1 PS8 Q1 (Full surplus extraction with correlated signals)

A seller has a single unit to sell at zero opportunity cost, $\theta_0 = 0$, to one of two bidders. Each bidder has two possible types, $\{30, 60\}$, but the valuations are correlated according to the joint probability function, $f(\theta_1, \theta_2)$. Specifically, the probability that both are high or low is $f(30, 30) = f(60, 60) = \frac{1}{3}$ and the probability that the valuations are different is $f(30, 60) = f(60, 30) = \frac{1}{6}$.
Without loss of generality, suppose that the seller offers a direct mechanism $\{\phi_i, t_i\}_{i=1,2}$. Further, suppose that the seller wants to implement the efficient allocation of the good:

$$\phi_i(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_i > \theta_j \\ 0 & \text{if } \theta_i < \theta_j \\ \frac{1}{2} & \text{if } \theta_i = \theta_j \end{cases}$$

Restrict attention to symmetric payments which are paid as a function of the bidder's reported type and are independent from whether or not the bidder gets the good if there is a tie. Thus, the seller needs to determine four numbers: $t_1(30, 30)$, $t_1(30, 60)$, $t_1(60, 30)$ and $t_1(60, 60)$.

Problem 1.1. Show that the seller can implement the efficient allocation while simutaneously extracting all of the bidders' surplus.

Solution. Following the hint, there are two IC constraint for bidder i as a function of the four transfer payments

$$\mathbb{P}(\theta_j = 60 \mid \theta_i = 60) \times \left(\frac{1}{2} \times 60 - t(60, 60)\right) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 60) \times (60 - t(30, 60))$$
$$\geq \mathbb{P}(\theta_j = 60 \mid \theta_i = 60) \times (0 - t(60, 30)) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 60) \times \left(\frac{1}{2} \times 60 - t(30, 30)\right) \quad \text{(IC High)}$$

$$\mathbb{P}(\theta_j = 60 \mid \theta_i = 30) \times (0 - t(60, 30)) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 30) \times \left(\frac{1}{2} \times 30 - t(30, 30)\right)$$
$$\geq \mathbb{P}(\theta_j = 60 \mid \theta_i = 30) \times \left(\frac{1}{2}30 - t(60, 60)\right) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 30) \times (30 - t(30, 60)) \quad \text{(IC Low)}$$

$$\mathbb{P}(\theta_j = 60 \mid \theta_i = 60)\left(\frac{1}{2} \times 60 - t(60, 60)\right) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 60)(60 - t(30, 60)) \geq 0 \quad \text{(IR High)}$$

$$\mathbb{P}(\theta_j = 60 \mid \theta_i = 30)(0 - t(60, 30)) + \mathbb{P}(\theta_j = 30 \mid \theta_i = 30)\left(\frac{1}{2}30 - t(30, 30)\right) \geq 0 \quad \text{(IR Low)}$$

Now, to solve this, we want to first set the transfers such that (IC High) binds. Plugging in the relevant conditional probabilities, this yields that

$$\begin{aligned}
\frac{2}{3}\times\left(\frac{1}{2}\times 60-t\left(60,60\right)\right)+\frac{1}{3}\times\left(60-t\left(30,60\right)\right)&=\frac{2}{3}\times\left(0-t\left(60,30\right)\right)+\frac{1}{3}\times\left(\frac{1}{2}\times 60-t\left(30,30\right)\right)\\
2\left(30-t\left(60,60\right)\right)+\left(60-t\left(30,60\right)\right)&=-2t\left(60,30\right)+\left(30-t\left(30,30\right)\right)\\
60-2t\left(60,60\right)+60-t\left(30,60\right)&=-2t\left(60,30\right)+30-t\left(30,30\right)\\
120-2t\left(60,60\right)-t\left(30,60\right)&=30-2t\left(60,30\right)-t\left(30,30\right)
\end{aligned}$$

Now, to solve this, we need to plug in values such that the high type IR constraint binds. In particular,

$$\begin{aligned}
\frac{2}{3}\left(\frac{1}{2}\times 60-t\left(60,60\right)\right)+\frac{1}{3}\left(60-t\left(30,60\right)\right)&=0\\
2\left(30-t\left(60,60\right)\right)+60-t\left(30,60\right)&=0\\
60-2t\left(60,60\right)+60-t\left(30,60\right)&=0\\
120-2t\left(60,60\right)&=t\left(30,60\right)
\end{aligned}$$

and using the second IR constraint we have that

$$\begin{aligned}
\frac{1}{3}\left(0-t\left(60,30\right)\right)+\frac{2}{3}\left(\frac{1}{2}30-t\left(30,30\right)\right)&=0\\
-t\left(60,30\right)+30-2t\left(30,30\right)&=0\\
30-2t\left(30,30\right)&=t\left(60,30\right)
\end{aligned}$$

Now we have three equations in four unknowns. We now set $t\left(60,60\right)=30$ and solve for the other quantities. From the High Type's IR constrant this yields that

$$\begin{aligned}
120-2\times 30&=t\left(30,60\right)\\
60&=t\left(30,60\right)
\end{aligned}$$

From the high type's IC constraint we get that

$$\begin{aligned}
120-2\times 30-60&=30-2t\left(60,30\right)-t\left(30,30\right)\\
t\left(30,30\right)&=30-2t\left(60,30\right)
\end{aligned}$$

combining the expressions from the low type's IR constraint and the high type's IC constraint we get that

$$\begin{aligned}
t\left(30,30\right)&=10\\
t\left(60,30\right)&=10
\end{aligned}$$

All that's left is to verify that the low type's IC constraint is not violated. Plugging in the relevant conditional probabilities, we have that

$$\begin{aligned}
\frac{1}{3} \times (0 - t(60, 30)) + \frac{2}{3}\left(\frac{1}{2} \times 30 - t(30, 30)\right) &\geq \frac{1}{3} \times \left(\frac{1}{2} 30 - t(60, 60)\right) + \frac{2}{3} \times (30 - t(30, 60)) \\
(0 - 10) + 2 \times (15 - 10) &\geq (15 - 30) + 2 \times (30 - 60) \\
-10 + 2 \times 5 &\geq -15 + 2 \times (-30) \\
-10 + 10 &\geq -30 - 60 \\
0 &\geq -90
\end{aligned}$$

and so the low type's IC holds. Notice that the efficient allocation is implemented as the highest type always ends up with the object. Notice also that both IR constraints bind and so the seller extracts all of the bidder's surplus.

Problem 1.2. Reinterpret the answer in (a) as the following game. Each bidder is required to bid either 30 or 60. The winning bidder must pay her bid. In addition, any bidder who bids 30 is also required to take a side bet with the seller which has zero expected payoff if the bidder's true type i s$\theta_i = 30$ and a large negative payoff if the bidder's type is actually $\theta_i = 60$. Describe the side bet for this reinterpretation of the mechanism in (a).

Solution. All that's left is to precisely describe the side bet. First, notice that the surplus of the conditional surplus of the low type bidder (from part A) is given by

$$\text{Surplus} = \begin{cases} -10 & \theta_i = 30, \theta_j = 60 \\ 5 & \theta_i = 30, \theta_j = 30 \end{cases}$$

So we can replicate this payoff structure as a zero expected payoff side bet that pays 5 if both bidders are the low type and -10 if bidder i is the low type and bidder j is the high type. Now, notice that (from Part A) if the bidder is the high type and reports being the low type, the the high has the following payoff

$$\begin{aligned}
\frac{2}{3} \times (0 - 10) + \frac{1}{3} \times (30 - 10) &= 2 \times (-10) + 20 \\
&= 0
\end{aligned}$$

to replicate this payoff we can impose a penalty on the high type, pretending to be the low type

$$\begin{aligned}
60 - 30 &= 30 \\
&= 30
\end{aligned}$$

and so conditional on winning the high type will pay a penalty of -30.

2 PS8 Q3 (Impossibility of efficient public good provision.)

Consider a public goods setting in which two citizens must collectively decide whether or not to produce a public good which costs c (known to both parties). The value of the public good to each citizen is θ_i, distributed according to $F(\cdot)$ on $[0,1]$. An ex post efficient mechanism produces the public good if $\theta_1+\theta_2 \geq c$ and does not otherwise. Suppose that $c \in (0,2)$ which implies that it is efficient to build the public good for some type profiles, but not for all.
Suppose that the government designs a direct revelation mechanism to implement the public good with probability $\phi(\theta_1,\theta_2)$ and taxes each citizen the amount $t_i(\theta_1,\theta_2)$, where we require that $t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) = c$if the public good is built and $t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) = 0$ otherwise. (This is a form of ex post budget balance.) Assume that each agent has the right to refuse to participate in the mechanism after learning type θ_i, in which case the public good is not built and no taxes are charged to either agent.
Prove that there does not exist an IC, IR mechanism which is ex post efficient.
[Hint: Convert the public-goods problem into the Myerson-Satterthwaite bilateral trading problem. Imagine agent 1 in the role of buyer, $t_1 = t$ is a transfer from agent 1 to agent 2, and assume that agent 2 must build the public good using personal funds, c, if $\phi = 1$. Check all of the MS conditions. Then apply the MS impossibility theorem.]

Solution. Before converting the problem into a bilateral trading problem, first note that individual i's ex post utility (assuming truth-telling) is

$$U_i(\theta_1,\theta_2) = \phi(\theta_1,\theta_2)\theta_i - t_i(\theta_1,\theta_2).$$

Note that if $\phi = 1$, we have

$$t_1 + t_2 = c \leftrightarrow t_2 = c - t_1,$$

and if $\phi = 0$ we have

$$t_1 + t_2 = 0 \leftrightarrow t_2 = -t_1.$$

So we can rewrite individual 2's utility (assuming truth-telling) as:

$$U_2(\theta_1,\theta_2) = \phi(\theta_1,\theta_2)(\theta_2 - c) + t_1(\theta_1,\theta_2) = t_1(\theta_1,\theta_2) - \phi(\theta_1,\theta_2)(c-\theta_2),$$

while individual 1 still has utility

$$U_1(\theta_1,\theta_2) = \phi(\theta_1,\theta_2)\theta_1 - t_1(\theta_1,\theta_2)$$

Thus, to convert this problem into a bilateral trading problem, define the effective (tilde) type for each player as

$$\tilde{\theta}_1 = \theta_1$$
$$\tilde{\theta}_2 = c - \theta_2$$

We now define the trading rule and transfer function as functions of the effective types

$$\tilde{t}(\tilde{\theta}_1, \tilde{\theta}_2) = t_1(\tilde{\theta}_1, c - \tilde{\theta}_2)$$
$$\tilde{\phi}(\tilde{\theta}_1, \tilde{\theta}_2) = \phi(\tilde{\theta}_1, c - \tilde{\theta}_2).$$

We then get expected equilibrium quantities:

$$\bar{\tilde{t}}(\hat{\tilde{\theta}}_i) = \mathbb{E}_{\tilde{\theta}_{-i}}[\tilde{t}(\hat{\tilde{\theta}}_i, \tilde{\theta}_{-i})]$$
$$\bar{\tilde{\phi}}(\hat{\tilde{\theta}}_i) = \mathbb{E}_{\tilde{\theta}_{-i}}[\tilde{\phi}(\hat{\tilde{\theta}}_i, \tilde{\theta}_{-i})]$$

So in our usual Myerson-Satterthwaite notation, we now have interim expected utilities

$$U_1(\hat{\tilde{\theta}}_1|\tilde{\theta}_1) = \bar{\tilde{\phi}}(\hat{\tilde{\theta}}_1)\tilde{\theta}_1 - \bar{\tilde{t}}(\hat{\tilde{\theta}}_1)$$
$$U_2(\hat{\tilde{\theta}}_2|\tilde{\theta}_2) = \bar{\tilde{t}}(\hat{\tilde{\theta}}_2) - \bar{\tilde{\phi}}(\hat{\tilde{\theta}}_2)\tilde{\theta}_2.$$

Now note that the supports of the effective types overlap:

$$[\underline{\tilde{\theta}}_1, \bar{\tilde{\theta}}_1] = [0, 1]$$
$$[\underline{\tilde{\theta}}_2, \bar{\tilde{\theta}}_2] = [-1, 2]$$
$$\leftrightarrow \underline{\tilde{\theta}}_1 < \bar{\tilde{\theta}}_2, \quad \underline{\tilde{\theta}}_2 < \bar{\tilde{\theta}}_1.$$

Thus, since we only consider budget balanced mechanisms here, by the Myerson-Satterthwaite impossibility theorem there is no IC, IR, ex-post efficient mechanism.

3 PS8 Q5 (Bilateral trading game, Chatterjee-Samuelson (1983).) (Xindi)

Consider a setting of bilateral trade in which the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$. Suppose the buyer and seller play the following game. Both buyer and seller simultaneously bid a price for trade. If the buyer's bid, p_b, weakly exceeds the seller's bid, p_s, then trade takes place at the average price bid, $p = \frac{p_b+p_s}{2}$. Otherwise, trade does not take place. A pure-strategy Bayesian-Nash equilibrium is a pair of functions, $\{\overline{p}_b(\cdot), \overline{p}_s(\cdot)\}$,which is a map from a player's type to a bid.

Problem 3.1. Given $\overline{p}_s(\cdot)$ is played by the seller, the Buyer's optimization program is

$$\max_{p_b}\left(\theta_b - \frac{p_b + \mathbb{E}_{\theta_s}\left[\overline{p}_s(\theta_s) \mid p_b \geq \overline{p}_s(\theta_s)\right]}{2}\right) \times \mathbb{P}_{\theta_s}\left(p_b \geq \overline{p}_s(\theta_s)\right)$$

Write down the Seller's optimization program, given $\overline{p}_b(\cdot)$.

Solution. The seller's optimization program, given $\overline{p}_b(\cdot)$, is

$$\max_{p_s}\left(\frac{p_s + \mathbb{E}_{\theta_b}\left[\overline{p}_b(\theta_b) \mid \overline{p}_b(\theta_b) \geq p_s\right]}{2} - \theta_s\right) \times \mathbb{P}_{\theta_b}\left(\overline{p}_b(\theta_b) \geq p_s\right)$$

Problem 3.2. Show that the following is a discontinuous equilibrium to the game for any $x \in (0, 2)$

$$\overline{p}_b(\theta_b) = \begin{cases} x & \text{if } \theta_b \geq x \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{p}_s(\theta_s) = \begin{cases} x & \text{if } \theta_s \leq x \\ 2 & \text{otherwise} \end{cases}$$

Solution. ▷ First, let's check that the buyer has no incentive to deviate

* Given

$$\overline{p}_s(\theta_s) = \begin{cases} x & \text{if } \theta_s \leq x \\ 2 & \text{otherwise} \end{cases}$$

the minimum p_s that buyer can face is x, and use the fact that it is weakly dominant to never bid over his type, then we have

- when buyer's type $\theta_b < x$, he never gets the good, so he has no incentive to deviate
- when $\theta_b \geq x$, the price seller offers is either x or 2, which is determined by the seller's type. If $p_s = 2$, then buyer's maximum surplus is 0, so he won't deviate. If $p_s = x$, then it is strictly dominant for the buyer to bid x (as opposed to bid higher or lower) to get the good at the lowest possible price.

* So, the buyer has no incentive to deviate.

▷ Then, let's check that the seller has no incentive to deviate

* Given

$$\overline{p}_b(\theta_b) = \begin{cases} x & \text{if } \theta_b \geq x \\ 0 & \text{otherwise} \end{cases}$$

the maximum p_b that seller can face is x, and use the fact that it is weakly dominant to never bid lower than his type, then we have

- when seller's type $\theta_s > x$, his best payoff is zero by not trading, so he have no incentive to deviate from bidding 2 which guarantees no trading.
- when seller's type $\theta_s \leq x$, the price buyer offers is either 0 or x. If $p_b = 0$, then the seller's maximum payoff is 0, so he won't deviate. If $p_b = x$, then it is strictly dominant for the seller to bid x to sell the good at the highest possible price.

* So, the seller has no incentive to deviate.

Problem 3.3. Now consider only linear equilibria: $\overline{p}_b(\theta_b) = \alpha_b + \beta_b\theta_b$ and $\overline{p}_s(\theta_s) = \alpha_s + \beta_s\theta_s$. Show that a linear equilibrium exists and compute the equilibrium bidding functions.

Solution. ▷ First of all, since it is weakly dominant to never bid outside the common support, we will concentrate on the case where $\overline{p}_b(\theta_b), \overline{p}_s(\theta_s) \in [0, 2]$.

▷ Then, consider the seller's problem given $\overline{p}_b(\theta_b) = \alpha_b + \beta_b\theta_b$:

* The maximization problem is

$$\begin{aligned}
&\max_{p_s} \left(\frac{p_s + \mathbb{E}_{\theta_b}\left[\alpha_b + \beta_b\theta_b \mid \alpha_b + \beta_b\theta_b \geq p_s\right]}{2} - \theta_s \right) \times \mathbb{P}_{\theta_b}\left(\alpha_b + \beta_b\theta_b \geq p_s\right) \\
\Leftrightarrow &\max_{p_s} \left(\frac{p_s + \mathbb{E}_{\theta_b}\left[\alpha_b + \beta_b\theta_b \mid \theta_b \geq \frac{p_s - \alpha_b}{\beta_b}\right]}{2} - \theta_s \right) \times \mathbb{P}_{\theta_b}\left(\theta_b \geq \frac{p_s - \alpha_b}{\beta_b}\right) \\
\Leftrightarrow &\max_{p_s} \left(\frac{p_s + \alpha_b + \beta_b\mathbb{E}_{\theta_b}\left[\theta_b \mid \theta_b \geq \frac{p_s - \alpha_b}{\beta_b}\right]}{2} - \theta_s \right) \times \left(1 - \frac{p_s - \alpha_b}{2\beta_b}\right) \\
\Leftrightarrow &\max_{p_s} \left(\frac{p_s + \alpha_b + \beta_b\left(1 + \frac{p_s - \alpha_b}{2\beta_b}\right)}{2} - \theta_s \right) \times \left(1 - \frac{p_s - \alpha_b}{2\beta_b}\right) \\
\Leftrightarrow &\max_{p_s} \left(\frac{3}{4}p_s + \frac{\alpha_b}{4} + \frac{\beta_b}{2} - \theta_s \right) \times \left(1 - \frac{p_s - \alpha_b}{2\beta_b}\right)
\end{aligned}$$

* The f.o.c. for p_s is

$$\begin{aligned}
\frac{3}{4}\left(1-\frac{p_s-\alpha_b}{2\beta_b}\right)-\frac{1}{2\beta_b}\left(\frac{3}{4}p_s+\frac{\alpha_b}{4}+\frac{\beta_b}{2}-\theta_s\right)&=0\\
\Rightarrow \frac{3}{4}-\frac{3}{4}\frac{p_s}{2\beta_b}+\frac{3}{4}\frac{\alpha_b}{2\beta_b}-\frac{1}{2\beta_b}\frac{3}{4}p_s-\frac{1}{2\beta_b}\frac{\alpha_b}{4}-\frac{1}{2\beta_b}\frac{\beta_b}{2}+\frac{1}{2\beta_b}\theta_s&=0\\
\Rightarrow \frac{3}{4}-\frac{3}{4}\frac{p_s}{\beta_b}+\frac{3}{8}\frac{\alpha_b}{\beta_b}-\frac{1}{8}\frac{\alpha_b}{\beta_b}-\frac{1}{4}+\frac{1}{2\beta_b}\theta_s&=0\\
\Rightarrow \frac{1}{2}-\frac{3}{4}\frac{p_s}{\beta_b}+\frac{1}{4}\frac{\alpha_b}{\beta_b}+\frac{1}{2\beta_b}\theta_s&=0\\
\Rightarrow \frac{3}{4}\frac{p_s}{\beta_b}&=\frac{1}{2}+\frac{1}{4}\frac{\alpha_b}{\beta_b}+\frac{1}{2\beta_b}\theta_s\\
\Rightarrow p_s&=\frac{4\beta_b}{3}\left(\frac{1}{2}+\frac{1}{4}\frac{\alpha_b}{\beta_b}+\frac{1}{2\beta_b}\theta_s\right)\\
\Rightarrow \overline{p}_s\left(\theta_s\right)&=\underbrace{\frac{2\beta_b}{3}+\frac{\alpha_b}{3}}_{\alpha_s}+\underbrace{\frac{2}{3}}_{\beta_s}\theta_s
\end{aligned}$$

▷ Now, consider the buyer's problem given $\overline{p}_s\left(\theta_s\right)=\alpha_s+\beta_s\theta_s$:

* The maximization problem is

$$\begin{aligned}
&\max_{p_b}\left(\theta_b-\frac{p_b+\mathbb{E}_{\theta_s}\left[\alpha_s+\beta_s\theta_s \mid p_b\geq\alpha_s+\beta_s\theta_s\right]}{2}\right)\times\mathbb{P}_{\theta_s}\left(p_b\geq\alpha_s+\beta_s\theta_s\right)\\
\Leftrightarrow &\max_{p_b}\left(\theta_b-\frac{p_b+\alpha_s+\beta_s\mathbb{E}_{\theta_s}\left[\theta_s \mid \theta_s\leq\frac{p_b-\alpha_s}{\beta_s}\right]}{2}\right)\times\mathbb{P}_{\theta_s}\left(\theta_s\leq\frac{p_b-\alpha_s}{\beta_s}\right)\\
\Leftrightarrow &\max_{p_b}\left(\theta_b-\frac{p_b+\alpha_s+\beta_s\frac{p_b-\alpha_s}{2\beta_s}}{2}\right)\times\frac{p_b-\alpha_s}{2\beta_s}\\
\Leftrightarrow &\max_{p_b}\left(\theta_b-\frac{p_b+\frac{1}{2}\alpha_s+\frac{p_b}{2}}{2}\right)\times\frac{p_b-\alpha_s}{2\beta_s}\\
\Leftrightarrow &\max_{p_b}\left(\theta_b-\frac{\alpha_s}{4}-\frac{3p_b}{4}\right)\times\frac{p_b-\alpha_s}{2\beta_s}
\end{aligned}$$

* The f.o.c. for p_b is

$$\begin{aligned}
-\frac{3}{4}\frac{p_b-\alpha_s}{2\beta_s}+\frac{1}{2\beta_s}\left(\theta_b-\frac{\alpha_s}{4}-\frac{3p_b}{4}\right)&=0\\
\Rightarrow -\frac{3}{4}\frac{p_b}{2\beta_s}+\frac{3}{4}\frac{\alpha_s}{2\beta_s}+\frac{1}{2\beta_s}\theta_b-\frac{1}{2\beta_s}\frac{\alpha_s}{4}-\frac{1}{2\beta_s}\frac{3p_b}{4}&=0\\
\Rightarrow -\frac{3}{4}\frac{p_b}{\beta_s}+\frac{3}{8}\frac{\alpha_s}{\beta_s}+\frac{1}{2\beta_s}\theta_b-\frac{\alpha_s}{8\beta_s}&=0\\
\Rightarrow -\frac{3}{4}p_b+\frac{1}{4}\alpha_s+\frac{1}{2}\theta_b&=0\\
\Rightarrow \frac{3}{4}p_b&=\frac{1}{4}\alpha_s+\frac{1}{2}\theta_b\\
\Rightarrow \overline{p}_b\left(\theta_b\right)&=\underbrace{\frac{1}{3}\alpha_s}_{\alpha_b}+\underbrace{\frac{2}{3}}_{\beta_b}\theta_b
\end{aligned}$$

▷ Combining the results to both the seller and the buyer, we have

$$\frac{1}{3}\alpha_s=\alpha_b \text{ and } \frac{2\beta_b}{3}+\frac{\alpha_b}{3}=\alpha_s$$

i.e.

$$\begin{aligned}
\frac{4}{9}+\frac{1}{9}\alpha_s&=\alpha_s\\
\Rightarrow \alpha_s&=\frac{1}{2}\\
\Rightarrow \alpha_b&=\frac{1}{6}
\end{aligned}$$

▷ So, the equilibrium bidding functions are:

$$\overline{p}_b\left(\theta_b\right)=\frac{1}{6}+\frac{2}{3}\theta_b$$

$$\overline{p}_s\left(\theta_s\right)=\frac{1}{2}+\frac{2}{3}\theta_s$$

Problem 3.4. In the equilibrium in (c), trade takes place if and only if $\theta_b-\theta_s\geq\alpha$. What is the value of the gap α?

Solution.

$$\begin{aligned}
\overline{p}_b\left(\theta_b\right) &\geq \overline{p}_s\left(\theta_s\right) \\
\Rightarrow \frac{1}{6}+\frac{2}{3}\theta_b &\geq \frac{1}{2}+\frac{2}{3}\theta_s \\
\Rightarrow \frac{2}{3}\theta_b-\frac{2}{3}\theta_s &\geq \frac{1}{2}-\frac{1}{6} \\
\Rightarrow \theta_b-\theta_s &\geq \frac{1}{3}\frac{3}{2} \\
\Rightarrow \theta_b-\theta_s &\geq \frac{1}{2}
\end{aligned}$$

So, the gap $\alpha = \frac{1}{2}$.

4 PS8 Q7 (Maximizing profit for a bilateral trading platform.) (Young Soo)

Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.
Suppose that a trading platform designs a profit maximizing mechanism which allocates the seller's good to the buyer with probability $\phi(\theta_b, \theta_s)$, the buyer pays the platform $t_b(\theta_b, \theta_s)$, the seller receives $t_s(\theta_b, \theta_s)$ and the platform keeps the difference in payments as profit:

$$t_b(\theta_b, \theta_s) - t_s(\theta_b, \theta_s).$$

Problem 4.1. Write down the monotonicity and integral conditions which are necessary and sufficient for the platform's mechanism to be incentive compatible.

Solution. Define the expected outcome of agent i, $\{\bar{\phi}_i(\hat{\theta}_i), \bar{t}_i(\hat{\theta}_i)\}_{i=b,s}$, from reporting $\hat{\theta}_i$:

$$\begin{aligned}
\bar{\phi}_b(\hat{\theta}_b) &:= E_{\theta_s}[\phi(\hat{\theta}_b, \theta_s)],\\
\bar{\phi}_s(\hat{\theta}_s) &:= E_{\theta_b}[\phi(\theta_b, \hat{\theta}_s)],\\
\bar{t}_b(\hat{\theta}_b) &:= E_{\theta_s}[t_b(\hat{\theta}_b, \theta_s)],\\
\bar{t}_s(\hat{\theta}_s) &:= E_{\theta_b}[t_s(\theta_b, \hat{\theta}_s)].
\end{aligned}$$

Then, the expected utility of buyer of type θ_b reporting $\hat{\theta}_b$ is given as:

$$U(\hat{\theta}_b|\theta_b) := \bar{\phi}_b(\hat{\theta}_b)\theta_b - \bar{t}_b(\hat{\theta}_b),$$

and the expected utility of seller of type θ_s reporting $\hat{\theta}_s$ is given as:

$$U(\hat{\theta}_s|\theta_s) := \bar{t}_s(\hat{\theta}_s) - \bar{\phi}_s(\hat{\theta}_s)\theta_s.$$

Lastly, define the expected utility of reporting one's true type as:

$$\begin{aligned}
U_b(\theta_b) &:= U_b(\theta_b|\theta_b),\\
U_s(\theta_s) &:= U_s(\theta_s|\theta_s).
\end{aligned}$$

For $\{\phi, t_b, t_c\}$ be incentive-compatible, we must have (truth-telling is optimal):

$$\begin{aligned}
U_b(\theta_b) \geq U_b(\hat{\theta}_b|\theta_b) \quad \forall \theta_b, \hat{\theta}_b \in [0, 2].\\
U_s(\theta_s) \geq U_s(\hat{\theta}_s|\theta_s) \quad \forall \theta_s, \hat{\theta}_s \in [0, 2].
\end{aligned}$$

The two necessary and sufficient conditions for IC are:

1. $\bar{\phi}_b(\cdot)$ is nondecreasing and $\bar{\phi}_s(\cdot)$ is nonincreasing

2. For $\forall \theta_b, \theta_s \in [0, 2]$,

$$
\begin{aligned}
U_b(\theta_b) &= U_b(0) + \int_0^{\theta_b} \bar{\phi}_b(x)dx \\
U_s(\theta_s) &= U_s(2) + \int_{\theta_s}^{2} \bar{\phi}_s(x)dx.
\end{aligned}
$$

Now, we prove both sufficiency and necessity.

Sufficiency (1,2 $\Rightarrow$ IC):

Proof. Given the above two conditions, we want to show that truth-telling is optimal for both buyer and seller of any type.

First, consider the case where $\theta_b < \hat{\theta}_b$. Then,

$$
\begin{aligned}
\bar{\phi}_b(\theta_b) &\leq \bar{\phi}_b(\hat{\theta}_b) && \because \bar{\phi}_b(\cdot) \text{ nondecreasing} \\
\int_{\theta_b}^{\hat{\theta}_b} \bar{\phi}_b(x)dx &\leq \int_{\theta_b}^{\hat{\theta}_b} \bar{\phi}_b(\hat{\theta}_b)dx \\
U_b(\hat{\theta}_b) - U_b(\theta_b) &\leq (\hat{\theta}_b - \theta_b)\bar{\phi}_b(\hat{\theta}_b) && \because \text{ second condition} \\
U_b(\theta_b) &\geq U(\hat{\theta}_b) - (\hat{\theta}_b - \theta_b)\bar{\phi}_b(\hat{\theta}_b) \\
&= \bar{\phi}_b(\hat{\theta}_b)\hat{\theta}_b - \bar{t}_b(\hat{\theta}_b) - (\hat{\theta}_b - \theta_b)\bar{\phi}_b(\hat{\theta}_b) \\
&= \bar{\phi}_b(\hat{\theta}_b)\theta_b - \bar{t}_b(\hat{\theta}_b) = U(\hat{\theta}_b|\theta_b) \\
\Rightarrow U(\theta_b) &\geq U(\hat{\theta}_b|\theta_b),
\end{aligned}
$$

where in the last inequality we obtained the desired result. The symmetric argument holds for $\theta_b > \hat{\theta}_b$, so we are done for the buyer.

Now, consider $\theta_s < \hat{\theta}_s$. Then,

$$
\begin{aligned}
\bar{\phi}_s(\theta_s) &\geq \bar{\phi}_s(\hat{\theta}_s) && \because \bar{\phi}_s(\cdot) \text{ nonincreasing} \\
\int_{\theta_s}^{\hat{\theta}_s} \bar{\phi}_s(x)dx &\geq \int_{\theta_s}^{\hat{\theta}_s} \bar{\phi}_s(\hat{\theta}_s)dx \\
U_s(\theta_b) - U_s(\hat{\theta}_b) &\geq (\hat{\theta}_s - \theta_s)\bar{\phi}_s(\hat{\theta}_s) && \because \text{ second condition} \\
U_s(\theta_s) &\geq U(\hat{\theta}_s) + (\hat{\theta}_s - \theta_s)\bar{\phi}_s(\hat{\theta}_s) \\
&= \bar{t}_s(\hat{\theta}_s) - \bar{\phi}_s(\hat{\theta}_s)\hat{\theta}_s + (\hat{\theta}_s - \theta_s)\bar{\phi}_s(\hat{\theta}_s) \\
&= \bar{t}_s(\hat{\theta}_s) - \bar{\phi}_s(\hat{\theta}_s)\theta_s = U(\hat{\theta}_s|\theta_s) \\
\Rightarrow U(\theta_s) &\geq U_s(\hat{\theta}_s|\theta_s).
\end{aligned}
$$

Again, the symmetric argument holds for $\theta_s > \hat{\theta}_s$, so we are done.

■

Necessity (IC ⇒1,2):

Proof. To avoid confusion when taking partial derivatives, we will switch notation: $U_i \Rightarrow U^i$ for $i \in \{b, s\}$. For truth-telling be optimal, we have the following FOC conditions:

$$\begin{aligned}
0 &= U_1^b(\theta_b|\theta_b) = \bar{\phi}_b'(\theta_b)\theta_b - \bar{t}_b'(\theta_b) && \forall \theta_b \in [0,2] \\
0 &= U_1^s(\theta_s|\theta_s) = \bar{t}_s'(\theta_s) - \bar{\phi}_s'(\theta_s)\theta_s && \forall \theta_s \in [0,2]
\end{aligned}$$

Totally differentiating above:

$$\begin{aligned}
0 &= U_{11}^b(\theta_b|\theta_b) + U_{12}^b(\theta_b|\theta_b) \\
&\leq \bar{\phi}_b'(\theta_b) && \because U^b(\cdot|\cdot),\ \text{concave in first argument} \\
0 &= U_{11}^s(\theta_s|\theta_s) + U_{12}^s(\theta_s|\theta_s) \\
&\leq -\bar{\phi}_b'(\theta_b) && \because U^s(\cdot|\cdot),\ \text{convex in first argument,}
\end{aligned}$$

Hence, we have shown that $\bar{\phi}_b$ is nondecreasing and $\bar{\phi}_s$ nonincreasing. To show the second condtion, note that by envelope theorem,

$$\begin{aligned}
\frac{d}{d\theta_b} U^b(\theta_b) &= \underbrace{U_1^b(\theta_b|\theta_b)}_{=0} + U_2^b(\theta_b|\theta_b) \\
&= U_2^b(\theta_b|\theta_b) \\
&= \phi_b(\theta_b) \geq 0, \\
\frac{d}{d\theta_b} U^s(\theta_s) &= \underbrace{U_1^s(\theta_s|\theta_s)}_{=0} + U_2^s(\theta_s|\theta_s) \\
&= U_2^s(\theta_s|\theta_s) \\
&= -\phi_s(\theta_s) \leq 0.
\end{aligned}$$

Since expected utility is increasing (decreasing) in type for buyer (seller), the optimal mechanism should only bind the lowest type buyer's (highest type seller's) IR to ensure that the IR constraints for all other types are satisfied. Hence, integrating both sides yields of the above equation (second equality) for both buyer and seller yields the desired equations:

$$\begin{aligned}
U_b(\theta_b) &= U_b(0) + \int_0^{\theta_b} \bar{\phi}_b(x)dx \\
U_s(\theta_s) &= U_s(2) + \int_{\theta_s}^2 \bar{\phi}_s(x)dx.
\end{aligned}$$

■

Problem 4.2. Using your answer in (a), compute $E[U_b(\theta_b)]$ and $E[U_s(\theta_s)]$ as functions of $\phi(\theta_b, \theta_s)$, $U_b(0)$, and $U_s(2)$.

Solution. Using the expressions obtained in part (a):

$$\begin{aligned}
U_b(\theta_b) &= U_b(0) + \int_0^{\theta_b} \bar{\phi}_b(x)dx \\
\Rightarrow E[U_b(\theta_b)] &= E_{\theta_b}[U_b(\theta_b)] && \because \theta_b \perp\!\!\!\perp \theta_s \\
&= U_b(0) + \int_0^2 \left(\int_0^{\theta_b} \bar{\phi}_b(x)dx\right) f_b(\theta_b)d\theta_b \\
&= U_b(0) + \int_0^2 \left(\int_x^2 f_b(\theta_b)d\theta_b\right) \bar{\phi}_b(x)dx && \because \text{changing integration order} \\
&= U_b(0) + \int_0^2 (1 - F_b(x))\bar{\phi}_b(x)dx \\
&= U_b(0) + \int_0^2 \frac{1 - F_b(x)}{f_b(x)} \bar{\phi}_b(x) f_b(x)dx \\
&= U_b(0) + E\left[\frac{1 - F_b(\theta_b)}{f_b(\theta_b)} \bar{\phi}_b(\theta_b)\right] \\
&= U_b(0) + E\left[\frac{1 - F_b(\theta_b)}{f_b(\theta_b)} E_{\theta_s}[\phi(\theta_b, \theta_s)]\right] \\
&= U_b(0) + E\left[\frac{1 - F_b(\theta_b)}{f_b(\theta_b)} \phi(\theta_b, \theta_s)\right] = U_b(0) + E[(2 - \theta_b)\phi(\theta_b, \theta_s)] && \because \theta_b \sim unif[0, 2].
\end{aligned}$$

Similarly,

$$\begin{aligned}
U_s(\theta_s) &= U_s(2) + \int_{\theta_s}^2 \bar{\phi}_s(x)dx \\
\Rightarrow E[U_s(\theta_s)] &= E_{\theta_s}[U_s(\theta_s)] && \because \theta_b \perp\!\!\!\perp \theta_s \\
&= U_s(2) + \int_0^2 \left(\int_{\theta_s}^2 \bar{\phi}_s(x)dx\right) f_s(\theta_s)d\theta_s \\
&= U_s(2) + \int_0^2 \left(\int_0^x f_s(\theta_s)d\theta_s\right) \bar{\phi}_s(x)dx && \because \text{changing integration order} \\
&= U_s(2) + \int_0^2 \frac{F_s(x)}{f_s(x)} \bar{\phi}_s(x) f_s(x)dx \\
&= U_s(2) + E\left[\frac{F_s(\theta_s)}{f_s(\theta_s)} E_{\theta_b}[\phi(\theta_b, \theta_s)]\right] \\
&= U_s(2) + E\left[\frac{F_s(\theta_s)}{f_s(\theta_s)} \phi(\theta_b, \theta_s)\right] = U_s(2) + E[\theta_s \phi(\theta_b, \theta_s)] && \because \theta_b \sim unif[0, 2].
\end{aligned}$$

Problem 4.3. Using our result in (b), write the platform's objective entirely in terms of $\phi(\theta_b, \theta_s)$, $U_b(0)$, and $U_s(2)$.

Solution. The trading platform maximizes $E[t_b(\theta_b, \theta_s) - t_s(\theta_b, \theta_s)]$. Notice that, because θ_b and θ_s are independently distributed,

$$E[t_b(\theta_b, \theta_s)] = E[\bar{t}_b(\theta_b)],$$
$$E[t_s(\theta_b, \theta_s)] = E[\bar{t}_s(\theta_s)].$$

Recall that

$$\bar{t}_b(\theta_b) = \bar{\phi}_b(\theta_b)\theta_b - U_b(\theta_b),$$
$$\bar{t}_s(\theta_s) = U_s(\theta_s) + \bar{\phi}_s(\theta_s)\theta_s.$$

Substituting in the expressions obtained in part (b), we get:

$$\begin{aligned}
E[t_b(\theta_b, \theta_s) - t_s(\theta_b, \theta_s)] &= E\left[\bar{\phi}_b(\theta_b)\theta_b - U_b(\theta_b) - U_s(\theta_s) - \bar{\phi}_s(\theta_s)\theta_s.\right] \\
&= E\left[\phi(\theta_b, \theta_s)\theta_b - U_b(\theta_b) - U_s(\theta_s) - \bar{\phi}(\theta_b, \theta_s)\theta_s.\right] \\
&= -U_b(0) - U_s(2) \\
&+ E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right].
\end{aligned}$$

Note that $\theta_b, \theta_s \sim unif[0, 2]$, so for $\forall \theta_i \in [0, 2]$ where $i \in \{b, s\}$,

$$F_i(\theta_i) = \frac{\theta_i}{2}$$
$$f_i(\theta_i) = \frac{1}{2}$$

Hence, the platform's objective becomes

$$\max_{\phi(\cdot)} \left\{ - U_b(0) - U_s(2) + E\left[\phi(\theta_b, \theta_s)(2(\theta_b - \theta_s) - 2)\right]\right\},$$

subject to $\bar{\phi}_b(\cdot)$ nondecreasing and $\bar{\phi}_s(\cdot)$ nonincreasing.

Problem 4.4. Solve for the profit-maximizing trading rule, ϕ.

Solution. Using the platform's objective function derived in part (c), the optimal trading rule, ϕ, would be such that allows trade whenever the pair (θ_b, θ_s) generates a positive expected profit, that is, when $2(\theta_b - \theta_s) - 2 \geq 0$. Hence, the optimal trading rule is given as:

$$\phi(\theta_b, \theta_s) = \begin{cases} 1 & \text{if } \theta_b - \theta_s \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4.5. How does your answer in (d) compare to the welfare-maximizing rule found in Exercise 6.

Solution. Compared to the welfare-maximizing rule found in Exercise 6, where trade occurs when $\theta_b - \theta_s \geq \frac{1}{2}$, the trade wedge under the profit-maximizing rule is larger ($\theta_b - \theta_s \geq 1$).