Price Theory II
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(Phil Reny)

Notes on Contingent Plans

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1 Contingent plans

Here's quick summary of Ch. 5.4 in JR3. The idea is to extend the multiple good general equilibrium framework to include time and uncertainty. To keep things simple, we assume everything is finite. To account for time and uncertainty, we index each good by the time t of consumption as well as the state s in which good is consumed.

Specifically, assume that there are T periods and that, at each date $t \in \{1, 2, \ldots, T\}$, there are S_t mutually exclusive and exhaustive events that might occur; i.e. $s_t \in \{1, 2, \ldots, S_t\}$. The vector $\{s_1, s_2, \ldots, s_t\}$ is a vector that gives a history of realised states up to period t. We now define consumption bundle as

$$\mathbf{x} = \{x_{kts}\},$$

where k denotes the good (from 1 to N), t denotes the time (from 1 to T) and $s = (s_1, s_2, \ldots, s_t)$ denotes one of $S_1 \times S_2 \times \cdots \times S_t$ states of the world describing the events occurred up to period t.

Observe that, for date $t = 1$, we have S_1 possible states of the world, and for date $t = 2$, we have $S_1 \times S_2$ possible states of the world. Hence, the total number of date-state pair is

$$M = S_1 + S_1 \times S_2 + \cdots + S_1 \times S_2 \times \cdots \times S_T.$$

Thus,

$$\mathbf{x} = \{x_{kts}\} \in \mathbb{R}_+^{NM}.$$

Firms and consumers are defined in the same manner as before; i.e. we would have a private ownership economy with NM goods. But, note that the utility function u^i defined over $\mathbf{x}$ includes attitude toward time (e.g. discounting), risk as well as potentially heterogeneous subjective probability assessments of different states of the world.

Let's just double check the budget constraints. Since this is just an economy with NM goods, the budget constraint for individual $i \in \mathcal{I}$ can be written as

$$\begin{aligned} \mathbf{p} \cdot \mathbf{x}^i &= \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \mathbf{p} \cdot \mathbf{y}^j, \ \forall i \in \mathcal{I}, \\ \Leftrightarrow \sum_{k,t,s} p_{kts} x^i_{kts} &= \sum_{k,t,s} p_{kts} e^i_{kts} + \sum_{j \in \mathcal{J}} \sum_{k,t,s} p_{kts} y^j_{kts}, \ \forall i \in \mathcal{I}. \end{aligned}$$

It's quite possible that in some time/state, expenditure exceeds income. This just means that consumers are borrowing (from other consumers). In other words, there is insurance being provided in the economy (think back to Theory of Income I!).

Market clearing condition is now

$$\begin{aligned} \sum_{i \in \mathcal{I}} \mathbf{x}^i &= \sum_{i \in \mathcal{I}} \mathbf{e}^i + \sum_{j \in \mathcal{J}} \mathbf{y}^j \\ \Leftrightarrow \sum_{i \in \mathcal{I}} x^i_{kts} &= \sum_{i \in \mathcal{I}} e^i_{kts} + \sum_{j \in \mathcal{J}} y^j_{kts}, \ \forall k, t, s. \end{aligned}$$

How does this work?

▷ At date 0, all firms and consumers participate in a (contingent) market for binding (i.e. enforceable) contracts regarding production and consumption at *all possible dates* and under *all possible states* at listed prices.

▷ When a date/state is realised, each agent/firm in the economy acts according to the plan/contracts.

We saw in the problem set that with this setup and assuming perfect foresight and time consistency on the part of the agent, no further trade would take place. But we also need perfect foresight on the part of the firms in terms of their production capabilities—they cannot claim to be able to supply more than they actually can in any contingency. There is also assumption that everyone agrees on the realisations of the state of the world and that these are common knowledge. We also need these contracts to be enforceable.

2 Core exam questions on general equilibrium

2.1 2013/14

Consider an exchange economy with two consumer and two goods. Consumer 1's continuous utility function is $u\left(x_1^1\right)+u\left(x_2^1\right)$ and consumer 2's continuous utility function is $v\left(x_1^2\right)+v\left(x_2^2\right)$, where $x_k^i \geq 0$ is the amount of good k consumed by consumer i. Assume that $u', v' > 0$ and $u'', v'' < 0$. Consumer 1's endowment vector is $(\alpha, 0)$ and consumer 2's endowment vector is $(0,1)$, where $\alpha > 0$. Thus, consumer 1 is endowed with α units of good 1 and zero units of good 2, and consumer 2 is endowed with zero units of good 1 and one unit of good 2.

2.1.1 Part (i)

State and verify the conditions of a theorem that guarantees the existence of a Walrasian equilibrium and prove that there is a unique Walrasian equilibrium when $\alpha = 1$.

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Existence The condition required for existence is: each consumer's utility function is continuous, strongly increasing, and strictly quasiconcave, and that aggregate endowment for each good is strictly positive.

Continuity is given as an assumption. We can see that each utility is strongly increasing given additively separable utility functions and that $u', v' > 0$. Since u and v are strictly concave and sum of strictly concave functions are also strictly concave, it follows that utility functions are also strictly quasiconcave. Finally, given $\alpha = 1$, aggregate endowment is given by $(\alpha, 1) = (1,1) \gg \mathbf{0}$.

Uniqueness The idea is to use exploit the symmetry in the problem to argue for uniqueness.

First, since utilities are strongly increasing, equilibrium prices must be strictly positive. Note that each consumer's utility functions are symmetric; i.e.

$$U\left(x_1^1, x_1^2\right) \coloneqq u\left(x_1^1\right) + u\left(x_2^1\right) \equiv U\left(x_1^2, x_1^1\right),$$
$$V\left(x_1^1, x_1^2\right) \coloneqq v\left(x_1^1\right) + v\left(x_2^1\right) \equiv V\left(x_1^2, x_1^1\right).$$

Suppose $p_1 = p_2 = 1$, then

$$MRS_{1,2}^1 = \frac{u'\left(x_1^1\right)}{u'\left(x_2^1\right)} = 1 = \frac{v'\left(x_1^2\right)}{v'\left(x_2^2\right)} = MRS_{1,2}^2,$$

in equilibrium, and since $u', v' > 0$, it must be that

$$x_1^1 = x_2^1,\ x_1^2 = x_2^2.$$

Substituting this into the budget constraint, we obtain

$$x_1^1 = x_2^1 = x_1^2 = x_2^2 = \frac{1}{2}.$$

Thus, there is a unique WEA allocation given $p_1/p_2 = 1$. It remains to show that there is no Walrasian equilibrium with $p_1/p_2 \neq 1$.

Consider the case when $p_1 > p_2$. Normalise $p_1 = 1 > p_2 = p$ This means that

$$MRS_{1,2}^1 = \frac{u'\left(x_1^1\right)}{u'\left(x_2^1\right)} = \frac{1}{p} = \frac{v'\left(x_1^2\right)}{v'\left(x_2^2\right)} = MRS_{1,2}^2 > 1$$

so that

$$u'\left(x_1^1\right) > u'\left(x_2^1\right) \Rightarrow x_1^1 < x_2^1,$$
$$v'\left(x_1^2\right) > v'\left(x_2^2\right) \Rightarrow x_1^2 < x_2^2,$$

where the implications follow from the fact that $u'', v'' < 0$. Since utility functions are strongly increasing, the budget constraint must bind in equilibrium, so, for consumer 1,

$$\begin{aligned} x_1^1 + p x_2^1 &= 1 \\ \Rightarrow x_2^1 + p x_2^1 &> 1 \because x_2^1 > x_1^1 \\ \Leftrightarrow x_2^1 &> \frac{1}{1+p}. \end{aligned}$$

Similarly, for consumer 2,

$$\begin{aligned} x_1^2 + p x_2^2 &= p \\ \Rightarrow x_2^2 + p x_2^2 &> p \because x_2^2 > x_1^1 \\ \Leftrightarrow x_2^2 &> \frac{p}{1+p}. \end{aligned}$$

Summing the two together, we obtain

$$x_2^1 + x_2^2 > \frac{1}{1+p} + \frac{p}{1+p} = 1.$$

Thus, good market 2 does not clear. Similar argument for $p_1 < p_2$ shows that goods market will not clear in this case. Thus, we conclude that WEA we found is unique.

2.1.2 Part (ii)

Suppose that consumer 1's endowment of good 1 increases to $\alpha > 1$. Prove that, in every Walrasian equilibrium, the price of good 2 exceeds the price of good 1. Prove also that consumer 2 is now strictly better off than in part (i). Is it clear whether consumer 1 is now better off than in part (i)? Why or why not?

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For the same reason as above, we need only to focus on strictly positive prices. Normalise $p_2 = p$ and $p_1 = 1$. Suppose $p \leq 1$. Then,

$$MRS_{1,2}^1 = \frac{1}{p} = MRS_{1,2}^1 \geq 1$$

so that

$$\begin{aligned} u\left(x_1^1\right) \geq u\left(x_2^1\right) \Rightarrow x_1^1 \leq x_2^1, \\ v\left(x_1^2\right) \geq v\left(x_2^2\right) \Rightarrow x_1^2 \leq x_2^2. \end{aligned}$$

Now, consider consumer 1's budget constraint:

$$\begin{aligned} x_1^1 + p x_2^1 = \alpha \\ \Rightarrow x_2^1 + p x_2^1 \geq \alpha \because x_2^1 \geq x_1^1 \\ \Leftrightarrow x_2^1 \geq \frac{\alpha}{1+p} \end{aligned}$$

and for consumer 2,

$$\begin{aligned} x_1^2 + p x_2^2 = p \\ \Rightarrow x_2^2 + p x_2^2 \geq p \because x_2^2 \geq x_1^2 \\ \Leftrightarrow x_2^2 \geq \frac{p}{1+p}. \end{aligned}$$

Thus,

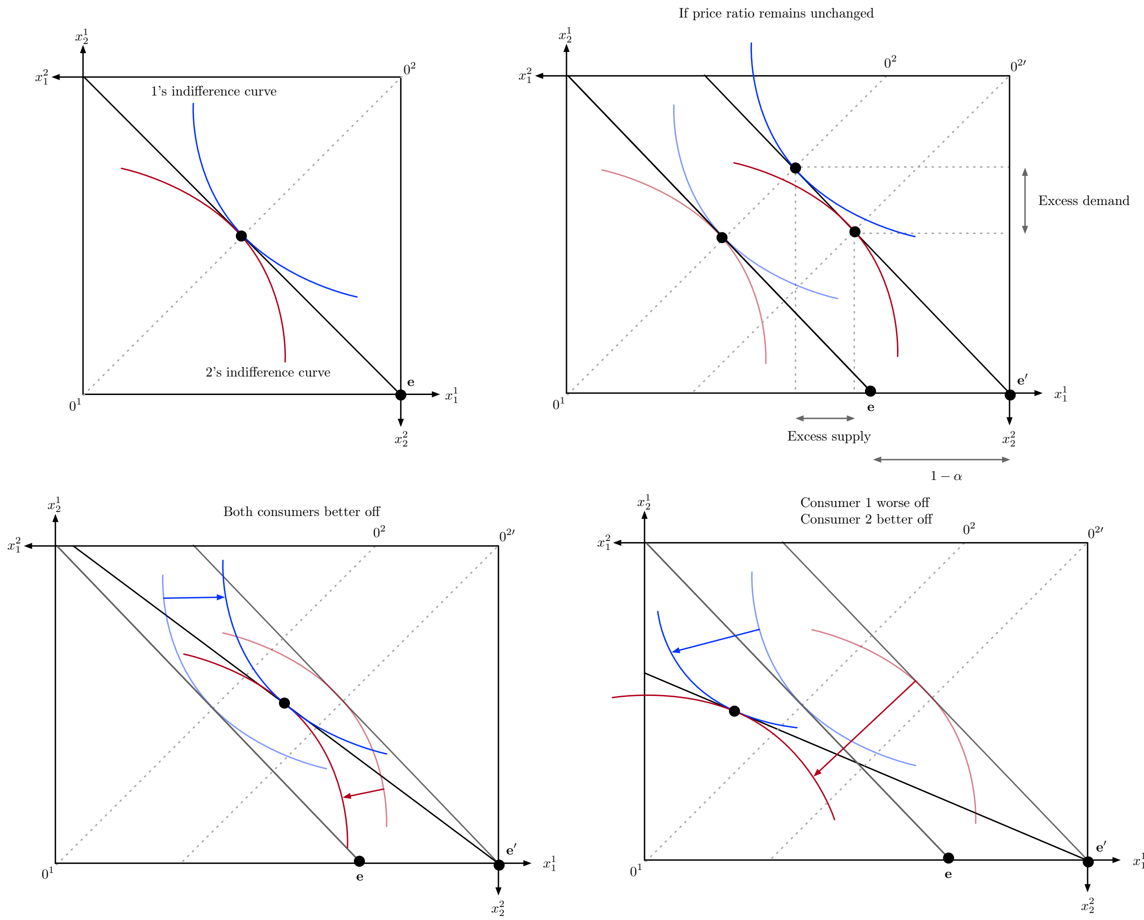
$$x_2^1 + x_2^2 \geq \frac{\alpha + p}{1+p} > 1 \because \alpha > 1.$$

Then, good-2 market does not clear if $p \leq 1$. Since we know that equilibrium exists, it follows that $p > 1$ in any Walrasian equilibrium.

Notice that in both cases, we normalised $p_1 = 1$ and we showed here that $p > 1$ whereas in part (i), we had $p = 1$. Thus, we realise that consumer 2's income is strictly greater here than in part (i). His budget set includes everything there as before and more—it thus follows that consumer 2 is strictly better off in this question than in part (a).

For consumer 1, the effect is ambiguous—he experiences two opposing effects. His relative income falls since there is more supply of good 1 in the economy which has a negative effect on utility. However, the fact that there is more of good 1 in the economy, which he values, has a positive effect. The net effect, as always, depends on the relative strength between these forces.

See figure.



2.2 2017/18

Consider the following private ownership economy with two goods. There is one firm with production set

$$Y = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \geq 0, y_1 + y_2 \leq 0\right\},$$

where negative quantities are inputs and positive quantities are outputs. There are two consumers. Consumer 1 has utility function and endowment vector, $u_1(x_1, x_2) = 2x_1 + x_2$ and $e^1 = (1, 0)$, and consumer 2 has utility function and endowment vector $u_2(x_1, x_2) = x_1 + 2x_2$ and $e^2 = (1, 0)$. Each consumer has one-half ownership share of the firm.

2.2.1 Part (a)

Let p_i denote the price of good i. What are the firm's maximum profits if $p_1 > p_2$? If $p_1 < p_2$?

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Let's first understand the production set Y. Since $y_1 \leq 0$, this is the input used to produce the output $y_2 \geq 0$. The last inequality means that the production function is linear. We can then write the firm's profit maximisation problem as

$$\Pi(\mathbf{p}) \coloneqq \max_{(\hat{y}_1, y_2) \in \mathbb{R}^2_+} p_2 y_2 - p_1 y_1 \text{ s.t. } y_2 \leq \hat{y}_1,$$

where I have defined $\hat{y}_1 \coloneqq -y_1$. We can simplify this one step further. Provided that $p_2 > 0$, the firm maximises its profits by ensuring that the constraint binds so that $y_2 = \hat{y}_1$ at any optimum with $p_2 > 0$. We will indeed assume that $p_2 > 0$. Hence,

$$\Pi(\mathbf{p}) \coloneqq \max_{y_2 \in \mathbb{R}_+} (p_2 - p_1) y_2.$$

It is then clear that

$$\Pi(\mathbf{p}) = \begin{cases} 0 & \text{if } p_2 \leq p_1 \\ \infty & \text{if } p_2 > p_1 \end{cases}.$$

Note that

$$y_2(\mathbf{p}) = -y_1(\mathbf{p}) \begin{cases} = 0 & p_2 < p_1 \\ \in \mathbb{R}_+ & p_2 = p_1 \\ \infty & p_1 = p_2 \end{cases}.$$

2.2.2 Part (b)

Find a Walrasian equilibrium price vector and the associated Walrasian equilibrium allocation for this economy. What is the profit income of each consumer?

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First, observe that utility functions are both strongly increasing. Thus, in any equilibrium, prices must be strictly positive (otherwise each individual will demand infinite amount of the good whose price is zero).

Second, from the previous part, for the equilibrium to exist, it must be that $p_2 \leq p_1$; otherwise, the firm will demand an infinite amount of input, which contradicts the fact that there is only a finite endowment of the input. This also implies that consumers do not obtain any profits from their ownership in the firm.

Together, in any equilibrium, the price vector satisfies

$$0 < p_2 \leq p_1.$$

Consumer 1 solves the following problem:

$$\max_{\left(x_1^1, x_2^1\right) \in \mathbb{R}_+^2} \quad 2x_1^1 + x_2^1 \ s.t. \ p_1 x_1^1 + p_2 x_2^1 \leq m^1 \left(\mathbf{p}\right).$$

This is an example of perfect substitutes, and the demand functions are

$$x_1^1 \left(\mathbf{p}\right) = \begin{cases} 0 & \text{if } \frac{p_1}{p_2} > 2 \\ \hat{x}_1^1 \in \left[0, \frac{m^1(\mathbf{p})}{p_1}\right] & \text{if } \frac{p_1}{p_2} = 2 \\ \frac{m^1(\mathbf{p})}{p_1} & \text{if } \frac{p_1}{p_2} < 2 \end{cases}, \ x_2^1 \left(\mathbf{p}\right) = \begin{cases} \frac{m^1(\mathbf{p})}{2} & \text{if } \frac{p_1}{p_2} > 2 \\ \frac{m^1(\mathbf{p}) - p_1 \hat{x}_1^1}{p_2} & \text{if } \frac{p_1}{p_2} = 2 \\ 0 & \text{if } \frac{p_1}{p_2} < 2 \end{cases}. \tag{2.1}$$

By symmetry, consumer 2's demand is given by

$$x_1^2 \left(\mathbf{p}\right) = \begin{cases} 0 & \text{if } \frac{p_1}{p_2} > \frac{1}{2} \\ \hat{x}_1^2 \in \left[0, \frac{m^2(\mathbf{p})}{p_1}\right] & \text{if } \frac{p_1}{p_2} = \frac{1}{2} \\ \frac{m^2(\mathbf{p})}{p_1} & \text{if } \frac{p_1}{p_2} < \frac{1}{2} \end{cases}, \ x_2^2 \left(\mathbf{p}\right) = \begin{cases} \frac{m^2(\mathbf{p})}{p_2} & \text{if } \frac{p_1}{p_2} > \frac{1}{2} \\ \frac{m^2(\mathbf{p}) - p_1 \hat{x}_1^2}{p_2} & \text{if } \frac{p_1}{p_2} = \frac{1}{2} \\ 0 & \text{if } \frac{p_1}{p_2} < \frac{1}{2} \end{cases}. \tag{2.2}$$

Guess that our equilibrium prices satisfy

$$p_1 = p_2 = p^*,$$

which is the only price vector that allows for (finite) positive production of good 2. In this case,

$$\Pi \left(\mathbf{p}^*\right) = 0 \Rightarrow m^1 \left(\mathbf{p}^*\right) = m^2 \left(\mathbf{p}^*\right) = p^*,$$

where $\mathbf{p}^* := (p^*, p^*)$. From (2.1) and (2.2), we have

$$\begin{aligned} x_1^1 \left(\mathbf{p}^*\right) &= \frac{m^1 \left(\mathbf{p}^*\right)}{p^*} = 1, \quad & x_2^1 \left(\mathbf{p}^*\right) &= 0, \\ x_1^2 \left(\mathbf{p}^*\right) &= 0, & x_2^2 \left(\mathbf{p}^*\right) &= \frac{m^2 \left(\mathbf{p}^*\right)}{p_2} = 1. \end{aligned}$$

We now use the market clearing condition to back out the equilibrium production plan:

$$\begin{aligned} x_1^1 \left(\mathbf{p}^*\right) + x_1^2 \left(\mathbf{p}^*\right) &= e_1^1 + e_1^2 + y_1 \left(\mathbf{p}^*\right) \\ \Leftrightarrow 1 &= 2 + y_1 \left(\mathbf{p}^*\right) \\ \Leftrightarrow y_1 \left(\mathbf{p}^*\right) &= -1 \\ \Rightarrow y_2 \left(\mathbf{p}^*\right) &= -y_1 \left(\mathbf{p}^*\right) = 1. \end{aligned}$$

Thus, a Walrasian equilibrium price is $\mathbf{p}^* = (1, 1)$ in which case the associated Walrasian equilibrium allocation is given by

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}\right\} = \left\{(1, 0), (0, 1), (-1, 1)\right\}.$$