1 PS7 Q1

There are n bidders participating in a first-price auction. Each bidder's value is independently drawn from $[0, 1]$ according to the distribution function F, having continuous and strictly positive density f. If a bidder's value is θ_i and he wins the object with a bid of $b_i < \theta_i$, then his von Neumann-Morgenstern utility is $(\theta_i - b_i)^{\frac{1}{\alpha}}$, where $\alpha \geq 1$ is fixed and common to all bidders. Consequently, the bidders are risk averse when $\alpha > 1$ and risk neutral when $\alpha = 1$. Given the risk-aversion parameter α, let $\bar{b}_\alpha(\theta)$ denote the (symmetric) equilibrium bid of a bidder when his value is θ. The following parts will guide you toward finding $\bar{b}_\alpha(\theta)$ and uncovering some of its implications.

Problem 1.1. Let $U\left(\hat{\theta} \mid \theta\right)$ denote a bidder's expected utility from bidding $\bar{b}_\alpha\left(\hat{\theta}\right)$, given that all other bidders employ $\bar{b}_\alpha(\theta)$. Show that

$$U\left(\hat{\theta} \mid \theta\right) = F\left(\hat{\theta}\right)^{n-1}\left(\theta - \bar{b}_\alpha\left(\hat{\theta}\right)\right)^{\frac{1}{\alpha}}$$

Why must $U\left(\hat{\theta} \mid \theta\right)$ be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$?

Solution. Same as in the risk neutral case, we assume that the symmetric bidding function $\bar{b}_\alpha(\theta)$ is strictly increasing, which can be verified later when we actually obtain the expression for $\bar{b}_\alpha(\theta)$. Then, given that all other bidders employ $\bar{b}_\alpha(\theta)$, we have that

$$\begin{aligned}
\mathbb{P}\left(\bar{b}_\alpha\left(\hat{\theta}_i\right) \geq \max_{j\neq i} \bar{b}_\alpha(\theta_j)\right) &= \mathbb{P}\left(\hat{\theta}_i \geq \max_{j\neq i}\theta_j\right) \\
&= \mathbb{P}\left(\max_{j\neq i}\theta_j \leq \hat{\theta}_i\right) \\
\text{[by independence]} &= \prod_{j\neq i}\mathbb{P}\left(\theta_j \leq \hat{\theta}_i\right) \\
&= F\left(\hat{\theta}_i\right)^{n-1}
\end{aligned}$$

So, if a bidder bids $\hat{\theta}$, his probability of winning is $F\left(\hat{\theta}\right)^{n-1}$. Since bidders have common von Neumann-Morgenstern utility $(\theta_i - b_i)^{\frac{1}{\alpha}}$, the utility of bidding $\bar{b}_\alpha\left(\hat{\theta}\right)$ with true type θ is

$$U\left(\hat{\theta} \mid \theta\right) = F\left(\hat{\theta}\right)^{n-1}\left(\theta - \bar{b}_\alpha\left(\hat{\theta}\right)\right)^{\frac{1}{\alpha}}$$

$U\left(\hat{\theta} \mid \theta\right)$ must be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$ because this condition ensures that no bidder will have incentive to deviate in equilibrium. Again, this is the incentive compatibility condition imposed on any candidate bidding function, which can be verified after we actually obtain the expression for $\bar{b}_\alpha(\theta)$.

Problem 1.2. Use part (a) to argue that

$$U\left(\hat{\theta} \mid \theta\right)^{\alpha} = F\left(\hat{\theta}\right)^{\alpha(n-1)}\left(\theta - \overline{b}_{\alpha}\left(\hat{\theta}\right)\right)$$

must be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$.

Solution. Given the result in (a), we know that any candidate bidding function $\overline{b}_{\alpha}(\theta)$ must have the property that

$$U\left(\hat{\theta} \mid \theta\right) = F\left(\hat{\theta}\right)^{n-1}\left(\theta - \overline{b}_{\alpha}\left(\hat{\theta}\right)\right)^{\frac{1}{\alpha}}$$

is maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$. Note that $\alpha \geq 1$ implies that $U\left(\hat{\theta} \mid \theta\right)^{\alpha}$ is a strictly increasing transformation of $U\left(\hat{\theta} \mid \theta\right)$. So, $U\left(\hat{\theta} \mid \theta\right)^{\alpha}$ is also maximized when $U\left(\hat{\theta} \mid \theta\right)$ is maximized, i.e.

$$U\left(\hat{\theta} \mid \theta\right)^{\alpha} = F\left(\hat{\theta}\right)^{\alpha(n-1)}\left(\theta - \overline{b}_{\alpha}\left(\hat{\theta}\right)\right)$$

must be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$.

Problem 1.3. Use part (b) to argue that a first-price auction with the n risk-averse bidders above whose values are each independently distributed according to $F(\theta)$, is equivalent to a first-price auction with n risk neutral bidders whose values are each independently distributed according to $F(\theta)^{\alpha}$. Use the solution for the risk-neutral case to conclude that

$$\overline{b}_{\alpha}(\theta) = \theta - \int_{0}^{\theta}\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(n-1)} dx$$

Solution. Note that if $F(\theta)$ is a c.d.f., then $\tilde{F}(\theta) \equiv F(\theta)^{\alpha}$ is also a c.d.f. because $F(0) = \tilde{F}(0) = 0$, $F(1) = \tilde{F}(1) = 1$ and both functions are right continuous. Then, denote the utility of a hypothetical risk neutral bidder corresponding to a risk averse bidder in our case by

$$\tilde{U}\left(\hat{\theta} \mid \theta\right) = \tilde{F}\left(\hat{\theta}\right)^{(n-1)}\left(\theta - \overline{b}_{\alpha}\left(\hat{\theta}\right)\right)$$

By the result in part (b), we know that

$$\tilde{U}\left(\hat{\theta} \mid \theta\right) = U\left(\hat{\theta} \mid \theta\right)^{\alpha}$$

is also maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$. Now, with this tilde notation, the problem resembles the problem for the risk neutral bidders in class, and according to the result we derived in class, the optimal bidding function is

$$\bar{b}_{\alpha}(\theta) = \theta - \int_0^{\theta} \left(\frac{\tilde{F}(x)}{\tilde{F}(\theta)} \right)^{(n-1)} dx$$

which, as we have checked in class, is strictly increasing in θ and incentive compatible.

Now, plug in the expression for $\tilde{F}(\theta)$ we have

$$\begin{aligned}
\bar{b}_{\alpha}(\theta) &= \theta - \int_0^{\theta} \left(\frac{\tilde{F}(x)}{\tilde{F}(\theta)} \right)^{(n-1)} dx \\
&= \theta - \int_0^{\theta} \left(\frac{F(x)^{\alpha}}{F(\theta)^{\alpha}} \right)^{(n-1)} d \\
&= \theta - \int_0^{\theta} \left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} dx
\end{aligned}$$

Problem 1.4. Prove that $\bar{b}_{\alpha}(\theta)$ is strictly increasing in α. Does this make sense? Conclude that as bidders become more risk averse, the seller's revenue from a first-price auction increases.

Solution. In order to have the strictly increasing property of the bidding funciton, we could impose an assumption that the p.d.f. is strictly positive on the support. Then, take derivative for $\bar{b}_{\alpha}(\theta)$ w.r.t. α:

$$\begin{aligned}
\frac{\partial \bar{b}_{\alpha}(\theta)}{\partial \alpha} &= \frac{\partial}{\partial \alpha} \left[\theta - \int_0^{\theta} \left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} dx \right] \\
&= - \int_0^{\theta} \frac{\partial}{\partial \alpha} \left(\left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} \right) dx \\
&= - \int_0^{\theta} (n-1) \left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} \underbrace{\ln \underbrace{\left(\frac{F(x)}{F(\theta)} \right)}_{\leq 1}}_{\leq 0} dx \\
&= \int_0^{\theta} (n-1) \left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} \underbrace{\left(-\ln \left(\frac{F(x)}{F(\theta)} \right) \right)}_{\geq 0} dx
\end{aligned}$$

which is strictly positive because all terms in the integrand are strictly positive except for when $x = \theta$, but the sum is still strictly positive.

This result is very intuitive because as bidders become more risk averse, they will prefer to have a more certain even with lower value outcome, which is achieved by raising the bid closer to his true valuation (probability of winning is higher while the surplus is lower if he wins the good).

Notice that the bidding function is strictly increasing in type, so the seller still awards the good to the highest type (also the highest bidder in our case). However, since we have shown that every bidder bids strictly higher than they would in risk neutral case, the expected revenue to the seller is strictly higher when bidders are risk averse.

Problem 1.5. Use part (d) and the revenue equivalence result for the standard auctions in the risk-neutral case to argue that when bidders are risk averse as above, a first-price auction raises more revenue for the seller than a second-price auction. Hence, these two standard auctions no longer generate the same revenue when bidders are risk averse.

Solution. First of all, following exactly the same logic as in class, we can show that even in the risk averse case, it is still a dominant strategy for agents to bid the true valuation in the second-price auction. Thus, the expected revenue from the second-price auction with risk averse agents, denoted by ER^{II}_{ra}, are exactly the same as from the second-price auction with risk neutral agents, denoted by ER^{II}_{rn}, i.e.

$$ER^{\mathrm{II}}_{ra} = ER^{\mathrm{II}}_{rn}$$

From the result in class, we know that

$$ER^{\mathrm{II}}_{rn} = ER^{\mathrm{I}}_{rn}$$

where ER^{I}_{rn} is the expected revenue from the second-price auction with risk neutral agents. So, we have $ER^{\mathrm{II}}_{ra} = ER^{\mathrm{I}}_{rn}$.

Notice that in (d) we showed that the expected revenue from the first-price auction with risk averse agents ER^{I}_{ra} is higher than in risk neutral case, so

$$\begin{aligned} & ER^{\mathrm{I}}_{ra} > ER^{\mathrm{I}}_{rn} \\ \Rightarrow\ & ER^{\mathrm{I}}_{ra} > ER^{\mathrm{II}}_{ra} \end{aligned}$$

i.e. the expected revenue is higher in first-price auction than second-price auction with risk averse agents.

Problem 1.6. What happens to the seller's revenue as the bidders become infinitely-risk averse (i.e., as $\alpha \to \infty$)?

Solution. From the expression of the optimal bidding function

$$\overline{b}_{\alpha}(\theta) = \theta - \int_0^{\theta} \left(\frac{F(x)}{F(\theta)} \right)^{\alpha(n-1)} dx$$

we can see that

$$\lim_{\alpha \to \infty} \bar{b}_\alpha(\theta) = \theta$$

So, in that extreme case, all agents will bid their true type, but since this is a first-price auction, the winner will pay his own bid, which means that the expected revenue to the seller is

$$\mathbb{E}\left[\max\left\{\theta_1, \ldots, \theta_n\right\}\right]$$

which is the first best result to the seller, i.e. the person who values the good most gets the good, and pays all his valuation to the seller, leaving himself zero surplus.

2 PS7 Q3

Consider a common-value auction with n bidders. Each bidder privately learns a signal, θ_i, independently distributed according to $F(\theta)i)$ on $[0, 1]$ (with positive density, $f(\theta_i)$). For now, notice we are allowing distributions to differ across bidders and the signals are independently distributed. Each bidder i has a common value of the good given by

$$v(\theta) = \frac{1}{n} \sum_{j=1} \theta_j$$

The sellerâs value for the good is $\theta_0 = 0$, and does not depend upon the signals of the bidders. We first want to compute the optimal auction using Myersonâs framework

Problem 2.1. Write down the two conditions which characterize an incentive-compatible direct mechanism, $\{\phi_i(), t_i()\}_i$.

Solution. Let

$$\begin{aligned}
\bar{\phi}_i(\theta_i) &= \mathbb{E}_{\theta_{-i}}[\phi_i(\theta_i, \theta_{-i})] \\
\bar{t}_i(\theta_i) &= \mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] \\
U_i(\hat{\theta}_i|\theta_i) &= \bar{\phi}(\hat{\theta}_i)\mathbb{E}_{\theta_{-i}}[v(\theta_i, \theta_{-i})] - \bar{t}_i(\hat{\theta}_i, \theta_{-i})
\end{aligned}$$

So the two conditions characterizing IC direct mechanisms are:

- ▷ $\bar{\phi}_i(\theta_i)$ non-decreasing, $\forall i$
- ▷ $U_i(\theta_i) = U(\theta_i|\theta_i) = U_i(0) + \int_0^{\theta_i} \bar{\phi}_i(s)ds$.

Problem 2.2. Using the conditions in (a), find an expression for $E_\theta[U_i(\theta_i)]$.

Solution.

$$\begin{aligned}
\mathbb{E}_\theta[U_i(\theta_i)] &= \mathbb{E}_\theta\left[U_i(0) + \int_0^{\theta_i} \bar{\phi}_i(s)ds\right] \\
&= U_i(0) + \int_{\theta\in\Theta}\int_0^{\theta_i} \bar{\phi}_i(s)dsdG(\theta), \text{ where } G \text{ is the CDF of the joint type vector} \\
&= U_i(0) + \int_0^1\int_0^{\theta_i} \bar{\phi}_i(s)dsdF_i(\theta_i), \text{ since types are independent} \\
&= U_i(0) + \int_0^1\int_s^1 dF_i(\theta_i)\bar{\phi}_i(s)ds \\
&= U_i(0) + \int_0^1 (1-F_i(s))\bar{\phi}_i(s)ds \\
&= U_i(0) + \mathbb{E}_{\theta_i}\left[\bar{\phi}_i(\theta_i)\frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right] \\
&= U_i(0) + \mathbb{E}_\theta\left[\phi_i(\theta)\frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right].
\end{aligned}$$

Problem 2.3. Using (b), write the sellerâs objective function expressed only in terms of $\phi_i()$ and $U(0)$.

Solution. The seller's problem is:

$$\begin{aligned}
&\max_{\phi,U_i}\mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta)(v(\theta)-\theta_0)+\theta_0-\sum_{i=1}^n U_i(\theta_i)\right] \\
\leftrightarrow &\max_{\phi}\mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta)v(\theta)-\sum_{i=1}^n\left(U_i(0)+\phi_i(\theta)\frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)\right], \text{ since } \theta_0=0 \\
\leftrightarrow &\max_{\phi}\mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta)\left(v(\theta)-\frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)-U_i(0)\right]
\end{aligned}$$

Note that as usual, since U_i is increasing in type, setting $U_i(0)=0$ to ensure that all types' IR constraints bind.

Problem 2.4. Using (c), find the optimal $\phi_i()$ which maximizes the sellerâs expected profit and find an expression determining the reservation type for each i. Make any regularity assumption(s) that you use explicit.

Solution. The regularity condition we need is that virtual type

$$\begin{aligned}
J_i(\theta) &= v(\theta) - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)} \\
&= \frac{\theta_i}{n} + \sum_{j\neq i}^n \frac{\theta_j}{n} - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)},
\end{aligned}$$

is non-decreasing so that the point-wise solution to the maximization problem is indeed optimal.

Following Myerson's optimal auction result, the optimal set of $\phi_i()$ is $\forall i$

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } J_i(\theta) > \max_{j\neq i} J_j(\theta) \text{ and } \mathrm{J}_i(\theta) > 0 \\ 0 & \text{if } J_i(\theta) < \max_{j\neq i} J_j(\theta) \text{ or } \mathrm{J}_i(\theta) < 0 \\ 1/k & \text{if have } k \text{ ties above } \theta_0 = 0 \end{cases}.$$

(Note that we use that $\theta_0 = 0$ here so that we don't have to worry about the winning virtual type being higher than θ_0).

The reservation type for bidder i is r_i^* such that $J_i(r_i^*, \theta_{-i}) = \theta_0 = 0$, which simplifies to

$$\frac{r_i^*}{n} + \sum_{j\neq i}^{n} \frac{\theta_j}{n} = \frac{(1 - F_i(r_i^*))}{f_i(r_i^*)}.$$

Note that the reservation price for bidder i will depend on the types of all other players.

Problem 2.5. Now further assume that the signal distributions are symmetric across bidders and uniform on [0, 1]. Suppose that the seller uses a first-price auction without reserve price to sell the good. Find the symmetric equilibrium bid function $\bar{b}()$. [Hint: it is linear.]

Solution. We conjecture

$$\bar{b}(\theta_i) = \alpha + \beta\theta_i.$$

Since $\bar{b}(\theta_i)$ is strictly increasing we have

$$U_i(\hat{\theta}_i|\theta_i) = F^{n-1}(\hat{\theta}_i)(\mathbb{E}_{\theta_{-i}}[v(\theta_i, \theta_{-i})] - \bar{b}(\hat{\theta}_i)) = \hat{\theta}_i^{n-1}\left(\frac{\theta_i}{n} + \frac{n-1}{2n} - \bar{b}(\hat{\theta}_i)\right).$$

The FOC for truth-telling asserts

$$U_{i1}(\theta_i, \theta_i) = 0,$$

which implies

$$\begin{aligned} 0 &= (n-1)\theta_i^{n-2}\left(\frac{\theta_i}{n} + \frac{n-1}{2} - \bar{b}(\theta_i)\right) - \theta_i^{n-1}\bar{b}'(\theta_i) \qquad (1) \\ &= (n-1)\theta_i^{n-2}\left(\frac{\theta_i}{n} + \frac{n-1}{2} - (\alpha + \beta\theta_i)\right) - \theta_i^{n-1}\beta \\ \leftrightarrow \theta_i\beta &= (n-1)\left(\frac{\theta_i}{n} + \frac{n-1}{2} - (\alpha + \beta\theta_i)\right) \\ \beta\theta_i &= (n-1)\theta_i\left(\frac{1}{n} - \beta\right) + \frac{(n-1)^2}{2} - (n-1)\alpha \\ \leftrightarrow \theta\left(\beta - \frac{n-1}{n} + (n-1)\beta\right) &= \frac{(n-1)^2}{2} - (n-1)\alpha. \end{aligned}$$

Letting

$$\alpha = \frac{n-1}{2n}$$

we get that

$$\beta = \frac{n-1}{n^2}.$$

Note that we get the same solution if we solve the differential equation in (1). Thus

$$\bar{b}(\theta_i) = \frac{n-1}{2n} + \frac{n-1}{n^2}\theta_i.$$

Problem 2.6. Is the auction in (e) optimal? Why or why not?

Solution. No the auction in e is not optimal. Since everyone has the same type distribution, the highest type bidder also has the highest virtual type, which is what we want. However, note that in this case

$$J_i(\theta) = \frac{\theta_i}{n} + \sum_{j\neq i}^{n} \frac{\theta_j}{n} - 1 + \theta_i$$

can be negative, in which case it is optimal to for the seller not to sell the item. Thus, without setting a positive reservation price, the first price auction is not optimal here.

Problem 2.7. Show that for the equilibrium in (e), if n > 4, the bid function declines as the number of bidders increases. Explain.

Solution. We examine $\bar{b}_{n+1}(\theta_i) - \bar{b}_n(\theta_i)$.

$$\begin{aligned}\bar{b}_{n+1}(\theta_i) - \bar{b}_n(\theta_i) &= \left(\frac{n}{(n+1)^2}\theta_i + \frac{n}{2(n+1)}\right) - \left(\frac{n-1}{n^2}\theta_i + \frac{n-1}{2n}\right)\\ &= \theta_i\left(\frac{-n^2+n+1}{n^2(n+1)^2}\right) + \frac{1}{2n(n+1)}.\end{aligned}$$

Note that the first term is negative since $n > 4$ and the second term is positive. Thus, for sufficiently large θ_i:

$$\theta_i > \frac{n(n+1)}{2(n^2-n-1),}$$

$\bar{b}_n(\theta_i)$ is decreasing in n for $n > 4$. On the other hand, note that for $n < 4$ the bid function is increasing for all $\theta \in [0,1]$. Note that we need only prove this statement for $\theta = 1$ and it will be true $\forall\theta \in [0,1]$ (because

$\theta_i = 1$ puts the max weight on the negative term):

$$\begin{aligned} &\bar{b}_{n+1}(1) - \bar{b}_n(1) > 0 \\ \leftrightarrow\ &\frac{n^2 - n - 1}{n^2(n+1)^2} < \frac{1}{2n(n+1)} \\ \leftrightarrow\ &n^2 - n - 1 < \frac{n(n+1)}{2} \\ \leftrightarrow\ &\frac{1}{2}n^2 - \frac{3}{2}n - 1 < 0 \\ \leftrightarrow\ &n^2 - 3n - 2 < 0, \end{aligned}$$

which is true for $n < 4$.

The intuition here is that if your type is relatively high, since types are independent it's likely that everyone else's type is relatively low. Your value is the average of your type and the types of others. So as n increases, your value depends less on your type, which is relatively high, and more on the types of others, which are relatively low, and so your value falls and you lower your bid accordingly.

3 PS7 Q5

Consider an IPV model of auctions with a large number of bidders, each of whom must pay k in order to learn their type before they can participate in the auction. The timing is as follows.

1. Seller offers an auction mechanism and offers invitations to bid to some subset of the population of bidders; each bidder sees how many bidders in total, n, are invited to the auction.
2. bidders with invitations to attend the auction decide whether or not to spend k to learn their type and participate in the auction; those that decide to do so learn their type θ_i which is distributed i.i.d. according to $F(\cdot)$ on $[\underline{\theta}, \bar{\theta}]$. The seller's cost of the good is θ_0.
3. bidders report their types into the mechanism and the winner and transfers are determined accordingly.

Problem 3.1. Characterize incentive compatibility for those who decide to participate in the mechanism. As usual, you should have a monotonicity condition and an integral condition for the bidder's indirect utility function, $U_i(\theta)$.

Solution. Define

$$\bar{\phi}_i = E_{\theta_{-i}}[\phi(\theta_i, \theta_{-i})].$$

Then, we have incentive compatibility when

1. $\forall i, \bar{\phi}_i(\theta_i)$ is increasing in θ_i
2. $\forall i.$ we have

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{\phi}_i(s)ds.$$

Problem 3.2. What is the relevant individual rationality constraint for the participating bidders that ensures all the invited bidders are willing to spend k to learn their type? Note that the bidders decide whether or not to participate *before* they learn their type, so this will involve the bidder's expected utility, $E[U_i(\theta)]$. In other words, the seller must satisfy ex ante IR constraints and not interim IR constraints for the bidders.

Solution. The IR constraints is, $\forall i$,

$$E[U_i(\theta)] - k \geq 0.$$

Problem 3.3. Suppose that it is optimal for the seller to invite n bidders to the auction. Using (a) and (b), write the seller's program in terms of $\phi_i(\cdot)$ and $U_i(\underline{\theta})$.

Solution. We use the IC constraint

$$E[U_i(\theta)] = U_i(\underline{\theta}_i) + E[\int_{\underline{\theta}_i}^{\theta_i} \bar{\phi}_i(s)ds]$$

and by our usual integration by parts trick, we know

$$U_i(\underline{\theta}_i) + E[\int_{\underline{\theta}_i}^{\theta_i} \bar{\phi}_i(s)ds] = U_i(\underline{\theta}_i) + E[\phi_i(\theta)\frac{1-F(\theta_i)}{f(\theta_i)}]$$

and thus the sellers program becomes

$$\max_{\phi, U_i(\underline{\theta})} E[\sum_{i=1}^{n} \phi_i(\theta)(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \theta_0)] + \theta_0 - \sum_{i=1}^{n} U_i(\underline{\theta}_i)$$

$s.t. \bar{\phi}_i$ is increasing.

$$U_i(\underline{\theta}_i) \geq k - E[\phi_i(\theta)\frac{1-F(\theta_i)}{f(\theta_i)}]$$

Problem 3.4. What is the optimal allocation for the seller? What is the optimal reserve type, θ^*? How does your answer differ from the standard optimal auction of Myerson? [Hint: to solve for the optimal auction, use a Lagrange multiplier, λ, for the participation constraint in (b), incorporate it into the seller's objective function, and prove that at the optimum the multiplier is $\lambda = 1$.]

Solution. Since the IR constraint on the RHS does not depend on $\underline{\theta}_i$, it is revenue maximizing to have it bind. Thus, we plug the IR constraint into the program to get

$$\max_{\phi,} E[\sum_{i=1}^{n} \phi_i(\theta)(\theta_i - \theta_0)] + \theta_0 - nk.$$

Thus, the seller's optimal allocation is

$$\phi_i(\theta_i) = 1\{\theta_i > \max_{j \neq i} \theta_j, \theta_i \geq \theta_0\}$$

and the optimal reserve type $\theta^* = \theta_0$. The answer differ's from the standard optimal auction of Myerson because now instead of giving it to the bidder with the highest virtual type we are giving it to the bidder with the highest actual type.

Problem 3.5. Using you answer from (d), show that the value function of the seller (as a function of n) is simply

$$\Pi(n) = E[\max\{\theta_1, ..., \theta_n, \theta_0\} - nk].$$

Conclude that the seller will want to invite the socially optimal number of bidders to the auction when there is a cost to entry.

Solution. By the nature of our optimal allocation and reserve type, the value function will be $\Pi(n)$ by construction. The seller can choose n and therefore maximizes $\Pi(n)$ with respect to n. Total surplus is maximized by solving

$$\max_{\phi,} E[\sum_{i=1}^{n} \phi_i(\theta)(\theta_i - \theta_0)] - nk.$$

Therefore, the optimal n chosen also maximizes total surplus and thus it is the socially optimal nuber of bidder.s