Price Theory II
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(Phil Reny)

Past exam questions

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Contents

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1 Coefficient of absolute and relative risk aversion

1.1 Core 2011/12

Suppose that $u : \mathbb{R}_+ \to \mathbb{R}$ is a consumer's von Neumann-Morgenstern utility function over wealth. Suppose also that u is twice differentiable and that $u' > 0$. Prove that if this consumer's preferences over wealth gambles are independent of her initial wealth, then she must display constant absolute risk aversion, i.e. the ratio $u''(w)/u'(w)$ must be independent of w. (Hint: Consider, for any $w_0 > 0$, the two vNM utility functions $u(w + w_0)$ and $u(w)$ over wealth w. How are they related?)

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Let w_1 and w_2 be any two gambles and $\succsim$ preference over these gambles that satisfy the usual axioms. Then,

$$w_1 \succsim w_2 \Leftrightarrow u(w_1) \geq u(w_2).$$

That consumer's preferences are independent of the initial wealth means

$$w_1 \succsim w_2 \Leftrightarrow u(w_1) \geq u(w_2) \Leftrightarrow u(w_1 + w_0) \geq u(w_2 + w_0), \ \forall w_0 > 0. \tag{1.1}$$

Fix some $w_0 > 0$. Define

$$v(w) \coloneqq u(w + w_0).$$

(1.1) means that u and v represent the same preferences, and we know from class that vNM utility functions are unique up to positive affine transformations, there must exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$such that

$$v(w) \equiv \alpha + \beta u(w).$$

Since this is an identify, we may differentiate both sides with respect to w once and twice to obtain

$$v'(w) \equiv \beta u'(w),$$
$$v''(w) \equiv \beta u''(w).$$

Since $u' > 0$ by assumption, we can divide one by the other to obtain that

$$\frac{u''(w)}{u'(w)} = \frac{v''(w)}{v'(w)} = \frac{u''(w + w_0)}{u'(w + w_0)}, \ \forall w.$$

But this can hold only if $u''(w)/u'(w)$ is constant with respect to w. Since w_0 was arbitrary, we have the desired result.

here's

Proposition 1.1. *The general form of utility function with constant absolute risk aversion is given by*

$$u(x) \coloneqq B - A \exp[-Rx],$$

where B and $A > 0$ are some constants and $R \equiv ra(x)$.

Proof. Define $R \equiv ra(x) > 0$. Then, $u(\cdot)$ must satisfy the following differential equation:

$$R = -\frac{u''(x)}{u'(x)}.$$

We can write above as

$$\frac{d}{dx}\left[\ln u'(x)\right] = -R.$$

The solution is given by

$$\begin{aligned}
\ln u'(x) &= -\int R dx = -Rx + C_1 \\
\Rightarrow u'(x) &= \exp\left[C_1\right]\exp\left[-Rx\right] \\
\Rightarrow \int u'(x)\, dx &= \int \exp\left[C_1\right]\exp\left[-Rx\right] dx \\
\Rightarrow u(x) &= C_2 - \frac{\exp\left[C_1\right]}{R}\exp\left[-Rx\right],
\end{aligned}$$

where C_1 and C_2 are some constants of integration. Define

$$B \coloneqq C_2,\ A \coloneqq \frac{\exp\left[C_1\right]}{R} > 0$$

and we are done. ■

1.2 Core 2012/13

Suppose that $u : \mathbb{R}_+ \to \mathbb{R}$ is a consumer's von Neumann-Morgenstern utility function over wealth. Suppose also that u is twice differentiable and that $u' > 0$. Suppose that, whenever this consumer prefers one wealth gamble over another, she also prefers the one gamble over the other when all outcomes of the two gambles are multiplied by any positive constant $\lambda > 0$. Prove that his consumer must display constant relative aversion, i.e., prove that the ratio $wu''(w)/u'(w)$ must be independent of w. (Hint: Consider, for any $\lambda > 0$, the two vNM utility functions $u(\lambda w)$ and $u(w)$ over wealth w. How are they related?) Can you describe her utility function?

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Let w_1 and w_2 be any two gambles and $\succsim$ preference over these gambles that satisfy the usual axioms. Then,

$$w_1 \succsim w_2 \Leftrightarrow u(w_1) \geq u(w_2).$$

The described characteristics of the consumer preference means that

$$w_1 \succsim w_2 \Leftrightarrow u(w_1) \geq u(w_2) \Leftrightarrow u(\lambda w_1) \geq u(\lambda w_2), \ \forall \lambda > 0. \tag{1.2}$$

Fix some $\lambda > 0$. Define

$$v(w) := u(\lambda w).$$

(1.2) means that u and v represent the same preferences, and we know from class that vNM utility functions are unique up to positive affine transformations, there must exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$such that

$$v(w) \equiv \alpha + \beta u(w).$$

Since this is an identify, we may differentiate both sides with respect to w once and twice to obtain

$$v'(w) \equiv \beta u'(w),$$
$$v''(w) = \beta u''(w).$$

Since $u' > 0$ by assumption, we can divide one by the other to obtain that

$$\frac{u''(w)}{u'(w)} = \frac{v''(w)}{v'(w)} = \frac{\lambda u''(\lambda w)}{u'(\lambda w)}, \ \forall w.$$

Multiplying the expression by w yields:

$$w\frac{u''(w)}{u'(w)} = \lambda w\frac{\lambda u''(\lambda w)}{u'(\lambda w)}, \ \forall w.$$

But for above to hold, it must be that $wu''(w)/u'(w)$ is constant. Since $\lambda > 0$ was arbitrary, we have the desired result

Here's the converse.

Proposition 1.2. *The general form of utility function with constant relative risk aversion is given* *by*

$$u(x) \coloneqq A\frac{x^{1-r}}{1-r}+B,$$

where $A>0$ and B are constants and $r \equiv rra(x)$. Moreover, if $r=1$, then

$$u(x) \coloneqq A\ln x + B.$$

Proof. Let $r \equiv rra(x) > 0$. Then, $u(\cdot)$ must satisfy the following differential equation:

$$r = -\frac{u''(x)}{u'(x)}x.$$

Rewrite above as

$$\begin{aligned}
\frac{d}{dx}\left[\ln u'(x)\right] &= -\frac{r}{x} \\
\Rightarrow \ln u'(x) &= -r\int \frac{1}{x}dx \\
&= -r\left(\ln x + C_1\right) \\
&= \ln x^{-r} - rC_1 \\
\Rightarrow u'(x) &= x^{-r}\exp\left[-rC_1\right] \\
\Rightarrow \int u'(x)\,dx &= \exp\left[-rC_1\right]\int x^{-r}dx \\
\Rightarrow u(x) &= \exp\left[-rC_1\right]\left(\frac{1}{1-r}x^{1-r}+C_2\right) \\
&= \exp\left[-rC_1\right]\frac{1}{1-r}x^{1-r} + \exp\left[-rC_1\right]C_2.
\end{aligned}$$

Hence, a general form of utility function with constant relative risk aversion is given by

$$u(x) = A\frac{x^{1-\alpha}}{1-\alpha}+B,$$

where $A>0$ and B are constants.

If $\alpha = 1$, then

$$\begin{aligned}
u'(x) &= \frac{1}{x}\exp\left[-C_1\right] \\
\Rightarrow \int u'(x)\,dx &= \exp\left[-C_1\right]\left(\ln x + C_2\right) \\
\Rightarrow u(x) &= \exp\left[-C_1\right]\ln x + \exp\left[-C_1\right]C_2.
\end{aligned}$$

So the expression is

$$u(x) = A\ln x + B,$$

where $A>0$ and B are again constants. ■

Remark 1.1. We said earlier that with relative risk aversion, people often assume

$$u(x) = \frac{x^{1-r}-1}{1-r}, \tag{1.3}$$

which of course is consistent with the general form we derived $(A \coloneqq 1,\ B \coloneqq -1/(1-r))$. But why do we we simply not set $B = 0$? We do this to include the special case in which $r = 1$ and the utility is log. To see this, take the limit of $u(x)$ above as r tends to 1:

$$\begin{aligned}
\lim_{r\to 1} \frac{x^{1-r}-1}{1-r} &= \lim_{r\to 1} \frac{e^{(1-r)\ln x}-1}{1-r} \\
\text{[L'Hôpital]} &= \lim_{r\to 1} \frac{-(\ln x)\, e^{(1-r)\ln x}}{-1} \\
&= \ln x.
\end{aligned}$$

Without the -1 in the numerator, we won't have $0/0$ when we set $r = 1$ so that we won't be able to use L'Hôpital's rule. Put differently, (1.3) is a generalisation of log utility.

2 Social choice

2.1 2015 Core

Show that the conclusion of the Gibbard-Satterthwaite theorem need not hold when there are just two alternatives. [Hint: You may assume an odd number of individuals if this is helpful.]

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Let $X = \{x, y\}$ and consider a social choice function based on majority voting rule. We want to show that this social choice function is (i) strategy proof; and (iii) non-dictatorial.

Strategy proof requires that if reporting untruthfully one's preference leads to a change in the social choice, then

$$\begin{bmatrix} c\left(R^i, R^{-i}\right) = x \\ c\left(\tilde{R}^i, R^{-i}\right) = y \end{bmatrix} \Rightarrow xR^iy$$

for all i, R^i, $\tilde{R}^i$ and all R^{-i}. Assume odd number of individuals. Under majority voting rule, the social states change when i misreports if and only if i is the pivotal voter—i.e. if and only if there are $(N-1)/2$ votes in favour of each of the two social states. If i reports according to his true preference, then the social state will be the one that he prefers. If he lies, then the social state will be that one that he does not prefer. Hence, i has no incentive to misreport. We can therefore conclude that the social choice function is strategy proof.

Non-dictatorship requires that there does not exists i such that

$$xR^iy \Rightarrow c\left(R^i, R^{-i}\right) = x,\ \forall R^{-i}, \forall x, y \in X.$$

Suppose xR^iy and $c\left(R^i, R^{-i}\right) = x$. By the fact that we have unrestricted domain, we may consider $\tilde{R}^{-i}$ in which we change all other preference relation to prefer y over x. Denote the modified preference relation as $\tilde{R}^{-i}$. Then, $c\left(R^i, \tilde{R}^{-i}\right) = y$. Hence, i cannot be a dictator. Since this holds for all i, we may conclude that the social choice function here is non-dictatorial.

2.2 2016 Core

There are three individuals $i = 1, 2, 3$, and the set of social alternatives is $A = \{1, 2, \ldots, K\}$. The preferences of the three individuals come from the following *restricted domain*. Each individual i has preferences over the K alternatives that can be represented by a utility function $u_i : \{1, \ldots, K\} \to \mathbb{R}$ that assigns distinct utility numbers to distinct alternatives (i.e. no indifference) and that satisfies the following strict quasiconcavity condition::

$$u_i(k) > \min\{u_i(j), u_i(\ell)\}, \text{ for all alternatives } j < k < \ell.$$

Let $\mathcal{D}$ denote the set of utility functions on A that satisfy the above conditions (i.e. no indifference and strictly quasiconcave).

Define the social choice function, $f : \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to A$ as follows. For any three utility functions $u_1, u_2, u_3 \in \mathcal{D}$, let $a_i^* \in A$ be the most preferred social alternative according to u_i. If $a_i^* = a_{i'}^*$ for at least two individuals $i \neq i'$, then the social choice is $f(u_1, u_2, u_3) = a_i^*$. But if all three most preferred alternatives are distinct, i.e. $a_i^* < a_{i'}^* < a_{i''}^*$, then the social choice is $f(u_1, u_2, u_3) = a_{i'}^*$; i.e. the social choice is the alternatives that lies "between" the other two.

Prove that f is strategy proof on the restricted domain $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$.

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Preliminaries A social choice function is strategy proof if each individual has no incentive to lie about the preference relation. Formally, let $R = (R^1, R^2, \ldots, R^N)$ denote the true preference ordering for the individuals. Suppose $f(R) = x$. Suppose that if i reports $\tilde{R}^i$ instead, then $f(\tilde{R}^i; R^{-i}) = y \neq x$, where R and $\tilde{R}$ differ only with respect to i's preference. Then, to ensure that i does not misreport, it must be that xR^iy

$$\begin{bmatrix} f(R^i; R^{-i}) = x \\ f(\tilde{R}^i; R^{-i}) = y \end{bmatrix} \Rightarrow [xR^iy]$$

for all i, for all R^i and $\tilde{R}^i$ and for all $R^{-i} \in \mathcal{R}^{-i}$.

Proof Consider first the case in which, for at least two individuals, their most preferred social alternatives are the same, and given by $a^* \in A$. Clearly, the individuals for whom a^* is the most preferred social alternative have no incentive to misreport their preferences since doing so makes them (weakly) worse off.

Suppose now that the most preferred alternatives are distinct. Without loss of generality, suppose that

$$a_1^* < a_2^* < a_3^*.$$

Since the social choice in this case is a_2^*, individual 2 does not have an incentive to misreport. So we focus on individuals 1 and 3.

Consider possible misreporting by a_1^*. If he reports instead $\tilde{a}_1 < a_1^*$, $\tilde{a}_1 = a_2^*$, or $a_1^* < \tilde{a}_1 < a_2^*$ then the social choice does not change so this does not make individual 1 better off. So consider the case in which he reports $a_2^* < \tilde{a}_1 < a_3^*$. The social choice is then $\tilde{a}_1$. But, by quasiconcavity of his preferences,

$$a_1^* < a_2^* < \tilde{a}_1 \Rightarrow u_i\left(a_2^*\right) > \min\left\{u_i\left(a_1^*\right), u_i\left(\tilde{a}_1\right)\right\} = u_i\left(\tilde{a}_1\right).$$

(the last equality uses the fact that preferences are strict.) So that he does not want to misreport. Suppose that individual 1 reports $\tilde{a}_1 = a_3^*$, the social choice becomes a_3^*. But

$$a_1^* < a_2^* < a_3^* = \tilde{a}_1 \Rightarrow u_i\left(a_2^*\right) > \min\left\{u_i\left(a_1^*\right), u_i\left(a_3^*\right)\right\} = u_i\left(a_3^*\right).$$

Thus, he does not want to misreport in this way. The same argument holds if he reports $\tilde{a}_1 > a_3^*$. Hence, we conclude that individual 1 does not wish to misreport his preferences.

The argument for individual 3 is symmetric.

We therefore conclude that f is strategy proof on the restricted domain $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$.

3 Matching

3.1 2017/18 Final

Assume all agents have strict preferences. Suppose that μ and μ' are stable matchings and that, for all $m \in M$,

$$\mu(m) \neq \mu'(m) \Rightarrow \mu(m) \succ_m \mu'(m).$$

Prove that, for all $w \in W$,

$$\mu(w) \neq \mu'(w) \Rightarrow \mu'(w) \succ_w \mu(w).$$

That is, prove that, if all of the men weakly prefer the matching μ to the matching μ', then all the women weakly prefer the matching μ' to the matching μ.

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By way of contradiction, suppose that, for some $w \in W$,

$$\mu(w) \neq \mu'(w) \Rightarrow \mu(w) \succ_w \mu'(w).$$

That left-hand side implies that w, who is matched to (say) $m = \mu(w)$ is matched to another man $m' = \mu'(w)$. The implication then is that w strictly prefers m over m'; i.e.

$$m \succ_w m'. \tag{3.1}$$

That w is matched to a different man means that $m = \mu(w)$ is matched to another woman (or himself); i.e. $\mu(m) \neq \mu'(m)$. But, by the assumption of the question, this implies that

$$w = \mu(m) \succ_m \mu'(m). \tag{3.2}$$

But (3.1) and (3.2) together means that (m, w) would form a blocking pair under μ', contradicting the stability of μ'.

Remark 3.1. In the notation, I have assumed that you are never matched to yourself but the proof works even if that were not the case.

3.2 Core 2015/16

Consider a marriage market with N men and N women in which each individual has strict preferences over the individuals on the other side of the market. Suppose also that all individuals strictly prefer being matched with any individual on the other side of the market to being single. Prove that the matching that results from the male-proposing Gale-Shapley algorithm is the worst stable matching for all of the women.

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Let μ denote the stable matching arising from the male-proposing Gale-Shapley algorithm. By way of contradiction, suppose that there exists a stable μ' such that, for some $w \in W$,

$$m \coloneqq \mu(w) \succ_w \mu'(w) \eqqcolon m'. \tag{3.3}$$

That is, μ' matches w with strictly less preferred man than under μ. Clearly, $m \neq m'$.

We know that all men prefers matching produced by male-proposing DAA over any other stable matching (since preferences are restricted to be strict here, to the extent that stable matching produces a different match for a man, he must strictly prefer the matching produced by the male-proposing DAA). Since m' is now matched to a different woman (w) from that under the male-proposing DAA, and m is also matched to to a different woman, we must have

$$\begin{aligned} \mu(m') \succ_{m'} \mu'(m') &= w \\ w = \mu(m) &\succ_m \mu'(m). \end{aligned}$$

Note that, under μ', the matching is (m', w) and $(m, \mu'(m))$ and under μ the matching is (w, m) and $(m', \mu(m'))$. But above tells us that m strictly prefers w over $\mu'(m)$ and that w strictly prefers m over m'. That is, they would form a blocking pair—contradicting that μ' is stable.

We can also use the result from the previous exercise to prove this. As argued above, (3.3) (and that there are same number of men and women and all preferred to be matched) implies that $\mu(m) \neq \mu'(m)$ and since μ (male-proposing DAA) is strictly preferred by any other stable matching μ', we have $\mu(m) \succ_m \mu'(m)$. We also have that $\mu(w) \neq \mu'(w)$ from above. Then from the previous exercise, we know that, for all $w \in W$,

$$\mu(w) \neq \mu'(w) \Rightarrow \mu'(w) \succ_w \mu(w).$$

Since this holds for all stable non-male-proposing DAA, μ', it follows that μ is the least preferred stable matching by the women.