Macroeconomics Review Sheet: Laibson

Matthew Basilico

Spring, 2013

Course Outline:

1. **Discrete Time Methods**
   (a) Bellman Equation, Contraction Mapping Theorem, Blackwell's Sufficient Conditions, Numerical Methods
      i. Applications to growth, search, **consumption**, asset pricing
2. **Continuous Time Methods**
   (a) Bellman Equation, Brownian Motion, Ito Proccess, Ito's Lemma
      i. Application to search, consumption, price-setting, **investment**, I.O., asset-pricing

Lecture 1

Sequence Problem
What is the sequence problem?

Find $v(x_0)$ such that

$$v(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1})$$

subject to $x_{t+1} \in \Gamma(x)$ with x_0 given

Variable definitions

- x_t is the state vector at date t
- $F(x_t,\ x_{t+1})$ is the flow payoff at date t
  - function F is stationary
  - a.k.a. the flow utility function
- δ^t is the expontential discount function

Discount Variable Definitions
What are the definitions of the different "discount" variables?

- δ^t is the exponential discount *function*
- δ is referrred to as the *exponential discount factor*
- ρ is the *discount rate*
  - which is the rate of decline of the discount function, so
    * $\rho \equiv -\ln\delta = -\frac{\frac{d\delta^t}{dt}}{\delta^t}$
      · so $e^{-\rho} = \delta$

Bellman Equation
What is the Bellman Equation? What are its components?

Bellman equation expresses the value function as a combination of a flow payoff and a discounted continuation payoff

$$v(x) = \sup_{x_{+1}\in\Gamma(x)} \{F(x,\ x_{+1}) + \delta v(x_{+1})\}$$

Components:

- Flow payoff is $F(x,\ x_{+1})$
- Current value function is $v(x)$. Continuation value function is $v(x_{+1})$
- Equation holds for all (feasible) values of x
- $v(\cdot)$ is called the *solution* to the Bellman Equation
  - Any old function won't solve it

Policy Function
What is a policy function? What is an optimal policy?

A *policy* is a mapping from x to the action space (which is equivalent to the choice variables)
An *optimal policy* achieves payoff $v(x)$ for all feasible x

Relationship between Bellman and Sequence Problem

A solution to the Bellman will also be a solution to the sequence problem, and vice versa.
1. A solution to the Sequence Problem is a solution to the Bellman

$$
\begin{aligned}
v(x_0) &= \sup_{x_{+1}\in\Gamma(x)} \sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1}) \\
&= \sup_{x_{+1}\in\Gamma(x)} \left\{ F(x_0, x_1) + \sum_{t=1}^{\infty} \delta^t F(x_t, x_{t+1}) \right\} \\
&= \sup_{x_{+1}\in\Gamma(x)} \left\{ F(x_0, x_1) + \delta \sum_{t=1}^{\infty} \delta^{t-1} F(x_t, x_{t+1}) \right\} \\
&= \sup_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \delta \sup_{x_{+1}\in\Gamma(x)} \sum_{t=0}^{\infty} \delta^t F(x_{t+1}, x_{t+2}) \right\} \\
&= \sup_{x_1\in\Gamma(x_0)} \{F(x_0, x_1) + \delta v(x_1)\}
\end{aligned}
$$

2. A solution to the Bellman Equation is also a solution to the sequence problem

$$
\begin{aligned}
v(x_0) &= \sup_{x_1\in\Gamma(x_0)} \{F(x_0, x_1) + \delta v(x_1)\} \\
&= \sup_{x_{+1}\in\Gamma(x)} \{F(x_0, x_1) + \delta [F(x_1, x_2) + \delta v(x_2)]\} \\
&\vdots \\
&= \sup_{x_{+1}\in\Gamma(x)} \left\{ F(x_0, x_1) + \cdots + \delta^{n-1} F(x_{n-1}, x_n) + \delta^n v(x_n) \right\} \\
&= \sup_{x_{+1}\in\Gamma(x)} \sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1})
\end{aligned}
$$

Final step requires $\lim_{n\to\infty} \delta^n v(x_n) = 0 \;\; \forall$ feasible x sequences (Stokey and Lucas Thm. 4.3).

Notation with Sequence Problem and Bellman example: Optimal Growth with Cobb-Douglas Technology

How do we write optimal growth using log utility and Cobb-Douglas technology? How can this be translated into sequence problem and Bellman Equation notation?

Optimal growth:

$$\sup_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \ln(c_t)$$

subject to the constraints $c, k \geq 0,\; k^{\alpha} = c + k_{+1},$ and k_0 given

Optimal growth in Sequence Problem notation: [∞- horizon]

$$v(k_0) = \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \ln(k_t^{\alpha} - k_{t+1})$$

such that $k_{t+1} \in [0, k^{\alpha}] \equiv \Gamma(k)$ and $k_0$given

Optimal growth in Bellman Equation notation: [2-period]

$$v(k) = \sup_{k_{+1}\in[0,k^{\alpha}]} \{\ln(k^{\alpha} - k_{+1}) + \delta v(k_{+1})\} \qquad \forall k$$

Methods for Solving the Bellman Equation

What are the 3 methods for solving the Bellman Equation?

1. Guess a solution
2. Iterate a functional operator analytically (This is really just for illustration)
3. Iterate a functional operator numerically (This is the way iterative methods are used in most cases)

FOC and Envelope
What are the FOC and envelope conditions?

First order condtion (FOC):
Differentiate with respect to choice variable

$$0 = \frac{\partial F(x, x_{+1})}{\partial x_{+1}} + \delta v'(x_{+1})$$

Envelope Theorem:
Differentiate with respect to state variable

$$v'(x) = \frac{\partial F(x, x_{+1})}{\partial x}$$

Optimal Stopping using G&C
How do we approach optimal stopping using Guess and Check?

1. Write the Bellman Equation
2. Propose Threshold Rule as Policy Function
3. Use Policy Function to Write Value Function [Bellman Guess]
4. Refine Value Function using continuity of $v(\cdot)$
5. Find value of threshold x^* so that Value Function solves Bellman (using indifference)

Expanded:
Agent draws an offer x from a uniform distribution with support in the unit interval

1. Write the Bellman Equation

$$v(x) = \max \{x, \delta E v(x_{+1})\}$$

2. Propose Threshold Rule as Policy Function

   (a) Stationary threshold rule, with threshold x^*:

$$\begin{array}{lll} Accept & if & x \geq x^* \\ Reject & if & x < x^* \end{array}$$

3. Use Policy Function to Write Value Function [Bellman Guess]

   (a) Stationary threshold rule implies that there exists some constant $\underline{v}$ such that

$$v(x) = \left\{ \begin{array}{lll} x & if & x \geq x^* \\ \underline{v} & if & x < x^* \end{array} \right\}$$

4. Refine Value Function using continuity of $v(\cdot)$

   (a) By continuity of the Value Function, it follows that $\underline{v} = x^*$

$$v(x) = \left\{ \begin{array}{lll} x & if & x \geq x^* \\ x^* & if & x < x^* \end{array} \right\}$$

5. Find value of threshold x^* so that Value Function solves Bellman

   (a) Use indifference condition

$$v(x^*) = x^* = \delta E v(x_{+1})$$

$$\text{so } x^* = \delta \int_{x=0}^{x=x^*} x^* f(x)dx + \delta \int_{x=x^*}^{x=1} x f(x)dx = \delta \frac{1}{2}(x^*)^2 + \delta \frac{1}{2}$$

$$\text{Giving solution } x^* = \delta^{-1}\left(1 - \sqrt{1-\delta^2}\right)$$

Lecture 2

Bellman Operator
What is the Bellman operator? What are its properties as a functional operator?

Bellman operator B, operating on a function w, is defined

$$(Bw)(x) \equiv \sup_{x_{+1} \in \Gamma(x)} \{F(x,\ x_{+1}) + \delta w(x_{+1})\} \quad \forall x$$

Note: The Bellman operator is a contraction mapping that can be used to iterate until convergence to a fixed point, a function that is a solution to the Bellman Equation.
Having Bw on the LHS frees the RHS continuation value function w from being the same function. Hence if $\sup Bw(x) \neq w(x)$, then keep getting a better ("closer") function to the fixed point where $Bw(x) = w(x)$.
Notation: When you iterate, *RHS value function lags* (one period). When you are not iterating, it doesn't lag.

Properties as a functional operator

- Definition is expressed pointwise - for one value of x - but applies to all values of x
- Operator B maps *function* w to new *function* Bw
  - Hence operator B is a *functional operator* since it maps functions

Properties at solution to Bellman Equation

- If v is a solution to the Bellman Equation, then $Bv = v$ (that is, $(Bv)(x) = B(x)\ \forall x$)
- Function v is a *fixed point* of B (B maps v to v)

Contraction Mapping
Why does $B^n w$ converge as $n \to \infty$? What is a contraction mapping? What is the contraction mapping theorem?

The reason $B^n w$ converges as $n \to \infty$ is that B is a *contraction mapping*.

Definition: Let (S, d) be a metric space and $B:\ S \to S$ be a function mapping S onto itself. B is a contraction mapping if for some $\delta \in (0, 1)$, $d(Bf,\ Bg) \leq \delta d(f, g)$ for any two functions f and g.

- Intuition: B is a contraction mapping if operating B on any two functions moves them strictly closer together. (Bf and Bg are strictly closer together than f and g)
- A metric (distance function) is just a way of representing the distance between two functions (e.g. maximum pointwise gap between two functions)

Contraction Mapping Theorem:
If (S, d) is a complete metric space and $B: S \to S$ is a contraction mapping, then:

1. B has exactly one fixed point $v \in S$
2. for any $v_0 \in S$, $\lim B^n v_0 = v$
3. $B^n v_0$ has an exponential convergence rate at least as great as $-\ln \delta$

Blackwell's Theorem
What is Blackwell's Theorem? How is it proved?

Blackwell's Sufficient Conditions **-** these are *sufficienct* (but not necessary) for an operator to be a contraction mapping
Let $X \subseteq \mathcal{R}^l$ and let $C(X)$ be a space of bounded functions $f: X \to \mathcal{R}$, with the sup-metric.
Let $B: C(X) \to C(X)$ be an operator satisfying:

1. **(Monotonicity)** if $f, g \in C(X)$ and $f(x) \leq g(x) \forall x \in X$, then $(Bf)(x) \leq (Bg)(x) \forall x \in X$
2. **(Discounting)** there exists some $\delta \in (0, 1)$ such that

$$[B(f + a)](x) \leq (Bf)(x) + \delta a \qquad \forall f \in C(X),\ a \geq 0,\ x \in X$$

Then B is a contraction with modulus δ.
[Note that a is a constant, and $f + a$ is the function generated by adding a constant to the function f]

Proof:
For any $f, g \in C(X)$ we have $f \leq g + d(f, g)$
Properties 1 & 2 of the conditions imply

$$Bf \leq B(g + d(f, g)) \leq Bg + \delta d(f, g)$$

$$Bg \leq B(f + d(f, g)) \leq Bf + \delta d(f, g)$$

Combining the first and last terms:

$$Bf - Bg \leq \delta d(f, g)$$

$$Bg - Bf \leq \delta d(f, g)$$

$$|(Bf)(x) - (Bg)(x)| \leq \delta d(f, g) \ \ \forall x$$

$$\sup_x |(Bf)(x) - (Bg)(x)| \leq \delta d(f, g)$$

$$d(Bf, Bg) \leq \delta d(f, g)$$

Checking Blackwell's Conditions
Ex 4.1: Check Blackwell conditions for a Bellman operator in a consumption problem

Consumption problem with stochastic asset returns, stocastic labor income, and a liquidity constraint

$$(Bf)(x) = \sup_{c\in[0,x]} \left\{ u(c) + \delta E f\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right)\right\} \quad \forall x$$

1. Monotonicity:
Assume $f(x) \leq g(x) \forall x$. Suppose c_f^* is the optimal policy when the continuation value function is f.

$$(Bf)(x) = \sup_{c\in[0,x]} \left\{ u(c) + \delta E f\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right)\right\} = u(c_f^*) + \delta E f\left(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}\right)$$

$$\leq u(c_f^*) + \delta E g\left(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}\right) \leq \sup_{c\in[0,x]} \left\{ u(c) + \delta E g\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right)\right\} = (Bg)(x)$$

[Note (in top line) elimination of sup by using optimal policy c_f^*. In bottom line, can use any policy to add in sup]

2. Discounting
Adding a constant (Δ) to an optimization problem does not effect optimal choice, so:

$$[B(f+\Delta)](x) = \sup_{c\in[0,x]} \left\{ u(c) + \delta E \left[f\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right) + \Delta\right]\right\}$$

$$= \sup_{c\in[0,x]} \left\{ u(c) + \delta E f\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right)\right\} + \delta\Delta = (Bf)(x) + \delta\Delta$$

Iteration Application: Search/Stopping
What is the general notation? Taking lecture 1 example, can you iterate from an initial guess of $v_0(x) = 0$ for 2 steps? How about proving convergence?

Notation: Iteration of the Bellman operator B:

$$v_n(x) = (B^n v_0)(x) = B(B^{n-1}v_0)(x) = \max\left\{x,\ \delta E(B^{n-1}v_0)(x)\right\}$$

Let $x_n \equiv \delta E(B^{n-1}v_0)(x_{+1})$, which is the continuation payoff for $v_n(x)$.
x_n is also the cutoff threshold associated with $v_n(x) = (B^n v_0)(x)$
x_n is sufficient statistic for $v_n(x) = (B^n v_0)(x)$

Bellman Operator for the Problem: $(Bw)(x) \equiv \max\{x, \delta E w(x_{+1})\}$. *Note there is no lag here in the w index since we are not iterating
Iterating Bellman Operator once on $v_0(x) = 0$:

$$v_1(x) = (Bv_0)(x) = \max\{x, \delta E v_0(x_{+1})\} = \max\{x, 0\} = x$$

Iterating again on $v_1(x) = x$:

$$v_2(x) = \left[(B^2 v_0)(x)\right] = (Bv_1)(x) = \max\{x, \delta E v_1(x_{+1})\} = \max\{x, \delta E x_{+1}\}$$

$x_2 = \delta E x_{+1} = \frac{\delta}{2}$

$$v_2 = \left\{ \begin{array}{ccc} x_2 & if & x \leq x_2 \\ x & if & x \geq x_2 \end{array} \right\}$$

Proof at limit:
We have:

$$(B^{n-1}v_0)(x) = \left\{ \begin{array}{ccc} x_{n-1} & if & x \leq x_{n-1} \\ x & if & x \geq x_{n-1} \end{array} \right\}$$

$$x_n = \delta E(B^{n-1}v_0)(x) = \delta \left[\int_{x=0}^{x=x_{n-1}} x_{n-1} f(x)dx + \int_{x=x_{n-1}}^{x=1} x f(x)dx \right] = \frac{\delta}{2}\left(x_{n-1}^2 + 1\right)$$

We set $\underline{x_n = x_{n-1}}$ to confirm that this sequence converges to

$$\lim_{n\to\infty} x_n = \delta^{-1}\left(1 - \sqrt{1-\delta^2}\right)$$

Lecture 3

Classical Consumption Model
1. Consumption; 2. Linearization of the Euler Equation; 3. Empirical tests without "precautionary savings effects"

Classical Consumption

What is the classical consumption problem, in Sequence Problem and Bellman Equation notation?

1. Sequence Problem Representation

Find $v(x)$ such that

$$v(x_0) = \sup_{\{c_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \delta^t u(c_t)$$

$$\textit{subject to} \left\{ \begin{array}{cc} \text{a static budget constraint for consumption:} & c_t \in \Gamma^C(x) \\ \text{a dynamic budget constraint for assets:} & x_{t+1} \in \Gamma^X\left(x_t, c_t, \tilde{R}_{t+1}, \tilde{y}_{t+1}, \ldots\right) \end{array} \right\}$$

Variables: x is vector of assets, c is consumption, R is vector of financial asset returns, y is vector of labor income

Common Example:
The only asset is cash on hand, and consumption is constrained to lie between 0 and x:

$$c_t \in \Gamma^C(x_t) \equiv [0, x_t]; \quad x_{t+1} \in \Gamma^X\left(x_t, c_t, \tilde{R}_{t+1}, \tilde{y}_{t+1}, \ldots\right) \equiv \tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1}; \quad x_0 = y_0$$

Assumptions: $\tilde{y}$ is exogenous and iid; u is concave; $\lim_{c\downarrow 0} u'(c) = \infty$ (so $c > 0$ as long as $x > 0$)

2. Bellman Equation Representation

[It is more convenient to think about c as the choice variable. The state variable, x, is stochastic, so it is not directly chosen (rather a distribution for x_{t+1} is chosen at time t)]

$$v(x_t) = \sup_{c_t \in [0,x_t]} \{u(c_t) + \delta E_t v(x_{t+1})\} \quad \forall x$$

$$x_{t+1} = \tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1}$$

$$x_0 = y_0$$

Euler Equation 1: Optimization

How is the Euler equation derived from the Bellman Equation of the consumption problem above using optimization?

$$v(x_t) = \sup_{c_t \in [0,x_t]} \left\{u(c_t) + \delta E_t v\left(\tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1}\right)\right\} \quad \forall x$$

1. First Order Condition:

$$\begin{array}{llll} u'(c_t) = \delta E_t \tilde{R}_{t+1} v'(x_{t+1}) & if & 0 < c_t < x_t & \text{(interior)} \\ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} v'(x_{t+1}) & if & c_t = x_t & \text{(boundary)} \end{array}$$

$\left[FOC_{c_t}: \quad 0 = u'(c_t) + \delta E_t v'(x_{t+1}) \times (-\tilde{R}_{t+1}) \text{ using chain rule}\right]$

2. Envelope Theorem

$$v'(x_t) = u'(c_t)$$

$\left[\text{Differentiate with respect to state variable } x_t: \ v'(x) = u'(c_t)\frac{\partial c_t}{\partial x_t} = u'(c_t)\right]$

$\left[\text{Note: } x_{t+1} = \tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1} \implies c_t\tilde{R}_{t+1} = \tilde{R}_{t+1}x_t - x_{t+1} + \tilde{y}_{t+1} \implies c_t = x_t - \frac{x_{t+1} + \tilde{y}_{t+1}}{\tilde{R}_{t+1}}\right]$

Putting the FOC and Envelope Condition together, and moving the Envelope Condition forward one period, we get:

▶***Euler Equation:***

$$\begin{array}{llll} u'(c_t) = \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) & if & 0 < c_t < x_t & \text{(interior)} \\ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) & if & c_t = x_t & \text{(boundary)} \end{array}$$

Euler Equation 2: Perturbation

How is the Euler equation derived from the Bellman Equation of the consumption problem above using perturbation?

[*Prefers that we know it this way*]

1. General Intuition

- "Cost of consuming one dollar less today?"

  - Utility loss today $= u'(c_t)$

- "Value of saving an extra dollar" – Discounted, expected utility gain of consuming $\tilde{R}_{t+1}$ more dollars tomorrow:
  - Utility gain tomorrow $= \delta E_t \tilde{R}_{t+1} u'(c_{t+1})$

2. Perturbation
***Idea: at an optimum, there cannot be a perturbation that improves welfare of the agent.*
If there is such a feasible pertubation, then we've proven that whatever we wrote down is not an optimum.

Proof by contradiction: [Assume inequalities, and show they produce contradictions so cannot hold]

1. Suppose $u'(c_t) < \delta E_t \tilde{R}_{t+1} u'(c_{t+1})$.

   [If $u'(c_t)$ is less than the discounted value of a dollar saved, then what should I do? Save more, consume less.]

   (a) Then we can reduce c_t by ε and raise c_{t+1} by $\tilde{R}_{t+1} \cdot \varepsilon$ *to generate a net utility gain*:

$$0 < \Big[\delta E_t \tilde{R}_{t+1} u'(c_{t+1}) - u'(c_t)\Big] \cdot \varepsilon$$

   $\Longrightarrow$ this perturbation raises welfare [Q? - defining welfare here]

   [Note the expression is from simply moving $u'(c_t)$ to RHS from above and multiplying by ε]

   (a) If this perturbation is possible that raises welfare, then can't be true that we started this analysis at an optimum.
      i. This perturbation is always possible along the equilibrium path.
      ii. Hence, if this cannot be an optimum, we've shown that at an optimum, we must have:

$$u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(c_{t+1})$$

2. Suppose $u'(c_t) > \delta E_t \tilde{R}_{t+1} u'(c_{t+1})$

   (a) Then we can raise c_t by ε and reduce c_{t+1} by $\tilde{R}_{t+1} \cdot \varepsilon$ *to generate a net utility gain*:
      i. [If $u'(c_t)$ is greater than the value of a dollar saved, then what should I do? Save less, consume more.]

$$\varepsilon \cdot \Big[u'(c_t) - \delta E_t \tilde{R}_{t+1} u'(c_{t+1})\Big] > 0$$

      $\Longrightarrow$ this perturbation raises welfare [Q? - defining welfare here]

   (b) If this perturbation is possible that raises welfare, then can't be true that we started this analysis at an optimum.
      i. This perturbation is always possible along the equilibrium path, as long as $c_t < x_t$ (liquidity constraint not binding).
      ii. Hence, if this cannot be an optimum, we've shown that at an optimum, we must have:

$$u'(c_t) \leq \delta E_t \tilde{R}_{t+1} u'(c_{t+1}) \quad \text{as long as } c_t < x_t$$

It follows that:

$$\begin{array}{llll} u'(c_t) = \delta E_t \tilde{R}_{t+1} u'(x_{t+1}) & if & 0 < c_t < x_t & \text{(interior)} \\ u'(c_t) \geq \delta E_t \tilde{R}_{t+1} u'(x_{t+1}) & if & c_t = x_t & \text{(boundary)} \end{array}$$

Linearizing the Euler Equation

How do you linearize the Euler Equation?

Goal: Can we linearize this and make it operational without unrealistic assumptions?
Equation tells us what I should expect *consumption growth* $\Delta \ln c_{t+1}$ to do

1. Assume u is an isoelastic (i.e. CRRA) utility function

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

   Note: $\lim_{\gamma \to 1} \frac{c^{\gamma-1}-1}{1-\gamma} = \ln c$. Important special case.

2. Assume R_{t+1} is known at time t

3. Rewrite the Euler Equation [using functional form]:

$$c_t^{-\gamma} = E_t \delta R_{t+1} c_{t+1}^{-\gamma}$$

4. [Algebra] Divide by $c_t^{-\gamma}$

$$1 = E_t \delta R_{t+1} c_{t+1}^{-\gamma} c_t^{\gamma}$$

5. [Algebra] Rearrange, using exp[ln]:

$$1 = E_t \exp \left[\ln \left(\delta R_{t+1} c_{t+1}^{-\gamma} c_t^{\gamma} \right) \right]$$

6. [Algebra] Distribute ln:

$$1 = E_t \exp \left[r_{t+1} - \rho + (-\gamma) \ln c_{t+1}/c_t \right]$$

   [Note: $\ln \delta = -\rho$; $\ln R_{t+1} = r_{t+1}$]

7. [Algebra] Substitute notation for consumption growth:

$$1 = E_t \exp \left[r_{t+1} - \rho - \gamma \Delta \ln c_{t+1} \right]$$

   [Note: $\ln(c_{t+1}/c_t) = \ln c_{t+1} - \ln c_t \equiv \Delta \ln c_{t+1}$]

8. [Stats] Assume $\Delta \ln c_{t+1}$ is conditionally normally distributed. Apply expectation operator:

$$1 = \exp \left[E_t r_{t+1} - \rho - \gamma \Delta \ln c_{t+1} + \frac{1}{2} \gamma^2 V_t \Delta \ln c_{t+1} \right]$$

   $\Big[$Note: $Ee^{\tilde{a}} = e^{E\tilde{a} + \frac{1}{2} Var \tilde{a}}$ where $\tilde{a}$is a random variable$\Big]$

   $\Big[$Here we have: $-\gamma \Delta \ln c_{t+1} \sim \mathcal{N}(-\gamma \Delta \ln c_{t+1}, \gamma^2 V_t \Delta \ln c_{t+1})\Big]$

9. [Algebra] Take ln:

$$0 = E_t r_{t+1} - \rho - \gamma \Delta \ln c_{t+1} + \frac{1}{2}\gamma^2 V_t \Delta \ln c_{t+1}$$

10. [Algebra] Divide by γ, rearrange:

$$(*) \quad \Delta \ln c_{t+1} = \frac{1}{\gamma}\left(E_t r_{t+1} - \rho\right) + \frac{1}{2}\gamma V_t \Delta \ln c_{t+1}$$

Terms of linearized Euler Equation $(*)$*:*
$\Delta \ln c_{t+1}$ = consumption growth
$V_t \Delta \ln c_{t+1}$ =conditional variance in consumption growth – *conditional on information known at time t.*
(i.e. conditional on being in year 49, what is the variance in consumption growth between 49 and 50)
It is also known as the *"precautionary savings term"* as seen in the regression analysis

$\Delta \ln c_{t+1}$
Why is $\Delta \ln c_{t+1}$ *consumption growth?*

$$\frac{c_{t+1} - c_t}{c_t} \approx \ln\left(1 + \frac{c_{t+1} - c_t}{c_t}\right) = \ln\left(1 + \frac{c_{t+1}}{c_t} - 1\right) = \ln\left(\frac{c_{t+1}}{c_t}\right) \equiv \Delta \ln c_{t+1}$$

[the second step uses the log approximation, $\Delta x \approx \ln(1 + \Delta x)$]

Important Consumption Models

What are the two important consumption models?

In 1970s, began recognizing that what is interesting is *change* in consumption, as opposed to just levels of consumption. Since then have stayed with this approach.

1. **Life Cycle Hypothesis (Modigliani & Brumberg, 1954; Friedman) = Eat the Pie Problem**

   (a) Assumptions

      i. $\tilde{R}_t = R$

      ii. $\delta R = 1$ $[\delta < 1; R > 1]$

      iii. Perfect capital markets, no moral hazard, so future labor income can be exchanged for current capital

   (b) Bellman Equation

$$v(x) = \sup_{c \le x}\{u(c) + \delta E v(x_{+1})\} \quad \forall x$$

$$x_{+1} = R(x - c)$$

$$\underline{x_0 = E\sum_{t=0}^{\infty} R^{-t}\tilde{y}_t}$$

[At date 0, sells all his labor income for a lump sum, discounted at rate r. This sum is his only source of wealth now]

["Eat the Pie," except every piece he leaves grows by a factor r]

(c) Euler Equation

i. $u'(c_t) = \delta R u'(c_{t+1}) = 1 \cdot u'(c_{t+1}) = u'(c_{t+1})$

ii. $\implies$ ***consumption is constant*** in all periods

(d) Budget Constraint

$$\sum_{t=0}^{\infty} R^{-t} c_t = E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t$$

[(Discounted sum of his consumption) must be equal to [the expectation at date 0 of (the discounted sum of his labor income)]]

(e) Subsitute Euler Equation

$$\sum_{t=0}^{\infty} R^{-t} c_0 = E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t$$

i. To give final form: [Euler Equation + Budget Constraint] $\left[\text{Note: } \sum_{t=0}^{\infty} R^{-t} = \frac{1}{1-\frac{1}{R}}\right]$

$$c_0 = \left(1 - \frac{1}{R}\right) E_0 \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t \quad \forall t$$

Intuition: Consumption is an annuity. Consumption is equal to interest rate scaling my total net worth at date 0

$\left(1 - \frac{1}{R}\right)$ is the annuity scaling term; approximately equal to the real net interest rate

2. **Certainty Equivalence Model (Hall, 1978)**

(a) Assumptions

i. $\tilde{R}_t = R$

ii. $\delta R = 1$ $[\delta < 1; R > 1]$

iii. ***Can't sell claims to labor income***

iv. ***Quadratic utility***: $u(c) = \alpha c - \frac{\beta}{2} c^2$

A. Note: This admits negative consumption, and does not imply that $\lim_{c \downarrow 0} u'(c) = \infty$

B. [Hence it is possible to consume a negative amount]

(b) Bellman Equation

$$v(x) = \sup_c \{u(c) + \delta E v(x_{+1})\} \quad \forall x$$

$$x_{+1} = R(x - c) + \tilde{y}_{+1}$$

$$x_0 = y_0$$

(c) Euler Equation

i. Euler becomes: $c_t = E_t c_{t+1} = E_t c_{t+n}$
$u'(c) = \alpha - \beta c$. $u'(c_t) = \delta E_t \tilde{R}_{t+1}$; $u'(c_{t+1}) = E_t u'(c_{t+1})$ [Since $\delta R = 1$]
$\implies \alpha - \beta c_t = E_t[\alpha - \beta c_{t+1}] = \alpha - \beta E_t c_{t+1} \implies -\beta c_t = -\beta E_t c_{t+1} \implies c_t = E_t c_{t+1}$

ii. So $c_t = E_t c_{t+1} = E_t c_{t+n}$ implies consumption is ***a random walk***:

$$c_{t+1} = c_t + \varepsilon_{t+1}$$

iii. ***So*** Δc_{t+1} ***can not be predicted by any information available at time*** t

A. "If I know c_t, no other economic variable should have predictive power at c_{t+1}; any other information should be *orthogonal*"

(d) Budget Constraint

$$\sum_{t=0}^{\infty} R^{-t} c_t \leq \sum_{t=0}^{\infty} R^{-t} \tilde{y}_t$$

[Sum from t to ∞ of discounted consumption is $\leq$the discounted value of stochastic labor income]

i. Budget Constraint at time t:

$$\sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}$$

[Discounted sum of remaining consumption = cash on hand (at date t) and the discounted sum of remaining labor income]

ii. Now applying expectation: ["if it's true on every path, it's true in expectation"]

$$E_t \sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}$$

(e) Subsitute Euler Equation [$c_t = E_t c_{t+s}$]

$$\sum_{s=0}^{\infty} R^{-s} c_t = x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}$$

i. To give final form: [Euler Equation + Budget Constraint]$\left[\text{Note: } \sum_{t=0}^{\infty} R^{-t} = \frac{1}{1-\frac{1}{R}}\right]$

$$c_t = \left(1 - \frac{1}{R}\right)\left(x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s}\right) \quad \forall t$$

Intuition: Just like in Modigliani, consumption is equal to interest rate scaling my total net worth at date t

i. (but get there just by assuming a quadratic utility function)

$\left(1 - \frac{1}{R}\right)$ is the annuity scaling term (approximately equal to the real net interest rate)
$(x_t + E_t \sum_{s=1}^{\infty} R^{-s} \tilde{y}_{t+s})$ is total net worth at date t

Empirical Tests of Linearized Euler Equation: Without Precautionary Savings Effects

Want to prove that consumption growth is unpredictable between t and $t+1$
A. Testing Linearized Euler Equation: Without Precautionary Savings Effects

1. Start with: Linearized Euler Equation, in regression form:

$$\Delta \ln c_{t+1} = \frac{1}{\gamma}\left(E_t r_{t+1} - \rho\right) + \frac{1}{2}\gamma V_t \Delta \ln c_{t+1} + \varepsilon_{t+1}$$

[where ε_{t+1} is orthogonal to any information known at time t.]

2. Assume (counterfactually) "precautionary savings" is constant

   (a) Euler equation reduces to: $\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \varepsilon_{t+1}$

[Note: when we replace precautionary term with a constant, we are effectively ignoring its effect (since it is no longer separately identified from the other constant term $\frac{\rho}{\gamma}$)]

B. Estimating the Linearized Euler Equation: 2 Goals and Approaches

1. Goal 1: Estimate $\frac{1}{\gamma}$

   (a) $\frac{1}{\gamma}$ is EIS, the elasticity of intertemporal substitution [for this model, EIS is the inverse of the CRRA; $\therefore$ in other models EIS not $\frac{1}{\gamma}$]

   (b) Estimate using $\frac{\partial \Delta \ln c_{t+1}}{\partial E_t r_{t+1}}$, with equation $\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \varepsilon_{t+1}$

2. Goal 2: Test the orthagonality restriction $\Omega_t \perp \varepsilon_{t+1}$

   (a) This means, test the restriction that information available at time t does not predict consumption growth in the following regression [new βX_t term, for any variable]:

$$\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \beta X_t + \varepsilon_{t+1}$$

   (a) For example, does the date t expectation of income growth, $E_t \Delta \ln Y_{t+1}$ predict date $t+1$ consumption growth [for Shea, etc test below]?

      i. [Model is trying to say that an expectation of rising income–i.e. being hired on job market–doesn't translate into a rise in consumption. α should be 0.]

$$\Delta \ln c_{t+1} = \text{constant} + \frac{1}{\gamma} E_t r_{t+1} + \alpha E_t \Delta \ln Y_{t+1} + \varepsilon_{t+1}$$

C. Summary of Results:

1. $\frac{1}{\gamma} \in [0, 0.2]$ (Hall, 1988)

   (a) EIS is very small–people are fairly unresponsive to expected movements in the interest rate; this doesn't seem to be a big driver of consumption growth

      i. "the parameter that is scaling the expected value of the interest rate" is estimated to be close to 0

(b) But estimate may be messed up once you add in liquidity contraints. Someone who is very liquidity constrained $\rightarrow$ EIS of 0.

(c) So what exactly are we learning when we see American consumers are not responsive to the interest rate? [Is it just about liquidity constraints, or does it mean that even in their absence, something deep about responsiveness to interest rate]

2. $\alpha \in [0.1, 0.8]$ (Campell and Mankiw 1989, Shea 1995)

   (a) $\implies$ expected income growth (predicted at date t) predicts consupmtion growth at date $t+1$

   (b) $\implies$ the assumptions (1) the Euler Equation is true, (2) the utility function is CRRA, (3) the linearization is accurate, and (4) $V_t \Delta \ln c_{t+1}$ is constant, are *jointly rejected*

   (c) Theories on why does expected income predict consumption growth:

      i. Leisure and consumption expenditure are substitutes
      ii. Work-based expenses
      iii. Households support lots of dependents in mid-life when income is highest
      iv. Houesholds are liquidity-constrained and impatient
      v. Some consumers use rules of thumb: $c_{it} = Y_{it}$
      vi. Welfare costs of smoothing are second-order

Lecture 4

Precautionary Savings and Liquidity Contraints
1. Precautionary Savings Motives; 2. Liquidity Constraints; 3. Application: Numerical solution of problem with liquidity constraints; 4. Comparison to "eat-the-pie" problem

Precautionary Motives

Why do people save more in response to increased uncertainty?

1. Uncertainty and the Euler Equation

$$E_t \Delta \ln c_{t+1} = \frac{1}{\gamma}(E_t r_{t+1} - \rho) + \frac{\gamma}{2} V_t \Delta \ln c_{t+1}$$

   $\left[\text{where } V_t \Delta c_{t+1} = E_t[\Delta \ln c_{t+1} - E_t \Delta \ln c_{t+1}]^2\right]$

   (a) *Increase in economic uncertainty raises $V_t \Delta c_{t+1}$, raising $E_t \Delta \ln c_{t+1}$*

      i. Note: this is not a general result (i.e. doesn't apply to quadratic)

2. Key reason is the *convexity of marginal utility*

   (a) An increase in uncertainty raises the expected value of marginal utility

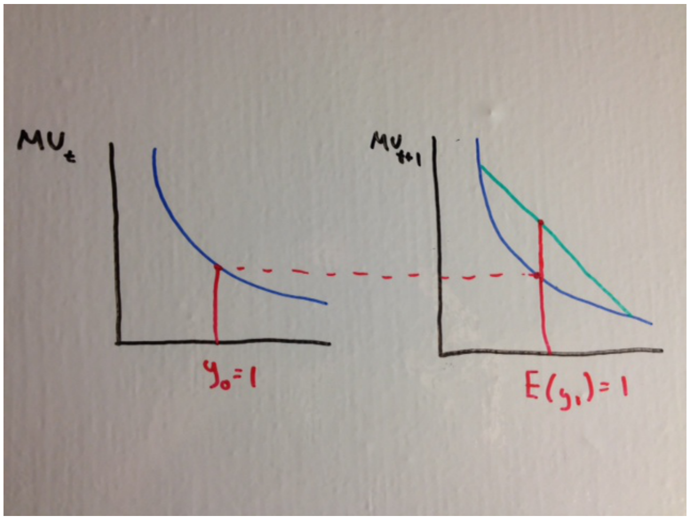
      i. Convexity of marginal utility curve raises marginal utility <u>in expectation</u> in the next period

         A. Euler equation (for CRRA): $u'(c_t) \neq u'(E_t(c_{t+1}))$ but rather $u'(c_t) = E_t u'(c_{t+1})$

B. (Since by Jensen's inequality $\implies \frac{E_t(c_{t+1})}{c_t} > 1$)

(b) This increase the motive to save

i. $\implies$ referred to as the "precautionary savings effect"



3. Definition: Precautionary Savings

(a) Precautionary saving is the reduction in consumption due to the fact that future labor income is uncertain instead of being fixed at its mean value.

Liquidity Constraints

Buffer Stock Models

What are the two key assumptions of the "buffer stock" models? Qualitatively, what are some of their predictions

Since the 1990s, consumption models have emphasized the role of liquidity constraints (Zeldes, Caroll, Deaton).

1. Consumers face a borrowing limit: e.g. $c_t \leq x_t$

(a) This matters whether or not it actually binds in equilibrium (since desire to preserve buffer stock will affect marginal utilty as you get close to full-borrowing level)

2. Consumers are impatient: $\rho > r$

- Predictions
  - Consumers accumulate a small stock of assets to buffer transitory income shocks
  - Consumption weakly tracks income at high frequencies (even predictable income)
  - Consumption strongly tracks income at low frequencies (even predictable income)

Eat the Pie Problem

A model in which the consumer can securitize her income stream. In this model, labor can be transformed into a bond.

1. Assumptions

   (a) If consumers have exogenous idiosyncratic labor income risk, then there is no risk premium and consumers can sell their labor income for

   $$W_0 = E_0 \sum_{t=0}^{\infty} R^{-t} y_t$$

   (b) Budget constraint

   $$W_{+1} = R(W - c)$$

2. Bellman Equation

$$v(W) = \sup_{c \in [0,W]} \{u(c) + \delta E v(R(W - c))\} \quad \forall x$$

3. Solve Bellman: Guess Solution Form

$$v(W) = \left\{ \begin{array}{ccc} \psi \frac{W^{1-\gamma}}{1-\gamma} & \text{if} & \gamma \in [0, \infty], \ \gamma \neq 1 \\ \phi + \psi \ln W & \text{if} & \gamma = 1 \end{array} \right\}$$

4. Confirm that solution works (problem set)

5. Derive optimal policy rule (problem set)

$$c = \psi^{-\frac{1}{\gamma}} W$$

$$\psi^{-\frac{1}{\gamma}} = 1 - (\delta R^{1-\gamma})^{\frac{1}{\gamma}}$$

Lecture 5

Non-stationary dynamic programming

Non-Stationary Dynamic Programming

What type of problems does non-stationary dynamic programming solve? What is different about the approach generally?

Non-Stationary Problems:

- So far we have assumed problem is stationary - that is, value function does not depend on time, only on the state variable
- Now we apply to *finite horizon* problems
  - E.g. Born at $t = 1$, terminate at $t = T = 40$

Two changes for finite horizon problems: ***backwards induction***

1. We index the value function, ***since each value function is now date-specific***
   - $v_t(x)$ represents value function for period t
   - E.g., $v_t(x) = E_t \sum_{s=t}^{T} \delta^{s-t} u(c_s)$
2. We *don't use an arbitrary* starting function for the Bellman operator
   - Instead, *iterate the Bellman operator on* $v_T(x)$, where T is the last period in the problem
     - In most cases, the last period is easy to calculate, (i.e. "end of life" with no bequest motive: $v_T(x) = u(x)$)

Backward Induction

How does backward induction proceed?

- Since you begin in the final period, the RHS equation is one period ahead of the LHS side equation (opposite of previous notation with operator)
  - Each iteration takes on more step back in time
- Concretely, we have the Bellman Equation

$$v_{t-1}(x) = \sup_{c \in [0,x]} \{u(c) + \delta E_t v_t (R(x-c) + \tilde{y})\} \quad \forall x$$

- ****To generate $v_{t-1}(x)$, apply the Bellman Operator***

$$v_{t-1}(x) = (Bv_t)(x) = \sup_{c \in [0,x]} \{u(c) + \delta E_t v_t (R(x-c) + \tilde{y})\} \quad \forall x$$

  - (As noted above, generally start at termination date T and work backwards)
- Generally we have:

$$v_{T-n}(x) = (B^n v_T)(x)$$

Lecture 6

Hyperbolic Discounting

Context:
For $\beta < 1$, $U_t = u(c_t) + \beta\left[\delta u(c_{t+1}) + \delta^2 u(c_{t+2}) + \delta^3 u(c_{t+3}) + \ldots\right]$

Hyperbolic Bellman Equations

Bellman Equation in set up in the following way. Define three equations where:
V is the continuation value function
W is the current value function
C is the consumption function

1. $V(x) = U(C(x)) + \delta E[V(R(x - C(x)) + y)]$
2. $W(x) = U(C(x)) + \beta\delta E[V(R(x - C(x)) + y)]$
3. $C(x) = \arg\max_c U(c) + \beta\delta E[V(R(x - c) + y)]$

Three further identities:

- ***Identity linking*** V ***and*** W

$$\beta V(x) = W(x) - (1 - \beta)U(C(x))$$

- Envelope Theorem

$$W'(x) = U'(C(x))$$

- FOC

$$U'(C(x)) = R\beta\delta E\left[V'(R(x - C(x)) + y)\right]$$

Hyperbolic Euler Equation

How do we derive an Euler equation for the hyperbolic scenario? What is the intuition behind the equation?

1. We have the FOC

$$u'(c_t) = R\beta\delta E_t\left[V'(x_{t+1})\right]$$

2. Use the *Identity linking* V *and* W to subsitute for V'

$$= R\delta E_t\left[W'(x_{t+1}) - (1 - \beta)u'(c_{t+1})\frac{dC_{t+1}}{dx_{t+1}}\right]$$

3. Use envelope theorem

(a) [substitute $u'(x_{t+1})$ for $W'(c_{t+1})$]

$$= R\delta E_t \left[u'(x_{t+1}) - (1-\beta) u'(c_{t+1}) \frac{dC_{t+1}}{dx_{t+1}} \right]$$

4. Distribute

$$= RE_t \left[\beta\delta \left(\frac{dC_{t+1}}{dx_{t+1}} \right) + \delta \left(1 - \frac{dC_{t+1}}{dx_{t+1}} \right) \right] u'(c_{t+1}) \qquad (*)$$

Interpretation of Hyperbolic Euler Equation $(*)$

- First, note that $\frac{dC_{t+1}}{dx_{t+1}}$ is the *marginal propensity to consume ("MPC").*
- We then see that the main difference in this Euler equation is that it is a ***weighted average of discount factors***

$$= RE_t \left[\beta\delta \underbrace{\left(\frac{dC_{t+1}}{dx_{t+1}} \right)}_{\text{short run}} + \delta \underbrace{\left(1 - \frac{dC_{t+1}}{dx_{t+1}} \right)}_{\text{long run}} \right] u'(c_{t+1})$$

- More cash on hand will shift to to the long-run term (less hyperbolic discounting)

Sophisticated vs. Naive Agents

[From Section Notes]

- Sophisticated Agents
  - The above analysis has assumed sophisticated agents
  - A sophisticated agent knows that in the future, he will continue to be quasi-hyperbolic (that is, he knows that he will continue to choose future consumption in such a way that he discounts by $\beta\delta$ between the current and next period, but by δ between future periods.)
  - This is reflected in the specification of the consumption policy function (as above)
- Naive Agents
  - Doesn't realize he will discount in a quasi-hyperbolic way in future periods
  - He wrongly believes that, starting from the next period, he will discount by δ between all periods.
    - (Another way of saying: he wrongly believes that he will be willing to commit to the path that he currently sets)
  - He will decide today, believing that he will choose from tomorrow forward by the exponential discount function. (^{e} superscript); however, when he reaches the next date, he again reoptimizes:

1. Define
   (a) *Continuation* value function, as given by standard *exponential* discounting Bellman equation:

$$V^e_{t+1}(x) = \max_{c^e_{t+1}\in[0,x]} \left\{u\left(c^e_{t+1}\right) + \delta E_{t+1} V^e_{t+2}\left(R(x - c^e_{t+1}) + \tilde{y}\right)\right\}$$

   (b) *Current* value function, as given by the hyperbolic equation:

$$W^n_t(x) = \max_{c^e_t\in[0,x]} \left\{u\left(c^n\right) + \delta E_t V^e_{t+1}\left(R(x - c^e_t) + \tilde{y}\right)\right\}$$

2. Since he is not able to commit to the exponential continuation function, realized consumption path will be given:
   (a) Realized consumption path: $\{c^n_\tau(x_\tau)\}^T_{\tau=t}$
   (b) Solution at each time t is characterized by

$$u'(c^e_{t+1}) = \delta R E_{t+1} u'\left(c^e_{t+2}\right)$$

$$u'(c^n_t) = \beta\delta R E_t V^{'e}(x_{t+1}) = \beta\delta R E_t u'\left(c^e_{t+1}\right)$$

Lecture 7

Asset Pricing
1. Equity Premium Puzzle; 2. Calibration of Risk Aversion; 3. Resolutions to the Equity Premium Puzzle

Intuition

- Classical economics says that you should take an arbitrarily small bet with any positive return...***as long as the payoffs of the bet are uncorrelated with your consumption path*** (punchline)
- Don't want to make a bet where you will give up resources in a low state of consumption (when the marginal utility of consumption is higher) for resources in a high state of consumption (where marginal utility of consumption is lower) [all because of curvature of the utility function]
- "Not variance-based risk, *covariance* based risk"

Asset Pricing Equation

What is the asset pricing equation? How is it derived?

Asset Pricing Equation
General Form:

$$\pi^{ij} = \gamma\sigma_{ic} - \gamma\sigma_{jc}$$

where $\pi^{ij} = r^i - r^j$ (the gap in rate of returns between assets $i$and j)

$$\sigma_{ic} = Cov(\sigma^i \varepsilon^i, \Delta \ln c)$$

Risk-Free Asset:
Definition: we denote the risk free return: R_t^f. Specifically, we assume that at time $t-1$, R_t^f is known. When i =equities and j =risk free asset, we have the simple relationship

$$\pi^{equity,f} = \gamma \sigma_{equity,c}$$

Intuition:

- The equity premium $\pi^{equity,f}$, is equal to the amount of risk, $\sigma_{equity,c}$, multiplied by the price of risk, γ.

Asset Notation and Statistics:

1. Allowing for many different types of assets: $R_{t+1}^1, R_{t+1}^2, \ldots, R_{t+1}^i, \ldots, R_{t+1}^I$
2. Assets have stochastic returns:

$$R_{t+1}^i = e^{\left(r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}[\sigma^i]^2\right)} \text{ where } \varepsilon_{t+1}^i \text{has unit variance}$$

ε_{t+1}^i has unit variance $\implies$ ε_{t+1}^i is a normally distributed random variable: $\varepsilon_{t+1}^i \sim \mathcal{N}(0,1)$

$x \sim N(\mu, \sigma^2) \longrightarrow Ax + B \sim N(A\mu + B, A^2\sigma^2)$
$\longrightarrow E[\exp(Ax + B)] = \exp\left(A\mu + B + \frac{1}{2}A^2\sigma^2\right)$
[Since $x \sim N(\mu, \sigma^2) \longrightarrow E(\exp(x)) = \exp(E(x) + \frac{1}{2}Var(x))$; so if standard normal:
$= exp(\mu + \frac{1}{2}\sigma^2)$]

Hence: $\sigma^i \varepsilon_{t+1}^i + r_{t+1}^i - \frac{1}{2}[\sigma^i]^2 \sim N(\sigma^i * 0 + r_{t+1}^i - \frac{1}{2}[\sigma^i]^2, \sigma^{i2} * 1^2) = N(r_{t+1}^i - \frac{1}{2}[\sigma^i]^2, \sigma^{i2})$
$E[R_i^t] = E\left[\exp(r_{t+1}^i + \sigma^i \varepsilon_{t+1}^i - \frac{1}{2}[\sigma^i]^2)\right]$ which as we've seen is now exp("a random variable") so we can use
$= \exp(mean + \frac{1}{2}Var) = \exp(r_{t+1}^i - \frac{1}{2}[\sigma^i]^2 + \frac{1}{2}\sigma^{i2}) = \exp(r_{t+1}^i)$
Finally we have that since for small x: $\ln(1+x) \approx x \quad \implies \quad 1 + x \approx e^x$
$\therefore \exp(r_t^i) \approx 1 + r_{t+1}^i$.

Derivation of Asset Pricing Equation: $\pi^{ij} = \gamma\sigma_{ic} - \gamma\sigma_{jc}$
How is the Asset Pricing Equation derived?

Use Euler, Substitute, Take Expectation (Random Variable), Difference Two Equations, Apply Covariance Formula

1. Start with the Euler Equation

$$u'(c_t) = E_t\left[\delta R_{t+1}^i u'(c_{t+1})\right]$$

2. Assume u is isoelastic (CRRA):

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

3. Substitute into the Euler equation
   (a) $\delta = \exp(-\rho)$
   (b) $R^i_{t+1} = e^{\left(r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}[\sigma^i]^2\right)}$ where ε^i_{t+1}has unit variance
   (c) $u'(c) = c^{-\gamma}$ (given CRRA)

$$c^{-\gamma} = E_t\left[\delta R^i_{t+1} c^{-\gamma}_{t+1}\right]$$

$$c^{-\gamma} = E_t\left[\exp\left(-\rho + r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2\right) c^{-\gamma}_{t+1}\right]$$

$$1 = E_t\left[\exp\left(-\rho + r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2\right)\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]$$

4. Re-write in terms of $\exp(\cdot)$, using $\exp(\ln)$

$$1 = E_t\left[\exp\left(-\rho + r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2\right)\exp\left[-\gamma \ln\left(\frac{c_{t+1}}{c_t}\right)\right]\right]$$

$$1 = E_t\left[\exp\left(-\rho + r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2 - \gamma \Delta \ln\left(c_{t+1}\right)\right)\right]$$

   (a) Rearrange, and note

$$1 = E_t\left[\exp\left(\underbrace{-\rho + r^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2}_{\text{non-stochastic}} + \underbrace{\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)}_{\text{random variable}}\right)\right]$$

5. Take Expectation, applying the formula for the expectation of a Random Variable
   (a) That is, use: $Sx \sim N(S\mu, S^2\sigma^2) \longrightarrow E\left[\exp\left(Sx\right)\right] = \exp\left(S\mu + \frac{1}{2}S^2\sigma^2\right)$
      i. For the random variable $\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right]$
      ii. That is, $E\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right] = \exp\left[-\gamma E_t\left[\Delta \ln\left(c_{t+1}\right)\right] + \frac{1}{2}V\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right]\right]$

$$1 = \left[\exp\left(-\rho + r^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2 - \gamma E_t\left[\Delta \ln\left(c_{t+1}\right)\right] + \frac{1}{2}V\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right]\right)\right]$$

6. Take Ln

$$0 = -\rho + r^i_{t+1} - \frac{1}{2}\left(\sigma^i\right)^2 - \gamma E_t\left[\Delta \ln\left(c_{t+1}\right)\right] + \frac{1}{2}V\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right]$$

7. Difference Euler Equation for assets i and j, and Re-write:

$$0 = r^i_{t+1} - r^j_{t+1} - \frac{1}{2}\left[\left(\sigma^i\right)^2 - \left(\sigma^j\right)^2\right] + \frac{1}{2}\left[V\left[\sigma^i \varepsilon^i_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right] - V\left[\sigma^j \varepsilon^j_{t+1} - \gamma \Delta \ln\left(c_{t+1}\right)\right]\right]$$

(a) Note that $-\rho, \gamma E_t\left[\Delta \ln\left(c_{t+1}\right)\right]$ drop out of the equation

(b) Re-write with interest rate difference on LHS:

$$r_{t+1}^i - r_{t+1}^j = \frac{1}{2}\left[\left(\sigma^i\right)^2 - \left(\sigma^j\right)^2\right] - \frac{1}{2}\left[V\left[\sigma^i \varepsilon_{t+1}^i - \gamma\Delta \ln\left(c_{t+1}\right)\right] - V\left[\sigma^j \varepsilon_{t+1}^j - \gamma\Delta \ln\left(c_{t+1}\right)\right]\right]$$

8. Apply formula for $V(A+B)$

   (a) $[V(A+B) = V(A) + V(B) + 2Cov(A,B)]$

      i. Also note $2Cov(A, -\alpha B) = -2\alpha Cov(A,B)$

   Note, in our case: $V\left[\sigma^i \varepsilon_{t+1}^i - \gamma\Delta \ln\left(c_{t+1}\right)\right] = V\left(\sigma^i \varepsilon_{t+1}^i\right) + V\left(-\gamma\Delta \ln\left(c_{t+1}\right)\right) + 2Cov\left(\sigma^i \varepsilon_{t+1}^i, \gamma\Delta \ln\left(c_{t+1}\right)\right)$
   $= \left(\sigma^i\right)^2 + \gamma^2 V \Delta \ln\left(c_{t+1}\right) - 2\gamma\sigma_{ic}$

$$r_{t+1}^i - r_{t+1}^j = \frac{1}{2}\left[\left(\sigma^i\right)^2 - \left(\sigma^j\right)^2 - \left[\left(\sigma^i\right)^2 + \gamma^2 V \Delta \ln\left(c_{t+1}\right) - 2\gamma\sigma_{ic}\right] + \left[\left(\sigma^j\right)^2 + \gamma^2 V \Delta \ln\left(c_{t+1}\right) - 2\gamma\sigma_{jc}\right]\right]$$

   Just cancel terms: $r_t^i - r_t^j = \frac{1}{2}\left[\left(\sigma^i\right)^2 - \left(\sigma^i\right)^2 + \left(\sigma^j\right)^2 - \left(\sigma^j\right)^2 + \gamma^2 V \Delta \ln\left(c_{t+1}\right) - \gamma^2 V \Delta \ln\left(c_{t+1}\right) + 2\gamma\sigma_{ic} - 2\gamma\sigma_{jc}\right]$

   To get out asset pricing equation:

$$\pi^{ij} = r_{t+1}^i - r_{t+1}^j = \gamma\sigma_{ic} - \gamma\sigma_{jc}$$

Equity Premium Puzzle

What is the equity premium puzzle?

We can rearrange the asset pricing equation for the risk free asset $\left[\pi^{equity,f} = \gamma\sigma_{equity,c}\right]$ to isolate γ:

$$\gamma = \frac{\pi^{equity,f}}{\sigma_{equity,c}}$$

Empirical Date: In the US post-war period

- $\pi^{equity,f} \approx .06$
- $\sigma_{equity,c} \approx .0003$

$$\gamma = \frac{.06}{.0003} = 200$$

Lecture 8

Summary Equations

- Brownian Motion; Discrete to Continuous
  $\Delta x \equiv x(t+\Delta t) - x(t) = \left\{ \begin{array}{lll} +h & \text{with prob} & p \\ -h & \text{with prob} & q = 1-p \end{array} \right\}$
  $E\left[x(t) - x(0)\right] = n\left((p-q)h\right) = \frac{t}{\Delta t}(p-q)h$ **(*)**
  $V\left[x(t) - x(0)\right] = n\left(4pqh^2\right) = \frac{t}{\Delta t}4pqh^2$ **(*)**
- Calibration
  $h = \sigma\sqrt{\Delta t}$
  $p = \frac{1}{2}\left[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right], \Rightarrow (p-q) = \frac{\alpha}{\sigma}\sqrt{\Delta t}$
  - $E\left[x(t) - x(0)\right] = \frac{t}{\Delta t}(p-q)h = \frac{t}{\Delta t}\left(\frac{\alpha}{\sigma}\sqrt{\Delta t}\right)\left(\sigma\sqrt{\Delta t}\right) = \underbrace{\alpha t}_{\text{linear drift}}$
  - $V\left[x(t) - x(0)\right] = \underbrace{\sigma^2 t}_{\text{linear variance}}$
- Weiner
  1. $\underline{\Delta z = \varepsilon\sqrt{\Delta t} \text{ where } \varepsilon \sim N(0,1)}$, [also stated $\Delta z \sim N(0, \Delta t)$] *Vertical movements* $\propto\sqrt{\Delta t}$
  2. Non overlapping increments of Δz are *independent*
- Ito:
  $dx = a(x,t)dt + b(x,t)dz$; $dx = \underbrace{a(x,t)dt}_{\text{drift}} + \underbrace{b(x,t)dz}_{\text{standard deviation}}$
  1. Random Walk: $dx = \alpha dt + \sigma dz$ [with drift α and variance σ^2]
  2. Geometric Random Walk: $dx = \alpha x dt + \sigma x dz$ [with proportional drift α and proportional variance σ^2]
- Ito's Lemma: $z(t)$ is Wiener, $x(t)$ is Ito with $dx = a(x,t)dt + b(x,t)dz$. Let $V = V(x,t)$, then
  $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt$
  $= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dz$
- Value Function as Ito Process:
  - **Value Function is Ito Process** (V) **of an Ito Process** (x) **or a Weiner Process** (z) (need to check this)

$$dV = \hat{a}(x,t)dt + \hat{b}(x,t)dz = \underbrace{\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]}_{\hat{a}}dt + \underbrace{\frac{\partial V}{\partial x}b(x,t)}_{\hat{b}}dz$$

- Proof of Ito's Lemma:
  $dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial^2 V}{\partial x^2}dxdt + \text{h.o.t.}$‘
  - $(dx)^2 = b(x,t)^2(dz)^2 + \text{h.o.t.} = \mathbf{b(x,t)^2dt} + \text{h.o.t.}$

Motivation: Discrete to Continuous Time

*Brownian Motion Approximation with **discrete intervals** in **continuous time***

Have continuous time world, but use **discrete time steps Δt.** Want to see what happens as the step length goes to zero, that is $\Delta t \to 0$.

Process with jumps at discrete intervals:

- At every Δt intervals, a process $x(t)$ either goes up or down:

$$\Delta x \equiv x(t+\Delta t) - x(t) = \left\{ \begin{array}{ccc} +h & \text{with prob} & p \\ -h & \text{with prob} & q = 1-p \end{array} \right\}$$

- Expectation of this process:
  $E(\Delta x) = ph + q(-h) = (p-q)h$
- Variance of this process
  Definition of Variance of a random variable: $V(\Delta x) = E\left[\Delta x - E\Delta x\right]^2$
  We have: $V(\Delta x) = E\left[\Delta x - E\Delta x\right]^2 = E\left[(\Delta x)^2\right] - \left[E\Delta x\right]^2 = 4pqh^2$
  (Since)$E\left[(\Delta x)^2\right] = ph^2 + q\left(-h\right)^2 = h^2$

Analysis of $x(t) - x(0)$ in this process

- Time span of length t implies $n = \frac{t}{\Delta t}$ steps in $x(t) - x(0)$
  - $\implies x(t) - x(0)$ is a *binomial random variable*
- General form for probability that $x(t) - x(0)$ is a certain combination of h and $-h$:
  Probability that $x(t) - x(0) = (k)(h) + (n-k)(-h)$
  is $\binom{n}{k} p^k q^{n-k}$
  (where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$)
- **Expectation and Variance:**
  - $E\left[x(t) - x(0)\right] = n\left((p-q)h\right) = \frac{t}{\Delta t}(p-q)h$ **(*)**
    * (n times the individual step Δx. Since these are iid steps, the sum is n times each element in the sum)
  - $V\left[x(t) - x(0)\right] = n\left(4pqh^2\right) = \frac{t}{\Delta t}4pqh^2$ **(*)**
    * (Since all steps are iid: when we look at sum of random variables, where random variables are all independent, the variance of the sum is the sum of the variances)

We have binomial random variable. As we chop into smaller and smaller time steps, will converge to normal density.

So far: we've taken the discrete time process, discussed the individual steps Δx given a time step Δt, and then we've aggregated across many time steps to an interval of length t, during which time there are n steps $\left(n = \frac{t}{\Delta t}\right)$, and we've observed that this random variable, $x(t) - x(0)$, is a binomial random variable with mean $\frac{t}{\Delta t}(p-q)h$ and variance $\frac{t}{\Delta t}4pqh^2$. Our next job is to move from discrete number of steps to smaller and smaller steps.

***Calibrate** h and p to give us properites we want as we take $\Delta t \to 0$:* **[Linear drift and variance]**

- Let:
  $h = \sigma\sqrt{\Delta t}$ (*)
  $p = \frac{1}{2}\left[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$ (*)
  - These Follow:

  $q = 1 - p = \frac{1}{2}\left[1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$
  $(p-q) = \frac{\alpha}{\sigma}\sqrt{\Delta t}$
- "Where did this come from?" (linking Δt to h, p, q) **We are trying to get to continuous time random walk with drift.** Random walk has variance that increases linearly with t "forcasting horizon." Would like this property to emerge in continuous time.
  - Consider the variance (of the aggregate time random variable): $\frac{t}{\Delta t}4pqh^2$.
  - Δt is going to zero; p and q are numbers that are around .5 so can think of them as constants in the limit, 4 is constant, t is the thing we'd like everything to be linear with respect to
    * So given all this, h must be proportional to $\sqrt{\Delta t}$ [then put in scaling parameter σ]
      · But then, what must $(p-q)$ be for drift to be linear in t $\left\{E\left[x(t) - x(0)\right] = \frac{t}{\Delta t}(p-q)h\right\}$? $(p-q)$ must also be proportional to $\sqrt{\Delta t}$, which (subject to affine transformations) pins down p, q and $(p-q)$.
    * "No other way to calibrate this for it to make sense as a continuous time random walk"
- Expectation and Variance

$$E\left[x(t) - x(0)\right] = \frac{t}{\Delta t}(p-q)h = \frac{t}{\Delta t}\left(\frac{\alpha}{\sigma}\sqrt{\Delta t}\right)\left(\sigma\sqrt{\Delta t}\right) = \underbrace{\alpha t}_{\text{linear drift}}$$

$$V\left[x(t) - x(0)\right] = \frac{t}{\Delta t}4pqh^2 = \frac{t}{\Delta t}4\left(\frac{1}{4}\right)\left(1 - \left(\frac{\alpha}{\sigma}\right)^2\Delta t\right)\left(\sigma^2\Delta t\right) = t\sigma^2\left(1 - \left(\frac{\alpha}{\sigma}\right)^2\Delta t\right) \longrightarrow$$

$$\underbrace{\sigma^2 t}_{\text{linear variance}} \qquad (\text{as } \Delta t \to 0)$$

Intuition as $\Delta t \to 0$

- As take Δt to 0, mathematically converging to a *continuous time random walk.*
- At a point t,we have a **binomial random variable**. It is converging to a **normal random variable** with mean αt and var $\sigma^2 t$.
- [These are the four graphs simulating the process, which appear closer and closer to Brownian motion as Δt gets smaller]

- Hence this Random Walk, a *continuous time stochastic process*, is the limit case of a family of stochastic processes

Properties of Limit Case (Random Walk):

1. Vertical movements proportional to $\sqrt{\Delta t}$ **(not** Δt**)**
2. $[\mathbf{x(t)} - \mathbf{x(0)}] \rightarrow^{D} N(\alpha t, \sigma^2 t)$, since *Binomial* $\rightarrow^{D}$ *Normal*
3. "Length of curve during 1 time period" $> nh = \frac{1}{\Delta t}\sigma\sqrt{\Delta t} = \sigma\frac{1}{\sqrt{\Delta t}} \rightarrow \infty$
   (a) (Since length will be greater than the number of step sizes, time the length of each step size)
   (b) Hence length of curve is infinity over any finite period of time
4. $\frac{\Delta x}{\Delta t} = \frac{\pm\sigma\sqrt{\Delta t}}{\Delta t} = \frac{\pm\sigma}{\Delta t} \rightarrow \pm\infty$
   (a) Time derivative, $E\left(\frac{dx}{dt}\right)$, doesn't exist
   (b) So we can't talk about expectation of the derivative. However, we can talk about the following "derivative-like objects" (5,6):
5. $\frac{E(\Delta x)}{\Delta t} = \frac{(p-q)h}{\Delta t} = \frac{\left(\frac{\alpha}{\sigma}\sqrt{\Delta t}\right)\left(\sigma\sqrt{\Delta t}\right)}{\Delta t} = \alpha$
   (a) So we write $E\left(dx\right) = \alpha dt$
      i. That is, in a period Δt, you expect this process to increase by α
6. $\frac{V(\Delta x)}{\Delta t} = \frac{4\left(\frac{1}{4}\right)\left(1-\left(\frac{\alpha}{\sigma}\right)^2\Delta t\right)\sigma^2\Delta t}{\Delta t} \rightarrow \sigma^2$
   (a) So we write $V\left(dx\right) = \sigma^2 dt$

- Finally we see what the process is *everywhere continuous, nowhere differentiable*

When we let Δt converge to zero, the limiting process is called a continuous time random walk with (instantaneous) drift α and (instantaneous) variance σ^2.
We generated this continuous-time stochastic process by building it up as a limit case.
We could have also just defined the process directly as opposed to converging to it:

***Note that* "x" *above becomes* "z" *for following*:**

Wiener Process

Weiner processes: Family of processes described before with zero drift, and unit variance per unit time
Definition

If a continuous time stoachastic process $z(t)$ is a *Wiener Process*, then any change in z, Δz, corresponding to a time interval Δt, satisfies the following conditions:

1. $\Delta z = \varepsilon\sqrt{\Delta t}$ where $\varepsilon \sim N(0,1)$

   (a) [often also stated $\Delta z \sim N(0, \Delta t)$]
   (b) *ε is the gausian piece, and $\sqrt{\Delta t}$ is the scaling piece*

2. <u>Non overlapping increments of Δz are *independent*</u>
   (a) A.K.A. If $t_1 \leq t_2 \leq t_3 \leq t_4$ then $E\left[\left(z(t_2) - z(t_1)\right)\left(z(t_4) - z(t_3)\right)\right] = 0$

A process fitting these two properties is a Weiner process. It is the same process converged to in the first part of the lecture above.

Properties

- A Wiener process is a continuous time random walk with zero drift and unit variance
- $z(t)$ has the Markov property: the current value of the process is a sufficient statistic for the distribution of future values
  - History doesn't matter. All you need to know to predict in y periods is where you are right now.
- $z(t) \sim N(z(0), t)$ so the **variance of $z(t)$ rises linearly with t**
  - (Zero drift case)

Note on terminology:
Wiener processes are a subset of Brownian motion (processes), which are a subset of Ito Processes.

Ito Process

Now we generalize. Let drift (which we'll think of as a with arguments x and t, $a(x, t)$) depend on **time t and state x**.
[Let $\lim_{\Delta t \to 0} \frac{E\Delta x}{\Delta t}$ equal to a function a with those two arguments]
Similarly, generalize variance.

From Wiener to Ito Process

Let $z(t)$ be a Wiener Process. Let $x(t)$ be another continuous time stocastic process such that

$$\begin{array}{lll} \lim_{\Delta t \to 0} \frac{E\Delta x}{\Delta t} = a\left(x, t\right) & \text{i.e. } E\left(dx\right) = a\left(x, t\right)dt & \text{"drift"} \\ \lim_{\Delta t \to 0} \frac{V\Delta x}{\Delta t} = b\left(x, t\right)^2 & \text{i.e. } V\left(dx\right) = b\left(x, t\right)^2 dt & \text{"variance"} \end{array}$$

Note that the "i.e." equations are formed by simply bringing the dt to the RHS.
We summarize these properties by writing the expression for an ***Ito Process***:

$$dx = a(x, t)dt + b(x, t)dz$$

- This is not technically an equation, but a way of saying
- Drift and Standard Deviation terms: $dx = \underbrace{a(x, t)dt}_{\text{drift}} + \underbrace{b(x, t)dz}_{\text{standard deviation}}$

  - We think of dz as increments of our Brownian motion from Weiner process/limit case of discrete time process.
  - What is telling us about variance is whatever term is multiplying dz. The process changes as z changes, and what you want to know is how does a little change in z–this background Weiner process–cause a change in x. In this case, an instantaneous change in z causes a change in x with scaling factor b.

- Deterministic and Stochastic terms: $dx = \underbrace{a(x,t)dt}_{\text{deterministic}} + \underbrace{b(x,t)dz}_{\text{stochastic}}$

Ito Process: 2 Key Examples

- Random Walk

$$dx = \alpha dt + \sigma dz$$

  - Random Walk with drift α and variance σ^2

- Geometric Random Walk

$$dx = \alpha x dt + \sigma x dz$$

  - Geometric Random Walk with proportional drift α and proportional variance σ^2
    * Drift and variance are now dependent on the state variable x
    * Can also think of as $\frac{dx}{x} = \alpha dt + \sigma dz$, so percent change in dx is a random walk
  - **Note on Geometric Random Walk: If it starts above zero, if will never become negative (or even become zero): even with strongly negative drift, given its proportionality to x the process will be driven asymptotically to zero at the lowest.
  - Laibson: Asset returns are well approximated by Geometric Random Walk

Ito's Lemma

Motivation is to work with functions that take Ito Process as an argument

Ito's Lemma

"Basically just a taylor expansion"

Theorem: Let $z(t)$ be a Wiener Process. Let $x(t)$ be an Ito Process with $dx = a(x,t)dt + b(x,t)dz$. Let $V = V(x,t)$, then

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt$$

$$= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dz$$

[the second line simply follows from expanding $dx = a(x,t)dt + b(x,t)dz$]

Motivation: Work with Value Functions (which will be Ito Processes) that take Ito Processes as Arguments

- We are going to want to work with functions (value functions–the solutions of Bellman equations) that are going to *take as an argument, an Ito Process*
- Motivation is to know what these functions look like, that take x as an argument, where x is an Ito Process
  - More specifically, we'd like to know what the *Ito Process* looks like that characterizes the value function V.
    - Value function V is 2^{nd} Ito Process, which takes an Ito Process (x) as an argument
      - Not only want to talk about Ito Process x, but also about Ito Process V
- Since we are usually looking for a solution to a question: i.e. Oil Well
  - Price of oil is given by x
  - The Ito Process $dx = a(x,t)dt + b(x,t)dz$ describes the path of x, the price of oil (not the oil well)
    - e.g. price of oil is characterized by geometric Brownian motion: $dx = \alpha x dt + \sigma x dz$
  - $V(x,t)$ is value of Oil *Well,* which depends on the price of oil x and time
    - Want to characterize the value function of the oil well, $V(x,t)$, *where x follows an Ito Process*
      - Ito Process that characterizes value function over time: $dV = \hat{a}(V,x,t)dt + \hat{b}(V,x,t)dz$
      - But we will see that the dependence of $\hat{a}$ and $\hat{b}$ on V will drop
  - We want to write the stochastic process which describes the evolution of V:

$$dV = \hat{a}(x,t)dt + \hat{b}(x,t)dz$$

    - which is called the total differential of V
    - Note that the hat terms $(\hat{a}, \hat{b})$ are what we are looking for in this stochastic process, which are different than the terms in the Ito Process (a, b)
- Ito's Lemma gives the solution to this problem in the following form:
  -

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dz \text{ where:}$$

$$dV = \underbrace{\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]}_{\hat{a}} dt + \underbrace{\frac{\partial V}{\partial x}b(x,t)}_{\hat{b}}\, dz$$

Proof of Ito's Lemma

Proof: Using a Taylor Expansion

$$dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial^2 V}{\partial x^2}dxdt + \text{h.o.t.}$$

- Any term of order $(dt)^{\frac{3}{2}}$ or higher is small relative to terms of order dt.
  - [This is because $(\Delta t)^2$ is infinitely less important than $(\Delta t)^{\frac{3}{2}}$, which is infinitely less important than (Δt)]
- Note that $(dz)^2 = dt$. Since $\Delta z \propto \sqrt{\Delta t}$, recalling $\Delta z \approx h$ [there's a Law of Large Numbers making this true, this is "math PhD land" and not something Laibson knows or expects us to know]
  - Hence we can eliminate:
    * $(dt)^2 = \text{h.o.t.}$
    * $dxdt = a(x,t)(dt)^2 + b(x,t)dzdt = \text{h.o.t.}$
    * $(dx)^2 = b(x,t)^2(dz)^2 + \text{h.o.t.} = \mathbf{b(x,t)^2dt} + \text{h.o.t.}$
  - Combining these results we have:

$$\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt$$

dz is a Wiener process that depends only on t through random motion
dx is a Ito process that depends on x and t through random motion
$V(x,t)$ is a value function (and an Ito Process) that depends on a Ito process dx and directly on t

Intuition: Drift in V through Curvature of V:

- Even if we assume $a(x,t) = 0$ [no drift in Ito Process] and assume $\frac{\partial V}{\partial t} = 0$ [holding x fixed, V doesn't depend on t]
- We still have $E(dV) = \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt \neq 0$
  - If V is concave, V is expected to fall due to variation in x [convex $\rightarrow$ rise]
    * Ex: $V(x) = \ln x$, hence $V' = \frac{1}{x}$ and $V'' = -\frac{1}{x^2}$
      · $dx = \alpha x dt + \sigma x dz$
      · $dV = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dz$
      · $= \left[0 + \frac{1}{x}\alpha x - \frac{1}{2x^2}(\sigma x)^2\right]dt + \frac{1}{x}\sigma x dz$
      · $= \left[\alpha - \frac{1}{2}\sigma^2\right]dt + \sigma dz$
  - $\implies$ ***Growth in V falls below α due to concavity of V***
- *Intuition for proof*: for Ito Processes, $(dx)^2$ behaves like $b(x,t)^2 dt$, so the effect of concavity is of order dt and can not be ignored when calculating the differential of V
  - Because of Jensen's inequality, if you are moving on independent variable (x) axis randomly, expected value will go down

Lecture Summary
Talked about a continuous time process (Brownian motion, Weiner process or its generalization as an Ito Process) that has these wonderful properties that is kind of like the continuous time analog of things in discrete time are thought of as random walks. But it is even more general that this:
Has perverse property that it moves infinitely far over unit time, but when you look at how it changes over any discrete interval of time, it looks very natural.
Also learned how to study functions that take Ito Processes as arguments: key tool in these processes is Ito's Lemma. It uses a 2nd order taylor expansion and throws out higher order terms. So we can characterize drift in that function V very simply and powerfully, which we put to use in next lectures.

Lecture 9

Continuous Time Bellman Equation & Boundary Conditions

Continuous Time Bellman Equation

Final Form:

$$\rho V(x,t)dt = \max_u \{w(u,x,t)dt + \mathbb{E}[dV]\}$$

Inserting Ito's Lemma

$$\rho V(x,t)dt = \max_u \left\{ w(u,x,t)dt + \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 \right] dt \right\}$$

Next 1.5 lectures about solving for V using this equation with Ito's Lemma. First, Merton. Then, stopping problem.

- Intuition

$$\underbrace{\rho V(x,t)dt}_{\text{required rate of return}} = \max_u \left\{ \underbrace{w(u,x,t)dt}_{\text{instantaneous dividend}} + \underbrace{\mathbb{E}[dV]}_{\text{instantaneous capital gain}} \right\}$$

  - required rate of return = instantaneous return; instantaneous capital gain = instantaneous change in the program.

- Derivation
  - Towards Condensed Bellman
    - Let $w(x,u,t)$ = instantaneous payoff function
      - Arguments: x is state variable, u is control variable, t is time
    - Let $x' = x + \Delta x$ and $t' = t + \Delta t$. We'll end up thinking about what happens as $\Delta t \to 0$
      - "Just write down Bellman equation as we did in discrete time"

$$V(x,t)=\max_u\left\{w(x,u,t)\Delta t+(1+\rho\Delta t)^{-1}EV(x',t')\right\}$$

$$V(x,t)=\max_u\left\{\begin{array}{ccc}\underbrace{w(x,u,t)\Delta t}_{\text{flow payoff }\times\text{time step}} & + & \underbrace{(1+\rho\Delta t)^{-1}EV(x',t')}_{\text{discounted value of continuation}}\end{array}\right\}$$

* 3 simple steps algebra with $(1+\rho\Delta t)$ term:

$$(1+\rho\Delta t)\,V(x,t)=\max_u\left\{(1+\rho\Delta t)\,w(x,u,t)\Delta t+EV(x',t')\right\}$$

$$\rho\Delta tV(x,t)=\max_u\left\{(1+\rho\Delta t)\,w(x,u,t)\Delta t+EV(x',t')-V(x,t)\right\}$$

$$\rho\Delta tV(x,t)=\max_u\left\{w(x,u,t)\Delta t+\rho w(x,u,t)\,(\Delta t)^2+EV(x',t')-V(x,t)\right\}$$

* Let $\Delta t\to 0$. Terms of order $(dt)^2=0$

$$\rho V(x,t)dt=\max_u\left\{w(x,u,t)dt+\mathbb{E}\left[dV\right]\right\}\qquad \textbf{(*)}$$

- Substitute using Ito's Lemma
  * Substitute for $\mathbb{E}\left[dV\right]$
    · where

$$dV=\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}a(x,u,t)+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,u,t)^2\right]dt+\frac{\partial V}{\partial x}b(x,u,t)dz$$

$$\left(\text{since }\mathbb{E}\left[\frac{\partial V}{\partial x}bdz\right]=0\right)$$

$$\mathbb{E}\left[dV\right]=\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}a+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}b^2\right]dt$$

  * Substituting this expression into (*) we get

$$\rho V(x,t)dt=\max_u\left\{w(u,x,t)dt+\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}a+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt\right\}$$

    · which is a partial differential equation (in x and t)

- Interpretation
  - **Sequence Problem:**

$$\int_t^\infty e^{-\rho t}w\left(x(t),u(t),t\right)dt$$

  - Interpretation of Bellman (*) before substuting for Ito's Lemma:

$$\underbrace{\rho V(x,t)dt}_{\text{required rate of return}}=\max_u\left\{\begin{array}{ccc}\underbrace{w(x,u,t)}_{\text{instantaneous dividend}} & + & \underbrace{\mathbb{E}\left[dV\right]}_{\text{instantaneous capital gain}}\end{array}\right\}$$

    * note that instantaneous capital gain $\equiv$ instantaneous change in the program
    * And when we plug in Ito's Lemma
    *

$$\text{Instantaneous capital gain } = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]$$

Application 1: Merton's Consumption

- Merton's Consumption
  - Consumer has 2 assets
    * Risk free: return r
    * Equity: return $r + \pi$ with proportional variance σ^2
  - Invests share θ in assets, and consumes at rate c
- Ito Process that characterizes dynamics of x, her cash on hand:

$$dx = [rx + \pi x\theta - c]\, dt + \theta x\sigma dz$$

  - Intuition
    * Drift term: earn $rxdt$ every period on assets. A fraction θx is getting an additional equity premium π $(\pi x\theta dt)$. Consuming at rate c, so depletes assets at rate cdt.
    * Standard Deviation term: cost to holding equities is volatility. Volatility scales with the amount of assets in risky category: $\theta x\sigma dz$.
  - So, $a = [rx + \pi x\theta - c]$ and $b = \theta x\sigma$
  - aka "***equation of motion for the state variable***"
- Distinguishing Variables
  - x is a stock (not in the sense of equity, but a quantity)
  - c is a flow (flow means rate).
    * Need to make sure units match (usually use annual in economics)
    * i.e. spending \$1000/48hrs is ~\$180,000/year
      · He can easily consume at a rate that is above his total stock of wealth
    * c is bounded between 0 and ∞
      · *Not between* 0 *and* x *as it was in discrete time*

- Bellman Equation

$$\rho V(x,t)dt = \max_{c,\theta}\left\{w(u,x,t)dt + \left[\frac{\partial V}{\partial x}\left[rx + \pi x\theta - c\right] + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\theta x\sigma)^2\right]dt\right\}$$

  - Note the ***maximization over both*** c ***and*** θ***,*** so two FOCs
    * In world of continuous time, these can be changed instantaneously.

· (In principle, in every single instant, could be picking a new consumption level c and asset allocation level θ)

· It will turn out that in CRRA world, he will prefer a constant asset allocation θ, even though he is constantly changing his consumption

– V doesn't depend directly on t, so first term was removed

  * If someone was finitely lived, then function would depend on t

– "dt" can be removed since notationally redundant

Solving with CRRA utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ [Lecture]

- It turns out that in this problem, the value function inherits similar properties to the utilty function: $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$
- If given utility function is CRRA $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, then it can be proved that $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$
  - This is just a scaled version of the utility function (by ψ)
  - This gives us single solution to Bellman equation
- Solving for Policy Functions [given value function $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$]
- Assume $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$: find values for policy functions that solve Bellman Equation
  - We need to know ψ, θ and optimal consumption policy rule for c
  - The restriction $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$ allows us to do this
- Taking $V(x) = \psi \frac{x^{1-\gamma}}{1-\gamma}$, re-write Bellman plugging in for $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\frac{\partial V}{\partial x}$, $\frac{\partial^2 V}{\partial x^2}$

$$\rho\psi \frac{x^{1-\gamma}}{1-\gamma} dt = \max_{c,\theta} \left\{ u(c)dt + \left[\psi x^{-\gamma}[(r+\theta\pi)x - c] - \frac{\gamma}{2}\psi x^{-\gamma-1}(\theta\sigma x)^2 \right] dt \right\}$$

$$\rho\psi \frac{x^{1-\gamma}}{1-\gamma} = \max_{c,\theta} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \left[\psi x^{-\gamma}[(r+\theta\pi)x - c] - \frac{\gamma}{2}\psi x^{-\gamma-1}(\theta\sigma x)^2 \right] \right\}$$

- Two FOCs, θ and c :
  - $FOC_\theta :\ \psi x^{-\gamma}\pi x - 2\left(\frac{\gamma}{2}\right)\psi x^{-\gamma-1}\theta(\sigma x)^2 = 0$
  - $FOC_c :\ u'(c) = c^{-\gamma} = \psi x^{-\gamma}$
- Simplifying

  $c = \psi^{-\frac{1}{\gamma}} x$

  $\theta = \frac{\pi}{\gamma\sigma^2}$

  - Interpretation:

    Nice result: Put more into equities the higher the equity premium π; put less into equities with higher RRA γ and higher variability of equities σ^2

    Why is variance of equities replacing familiar covariance term from Asset Pricing lecture? In Merton's world, all the stocasticity is coming from equity returns.

- Plugging back in, we get:
  $\psi^{-\frac{1}{\gamma}} = \frac{\rho}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\pi^2}{2\gamma\sigma^2}\right)$
- Special case: $\gamma = 1$ implies $c = \rho x$
  - So $MPC \simeq .05$ (Marginal Propensity to Consume)
  - But we also get $\theta = \frac{\pi}{\gamma\sigma^2} = \frac{0.06}{1(0.16)^2} = 2.34$

Saying to borrow money so have assets of 2.34 in equity [and therefore 1.34 in liabilities]
We don't know what's wrong here. This model doesn't match up with what "wise souls" think you should do.

- Can we fix by increasing γ? Remember no one in classroom had $\gamma \geq 5$
  - With $\gamma = 5$, $\theta = \frac{0.06}{5(0.16)^2} = 0.47$
    - Recommends 47% allocation into equities

Typical retiree:
No liabilities; \$600,000 of social security (bonds), \$200,000 in house, less than \$200,000 in 401(k)
401(k) usually 60% bonds and 40% stocks
Equity allocation is 8% of total assets

Solving with Log utility $u(c) = \ln(c)$ [PS5]

- Given that guess for value function is $V(x) = \psi \ln(x) + \phi$
- Set up is the same: $\rho V(x,t)dt = \max_{c,\theta} \left\{ w(u,x,t)dt + \left[\frac{\partial V}{\partial x}\left[rx + \pi x\theta - c\right] + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\theta x\sigma)^2\right] dt \right\}$
- When substitute for $V(x)$ and $u(c)$:

$$\rho\left[\phi + \psi \ln(x)\right] = \max_{c,\theta}\left\{\ln(c) + \left[(r + \theta\pi)x - c\right]\psi x^{-1} - \frac{1}{2}\psi(\theta\sigma x)^2\psi x^{-2}\right\}$$

- FOCs:
  - $FOC_c:\ c = \frac{1}{\psi}x$
  - $FOC_\theta:\ \theta = \frac{\pi}{\sigma^2}$
- Plugging back into Bellman
  $\rho\phi + \rho\psi\ln(x) = \psi r - \ln(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2} + \ln(x)$
- Observation to solve for parameters
  - We can re-write the above with just constants on RHS: $\rho\psi\ln(x) - \ln(x) = \psi r - \ln(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2} - \rho\phi$
    - Given that the RHS is constant, this will only hold if $\rho\psi = 1$. [Otherwise LHS would be increasing in x, while RHS won't increase, a contradiction]
      - Hence, $\rho\psi = 1$ and $0 = \psi r - \ln(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2} - \rho\phi$. This means:
  - The Bellman equation is satisfied $\forall x$ iff
    - $\rho\psi = 1$ and $\rho\phi = \psi r - \ln(\psi) - 1 + \frac{\psi\pi^2}{2\sigma^2}$

- Giving us
  - $\psi = \frac{1}{\rho}$
  - $\phi = \frac{1}{\rho}\left(\frac{r}{\rho} + \ln(\rho) - 1 + \frac{\pi^2}{2\rho\sigma^2}\right)$

- Optimal Policy
  - Plugging back into FOCs
    - $c = \rho x$ and $\theta = \frac{\pi}{\sigma^2}$

General Continuous Time Problem

Most General Form

- Equation of motion of the state variable
  - $dx = a(x, u, t)dt + b(x, u, t)dz,$
  - Note dx depends on choice of control u
- Using Ito's Lemma, Continuous Time Bellman Equation
  - $\rho V(x, t) = w(x, u^*, t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} a(x, u^*, t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} b(x, u^*, t)^2$
  - $u^* = u(x, t)$ =optimal value of control variable

Properties: PDE and Boundary Conditions

- Value function is a 2nd order PDE
- V is the 'dependent' variable; x and t are the 'independent' variables
- PDE will have continuum of solutions

  Need structure from economics to find the right solution to this equation–the solution the makes economic sense–from the infinite number of solutions that make mathematical sense.

  $\Longrightarrow$ These restrictions are called *boundary conditions*, which we need to solve PDE

  $\Longrightarrow$ *Need to use economic logic to derive boundary conditions*

  - Name of the game in these problems is to use economic logic to find right solution, and eliminate all other solutions
  - (Any policy generates a solution, we're not looking for any solution that corresponds to any policy, but for *the* solution that corresponds to the optimal policy)
- [In Merton's problem, we exploited 'known' functional form of value function, which imposes boundary conditions]

Two key examples of boundary conditions in economics

1. Terminal Condition
   - Suppose problem ends at date T

- i.e. have already agreed to sell oil well on date T

- Then we know that $V(x,T) = \Omega(x,T)\ \forall x$ [Note this is at date T]
  - Where $\Omega(x,t)$ is (some known, exogenous) termination function. Note that it can be state dependent–i.e. change with the price of oil.
- Can solve problem using techniques analogous to backward induction
- This restriction gives enough structure to pin down solution to the PDE

2. Stationary $\infty-$horizon problem

   - Value function doesn't depend on $t \implies$ becomes an ODE

$$\rho V(x) = w(x,u^*) + a(x,u^*)V' + \frac{1}{2}b(x,u^*)^2V''$$

   - Note the use of V' since now only one argument for V, can drop the ∂ notation
   - We will use Value Matching and Smooth Pasting

Motivating Example for Value Matching and Smooth Pasting:

- Consider the *optimal stopping problem*:

$$V(x,t) = \max\left\{w(x,t)\Delta t + (1+\rho\Delta t)^{-1}EV(x',t'), \Omega(x,t)\right\}$$

- where $\Omega(x,t)$ is the termination payoff in the stopping problem
  - the solution is characterized by the stopping rule:
    * if $x > x^*(t)$ continue; if $x \leq x^*(t)$ stop
    * Motivation: i.e. irreversibly closing a production facility
  - Assume that $dx = a(x,t)dt + b(x,t)dz$
  - In continuation region, value function characterized using Ito's Lemma:
    * $\rho V(x,t)dt = w(x,t)dt + \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt$

Value Matching

- Value Matching:
  All continuous problems must have continuous value functions

$$V(x,t) = \Omega(x,t)$$

for all x,t such that $x(t) \leq x^*(t)$

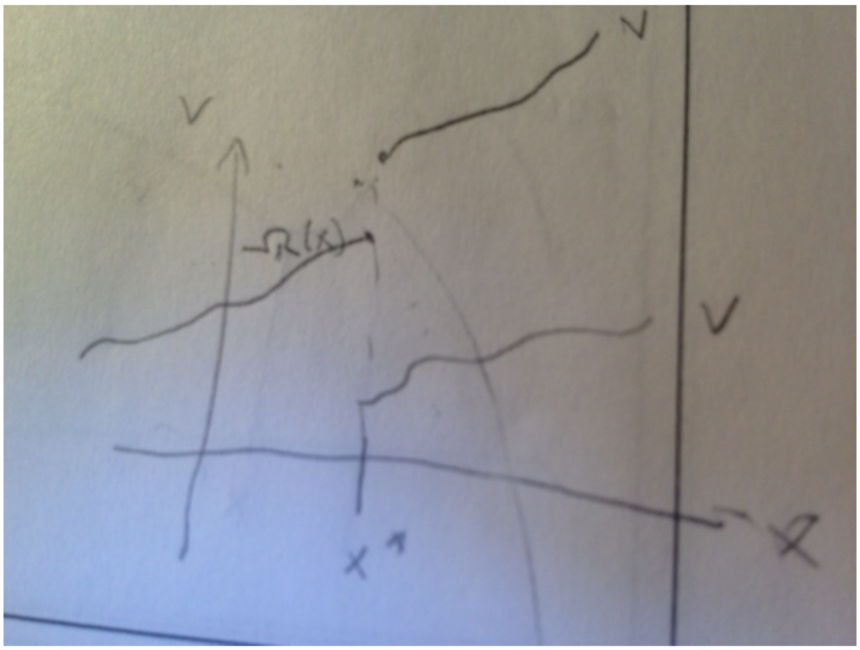
(at boundary and to left of boundary)

– A discontinuity is not permitted, otherwise one could get a better payoff from not following the policy

– Intuition: an optimal value function has to be continuous at the stopping boundary.

  * If there was a discontinuity, can't be optimal to quit (if V is above Ω) or to have continued so far (if V below Ω)

– See graph [remember V on y axis, x on x-axis]



Smooth Pasting

Derivatives with respect to x must be equal at the boundary.
Using problem above:

- Smooth Pasting:

$$V_x\left(x^*(t), t\right) = \Omega_x\left(x^*(t), t\right)$$

  – Intuition

    * V and Ω must have same slope at boundary.

      · Proof intuition: if slopes are not equal at convex kink, then can show that the policy of stopping at the boundary is dominated by the policy of continuing for just another instant and then stopping. (It doesn't mean the latter is the optimal policy, but shows that initial policy of stopping at the boundary is sub-optimal)

    * If value functions don't smooth paste at $x^*(t)$, then stopping at $x^*(t)$ can't be optimal:

    * Better to stop an instant earlier or later

      · If there is a (convex) kink at the boundary, then the gain from waiting is in $\sqrt{\Delta t}$ and the cost from waiting is in Δt. So there can't be a kink at the boundary.

- * Think of graph [remember V on y axis, x on x-axis]

Will need a 3rd boundary condition from the following lecture

Lecture 10

Third Boundary Condition, Stopping Problem and ODE Strategies

Statement of Stopping Problem

- Assume that a continuous time stochastic process $x(t)$ is an Ito Process,

$$dx = adt + bdz$$

  - i.e. x is the price of a commodity produced by this firm

- While in operation, firm has flow profit
‘

$$w(x) = x$$

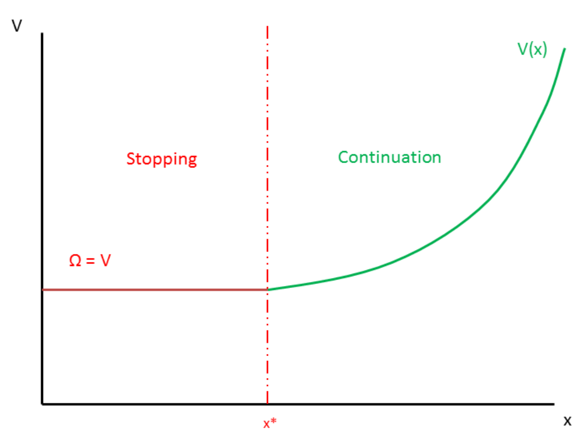
  - (Or $w(x) = x - c$ if there is a cost of production)

- Assume firm can costlessly (permanently) exit the industry and realize termination payoff

$$\Omega = 0$$

- Intuitively, firm will have a stationary stopping rule
  - if $x > x^*$ continue
  - if $x \leq x^*$ stop (exit)

- Draw the Value Function ($\Omega = 0$ in below graph)



Characterizing Solution as an ODE

- Overall Value Function

$$V(x) = \max\left\{x\Delta t + (1+\rho\Delta t)^{-1}EV(x'), 0\right\}$$

  - Where $x' = x(t+\Delta t)$, and ρ is interest rate

- In stopping region
  - $V(x) = 0$
- In continuation region
  - General

$$\rho V(x)dt = xdt + E(dV)$$

Since: $V(x) = x\Delta t + (1+\rho\Delta t)^{-1}EV(x')$
$\implies (1+\rho\Delta t)V(x) = (1+\rho\Delta t)x\Delta t + EV(x')$
$\implies \rho\Delta tV(x) = (1+\rho\Delta t)x\Delta t + EV(x') - V(x)$.
Let $\Delta t \to 0$ and multiply out. Terms of order $(dt)^2 = 0$.

  - Substitute for $E(dV)$ using Ito's Lemma
    - 2nd Order ODE:

$$\rho V = x + aV' + \frac{b^2}{2}V''$$

  - Interpretation
    - Value function must satisfy this differential equation in the continuation region
      - We've derived a 2nd order ODE that restricts the form of the value function V in the continuation region
      - 2nd order because highest order derivative is 2nd order, ordinary because only depends on one variable (x),

$$\underbrace{\rho V}_{\text{required return}} = \underbrace{x}_{\text{dividend}} + \underbrace{\overbrace{aV'}^{\text{drift in argument}} + \overbrace{\frac{b^2}{2}V''}^{\text{drift from jensen's inequality}}}_{\text{capital gain =how } V \text{is instantly drifting}}$$

    - Term $\frac{b^2}{2}V''$ is drift resulting from brownian motion–'jiggling and jaggling': it is negative if concave, and positive if convex (as in this case).

- Intuition: <u>x^* almost always negative.</u> [*don't confuse with $V(x) = \Omega = 0$. This is the value you get when stopping; but can decide on some policy of x for when to stop: x^**]
  - Why? Option value – love the ability take the chance at future profits. Even if drift is negative, will still stay in, since through Brownian motion may end up profitable again.
- And now, we have continuum of solutions to ODE, so need restrictions to get to single solution

Third Boundary Condition

Boundary Conditions
*Three variables we need to pin down: two constants in 2nd order ODE, and free boundary (x^*)

- Value Matching: $V(x^*) = 0$
- Smooth Pasting: $V'(x^*) = 0$
- ***Third Boundary Condition for large*** x
  - As $x \to \infty$ the option value of exiting goes to zero
    - It is so unlikely that profits would become negative (and if it happened, would likely be very negatively discounted) that value of being able to shut down is close to zero
    - As x gets very large, value function approaches that of the firm that does not have the option to shut down (which would mean it would theoretically need to keep running at major losses)
  - Hence V converges to the value associated with the policy of never exiting the industry [equation derived later]

$$\lim_{x\to\infty} \frac{V(x)}{\frac{1}{\rho}\left(x + \frac{a}{\rho}\right)} = 1$$

    - Denominator is value function of firm that can't shut down
      - [Or can think of it as $\lim_{x\to\infty} V(x) = \frac{1}{\rho}\left(x + \frac{a}{\rho}\right)$]
  - We can also write:

$$\lim_{x\to\infty} V'(x) = \frac{1}{\rho}$$

    - Intuition: at large x, how does value function change with one more unit of price x?
      - *Use sequence problem:*
      - Having one more unit of price produces same exact sequence shifted up by one unit all the way to infinity. What is the additional value of a firm with price one unit higher?
      -

$$\underbrace{\int e^{-\rho t} x(t) dt}_{\text{old oil well}} - \underbrace{\int e^{-\rho t} (x')(t) dt}_{\text{new oil well}} = \int_0^\infty e^{-\rho t} \cdot 1 \cdot dt = \frac{1}{\rho}$$

      - So if I raise price by 1, I raise value of firm by $\frac{1}{\rho}$

Solving 2nd Order ODEs

Definitions

- Goal: Solve for F with independent variable x
- Complete Equation
  - Generic 2nd order ODE:

$$F''(x) + A(x)F'(x) + B(x)F(x) = C(x)$$

- Reduced Equation
  - $C(x)$ is replaced by 0

$$F''(x) + A(x)F'(x) + B(x)F(x) = 0$$

Theorem 4.1

- Any solution $\hat{F}(x)$, of the reduced equation can be expressed as a *linear combination of any two linearly independent solutions* of the reduced equation, F_1 and F_2

$$\hat{F}(x) = C_1F_1(x) + C_2F_2(x) \quad \text{provided } F_1,\ F_2 \text{linearly independent}$$

  - Note that two solutions are linearly independent if there do not exist constants A_1 and A_2 such that

$$A_1F_1(x) + A_2F_2(x) = 0 \quad \forall x$$

Theorem 4.2

- The general solution of the complete equation is the sum of
  - any particular solution of the complete equation
  - and the general solution of the reduced equation

Solving Stopping Problem

1. State Complete and Reduced Equation

- Complete Equation
  - Differential equation that characterizes continuation region

$$\rho V = x + aV' + \frac{b^2}{2}V''$$

- Reduced equation

$$0 = -\rho V + aV' + \frac{b^2}{2}V''$$

2. Guess Form of Solution to find General Solution to Reduced Equation

- First challenge is to find solutions of the reduced equation
- Consider class

$$e^{rx}$$

- To confirm this is in fact a solution, differentiate and plug in to find:

$$0 = -\rho e^{rx} + are^{rx} + \frac{b^2}{2} r^2 e^{rx}$$

  - This implies (dividing through by e^{rx})

$$0 = -\rho + ar + \frac{b^2}{2} r^2$$

- Apply quadratic formula

$$r = \frac{-a \pm \sqrt{a^2 + 2b^2 \rho}}{b^2}$$

- Let r^+ represent the positive root and let r^- represent the negative root
- *General solution to reduced equation*
  - Any solution to the reduced equation can be expressed

$$C^+ e^{r^+} + C^- e^{r^-}$$

3. *Find Particular Solution to Complete Equation*

- Want a solution to the complete equation $\rho V = x + aV' + \frac{b^2}{2} V''$
- Consider specific example: payoff function of policy "never leave the industry"
- The value of this policy is

$$E \int_0^\infty e^{-\rho t} x(t) dt$$

- In solving for value of the policy without the expectation, we'll also need to know $E[x(t)]$:

$$E[x(t)] = E\left[x(0) + \int_0^t dx(t)\right]$$

$$= E\left[x(0) + \int_0^t [adt + bdz(t)]\right] = E\left[x(0) + at + \int_0^t bdz(t)\right]$$

[$\int_0^t bdz(t) = 0$ as it is the Brownian motion term]. Hence:

$$E[x(t)] = x(0) + at$$

- The value of the policy in "$V(x) =$" form, *without the expectation*

– Derived by integration by parts

$$\int udv = uv - \int vdu$$

– Taking $E\int_0^\infty e^{-\rho t}x(t)dt$:

$$E\int_0^\infty e^{-\rho t}x(t)dt = E\int_0^\infty e^{-\rho t}\left[x(0)+at\right]dt$$

$$= E\int_0^\infty e^{-\rho t}\left[x(0)\right]dt + \int_0^\infty e^{-\rho t}\left[at\right]dt = \frac{x(0)}{\rho} + E\int_0^\infty e^{-\rho t}\left[at\right]dt$$

Think of $dv = e^{-\rho t}$ and $u = at$
Then $v = -\frac{1}{\rho}e^{-\rho t}$ and $du = adt$
So if $\int udv = \int_0^\infty e^{-\rho t}(at)dt$ then $uv - \int vdu = [-at\frac{1}{\rho}e^{-\rho t}]_0^\infty - \int_0^\infty -\frac{1}{\rho}e^{-\rho t}adt$
Note $-\frac{1}{\rho}e^{-\rho t}]_0^\infty = \frac{1}{\rho}$
$= [-at\frac{1}{\rho}e^{-\rho t}]_0^\infty - [\frac{1}{\rho^2}e^{-\rho t}a]_0^\infty = 0 + a\frac{1}{\rho^2}$

$$= \frac{1}{\rho}\left(x(0)+\frac{a}{\rho}\right)$$

- Hence our candidate particular solution takes the form

$$V(x) = \frac{1}{\rho}\left(x+\frac{a}{\rho}\right)$$

- Confirm that this is a solution to the complete equation $\rho V = x + aV' + \frac{b^2}{2}V''$

$$\rho\left[\frac{1}{\rho}\left(x+\frac{a}{\rho}\right)\right] = x + \frac{a}{\rho} = x + a\left(\frac{1}{\rho}\right) + \frac{b^2}{2}\cdot 0$$

4. Put Pieces Together

- General solution to reduced equation

$$C^+e^{r^+x} + C^-e^{r^-x} \text{ with roots } r^+, r^- = \frac{-a\pm\sqrt{a^2+2b^2\rho}}{b^2}$$

- Particular solution to general equation

$$\frac{1}{\rho}\left(x+\frac{a}{\rho}\right)$$

- Hence the general solution to the complete equation is
  – [Sum of particular solution to complete equation and general solution to reduced equation]

$$V(x) = \frac{1}{\rho}\left(x+\frac{a}{\rho}\right) + C^+e^{r^+x} + C^-e^{r^-x}$$

- Now just need to apply boundary conditions

5. *Apply Boundary Conditions*

- 3 Conditions
  - Value Matching: $V(x^*) = 0$
  - Smooth Pasting: $V'(x^*) = 0$
  - $\lim_{x\to\infty} V'(x) = \frac{1}{\rho}$
- This gives us three conditions on the general solution
  - $C^+ = 0$ [from $\lim_{x\to\infty} V'(x) = \frac{1}{\rho}$]
    * Which then implies:
  - $V(x^*) = \frac{1}{\rho}\left(x^* + \frac{a}{\rho}\right) + C^- e^{r^- x^*} = 0$ [Value Matching] (1)
  - $V'(x^*) = \frac{1}{\rho} + r^- C^- e^{r^- x^*} = 0$ [Smooth Pasting] (2)
- Simplifying from (1) and (2)
  - Plugging (1) into (2)
    * (1) implies $C^- e^{r^- x^*} = -\frac{1}{\rho}\left(x^* + \frac{a}{\rho}\right)$
      · Plugging this into (2) we get
    * $\frac{1}{\rho} - r^- \frac{1}{\rho}\left(x^* + \frac{a}{\rho}\right) = 0$
  - Simplifying: $1 = r^-\left(x^* + \frac{a}{\rho}\right) \implies \frac{1}{r^-} = x^* + \frac{a}{\rho}$
    * Gives us $x^* = \frac{1}{r^-} - \frac{a}{\rho}$
  - Plugging in for $r^- = \frac{-a-\sqrt{a^2+2b^2\rho}}{b^2}$
  - We get our solution for x^*

$$x^* = -\frac{b^2}{a + \sqrt{a^2 + 2b^2\rho}} - \frac{a}{\rho} < 0 \qquad (*)$$

The general solution to the complete equation can be expressed as:

- the sum of a particular solution to the complete differential equation and the general solution to the reduced equation
  - Note: this is usually done in search problems by looking for the particular solution of the policy "never leave." Then you can get a general solution, and then use value matching and smooth pasting to come up with the threshold value.

Lecture 12

Discrete Adjustment - Lumpy Investment

Lumpy Investment Motivation

- Plant level data suggests individual establishments do not smooth adjustment of their capital stocks.
  - Instead, investment is lumpy. Ex firms build new plant all at once, not a little bit each year
- Doms & Donne 1993: 1972-1989: $\frac{\text{largest investment episode, firm } i}{\text{total investment, firm } i}$. If investment were spread this ratio would be $\frac{1}{18}$.
  - Instead average value was $\frac{1}{4}$. Hence, on average firms did 25% of investment in 1 year over 18 year period
- We go from convex to affine cost functions:
  - smooth convex cost functions assume small adjustments are costlessly reversible
  - use affine functions, where small adjustments are costly to reverse

Lumpy Investment

Capital Adjustment Costs Notation

- Capital adjustment has fixed and variable costs
  - Upward Adjustment Cost: $C_U + c_u I$
    - C_U is fixed cost, c_U is variable cost
  - Downward Adjustment Cost: $C_D + c_D(-I)$.
    - $C_D + c_D |I|$ is the general case. Here we usually assume $c_D > 0$ and $I < 0$, so we get $C_D + c_D(-I)$
    - C_D is fixed cost, c_D is variable cost

Firm's Problem

- Firm loses profits if the actual capital stock deviates from target capital stock x^*
  - Deviations $(x - x^*)$ generate instantaneous negative payoff
- Functional form:

$$-\frac{b}{2}(x - x^*)^2 = -\frac{b}{2}X^2$$

  - where $X = (x - x^*)$.

- X is an Ito Process between adjustments:

$$dX = \alpha dt + \sigma dz$$

  - where dz are Brownian increments

Value Function Representation

- Overall Value Function

$$V(X) = \max \mathbb{E}\left\{\int_{\tau=t}^{\infty} e^{-\rho(\tau-t)}\left(-\frac{b}{2}X_{\tau}^{2}\right)d\tau - \sum_{n=1}^{\infty} e^{-\rho(\tau(n)-t)}A(n)\right\}$$

$$A(n) = \left\{\begin{array}{ccc} C_U + c_U I_n & \text{if} & I_n > 0 \\ C_D + c_D\left|I_n\right| & \text{if} & I_n < 0 \end{array}\right\}$$

  - Notation:
    $\tau(n)$ = date of nth adjustment; $A(n)$ = cost of nth adjustment; I_n = investment in nth adjustment
    * Below x^*: Adjust capital at $X = U$. Adjust to $X = u$.
    * Above x^*: Adjust capital at $X = D$. Adjust to $X = d$.

- **Value Function in all 3 Regions**

  1. Between Adjustments:
     - *Bellman Equation*

$$\rho V(X) = -\frac{b}{2}X^2 + \alpha V'(X) + \frac{1}{2}\sigma^2 V''(X)$$

       * Using Ito's Lemma: $\mathbb{E}[dV] = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b^2\right]dt$
     - Functional form solved for below *[PS6 2.d]*
  2. Action Region Below
     - [If $X \leq U$]

$$V(X) = V(u) - \left[C_U + c_U(u - X)\right]$$

  - ***This implies***

$$V'(X) = c_U \quad \forall X \leq U$$

  1. Action Region Above
     - [If $X \geq D$]

$$V(X) = V(d) - \left[C_D + c_D(X - d)\right]$$

  - ***This implies***

$$V'(X) = -c_D \quad \forall X \geq D$$

Boundary Conditions

1. Value Matching

$$\lim_{X \uparrow U} V(X) = V(u) - [C_U + c_U (u - U)] = V(U) = \lim_{X \downarrow U} V(X)$$

$$\lim_{X \downarrow D} V(X) = V(d) - [C_D + c_D (D - d)] = V(D) = \lim_{X \uparrow D} V(X)$$

2. Smooth Pasting

$$\lim_{X \uparrow U} V'(X) = c_U = V'(U) = \lim_{X \downarrow U} V'(X)$$

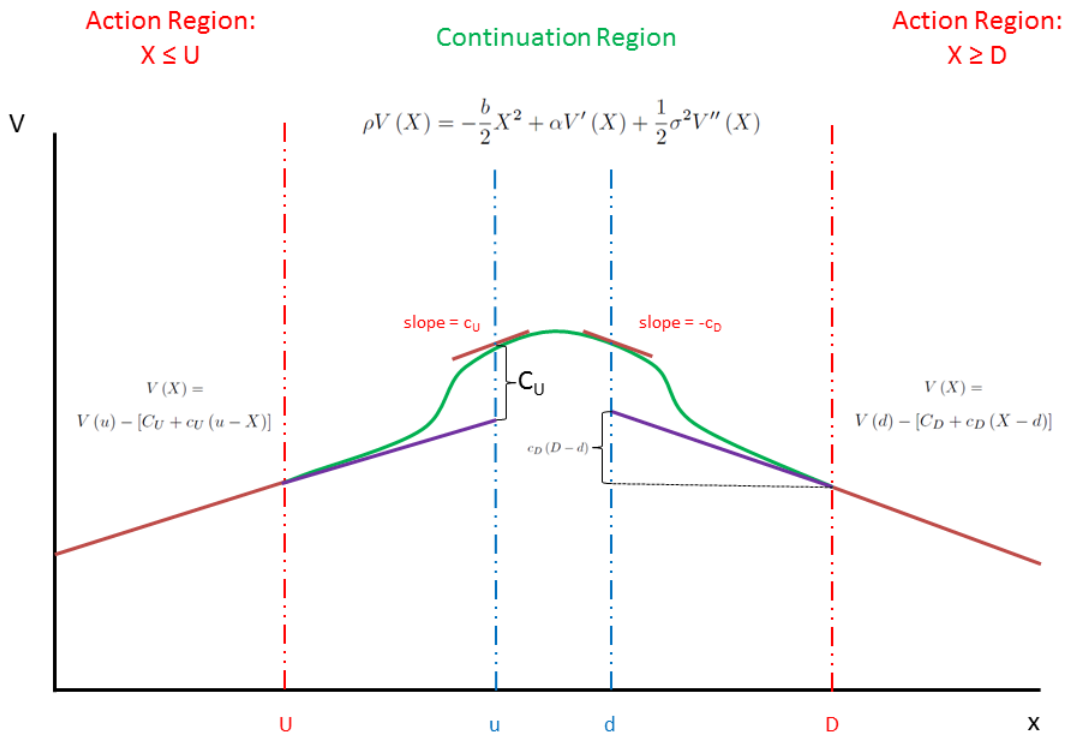
$$\lim_{X \downarrow D} V'(X) = -c_D = V'(D) = \lim_{X \uparrow D} V'(X)$$

First Order Conditions

- ***Intuition:*** *When making capital adjustment, willing to move until the marginal value of moving = marginal cost*
  - This gives us two important optimality condtitions:

$$V'(u) = c_U$$

$$V'(d) = -c_D$$



Solving for Functional Form of $V(X)$ in Continuation Region
PS6 2.d

Question: Show that $V(X) = -\frac{b}{2}\left(\frac{X^2}{\rho} + \frac{\sigma+2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3}\right)$ is a solution to the continuation region equation: $\rho V(X) = -\frac{b}{2}X^2 + \alpha V'(X) + \frac{1}{2}\sigma^2 V''(X)$.
Show that this is the expected present value of the firm's payoff stream assuming that adjustment costs are infinite.

This is "guess and check," where we've been provided the guess:

$$V(X) = -\frac{b}{2}\left(\frac{X^2}{\rho} + \frac{\sigma + 2\alpha X}{\rho^2} + \frac{2\alpha^2}{\rho^3}\right)$$

Calculate $V'(X)$ and $V''(X)$:

$$V'(X) = -\frac{b}{2}\left(\frac{2X}{\rho} + \frac{2\alpha^2}{\rho^2}\right), \quad V''(X) = -\frac{b}{\rho}$$

We plug back into the continuous time Bellmann Equation from $V(X) = \frac{1}{\rho}\left[\frac{-b}{2}X^2 + \alpha V'(X) + \frac{1}{2}\sigma^2 V''(X)\right]$:

$$V(X) = \frac{1}{\rho}\left[\frac{-b}{2}X^2 - \alpha\frac{b}{2}\left(\frac{2X}{\rho} + \frac{2\alpha^2}{\rho^2}\right) - \frac{1}{2}\sigma^2\frac{b}{\rho}\right] = -\frac{b}{2\rho}\left[X^2 + \frac{2\alpha X}{\rho} + \frac{\sigma^2}{\rho} + \frac{2\alpha}{\rho^2}^2\right]$$

$$= -\frac{b}{2}\left[\frac{X^2}{\rho} + \frac{\sigma^2 + 2\alpha X}{\rho^2} + \frac{2\alpha}{\rho^3}\right]$$

So our guessed solution works as a solution to the continuous Bellmann.

We also know that this works as the expected present value of the firm's payoff stream, given that this is the equivalent to the solution to the continuous time Bellman for the continuation region, because the infinite adjustment costs mean that $u = -\infty$ and $d = \infty$, hence the solution will hold for $X \in \mathbb{R}$.

Appendix: Math Reminders

Infinite Sum of a Geometric Series with $|x| < 1$:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

L'Hopital's Rule

$$\text{If } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty$$

$$\text{and } \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists}$$

$$\text{and } g'(x) \neq 0$$

$$\text{then } \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Probability

Expectation of RV:
$E(\exp(\tilde{a})) = \exp[E[\tilde{a}] + \frac{1}{2}Var[\tilde{a}]]$.
When $\tilde{a}$ is a random variable, or anything
$x \sim N(\mu, \sigma^2) \longrightarrow Ax + B \sim N(A\mu + B, A^2\sigma^2)$
$\longrightarrow E\left[\exp\left(Ax + B\right)\right] = \exp\left(A\mu + B + \frac{1}{2}A^2\sigma^2\right)$
[Since $x \sim N(\mu, \sigma^2) \longrightarrow E(\exp(x)) = \exp(E(x) + \frac{1}{2}Var(x))$; so if standard normal: $= exp(\mu + \frac{1}{2}\sigma^2)$]

In case of Asset pricing
R_i^t has stochastic returns
We have: $R_i^t = \exp(r_i^t + \sigma^i \varepsilon_t^i - \frac{1}{2}[\sigma^i]^2)$
where ε_t^i has unit variance. Hence ε_t^i is a normally distributed random variable: $\varepsilon_t^i \sim N(0, 1)$
From above we see that this implies: $\sigma^i \varepsilon_t^i + r_i^t - \frac{1}{2}[\sigma^i]^2 \sim N(\sigma^i * 0 + r_i^t - \frac{1}{2}[\sigma^i]^2, \sigma^{i2} * 1^2) = N(r_i^t - \frac{1}{2}[\sigma^i]^2, \sigma^{i2})$

Then:
$E[R_i^t] = E[\exp(r_i^t + \sigma^i \varepsilon_t^i - \frac{1}{2}[\sigma^i]^2)]$ which as we've seen is now exp("a random variable") so we can use
$= exp(mean + \frac{1}{2}Var) = exp(r_i^t - \frac{1}{2}[\sigma^i]^2 + \frac{1}{2}\sigma^{i2})$
$= exp(r_i^t)$

Covariance:
If A and B are random variables, then

$$V(A+B) = V(A) + V(B) + 2Cov(A,B)$$

Log Approximation:
For small x:
$\ln(1+x) \approx x$
$1 + x \approx e^x$