Theory of Income, Fall 2009
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Problem Set 4

1 Neoclassical Growth Model: Exogenous Growth

Let the exogenous time augmenting productivity growth gross rate be $\lambda > 1$. Feasibility is given by

$$c_t + k_{t+1} = F\left(k_t, \lambda^t n_t\right) - (1-\delta) k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. The endowment of time per period is normalized to 1, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v\left(c_t, 1-n_t\right).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t\left[c_t + x_t\right] = \sum_{t=0}^{\infty} p_t\left[w_t n_t + k_t v_t\right],$$

and the law of motion of capital is

$$k_{t+1} = x_t + (1-\delta) k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the real wage and rental rate of capital in terms of consumption goods at period t.

The firm's problem is

$$\max_{k_t, n_t} \; F\left(k_t, \lambda^t n_t\right) - w_t n_t - v_t k_t.$$

Exercise 1. Let r_t be the time t interest rate, i.e. $p_t/p_{t+1} = 1 + r_t$. Use the budget constraint of the household, and the law of motion of capital to show that, as long as $x_t > 0$,

$$v_{t+1} = r_t + \delta.$$

[Hint: Consider an investment of 1 at t, renting it on $t+1$ and consuming the undepreciated capital at $t+1$].

Ans: In fact, there is no need to use the budget constraint of the household to obtain this result. Consider instead the following arbitrage argument: the household borrows one unit of good at period t and invests it to construct capital and rent it to the firm the following period. The next period the household has to repay $(1 + r_t)$ and receives $(1 - \delta) + v_{t+1}$. That is, since a fraction δ of the capital depreciates, the household receives $(1 - \delta)$ units of capital plus the rental rate v_{t+1}. Since net cash-flows at t are zero, they have to be zero at $t + 1$ as well. Thus,

$$1 - \delta + v_{t+1} - (1 + r_t) = 0$$

or

$$v_{t+1} = r_t + \delta,$$

as desired.

Exercise 2. Write down the first order condition w.r.t. c_t, n_t and k_{t+1}. Use μ for the multiplier on the budget constraint [Hint: Replace x_t in the household budget constraint using the law of motion of capital]. Combine the FOC for c_t for two consecutive periods to obtain a relationship between the marginal rate of substitution between c_t and c_{t+1} and r_t. Combine the FOC with respect to c_t and n_t to obtain a relationship between the marginal rate of substitution between n_t and c_t and w_t.

Ans: Replacing x_t into the household budget constraint, we note that the latter becomes

$$\sum_{t=0}^{\infty} p_t \left[w_t n_t - c_t + (v_t + 1 - \delta) k_t - k_{t+1} \right] = 0.$$

Thus, the Lagrangian of the household's problem is given by

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) + \mu \left[\sum_{t=0}^{\infty} p_t \left[w_t n_t - c_t + (v_t + 1 - \delta) k_t - k_{t+1} \right] \right].$$

The FOCs w.r.t. c_t, n_t and k_{t+1} are, respectively

$$\beta^t U_c(c_t, 1 - n_t) = \mu p_t, \quad t = 0, 1, 2, .. \tag{1}$$

$$\beta^t U_{1-n}(c_t, 1 - n_t) = \mu p_t w_t, \quad t = 0, 1, 2, .. \tag{2}$$

and

$$-p_t + p_{t+1}(v_{t+1} + 1 - \delta) = 0, \quad t = 0, 1, 2, .. \tag{3}$$

So, we see that the last equation implies the no-arbitrage condition $(1 + r_t) \equiv \frac{p_t}{p_{t+1}} = v_{t+1} + 1 - \delta$.

Now divide (1) at period $t+1$ with the same equation at period t to get

$$\frac{\beta^{t+1}U_c\left(c_{t+1},1-n_{t+1}\right)}{\beta^{t}U_c\left(c_{t},1-n_{t}\right)}=\frac{p_{t+1}}{p_t}.$$

Canceling terms and using $(1+r_t)\equiv\frac{p_t}{p_{t+1}}$ we get

$$\frac{\beta U_c\left(c_{t+1},1-n_{t+1}\right)}{U_c\left(c_{t},1-n_{t}\right)}=\frac{1}{1+r_t}, \tag{4}$$

that is, the marginal rate of substitution between consumption at $t+1$ and t must equal its relative price. This is the standard Euler equation.

Dividing (1) with (2) we obtain

$$\frac{U_c\left(c_{t},1-n_{t}\right)}{U_{1-n}\left(c_{t},1-n_{t}\right)}=\frac{1}{w_t}, \tag{5}$$

that is, the marginal rate of substitution between consumption and leisure must equal the relative price of consumption versus leisure.

Finally, the firm's problem is

$$\max_{k_t,n_t} F\left(k_t,\lambda^t n_t\right)-w_t n_t-\upsilon_t k_t,$$

which gives the following optimality conditions:

$$F_k\left(k_t,\lambda^t n_t\right)=\upsilon_t, \tag{6}$$

and

$$F_n\left(k_t,\lambda^t n_t\right)\lambda^t=w_t. \tag{7}$$

Exercise 3. Use the expression for the rental rate of capital υ_t for $t\geq 1$ and the law of motion of capital for $t\geq 0$ (solving for x_t) to show that the household's budget constraint can be written as

$$\sum_{t=0}^{\infty}p_t c_t=p_0 k_0\left(\upsilon_0+1-\delta\right)+\sum_{t=0}^{\infty}p_t w_t n_t.$$

Ans: Start with the budget constraint

$$\sum_{t=0}^{\infty}p_t\left[c_t+k_{t+1}-(1-\delta)k_t\right]=\sum_{t=0}^{\infty}p_t\left[w_t n_t+\upsilon_t k_t\right].$$

Consider the terms involving only capital:

$$
\begin{aligned}
\sum_{t=0}^{\infty} p_t \left[k_{t+1} - (1-\delta+\upsilon_t)\, k_t\right] &= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_t \left(1-\delta+\upsilon_t\right) k_t \\
&= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=1}^{\infty} p_t \left(1-\delta+\upsilon_t\right) k_t - p_0 \left(1-\delta+\upsilon_0\right) k_0 \\
&= \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_{t+1} \left(1-\delta+\upsilon_{t+1}\right) k_{t+1} - p_0 \left(1-\delta+\upsilon_0\right) k_0 \\
&= \sum_{t=0}^{\infty} \left[p_t - p_{t+1}\left(1-\delta+\upsilon_{t+1}\right)\right] k_{t+1} - p_0 \left(1-\delta+\upsilon_0\right) k_0.
\end{aligned}
$$

Thus, the budget constraint becomes

$$
\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t w_t n_t + \sum_{t=0}^{\infty} \left[p_{t+1}\left(1-\delta+\upsilon_{t+1}\right) - p_t\right] k_{t+1} + p_0 \left(1-\delta+\upsilon_0\right) k_0.
$$

Using the equilibrium condition $p_{t+1}\left(1-\delta+\upsilon_{t+1}\right) = p_t$, this equation reduces to

$$
\sum_{t=0}^{\infty} p_t c_t = p_0 \left(1-\delta+\upsilon_0\right) k_0 + \sum_{t=0}^{\infty} p_t w_t n_t,
$$

as was to be shown.

Definition: A *balanced growth path* is given by an initial capital k_0 and λ such that it is optimal to set

$$
\begin{aligned}
c_t &= c_0 \lambda^t, \\
k_t &= k_0 \lambda^t, \\
n_t &= n_0, \\
w_t &= \lambda^t w_0, \\
r_t &= r_0,
\end{aligned}
$$

for all $t \geq 0$.

Exercise 4. Write down the FOC for the household imposing a balanced growth path. Use the FOC for the household and firm's problem as well as feasibility to write down the system of equations in c_0, n_0, k_0, w_0, r_0 that a balanced growth path must satisfy.

Ans: Evaluating the firm's FOCs at the balanced growth path we get

$$
F_k\left(k_0\lambda^t, n_0\lambda^t\right) = r_0 + \delta,
$$

and

$$F_n\left(k_0\lambda^t, n_0\lambda^t\right)\lambda^t = w_0\lambda^t.$$

Since $F(k,n)$ is homogeneous of degree one, all its partial derivatives are homogeneous of degree zero. Thus, we can rewrite the above equations as

$$F_k(k_0, n_0) = r_0 + \delta, \tag{8}$$

and

$$F_n(k_0, n_0) = w_0. \tag{9}$$

The household's FOCs evaluated at the balanced growth path are

$$U_c\left(c_0\lambda^t, 1-n_0\right) = \beta(1+r_0)\,U_c\left(c_0\lambda^{t+1}, 1-n_0\right), \tag{10}$$

and

$$\frac{U_{1-n}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)} = w_0\lambda^t. \tag{11}$$

Finally, feasibility in the balanced growth path is given by

$$c_0\lambda^t + k_0\lambda^{t+1} = F\left(k_0\lambda^t, \lambda^t n_0\right) + (1-\delta)\,k_0\lambda^t,$$

or, using that $F\left(k_0\lambda^t, \lambda^t n_0\right) = \lambda^t F(k_0, n_0)$,

$$c_0 + k_0\lambda = F(k_0, n_0) + (1-\delta)\,k_0. \tag{12}$$

Note that we have 5 equations, (8), (9), (10), (11) and (12), in 4 unknowns: (c_0, n_0, w_0, r_0) (recall that k_0 is given). Does that mean that the model is overidentified? No, in fact the equations that come from the firm's problem determine only one equilibrium quantity: the capital to labor ratio. To see this, dividing (8) and (9) and using the homogeneity of degree zero of the partial derivatives we obtain

$$\frac{F_k(k_0/n_0, 1)}{F_n(k_0/n_0, 1)} = \frac{r_0+\delta}{w_0}, \tag{13}$$

so we can only obtain the capital to labor ratio. The intuition for this is that since technology displays constant returns to scale, the scale of the firm remains undetermined in equilibrium. In other words, the absolute demand for capital and labor is not determined by looking at the firm's problem.

Therefore, the quantities (c_0, n_0, w_0, r_0) are pinned down by the system of equations given by (10), (11), (12) and (13).

Exercise 5. Show that if preferences are of the form

$$v(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} h(1-n), \tag{14}$$

for $\gamma \neq 1$, or

$$v(c, 1-n) = \log c + h(1-n),$$

then there is a balanced growth path.

Ans: We must show that the above system of equations has a solution. Let's focus on the case $\gamma \neq 1$. From the household's conditions (10) and (11) we obtain

$$\left(c_0\lambda^t\right)^{-\gamma} \upsilon(1-n_0) = \beta(1+r_0)\left(c_0\lambda^{t+1}\right)^{-\gamma} \upsilon(1-n_0),$$

and

$$\frac{(1-\gamma)^{-1}\left(c_0\lambda^t\right)^{1-\gamma} \upsilon'(1-n_0)}{\left(c_0\lambda^t\right)^{-\gamma} \upsilon(1-n_0)} = w_0\lambda^t.$$

The first equation gives

$$1 = \beta(1+r_0)\lambda^{-\gamma}, \tag{15}$$

so we solved for the equilibrium interest rate $r_0 = \lambda^\gamma/\beta - 1$. The second equation implies

$$\frac{c_0}{1-\gamma}\frac{\upsilon'(1-n_0)}{\upsilon(1-n_0)} = w_0,$$

which can be used to solve for consumption as a function of labor and the wage rate:

$$c_0 = f(n_0, w_0) \equiv (1-\gamma) w_0 \frac{\upsilon(1-n_0)}{\upsilon'(1-n_0)}.$$

Using equations (15) and (13) we can solve for the capital to labor ratio as a function of w_0 and parameters:

$$\frac{F_k(k_0/n_0, 1)}{F_n(k_0/n_0, 1)} = \frac{\lambda^\gamma/\beta + \delta - 1}{w_0},$$

or

$$k_0/n_0 \equiv g(w_0),$$

where $g(w_0)$ solves

$$\frac{F_k(g(w_0), 1)}{F_n(g(w_0), 1)} = \frac{\lambda^\gamma/\beta + \delta - 1}{w_0}.$$

Using feasibility and the definition of c_0 and k_0/n_0 we notice that

$$c_0 + k_0(\lambda - (1-\delta)) = F(k_0, n_0),$$

or

$$f(n_0, w_0) + \frac{k_0}{n_0} n_0 (\lambda - (1-\delta)) = n_0 F\left(\frac{k_0}{n_0}, 1\right),$$

or

$$f(n_0, w_0) + g(w_0) n_0 (\lambda - (1-\delta)) = n_0 F(g(w_0), 1).$$

Consider solving this equation for n_0 as a function of w_0:

$$n_0 \equiv h(w_0).$$

Finally, since k_0 is given, we can use the equation $k_0/n_0 \equiv g(w_0)$ or $k_0 = h(w_0) g(w_0)$ to solve for w_0 as a function of k_0 (one equation in one unknown). Once we know w_0 we can solve for the rest of the equilibrium variables.

The case of log utility can be done in exactly the same way and is left as an exercise.

Exercise 6. Assume that $v(c,l)$ is strictly concave and increasing in (c,l) and have the form described in (14). Consider first the case where $\gamma \in (0,1)$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex? Next, consider the case where $\gamma = 1$. What are the properties of $h(l)$? i.e. is it increasing or decreasing, concave or convex? Finally, consider the case where $\gamma > 1$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex?

Ans: Consider the following first and second order derivatives (here $l = 1-n$ is leisure). For $\gamma \neq 1$,

$$\begin{aligned}
v_c(c,l) &= c^{-\gamma} h(l) > 0 \\
v_l(c,l) &= \frac{c^{1-\gamma}}{1-\gamma} h'(l) > 0 \\
v_{cc}(c,l) &= -\gamma c^{-\gamma-1} h(l) < 0 \\
v_{ll}(c,l) &= \frac{c^{1-\gamma}}{1-\gamma} h''(l) < 0,
\end{aligned}$$

and for $\gamma = 1$,

$$\begin{aligned}
v_c(c,l) &= 1/c > 0, \\
v_l(c,l) &= h'(l) > 0, \\
v_{cc}(c,l) &= -1/c^2, \\
v_{ll}(c,l) &= h''(l).
\end{aligned}$$

i) If $\gamma \in (0,1)$, from $v_c > 0$ we have $h(l) > 0$, from $v_l > 0$ we require $h'(l) > 0$ and from $v_{ll} < 0$ we obtain $h''(l) < 0$ ($h(l)$ is positive, increasing and concave).

ii) If $\gamma = 1$, $h(l)$ can be positive or negative. From $v_l > 0$ we obtain $h'(l) > 0$ and from $v_{ll} < 0$ we require $h''(l) < 0$ ($h(l)$ is increasing and concave).

iii) If $\gamma > 1$ we have $(1-\gamma) < 0$. From $v_c > 0$ we require $h(l) > 0$, from $v_l > 0$ we obtain $h'(l) < 0$ and from $v_{ll} < 0$ we require $h''(l) > 0$ ($h(l)$ is positive, decreasing and convex).

Exercise 7. Let $v(c, 1-n)$ be

$$v(c, 1-n) = g(c - n^{\sigma}/\sigma),$$

for $\sigma > 1$ and g strictly increasing and concave. What is the income elasticity of the labor supply for this utility function? Show that this preferences are inconsistent with a balanced growth path. [Hint: In a BGP we must have

$$\frac{v_l\left(c_0\lambda^t, 1-n_0\right)}{v_c\left(c_0\lambda^t, 1-n_0\right)} = w_0\lambda^t,$$

but

$$\frac{v_l\left(c_0\lambda^t, 1-n_0\right)}{v_c\left(c_0\lambda^t, 1-n_0\right)} = \frac{g'\left(c_0\lambda^t - n_0^{\sigma}/\sigma\right)}{g'\left(c_0\lambda^t - n_0^{\sigma}/\sigma\right)} n_0^{\sigma-1} = n_0^{\sigma-1} = w_0\lambda^t,$$

so compare the LHS and RHS of the last equality].

Ans: Following the hint, if there exists a BGP, it must satisfy

$$n_0^{\sigma-1} = w_0\lambda^t.$$

But this equation implies that labor is not constant over time, so these preferences are inconsistent with a BGP. In general, the condition equating the marginal rate of substitution with the wage rate is

$$\frac{g'\left(c_t - n_t^{\sigma}/\sigma\right)}{g'\left(c_t - n_t^{\sigma}/\sigma\right)} n_t^{\sigma-1} = w_t,$$

so that the elasticity of labor supply with respect to the wage rate is $1/(\sigma-1)$. For a balanced growth path to exist, this elasticity must be zero.

Exercise 8. Show that if the economy admits a balanced growth path for an open set of parameters β, λ and δ, preferences must be of the form in (14). [Hint: Write down an Euler equation-like expression relating the marginal rate of substitution of consumption between t and $t+1$ with r. Impose the balanced growth condition on this, noticing that this expression must be satisfied for all t. Differentiate this expression with respect to t to obtain a differential equation, whose solution implies that v is of the form $B(1-n) + c^{1-\gamma(1-n)}h(1-n)$ or $B(1-n) + \log c + h(1-n)$ where γ is a function of $(1-n)$. Use this, and the condition that marginal rates of substitution equal relative prices on a balanced growth path to establish the required result].

Ans: First, we know that in any balanced growth path the following equation must hold for all t

$$U_c\left(c_0\lambda^t, 1-n_0\right) = \beta\left(1+r_0\right)U_c\left(c_0\lambda^{t+1}, 1-n_0\right). \tag{16}$$

We want to differentiate the last expression with respect to t. To do that, recall that

$$\lambda^t = e^{\log \lambda^t} = e^{t \log \lambda},$$

so that $d\lambda^t/dt = \lambda^t \log \lambda$. Taking the derivative w.r.t. t we obtain

$$U_c\left(c_0\lambda^t, 1-n_0\right) c_0\lambda^t \log \lambda = \beta\left(1+r_0\right)U_{cc}\left(c_0\lambda^{t+1}, 1-n_0\right) c_0\lambda^{t+1} \log \lambda.$$

Dividing the last expression by equation (16) we find

$$\frac{U_{cc}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)} c_0\lambda^t = \frac{U_{cc}\left(c_0\lambda^{t+1}, 1-n_0\right)}{U_c\left(c_0\lambda^{t+1}, 1-n_0\right)} c_0\lambda^{t+1}.$$

Moreover, noting that $c_t = c_0\lambda^t$, this expression implies that any BGP must satisfy

$$\frac{U_{cc}\left(c_t, 1-n_0\right)}{U_c\left(c_t, 1-n_0\right)} c_t = \frac{U_{cc}\left(c_{t+1}, 1-n_0\right)}{U_c\left(c_{t+1}, 1-n_0\right)} c_{t+1},$$

for all c_t. Equivalently,

$$\frac{U_{cc}\left(c, 1-n\right)}{U_c\left(c, 1-n\right)} c = \text{constant (independent of } c\text{)}.$$

Note that it could be the case that the constant may depend on n. So, let us rewrite the last condition, without loss of generality, as

$$\frac{U_{cc}\left(c, 1-n\right)}{U_c\left(c, 1-n\right)} c = -\gamma\left(1-n\right)$$

where $\gamma\left(1-n\right)$ means that the constant is function of $(1-n)$. To avoid cumbersome notation, keep n implicit and rewrite the last equation as

$$\frac{u''(c)}{u'(c)} c = -\gamma$$

where $u(c) \equiv U(c, 1-n)$. That's a second order differential equation, so we will find its solution. Rewrite it as

$$u''(c)\, c = -\gamma u'(c)$$

Now we will integrate both sides of the equation with respect to c. For the left side expression, use integration by parts, i.e.,

$$\int u''(c)\, c dc = u'(c)\, c - \int u'(c)\, dc.$$

Then

$$u'(c)\, c - \int u'(c)\, dc = -\gamma \int u'(c)\, dc,$$

or

$$u'(c)\, c = (1-\gamma) \int u'(c)\, dc$$

or

$$u'(c)\, c - (1-\gamma)\, u(c) = A,$$

where A is a constant of integration. Multiply both sides of the equation by $c^{-(2-\gamma)}$ to obtain

$$u'(c)\, c^{-(1-\gamma)} - (1-\gamma)\, u(c)\, c^{-(2-\gamma)} = Ac^{-(2-\gamma)}.$$

Now note that

$$\frac{d\left(u(c)\, c^{-(1-\gamma)}\right)}{dc} = u'(c)\, c^{-(1-\gamma)} - (1-\gamma)\, u(c)\, c^{-(2-\gamma)}.$$

Thus, we have

$$\frac{d\left(u(c)\, c^{-(1-\gamma)}\right)}{dc} = Ac^{-(2-\gamma)}.$$

Finally, integrate the last equation with respect to c to obtain

$$u(c)\, c^{-(1-\gamma)} = -\frac{Ac^{-(1-\gamma)}}{(1-\gamma)} + \frac{\upsilon}{1-\gamma}$$

where $\upsilon/(1-\gamma)$ is another constant of integration. Now solving for $u(c)$ we get

$$u(c) = B + \frac{c^{(1-\gamma)}}{(1-\gamma)}\upsilon,$$

or, making n explicit,

$$U(c, 1-n) = B(1-n) + \frac{c^{(1-\gamma(1-n))}}{(1-\gamma(1-n))}\upsilon(1-n)$$

This is the solution of the differential equation given by the intertemporal condition, where we make explicit the fact that in general the constants will depend on n. Now we must also make sure that the intratemporal condition is satisfied. Recall the intratemporal condition:

$$\frac{U_{1-n}\left(c_0\lambda^t, 1-n_0\right)}{U_c\left(c_0\lambda^t, 1-n_0\right)} = w_0\lambda^t.$$

Using the U obtained above and letting $\tilde{v}(1-n) \equiv \frac{v(1-n)}{1-\gamma(1-n)}$ the last equation reads

$$\frac{B'(1-n_0) + \left(c_0\lambda^t\right)^{1-\gamma(1-n_0)}\left[\tilde{v}'(1-n_0) - \gamma'(1-n_0)\,\tilde{v}(1-n_0)\log(c_0\lambda^t)\right]}{(1-\gamma(1-n_0))\,\tilde{v}(1-n_0)\left(c_0\lambda^t\right)^{-\gamma(1-n)}} = w_0\lambda^t.$$

Now, notice that the above equation must hold for all t. It follows that it is necessary and sufficient that

$$B'(1-n_0) = \gamma'(1-n_0) = 0,$$

otherwise the equation will depend on t, concluding that the only preferences consistent with a BGP are

$$U(c, 1-n) = a + \frac{c^{1-\gamma}}{1-\gamma}v(1-n),$$

where a and γ are constants and $\gamma \neq 1$.

The case of log utility can be done in exactly the same way and is left as an exercise.

2 Deriving the Euler Equation in Continuous Time

In this question we obtain the continuous time Euler Equation by taking limits of the discrete time Euler equation. The point of this is to realize that although the expression for the continuous time counterpart is less intuitive that the one for the discrete time -which has the natural interpretation of equating marginal cost to marginal benefit- they are really the same.

The idea is to consider a sequence of discrete time dynamic programing problems. In each problem the length of time between periods where the state is decided is denoted by Δ. Decisions are taken at times $0,\ \Delta,\ 2\Delta,\ 3\Delta, ...$ The sequence of states to be chosen is

$$\left\{x_{\Delta(t+1)}\right\}_{t=0}^{\infty} = \left\{x_\Delta, x_{2\Delta}, x_{3\Delta}, ...\right\},$$

where x_0 is given. Setting $\Delta = 1$ we obtain the standard problem analyzed in the class notes. We adjust the discount factor for each problem accordingly letting

$$\beta = \frac{1}{1+\Delta\rho},$$

so that ρ has the interpretation of a discount rate.

For each Δ we write the period return function during the interval of time of length Δ as

F, and the corresponding return function per unit of time as $\hat{F}$. They satisfy:

$$F\left(x_t, x_{t+\Delta}\right) \equiv \Delta\ \hat{F}\left(x_t, \frac{1}{\Delta}\left(x_{t+\Delta} - x_t\right)\right),$$

where $t = i\Delta$ for some integer i. It helps to write these return functions as

$$F\left(x, y\right) \equiv \Delta\ \hat{F}\left(x, \frac{1}{\Delta}\left(y - x\right)\right),$$

or

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

where we define $\dot{x}$ as the change per unit of time on the state:

$$\dot{x} \equiv \frac{y - x}{\Delta},$$

or using time subscripts:

$$\dot{x}_t = \frac{x_{t+\Delta} - x_t}{\Delta},$$

for $t = i\Delta$ and any integer i.

Likewise we can define the feasible correspondence $\hat{\Gamma}$ for the change per unit of time $\dot{x}$ in terms of the feasible correspondence for levels Γ as

$$\hat{\Gamma}\left(x\right) = \left\{\dot{x} : y \in \Gamma\left(x\right),\ y = \dot{x}\Delta + x\right\}.$$

Thus for each Δ we consider the problem

$$\max_{\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ F\left(x_{t\Delta},\ x_{(t+1)\Delta}\right),$$

subject to

$$x_{(t+1)\Delta} \in \Gamma\left(x_{t\Delta}\right)\ ,$$

for all $t \geq 0$, where x_0 given.

Equivalently, we can write this problem as a choice of the sequence of discrete time changes $\left\{\dot{x}_{t\Delta}\right\}_{t=0}^{\infty}$:

$$\max_{\left\{\dot{x}_{t\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ \Delta\ \hat{F}\left(x_{t\Delta},\ \dot{x}_{t\Delta}\right),$$

subject to

$$\dot{x}_{t\Delta} \in \hat{\Gamma}\left(x_{t\Delta}\right),$$

$$x_{t\Delta+\Delta} = x_{t\Delta} + \dot{x}_{t\Delta}\ \Delta,$$

for all $t \geq 0$, and for given x_0.

We emphasize that the optimal sequence $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$ that solves the problem with interval of length Δ is a function of Δ.

For future reference, we also introduce the notation for the changes per unit of time on the change per unit of time of the state, denoting it by $\ddot{x}_t$:

$$\ddot{x}_t \equiv \frac{1}{\Delta}\left(\dot{x}_{t+\Delta} - \dot{x}_t\right),$$

for all $t = i\Delta$ and an integer i.

Exercise 1. Derive a formula for F_y and F_x in terms of $\partial \hat{F}/\partial x$ and $\partial \hat{F}/\partial \dot{x}$. In particular use the relationship between F and $\hat{F}$ to show that

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial \dot{x}}\hat{F}\left(x, \dot{x}\right),$$

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Ans: Differentiating

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

w.r.t. x and $\dot{x}$ we obtain:

$$F_x\left(x, \dot{x}\Delta + x\right) + F_y\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right),$$

and

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial \dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Inserting this in the previous expression we have

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Exercise 2. Write the Euler Equations for the problem where we chose the sequence of levels of the state: $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$. Your Euler equation should involve $F_y,\ F_x,\ \Delta,\ \rho$ and be evaluated at $x_t,\ x_{t+\Delta}$ and $x_{t+2\Delta}$. [Hint: This is the standard problem].

Ans: The Euler equation is

$$F_y\left(x_t, x_{t+\Delta}\right) + \left(\frac{1}{1+\Delta\rho}\right) F_x\left(x_{t+\Delta}, x_{t+2\Delta}\right) = 0.$$

Exercise 3. Rewrite the Euler equation obtained in 2 replacing the $x_{t+\Delta}$ in F_y in terms of Δ , x_t and $\dot{x}_t$, and replacing the $x_{t+2\Delta}$ in F_x in terms of Δ , $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$.

Ans: Using the definition of $\dot{x}_t$ we can write the EE as

$$F_y\left(x_t, \dot{x}_t\Delta + x_t\right) + \left(\frac{1}{1+\Delta\rho}\right) F_x\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\Delta + x_{t+\Delta}\right) = 0.$$

Exercise 4. Use the relationship between the derivatives of F and $\hat{F}$ found in 1 into your expression for the Euler equation found in exercise 3.

Ans: Using the relationship between the derivatives of F and $\hat{F}$ we can write this EE as

$$\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)\left[\Delta\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right)\right] = 0.$$

Exercise 5. Show that by rearranging the terms in the expression found in 4, the Euler equation can be written as:

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Ans: Multiplying both sides of the answer in 4 by $(1+\Delta\rho)$ yields

$$(1+\Delta\rho)\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \left[\Delta\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right)\right] = 0,$$

or rearranging,

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Assumptions. The next steps consists on taking the limit of the above expression as $\Delta \to 0$. For this we will assume that as we take the limit as $\Delta \to 0$, the solutions are such that the resulting path $x(t)$ is twice differentiable with respect to time, so that the following limits are well defined and given by the corresponding expressions:

$$\begin{aligned}
\lim_{\Delta\to 0} x_{t+\Delta} &= x_t, \\
\lim_{\Delta\to 0} \frac{x_{t+\Delta} - x_t}{\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \dot{x}_{t+\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \frac{\dot{x}_{t+\Delta} - \dot{x}_t}{\Delta} &= \ddot{x},
\end{aligned}$$

for all t.

Exercise 6. Use the Assumptions to show that the limit of the RHS of the expression in 5 is

$$\lim_{\Delta\to 0}\left[\rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)\right]=\rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_t,\dot{x}_t\right).$$

Ans: Use the assumption that $x(t)$ is twice differentiable and hence the expressions for the limits written above.

Exercise 7. Taking the limit of the LHS of the expression in 5 is more subtle. Use the expressions for the limits as $\Delta\to 0$ in the Assumptions to show that the limit as $\Delta\to 0$ of the LHS of the EE derived in 5

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right],$$

requires the use of L'Hôpital's rule for its evaluation.

Ans: The LHS is

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right] &= \frac{\lim_{\Delta\to 0}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right]}{\lim_{\Delta\to 0}\ \Delta}\\ &= \frac{\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)}{0}=\frac{0}{0}.\end{aligned}$$

Thus we use L'Hôpital's rule and compute

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right] &= \lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right]\\ &= \lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right).\end{aligned}$$

Exercise 8. We now apply L'Hôpital's rule to evaluate the limit as $\Delta\to 0$ of the LHS of the EE derived in 5. To do so use the definitions

$$\begin{aligned}x_{t+\Delta} &= x_t+\dot{x}_t\Delta,\\ \dot{x}_{t+\Delta} &= \dot{x}_t+\ \ddot{x}_t\Delta,\end{aligned}$$

so that

$$\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)=\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right),$$

in computing the derivative

$$\frac{\partial}{\partial\Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right).$$

Show that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)\right]=\frac{\partial^{2}}{\partial x\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ \dot{x}_{t}+\frac{\partial^{2}}{\partial\dot{x}\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ddot{x}_{t}.$$

[Hint: Use the assumptions to take the limit].

Ans: Applying L'Hôpital's rule, and replacing $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$ by the expressions above we can compute the derivative

$$\begin{aligned}\frac{\partial}{\partial\Delta}\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right) &= \frac{\partial}{\partial\Delta}\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t}+\dot{x}_{t}\Delta,\ \dot{x}_{t}+\ddot{x}_{t}\Delta\right)\\ &= \frac{\partial^{2}}{\partial x\partial\dot{x}}\hat{F}\left(x_{t}+\dot{x}_{t}\Delta,\ \dot{x}_{t}+\ddot{x}_{t}\Delta\right)\dot{x}_{t}+\frac{\partial^{2}}{\partial\dot{x}\partial\dot{x}}\hat{F}\left(x_{t}+\dot{x}_{t}\Delta,\ \dot{x}_{t}+\ddot{x}_{t}\Delta\right)\ddot{x}_{t},\end{aligned}$$

and taking the limit

$$\lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)=\frac{\partial^{2}}{\partial x\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ \dot{x}_{t}+\frac{\partial^{2}}{\partial\dot{x}\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ddot{x}_{t}.$$

Summarizing,

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)\right]=\frac{\partial^{2}}{\partial x\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ \dot{x}_{t}+\frac{\partial^{2}}{\partial\dot{x}\partial\dot{x}}\hat{F}\left(x_{t},\ \dot{x}_{t}\right)\ddot{x}_{t}.$$

Exercise 9. Use your answers to question 5, 6 and 8 to obtain the continuous time Euler equation:

$$\frac{\partial^{2}}{\partial\dot{x}\partial x}\hat{F}\left(x_{t},\dot{x}_{t}\right)\ \dot{x}_{t}+\frac{\partial^{2}}{\partial\dot{x}\partial\dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)\ddot{x}_{t}=\rho\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t},\dot{x}_{t}\right)+\frac{\partial}{\partial x}\hat{F}\left(x_{t},\dot{x}_{t}\right).$$

Ans: It follows from taking the limits in 5 as $\Delta\to 0$, which is done in 6 and 8.

3 Continuous time agent's problem

Exercise 1. Let the discrete time budget constraint for a problem with length of time period Δ be

$$a_{t+\Delta}+\Delta c_{t}+\Delta\tau_{t}=\Delta w_{t}\left(1-\bar{\tau}_{t}\right)+\left(1+\Delta r_{t}\left(1-\bar{\tau}_{t}\right)\right)a_{t},$$

where a_t are assets, w_t wages, τ_t lump sum taxes, $\bar{\tau}_t$ income tax rate, and r_t the interest rate. Show that as Δ goes to zero this gives the following asset accumulation equation:

$$\dot{a}\left(t\right)+c\left(t\right)+\tau\left(t\right)=\left(1-\bar{\tau}\left(t\right)\right)\left[w\left(t\right)+r\left(t\right)a\left(t\right)\right].$$

[Hint: Rearrange the discrete time expression, divide by Δ, and take limits].

Ans: Subtracting a_t from both sides of the budget constraint and dividing by Δ we obtain

$$\frac{a_{t+\Delta} - a_t}{\Delta} + c_t + \tau_t = w_t \left(1 - \bar{\tau}_t\right) + r_t \left(1 - \bar{\tau}_t\right) a_t.$$

Let $\Delta \to 0$ and use $\lim_{\Delta \to 0} \left(a_{t+\Delta} - a_t\right)/\Delta = \dot{a}\left(t\right)$ to obtain

$$\dot{a}\left(t\right) + c\left(t\right) + \tau\left(t\right) = \left(1 - \bar{\tau}\left(t\right)\right)\left[w\left(t\right) + r\left(t\right) a\left(t\right)\right].$$

Exercise 2. Show that the following present value budget constraint

$$\int_t^{\infty} \left[c\left(s\right) + \tau\left(s\right) - w\left(s\right)\left(1 - \bar{\tau}\left(s\right)\right)\right] e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds = a\left(t\right),$$

is a solution of the previous asset accumulation equation. [Hint: Differentiate this expression with respect to time].

Ans: We will differentiate the above expression with respect to t. To that end, recall Leibniz's rule: Let $I\left(s\right) = \int_{a(s)}^{b(s)} f\left(x, s\right) dx$, then,

$$I'\left(s\right) = f\left(b, s\right) b'\left(s\right) - f\left(a, s\right) a'\left(s\right) + \int_{a(s)}^{b(s)} \frac{\partial f\left(x, s\right)}{\partial s} dx.$$

Using that formula, compute the derivative with respect to t of the above expression:

$$\begin{aligned} \dot{a}\left(t\right) &= -\left[c\left(t\right) + \tau\left(t\right) - w\left(t\right)\left(1 - \bar{\tau}\left(t\right)\right)\right] \underbrace{e^{-\int_t^t r(u)(1-\bar{\tau}(u))du}}_{=1} \\ &\quad + \underbrace{\left[\int_t^{\infty} \left[c\left(s\right) + \tau\left(s\right) - w\left(s\right)\left(1 - \bar{\tau}\left(s\right)\right)\right] e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds\right]}_{=a(t)} \times r\left(t\right)\left(1 - \bar{\tau}\left(t\right)\right) \end{aligned}$$

or, rearranging,

$$\dot{a}\left(t\right) + c\left(t\right) + \tau\left(t\right) = \left(1 - \bar{\tau}\left(t\right)\right)\left[w\left(t\right) + r\left(t\right) a\left(t\right)\right].$$

Aside: Let us follow the backward procedure by solving the differential equation directly. To this end, let $R\left(s\right)$ be defined as

$$R\left(s\right) \equiv e^{-\int_t^s r(u)(1-\bar{\tau}(u))du}.$$

Then, integrating the budget constraint from period t to some period T gives

$$\int_t^T [c(s) + \tau(s) - (1-\bar{\tau}(s)) w(s)] R(s) ds = \int_t^T [r(s)(1-\bar{\tau}(s)) a(s) - \dot{a}(s)] R(s) ds. \tag{17}$$

Now, note that

$$\begin{aligned} \frac{d(a(s) R(s))}{ds} &= \dot{a}(s) R(s) + a(s) \dot{R}(s) = \dot{a}(s) R(s) - r(s)(1-\bar{\tau}(s)) a(s) R(s) \\ &= -[r(s)(1-\bar{\tau}(s)) a(s) - \dot{a}(s)] R(s), \end{aligned}$$

where we have used the fact that

$$\frac{\partial\left(-\int_t^s r(u)(1-\bar{\tau}(u)) du\right)}{\partial s} = -r(s)(1-\bar{\tau}(s)).$$

Thus, the RHS of equation (17) can be written as

$$-\int_t^T \frac{\partial(a(s) R(s))}{\partial s} ds = -\left. a(s) R(s)\right|_t^T = -a(T) R(T) + a(t),$$

where we have used the fact that

$$R(t) = e^{-\int_t^t r(u)(1-\bar{\tau}(u)) du} = 1.$$

Using these results, and taking the limit as T goes to ∞ we find that expression (17) can be written as

$$\begin{aligned} \lim_{T\to\infty} \int_t^T [c(s) + \tau(s) - (1-\bar{\tau}(s)) w(s)] R(s) ds &= -\lim_{T\to\infty} a(T) R(T) + a(t) \\ \int_t^\infty [c(s) + \tau(s) - (1-\bar{\tau}(s)) w(s)] e^{-\int_t^s r(u)[1-\bar{\tau}(u)] du} ds &= a(t), \end{aligned} \tag{18}$$

as long as the following no-Ponzi-game condition holds:

$$\lim_{T\to\infty} a(T) e^{-\int_t^T r(u)(1-\bar{\tau}(u)) du} = 0.$$

Exercise 3. Formulate the problem of an agent with utility

$$\int_0^\infty e^{-\rho t} U(c(t)) dt,$$

of choosing consumption subject to the present value budget constraint (at time $t = 0$) obtained in the previous exercise. Write the Lagrangian using λ for the multiplier of the present value budget constraint. Show that the FOC with respect to $c(t)$ is:

$$e^{-\rho t} U'(c(t)) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Show that this equation implies

$$\frac{\dot{c}(t)}{c(t)} = [(1-\bar{\tau}(t)) r(t) - \rho] \ / \ \left[-c(t)\frac{U''(c(t))}{U'(c(t))}\right].$$

[Hint: Differentiate both sides of the FOC with respect to time].

Ans: The Lagrangian is

$$\int_0^\infty e^{-\rho t} U(c(t))\,dt + \lambda\left[a(0) - \int_0^\infty [c(t) + \tau(t) - w(t)(1-\bar{\tau}(t))]\, e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds} dt\right].$$

The FOC w.r.t. $c(t)$ is then

$$e^{-\rho t} U'(c(t)) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Differentiating the first order condition w.r.t. t we obtain

$$-\rho e^{-\rho t} U'(c(t)) + e^{-\rho t} U''(c(t))\,\dot{c}(t) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}\left[-r(t)(1-\bar{\tau}(t))\right].$$

Using the FOC again to eliminate λ we find

$$-\rho e^{-\rho t} U'(c(t)) + e^{-\rho t} U''(c(t))\,\dot{c}(t) = e^{-\rho t} U'(c(t))\left[-r(t)(1-\bar{\tau}(t))\right],$$

or

$$-\rho + \frac{U''(c(t))\,c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)} = \left[-r(t)(1-\bar{\tau}(t))\right],$$

or

$$\frac{\dot{c}(t)}{c(t)} = \frac{-U'(c(t))}{U''(c(t))\,c(t)}\left[r(t)(1-\bar{\tau}(t)) - \rho\right],$$

as was to be shown.

Exercise 4. Consider the budget constraint of the government with purchases g_t, lump sum taxes τ_t and income taxes at rate $\bar{\tau}(t)$, and government assets (i.e. minus government debt) b_t :

$$b_{t+\Delta} + \Delta g_t = \bar{\tau}_t(\Delta w_t + \Delta r_t a_t) + \Delta\tau_t + b_t(1+\Delta r_t).$$

Show that, as Δ goes to zero it implies:

$$\dot{b}(t) + g(t) = \bar{\tau}(t)(w(t) + r(t)\,a(t)) + \tau(t) + b(t)\,r(t),$$

and that it corresponds to the following present value budget constraint:

$$b(t)+\int_t^{\infty}\left[\tau(s)+\bar{\tau}(s)(w(s)+r(s)a(s))-g(s)\right]e^{-\int_t^s r(u)du}ds=0.$$

Ans: Repeat the same steps of exercises 1 and 2.

Exercise 5. Walras' law. Show that if i) $a,c,\tau,\bar{\tau},w,r$ satisfy the asset accumulation equation for the households, ii) $b,g,\tau,\bar{\tau},w,a,r$ satisfy the asset accumulation equation for the government, iii) there is equilibrium in the asset market, i.e.

$$a(t)+b(t)=k(t),$$

for all $t\geq 0$, and iv) firms maximize profits, so that:

$$\begin{aligned} r(t) &= f'(k(t)),\\ w(t) &= f(k(t))-f'(k(t))k(t), \end{aligned}$$

for all $t\geq 0$. Then the allocation is feasible, i.e.

$$\dot{k}(t)+c(t)+g(t)=f(k(t)),$$

holds for all $t\geq 0$.

Ans: Adding the asset-accumulation equations for the household and the government we obtain the asset-accumulation equation for the economy as a whole:

$$\begin{aligned} \dot{a}(t)+\dot{b}(t)+c(t)+g(t)+\tau(t) &= [1-\bar{\tau}(t)][w(t)+r(t)a(t)]+\bar{\tau}(t)[w(t)+r(t)a(t)]\\ &\quad +\tau(t)+b(t)r(t)\\ \dot{a}(t)+\dot{b}(t)+c(t)+g(t) &= w(t)+r(t)[a(t)+b(t)]. \end{aligned}$$

If there is equilibrium in the asset markets, i.e.,

$$a(t)+b(t)=k(t)\qquad \forall t\geq 0,$$

and firms maximize profits, so that

$$\begin{aligned} r(t) &= f'(k(t))\qquad \forall t\geq 0\\ w(t) &= f(k(t))-f'(k(t))k(t)\qquad \forall t\geq 0, \end{aligned}$$

then it follows that

$$\begin{aligned}
\dot{k}(t)+c(t)+g(t) &= f(k(t))-f'(k(t))\,k(t)+f'(k(t))\,k(t) \\
\dot{k}(t)+c(t)+g(t) &= f(k(t)) \qquad \forall t\geq 0, \qquad (19)
\end{aligned}$$

that is, the equilibrium allocation must be feasible.

Exercise 6. Ricardian Equivalence. Let a, b, τ, g, r, w, k be an equilibrium with lump sum taxes, so $\bar{\tau}(t)=0$ all t. Consider the following fiscal policies with lump sum taxes τ' and debt b' satisfying:

$$\int_0^\infty \tau'(t)\,e^{-\int_0^t r(s)ds}dt=\int_0^\infty \tau(t)\,e^{-\int_0^t r(s)ds}dt,$$

and $b'(0)=b(0)$. Show that $a', b', \tau', g, r, w, k$ is also an equilibrium with lump sum taxes for some path of assets a' such that $a'(0)=a(0)$. [Hint: You must show that agents still maximize with the same choices c given τ', r, w, for some path of assets a' with $a'(0)=a(0)$ given, that firms maximize their profits, and that the government budget constraint also holds].

Ans: First let's define a competitive equilibrium with lump sum taxes: It is a set of allocations $(a(t), c(t), g(t), b(t), \tau(t), r(t), w(t), k(t))$ for all t such that:

1. Given $\{\tau(t), r(t), w(t)\}$ and $a(0)$, $\{a(t), c(t)\}$ solve the consumer's problem,
2. Given $\{g(t)\}$ and $b(0)$, $\{b(t), \tau(t)\}$ are such that the government budget constraint is satisfied,
3. Firm's optimize given $\{r(t), w(t)\}$,
4. The asset market clears: $k(t)=a(t)+b(t)$, $\forall t\geq 0$, and
5. The goods market clears: $\dot{k}(t)+c(t)+g(t)=w(t)+r(t)\,k(t)$, $\forall t\geq 0$.

As we showed in the previous question, if the first four conditions are satisfied, the fifth is automatically satisfied.

Now suppose that government chooses another fiscal policy (τ', b') such that

$$\int_0^\infty \tau'(t)\,e^{-\int_0^t r(s)ds}dt=\int_0^\infty \tau(t)\,e^{-\int_0^t r(s)ds}dt \qquad (20)$$

and $b'(0)=b(0)$.

We must show that $(a'(t), c(t), g(t), b'(t), \tau'(t), r(t), w(t), k(t))$ for all t is also an equilibrium with lump-sum taxes. To do this let's see that the above conditions 1 to 4 are still satisfied for those allocations and prices where a' will be defined below.

1. Given $\{\tau'(t), r(t), w(t)\}$ and $a(0)$, $\{c(t)\}$ still solves the consumer's problem for

some a' (to be found below). The consumer's problem is:

$$\max_{c(t)} \int_0^\infty e^{-\rho t} U(c(t))\, dt,$$

$$\text{s.t.} : a(0) = \int_0^\infty [c(t) - w(t)]\, e^{-\int_0^t r(s) ds} dt + \int_0^\infty \tau'(t)\, e^{-\int_0^t r(s) ds} dt.$$

Using (20), the budget constraint is equivalent to

$$a(0) = \int_0^\infty [c(t) - w(t)]\, e^{-\int_0^t r(s) ds} dt + \int_0^\infty \tau(t)\, e^{-\int_0^t r(s) ds} dt.$$

Thus, if the new tax policy satisfies (20), the budget set of the consumer is exactly the same under the two fiscal regimes. Therefore, the optimal consumption choice will be the same (otherwise $\{c(t)\}$ wouldn't be an optimal choice for the initial fiscal policy). Notice, though, that the asset allocation need not be the same.

2. Given $\{g(t)\}$ (the same in both regimes) and using $b'(0) = b(0)$, by construction $\{b'(t), \tau'(t)\}$ satisfies the government budget constraint.

3. If $\{r(t), w(t)\}$ do not change, the firm's problem is exactly the same as before and the same policies are chosen.

4. Market clearing: Given the same $\{k(t)\}$, this condition reads

$$a'(t) + b'(t) = k(t)$$

that is, the level of assets held by the consumers adjust to the change in the government's assets in such a way that the total stock of capital remains unmodified.

In other words, we have shown that for the new fiscal policy (τ', b'), the allocation and prices $(a'(t), c(t), g(t), b'(t), \tau'(t), r(t), w(t), k(t))$ for all t constitutes a competitive equilibrium, where $a'(t) = k(t) - b'(t)$.

4 Convergence after a permanent productivity shock in the Neoclassical Growth Model

Let $f(k) = \varepsilon\, k^\alpha$ be a neoclassical production function with productivity parameter $\varepsilon > 0$ and capital share parameter $\alpha \in (0,1)$. The law of motion of capital is given by $dk/dt = f(k) - c$ (we set depreciation rate to zero). Preferences are $\int_0^\infty e^{-\rho t} U(c(t))\, dt$ for $U(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma > 0$.

Exercise 1. Continuous time Euler Equation . Write down the differential equation that c must satisfy in an optimal path. Your solution should contain, ρ, $f'(k)$, dc/dt, c and γ.

Ans: The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\gamma}}{1-\gamma} + \lambda \left[f(k) - c \right],$$

where λ is the co-state variable. The optimality conditions are, as usual, $\mathcal{H}_c = 0$, and $\dot{\lambda} = \rho\lambda - \mathcal{H}_k$. That is,

$$\begin{aligned} c &: \; c(t)^{-\gamma} = \lambda(t) \\ k &: \; \frac{\dot{\lambda}}{\lambda} = \left(\rho - f'(k)\right). \end{aligned}$$

Taking logs of the first equation and differentiating w.r.t. time we obtain

$$\frac{\dot{c}}{c} = -\frac{1}{\gamma}\frac{\dot{\lambda}}{\lambda} = \frac{1}{\gamma}\left[f'(k) - \rho\right],$$

or, using the definition of f,

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}\left[\alpha\varepsilon k^{\alpha-1} - \rho\right]. \tag{21}$$

Exercise 2. Steady states . Write down two equations in two unknowns (c^* and k^*) that determine the steady state values of consumption and capital.

Ans: The dynamic system that characterizes the solution of the problem is given by (21) and the capital accumulation equation

$$\dot{k} = \varepsilon k^{\alpha} - c. \tag{22}$$

In steady state, $\dot{c} = \dot{k} = 0$. The first equation becomes

$$\alpha\varepsilon k^{*(\alpha-1)} = \rho, \text{ (i.e. } f'(k^*) = \rho),$$

and the second

$$\varepsilon k^{*\alpha} = c^*, \text{ (i.e. } f(k^*) = c^*),$$

where (c^*, k^*) denotes steady state values. Solving for k^* and c^* we get

$$k^*(\varepsilon) = \left(\frac{\alpha\varepsilon}{\rho}\right)^{\frac{1}{1-\alpha}}, \tag{23}$$

and

$$c^*(\varepsilon) = \varepsilon^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \tag{24}$$

(making the dependence on ε explicit).

Exercise 3. Steady state consumption-capital ratio . Write an expression for the steady state value of consumption over capital, i.e. c^*/k^* as a function of ρ and α.

Ans: Dividing (24) by (23) we find the steady state consumption to capital ratio:

$$\frac{c^*}{k^*} = \frac{\rho}{\alpha}. \tag{25}$$

Note that this ratio is independent of the technology level ε.

Exercise 4. Slope of the saddle path or optimal consumption function . Let $c(k)$ be the value of consumption that belong to the saddle path for a given capital k. In this exercise you need to find an expression for the slope of the saddle path evaluated at (c^*, k^*), i.e. you need to find $c'(k^*)$. Notice that $c(k)$ is also the optimal decision rule for consumption for a given k.

4.1) Derive the following quadratic equation:

$$c'(k^*)\left[\rho - c'(k^*)\right] = -\frac{\rho^2}{\gamma}\frac{1-\alpha}{\alpha},$$

whose positive solution is $c'(k^*)$.

4.2) Give an intuitive explanation of why $c'(k^*)$ is decreasing in γ. Your explanation should contain, *at most,* three lines.

[Hint for 4.1: To obtain the quadratic equation, evaluate

$$\frac{dc(k^*)}{dk} = \lim_{k\to k^*}\frac{dc/dt}{dk/dt},$$

where dc/dt is the expression obtained in 1) and dk/dt is the law of motion of capital, and where c is written as a function of k, i.e. $c = c(k)$ in both expressions. You need to use L'Hôpital's rule to find an expression for the right hand side, since at the steady state $dc/dt = dk/dt = 0$. To use L'Hôpital, differentiate the denominator and the numerator with respect to k, and evaluate them at the steady state value k^*].

Ans: Note that

$$c'(k^*) \equiv \frac{dc(k^*)}{dk} = \lim_{k\to k^*}\frac{\dot{c}(t)}{\dot{k}(t)} = \lim_{k\to k^*}\frac{\frac{1}{\gamma}\left(f'(k(t)) - \rho\right)c(k(t))}{f(k(t)) - c(k(t))} = \frac{0}{0}.$$

Thus, using L'Hôpital's rule we obtain

$$\begin{aligned}\lim_{k\to k^*}\frac{\frac{1}{\gamma}\left(f'\left(k\left(t\right)\right)-\rho\right)c\left(k\left(t\right)\right)}{f\left(k\left(t\right)\right)-c\left(k\left(t\right)\right)} &= \lim_{k\to k^*}\frac{\frac{1}{\gamma}f''\left(k\left(t\right)\right)c\left(k\left(t\right)\right)+\frac{1}{\gamma}\left(f'\left(k\left(t\right)\right)-\rho\right)c'\left(k\left(t\right)\right)}{f'\left(k\left(t\right)\right)-c'\left(k\left(t\right)\right)} \\ &= \frac{\frac{1}{\gamma}f''\left(k^*\right)c\left(k^*\right)+\frac{1}{\gamma}\left[f'\left(k^*\right)-\rho\right]c'\left(k^*\right)}{f'\left(k^*\right)-c'\left(k^*\right)}.\end{aligned}$$

Given that $f'\left(k^*\right)=\rho$, from the above expression we can write

$$\begin{aligned}c'\left(k^*\right)\left[\rho-c'\left(k^*\right)\right] &= \frac{1}{\gamma}f''\left(k^*\right)c\left(k^*\right) \\ &= \frac{1}{\gamma}\left[\alpha\left(\alpha-1\right)\varepsilon\ \left(k^*\right)^{\alpha-2}\right]\varepsilon\left(k^*\right)^{\alpha} \\ &= \frac{1}{\gamma}\alpha\left(\alpha-1\right)\left[\varepsilon\left(k^*\right)^{\alpha-1}\right]^2 \\ &= \frac{1}{\gamma}\alpha\left(\alpha-1\right)\left[\varepsilon\left(\frac{\rho}{\alpha\varepsilon}\right)\right]^2 \\ c'\left(k^*\right)\left[\rho-c'\left(k^*\right)\right] &= -\frac{\rho^2}{\gamma}\frac{1-\alpha}{\alpha},\end{aligned}$$

as desired. Note that $c'\left(k^*\right)$ is independent of the technology level ε.

The quadratic equation $x\left[\rho-x\right]=-\frac{\rho^2}{\gamma}\frac{(1-\alpha)}{\alpha}$ has solutions

$$x=\frac{\rho}{2}\left[1\pm\sqrt{1+\frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right].$$

Since we also know that $c'\left(k^*\right)>0$, and $\sqrt{1+\frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}>1$, then the solution is

$$c'\left(k^*\right)=\frac{\rho}{2}\left[1+\sqrt{1+\frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right].$$

Note that

$$\begin{aligned}\left\{\left[\rho-c'\left(k^*\right)\right]-c'\left(k^*\right)\right\}dc'\left(k^*\right) &= \frac{\rho^2}{\gamma^2}\frac{1-\alpha}{\alpha}d\gamma \\ \frac{dc'\left(k^*\right)}{d\gamma} &= \frac{\rho^2}{\gamma^2}\frac{1-\alpha}{\alpha}\frac{1}{\rho-2c'\left(k^*\right)}.\end{aligned}$$

Thus,

$$\text{sgn}\,\frac{dc'(k^*)}{d\gamma} = \text{sgn}\left[\rho - 2c'(k^*)\right] = \text{sgn}\left[\rho - 2\frac{\rho}{\alpha}\right] = \text{sgn}\left[\alpha - 2\right] < 0,$$

which implies that the slope of the saddle path is decreasing in γ (which is the inverse of the elasticity of substitution); that is, the lower is γ the higher is the speed of convergence towards the steady. The intuition for this results is as follows: the lower is γ, the higher is the elasticity of substitution, that is, the more willing people are to accept low consumption early on in exchange for higher consumption in the future. Thus, as γ falls, capital accumulates more rapidly and the economy converges quicker to the steady state.

Exercise 5. Impact effect on consumption of a permanent change in productivity . Suppose that at time $t = 0$ the economy is in a steady state corresponding to productivity ε. Suppose that at $t = 0$ we learn that productivity will immediately and permanently increase by a very small amount from ε to $\varepsilon' (> \varepsilon)$. We let $c^*(\cdot)$ and $k^*(\cdot)$ be the steady state consumption and capital levels and $c(k, \cdot)$ be the optimal consumption decision rules -as a function of capital- that corresponding to each productivity level ε and ε'.

It turns out that consumption at time $t = 0$ may increase, decrease or stay the same value relative to the old steady state, i.e. that $c(k^*(\varepsilon), \varepsilon') \geq (\leq)\ c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$.

5.1. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c(k^*(\varepsilon), \varepsilon') > c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$. Indicate $c(k^*(\varepsilon), \varepsilon')$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.2. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c(k^*(\varepsilon), \varepsilon') < c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$. Indicate $c(k^*(\varepsilon), \varepsilon')$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.3. Explain why it may be the case that consumption does not increase in impact (i.e. explain case 5.2). Make sure to mention the income effect and intertemporal substitution effects of the increase in productivity in your explanation.

5.4. Given your previous explanation, for which values of γ do you think that case 5.2 will occur ? Explain.

Hints. Use 3), 4) to argue that $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$. Use a phase diagram to argue that whether $c(k^*(\varepsilon), \varepsilon') \geq (\leq)\ c^*(\varepsilon)$ depend on whether $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$ is smaller (higher) than $c^*(\varepsilon)/k^*(\varepsilon) = c^*(\varepsilon')/k^*(\varepsilon')$.

Ans: If ε increases, both curves describing the steady state move. The curve $\dot{c} = 0$ moves to the right and the curve $\dot{k} = 0$ moves up for all levels of capital. Therefore, a higher ε

means that both the steady state level of capital and consumption increase. Moreover, since the ratio c^*/k^* is independent of ε, both steady states lie in the same ray:

$$\frac{c^*(\varepsilon)}{k^*(\varepsilon)} = \frac{c^*(\varepsilon')}{k^*(\varepsilon')} = \frac{\rho}{\alpha}.$$

Also, we must have that

$$c'(k^*(\varepsilon), \varepsilon) = c'\left(k^*\left(\varepsilon'\right), \varepsilon'\right),$$

since the slope of the saddle path is invariant to changes in the productivity parameter.

5.1) Figure 1 below depicts the case where $c'[k^*(\cdot), \cdot] < c^*(\cdot)/k^*(\cdot)$. Before the permanent change in productivity, the system is in the steady state $(c^*(\varepsilon), k^*(\varepsilon))$. The slope of the associated saddle path is $c'(k^*(\varepsilon), \varepsilon)$. Just after the unexpected increase in ε occurs, consumption jumps up to $c(k^*(\varepsilon), \varepsilon')$ and the stock of capital remains at $k^*(\varepsilon)$. After that, the system starts converging to the new steady state $(c^*(\varepsilon'), k^*(\varepsilon'))$ through the saddle path with slope $c'(k^*(\varepsilon'), \varepsilon')$.

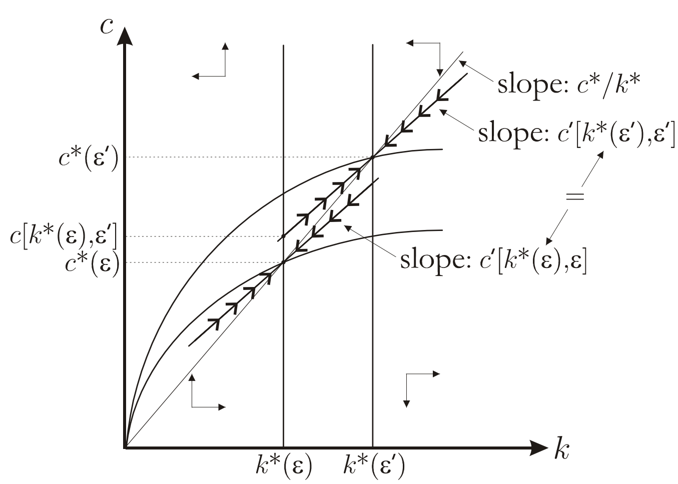

Figure 1. The slope of the saddle path is less than c^*/k^* and, thus, consumption jumps on impact.

5.2) Figure 2 depicts the case where $c'[k^*(\cdot), \cdot] > c^*(\cdot)/k^*(\cdot)$. Here consumption decreases when the productivity change takes place. The rest of the dynamics are identical to the previous case.

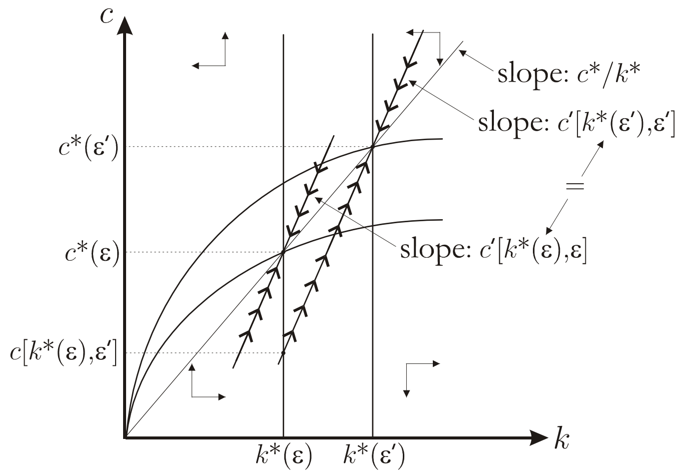

Figure 2. The slope of the saddle path is greather than c^*/k^* and, thus, consumption falls on impact.

5.3) A higher ε means that the economy is wealthier, so one might suspect that consumption should increase. It does in the long run, for sure. However it can be the case that it first decreases. The reason is that there are offsetting income and substitution effects: the income effect comes from the fact that the agent is wealthier, the present value of earnings is higher, so that is a force towards increasing current consumption. However a higher ε means that for a given level of k, the marginal productivity of capital (and therefore the interest rate) is higher. And a higher interest rate means that future consumption is cheaper relative to today's consumption. So, that's a force toward reducing current consumption in favor of future consumption. If the substitution effect is stronger than the income effect, we will see that current consumption will decline.

5.4) Of course, which effect dominates depends not only on the change in the relative price, but also on the willingness to substitute intertemporally. A higher γ means that the agent is less willing to substitute intertemporally, in this case the income effect will tend to dominate and consumption will increase. As we saw in the figures, for flatter slopes of the saddle path, the income effect dominates and consumption increases, and for steeper slopes, the substitution effect dominates and consumption decreases.

Notice that if the saddle paths have constant slopes, and using that in both steady states the consumption to capital ratio is constant, we have an easy way to see for what values of γ which effect dominates:

$$\begin{aligned}
\text{if } c'(k^*) &< \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ income effect dominates} \rightarrow c(k^*(\varepsilon_0), \varepsilon_1) > c^*(\varepsilon_0), \\
\text{if } c'(k^*) &> \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ substitution effect dominates} \rightarrow c(k^*(\varepsilon_0), \varepsilon_1) < c^*(\varepsilon_0), \\
\text{if } c'(k^*) &= \frac{c^*}{k^*} = \frac{\rho}{\alpha} \rightarrow \text{ income and subst. effects cancel out} \rightarrow c(k^*(\varepsilon_0), \varepsilon_1) = c^*(\varepsilon_0).
\end{aligned}$$

Here is when the assumption that the productivity change is small becomes crucial. For infinitesimal changes in ε, we can assume that the slope of the saddle path is, indeed, constant, and the above analysis goes through. For larger changes in ε, the above analysis turns out to be 'approximately' right.

But we can go a little bit further. We know the slope of the saddle path, and we can solve for the γ that satisfies the above conditions. Since $c'(k^*) = \frac{\rho}{2}\left[1+\sqrt{1+\frac{4}{\gamma}\left(\frac{1-\alpha}{\alpha}\right)}\right]$, the conditions are

$$\begin{aligned}
\text{if } \gamma &> \alpha \rightarrow \text{ income effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) > c^*\left(\varepsilon_0\right), \\
\text{if } \gamma &< \alpha \rightarrow \text{ substitution effect dominates} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) < c^*\left(\varepsilon_0\right), \\
\text{if } \gamma &= \alpha \rightarrow \text{ income and subst. effects cancel out} \rightarrow c\left(k^*\left(\varepsilon_0\right), \varepsilon_1\right) = c^*\left(\varepsilon_0\right).
\end{aligned}$$

Note that there is a trade-off between the concavity of the utility function, as described by γ, and the concavity of the production function, as described by α.

5 Investment in the Neoclassical Growth Model and Business Cycles

This exercise examines the dynamic behavior of gross investment in the neoclassical growth model. Specifically, it considers the adjustment path to the steady state, starting with a capital stock below its steady state value. Since capital is the state variable and since the dynamics of the neoclassical growth model are stable, net investment (i.e. the change in the stock of capital) must be decreasing in its adjustment path. Nevertheless, the behavior of gross investment (i.e. the change in capital plus the value of depreciation) can be different. In particular, below it is shown that gross investment could either increase or decrease in the adjustment path to steady state. Moreover, the investment to GDP ratio could be increasing.

This question is interesting because the neoclassical growth model is used as a simple business cycles model where unanticipated permanent productivity shocks are the source of fluctuations. In particular, we will like to see if the adjustment path after a permanent unexpected increase in productivity is consistent with the increase in investment typical of an expansion.

The planner's problem is to maximize

$$\int_0^\infty e^{-\rho t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt,$$

subject to

$$\dot{k} = Ak^{\alpha} - c - \delta k.$$

That is, the production function is Cobb-Douglas, preferences are of the CRRA form with relative risk aversion γ, ρ is the discount rate and δ is the depreciation rate of capital. The solution to this problem is given by the Euler Equation and the law of motion of capital:

$$\begin{aligned} \dot{c} &= \frac{c}{\gamma}\left(\alpha A k^{\alpha-1} - (\rho+\delta)\right), \\ \dot{k} &= Ak^{\alpha} - c - \delta k. \end{aligned}$$

Exercise 1. Letting x be gross investment, i.e. $x = \dot{k} + \delta k$, show that the linear approximation of $x(t)$ around the steady state satisfies

$$x(t) - x^* = (k(0) - k^*) \exp(\lambda_1 t) [\lambda_1 + \delta],$$

for all $t \geq 0$,where λ_1 solves

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2},$$

and where x^* and k^* are steady state values. [**Hint**: the following steps will be useful:

Step 1 : Differentiate the equation for $\dot{k}$ with respect to t and insert the $\dot{c}$ equation into the result you obtain. This gives a second order differential equation in k (the Euler equation!). Linearize the resulting equation, that is, obtain

$$\ddot{k} = g\left(\dot{k}, k\right) \cong g(0, k^*) + g_{\dot{k}}(0, k^*)\,\dot{k} + g_k(0, k^*)(k - k^*),$$

and compute the coefficients. You should obtain $\ddot{k} = \rho\dot{k} - f''(k^*)(c^*/\gamma)(k - k^*)$. Moreover, denoting $z(t) = k(t) - k^*$ the last equation becomes $\ddot{z} = \rho\dot{z} - f''(k^*)(c^*/\gamma)\,z$

Ans: Differentiating $\dot{k}$ w.r.t. t and inserting the $\dot{c}$ equation:

$$\begin{aligned} \ddot{k} &= \alpha A k^{\alpha-1}\dot{k} - \dot{c} - \delta\dot{k}, \\ \ddot{k} &= \left(\alpha A k^{\alpha-1} - \delta\right)\dot{k} - \left[Ak^{\alpha} - \delta k - \dot{k}\right](1/\gamma)\left(\alpha A k^{\alpha-1} - (\rho+\delta)\right) \equiv g\left(\dot{k}, k\right). \end{aligned}$$

Consider the linearized version:

$$\ddot{k} = g\left(\dot{k}, k\right) \cong g(0, k^*) + g_{\dot{k}}(0, k^*)\,\dot{k} + g_k(0, k^*)(k - k^*),$$

$$
\begin{aligned}
g\left(0, k^{*}\right) &= 0, \\
g_{\dot{k}}\left(0, k^{*}\right) &= \rho, \\
g_{k}\left(0, k^{*}\right) &= -f''\left(k^{*}\right)\left(c^{*}/\gamma\right).
\end{aligned}
$$

Thus,

$$
\ddot{k} = \rho\dot{k} - f''\left(k^{*}\right)\left(c^{*}/\gamma\right)\left(k - k^{*}\right),
$$

or denoting $z\left(t\right) = k\left(t\right) - k^{*}$,

$$
\ddot{z} = \rho\dot{z} - f''\left(k^{*}\right)\left(c^{*}/\gamma\right)z.
$$

Step 2 : Argue that the solution to the last differential equation is $z\left(t\right) = B\exp\left(\lambda t\right)$ where λ solves the quadratic equation $\lambda^{2} - \rho\lambda + \left(c^{*}/\gamma\right)f''\left(k^{*}\right) = 0$. Pick the stable solution (i.e. the negative root of λ), which satisfies

$$
\lambda_{1} = \frac{\rho - \sqrt{\rho^{2} - 4f''\left(k^{*}\right)\left(c^{*}/\gamma\right)}}{2} < 0
$$

Ans: The solution to the differential equation $\ddot{z} = \rho\dot{z} - f''\left(k^{*}\right)\left(c^{*}/\gamma\right)z$ is

$$
z\left(t\right) = B\exp\left(\lambda t\right),
$$

since

$$
\begin{aligned}
\dot{z} &= B\lambda\exp\left(\lambda t\right) \\
\ddot{z} &= B\lambda^{2}\exp\left(\lambda t\right),
\end{aligned}
$$

$$
B\lambda^{2}\exp\left(\lambda t\right) = \rho B\lambda\exp\left(\lambda t\right) - f''\left(k^{*}\right)\left(c^{*}/\gamma\right)B\exp\left(\lambda t\right),
$$

or

$$
\lambda^{2} = \rho\lambda - f''\left(k^{*}\right)\left(c^{*}/\gamma\right),
$$

or

$$
\lambda^{2} - \rho\lambda + \left(c^{*}/\gamma\right)f''\left(k^{*}\right) = 0.
$$

This quadratic equation has solution:

$$
\begin{aligned}
\lambda &= \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a} \\
&= \frac{\rho \pm \sqrt{\rho^{2} - 4f''\left(k^{*}\right)\left(c^{*}/\gamma\right)}}{2}
\end{aligned}
$$

with

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2} < 0$$

and

$$\lambda_2 = \frac{\rho + \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2} > 0$$

Thus, focusing on the stable solution,

$$\begin{aligned} z(t) &= B\exp(\lambda_1 t) \\ k(0) - k^* &= B\exp(\lambda_1 0) = B, \end{aligned}$$

or

$$B = k(0) - k^*,$$

and

$$k(t) - k^* = (k(0) - k^*)\exp(\lambda_1 t).$$

Step 3 : Show that for the functional forms we are using:

$$f''(k^*)(c^*/\gamma) = (\alpha - 1)\frac{(\rho + \delta)}{\gamma}\left(\frac{\rho + \delta(1 - \alpha)}{\alpha}\right)$$

which implies

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1 - \alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2}$$

Ans: Notice that with the assumed functional form for $f(k)$ we have that $f(k) = kf'(k)/\alpha$ and $f''(k) = (\alpha - 1) f'(k)/k$. Thus,

$$\begin{aligned} f''(k^*)(c^*/\gamma) &= f''(k^*)\frac{1}{\gamma}(f(k^*) - \delta k^*) \\ &= (\alpha - 1) f'(k^*)\frac{1}{\gamma}\left(\frac{f'(k^*)}{\alpha} - \delta\right) \\ &= (\alpha - 1)\frac{(\rho + \delta)}{\gamma}\left(\frac{\rho + \delta(1 - \alpha)}{\alpha}\right), \end{aligned}$$

where the last line uses that in the steady state $f'(k^*) = \rho + \delta$. Then,

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1 - \alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2}.$$

Step 4 : Use

$$\begin{aligned} x(t) &= \dot{k}(t) + \delta k(t) \\ &= \dot{z}(t) + \delta(z(t) + k^*) \end{aligned}$$

and replace the linearized solution to obtain the result].

Ans: Using

$$x(t) = \dot{k}(t) + \delta k(t),$$

and substituting the linearized version

$$\begin{aligned} x(t) &= \dot{z}(t) + \delta(z(t) + k^*) \\ &= B\lambda_1 \exp(\lambda_1 t) + \delta(B \exp(\lambda_1 t) + k^*) \\ &= \delta k^* + B \exp(\lambda_1 t)(\lambda_1 + \delta), \end{aligned}$$

or

$$x(t) - x^* = (k(0) - k^*) \exp(\lambda_1 t)(\lambda_1 + \delta).$$

From

$$x(t) - x^* = (k(0) - k^*) \exp(\lambda_1 t)[\lambda_1 + \delta]$$

it is clear that the parameter of interest that determines the dynamics of investment is $\lambda_1+\delta$. In particular, whether $\lambda_1 + \delta < 0$ or $\lambda_1 + \delta > 0$.

Exercise 2. Let $\lambda_1(p)$ be the solution of λ_1 as a function of the parameters $p = (\rho, \delta, \gamma, \alpha)$. Show that

$$\lambda_1(p) + \delta \geq 0,$$

if and only if

$$\gamma \geq (1 - \alpha)\left[\frac{\rho/\delta + (1 - \alpha)}{\alpha}\right].$$

What does this imply for the dynamics of investment x close to the steady state if:

a) γ is large enough so that $\gamma > (1 - \alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

b) if δ is small enough so that $\gamma < (1 - \alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

c) if α is small enough so that $\gamma < (1 - \alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this? [Hint: how does the marginal productivity of capital vary with α *close* to the steady state?]

[Hint: Let $F(p) \equiv 2(\lambda_1(p) + \delta)$, and use the explicit expression for λ_1 derived above.

Show that

$$F(p) = 2\delta + \rho - \sqrt{(2\delta + \rho)^2 + \left(4(\rho + \delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta + (1-\alpha)}{\alpha}\right) - \gamma\right]\right)}$$

].

Ans: We have

$$\begin{aligned} F(p) &= 2(\lambda_1(p) + \delta) \\ &= 2\delta + \rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho + \delta(1-\alpha)}{\alpha}\right)} \\ &= 2\delta + \rho - \sqrt{\rho^2 + 4(\rho+\delta)\delta + \left(4(\rho + \delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta + (1-\alpha)}{\alpha}\right) - \gamma\right]\right)}, \end{aligned}$$

or

$$F(p) = 2\delta + \rho - \sqrt{(2\delta + \rho)^2 + \left(4(\rho + \delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta + (1-\alpha)}{\alpha}\right) - \gamma\right]\right)}.$$

Thus, $F(p) \geq 0$ (and hence $\lambda(p) + \delta$) iff $\gamma \geq (1-\alpha)\left[\frac{\rho/\delta + (1-\alpha)}{\alpha}\right]$.

a) γ is the inverse of the elasticity of substitution. Everything else equal, a higher γ implies that the agent is less willing to substitute intertemporally. She wants a flat consumption path, as close to the steady state consumption as possible. This implies that she will invest little (even starting with investment below its steady state value) and it will take longer to converge to the steady state.

b) In the limit, we know that if $\delta = 0$ then investment is zero in the steady state. Thus, along the transition path, investment is always coming from above its steady state value. For δ close enough to zero this is still true.

c) The marginal productivity of capital is $f'(k) = \alpha A k^{\alpha - 1}$. Close to the steady state (i.e. linearizing it around k^*) it satisfies

$$f'(k) \simeq f'(k^*) + f''(k^*)(k - k^*).$$

Thus

$$(\alpha - 1) f'(k)/k$$

$$
\begin{aligned}
f'(k) &\simeq f'(k^*) + (\alpha-1) f'(k^*) \frac{k-k^*}{k^*} \\
&= f'(k^*) \left(1 + (1-\alpha) \frac{k^*-k}{k^*}\right) \\
&= (\rho+\delta) \left(1 + (1-\alpha) \frac{k^*-k}{k^*}\right),
\end{aligned}
$$

where the last line uses that in the steady state $f'(k^*) = \rho + \delta$. Thus, close to the steady state, for $k < k^*$ we have $\partial f'(k^*)/\partial\alpha < 0$. This means that as α declines, the marginal productivity of capital increases (close to k^*) and hence, the benefits from investing increases. Hence investment will come from above its steady state value.

Exercise 3. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

Compute $\lambda_1 + \delta$ for these two set of parameters.

Ans: For the first set of parameters, $\lambda_1 + \delta = -0.04$. For the second set, $\lambda_1 + \delta = 0.01$.

Now we examine the behavior of the ratio x/y, where y denotes GDP. This ratio is a natural business cycles indicator, high in the expansions and low in the recessions. It has the added advantage of being independent of common "trends" in x and y.

We focus in the following quantity:

$$\mu(k) = \left(\frac{k}{x(k)/y(k)} d\frac{x(k)/y(k)}{dk}\right)$$

where $x(k)$, and $y(k)$ are investment and GDP as function of capita (i.e. $\mu(k)$ is the elasticity of the investment to GDP ratio with respect to capital). We are interested in this quantity evaluated at $k = k^*$ which we simply refer to as μ. This can be written as

$$
\begin{aligned}
\mu(k(t)) &= \left(\frac{k(t)}{x(k(t))/y(k(t))} d\frac{x(k(t))/y(k(t))}{dk}\right) \\
&= \left(\frac{1}{x(t)/y(t)} d\frac{x(t)/y(t)}{dt}\right) \Big/ \left(\frac{1}{k(t)} \frac{dk(t)}{dt}\right)
\end{aligned}
$$

Then we study

$$\upsilon(t) = \frac{1}{x(t)/y(t)} d\frac{x(t)/y(t)}{dt}$$

for the linear approximation to the solution of the model.

Exercise 4. Show that

$$\mu = \frac{\lambda_1}{\delta} + 1 - \alpha$$

[Hint: Start by finding the linearized approximation of $\mu(k(t))$. To that end, note that $\upsilon(t) = \frac{1}{x(t)}\frac{dx(t)}{dt} - \frac{1}{y(t)}\frac{dy(t)}{dt}$. Then use the linear approximations

$$\begin{aligned}
x(t) - x^* &= (\lambda_1 + \delta)(k(t) - k^*), \\
y(t) - y^* &= (\rho + \delta)(k(t) - k^*), \\
k(t) - k^* &= (k(0) - k^*)\exp(\lambda_1 t), \\
dk(t)/dt &= (k(0) - k^*)\exp(\lambda_1 t)\lambda_1
\end{aligned}$$

to obtain the derivatives. Replace them into your expression for $\mu(k(t))$. Lastly, take the limit as t goes to infinity and use the equations relating x^*, k^* and y^* to obtain μ].

Ans: We have

$$\upsilon(t) = \frac{1}{x(t)}\frac{dx(t)}{dt} - \frac{1}{y(t)}\frac{dy(t)}{dt},$$

where

$$\begin{aligned}
\frac{1}{x(t)}\frac{dx(t)}{dt} &= \frac{1}{x(t)}[k(0) - k^*]\exp(\lambda_1 t)[\lambda_1 + \delta]\lambda_1 \\
\frac{1}{y(t)}\frac{dy(t)}{dt} &= \frac{1}{y(t)}[k(0) - k^*]\exp(\lambda_1 t)[\rho + \delta]\lambda_1,
\end{aligned}$$

where we used the linear approximations:

$$\begin{aligned}
x(t) - x^* &= (\lambda_1 + \delta)(k(t) - k^*), \\
y(t) - y^* &= (\rho + \delta)(k(t) - k^*), \\
k(t) - k^* &= (k(0) - k^*)\exp(\lambda_1 t), \\
dk(t)/dt &= (k(0) - k^*)\exp(\lambda_1 t)\lambda_1.
\end{aligned}$$

Thus

$$\upsilon(t) = [k(0) - k^*]\exp(\lambda_1 t)\lambda_1\left(\frac{\lambda_1 + \delta}{x(t)} - \frac{\rho + \delta}{y(t)}\right).$$

Consider

$$\mu(k(t)) \equiv k(t)\frac{\upsilon(t)}{dk(t)/dt} = k(t)\frac{\upsilon(t)}{[k(0) - k^*]\exp(\lambda_1 t)\lambda_1} = k(t)\left(\frac{\lambda_1 + \delta}{x(t)} - \frac{\rho + \delta}{y(t)}\right),$$

and let $t \to \infty$, so that

$$\mu = k^* \left(\frac{\lambda_1 + \delta}{x^*} - \frac{\rho + \delta}{y^*} \right).$$

Using

$$\begin{aligned} \alpha y^* &= (\rho + \delta) k^* \\ x^* &= \delta k^*, \end{aligned}$$

we arrive at

$$\mu = \left(\frac{\lambda_1 + \delta}{\delta} - \frac{(\rho + \delta)}{(\rho + \delta)/\alpha} \right) = \frac{\lambda_1}{\delta} + 1 - \alpha.$$

Exercise 5. Show that

$$\lim_{\rho/\delta \to 0} \mu = (1 - \alpha) \left[1 - \frac{1}{\sqrt{\gamma \alpha}} \right].$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$, ρ/δ is small, and $\gamma\alpha > 1$? Does this setting of parameters made investment/GDP in the model procyclical?

Ans: Since

$$\mu = \frac{\lambda_1}{\delta} + 1 - \alpha = \frac{\frac{\rho}{\delta} - \sqrt{\left(\frac{\rho}{\delta}\right)^2 + \frac{4(1-\alpha)}{\gamma\alpha} \left(\frac{\rho}{\delta} + 1\right) \left(\frac{\rho}{\delta} + (1 - \alpha)\right)}}{2} + 1 - \alpha,$$

we have

$$\lim_{\rho/\delta \to 0} \mu = \frac{-\sqrt{\frac{4(1-\alpha)}{\gamma\alpha} (1 - \alpha)}}{2} + 1 - \alpha = (1 - \alpha) \left[1 - \frac{1}{\sqrt{\alpha\gamma}} \right].$$

If $k(0) < k^*$, ρ/δ is small, and $\gamma\alpha > 1$, then $\mu > 0$ and hence, close to the steady state

$$\frac{d(x(k)/y(k))}{dk} > 0,$$

which implies that x/y is increasing in the transition. Since the transition is interpreted as an expansion, this parameter setting is not consistent with the procyclicality of investment.

Exercise 6. Show that

$$\lim_{\rho/\delta \to \infty} \mu = -\infty.$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$ and ρ/δ is very large? Is this consistent with the interpretation of the adjustment to steady state as an expansion?

Ans: Letting $\zeta = \rho/\delta$ and $L(\zeta) = \lambda_1/\delta$:

$$\frac{\lambda_1}{\delta} + 1 - \alpha = L(\zeta) + 1 - \alpha = \frac{\zeta - \sqrt{\zeta^2 + \frac{4(1-\alpha)}{\gamma\alpha}(\zeta+1)(\zeta+(1-\alpha))}}{2} + 1 - \alpha.$$

Since $L(\zeta) < 0$, and

$$\begin{aligned} \lim_{\zeta\to\infty} \frac{L(\zeta)}{\zeta} &= \lim_{\zeta\to\infty} \frac{1 - \sqrt{1 + \frac{4(1-\alpha)}{\gamma\alpha}(1+1/\zeta)(1+(1-\alpha)/\zeta)}}{2} \\ &= \frac{1-\sqrt{1+\frac{4(1-\alpha)}{\gamma\alpha}}}{2} < 0, \end{aligned}$$

then $L(\zeta) \to -\infty$ as $\zeta \to \infty$, and thus $\mu \to -\infty$ as $\rho/\delta \to \infty$.

If $k(0) < k^*$, and ρ/δ is large then $\mu < 0$ and hence, close to the steady state

$$\frac{dx(k)/y(k)}{dk} < 0$$

which implies that x/y is decreasing in the transition. Since the transition is interpreted as an expansion, this parameter setting is consistent with the procyclicality of investment.

Exercise 7. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

For each setting of parameters compute $\lambda_1/\delta + (1-\alpha)$ and comment if they are consistent with the procyclicality of x/y.

Ans: For the first set of parameters $\mu = \lambda_1/\delta + (1-\alpha) = -0.85$. For the second set, $\mu = \lambda_1/\delta + (1-\alpha) = -0.29$. Both are consistent with the procyclicality of x/y.

Exercise 8. Suppose that the economy is in steady state with $A = 1$. Use the first set of parameters ($\alpha = 0.3,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$) for all the calculations.

i. What is the steady state investment to GDP ratio x^*/y^*?

Assume that A unexpectedly changes to $A' = (1+\varepsilon)^{1-\alpha}$, or approximately $(1-\alpha)\varepsilon/100\,\%$, for small ε. Assume that A' will stay at that level forever.

ii. What is the % change in the steady state capital? Denote this capital by k^{**}. Does the steady state value x/y depend on A?

iii. Use the definition of μ above to compute the change in the investment/GDP ratio x/y on impact. Your answer should be a function of μ, (x^*/y^*) and ε. [Hint. Let $z(k) = x(k)/y(k)$, use a first order approximation for z around k^{**} and evaluate it at $k = k^*$].

iv. Using the reference numerical values for all parameters, compute the new value of x/y just after the change in productivity if $\varepsilon = 0.1$ (10%).

Ans:

i.

$$\frac{x^*}{y^*} = \frac{\delta k^*}{(\rho+\delta) k^*/\alpha} = \alpha \frac{\delta}{\delta+\rho} = 0.3 \; \frac{.075}{.075+.075} = .15.$$

ii.

$$\begin{aligned} \rho+\delta &= \alpha A (k^*)^{\alpha-1} \\ \rho+\delta &= \alpha A (1+\varepsilon)^{1-\alpha} (k^{**})^{\alpha-1}. \end{aligned}$$

Thus

$$1 = \left(\frac{1}{1+\varepsilon}\right)^{1-\alpha} \left(\frac{k^*}{k^{**}}\right)^{a-1},$$

or

$$\left(\frac{k^*}{k^{**}}\right)^{1-\alpha} = \left(\frac{1}{1+\varepsilon}\right)^{1-\alpha},$$

or

$$k^{**} = (1+\varepsilon) k^*,$$

so the steady state capital increases by $\varepsilon/100\%$. x^*/y^* is independent of A.

iii. Let $z(k) = x(k)/y(k)$.

$$\mu = \frac{k^{**}}{z(k^{**})} \frac{dz(k^{**})}{dk},$$

so

$$\begin{aligned} z(k^*) &\cong z(k^{**}) + \frac{dz(k^{**})}{dk}(k^* - k^{**}) \\ &= z(k^{**}) + z(k^{**}) \frac{k^{**}}{z(k^{**})} \frac{dz(k^{**})}{dk} \frac{k^*-k^{**}}{k^{**}} \\ &= z(k^{**}) \left(1 + \mu \frac{k^* - (1+\varepsilon) k^*}{(1+\varepsilon) k^*}\right) \\ &= z(k^{**}) \left(1 - \frac{\mu\varepsilon}{1+\varepsilon}\right). \end{aligned}$$

Hence

$$z(k^*) = \left(\frac{x^*}{y^*}\right) \left(1 - \frac{\mu\varepsilon}{1+\varepsilon}\right).$$

iv.

$$0.15 \ \times \left(1 - \frac{(-.85) \ \times 0.1}{1 + 0.1}\right) = 0.162,$$

so on impact investment/GDP is about 16% (an increase of about 8%).

6 Investment Specific Technological Progress

Consider the following version of the neoclassical growth model

$$\max \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt$$

$$\begin{aligned}
\text{s.t.} \quad p_k x\left(t\right) + c\left(t\right) &= F\left[k(t), e^{\gamma t}\right], \\
\dot{k}\left(t\right) &= e^{\eta t} x\left(t\right) - \delta k\left(t\right), \quad \text{all } t \geq 0, \\
\text{given } k\left(0\right) &= k_0 > 0,
\end{aligned}$$

where F has constant returns to scale, and (inelastically supplied) labor is normalized at unity. Labor-augmenting technical change occurs at the rate $\gamma \geq 0$, and investment-specific technical change at the rate η. The constant p_k corresponds to the initial (time $t = 0$) price of investment in terms of consumption.

Exercise 1. Formulate the Hamiltonian using c as the only control variable and write the first order conditions for an optimum.

Ans: Using the feasibility constraint in the law of motion for capital we obtain

$$\dot{k}_t = e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t.$$

Thus, the Hamiltonian is

$$H = u\left(c_t\right) + \lambda_t \left[e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t\right],$$

where λ_t is the co-state variable. The FOCs for an optimum are

$$0 = \frac{\partial H}{\partial c} = u'\left(c_t\right) - \lambda_t \frac{e^{\eta t}}{p_k}, \tag{26}$$

and

$$\dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial k} = \lambda_t \rho + \lambda_t \delta - \lambda_t \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right),$$

or

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho + \delta - \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right). \tag{27}$$

Exercise 2. Reduce the system obtained in 1 to a pair of differential equations in (c, k).

Ans: Take logs to (26)

$$\log u'(c_t) = \log \lambda_t + \eta t - \log p_k.$$

Differentiate w.r.t. time,

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \frac{\dot{\lambda}_t}{\lambda_t} + \eta.$$

Solving for $\dot{\lambda}/\lambda$ and introducing the result into (27) we obtain

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \eta + \rho + \delta - \frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right).$$

Thus, the pair of differential equations in (c, k) are

$$\frac{\dot{c}_t}{c_t} = -\frac{1}{c_t u''(c_t)/u'(c_t)} \left[\frac{e^{\eta t}}{p_k} F_k\left(k_t, e^{\gamma t}\right) - (\rho + \delta + \eta)\right],$$

$$\dot{k}_t = e^{\eta t} \frac{F\left(k_t, e^{\gamma t}\right) - c_t}{p_k} - \delta k_t.$$

Use the following functional form for the remaining of this question:

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \\ F\left(k, e^{\gamma t}\right) &= A\ k^{\alpha} e^{(1-\alpha)\gamma t}, \quad 0 < \alpha < 1. \end{aligned}$$

Balanced Growth Path. Suppose $\gamma, \eta > 0$. Consider the growth rates of capital and consumption in the long run. Conjecture that in the long run the ratio of consumption to capital falls at the rate η. That is, conjecture that along the balanced growth path $c(t)/k(t) = x^* e^{-\eta t}$, where x^* is a constant that must be determined.

Exercise 3. Calculate the long run growth rates for capital (g_k), consumption (g_c) and output as function of the parameters $\alpha, \sigma, \delta, \gamma$ and η.

Ans: Using the above functional forms the system of differential equations becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left[\frac{e^{\eta t} \alpha A k_t^{\alpha-1} \left(e^{\gamma t}\right)^{1-\alpha}}{p_k} - (\rho + \delta + \eta)\right], \tag{28}$$

$$\dot{k}_t = \frac{e^{\eta t}}{p_k}\left[Ak_t^{\alpha}\left(e^{\gamma t}\right)^{1-\alpha} - c_t\right] - \delta k_t,$$

or, dividing by k_t, the last equation becomes

$$\frac{\dot{k}_t}{k_t} = \frac{e^{\eta t}}{p_k}\left[Ak_t^{\alpha-1}\left(e^{\gamma t}\right)^{1-\alpha} - \frac{c_t}{k_t}\right] - \delta. \tag{29}$$

Rewrite (28) as

$$g_c = \frac{1}{\sigma}\left[\frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1} - (\rho+\delta+\eta)\right]. \tag{30}$$

Since in a balanced growth path the left hand side is constant (i.e. independent of t), so has to be the right hand side. But the RHS is constant if and only if

$$e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1},$$

is constant. Taking logs of that expression we find

$$[\eta+\gamma(1-\alpha)]\,t - (1-\alpha)\,k_t = \text{constant},$$

and differentiating w.r.t. time we have

$$\eta+\gamma(1-\alpha) - (1-\alpha)\frac{\dot{k}_t}{k_t} = 0,$$

or

$$g_k = \gamma + \frac{\eta}{1-\alpha}.$$

That is, we pinned down the growth rate of capital in the balanced growth path.

Now using (29) we obtain

$$g_k + \delta = \left[\frac{1}{\alpha}\frac{\alpha A}{p_k}e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1} - \frac{e^{\eta t}}{p_k}\frac{c_t}{k_t}\right]. \tag{31}$$

Moreover, as we showed above the term $e^{[\eta+\gamma(1-\alpha)]t}k_t^{\alpha-1}$ is constant in a balanced growth path. Thus it immediately follows that

$$e^{\eta t}\frac{c_t}{k_t} = x^* = \text{constant}$$

in a steady state (noticed that we didn't use the conjecture, we proved it!). Hence (taking logs and differentiating w.r.t. t)

$$\eta + \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t},$$

or

$$g_c = \gamma + \frac{\eta}{1-\alpha} - \eta,$$

or

$$g_c = \gamma + \frac{\alpha}{1-\alpha}\eta.$$

Furthermore (even though not asked), we can obtain x^*. Rewrite (31) as

$$\gamma + \frac{\eta}{1-\alpha} + \delta = \frac{1}{\alpha}\frac{\alpha A}{p_k} e^{[\eta+\gamma(1-\alpha)]t} k_t^{\alpha-1} - \frac{x^*}{p_k}.$$

Use (30) to get

$$\left(\gamma + \frac{\eta\alpha}{1-\alpha}\right)\sigma + \rho + \delta + \eta = \frac{\alpha A}{p_k} e^{[\eta+\gamma(1-\alpha)]t} k_t^{\alpha-1},$$

and introducing the last equation into the previous one

$$\frac{x^*}{p_k} = \frac{1}{\alpha}\left[\left(\gamma + \frac{\eta}{1-\alpha} - \eta\right)\sigma + \rho + \delta + \eta\right] - \left(\gamma + \frac{\eta}{1-\alpha} + \delta\right),$$

which gives x^*.

Exercise 4. How would you measure the contributions to long term growth of labor-augmenting technical change and investment-specific technical change for the US economy? [Sketch only, 10 lines maximum. Hint: A good answer should mention measuring TFP (total factor productivity) being careful on the units on which GDP and factors are measured, and the rate at which the investment deflator and the consumption deflator grow. Notice also that labor-augmenting technical change is closely related to TFP when the production function is Cobb-Douglas].

Ans: Taking logs of the production function we obtain

$$\log y_t = (1-\alpha)\gamma t + \alpha \ \log k_t.$$

Differentiating this expression w.r.t. time yields

$$g_y = (1-\alpha)\gamma + \alpha g_k,$$

or, rearranging,

$$g_{\text{TFP}} \equiv (1-\alpha)\gamma = g_y - \alpha g_k,$$

which gives an expression for TFP growth in terms of measurable quantities: the rate of growth of real GDP, g_y, the rate of growth of the real stock of capital, g_k and capital income's share in total production, α. Dividing this expression by $(1-\alpha)$ we obtain the rate of labor-augmenting technical change, γ. Moreover, recall that

$$g_k = \gamma + \frac{\eta}{1-\alpha}.$$

Hence

$$\eta = (1-\alpha)(g_k - \gamma),$$

gives the rate of investment-specific technical change. Finally, since

$$g_c = g_y = \gamma + \frac{\alpha}{1-\alpha}\eta,$$

then γ/g_y and $\frac{\alpha}{1-\alpha}\eta/g_y$ are the contributions to long term growth of labor-augmenting technical change and investment-specific technical change.

For the rest of the question consider the case where $\eta = \gamma = 0$.

Exercise 5. Draw the phase diagram corresponding to the system in b. Have c in the vertical axis and k in the horizontal axis. Indicate the $\dot{k} = 0$ and $\dot{c} = 0$ locus, the steady state values k^*, c^*. Include arrows indicating the direction of movement in each relevant quadrant, display the saddle path clearly, and indicate typical paths of trajectories that start close but not on the saddle path.

Ans: The system of differential equations for $\eta = \gamma = 0$ is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left[\alpha A k_t^{\alpha-1}/p_k - (\rho+\delta)\right], \tag{32}$$

$$\dot{k}_t = (Ak_t^{\alpha} - c_t)/p_k - \delta k_t. \tag{33}$$

The dynamics of equations (33) and (32) are shown in the phase diagram of Figure 3 below. As usual, the $\dot{k}_t = 0$ locus is strictly concave and strictly increasing (decreasing) for all k that satisfy $f'(k) > (<) p_k \delta$. In turn, the $\dot{c}_t = 0$ locus is a vertical line in the (k, c) plane. Starting from a situation in which $\dot{c}_t = 0$, if we increase (decrease) the stock of capital then consumption will start falling (growing), as can be seen directly from equation (32). We summarize this with the arrows pointing south (north) to the right (left) of the $\dot{c}_t = 0$ locus. Similarly, starting from a situation in which $\dot{k}_t = 0$, if we increase (decrease) consumption then the stock of capital will start falling (growing), as can be seen directly from equation (33). This is summarized with the arrows pointing west (east) above (below) the $\dot{k}_t = 0$ locus.

Starting from any initial level of capital there exists a unique trajectory (the saddle path) that converges to the steady state, (k^*, c^*). The trajectory is unique because any other (k, c) pair off the saddle path would lead to a trajectory that eventually violates the necessary conditions for optimality. For instance, if the economy starts with a capital stock of k_0

then the optimal level of consumption is c_0. Any level of consumption above c_0 will lead to zero consumption and zero capital stock in finite time. This would clearly violate the Euler equation at that time. In turn, any level of consumption above c_0 will lead to a very large amount of capital, which would violate the transversality condition.

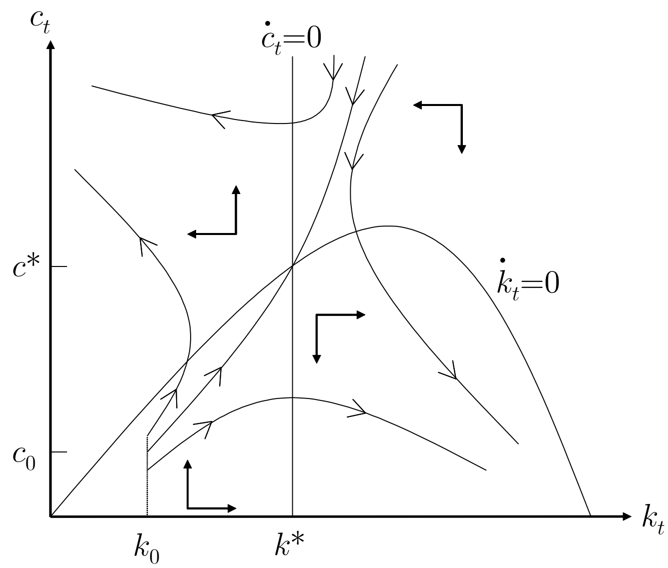

Figure 3. Phase diagram of the neoclassical growth model.

Exercise 6. Find an expression for the steady state capital stock k^* as a function of p_k, α, ρ and δ. Find an expression for the value of capital relative to GDP, i.e. $p_k k^*/y^*$ where y^* is the steady state output.

Ans: (32) at the steady state becomes

$$0 = \frac{1}{\sigma}\left[\alpha A\left(k^*\right)^{\alpha-1}/p_k - (\rho+\delta)\right].$$

Thus

$$k^* = \left[\frac{\alpha A}{p_k\left(\rho+\delta\right)}\right]^{1/(1-\alpha)}.$$

To obtain $p_k k^*/y^*$, recall that $f\left(k^*\right) = k^* f'\left(k^*\right)/\alpha$. Thus,

$$\begin{aligned}\frac{p_k k^*}{y^*} &= \frac{p_k k^*}{k^* f'\left(k^*\right)/\alpha} \\ &= \frac{\alpha}{f'\left(k^*\right)/p_k} \\ &= \frac{\alpha}{\rho+\delta},\end{aligned}$$

where the last line uses that in the steady state the marginal product of capital equals $\rho + \delta$.

Exercise 7. Find an expression for the steady state consumption c^* as a function of p_k, α, ρ and δ. Find an expression for the value of consumption relative to GDP, i.e. c^*/y^* where y^* is the steady state output.

Ans: Feasibility at the steady state is

$$p_k \delta k^* + c^* = f(k^*).$$

Thus

$$\begin{aligned} c^* &= f(k^*) - p_k \delta k^* \\ &= \frac{k^* f'(k^*)}{\alpha} - p_k \delta k^* \\ &= \left(\frac{f'(k^*)/p_k}{\alpha} - \delta \right) p_k k^*, \end{aligned}$$

or

$$c^* = \left(\frac{\rho + \delta}{\alpha} - \delta \right) p_k^{-\alpha/(1-\alpha)} \left(\frac{\alpha A}{\rho + \delta} \right)^{1/(1-\alpha)}.$$

On the other hand,

$$\frac{c^*}{y^*} = 1 - \frac{p_k \delta k^*}{y^*},$$

or, using the result in exercise 6,

$$\frac{c^*}{y^*} = 1 - \frac{\alpha \delta}{\rho + \delta}.$$

For the remaining of this question, consider the case of no depreciation, i.e. $\delta = 0$.

Exercise 8. What is the steady state value of investment x^*? What is the steady state value of investment $p_k x^*/y^*$ relative to GDP.

Ans: $x^* = \delta k^* = 0$ and $p_k x^*/y^* = 0$.

Exercise 9. Assume that $k(0)$ is smaller than the its steady state value k^*. Draw a figure with time t in the horizontal axis, and output $y(t)$, investment $x(t)$, and consumption $c(t)$ in the vertical axis. Label the steady state values for output, investment and consumption, y^*, x^* and c^* in the vertical axis. Make sure $y(t) = c(t) + p_k x(t)$ and that x^* is as in exercise 8.

Ans: The dynamics of capital and consumption can be read off the phase diagram in Figure 3. Figure 4 below plots the associated dynamics. We see that 1) investment converges from above to $x^* = 0$, and 2) consumption and output converge from below to $c^* = y^*$, with consumption being less than output along the transition path (since investment is strictly positive).

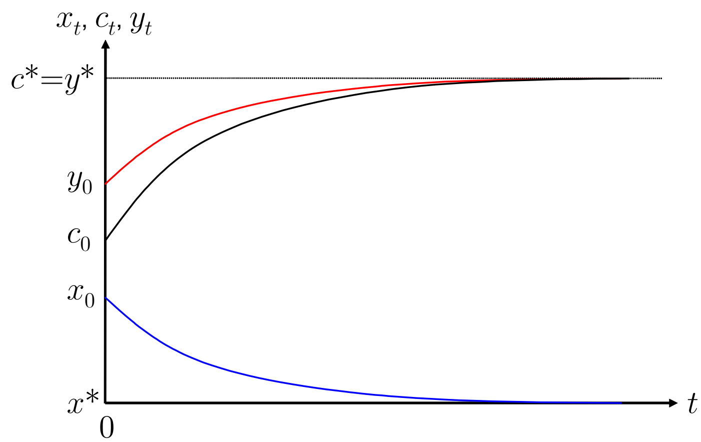

Figure 4. Dynamics of investment, consumption and output for $k(0) < k^*$.

Exercise 10. Assume that $k(0) = k^*$ is the steady state value of capital for the price p_k. Assume that at time $t = 0$, the price of capital decreases to p_k' with $p_k' < p_k$ and will remain there forever. Denote the new steady state values as y^{**}, c^{**} and x^{**}. Draw a figure with time t in the horizontal axis, and output $y(t)$, investment $x(t)$ and consumption $c(t)$ in the vertical axis. Make sure $y(t) = c(t) + p_k' x(t)$ and that x^* and x^{**} are as in exercise 8. Label the steady states values for output y^* and y^{**}, investment x^* and x^{**} and consumption c^{**} and c^* in the vertical axis. Show whether $c^* < c(0)$ or $c^* \geq c(0)$ and whether $y^* < y(0)$ or $y^* \geq y(0)$.

What do you learn about the "cyclicality" of consumption? That is, would the transition caused by a permanent decrease in p_k starting from a steady state capital look like an economic boom? (recall that in a boom consumption and GDP both increase together). How do you think your answer will change for a higher (positive) value of the depreciation δ?

Ans: The dynamics of consumption and capital can be read off the phase diagram in Figure 5. We see that a fall in the price of capital shifts the $\dot{c}_t = 0$ locus to the right (to $\dot{c}_t' = 0$) while it leaves the $\dot{k}_t = 0$ locus unmodified. Figure 6 plots the associated dynamics. In particular, 1) investment converges from above to $x^* = x^{**} = 0$, 2) consumption and output converge from below to $c^{**} = y^{**} > c^* = y^*$, with consumption being less than output along the transition path (since investment is strictly positive), and 3) $y(0) = y^* = c^*$ (since the capital stock is given at time zero) and $c(0) < y(0) = c^*$ (consumption must fall on impact to free up resources for investment).

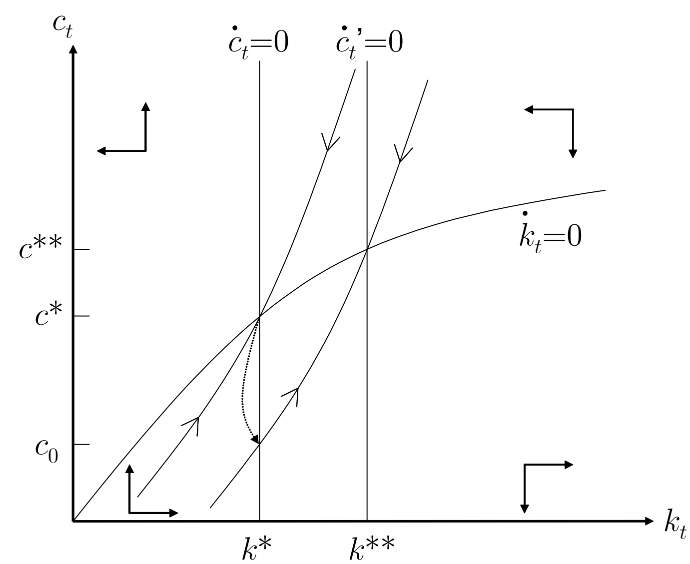

Figure 5. The impact of a fall in the price of capital, p_k, when $\delta = 0$.

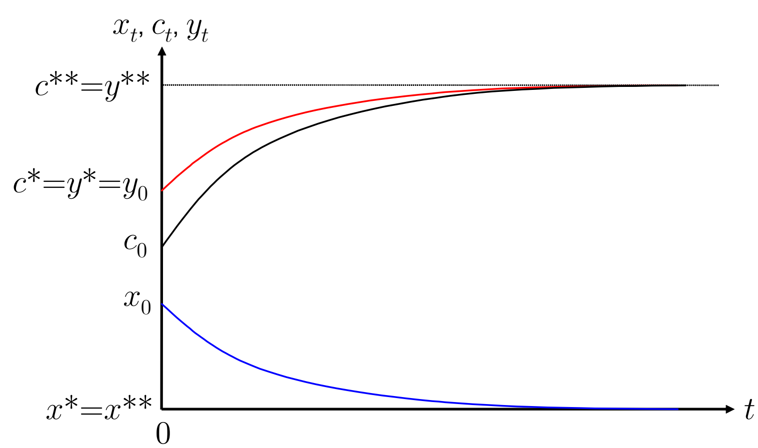

Figure 6. Dynamics of investment, consumption and output due to a fall in the price of capital, p_k, when $\delta = 0$.

Since consumption and output move together along the transition path (i.e., consumption is procyclical), a fall in the price of capital will look like an economic boom. The case of a strictly positive depreciation rate, $\delta > 0$, is left as an exercise.

7 Habit Formation model

Assume that agents select $c(t)$ and $l(t)$ for $t \geq 0$ to maximize

$$\int_0^{\infty} e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - Al(t) \right] dt,$$

subject to

$$\begin{aligned} c(t) &= wl(t), \\ \dot{s}(t) &= c(t) - \delta s(t), \end{aligned}$$

for all $t \geq 0$, for given $s(0)$.

In this problem c is consumption, l labor, w the real wage, ρ the discount factor, s the undepreciated stock of past consumption or the habit level, δ the depreciation rate of consumption in the habit level, and $u(c - \delta s) - Al$ the period utility function.

As it is clear from the expression above the agent has no access to a savings account or a storage technology. At each point in time the agent must decide how much to work (or consume). The problem is not static because we assume that preferences depend on labor l, and consumption c *relative to the depreciated stock of consumption* s.

We can write this problem as

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - \frac{A}{w} c(t) \right] dt,$$

subject to

$$\dot{s}(t) = c(t) - \delta s(t),$$

for all $t \geq 0$ given $s(0)$. We will ignore the non-negativity conditions for $l(t)$.

We will assume that $0 < \eta < \delta$, $\delta > 0, \rho > 0$ and $w > 0$. At some points we will also assume that u has constant relative risk aversion γ, i.e.

$$u(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma},$$

for $\gamma > 0$.

Exercise 1. Write the Hamiltonian $H(c, s, \lambda)$. Which is the control and which is the state?

Ans:

$$H(c, s, \lambda) = u(c - \eta s) - \frac{A}{w} c + \lambda (c - \delta s).$$

The control is c and the state is s.

Exercise 2. Write the first order conditions (these are two equations $H_c = 0$ and $\dot{\lambda} = ...$). Your answer should be in terms of the parameters ρ, δ, η, w and the function u.

Ans:

$$\begin{aligned} 0 &= u' - \frac{A}{w} + \lambda \\ \dot{\lambda} &= \lambda\rho - H_s = \lambda\rho + u'\left(c - \eta s\right)\eta + \delta\lambda, \end{aligned}$$

or

$$\begin{aligned} 0 &= u' - \frac{A}{w} + \lambda \\ \dot{\lambda} &= \lambda\left(\rho + \delta\right) + u'\left(c - \eta s\right)\eta. \end{aligned}$$

Exercise 3. Use the previous equations, including the one for $\dot{s}$, to solve for steady state c as a function of the parameters of the model.

Ans:

$$\begin{aligned} \dot{s} &= 0 \text{ gives } c^* = \delta s^*. \\ \dot{\lambda} &= 0 \text{ gives } -\lambda^*\left(\rho + \delta\right) = u'\left(c^* - \eta s^*\right)\eta, \end{aligned}$$

or

$$\left[u'\left(c^* - \eta s^*\right) - \frac{A}{w}\right]\left(\rho + \delta\right) = u'\left(c^* - \eta s^*\right)\eta,$$

or

$$u'\left(c^* - \eta s^*\right) = \frac{A}{w}\frac{\left(\rho + \delta\right)}{\rho + \delta - \eta},$$

or

$$u'\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right) = \frac{A}{w}\frac{\left(\rho + \delta\right)}{\rho + \delta - \eta},$$

which gives the steady state level of consumption c^*.

Exercise 4. Show that the long run elasticity of consumption and labor supply is

$$\frac{w}{c^*}\frac{dc^*}{dw} = \left[\frac{u'}{-cu''}\left(c^*\left(1 - \frac{\eta}{\delta}\right)\right)\right] \ / \ \left[1 - \frac{\eta}{\delta}\right],$$

i.e. this is the percentage change in steady state consumption for a 1% change in wages.

Ans:

$$
\begin{aligned}
u''\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\left[1-\frac{\eta}{\delta}\right]\frac{dc^*}{dw} &= -\frac{A}{w^2}\frac{(\rho+\delta)}{\rho+\delta-\eta} \\
u''\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\left[1-\frac{\eta}{\delta}\right]\frac{w}{c^*}\frac{dc^*}{dw} &= -\frac{A}{w}\frac{(\rho+\delta)}{\rho+\delta-\eta}\frac{1}{c^*} = -u'\left(c^*\left[1-\frac{\eta}{\delta}\right]\right)\frac{1}{c^*},
\end{aligned}
$$

or

$$
\frac{w}{c^*}\frac{dc^*}{dw} = \left[\frac{u'}{-cu''}\left(c^*\left(1-\frac{\eta}{\delta}\right)\right)\right] \ / \ \left[1-\frac{\eta}{\delta}\right].
$$

Exercise 5. To be able to draw the phase diagram for this model we want to eliminate λ from the system of three equations (one for $\dot{\lambda}$, one for $\dot{s}$ and $H_c = 0$). This is analogous to what we did in the neoclassical growth model. You must show that you can solve for λ and express the system of two differential equations in $\dot{s}$ and $\dot{c}$ as

$$
\begin{aligned}
\dot{s} &= c - \delta s, \\
\dot{c} &= \frac{u'}{-u''}\left[\eta-(\rho+\delta)\right] + \left[\frac{1}{-u''}\right]\frac{A}{w}(\rho+\delta) + \eta\left(c-\delta s\right).
\end{aligned}
$$

Ans: Differentiating $H_c = 0$ w.r.t. time:

$$
\dot{\lambda} + u''\left(c-\eta s\right)\left(\dot{c}-\eta\dot{s}\right) = 0.
$$

Using $\dot{s}$:

$$
\dot{\lambda} = -u''\left(c-\eta s\right)\left(\dot{c}-\eta\left(c-\delta s\right)\right).
$$

Inserting it in $\dot{\lambda}$:

$$
-u''\left(c-\eta s\right)\left(\dot{c}-\eta\left(c-\delta s\right)\right) = \left[-u'\left(c-\eta s\right)+\frac{A}{w}\right](\rho+\delta) + u'\left(c-\eta s\right)\eta,
$$

or

$$
-u''\left(c-\eta s\right)\dot{c} = u'\left(c-\eta s\right)\left[\eta-(\rho+\delta)\right] + \frac{A}{w}(\rho+\delta) - u''\left(c-\eta s\right)\left(c-\delta s\right)\eta,
$$

or

$$
\dot{c} = \frac{u'\left(c-\eta s\right)}{-u''\left(c-\eta s\right)}\left[\eta-(\rho+\delta)\right] + \left[\frac{1}{-u''\left(c-\eta s\right)}\right]\frac{A}{w}(\rho+\delta) + \eta\left(c-\delta s\right). \tag{34}
$$

Exercise 6. Define the function $\theta(s)$ as giving the combinations of $(c, s) = (\theta(s), s)$ such that $\dot{c} = 0$, where $\dot{c}$ is given in question 5. Show that, at the steady state level s^*

$$
\theta'\left(s^*\right) = \frac{\eta\left[(\rho+2\delta)-\eta\right]}{(\rho+\delta)}.
$$

Ans: Evaluating (34) at $\dot{c} = 0$ and multiplying by $-u''$ we obtain (note that at $\dot{c} = 0$, we have $c = \theta(s)$)

$$0 = u'(\theta(s) - \eta s)[\eta - (\rho + \delta)] + \frac{A}{w}(\rho + \delta) - u''(\theta(s) - \eta s)(\theta(s) - \delta s)\eta.$$

Differentiating the last expression w.r.t. s we find

$$\begin{aligned} 0 &= u''(\theta(s) - \eta s)[\eta - (\rho + \delta)]\left(\theta'(s) - \eta\right) - u'''(\theta(s) - \eta s)(\theta(s) - \delta s)\eta\left(\theta'(s) - \eta\right) \\ &\quad -u''(\theta(s) - \eta s)\left(\theta'(s) - \delta\right)\eta. \end{aligned}$$

Evaluating at the steady state $c^* = \theta(s^*) = \delta s^*$, we find

$$0 = u''\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right)[\eta - (\rho + \delta)]\left(\theta'(s^*) - \eta\right) - u''\left(c^*\left[1 - \frac{\eta}{\delta}\right]\right)\left(\theta'(s^*) - \delta\right)\eta,$$

or, dividing by u'',

$$[\eta - (\rho + \delta)]\left(\theta'(s^*) - \eta\right) - \left(\theta'(s^*) - \delta\right)\eta = 0.$$

Thus

$$\theta'(s^*) = \frac{\eta[(\rho + 2\delta) - \eta]}{(\rho + \delta)}.$$

Exercise 7. Show that, since $\eta < \delta$, then

$$0 < \theta'(s^*) < \delta.$$

Ans: Since $\eta < \delta$, then $\theta'(s^*) > 0$. For the other inequality, we will show that $\theta'(s^*) - \delta < 0$:

$$\begin{aligned} \theta'(s^*) - \delta &= \frac{\eta[\rho + 2\delta - \eta]}{(\rho + \delta)} - \delta \\ &= \frac{\eta[\rho + \delta + (\delta - \eta)]}{(\rho + \delta)} - \delta \\ &= \eta\left[1 + \frac{(\delta - \eta)}{\rho + \delta}\right] - \delta \\ &= \eta\frac{(\delta - \eta)}{\rho + \delta} - (\delta - \eta) \\ &= (\delta - \eta)\left[\frac{\eta}{\rho + \delta} - 1\right] \\ &= (\delta - \eta)\left[\frac{\eta - \delta - \rho}{\rho + \delta}\right] < 0, \end{aligned}$$

since $\delta > \eta$.

From now on assume that the utility function u has CRRA γ.

Exercise 8. Show that $d\dot{c}/dc > 0$, where $\dot{c}$ is the equation found in exercise 5 evaluated at steady state values c^* and s^*. [Hint. Using that u has CRRA it is easy to show that $d(\dot{c}/c)/dc > 0$. This, of course, implies that $d\dot{c}/dc > 0$].

Ans: We have

$$\begin{aligned}
\dot{c} &= \frac{u'(c-\eta s)}{-u''(c-\eta s)}[\eta - (\rho+\delta)] + \left[\frac{1}{-u''(c-\eta s)}\right]\frac{A}{w}(\rho+\delta) + \eta(c-\delta s) \\
&= (c-\eta s)\frac{u'(c-\eta s)}{-u''(c-\eta s)(c-\eta s)}\left[\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)}\right] + \eta(c-\delta s).
\end{aligned}$$

But with the CRRA preferences we have $-u''(c-\eta s)(c-\eta s)/u'(c-\eta s) = \gamma$. Hence

$$\dot{c} = (c-\eta s)\frac{1}{\gamma}\left[\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)}\right] + \eta(c-\delta s).$$

Differentiating this expression w.r.t c we obtain

$$\begin{aligned}
\frac{d\dot{c}}{dc} &= \frac{1}{\gamma}\left[\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)}\right] - (c-\eta s)\frac{1}{\gamma}\frac{A}{w}(\rho+\delta)\left[\frac{u''(c-\eta s)}{(u'(c-\eta s))^2}\right] + \eta \\
&= \frac{1}{\gamma}\left[\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)}\right] + \left[-\frac{u''(c-\eta s)(c-\eta s)}{u'(c-\eta s)}\right]\frac{1}{\gamma}\frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)} + \eta \\
&= \frac{1}{\gamma}\left[\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)}\right] + \frac{A}{w}\frac{(\rho+\delta)}{u'(c-\eta s)} + \eta.
\end{aligned}$$

But, from question 3, we know that in the steady state

$$\eta - (\rho+\delta) + \frac{A}{w}\frac{(\rho+\delta)}{u'(c^*-\eta s^*)} = 0.$$

Thus,

$$\begin{aligned}
\left.\frac{d\dot{c}}{dc}\right|_{c^*,s^*} &= \frac{A}{w}\frac{(\rho+\delta)}{u'(c^*-\eta s^*)} + \eta \\
&= (\rho+\delta) > 0.
\end{aligned}$$

Exercise 9. Draw the phase diagram. To simplify draw the phase diagram as if both the $\dot{s}=0$ and the $\dot{c}=0$ loci were linear. Put s in the x-axis and c in the y-axis. Make sure to draw arrows with the directions in which s and c will move in all relevant quadrants. Make sure that you include some arrows that intersect the $\dot{s}=0$ and $\dot{c}=0$ loci at points different from the steady state, and that the slope of these arrows are consistent with the $\dot{c}$ and $\dot{s}$ equations (either they cross vertically or horizontally). Make sure your graph is readable, the points you'll obtain depend on it! Plot the saddle path, make sure that it crosses the right quadrants, and that it has the right slope (i.e. that it is stepper or flatter than the relevant $\dot{c} = 0$ or $\dot{s} = 0$ loci).

Ans: The system of differential equations is

$$\dot{c} = (c - \eta s) \frac{1}{\gamma} \left[\eta - (\rho + \delta) + \frac{A}{w} \frac{(\rho + \delta)}{u'(c - \eta s)} \right] + \eta (c - \delta s), \tag{35}$$

and

$$\dot{s} = c - \delta s. \tag{36}$$

The dynamics of these equations are shown in the phase diagram of Figure 7 below.

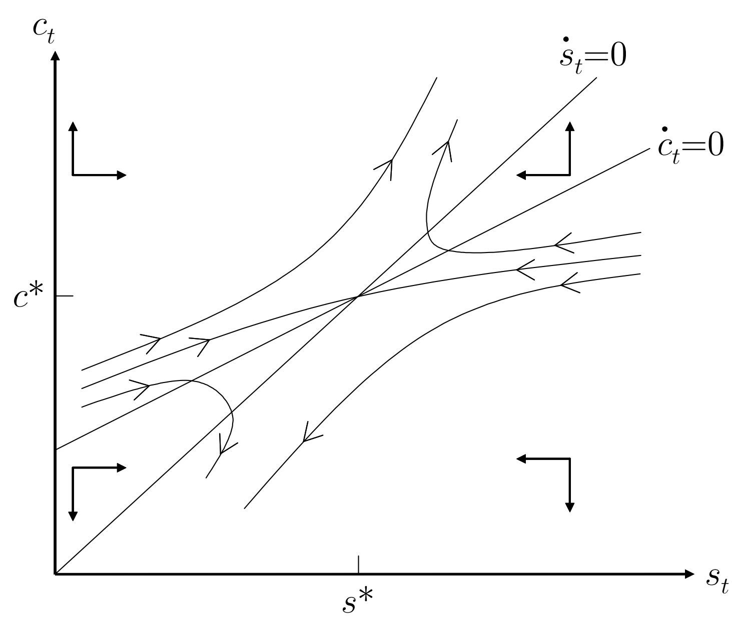

Figure 7. Phase diagram of the habit formation model.

The $\dot{s}_t = 0$ locus is a line emanating from the origin with slope $\delta > 0$. In turn, we draw the $\dot{c}_t = 0$ locus as a line with slope $0 < \theta'(s) < \delta$. Starting from a situation in which $\dot{c}_t = 0$, if we increase (decrease) the habit level then consumption will start falling (growing), as can be checked from equation (35). We summarize this with the arrows pointing south (north) to the right (left) of the $\dot{c}_t = 0$ locus. Similarly, starting from a situation in which $\dot{s}_t = 0$, if we increase (decrease) consumption then the habit level will start growing (falling), as can be seen directly from equation (36). This is summarized with the arrows pointing east (west) above (below) the $\dot{s}_t = 0$ locus.

Exercise 10. Assume that at time $t = 0$, the habit levels $s(0)$ is given by the steady state value s^*. Then, "unexpectedly", agents learn at time $t = 0$ that w will be higher, say $\bar{w} > w$ for T periods, and then it will return to the steady state value w. That is, $w(t) = \bar{w}$ for $t = (0, T)$, and $w(t) = w$ for $t \geq T$. You have to analyze the path for optimal consumption and habit for the case of this transitory increase in wages. In this page you must draw a phase diagram with both saddle path corresponding to w and $\bar{w}$, but the flow (arrows) should correspond to $\bar{w}$. Remember that s cannot "jump" at time $t = 0$ but consumption can. Remember also that at time $t = T$ the system must land continually (with respect to time) in the saddle path corresponding to w. In this phase diagram you must draw the trajectory in the c-s space that the agent will chose. Make sure that this trajectory is clearly labeled, that includes arrows showing the direction of movement, that it starts at the right height (i.e. the qualitatively correct level of c) and that if it crosses any of the $\dot{c} = 0$ or $\dot{s} = 0$ it does so with the right slope. In the next page you must also draw two figures with time t in the horizontal axis and with the optimal path of c (one figure) and s in the vertical axis. Start these figures at some time $t < 0$ and clearly label the time periods $t = 0$, $t = T$ and include horizontal lines for the steady state values for consumption $c^*$and $\bar{c}^*$ and habit $s^*, \bar{s}^*$ that correspond to the values of w and $\bar{w}$, respectively. You should obtain the qualitative features of these time trajectories from your previous figure. Make sure your graphs are readable, the points you'll obtain depend on it!

Ans: From exercise 4, we know that in the steady state

$$\frac{dc^*}{dw} > 0.$$

Since $c^* = \delta s^*$, it follows that

$$\frac{ds^*}{dw} > 0,$$

as well. That is, a (permanent) rise in the wage rate will unambiguously increase the steady state levels of consumption and the habit level. This can be seen graphically in Figure 8 below. Indeed, a rise in w will shift the $\dot{c}_t = 0$ locus upward, and will have no effect on the $\dot{s}_t = 0$ locus. Thus, the new steady state values of c and s, $(\bar{c}^*, \bar{s}^*)$ are such that $\bar{c}^* > c^*$ and $\bar{s}^* > s^*$. The intuition for this result is clear: a permanent rise in w makes the individual richer, so it is reasonable to expect that he will enjoy a higher level of consumption (and habit level) in the long-run.

Notice that according to Figure 8, if the wage rise were permanent, consumption would rise on impact to $\bar{c}$ and settle immediately in the saddle path that characterizes the new dynamic system. Thereafter, consumption and the habit level would increase monotonically towards their new steady-state values. However, in our case the individual knows that at time $t = T$ the wage rate will fall back to its previous value. Thus, the dynamics of the system will be dictated by the $\dot{c}' = 0$ and $\dot{s} = 0$ loci for $t < T$, and by the old $\dot{c} = 0$ and $\dot{s} = 0$ loci for $t \geq T$. Notice that consumption cannot jump at time $t = T$ since this would violate the Euler equation at that time. Hence, the dynamics of capital and consumption for $t \in [0, T)$ must be such that the system reaches the old saddle path at time $t = T$ in a continuous fashion.

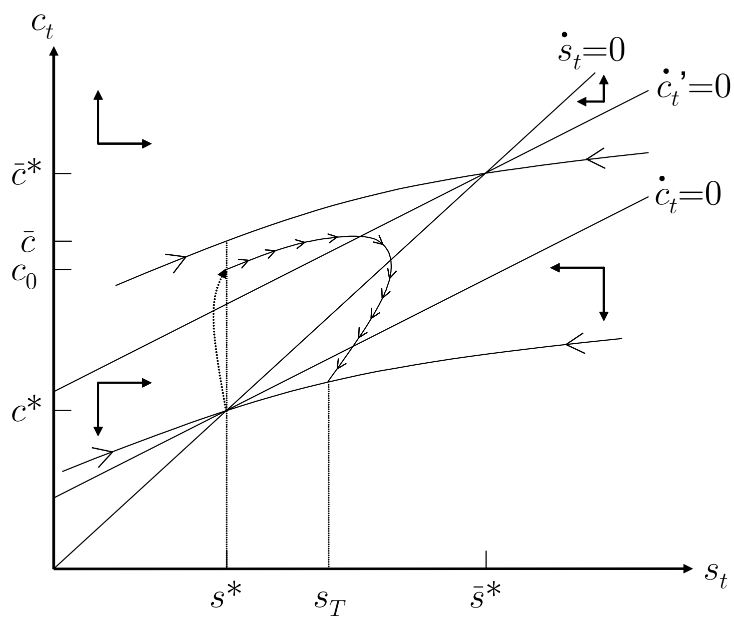

Figure 8. The impact of a temporary rise in the wage rate, w, if c_0 is above the $\dot{c}' = 0$ locus.

Moreover, consumption cannot stay constant or fall on impact. Otherwise, consumption would be decreasing afterwards and then will have to jump upwards at time $t = T$ so that the economy reaches the old saddle path. However, consumption cannot rise on impact all the way up to $\bar{c}$ either. If it did, consumption would follow the non-decreasing dynamics dictated by the new saddle path for $t \in [s, T)$ and then would have to jump downwards at time $t = T$ so that the economy lands in the old saddle path. These two observations imply that consumption must *rise on impact* to someplace strictly between c^* and $\bar{c}$, like c_0 as shown in Figure 8 above. This figure depicts the case where c_0 is above the $\dot{c}' = 0$ locus. After the initial jump in consumption, consumption and the habit level rise until the path crosses the $\dot{c}' = 0$ locus, when consumption starts to fall and the habit level keeps rising. Once the path crosses the $\dot{s} = 0$ locus, both consumption and the habit level fall until at time T the trajectory lands continuously in the old saddle path. Afterwards, consumption and the habit level monotonically decrease towards their old steady-state levels. Figure 9 shows the case where c_0 is below the $\dot{c}' = 0$ locus. The dynamics of consumption and the habit level are similar to that of the previous case once the trajectory crossed the $\dot{c}' = 0$ locus.

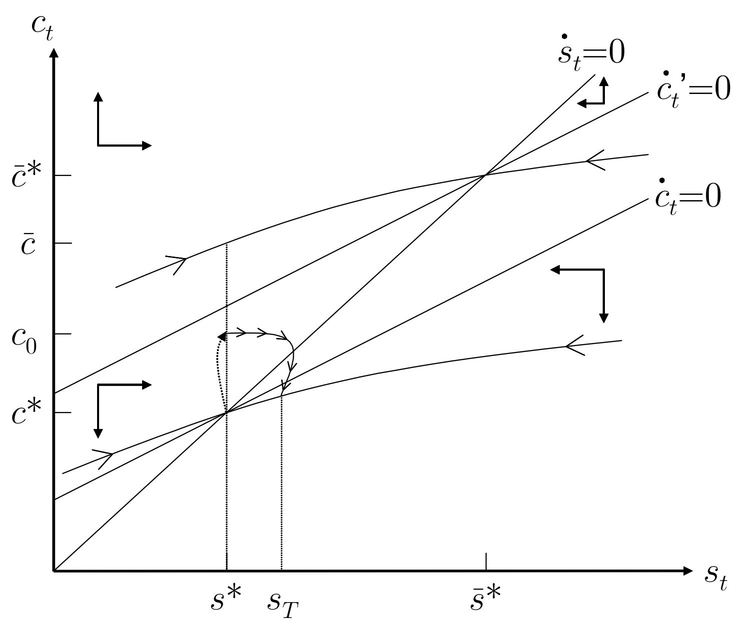

Figure 9. The impact of a temporary rise in the wage rate, w, if c_0 is below the $\dot{c}' = 0$ locus.

Notice that how much consumption falls on impact depends on how farther away in the future the wage rate will return to its previous level, T. In the limit as $T \to 0$ (i.e., the wage rate is constant), $c_0 \to c^*$, whereas as $T \to +\infty$ (i.e., the wage increase is permanent), $c_0 \to \bar{c}$.

The dynamics of consumption and the habit level for each case can be easily read off the applicable saddle path diagram. This is left as an exercise.