Theory of Income, Fall 2008
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Problem Set 7

1 Varieties of Growth Models

In this problem set we consider three different models. One with no steady states, the Lucas-Uzawa endogenous growth model,a growth model with steady states that are not necessarily stable and a model of durable and non durable goods.

1.1 Lucas-Uzawa endogenous growth model

Let $N(t)$ be the size of population at time t and let

$$\dot{N}(t) = nN(t),$$

so the population growth rate is n. Let the "dynasty" preferences be given by

$$\int_0^\infty N(t) e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt,$$

where $c(t)$ is per-capita consumption. Production is done using a Cobb-Douglas function of capital and time, in efficiency units, i.e.,

$$\dot{K}(t) + N(t) c(t) = A [K(t)]^{\alpha} [h(t) u(t) N(t)]^{1-\alpha},$$

where $K(t)$ is the stock of physical capital, $h(t)$ is the per-person stock of human capital, $u(t)$ is the fraction of time devoted to production. Thus, $\dot{K}$ is investment (we assume that the depreciation rate is zero) and hN is the total units of time available for production, in efficiency units. The accumulation of human capital is as follows

$$\dot{h}(t) = \delta h(t) [1 - u(t)],$$

where $1 - u(t)$ is the fraction of time devoted to learning.

1. Write down the Hamiltonian and all the relevant first order conditions for this problem. [Hint: These are two differential equations for the co-states of K and h, and two FOCs w.r.t. to the controls c, and u].

Definition. We say that the numbers N, K, h, c, u and κ are a balanced growth path if

$$\begin{aligned} K(t) &= e^{(n+\kappa)t} K, \\ h(t) &= e^{\kappa t} h, \\ c(t) &= e^{\kappa t} c, \\ u(t) &= u, \end{aligned}$$

is optimal for initial conditions $K(0) = K$, $h(0) = h$, $c(0) = 0$ and $N(0) = N$.

2. Write down a formula for the growth rate κ, the time devoted to production u, and the savings rate $\dot{K}/\left(AK^{\alpha}[huN]^{1-\alpha}\right)$ in a balanced growth path. Your expressions should be a function of the parameters ρ, n, δ, σ, α. How does κ change as a function of ρ, δ, σ and n? How does u change as function of α? Give an intuitive explanation for each of these cases.

3. Write down the relevant transversality conditions. What conditions are needed on the parameters ρ, n, δ, σ, α, so that the balanced growth path characterized in 2 satisfies the transversality conditions?

1.2 A one sector growth model where capital is produced with labor only

Consider the following discrete time growth model, where labor, inelastically supplied, is used to produced capital. Feasibility is given by

$$
\begin{aligned}
c_t &= n_t f\left(k_t/n_t\right), \\
k_{t+1} &= 1 - n_t, \\
0 &\leq n_t \leq 1,
\end{aligned}
$$

where f is a neoclassical production function (strictly increasing and strictly concave, C^2, with $f(0) = 0$, and satisfying Inada conditions) and n_t is labor used in production of consumption goods, so that $1 - n_t$ is the time used in production of capital goods, which we assume equal to next period capital stock (i.e. 100% depreciation). Preferences are standard discounted utility, with period utility function U (strictly increasing, strictly concave, satisfying Inada conditions, C^2, and $U(0) = 0$), and the discount rate is $\beta \in (0, 1)$.

The corresponding Bellman Equation is

$$v(k) = \max_{y \in [0,1]} \{U[(1-y)\ f(k/(1-y))] + \beta v(y)\},$$

for any $k \in [0,1]$, where we use that $y = 1 - n$.

Preliminaries. Notice that the period return function is

$$F(k,y) = U[(1-y)\ f(k/(1-y))].$$

Moreover, letting $\kappa \equiv k/n$ denote the capital labor ratio, we know that the production function satisfies the following conditions:

$$\begin{aligned}
f'(\kappa) &> 0, \qquad f''(\kappa) < 0, \\
\lim_{\kappa \to 0} f'(\kappa) &= \lim_{\kappa \to \infty} \left[f(\kappa) - \kappa f'(\kappa)\right] = \infty, \\
\lim_{\kappa \to \infty} f'(\kappa) &= \lim_{\kappa \to 0} \left[f(\kappa) - \kappa f'(\kappa)\right] = 0.
\end{aligned}$$

The last four conditions are the Inada conditions written in intensive form, and say that the marginal product of capital and labor go to infinity (zero) as capital and labor tend to zero (infinity), respectively. To check that $f(\kappa) - \kappa f'(\kappa)$ is indeed the marginal product of labor, simply differentiate $G(k,n) \equiv nf(k/n)$ w.r.t. n.

We will now show that the value function is strictly increasing, strictly concave and differentiable.

$v(k)$ is strictly increasing. The feasibility correspondence is $\Gamma(k) = [0,1]$, whose graph is the square $[0,1] \times [0,1]$. Evidently, Γ is monotone, since for any $k' > k$ we have that $\Gamma(k) \subseteq \Gamma(k')$ (in fact, in our case $\Gamma(k) = \Gamma(k') = [0,1]$ since Γ is independent of k). Moreover, for each y, $F(k,y)$ is strictly increasing in k, since

$$F_x(k,y) = U'(c)\ f'(\kappa) > 0,$$

given that U and f are strictly increasing. Thus, by Theorem 4.7 of RMED we conclude that $v(k)$ is strictly increasing.

$v(k)$ is strictly concave. Γ is clearly convex, since for any $\theta \in [0,1]$, $y \in \Gamma(k) = [0,1]$ and $y' \in \Gamma(k') = [0,1]$ we have that

$$\theta y + (1-\theta) y' \in \Gamma\left(\theta k + (1-\theta) k'\right) = [0,1].$$

It only rests to show that $F(k,y)$ is strictly concave. To do this, it suffices to show that

$G(k,n) = nf(k/n)$ is concave, since U is strictly increasing and strictly concave. But, since the production function is neoclassical, we know that $G(k,n)$ is indeed concave. Thus, by Theorem 4.8 of RMED we conclude that $v(k)$ is strictly concave. This also implies that the optimal policy $g(k)$ is a continuous, single-valued function.

$v(k)$ *is differentiable.* F is differentiable since U and f are twice differentiable. Thus, by Theorem 4.9 of RMED we conclude that $v(k)$ is differentiable as well.

1. Write down the Euler Equation and the Envelope condition for this problem. Denote the optimal decision rule by g, i.e., $k' = g(k)$.

2. Write down the equation that an interior steady state k^* must satisfy. Show that a solution k^* exists and it is unique.

3. Show that $g(0) = 1$.

4. Let $U(c) = c^{\alpha}$ for $0 < \alpha < 1$, $f(z) = z^{\theta}$, $0 < \theta < 1$. Show that g is strictly decreasing in $k \in [0, 1]$. [Hint: In a previous problem set or exercise you showed that this depends on the sign of F_{xy}]. What is the intuition for this result? Why is this so different from the neoclassical growth model?

5. Show that the linearized Euler equation is

$$\left(k_{t+2}-k^*\right)+B\left(k_{t+1}-k^*\right)+\frac{1}{\beta}\left(k_t-k^*\right)=0,$$

where

$$B=\frac{1-\alpha\left(1-\theta\right)}{\alpha\left(1-\theta\right)}+\frac{1-\alpha\theta}{\alpha\theta\beta}.$$

6. Let $\beta = 0.1$, $\alpha = 0.8$ and $\theta = 0.75$. Show that $k^* = 0.23$, approximately. Show that this steady state is unstable. [Hint: Show that both roots of the previous equation are larger than one in absolute value].

7. Define a two-period cycle as a sequence $\{k_t\}$ such that

$$\begin{aligned} k_t &= x \text{ if } t \text{ is odd}, \\ k_t &= y \text{ if } t \text{ is even}, \end{aligned}$$

for $x, y \in [0,1]$, $x \neq y$. Show that a two-period cycle is optimal if $k_0 = x$ and

$$\begin{aligned} 0 &= F_y(x,y) + \beta F_x(y,x), \\ 0 &= F_y(y,x) + \beta F_x(x,y), \end{aligned}$$

for F the period return function, i.e. $F(x,y) = U[(1-y)\ \ f(k/(1-y))]$.

8. Show that $(0.29, 0.18)$ is a two-period optimal cycle for the previous parameters.

2 A model of durabe and non-durable goods

Consider an economy where in each period every one of the consumers has an endowment y. This endowment can be used for investment in durable goods or for consumption of non-durables. Then the technology for this economy is:

$$x(t) + c(t) = y$$

for all $t \geq 0$, where $x(t)$ denote the investment in durables and $c(t)$ the consumption of non-durables. The stock of durable goods have a law of motion:

$$\dot{d}(t) = x - \delta d(t)$$

where δ is the depreciation rate of durables per unit of time.

The period utility function depends on the flow of nondurable purchases and on the stock of durables, and is given by $U(c, d)$. We assume that U is strictly quasi-concave in (c, d). In some cases we will specialize to

$$U(c, d) = \frac{[h(c, d)]^{1-\gamma} - 1}{1 - \gamma} \tag{1}$$

for $\gamma \geq 0$, and where

$$h(c, d) = \left[c^{-\theta} + \frac{1}{A} d^{-\theta}\right]^{-1/\theta}$$

for $\theta \geq -1$.The agent's utility is the discounted value of $U(c, d)$, using discount rate ρ. With this parameterization the elasticity of substitution between c and d is $1/(1+\theta)$, and the inter-temporal elasticity of substituion between the bundle h of (c, d) is $1/\gamma$.

Thus problem of the planner for this economy is

$$\max_{c,d} \int_0^\infty e^{-\rho t} U(c(t), d(t))\, dt$$

subject to

$$\dot{d}(t) + c(t) = y - \delta d(t),$$

and $d(0) > 0$ given.

Q0. To better understand the utility function in 1. show that if

$$\begin{aligned} U_{cd} &> 0 \text{ if } \frac{1}{\gamma} > \frac{1}{1+\theta}, \\ U_{cd} &= 0 \text{ if } \frac{1}{\gamma} = \frac{1}{1+\theta}, \\ U_{cd} &< 0 \text{ if } \frac{1}{\gamma} < \frac{1}{1+\theta}. \end{aligned}$$

And hence that if $\sigma \equiv 1/\gamma = 1/(1+\theta)$ the utility function is additively separable in c, d :

$$U(c, d) = \frac{c^{1-\frac{1}{\sigma}} + \frac{1}{A} d^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}}.$$

so

$$U(c,d)=\frac{\left[c^{-\theta}+\frac{1}{A}d^{-\theta}\right]^{-\frac{1-\gamma}{\theta}}}{1-\gamma}=\frac{c^{1-\frac{1}{\sigma}}+\frac{1}{A}d^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}$$

Q1. Write the Hamiltonian of the problem, using λ for the co-state, d for the state, and c for the control.

Q2. Write the f.o.c. w.r.t c and d.

Q3. Let $\bar{v}\equiv\delta+\rho$ be the steady state user cost of the durable good. Write two equations in two unknowns for the steady state values of $\left(\bar{c},\bar{d}\right)$ in terms of $U_d,\ U_c,\ \rho$ and $\bar{v}$..

Q4. Use the equation $H_c=0$ to obtain a differential equation linking $\dot{\lambda},\dot{c}$ and $\dot{d}$.

Q5. Using this last expression, replace the law of motion for the co-state variable and the law of motion of the state variable to find the law of motion of the control $\dot{c}$ as a function of parameters c and d.

Q6. Linearize this last ODE around the steady state, i.e.: $\left(\dot{c}, \dot{d}, c, d\right) = \left(0, 0, \bar{c}, \bar{d}\right)$ and replacing $\dot{d}$ by using the resource constraint of the economy.Your answer should be of the type $\dot{c} = a_{11} \left(c - \bar{c}\right) + a_{12} \left(d - \bar{d}\right)$ for two constants a_{11} and a_{12}. This constant are functions of the second derivatives of U evaluated at the steady state, and of δ and ρ.

Note on linearization: Suppose we want to linearize the function $g\left(x, y\right)$ around $\bar{x}$ and $\bar{y}$. We get:

$$g\left(x, y\right) = g_x\left(\bar{x}, \bar{y}\right)\left(x - \bar{x}\right) + g_x\left(\bar{x}, \bar{y}\right)\left(y - \bar{y}\right)$$

Q7. Using your previous answer to write the two **linear** differential equations that characterize the dynamics of this economy, one for $\dot{c} = a\ (c, d)$ and one for $\dot{d} = b\,(c, d)\,.$

Q8. Assume that $U_{cd}\,(\delta + \bar{v}) - U_{dd} > 0$. Draw the phase diagram with c in the y-axis and d in the x-axis. Label the axis, label the steady states, draw all the arrows for the field, and indicate clearly where the the stable arm (saddle-path) is.

Q9. We are looking for a solution of the form

$$c = \psi\,(d) = \bar{c} + \psi'\left(\bar{d}\right)\left(d - \bar{d}\right)$$

thus, we are looking for the value of the constant $\psi'\,(\bar{m})\,.$ Use the method of undetermined coefficients to find a quadratic equation for ψ' as a function of: $[\bar{v} + \delta]$ and

$$\Delta \equiv \left[\frac{U_{cd}\,(\delta + \bar{v}) - U_{dd}}{-U_{cc}}\right]$$

where the second derivatives are evaluated at the steady state. Hint: You need to use L'Hopital's rule. Recall what we did in pset 5 problem 1.

Q10. Show that the stable solution is given by

$$\psi' = \frac{-\left[\bar{v}+\delta\right] + \sqrt{\left[\bar{v}+\delta\right]^2 + 4\frac{U_{cd}(\bar{v}+\delta)-U_{dd}}{(-U_{cc})}}}{2},$$

(hint: this is trivial once you have the solution of Q9, and the figure for the saddle path, that help you to find the "right" solution of the quadratic equation).

Elasticity of the optimal consumption function.

For this we specialize the utility function U to (1).

We will show how the elasticity of the policy function is related to the inter-temporal and intra-temporal elasticities of substitution. Recall that the intra-tempral elasticity of substitution between c and d is $1/(1+\theta)$, and the inter-temporal elasticity of substituion between bundles of (c, d) given by h is $1/\gamma$.

We will parameterized the problem as a function of $\left(\theta, \gamma, \delta, \bar{v}, \bar{d}/\bar{c}\right)$. The interpretation of $\bar{d}/\bar{c}$ as a parameter, is that we solve for the constant A using the steady-state equation derived above as a function of the paramters $\theta, \bar{v}$, so that $\bar{d}/\bar{c}$. We obtain the following result:

Keeping the steady state value $\bar{d}/\bar{c}$ fixed, the elasticity of the optimal consumption function evaluated at steady state is a function of $\gamma/\left(1+\theta\right)$ and satisfies

$$\frac{d}{c}\frac{\partial c\left(d\right)}{\partial d}|_{d=\bar{d}} \equiv \frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right) = \begin{cases} 1 & \text{for } \frac{\gamma}{1+\theta} = 0 \\ < 1 & \text{for } \frac{\gamma}{1+\theta} > 0 \end{cases}$$

and $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

As an intermediate step to see why $\left(\bar{d}/\bar{c}\right)\left(\partial c\left(\bar{d}\right)/\partial d\right)$ depends on the ratio of γ to $1+\theta$ only, and to develop a formulat for $\Delta\left(\gamma/\left(1+\theta\right)\right)$ do the following:

Q11. To show this, first show that when h is a CES we have that:

$$\begin{aligned} \frac{h_{dd}}{h_{cc}} &= \frac{1}{\left(d/c\right)^2}, \\ \frac{h_{cd}}{h_{cc}} &= -\frac{1}{d/c}, \end{aligned}$$

$$\frac{h_c h_c}{-h h_{cc}} = \frac{1}{\left(1+\theta\right)\bar{v}\left(d/c\right)},$$

and that for $U\left(c,d\right) = h\left(c,d\right)^{1-\gamma}/\left(1-\gamma\right)$

$$\begin{aligned} \frac{U_{dd}}{U_{cc}} &= \frac{h_{dd}/h_{cc} + \gamma\bar{v}^2\left(h_c h_c\right)/\left(-h h_{cc}\right)}{1+\gamma\left(h_c h_c\right)/\left(-h h_{cc}\right)}, \\ \frac{U_{cd}}{U_{cc}} &= \frac{h_{cd}/h_{cc} + \gamma\bar{v}\left(h_c h_c\right)/\left(-h h_{cc}\right)}{1+\gamma\left(h_c h_c\right)/\left(-h h_{cc}\right)} \end{aligned}$$

and

$$\frac{h_c h_c}{-h h_{cc}} = \frac{1}{\left(1+\theta\right)\bar{v}\left(d/c\right)} .$$

Q12. First assume that $\gamma = 0$. Using part of the results of Q11 show that

$$\psi'\left(\bar{d}\right) = \frac{\bar{c}}{\bar{d}}$$

Q13. Assume that $\gamma > 0$ and that $1/\gamma = \sigma$ and $-\theta = 1 - \frac{1}{\sigma}$, or $\frac{\gamma}{1+\theta} = 1$ so that U is additively separable. What is the value of $\Delta\left(\frac{\gamma}{1+\theta}\right) = \Delta(1)$ for this case? (hint: compute U_{cd}, U_{dd} and U_{cc} at the steady state values of c, d). Verify that Δ, and hence $-(d/c)\,\partial c/\partial d$ does depend on the particular value of σ, given $\bar{c}/\bar{d}$ and $\bar{v}$. Show that the value of $(d/c)\,\psi'$ is smaller than the one for $\gamma = 0$ and $\theta > -1$.

Q14. Assume that $\gamma > 0$, what assumptions are required for $1/(1+\theta)$ such that you also find

$$\psi'\left(\bar{d}\right) = \frac{\bar{c}}{\bar{d}}$$

Hint: look at the formula for ψ'.

Q15. Give an intuitive interpretation for this last two results. (2 lines max).

Q16. Assume that $\gamma > 0$. We will like to show that $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

For this, show that $\Delta\left(\frac{\gamma}{1+\theta}\right)$ is decreasing in γ provided that $\delta > 0$, where $\Delta\left(\gamma/(1+\theta)\right)$ is given by

$$\Delta\left(\frac{\gamma}{1+\theta}\right) \equiv \frac{U_{cd}\left(\delta+\bar{v}\right) - U_{dd}}{\left(-U_{cc}\right)}.$$

Q17. Argue that if $\Delta\left(\gamma/\left(1+\theta\right)\right)$ is decreasing in γ then $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

Q18. Give an intuitive interpretation of this result. (2 lines max).