1 Capital Utilization Rate in the Neoclassical Growth Model

Let output per person at time t be given by $f(k,u)$ where k is the capital stock at time t, and u is the capital utilization at time t. The net change of capital per unit of time is given by:

$$\dot{k} = x - \delta(u)k \ , \tag{1}$$

where x is investment and δ the depreciation rate, which depends on the capital utilization. Output can used for consumption or gross investment as follows

$$c + x = f(k,u) \ . \tag{2}$$

We maintain the following **Assumptions** throughout

(a) For every fixed level of capital k, the function $f(k,\cdot)$ is increasing and concave in the capital utilization rate.

(b) For every fixed level of the capital utilization rate $u > 0$, the function $f(\cdot,u)$ is strictly increasing and strictly concave in capital. We assume that it satisfies standard Inada conditions.

(c) $f_{uk} < 0$, so the marginal productivity of capital decreases as its utilization rate increases.

(d) $\delta(u)$ is positive, increasing and convex in the utilization rate u.

1. Question (5 points). Combining equations (1)-(2), obtain a function $c(k,u,\dot{k})$ for consumption.

   Answer

   $$c(k,u,\dot{k}) \equiv f(k,u) - \delta(u)k - \dot{k}$$

   We study first the problem of maximizing consumption, for fixed levels of capital k and of net change of capital $\dot{k}$ by choice of the utilization rate u. We let $u^* \equiv \arg\max_u c(k,u,\dot{k})$.

2. Question (5 points). Is u^* a function of $\dot{k}$?

   Answer. No, $\dot{k}$ has a level effect on the objective function.

$$u^*(k) \equiv \arg\max_u f(k,u) - \delta(u)k - \dot{k}$$

3. Question (10 points). Write down the first order conditions for u^*. (Use f_u to refer to the derivative of f, and δ' to refer to the derivative of δ)

   Answer.

$$f_u(k,u) = \delta'(u)k$$

4. Question (15 points). Is u^* increasing or decreasing in k? Derive an expression for its derivative. (Use f_u, f_k, f_{uu} and so on to refer to the derivatives of f, and δ' and δ'' to refer to the derivatives of δ). Give one line intuitive explanation of its sign.

   Answer. The function u^* is decreasing in k. Totally differentiating the first order condition:

$$f_{uk}(k,u) - \delta'(u) = [\delta''(u)k - f_{uu}(k,u)]u_k$$

   or

$$u_k = \frac{f_{uk}(k,u) - \delta'(u)}{\delta''(u)k - f_{uu}(k,u)} < 0$$

   The reason why u^* is decreasing is that for small k, the cost of high capital utilization, given by $\delta(u)k$ is small, so it pays to use the capital more.

5. Question (15 points). Define net output as

$$g(k) = \max_u f(k,u) - \delta(u)k \equiv f\left(k, u^*\left(k\right)\right) - \delta\left(u^*\left(k\right)\right)k \tag{3}$$

   Under assumptions (a) to (d) show that g is strictly concave in k, and that $g'(0) > 0$. Make sure you explain how do you use assumptions or answers to previous questions in your answer.

Answer. By the envelope we have: $g' = f_k - \delta$. Evaluated at zero we have $f_k(0, u^*(0)) - \delta(u^*(0)) > 0$. The Inada conditions of f ensures that $g'(0) > 0$. Differentiating this expression we obtain:

$$g''(k) = f_{kk} + [f_{ku} - \delta'(u)]u_k$$

We have that $f_{kk} < 0$ by assumption. We have shown that $u_k < 0$. Finally we assume that $f_{ku} < 0$ and that $\delta' > 0$.

6. Question (15 points). Multiplicative utilization in a neoclassical production function. Assume that

$$f(k, u) = F(ku, 1) \tag{4}$$

where $F(x, y)$ is a constant returns to scale, strictly quasi-concave, neoclassical production function. Does this specification of f satisfies assumptions (a), (b) and (c)? (Use F_1, F_2 and so on to refer to the derivatives of F in your answer).

Answer. No

$$\begin{aligned} f_k(k,u) &= F_1(ku,1)u > 0,\ f_u(k,u) = F_1(ku,1)k > 0, \\ f_{kk}(k,u) &= F_{11}(ku,1)u^2 < 0, \\ f_{uu}(k,u) &= F_{11}(ku,1)k^2 < 0,\ f_{uk}(k,u) = F_1(ku,1) + F_{11}(ku,1)uk \end{aligned}$$

it requires that $f_{uk}(k,u) = F_1(ku,1) + F_{11}(ku,1)uk < 0$, a condition on the elasticity of $F(\cdot, 1)$ being smaller than one.

7. Question (5 points). Show that under the assumption of the previous question, at an optimum, $f_{uk}(k,u) - f_u(k,u)/k < 0$.

Answer. If follows from replacing the foc from u into the expressions found for the derivatives of f in terms of F.

8. Question (5 points). Show that under the assumption of the previous two questions (the multiplicative CRTS case), and using the answers

to them, as well as the characterization of u_k, show that $f_{ku} - \delta' \geq 0$ at an optimum, and hence that $u_k < 0$ and that g is concave.

Answer. Using the expression for the numerator of u_k, the one of f_{uk}, and the foc for u^*:

$$f_{uk}(k,u) - \delta' = F_1(ku,1) + F_{11}(ku,1)uk - F_1(ku,1)k/k = F_{11}(ku,1)uk < 0$$

and hence $u_k < 0$.

9. Question (15 points). Parametric Example. Assume that f is defined as in the previous questions with

$$f(x,y) = (x)^\alpha (yA)^{1-\alpha}, \ 0 < \alpha < 1, \ 0 < A \tag{5}$$

and assume that

$$\delta(u) = d \ u^{1+\psi}/(1+\psi), \ \psi \geq 0 \ . \tag{6}$$

Solve for $u^*(k)$, $f(k, u^*(k))$ and $k\delta(u^*(k))$ in this case (collect the powers of α/d, and k/A separately for the first expression, those of α/d, k and d separately for the second, and those for α/d, k and A for the third). Show that $f(k, u^*(k))$ and $k\delta(u^*(k))$ are jointly homogeneous of degree one in k and A.

Answer.

$$\alpha (ku)^\alpha A^{1-\alpha}/u = \ d \ k \ u^\psi$$

or

$$u = \left(\frac{\alpha}{d} \left(\frac{A}{k} \right)^{1-\alpha} \right)^{1/(1+\psi)} = \left(\frac{\alpha}{d} \right)^{1/(1+\psi)} \left(\frac{k}{A} \right)^{-\frac{1-\alpha}{1+\psi}}$$

We then have:

$$\begin{aligned} f(ku) &= k^\alpha \left(\frac{\alpha}{d} \right)^{1/(1+\psi)} \left(\frac{k}{A} \right)^{-\frac{\alpha(1-\alpha)}{1+\psi}} A^{1-\alpha} \\ &= \left(\frac{\alpha}{d} \right)^{1/(1+\psi)} k^{\alpha\left(1 - \frac{(1-\alpha)}{1+\psi}\right)} A^{(1-\alpha)\left(\frac{\alpha}{1+\psi} + 1\right)} \end{aligned}$$

The sum of the exponents of k and A is 1, so it is hod 1.

$$k\delta(k) = (k)^{1-\frac{1-\alpha}{1+\psi}} \left(\frac{\alpha}{d}\right)^{1/(1+\psi)} (A)^{\frac{1-\alpha}{1+\psi}}$$

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Intertemporal problem. Consider the problem of an agent maximizing the present discounted value of utility, with discount rate ρ and initial capital $k(0)$. Assume that the production function $f(u(t), k(t); A(t))$ is shifted by a time varying parameter $A(t)$ whose path the agent takes as given. The agent solves:

$$\max_{c(t),u(t)\ \ t\geq 0} \int_0^\infty \exp(-\rho t)\ U(c(t))\ dt \tag{7}$$

subject to

$$\dot{k}(t) = f\left(k\left(t\right), u\left(t\right); A\left(t\right)\right) - k\left(t\right)\delta\left(u\left(t\right)\right) - c(t), \tag{8}$$

for $k(0) > 0$ given.

10. Question (10) points. Write the Hamiltonian and the law of motion for the co-state for this problem. Use λ for the costate. Write the problem so that the only controls are c and u.

    Answer:

$$H(k, u, c, \lambda) = U(c) + \lambda(f(k, u, A) - c - \delta(u)k)$$
$$\dot{\lambda} = \rho\lambda - \lambda(f_k(k, u, A) - \delta(u))$$

11. Question (10) points. Write the first order conditions of the Hamiltonian with respect to the controls c and u.

    Answer:

$$c:\ \ 0 = U'(c) - \lambda$$
$$u:\ \ 0 = \lambda(\ f_u(k, u, A) - \delta'(u)k\ )$$

12. Question (5 points) Write the transversality condition for this problem.

    Answer:

$$\lim_{t\to\infty} \exp(-\rho t)\lambda(t)k(t) = \lim_{t\to\infty} \exp(-\rho t)U'\left(c\left(t\right)\right)k(t)$$

13. Questions (15 points) We define a balanced growth path as an optimal solution with

$$c(t) = c(0)\exp(\mu t),\ \ k(t) = k(0)\exp(\mu t),\ \ u(t) = u(0)$$

    for all $t \geq 0$. Assume that

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

    for $\gamma > 0$ and that

$$A(t) = A(0)\exp(\eta t)$$

    for some $\eta > 0$, and furthermore that $f(k, u; A) = F(ku, A)$, where $F(x, y)$ is a constant returns to scale, strictly quasi-concave, neoclassical production function. Write down the law of motion of the co-state, low of motion of the state, and first order condition for u, and use the f.o.c of the co-state to replace $\lambda(t)$ by a function of $c(t)$). Evaluate these three equations in a balance growth path. What should be the value of μ?

    Answer. We guess that $\mu = \eta$, since F is CRTS. Using the law of motion of the co-state λ at t and the foc w.r.t. $c(t)$ and that F is CRST:

$$\begin{aligned}
\gamma\dot{c}(t)/c(t) = \gamma\eta &= F_1\left(u\left(t\right)k\left(t\right), A\left(t\right)\right) - \left(\delta(u(t)) + \rho\right) \\
&= F_1\left(\frac{u(t)k(t)}{A(t)}, 1\right) - \left(\delta(u(t)) + \rho\right) \\
&= F_1\left(\frac{u(0)k(0)}{A(0)}, 1\right) - \left(\delta(u(0)) + \rho\right)
\end{aligned}$$

using feasibility and that F is CRST

$$\begin{aligned} \dot{k}(t) + c(t) + \delta(u(t))k(t) &= \exp(\eta t)[\eta k(0) + c(0) + \delta(u(0))k(0)] \\ &= F(u(t)k(t), A(t)) \\ &= F(u(0)k(0), A(0))\exp(\eta t) \end{aligned}$$

using the foc w.r.t. $u(t)$ and that F is CRST:

$$\begin{aligned} &\delta'(u(t))k(t) = \delta(u(0))k(0)\exp(\eta t) \\ &F_1\left(\frac{k(0)u(0)}{A(0)}, 1\right)k(t) = F_1\left(\frac{k(t)u(t)}{A(t)}, 1\right)k(t) \\ &= F_1\left(k(t)u(t), A(t)\right)k(t) = \delta'(u(t))k(t) = \delta'(u(0))k(0)\exp(\eta t) \end{aligned}$$

14. Question (15 points). Use the conditions for a balanced growth path you have derived in the answer to the previous question to write 3 equations that determine the values for $u(0), k(0)/A(0)$ and $c(0)/A(0)$. These equations should involve the numbers η, γ, ρ as well as the functions $F_1(\cdot, 1), F(\cdot, 1), \delta(\cdot)$ and $\delta'(\cdot)$. Time should NOT appear in these equations.

    Answer:

$$\begin{aligned} \gamma\eta &= F_1\left(u(0)\frac{k(0)}{A(0)}, 1\right) - (\delta(u(0)) + \rho) \\ \eta\frac{k(0)}{A(0)} + \frac{c(0)}{A(0)} + \delta(u(0))\frac{k(0)}{A(0)} &= F\left(u(0)\frac{k(0)}{A(0)}, 1\right) \\ \delta'(u(0)) &= F_1\left(u(0)\frac{k(0)}{A(0)}, 1\right) \end{aligned}$$

15. Question (5 points) What inequality on the parameters η, γ and ρ is implied by the transversality condition in a balanced growth path?

    Answer.

$$\begin{aligned} 0 &= \lim_{t\to\infty} \exp(-\rho t)\lambda(t)k(t) \\ &= \lim_{t\to\infty} \exp\left((-\rho + \eta)\, t\right) u'(c(t))k(0) \\ &= \lim_{t\to\infty} \exp\left((-\rho + \eta - \gamma\eta)\, t\right) c(0)^{-\gamma}k(0)\ , \end{aligned}$$

or $\rho > \eta(1-\gamma)$.

From now on we will assume that $\eta = 0$, so there is no growth in the economy, and $A(0) = A > 0$ for all $t \geq 0$. We can also let $U(\cdot)$ be any strictly increasing and strictly concave utility function. We maintain the specification of $f(k, u; A) = F(kh, A)$ as above.

16. Question (15 points) Rewrite the problem described by the objective function (7) and the law of motion (8) using the definition of net output: $g(k, A) = \max_u F(uk, A) - \delta(u)k$ analyzed in a previous question, instead of using the function f and explicitly maximizing u. s Which are (is) the control(s)?

    Answer
$$\max_{c(t)\ t\geq 0} \int_0^\infty \exp(-\rho t)\ U(c(t))\ dt$$
    subject to
$$\dot{k}(t) = g\left(k\left(t\right), A\right) - c(t),$$
    for $k(0) > 0$ given. Consumption is the only control.

17. Question (15) Is the problem described in the previous question the same as the Neoclassical growth model with fixed labor supply (as in Cass-Koopmans)? Does this model has a unique interior steady state? If an interior steady state exist, is it locally stable? (Answer yes or no to each of these questions)

    Answer. Yes it is the same as the Cass-Koopmans. It has a unique interior steady state. The steady state is locally stable.

18. Question (10 points). Assume that $k(0) > 0$ is below the steady state level of capital $\bar{k}$, i.e. $k(0) < \bar{k}$. How is the time path of capital utilization rate $u(t)$, in particular is it increasing or decreasing as times goes by? Hint: use your answer to the previous question, and the characterization of u^* obtained above.

Answer. Since $\bar{k}$ is locally stable, then $k(t)$ converges to $\bar{k}$ from below, i.e. its path is increasing in time t. Since $u^*(k)$ is decreasing in k, then $u(t)$ is decreasing in time t.

19. Question (30 points). Consider an economy that starts with a capital $k(0) = \bar{k}$, the steady state corresponding to the constant value of the labor augmenting productivity A. Imagine that this economy is now subject to an unexpected and permanent increase to a higher level of labor productivity, say $A' > A$. Denote the new steady state by $\bar{k}'$. Is $\bar{k} > \bar{k}'$? How are the values of the utilization rates corresponding to the two steady states? If one interprets the transition to the new steady state as an expansion, is the behavior of the capital utilization rate pro-cyclical or counter-cyclical?

    Answer. The steady state is increasing in productivity: $\bar{k} < \bar{k}'$. The utilization rate is the same across steady sates, a particular case of the problem studied above. The utilization rate is pro-cyclical, it is higher than its steady state value during the expansion.