Theory of Income I
Fall 2018
Solution to the Final

December 20, 2018

Agustin Gutierrez and Takuma Habu

1 True or False (18 points)

Answer True or False. No explanation is required.

......

(a) If the social planner's problem for an Arrow-Debreu economy has a unique solution, then the competitive equilibrium—for given endowments and ownership of firms—has a unique equilibrium. (3 points)

True. Recall the one-to-one relationship we found in class between solving the planner's problem and solving for the competitive equilibrium directly. (Note: relative price is unique).

(b) An economy where all agents have identical utility function has a representative agent in the strong sense of Gorman aggregation (i.e. redistribution of income or endowments across households does not affect the demand). (3 points)

False. Remember our TA session on aggregation. With only identical utility, the income effect can differ depending on the level of income when we redistribute.

(c) If markets are incomplete, the competitive allocation cannot be Pareto optimal. (3 points)

False. Initial endowments could be Pareto optimal.

(d) Take an economy with two agents with utility functions $u^i : X^i \to \mathbb{R}_+$. Suppose you maximise $u^1\left(x^1\right) + \log\left[u^2\left(x^2\right)\right]$ subject to x^1 and x^2 being feasible (assume maximum exists). Then, the resulting allocation must be Pareto optimal. (3 points)

True. Think of the objective as a social welfare function. Any increasing transformation of the utility function does not alter the Pareto optimal allocation. To see this explicitly, write the planner's problem as

$$\max_{x^1, x^2} \; u^1\left(x^1\right) + \log\left[u^2\left(x^2\right)\right] \;\; s.t. \; x^1 + x^2 = e,$$

where e is aggregate endowment vector. Let γ_ℓ denote the Lagrange multiple for each good. The first-order condition with respect to x_ℓ^i is

$$\frac{\partial u^1\left(x^1\right)}{\partial x_\ell^1} = \gamma_\ell,$$
$$\frac{1}{u^2\left(x^2\right)} \frac{\partial u^2\left(x^2\right)}{\partial x_\ell^2} = \gamma_\ell.$$

Marginal rate of substitution between any two goods for each individual is thus

$$\frac{\partial u^i\left(x^i\right)/\partial x_\ell^i}{\partial u^i\left(x^i\right)/\partial x_k^i} = \frac{\gamma_\ell}{\gamma_k}.$$

Note, in particular, that the $\left(u^2\left(x^2\right)\right)^{-1}$ term cancels out. Since the right-hand side is independent of i, we realise that any solution to the problem will be Pareto optimal.

(e) Consider a pure-endowment OLG model where agents live for two periods and use expected utility. Agents are indexed by the period in which they are born, $t = 0, 1, \ldots, \infty$. Agents born in period t have an endowment e_t^t when young and e_{t+1}^t when old. Normalize the size of each cohort to 1. The economy starts in period $t = 1$ and goes on forever. The aggregate endowment at time t is $\overline{e}_t$. The endowment of each young and old are proportional to aggregate endowment so that $e_t^t = (1 - \alpha)\,\overline{e}_t$ and $e_{t+1}^t = \alpha\overline{e}_{t+1}$. Assume that $\overline{e}_t$ is a random variable, so this economy has aggregate uncertainty. In particular, $\overline{e}_{t+1} = \overline{e}_t\epsilon_{t+1}$, where $\{\epsilon_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. random variables, so that the aggregate endowment follows a random walk. While it is not needed to answer this question, you can assume any (non-degenerate) distribution for ϵ_t; e.g. it takes two values, it is log-normal etc. The economy starts after $\overline{e}_1$ is realized. Agents born in period $t \geq 1$ use expected utility with a separable utility function; i.e. their utility is

$$u\,(c_t) + \beta\mathbb{E}_t\left[u\left(c_{t+1}^t\right)\right],$$

where $\beta > 0$, the expectation is taken with the knowledge of all the information as of period t, and

$$u\,(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

The initial old simply has utility function $u\left(c_1^0\right)$. Assume that there are complete markets; i.e. agents alive at time t can trade in a complete set of Arrow securities which pay at date $t+1$ indexed to each realization of $\overline{e}_{t+1}$.

(i) The equilibrium is autarky (i.e. $c_t^t = e_t^t$ and $c_{t+1}^t = e_{t+1}^t$ for all $t \geq 1$ and $c_1^0 = \overline{e}_1 - e_1^1$). (3 points)

True. The initial old consumes his endowment. By the feasibility constraint, the young in period 1 must consume his endowment. Then, the budget constraint again would imply the young in period 2 must consume his endowment. And so on.

(ii) Suppose that the expected growth rate of the economy is 2% per period; i.e. $\mathbb{E}_t\left[\epsilon_{t+1}\right] = 1.02$ for all $t \geq 1$. Assume also that the one period (net) interest rate is constant, say equal to 1%; i.e. the price in period t of consumption good at $t+1$ delivered in all states of nature is equal to $1/1.01$. Furthermore, assume that, for all t, the price at time t of a security that gives $\overline{e}_{t+1}/\overline{e}_t$ units of consumption when aggregate endowment is $\overline{e}_{t+1}$ is $1/1.05$ so that the multiplicative excess return of the aggregate (one period) portfolio is 4%. Given these assumptions, the allocation is Pareto optimal. (3 points)

True. Recall PS3, Q5, Uncertain in an OLD model, Part 11, which told us that the economy is efficient if the expected return of an asset consisting of aggregate consumption is higher than the expected growth rate of consumption. Although the set up is different, the intuition carries over to this question. Here, the excess return (of asset with return based on aggregate consumption relative to the risk-free rate) is 4% while the expected growth rate is 2%. So the allocation is Pareto optimal.

2 Euler Equation and slope of optimal rule (12 points)

Consider the problem of maximizing

$$\sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right)$$

subject to the initial condition x_0 by a choice of sequence $\{x_{t+1}\}_{t=0}^{\infty}$, where $x_t \in \mathbb{R}$ and $\Gamma(x) = \mathbb{R}_+$. Assume that F is strictly concave, strictly increasing in its first argument and strictly decreasing in the second. Furthermore, assume that F is twice differentiable. Also assume the following Inada condition to ensure that the solution is interior, $F_x(0, y) = \infty$, where F_x is the partial derivative of F with respect to the first argument (and F_y is the partial derivative of F with respect to the second argument).

(a) Write the Euler Equation for this problem. (3 points)

$F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_{t+1}, x_{t+2}\right) = 0, \ \forall t \geq 0.$

(b) Write the transversality condition. (3 points)

$\lim_{T\to\infty} \beta^T F_x\left(x_T, x_{T+1}\right) x_T = 0.$

(c) Write the equation that the steady state has to satisfy. Denote the steady-state value of x_t as $\overline{x}$. (3 points)

In the steady state $x_t = x_{t+1} = x_{t+2} = \overline{x}$ and so the Euler equation becomes

$$F_y\left(\overline{x}, \overline{x}\right) + \beta F_x\left(\overline{x}, \overline{x}\right) = 0.$$

(d) Let g be the optimal decision rule, i.e. $x_{t+1} = g(x_t)$. Write the equation that $g'(\overline{x})$ has to satisfy in terms of the discount factor and the relevant second derivatives. [Hint: It is a quadratic equation.] (3 points)

Write the Euler equation using the optimal decision rule:

$$F_y\left(x_t, g\left(x_t\right)\right) + \beta F_x\left(g\left(x_t\right), g\left(g\left(x_t\right)\right)\right) = 0, \ \forall t \geq 0.$$

Differentiating with respect to x_t yields

$$\begin{aligned} 0 = {} & F_{yx}\left(x_t, g\left(x_t\right)\right) + F_{yy}\left(x_t, g\left(x_t\right)\right) g'\left(x_t\right) \\ & + \beta \left[F_{xx}\left(g\left(x_t\right), g\left(g\left(x_t\right)\right)\right) g'\left(x_t\right) + F_{xy}\left(g\left(x_t\right), g\left(g\left(x_t\right)\right)\right) \left(g'\left(x_t\right)\right)^2\right]. \end{aligned}$$

Evaluating above at $x_t = \overline{x}$, and using the fact that $\overline{x} = g(\overline{x}) = g(g(\overline{x}))$, we obtain

$$0 = \beta F_{xy}\left(g'\left(\overline{x}\right)\right)^2 + \left(F_{yy} + \beta F_{xx}\right) g'\left(\overline{x}\right) + F_{yx},$$

which is quadratic in $g'(\overline{x})$.

3 Neoclassical Growth Model with lump-sum taxes (25 points)

Assume that all agents have utility

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{t} v\left(c_t,\ell_t\right)$$

over consumption and leisure sequences. Feasibility is given by

$$i_t + c_t + g_t = F\left(k_t, n_t\right),$$

where g_t stands for government purchases, and the law of motion for capital is given by

$$k_{t+1} = i_t + (1-\delta)\, k_t,\ \forall t \geq 0.$$

Labour is $n_t = 1 - \ell_t$. We will write down the competitive equilibrium with lump-sum taxes.

Assume that v is increasing and concave, and assume that F is increasing, strictly quasiconcave and homogenous of degree one.

We will assume that households own and accumulate capital, which they then rent to the firms. We will also assume that government purchases g_t are financed with lump-sum taxes.

Use w_t for wages and υ_t for the rental rate of capital at time t, both in units of time-t consumption. Use p_t for the price of consumption (and investment) at time t in units of consumption at time 0. Use r_t for the net interest rate; i.e. the net return in terms of consumption of investing one unit between time t and $t+1$. Use τ_t for lump-sum taxes charged in period t in units of time-t consumption.

(a) Write down the Arrow-Debreu present-value budget constraint for the household. Make sure that k_t, i_t, ℓ_t and c_t for different t's are shown in your expression. Write it using prices p_t, w_t and υ_t. (2 points)

In each period, the household spends its income on consumption, taxed at τ_t, leisure and investment. Its income, on the other hand, is the wages it gets by supply its endowment of time (of one) and the rental income. Summing across all periods, we obtain

$$\sum_{t=0}^{\infty} p_t\left(c_t + w_t\ell_t + i_t\right) = \sum_{t=0}^{\infty} p_t\left(w_t + \upsilon_t k_t - \tau_t\right).$$

(b) Write down the problem for the firm using prices p_t, w_t and υ_t. (2 points)

Firms maximise its profits in each period t. Hence, its problem is

$$\max_{n_t,k_t}\ p_t\left[F\left(k_t,n_t\right) - w_t n_t - \upsilon_t k_t\right],\ \forall t \geq 0.$$

Equivalently, we may also write

$$\max_{\{n_t,k_t\}_{t=0}^{\infty}}\ \sum_{t=0}^{\infty} p_t\left[F\left(k_t,n_t\right) - w_t n_t - \upsilon_t k_t\right].$$

Note that the question tells you that w_t and υ_t are in units of time-t consumption. So we cannot write, for example, $p_t F(k_t, n_t) - w_t n_t - \upsilon_t k_t$!]

(c) Write down the Arrow-Debreu present-value budget constraint for the government. (2 points)

The government must finance its purchases using its tax revenue.

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \tau_t.$$

(d) Write down the relationship between the Arrow-Debreu prices p_t and p_{t+1} and the net interest rate r_t. Also write down the relationship between r_t, υ_{t+1} and δ that must hold so that the household is indifferent among interior allocations of investment. (3 points)

As always,

$$\frac{p_{t+1}}{p_t} = \frac{1}{1+r_t}.$$

When the household invests, it foregoes one unit of consumption today but gains $1 - \delta + \upsilon_{t+1}$ units of consumption tomorrow. The household must be indifferent between the two in equilibrium so that

$$p_t = p_{t+1}(1 - \delta + \upsilon_{t+1}).$$

Combining the two expressions, we obtain that

$$\begin{aligned} 1 + r_t &= 1 - \delta + \upsilon_{t+1} \\ \Leftrightarrow \upsilon_{t+1} &= r_t + \delta. \end{aligned}$$

(e) Is the equilibrium Pareto optimal? Maximum two lines of explanation. (2 points)

Since the tax is lump sum, the allocation is Pareto optimal (observe that there are no 'wedges' in the Euler equation or in the intratemporal conditions). [Some of you mentioned that you can improve everyone by setting $g = 0$—I'm afraid that wasn't the point of the question!]

(f) Suppose that $g_t = g$ for all $t \geq 0$. Write down 8 equations in 8 unknowns characterizing the steady state of this economy. [Hint: the unknowns are: c, ℓ, n, i, k, w, r and υ.] (8 points)

Firm problem *yields the following two conditions:*

$$\begin{aligned} F_k(k_t, n_t) &= \upsilon_t, \\ F_n(k_t, n_t) &= w_t. \end{aligned}$$

Household problem *Before writing the problem of the household, let us rewrite its budget constraint using the law of motion for capital:*

$$\sum_{t=0}^{\infty} p_t [c_t + \ell_t + k_{t+1} - (1-\delta) k_t] = \sum_{t=0}^{\infty} p_t (w_t + \upsilon_t k_t - \tau_t)$$

The household's problem is then

$$\max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t v\left(c_t,\ell_t\right)$$
$$s.t. \quad \sum_{t=0}^{\infty} p_t\left[c_t+w_t\ell_t+k_{t+1}-(1-\delta)\,k_t\right]=\sum_{t=0}^{\infty} p_t\left(w_t+\upsilon_t k_t-\tau_t\right)$$
$$k_0 \text{ given.}$$

Letting λ be the Lagrangian, the first-order conditions are

$$\begin{aligned}
\{c_t\}: &\quad v_c\left(c_t,\ell_t\right)=\lambda p_t\\
\{\ell_t\}: &\quad v_\ell\left(c_t,\ell_t\right)=\lambda p_t w_t,\\
\{k_{t+1}\}: &\quad p_t=p_{t+1}\left(1-\delta+\upsilon_{t+1}\right).
\end{aligned}$$

The Euler equation is

$$\begin{aligned}
\frac{1}{1+\rho}\frac{v_c\left(c_{t+1},\ell_{t+1}\right)}{v_c\left(c_t,\ell_t\right)} &= \frac{p_{t+1}}{p_t}\\
\Leftrightarrow \frac{v_c\left(c_{t+1},\ell_{t+1}\right)}{v_c\left(c_t,\ell_t\right)}\left(1+r_t\right) &= 1.
\end{aligned}$$

The intratemporal condition is

$$\frac{v_\ell\left(c_t,\ell_t\right)}{v_c\left(c_t,\ell_t\right)}=w_t.$$

Finally, we also have

$$\upsilon_{t+1}=r_t+\delta.$$

Market clearing

$$i_t+c_t+g_t=F\left(k_t,n_t\right).$$

Steady state *Let variables without subscript t denote their steady-state values. We are given that $g_t=g$ for all $t\geq 0$.*

From the firm's problem, we obtain

$$\begin{aligned}
F_k\left(k,n\right) &= \upsilon,\\
F_n\left(k,n\right) &= w.
\end{aligned}$$

The Euler condition gives

$$r=\rho.$$

and the intratemporal condition gives

$$\frac{v_\ell(c,\ell)}{v_c(c,\ell)} = w$$

and we also have

$$\upsilon = r + \delta = \rho + \delta.$$

Law of motion for capital gives

$$i = \delta k.$$

We also have

$$n = 1 - \ell.$$

Finally, market clearing gives

$$i + c + g = F(k,n).$$

(g) Assume that utility function is given by $v(c,\ell) = U(u(c) + A\ell)$, where U and u are both strictly concave and strictly increasing. Do the steady-state values of consumption and leisure strictly increase, strictly decrease or stay constant as g increases? (3 points)

Recall that c is a normal good if

$$\frac{\partial}{\partial \ell}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) < 0$$

while ℓ is a normal good if

$$\frac{\partial}{\partial c}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) > 0.$$

Here, marginal rate of substitution is given by

$$\frac{v_\ell(c,\ell)}{v_c(c,\ell)} = \frac{A}{u'(c)}$$

so that

$$\begin{aligned}\frac{\partial}{\partial \ell}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) &= 0, \\ \frac{\partial}{\partial c}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) &= -Au''(c)\left(\frac{1}{u'(c)}\right)^2 > 0.\end{aligned}$$

Hence, ℓ is a normal good while c does not change with wealth.

Since $g_t = g$ for all $t \geq 0$, we must have $\tau_t = \tau = g$ for all $t \geq 0$. The household's budget constraint in the steady state is

$$\begin{aligned}\sum_{t=0}^{\infty} p_t\left[c_t + w_t\ell_t + k_{t+1} - (1-\delta)k_t\right] &= \sum_{t=0}^{\infty} p_t\left(w_t + \upsilon_t k_t - \tau_t\right) \\ \Rightarrow c + w\ell + k - (1-\delta)k &= w + \upsilon k - \tau \\ \Leftrightarrow c + w\ell + \delta k &= w + (\rho+\delta)k - g \\ \Leftrightarrow c + w\ell &= w + \rho k - g.\end{aligned}$$

Thus, we can see that an increase in g is equivalent to a reduction in wealth. It follows that steady-state leisure is decreasing in g while steady-state consumption is constant with respect to changes in g.

Put differently, the utility function is quasilinear in leisure so that there is no income effect for consumption. Quasilinearity means that the MRS (i.e. the slope of the indifference curve) is the same for a given level of c for any level of ℓ. So when income shifts (which is a parallel movement of the budget constraint), the variable that the utility function is quasilinear with respect to does not change.

(h) Assume that utility function is given by $v(c,\ell) = U(c + H(\ell))$, where U and $H(\ell)$ are both strictly concave and strictly increasing. Do the steady-state values of consumption and leisure strictly increase, strictly decrease or stay constant as g increases? (3 points)

The marginal rate of substitution is given by

$$\frac{v_\ell(c,\ell)}{v_c(c,\ell)} = H'(\ell)$$

so that

$$\begin{aligned}\frac{\partial}{\partial \ell}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) &= H''(\ell) < 0,\\ \frac{\partial}{\partial c}\left(\frac{v_\ell(c,\ell)}{v_c(c,\ell)}\right) &= 0.\end{aligned}$$

Hence, c is a normal good while ℓ does not change with wealth. It follows that steady-state consumption is decreasing in g while steady-steady leisure is constant with respect to changes in g.

Put differently, the utility function is now quasilinear in consumption so that there is no income effect for leisure.

4 Extraction of natural resources

Let x be the stock of a non-renewable natural resource. Let $a \geq 0$ denote the rate at which the resource is extracted per unit of time. Then,

$$\dot{x}(t) = -a(t), \ \forall t \geq 0.$$

Let $u(a(t))$ be the utility from consuming an outside good produced using the extracted natural resource in period t.

Assume that u is strictly increasing, strictly concave and satisfies the Inada conditions (in particular, $\lim_{a \to 0} u'(a) = \infty$). Let $h = u'^{-1}(\cdot)$ denote the inverse function of u'. The planner's problem is

$$\begin{aligned} \max_{\{a(t)\}_{t \in [0,\infty)}} \quad & \int_0^\infty u(a(t)) e^{-\rho t} dt \\ s.t. \quad & \dot{x}(t) = -a(t), \ \forall t \geq 0, \\ & x(0) > 0 \text{ given.} \end{aligned}$$

(a) Write the Hamiltonian for the planner's problem.

The state variable is x while the control is a. Letting λ denote the co-state as usual, we have

$$H(x, a, \lambda) = u(a) - \lambda a.$$

(b) Write the first-order conditions of the Hamiltonian with respect to the control and the law of motion for the co-state.

The first-order condition with respect to the control is

$$H_a(x, a, \lambda) = 0 : \quad u'(a) - \lambda = 0,$$

while the first-order condition with respect to the law of motion of the co-state is:

$$\dot{\lambda} = \rho\lambda - H_x(x, a, \lambda) : \quad \dot{\lambda} = \rho\lambda.$$

(c) *Write the appropriate transversality condition for this problem.*

$\lim_{t \to \infty} e^{-\rho t} \lambda(t) x(t) = 0.$

(d) Given an economic interpretation of $\lambda(t)$.

The co-state is that $\lambda(t)$ is the marginal value (in period t) of an infinitesimal extraction of the natural resource.

(e) Does the solution to this problem have a steady state? Maximum one line of explanation.

No, since the natural resource will be depleted in finite time.

(f) Use the previous expression to write an expression for $\lambda(t)$ as a function of $\lambda(0)$.

We have

$$\begin{aligned} \dot{\lambda}(t) &= \rho\lambda(t) \\ \Leftrightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{d\ln\lambda(t)}{dt} &= \rho. \end{aligned}$$

Integrating both sides with respect to t yields

$$\begin{aligned} &\ln\lambda(t) = \rho t + c \\ &\Leftrightarrow \lambda(t) = \exp[c]\exp[\rho t], \end{aligned}$$

where c is a constant of integration. We know that, at $t = 0$, the co-state equals $\lambda(0)$ and so we must have

$$\lambda(t) = \lambda(0)\, e^{\rho t}.$$

(g) Is there a steady-state value for the co-state? Maximum one line of explanation.

No, because $\rho > 0$.

(h) What is the limit of $x(t)$ as $t \to \infty$?

Substituting the expression from Part (f) into the transversality condition yields

$$0 = \lim_{t\to\infty} e^{-\rho t}\lambda(0)\, e^{\rho t} x(t) = \lim_{t\to\infty} \lambda(0)\, x(t).$$

Since $\lambda(0)$ is a constant, above implies that $x(t) \to 0$ as $t \to \infty$.

(i) Given your answer to the previous two parts, use the law of motion for the stock of the natural resource to obtain a (very simple) integral equation relating $x(0)$ with the path of the optimal extraction $a(t)$ from $t = 0$ to ∞.

Integrating the law of motion from 0 to ∞ yields

$$\begin{aligned} \int_0^\infty \dot{x}(t)\, dt &= \int_0^\infty -a(t)\, dt \\ \Leftrightarrow x(\infty) - x(0) &= \int_0^\infty -a(t)\, dt \\ \Leftrightarrow x(0) &= \int_0^\infty a(t)\, dt, \end{aligned}$$

where we used the fact that $x(\infty) = \lim_{t\to\infty} x(t) = 0$ from Part (h).

(j) Use the previous expression for the first-order condition of the Hamiltonian to write a as a function of λ using h.

$u'(a) = \lambda \Leftrightarrow a = u'^{-1}(\lambda) = h(\lambda)$.

(k) Write an expression for $a(t)$ in the optimal path as a function of $\lambda(0)$, the parameter ρ, and the function h. You must combine at least two of your previous answers.

Combining the expression from Parts (f) and (j) yields

$$a(t) = h\left(\lambda(0)\, e^{\rho t}\right).$$

(l) Use the integral expression linking $x(0)$ with the path of $a(t)$ as well as the expression for each $a(t)$ in terms of $\lambda(0)$ to obtain an expression for $x(0)$ as a function of the initial value of co-state $\lambda(0)$. Use this expression to implicitly define $\Lambda(x) = \lambda(0)$ when $x = x(0)$ on the optimal path.

Combining expressions from Parts (i) and (k) yields

$$x(0) = \int_0^\infty h\left(\lambda(0)\, e^{\rho t}\right) dt.$$

and so when $x = x(0)$ on the optimal path,

$$x = \int_0^\infty h\left(\Lambda(x)\, e^{\rho t}\right) dt.$$

(m) Find an expression for the derivative $h'(\lambda)$. What is its sign?

$$h'(\lambda) = \frac{du'^{-1}(\lambda)}{d\lambda} = \frac{1}{u''\left(u'^{-1}(a)\right)} = \frac{1}{u''\left(h(\lambda)\right)} < 0$$

since u is strictly concave.

(n) Find an expression for the derivative of the co-state with respect to x on the optimal path, i.e. $\Lambda'(x)$. What is its sign?

Differentiating the expression from Part (l) with respect to x yields

$$\begin{aligned} 1 &= \int_0^\infty \Lambda'(x)\, h'\left(\Lambda(x)\, e^{\rho t}\right) e^{\rho t} dt \\ &= \Lambda'(x) \int_0^\infty h'\left(\Lambda(x)\, e^{\rho t}\right) e^{\rho t} dt \\ \Leftrightarrow \Lambda'(x) &= \left(\int_0^\infty \frac{e^{\rho t}}{u''\left(\Lambda(x)\, e^{\rho t}\right)} dt\right)^{-1}. \end{aligned}$$

Since u is strictly concave, $\Lambda'(x) < 0$.

(o) Give an economic interpretation to the sign of $\Lambda'(x)$ found in the previous part.

Marginal value from using an infinitesimal amount of resource from a level of stock of natural resource is declining as the level of stock declines. In other words, this means that the value function is concave.