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# Optimal Cartel Trigger Price Strategies*

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A dynamical model of industry equilibrium is described in which a cartel deters deviations from collusive output levels by threatening to produce at Cournot quantities for a period of fixed duration whenever the market price falls below some trigger price. In this model firms can observe only their own production level and a common market price. The market demand curve is assumed to have a stochastic component, so that an unexpectedly low price may signal either deviations from collusive output levels or a "downward" demand shock. *Journal of Economic Literature* Classification Numbers: 611, 026.

## 1. INTRODUCTION

Consider an industry trade association which incurs costs in the process of collecting and disseminating member price and quantity data, as well as costs of verifying or monitoring the accuracy of these data. In the absence of such transaction costs, Friedman [1] and Osborne [3] have demonstrated that adherence to a perfectly collusive agreement could be monitored and enforced internally without having to revert to "punishing" behavior.

This paper examines situations in which these costs do exist, and so the firms of the industry are faced with a problem of detecting and deterring cheating on an agreement. In particular, it is assumed that firms observe only their own production level and the market price, but not the quantity produced by any other firm. (Their output is assumed to be of homogeneous quality, and so they face a common market price.) If the market demand curve has a stochastic component, an unexpectedly low price may signal either deviations from collusive output levels or a "downward" demand shock.

Green and Porter [2] have shown that under these circumstances, participating firms can deter deviations by threatening to produce at Cournot

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levels for a period of fixed duration whenever they observe market price below some trigger price. (They go on to propose econometric techniques employing aggregate industry time series data which are designed to detect such an enforcement mechanism. The equilibrium conditions of the theoretical model determine the estimating equations and stochastic structure of the econometric model. This model exploits the fact that there will be periodic switches between the Cournot and collusive output levels when such a noncooperative equilibrium exists.) More generally, they have developed a model in which price wars can be viewed as an equilibrium phenomenon. Their paper is an extension and reinterpretation of Stigler's [5] model of detection of secret price cutting in an oligopolistic market. This extension of Stigler's model offers an explanation that what looks like collusive behavior at a point in time is actually the noncooperative equilibrium outcome of a regularly repeated market game. A firm which considers a secret expansion of output above the collusive level must trade off immediate profit gains with the increased probability that the market price will fall below the trigger price, thereby increasing the likelihood of lower profits in future periods as the industry reverts to Cournot output levels.

The purpose of this paper is to characterize the optimality properties of the Green and Porter model from the standpoint of the firms participating in the cartel. We calculate which values of the trigger price and punishment period length maximize expected industry discounted value, subject to the requirement that firms have no incentive to deviate from cooperative period output levels. The implications for the equilibrium quantity vector of setting the trigger price and punishment period length at their optimal values are assessed.

A version of the Green and Porter model is presented in Section 2 and general properties of the equilibrium are outlined. Optimal cartel strategies are examined in Section 3 and 4. It is demonstrated that, in general, the optimal quantity in cooperative periods will exceed that which would maximize expected joint net returns in any single period. The optimal aggregate quantity in cooperative periods is shown to be a nondecreasing function of the number of firms, equaling the aggregate Cournot level in the limit, and a nondecreasing function of a term which reflects the "noisiness" of the stochastic demand component. There is also a discussion of some comparative statics results. Section 5 provides some summary remarks.

## 2. The Model

Consider an oligopoly of $N$ risk neutral firms. The industry output vector is denoted by $\bar{q}_t = (q_{1t}, \ldots, q_{Nt})$, where $q_{it}$ is the output of firm $i$ in period $t$. (Throughout the paper, a bar over a variable indicates that it is an $N$-

dimensional vector.) Total industry output is then $Q_t = \sum_{i=1}^{N} q_{it}$. It is assumed that the firms of the industry produce an output of homogeneous quality. As a result, they face a common market price which is determined by an inverse demand function depending solely on total industry output, and not on the distribution of outputs across firms. It is assumed that the inverse demand function has a multiplicative stochastic component, so that the market price firms observe $\hat{p}_t$ is determined by

$$\hat{p}_t = p(Q_t)\,\theta_t,$$

where $p: R_+ \to R_+$ is the inverse demand function and $\{\theta_t\}$ is an identically and independently distributed sequence of random variables not directly observed by firms, with mean $\mu$, density function $f$, and cumulative distribution $F$. It is assumed that $F(0) = 0$, $F(\infty) = 1$, and both $F$ and $f$ are continuously differentiable. (Section 4 discusses the properties of a model where the inverse demand function has an additive stochastic component, i.e., where $\hat{p}_t = p(Q_t) + \theta_t$.) Throughout this paper it is assumed that the inverse demand function is linear and decreasing in output, so that

$$p(Q_t) = a - bQ_t, \tag{1}$$

where $a$ and $b$ are positive constants.

Firms are hypothesized to be symmetric in that they each face the cost function

$$C(q_i) = c_0 + c_1 q_i, \tag{2}$$

where $c_0$ and $c_1$ are constants, and where the time subscripts have been dropped for notational simplicity. It is assumed that

$$0 < c_1 < \mu a, \tag{3}$$

and that the number of firms $N$ is small enough so that

$$0 < c_0 < (\mu a - c_1)^2/\mu b(N+1)^2. \tag{4}$$

Given $Q_i = Q - q_i = \sum_{j \neq i} q_j$, the total output of the other firms, firm $i$ faces the single period expected profit function

$$\pi_i(\bar{q}) = [A - B(Q_i + q_i)]\, q_i - c_0, \tag{5}$$

where $A = \mu a - c_1$ and $B = \mu b$. Equation (3) and $b > 0$ imply that $A$ and $B$ are positive constants.

Let $\bar{s} = (s_1, \ldots, s_N)$ denote the Cournot output vector. Then, given

$Q_i = \sum_{j \neq i} s_j$, $s_i$ maximizes $\pi_i(\bar{q})$ with respect to $q_i$ for every firm $i$. It is easy to show that

$$s_i = A/B(N+1), \qquad \text{for} \quad i = 1, \ldots, N,$$
$$= s, \qquad \text{say,}$$

and that

$$\pi_i(\bar{s}) = [A^2/B(N+1)^2] - c_0, \qquad \text{for} \quad i = 1, \ldots, N. \tag{6}$$

Equation (4) guarantees that firms earn positive profits at Cournot output levels.

Denote the output vector which maximizes expected joint net returns in a single period by $\bar{r} = (r_1, \ldots, r_N)$. Then, given $Q_i = \sum_{j \neq i} r_j$, $r_i$ maximizes $\sum_{j=1}^{N} \pi_j(\bar{q})$ with respect to $q_i$ for every firm $i$. It can be shown that

$$r_i = A/2BN, \qquad \text{for} \quad i = 1, \ldots, N$$
$$= r, \qquad \text{say,}$$

and that

$$\pi_i(\bar{r}) = (A^2/4BN) - c_0 \qquad \text{for} \quad i = 1, \ldots, N.$$

As long as there is more than one firm, single firm expected profits will be higher when $\bar{q} = \bar{r}$ than at Cournot output levels.

Now suppose that the industry behaves in the manner posited by Green and Porter. Consider, therefore, firms which follow a strategy of reversion to Cournot output levels for a finite number of periods, say $T-1$, whenever market price $\hat{p}_t$ falls below a predetermined trigger price, say $\tilde{p}$. Then cooperative and noncooperative periods are defined recursively by

(a) $t = 0$ is a cooperative period;

(b) if $t$ is a cooperative period and $\tilde{p} \leqslant \hat{p}_t$, then $t+1$ is a cooperative period;

(c) if $t$ is a cooperative period and $\tilde{p} > \hat{p}_t$, then $t+1, \ldots, t+T-1$ are noncooperative periods with $\bar{q}_t = \bar{s}$, and $t+T$ is a cooperative period.

Output levels in cooperative periods are determined below.

Since we will later allow the cartel to maximize joint value subject to enforcement constraints, the restriction to simple trigger strategies may not be desirable, given that more general strategies may lead to better outcomes. For example, the length of the punishment period could depend on the amount by which the price level fell below the trigger price. Unfortunately, models with strategies such as these are extremely complicated for computational purposes, and it is difficult to obtain any interesting results.

Note that firms will have no incentive to deviate from the punishing response of Cournot output levels, so that this threat strategy is self-enforcing. When all other firms produce at that level, then it is optimal for any particular firm to adhere to the agreement and behave in a similar manner.

Suppose that firms face a common discount factor $\beta$, where $0 < \beta < 1$. Then, given an industry output vector $\bar{q}$ in cooperative periods, the expected present discounted value of firm $i$ is defined by

$$V_i(\bar{q}) = \pi_i(\bar{q}) + \Pr\{\tilde{p} \leqslant \theta p(Q)\}\, \beta V_i(\bar{q}) + \Pr\{\tilde{p} > \theta p(Q)\} \left[ \sum_{\tau=1}^{T-1} \beta^\tau \pi_i(\bar{s}) + \beta^T V_i(\bar{q}) \right]. \tag{7}$$

Thus the expected discounted value of firm $i$ satisfies a recursive relation. This value equals present period expected profits plus the expected value next period. If present period market price exceeds the trigger price, next period's value equals that of the present period, discounted once. Otherwise, firms earn expected Cournot profits for $T-1$ periods, at which point the industry returns to cooperative behavior.

Equation (7) can be rewritten as

$$V_i(\bar{q}) = \frac{\pi_i(\bar{q}) + F(\tilde{p}/p(Q))[(\beta - \beta^T)/(1-\beta)]\, \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)\, F(\tilde{p}/p(Q))} = \frac{\pi_i(\bar{s})}{1-\beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)\, F(\tilde{p}/p(Q))}. \tag{8}$$

Thus expected firm value equals that when firms produce at Cournot levels in every period, plus the gain in profit in cooperative periods versus Cournot periods, appropriately discounted. Not surprisingly, the appropriate discount factor depends on both the trigger price and the length of noncooperative episodes.

A noncooperative equilibrium is charactrized by $T$, $\tilde{p}$, and an equilibrium output vector $\bar{q}^* = (q_1^*, ..., q_N^*)$ if

$$V_i(\bar{q}^*) = \mathrm{Max}\{V_i(\bar{q}) \mid q_j = q_j^* \text{ for } j \neq i, \text{ and } q_i \geqslant 0\}.$$

The first-order necessary conditions are $0 = V_i^i(\bar{q}^*)$, where $V_i^i(\bar{q}) = \partial V_i(\bar{q})/\partial q_i$, or

$$0 = [1 - \beta + (\beta - \beta^T)\, F]\, \pi_i^i(\bar{q}^*) + (\beta - \beta^T)\, [\tilde{p} p'(Q^*)/p(Q^*)^2]\, f\, [\pi_i(\bar{q}^*) - \pi_i(\bar{s})] \tag{9}$$

for $i = 1, ..., N$, where $\tilde{p}/p(Q^*)$, the argument of $F$ and $f$, has been suppressed and where $\pi_i^i(\bar{q}) = \partial \pi_i(\bar{q})/\partial q_i$, $p'(Q) = dp(Q)/dQ$, and $Q^* = \sum_{i=1}^N q_i^*$.

Note that there is no presumption that $\bar{q}^* = \bar{r}$, the single period joint profit maximizing vector. In Section 3, it is shown that in general $\bar{q}^*$ will *not* equal $\bar{r}$ if $\tilde{p}$ and $T$ are chosen to maximize total industry expected value.

An immediate result is:

PROPOSITION 2.1. *For any strictly concave profit function, the single period Cournot output vector $\bar{s}$ is an equilibrium quantity vector in cooperative periods for any values of $\tilde{p}$ and $T$.*

*Proof.* This is easily verified by realizing that $\pi_i^i(\bar{s}) = 0$ for $i = 1,..., N$, by the definition of $\bar{s}$, and letting $\bar{q}^* = \bar{s}$ in Eq. (9). Furthermore,

$$\frac{\partial^2 V_i(\bar{s})}{\partial q_i^2} = \frac{\partial^2 \pi_i(\bar{s})}{\partial q_i^2} \Big/ [1 - \beta + (\beta - \beta^T) F]$$

so that concavity of the profit function implies concavity of the value function at $\bar{s}$. Q.E.D.

As a result, the problems posed in this framework are not those of existence of equilibrium, but whether, among the equilibria which exist, there is one which dominates all the others. This is the motivating force behind the calculus of the following sections, which try to select the values of $\tilde{p}$ and $T$ which maximize total industry expected discounted value subject to the satisfaction of Eq. (9). These optimal values in turn imply an optimal value of $\bar{q}^*$, given Eq. (9).

Before turning to these issues, it is useful to derive restrictions on the range of those values of $\bar{q}^*$ which will be of iinterest.

PROPOSITION 2.2. *Given the symmetric linear specification of the demand and cost functions, all firms which have positive equilibrium output levels will produce exactly the same quantity in cooperative periods.*

*Proof.* If $q_k^*$ and $q_l^*$ are positive for some $k$ and $l$, then we can rewrite Eq. (9) as

$$0 = \alpha \pi_i^i(\bar{q}^*) + \gamma[\pi_i(\bar{q}^*) - \pi_i(\bar{s})] \qquad \text{for} \quad i = k, l, \tag{10}$$

where $\alpha = 1 - \beta + (\beta - \beta^T) F$ and $\gamma = (\beta - \beta^T) f\tilde{p}p'/p^2$ are common to both firms. Note that $\alpha > 0$ (since $0 < \beta < 1$) and $\gamma < 0$ (since $p' < 0$). But

$$\pi_i^i(\bar{q}^*) = A - BQ^* - Bq_i^* \qquad \text{for} \quad i = k, l, \tag{11}$$

and, from (5) and (6),

$$\pi_i(\bar{q}^*) - \pi_i(\bar{s}) = q_i^*(A - BQ^*) - A^2/(B(N+1)^2) \qquad \text{for} \quad i = k, l. \tag{12}$$

If we subtract Eq. (10) for firm $k$ from that of firm $l$, then (11) and (12) can be used to obtain

$$0 = \alpha B(q_k^* - q_l^*) + \gamma(A - BQ^*)(q_l^* - q_k^*). \tag{13}$$

Since $\alpha > 0$, $\gamma < 0$, and $q_k^*$ and $q_l^*$ are positive, Eq. (13) is satisfied if and only if $q_k^* = q_l^*$. Q.E.D.

We have chosen to work with a symmetric linear structure of inverse demand and costs precisely because of the resultant symmetric equilibrium, which enables a detailed examination of the nature of the effects of the probability distribution function of the stochastic demand component $F(\theta)$ on the optimal equilibrium solution. Also, the fact that the firms are symmetric implies that they will unanimously agree upon the optimal solution so that we can abstract from problems the cartel may face in trying to reconcile disparate interests.

Denote the equilibrium output in cooperative periods for a single firm by $q^*$. Then aggregate output $Q^* = Nq^*$. We can then rewrite the first-order necessary condition (Eq. (9)) as

$$\begin{aligned}[1 - \beta + (\beta - \beta^T)\,F][A - (N+1)\,Bq^*]\\ = (\beta - \beta^T) f[\tilde{p}b/(a - Nbq^*)^2][q^*(A - NBq^*) - A^2/(B(N+1)^2)], \end{aligned} \tag{14}$$

where the argument of both $F$ and $f$ is $\tilde{p}/(a - Nbq^*)$.

Because the Cournot output vector is an equilibrium, we can further restrict our attention to values of $q^*$ where firms do no worse than they would by producing at $s$.

Proposition 2.3. *In the symmetric linear structure of this section, a noncooperative equilibrium is characterized by* $q^*$, $\tilde{p}$, *and* $T$ *satisfying Eq.* (14), *where* $q^*$ *lies within* $(s/N, s]$.

*Proof.* From Eq. (8), we know that $V_i(\bar{q}) \geqslant V_i(\bar{s})$ if and only if $\pi_i(\bar{q}) \geqslant \pi_i(\bar{s})$, or, in the symmetric linear structure outlined above,

$$q(A - NBq) - c_0 \geqslant A^2/(B(N+1)^2) - c_0 \tag{15}$$

when $q_i = q$ for $i = 1,\ldots, N$. Simple manipulation of Eq. (15) reveals that it is equivalent to

$$A/N(N+1)\,B \leqslant q \leqslant A/(N+1)\,B$$

or

$$s/N \leqslant q \leqslant s,$$

where, as before, $s$ is the Cournot output of a single firm. We can therefore restrict our attention to values of $q^*$ in the closed interval $[s/N, s]$. It can be shown that $\pi_i^i(\bar{s}/N) > 0$ and $\pi_i(\bar{s}/N) = \pi_i(\bar{s})$ by construction, so that $V_i^i(\bar{s}/N) > 0$ for all $i$. Therefore $\bar{s}/N$ itself is not an equilibrium. Q.E.D.

In the proof of Proposition 2.1, we showed that the value function of firm $i$, $V_i(\bar{q})$, is concave in $q_i$ at the Cournot output vector $\bar{s}$ as long as the profit function $\pi_i(\bar{q})$ is concave. This condition is satisfied by our linear/quadratic system. To ensure concavity of $V_i(\bar{q})$ at $\bar{q}^*$, we must restrict the shape of the distribution function of the demand shock.

PROPOSITION 2.4. *If $F(\theta)$ is convex, then $V_i(\bar{q})$ is concave in $q_i$, given the symmetric linear structure of this section.*

*Proof.* If $F$ is a convex function, then $F(\tilde{p}/p(Q))$ is convex in $q_i$ when $p(Q)$ is linear. Since $\pi_i(\bar{q})$ is concave in $q_i$, $V_i(\bar{q})$ will then be concave in $q_i$. Q.E.D.

Convexity of $F(\theta)$ is sufficient to ensure concavity of the value function, but not necessary, as is demonstrated by the example of Section 3.4. With that exception, we shall henceforward assume that $F$ is a convex function. Convexity of the distribution function implies that the density function $f(\theta)$ is increasing over its support, so that we must now restrict the support of $F$ to the interval $(0, \theta_0)$, where $F(0) = 0$ and $F(\theta_0) = 1$, and where $\theta_0$ is finite. Under these conditions, $q_i^*$ will be the unique solution to $V_i^i(\bar{q}^*) = 0$, given $\tilde{p}$ and $T$.

It is important to note that in equilibrium, the incentive structure is designed to guarantee that firms do not deviate from $q^*$, the output level in cooperative periods. If they are rational, firms will then realize that, in equilibrium, the observed market price will be less than the trigger price only when there is a sufficiently small realization of the random demand component $\theta$, and never because of individual firm deviations. When this event occurs, rational firms will nevertheless revert to Cournot levels for $T-1$ periods, precisely because the enforcement mechanism must be employed on these occasions if firms are to have the correct incentives to produce $q^*$ in cooperative periods. If firms do not believe that the enforcement mechanism will be employed, then they have an incentive to cheat on the agreement. A corollary is that in equilibrium, observed "price wars" will always be preceded by a period with an unexpectedly low realization of $\theta$. In other words, in this framework price wars are precipitated by an unexpected fall in demand, rather than punishment of actual cheating.

A related point is that, if firms know the parameters of the inverse demand and cost functions, this analysis could easily be couched in terms of observed values of $\theta$, where in equilibrium firms assume that $\bar{q} = \bar{q}^*$ and then compare realizations of $\theta$ with some trigger value, say $\tilde{\theta}$. This approach will lead to first-order necessary conditions which are similar to those derived above, with similar implications.

Finally, note that any equilibrium quantity $q^*$, given some value of $\tilde{p}$ and $T$, will also represent the cooperative period output of a noncooperative equilibrium in $(\tilde{p}, T, q)$ strategy space. If all other firms produce $q^*$ in cooperative periods and revert to Cournot output levels for $T-1$ periods when market price is below $\tilde{p}$, then any firm will maximize its expected discounted value by behaving in a similar fashion.

## 3. Optimal Cartel Strategies

In general terms, Section 2 presented a model in which the expected present discounted value of firm $i$ is represented by $V_i(\bar{q}; \tilde{p}, T)$ for each of the $N$ firms in the industry (as in Eq. (8)). The solution to the first-order necessary conditions (14) is the Nash equilibrium quantity vector $\bar{q}^*(\tilde{p}, T)$, which is a function of the trigger price $\tilde{p}$ and of $T$, where $T-1$ is the length of episodes during which firms revert to Cournot output levels. The equilibrium value of firm $i$ will then be $V_i(\bar{q}^*(\tilde{p}, T); \tilde{p}, T)$, or $V_i^*(\tilde{p}, T)$, for $i = 1,..., N$, which is a function of $\tilde{p}$ and $T$. From firm $i$'s perspective, then, the optimal values of $\tilde{p}$ and $T$ are those which maximize $V_i^*(\tilde{p}, T)$ subject to $\tilde{p} \geqslant 0$ and $T \geqslant 1$. (Recall that $T-1$ is the length of reversionary episodes, rather than $T$.) Because firms are symmetric, these optimal values will be common to all firms. Also, these values will imply, via Eq. (14), the optimal Nash equilibrium quantity in cooperative periods.

Depending on the values of $\tilde{p}$ and $T$, there will be a range of possible equilibrium quantities. This range will depend on the discount factor, the distribution of the stochastic demand component, and the number of firms. In considering optimal choices of $\tilde{p}$ and $T$, this section focusses on an analysis of the best cooperative equilibrium that can be enforced. One should keep in mind that other equilibria are available to the cartel, but are inferior in the sense that they imply lower industry expected value.

For reasons which will become obvious later in the paper, for the purposes of this section we have chosen to regard $T$ as a continuous variable and assume that the optimal value happens to be an integer.

In this case the optimal values of $\tilde{p}$ and $T$, if they are interior solutions, will satisfy the first-order necessary conditions

$$0 = dV_i^* / d\tilde{p} \tag{16}$$

and

$$0 = dV_i^* / dT, \tag{17}$$

which will have the same solution for each firm $i$ because of symmetry.

Equations (16) and (17) can be written as

$$0=\sum_{j=1}^{N}\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^*}{\partial \tilde{p}}+\frac{\partial V_i}{\partial \tilde{p}}=\sum_{j\neq i}\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^*}{\partial \tilde{p}}+\frac{\partial V_i}{\partial \tilde{p}},$$

since $V_i$ is evaluated at $\bar{q}^*$ and $V_i^i(\bar{q}^*)=0$ by construction, and

$$0=\sum_{j\neq i}\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^*}{\partial T}+\frac{\partial V_i}{\partial T}.$$

Because of the symmetry results, these equations can in turn be rewritten as

$$0=(N-1)\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^*}{\partial \tilde{p}}+\frac{\partial V_i}{\partial \tilde{p}} \qquad \text{for} \quad j\neq i, \tag{18}$$

and

$$0=(N-1)\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^*}{\partial T}+\frac{\partial V_i}{\partial T} \qquad \text{for} \quad j\neq i. \tag{19}$$

The remainder of this section is divided into four parts and studies the solutions to these equations given the linear structure of the previous section. The first part examines interior solutions of these equations, and the second presents an example. The third subsection then demonstrates that for a wide class of distributions the optimal value of $T$ is infinite, and the fourth presents a further example which illustrates this point. To simplify the algebra of this section, we shall assume that $\mu$, the mean of $\theta$, equals one, so that $b=B$.

### 3.1. *Interior Solutions*

This section calculates the implications of choosing optimal values of the trigger price and punishment period length, assuming that they are interior. By interior we mean that the optimal value of $\tilde{p}$ is positive and finite (more accurately, that $\tilde{p}/p(Q^*)$ lies in the interior of the support of the distribution function $F$) and that $T$ is greater than unity and finite. Some comparative statics results are presented near the end of the section.

The first result concerns the cooperative period output implied by the optimal choice of $\tilde{p}$ and $T$.

PROPOSITION 3.1. *If $\tilde{p}$ and $T$ are chosen to maximize expected value of the firms of the industry and the optimal values are an interior solution, then the equilibrium output level will exceed the single period joint profit maximizing output. As a corollary, the equilibrium price will be lower.*

*Proof.* We begin by solving Eqs. (18) and (19) for the optimal values of $\tilde{p}$ and $T$. Note that if Eq. (14) is employed,

$$\frac{\partial V_i(\bar{q}^*)}{\partial q_j} = \frac{-(A - NBq^*)}{[1-\beta+(\beta-\beta^T)F]} \qquad \text{for} \quad j \neq i$$

and if we denote $p(Nq^*) = a - Nbq^*$ by $p^*$, then

$$\frac{\partial V_i(\bar{q}^*)}{\partial T} = \frac{\beta^T \ln \beta F[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1-\beta+(\beta-\beta^T)F]^2}$$
$$= \frac{p^{*2}[A-(N+1)Bq^*]\beta^T \ln \beta F}{(\beta-\beta^T)f[1-\beta+(\beta-\beta^T)F]\tilde{p}b},$$

where the second equality follows from Eq. (14), and

$$\frac{\partial V_i(\bar{q}^*)}{\partial \tilde{p}} = \frac{-(\beta-\beta^T)f[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1-\beta+(\beta-\beta^T)F]^2 p^*}$$
$$= \frac{-p^*[A-(N+1)Bq^*]}{\tilde{p}b[1-\beta+(\beta-\beta^T)F]},$$

where we have again used Eq. (14). Note that these derivatives are identical for all firms $i$. Again, $\tilde{p}/p^*$, the argument of $F$ and $f$, has been suppressed.

Totally differentiating the first-order necessary condition, Eq. (14), and employing Proposition 2.2, which shows that $q_j^* = q^*$ for all $j$, yields

$$\frac{\partial q^*}{\partial \tilde{p}} = \frac{[A-(N+1)Bq^* - b\,\Delta\eta]\,p^*}{\tilde{p}bK}$$

and

$$\frac{\partial q^*}{\partial T} = \frac{\beta^T \ln \beta p^*[\tilde{p}b\,\Delta f - Fp^*(A-(N+1)Bq^*)]}{\tilde{p}fb(\beta-\beta^T)K},$$

where

$$\Delta = \pi_i(\bar{q}^*) - \pi_i(\bar{s}), \tag{20}$$

which is common to all firms $i$,

$$\eta = 1 + \frac{f'(\tilde{p}/p^*)}{f(\tilde{p}/p^*)}\frac{\tilde{p}}{p^*}, \tag{21}$$

and

$$K = \frac{B\Delta(N+1)p^*}{A-(N+1)Bq^*} + bN\Delta(\eta+1) - (N-1)(A-NBq^*).$$

For each firm $i$, solution of Eq. (18), the optimality condition for $\tilde{p}$, yields

$$q^* = \frac{A}{2NB}\left(\frac{N+\eta+(N+1)\,a/A}{N+1+\eta}\right) = r\left(\frac{N+\eta+(N+1)\,a/A}{N+1+\eta}\right), \quad (22)$$

where $r$ is the value of $q_i$ which maximizes total single period expected profits of the industry, given $Q_i = (N-1)\,r$. Solution of Eq. (19), the optimality condition for $T$, taking $\tilde{p}$ as the optimal value of $\tilde{p}$ and using Eqs. (14) and (22), yields

$$\frac{f(\tilde{p}^*/p^*)}{F(\tilde{p}^*/p^*)}\frac{\tilde{p}^*}{p^*} - \frac{f'(\tilde{p}^*/p^*)}{f(p^*/p^*)}\frac{\tilde{p}^*}{p^*} = 1 \quad (23)$$

or

$$\frac{f(\tilde{p}^*/p^*)}{F(\tilde{p}^*/p^*)}\frac{\tilde{p}^*}{p^*} = \eta^*, \quad (24)$$

where

$$\eta^* = 1 + \frac{f'(\tilde{p}^*/p^*)}{f(\tilde{p}^*/p^*)}\frac{\tilde{p}^*}{p^*}.$$

Equation (24) implies that $\eta^*$ is nonnegative, and so Eq. (22) implies that $q^*$ is greater than $r$. Since $q^* > r$, $p(Nq^*) < p(Nr)$. Q.E.D.

One reason why $q^*$ will exceed $r$ in uncertain environments is that the gains to cheating decrease as $q^*$ increases, as measured by the left-hand side of Eq. (14), and so less severe penalties for suspected cheating (in terms of a larger $T$ or $\tilde{p}$, as in the right-hand side of (14)) can be employed.
Another implication of Eq. (23) is a comparative statics result.

PROPOSITION 3.2. *The optimal trigger price $\tilde{p}^*$ will adjust in exactly the same proportion as $p(Nq^*)$ in response to changes in the parameters $a$, $A$, $B$, and $N$. Neither $\tilde{p}^*$ nor $q^*$ will be affected by small changes in the discount factor $\beta$ if $\theta^*$ is unique.*

*Proof.* Let $\theta^* = \tilde{p}^*/p^*$. Then Eq. (23) can be rewritten as

$$\frac{f(\theta^*)}{F(\theta^*)}\theta^* - \frac{f'(\theta^*)}{f(\theta^*)}\theta^* = 1. \quad (25)$$

Thus the difference between the elasticity of $F$ with respect to $\theta$ and that of $f$ with respect to $\theta$, evaluated at $\theta^*$, is equal to unity. Thus $\theta^*$, and so $\eta^*$, is unaffected by changes in $a, A$, $B$, and $N$. (It is a function solely of the parameters of the distribution function $F$.) If there is a unique solution to this equation at $\theta^*$, $q^*$ will be determined by Eq. (22) with $\eta = \eta^*$, and then the optimal trigger price will be uniquely determined as $\tilde{p}^* = \theta^* p(Nq^*)$. (Section 3.3 discusses cases in which there is no solution for Eq. (25).) As the parameters mentioned above change, then, $\tilde{p}^*$ will adjust in the same proportion as $p(Nq^*)$. Furthermore, as long as $q^*$ is supportable by $\tilde{p}^*$ and some value of $T$, it will be determined by Eq. (22), and so not be affected by small changes in $\beta$. (Larger movements in $\beta$ may lead to situations in which $q^*$ is not supportable by $\tilde{p}^*$ for any value of $T$.) Q.E.D.

But how do $q^*$ and $\tilde{p}^*$ adjust in response to parameter movements? Here $dp^*/dq^* = -bN$, and so $d\tilde{p}^*/dq^* = -bN\theta^*$. But

$$\frac{dq^*}{dA} = \frac{N+\eta^*}{2NB(N+1+\eta^*)} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dA} = -\frac{(N+\eta^*)\,\theta^*}{2(N+1+\eta^*)} < 0;$$

$$\frac{dq^*}{da} = \frac{2N+1+\eta^*}{2NB(N+1+\eta^*)} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{da} = -\frac{(2N+1+\eta^*)\,\theta^*}{2(N+1+\eta^*)} < 0;$$

$$\frac{dq^*}{dB} = -\frac{q^*}{B} < 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dB} = N\theta^* q^* > 0;$$

and

$$\frac{d(Nq^*)}{dN} = \frac{A+a\eta^*}{2B(N+1+\eta^*)^2} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dN} = -\frac{(A+a\eta^*)\,\theta^*}{2(N+1+\eta^*)^2} < 0.$$

Thus the equilibrium quantity (trigger price) is a decreasing (increasing) function of the absolute value of the slope $b$ of the inverse demand curve and an increasing (decreasing) function of the intercept $a$ of the inverse demand curve, and $A = a - c_1$. (Recall Eqs. (1) and (2).) As the number of firms $N$ increases, total output increases and so the trigger price will fall. Also, $q^*$ is a decreasing function of $N$ since

$$\frac{dq^*}{dN} = -\frac{q^*}{N} + \frac{A+a\eta^*}{2BN(N+1+\eta^*)^2} < 0.$$

It is easy to show that as $N$ becomes infinitely large (suppose that fixed costs $c_0$ are zero, so that firms earn positive profits), $q^*$, $r$, and $s$ all tend to zero. However,

PROPOSITION 3.3. *Suppose that there are no fixed costs. Then, if the number of firms is large enough, optimal output in cooperative periods equals that of Cournot equilibrium.*

*Proof.* From Eq. (22), $q^*$ will not exceed $s$ if $N \leqslant N^0$, where $N^0$ is defined implicitly by

$$N^0(N^0 + 1 + \eta^*) = (N^0 + 1)^2(a/A) + \eta^*. \tag{26}$$

If $N > N^0$, then $q^*$ as given by (22) exceeds $s$, and so firms will set $q^* = s$ to maximize their discounted value. Proposition 2.1 shows that $s$ is always supportable as an equilibrium, and Proposition 2.3 shows that it dominates any symmetric quantity vector $q$ when $q > s$. Q.E.D.

Again, this proposition assumes that there are zero fixed costs. If not, then there will be some value of $N$, say $\hat{N}$, for which $\pi_i(\bar{q}^*(\hat{N})) \geqslant 0 > \pi_i(\bar{q}^*(\hat{N}+1))$. Then $\hat{N}$ will be the maximum number of firms in equilibrium. (In the concluding section of the paper there is a discussion of the effects of allowing $N$ to be endogenously determined.) In this case equilibrium output in cooperative periods would be $s$ if $\hat{N}$ exceeds $N^0$, but less than $s$ otherwise.

As the parameters of the distribution function $F$ change, $\theta^*$ changes according to Eq. (25) and the equilibrium value of $\eta$, $\eta^*$, is uniquely determined by $\theta^*$ according to

$$\eta^* = 1 + f'(\theta^*)\,\theta^*/f(\theta^*) = f(\theta^*)\,\theta^*/F(\theta^*), \tag{27}$$

so that $\eta^* \geqslant 0$. But then

PROPOSITION 3.4. *Equilibrium quantity $q^*$ is a nonincreasing function of $\eta^*$ with slope given by*

$$\frac{dq^*}{d\eta^*} = \frac{A-(N+1)\,a}{2BN(N+1+\eta^*)^2} < 0, \qquad \textit{when} \quad \eta^* > \eta^0, \tag{28}$$

$$= 0, \qquad\qquad \textit{otherwise,}$$

*where*

$$\eta^0 = \frac{(N+1)[(N+1)(a/A) - N]}{(N-1)},$$

*an increasing function of $N$. As $\eta^*$ goes to infinity, $q^*$ converges to $r$, the single period joint profit maximizing output.*

*Proof.* If $\eta = \eta^*$, $q^*$ is given by

$$q^* = \frac{A}{2BN}\left(\frac{N+\eta^* + (N+1)\,a/A}{N+1+\eta^*}\right). \tag{29}$$

Since $\eta^* > 0$, $q^* > s/N$ as required by Proposition 2.3. Also, $q^* \leqslant s$ only if

$n^* \geqslant \eta^0$. If $\eta^* < \eta^0$, then $q^* > s$, but the industry can do better by producing at $\bar{s}$, and will do so since it is an equilibrium. Since $\eta^0$ is an increasing function of $N$, the larger the number of firms, the less likely is the event of $\eta^* \geqslant \eta^0$, and so the less likely is $q^* < s$. Finally, $\lim_{\eta^* \to \infty} q^* = r$, and Eq. (28) can be derived by differentiating (29). Q.E.D.

Note that the requirement that $\eta^*$ be greater than $\eta^0$ is exactly equivalent to that which dictates that $N$ be less than $N^0$.

Recall that $\mu$, the mean of $\theta$, is being held constant. Thus an increase in $\eta^*$ represents a mean-preserving contraction of the density function $f$ about $\theta^* = \tilde{p}^*/p^*$. Heuristically, as $\eta^*$ increases without bound, the density function converges to a mass point at the mean $\mu$, so that $\theta^*$ must converge to $\mu$. If $\mu$ is unity, then $\tilde{p}^* = p(Nq^*) = p(Nr)$. Then we essentially have the certainty world of Friedman [1], in which $\bar{r}$ can be supported as an equilibrium. If $\eta^* < \eta^0$, then in some sense the distribution of $\theta$ is too noisy, so that no equilibrium output less than $s$ can be supported.

The optimal value of $T$, say $T^*$, can be obtained by substituting the values of $\tilde{p}^*$ and $q^*$ obtained from Eqs. (23) and (29) into Eq. (14), the first-order necessary condition. Thus $T^*$ solves

$$\begin{aligned}&[1-\beta+(\beta-\beta^{T^*})F(\theta^*)][A-(N+1)Bq^*]\\ &\quad=(\beta-\beta^{T^*})f(\theta^*)[b\theta^*/p(Nq^*)][q^*(A-NBq^*)-A^2/B(N+1)^2]. \qquad (30)\end{aligned}$$

Inversion of Eq. (30) yields

$$T^* = \left(\frac{1}{\ln\beta}\right)\ln\left\{\beta - \frac{(1-\beta)[A-(N+1)Bq^*]}{f(\theta^*)(b\theta^*/p^*)\Delta - F(\theta^*)[A-(N+1)Bq^*]}\right\}, \qquad (31)$$

where $\Delta$ is given by Eq. (20). For interior solutions of $\tilde{p}^*$ it can be shown that the expression in { } brackets will be positive and less than $\beta$, which in turn is less than unity, so that $T^*$ will be greater than unity, as required. To summarize, we have

PROPOSITION 3.5. *If the optimal values of $\tilde{p}^*$ and $T^*$ are interior solutions, then $q^*$, $\tilde{p}^*$, and $T^*$ will be determined by*

$$q^* = \frac{A}{2BN}\left(\frac{N+\eta^*+(N+1)(a/A)}{N+1+\eta^*}\right), \quad \textit{if} \quad \eta^* > \eta^0, \qquad (29)$$

$$= s, \qquad \textit{otherwise};$$

$$\frac{f(\tilde{p}^*/p^*)}{F(\tilde{p}^*/p^*)}\frac{\tilde{p}^*}{p^*} - \frac{f'(\tilde{p}^*/p^*)}{f(\tilde{p}^*/p^*)}\frac{\tilde{p}^*}{p^*} = 1; \qquad (23)$$

*and*

$$T^* = \frac{1}{\ln\beta}\left\{\ln\beta - \frac{(1-\beta)[A-(N+1)Bq^*]}{f(\theta^*)(b\theta^*/p^*)\Delta - F(\theta^*)[A-(N+1)Bq^*]}\right\}. \quad (31)$$

### 3.2. *An Example*

Suppose that $\theta$ is distributed according to

$$F(\theta) = [\alpha\theta/(\alpha+1)]^\alpha \qquad \text{for} \quad 0 \leqslant \theta \leqslant (\alpha+1)/\alpha, \quad \alpha > 0,$$

so that

$$f(\theta) = \frac{\alpha F(\theta)}{\theta}, \quad f'(\theta) = \frac{(\alpha-1)f(\theta)}{\theta}, \quad E[\theta] = 1, \quad \text{Var}\,[\theta] = \frac{1}{\alpha(\alpha+2)}.$$

Then decreases in $\alpha$ represent a mean-preserving spread in $f(\theta)$. Note that if $\alpha = 1$, $\theta$ is uniformly distributed. Also, $F(\theta)$ is convex when $\alpha$ exceeds one. Now

$$\frac{f(\theta)\,\theta}{F(\theta)} = \frac{f'(\theta)\,\theta}{f(\theta)} + 1 = \alpha \qquad \text{for all } \theta \in [0, (\alpha+1)/\alpha].$$

Then Eq. (25) is satisfied for all possible values of $\theta$, and equilibrium output is given by

$$\begin{aligned} q^* &= \frac{A}{2BN}\left(\frac{N+\alpha+(N+1)\,a/A}{N+1+\alpha}\right), && \text{if} \quad \alpha \geqslant \alpha^0, \\ &= A/B(N+1), && \text{if} \quad \alpha < \alpha^0, \end{aligned}$$

where $\alpha^0 = (N+1)[(N+1)(a/A) - N]/(N-1) = \eta^0$.

For finite $\alpha$, as $N$ goes to infinity, $\alpha^0$ also goes to infinity, and so $Nq^*$ converges to the same limit as $Ns$, namely, $A/B$. Since $\lim_{N\to\infty} Nq^* = A/B$, $\lim_{N\to\infty} p(Nq^*) = c_1$, so that in the limit price equals marginal cost of production. Also, as $\alpha$ approaches infinity, $q^*$ converges to $r$. As $\alpha$ falls, $q^*$ increases towards $s$, and $q^* = s$ once $\alpha$ falls to $\alpha^0$. Thus $q^*$ is an increasing function of the "noisiness" of $\theta$. If there is more than one firm in the industry (i.e., $N > 1$), $\alpha^0 > 1$ because $a \geqslant A$. In these cases, $F$ is convex and so $V_i(\bar{q})$ is concave in $q_i$. If $\theta$ is uniformly distributed, then $q^* = s$. This could alternatively be verified by examining the first-order necessary conditions (14) for any uniform distribution, and seeing that they can only be satisfied when $q^* = s$ if we restrict $q^*$ to $(s/N, s]$.

The optimal interior values of $\tilde{p}^*$ and $T^*$ must be chosen to satisfy the first-order necessary conditions, given $q^*$. There is a degree of indeterminancy in that if $\tilde{p}^*$ adjusts with $T^*$ so that the necessary conditions continue to hold, it does not matter what value $T^*$ takes on.

For example, consider the case of $a = A = 2$, $B = 1$, and $N = 2$. Then $r = \frac{1}{2}$, $p(\bar{r}) = 1$, $s = \frac{2}{3}$, $p(\bar{s}) = \frac{2}{3}$, and

$$q^* = \frac{1}{2}\frac{(5+\alpha)}{(3+\alpha)} \quad \text{and} \quad p(\bar{q}^*) = \frac{1+\alpha}{3+\alpha} \qquad \text{if} \quad \alpha \geqslant 3,$$

and

$$q^* = \tfrac{2}{3} \qquad\qquad \text{and} \quad p(\bar{q}^*) = \tfrac{2}{3} \qquad\qquad \text{if} \quad \alpha < 3.$$

Simple manipulation of the first-order conditions reveals that $\tilde{p}^*$ and $T^*$ must be chosen so that

$$\tilde{p}^* = p(\bar{q}^*)\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{(1-\beta)}{(\beta-\beta^{T^*})}\frac{9(1+\alpha)}{(\alpha^2-9)}\right)^{1/\alpha} \qquad \text{if} \quad \alpha > 3.$$

If $\tilde{p}^*$ is chosen to satisfy this equation, it does not matter what the value of $T^*$ is.

In this case, the single period joint profit maximizing output $r$ will be supportable as an equilibrium if, using the first-order conditions,

$$1-\beta = (\beta-\beta^T)\,F(\tilde{p})(\alpha-9)/9.$$

But this is only possible for interior values of $\tilde{p}$ and $T$ if $\alpha > 9$, which is more restrictive than $\alpha > 3$, the condition under which a $q^*$ less than $s$ can be supported. Thus $q^*$ is supportable as an equilibrium over a wider range of values of $\alpha$ ($\alpha > 3$) than is $r$ ($\alpha > 9$).

### 3.3. "*Corner*" *Solutions*

We now turn to cases in which the optimal values of $\tilde{p}$ or $T$, or both, may not be interior solutions.

For some distributions of $\theta$, no symmetric quantity vector less than the Cournot output vector is supportable as an equilibrium, e.g., the uniform distribution, as shown in Section 3.2. (Recall that linear structure of the inverse demand and cost functions guanratees that all equilibrium quantity vectors be symmetric.) When this is not the case, we can show that $\tilde{p}^*$ must be an interior solution and that $T^* > 1$. (We shall demonstrate that $\tilde{p}^*$ is positive and finite. More precisely, $\tilde{p}^*/p(Nq^*)$ must lie in the interior of the support of $F$.)

PROPOSITION 3.6. *Suppose that some symmetric $\bar{q}^0 = (q^0,..., q^0)$, where $q^0$ lies in the open interval $(s/N, s)$ and is supportable as an equilibrium by some $(\tilde{p}, T)$ combination, say $(\tilde{p}^0, T^0)$. Then the optimal trigger price $\tilde{p}^*$ must be positive and finite, and the optimal punishment period length $T^*$*

*must be greater than one. If there is no value of $\theta$ on the support of $F$ which satisfies Eq. (25), then $T^*$ is infinite.*

*Proof.* If $s/N < q^0 < s$, then $\pi_i(\bar{q}^0) > \pi_i(\bar{s})$ for all $i$, and so $V_i(\bar{q}^0, \tilde{p}^0, T^0) > V_i(\bar{s}, \tilde{p}, T)$ for all $i$, $\tilde{p}$, and $T$. Then we know that $\bar{q}^*$ cannot equal $\bar{s}$, since $\bar{q}^0$ is preferable to $\bar{s}$ for all firms, and is supported by $(\tilde{p}^0, T^0)$.

In this case, $\tilde{p}^*$ must be greater than zero. Suppose not, i.e., suppose that $\tilde{p}^* = 0$. Then

$$V_i(\bar{q}, 0, T) = \frac{\pi_i(\bar{s})}{1-\beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1-\beta+(\beta-\beta^T)F(0)} = \frac{\pi_i(\bar{q})}{1-\beta}$$

since $F(0) = 0$, and so $\bar{q}^* = \bar{s}$, which contradicts the fact that $\bar{q}^*$ cannot equal $\bar{s}$. Hence $\tilde{p}^* > 0$.

Also $\tilde{p}^*$ must be finite. If not, since $F(\infty) = 1$,

$$V_i(\bar{q}, \infty, T) = \frac{(\beta-\beta^T)}{(1-\beta^T)} \frac{\pi_i(\bar{s})}{(1-\beta)} + \frac{\pi_i(\bar{q})}{1-\beta^T},$$

and so $\bar{q}^* = \bar{s}$, a contradiction.

Further, $T^*$ must be greater than one. If $T = 1$, then

$$V_i(\bar{q}, \tilde{p}, 1) = \pi_i(\bar{q})/(1-\beta),$$

and so $\bar{q}^* = \bar{s}$, which is again a contradiction.

Since $\tilde{p}^*$ must be an interior solution, the optimal quantity $q^*$ will continue to be determined by Eq. (22), as in Section 3.1. The problem, however, is that for many distributions the optimal value of $T^*$ is infinite. For these distributions there is no value $\theta^*$ of $\theta$ in the support of $F$ which will satisfy Eq. (25). Included in this class are the exponential, Rayleigh, and lognormal distributions. To see that $T^*$ must be infinite in these cases, note that

$$\begin{aligned}
\frac{dV_i^*}{dT} &= (N-1)\frac{\partial V_i}{\partial q_j}\frac{\partial q_i^*}{\sigma T} + \frac{\partial V_i}{\partial T} \qquad \text{for} \quad j \neq i \\
&= \frac{\beta^T \ln \beta p^{*2}}{\tilde{p} b f (\beta-\beta^T)[1-\beta+(\beta-\beta^T)F]} \\
&\quad \times \left\{ -\frac{(N-1)[\tilde{p} b f \Delta - F p^*(A-(N+1)Bq^*)]}{K} \right. \\
&\quad \left. + [A-(N+1)Bq^*]F \right\}.
\end{aligned}$$

Equation (23) is derived by setting the expression in { } brackets equal to zero, and making use of Eqs. (14) and (22), with the implicit assumption that $T$ is finite. But if there is no value of $\theta^* = \tilde{p}^*/p^*$ which allows Eq. (23) to be satisfied, as will be the case for the aforementioned distributions, then $T^* = \infty$ guarantees that $dV_i^*/dT = 0$, since $0 < \beta < 1$. Thus we have the desired result. Q.E.D.

For each distribution where $T^*$ is infinite, one can construct examples in which some symmetric vector $\bar{q}^0 = (q^0, ..., q^0)$, where $q^0$ lies in the open interval $(s/N, s)$, is supportable as an equilibrium by some $(\tilde{p}^0, T^0)$ combination where $T^0$ is finite. In other words, for each of these distributions, even though $T^*$ is infinite, finite values of $T$ can support quantities less than Cournot output levels. The cartel can do better than just following single period Cournot strategies by producing $\bar{q}^*$ or $\bar{q}^0$.

If we set $T = T^*$ when $T^*$ is infinite, then $q^*$ and $\tilde{p}^*$ must be chosen to simultaneously satisfy Eq. (22) and

$$[1 - \beta + \beta F^*](A - (N+1)Bq^*) = \frac{\beta b f^* \tilde{p}^*}{p^{*2}} [q^*(A - NBq^*) - A^2/B(N+1)^2], \tag{32}$$

where $p^* = p(Nq^*)$, $F^* = F(\tilde{p}^*/p^*)$, and $f^* = f(\tilde{p}^*/p^*)$. By substituting Eq. (22) into (32), and by letting $\theta^* = \tilde{p}^*/p^*$ as before, one obtains, for example when $A = a$,

$$(1 + \eta^*)(N+1)^2(1 - \beta + \beta F^*) = \beta f^* \theta^* [2N^2 + (N-1)(1 + \eta^*)], \tag{33}$$

where $f^* = F(\theta^*)$, $f^* = f(\theta^*)$, and, as before,

$$\eta^* = 1 + f'(\theta^*)\,\theta^*/f^*. \tag{34}$$

Thus $\theta^*$ is chosen to satisfy Eq. (33), which yields $\eta^*$ via Eq. (34). Then $q^*$ is given by Eq. (29) and $\tilde{p}^*$ will again vary as the parameters $A$ and $B$ vary to exactly offset any proportional changes in $p^*$, since $A$ and $B$ do not enter Eq. (33) and so do not affect $\theta^*$. Now the derivatives of $q^*$ and $\tilde{p}^*$ with respect to $a$, $A$, and $B$ are exactly the same as those given in Section 3.1. Thus we again have that the equilibrium quantity (trigger price) is an increasing (decreasing) function of the intercept of the demand curve $a$, and $A = a - c_1$, where $c_1$ is the constant portion of marginal costs, and a decreasing (increasing) function of the absolute value of the slope of the inverse demand curve $b$.

Now, however, changes in the number of firms $N$ as well as variations in the discount factor $\beta$ and the parameters of the distribution function $F$ will alter $\theta^*$ in a nontrivial manner, so that the comparative statics results of Section 3.1 with respect to changes in $N$ will no longer hold. It is also important to note that we continue to have

$$\begin{aligned} q^* &= \frac{A}{2NB}\left(\frac{N+\eta^*+(N+1)\,a/A}{N+1+\eta^*}\right), && \text{for} \quad \eta^* \geqslant \eta^0, \\ &= \frac{A}{B(N+1)} = s, && \text{otherwise.} \end{aligned} \tag{35}$$

Under our assumption that $F$ is convex, when $q^* < s$ the second-order equilibrium conditions will be satisfied. Now $q^*$ will approach $r = A/2NB$ as $\eta^*$ approaches infinity. When $\eta^*$ is greater than $\eta^0$, $q^*$ is greater than $r$ is a nonincreasing fuction of $\eta^*$. We summarize in

PROPOSITION 3.7. *If the optimal value of $T$ is infinite, then $q^*$ will be determined by Eq.* (35). *As the parameters of the distribution function $F$ and the number of firms change, $\theta^*$ and $\eta^*$ will vary according to Eqs.* (33) *and* (34), *respectively. Then the optimal trigger price is given by $\tilde{p}^* = \theta^* p(Nq^*)$. Equilibrium quantity $q^*$ is a nonincreasing function of $\eta^*$. As $\eta^*$ goes to infinity, $q^*$ converges to $r$, the single period joint profit maximizing output.*

When $T$ is infinite, the cartel will behave cooperatively until the market price is less than the trigger price in some period, and then the firms will revert to Cournot behavior permanently beginning with the next period. We can then talk about the expected life of the cartel, which be denoted by $L$. Then $L$ is the expected number of cooperative periods before the "breakdown" of the cartel.

PROPOSITION 3.8. *If $T$ is infinite and $\tilde{p}$ is chosen optimally, then the expected life of the cartel will be $F(\theta^*)^{-1}$, which is a decreasing function of $\theta^* = \tilde{p}^*/p(Nq^*)$.*

*Proof.* The probability that the cartel has a life of $j$ periods is equal to $[1-F]^{j-1}F$, where the argument of $F$ is $\tilde{p}/p(\bar{q})$. In this case the market price has exceeded the trigger price in the previous $(j-1)$ periods, which occurs with probability $[1-F]^{j-1}$, and then market price falls below $\tilde{p}$ in the $j$th period, which occurs with probability $F$. (Note that we are making use of the assumption that the sequence $\{\theta_t\}$ is independently distributed.) Then the expected life of the cartel will be

$$L = \sum_{j=1}^{\infty} F[1-F]^{j-1} j = 1/F.$$

If $\tilde{p}$ and $q$ equal $\tilde{p}^*$ and $q^*$, respectively, then

$$L = 1/F(\theta^*), \tag{36}$$

where $\theta^* = \tilde{p}^*/p(Nq^*)$. Q.E.D.

But then $L$ depends only on the number of firms, the discount factor $\beta$, and the parameters of the distribution function $F$, which affect $\theta^*$, and not on the parameters of the inverse demand and cost functions, which do not affect $\theta^*$.

### 3.4. *Another Example*

Suppose now that $\ln \theta$ is normally distributed with mean $-\sigma^2/2$ and variance $\sigma^2$. Then $\theta$ will follow a lognormal distribution with mean one and variance $\exp(\sigma^2) - 1$, so that increases in $\sigma^2$ result in a mean-preserving spread in the distribution of $\theta$. If we denote the density function of a standard normal distribution by $\phi$ and its cumulative by $\Phi$, then $\theta f(\theta)/F(\theta) = \phi(z)/\Phi(z)\,\sigma$ and $\eta = z/\sigma$, where $z = (\ln \theta + \sigma^2/2)/\sigma$. Since there is no value of $z$ for which $\phi(z) = z\Phi(z)$, there will be no value of $\theta$ satisfying Eq. (25), which requires that $\eta$ equal $\theta f(\theta)/F(\theta)$. By the reasoning of Section 3.3, then, the optimal value of $T$ is infinite. Equation (36) then tells us that the expected life of the cartel will be given by

$$L = \Phi^{-1}((\ln \theta^* + \sigma^2/2)/\sigma)$$

when the trigger price is chosen optimally.

For example, suppose that $N = 2$, $a = A = 2$, and $B = 1$. Then $s = \frac{2}{3}$, $p(\bar{s}) = \frac{2}{3}$, $\pi_i(\bar{s}) = \frac{4}{9}$, $r = \frac{1}{2}$, $p(\bar{r}) = 1$, and $\pi_i(\bar{r}) = \frac{1}{2}$. Also, $s/N = \frac{1}{3}$. If the trigger price is chosen optimally, $q^*$ will be given by

$$q^* = \frac{1}{2}\left(\frac{9\sigma^2 - 2\ln \theta^*}{7\sigma^2 - 2\ln \theta^*}\right).$$

In this example, $F$ is not convex for all possible realizations of $\theta$ and so we cannot rely on Proposition 2.4 to show that the first-order necessary conditions for $\bar{q}^*$ to be a Nash equilibrium are also sufficient. However, $F(p/p(Q^*))$ will be convex in $q_i$ at $q^*$ if $\theta^* < \exp(\sigma^2/2)$. Examination of the expression for $q^*$ reveals that this is exactly the restriction on $\theta^*$ which ensures that $q^*$ is less than $s$. Furthermore, $F$ will then be convex in $q_i$ for all $q_i$ less than $q^*$. Thus, if $q^*$ is less than $s$, we can check whether $q_i = q^*$ maximizes $V_i(\bar{q}^*)$ on the interval $(0, 2 - q^*)$—i.e., the interval on which expected price is positive given that the other firm chooses to produce $q^*$—by directly checking the concavity of $V_i(\bar{q}^*)$ for $q_i$ on the interval $(q^*, 2 - q^*)$. If, for example, $q_1$ exceeds $(2 - q^*)/2$, then $\partial\pi(q_1, q^*)/\partial q_1$ is negative, and so $\partial V_i(q_1, q^*)/\partial q_1$ is negative. Thus we just have to check whether $q^*$ is the unique solution to $\partial V_i(\bar{q})/\partial q_i = 0$ when $q_j = q^*$ for $j \neq i$. Since this is in fact the case for the numerical examples presented below, in those cases $\bar{q}^*$ is indeed the Nash equilibrium quantity choice in cooperative periods when $(\tilde{p}, T)$ equals $(\tilde{p}^*, \infty)$. In other words, any $(\theta^*, q^*)$

combination satisfying both the expression for the optimal quantity (such that $q^* < s$) and the first-order necessary conditions will also satisfy the second-order conditions.

If $z^* = (\ln \theta^* + \sigma^2/2)/\sigma$, then we can write the first-order conditions, taking $T$ as infinite, as

$$1 - \beta + \beta\Phi(z^*) = \frac{\beta\phi(z^*)(13\sigma^2 - 2\ \ln \theta^*)}{9\sigma(5\sigma^2 - 2 \ln \theta^*)},$$

where we have employed the expression for $q^*$. As $\sigma^2$ and $\beta$ vary, we obtain the following values:

| $\sigma^2$ | $\beta$ | $q^*$ | $p^*$ | $\theta^*$ | $\tilde{p}^*$ | $L$ |
|---|---|---|---|---|---|---|
| 0.13 | 0.72 | 0.65 | 0.69 | 1.03 | 0.72 | 1.85 |
| 0.13 | 0.75 | 0.64 | 0.71 | 1.00 | 0.71 | 1.90 |
| 0.13 | 0.76 | 0.63 | 0.73 | 0.97 | 0.71 | 2.00 |
| 0.10 | 0.63 | 0.63 | 0.73 | 0.98 | 0.72 | 2.00 |

Thus increases in the discount factor, keeping $\sigma^2$ constant, result in decreases in the optimal quantity $q^*$ towards $r$, and decreases in $\tilde{p}^*$. Here decreases in $\theta^*$ more than offset the increases in $p^*$. The decreases in $\theta^*$ lead to increases in $L$. Decreases in $\sigma^2$, with $\beta$ falling to ensure that $q^*$ remains constant, result in increases in $\theta^*$ and $\tilde{p}^*$, but no change in $L$. (These examples were calculated by setting $\ln \theta^*$ equal to $k\sigma^2/2$, and letting $k$ equal $\frac{1}{2}$, 0, and $-\frac{1}{2}$. Here $q^*$ lies between $s/N$ and $s$ as long as $k$ is less than 1 and greater than $-4$.)

## 4. Additive Uncertainty

Suppose now that the industry inverse demand curve has an additive stochastic component, so that firms observe market price $\hat{p}_t$ in period $t$ according to

$$\hat{p}_t = p(Q_t) + \theta_t,$$

where $Q_t$ is aggregate output and $\{\theta_t\}$ is an identically and independently distributed sequence of random variables with zero mean, density function $f$, and cumulative distribution $F$. We shall continue to assume that $F$ is convex. It is assumed that the inverse demand function is given by Eq. (1) and that firms face the cost function described by Eq. (2), where we now assume that

$$0 < c_1 < a \tag{37}$$

and that the number of firms $N$ is small enough to ensure that

$$0 < c_0 < (a - c_1)^2/b(N+1)^2. \tag{38}$$

Then if $A = a - c_1$ and $B = b$, both of which will be positive because of assumption (37) and $b > 0$, Cournot output and the corresponding value of expected profits are given by $A/B(N+1)$ and Eq. (6), respectively. Equation (38) guarantees that expected profits are positive at Cournot output levels. Similarly, single period expected joint net returns are maximized if output is given by $A/2BN$.

If the industry employs the Green–Porter enforcement mechanism, the expected present discounted value of firm $i$, given an industry output vector $\bar{q}$ in cooperative periods, can be expressed as

$$V_i(\bar{q}) = \frac{\pi_i(\bar{s})}{1-\beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)F(\tilde{p} - p(Q))} \quad \text{for} \quad i = 1, \ldots, N,$$

where time subscripts have been suppressed. A noncooperative equilibrium at $\bar{q}^*$, $\tilde{p}$, $T$, and $\bar{s}$ must then satisfy the first-order necessary condition

$$\begin{aligned}[1 - \beta + (\beta - \beta^T)F][A - (N+1)Bq^*] \\ = (\beta - \beta^T)fb[q^*(A - NBq^*) - A^2/B(N+1)^2],\end{aligned} \tag{39}$$

where $\tilde{p} - p(Nq^*)$, the argument of $F$ and $f$, has been suppressed. Note that this formulation of the first-order conditions has implicitly assumed that the equilibrium output vector $\bar{q}^*$ is symmetric. Algebraic manipulation analogous to that underlying Proposition 2.2 demonstrates that this will be the case. It is also clear that the single period Cournot output vector will continue to be an equilibrium. To ensure that $V_i(\bar{q}^*) \geqslant V_i(\bar{s})$, any other symmetric equilibrium output must lie in the interval $(s/N, s]$. Thus Proposition 2.3 continues to hold, where Eq. (39) is now the first-order equilibrium condition. Again, convexity of $F$ guarantees that $V_i(\bar{q})$ is concave in $q_i$.

In this case the optimal values of $\tilde{p}$ and $T$, if they are interior solutions (so that $\tilde{p}$ is finite and $T$ is finite and greater than unity), will again satisfy the first-order necessary conditions given by Eqs. (18) and (19). Solution of these equations yields

PROPOSITION 4.1. *If there is an additive stochastic demand component and the optimal values of $\tilde{p}$ and $T$ are interior solutions, then the optimal output of an individual firm is given by*

$$q^* = \frac{A + (N+1)(f/f')}{2BN} = r + (N+1)f/(2BNf') \tag{40}$$

*when $f(\theta^*)/F(\theta^*)$ exceeds $(N+1)^2/(N-1)A$, and equals the single period Cournot quantity otherwise. Here $\theta^*$ satisfies*

$$f(\theta^*)/F(\theta^*) = f'(\theta^*)/f(\theta^*), \tag{41}$$

*where $\theta^* = \tilde{p}^* - p(Nq^*)$. In either event $q^*$ will be strictly greater than $r$ when $f(\theta^*)$ is finite.*

*Proof.* See Porter [4].

Also, we have the same comparative statics results as those derived in Section 3.1.

If $f$, the density function of $\theta$, is symmetric about zero with $f'(\theta) > 0$ when $\theta < 0$ and $f'(\theta) < 0$ when $\theta > 0$, then the nonnegativity of $f'(\theta^*)$ implies that $\theta^*$ is negative. But then $\tilde{p}^* < p(Nq^*)$. Thus in this special case, the trigger price will be strictly less than the expected price in cooperative periods.

Suppose instead that $\theta$ is distributed according to

$$F(\theta) = \exp(\alpha\theta - 1) \qquad \text{for} \quad \theta \leqslant 1/\alpha, \quad \text{where} \quad \alpha > 0,$$

so that

$$f(\theta) = \alpha F(\theta), \qquad f'(\theta) = \alpha f(\theta), \qquad \text{and} \qquad E[\theta] = 0.$$

Then decreases in $\alpha$ represent a mean-preserving spread in $f(\theta)$. Also, $F$ is convex, ensuring concavity of the value function. Now Eq. (41) is satisfied for all possible values of $\theta$ and optimal output is given by

$$\begin{aligned} q^* &= \frac{A + (N+1)/\alpha}{2BN}, \qquad &&\text{when} \quad \alpha \geqslant (N+1)^2/(N-1)A, \\ &= A/B(N+1), &&\text{otherwise.} \end{aligned}$$

Thus $q^*$ is a nonincreasing function of $\alpha$ and converges to $r$, the single period collusive output level, as $\alpha$ increases to infinity.

The problem with the derivations above is that for most potential distributions of $\theta$, there is no possible value of $\theta^*$ that satisfies Eq. (41). Reasoning exactly analogous to that of Section 3.3 demonstrates that if Eq. (41) cannot be satisfied, then $dV_i^*/dT = 0$ only if $T$ is infinite. In cases such as these, $q^*$ will now be determined by Eq. (40) when $f'(\theta)/f(\theta)$ exceeds $(N+1)^2/(N-1)A$, (this ensures that $q^* < s$), and $\tilde{p}^*$ will be chosen to satisfy the first-order conditions, given $q^*$. The expected life of the cartel $L$ will again be given by Eq. (36), where now $\theta^* = \tilde{p}^* - p(Nq^*)$. Porter [4] contains an example of a distribution of this sort.

In general, then, the results of Section 3 hold up under a different specification of the stochastic nature of demand.

## 5. SUMMARY

For the expected discounted value function of firm $i$, $V_i: R_+ \to R$, if $V_i^i(\bar{q}^*) = 0$ for a unique argument $q_i^*$ and if $\partial^2 V_i(\bar{q}^*)/\partial q_i^2 < 0$, then $V_i$ is quasiconcave. This will be the case when $F$ is convex. Thus, the necessary conditions derived in this paper for a noncooperative equilibrium are also sufficient, and so non-Cournot equilibria have been shown to exist for a variety of parametric oligopoly models when price information is not too "noisy."

As the results of the previous sections have demonstrated, the optimal cartel trigger price strategy depends in a critical way on the nature and distribution of the stochastic demand component. If $\tilde{p}^*$ and $T^*$ are interior solutions and the stochastic component does not have a degenerate distribution, then the optimal quantity will be greater than the single period collusive level and less than the single period Cournot level if the distribution is not too "noisy." More typically, however, the optimal punishment period length $(T^* - 1)$ is infinite. Then the optimal quantity will again be less than Cournot levels if there is not too much "noise" in the distribution of $\theta$, and again it will be greater than the collusive rate. Unfortunately, the mathematics is so complicated that the distribution of $\theta$ must be specified if one is to make qualitative comparisons.

In the examples of this paper we have employed a number of specific distributions, all of which are one-parameter distributions whose means are independent of the variable parameter. All are modifications of more familiar distributions (in particular, the gamma, lognormal, and negative exponential). The examples above could have been stated in terms of these more familiar distributions, but only at the cost of a greater algebraic burden, since the mean of $\theta$ would now vary with the distributional parameter.

A natural extension of this model would examine cases in which the number of firms is endogenously determined. One possible model would specify fixed costs to be greater than net variable returns for Cournot firms, and then $N$ could be determined by requiring that expected discounted value be nonnegative for all firms which operate at positive output levels, and negative at all positive quantities for all potential entrants, so that their optimal output is zero. Then $\bar{q}^*$ would be a Nash equilibrium quantity vector in cooperative periods given the Green–Porter enforcement mechanism, with some components equal to zero. If one wanted to maintain the symmetric specification of this paper and yet allow the entry decisions of firms to be well-defined problems, it could be assumed that firms make their entry decisions sequentially before time zero given $\tilde{p}^*$, $T^*$, and the decisions of the firms choosing before. Also, there would have to be a "no exit" constraint, so that firms which decided to enter would have to operate at positive output levels during noncooperative episodes. For the purposes of the present paper, however, models of this sort must be categorized as the subject of future research.

Finally, note that this paper brings moral hazard considerations into the cartel enforcement problem. To use the vocabulary of the moral hazard literature, the simple trigger price strategy studied here restricts the reward function to be a step function. In different contexts it has been shown that this will be optimal when the acting agent is risk neutral, as are the firms in the model of this paper. Furthermore, in many cases it will be optimal to use the maximum degree of punishment. In some sense this corresponds to an infinite punishment period length.

The moral hazard problem studied here is unconventional in that it is dynamic, multi-agent without a principal, and without possibilities for explicit contracting. These features may be interesting enough in their own right to merit further study in a general moral hazard context.

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