Price Theory, III
Econ 30300

Professor Lars Stole
Price Theory, III, Spring 2019

Final Examination, Econ 30300:

(Thursday, June 13, 2019)

MATERIALS: You are allowed to use two sheets of notes (A4 or US letter), both sides, when taking this exam. No other materials are allowed.

INSTRUCTIONS: Read all questions carefully. Be concise – irrelevant remarks will not count for anything. Be careful to budget your time between various parts, taking into account the points allocated to each question (the number in parentheses); these points vary depending upon the question.

There are 4 questions with a total of 120 points possible. Some questions are worth more than others. You have 120 minutes for the examination, so allocate about one minute per point. Allocate your time wisely.

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Question 1 (30 points) Consider a restaurant selling to a diner with unknown demand. The consumer desires at most a single meal, and has a payoff of $u(q,\theta) - p$ for a meal of quality $q \geq 0$ and price p, where

$$u(q,\theta) \equiv \theta q.$$

The consumer's type is privately known to the consumer, but distributed uniformly on $[\underline{\theta}, \overline{\theta}]$.

The restaurant's costs are strictly increasing and convex in quality. In addition, consumers with higher θ are also more demanding to serve and increase the restaurant's costs. Let costs for serving quality q to consumer with type θ be

$$C(q,\theta) \equiv c(\theta)q + \frac{1}{2}q^2,$$

where $c(\theta)$ is a strictly increasing function.

(a). What is the full-information, efficient allocation, $q^{fb}(\theta)$?

Assume in parts (b)–(e) that that **the allocation in (a) is strictly increasing in** θ. The firm wishes to maximize profits by offering a menu (literally) of prices and qualities, indexed by the different types of consumer: $\{p(\cdot), q(\cdot)\}_{\theta \in [\underline{\theta}, \overline{\theta}]}$. Define the type-$\theta$ consumer's indirect utility when choosing the menu item designed for θ as

$$U(\theta) \equiv \theta q(\theta) - p(\theta).$$

(b). Provide the standard representation of incentive-compatibility for $\{p(\cdot), q(\cdot)\}_{\theta \in [\underline{\theta}, \overline{\theta}]}$ in terms of a condition for $q(\cdot)$ and an integral condition for the consumer's indirect utility, $U(\cdot)$.

(c). Compute an expression for $E[U(\theta)]$ and use your result to reduce the restaurant's objective function to an object that depends only on the choice variables $q(\cdot)$ and $U(\underline{\theta})$.

(d). Characterize the optimal allocation $q(\cdot)$.

(e). Does the price schedule which implements this allocation exhibit quality discounts or quality premia over the qualities that are implemented? [Hint: Don't solve for $P(q)$, but instead consider the nature of the consumer's utility function.]

(f). Now suppose that $c'(\theta) > 2$ which implies, among other things, that the allocation in (a) is strictly decreasing in θ. Also suppose that $\overline{\theta} \geq c(\overline{\theta})$ so that it is efficient for type $\overline{\theta}$ to consume a meal in the full-information setting. How does this change your analysis in (b)–(d). Be specific. Then explain briefly (without mathematical proof) how the optimal $q(\cdot)$ will depend upon θ.

Question 2 (20 points) There are n firms involved in an R&D race in which a single firm will win. The winning firm i will obtain a payoff of $\pi_i > 0$, which is independently and identically distributed across the firms according to a continuous cumulative distribution $F(\cdot)$ on $[0, 1]$. π_i is private information to firm i, although F is commonly known by all firms. The firms simultaneously choose their R&D expenditures, r_i. The firm with the highest expenditure wins the race.

Assume that there is a symmetric equilibrium in which each firm chooses expenditures according to $\overline{r} : [0, 1] \to \Re_+$ and $\overline{r}$ is strictly increasing in π_i.

Derive the equilibrium R&D expenditure function. (You may assume that $F(\cdot)$ has a pdf $f(\cdot)$, but that is not necessary.)

Question 3 (30 points) Consider a two-person, public goods setting. There is a single public good that can be implemented, $x \in \{0, 1\}$; let $x = 1$ represent that the public good is chosen. Each person i values the public good at θ_i, which is uniformly (and independently) distributed on $[0, 1]$. If the public good is chosen, however, each person must contribute personal labor to build the public good equal to a cost of $\frac{1}{2}$. Thus, player i's value of the public good is $\theta_1 - \frac{1}{2}$ and player 2's value of the public good is $\theta_2 - \frac{1}{2}$.

(a). Characterize an ex post efficient public good allocation rule $\hat{x}(\cdot)$.

(b). Design a dominant-strategy incentive compatible (DSIC), direct-revelation mechanism, $\{\hat{x}, t_1, t_2\}$, that implements the ex post efficient allocation in (a) and has the property that each player pays a transfer equal to their externality on the other player whenever they are pivotal in the public good decision. Verify that your mechanism is DSIC for each player i.

(c). Design a Bayesian-incentive compatible (BIC), direct-revelation mechanism, $\{\hat{x}, t_1, t_2\}$, that implements the ex post efficient allocation in (a) and has the properties that each player pays a transfer equal to their *expected externality* on the other player and the total transfers are (ex post) budget balanced (i.e., $t_1(\theta_1, \theta_2) = -t_2(\theta_1, \theta_2)$ for all θ_1, θ_2.) Verify that this mechanism is budget balanced and is BIC for each player i.

(d). Suppose that the players have the ability to refuse to play in the mechanism. Importantly, if player i refuses, player i cannot be forced to contribute to the public good and thus $x = 0$. What are the smallest fixed payments that need to be added to the transfers of the DSIC mechanism you found in (b) to make that mechanism individually rational?

(e). Using you answer in (d), is it possible to implement the the ex post efficient allocation using an ex post budget-balanced (BB), Bayesian-incentive compatible (BIC) and individually rational (IR) mechanism? You may use any result from class other than Myerson-Satterthwaite (1983) to establish your result.

Question 4 (40 points) A seller has up to Q identical units for sale in an n-person auction. Each bidder i has a private value of $\theta_i \in [0,1]$ for one unit of the good (and no value for any additional units). θ_i is independently distributed on $[0,1]$ according to the cumulative distribution function $F(\cdot)$. We assume the distribution satisfies the monotone-hazard-rate condition. The seller's cost of each of the Q units is 0; the seller cannot sell more than Q units in total.

(a). Assume that $Q = 2$ and $n > 2$. Suppose that the seller uses a third-price auction: the two highest bidders win a unit and each pays the third-highest bid. Show that there is a dominant-strategy equilibrium in which each player i bids θ_i.

(b). Assume that $Q = 1$ and $n > 2$. Suppose as in (a) that the seller uses a third-price auction: the highest bidder wins the single unit and pays the third-highest bid. Does the result in (a) still apply? Explain.

Let $\{\phi_i(\cdot), t_i(\cdot)\}_{i=1}^n$ represent a direct-revelation auction mechanism. Let $\overline{\phi}_i(\hat{\theta}_i)$ be the expected probability that player i obtains a unit of the good given report $\hat{\theta}_i$ and all other players report truthfully. Similarly, let $\overline{t}_i(\hat{\theta}_i)$ represent the expected payment of player i to the seller given report $\hat{\theta}_i$ and all other players report truthfully.

(c). State a condition on $\overline{\phi}_i(\cdot)$ and an integral condition characterizing

$$U_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\theta_i - \overline{t}_i(\theta_i),$$

that are necessary and sufficient conditions for the mechanism to be incentive compatible.

(d). Using your result in (c), and the fact that

$$\overline{t}_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\theta_i - U_i(\theta_i),$$

state and sketch the proof to the Revenue Equivalence Theorem applied to this multi-unit auction setting.

(e). Using your result in (c), construct the seller's expected payoff as a function that depends only on the choice variables $\{\phi_i(\cdot)\}_{i=1}^n$ and $\{U_i(0)\}_{i=1}^n$.

(f). Find the optimal allocation, $\{\phi_i(\cdot)\}_{i=1}^n$. [You may write it in terms of the multiplier, $\lambda(\theta)$, on the capacity constraint.]

(g). Suppose that $Q > n$. Is there a simple way to implement the optimal mechanism? Explain.

(h). Return to the case where $Q = 2$ and $n > 2$, as in (a). Is the third-price auction optimal? Why or why not?

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Answers to examination

1 (a). The efficient allocation is the pointwise solution to

$$\max_{q\geq 0}\ \theta q - c(\theta)q - \frac{1}{2}q^2,$$

and thus

$$q^{fb}(\theta) = \max\{0, \theta - c(\theta)\}.$$

(b). A mechanism $\{p(\cdot), q(\cdot)\}$ is incentive compatible iff $q(\cdot)$ is nondecreasing and

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)ds, \forall\theta \in [\underline{\theta}, \overline{\theta}]$$

(c). Taking expectations of $U(\theta)$ using (b) and integrating by parts yields

$$E[U(\theta)] = U(\underline{\theta}) + E\left[\frac{1-F(\theta)}{f(\theta)}q(\theta)\right].$$

Incorporating this into the restaurant owner's objective function, we have

$$E\left[(\theta - c(\theta))q(\theta) - \frac{1}{2}q(\theta)^2 - \frac{1-F(\theta)}{f(\theta)}q(\theta) - U(\underline{\theta})\right],$$

or using the uniform distribution assumption,

$$E\left[(\theta - c(\theta))q(\theta) - \frac{1}{2}q(\theta)^2 - (\overline{\theta} - \theta)q(\theta) - U(\underline{\theta})\right].$$

(d). We maximize the expression in (c) pointwise, ignoring the required monotonicity condition. The resulting solution to the relaxed program is

$$q(\theta) = \max\left\{0, 2\theta - c(\theta) - \overline{\theta}\right\},$$

Because $\theta - c(\theta)$ is increasing in θ by assumption, we conclude that $q(\theta)$ is weakly increasing, as needed. The relaxed solution is also the solution to the unrelaxed program.

(e). Because the utility function is linear in q, incentive compatibility requires that $P(q)$ is convex. If it were not, then $u(q, \theta) - P(q) = \theta q - P(q)$ would fail to be a concave program around the qualities that are implemented.

(f). If $c'(\theta) > 2$, then the results in (b) and (c) are unaffected. The solution to the relaxed program in (d), however, is now weakly decreasing. Indeed, because $\overline{\theta} \geq c(\overline{\theta})$, the relaxed solution is

$$q(\theta) = 2\theta - c(\theta) - \overline{\theta},$$

which is strictly decreasing over $[\underline{\theta}, \overline{\theta}]$. Hence, the solution to the unrelaxed program must have a binding monotonicity condition and there must be pooling of types consuming same quality in the optimal mechanism. Thus, $q(\theta)$ will be constant in θ. This is because of the adverse selection in $C(q, \theta)$.

2 Because $\overline{r}$ is strictly increasing, the probability of i having the highest bid is $F(\pi_i)^{n-1}$. In equilibrium, firm i's payoff is

$$U(\pi_i) = G(\pi_i)\pi_i - \overline{r}(\pi_i),$$

where

$$G(\pi_i) \equiv F(\pi_i)^{n-1},$$

is the probability of winning for type π_i. Firm i solves the following program

$$U(\pi_i) = \max_{r_i} \text{ Prob}[\max_{\pi_j \neq \pi_i} \overline{r}(\pi_j) \leq r_i]\pi_i - r_i = \max_{r_i} G(\overline{r}^{-1}(r_i))\pi_i - r_i.$$

[You could have proceeded by finding the first-order condition to this program, imposing symmetry, and integrating the solution. Instead, let's do this the easier way with the envelope theorem.] Using the envelope theorem, we know that the expected payoff to bidder i is absolutely continuous and has a derivative (a.e.) equal to the probability of winning, $G(\pi_i)$. Hence,

$$U(\pi_i) = U(0) + \int_0^{\pi_i} G(s)ds.$$

Because $U(0) = 0$ (no firm will bid if their value from bidding is zero), we can combine our two representations of $U(\pi_i)$ to obtain

$$G(\pi_i)\pi_i - \overline{r}(\pi_i) = \int_0^{\pi_i} G(s)ds,$$

or

$$\overline{r}(\pi_i) = G(\pi_i)\pi_i - \int_0^{\pi_i} G(s)ds,$$

or using the original CDF,

$$\overline{r}(\pi_i) = F(\pi_i)^{n-1}\pi_i - \int_0^{\pi_i} F(s)^{n-1}ds.$$

If you took a slightly different route, you may have concluded

$$\overline{r}(\pi_i) = \int_0^{\pi_i} sg(s)ds = \int_0^{\pi_i} s(n-1)f(s)F(s)^{n-2}ds,$$

which is equal to the previous expression. (You can verify this by integrating by parts.)

3 (a). An ex post efficient rule is

$$\hat{x}(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \geq c \\ 0 & \text{otherwise.} \end{cases}$$

Note that you could resolve the case where $\theta_1 + \theta_2 = c$ either with $x = 0$ or $x = 1$. Both outcomes are ex post efficient at that measure-zero point.

(b). We are looking for the VCG mechanism which is DSIC, ex post efficient and has the property that a player pays their externality to the offers whenever they are pivotal. Define $\hat{x}_j(\theta_j)$ to be the public good decision absent player i. Hence, $\hat{x}_j(\theta_j) = 1$ iff $\theta_j \geq \frac{1}{2}$ and $\hat{x}_j = 0$ otherwise.

The VCG transfers are therefore (for $i \neq j$)

$$t_i^{vcg}(\theta_1, \theta_1) = u_j(\hat{x}_j(\theta_j), \theta_j) - u_j(\hat{x}(\theta_1, \theta_2), \theta_j) = \hat{x}_j(\theta_j)(\theta_j - \frac{1}{2}) - \hat{x}(\theta_1, \theta_2)(\theta_j - \frac{1}{2}) \geq 0.$$

To verify DSIC for player i, note that player i's payoff is

$$\hat{x}(\hat{\theta}_1, \hat{\theta}_2)(\theta_i - \frac{1}{2}) - \hat{x}_j(\hat{\theta}_j)(\hat{\theta}_j - \frac{1}{2}) + \hat{x}(\hat{\theta}_1, \hat{\theta}_2)(\hat{\theta}_j - \frac{1}{2})$$

which simplifies to

$$\hat{x}(\hat{\theta}_1, \hat{\theta}_2)(\theta_i + \hat{\theta}_j - 1) - \hat{x}_j(\hat{\theta}_j)(\hat{\theta}_j - \frac{1}{2}).$$

The first term is maximized by setting $\hat{\theta}_1 = \theta_1$ because of the construction of $\hat{x}(\cdot)$. (I.e., because $\hat{x}(\theta_1, \hat{\theta}_2) = 1$ iff $\theta_1 + \hat{\theta}_2 \geq 1$, reporting $\hat{\theta}_1 = \theta_1$ maximizes the first term.) The second term is independent of i's report and hence has no impact the choice of report. Thus truthful reporting is optimal for all $\hat{\theta}_2$; hence reporting the truth is DSIC.

(c). We build the expected externality mechanism using the VCG transfers from (b). To this end, we first take interim expectations:

$$\bar{t}_i^{vcg}(\theta_i) = \int_0^1 \hat{x}_j(\theta_j))(\theta_j - \frac{1}{2})d\theta_j - \int_0^1 \hat{x}(\theta_1, \theta_2)(\theta_j - \frac{1}{2})d\theta_j = \int_{\frac{1}{2}}^1 (\theta_j - \frac{1}{2})d\theta_j - \int_{1-\theta_i}^1 (\theta_j - \frac{1}{2})d\theta_j.$$

This integrates to

$$\bar{t}_i^{vcg}(\theta_i) = \frac{1}{8} - \frac{1}{2}\theta_i(1 - \theta_i).$$

Taking expectations again, we have

$$\bar{t}_i^{vcg} = \int_0^1 \bar{t}_i^{vcg}(\theta_i)d\theta_i = \frac{1}{8} - \frac{1}{2}\int_0^1 \theta_i(1 - \theta_i)d\theta_i = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}.$$

Following our recipe for the EE mechanism,

$$t_i^{ee}(\theta_1, \theta_2) = \bar{t}_i^{vcg}(\theta_i) + (\bar{t}_j^{vcg} - \bar{t}_j^{vcg}(\theta_j)) - \frac{1}{2}(\bar{t}_1^{vcg} + \bar{t}_2^{vcg}),$$

we have that

$$t_i^{ee}(\theta_1, \theta_2) = \frac{1}{8} - \frac{1}{2}\theta_i(1 - \theta_i) + \left(\frac{1}{24} - \frac{1}{8} + \frac{1}{2}\theta_j(1 - \theta_j)\right) - \frac{1}{2}\left(\frac{1}{24} + \frac{1}{24}\right).$$

Simplifying, we have

$$t_i^{ee}(\theta_1, \theta_2) = -\frac{1}{2}\theta_i(1 - \theta_i) + \frac{1}{2}\theta_j(1 - \theta_j).$$

Clearly $t_1^{ee}(\theta_1, \theta_2) + t_2^{ee}(\theta_1, \theta_2) = 0$, and budget balance is immediate.

To verify BIC for player i, note that

$$\bar{t}_i^{ee}(\theta_i) = \int_0^1 t_i^{ee}(\theta_1, \theta_2)d\theta_j = -\frac{1}{2}\theta_i(1 - \theta_i) + \frac{1}{12}.$$

When player i reports $\hat{\theta}_i$, the public good is implemented with probability Prob$[\theta_i \geq 1 - \hat{\theta}_j] = 1 - (1 - \hat{\theta}_i) = \hat{\theta}_i$. Thus, when player i reports $\hat{\theta}_i$ and his true type is θ_i, (and player j is truthful), player i receives payoff

$$U_i^{ee}(\hat{\theta}_i | \theta_i) = \hat{\theta}_i(\theta_i - \frac{1}{2}) + \frac{1}{2}\theta_i(1 - \theta_i) - \frac{1}{12}.$$

Expanding and simplifying this quadratic, we have

$$U_i^{ee}(\hat{\theta}_i | \theta_i) = \hat{\theta}_i\theta_i - \frac{1}{2}\hat{\theta}_i^2 - \frac{1}{12}.$$

This expression is strictly concave and uniquely maximized at $\hat{\theta}_i = \theta_i$. Hence, we have established BIC for player i.

(d). We need to first compute the indirect utilities of each player in the VCG mechanism when both players are truthful.

$$U_i^{vcg}(\theta_i) = \theta_i(\theta_i - \frac{1}{2}) + \frac{1}{2}\theta_i(1 - \theta_i) - \frac{1}{8} = \frac{1}{2}\theta_i^2 - \frac{1}{8}.$$

The smallest payment that guarantees individual rationality for i is

$$\psi_i^* = \max_{\theta_i \in [0,1]} 0 - \frac{1}{2}\theta_i^2 + \frac{1}{8} = \frac{1}{8}.$$

Thus, the IR-VCG payments are

$$t_i^{ir}(\theta_1, \theta_2) = t_i^{vcg}(\theta_1, \theta_2) - \frac{1}{8}.$$

(e). We want to show that the it is impossible to have an BIC, BB, ex post efficient public goods allocation that is also individually rational. We appeal to the final theorem in the course in which we established that having an expected budget surplus in the IR-VCG mechanism is necessary and sufficient for the existence of a BIC, BB, ex post efficient, IR public mechanism. To determine if there is an ex post expected surplus, take expectations of the IR-VCG payments. We have

$$\bar{t}_i^{ir} = \bar{t}_i^{vcg} - \frac{1}{8} = \frac{1}{24} - \frac{1}{8} = -\frac{1}{12}.$$

Adding these together for $i = 1, 2$, we have an expected budget deficit of $-\frac{1}{12} - \frac{1}{12} = -\frac{1}{6} < 0$. Thus, we conclude it is impossible to obtain a mechanism with the desired properties.

4 (a). Consider a bidder whose type is θ_i and suppose that the second-highest bid of the other $(n-1)$ bidders is $b^{(2)}$. There are two cases:

(i) Suppose $\theta_i > b^{(2)}$, then bidding θ_i will lead to i getting the good with a surplus of $\theta_i - b^{(2)} > 0$. Any other bid $b_i > b^{(2)}$ would also yield the same surplus, but no such bid would not do better. A bid of $b_i = b^{(2)}$ would either lead to the bidder getting the good (and the same surplus) or would possibly lead to someone else getting the good in a tie breaker (and thus zero surplus). Any bid $b_i < b^{(2)}$ would lead to a payoff of zero.

(ii) Suppose that $\theta_i \leq b^{(2)}$. Then bidding θ_i leads to zero surplus. Bidding any amount $b_i < \theta_i$ also leads to zero surplus. Bidding $b_i > \theta_i$ either leads to zero surplus (if $b_i < b^{(2)}$) or a loss (if $b_i > b^{(2)}$). Hence, bidder i can do no better than bidding θ_i.

(b). When $Q = 1$ and there is a third price auction, it is no longer DSIC to bid ones type. To see this, consider bidder i and suppose that the second-highest bid among the other $(n-1)$ bidders is $b^{(2)}$ and the highest bid among the other $(n-1)$ bidders is $b^{(1)}$. Suppose that $b^{(1)} > \theta_i > b^{(2)}$. In this case, bidder i would want to bid $b^{(1)} + \varepsilon$, win the auction and earn $\theta_i - b^{(2)} > 0$. Hence, truth telling cannot be a weakly dominant strategy. The reason is that a DSIC auction requires that bidder i pays the externality she imposes on the others when she wins the good. In the third-price auction with 2 goods, the externality caused by bidder i winning is the lost value that the third-highest bidder would have obtained consuming the good. In contrast, with only one good, the externality is the value that the second-highest bidder would have obtained, not the third highest.

(c). An auction mechanism is incentive compatible iff $\overline{\phi}_i(\cdot)$ is nondecreasing in θ_i and

$$U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} \overline{\phi}_i(s) ds.$$

(d).

Revenue equivalence theorem. *Any two auctions which deliver the same expected utility to the lowest type, $U_i(0)$, and which have the same interim probabilities of delivering the good to bidder i, $\overline{\phi}_i(\cdot)$, generate the same expected revenue.*

Proof: Because

$$\bar{t}_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\theta_i - U_i(\theta_i),$$

we can use the result in (c) and write

$$\bar{t}_i(\theta_i) = \overline{\phi}_i(\theta_i)\theta_i - U_i(0) - \int_0^{\theta_i} \overline{\phi}_i(s) ds.$$

Hence, the expected payment of bidder i with type θ_i is entirely determined by $U(0)$ and the function $\overline{\phi}_i(\cdot)$. Because this is true for all i, we conclude that the expected revenues across the auctions are the same. QED (e). Using (c), we can integrate by parts and write

$$E[U_i(\theta_i)] = U_i(0) + E_{\theta_i}\left[\frac{1-F(\theta_i)}{f(\theta_i)}\overline{\phi}_i(\theta_i)\right] = U_i(0) + E_\theta\left[\frac{1-F(\theta_i)}{f(\theta_i)}\phi_i(\theta)\right].$$

Substituting this into the seller's objective function, we have

$$E_\theta\left[\sum_{i=1}^n \phi_i(\theta)\theta_i - U_i(\theta_i)\right] = E_\theta\left[\sum_{i=1}^n \phi_i(\theta)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) - U_i(0)\right].$$

(f). We maximize the objective in (e) subject to monotonicity (i.e., $\overline{\phi}_i(\cdot)$ is nondecreasing), individual rationality (i.e., $U_i(\theta_i) \geq 0$) and the capacity constraint, $\sum_i \phi_i(\theta) \leq Q$. We consider the relaxed program (ignoring monotonicity). Because $U_i(\theta_i)$ is nondecreasing, it is necessary and sufficient for individual rationality that $U_i(0) \geq 0$. Because reducing $U_i(0)$ increases the seller's objective without impacting any other constraint, the optimal solution will set $U_i(0) = 0$ for each i. Let $\lambda(\theta)$ be the Lagrange multiplier on the capacity constraint. The Lagrangian can be written as

$$\mathcal{L} = E_\theta\left[\left(\sum_{i=1}^n \phi_i(\theta)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right) + \lambda(\theta)\left(Q - \sum_{i=1}^n \phi_i(\theta)\right)\right].$$

Given λ, maximizing this objective pointwise over ϕ_i yields

$$\phi_i(\theta) = \begin{cases} 1 & \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The multiplier, $\lambda(\theta) \geq 0$, is chosen to satisfy complementary slackness,

$$\lambda(\theta)\left(Q - \sum_{i=1}^n \phi_i(\theta)\right) = 0.$$

Lastly, note that $\phi_i(\theta)$ must be weakly increasing in θ_i. To see this, note that if $\lambda(\theta) = 0$, then ϕ_i is increasing because $J(\theta_i) \equiv \theta_i - (1 - F(\theta_i)/f(\theta_i)$ is increasing. If $\lambda(\theta) > 0$, then $\sum_j \phi_j(\theta) = Q$. In this case, an increase in θ_i increases $J(\theta_i)$ but has no impact on $J(\theta_j)$ for $j \neq i$. Because all ϕ_j share the same multiplier in their construction, $\lambda(\theta)$, ϕ_i must weakly increase (and $\sum_{j\neq i}\phi_j$ must weakly decrease) in θ_i. We conclude that $\overline{\phi}_i(\cdot)$ from the relaxed program is nondecreasing.

(g). $Q > n$ implies that the multiplier is always slack, $\lambda(\theta) = 0$. Hence, the optimal rule is to sell to bidder i if and only if

$$\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \geq 0.$$

This can be implemented by simply setting a price p^* such that

$$p^* = \frac{1-F(p^*)}{f(p^*)},$$

and offering each bidder the option to buy one uint at p^*. No "auction" is necessary.

(h). In the case where $Q = 2$ and $n > 2$, the optimal auction gives the good to the two highest bidders, providing that their values exceed the optimal reserve price, p^*, determined in (g). The third-price auction gives the two units to the two highest bidders, without a reserve price. Hence, the auction in (a) is not optimal.