Theory of Income
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(Fernando Alvarez)

Notes on Uncertainty
by Takuma Habu

University of Chicago

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For typos/comments, email me at takumahabu@uchicago.edu.

1 Risk aversion and certainty equivalents

1.1 Absolute and constant risk aversion utility functions

1.1.1 Definitions

Definition 1.1. (*Certainty equivalent*) Let x be a random variable and u the agent's utility function. A *certainty equivalent* of x, denoted $c_e(x)$, is characterised by

$$u(c_e) = \mathbb{E}[u(x)].$$

Hence, c_e is the sure (deterministic) amount of consumption that will be equivalent to a given random variable x.

Definition 1.2. *(Arrow-Pratt Coefficient of Absolute Risk Aversion)* This coefficient is a measure of the curvature of the utility function around the point x, and is given by

$$ra(x) = -\frac{u''(x)}{u'(x)}.$$

The higher the coefficient, the greater is the curvature and, hence, the more *risk averse* the agent is.

Definition 1.3. (*Risk tolerance*) The reciprocal of the Arrow-Pratt coefficient of absolute risk aversion is called the risk tolerance; i.e.

$$\tau(x) = \frac{1}{ra(x)}.$$

The risk tolerance index has the advantage of being measured in the same units as money (dollars). To see this, note that u' has units utility per dollar and u'' has unit utils. Hence,

$$\frac{u''(x)}{u'(x)} \overset{\text{unit}}{=} \frac{\text{util}/\$^2}{\text{util}/\$} = \frac{1}{\$} \Rightarrow \tau(x) \overset{\text{unit}}{=} \$.$$

As we saw in my last TA class (when we solve 2013/14 Midterm 1 Q2), risk tolerance also has the advantage of being the right measure for aggregating agents with constant absolute risk aversion!

Definition 1.4. (*Coefficient of Relative Risk Aversion*).

$$rra(x) = -\frac{u''(x)}{u'(x)}x.$$

Remark 1.1. Suppose that the agent's utility function exhibits constant relative risk aversion. Then, the agent's utility function exhibits *decreasing absolute risk aversion* (*DARA*). To see this observe that, in this case, we can write

$$r = ra(x)\,x,$$

where r is the coefficient of constant relative risk aversion. If r is fixed, then as x increases, $ra(x)$ must fall; i.e. the coefficient of absolute risk aversion must be falling as x increases. You will see that some macro models require DARA for the problems to be well-defined—e.g. Aiyagari's model which you will cover Stokey in Theory of Income II.

1.1.2 Deriving the functional forms

When you assume constant absolute risk aversion, you will often assume the functional form:

$$u(x) = -\exp[-Rx] = -\exp\left[-\frac{x}{T}\right],$$

where R is the coefficient of absolute risk aversion and T is the agent i's risk tolerance ($R = 1/T$).

If you instead assume constant relative risk aversion, you will often assume:

$$u(x) = \frac{x^{1-r} - 1}{1-r}.$$

But where do these come from?

Proposition 1.1. *The general form of utility function with constant absolute risk aversion is given by*

$$u(x) := B - A\exp[-Rx],$$

where B and $A > 0$ are some constants and $R \equiv ra(x)$.

Proof. Define $R \equiv ra(x) > 0$. Then, $u(\cdot)$ must satisfy the following differential equation:

$$R = -\frac{u''(x)}{u'(x)}.$$

We can write above as

$$\frac{d}{dx}[\ln u'(x)] = -R.$$

The solution is given by

$$\begin{aligned}
\ln u'(x) &= -\int R dx = -Rx + C_1 \\
\Rightarrow u'(x) &= \exp[C_1]\exp[-Rx] \\
\Rightarrow \int u'(x)\,dx &= \int \exp[C_1]\exp[-Rx]\,dx \\
\Rightarrow u(x) &= C_2 - \frac{\exp[C_1]}{R}\exp[-Rx],
\end{aligned}$$

where C_1 and C_2 are some constants of integration. Define

$$B := C_2,\ A := \frac{\exp[C_1]}{R} > 0$$

and we are done. ■

Proposition 1.2. *The general form of utility function with constant relative risk aversion is given* *by*

$$u(x) \coloneqq A\frac{x^{1-r}}{1-r} + B,$$

where $A > 0$ and B are constants and $r \equiv rra(x)$. Moreover, if $r = 1$, then

$$u(x) \coloneqq A \ln x + B.$$

Proof. Let $r \equiv rra(x) > 0$. Then, $u(\cdot)$ must satisfy the following differential equation:

$$r = -\frac{u''(x)}{u'(x)}x.$$

Rewrite above as

$$\begin{aligned}
\frac{d}{dx}[\ln u'(x)] &= -\frac{r}{x} \\
\Rightarrow \ln u'(x) &= -r\int \frac{1}{x}dx \\
&= -r(\ln x + C_1) \\
&= \ln x^{-r} - rC_1 \\
\Rightarrow u'(x) &= x^{-r}\exp[-rC_1] \\
\Rightarrow \int u'(x)\,dx &= \exp[-rC_1]\int x^{-r}dx \\
\Rightarrow u(x) &= \exp[-rC_1]\left(\frac{1}{1-r}x^{1-r} + C_2\right) \\
&= \exp[-rC_1]\frac{1}{1-r}x^{1-r} + \exp[-rC_1]C_2.
\end{aligned}$$

Hence, a general form of utility function with constant relative risk aversion is given by

$$u(x) = A\frac{x^{1-\alpha}}{1-\alpha} + B,$$

where $A > 0$ and B are constants.

If $\alpha = 1$, then

$$\begin{aligned}
u'(x) &= \frac{1}{x}\exp[-C_1] \\
\Rightarrow \int u'(x)\,dx &= \exp[-C_1](\ln x + C_2) \\
\Rightarrow u(x) &= \exp[-C_1]\ln x + \exp[-C_1]C_2.
\end{aligned}$$

So the expression is

$$u(x) = A\ln x + B,$$

where $A > 0$ and B are again constants. ■

Remark 1.2. We said earlier that with relative risk aversion, people often assume

$$u(x) = \frac{x^{1-r} - 1}{1-r}, \tag{1.1}$$

which of course is consistent with the general form we derived ($A \coloneqq 1$, $B \coloneqq -1/(1-r)$). But why do we we simply not set $B = 0$? We do this to include the special case in which $r = 1$ and the utility is log. To see this, take the limit of $u(x)$ above as r tends to 1:

$$\begin{aligned} \lim_{r\to 1} \frac{x^{1-r}-1}{1-r} &= \lim_{r\to 1} \frac{e^{(1-r)\ln x}-1}{1-r} \\ \text{[L'Hôpital]} &= \lim_{r\to 1} \frac{-(\ln x)\, e^{(1-r)\ln x}}{-1} \\ &= \ln x. \end{aligned}$$

Without the -1 in the numerator, we won't have $0/0$ when we set $r = 1$ so that we won't be able to use L'Hôpital's rule. Put differently, (1.1) is a generalisation of log utility.

1.2 Log normal distribution

You will see this all of the time in the first year: e.g. Price Theory II/III, Empirical Analysis II. So better convince yourself of this result!

Proposition 1.3. *Suppose X is log normally distributed with mean μ and variance σ^2; i.e.*

$$\ln X \sim N\left(\mu, \sigma^2\right).$$

Then,

$$\mathbb{E}\left[X^{-\gamma}\right] = \exp\left[-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right].$$

Remark 1.3. It's good to remember in terms of X to the power of $-\gamma$ just so you don't get confused by the signs.

Proof. Define $x \coloneqq \ln X$. Then,

$$\begin{aligned}
\mathbb{E}\left[X^{-\gamma}\right] &= \mathbb{E}\left[\exp\left(\ln\left(X^{-\gamma}\right)\right)\right] = \mathbb{E}\left[\exp\left[-\gamma x\right]\right] \\
&= \int_{-\infty}^{\infty} \exp\left[-\gamma x\right] \phi\left(x\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\gamma x\right] \exp\left[-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{2\sigma^2\gamma x + x^2 + \mu^2 - 2x\mu}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2 + 2\left(-\mu + \sigma^2\gamma\right)x + \mu^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\left(x + \left(-\mu + \sigma^2\gamma\right)\right)^2 - \left(-\mu + \sigma^2\gamma\right)^2 + \mu^2}{2\sigma^2}\right] dx \\
&= \exp\left[-\frac{-\left(\mu - \sigma^2\gamma\right)^2 + \mu^2}{2\sigma^2}\right] \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\left(x - \left(\mu - \sigma^2\gamma\right)\right)^2}{2\sigma^2}\right] dx}_{=1} \\
&= \exp\left[-\frac{-\mu^2 - \sigma^4\gamma^2 + 2\mu\sigma^2\gamma + \mu^2}{2\sigma^2}\right] \\
&= \exp\left[-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right].
\end{aligned}$$

■

Now why is this handy?

Proposition 1.4. *Suppose that the agent has constant absolute risk aversion utility function:*

$$u\left(x\right) \coloneqq -\exp\left[-\frac{x}{T}\right],$$

where T is the risk tolerance. Let $x \sim N\left(\mu, \sigma^2\right)$. Then,

$$c_e\left(x\right) = \mu - \frac{1}{2}\frac{\sigma^2}{T}.$$

Proof. Define $X \coloneqq -u\left(x\right)$. Since $x \sim N\left(\mu, \sigma^2\right)$,

$$\begin{aligned}
\ln\left(X\right) &= \ln\left(-u\left(x\right)\right) = \ln\left(\exp\left[-\frac{x}{T}\right]\right) \\
&= -\frac{x}{T} \\
&\sim N\left(-\frac{\mu}{T}, \frac{\sigma^2}{T^2}\right).
\end{aligned}$$

Then, by the proposition above,

$$\mathbb{E}\left[X\right] = \exp\left[-\frac{\mu}{T} + \frac{1}{2}\frac{\sigma^2}{T^2}\right].$$

Now, recall the definition of c_e:

$$\begin{aligned}
u\left(c_e\left(x\right)\right) &= \mathbb{E}\left[u\left(x\right)\right] \\
\Leftrightarrow -u\left(c_e\left(x\right)\right) &= \mathbb{E}\left[-u\left(x\right)\right] \\
\Leftrightarrow \exp\left[-\frac{c_e\left(x\right)}{T}\right] &= \exp\left[-\frac{\mu}{T} + \frac{1}{2}\frac{\sigma^2}{T^2}\right] \\
\Leftrightarrow c_e\left(x\right) &= \mu - \frac{1}{2}\frac{\sigma^2}{T}.
\end{aligned}$$

■

Remark 1.4. $\sigma^2/2T$ we found above is what Fernando calls the absolute insurance premium.

Remark 1.5. By Proposition 1.4, we can work with certainty equivalents or with $u\left(x\right)$ directly when maximising. Moreover, together with our observation that we have aggregation when agents have constant absolute risk aversion utility functions (with possibly different coefficient of absolute risk aversions), this result also tells us that, rather than maximising the aggregate utility function (who's risk tolerance is simply the sum of everybody's risk tolerances), we can maximise the certainty equivalent, which is sometimes easier!

Proposition 1.5. *Suppose that the agent has constant relative risk aversion utility function:*

$$u\left(x\right) \coloneqq \frac{x^{1-r}-1}{1-r},\ r>0$$

where γ is the coefficient of relative risk aversion. Let $\ln x \sim N\left(\mu, \sigma^2\right)$. Then,

$$c_e\left(x\right) = \exp\left[\mu + \frac{1}{2}\left(1-r\right)\sigma^2\right].$$

Proof. The certainty equivalent in this case is characterised by

$$\begin{aligned}
\frac{\left(c_e\left(x\right)\right)^{1-r}-1}{1-r} &= \mathbb{E}\left[\frac{x^{1-r}-1}{1-r}\right] \\
\Leftrightarrow \left(c_e\left(x\right)\right)^{1-r} &= \mathbb{E}\left[x^{1-r}\right]
\end{aligned}$$

Then, setting $-\gamma = 1-r$ in Proposition 1.3, we obtain

$$\mathbb{E}\left[x^{1-r}\right] = \exp\left[\left(1-r\right)\mu + \frac{1}{2}\left(1-r\right)^2\sigma^2\right].$$

Hence,

$$\begin{aligned}
c_e\left(x\right) &= \left(\exp\left[\left(1-r\right)\mu + \frac{1}{2}\left(1-r\right)^2\sigma^2\right]\right)^{\frac{1}{1-r}} \\
&= \exp\left[\mu + \frac{1}{2}\left(1-r\right)\sigma^2\right].
\end{aligned}$$

■

As an aside, we show below that, with quadratic utility, only the mean and the variance matter so this gives us more freedom for distribution of x.

Proposition 1.6. *Suppose that the agent has a quadratic utility function:*

$$u(x) \coloneqq x - \frac{\alpha}{2}x^2.$$

Let $\mathbb{E}[x] = \mu$ *and* $\mathrm{Var}[x] = \sigma^2$*. Then,*

$$c_e(x) = \frac{1 \pm \sqrt{(1-\alpha\mu)^2 + \alpha^2\sigma^2}}{\alpha}.$$

Proof. The certainty equivalent in this case is characterised by

$$\begin{aligned}
c_e(x) - \frac{\alpha}{2}(c_e(x))^2 &= \mathbb{E}\left[x - \frac{\alpha}{2}x^2\right] \\
&= \mathbb{E}[x] - \frac{\alpha}{2}\left(\mathrm{Var}[x] + \mathbb{E}[x]^2\right) \\
&= \mu - \frac{\alpha}{2}\left(\sigma^2 + \mu^2\right) \\
\Leftrightarrow 0 &= \frac{\alpha}{2}(c_e(x))^2 + (-1)\,c_e(x) + \left[\mu - \frac{\alpha}{2}\left(\sigma^2 + \mu^2\right)\right].
\end{aligned}$$

Using the quadratic formula,

$$\begin{aligned}
c_e(x) &= \frac{1 \pm \sqrt{1 - 2\alpha\left[\mu - \frac{\alpha}{2}\left(\sigma^2 + \mu^2\right)\right]}}{\alpha} \\
&= \frac{1 \pm \sqrt{1 - 2\alpha\mu + \alpha^2\sigma^2 + \alpha^2\mu^2}}{\alpha} \\
&= \frac{1 \pm \sqrt{(1-\alpha\mu)^2 + \alpha^2\sigma^2}}{\alpha}.
\end{aligned}$$

■

2 Midterm question

Remember this from the midterm?

> Recall that we argued that [reciprocal of the curvature of $U_i(c)$] describes the slope of the Pareto optimal allocations with respect to different values of the aggregate endowment.

Let's go through the maths to confirm this.[1]

Fix $\{\overline{e}_s\}_{s\in\mathcal{S}}$. The social's planner's problem is:

$$\begin{aligned}\max_{\{x_s^i\}} \quad & \sum_{i\in\mathcal{I}} \lambda_i \left(\sum_{s\in\mathcal{S}} v_i\left(x_s^i\right)\pi_s\right)\\ s.t. \quad & \sum_{i\in\mathcal{I}} x_s^i = \overline{e}_s,\ \forall s\in\mathcal{S}.\end{aligned}$$

The first-order condition with respect to x_s^i is

$$\lambda_i v_i'\left(x_s^i\right)\pi_s - \gamma_s\left(\overline{e}_s\right) = 0,\ \forall s\in\mathcal{S},\ \forall i\in\mathcal{I},$$

Rearrange above and write

$$\lambda_i v_i'\left(x_s^i\right) = \frac{\gamma_s\left(\overline{e}_s\right)}{\pi_s} =: p_s\left(\overline{e}_s\right),$$

where p_s will be proportional to the Arrow-Debrue price of consumption in state s. Recall the aggregation result above that we can write $x_s^i = g_i\left(\overline{e}_s\right)$, where g_i is an increasing function. Thus, we can write the first-order condition as

$$\lambda_i v_i'\left(g_i\left(\overline{e}_s\right)\right) = \frac{\gamma_s\left(\overline{e}_s\right)}{\pi_s} = p_s\left(\overline{e}_s\right),\ \forall s\in\mathcal{S},\ \forall i\in\mathcal{I}.$$

Since this holds for all s and i and any $\{\overline{e}_s\}_{s\in\mathcal{S}}$, it is, in fact, an identity; i.e.

$$\lambda_i v_i'\left(g_i\left(\overline{e}_s\right)\right) \equiv p_s\left(\overline{e}_s\right).$$

Differentiating both sides with respect to $\overline{e}_s$ yields

$$\lambda_i v_i''\left(g_i\left(\overline{e}_s\right)\right) g_i'\left(\overline{e}_s\right) = p_s'\left(\overline{e}_s\right). \tag{2.1}$$

From the first-order condition, we have an expression for λ_i:

$$\lambda_i = \frac{p_s\left(\overline{e}_s\right)}{v_i'\left(x_s^i\right)} = \frac{p_s\left(\overline{e}_s\right)}{v_i'\left(g_i\left(\overline{e}_s\right)\right)}.$$

Substituting into (2.1) yields

$$\frac{p_s\left(\overline{e}_s\right)}{v_i'\left(g_i\left(\overline{e}_s\right)\right)} v_i''\left(g_i\left(\overline{e}_s\right)\right) g_i'\left(\overline{e}_s\right) = p_s'\left(\overline{e}_s\right),\ \forall s\in\mathcal{S},\ \forall i\in\mathcal{I}$$

[1]This is based on Q1 from the 2016/17 Final.

which we can rearrange as

$$\begin{aligned} g_i'(\overline{e}_s) &= \frac{v_i'(g_i(\overline{e}_s))}{v_i''(g_i(\overline{e}_s))}\frac{p_s'(\overline{e}_s)}{p_s(\overline{e}_s)},\ \forall s\in\mathcal{S},\ \forall i\in\mathcal{I} \\ &= -T_i(g_i(\overline{e}_s))\frac{p_s'(\overline{e}_s)}{p_s(\overline{e}_s)},\ \forall s\in\mathcal{S},\ \forall i\in\mathcal{I}, \end{aligned} \tag{2.2}$$

where T_i is the risk tolerance for agent i—equivalently, (minus) the reciprocal of the curvature of v_i—evaluated at $g_i(\overline{e}_s)$. So we are already getting there!

But let's worth with p_s'/p_s. From the the feasibility condition, we have

$$\sum_{i\in\mathcal{I}} g_i(\overline{e}_s) = \overline{e}_s,\ \forall s\in\mathcal{S},$$

which is an identity so that we may differentiate with respect to $\overline{e}_s$ to obtain:

$$\sum_{i\in\mathcal{I}} g_i'(\overline{e}_s) = 1.$$

Substituting (2.2) yields

$$\begin{aligned} 1 &= \sum_{i\in\mathcal{I}} -T_i(g_i(\overline{e}_s))\frac{p_s'(\overline{e}_s)}{p_s(\overline{e}_s)} \\ &= \frac{p_s'(\overline{e}_s)}{p_s(\overline{e}_s)}\sum_{i\in\mathcal{I}} -T_i(g_i(\overline{e}_s)) \\ \Leftrightarrow \frac{p_s'(\overline{e}_s)}{p_s(\overline{e}_s)} &= \frac{1}{\sum_{i\in\mathcal{I}} -T_i(g_i(\overline{e}_s))},\ \forall s\in\mathcal{S}. \end{aligned}$$

Hence, we can rewrite (2.2) as

$$g_i'(\overline{e}_s) = \frac{T_i(g_i(\overline{e}_s))}{\sum_{i\in\mathcal{I}} T_i(g_i(\overline{e}_s))}.$$

Does this imply aggregation? Not quite. Recall

$$\lambda_i = \frac{p_s(\overline{e}_s)}{v_i'(g_i(\overline{e}_s))}.$$

Hence, if λ_i (for a given $\{\overline{e}_s\}_{s\in\mathcal{S}}$) change, in general, we would expect $g_i'(\overline{e}_s)$ to change. In other words, we aren't guaranteed that we have aggregation without further restriction on the utility function.

Exercise 2.1. Suppose

$$v_i(x) := -\exp\left[-\frac{x}{\tau_i}\right],$$

where $\tau_i > 0$. What is the risk tolerance function, $T_i(x)$, in this case? What is the value of $-p'(\overline{e}_s)/p(\overline{e})$? What is the value of $g_i'(\overline{e}_s)$?

Exercise 2.2. Now suppose that

$$v_i(x) := \frac{\sigma}{1-\sigma}\left(\frac{x}{\sigma}+\tau_i\right)^{1-\sigma},$$

where the consumption feasibility set of the agent i is

$$\left\{x : \frac{x}{\sigma} + \tau_i > 0\right\}.$$

What is the risk tolerance function, $T_i(x)$, for this case. Show that it is a linear function of c with common slope but potentially different intercept. Is the risk tolerance increasing or decreasing in x? What does this depend on? Is the risk tolerance increasing or decreasing on τ_i?

Remark 2.1. I'll also upload some notes from Myerson's Price Theory III class that provides a different treatment of essentially the same problem (since you won't have Myerson this year!)

3 Uncertainty

3.1 Arrow-Debreu economy

In studying the Welfare Theorems (and aggregation), we set up a general equilibrium model in which we had different agents and different goods. We can introduce uncertainty into this world by relabelling the goods as *contingent goods*.

That is, instead of thinking of goods 1 and 2 as, say, ice cream and potatoes, we can think of these goods instead as: ice cream when the weather is hot and ice cream when the weather is cold. In other words, when we used x_ℓ^i to denote consumption of good ℓ for agent i, we can now think of there being just one good in the economy and ℓ indexing the different states of the world (i.e. weather it's hot or cold).[2]

To make this clearer, let's denote the different states of the world with $s \in \{1, 2, \ldots, m\} =: \mathcal{S}$. Then, the budget constraint of agent i is

$$\sum_{s=1}^{m} p_s x_s^i = \sum_{s=1}^{m} p_s e_s^i.$$

We refer to p_s under this interpretation as *state* or *Arrow-Debreu prices.* It is the price of an *Arrow security* that pays one unit of the numeraire in state s and zero otherwise. The budget constraint is therefore telling us that the agent can buy consumption contingent on the state, and they finance their consumption by selling their endowment contingent on state.

In effect, we are assuming here that agents trade contingent plans before the state is realised. Contingent plan is plan that specifies what the agent will do under all possible states of the world. Once the state is realised, everyone simply follows the contingent plan they made.

Let's define a competitive equilibrium in this setting.

Definition 3.1. (*Competitive equilibrium in AD economy*) A competitive equilibrium is an allocation $\{x_s^i\}$ and state prices $\{p_s\}$ such that:

(i) taking as given state prices, each agent $i \in \{1, 2, \ldots, I\}$ maximises his utility:

$$\max_{\{x_s^i\}_{s=1}^m} \quad \sum_{s \in \mathcal{S}}^{m} v^i\left(x_s^i\right)\pi_s$$
$$s.t. \quad \sum_{s=1}^{m} p_s x_s^i = \sum_{s=1}^{m} p_s e_s^i;$$

(ii) each contingent market clears:

$$\sum_{i=1}^{I} x_s^i = \sum_{i=1}^{I} e_s^i, \ \forall s \in \{1, 2, \ldots, m\}.$$

[2] More generally, we can index each good as $x_{\ell m t}^i$, which denotes the consumption by agent i of good ℓ in state m in period t. In this case, there will be price for each (ℓ, m, t) triple; i.e. $p_{\ell m t}$. For a treatment of this more general case, see Jehle & Reny (3rd edition), Chapter 5.4, although I would suggest that you wait until you go through this in Price Theory II in the Winter quarter with Reny.

Note that we've assumed expected utility so that

$$u^i\left(\left\{x_s^i\right\}_{s=1}^m\right) = \sum_{s=1}^m v^i\left(x_s\right)\pi_s,$$

where v^i is agent's i's subutility function and π_s is the (common) probability that state s is realised.

3.2 Security market economy

Although above set up allows us to use the Welfare Theorems and the aggregation results we studied already by a simple relabelling, you may not like it because there no securities that agents trade.

We now change the set up. Let's suppose that, instead of agents trading contingent plans at the beginning of time, they get to trade securities—each security $k \in \{1, 2, \ldots, K\}$ pays d_{ks} as dividends in state s. After agents trade securities, the state of the world is realised, dividends are paid, and the usual goods trading occurs.

We can think of this as a two-period model in which, in period 1, agents consume nothing and simply trade securities, and, in period 2, after the state of the world is realised, they trade goods and consume on the basis of their decisions in the previous period. This is a clever trick that allows us to abstract from intertemporal issues.

Since there are two "periods", we need two budget constraints: one for trading of securities, and another for trading of goods:

$$\sum_{k=1}^K h_k^i q_k = \sum_{k=1}^K \theta_k^i q_k,$$
$$x_s^i = \sum_{k=1}^K h_k^i d_{ks} + \hat{e}_s^i,$$

where q_k denotes the price of security k, h_k^i is the quantity of security k that agent i purchases (negative means selling), θ_k^i is the endowment of security k that agent i has, and $\hat{e}_s^i$ denotes the endowments of the good in state s for agent i. Observe that the budget constraint for the goods has no prices—because we only have one good (for each state)!

Just as we did before, let's define a competitive equilibrium in this setting.

Definition 3.2. (*Competitive equilibrium in security market economy*) A competitive equilibrium is an allocation $\{\{x_s^i\}, \{h_k^i\}\}$ and prices $\{q_k\}$ such that:

(i) taking as given prices, each agent $i \in \{1, 2, \ldots, I\}$ maximises his utility:

$$\begin{aligned}
\max_{\left\{x_s^i, h_k^i\right\}_{s=1}^m} \quad & \sum_{s\in\mathcal{S}}^m v^i\left(x_s\right)\pi_s \\
s.t. \quad \sum_{k=1}^K h_k^i q_k \quad & = \sum_{k=1}^K \theta_k^i q_k, \\
x_s^i & = \sum_{k=1}^K h_k^i d_{ks} + \hat{e}_s^i,\ \forall s \in \{1, 2, \ldots, m\}
\end{aligned}$$

(ii) each security market clears:

$$\sum_{i=1}^{I} h_k^i = \sum_{i=1}^{I} \theta_k^i,\ \forall k \in \{1, 2, \dots, K\}\,;$$

(iii) each contingent goods market clears:

$$\sum_{i=1}^{I} x_s^i = \sum_{i=1}^{I} \left(\hat{e}_s^i + \sum_{k=1}^{K} d_{ks}\theta_k^i \right),\ \forall s \in \{1, 2, \dots, m\}\,.$$

Definition 3.3. *(Incomplete and complete markets)* If the payoff matrix, $\{d_{ks}\}$, does not have full rank, we refer to the economy as one with *incomplete markets*. If $\{d_{ks}\}$ has full rank, we refer to the economy as having *complete markets*.

3.3 Relationship between AD economy and security market economy

This is straight from the notes and I omit the proofs!

3.3.1 Budget feasibility

We first establish the relationship between a budget feasible allocation in the security market and the AD economy.

Definition 3.4. *(Price consistency)* We say that security prices and payoffs $\{q_k, \{d_{ks}\}\}$ are *consistent* with state price $\{p_s\}$, if

$$q_k = \sum_{s=1}^{m} p_s d_{ks},\ \forall k \in \{1, 2, \dots, K\}\,.$$

Definition 3.5. *(Endowment equivalence)* The endowment $\{e_s^i\}$ and $\{\{\hat{e}_s^i\}, \theta^i\}$ are *equivalent* if

$$\hat{e}_s^i + \sum_{k=1}^{K} d_{ks}\theta_k^i = e_s^i,\ \forall s \in \{1, 2, \dots, m\}\,.$$

Proposition 3.1. *Suppose that: (i) $\{q_k, \{d_{ks}\}\}$ are consistent with state prices $\{p_s\}$; and (ii) endowments $\{e_s^i\}$ and $\{\{\hat{e}_s^i\}, \theta^i\}$ are* equivalent.

(i) If $\{\{x_s^i\}, \{h_k^i\}\}$ is budget feasible in the security market economy, then $\{x_s^i\}$ is budget feasible in the A-D economy.

(ii) If $\{x_s^i\}$ is budget feasible in the A-D economy, then it must be budget feasible in the security market economy, provided that the payoff matrix $\{d_{ks}\}$ has full rank.

3.3.2 Equilibrium

The following establishes the relationship between equilibrium in the security market with that of the A-D economy.

Proposition 3.2. *Assume that the endowments* $\{e_s^i\}$ *and* $\{\{\hat{e}_s^i\}, \theta^i\}$ *are equivalent.*

(i) If $\{\{x_s^i\}, \{h_k^i\}\}$ *clears the markets in the security market economy, then* $\{x_s^i\}$ *clears the markets in the A-D economy.*

(ii) Assume also that $\{d_{ks}\}$ *has full rank. If* $\{x_s^i\}$ *clears the market in the A-D economy, then* $\{\{x_s^i\}, \{h_k^i\}\}$ *clears the market in the security market economy.*