# Theory of Income
# Autumn 2018

## (Fernando Alvarez)

Notes on Convergence of
Dynamic Systems
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# Contents

For typos/comments, email me at takumahabu@uchicago.edu.

Version control:

v1.1 Fixed a typo in the postliminaries, one of the elements of $\mathbf{P}$ had a minus sign missing. Also, I had repeated the guess-and-verify method of solving differential equations so removed the duplication.

# 1 Dynamic programming cheat sheet

### Discrete time

#### State formulation

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right)$$
$$s.t. \quad x_{t+1} \in \Gamma\left(x_t\right), \ \forall t \geq 0,$$
$$x_0 \text{ given.}$$

*Necessary and sufficient conditions*[1]

Euler equation:

$$F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_t, x_{t+1}\right) = 0, \ \forall t \geq 0.$$

Transversality condition:

$$\lim_{T\to\infty} \beta^T F_x\left(x_T, x_{T+1}\right) x_T = 0.$$

#### Control-state formulation

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t h\left(x_t, u_t\right)$$
$$s.t. \quad x_{t+1} = g\left(x_t, u_t\right), \ \forall t \geq 0,$$
$$u_t \in U$$
$$x_0 \text{ given.}$$

We can return to the state formulation by letting

$$F\left(x, y\right) = \max\left\{h\left(x, u\right) : u \in U, y = g\left(x, u\right)\right\},$$
$$\Gamma\left(x\right) = \left\{y : \exists u \in U \ s.t. \ y = g\left(x, u\right)\right\}.$$

### Continuous time

#### State formulation

$$\max_{\{\dot{x}(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} F\left(x\left(t\right), \dot{x}\left(t\right)\right) dt$$
$$s.t. \quad \dot{x}\left(t\right) \in \Gamma\left(x\left(t\right)\right), \ \forall t \geq 0,$$
$$x_0 \text{ given.}$$

*Necessary and sufficient conditions*

Euler equation:

$$F_x + \rho F_{\dot{x}} = F_{\dot{x}x}\dot{x} + F_{\dot{x}\dot{x}}\ddot{x}, \ \forall t \geq 0$$

Transversality condition:

$$\lim_{T\to\infty} e^{-\rho T} F_{\dot{x}}\left(x\left(T\right), \dot{x}\left(T\right)\right) x\left(T\right) = 0.$$

#### Control-state formulation

$$\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h\left(x\left(t\right), u\left(t\right)\right)$$
$$s.t. \quad \dot{x}\left(t\right) = g\left(x\left(t\right), u\left(t\right)\right), \ \forall t \geq 0,$$
$$u\left(t\right) \in U$$
$$x_0 \text{ given.}$$

*Necessary and sufficient conditions*

First-order conditions:

$$H\left(x, u, \lambda\right) = h\left(x, u\right) + \lambda g\left(x, u\right),$$
$$H_u\left(x, u, \lambda\right) = 0,$$
$$\dot{\lambda} = \rho\lambda - H_x\left(x, u, \lambda\right),$$
$$\dot{x} = g\left(x, u\right).$$

Transversality condition:

$$\lim_{T\to\infty} e^{-\rho T} \lambda\left(T\right) x\left(T\right) = 0.$$

[1] $F$ concave and $C^1$.

## 2 Convergence cheat sheet: One dimension

### Discrete time

**Sequence problem:**

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right)$$
$$s.t. \quad x_{t+1} \in \Gamma\left(x_t\right), \ \forall t \geq 0,$$
$$x_0 \text{ given.}$$

**Euler equation:**

$$F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_t, x_{t+1}\right) = 0, \ \forall t \geq 0.$$

**Transversality condition:**

$$\lim_{T\to\infty} \beta^T F_x\left(x_T, x_{T+1}\right) x_T = 0.$$

**Linearisation and steady state:**

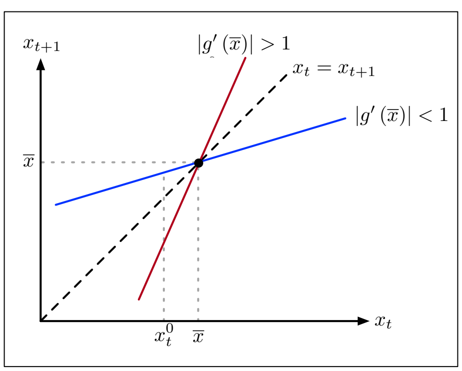
$$x_{t+1} = g\left(x\right) \simeq g\left(\overline{x}\right) + g'\left(\overline{x}\right)\left(x_t - \overline{x}\right).$$

At the steady state, $\overline{x} = g\left(\overline{x}\right)$. So,

$$x_{t+1} - \overline{x} \simeq g'\left(\overline{x}\right)\left(x_t - \overline{x}\right).$$

**Condition for convergence:**

$$\left|g'\left(\overline{x}\right)\right| < 1.$$




$|g'(\overline{x})| < 1$: $x_{t+1} > x_t^0 \Rightarrow$ move towards $\overline{x}$
$|g'(\overline{x})| > 1$: $x_{t+1} < x_t^0 \Rightarrow$ move away from $\overline{x}$
If $g'(x) < 0$, then oscillates.

### Continuous time

**Sequence problem:**

$$\max_{\{\dot{x}(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} F\left(x\left(t\right), \dot{x}\left(t\right)\right) dt$$
$$s.t. \quad \dot{x}\left(t\right) \in \Gamma\left(x\left(t\right)\right), \ \forall t \geq 0,$$
$$x_0 \text{ given.}$$

**Euler equation:**

$$F_x + \rho F_{\dot{x}} = F_{\dot{x}x}\dot{x} + F_{\dot{x}\dot{x}}\ddot{x}, \ \forall t \geq 0.$$

**Transversality condition:**

$$\lim_{T\to\infty} e^{-\rho T} F_{\dot{x}}\left(x\left(T\right), \dot{x}\left(T\right)\right) x\left(T\right) = 0.$$

**Linearisation and steady state:**

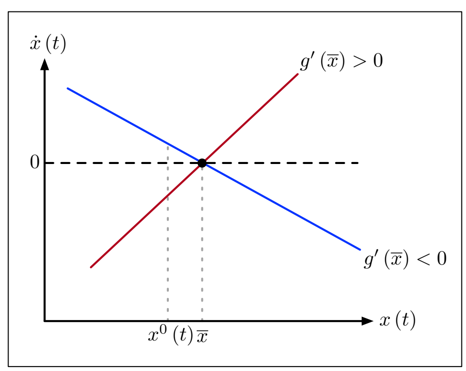
$$\dot{x}\left(t\right) = g\left(x\left(t\right)\right) \simeq g\left(\overline{x}\right) + g'\left(\overline{x}\right)\left(x - \overline{x}\right).$$

At steady state, $0 = g\left(\overline{x}\right)$. So,

$$\dot{x}\left(t\right) \simeq g'\left(\overline{x}\right)\left(x\left(t\right) - \overline{x}\right).$$

**Condition for convergence:**

$$g'\left(\overline{x}\right) < 0.$$




$g'(\overline{x}) < 0$: $\dot{x}(t) > x^0(t) \Rightarrow$ move towards $\overline{x}$
$g'(\overline{x}) > 0$: $\dot{x}(t) < x^0(t) \Rightarrow$ move away from $\overline{x}$
Cannot oscillate since $x(t)$ is continuous.

### Discrete time

**Obtaining $g'(x)$:**
Differentiate EE with respect to $x$ and evaluate at $\overline{x}$.

Quadratic $Q(\lambda)$ with $\lambda \coloneqq g'(\overline{x})$:

$$\tilde{Q}(\lambda) = \beta F_{xy}\lambda^2 + (F_{yy} + \beta F_{xx})\lambda + F_{yx} = 0.$$

Almost-reciprocal roots:

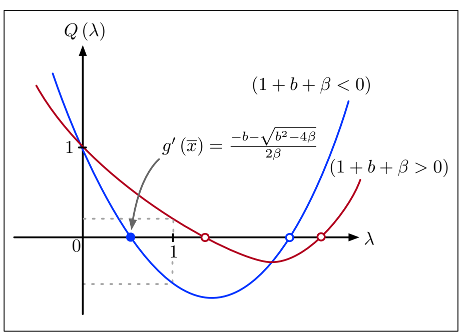
$$\tilde{Q}(\lambda_1) = 0 \Rightarrow \lambda_2 = \frac{1}{\lambda_1 \beta}.$$

If $|\lambda_1| < 1$ , then $|\lambda_2| > 1$.
Plot: Assume $F_{xy} > 0$, $F$ concave.

$$Q(\lambda) = \beta\lambda^2 + b\lambda + 1,$$

where $b \coloneqq (F_{yy} + \beta F_{xx})/F_{xy} < 0$. Note:

$$\begin{aligned} Q(0) &= 1, \\ Q(1) &= 1 + b + \beta, \\ Q'(0) &= b < 0, \\ Q''(\lambda) &= 2\beta > 0, \end{aligned}$$




Roots:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\beta}}{2\beta}.$$

$1 + b + \beta < 0 \Rightarrow 0 < \lambda_1 < 1 < \lambda_2$. Use "$-$".
$1 + b + \beta > 0 \Rightarrow 1 < \lambda_1 < \lambda_2$ . Locally unstable.

**Speed of convergence:**

$$x_t - \overline{x} = (g'(\overline{x}))^t (x_0 - \overline{x}).$$

$g'(x)$ closer to zero $\Rightarrow$ faster convergence
Lower $b \Rightarrow$ Lower $\lambda_1 \Rightarrow$ Faster convergence.

### Continuous time

**Obtaining $g'(x)$:**
Differentiate EE with respect to $x$ and evaluate at $\overline{x}$.

Quadratic $Q(\lambda)$ with $\lambda \coloneqq g'(\overline{x})$:

$$Q(\lambda) = -F_{\dot{x}\dot{x}}\lambda^2 + \rho F_{\dot{x}\dot{x}}\lambda + (F_{xx} + \rho F_{\dot{x}x}) = 0.$$

Almost-reciprocal roots:

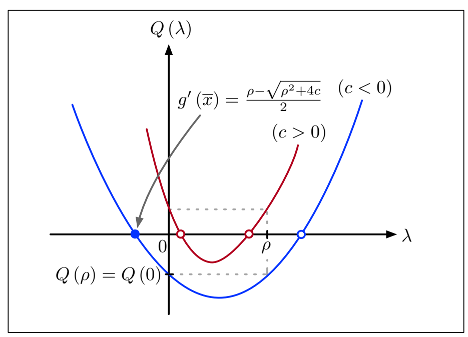
$$Q(\lambda_1) = 0 \Rightarrow \lambda_2 = -\lambda_1 + \rho.$$

If $\lambda_1 < 0$, then $\lambda_2 > 0$.
Plot: Assume $F$ concave.

$$\tilde{Q}(\lambda) = -F_{\dot{x}\dot{x}}\left(\lambda^2 - \rho\lambda - c\right),$$

where $c \coloneqq (F_{xx} + \rho F_{\dot{x}x})/F_{\dot{x}\dot{x}}$. Note:

$$\begin{aligned} Q(0) &= F_{xx} + \rho F_{\dot{x}x}, \\ Q(\rho) &= Q(0), \\ Q'(0) &= F_{\dot{x}\dot{x}}\rho < 0, \\ Q''(\lambda) &= -2F_{\dot{x}\dot{x}} > 0. \end{aligned}$$




Roots:

$$\lambda = \frac{\rho \pm \sqrt{\rho^2 + 4c}}{2}.$$

$c > 0 \Rightarrow \lambda_1 < 0 < \rho < \lambda_2$. Use "$-$".
$c < 0 \Rightarrow 0 < \lambda_1 < \lambda_2 < \rho$. Locally unstable.

**Speed of convergence:**

$$x(t) = \overline{x} + (\overline{x} - x_0)\exp[g'(\overline{x})t]$$

More negative $g'(x) \Rightarrow$ faster convergence
Higher $c \Rightarrow$ Lower $\lambda_1 \Rightarrow$ Faster convergence.

# 3 Convergence cheat sheet: Multiple dimensions

**Discrete time**

**Dynamic system:**

$$\mathbf{x}_{t+1} = m(\mathbf{x}_t), \forall t \geq 0,$$

where $\mathbf{x} \in \mathbb{R}^n$ and $m : \mathbb{R}^n \to \mathbb{R}^n$.

**Linearisation and steady state:**

$$\mathbf{x}_{t+1} \simeq m(\overline{\mathbf{x}}) + m'(\overline{\mathbf{x}})(\mathbf{x}_t - \overline{\mathbf{x}}).$$

At the steady state, $\overline{\mathbf{x}} = m(\overline{\mathbf{x}})$. So,

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t,$$

where $\mathbf{y}_t \coloneqq \mathbf{x}_t - \overline{\mathbf{x}}$ and $\mathbf{A}$ is Jacobian $m'(\overline{\mathbf{x}})$.

Diagonalise $\mathbf{A}$ :

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P},$$

where $\mathbf{\Lambda}$ are eigenvalues (diagonal) and $\mathbf{P}$ eigenvectors. Then,

$$\mathbf{z}_{t+1} = \mathbf{\Lambda}\mathbf{z}_t = \mathbf{\Lambda}^{t+1}\mathbf{z}_0,$$

where $\mathbf{z}_t \coloneqq \mathbf{P}\mathbf{y}_t$.

**Stability:**

For any $|\lambda_i| \geq 1$, $z_{i,0} = 0$.

For any $|\lambda_i| < 1$, $z_{i,0}$ arbitrary.

**Speed of convergence:**

For $|\lambda_i| < 1$, faster convergence if $\lambda_i$ is closer to zero.

**Slope of saddle path:**

Let $n = 2$, $|\lambda_1| < 1$, $|\lambda_2| \geq 1$.

$$\begin{pmatrix} \lambda^t z_{1,0} \\ 0 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$$

$$\Rightarrow 0 = p_{21}y_{1,t} + p_{22}y_{2,t}$$

$$\Leftrightarrow y_{1,t} = -\frac{p_{22}}{p_{21}}y_{2,t}.$$

So slope is $-p_{22}/p_{21}$; i.e. depends on eigenvectors.

For $n > 2$, group the eigenvalues to stable and and unstable.

**Continuous time**

**Dynamic system:**

$$\dot{\mathbf{x}}(t) = m(\mathbf{x}(t)), \ \forall t \geq 0.$$

where $\mathbf{x} \in \mathbb{R}^n$ and $m : \mathbb{R}^n \to \mathbb{R}^n$.

**Linearisation and steady state:**

$$\dot{\mathbf{x}}(t) \simeq m'(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*).$$

At the steady state, $\mathbf{0} = m(\overline{\mathbf{x}})$. So,

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t),$$

where $\mathbf{y}(t) \coloneqq \mathbf{x}(t) - \overline{\mathbf{x}}$ and $\mathbf{A}$ is Jacobian $m'(\overline{\mathbf{x}})$.

Diagonalise $\mathbf{A}$ :

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P},$$

where $\mathbf{\Lambda}$ are eigenvalues (diagonal) and $\mathbf{P}$ eigenvectors. Then,

$$\dot{\mathbf{z}}(t) = \mathbf{\Lambda}\mathbf{z}(t) = e^{\mathbf{\Lambda}t}\mathbf{z}(0),$$

where $\mathbf{z}(t) \coloneqq \mathbf{P}\mathbf{y}(t)$.

**Stability:**

For any $\lambda_i \geq 0$, $z_i(0) = 0$.

For any $\lambda_i < 0$, $z_{i,0}$ arbitrary.

**Speed of convergence:**

For $\lambda_i < 0$, faster convergence if $\lambda_i$ is more negative.

**Slope of saddle path:**

Let $n = 2$, $\lambda_1 < 0$, $\lambda_2 \geq 0$.

$$\begin{pmatrix} e^{\lambda_1 t} z_1(0) \\ 0 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\Rightarrow 0 = p_{21}y_1(t) + p_{22}y_2(t)$$

$$\Leftrightarrow y_1(t) = -\frac{p_{22}}{p_{21}}y_2(t).$$

So slope is $-p_{22}/p_{21}$; i.e. depends on eigenvectors.

For $n > 2$, group the eigenvalues to stable and and unstable.

# 4 Neoclassical growth model

We take the case when labour supply is perfectly inelastic—this means, in particular, that consumers do not face a trade off between consumption and leisure, which simplifies the problem.

## 4.1 Discrete time

In the neoclassical growth model, output can be used for consumption and investment:

$$C_t + I_t = G(k_t, 1)\, l$$

and the next-period capital is given by

$$k_{t+1} = k_t (1 - \delta) + I_t.$$

$G(\cdot, \cdot)$ is a neoclassical constant returns production function (i.e. satisfies Inada conditions) and $\delta$ is the depreciation rate. Combining the two gives the law of motion in terms o consumption and capital

$$\begin{aligned} C_t &= G(k_t, 1) + k_t (1 - \delta) - k_{t+1} \\ &= f(k_t) - k_{t+1}, \end{aligned}$$

where we defined $f(k) \coloneqq G(k, 1) + (1 - \delta) k$. The consumer then chooses an infinite sequence of $k_{t+1}$ to maximise his utility (in any given period $t$, $k_t$ is given). Thus, the neoclassical growth model takes the following from

$$\begin{aligned} V^*(k_0) \coloneqq \max_{\{k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ s.t. \quad & 0 \leq k_{t+1} \leq f(k_t), \\ & k_0 \text{ given.} \end{aligned}$$

In setting the lower bound for $k_{t+1}$ to be zero, we are implicitly assuming that capital can be dismantled at no cost and consumed. If capital is irreversible, then the lower bound for $k_{t+1}$ should be set to $k_t$.

We can fit the problem above into our general notation:

$$\begin{aligned} F(x, y) &= U(f(x) - y), \\ \Gamma(x) &= [0, f(x)]. \end{aligned}$$

### 4.1.1 EE and TC

We have

$$\begin{aligned} F_x(x, y) &= U'(f(x) - y) f'(x), \\ F_y(x, y) &= -U'(f(x) - y), \end{aligned}$$

so that the Euler equations for the neoclassical growth model is given by:

$$\left[-U'\left(f\left(k_t\right)-k_{t+1}\right)\right]+\beta\left[U'\left(f\left(k_{t+1}\right)-k_{t+2}\right)f'\left(k_{t+1}\right)\right]=0,\ \forall t\geq 0.$$

The Transversality Condition is given by

$$\lim_{t\to\infty}\beta^t U'\left(f\left(k_t\right)-k_{t+1}\right)f'\left(k_t\right)k_t=0.$$

### 4.1.2 Steady state

The steady state solves

$$\begin{aligned}\beta U'\left(f\left(\bar{k}\right)-\bar{k}\right)f'\left(\bar{k}\right)&=U'\left(f\left(\bar{k}\right)-\bar{k}\right)\\ \Leftrightarrow f'\left(\bar{k}\right)&=\frac{1}{\beta}.\end{aligned}$$

Note that $f'(k)$ equals the capital rental rate and so this equation tells us that the rental rate constant in the steady state. That is, capital is supplied perfectly elastically in the steady state.

### 4.1.3 Speed of convergence

Since

$$F\left(x,y\right)=U\left(f\left(x\right)-y\right),$$

we have

$$\begin{aligned}F_x&=f'\left(x\right)U'\left(f\left(x\right)-y\right)\\ F_y&=-U'\left(f\left(x\right)-y\right)\\ F_{xx}&=f''\left(x\right)u'\left(f\left(x\right)-y\right)+\left[f'\left(x\right)\right]^2U''\left(f\left(x\right)-y\right)\\ F_{yy}&=U''\left(f\left(x\right)-y\right)\\ F_{xy}&=-f'\left(x\right)U''\left(f\left(x\right)-y\right).\end{aligned}$$

Recall that the steady-state capital solves $1 = \beta f'(k^*)$ . Evaluating these at the steady-state values and substituting into the derivative of the Euler equation with respect to $x$ gives

$$\begin{aligned}
0 &= F_{yx} + F_{yy} g' + \beta \left( F_{xx} g' + F_{xy} (g')^2 \right) \\
&= -f'U'' + U'' g' + \beta \left( \left( f'U' + (f')^2 U'' \right) g' - f'U'' (g')^2 \right) \\
&= -f'U'' + \left( U'' + \beta f'' Y' + \beta (f')^2 U'' \right) g' - \beta f' U'' (g')^2 \\
&= -U'' \left[ f' - \left( 1 + \beta f'' \frac{U'}{U''} + \beta (f')^2 \right) g' + \beta f' (g')^2 \right] \\
&= -U'' \left[ \frac{1}{\beta} - \left( 1 + \frac{1}{\beta} + \beta f'' \frac{U'}{U''} \right) g' + (g')^2 \right] \because f' = 1/\beta \\
&= -U'' \left[ \frac{1}{\beta} - \left( 1 + \frac{1}{\beta} + \frac{f''}{f'} \frac{U'}{U''} \right) g' + (g')^2 \right] \because \beta = 1/f' \\
&= -U'' \left[ \frac{1}{\beta} - \left( 1 + \frac{1}{\beta} + \left( \frac{f''}{f'} / \frac{U''}{U'} \right) \right) g' + (g')^2 \right].
\end{aligned}$$

In this discrete-time case, we notice that the expression depends on the curvature of the production function, $f''/f'$ (related to the elasticity of production) and the curvature of the utility function, $U''/U'$ (related to the elasticity of intertemporal substitution).

The quadratic equation is given by

$$Q(\lambda) = \lambda^2 - \left( 1 + \frac{1}{\beta} + \left( \frac{f''}{f'} / \frac{U''}{U'} \right) \right) \lambda + \frac{1}{\beta}.$$

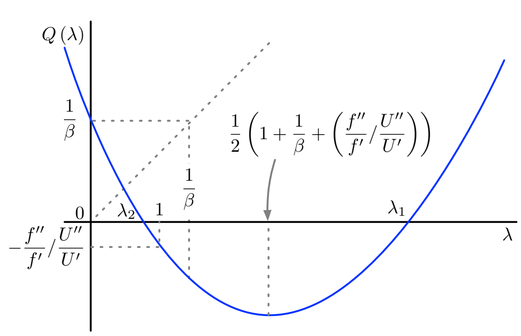
Notice that

$$\begin{aligned}
Q(0) &= \frac{1}{\beta} > 0, \\
Q(1) &= -\frac{f''}{f'} / \frac{U''}{U'} < 0, \\
Q\left(\frac{1}{\beta}\right) &= -\left( \frac{f''}{f'} / \frac{U''}{U'} \right) \frac{1}{\beta} < 0,
\end{aligned}$$

and $Q(\lambda)$ attains a minimum at $Q'(\lambda^*) = 0$, where

$$\lambda^* = \frac{1}{2} \left( 1 + \frac{1}{\beta} + \left( \frac{f''}{f'} / \frac{U''}{U'} \right) \right) > 1.$$

See figure below.



Therefore, we have that

$$0 < \lambda_1 < 1 < \frac{1}{\beta} < \lambda_2.$$

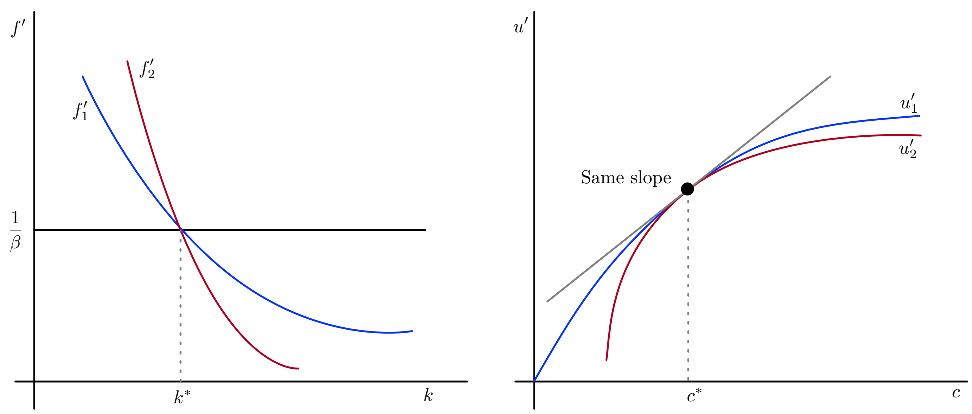
The solution is given by

$$\lambda = \frac{1}{2}\left(\left(1 + \frac{1}{\beta} + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right) \pm \sqrt{\left(1 + \frac{1}{\beta} + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right)^2 - \frac{4}{\beta}}\right).$$

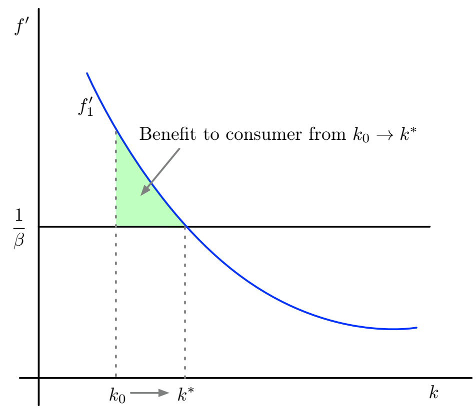
Thus, the root with absolute value less than one is given by

$$\lambda_1 = \frac{1}{2}\left(\left(1 + \frac{1}{\beta} + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right) - \sqrt{\left(1 + \frac{1}{\beta} + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right)^2 - \frac{4}{\beta}}\right).$$

Suppose we change the production function and the utility function such that the steady states do not change. That is, we change $f''/f'$ and $U''/U'$ but without changing the slope at the steady state. Then, the more curvature there is in $U$, the more agents prefer to smooth consumption. On the other hand, if $u$ is linear, then agents do not mind consuming today or tomorrow (except for the $\beta$ discount).




Suppose $k_0 < k^*$, then notice that benefit to the consumer from moving from $k_0$ to $k^*$ is represented by the shaded area in the figure below (note that, to increase capital stock, agents must abstain from consumption). We therefore see that a higher $f''$ implies a larger gain so that speed of convergence is faster. In case $f'$ is linear, then it has to coincide with $1/\beta$ and the benefit to consumer is zero. If the production is linear but there is curvature in $U$, it will take forever to converge. Thus, we see that the speed of convergence depends on the "fight" between the curvature of $f$ and $u$.




## 4.2 Continuous time

In the continuous-time neoclassical growth model, law of motion for capital is given by

$$\dot{k}(t) = f(k) - c(t) - \delta k(t).$$

Unlike in the discrete-time case, above $f$ is the production function gross of depreciation. The consumer then chooses an infinite path of $\dot{k}(t)$ to maximise his utility. Thus, the neoclassical growth model now takes the following from

$$\begin{aligned} V^*(k(0)) := \max_{\{\dot{k}(t)\}_{t=0}^{\infty}} \quad & \int_0^{\infty} e^{-\rho t} U\left(f(k) - \delta k(t) - \dot{k}(t)\right) \\ s.t. \quad & \dot{k} \in \mathbb{R}, \\ & k(0) \text{ given.} \end{aligned}$$

We can fit the problem above into our general notation:

$$\begin{aligned} F\left(k, \dot{k}\right) &= U\left(f(k) - \delta k - \dot{k}\right), \\ \Gamma(k) &= \mathbb{R}. \end{aligned}$$

### 4.2.1 EE and TC

Note

$$\begin{aligned} F_k &= \left(f'(k) - \delta\right) U'\left(f(k) - \delta k - \dot{k}\right), \\ F_{\dot{k}} &= -U'\left(f(k) - \delta k - \dot{k}\right), \\ F_{\dot{k}k} &= -\left(f'(k) - \delta\right) U''\left(f(k) - \delta k - \dot{k}\right), \\ F_{\dot{k}\dot{k}} &= U''\left(f(k) - \delta k - \dot{k}\right). \end{aligned}$$

The EE is therefore given by

$$\begin{aligned}
&\left(f'(k)-\delta\right)U' - \rho U' = -\left(f'(k)-\delta\right)U''\dot{k} + U''\ddot{k} \\
\Leftrightarrow\ &\left(f'(k)-\delta-\rho\right)U' = -\Big(\left(f'(k)-\delta\right)\dot{k} - \ddot{k}\Big)U''. \qquad (4.1)
\end{aligned}$$

### 4.2.2 Steady state

The steady state is given by

$$\left(f'(k)-\delta-\rho\right)U' = 0 \Rightarrow f'(k) = \rho + \delta.$$

### 4.2.3 Speed of convergence

Recall that

$$F\left(k,\dot{k}\right) = U\left(f(k) - \delta k - \dot{k}\right)$$

so that

$$\begin{aligned}
F_k\left(k,\dot{k}\right) &= \left(f'(k)-\delta\right)U', \\
F_{\dot{k}}\left(k,\dot{k}\right) &= -U', \\
F_{\dot{k}k}\left(k,\dot{k}\right) &= -\left(f'(k)-\delta\right)U'', \\
F_{\dot{k}\dot{k}}\left(k,\dot{k}\right) &= U'',
\end{aligned}$$

where we add one extra derivative

$$F_{kk} = f''(k)\,U' + \left(f'(k)-\delta\right)^2 U''.$$

Evaluating them at the steady state $k = \bar{k}$ using the fact that $\rho = f'\left(\bar{k}\right) - \delta$ gives

$$\begin{aligned}
F_k\left(\bar{k},0\right) &= \left(f'\left(\bar{k}\right)-\delta\right)U' = \rho U', \\
F_{\dot{k}}\left(\bar{k},0\right) &= -U', \\
F_{\dot{k}k}\left(\bar{k},0\right) &= -\rho U'', \\
F_{\dot{k}\dot{k}}\left(\bar{k},0\right) &= U'', \\
F_{kk}\left(\bar{k},0\right) &= f''\left(\bar{k}\right)U' + \rho^2 U''.
\end{aligned}$$

The quadratic equation, $Q(\lambda) := \left(-F_{\dot{k}k}\right)\lambda^2 + \left(\rho F_{\dot{k}k}\right)\lambda + \left(F_{kk} + \rho F_{\dot{k}k}\right)$, then becomes

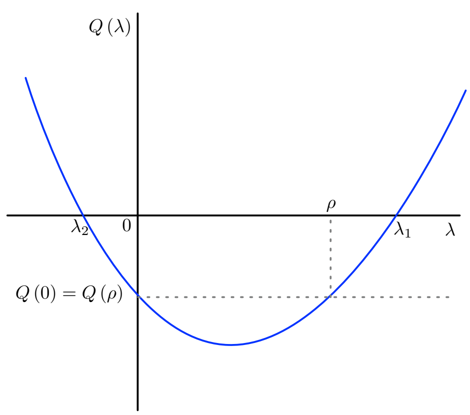
$$\begin{aligned} Q(\lambda) &= (-U'')\lambda^2 + (\rho U'')\lambda + \left(f''U' + \rho^2 U'' - \rho^2 U''\right) \\ &= (-U'')\lambda^2 + (\rho U'')\lambda + (f''U') \\ &= (-U'')\left[\lambda^2 - \rho\lambda - \frac{U'}{U''}f''\right] \\ &= (-U'')\left[\lambda^2 - \rho\lambda - \frac{U'}{U''}f''\left(\frac{f'}{f'}\frac{c}{c}\frac{k}{k}\right)\right] \\ &= (-U'')\left[\lambda^2 - \rho\lambda - \left(\frac{cf'}{k}\right)\left(-\frac{U''}{U'}c\right)^{-1}\left(-\frac{f''}{f'}k\right)\right]. \end{aligned}$$

Notice that $-\frac{U''}{U'}c$ is the elasticity of marginal utility (or, the reciprocal of the elasticity of intertemporal substitution) and $-\frac{f''}{f'}k$ is the elasticity of marginal productivity.

Now we check that one root is negative, say $\lambda_2$, and the other is then positive and larger than $\rho$. To do this, note that,

$$\begin{aligned} Q(0) &= (-U'')\left[-\left(\frac{cf'}{k}\right)\left(-\frac{U''}{U'}c\right)^{-1}\left(-\frac{f''}{f'}k\right)\right] < 0 \because U'' < 0, \\ Q(\rho) &= Q(0), \\ \frac{\partial Q(\lambda)}{\partial \lambda} &= -U''(2\lambda - \rho), \\ \frac{\partial^2 Q(\lambda)}{\partial \lambda^2} &= -2U'' > 0. \end{aligned}$$

Figure below plots $Q(\lambda)$, which visually shows that one root is negative and the other is positive and larger than $\rho$.




We can, in fact, solve for the roots using the quadratic formula:

$$\lambda = g'(k) = \frac{\rho \pm \sqrt{\rho^2 - 4\left(\frac{cf'}{k}\right)\left(-\frac{U''}{U'}c\right)^{-1}\left(-\frac{f''}{f'}k\right)}}{2},$$

and the negative is root is given by when $\pm$ is $-$.

### 4.3 Moving from discrete to continuous time

Let $\Delta$ denote the length of time between periods when the state is determined. Decisions are taken whenever the state is determined at times 0, $\Delta$, $2\Delta$, .... The sequence of state is then

$$\{x_{\Delta s}\}_{s=0}^{\infty} = \{x_0, x_\Delta, x_{2\Delta}, \ldots\},$$

where $x_0$, as usual, is given. We define the discount factor $\beta$ for an interval $\Delta$ using (the instantaneous) discount rate $\rho$ as

$$\beta = \frac{1}{1+\Delta\rho}.$$

We let $U$ denote the instantaneous utility from consuming $c_t$ amount of consumption. So, during an interval of length $\Delta$ in which consumption is given by $c_{s\Delta}$, the total utility is given by

$$\Delta U\left(c_{s\Delta}\right).$$

Market clearing condition must hold at all times (including during the interval $\Delta$). Instantaneous market clearing condition is thus

$$c_s + i_s = f\left(k_s\right),$$

where $i_s$ denotes investment. Market clearing condition for the interval of length $\Delta$ from period $t\Delta$ is thus

$$\Delta c_{s\Delta} + \Delta i_{s\Delta} = \Delta f\left(k_{s\Delta}\right).$$

Law of motion for capital is

$$k_{s\Delta+\Delta} = k_{(s+1)\Delta} = \Delta i_{s\Delta} + k_{s\Delta}\left(1-\Delta\delta\right),$$

where $\delta$ is the instantaneous depreciation rate. Thus, we can write the neoclassical growth model as

$$\begin{aligned}
\max_{\{c_t, i_t\}_{t=0}^{\infty}} \quad & \sum_{s=0}^{\infty}\left(\frac{1}{1+\Delta\rho}\right)^s \Delta U\left(c_{s\Delta}\right), \\
s.t. \quad & \Delta c_t + \Delta i_t = \Delta f\left(k_t\right),\ \forall t \geq 0, \\
& k_{t+\Delta} = \Delta i_t + k_t\left(1-\Delta\delta\right),\ \forall t \geq 0, \\
& k_0 \text{ given},
\end{aligned}$$

where $t = s\Delta$ for some integer $s$. In other words, we may write the problem in an equivalent manner as

$$\begin{aligned}
\max_{\{c_{s\Delta}, i_{s\Delta}\}_{s=0}^{\infty}} \quad & \sum_{s=0}^{\infty}\left(\frac{1}{1+\Delta\rho}\right)^s \Delta U\left(c_{s\Delta}\right), \\
s.t. \quad & \Delta c_{s\Delta} + \Delta i_{s\Delta} = \Delta f\left(k_{s\Delta}\right),\ \forall s \geq 0, \\
& k_{(s+1)\Delta} = \Delta i_{s\Delta} + k_{s\Delta}\left(1-\Delta\delta\right),\ \forall s \geq 0, \\
& k_0 \text{ given}.
\end{aligned}$$

Letting $\Delta = 1$, the problem reduces to the standard one.

**Continuous-time version of the law of motion for capital** Fix $\Delta$. First, we eliminate $\Delta i_t$ from the constraint to obtain the law of motion for capital:

$$k_{t+\Delta} = \Delta f(k_t) + k_t(1 - \Delta\delta) - \Delta c_t. \tag{4.2}$$

Rearranging above and dividing through by $\Delta$:

$$\frac{k_{t+\Delta} - k_t}{\Delta} = f(k_t) - \delta k_t - c_t. \tag{4.3}$$

Taking limits as $\Delta \downarrow 0$:

$$\lim_{\Delta \downarrow 0} \frac{k_{t+\Delta} - k_t}{\Delta} := \dot{k}(t) = f(k(t)) - \delta k(t) - c(t),$$

where we change notation following the convention (capital at time $t$ is written $k_t$ in discrete time and $k(t)$ in continuous time). This gives the continuous time version of the law of motion for capital.

**Continuous-time version of EE** Since

$$\dot{k}_t := \frac{k_{t+\Delta} - k_t}{\Delta} \Rightarrow k_{t+\Delta} = \dot{k}_t\Delta + k_t.$$

We can rewrite (4.3), while noting that $t = s\Delta$, as

$$\begin{aligned} c_t = c_{s\Delta} &= f(k_t) - \delta k_t - \frac{k_{t+\Delta} - k_t}{\Delta} \\ &= f(k_t) - \delta k_t - \dot{k}_t \end{aligned}$$

Using (4.2), we can then write the problem as

$$\max_{\{\dot{k}_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t \Delta U\left(f(k_t) - \delta k_t - \dot{k}_t\right).$$

Writing the sequence as

$$\begin{aligned} &\cdots + \left(\frac{1}{1+\Delta\rho}\right)^t \Delta U\left(f(k_t) - \delta k_t - \dot{k}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)^{t+1} \Delta U\left(f(k_{t+\Delta}) - \delta k_{t+\Delta} - \dot{k}_{t+\Delta}\right)\cdots \\ = &\cdots + \left(\frac{1}{1+\Delta\rho}\right)^t \Delta U\left(f(k_t) - \delta k_t - \dot{k}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)^{t+1} \Delta U\left(f\left(\dot{k}_t\Delta + k_t\right) - \delta\left(\dot{k}_t\Delta + k_t\right) - \frac{k_{t+2\Delta} - k_{t+\Delta}}{\Delta}\right)\cdots \\ = &\cdots + \left(\frac{1}{1+\Delta\rho}\right)^t \Delta U\left(f(k_t) - \delta k_t - \dot{k}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)^{t+1} \Delta U\left(f\left(\dot{k}_t\Delta + k_t\right) - \delta\left(\dot{k}_t\Delta + k_t\right) - \frac{k_{t+2\Delta} - \left(\dot{k}_t\Delta + k_t\right)}{\Delta}\right. \\ = &\cdots + \left(\frac{1}{1+\Delta\rho}\right)^t \Delta U\left(f(k_t) - \delta k_t - \dot{k}_t\right) + \left(\frac{1}{1+\Delta\rho}\right)^{t+1} \Delta U\left(f\left(\dot{k}_t\Delta + k_t\right) - \delta\left(\dot{k}_t\Delta + k_t\right) - \frac{k_{t+2\Delta} - k_t}{\Delta} + \dot{k}_t\right)\cdots \end{aligned}$$

The first-order condition with respect to $\dot{k}_t$ is

$$-\left(\frac{1}{1+\Delta\rho}\right)^t \Delta U'(c_t) + \left(\frac{1}{1+\Delta\rho}\right)^{t+1} \Delta U'(c_{t+\Delta})\left(\Delta f'(k_{t+\Delta}) - \Delta\delta + 1\right) = 0.$$

Rearranging this gives the EE:

$$U'(c_t) = \left(\frac{1}{1+\Delta\rho}\right) U'(c_{t+\Delta})\left(\Delta f'(k_{t+\Delta}) + (1-\Delta\delta)\right).$$

Taylor expansion of $U'(c_{t+\Delta})$ around $c_t$ gives

$$U'(c_{t+\Delta}) = U'(c_t) + U''(c_t)(c_{t+\Delta} - c_t) + R(c_{t+\Delta} - c_t), \tag{4.4}$$

where $R(c_{t+\Delta} - c_t) = o(\Delta)$ as $\Delta \to 0$; i.e.

$$\lim_{\Delta\downarrow 0} \frac{R(c_{t+\Delta} - c_t)}{\Delta} = 0.$$

Substituting (4.4) into the EE:

$$(1+\Delta\rho)U'(c_t) = \left(U'(c_t) + U''(c_t)(c_{t+\Delta} - c_t) + R(c_{t+\Delta} - c_t)\right)\left(\Delta f'(k_{t+\Delta}) + (1-\Delta\delta)\right).$$

Collecting $U'(c_t)$ together

$$\begin{aligned}
&\left[(1+\Delta\rho) - \Delta f'(k_{t+\Delta}) - (1-\Delta\delta)\right] U'(c_t) \\
&= \left[U''(c_t)(c_{t+\Delta} - c_t) + R(c_{t+\Delta} - c_t)\right]\left[\Delta f'(k_{t+\Delta}) + (1-\Delta\delta)\right].
\end{aligned}$$

Dividing through by $\Delta$:

$$\begin{aligned}
&\left[\rho - f'(k_{t+\Delta}) + \delta\right] U'(c_t) \\
&= \left(U''(c_t)\frac{(c_{t+\Delta} - c_t)}{\Delta} + \frac{R(c_{t+\Delta} - c_t)}{\Delta}\right)\left(\Delta f'(k_{t+\Delta}) + (1-\Delta\delta)\right).
\end{aligned}$$

Taking limits as $\Delta \downarrow 0$ of each side

$$\lim_{\Delta\downarrow 0}\left[\rho - f'(k_{t+\Delta}) + \delta\right] U'(c_t) = \left[\rho - f'(k(t)) + \delta\right] U'(c(t)),$$

and

$$\lim_{\Delta\downarrow 0}\left(U''(c_t)\frac{(c_{t+\Delta} - c_t)}{\Delta} + \frac{R(c_{t+\Delta} - c_t)}{\Delta}\right)\left(\Delta f'(k_{t+\Delta}) + (1-\Delta\delta)\right) = U''(c(t))\,\dot{c}(t),$$

where

$$\dot{c}(t) := \lim_{\Delta\downarrow 0}\frac{(c_{t+\Delta} - c_t)}{\Delta}.$$

Combining the two and rearranging, we obtain

$$\left(f'(k(t)) - \delta - \rho\right) U'(c(t)) = -U''(c(t))\,\dot{c}_t,$$

where

$$\begin{aligned} c(t) &= f(k(t)) - \delta k(t) - \dot{k}(t) \\ \Rightarrow \dot{c}(t) &= (f'(k(t)) - \delta)\dot{k}(t) - \ddot{k}(t). \end{aligned}$$

Therefore, we have

$$(f'(k(t)) - \delta - \rho) U'(c(t)) = -U''(c(t)) \left[ (f'(k(t)) - \delta)\dot{k}(t) - \ddot{k}(t) \right].$$

as we had in (4.1).

*Remark* 4.1. See also PS4, Q2.

## 4.4 Hamiltonian method

We can also analyse the neoclassical model using the Hamiltonian. The period-return function is $U(c)$ and the law of motion is given by $\dot{k} = f(k) - \delta k - c$ (note that $k(0)$ is given). That is,

$$\begin{aligned} h(k,c) &= U(c), \\ g(k,c) &= f(k) - \delta k - c, \\ \Rightarrow H(k,c,\lambda) &= U(c) + \lambda (f(k) - \delta k - c). \end{aligned}$$

Then,

$$\begin{aligned} H_u = 0 &\Rightarrow U'(c) = \lambda, \\ \dot{\lambda} = \rho\lambda - H_x &\Rightarrow \dot{\lambda} = \lambda(\rho - (f'(k) - \delta)), \\ \dot{x} = g &\Rightarrow \dot{k} = f(k) - \delta k - c. \end{aligned}$$

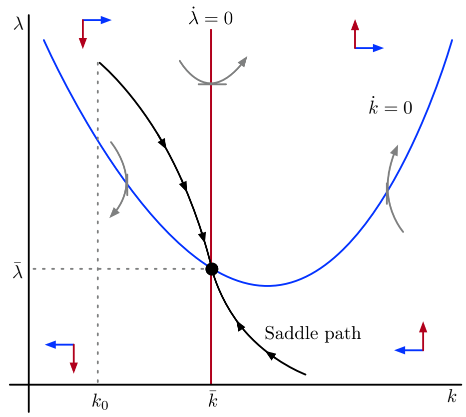
We therefore have the following dynamic equations:

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho - (f'(k) - \delta)), \\ \dot{k} &= f(k) - \delta k - c. \end{aligned}$$

**Phase diagram in $(k,\lambda)$ space** To draw the phase diagram in $(k,\lambda)$ space, note that

- ▷ $\dot{\lambda} = 0$: $f'(\bar{k}) - \delta = \rho$. In $(k,\lambda)$ space, this is a vertical line. Dynamics: From $\dot{\lambda} = 0$ if $k > \bar{k}$, then $f'(k)$ is lower and $-f'(k)$ is higher so that $\dot{\lambda} > 0$. And if $k < \bar{k}$ then $\dot{\lambda} < 0$.
- ▷ $\dot{k} = 0$: $c = f(k)$. Since $\lambda = U'(c)$, we have that $\lambda = U'(f(k))$. Note that $U' > 0$ and $U'' < 0$ so that $U'(\cdot)$ is a strictly decreasing transformation. Thus, when $f(k) - \delta k$ achieves its maximum (at $\hat{k}$ such that $f'(\hat{k}) = \delta$), $U'(f(k) - \delta)$ is at its minimum. As $k \to 0$, $f(k) \to 0$ and $U'(f(k) - \delta) \to \infty$. As $k > \hat{k}$, $f(\hat{k})$ falls so that $\lambda$ increases. These observations imply that $\dot{k} = 0$ locus is U-shaped in $(k,\lambda)$ space. Dynamics: If $c > \bar{c}$, then $\dot{k} < 0$ and if $c < \bar{c}$, then $\dot{k} > 0$. Note that $c > \bar{c}$ represents points below the $\dot{k} = 0$ locus.

This gives the following phase diagram.




We see that low value of $k$ corresponds to a higher value of $\lambda$. Does it make sense? Recall that $\lambda$ is the marginal value of a unit of $k$ and, at the optimal, marginal benefit from $c$ is equated to $\lambda$, the marginal value of capital. In a similar way that higher $c$ implies lower marginal benefit, a higher $k$ implies lower marginal value of capital; i.e. it represents diminishing returns. Note also that a higher $k$ implies higher production, and since $c$ is a normal good (this is due to the fact that we have separable utility function), higher output implies higher income and $c$ increases in every period.

#### 4.4.1 Alternative way of computing slope of saddle path

Recall that in the Hamiltonian set up we have $\lambda$, the co-state variable, and $x$, the state variable. The first-order conditions were:

$$\begin{aligned}
\dot{\lambda} &= \rho\lambda - H_x\left(x, \mu\left(x, \lambda\right), \lambda\right), \\
\dot{x} &= g\left(x, \mu\left(x, \lambda\right)\right), \\
0 &= H_u\left(x, \mu\left(x, \lambda\right), \lambda\right).
\end{aligned}$$

Let $\phi$ be such that $\lambda \coloneqq \phi\left(x\right)$, where $\lambda$ is the co-state and $x$ is the state variable. Slope of the saddle is

$$\frac{d\lambda}{dx} = \phi'\left(\overline{x}\right).$$

The "trick":

$$\phi' = \frac{d\lambda}{dx} = \frac{d\lambda/dt}{dx/dt} \equiv \frac{\dot{\lambda}\left(\lambda, x\right)}{\dot{x}\left(\lambda, x\right)}.$$

But in the steady state, $\dot{\lambda} = \dot{x} = 0$ which implies that $\phi' =$ "0/0". So use L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\overline{x}} \phi'\left(x\right) \equiv \left.\frac{d\lambda}{dx}\right|_{x=\overline{x}} &= \left.\frac{\dot{\lambda}\left(\lambda, x\right)}{\dot{x}\left(\lambda, x\right)}\right|_{x=\overline{x}} \\
&= \left.\frac{\frac{d}{dx}\dot{\lambda}\left(\lambda, x\right)}{\frac{d}{dx}\dot{x}\left(\lambda, x\right)}\right|_{x=\overline{x}} = \left.\frac{\frac{d\dot{\lambda}}{d\lambda}\frac{d\lambda}{dx} + \frac{d\dot{\lambda}}{dx}}{\frac{d\dot{x}}{d\lambda}\frac{d\lambda}{dx} + \frac{d\dot{x}}{dx}}\right|_{x=\overline{x}} \\
&= \left.\frac{\frac{d\dot{\lambda}}{d\lambda}\phi' + \frac{d\dot{\lambda}}{dx}}{\frac{d\dot{x}}{d\lambda}\phi' + \frac{d\dot{x}}{dx}}\right|_{x=\overline{x}},
\end{aligned}$$

where

$$\frac{d\dot{\lambda}}{d\lambda} = \rho - H_{xu}\mu_\lambda - H_{x\lambda}, \quad \frac{d\dot{x}}{dx} = g_x + g_u\mu_x,$$
$$\frac{d\dot{\lambda}}{dx} = -H_{xx} - H_{xu}\mu_x, \qquad 0 = H_{ux} + H_{uu}\mu_x,$$
$$\frac{d\dot{x}}{d\lambda} = g_u\mu_\lambda, \qquad 0 = H_{u\lambda} + H_{uu}\mu_\lambda.$$

This gives a quadratic:

$$(\phi')^2 \frac{d\dot{x}}{d\lambda} + \phi\left(\frac{d\dot{x}}{dx} - \frac{d\dot{\lambda}}{d\lambda}\right) - \frac{d\dot{\lambda}}{dx} = 0.$$

One of the two roots of this quadratic equation is the slope of the saddle path. Unlike when we were analysing convergence, there is no “rule” as to which root will represent the stable steady state. Thus, when using this method, we would usually plot the phase diagram to know the slope of the saddle path from the diagram, and pick the appropriate root using the equation above accordingly.

Recall that

$$\dot{\lambda} = \lambda\left(\rho - \left(f'(k) - \delta\right)\right)$$
$$\dot{k} = f(k) - \delta k - c(k).$$

Since $U'(c) = \lambda$, we can write

$$\dot{k} = f(k) - \delta k - (U')^{-1}(\lambda).$$

Then

$$\phi'(k) \equiv \frac{d\lambda}{dk} = \frac{\frac{d\dot{\lambda}}{d\lambda}\phi' + \frac{d\dot{\lambda}}{dk}}{\frac{d\dot{k}}{d\lambda}\phi' + \frac{d\dot{k}}{dk}},$$

where

$$\frac{d\dot{\lambda}}{d\lambda} = \rho - \left(f'(k) - \delta\right)$$
$$\frac{d\dot{\lambda}}{dk} = \frac{d\lambda}{dk}\left(\rho - f'(k)\right) - \lambda f''(k) = \phi'(k)\left(\rho - f'(k)\right) - \lambda f''(k)$$
$$\frac{d\dot{k}}{d\lambda} = -\frac{d\,(U')^{-1}}{d\lambda} = -\frac{d\,(U')^{-1}(\lambda)}{d\lambda}$$
$$\frac{d\dot{k}}{dk} = f''(k) - \delta - \frac{d\,(U')^{-1}(\lambda)}{d\lambda}\frac{d\lambda}{dk} = f''(k) - \delta - \frac{d\,(U')^{-1}(\lambda)}{d\lambda}\phi'(k).$$

Hence,

$$\phi'\left(\bar{k}\right) = \frac{\left(\rho - f'\left(\bar{k}\right) - \delta\right)\phi'\left(\bar{k}\right) + \phi'\left(\bar{k}\right)\left(\rho - f'\left(\bar{k}\right)\right) - \lambda f''\left(\bar{k}\right)}{-\frac{d(U')^{-1}(\lambda)}{d\lambda}\phi'\left(\bar{k}\right) + f''\left(\bar{k}\right) - \delta - \frac{d(U')^{-1}(\lambda)}{d\lambda}\phi'\left(\bar{k}\right)}.$$

In the steady state, $\rho = f'\left(\bar{k}\right) - \delta$, so we can simplify above to

$$\phi'\left(\bar{k}\right) = \frac{\delta\phi'\left(\bar{k}\right) + \lambda f''\left(\bar{k}\right)}{2\frac{d(U')^{-1}(\lambda)}{d\lambda}\phi'\left(\bar{k}\right) - f''\left(\bar{k}\right) + \delta}$$

and so

$$2\frac{d\left(U'\right)^{-1}(\lambda)}{d\lambda}\left(\phi'\left(\bar{k}\right)\right)^{2}-\left(f''\left(\bar{k}\right)+\delta\right)\phi'\left(\bar{k}\right)+\left(\delta-\lambda f''\left(\bar{k}\right)\right)=0$$

This is a quadratic equation in $\phi'(k)$ which we can solves to obtain the slope of the saddle path in $(k, \lambda)$ space.

$$\phi'\left(\bar{k}\right)=\frac{f''\left(\bar{k}\right)+\delta\pm\sqrt{\left(f''\left(\bar{k}\right)+\delta\right)^{2}-8\frac{d(U')^{-1}(\lambda)}{d\lambda}\left(\delta-\lambda f''\left(\bar{k}\right)\right)}}{4\frac{d(U')^{-1}(\lambda)}{d\lambda}}.$$

Recall from earlier that when we drew the phase diagram that the slope of the saddle path is negative in $(k, \lambda)$ space. Since $U'' < 0$, it follows that $d\left(U'\right)^{-1}(\lambda)/d\lambda < 0$ and so the negative root is given by when $\pm$ is $+$.[2] That is,

$$\phi'\left(\bar{k}\right)=\frac{f''\left(\bar{k}\right)+\delta+\sqrt{\left(f''\left(\bar{k}\right)+\delta\right)^{2}+8\frac{d(U')^{-1}(\lambda)}{d\lambda}\lambda f''\left(\bar{k}\right)}}{4\frac{d(U')^{-1}(\lambda)}{d\lambda}}.$$

[2]For example, let $U(c) = \ln c$. Then $U'(c) = 1/c$ so that

$$1/c=\lambda\Leftrightarrow\lambda=\frac{1}{c};$$

i.e. $\left(U'\right)^{-1}(\lambda) = 1/\lambda$. Then, $d\left(U'\right)^{-1}(\lambda)/d\lambda = -1/\lambda^2 < 0$.

# 5 Exercises

## 5.1 Problem set 6

- ▷ Q1: Adjustment cost model. Skip the Bellman equation parts (Parts 1–7).
- ▷ Q2: Speed of convergence and intertemporal substitution. Note that this was a question in the Final in 2003/04.
- ▷ Q3: Speed of convergence versus slope of saddle path (perhaps cover after Fernando/we cover slope of saddle path—as this question illustrates, the two things are different things!)

## 5.2 2016/17 Core exam: A quadratic dynamic programming problem

[This was also the second midterm last year.]

Consider a single-agent, deterministic, discrete-time dynamic programming problem where the decision maker seeks to maximise the discounted value of the period-$t$ return which is given by

$$F\left(x_t, x_{t+1}\right) = -\frac{a}{2}\left(x_t - \bar{x}\right)^2 - \frac{b}{2}\left(x_{t+1} - \bar{x}\right)^2 - \frac{c}{2}\left(x_{t+1} - x_t\right)^2,$$

where the scalar state is given by $x_t$. The agent uses a discount factor $\beta \in (0, 1)$. The initial condition at time $t = 0$ is given by $x_0$. The parameters, $a, b, \bar{x} > 0$ are given. Since $a$ and $b$ are positive, the decision maker likes the state to be close to $\bar{x}$. The term $\left(x_{t+1} - x_t\right)^2$ measures the changes, either increases or decreases, in the states. When $c > 0$, the agent dislikes changes to the state. When $c < 0$, the agent gets extra utility out of changing the state. We will analyse the differences in the dynamics between the two forces; i.e. the desire to be close to $\bar{x}$ and the taste (or distaste) for changing the state.

### 5.2.1 Part 1

Write down the first derivatives of $F$ with respect to $x_t$ and with respect to $x_{t+1}$.

......

$$\begin{aligned} F_{1,t} &\coloneqq \frac{dF\left(x_t, x_{t+1}\right)}{dx_t} = -\,a\left(x_t - \bar{x}\right) + c\left(x_{t+1} - x_t\right), \\ F_{2,t} &\coloneqq \frac{dF\left(x_t, x_{t+1}\right)}{dx_{t+1}} = -\,b\left(x_{t+1} - \bar{x}\right) - c\left(x_{t+1} - x_t\right). \end{aligned}$$

### 5.2.2 Part 2

Write down the second derivatives of $F$ with respect to $x_t$, with respect to $x_{t+1}$, and its cross derivative.

......

$$\begin{aligned} F_{11,t} \coloneqq \frac{d^2 F\left(x_t, x_{t+1}\right)}{dx_t dx_t} =& - a - c, \\ F_{22,t} \coloneqq \frac{dF\left(x_t, x_{t+1}\right)}{dx_{t+1}} =& - b - c, \\ F_{21,t} \coloneqq \frac{dF\left(x_t, x_{t+1}\right)}{dx_{t+1} dx_t} =& c. \end{aligned}$$

#### 5.2.3 Part 3

Is the period return function strictly concave in $(x_t, x_{t+1})$? What are the required conditions on the constants $a$, $b$ and $c$? [Recall that strict concavity for a differentiable function of two variables is equivalent to three strict inequalities involving second-order derivatives.]

. . . . . .

The conditions are

$$\begin{aligned} F_{11,t} &< 0, \\ F_{22,t} &< 0, \\ F_{11,t} F_{22,t} - \left(F_{21,t}\right)^2 &> 0. \end{aligned}$$

Let us check each one in turn.

$$\begin{aligned} F_{11,t} < 0 &\Leftrightarrow -a - c < 0 \\ &\Leftrightarrow -c < a, \\ F_{22,t} < 0 &\Leftrightarrow -b - c < 0 \\ &\Leftrightarrow -b < c. \end{aligned}$$

Moreover,

$$\begin{aligned} 0 &< F_{11,t} F_{22,t} - \left(F_{21,t}\right)^2 \\ &= \left(-a - c\right)\left(-b - c\right) - c^2 \\ &= ab + ac + bc + c^2 - c^2 \\ &= ab + ac + bc. \end{aligned}$$

Since $ab > 0$,

$$\begin{aligned} 0 &< 1 + \frac{c}{b} + \frac{c}{a} \\ &= 1 + c\left(\frac{1}{b} + \frac{1}{a}\right) \\ \Rightarrow -\frac{ab}{a+b} &< c. \end{aligned}$$

### 5.2.4 Part 4

From now on, assume that $(a, b, c)$ are such that the function $F$ is strictly concave in $(x_t, x_{t+1})$.

Write down explicitly the Euler equation that applies to the optimal choice of $x_{t+1}$. This equation should involve the quantities, $x_t$, $x_{t+1}$ and $x_{t+1}$ and the constants, $a$, $b$, $c$, $\bar{x}$ and $\beta$.

......

The Euler equation is

$$\begin{aligned} 0 &= F_{2,t} + \beta F_{1,t+1} \\ &= -b\left(x_{t+1} - \bar{x}\right) - c\left(x_{t+1} - x_t\right) + \beta\left[-a\left(x_{t+1} - \bar{x}\right) + c\left(x_{t+2} - x_{t+1}\right)\right] \\ &= -\left(b + \beta a\right)\left(x_{t+1} - \bar{x}\right) - c\left(x_{t+1} - x_t\right) + \beta c\left(x_{t+2} - x_{t+1}\right). \end{aligned}$$

### 5.2.5 Part 5

Rewrite the Euler equation so that, instead of having $x_t$, $x_{t+1}$, $x_{t+2}$, and $\bar{x}$ enter separately, the equation involves the variables $z_t \equiv x_t - \bar{x}$, $z_{t+1} \equiv x_{t+1} - \bar{x}$ and $z_{t+2} \equiv x_{t+2} - \bar{x}$ and the constants $a$, $b$, $c$ and $\beta$. There should be no $\bar{x}$ in your expression.

......

Adding and subtracting $\bar{x}$ in the relevant places, we may write

$$\begin{aligned} 0 &= -\left(b + \beta a\right)\left(x_{t+1} - \bar{x}\right) - c\left[\left(x_{t+1} - \bar{x}\right) + \left(\bar{x} - x_t\right)\right] + \beta c\left[\left(x_{t+2} - \bar{x}\right) + \left(\bar{x} - x_{t+1}\right)\right] \\ &= -\left(b + \beta a\right) z_{t+1} - c\left[\left(z_{t+1} - z_t\right) - \beta\left(z_{t+2} - z_{t+1}\right)\right]. \end{aligned}$$

### 5.2.6 Part 6

What is the steady state of this system? Answer it in terms of $x$ and $z$.

......

In the steady state,

$$x_t = x_{t+1} = x_{t+2} = x,$$

which, in turn, implies

$$z_t = z_{t+1} = z_{t+2} = z.$$

Substituting into the expression from part 5 yields

$$z = 0.$$

Hence,

$$x = \bar{x}.$$

#### 5.2.7 Part 7

Assume that the optimal decision rule takes the form

$$z_{t+1} = \gamma z_t$$

for some constant $\gamma$—and likewise recursively solve for $z_{t+2}$. Replace this equation into the Euler equation and obtain a quadratic equation for $\gamma$. [Hint: the equation must hold for all values of $z_t$.]

......

First, note that

$$z_{t+2} = \gamma z_{t+1} = \gamma^2 z_t.$$

Hence,

$$\begin{aligned} z_{t+1} - z_t &= \gamma z_t - z_t = (\gamma - 1) z_t, \\ z_{t+2} - z_{t+1} &= \gamma^2 z_t - \gamma z_t = \gamma (\gamma - 1) z_t. \end{aligned}$$

Substituting into the Euler equation, we have

$$\begin{aligned} 0 &= -(b + \beta a) \gamma z_t - c [(\gamma - 1) z_t - \beta\gamma (\gamma - 1) z_t] \\ &= z_t [-(b + \beta a) \gamma - c (\gamma - 1)(1 - \beta\gamma)]. \end{aligned}$$

For this to hold for all $z_t$, it must be that

$$0 = -(b + \beta a) \gamma + c (1 - \gamma)(1 - \beta\gamma)$$

#### 5.2.8 Part 8

What is the only solution for $\gamma$ of this equation if $c = 0$? What is the economic interpretation of $c = 0$? What is the interpretation of $\gamma$ for the dynamics of the state under the optimal decision rule?

......

Imposing that $c = 0$, the Euler equation implies

$$0 = -(b + \beta a) \gamma.$$

Since $a, b, \beta \neq 0$, it follows that

$$\gamma = 0.$$

$c = 0$ means that there is no adjustment cost—(convex) cost associated with changing the state from the current value to a new value.

Since

$$z_t = \gamma^t z_0,$$

$\gamma$ captures the speed of convergence. That $\gamma = 0$ means that we are always at the steady state.

#### 5.2.9 Part 9

Assume that $c > 0$. Furthermore, assume that $a = b > 0$. Denote $\alpha = a/c = b/c$.

**Part (a)** What is the economic interpretation of $\alpha$? One line.

......

$\alpha$ describes the relative preference between having the state closer to $\bar{x}$ and having the next-period state closer to the current-period state.

**Part (b)** Rewrite the coefficients of the quadratic equation dividing them by $c$. The coefficients of your equation should be a function only of the constants $\beta$ and $\alpha$. Denote this quadratic equation by $Q(\gamma)$. Note $Q(\gamma) = 0$ is solved by the coefficient of the optimal decision rule. Write the quadratic equation so that the coefficient for $\gamma^2$ is $\beta$.

......

$$\begin{aligned}
Q(\gamma) &= -\left(\frac{b}{c} + \beta\frac{a}{c}\right)\gamma + (1-\gamma)(1-\beta\gamma) \\
&= -\alpha(1+\beta)\gamma + (1-\gamma)(1-\beta\gamma) \\
&= -\alpha(1+\beta)\gamma + 1 - \beta\gamma - \gamma + \beta\gamma^2 \\
&= \beta\gamma^2 - \alpha(1+\beta)\gamma - (1+\beta)\gamma + 1 \\
&= \beta\gamma^2 - (1+\alpha)(1+\beta)\gamma + 1.
\end{aligned}$$

**Part (c)** Compute $Q(0)$ and $\partial Q(0)/\partial\gamma$. Are they positive or negative?

......

$$\begin{aligned}
Q(0) &= 1 > 0, \\
\left.\frac{\partial Q(\gamma)}{\partial\gamma}\right|_{\gamma=0} &= \left. 2\beta\gamma - (1+\alpha)(1+\beta)\right|_{\gamma=0} \\
&= -(1+\alpha)(1+\beta) < 0.
\end{aligned}$$

**Part (d)** Compute $Q(1)$. Is it positive or negative?

......

$$\begin{aligned}
Q(1) &= \beta - (1+\alpha)(1+\beta) + 1 \\
&= -\alpha(1+\beta) < 0.
\end{aligned}$$

**Part (e)** Compute $\partial^2 Q/\partial\gamma^2$. Is it positive or negative?

......

$$\frac{\partial^2 Q(\gamma)}{\partial\gamma^2} = 2\beta > 0.$$

**Part (f)** Use the information obtained above to sketch $Q(\gamma)$, with $\gamma$ in the horizontal axis. Make sure you include the value of $\gamma = 0$ and $\gamma = 1$ and its corresponding values in the vertical axis.

......

$Q(\gamma)$ cuts the $y$-axis at value 1 and is downward sloping at this point. Since $Q(1)$ is negative and $Q(\gamma)$ is strictly convex.

**Part (g)** What can you conclude about the roots of this equation? Are they both smaller than one, one smaller and one larger than one, or both larger than one, etc. In particular, how is the root with the smaller absolute value.

......

See figure above. One root lies between $(0,1)$ while the other root is larger than one.

### 5.2.10 Part 10

Consider the smallest root of this equation, which gives the solution of the optimal decision rule.

**Part (a)** Write an explicit solution of this root, as a function of $\alpha$ and $\beta$. [Hint: use the expression for the root of a quadratic equation.]

......

We are solving

$$0 = \beta\gamma^2 - (1+\alpha)(1+\beta)\gamma + 1.$$

So

$$\gamma = \frac{(1+\alpha)(1+\beta) \pm \sqrt{(1+\alpha)^2(1+\beta)^2 - 4\beta}}{2\beta}.$$

The smaller root, $\gamma_1$, is then given by

$$\gamma_1 = \frac{(1+\alpha)(1+\beta) - \sqrt{(1+\alpha)^2(1+\beta)^2 - 4\beta}}{2\beta}.$$

**Part (b)** Is this root increasing or decreasing in $\alpha$? You don't need to write a proof, just state the correct result. Hint: differentiate the expression for the smallest root with respect to $\alpha$.

......

We can use the figure. A higher $\alpha$ implies that $Q(1)$ is more negative so that the smaller root becomes smaller. That is, the root is decreasing in $\alpha$.

**Part (c)** What is the limit of this root when $\alpha \to \infty$? You don't need to write a proof, just state the correct result. [Hint: you can define $x = 1/(1+\alpha)$, take $x \to 0$ and use L'Hôpital's rule.]

......

You can see from the figure that the limit is zero.

More formally, let us write the expression for the smallest root in terms of $x$ as defined in the question:

$$\begin{aligned}\gamma_1 &= \frac{\frac{1+\beta}{x} - \sqrt{\frac{1}{x^2}(1+\beta)^2 - 4\beta}}{2b} \\ &= \frac{\frac{1+\beta}{x} - \frac{1}{x}\sqrt{(1+\beta)^2 - 4\beta x^2}}{2b} \\ &= \frac{1+\beta - \sqrt{(1+\beta)^2 - 4\beta x^2}}{2bx}.\end{aligned}$$

Consider

$$\begin{aligned}\lim_{x\to 0}\gamma_1 &= \lim_{x\to 0}\frac{1+\beta - \sqrt{(1+\beta)^2 - 4\beta x^2}}{2bx} \\ &= \lim_{x\to 0} -\frac{\frac{1}{2}\frac{-8\beta x}{\sqrt{(1+\beta)^2-4\beta x^2}}}{2b} \\ &= \lim_{x\to 0}\frac{2\beta x}{b\sqrt{(1+\beta)^2 - 4\beta x^2}} = 0.\end{aligned}$$

**Part (d)** What is the limit of this root when $\alpha \to 0$? You don't need to write a proof, just state the correct result.

......

$$\begin{aligned}\lim_{\alpha\to 0}\gamma_1 &= \lim_{\alpha\to 0}\frac{(1+\alpha)(1+\beta) - \sqrt{(1+\alpha)^2(1+\beta)^2 - 4\beta}}{2\beta} \\ &= \frac{(1+\beta) - \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} \\ &= \frac{(1+\beta) - \sqrt{\beta^2 - 2\beta + 1}}{2\beta} = \frac{(1+\beta) - \sqrt{(\beta-1)^2}}{2\beta} \\ &= \frac{(1+\beta) - |\beta - 1|}{2\beta} = \frac{(1+\beta) - (1-\beta)}{2\beta} = 1.\end{aligned}$$

**Part (e)** What is the economic intuition of these results? What are the implication for the optimal dynamics? Is the convergence monotone or oscillatory?

......

Recall that $\gamma_1$ gives the speed of convergence. Higher $\gamma_1$ implies slower convergence. This means that $\alpha$ decreasing, which corresponds adjustment cost becoming relatively more important slows down the speed of adjustment to $\bar{x}$. The convergence is monotone since $\gamma_1$ is strictly positive.

#### 5.2.11 Part 11

Assume that $c < 0$ from now on. Rewrite the coefficients of the quadratic equation dividing them by $c$. As before, denote $a/c = b/c = \alpha < 0$, but notice the change on the sign. What is the range of values of $\alpha$ that is consistent with $F$ being concave? Recall we maintain the assumption that $a, b > 0$.

......

The quadratic equation remains the same as before:

$$Q(\gamma) = \beta\gamma^2 - (1+\alpha)(1+\beta)\gamma + 1$$

where $\alpha < 0$. Recall that for $F$ to be concave, we need

$$\begin{aligned} -c &< a, \\ -c &< b, \\ -\frac{ab}{a+b} &< c. \end{aligned}$$

Let us dividing these inequalities by $-c > 0$. First,

$$\begin{aligned} 1 &< -\frac{a}{c} = -\alpha, \\ 1 &< -\frac{b}{c} = -\alpha \end{aligned}$$

So we need that

$$\alpha < -1.$$

Similarly,

$$\begin{aligned} -1 &> \frac{1}{c}\frac{ab}{a+b} \\ &= \frac{\frac{a}{c}b}{a+b}\frac{\frac{1}{c}}{\frac{1}{c}} \\ &= \frac{\frac{a}{c}\frac{b}{c}}{\frac{a}{c}+\frac{b}{c}} = \frac{\alpha^2}{2\alpha} = \frac{\alpha}{2} \\ \Leftrightarrow -2 &> \alpha \\ \Leftrightarrow \alpha &< -2. \end{aligned}$$

Hence, for $F$ to be concave, we need

$$\alpha < -2.$$

#### 5.2.12 Part 12

Consider the smallest root (in absolute value) of this equation.

**Part (a)** What is the economic interpretation of $c < 0$?

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The agent gains utility from changing the state away from the current state.

**Part (b)** Using $Q(\gamma)$ as defined above, compute $Q(0)$ and $\partial Q(0)/\partial\gamma$. Are they positive or negative?

......

$$\begin{aligned} Q(0) &= 1 > 0, \\ \left.\frac{\partial Q(\gamma)}{\partial\gamma}\right|_{\gamma=0} &= 2\beta\gamma - (1+\alpha)(1+\beta)\big|_{\gamma=0} \\ &= -\Bigg(1\underbrace{+\alpha}_{<-2}\Bigg)(1+\beta) > 0. \end{aligned}$$

**Part (c)** Compute $Q(-1)$. Is it positive or negative?

......

$$\begin{aligned} Q(-1) &= 1 + \beta + (1+\alpha)(1+\beta) \\ &= \Bigg(2\underbrace{+\alpha}_{<-2}\Bigg)(1+\beta) < 0. \end{aligned}$$

**Part (d)** Compute $\partial^2 Q/\partial\gamma^2$. Is it positive or negative?

......

$$\frac{\partial^2 Q(\gamma)}{\partial\gamma^2} = 2\beta > 0.$$

**Part (e)** Use the information obtained above to sketch $Q(\gamma)$, with $\gamma$ in the horizontal axis. Make sure you include the value of $\gamma = 0$ and $\gamma = 1$ and its corresponding values in the vertical axis.

......

[Figure to be added.]

**Part (f)** What can you conclude about the roots of this equation? Are they both smaller than one, one smaller and one larger than one, or both larger than one, etc. In particular, how is the root with the smaller absolute value.

......

See figure above. One root lies between $(-1, 0)$ while the other root is less than $-1$.

#### 5.2.13 Part 13

Consider the smallest root of this equation (in absolute terms), which gives the solution of the optimal decision rule.

**Part (a)** Write an explicit solution of this root, as a function of $\alpha$ and $\beta$. [Hint: use the expression for the root of a quadratic equation.]

......

$$\gamma_1 = \frac{(1+\alpha)(1+\beta) + \sqrt{(1+\alpha)^2 (1+\beta)^2 - 4\beta}}{2\beta}.$$

**Part (b)** Is this root increasing or decreasing in $\alpha$? You don't need to write a proof, just state the correct result. Hint: differentiate the expression for the smallest root with respect to $\alpha$.

......

We can use the figure. Recall that

$$Q(-1) = \left(2 \underbrace{+\alpha}_{<-2}\right)(1+\beta) < 0.$$

A higher $\alpha$ (which implies less negative $\alpha$) means that $Q(-1)$ is less negative so $\gamma_1$ (so increasing) is less negative; i.e. the absolute value of $\gamma_1$ decreases. Thus, $\gamma_1$ is increasing in $\alpha$.

**Part (c)** What is the limit of this root when $\alpha \to -2$? You don't need to write a proof, just state the correct result.

......

You can see from the figure that the limit is $-1$.

$$\begin{aligned}
\lim_{\alpha \to -2} \gamma_1 &= \lim_{\alpha \to -2} \frac{(1+\alpha)(1+\beta) + \sqrt{(1+\alpha)^2 (1+\beta)^2 - 4\beta}}{2\beta} \\
&= \frac{-(1+\beta) + \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} \\
&= \frac{-(1+\beta) + |\beta - 1|}{2\beta} = \frac{-(1+\beta) + (1-\beta)}{2\beta} \\
&= \frac{-2\beta}{2\beta} = -1.
\end{aligned}$$

**Part (d)** What is the limit of this root when $\alpha \to -\infty$? You don't need to write a proof, just state the correct result. [Hint: you can define $x = -1/(1+\alpha)$, take $x \to 0$, use L'Hôpital's rule.]

......

From the figure, the limit is 0.

More formally, let us write the expression for the smallest root in terms of $x$ as defined in the question:

$$\begin{aligned}
\gamma_1 &= \frac{\frac{1+\beta}{x} + \sqrt{\frac{1}{x^2}\left(1+\beta\right)^2 - 4\beta}}{2b} \\
&= \frac{\frac{1+\beta}{x} + \frac{1}{x}\sqrt{\left(1+\beta\right)^2 - 4\beta x^2}}{2b} \\
&= \frac{1+\beta + \sqrt{\left(1+\beta\right)^2 - 4\beta x^2}}{2bx}.
\end{aligned}$$

Consider

$$\begin{aligned}
\lim_{x\uparrow 0} \gamma_1 &= \lim_{x\uparrow 0} \frac{1+\beta + \sqrt{\left(1+\beta\right)^2 - 4\beta x^2}}{2bx} \\
&= \lim_{x\to 0} \frac{\frac{1}{2}\frac{-8\beta x}{\sqrt{(1+\beta)^2 - 4\beta x^2}}}{2b} \\
&= \lim_{x\to 0} \frac{-2\beta x}{b\sqrt{\left(1+\beta\right)^2 - 4\beta x^2}} = 0.
\end{aligned}$$

**Part (e)** What is the economic intuition of these results? What are the implication for the optimal dynamics? Is the convergence monotone or oscillatory?

. . . . . .

When $c$ is negative, agents prefer to be moving away from the previous value, rather than staying at the constant value $\bar{x}$. Thus, as $\alpha \to -\infty$, i.e. when $c$ is small in absolute value, agent's desire to be away from the previous value is small so that he prefers to be at value $\bar{x}$. Hence, convergence is faster as $\alpha \to -\infty$.

Since the root is negative, convergence is oscillatory.

# 6 Maths postliminaries

## 6.1 Decomposition

Suppose we have a system of simultaneous first-order difference equations:

$$\begin{aligned} x_{t+1} &= 4x_t + 2y_t, \\ y_{t+1} &= -x_t + y_t, \end{aligned}$$

with initial values $x_0$ and $y_0$. We can write this in matrix form, $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are given by $k$ that solves $|\mathbf{A} - k\mathbf{I}| = 0$; i.e.

$$\begin{aligned} \begin{vmatrix} 4-k & 2 \\ -1 & 1-k \end{vmatrix} &= 0 \\ \Rightarrow (4-k)(1-k) + 2 &= 0 \\ \Rightarrow 6 - 5k + k^2 &= 0 \\ \Rightarrow (k-2)(k-3) &= 0. \end{aligned}$$

Thus, eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. The eigenvector corresponding to $\lambda_1$ is

$$\begin{aligned} \begin{pmatrix} 4-2 & 2 \\ -1 & 1-2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} &= 0 \\ \begin{pmatrix} 2x_t + 2y_t \\ -x_t - y_t \end{pmatrix} &= 0 \end{aligned}$$

so that the corresponding eigenvector is $(1, -1)$. For $\lambda_2$,

$$\begin{pmatrix} 4-3 & 2 \\ -1 & 1-3 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_t + 2y_t \\ -x_t - 2y_t \end{pmatrix} = 0$$

so that the corresponding eigenvector is $(2, -1)$. Hence

$$\begin{aligned} \mathbf{P}^{-1} &= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \\ \mathbf{\Lambda} &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \\ \mathbf{P} &= \frac{1}{1} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

To verify

$$\mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} = \mathbf{A}.$$

Notice that

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0, \\ \mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 \\ &= \mathbf{A}^2\mathbf{x}_0, \\ \vdots &= \vdots \\ \mathbf{x}_t &= \mathbf{A}^t\mathbf{x}_0. \end{aligned}$$

Since $\mathbf{A} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}$,

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}\mathbf{x}_0, \\ \mathbf{x}_2 &= \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}\mathbf{x}_1, \\ &= \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}\mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}\mathbf{x}_0 \\ &= \mathbf{P}^{-1}\mathbf{\Lambda}^2\mathbf{P}\mathbf{x}_0, \\ \vdots &= \vdots \\ \mathbf{x}_t &= \mathbf{P}^{-1}\mathbf{\Lambda}^t\mathbf{P}\mathbf{x}_0. \end{aligned}$$

Define $\mathbf{z}_t = \mathbf{P}\mathbf{x}_t$, then

$$\begin{aligned} \mathbf{z}_t &= \mathbf{\Lambda}^t\mathbf{z}_0 \\ \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} &= \begin{pmatrix} 2^t & 0 \\ 0 & 3^t \end{pmatrix}\begin{pmatrix} z_{1,0} \\ z_{2,0} \end{pmatrix}. \end{aligned}$$

That is,

$$\begin{aligned} z_{1,t} &= \lambda_1^t z_{1,0}, \\ z_{2,t} &= \lambda_2^t z_{2,0}. \end{aligned}$$

## 6.2 Solving a differential equation

[This will come up time and again. Make sure you know at least one way of solving!]

The magnitude of $\left|g'\left(\bar{k}\right)\right|$ describes the speed of convergence. To see this, recall the linearised law of motion:

$$\dot{k} = g'\left(\bar{k}\right)\left(k - \bar{k}\right).$$

Rewriting this as

$$\frac{dk}{dt} = -g'\left(\bar{k}\right)\bar{k} + g'\left(\bar{k}\right)k,$$

which is differential equation in $k$.

### 6.2.1 Guess and verify

To solve this, guess that the solution is of the form

$$k(t) = \exp\left[g'\left(\bar{k}\right)t\right]c(t),$$

where $c(t)$ is the constant of variation. Then,

$$\begin{aligned}\frac{dk(t)}{dt} &= g'\left(\bar{k}\right)\exp\left[g'\left(\bar{k}\right)t\right]c(t) + c'(t)\exp\left[g'\left(\bar{k}\right)t\right]\\ &= g'\left(\bar{k}\right)k(t) + c'(t)\exp\left[g'\left(\bar{k}\right)t\right].\end{aligned}$$

Substituting this into the expression for $dk/dt$,

$$\begin{aligned}-g'\left(\bar{k}\right)\bar{k} + g'\left(\bar{k}\right)k(t) &= g'\left(\bar{k}\right)k(t) + c'(t)\exp\left[g'\left(\bar{k}\right)t\right]\\ \Leftrightarrow c'(t) &= -g'\left(\bar{k}\right)\bar{k}\exp\left[-g'\left(\bar{k}\right)t\right].\end{aligned}$$

Integrating both sides with respect to $t$

$$\begin{aligned}\int c'(t)\,dt &= \int -g'\left(\bar{k}\right)\bar{k}\exp\left[-g'\left(\bar{k}\right)t\right]dt\\ \Rightarrow c(t) &= \bar{k}\exp\left[-g'\left(\bar{k}\right)t\right] + C.\end{aligned}$$

Hence,

$$\begin{aligned}k(t) &= \exp\left[g'\left(\bar{k}\right)t\right]\left(\bar{k}\exp\left[-g'\left(\bar{k}\right)t\right] + C\right)\\ &= \bar{k} + \exp\left[g'\left(\bar{k}\right)t\right]C.\end{aligned}$$

To pin down $C$, we need a boundary condition, which is that $k(0) = k_0$.

$$k_0 = \bar{k} + C \Leftrightarrow C = \bar{k} - k_0.$$

Thus, we obtain that

$$k(t) = \bar{k} + \left(\bar{k} - k_0\right)\exp\left[g'\left(\bar{k}\right)t\right].$$

#### 6.2.2 Using integrating factor

We can solve this first-order differential equation using an integrating factor.[3] The integrating factor is then given by

$$I(t) = \exp\left[\int^t -g'\left(\bar{k}\right) dt\right] = \exp\left[-g'\left(\bar{k}\right) t\right].$$

The solution is then given by

$$\begin{aligned}\exp\left[-g'\left(\bar{k}\right) t\right] k(t) &= \int^t -g'\left(\bar{k}\right) \bar{k} \exp\left[-g'\left(\bar{k}\right) t\right] dt + c \\ &= \bar{k} \exp\left[-g'\left(\bar{k}\right) t\right] + c.\end{aligned}$$

Initial condition is that $k(0) = k_0$. Notice that if $t = 0$, then above expression simplifies to

$$k_0 = \bar{k} + c \Rightarrow c = k_0 - \bar{k}.$$

This gives us a linearised version of the law of motion for capital:[4]

$$\begin{aligned}\exp\left[-g'\left(\bar{k}\right) t\right] k(t) &= \bar{k} \exp\left[-g'\left(\bar{k}\right) t\right] + \left(k_0 - \bar{k}\right) \\ \Rightarrow k(t) &= \bar{k} + \left(k_0 - \bar{k}\right) \exp\left[g'\left(\bar{k}\right) t\right].\end{aligned}$$

Thus, the if $g'\left(\bar{k}\right) < 0$, then $k(t) \to \bar{k}$. Moreover, the more negative $g'\left(\bar{k}\right)$, the faster is the convergence.

*Remark* 6.1. (*Half life*) Let us define $\tau$ as the time $\tau$ that it takes so that the system reaches close to the "half" of the difference between the initial point and the steady state.

$$k(\tau) - \bar{k} = \frac{1}{2}\left(k_0 - \bar{k}\right).$$

[3] Recall that if we have a differential equation of the form

$$\frac{dy}{dt} + P(t) y = Q(t),$$

then we can define the integrating factor to be $I(t)$ such that $I'(t) = P(t) I(t)$. Multiplying both sides by $I(t)$ gives

$$\begin{aligned}I(t) \frac{dy}{dt} + P(t) I(t) y = Q(t) I(t) &\Rightarrow \frac{dI(t) y}{dt} = Q(t) I(t) \\ &\Rightarrow I(t) y = \int^t Q(t) I(t) dt + c,\end{aligned}$$

where $c$ depends on the initial condition. To find $I(t)$:

$$\frac{I'(t)}{I(t)} = P(t) \Rightarrow \frac{d \ln(I(t))}{dt} = P(t) \Rightarrow \ln(I(t)) = \int^t P(t) dt \Rightarrow I(t) = \exp\left[\int^t P(t) dt\right].$$

[4] To verify this, differentiating the expression with respect to $t$ yields

$$\dot{k}(t) = g'\left(\bar{k}\right)\left[k(0) - \bar{k}\right] \exp\left[g'\left(\bar{k}\right) t\right] = g'\left(\bar{k}\right)\left(k(t) - \bar{k}\right).$$

Substituting for $k(\tau)$ the linearised law of motion for capital yields

$$\begin{aligned}
\bar{k} + \left(k_0 - \bar{k}\right) \exp\left[g'\left(\bar{k}\right)\tau\right] - \bar{k} &= \frac{1}{2}\left(k_0 - \bar{k}\right) \\
\Rightarrow \exp\left[g'\left(\bar{k}\right)\tau\right] &= \frac{1}{2} \\
\Rightarrow \tau &= -\frac{\ln 2}{g'\left(\bar{k}\right)}.
\end{aligned}$$