Sequential equilibria

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1 What *is* a sequential equilibrium?

In game theory, we are interested in finding equilibria. The leading equilibrium concept is the obvious Nash equilibrium which we covered extensively in class. However, we are often confronted with a multiplicity of equilibria in extensive form games. As a result, *refinements* of Nash equilibria are often employed. The "standard" refinement concept is due to Selten (1965) and is known as *subgame perfect* equilibrium. Informally, it is defined as follows.

Definition 1 (Subgame perfect equilibrium)**.** *A fully specified plan of actions or strategy π is subgame perfect if for every proper subgame the strategy π restricted to the subgame constitutes a Nash equilibrium for that subgame.*

However even this refinement is often *not* enough to get an *unique* equilibrium. In this note, we will look at an equilibrium notion that is even more strict than imposing subgame perfection, i.e. *sequential* equilibrium. To define this properly, we will unfortunately have to go through some terminology and notation.[1]

1.1 Background: notation and formal definitions

We will closely follow Kreps and Wilson (*Econometrica*, 1982) who came up with the idea of sequential equilibrium. The following features describe an *extensive form* game: (1) the physical order of play; (2) the choices available to a player whenever it is his turn to move; (3) the rules for determining whose move it is at any point; (4) the information a player has whenever it is his turn to move; (5) the payoffs to the players as functions of the moves they select; (6) the initial conditions that begin the game (that is, the actions of nature).

We will demonstrate our formal notation by way of a simple example. Consider the extensive form game defined as Γ in figure 1.

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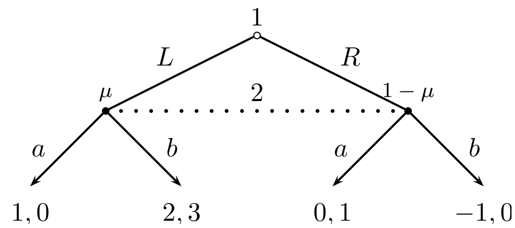
[1]The practical reader can skip this part and move straight to the informal definition at the end of section 1 or section 2.



The physical order of play is determined by the set of nodes T.[2] This set contains **7** points: one open circle, two closed circles and four arrows. The choices available to a player are characterized by the set of *decision* nodes X: the open and two closed circles. At the *open* circle, player 1 can choose L or R whereas at the two *closed* circles, player 2 can choose between a or b. Information sets $h \in H$ are characterized by these circles as well and the "non-trivial" information set is emphasized with a dashed line. Then, let $\iota(h)$ describe that player who moves at information set h.[3] Furthermore, payoffs are determined by the set of *terminal* nodes Z (i.e. the four arrows) and utility functions $(u_i)_{i \in N}$ which are summarized by the numbers at the bottom of the game tree.

The following definitions will prove to be useful.

Definition 2 (Behavioral strategy profile). *A behavioral strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a vector that specifies an information set contingent strategy for each player, i.e. it specifies a move probability $\sigma_i(d_i|h)$ for every possible move d_i at every possible information set h for every player $i \in N$ in the game.*

Definition 3 (Beliefs). *A system of beliefs μ is a function $\mu : X \to [0,1]$ such that $\sum_{x \in h} \mu(x) = 1$ for every information set $h \in H$, i.e. $\mu(x)$ is the probability assigned by that player i who reaches information set h to $x \in h$, conditional on h being reached.*

Definition 4 (Assessment). *An assessment is a pair (μ, π) consisting of a system of beliefs μ and a strategy π.*

Finally, denote Π^0 as the set of *strictly positive* strategies, i.e. the restricted set of actions where every player $i \in N$ randomizes at every decision node. Then, we denote Ψ^0 as the subset of assessments (μ, π) where $\pi \in \Pi^0$ and μ is consistent with Bayes' rule.

Definition 5 (Sequential rationality). *The assessment (μ, π) is sequentially rational if at all information sets $h \in H$:*

$$\mathbb{E}^{\mu,\pi}\left[u^{\iota(h)}(z)|h\right] \geq \mathbb{E}^{\mu,\overline{\pi}}\left[u^{\iota(h)}(z)|h\right]$$

for all strategies $\overline{\pi}$ such that $\overline{\pi}^j = \pi^j$ for $j \neq \iota(h)$.

[2] Formally, this is represented by T *and* a binary relationship $\succ$ that determines *precedence*, i.e. the direction the game moves from any node onwards. The binary relation $\succ$ must be a partial order, and $(T, \succ)$ must form an arborescence. In our example, this is obvious. We start from the top and proceed downwards.

[3] The function $\iota : H \to N$ is sometimes called the *player function*, see Osborne (2004) for example.

where $\mathbb{E}^{\mu,\pi}\big[u^{\iota(h)}(z)|h\big]$ denotes the expected utility at terminal node z to player $\iota(h)$ who adopts beliefs μ and plays strategy π conditional on arriving at information set h. Please keep in mind that saying a behavioral strategy is sequentially rational is meaningless without specifying the underlying beliefs!

Definition 6 (Consistency)**.** *An assessment* (μ,π) *is consistent if* $(\mu,\pi) = \lim_{n\to\infty}(\mu_n,\pi_n)$ *for some sequence* $\{(\mu_n,\pi_n)\}_{n=1}^{\infty} \subseteq \Psi^0$*.*

After all this work, we can finally define the concept of *sequential* equilibrium.

Definition 7 (Sequential equilibrium)**.** *A sequential equilibrium is an assessment* (μ,π) *in an extensive form game that is consistent and sequentially rational.*

1.2 Informal definition: intuition

Informally speaking, an assessment is a sequential equilibrium if its strategies are sensible given its beliefs *and* its beliefs are sensible given its strategies. With sensible, we mean sequential rationality and consistency.

Imposing sequential rationality on an assessment (μ,π) is loosely that a strategy π must maximize expected payoffs (based on beliefs μ) in every information set for every player $i \in N$. Furthermore, this strategy π is robust to *unilateral* deviations. Lastly, a consistent assessment roughly means that the strategy π is "robust to small errors" in beliefs around μ. This must hold *on-* and *off*-the-equilbrium path.

2 Cookbook recipe for finding sequential equilibria

Finding sequential equilibria in extensive form games can become quite an involved task. The important thing is to work *neatly* and *not* be in a rush. The following series of steps can help you avoid making unnecessary mistakes.[4]

1. Draw the game tree *large* in your blue book.[5]

2. Write down all the beliefs $\mu = (\mu_1,\mu_2,\ldots,\mu_b)$ and strategies $\pi = (\sigma_1,\sigma_2,\ldots,\sigma_N)$ next to the appropriate nodes in the game tree. Then, write down these beliefs and strategies $(\mu;\pi)$ under the game tree *again* so that you know what you need to determine *endogenously.*

3. Calculate the relationships between all the previously determined endogenous variables such that they are consistent with *Bayes' rule.*

[4] Obviously, I will claim nothing about whether this is the fastest and/or most efficient way.

[5] There is nothing more annoying than to squeeze a bunch of information on a small piece of paper: it is difficult for you to keep track of calculations and hard for graders to read.

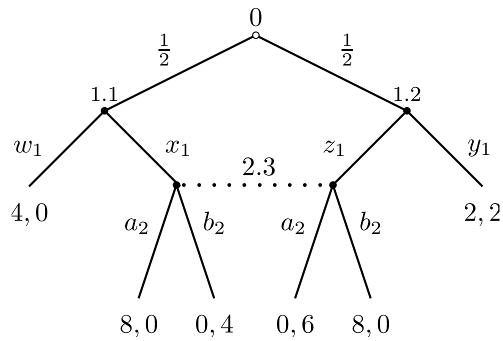
4. Starting at the bottom of the tree, calculate the conditions that induce utility maximizing choices.[6] Eliminate those actions that are inconsistent with those beliefs calculated in step **3**.

5. Determine the equilibrium with that information that was not eliminated in step **4**.

6. Write down the equilibrium (formally).

Example: following the cookbook recipe step-by-step

Let our extensive form game be summarized by figure 2.



Then, we have the following:

Claim 1. *The unique sequential equilibrium* σ^* *of the extensive form game* Γ *is given by:*

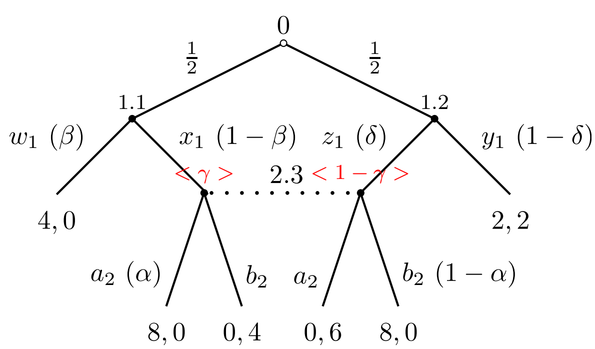
$$\sigma_1^* = \begin{cases} [x_1], & \text{if } h = 1.1, \\ \frac{1}{3}[y] + \frac{2}{3}[z_1], & \text{if } h = 1.2. \end{cases}$$

$$\sigma_2^* = \frac{3}{4}[a_2] + \frac{1}{4}[b_2], \quad \text{if } h = 2.3.$$

with beliefs $\gamma^* = \frac{3}{5}$.

Proof. Beliefs are denoted in red.

Step **1** and **2**.



[6]This should ring some bells: *backward induction*.

Our set of endogenous variables becomes: $(\gamma; \beta, \delta, \alpha)$.

Step **3**. In sequential games, we should interpret *Bayes' rule* in the following way. Let h be the event "having reached information set h" and denote ω as "being at node ω". Then, we simply have $\mathbb{P}(\omega|h) = \frac{\mathbb{P}(\omega\cap h)}{\mathbb{P}(h)}$. In our case, there is only one "non-trivial" information set. Reaching this set from the *left* side of the game happens with probability $\frac{1}{2}(1-\beta)$. Similarly, we derive that reaching h from the *right* side occurs with probability $\frac{1}{2}\delta$. Therefore, we obtain:

$$\gamma = \frac{\frac{1}{2}(1-\beta)}{\frac{1}{2}(1-\beta)+\frac{1}{2}\delta} = \frac{1-\beta}{1-\beta+\delta}$$

Step **4**. Before we started this question, we did not know what type of equilibrium to expect. Therefore, we have to verify every possibility at the bottom, i.e. player 2 can either play a *pure* or *mixed* strategy.

Suppose that $\sigma_2^* = [a_2]$ or $\alpha = 1$. Then, sequential rationality at information set 2.3 requires:

$$\begin{aligned} \mathbb{E}u_2(a_2|2.3) &\geq \mathbb{E}u_2(b_2|2.3) \\ 6(1-\gamma) &\geq 4\gamma \\ \gamma &\leq \tfrac{3}{5} \end{aligned} \tag{1}$$

Given that player 2 always choose $[a_2]$, it is straightforward to show that player 1.1 and 1.2 always choose $[x_1]$ and $[y_1]$ respectively. Equivalently, we could state that $\beta = \delta = 0$ then must be the case. However by equation (1), we get $\gamma = 1$, but we required $\gamma \leq \frac{3}{5}$. **Contradiction**.

Possible strategies for player 1

β	δ	$\gamma(\beta,\delta)$	Feasible?
0	0	1	$\times$
0	$\in(0,1)$	$\frac{1}{1+\delta}$	$\checkmark$
0	1	$\frac{1}{2}$	$\times$
$\in(0,1)$	0	1	$\times$
$\in(0,1)$	$\in(0,1)$	$\frac{1-\beta}{1-\beta+\delta}$	$\times$
$\in(0,1)$	1	$\frac{1-\beta}{2-\beta}$	–
1	0	Indeterminate	–
1	$\in(0,1)$	0	$\times$
1	1	0	$\times$

Suppose that $\sigma_2^* = [b_2]$ or $\alpha = 0$. Then by similar reasoning as above, we require:

$$\gamma \geq \frac{3}{5} \tag{2}$$

However, this induces $\beta = \delta = 1$. By equation (2), we get $\gamma = 0$. **Contradiction**.

Therefore in *any* sequential equilibrium, it must be the case that player 2 randomizes, i.e. $\alpha \in (0,1)$. By indifference of player 2, we can pin down γ as we then get $\gamma = \frac{3}{5}$. Recall that we also insisted that γ is derived through *Bayes' rule* as stated in step **3**. Combining these facts, we get:

$$\tfrac{3}{5} = \frac{1-\beta}{1-\beta+\delta}$$

The table above summarizes the possibilities for β, δ and whether these values can be consistent with $\gamma = \frac{3}{5}$ or not. We have *three* potential candidates. Let us start with $\beta = 1$ and $\delta = 0$. Note that γ is indeterminate but as long as we can construct a sequence of strategies that converge to $\gamma = \frac{3}{5}$ we are fine. In the following, we will observe that this is actually not necessary. If $\beta = 1$ and $\delta = 0$, then $[w_1]$ and $[y_1]$ must be optimal. This occurs if and only if:

$$\begin{aligned} &4 \geq 8\alpha \\ &2 \geq 8(1-\alpha) \end{aligned}$$

which can be reduced to $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{3}{4}$. **Contradiction**.

Let us check $\beta \in (0,1)$ and $\delta = 1$. We require the following conditions:

$$\begin{aligned} \mathbb{E}u_1(w_1|1.1) \geq \mathbb{E}u_1(x_1|1.1) \\ \mathbb{E}u_1(z_1|1.2) \geq \mathbb{E}u_1(y_1|1.2) \end{aligned}$$

These conditions can be reduced to $\alpha = \frac{1}{2}$ and $\alpha \leq \frac{3}{4}$. Even though these are not contradictory, we also require:

$$\tfrac{3}{5} = \tfrac{1-\beta}{2-\beta}$$

which happens if and only if $\beta = -\frac{1}{2}$. **Contradiction**.

Step **5**. We have only one more possibility left: $\beta = 0$ and $\delta \in (0,1)$. These conditions imply:

$$\begin{aligned} \mathbb{E}u_1(x_1|1.1) \geq \mathbb{E}u_1(w_1|1.1) \\ \mathbb{E}u_1(z_1|1.2) = \mathbb{E}u_1(y_1|1.2) \end{aligned}$$

which can be reduced to $8\alpha \geq 4$ and $2 = 8(1-\alpha)$. The intersection of these two statements are consistent and give us:

$$\alpha = \tfrac{3}{4}$$

We retrieve δ from the fact that $\gamma = \frac{3}{5}$. We get:

$$\tfrac{3}{5} = \tfrac{1}{1+\delta}$$

which obviously implies that $\delta = \frac{2}{3}$.

Step **6**. If the game tree with *all* the actions and beliefs are drawn well enough, it is sufficient to state:

$$(\gamma^*; \beta^*, \delta^*, \alpha^*) = (\tfrac{3}{5}; 0, \tfrac{2}{3}, \tfrac{3}{4})$$

This will give you *full* credit. Formally, the equilibrium is given by:

$$\sigma_1^* = \begin{cases} [x_1], & \text{if } h = 1.1, \\ \frac{1}{3}[y_1] + \frac{2}{3}[z_1], & \text{if } h = 1.2. \end{cases}$$

$$\sigma_2^* = \tfrac{3}{4}[a_2] + \tfrac{1}{4}[b_2], \quad \text{if } h = 2.3.$$

with beliefs $\gamma^* = \frac{3}{5}$. Therefore, the equilibrium is *unique*. This is exactly what we wanted to show. $\square$

Please note that this example went through basically *all* possibilities which is a bit cumbersome. In an exam however, you will usually get a hint in the form of "find a sequential equilibrium where player 1 randomizes" or "find a sequential equilibrium where player 2 randomizes". This information allows you to eliminate a lot of strategies straight away and makes matters much easier.