = F'(K(t))\, K(t) + F(K(t)) - F'(K(t))\, K(t) = F(K(t))$

$$\begin{aligned} \frac{dC(t)}{dt} &= [r(t) + \alpha - \theta]\, C(t) - (p+\alpha)(p+\theta)\, V(t)\ , \\ \frac{dV(t)}{dt} &= r(t)\ V(t) + Y(t) - C(t)\ , \end{aligned}$$

- Becomes

$$\begin{aligned} \frac{dC(t)}{dt} &= [F'(K(t)) + \alpha - \theta]\, C(t) - (p+\alpha)(p+\theta) K(t)\ , \\ \frac{dK(t)}{dt} &= F(K(t)) - C(t)\ . \end{aligned}$$

Capital Accumulation, Dynamics

- Phase diagram on (K, C) space.
- Locate (K, C) such that $\dot{C} \equiv \frac{dC(t)}{dt} = 0$. Line given by: $C = \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$
- Locate (K, C) such that $\dot{K} \equiv \frac{dK(t)}{dt} = 0$. Line given by: $C = F(K)$
- Locate steady state(s): $(\bar{K}, \bar{C})$.
- Find field (direction of movements everywhere)
- Discard paths that takes system to points that eventually are not feasible.

Dynamics, more details

- $\dot{K} = 0$: $C = F(K)$.
  - Concave, initially increasing. Maximum at $F'(\bar{K}) = 0$, for all $K \geq 0$.
  - Starts very step, and converges to zero with Inada conditions.
  - If $C > F(K)$, then capital decreases.
- $\dot{C} = 0 : C = \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$, for all $K \geq \hat{K}$
  - Convex increasing function of K, since $F'(\cdot)$ is decreasing.
  - Asymptotes to $+\infty$ as K converges to $\hat{K}$ defined as: $F'(\hat{K}) = \theta - \alpha$.
  - If $C > \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$, consumption decreases.
- Unique Interior Steady state solves: $F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta} \bar{K}$.

Dynamics, saddle path and field

- Locate the direction of change at the lines $\dot{C} = 0$ and $\dot{K} = 0$.
- Locate the direction of change close to the axes:$C = 0$ and $K = 0$.
- Follow the motion of the system in four quadrants defined by the intersections of the $\dot{C} = 0$ and $\dot{K} = 0$ lines.
- Argue that there is an upward slopping line $C = \mathbf{S}(K)$ such that :
  - If $C(0) = \mathbf{S}(K(0))$, and the system follows the ODE, converges to SS.
  - Any other path converges in finite time to negative C or K.
  - Thus, in a perfect foresight equilibrium $C(t) = \mathbf{S}(K(t))$ for all $t \geq 0$.
  - What is the relationship between $C = \mathbf{S}(K)$ and $C = (\theta + p)(K + H)$?.
  - Is $\mathbf{S}(\cdot)$ necessary linear?

Figure: Case of $\alpha = 0$

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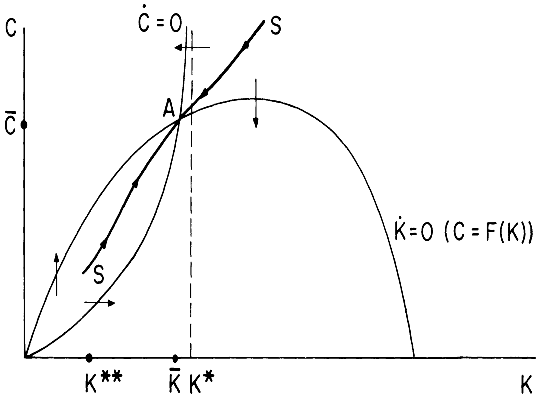


FIG. 2

$F'(K^*) = \theta$ and $F'(K^{**}) = p + \theta$

Adjustment to Steady State

- Assume that $0 < K(0) < \bar{K}$.
- What are the characteristics of the time path for $C(t)$ and $K(t)$?
  Are $C(t)$ and $K(t)$ monotone?
- What are the characteristics of the time path for labor income $Y(t)$?
  Are $Y(t)$ monotone?
- What are the characteristics of the time path for interest rates $r(t)$?
  Are $r(t)$ monotone?
- What are the characteristics of the time path for human wealth $H(t)$?
  Are $H(t)$ monotone?

Comparative Statics of Steady States

- Unique Interior Steady state $\bar{K} : F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta}\bar{K}$ is
  - Decreasing in p.
  - Increasing in α.
  - Decreasing in θ.
- Interest rate $r = F'(K)$. If $r > (<)\ \theta$, individual consumption $\uparrow\ (\downarrow)$.
- Set $\alpha = 0$. If individual consumption is constant ($r = \theta$), zero savings.
- If individual consumption increases, then agents save when young.
- Slope of consumption profile depends only on $r - \theta$: thus as $p \downarrow$, longer lifetime, smaller turn-around of capital across generations.
- If $\alpha > 0$, agents save, even if their consumption is constant.

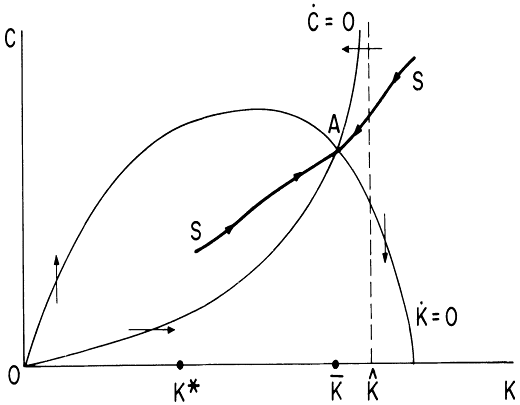
Comparative Statics of Steady States, cont.

- Unique Interior Steady state $\bar{K} : F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta}\bar{K}$
  - If $\alpha = 0$, then $\bar{K}$ is smaller than golden rule, and hence $r > 0$.
  - If $\alpha = 0$, then $\theta < F'(\bar{K}) < \theta + p$.
    - Define $F'(K^{**}) = \theta + p$. Replace on expression for $\bar{K}$.
    - Use that, by concavity of $F(\cdot) : F(K^{**}) < (p+\theta)K^{**} = F'(K^{**})K^{**}$.
  - If $\alpha > \theta$, then $\bar{K}$ can be larger than golden rule, and hence $r < 0$.
  - If $\alpha > \theta$, for p large enough, then $\bar{K}$ larger than golden rule.
- Recall that if $p = 0$, we must set $\alpha = 0$.
  - If $p = 0$ then $F'(\bar{K}) = \theta$, smaller than golden rule, and hence $r > 0$.
  - If $p = 0$ then $\dot{C} = 0$ is a vertical line.

Comparative Statics of Steady States, details

- Mathematical argument to show that $\bar{K}$ is increasing in α.
- Steady state solves: $F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta}\bar{K}$
- Suffices to show that $\frac{p+\alpha}{F'(\bar{K})+\alpha-\theta}$ is decreasing in α.
- We have $\frac{\partial}{\partial\alpha}\frac{p+\alpha}{F'(\bar{K})+\alpha-\theta} = \frac{F'(\bar{K})-\theta-p}{(F'(\bar{K})+\alpha-\theta)^2}$.
- Recall that if $\alpha = 0$ then $0 < F'(\bar{K}) < \theta + p$.
- Thus for $\alpha = 0$, steady state $\bar{K}$ is increasing in α.
- For higher values of α, capital $\bar{K}$ is higher, and thus $F'(\bar{K})$ even smaller.
- Hence, $\bar{K}$ is increasing for all non-negative values of α.

Figure: Case of $\alpha > 0$ large enough



$$K^* \mid r(K^*) = \theta$$
$$\hat{K} \mid r(\hat{K}) = \theta - \alpha$$

Infinite Lifetime, limit as $p \downarrow 0$

- If $p = 0$, stationarity requires $\alpha = 0$.
- System becomes:

$$\begin{aligned} \frac{dC(t)}{dt} &= \left[F'\left(K(t)\right) - \theta\right] C(t) , \\ \frac{dK(t)}{dt} &= F\left(K(t)\right) - C(t) . \end{aligned}$$

= neoclassical

- $\dot{C} = 0$ locus vertical.

infinite elastic supply of capital

- Steady state $F'(\bar{K}) = \theta$.
- Slightly different argument to rule out paths outside saddle path.
- Similar adjustment to steady state.

Government Budget Constraint

- Let $T(t)$ denote taxes levied to all agents alive at time t.
- Let $G(t)$ denote government purchases at time t.
- Let $B(t)$ denote government debt at time t.
- Sequential B.C.: $\frac{dB(z)}{dz} = G(z) - T(z) + r(z)B(z)$.
- Present value B.C: $B(t) = \int_t^{\infty} [T(z) - G(z)]\, e^{-\int_t^z r(\mu)d\mu} dz$ or
- Sequential B.C. and boundary $\lim_{T\to\infty} B(T)e^{-\int_T^z r(\mu)d\mu} = 0$ are equivalent to present value.

Ricarding Equivalence

- Effect on taxes and debt on determination of consumption $C(t)$ as a function of human wealth $H(t)$ and non-human wealth $V(t)$.
- To simplify notation and discussion, for now set $\alpha = 0$.
- Labor income $Y(t)$ becomes labor income after taxes $Y(t) - T(t)$:

$$
\begin{aligned}
C(t) &= (p+\theta)\,(H(t) + V(t)) \\
H(t) &= \int_t^{\infty} [Y(z) - T(t)]\; \exp\left\{-\int_t^z (p + r(\mu))\, d\mu\right\} dz \\
\frac{dV(t)}{dt} &= r(t)\; V(t) + Y(t) - T(t) - C(t)\;.
\end{aligned}
$$

- or ODE for agg. human wealth: $\frac{dH(t)}{dt} = [r(t) + p]\, H(t) - [Y(t) - T(t)]$.

Ricarding Equivalence, Partial Equilibrium

- Change in taxes at time t offset with change at time $t+s$
  Use Gov. B.C.: $dT(t+s) = -\exp\left[\int_t^{t+s} r(\mu)d\mu\right] dT(s)$
- $dT(t)$ and $dT(t+s)$ used heuristically, each has negligible impact.
- Effect on human wealth $H(t)$, and hence, $C(t)$: Use ODE for H:
  $dH(t) = -dT(t) - dT(t+s)\left\{\exp\left[-\int_t^{t+s}[r(\mu)+p]\,d\mu\right]\right\}$
- Replace into Gov. BC: $dH(t) = -dT(t)\,[1-\exp(-ps)]$.
- If $dT(t) < 0$, then $dH(t) > 0$ and $dC(t) > 0$ if $p > 0$:
  or, cut on taxes increases consumption if $p > 0$. Why?
  - Do you get same equilibrium $C's$ and $r's$? budget constraint
  - Is this a failure of Ricarding Equivalence? No.
  - Compare with experiment in problem set.

Government Debt on Pure Endowment case

- Still $Y(t) - G = C(t)$ for $G = 0$, but $V(t) = B(t)$:
- Private sector previous cohorts cumulated savings equal gov't debt

  $V(t) = \int_{-\infty}^{t} N(s,t)\; v(s,t)\; ds = B(t)$.
- $0 = \frac{dC(t)}{dt} = [r(t) + \alpha - \theta]\,(Y - G) - (p + \alpha)(p + \theta)B(t)$ implies
- $r(t) = (p + \alpha)(p + \theta)\frac{B(t)}{Y - G} + [\theta - \alpha]$.
- Evolution of debt $\frac{dB(t)}{dt} = r(t)B(t) + G - T$.
- Replacing interest rate into dynamics for debt:

$$\dot{B} = (p + \alpha)(p + \theta)\frac{B^2}{Y - G} + B\,[\theta - \alpha] + G - T \;.$$

Simple debt dynamics

- Set $T = G = 0$

$$\dot{B} = (p + \alpha)(p + \theta)\frac{B^2}{Y} + B\left[\theta - \alpha\right].$$

- Two steady states: $B^* = 0$ and $\bar{B} = \frac{\alpha-\theta}{(p+\alpha)(p+\theta)}Y$
- Autarky steady state interest rate $\bar{r} = \theta - \alpha$.
- If $\bar{r} < 0$, interior steady state $\bar{B}$ is positive.
- Interior positive steady state $\bar{B}$ is not stable:
  - If $B(0) < \bar{B}$, then $\lim_{t\to} B(t) = 0$.
  - If $B(0) > \bar{B}$, there is no equilibrium.
- Identical to the case of 2 period OLG model, log utility. Try it.
- How will the analysis change if $T - G \neq 0$? Try it.
- Which s.s. is stable when $T - G \neq 0$? How does it depend on $\theta - \alpha$?

Debt dynamics and capital

- Set $\alpha = 0$ for simplicity (but it rules out some important cases)
- Dynamical System with $B + K = V$, keeping T, G constant:

$$
\begin{aligned}
\frac{dC}{dt} &= (r-\theta)C - p(p+\theta)(B+K) \\
\frac{dK}{dt} &= F(K) - C - G\,, \\
\frac{dB}{dt} &= r\,B + G - T\,, \\
r &= F'(K)\,.
\end{aligned}
$$

- Steady States for given G, T:

$$
\begin{aligned}
[F'(K^*) - \theta]\,C^* &= p(p+\theta)(B^* + K^*)\,, \\
F(K^*) &= C^* + G\,, \qquad \text{before C = F(K)} \\
F'(K^*)B^* &= T - G\,.
\end{aligned}
$$

Debt and capital: steady states (fixed G, T)

- Steady States capital K^* solves:

$$F(K^*) - G \quad = \quad \frac{p(p+\theta)}{[F'(K^*) - \theta]} \left(\frac{T - G}{F'(K^*)} + K^* \right) .$$

- Interpretation of comparative static w.r.t. G and T.
  Does increases in taxes T increase the debt?
- Is there a unique steady state? Signs of B and r at s.s.
  Compare with pure endowment.
- How do K^*, B^* and r^* vary with G and or T?.
  What is the difference between increases in G and decreases in T.
- What is the effect on steady state quantities if p increases?
  What is the limit case as $p \downarrow 0$?

Debt dynamics and capital: balanced budget

- Consider Fiscal Policy of keeping debt constant, adjusting taxes.
- Dynamical System with $B + K = V$, keeping B, G constant:

$$
\begin{aligned}
\frac{dC}{dt} &= \left(F'(K) - \theta\right) C - p(p+\theta)(B+K) \\
\frac{dK}{dt} &= F(K) - C - G \,, \\
T &= F'(K)B + G \,.
\end{aligned}
$$

- Steady States for given G, B:

$$
\begin{aligned}
\left[F'(K^*) - \theta\right] C^* &= p(p+\theta)(B+K^*) \,, \\
F(K^*) &= C^* + G
\end{aligned}
$$

- Is there a unique steady state K^*?
- What is the effect on K^*, r^* of an increase in the initial level B?
- What is the effect on K^*, r^* of an increase on G?
- How do the answers depend on p?