TA session
Ramsey Problem with Noncontingent Debt and Optimal Maturity Structure

Takashi Onoda

February 22, 2019

Overview

1. Theory
2. Example
3. Simulations

Motivation of Angeletos (2002) and Buera and Nicolini (2004)

- In Lucas and Stokey (1983), the government is allowed to use contingent debt and can smooth distortion across states.
- In the real world, markets for state contingent government debt do not exit.
- Is it possible to achieve the allocations under complete market only with noncontinget debt?

What Angeletos (2002) and Buera and Nicolini (2004) do

- They theoretically show it is possible.
- They exploit variation in price changes across different maturity debt.
- Governments can replicate contingent debt and implement the Ramsey plan by carefully choosing maturity structure.

Economy

- We follow Buera and Nicolini (2002).
- A barter economy in Lucas and Stokey (1983).
- The representative agent is endowed with one unit of labor in every period.
- A flat rate labor tax τ_l is the only available tax.
- Technology is given by

$$c_t + g_t \leq s_t(1 - x_t) \qquad t = 0, 1, 2, \dots, \quad \text{all } g_t \tag{1}$$

where x_t, c_t, and g_t represent leisure and private and public consumption, respectively, and s_t is a productivity shock.

Economy

- g_t is exogenous and follows some stochastic process.
- There is a finite number N possible values for the pair $h_t = (s_t, g_t)$ at each t.
- Let $h^t = \{h_0, \ldots, h_t\}$ be the history of shocks up to t.

Economy

- The representative agent's preferences is

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t [U\left(c_t(h^t), x_t(h^t)\right)] \tag{2}$$

where $\beta \in (0,1)$, U is strictly increasing in both arguments and strictly concave.

- If complete market, the implementability constraint becomes

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[c_t \frac{U_{c,t}}{U_{c.0}} - (1 - x_t)\frac{U_{x,t}}{U_{c,0}}\right] = b_{-1} \tag{3}$$

where b_{-1} represents the value of liabilities inherited by the government in units of time zero consumption.

Contingent Debt Case

- First, we consider the case when one-period state contingent bonds is available.
- A condition corresponding (3) for period t is

$$z_{t-1}(h^{t-1}, h_t) = \mathrm{E_t} \sum_{t=0}^{\infty} \beta^t \left[c_t \frac{U'(c_s(h^s))}{U'(c_t(h^t))} - (1 - x_s(h^s)) \frac{V'(x_s(h^s))}{U'(c_t(h^t))} \right]$$

  where $U(c, x) = U(c) + V(x)$.
- The RHS is the expected discounted value of the government surplus of history h^t. Solving the Ramsey problem gives the RHS.
- To match the RHS value, the government issues contingent debt so that the debt level becomes $b_{t-1}(h^{t-1}, h_t) = z_{t-1}(h^{t-1}, h_t)$.
- This illustrates $b_{t-1}(h^{t-1}, h_t)$ needs to vary depending on h_t.

Non-contingent Debt Case

- If only noncontingent one-period bonds (e.g. Aiyagari et al. (2002)), it imposes the restriction $b_{t-1}(h^{t-1}, h^t) = b_{t-1}(h^{t-1})$.
- However, if the government can issue non-contingent bonds maturing at J consecutive dates and J is large, we can replicate the contingent debt result.
- The time t value of outstanding obligations, given h_t, is

$$V(h^{t-1}, h_t) = \sum_{j=0}^{j-1} p_t^j(h^{t-1}, h_t) b_t^j(h^{t-1})$$

  where $p_t^j(h^{t-1}, h_t)$ and $b_t^j(h^{t-1})$ are the price and the amount at period t of zero-coupon bond maturing at $t+j$, respectively.
- The dependence of $p_t^j(h^{t-1}, h_t)$ on h_t is the key.

Condition for Replication

- The condition that we can produce $z_{t-1}(h^{t-1}, h^t)$ for all $t, (h^{t-1}, h_t)$ is

$$z(h^{t-1}, h_t) = V(h^{t-1}, h_t) = \sum_{j=0}^{j-1} \frac{\mathrm{E_t}[\beta^j U'(c_{t+j}|h^{t-1}, h^t]}{U'(c_t)} b_t^j(h^{t-1})$$

for all $t, (h^{t-1}, h_t)$

- Let's put this into matrix form!

Condition for Replication

$$A_t(h^{t-1})b_{t-1}(h^{t-1}) = Z_t(h^{t-1})$$

where

$$A_t(h^{t-1}) = \begin{pmatrix} 1 & \beta\frac{\mathrm{E_t}[U'(c_{t+1}|h^{t-1},h^1]}{U'(c_t)} & \dots & \beta^{J-1}\frac{\mathrm{E_t}[U'(c_{t+J-1}|h^{t-1},h^1]}{U'(c_t)} \\ 1 & \beta\frac{\mathrm{E_t}[U'(c_{t+1}|h^{t-1},h^2]}{U'(c_t)} & \dots & \beta^{J-1}\frac{\mathrm{E_t}[U'(c_{t+J-1}|h^{t-1},h^2]}{U'(c_t)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \beta\frac{\mathrm{E_t}[U'(c_{t+1}|h^{t-1},h^N]}{U'(c_t)} & \dots & \beta^{J-1}\frac{\mathrm{E_t}[U'(c_{t+J-1}|h^{t-1},h^N]}{U'(c_t)} \end{pmatrix}$$

$$b_{t-1}(h^{t-1}) = \begin{pmatrix} b_t^0(h^{t-1}) \\ b_t^1(h^{t-1}) \\ \vdots \\ b_t^{J-1}(h^{t-1}) \end{pmatrix}, \; Z_t(h^{t-1}) = \begin{pmatrix} z_t(h^{t-1}, h_1) \\ z_t(h^{t-1}, h_2) \\ \vdots \\ z_t(h^{t-1}, h_N) \end{pmatrix}$$

Full-rank Condition

- If $A_t(h^{t-1})$ has rank N, the condition can be satisfied with any $Z_t(h^{t-1})$ by choosing appropriate $b_{t-1}(h^{t-1})$.
- A necessary condition is $J \geq N$.
- Even if $A_t(h^{t-1})$ is not full rank, τ_l can control bond prices (Proposition 3 in Angeletos (2002)) and perturbation by τ_l can force $A_t(h^{t-1})$ to be full rank.

Theorem 1 of Angeletos (2002)

Let $S \geq 1$ be the number of states and $N \geq 1$ the number of maturities. When $N \geq S$, then and only then any policy/allocation that is implementable with Arrow-Debreu markets can either be implemented with noncontingent debt or be approximated arbitrarily well.

History Independence and Markov Case

- In the Ramsey problem, optimal tax rates and optimal allocations are independent of history under complete market.
- This implies the following lemma and consequently the theorem where Q_t is A_t and V_t is Z_t in this slides.

Lemma 3 of Angeletos (2002)

Along the Ramsey plan, $Q_t(s^{t-1}) = \bar{Q}_t$, and $V_t(s^{t-1}) = \bar{V}_t$, $\forall s^{t-1}$, $t \geq 1$. If the economy is stationary and uncertainty follows a Markov process, then $\bar{Q}_t = \bar{Q}$ and $\bar{V}_t = \bar{V}$, $\forall t \geq 1$.

Theorem 2 of Angeletos (2002)

The optimal maturity structure at t is independent of s_t. If the economy is stationary and uncertainty follows a Markov process, then the optimal maturity structure is independent of t as well.

Example

- Assume c_t is negatively correlated with g_t.
- $J = N = 2$
- No productivity shock ($s_t = 1$ for all t).
- Two states for g_t ($= h_t$)
- Let $b_0^j = 0$, $j = 1, 2$ $g_0 = g_h$ and assume that government expenditure follows a Markov process with two states, $g_h > 0$ and $g_l = 0$
- In the Ramsey plan, $z(g_h) = 0$ and $z(g_l) > 0$. Note if g_h, the state is identical to the initial period.

Example

- The condition for replication becomes

$$\begin{pmatrix} 1 & \beta \mathrm{E}_h \frac{U'(c)}{U'(c_h)} \\ 1 & \beta \mathrm{E}_l \frac{U'(c)}{U'(c_l)} \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} 0 \\ z_l \end{pmatrix}$$

- $U'(c_h) > U'(c_l)$ ensures det is positive.

$$\begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} \frac{-U'(c_l)\mathrm{E}_h U'(c) z_l)}{det} \\ \frac{z_l}{det} \end{pmatrix} = \begin{pmatrix} - \\ + \end{pmatrix}$$

- When the government needs to more wealth in g_h than g_l, sell two-period bond and buy one-period bond.
- This is because $g \uparrow \Rightarrow c \downarrow \Rightarrow u'(c) \uparrow \Rightarrow$ interest rate $\uparrow \Rightarrow b^2$ price $\downarrow$ relatively. In order to transfer wealth from g_l to g_h, the government sells (issues) b^2 whose price goes down in g_l.

Simulation (1)

- Buera and Nicolini (2004) provides simulations to show the optimized level of debt.
- The simulated debt position is large even in only two state case.

Table 1
Debt positions in units of peace-time GDP and interest rates

Risk aversion	Debt positions $\left(\frac{p^j(g_{\text{low}})b^j}{y(g_{\text{low}})}\right)$		Short term interest rates (%)	
			g_{low}	g_{high}
(A) War example				
Quarterly calibration	$j = 0.25$	$j = 30$		
0.50	−18.20	19.33	1.88	3.96
2.00	−7.10	9.09	1.44	9.38
10.00	−5.56	7.22	1.40	9.77
Yearly calibration	$j = 1$	$j = 30$		
0.50	−4.53	4.81	1.90	4.00
2.00	−1.77	2.26	1.46	9.64
10.00	−1.38	1.80	1.42	10.06
(B) Business cycle				
Quarterly calibration	$j = 0.25$	$j = 30$		
0.50	−7.60	7.60	1.91	2.04
2.00	−2.88	2.90	1.64	2.15
10.00	−2.50	2.52	1.61	2.16

Note: In columns 2 and 3, we report the maturities implementing the Ramsey allocation that minimized the present value of the absolute value of the positions. In the war example the process for the government expenditures was calibrated to match the US experience during the 20th century (see the appendix for details). The business cycle examples were calibrated to match the US postwar experience. By quarterly (yearly) calibration we mean that the calibration and the simulation were done using a quarter (year) as

Simulation (2)

- The debt position becomes huge in four states case.

Table 2
Debt positions and interest rates for the four states examples

Risk aversion	Debt positions $\left(\frac{p^j(g_{low}, s_{high})b^j}{y_{g_{low}, s_{high}}}\right)$				Short term interest rates (%)			
					g_{low}, s_{high}	g_{low}, s_{low}	g_{high}, s_{high}	g_{high}, s_{low}
Business cycle examples								
Our calibration	$j=0.25$	$j=1$	$j=2$	$j=30$				
0.50	−21.83	146.08	322.70	198.46	1.03	3.93	1.15	4.06
2.00	−10.10	65.61	−140.41	84.92	−1.15	7.68	−0.66	8.21
10.00	−9.13	60.22	−129.22	78.13	−0.44	6.16	0.07	6.73
Calibration of Chari et al. (1995)	$j=1$	$j=4$	$j=13$	$j=30$				
1.00	−24.62	71.96	−171.89	125.12	1.81	2.04	1.99	2.23
2.00	−13.58	45.01	−125.93	95.07	0.93	2.08	1.90	3.12
9.00	−7.49	20.52	−43.39	30.00	0.41	2.58	1.25	3.54

Note: In columns 2–5, we report the maturities implementing the Ramsey allocation that minimized the average value of the absolute value of the positions. The business cycle examples were calibrated to match the US postwar experience. Our calibration was done taking a quarter as the time period, while Chari et al. (1995) use the year. In our (Chari et al. (1995)) calibration the short-term interest rate corresponds to a 3-month (1-year) bond. In both cases, interest rates correspond to yearly returns.

Conclusion

- Allocations in a Ramsey problem under complete markets can be (at least approximately) achieved by optimizing noncontingent debt maturities.
- The simulations suggest that the size of financial transactions the government must undertake each period is very large and increases dramatically with the number of states.
- For taxation under incomplete markets, check Aiyagari et al. (2002), where only one-period noncontingent debt is available.