The University of Chicago
Graduate School of Business

Business 400

Spring 2001 | Kevin M. Murphy

Reading List, Part I

Background Readings

Becker, Gary S. *Economic Theory* (Knopf 1971), Chapters 1 and 2.

Stigler, George J. *Theory of Price* (4th edition, Macmillan, 1988), Chapters 1 and 2.

Part I: Demand Theory

Becker, Gary S. *Economic Theory* (Knopf 1971) Chapters 3-11.

Deaton, Angus, and Muellbauer, John. *Economics and Consumer Behaviour*, especially Chapters 1-4 (use the rest as a basic reference for this section).

Varian, Hal. *Microeconomic Analysis* (2d Edition), Chapter 3, ATheory of the Consumer.@

Stigler, George J. *Theory of Price* (4th edition, Macmillan, 1988), Chapters 3 and 4.

Rosen, Sherwin. AHedonic Prices and Implicit Markets,@ *Journal of Political Economy* 82 (January 1974): 34-55.

Becker, Gary S., and Murphy, Kevin M. AA Theory of Rational Addiction,@ *Journal of Political Economy* 96 (August 1988): 675-700.

Becker, Gary S. AA Theory of the Allocation of Time,@ *Economic Journal* 75 (September 1965).

Lecture 0. Mathematical Essentials

Throughout this course we will be dealing with models involving constrained optimization. In this section we outline a rather straightforward method for finding solutions to these problems and analyzing the comparative statics of these solutions. The formal solution to these problems is most easily understood in terms of the method of lagrange multipliers. In general we will consider a problem of the form

$$(0.1) \qquad \max_{X} F(X_1 \ldots X_k, B) \quad \text{s.t} \quad G(X_1 \ldots X_k, B) \leq M$$

where X represents the individual's decision variables (how much of each good to consume for the generic consumer problem or how much of each type of product to produce) and B represent the parameters influencing the individual's decision such as the prices of individual products or the market rates of return on alternative investments. The function F represents the individual's objective while his choice of the X's is restricted to combinations that satisfy $G(X,B) \leq M$. M gives the level of this individual's constraint.

The solution to this problem can be found by setting up the Lagrangian function

$$(0.2) \qquad L = F(X_1 \ldots X_k, B) + \lambda\,(M - G(X_1 \ldots X_k, B))$$

The solution for the optimal levels of X will be a critical point of L (i.e. a point where the derivatives of the Lagrangian with respect to X and λ are simultaneously zero. Hence these conditions are

$$
\begin{array}{lll}
(0.3) & dL/dX_1 = dF/d X_1 - \lambda\, dG/ d X_1 = 0 & \Rightarrow dF/d X_1 = \lambda\, dG/ d X_1 \\
 & dL/dX_2 = dF/d X_2 - \lambda\, dG/ d X_2 = 0 & \Rightarrow dF/dX_2 = \lambda\, dG/dX_2 \\
 & . & \\
 & . & \\
 & . & \\
 & dL/dX_k = dF/ dX_k - \lambda\, dG/ dX_k = 0 & \Rightarrow dF/dX_k = \lambda\, dG/dX_k \\
 & & \\
 & dL/d\lambda = M - G(X,B) = 0 & \Rightarrow M = G(X,B)
\end{array}
$$

The first k equations tell us that the optimal solution requires the impact of each of the X variables on the objective function, dF/dX_j, to be proportionate to its effect on the constraint function, dG/dX_j (with λ being the factor of proportionality). The necessity of this condition follows from the following argument: if we increase X_i by some small amount dX_i then the value of our constraint function would fall be $dG/dX_i * dX_i$ which would then allow us to change X_j by an amount $- dG/dX_i / dG/dX_j\, dX_i$ (to keep the value of $G(X,B) = M$). The net effect of this on the objective function would be

(0.4) $dF = dF/dX_i\ dX_i - dF/dX_j\ dG/dX_i / dG/dX_j\ dX_i$.

At the optimum dF must be zero which implies that

(0.5) $dF/dX_i / dFdX_j = dG/dX_i / dG/dX_j$

or that the effects of the x variables on the objective must be proportional to their effect on the constraint. In fact it should be noted that the argument used here to derive equation (0.5), where we evaluated the change in the objective function by changing the X variables in such a way as to continue to satisfy the constraint, G(X,B) = M will often be used to derive necessary conditions for maximization when it is impractical or unnecessary to derive the full set of conditions for maximization.

The k+1 first order conditions in (0.3) give us k+1 equations to solve for the k+1 variables, $X_1 \ldots X_k$ and L. Since the first order conditions depend on the parameters of the model, B and M, the solutions for $X_1 \ldots X_k$ and L will also be functions of B and M. We will typically denote these solutions by

(0.6)
$$\begin{aligned} X_1 &= X_1{}^*(B,M) \\ X_2 &= X_2{}^*(B,M) \\ &. \\ &. \\ X_k &= X_k{}^*(B,M) \\ \lambda &= \lambda^*(B,M) \end{aligned}$$

and the optimized value of the Lagrangian by

(0.7) $L^*(B,M) = F(X_1(B,M)\ldots X_k(B,M),B) + \lambda(B,M)^*(M - G(X_1(B,M)\ldots X_k(B,M),B))$.

In practice we are often interested in how the X variables respond to changes in the underlying parameters B and M. These can be found by totally differentiating the equations in 0.3 to yield

(0.8)
$$\begin{pmatrix} Fxx - l\,Gx & -Gx \\ -Gx & 0 \end{pmatrix}\begin{pmatrix} dX \\ d\,l \end{pmatrix} = \begin{pmatrix} -Fxb + l\,Gxb & 0 \\ Gb & -1 \end{pmatrix}\begin{pmatrix} dB \\ dM \end{pmatrix}$$

which implies

(0.9)
$$\begin{pmatrix} dX \\ d\,l \end{pmatrix} = \begin{pmatrix} Fxx - l\,Gxx & -Gx \\ -Gx & 0 \end{pmatrix}^{-1}\begin{pmatrix} -Fxb + l\,Gxb & 0 \\ Gb & -1 \end{pmatrix}\begin{pmatrix} dB \\ dM \end{pmatrix}$$

Often we are also interested in how the value of the maximized the optimized Lagrangian, L* (equal in value to the optimized objective function since M - G(X1(B,M)...Xk(B,M),B) = 0) responds to changes in the underlying parameters, B. By the envelope theorem (or by total differentiation and wholesale cancellation using the equations in 0.3) we then have

(0.8) $dL^*/dB = dF/dB - \lambda dG/dB$ and
$dL^*/dM = \lambda$.

Hence, the derivatives of the maximized Lagrangian with respect to the parameters B and M are equal to the derivatives of the original Lagrangian. Intuitively since we have maximized with respect to the X variables (and hence the derivatives with respect to these variables are zero) any changes in the X variables (or L) have only second order effects on the objective. Understanding and being able to implement the techniques illustrated in this section should allow you to analyze most of the problem we cover in this course.

PART I. DEMAND AND CONSUMPTION DECISION

Lecture 1. The Basic Utility Maximizing Model

1. Characterizing the Solution to the Consumer's Problem

One of the most useful paradigms in economics is the theory of rational consumer choice. The analysis of the consumer's budget constraint and his objective as represented by a utility function or its associated indifference maps is probably the most useful of these formulations even if it is not the most general. In this section we beg many of the technical questions regarding existence, continuity and other considerations and focus instead on solutions to the most standard problems since they arise by far the most frequently. We will put the most emphasis on learning how to apply this approach to different problems.

We begin with a single consumer who consumes a collection of k goods, $X_1...X_k$. He purchases these goods in a market at prices $P_1...P_k$ and has money income M to spend on these goods. The statement that there exist market prices for this goods implies that the consumer can obtain as much as neessary of these goods without altering these prices. His budget set, or the set of goods that he can consume is thus

(1.1) $(X \mid \Sigma_k X_k P_k \leq M \}$

Before proceeding, we should note some things about these goods. First if the consumer is making decisions about consumption in various time periods (such as over the lifecycle) then the consumption of goods at various dates should be included as some of the potential X variables and shortfalls of $\Sigma(X_k P_k)$ below M should be regarded as resources not used and NOT as savings (how would you handle bequests to future generations or to charities?). Secondly the list $X_1...X_k$ can include goods that are not bought in the market such as whether to sleep in your bed or on the couch or whether to eat your French fries before or after your hamburger (why might these goods be important?). These goods would simply have $P_k = 0$ and would involve tradeoffs within the utility function rather than working through the budget constraint. It is important to be aware that we only observe these goods through their interaction with 'real goods' An example of the usefulness of such an analysis can be found in "A theory of Rational Addiction" (Becker and Murphy, 1988), in which the stock of past consumption is a key variable in explaining present and future consumption. Thirdly, we measure both income and the consumption of goods $X_1...X_k$ as flows per unit time (i.e. hours of watching television per week) or average weekly income. Finally, we can deal, as we shall see, with 'bads' (goods that provide negative utility) within this framework. We only need to transform them into (-goods), and the analysis proceeds unchanged.

Now consider the preference side of the problem. The consumer's preferences are represented by the utility function $U(X_1...X_k)$ which serves to order potential consumption bundles in terms of their desirability to the consumer. What is important about the utility function is the way it orders the bundles of X's and NOT the specific number attached to particular bundles. In fact if $U(X_1,...X_k)$ is a utility function and G() is any strictly monotonic function, then the utility function $G(U(X_1...X_k))$ will lead to identical consumer choices and hence is equivalent from the economic perspective. With these definitions in mind and using the tools of constrained optimization we can then write the consumers optimization problem as

(1.2) $\max U(X_1,...X_k)$ s.t. $\Sigma (X_k P_k) \leq M$.

Since we have included all current and future uses for income in $X_1...X_k$ it is natural to assume that consumer will fully utilize his resources and set $\Sigma (X_k P_k) = M$. We do so in what follows. The consumers optimization under these conditions leads to the Lagrangian

(1.3) $L = U(X_1...X_k) + \lambda (M - \Sigma(P_k X_k))$,

where the multiplier, λ, gives the increase in utility (the objective function) from a \$1 increase in income (the level of the constraint) or what we will term the **marginal utility of income**. The first order conditions associated with this problem are then

(1.4)
$$\partial U/\partial X_1 = \lambda P_1$$
$$\partial U/\partial X_2 = \lambda P_2$$
$$.$$
$$.$$
$$.$$
$$\partial U/\partial X_k = \lambda P_k$$

$$\Sigma_k X_k P_k = M.$$

If we divide two of the first k equations by one another we obtain

(1.5) $\partial U/\partial X_i \,/\, \partial U/\partial X_j = P_i / P_j$.

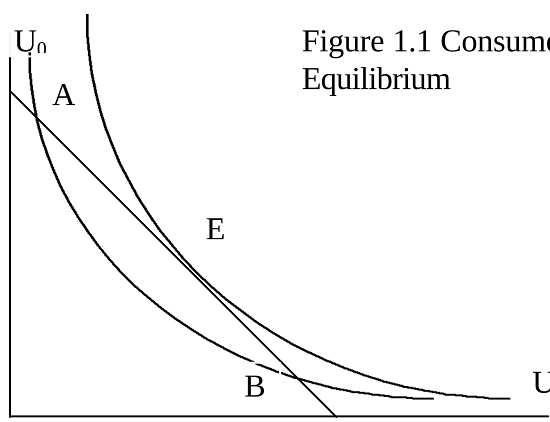
The left hand side of equation (5) represents the consumers (marginal) value of good i in terms of good j. To see this, consider how much of good X_j this consumer would be willing to give up to obtain a small increase in the consumption of good X_i, dX_i. The maximum he would be willing to give up would be the amount that just holds his utility constant which would require that

(1.6) $d\partial U = \partial U/\partial X_i \, dX_i + \partial U/\partial X_j \, dX_j = 0 \Rightarrow dX_j/dX_i = -\partial U/\partial X_i \,/\, \partial U/\partial X_j$.

and hence the value of good X_i expressed in terms of units of X_j is $\partial U/\partial X_i \,/\, \partial U/\partial X_j$. For example, if $\partial U/\partial X_i = 10$ and $\partial U/\partial X_j = 5$ then on the margin a unit of good i is worth twice as much to this consumer (the value of X_i in terms of X_j on the margin is 2) in the sense that increasing consumption of good i by a small amount d and decreasing consumption of good j by twice that amount would leave him as well off as before. Similarly the right hand side of equation (5) represents the COST of good X_i in terms of good X_j in the market place since buying a unit of good X_i costs P_i dollars which would require one to sacrifice P_i/P_j units of good j.

The implication of equation (5) is then that at the consumer's optimal choice, for any two goods, their relative values in consumption must be equal to their relative prices in the market place or simply the marginal value (defined by the consumer's preferences) of any one good i in terms of another good j must be equal to the marginal cost of good i in terms of good j. In the two good case (or equivalently if we restrict attention to two of the goods in the multi-good case) this is equivalent to saying that at the equilibrium point the slope of the individual's indifference curve (which measures his valuation of the good on the x-axis in terms of the good on the y-axis) must be equal to the slope of his budget line (which measures the cost of the x-axis good in terms of the y-axis good). This is illustrated in figure 1.1 where the consumer's equilibrium point is given by point E, where the indifference curve is tangent to the budget line and hence where both must have the same slope. At point A in the figure the marginal value of the x-axis good (measured in terms of units of the other good) exceeds its cost (again measured in terms of the other good) and hence it pays this consumer to move down along the budget line buying additional units of the x-axis good and less of the other good. The opposite conditions hold at point B in the figure.

Figure 1.1 Consumer's Equilibrium



To beat still further on a dead horse we can divide through by the prices in the FOC's for $X_1 ... X_k$ in (4) to obtain

(1.7) $\partial U/\partial X_i \,/\, P_i = \lambda$ for i=1,...K i.e., the marginal utility of an additional dollar spent on any one good must be the same for all goods and equal to the equilibrium marginal utility of income, L. At the risk of beating on a horse that is already buried we can express this same condition one final time as

(1.8) $\partial U/\partial X_i / \lambda = P_i$.

Since $\partial U/\partial X_i$ is the marginal utility of good i and λ is by definition the marginal utility of income the left hand side of equation (8) gives the consumer's marginal value of good i in terms of DOLLARS which according to equation (8) must be equal to he marginal cost of good i (again in terms of dollars), P_i in order for the consumer to be at his optimum.

Equations (5), (7), and (8), Figure 1, and for that matter the systems of equations in equation (4) are not alternative solutions to the consumer's problem or even alternative solution concepts. They are simply alternative ways of stating the same conditions. The reasons for stating all of them is that depending on the circumstances it may be easier to see a given result or exposit the conditions of equilibrium in one form rather than another. It pays to know many ways to think about the same problem.

2. The Demand Curves

The equations in (4) represent k+1 equations in the unknowns $X_1...X_k$ and λ. In general these equations can be solved (via the implicit function theorem) to yield the optimal consumption of $X_1...X_k$ as a function of the remaining variables: the prices, $P_1...P_k$, and nominal income, M. Mathematically this implies we have the k demand equations

$$
\begin{aligned}
(1.9)\quad & X_1 = X_1^d(P_1...P_k,M) \\
& . \\
& . \\
& . \\
& X_k = X_k^d(P_1...P_k,M).
\end{aligned}
$$

These demand equations give the consumer's optimal choices of $X_1...X_k$ as functions of the corresponding prices, $P_1...P_k$ and nominal income M. These demand equations are in essence a summary of the behavioral content of the consumer's utility function.[1]

[1]In practice when working with an explicit utility function this is often (but not always) done by inverting the k first order conditions for $X_1...X_k$ to find $X_1...X_k$ as a function of L and $P_1...P_k$ and then plugging all of these into the budget constraint to find L as a function of M and $P_1...P_k$. Substituting this solution back into the original solutions for $X_1...X_k$ yields the desired result of $X_1...X_k$ expressed as functions of the primitives, $P_1...P_k$ and M.

In practice one often starts with these demand equations rather than the underlying utility structure; but when doing so it is necessary to remember the framework from which these equations were derived and the resulting restrictions the framework places on these demand functions.

First, the dependence of say X_1 on $P_1...P_k$ and not directly on $X_2...X_k$ does not say that the consumption of good X_1 does not depend on the choices for goods $X_2...X_k$,but rather that the consumers optimal choices of $X_2...X_k$ are implicitly incorporated into the solution and have been replaced by their associated prices, $P_2...P_k$, and income, M. For example, in this framework the link between the number and size of cars and the demand for gasoline is captured not by expressing the demand for gasoline as a function of the number of cars of various types (and fuel efficiencies) being used but through the effects of the PRICES of these cars on the types of cars chosen and indirectly through to the choices on gasoline consumption. Hence, using equation (1) to measure the responsiveness of gasoline demand to a change in the price of gasoline implicitly incorporates the adjustment of both the number and characteristics of cars to any change in the price of gasoline (such as reducing the number of cars and switching to more fuel efficient cars when the price of gasoline rises). When forces other than the adjustment of the market prices for cars (such as individual level lags or costs in adjustment) prevent consumers from adjusting their stocks of cars to the levels they would desire at current prices, modeling the linkages between cars and gasoline consumption through controlling for the prices of cars may prove unsatisfactory (see Lecture 5 for an alternative methodology).

While the formulation above tells us much about the properties and conditions for consumers to be at their optimal choice unfortunately theory alone cannot tell us much about the exact values chosen or equivalently the exact locations and shape of the individual's indifference curves. Economic theory in its present state is simply not powerful enough to tell us why at some primitive level consumers in the United States spend 20-30 percent of their income on housing or why the average work week is 36 hours. Instead we typically try to focus on how the consumption vector, $X_1...X_k$, is affected by changes in income and prices. These responses of the consumer's equilibrium choice to prices and income is often summarized by various income and price elasticities. An elasticity is a unitless number which describes the percentage change in one quantity (in this case the consumption of a particular good) relative to a given percentage change in another quantity (in this case either a product price or the level of income) holding the other independent variables (in this case prices and/or income constant).

3. Elasticities and other useful concepts

The income elasticity for good j, N_j, measures the consumers percentage increase in the consumption of good X_j in response to a one percent increase in nominal income holding all goods prices constant. In mathematical terms we have

(1.10) $\eta_j = \partial X_j/\partial M\ (M/X_j)$.

or, equivalently,

(1.11) $\eta_j = \partial \log X_j / \partial \log M$

Income elasticities are used to categorize goods as normal, inferior, luxuries and necessities and are important for understanding how budget shares (i.e. the S_j) change with income. The categorizations are as follows:

If $\eta_j > 0$ then the consumption of X_j increases with income and X_j is called a **normal** good.

If $\eta_j < 0$ then the consumption of good X_j decreases with income and X_j is called an **inferior** good.

If $\eta_j < 1$ then X_j is called a **necesity**. All inferior goods are necessities. The share of income spent on necessities declines with the level of income.

If $\eta_j > 1$ then X_j is called a **luxury**. Luxuries are of course normal goods. The share of income spent on luxuries rises with the level of income.

It is important to realize that since $\eta_j = \partial X_j(P_1 ... P_k, M)/\partial M\ (M/X_j)$ is in general a function of both prices and income it is literally not correct to speak about THE income elasticity of demand for a particular good. The income elasticity of demand can and will often vary with the level of income as well as with the level of product prices. In fact no good can be inferior through its entire range since consumption must increase from zero at least for a while before it can decline. Instead goods tend to be inferior over particular income ranges and not at all levels of income.

In general broad categories of goods like food, housing, clothing etc. are normal goods throughout the range of income. Inferior goods typically arise when we examine particular goods within these categories. Often these goods go through a cycle The Engel curve represents the relation between consumption of a good and income level, and it usually has a similar shape. At low income levels many goods are not consumed at all. As incomes rise to a critical level consumer's begin consuming the good and consumption rises with income for some range. Eventually as incomes become high enough consumers switch their consumption to other goods within these categories (often to goods of higher quality) and consumption of this good begins to decline (the inferior good phase) and may eventually become zero again at high enough income levels. In general, the strength of this pattern will vary with how narrowly we define the good and the availability of good substitutes particularly similar goods of higher and lower quality.

The price elasticity of good j with respect to the price of good i is defined as the percentage change in the quantity consumed of good j when there is a one percent increase in the price of good i, holding all other prices constant. These Marshallian elasticities are also called uncompensated elasticities or gross price elasticities. Mathematically, (1.12) $\varepsilon_{ji}=\partial X_j/\partial X_i\ (P_i/X_j)$

or, equ$_i$valently,

(1.13) $\varepsilon_{ji}=d\log X_j/d\log P_i$

The price elasticity ε_{ii} is called the 'own price' elasticity, and ε_{ji} for i different than j is called the 'cross price' elasticity.

> If $|\varepsilon_{ii}|<1$, we call the demand for the good **inelastic** at that price.
> If $|\varepsilon_{ii}|>1$, we call the demand for the good **elastic** at that price.

What is important about this concept is the fact that if demand is elastic, expenditures in the good increase as the price decreases; if demand is inelastic, expenditure in the good increases as the price increses, since the response of the consumers will be smaller. When the demand is completely inelastic ($\varepsilon=0$) there is no response by the consumers to a price change. On the other hand, if the demand is completely elastic ($\varepsilon= \infty$), consumers stop buying the good as the price goes up.

The price elasticity of good i with respect to a change in the price of good j is called 'cross price' elasticity.

> If $\varepsilon_{ij}>0$, then an increase in the price of j increases the quantity consumed of i, and we call i and j **substitutes**.
> If $\varepsilon_{ij}<0$, an increase in the price of j decreases the quantity consumed of i, and we call i and j **complements**.

Lecture 2. Some Properties of the Marshallian Demand Functions. Price Indexes

1. Properties of Demand Curves

The demand framework outlined in Lecture 1 and summarized by the demand equations in (1.9) places some important restrictions on the forms and properties of these demand functions and their associated elasticities. Most of these restrictions in fact are implied by the budget constraint (1.1),

$$\Sigma_K P_k X_k = M.$$

The fact that the consumer must satisfy the budget constraint places a restriction on the demand functions, the *adding-up* restriction:

(2.1) $\Sigma P_k X_k^d (P_1 ... P_k, M) = M$

Furthermore, the budget constraint is linear and homogeneous in M and P, so the vector $(X_1 ... X_K)$ will also satisfy the budget constraint for any multiple of P and M. If both prices and nominal income are multiplied by the same factor, the constraint does not change. This implies that the demand functions are *homogeneous of degree zero* in prices and income.

Formally,

(2.2) $X_k^d (tP_1,...,tP_K,tM) = X_k^d (P_1,...,P_K,M)$, k=1...,K.

We can express this restriction in terms of elasticities, which is sometimes more convenient. Differentiate totally the demand function, $X_j = X_j d(P_1,...,P_K,M)$,

(2.3) $dX_j = (dX_j/dM)\, dM + \Sigma_k (dX_j/dP_k)\, dP_k$

Now use homogeneity restriction to obtain

(2.4) $\Sigma_k P_k\, dX_j/dP_k + M\, dX_j/dM = 0$,

or, in terms of elasticities,

(2.5) $\Sigma_k\, \varepsilon_{jk} + \eta_j = 0$

Sometimes it is also convenient to express the restrictions implied by the Budget Constraint in terms of elasticities. We can totally differentiate the budget constraint to obtain

(2.6) $dM = \Sigma\, (X_j\, dP_j) + \Sigma (P_j\, dX_j)$

which after dividing both sides by M yields

(2.7) $\quad dlogM = \Sigma_j S_j \, dlogX_j \; + \; \Sigma_j S_j \, dlogP_j \; = dlogX + dlogP$, where

$$dlogP = \Sigma_j S_j \, dlogP_j \text{ , and}$$

$$dlogX = \Sigma_j S_j \, dlogX_j.$$

Here S_j represents the share (equal to P_jX_j/M) of good j in total consumption and dlogP and dlogX are indexes of the percentage changes in prices and consumption respectively. Equation 2.7 partitions the percentage change in income into percentage change in consumption ($\Sigma_j S_j dlogX_j$) and percentage change in prices ($\Sigma_j S_j dlogP_j$). The two terms on the right hand side of equation 2.7 are extremely useful measures of changes in prices and real consumption and are in fact the theoretical analogs of the measures of consumer prices and real GNP components used in practice.

2. Price and Quantity Indexes

Since dlogM represents the percentage change in nominal resources and dlogP represented the percentage change in nominal prices, dlogM - dlogP is equal to the percentage change in REAL income or purchasing power. In fact, equation (2.7) can be rearranged to yield

(2.8) $\quad dlogM - \Sigma_j S_j \, dlogP_j = \Sigma_j S_j \, dlogX_j$

which simply points out the equality of the change in real income and the change in real value of consumption implied by the consumer's budget constraint.

How is this index related to the change in Utility? If the Utility function is $U = U(X_1,..., X_k)$, totally differentiating this would produce

$$dU \quad = U_1dX_1 + ...+ U_kdX_k$$

Since $U_i = \lambda P_i$, we can rewrite the total differential as

$$dU \quad = \lambda \, (\Sigma_i \, P_k \, dX_k)$$

Thus the change in Utility is proportional to the change in quantities at fixed prices.

In practice, we need an empirical analogue for (2.7). We usually have a set of prices and quantities at time t=1 and another set at t=2. Let $X_{11},..., X_{1K}$, and $P_{11},...,P_{1K}$, be the quantities and prices at t=1 and $X_{21},..., X_{2K}$ and $P_{21},...,P_{2K}$, at t=2. Then we can measure the price change as

$$PI \quad = (\Sigma_k \, X_{1k} \, P_{2k})/ \, (\Sigma_k \, X_{1k} \, P_{1k}).$$

This is a Consumer Price Index type index, called a Laspayres index. This price index is approximately equal to 1+ the inflation rate. The changes in this index are not infinitesimal changes anymore, so the weights we choose, whether beginning or ending quantities will affect the resulting number. We can rearrange the definition of PI to relate it to the theoretical concept we used above (2.7),

$$PI = (\Sigma_k X_{1k} P_{2k})/ (\Sigma_k X_{1k} P_{1k}) = [(\Sigma_k X_{1k} P_{1k} / M_1)(P_{2k} / P_{1k})] =$$

$$= \Sigma_k S_{1k} (P_{2k}/P_{1k})$$

Concerning the quantities, the analogue to the calculation above is,

$$QI= (\Sigma_k X_{2k} P_{1k}) / (\Sigma_k X_{1k} P_{1k}).$$

This is a real GDP measure. However, we cannot use the same base period for both indexes if we want to divide the nominal change into a price change and a quantity change.

For example, assume we want to obtain real GDP growth, and assume we define it according to the index we have called QI. We know $\Sigma_k X_{1k} P_{1k}$, since this is nominal GDP of period one. What kind of price index do we need to use in order to convert the nominal GDP of period 2 into a real (i.e., prices of period 1) GDP?

Nominal GDP = $\Sigma_k X_{2k} P_{2k}$

Real GDP = $\Sigma_k X_{2k} P_{1k}$

Change in Real GDP = Change in Nominal GDP / (some price index)

$$(2.9)\ 1 + g = (\Sigma_k X_{2k} P_{1k})/ (\Sigma_k X_{1k} P_{2k}) =$$

$$= \{[(\Sigma_k X_{2k} P_{2k}) / (\Sigma_k X_{1k} P_{1k})] \} / \{(\Sigma_k X_{2k} P_{2k}) / (\Sigma_k X_{2k} P_{1k})\}$$

So the price index that the quantity index we want to use defines implicitly is

Price Deflator = $\Sigma (X_{2k} P_{2k})/\Sigma (X_{2k} P_{1k})$

This index called the 'implicit price deflator'. Note that since we wanted to have fixed period 1 prices in the quantity index, we needed to use fixed period 2 quantities in the price index, in order for the changes to add up.

3. Some Implications of the Budget Constraint

We can obtain more implications of the budget constraint by differentiating equation 2.7. If we use equation 2.7 when prices are constant and we are simply changing the nominal income of the consumer we obtain (after dividing through by dlogM)

(2.10) $\Sigma_j S_j \eta_j = 1$

This equation tells us that, weighted by their budget shares, income elasticities must add up to one. In particular, this expression implies that not all goods can be inferior.

Another interesting representation of the budget constraint is obtained when we differentiate it with respect to one price,

(2.11) $0 = X_j + \Sigma_k P_k \, dX_k/dP_j$

After a change in the price of good j, the individual must rearrange consumption in a way that the budget constraint is not violated. After some manipulation, we can express this restriction in terms of elasticities,

(2.12) $0 = X_j P_j/M + \Sigma_k((P_k X_k/M)\,(dX_k/dP_j)\,(P_j/X_k))$

(2.13) $0 = S_j + \Sigma_k S_k \varepsilon_{kj})$ or, in a more illuminating way,

(2.14) $S_j (1 + \varepsilon_{jj}) = - \Sigma_{k \neq j} S_k \varepsilon_{kj}$

Thus the budget constraint implies that , if the demand for good j is elastic, then the other goods must be, on average, substitutes for good j, regardless of how they enter in the utility function. Suppose, for example, that the Utility function can be written as

$U = U_1 (X_1,...,X_{j-1}, X_{j+1}... X_K) + V_j;$

Equation 2.14 implies that ε_{jk} is not equal to zero. If $|\varepsilon_{jj}|<1$, then

$\Sigma_{k \neq j} S_k \varepsilon_{kj} < 0.$

Thus even though good j was separable from the rest in the utility function, all the other goods will be, on average complements of good j.

In general there are two effects running between two unrelated goods: first, all goods can produce utility for the consumer, so they are all substitutes in a sense. Second, unrelated goods will tend to be substitutes and complements due to the effect of a change in expenditure in this good in the total budget constraint as shown in 2.14 If the demand for the good is inelastic, an increase in price increases total expenditure in this good, so expenditure in the other goods must decrease, which, given fixed prices, means lower quantity of the other goods. So if demand for a good is inelastic, this good will behave as a complement of the rest, while if the demand is elastic, it will behave as a substitute.

Another implication of equation 2.14 is in predicting the reaction of expenditure in other goods due to a change of price in good j: it depends on the own price elasticity of good j. If the demand for good j is elastic, when the price of good j decreases the expenditure in other goods in the economy will decrease.

Questions:

It is often stated that the own price elasticity of demand for many products declines with the level of income. If this is so then how must the income elasticity of demand for these goods vary as we change their price?

Why do consumers switch to higher quality products as their income rises? can you come up with a simple model to illustrate this effect?

Lecture 3. Cost function and Duality. Reinterpreting the Price Indexes

1. The Cost Function

In Lecture I we examined the Marshallian demand curves, which are the solution to the problem: given the prices the consumer faces and the income available to him, what quantities will he consume to be as well off as possible, i.e., to maximize his utility?

Mathematically, we stated the problem as

(3.1) maximize $U(X_1...,X_n)$

given budget constraint $M = \Sigma_i X_i P_i$

The solutions were

$$(3.2) \qquad \begin{array}{l} X_1=X_1(P_1...P_k,M) \\ \quad ... \\ X_k=X_k(P_1...P_k,M) \end{array}$$

There is a useful variation of this problem that will help as to understand better the choice of the consumer. We restate the problem as: what is the least costly way to achieve a certain level of utility? This formulation is less intuitively appealing than the one before, but it is mathematically and conceptually equivalent.

The mathematical formulation, called the dual of the problem in 2.1, is

(3.3) min $C= \Sigma_j X_j P_j$, by choice of X_j ,

such that $U=U^o$ and prices are $P_1...P_k$ given.

The solutions to this problem will be functions of prices and utility of the form

$$(3.4) \qquad \begin{array}{l} X_1^*=X_1^*(P_1...P_k,U) \\ \quad ... \\ X_k^*=X_k^*(P_1...P_k,U) \end{array}$$

In order to solve mathematically this problem we must (as in 1.3.) set up a Lagrangian,

(3.5) $L = \Sigma_j X_j P_j + \mu (U^* - U(X_1,...,X_k))$

As before, there will be k+1 first order conditions. The first k will be identical to the ones we obtained in lecture one when we set up the dual problem. Only the k+1 condition will differ:

(3.6) $P_1 = \mu\partial U/\partial X_1$

....

$P_k = \mu\partial U/\partial X_1$

$U^{*}= U(X_1,...,X_k)$

Again we can interprete this FOCs as telling us that the marginal utility of the good must be proportional to its price. The functions $X_1^{*}(P_1,...,P_{k,}U^o),...,X_k^{*}(P_1,...,P_k,U^o)$ that solve this problem are called the *Hicksian* demand curves. If we substitute back this quantities into the budget constraint we obtain the *Cost function*, which is thus defined as:

(3.7) $C\ (P_1,...,,P_k)= \min \Sigma_j\ X_jP_j$ s.t. $U(X_1,...,X_k) = U^o$

What is the economic meaning of μ? μ represents the marginal cost of an additional unit of income. Given that the problem we are now solving is mathematically equivalent the one we set up before, there must be a relation between μ and λ, the Lagrange multiplier on the budget constraint in the initial problem (called the 'primal').

(3.8) $\mu_{\text{cost minimization}=}\ P_i/U_i$

But, in 3.1 the first order conditions are of the form

$\partial U/\partial X_i = P_i\,\lambda$, so

(3.9) $\mu_{\text{cost minimization}} = 1/\lambda_{\text{utility maximization}}$

2. Properties of the cost function

1. Non decreasing in prices, and increasing in at least one price.

It increases in prices since at least as much expenditure has to be made in order to be as well off when prices go up. We can show this by applying the envelope theorem,

(3.10) $\partial C/\partial P_i = \partial L/\partial P_i = X_i \geq 0$

2. Increasing in Utility

At given prices, the consumer has to spend more to be better off. We can prove this by using the envelope theorem on the cost minimization,

(3.11) $U = \partial L/\partial U = \mu \geq 0$

3 Homogeneous of decree one in prices

This is a direct consequence of the definition of cost function. If prices double, twice as much income is required to stay in the same indifference curve.

4. Concave in prices

This property is the most important one, since it will allow us to show important properties of the demand curves.

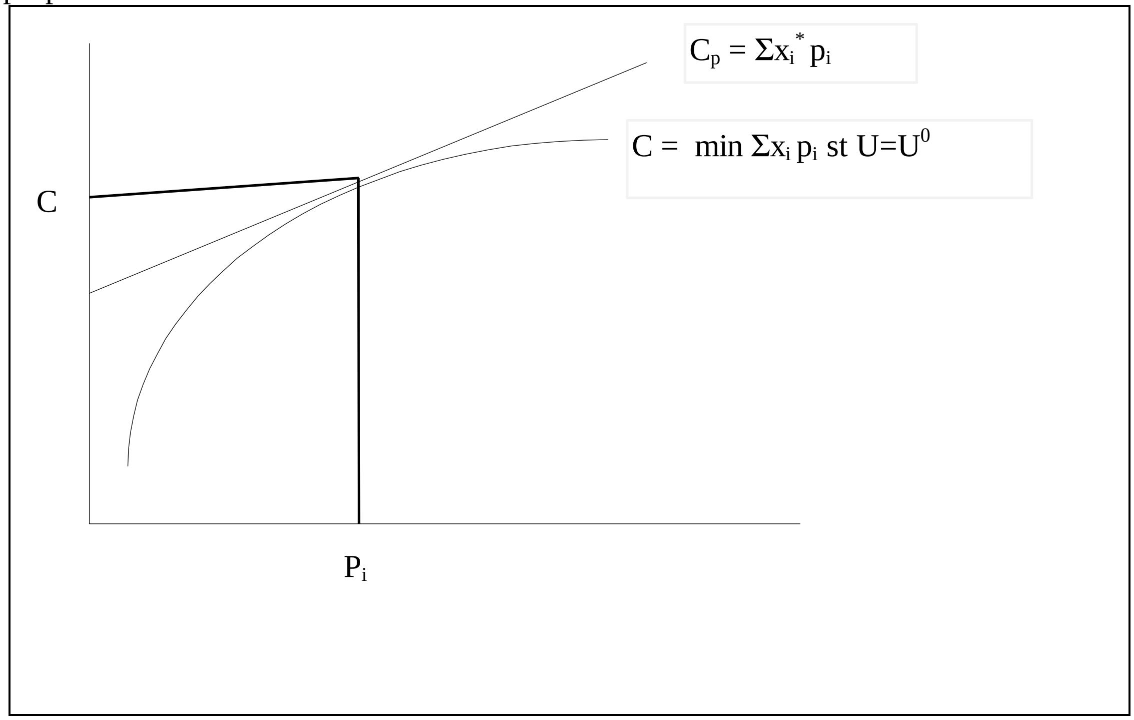


Figure 3.1 Concavity of the Cost Function

Observe Figure 3.1. The straight line represents a passive response to a change in prices. The consumer does not optimize and reacts passively to the price change. As a consequence, the slope of the cost curve is X_j , the quantity bought of good j. But the consumer *adapts the bundle she consumes* as the prices change, so the cost curve must be always under this line. When will the two lines touch each other? When the price is such that the optimal quantity consumed is X_j*, both the passive cost function and the one that involves optimally adapting the quantity consumed will represent the same cost, and they will be tangent to each other. In the rest of the domain of the function, the cost function will be under this straight-line. Thus it must be concave.

5. The derivatives of the cost function with respect to prices, where they exist, are the Hicksian demand functions. This property is know as Shephard’s lemma.

This is an important property, since it will allow as to obtain the demand functions given any cost function. To obtain it, we only need to apply the envelope theorem (see mathematical appendix), which says that when the derivative of an optimized function is taken with respect to a parameter at the optimum the only change that matters is the first order change.

(3.12) $\partial C/\partial P_j = X^*_j\ (P_1....P_k)$

Combining the properties of concavity and Sephard's lemma, we obtain an important property:

(3.13) $\partial^2 C(P_1,...,P_k,U^*)\ /\partial P_i^2 = \partial X_i\ (P_1,...,P_k,U^*)\ /\partial P_i \leq 0.$

Thus the Hicksian demand curves have a downward slope. This is one version of the 'Law of Demand'.The whole matrix of second derivatives of the cost function is, by concavity, negative semidefinite.

To show this, the same graphical derivation for one factor price can be done for some linear combination of prices $\Sigma\theta_k P_k$.

The key thing is thus that we have transformed a linear budget constraint into a concave cost function as a consequence of the minimization. No assumptions about functional form were needed. If the consumer minimizes expenditure to achieve a level of utility, the cost incurred will grow less than proportionally with a change in prices.

6. Symmetry

The change in the quantity demanded of good i when the price of good k changes is the same as the change in the quantity demanded of good k when the price of good i changes, holding utility constant. This is a consequence of the symmetry of the matrix of second derivatives of the cost function:

Since $\partial^2 C(P_1,...,P_k,U^*)\ /\partial P_i\,\partial P_k = \partial^2 C(P_1,...,P_k,U^*)/\partial P_k P_i$,

Applying Sephard's lemma,

(3.14) $\partial X_i\ (P_1,...,P_k,U^*)\ /\partial P_k = \partial X_k\ (P_1,...,P_k,U^*)\ /\partial P_i$.

3. Reinterpreting the Laspeyres price index

A price index tries to measure how a change in prices affects the level of utility attained. A logical price index would be one which compares the cost of a given level of utility under a set of prices and under another set of prices. We will say that the second set of prices is higher if the consumer has to incur a higher cost to achieve a certain level of utility. For example, if under prices $P_{11},...,P_{1k}$ it costs 100 to achieve utility U* and under prices $P_{21},...,P_{2k}$ it costs 200 to achieve that same level of utility, we can conclude prices $P_{21},...,P_{2k}$ are twice as high as prices $P_{11},...,P_{1k}$.

Price Index = $C(P_{21},...,P_{2k},U^*) / C(P_{11},...,P_{1k},U^*)$

Is there any relation between this price index an the usual consumer price index?

The consumer price index, a Laspeyres index, is defined as

(3.15) $CPI = \Sigma_k P_{k2} X_{k1} / \Sigma_k P_{k1} X_{k1}$

If we expand the cost function, to obtain a second order approximation,

(3.16) $C(U_1, P_2) \cong C(U_1,P_1) + \Sigma_k C_{k1}(P_{k2} - P_{k1}) + 1/2\ \Sigma_{kj} C_{kj1} (P_{k2} - P_{k1})(P_{j2} - P_{j1})$

Where $C(U_1,P_1) = \Sigma_k X_{k1} P_{k1}$, so, applying the Sephard's lemma $\partial C/\partial P_k = C_k = X_k$,

$C(U_1,P_2) = \Sigma_k X_{k1} P_{k2} + II$.

Where $II = 1/2\ \Sigma_{kj} C_{kj1} (P_{k2} - P_{k1})(P_{j2} - P_{j1})$

Then

(3.17) $C(P_{21},...,P_{2k},U^*) / C(P_{11},...,P_{1k},U^*) = \Sigma_k X_{k1} P_{k2} / \Sigma_k X_{k1} P_{k1} + II / \Sigma_k X_{k1} P_{k1}$.

Where (II) ≤ 0 by concavity.

Thus the CPI is a linear approximation to the actual change in cost of living. The change indicated by a Laspeyres index will always be larger than the true change in the cost of attaining the initial utility with the new prices.

For this reason, if we want to build a price index between 0 and date T, it is better to approximate it by a combination of price indexes:

$P_T/P_0 = (\Sigma P_{1k} X_{0k} / \Sigma P_{0k} X_{0k}) (\Sigma P_{2k} X_{1k} / \Sigma P_{1k} X_{1k}) \ldots (\Sigma P_{Tk} X_{(T-1)k} / \Sigma P_{(T-1)k} X_{(T-1)k})$

LECTURE 4. Hicksian Demand Functions

Hicksian demand and Marshallian demand: the Slutsky equation

For a set of prices and income, the solutions of the problem

$$\max U (X_1,...,X_k)$$

$$\text{s.t. } I = \Sigma_j P_j X_j$$

and the problem

$$\min C = \Sigma_j P_j X_j$$

$$\text{s.t. } U (X_1,...,X_k)$$

are the same.

Thus, we have the identity

(4.1) $X_i *(P_{1,}...,P_k,I) = X_i *(P_{1,}...,P_k,U^*)$.

Taking the derivative with respect to price on both sides of this identity,

(4.2) $MX_i *(P_{1,}...,P_k,I)/ MP_j +(MX_i *(P_{1,}...,P_k,I)/MI) \text{ x } (MI/MP_j)= MX_i *(P_{1,}...,P_k,U^*)/MP_j$

And substituting in the value of MI/MP_j ,

(4.3) $MX_i *(P_{1,}...,P_k,I)/ MP_j + X_j \quad MX_i *(P_{1,}...,P_k,I)/MI = MX_i *(P_{1,}...,P_k,U^*)/MP_j$

(4.4) $MX_i *(P_{1,}...,P_k,I)/ MP_j = MX_i *(P_{1,}...,P_k,U^*)/MP_j - X_j \quad MX_i *(P_{1,}...,P_k,I)/MI$

This expression is known as the ?Slutsky equation?. It allows us to relate the (observable) Marshallian demand with the unobservable Hicksian demand through the income effect. We can interprete it as meaning that the total effect of a price change is equal to the change in quantity demanded holding utility constant minus the change in real income, given by the share consumed of the good whose price changed, times the impact of that change on my consumption of ?i?. The first effect is called the ?substitution effect? and the second the ?income effect?.

After some algebraic manipulation, we can rewrite the Slutsky equation in terms of elasticities

(4.5) $E_{ij} = E_{ij}^* - K_j N_i$

where E_{ij}^* is the compensated elasticity of good i to price j, N_j is the income elasticity of good j, K_j is the share of good i in total consumption and E_{ij} the uncompensated price elasticity.

We proved above that, by symmetry,

$$MX_i{}^*(P_1,...,P_k,U^*)/MP_j = MX_i{}^*(P_1,...,P_k,U^*)/MP_i$$

Then the uncompensated effects will be equal when

$$X_j \quad MX_i{}^*(P_1,...,P_k,I)/MI = X_i \quad MX_j{}^*(P_1,...,P_k,I)/MI$$

or, in terms of elasticities,

(4.6) $I / X_i \; MX_i{}^*(P_1,...,P_k,I)/MI = I / X_j \; MX_j{}^*(P_1,...,P_k,I)/MI$

$$N_i = N_j$$

The uncompensated cross price effects are equal if both goods have the same income elasticity.

It is clear by symmetry, that the cross price derivatives are equal. What about the compensated elasticities?

$$(P_i X_i / I)\; P_j / X_i \; MX_i{}^*(P_1,...,P_k,U^*)/MP_j = MX_i{}^*(P_1,...,P_k,U^*)/MP_i \; P_i / X_j \; (P_j X_j / I), \text{ so}$$

(4.7) $K_i E_{ij}{}^* = K_j E_{ji}{}^*$

Thus symmetry implies that the compensated elasticities will not be equal, unless the share of income spent in the two goods is equal.

Some useful expressions relating uncompensated demand elasticities

The Slutsky equation allows us to transform some of the expressions we obtained in Lecture 2 relating uncompensated demand elasticities into restrictions on the compensated demand elasticities.

$$\Sigma_j E_{ij} = \Sigma_j E_{ij}{}^* - N_i \Sigma_j K_j$$

But we know that $\Sigma_i E_{ij} = N_i$, so

(4.8) $\Sigma_j E_{ij}{}^* = 0$

This useful expression tells us that the sum of the compensated price elasticities of all goods with respect to one price is 0.

Start again with the Slutsky equation, this time summing up in both sides with respect to good i,

$$\Sigma_i K_i E_{ij} = \Sigma_i K_i E_{ij}{}^* - \Sigma_i K_i K_j N_i$$

We know (see lecture 2) that

$$\Sigma_i K_i E_{ij} = K_j \text{ , and } \Sigma_i K_i N_i = 1\text{, so}$$

(4.9) $\Sigma_i K_i E_{ij}* = 0$

Observable implications of cost minimization: revealed preference

We have been able to show that the quantity demanded of a good falls when its price increases holding utility constant. This is due to the concavity of the cost function. Here we give a more intuitive argument in the two dimensional case.

Figure 4.1. Revealed Preference

When the price of X_1 goes down, the consumer must necessarily choose points in the region B. The reason is that the region A was available before and the consumer did not choose it. The cost of the initial bundle was

$$C^0 = P_{10} X_{10} + P_{20} X_{20}$$

The new bundle costs

$$C^1 = P_{11} X_{11} + P_{21} X_{21}$$

Since the consumer minimizes costs in each case, it must be true that that :

$$P_{10} X_{10} + P_{20} X_{20} \ \# \ P_{10} X_{11} + P_{20} X_{21} \text{ , and}$$
$$P_{11} X_{11} + P_{21} X_{21} \ \# \ P_{11} X_{10} + P_{21} X_{20}$$

So

(4.10) $(P_{10} - P_{11}) (X_{10} - X_{11}) + (P_{20} - P_{21}) (X_{20} - X_{21}) \ \# \ 0$

This is exactly the same result that we obtained before. The concavity of the cost function is equivalent to this revealed preference argument: the optimal adjustment, i.e., the one that cost minimizes, will require prices and quantities to be negatively related. The adjustment when prices increase will be such that the quantities demanded of the goods whose prices go up will decrease on average.

Restating the First law of demand

Recall the Slutsky equation

$$E_{ij} = E_{ij}* - K_j N_i$$

Note that, even though $E_{ij}*$ is unambiguously negative, as a consequence of the concavity of the cost function, E_{ij} may be positive if the good is inferior and the share of consumption of good i, K_i , is large enough.

Is this likely? No. In fact it is extremely unlikely, for two reasons. First, in order to find inferior goods we need to look at narrowly defined categories of goods. Consequently, they will be goods with small shares. Second, these goods are inferior because you substitute heavily towards other goods. Thus, E_{ij}^* will tend to be large. Consequently, these will be goods with a large E_{ii}^*.

Thus we state now the law of demand in terms of uncompensated own price effects,

(4.11) $E_{ii} < 0$

Lecture 5. More on demand functions: the second law of demand, empirical estimation, welfare analysis.

The second law of demand

We concluded Lecture 4 by generalizing the law of demand, from the strictly true statement that the compensated own price elasticity of demand is negative, to the statement that the uncompensated price elasticity of demand is almost certainly negative as well. We argued that, on theoretical grounds, it is implausible that the two conditions that are needed in order to observe a positive reaction to a change in price are highly unlikely to happen together. We will call this proposition the first law of demand. We can express it mathematically,

(5.1) $\partial X_i/\partial P_i \leq 0$

The second law of demand, which we will prove in what follows, is a corollary of the first one, and it says:

'The reaction of the quantity demanded to a price change is larger in the long run than

The reason for this is that, through time, it is easier to find substitutes to the goods whose price has increased. Furthermore, the adjustment of complements takes time and feeds back into the quantity demanded of the good whose price has changed. In the long run, whatever the adjustment that has to be made will feed back into the demand of the good and reinforce the initial effect.

If, for example, the price of gasoline goes up, the stock of cars (a complement) will take a long time to reach the new steady state level. The energy saving devices that will be incorporated will also take a long time to be available and incorporated to all of the stock of cars. Finally, the consumers will take time to substitute towards other forms of transport and towards less energy intensive habits.

The demand curve that we explored in Lecture II did not take into account all of these adjustments. It was a long run demand curve, since it reflected the decisions of a consumer who could optimally adjust all of the quantities consumed of all goods as the price of one good changed.

In order to obtain a short run demand curve, we need to specify where the adjustment costs that provoke a lapse between the initial change in price and the final quantity adjustment lie. In particular, in the case of cars and gasoline, the change in the price of gasoline has initially no impact on the quantity of cars circulating. For this reason, we model this adjustment by keeping the quantity of cars fixed in the short run.

Assume we have the long run demand system

$$X_1 = X_1^d (P_1,P_2,...,P_k, M)$$
$$\vdots$$
$$X_k = X_k^d (P_1,P_2,...,P_k, M)$$

Now assume that goods 1 and k are complements in consumption, but good k is durable. The price of good 1 shifts, but due to this adjustment costs the quantity of good k does not react in the short run. Thus, in the short run, as the price of good 1 shifts, the quantity of good k is fixed. In the short run all the adjustment in the market for good k will take place through its price. Log-linearizing the demand curves (see next section for a derivation),

(5.2) $\quad \text{dlog } X_1 = \varepsilon_{11} \text{ dlog } P_1 + \varepsilon_{1k} \text{ dlog } P_k$

$$\vdots$$

(5.3) $\quad \text{dlog } X_j = \varepsilon_{j1} \text{ dlog } P_1 + \varepsilon_{jk} \text{ dlog } P_k$

$$\vdots$$

(5.4) $\quad \text{dlog } X_k = 0 = \varepsilon_{k1} \text{ dlog } P_1 + \varepsilon_{kk} \text{ dlog } P_k$

So,

(5.5) $\quad \text{dlog } P_k = -(\varepsilon_{k1}/ \varepsilon_{kk}) \text{ dlog } P_1$

Substituting in equation (2),

(5.6) $\quad \text{dlog } X_1 = [\varepsilon_{11} - ((\varepsilon_{1k}\, \varepsilon_{k1})/ \varepsilon_{kk})] \text{ dlog } P_1$

The term $- ((\varepsilon_{1k}\, \varepsilon_{k1})/ \varepsilon_{kk})$ is unambiguously positive, since ε_{kk} is negative and ε_{1k} and ε_{k1} have the same sign (negative in the case of cars and gasoline, which are complements). Since ε_{11} is negative, the fact that the quantity of k is more or less fixed in the short run reduces the effects of the change in price of good one on the quantity of good one with respect to what they will be in the long run. In the long run, all of the effect of the change in P_1 will have gone through the quantities of good k and the price of good k will be unchanged with respect to its initial value.

If good goods k and 1 were substitutes instead of complements the whole argument goes through unchanged, since the sign of ε_{1k} does not play any role in this argument due to the fact that it appears squared.

What about the effects of this price changes in the market of other goods? Operating similarly from the demand of good j, (substitute (5) in equation (3)),

(5.7) $\quad \text{dlog } X_j = [\varepsilon_{j1} - ((\varepsilon_{jk}\, \varepsilon_{k1})/\varepsilon_{kk})] \text{ dlog } P_1$

This means that, in order to determine whether two goods are more or less complementary in the short run than in the long run have to determine whether the signs ε_{jk} and ε_{k1} are the same or opposite. If they are the same, and the goods 1 and k are complements, we are in a similar situation as above: as time passes, the quantity of good j reacts more to a change in the price of good 1. If they are opposite, the short run effect is larger.

We have been working, implicitly, with a demand curve for good j and 1 in which the quantity of good k is fixed. This demand curve,

$$X_1 = X_1^d (P_1,P_2,...,X_k, M)$$

is called a *conditional demand curve.* It is very often useful in empirical work, since the prices of some of the goods maybe difficult to estimate.

In general, it is always possible to reformulate the demand curve for a good as a function of prices of some goods and quantities of others. For example, in (7) instead of assuming that dX_k is 0, substitute for $P_{k,}$

$$(5.8) \quad d\log P_k = d\log X_k /\varepsilon_{kk} - (\varepsilon_{k1}/\varepsilon_{kk})d\log P_1$$

and now the change in the quantity of 1 is

$$(5.9) \quad d\log X_1 = [\varepsilon_{11} - ((\varepsilon_{1k}\, \varepsilon_{k1})/ \varepsilon_{kk})]\, d\log P_1 + (\varepsilon_{1k}/\varepsilon_{kk})\, d\log X_k$$

Note that, substituting in $d\log X_k = 0$, we are back to the result in equation (9).

2. Estimating Demand Curves: linearization and aggregation

Most of the empirical work on the estimation of demand functions has been concerned with a linearized version of the functions that we have been studying. It is also useful in order to be able to analyze the behavior of these curves to study linear forms of them, as we saw in the paragraph above. Differentiating totally the Marshallian demand function for good i,

$$dX_i = \partial X_i/\partial P_1\, dP_1 + \partial X_i/\partial P_2\, dP_2 + ... + \partial X_i/\partial P_k\, dP_k + \partial X_i/\partial M\, dM$$

$$dX_i/X_i = \partial X_i/\partial P_1\, P_1/X_i\, dP_1/P_1 + \partial X_i/\partial P_2\, P_2/X_i\, dP_2/P_2 + ...+ \partial X_i/\partial P_k\, P_k/X_i\, dP_k/P_k +$$

$$\partial X_i/\partial M\, M/X_i\, dM/M,$$

and, in terms of elasticities,

$$(5.10) \quad d\log X_i = \varepsilon_{i1}\, d\log P_1 + \varepsilon_{i2}\, d\log P_2 + ... + \varepsilon_{ik}\, d\log P_k + \eta_i\, d\log M$$

This log-linearized version of the demand curve for good i has very useful properties. In particular, the empirical coefficients on prices and income are the price and income elasticities.

If we want to obtain, from this curve, the Hicksian demand curve, we can write it by approximating the constant utility level by a constant real income

$$X_j^d\ (P_1,P_2,...,P_k,\ U) \approx X_j^d\ (P_1,P_2,...,P_k,\ M^*)$$

where

(5.11) $\text{dlog } M^* = \text{dlog } M - \Sigma_k\ S_k \text{ dlog } P_k$,

so

$$\text{dlog } X_i = \varepsilon_{i1} \text{ dlog } P_1 + \varepsilon_{i2} \text{ dlog } P_2 + ... + \varepsilon_{ik} \text{ dlog } P_k + \eta_i\ (\text{dlog } M - \Sigma_k\ S_k \text{ dlog } P_k\ ) + \eta_i\ (\Sigma_k\ S_k \text{ dlog } P_k)$$

$$\text{dlog } X_i = (\varepsilon_{i1} + \eta_i\ S_1) \text{ dlog } P_1 + (\varepsilon_{i2}+ \eta_i\ S_2)\text{dlog } P_2 + ... + (\varepsilon_{ik} + \eta_i\ S_k) \text{ dlog } P_k + \eta_i \text{ dlog} M^*$$

And applying Slutsky's formula,

(5.12) $\text{dlog } X_i = \varepsilon_{i1}^* \text{ dlog } P_1 + \varepsilon_{i2}^* \text{ dlog } P_2 + ... + \varepsilon_{ik}^* \text{ dlog } P_k + \eta_i \text{ dlog } M^*$

Thus when we estimate the change in the quantity demanded of good i holding real income constant, the coefficients on the prices are the compensated demand elasticities.

There is a further problem which is common to equations (9) and (11): working with demand curves would imply estimating all of the possible cross effects, making this approach impractical.

Usually, the best solution to this problem is to aggregate across goods. In order to do this, we have to make assumptions about the substitution pattern between goods.

If, for example, we assume that, in expression (12) all the cross price elasticities are equal we will obtain

(5.13) $\text{dlog } X_i = \varepsilon_{ii}^* \text{ dlog } P_i + \varepsilon_i^*\ (\Sigma_{j\neq i} \text{ dlog } P_j\ ) + \eta_i \text{ dlog } M^*$

If we assume that the compensated price elasticities are equal to the share of expenditure in that good times a constant, so that the reaction of a change in a price is proportional to the share spent in the good whose price has changed,

$$\varepsilon_{ik}^* = S_k / (1 - S_i)\ \varepsilon^*$$

where we have divided all of the weights by a normalizing constant so that summing them up in k gives 1.

We obtain

(5.14) $\text{dlog } X_i = \varepsilon_{ii}^* \text{ dlog } P_i + \varepsilon^*/(1-S_i)\ \Sigma\ S_{j\neq i} \text{ dlog } P_j + \eta_1 \text{ dlog } M^*$

or, in a more usual form,

(5.15) $\text{dlog } X_i = \varepsilon_{ii}^* \text{ dlog } P_i + \varepsilon_i^* \text{ dlog } P_z + \eta_i \text{ dlog } M^*$

Where P_z is

(5.16) $\text{dlog}P_z = (\Sigma_{j\neq i} S_j \text{ dlog } P_j)/(1-S_i) = (\Sigma_{j\neq i} S_j \text{ dlog } P_j)/\ \Sigma_{j\neq i} S_j$

This is our usual price index, with one important peculiarity: it *excludes* the price of the good whose demand curve we are calculating.

In (15) we are back to the two good case. We know that (see Lecture 3)

$\Sigma_k\ \varepsilon_{jk}^* = 0$

So

$\varepsilon_{ii}^* + \varepsilon_i^* = 0$,

and we can rewrite (15) as

(5.17) $\text{dlog } X_i = \varepsilon_{ii}^* (\text{dlog } P_i - \text{dlog } P_z) + \eta_i \text{ dlog } M^*$

It is important to note in this demand function that the price index P_z excludes the price of i, whereas the price index that is used to calculate M* takes all goods into account.

We can do a very similar derivation if there are some goods which we want to take explicitly into account because they are important complements or substitutes.

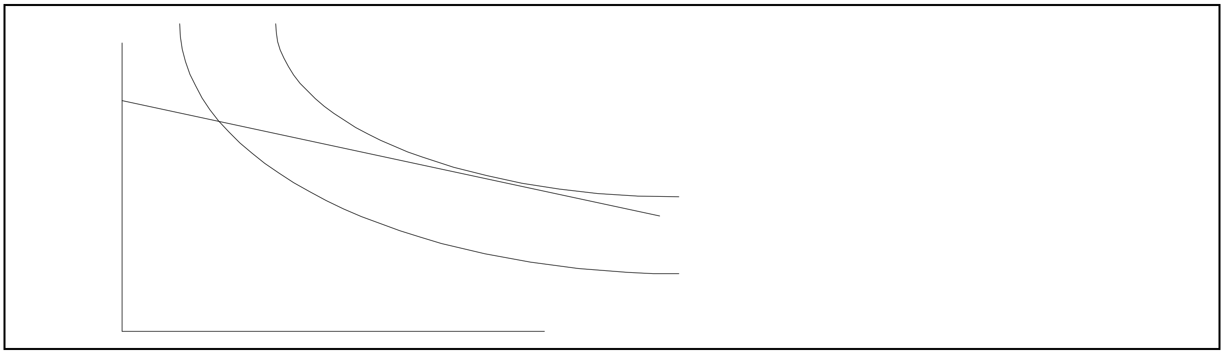
(5.18) $\text{dlog } X_i = \varepsilon_{ii}^* (\text{dlog } P_i - \text{dlog } P_z) + \eta_i \text{ dlog } M^* + \varepsilon_{i1} (\text{dlog}P_1 - \text{dlog}P_z) + \varepsilon_{i1} (\text{dlog}P_2 - \text{dlog}P_z)$

Where P_z is again a price index,

$$\text{dlog}P_z = (\Sigma_{j\neq i,1,2} S_j \text{ dlog } P_j)/(1-S_i) = (\Sigma_{j\neq i,1,2} S_j \text{ dlog } P_j)/\ \Sigma_{j\neq i,1,2} S_j$$

3. Welfare analysis

Figure 5.1. Gains From Trade



An important use of compensated demand use is to measure gains from trade. As we see in Figure 5.1, whatever the market clearing price vector the consumer is better off after trade becomes possible.

How much better off? In Figure 5.1 we see how the consumer values ΔX by more than the current market price in terms of Y. The difference between what he would have prepared to give in terms of Y for ΔX and the market value of those units is a measure of the increase in welfare.
In terms of Graph 5.2, where the compensated demand curve is represented, that value corresponds to the difference between the area under the demand curve, which is the total valuation by the consumer of the extra units consumed, and the straight line representing the market price. This quasi-triangular area is called the consumer surplus.

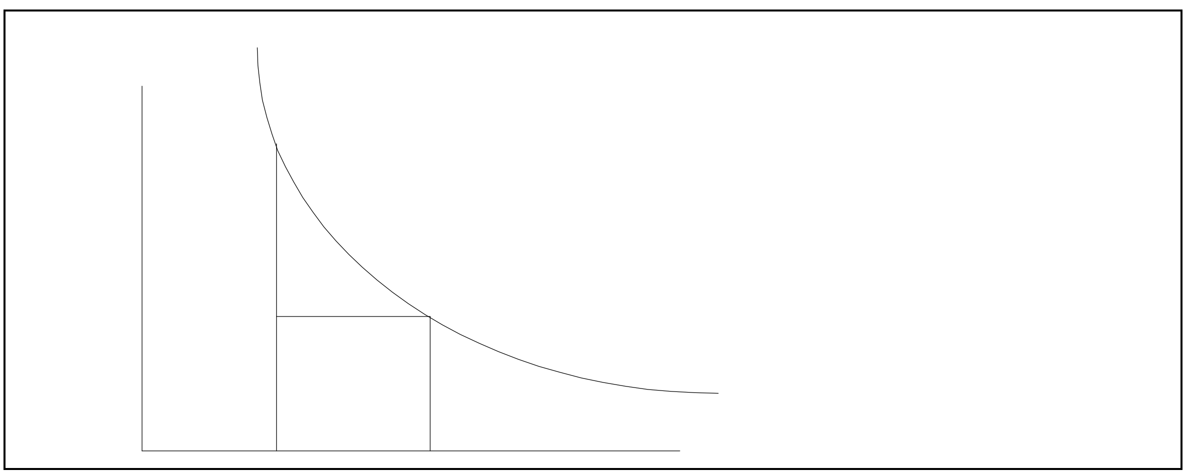


Figure 5.2. Compensated Demand Curve and Welfare Analysis

Often we will use the uncompensated demand curve to approximate this welfare improvement. What is the relation to this 'true' measure? If the good is normal, the Slutsky equation shows that the uncompensated demand curve is flatter, so by using the uncompensated demand curve we overestimate the effect a bit. In general, the income effect will not be significant, so we will not be far off.

In the next lecture we will introduce a new type of demand curve, the marginal utility of income constant demand curve or Frisch demand curve.

Lecture 6. The Life Cycle problem, Frisch Demand Curves and other useful demand curves

1. The Life-Cycle Model

Consider an intertemporal maximization problem. A person wants to maximize total discounted lifetime utility. Assume his preferences are additively separable across time, and the discount factor is $e^{-\rho t}$. Then his total utility over the lifetime is

(6.1) $\max \int_0^T e^{-\rho t} U(X_{1t},\ldots,X_{kt})$

What is the budget constraint? The present value of consumption cannot be higher than the present value of income. We can write this as

(6.2) s.t. $\int_0^T e^{-rt} [\, y(t) - \Sigma_k P_{kt} X_{kt} \,] = 0$

Why is future income and consumption discounted at a rate r? Imagine a two period models with borrowing and lending. Then we can write consumption in each period as

$$C_1 = Y_1 - S_1$$

If this savings are invested at rate r, then consumption in period 2 will be

$$C_2 = Y_2 + (1+r) S_1$$

Substituting in S_1 from period one, we can rewrite this equation as

$$C_1 + C_2/(1+r) = Y_1 + Y_{2/}/(1+r)$$

The period by period budget constraints imply that present value of consumption is equal to present value of income, as we wanted to show.

We can set up the Lagrangean of this problem and solve it as in Lecture 1,

(6.3) $L = \int_0^T e^{-\rho t} U(X_{1t},\ldots,X_{kt}) + \lambda \int_0^T e^{-rt} [\, y(t) - \Sigma_k P_{kt} X_{kt}$

The "within period" first order conditions to this problem have the form

(6.4) $\partial U / \partial X_{jt} = \lambda\, e^{-(r-\rho)t} P_{jt,}$

(6.5) $U_{it} / U_{kt} = P_{it} / P_{kt}$, for every i, k.

Condition 5 is true regardless of whether a capital market exists or whether the consumer can borrow or lend.

The between periods conditions are

(6.6) $[\partial U / \partial X_{jt}]/[\partial U / \partial X_{j\tau}] = \lambda\, e^{-(r-\rho)(t-\tau)}\, [P_{jt} / P_{jt\tau}]$

Thus the consumption in t relative to τ will depend on the relative prices between the two periods and on the market discount rate.

This system is rarely solved entirely. The usual way to proceed is to analyze the response of the optimal quantities in two types of excercise: change the price of good j at time t holding all other prices constant and change the price vector to analyze the changes in aggragate consumption over time.

We can gain more insights into the problem by writing it in terms of the cost minimization in order to obtain the compensated demand curves. This set up allows us to clearly differentiate the two stages in the consumer's problem. In each period, the consumer finds the last costly way to achieve a certain level of utility. Then the consumer redistributes expenditure between periods so as to equalize the marginal cost of an additional unit of utility in each period.

(6.6) $\min C(P_1,\ldots,P_k, U^*) = \min \int_0^T e^{-rt} \Sigma_k P_{kt} X_{kt}\, dt +$

$$+ 1/\lambda\, [\int_0^T e^{-\rho t} (U(X_{1t},\ldots,X_{kt}) - U^*)\, dt]$$

(Where we have kept the Lagrange multiplier of the primal problem, λ)

The first stage first order conditions will be given by the same expressions as above, thus the solutions to this first stage will be a set of compensated demand curves,

(6.7) $X_{it} = X_{it}^*(P_{1t},\ldots,P_{kt}, U^*)$, for every good i.

The second stage will give us the borrowing and lending decisions. The consumer will equalize the marginal cost of an additional unit of utility across all the periods. Using the envelop theorem on 6,

(6.8) $\partial C/\partial U^* = 1/\lambda\, e^{-(r-\rho)t}$

If prices are constant over time, the determinant of the intertemporal consumption pattern will clearly be $e^{-(r-\rho)t}$. It means that the consumer will consume more in periods where it costs less. If, for example, $r>\rho$, it is cheaper to consume in the future. In this case λ will decrease with age and consumption will grow with age. If $r=\rho$, then λ is constant, and we have the Frisch demand curve case we will study later. This is the easier case to deal with.

Differentiating (8) with respect to t,

(6.9) $\partial^2 C/\partial U^2 \; dU/dt + \; \Sigma_k \partial^2 C/\partial U \, \partial P_k \; dP_k / dt = 1/\lambda \, (r-\rho) \, e^{-(r-\rho)t}$

(6.10) $\partial^2 C/\partial U^2 \; dU/dt + \; \Sigma_k \partial^2 C/\partial U \, \partial P_k \; dP_k / dt = (r-\rho)\partial C/\partial U^*$

(6.11) $[\partial^2 C/\partial U^2 \; dU/dt] / \partial C/\partial U + [\; \Sigma_k \partial^2 C/\partial U \, \partial P_k \; dP_k / dt] / \partial C/\partial U = (r-\rho)$

Since

$dU/dt = \Sigma_k (\partial U/\partial X_k) dX_k/dt$

$(\partial X_k / \partial U) / (\partial C/\partial U) = \partial X_k / \partial M$

(6.12) $[\partial^2 C/\partial U^2 \; dU/dC \, \Sigma_k P_k \, dX_k /dt] / (\partial C/\partial U) + \; \Sigma_k (\partial X_k / \partial M) \, (dP_k / dt) = (r-\rho)$

(6.13) $\Sigma_k S_k \, dX_k /dt = 1/ \quad [r - \rho - \; \Sigma_k S_k \eta_k \, d\log P_k] = \; (r-\rho)$

Where $M \, [\partial^2 C/\partial U^2] / \; (\partial C/\partial U)^2 =$

This equation reveals the important restrictions imposed by the single good model. Most empirical implementations of the intertemporal consumption model are based on a model such as

(6.14) $\max \int_0^T e^{-\rho t} \, U(C_t) \, dt$

And use, as a measure of C and P,

$d\log C = \Sigma_k S_k \, d\log X_k$

$d\log P = \Sigma_k S_k \, d\log P_k$

Equation 6.13 reveals that while the measure for aggregate consumption generally used is correct, the measure for the price change is not. In 6.13 we can see that the price of each good must be weighted by its income elasticity. The reason is that in order to account for consumption growth, we are interested by changes in the price that matter. i.e., those of the goods with higher income elasticity. As consumption grows, increases in consummation go mostly towards foods with high income elasticity. Thus the prices of these goods need to be over-represented in the price index. c The aggregate in 6.13 includes this correction that allows us to distinguish between price changes which are important on average and those that matter on the margin.

2. Changes in consumption over time with constatn marginal utility of income

In the intertemporal problem, if there are transitory price changes the effect on λ is very small, since λ depends on the whole vector of prices and income for the entire lifetime. For these kind of problems, to think of a constant λ is a good approximation to the complete solution. This is very useful if we want to do the kind of comparative static excercise that we mentioned above: how does consumption over time change when the price of one good changes over time? Going back to the first order condition (4),

(6.15) $\partial U / \partial X_{jt} = \lambda e^{-(r-\rho)t} P_{jt,}$

We can express the changes of all quantities over time when the price of good 1 over time changes assuming the change is small enough that the marginal utility of income does not change,

(6.16) Σ_k (dlog U_1 / dlog X_j (t)) (dlog X_j (t) / dlog t)= - (r-ρ) + dllog P_1 (t) /dt

....

(6.17) Σ_k (dlog U_j / dlog X_k (t)) (dlog X_k (t) / dlogt t)= - (r-ρ) for all j ≠1

We can rewrite these equations using the concept of the elasticity of the marginal utility functions,θ

(6.18) Σ_k $\theta_{1\,k}$(t) (dlog X_k (t) / dlog t)= - (r-ρ) + dllog P_1 (t) /dt

....

(6.19) $\Sigma\theta_{jk}$ (dlog X_k (t) / dlogt t) = - (r-ρ) for all j ≠1

This is a set of K equations with K unknowns (the rates of change of the quantities) which can be solved in θ, P.

In a one good model, the first order condition is

(6.20) $\partial U / \partial X_t = \lambda e^{-(r-\rho)t}{}_,$

The rate of change of the quantites when λ is constant will be

(6.21) θ (t) (dlog X (t) / dlogt t) = - (r-ρ)

Then the rate of interest that clears the market for consumption X(t) will be

(6.21) r = ρ - θ dlog X(t) / dlogt

3. Frish Demand Curves

The demand curves that we could have obtained from the problem in 2, i.e. those that are obtained holding λ constant are called Frisch demand curves. Another way to motivate these curves is by thinking of a separable utility function over a continuum of (or over sufficiently many) goods.

(6.22) $U = \Sigma_j U(X_j)$

Finally, we can motivate these curves by writing a utility function which is separable between the goods we are considering and the rest, which we assume to have constant marginal utility,

(6.23) $\max U = U(X_1,...,X_k) + \lambda Y$

(6.24) s.t. $M = \Sigma_k P_k X_k + Y$

Where we assume that the marginal utility of good Y is constant. We can rewrite this budget constraint,

(6.25) $Y = M - \Sigma_k P_k X_k$

and we obtain the maximization problem

(6.26) $\max U = U(X_1,...,X_k) + \lambda (M - \Sigma_k P_k X_k)$

In all three examples we need to solve the following problem

(6.27) $\max U (X_{1t},...,X_{kt}) - \lambda \Sigma_k P_k X_k$

which has solutions

(6.28) $X_1 = X_1^d (P_1,..., P_k, \lambda)$
:
(6.29) $X_k = X_k^d (P_1,..., P_k, \lambda)$

We call these solutions Frisch demand curves or Marginal Utility of Income Constant demand curves.

λ is now the opportunity cost of money in terms of utility. If we substitute back the demands in a utility function as in (1), we obtain the surplus of spending today instead of spending another period.

(6.30) $S (P_1,..., P_k, \lambda) = \max U (X_{1t},...,X_{kt}) - \lambda \Sigma_k P_{kt} X_{kt}$

The Frisch demand curves allow us to relate directly the demand schedule with the marginal utility schedule, without intervening income effects.

(6.31) $U_x = \lambda P \Rightarrow U_{xx}\, dX = \lambda\, dP$,

so,

(6.32) $dX = U_{xx}^{-1}\ \lambda\, dP$

Thus changes in the demand function can be thought of as those which change the marginal utility of the good we are considering. Furthermore, since there are no income effects in these demand curves, welfare analysis is easy to carry out.

How do the elasticities measured on these curves compare with those measured on a compensated demand curve?

(6.33) $X_i^F{}^*(P_1,...,P_k, \lambda) = X_i^*(P_1,...,P_k, U(\lambda))$

(6.34) $\partial X_i^F{}^*(P_1,...,P_k, \lambda) / \partial P_i = \partial X_i^*(P_1,...,P_k, U(\lambda)) / \partial P_i^* +$

$+ \partial X_i^*(P_1,...,P_k, U(\lambda))/\partial U\ \ \partial U/\partial\lambda\ \ \partial\lambda / \partial P_i$

The sign of the second term in the right hand side of this expression is negative, and the Frisch demand curve will always be more elastic than the compensated demand function.

4. All or nothing demand curves

Another useful demand curve is motivated by giving bundles to the consumer in the form of a discrete choice between accepting the quantity offered at a given price or not. It is the indifference curve of the consumer in the price-quantity space that goes through the point $(q,p)=(0,P_0)$.

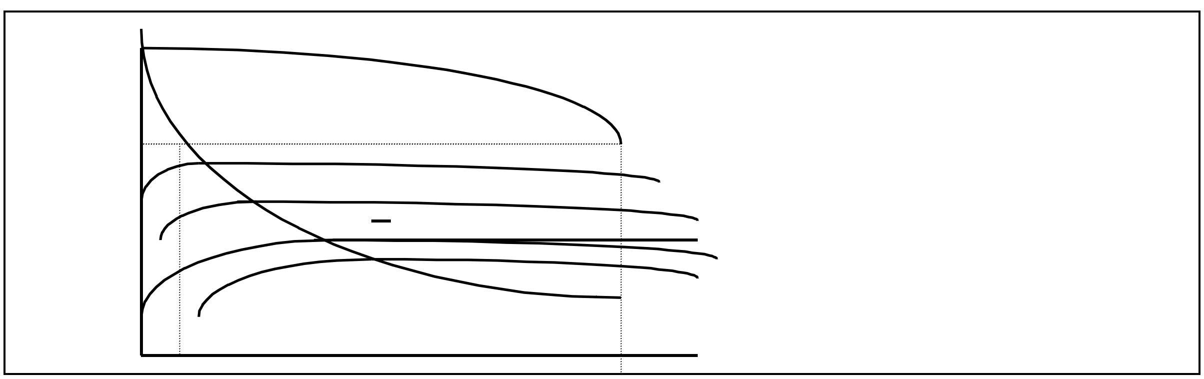


Figure 6.1. All or Nothing Demand Curves

In this graph we have drawn a Marshallian demand curve together with the consumer's indifference curves in the price quantity space. The curves peak at the demand curve, because for a given price the demand curve gives the optimal combination of price and quantities for the consumer. At P_1 the consumer will choose the quantity Q_1. But if he is offered the discrete choice (Q_1* P_1) on a take it or leave it basis he will accept, since this leaves him indifferent with the combination (0, P_0).

This demand curves show combinations of price and quantitiy for which the surplus of the consumer is 0. The entire consumer surplus along these demand curves is transferred to the producer. This demand curves are specially useful to deal with problems of monopoly pricing, particularly of price discrimination.

Lecture 7. Two Applications of the Utility Maximization Framework: Willingness to Pay for Public Goods and Consumption Based Asset Pricing

1. Demand curves as marginal value schedules

In the standard case, everybody gets different amounts of good X, depending on their individual willingness to pay. For some goods, however, no markets exist. In the case, for example, of the provision of eduction by a school district, every consumer gets the same amount regardless of their willingness to pay. In this case, the height of the demand curve for good X by the consumer 'i' at the available amount X_0 will give the marginal value of this quantity for consumer 'i'.

For example, imagine a community where all citizens have the same preferences over education and different income level. Assume they pay for schooling out of a flat rate income tax.. Will those who earn less or those who earn more prefer better schooling?

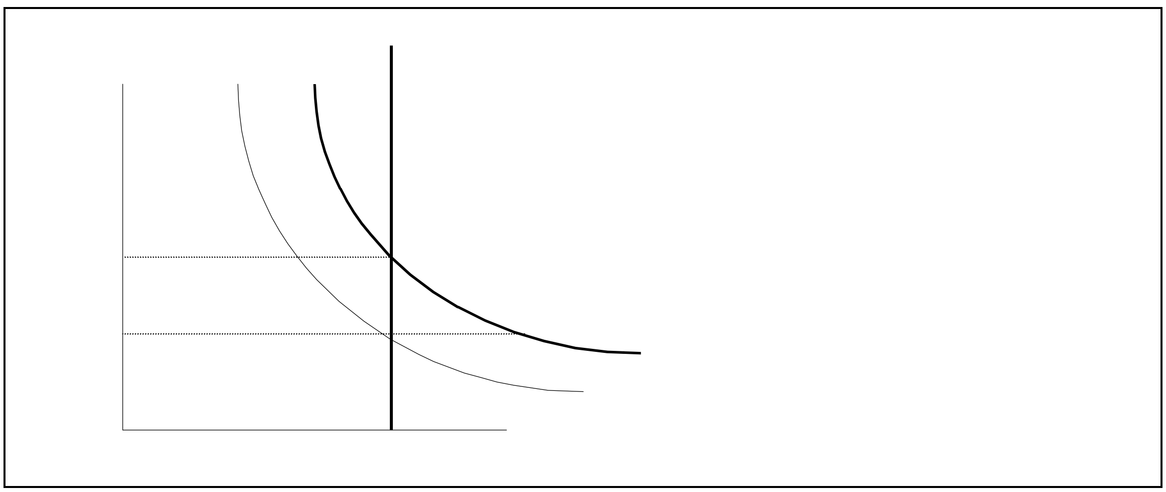


Figure 6.1. Marginal Value Schedule

Figure 6.1 shows the marginal valuation of X_0 at income level I_0. If we raise income to I_1, the marginal value schedule (demand curve) will shift to the right to D (I_1),

How large will the change be? We know that the quantity the consumer demands increases by the income elasticity of demand, when prices are fixed. Since the quantity is given, we have to transform this increase in quantity to an increase in price in the new demand curve. This is precisely the inverse of the price elasticity of good s.

Analytically,

(6.1) $dP_{si}^*/dM_i = (\partial P_{si}^*/\partial X_{si}) (\partial X_{si} / \partial M_i)$

(6.2) $d\log P_{si}^*/d\log M_i = (\partial \log X_{si} / \partial \log M_i) (\partial \log P_{si}^*/\partial \log X_{si})$

(6.3) $d\log P_{si}^*/d\log M_i = \eta_{si} / |\varepsilon_{ssi}|$

Now we can go back to our original question. When will the higher income citizens be willing to pay a higher proportional tax rate than lower income citizens? $d\log P_{si}^*/d\log M_i$ is the elasticity of valuation with respect to income. If it is bigger than one, it means that as income increases the valuation increases by *more* than income, meaning that those with higher income will be prepared to pay a higher tax rate.

Thus,

(6.4) $d\log P_{si}^*/d\log M_i > 1$ iff $\eta_{si} / |\varepsilon_{ssi}| > 1$

And the condition that we need is that

(6.5) $\eta_{si} > |\varepsilon_{ssi}|$

Thus higher income citizens will want a higher proportion of income going into public schools if the income elasticity of schooling is higher than the price elasticity of schooling.

Note that, at a high income level, η_{si} becomes negative, rich people do not send their kids to public schools. From this level onwards, the willingness to pay actually decreases as income increases.

What if, instead of a proportional tax, there is a progressive tax? To generalize equation (2), note that, in this example, the increase in the consumer's valuation of the good is $d\log P_{si}^*/d\log M_i$, while the increase in cost with a proportional tax rate is $d\log C / d\log M_i = 1$. In general, the condition needed for higher income people to prefer better schools is that their marginal valuation increases by more than their marginal cost,

(6.6) $d\log P_{si}^*/d\log M_i > d\log C / d\log M_i$

With a progressive income tax, $d\log C / d\log M_i = p$, and condition (5) becomes

(6.7) $\eta_{si} / |\varepsilon_{ssi}| > p$

If the school is funded by a proportional property tax, the increase in cost as income increases is given by

(6.8) $d\log C / d\log M_i = (d\log C / d\log H_i)(d\log H_i / d\log M_i)$, where H is property expenditure.

so,

(6.9) $d\log C / d\log M = 1\ \eta_{hi}$

and condition (5) becomes

(6.10) $\eta_{si} / |\varepsilon_{ssi}| > \eta_{hi}$

2. Consumption based asset pricing

Suppose an economy in which a representative consumer lives for two periods. There is only one consumption good, which the consumer can choose to invest in a stock or consume. In period one the consumer is endowed with Y_1 units of the consumption good, in period 2 with Y_2 units. If the consumer invests in the stock, he can buy Q units of it at a price of P per unit. Given the current technology, the stock will produce K times the amount invested. If the consumer has an additively separable utility function, his problem at the start of period 1 will be to choose how many units of stock to purchase to

(6.11) maximize $U(c_1) + \beta U(c_2)$

subject to

(6.12) $C_1 + PQ = Y_1$

(6.13) $C_2 = Y_2 + KQ$

Or, substituting in the Budget constraint,

(6.14) Maximize $U(Y_1 - PQ) + \beta U(Y_2 + KQ)$

The amount Q of stock to buy will be derived from the first order conditions

(6.15) $P U'(Y_1 - PQ) = \beta K U'(Y_2 + KQ)$

Now imagine we observe this economy. We know the consumption streams, so we can recover the price of the asset Q that makes this consumption streams optimal. This price is

(6.16) $P = \beta K U_2' / U_1'$

(where, for simplicity U_1 and U_2 are period one and two utility functions)

Equation 16 allows us to price an asset which produces units of consumption good in period 2 as a function of its productivity K and the endowments of the consumer in periods 1 and 2.

How does the price of the stock changes as it becomes more productive? Taking derivatives with respect to K in equation 15, and remembering that, by the implicit function theorem P = P(K),

(6.17) $U_1' \, dP/dK = \beta \, U_2' + \beta \, K \, Q \, U_2''$

(6.18) $dP/dK = \beta \, U_2'/ U_1' + \beta \, K \, Q \, U_2''/ U_1'$

(6.19) $d\log P \,/\, d\log K = 1 + Q \, K \, U_2'' \,/\, U_2'$

(6.20) $d\log P \,/\, d\log K = 1 + d\log U_2'/d\log K = 1 + \theta$

If $\theta = -1$, the stock price does not react to technological improvements.
If $-1 < \theta < 0$, as productivity increases the stock price increases.

So in trying to understand whether the stock market is worth more or less when productivity increases, a key role is played by the marginal utility of income schedule. There are two opposing effects: on the one hand, the increase in productivity allows for an increase in period 1 utility holding period two utility constant. This effect will always tend to increase the price of the asset. At the same time, there is a price effect going in the opposite direction: as the consumer gains more period two consumption thanks to the increase in productivity, he values less the assets that allow him to consume more in period 2.

Lecture 8. Heterogeneity and Market Demand

Think of the following problem: a population of N individuals can choose one among two possible places to live, ‘a’and ‘b’. In ‘a’there are N_a spots, and in ‘b’there are N_b spots. Each person in the population is characterized by a valuation for each spot, V_{ai} and V_{bi} ,, expressed in dollar amounts. Define

(8.1) $\Delta_i = V_{ai} - V_{bi}$

Then there will be a set of equilibrium prices in this economy depending on N_a , N_b and the distribution of Δ_i in the population.

Assume that $N < N_a + N_b$, i.e., there are more places than people. In this case, the prices of some of the places will be O.

Assume also that

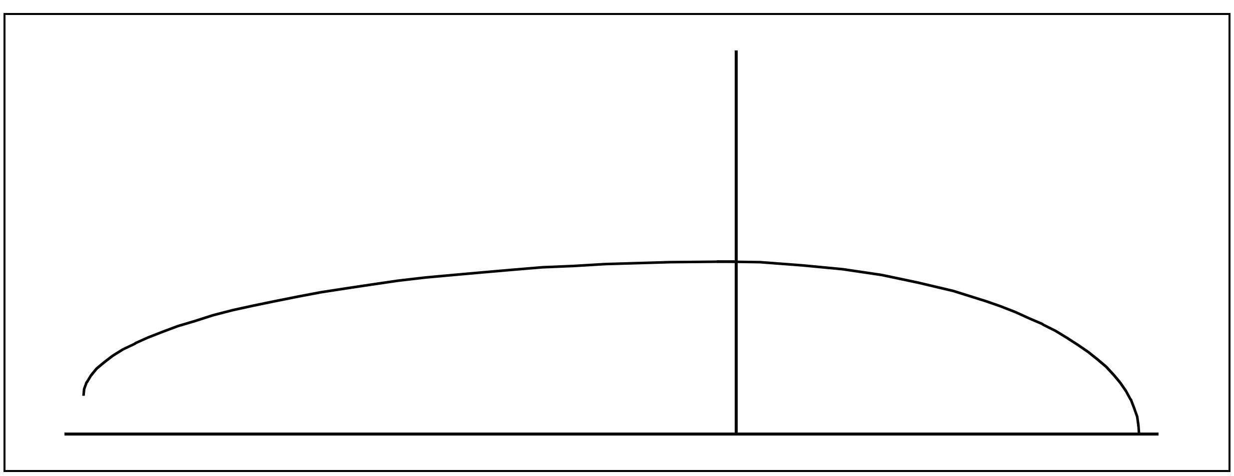
$\Delta_i > 0$ for every person i.

Then

$P_b = 0$,

and, for a given distribution of Δ_i, F_Δ, as the number of spots available in ‘a’increases the equilibrium price that clears the market will decrease. If only one spot is ready, only the person with the highest valuation will get it and the price will be $P = \Delta_{max}$,. If two spots are available the price will be that of the second highest valuation, etc. Thus a downward slopping demand curve originates, but for reasons entirely different than the decreasing Marginal Utility at individual level that we used before to obtain this property.

Figure 8.1. Distribution of Valuations in the Population



Assume a distribution of Δ_i like the one in figure 6.1. As the difference in price decreases, more and more people find their individual valuation of a spot in ‘a’over a spot in ‘b’higher than the market price. Thus a demand curve such as the one in 6.2 will be defined. Since the supply is given, an equilibrium, market clearing price differential for ‘a’, P* will be the one at which exactly $N^* = N_a$ individuals have a higher valuation than P*.

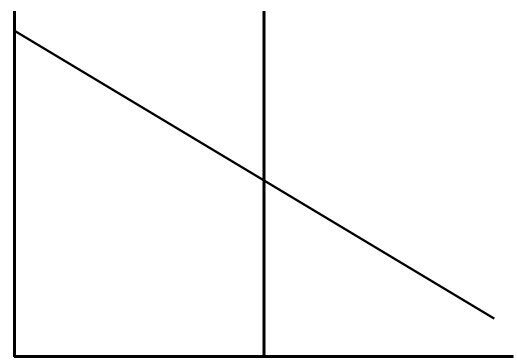


Figure 8.2. Demand for A

The individuals who value a over b by less than P* are $F_\Delta(P^*)$. Thus, at the equilibrium price,

(8.2) $1 - F_\Delta (P^*) = N^*/ N,$

(8.3) $N^*(P) = N (1 - F (P))$

Which is a demand curve for the demand for spots in A as a function of the price difference between A and B. The slope will be

(8.4) $dN^*/dP = -N f(p)$

which is negative: as the price goes up, less people have a valuation which is high enough to justify the acquisition of a.

Thus, the sorting of people according to their values produces a downward sloping demand curve. The valuation of the last person determines the market price.

What happens if people get more homogeneous? From equation (5),

(8.5) $P/N^* \, dN^*/dP = - f (P) / [1 - F(P)] \, P$

Where $f(p) / [1- F(P)]$ has the same expression as the *hazard rate* used in duration models. Equation (5) means that the more homogeneous the population, the closer together they are, and the larger the number of people that are on the edge. Thus the hazard rate is very high and the elasticity of demand will also be higher the more homogeneous the population.

We can generalize this analysis in several ways. In the examples that we will examine, there is a heterogeneous good for which a preference ordering can be established among consumers. In all of these cases it will be useful to work in a price quality map, drawing consumer indifference curves at different Utility levels. Utility level grows towards more quality for a given price for all consumers and decreases as prices for a given quality increase.

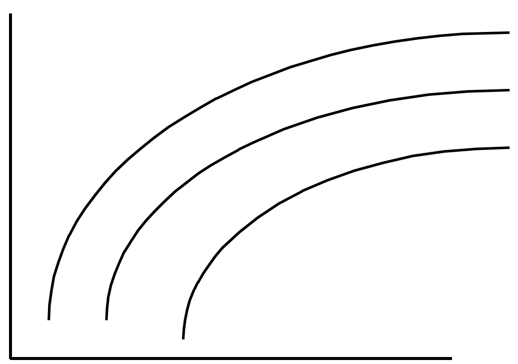


Figure 8.3. Preferences in the price quality space with a continuum of goods

We can differentiate several cases, which we will treat separately.

1. *Heterogeneous goods with fixed supply, homogeneous consumers*.

If all the consumers are the same, they must end up at the same utility level. The consumers must be indifferent between the goods remaining in the market, so the market equilibrium demand for quality is the indifference curve determined by the fixed supply of quality level.

2. *Heterogeneous goods elasticaly supplied by homogeneous producers, homogeneous consumers*

In this case tsince there is free entry of producers, all of them will have 0 profits. The 0 profit condition in the price quality space will determine one set of combinations along the isoprofit line of prices and qualities. The consumers will maximize utility subject to this 0 profit condition. Since they are all identical, only one quality will exist in the market.

3. *Heterogeneous good elastically supplied by homogeneous producers, heterogeneous consumers.*

Consumers have preferences over quality of the heterogeneous good and consumption of other goods. We can think either of different preferences for different consumers, as in the neighborhood case above, or of different budget constraints. If we think of consumers with the same preferences and different budget constraints,

(8.6) $C(Q) + O = Y$

and we can express their preferences

(8.7) $U(Q, O) = U(Q, Y-C(Q))$

Where, if there is free entry, $C(Q)$ is the cost function for the homogeneous firms.

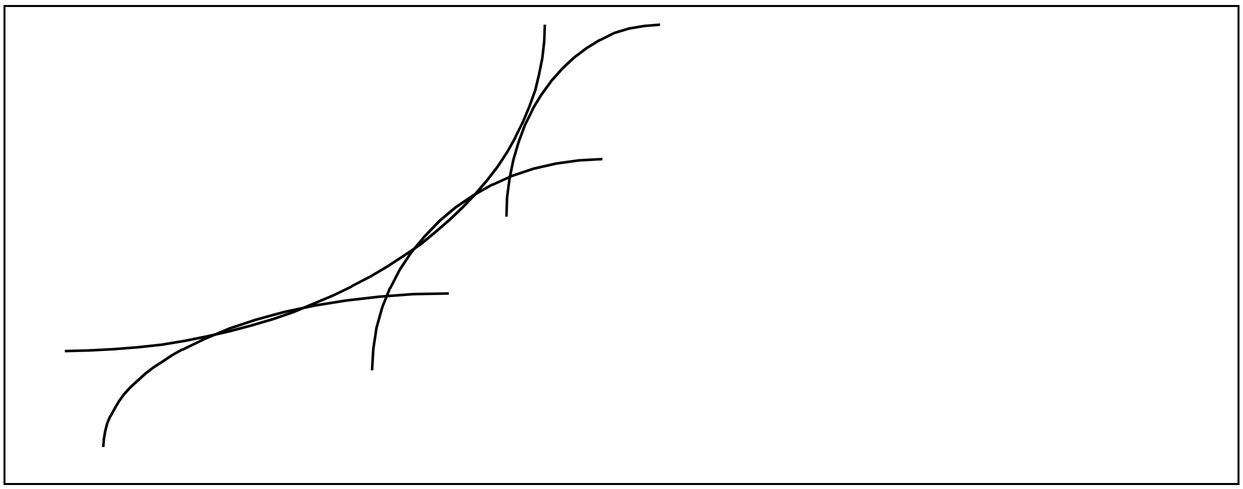


Figure 8.4. Heterogeneous consumers, homogeneous firms

Then, the consumers chose $Q^*(Y)$ to solve

$$\max_{Q} U(Q, Y-C(Q))$$

The first order condition for this problem is

(8.8) $U_1 / U_2 = C'$

We can use the implicit function theorem to find under what conditions will the quality chosen be increasing in Y.

$$\partial^2 U/\partial Q^2 \, dQ/dY = C'(Q) \, dQ/dY \, U_2 + C'(Q) \, \partial^2 U / \partial Y^2$$

(8.9) $dQ/dY = [C'(Q) \, \partial^2 U / \partial Y^2] / [\partial^2 U/\partial Q^2 - C''(Q) \, U_2]$

So,

(8.10) $\partial^2 U / \partial Y^2 < 0$ (decreasing marginal utility of income)

Then quality will be a normal good.

4. *Homogeneous consumers, heterogeneous producers*

Again the equilibrium condition will be given by the fact that all of the consumers must be indifferent between consuming one quality level or another. Thus, the cost functions of the different firms must all be tangent to the indifference curve of the consumers at that indifference level.

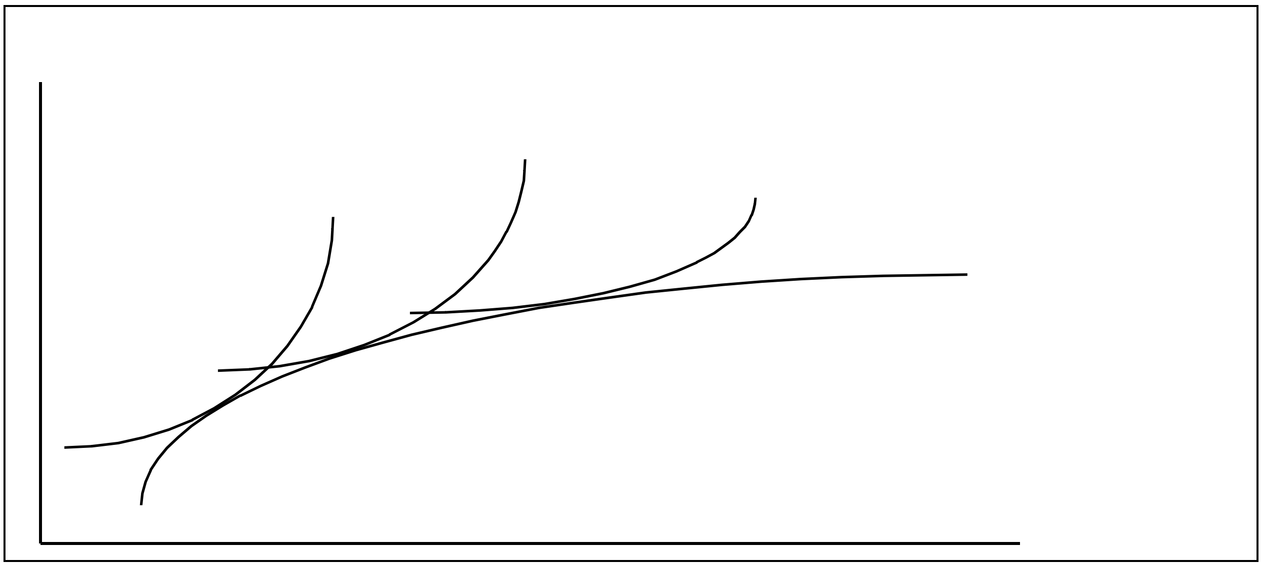


Figure 8.5. Homogeneous Consumers, Heterogenous Firms

5. *Both sides heterogeneous*

Figure 8.6. Heterogeneous Consumers and Producers

In order for the equilibrium to be possible, there must be a large enough number of producers and consumers so that they can match each other. In this case there will be a continuum quality level. The market schedule price-quality will represent neither demand conditions nor supply conditions. It will be an envelope of both the indifference curves of the consumers and the cost functions of the producers

In general, in order to set up this problem we will require a distribution of tastes or income levels on the consumer side, f(Y). On the cost side, we may use a linear cost function

$C(Q,s) = a + sQ$, with s distributed g(s)

S will be the source of heterogeneity on the firm size, and income or tastes on the consumer side.

PART II. SUPPLY AND PRODUCTION

Lecture 9. Technology and production function

1. Diminishing Returns and technology in the natural resources case

Assume there exist two plots of land in which we can produce the agricultural products X and Y. In plot A we can produce either 5 units of X or 8 units of Y. In plot B we can produce either 10 units of X or 10 units of Y.

	X	*Y*
A	*5*	*8*
B	*10*	*10*

Then the cost of X and Y in terms of the other product is

	Cost of X	*Cost of Y*
A	*8/5*	*5/8*
B	*1*	*1*

Note that what matters here is cost per unit produced. It would be the same if one plot produced 500 and the other 800. In plot A in order to produce a unit of Y we have to renounce to produce 5/9 units of X and in plot B we have to renounce to produce 1 unit of X. This means plot A is the low cost producer of Y, and plot B is the low cost produces of X. We can represent this technology by noting that, if we had to produce both X and Y, we would start by the low cost option, and only if we need to produce more we switch to the other (high cost) plot.

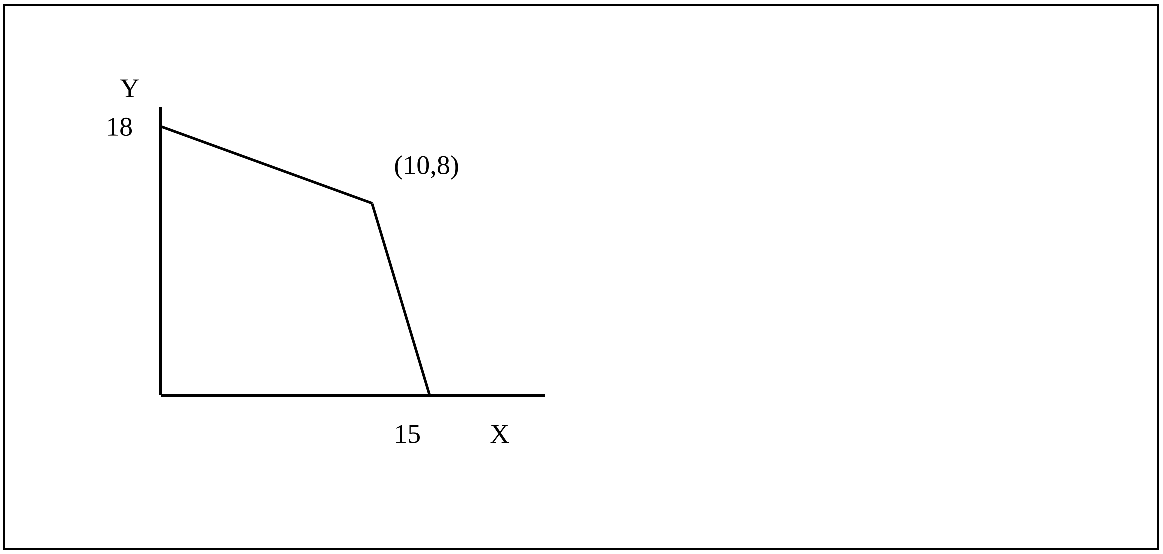


Figure 9.1. Agricultural Production Possibilities Frontier

Thus the cost of Y rises as we need to produce more Y. Note that to obtain this rising marginal cost result, we only need to assume that we produce first with the low cost technology. In the same way as heterogeneous valuations are a rationale for downward sloping demand curves, the existence of heterogeneous technologies is a rationale for diminishing returns. This is particularly clear, as in our example, in the case of industries of production or extraction of natural resources. In these industries, there are heterogeneous costs, and as production increases, there is a need to use the low cost resources first.

2. The production choice in a two good world

If there was a continuum of possible plots , we will obtain a smooth production possibility frontier as in figure 2,

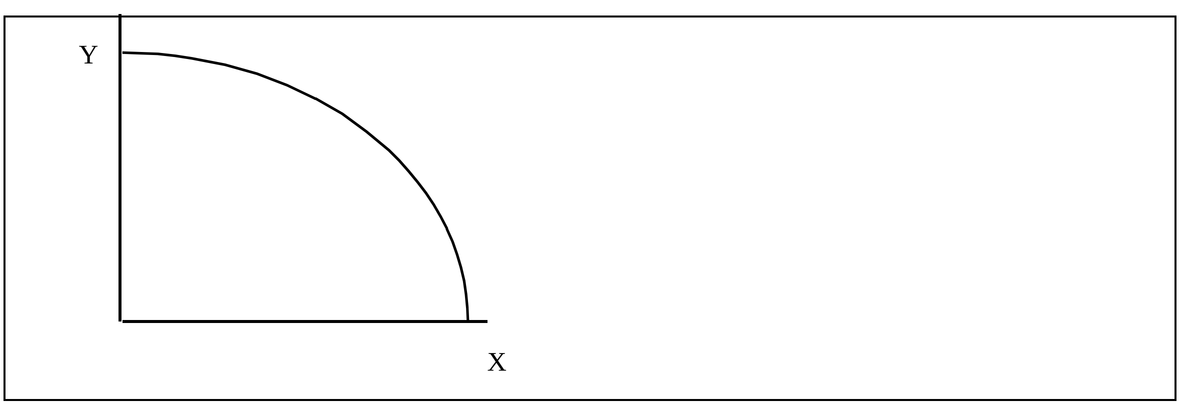


Figure 9.2 Production Possibilities Frontier with a Continuum of Technologies

If Robinson Crusoe was facing this production possibility frontier, he would maximize his utility subject to the resource constraint given by figure 2. Then he will have an equilibrium level of consumption of X Y given by (X_0,Y_0).

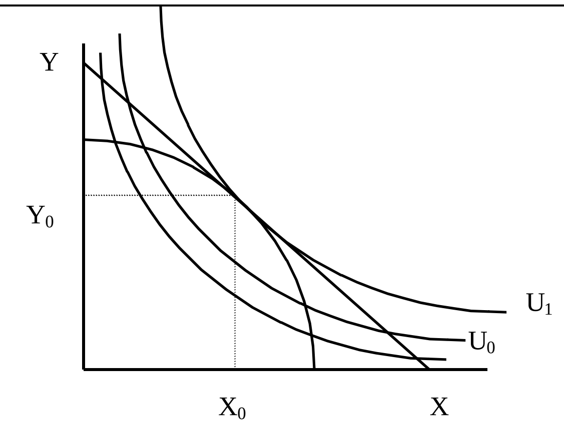


Figure 9.3. Production Decision of Robinson Crusoe

Point (X_0, Y_0) is characterized by a tangency condition. In it, the slope of the consumers indifference curve is equal to the slope of the production possibilities frontier. This means that the cost of an extra unit of X to the consumer, which is given by the amount of Y he must renounce to consume, is equal to the value to him of an extra unit of X in terms of Y.

If this economy opens up to the world economy, trade will emerge whenever the relative prices of goods X and Y diverge. Note that, in real terms, it is impossible for both goods to be cheaper in one country than in the other. The reason is that the price of one good in real terms is given by its cost in terms of units of Y which cannot be produced. Thus if the price of X in terms of Y is lower, the price of Y in terms of X must be higher.

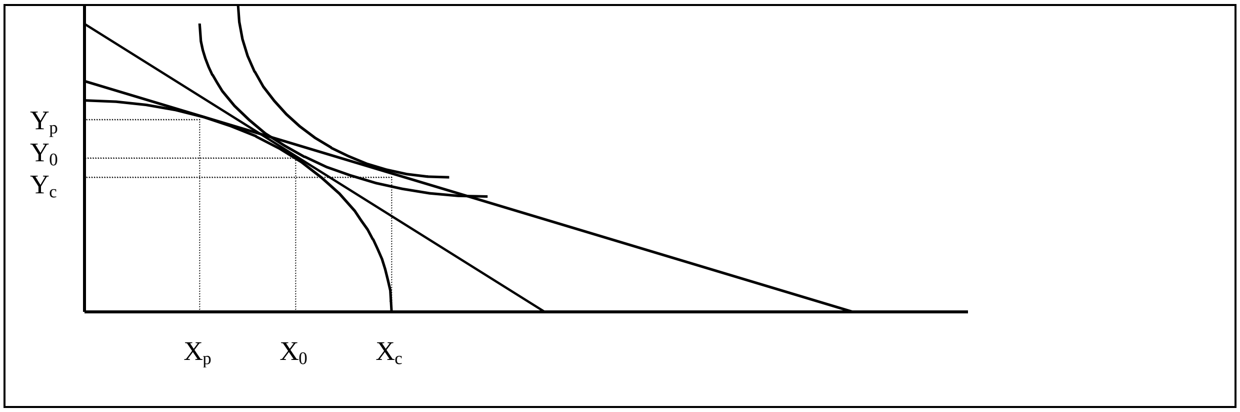


Figure 9.5 Production and Consumption when Trade is Possible

In the case illustrated in figure 5, the international price of Y is higher than the domestic price. Thus Robinson Crusoe will specialize in the production of Y and produce less X.

What about his consumption? If both X and Y are normal goods, the consumption of X unambiguously increases, since income and substitution effects go in the same direction. The direction of the consumption of Y is unclear, since both effects go in opposite directions.

Two important conclusions can be extracted form this discussion:

- Trade makes Robinson Crusoe unambiguously better off. Moreover, the more differences his internal prices from the international prices, the more his situation is improved by trade. The worst possible situation would be one in which the cost for him of producing X is the same as the international price.

- His preferences do not affect the production choice. No matter who owns this production technology, we are faced with production at point (X_p,Y_p). This point is given solely by the international prices and the technological possibilities of out consumer He will produce were cost of X in terms of Y is equal to P_x (price equal marginal cost).

For this reason, we can generally express the objective function in the supply side of the economy as ‘maximize profits’. Once income is maximized the production choices can be made.

This separation will not always hold. In particular, if the production choice enters the utility function or if the output and input choices affect prices (so that prices cannot be taken as given), preferences will affect output decisions.

3. Production function and profit maximization

We will generally assume that it is possible to represent the production choices by a production function that relates units of input required with units of output produced. For example, imagine a good that requires only one input, L, in its production. We represent the amount of output as a concave function of the amount of L used. Concavity is equivalent to diminishing returns: as we increase the amount of L we use, the production of Y increases less each time.

(9.1) $Y = F(L)$

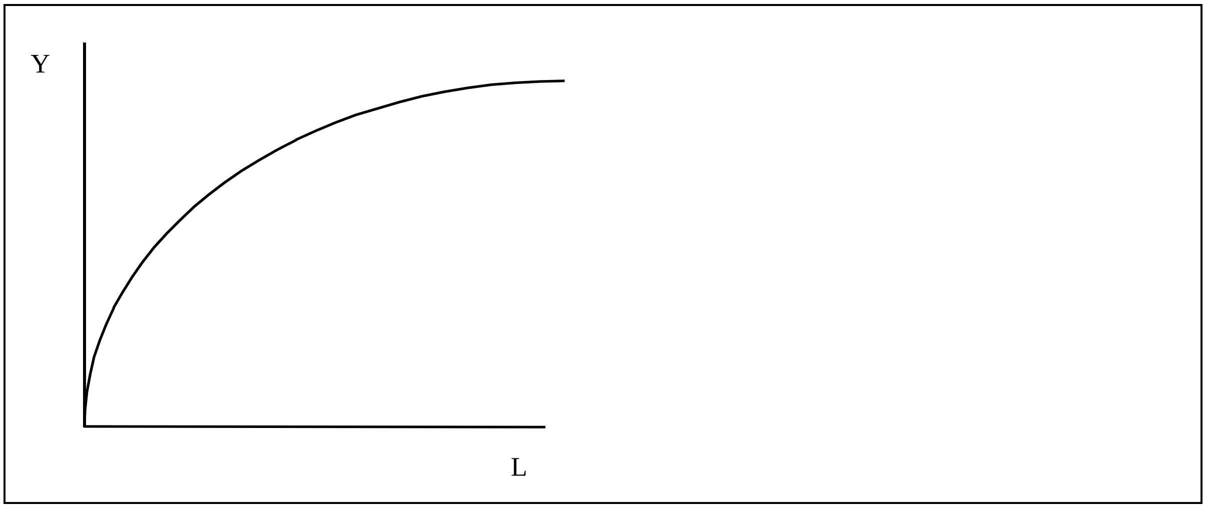


Figure 9.6. Production Function with one input

Profits for this production function will be:

(9.2) $\pi = p\,F(L) - w\,L$

Where p is the price of Y and w is the wage per hour. The maximum level of profit, will be achieved when[1]

(9.3) $p\,F'(L) = w$

This condition says that the optimal amount of L employed will be reached when its price (w) is equal to the value of the marginal product of the input. In terms of figure 7, this

[1] We are assuming neither the prices of the input nor the output price is affected when the level of demand of inputs or of supply of output of these production unit changes.

means the profit line is parallel to the wL line. We can also write this condition as price of output equals its marginal cost,

(9.4) $p = w / F'(L)$

Condition 9.4 will define an optimal amount of L for each level of p and w,

(9.5) $L^* = L(p,w)$

This is a demand schedule for L. It relates prices of inputs and outputs to quantity of input used

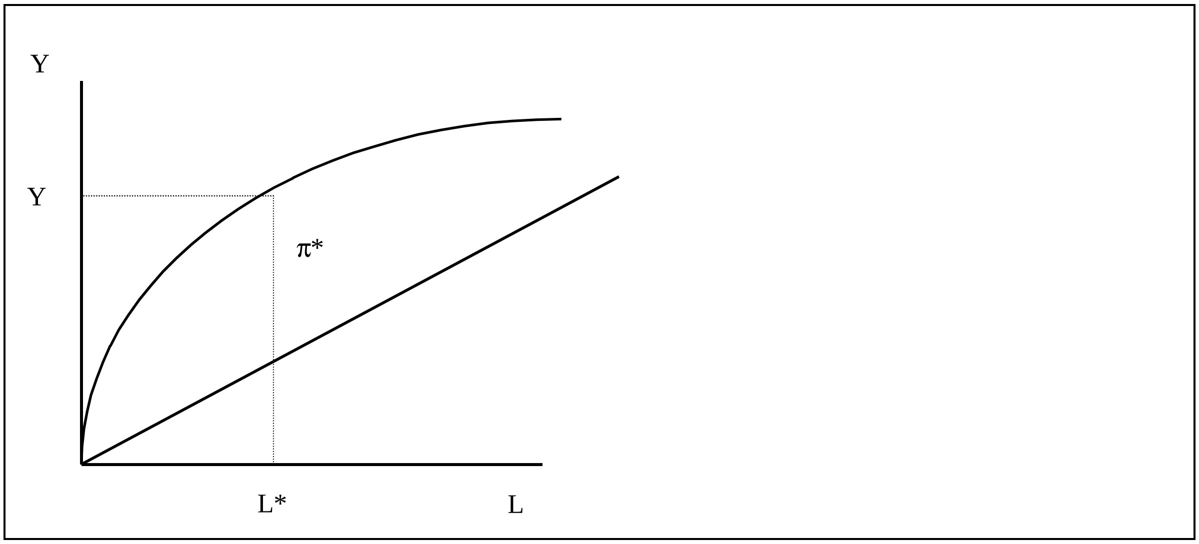


Figure 9.7. Profit Maximization

Condition (9.4) will also define a supply of Y,

(9.6) $Y^* = F(L^*) = F(L(p,w)) = S(p,w)$

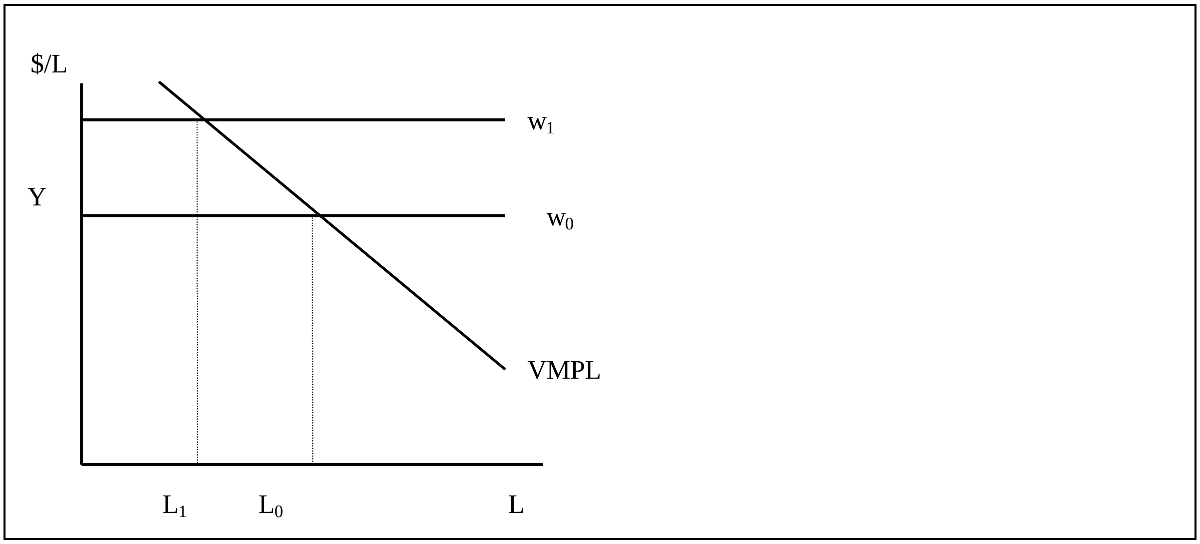


Figure 9.8 Demand of Labor

Figure 9.8 shows the amount of labor demanded at different wage levels. Since at the optimum, by equation 9.4, the price of the input must equal the value of its marginal product, the demand schedule coincides with the VMP schedule.

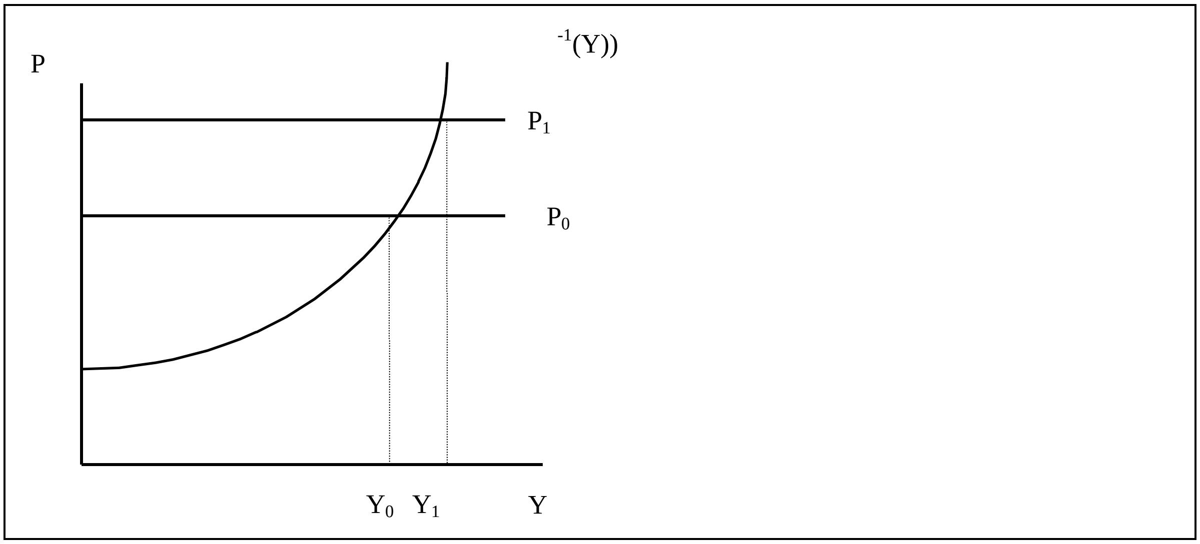


Figure 9.9 Supply Schedule

The supply schedule in figure 9.9 is a representation of the first order condition in 9.5, marginal cost equals price. Since there is only one input, figures 9.8 and 9.9 are the same curve which has been flipped over.

When there is another input, such as capital, the profit function becomes

(9.7) $\pi = p\, F(L,K) - wL - rK$

The rental price of capital is determined by the interest an depreciation. The depreciation is due to the physical depreciation and to the change of value in the market of the capital stock due to obsolescence.

Now the first order conditions are

$$\text{(9.8)} \qquad p\,F_L = w$$
$$p\,F_K = r$$

The effects of a change in the price of one input are now more complex. For example, if the wage rate falls, in the short run (with the level of capital fixed) there is a movement along the VMP_L schedule as before. Also the level of output increases as we increase the amount of L employed. IN the long run, we can vary capital as well. How does it change? It will depend on the cross derivative F_{KL}. If $F_{KL} > 0$, VMP_K will move to the right when the wage changes, and this will cause VMP_L also to shift up. Then the fall in the demand for labor will be larger for a given change in w in the long run than in the short run.

What happens if $F_{KL} < 0$? In the long run VMP_K will move to the left instead of to the right. But that will cause VMP_L to move up, and again we have a larger reaction in the long run than in the short run

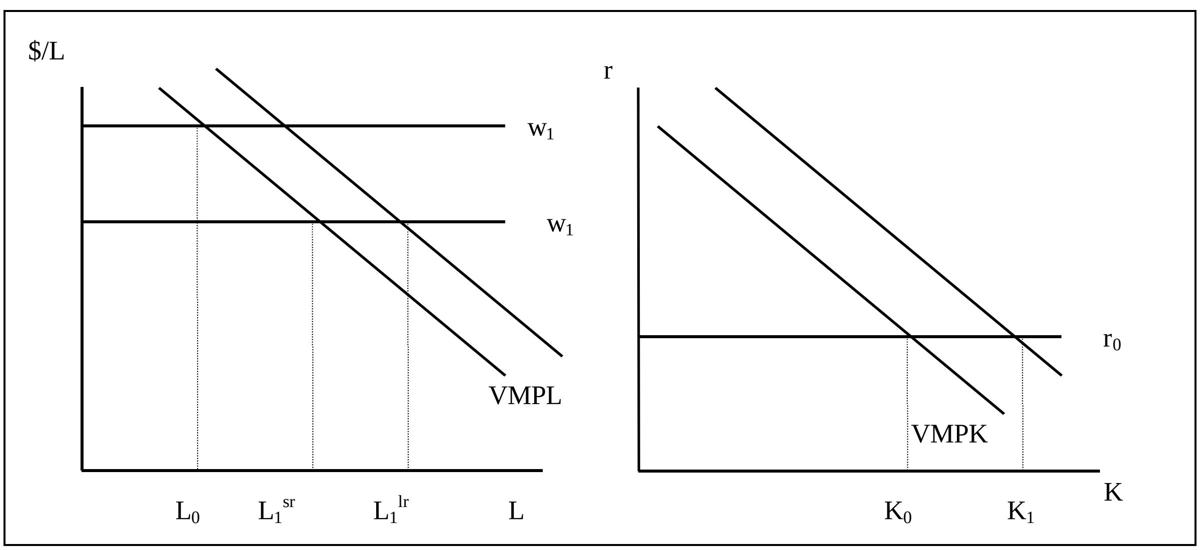


Figure 9.10. Long Run and Short Run Effects of a change in the price of an input

Thus we can conclude that the demand for an input is more elastic in the long run than in the short run regardless of whether the other inputs are complements or substitutes.

Lecture 10. Cost function and Derived Factor Demand

1. Profit maximization and Cost Minimization

In Lecture 9 we explored one way to set-up the firm's problem which consisted in maximizing profits by choice of the level of inputs.

(10.1) $\max \pi = pf(L,K) - wL - rK,$

with first order conditions

(10.2) $p\, F_L(L,K) = w$

$p\, F_K\,(L,K) = r$

The solutions to this problem are a demand of inputs and a supply of output as a function of the prices of the inputs and output:

(10.3) $L^* = L\,(p,w,r)$

$K^* = K\,(p,w,r)$

(10.4) $Y^* = F(L(p,w,r), K(p,w,r)) = Y\,(p,w,r)$

We could also substitute back this optimal quantities in the original problem to obtain the level of profit as a function of the prices of inputs and outputs. This function is called the profit function

(10.5) $\pi\,(p,w,r) = \max pF(L,K) - wL - rK = p\,F\,(L(p,w,r),K(p,w,r))$

This approach is correct as long as the firm is a price taker both in the product and in the input market. If, for example, the firm is a monopolist or oligopolist, its production decision will affect 'p' and this maximization will be invalid.

We can also set up the problem of the firm in two stages. In the first the firm finds the amount of each input that obtains the minimum cost for a given level of output and price of inputs, and in the second it decides on the output level depending on the market price. The assumption that the firm is a price taker in the product market is only important in the second stage.

The two steps in the two stage minimization are

(10.6) $C(w, r, y) = \min wL + rK + \lambda (y - F(L,K))$,

with first order conditions

$$w = \lambda F_L$$
$$r = \lambda F_K$$

Which we can rewrite as

$$\lambda = w / F_L$$
$$\lambda = r / F_K$$

λ is what it would cost to produce an extra unit of output using only one input. Thus the first order conditions in (10.6) mean that, at the optimum, the marginal cost is the same whether you change only labor, only capital, or both.

The solutions of this stage will be a set of 'conditional derived demand functions'. Conditional refers to the fact that they are demand functions for a given, fixed, level of output:

(10.7) $L^* = L(w,r,y)$

(10.8) $K^* = K(w,r,y)$

The second stage will be

(10.7) $\min py - C(w,r,y)$, with first order condition

$$p = \partial C(w, r, y(w, r, p)) / \partial y = C_y(w, r, y(w_1,...w_k,p))$$

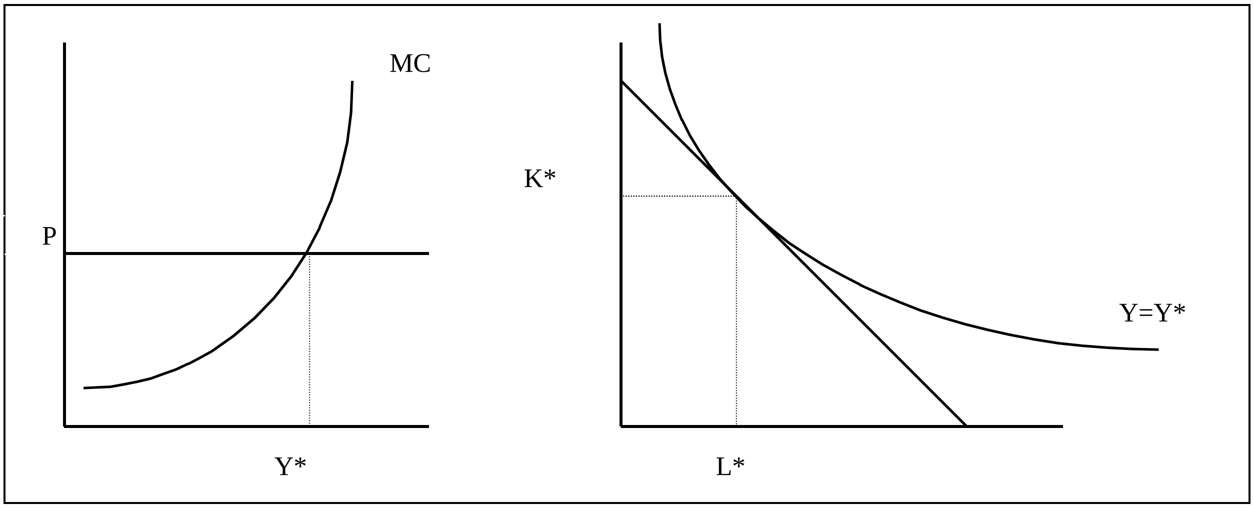


Figure 10.1 Two stage solution to profit maximization

The first order conditions in (10.6) are valid regardless of whether the firm is a price taker or it affects the price of the outputs. Note, however, that if the firms decisions affect the input prices this set up is not valid.

2. Properties of the Cost Function[1]

1. Concave in r,w.
2. Homogeneous of degree one in r,w.
3. Increasing in w,r.
4. $\partial C/\partial w = L^*(w,r,y)$.

The fourth property assumes r fixed, thus K is being freely adjusted as w changes. If K is held fixed, the marginal product schedule should be used. The marginal product schedule, as we saw in Lecture 9, is also a demand for labor, but it holds K fixed.

Contrary to what is the case in consumer theory, the conditional factor demand (i.e., the one that keeps y fixed, as in (10.7)) is more used in production theory than the unconditional one. The reason is that in production theory we usually have costs and output data, while it is generally difficult to obtain output price data.

Often in production theory both the cost function and the conditional derived demand function can be estimated econometrically, and property 4 is then tested. What is this test really testing? With all generality, the derivative of the cost function $C = \Sigma\, w\, x(w,y)$ is

(10.8) $$\partial C / \partial w_i = X_i + \Sigma\, w_j\, \partial X_j / \partial w_i$$

So what we are testing is that $\Sigma\, w_j\, \partial X_j / \partial w_i = 0$. Since output is constant, we know that $\Sigma\, F_j\, \partial X_j / \partial w_i = 0$. If firms are maximizing profits, $w_i = p\, F_i$. So this econometric test is testing whether firms are in fact optimizing.

3. Elasticity of substitution

<u>(a) Partial elasticity of substitution</u>

Since

$$C_i = \partial C / \partial w_i = X_i$$

(10.9) $$(w_j / X_i)\ (d X_i / dw_j) = (X_j\, w_j / C)\, (C_{ij}\, C) / (C_i C_j) = K_j\, (C_{ij}\, C) / (C_i\, C_j)$$

[1]For a mathematical derivation of these properties see Lecture 3, in which they are derived in the consumer's case.

Where $K_j = X_j w_j / C$ is the share of input j in the costs of production

If we define the partial elasticity of substitution $\ _{ij} = (C_{ij} C) / (C_i C_j)$, then the compensated demand elasticity (compensated in the sense of keeping income constant) will be

(10.10) $\varepsilon_{ij}^* = K_j \ _{ij}$

As we showed in Lecture 4 for the case of demand theory, the following identities hold

(10.11) $\Sigma_j \varepsilon_{ij}^* = 0$

(10.11) $\Sigma_i K_i \varepsilon_{ij}^* = 0$

<u>(b) Direct elasticity of Substitution</u>

We define the direct elasticity of substitution as

$$= d\ln (L/K) / d\log (r/w)$$

What is the relationship between the direct elasticity of substitution and the partial elasticity of substitution? In the two factor case they are the same,

(10.12) $= d\ln (L/K)/d\log (r/w) = d\ln (L/K) / d\log r =$

$d\log L/d\log r - d\log k/d\log r =$

$K_2 \ _{12} - K_2 \ _{22} = K_2 \ _{12} + K_1 \ _{12} = \ _{12}$

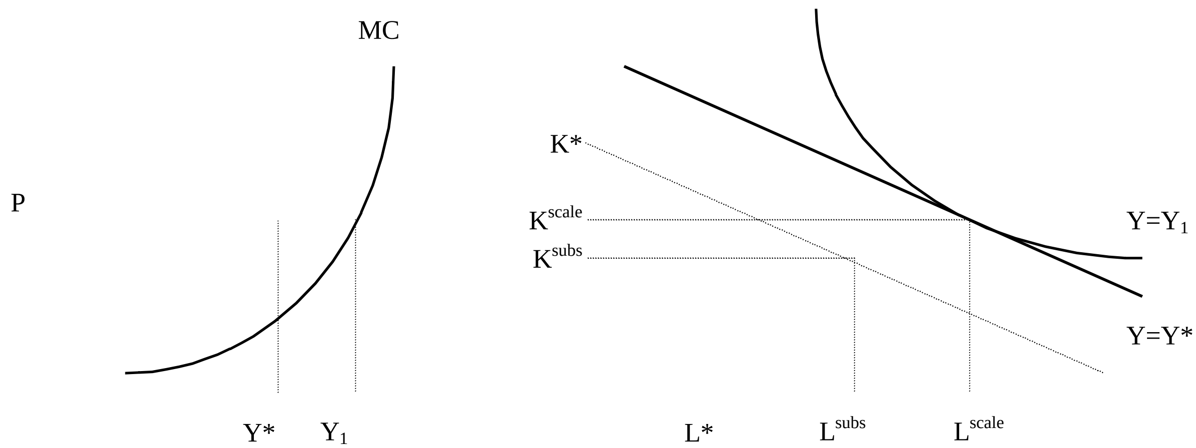
Where we have used the adding up identity (10.11) and the fact that a relative price change and an absolute change are the same in the tow factor case.

4. Derived demand for an input: Substitution and Scale effects

Using the conditional demand functions, we can decompose the change in the demand for an input when its price changes into a change in demand holding output constant and the change due to the change in output. This is analogous to the Slutsky decomposition in demand theory. We call here the price effect holding output constant the substitution effect, and the effect due to the change in the amount of output produced the scale effect.

(10.13) $dL / dw = dL^* / dw + (dL/dY) (dY/dw)$

Where dL^*/dw is the derivative holding output constant, i.e. the derivative of the conditional factor demand function.



MC
P
Y*
Y1
K*
Kscale
Ksubs
Y=Y1
Y=Y*
L*
Lsubs
Lscale

(10.19) $dX_i / dw_i = [C_{ii} - (C_{yi} C_{yi} / C_{yy})] = [C_{ii} - (C_{yi}^2 / C_{yy})] < 0$

This equation means that in the case of the derived demand of factors of production there is not even a theoretical possibility of 'Giffen good'. The change in the quantity demanded and in the price go unambiguously in opposite directions.

The intuition for this result is that if there is an increase in the price of a normal factor (i.e., the marginal cost increases when its price increases, $C_{iy} > 0$) the substitution effect means less input is used and the scale effect means also less output is needed because of the higher marginal cost. If the factor is inferior (i.e. $C_{iy} < 0$), substitution means less factor is used if its price increases and the scale effect means *more* output is produced because of the *decrease* in marginal cost. But as output increases *less* factor is used since it is inferior. This is the same result as in the normal factor case.

Is there a case in which we may consider a factor as inferior? Skilled labor might be an example. If its price increases substantially, there will be a jump to mass production, so that as its price increases production increases.

There is another way to obtain condition (10.18), which is based on observing the identity between the two approaches to profit maximization explored at the start of this lecture,

(10.20) $\pi(p,\mathbf{w}) = \max_x \; p\, f(\mathbf{x}) - \mathbf{w}'\mathbf{x} = \max_y \; py - C(\mathbf{w},y)$

(10.21) $\partial\pi / \partial w_i = x_i(\mathbf{w},p)$

(10.22) $\partial\pi / (\partial w_i \, \partial w_j) = \partial x_i(\mathbf{w},p)/\partial w_j = C_{ij} - (C_{yi}C_{jy} / C_{yy})$

Which is again the equivalent of the Slutsky equation as in (10.18)

Lecture 11. The Constant Returns to Scale Specification

1. Homogeneous and homothetic production functions

The constant returns to scale model is the basis of most microeconomic analysis. A usual two factor production function is written as

$$Y = F(L,K)$$

or it can be expanded, for example to

$$Y = F(L_1,L_2,K)$$

where L_1 can be production workers and L_2 can be non production workers.

Constant returns to scale means that the production function is linearly homogeneous, i.e.,

$$Y^* = F(tL_1 , tL_2, tK) = t\, F(L_1,L_2,K) = tY.$$

This implies that the slope of the isoquant curves are constant along rays that start at the origin. This is because the derivative of a function which is homogenous of degree one is homogeneous of degree 0, thus

$$(11.1) \qquad [\partial F(tL,tK) /\partial L] / [\partial F(tL,tK) /\partial K] = [\partial F(L,K) /\partial L] / [\partial F(L,K) /\partial K]$$

An homothetic production function is one which can be expressed as a positive monotonic transformation of a function that is homogenous of degree one , i.e., for the two inputs case,

$$Y = Z(L,K) = G(F(L,K))$$

The slopes of the isoquant curves of this kind of production function are also constant along rays starting from the origin. The monotonic transformation G thus amounts to relabelling the isoquant curves of a constant returns to scale production function. Furthermore, the slopes of the isoquant curves of the homothetic production function and of the homogeneous production function of which it is a transformation are identical,

$$(11.2) \qquad [\partial Z/ \partial L] / [\partial Z/\partial K] = \{G'[\partial F(L,K) /\partial L]\} / \{G'[\partial F(L,K) /\partial K]\} =$$

$$= [\partial F(L,K) /\partial L] / [\partial F(L,K) /\partial K]$$

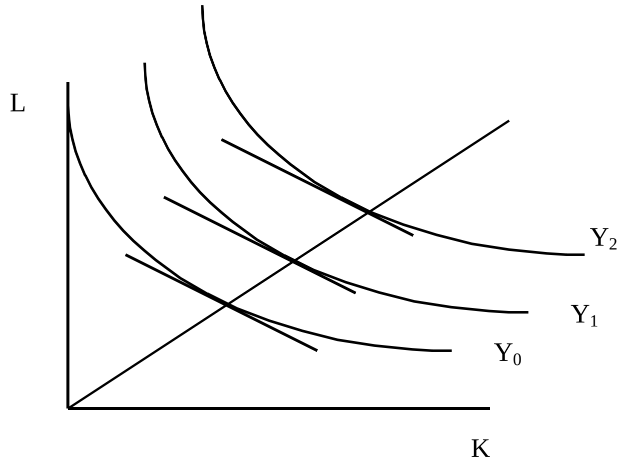


Figure 11.1 Constant Returns to Scale Production Function

The implications of (11.2) for economic analysis is that if we are dealing with homothetic production functions we can deal separately with the problem of the scale of production and the problem of the relative use of the inputs. Since the slopes of the homothetic function and the homogeneous function of which it is a transformation are the same, the first order conditions are the same in both problems,

(11.3) $$MP_L \,/\, MP_K = w/\,r \Rightarrow F_L \,/\, F_K = w/r$$

will be the first order condition both for Z and for F itself, as we proved in 11.2.

The cost function of a CRS production function can be written

$$C\ (Y,\, w_1,...,w_k) = Y\ \theta\ (w_1,...,w_k,1\ )$$

This means that the profit maximization is not well defined:

(11.4) $$P = C_y \Rightarrow P = \theta\ (w_1,...,w_k)$$

This equation cannot be solved for y, since C_y does not depend on y. The level of output will be either 0 or ∞. How can we avoid this problem? One way to solve it is to assume that the industry is in long term equilibrium so that P = MC = AC.

The cost curves for this kind of production functions are linear in output. U-shaped cost curves are usually interpreted as being the consequence of minimizing costs when capital (or other input) is fixed. For example, in an agricultural production function with constant returns to scale we obtain a U-shaped cost curve if land is fixed.

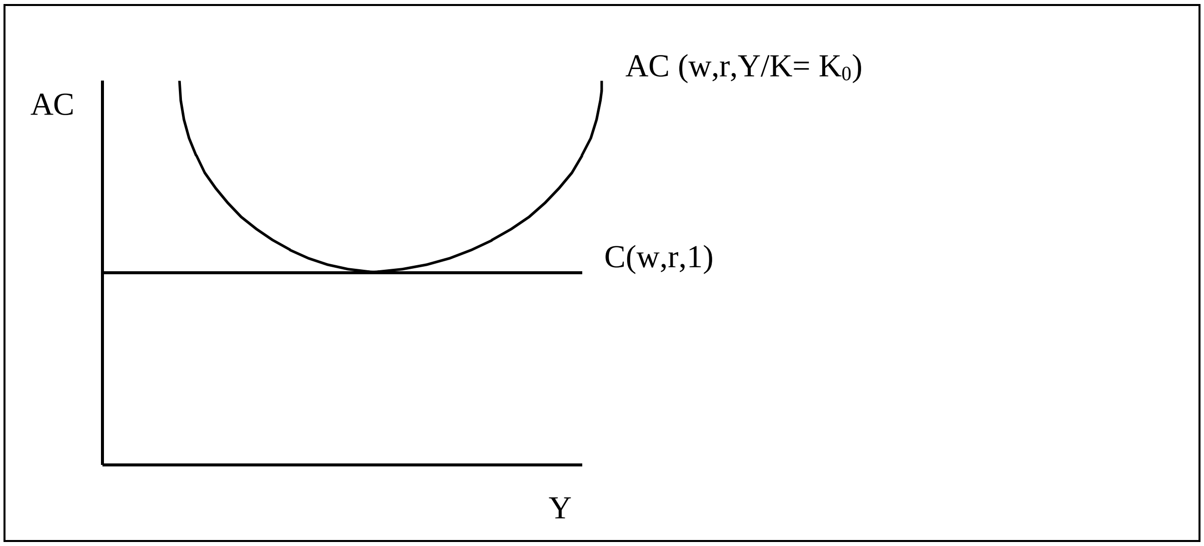


Figure 11.2. Long Run and Short Run Cost Curves with a CRS Technology

Another motivation for U-shaped cost curves is that we are cost minimizing in the level of Y/K rather than in the level of output. This amounts to scaling the axis by K and graphing the costs as a function of Y/K.

Finally, constant returns to scale implies that paying the factors their marginal products exhausts the total value of the product. By Euler's theorem, if F is linearly homogenous

(11.5) $$F(L,K) = L\, F_L + K\, F_K$$

Note that there is an interesting dichotomy in the CRS specification. The production side determines the price, since price is equal to marginal and average cost in the long run. Then, given this price, the demand side determines the quantity given this price. This is a useful framework to think of economic problems is many cases.

2. Reasons for using a CRS Production Function

The usual rationale for working with constant returns to scale production functions is that we do not want to worry about the scale of production, but only about the relative input levels. If we think that an homothetic representation is a good approximation to most production functions, then when we are willing to ignore the scale we can relate uniquely ratios of input prices to ratios of input quantities, even when there are increasing or decreasing returns to scale.

The cost function in the constant returns to scale case can be written

$$C\,(w_1,\ldots,w_k,y) = y\,\theta\,(w_1,\ldots,w_k)$$

Whereas in the homothetic case we can write it as

$$C(w_1,...,w_k,y) = g^{-1}(y)\,\theta(w_1,...,w_k)$$

Then, if we are willing to work with the relation of relative prices to relative quantities of inputs, we can disregard the first term in the right hand side and work as if the production function was homogeneous of degree one.

Also, we usually think that constant returns to scale is a good approximation to the way the economy works with free entry of firms. At industry level, replicability usually is a property of the data, so that if one firm produces 100 at C_0 two factories can produce 200 at the same cost. Even if there are decreasing returns to scale at firm level, each firm produces at its minimum, and the aggregate production function is approximately CRS if the technologies of the firms are more or less similar.

Furthermore, whereas at firm level if will not be the case in general that P=MC if the technology is CRS, at industry level P = MC is not a coincidence anymore. The reason is that at this level, P is not exogenous anymore. If P >MC, production will increase a lot until price decreases.

Finally, econometric tests of the constant returns to scale specification, conducted by fitting a function such as

$$C(y,w_1,...,w_k) = y\,\gamma\,\theta(w_1,...,w_k)$$

tend to give values of γ of around 1. The evidence seems thus to support the use of this specification.

3. Some useful production functions with CRS

1. <u>Cobb - Douglas</u>

(11.6) $$Y = A L^{\theta} K^{(1-\theta)}$$

This function has the following properties:

1. Unitary elasticity of substitution

2. Constant returns to scale

$$Y^* = A (tL)^{\theta}(tK)^{(1-\theta)} = t A L^{\theta} K^{(1-\theta)}$$

3. The labor share and the capital share are constant and independent of w and r.

$$F_L = A\theta (K/L)^{1-\theta}$$

So

$$(11.7) \qquad LF_L / Y = \theta$$

Analogously,

$$K F_K / Y = (1-\theta)$$

2. <u>Constant elasticity of scale production function (CES)</u>

$$(11.8) \qquad Y = (\theta_1 L^{\beta} + \theta_2 K^{\beta})^{1/\beta}$$

This is a generalization of the Cobb-Douglas case. The properties of this production function are:

1. Constant returns to scale

2. Other important production functions are special cases for different values of β. Cobb - Douglas is a special case when $\beta = 0$. When $\beta = 1$ we obtain the linear production function when. And we obtain the Leontief production function[1] when $\beta = -\infty$.

3. The elasticity of substitution is constant and depends only on β, which is the substitution parameter. It is easy to show that

$$= 1 / (1-\beta)$$

4. θ_1, θ_2 are the share parameters.

3. <u>Homothetic transformation of C - D</u>

Let

$$(11.9) \qquad Y = A L^{\theta} K^{\beta}$$

If $\theta + \beta > 1$ we have increasing returns to scale; if $\theta + \beta < 1$, we have decreasing returns to scale. The elasticity of substitution is still equal to one. This production function can be obtained by raising a Cobb -Douglas production function to a power.

4. <u>Homothetic generalization of CES</u>

$$(11.10) \qquad Y = (\theta_1 L^{\beta} + \theta_2 K^{\beta})^{\gamma/\beta}$$

---

[1]See next paragraph for an explanation of the Leontief production function.

Now θ_1 and θ_2 are again the share parameters and β is the elasticity of substitution parameter. γ is the scale parameter

5. Leontief Production Function

It is the limit of a CES when $\beta = -\infty$. Thus the elasticity of substitution is 0. It can be written

(11.12) $$Y = \min(K/K^*, L/L^*)$$

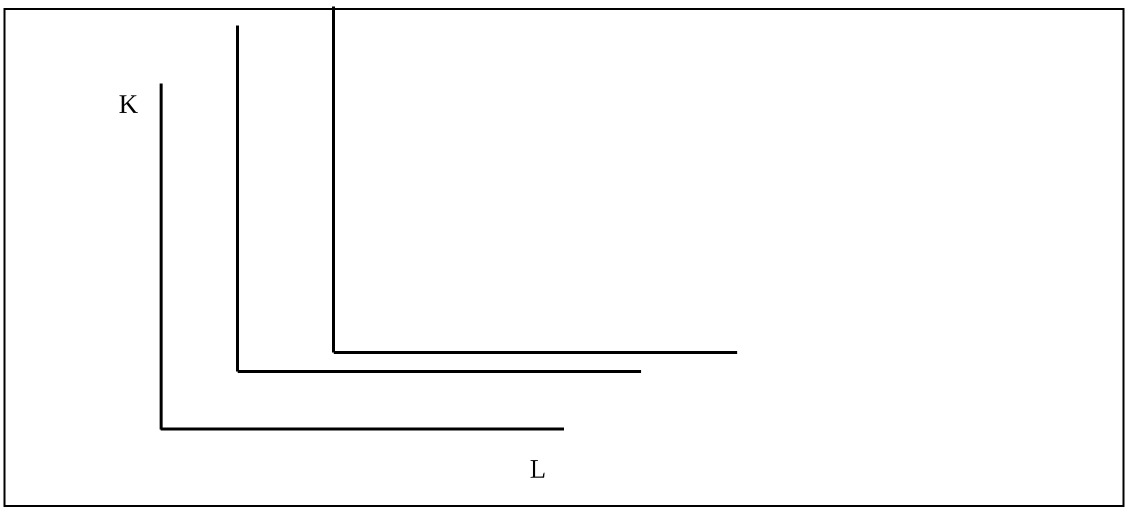


Figure 11.3. Leontief Production Function

4. Estimating the cost and factor demand functions in the case of CRS

The CRS assumption is most useful when dealing with the aggregate industry output. When the data is at the individual firm level, the best way to proceed is to estimate the conditional factor demand functions. The way to approach this problem econometrically can be described in the following two stages.

1. Normalize everything per unit of output, and obtain the unit cost function,

(11.13) $$C = Y\,\theta(w_1,\ldots,w_k) \Rightarrow c = \theta(w_1,\ldots,w_k)$$

2. Estimate the input demand equations,

Since

$$X_1 = Y\,\theta_1(w_1,\ldots,w_k)$$

Then,

(11.14) $x_1 = X_1 / Y = \theta_1 (w_1,...,w_k)$

3. Test if $x_1 = \partial C / \partial w_1$

Note that in an industry that is growing, the data on the input quantities will be non-stationary, so that we cannot infer which points correspond to a given production function. In this case, the CRS assumption allows us to set up the problem. By deflating everything by output we obtain comparable points.

Furthermore, even when the data are not stationary, the ratio of input quantities to input prices must be stable. Then another approach to estimation is to regress the ratio of input prices on the ratio of input quantities, to obtain the negative of the elasticity of substitution between the two factors,

$$\text{dln} (L/K) / \text{dln} (w/R) = - (\text{Elasticity of substitution between L and K})$$

5. Scale and substitution effects in the case of CRS

We know that the total effect on the quantity of a change in price is the sum of the substitution and the scale effects (see lecture 10).

$$\text{dlog } X_i / \text{dlog } w_j = \text{dlog } X_i^* / \text{dlog } w_j + \text{dlog } X_i / \text{dlog } Y \text{ dlog } Y / \text{dlog } w_i$$

$\Rightarrow$

(11.16) $\text{dlog } X_i / \text{dlog } w_j = k_j \quad _{ij} + 1 \text{ dlog } Y / \text{dlog } w_j$

(11.17) $\text{dlog } Y / \text{dlog } w_j = \text{dlog } Y / \text{dlog } P \text{ dlog } P / \text{dlog } w_i = \eta \text{ dlog } P / \text{dlog } w_i$

(11.18) $\text{dlog } P / \text{dlog } w_j = K_j$

Substituting in (11.17) and (11.18) in (11.16) we obtain

(11.19) $\varepsilon_{ij} = K_j (\quad _{ij} + \eta)$

Thus the scale effect is determined by the elasticity of the demand schedule. As the price of the factor increases, the price of the output rises depending on the share that the input has in the total cost. This price rise will cause a decrease in the demand of the factor depending on the slope of the demand for the final good produced.

Lecture 12. Technological Change

1. Analytical Specifications of Technical Change

There are several ways in which the possibility of technological change can be introduced in a production function. An obvious way is to add an argument to the usual production function,

(12.1) $Y = F(L,K,T)$

Another possibility is to write it as a shifter of the whole production function. This is called Hicks neutral technical change,

(12.2) $Y = A(T)F(L,K)$

This specification is very easy to analyze. The isoquants remain the same, but are only relabelled as a result of technical change. We can deal with this specification as if the data were stationary once we rename the variables.

A third possible specification of technological progress is labor augmenting technological progress.

(12.3) $Y = F(A(T)\, L\, ,\, K)$

In order for this specification to be valid it is not necessary that technical progress is embodied in the worker. If, for example, one worker does the work of two, it may be This specification, in the case of the Cobb-Douglas production function, turns out to give the same result as the Hicks neutral specification. For example,

(12.4) $Y = (LA)^{\theta}K^{1-\theta} = A^{\theta}L^{\theta}K^{1-\theta}$

2. Analyzing the consequences of labor augmenting technological progress

<u>1. Effect on the Marginal Product of Labor</u>

In general, it is uncertain how labor augmenting technological process affects MP_L and MP_k relatively. The reason is that

(12.5) $w = MPL \Rightarrow w = A\, F_L\, (AL,K)$

Thus when we change A, the direct effect is an increase in MP_L and the indirect effect is a reduction. Thus the ratio of rental price of capital to rental price of labor services moves in an ambiguous way,

(12.6) $$w/r = A\ g(AL/K)$$

The relative pattern of the marginal product of labor and capital with technological progress is called *technological bias*.

We say that if

$d\,[F_L/\,F_K\,]\,/dA > 0$ technological progress is biased towards labor,

and if
$d\,[F_L/\,F_K\,]\,/dA < 0$ technological progress is biased towards capital

2. Effect on the Demand for labor

What will be the consequence of labor augmenting technological progress for the demand for labor? The key to the response will be in the elasticity of the demand for labor. If the demand is elastic, more labor will be used, if it is inelastic less.

We can rename the variables and deal with effective units of labor. To do this, note that in the FOC in 5,

$$P\ A\ F_L(AL,K) = w \Rightarrow P\ F_L(AL,K) = w\ /\ A$$

We can call AL =L* the units of effective labor and w/A the wage per unit of effective labor,

$$P\ F_L(L^*,K) = w^*$$

So that the demand function for labor can be written as

$$L^* = D\ (w^*, P)$$

Now the elasticity of demand for labor is defined as (note that the meaningful concept for the firms is how much effective labor they have, not how many actual workers they have)

$$\text{dllog}\ L^*/\text{dlog}w^* = \varepsilon_D$$

Then the change in the number of workers when there is technological progress will be (where we assume that w is constant as A changes)

(12.7) $$\text{dlog}L/\text{dlog}A = \text{dlog}L^*/\text{dlog}A - 1 = \text{dlog}L^*/\text{dlog}\ w^*\ \text{dlog}w^*/\text{dlog}A -1 =$$

$$= -d\log L^*/d\log w^* -1 = \varepsilon_D - 1 = -K (\quad + \eta) -1$$

Where η is the elasticity of the demand for the product produced. For the actual use of labor to increase, it is necessary that the percentual increase in the demand for effective labor due to the effective decrease in its price be bigger than one. This will be the case when the demand is very elastic. For this to be the case, it is necessary that the final demand for the product be large, so that the scale effect of increasing production a lot will drive a large increase in the use of labor; or it is necessary that the substitutability between labor and capital be large, so that the increase in the substitution of capital for labor increases not only the use of labor in effective units (which is certain) but also in actual units.

Technological progress has historically reduced the need for the labor factor in the industry in which it takes place.

The key parameter in order to obtain a concrete prediction both for the demand and for the price of labor is the elasticity of substitution between capital and labor, ,

$$d\log (w/r) / d\log (L/K) = - 1/$$

We can graph the isoquants in the two factor case as a function of this parameter,

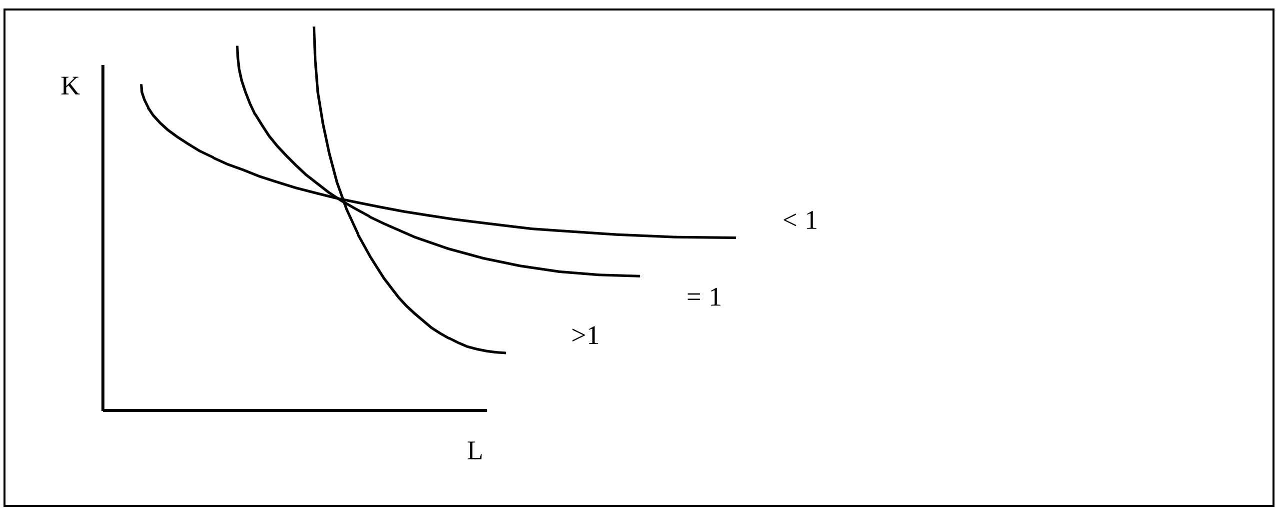


Figure 12.1 Isoquants and Elasticity of Substitution

If the elasticity of substitution is small, labor augmenting technological progress favors capital..

3. Measuring technological progress

There are two interesting parameters we would like to estimate: the *rate* of technological progress and the technological bias, i.e. how does technological progress affect the ratio of marginal product of labor to marginal product of capital.

1. Measuring the rate of technological progress: growth accounting

Let the production function have a general form as in 1,

$$Y(t) = F(L(t), K(t), A(t))$$

(12.8) $$dY/dt = F_L\, dL/dt + F_K\, dK/dt + F_A\, dA/dt$$

We call the last term in the RHS the total factor productivity or TFP,

(12.9) $$dY/dt = w\, dL/dt + R\, dK/dt + TFP$$

Then

$$[dY/dt]/Y = w L/Y\, [dL/dt]/L + RK/Y\, [dK/dt]/K + g_t$$

We call the last term in the RHS the total factor productivity or TFP,

(12.10) $$g_y = S_L\, g_L + S_K\, g_K + g_t$$

And the growth in total factor productivity would be

(12.11) $$g_t = g_y - S_L\, g_L + S_K\, g_K$$

This parameter g_t can be interpreted as the residual resulting from fitting a Cobb-Douglas production function to the output, labor and capital data. This residual is called the Solow residual, since it was he who proposed this way to measure technological progress.

CRS is not a required assumption. If we are in a situation with increasing returns to scale, the formula can be used with slight modification:

(12.13) $$d\log Y/dt = P/MR\, [S_L\, d\log L/dt + S_k\, d\log K/dt] + \text{residual}$$

Where the ratio P/MR can be obtained sector by sector and then aggregated for the entire economy.

2. Estimating technological bias

Estimating the rate of technological bias is more difficult, since it is necessary to know in order to identify it.

In the case of CRS, the production function is fully specified by the relation between relative prices and relative quantities. The elasticity of substitution can be estimated as

(12.14) $\quad$ (dlog w/r) /dt = -1/ (dlog L/K) / dt + dB/dt

Where dB/dt is the rate of 'demand shifts' in the relative price of labor, or technological bias.

4. Application : Changes in the Relative demand for College and High School graduates

The data on the relative demand for college versus high-school graduates during the last 30 years can be summarized by the following four data points

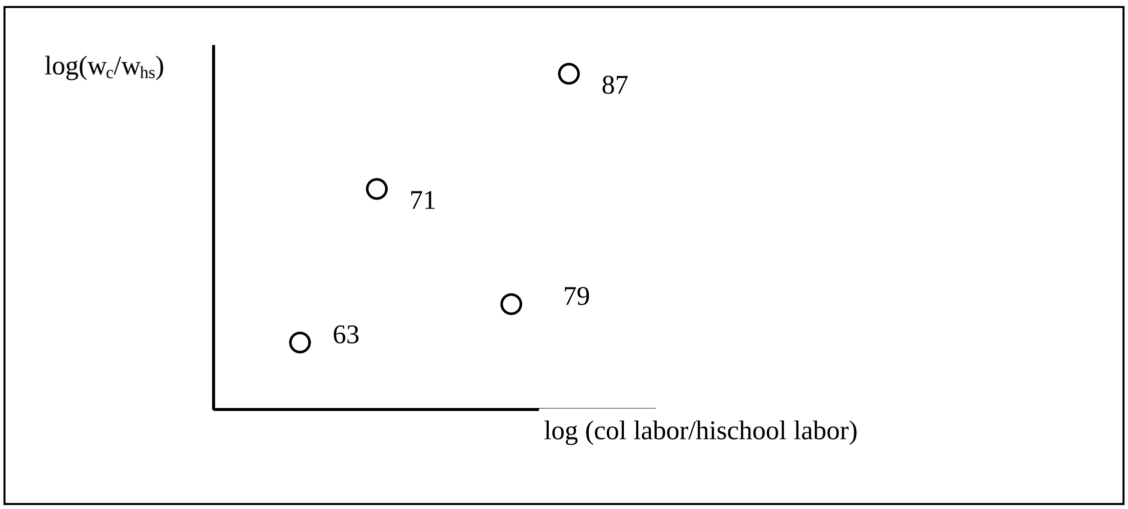


Figure 12.2. The relative demand for college vs. high school labor

The four data points in Figure 12.2 summarize the main trends in the evolution of the relative demand of college and high school graduates.

There are two stories that can explain the data. Both of them need of the existence of biased technical progress. However, the role that the demand and supply shifts play in each case in driving the fluctuations is completely different.

Hypotheses 1. Supply changes are the cause of the fluctuation, demand is shifting at constant rates

If the elasticity of substitution between college and high school labor is low, (so that the relative derived demands are steep) then a shift in B at a constant rate is enough to create the pattern in figure 12.2

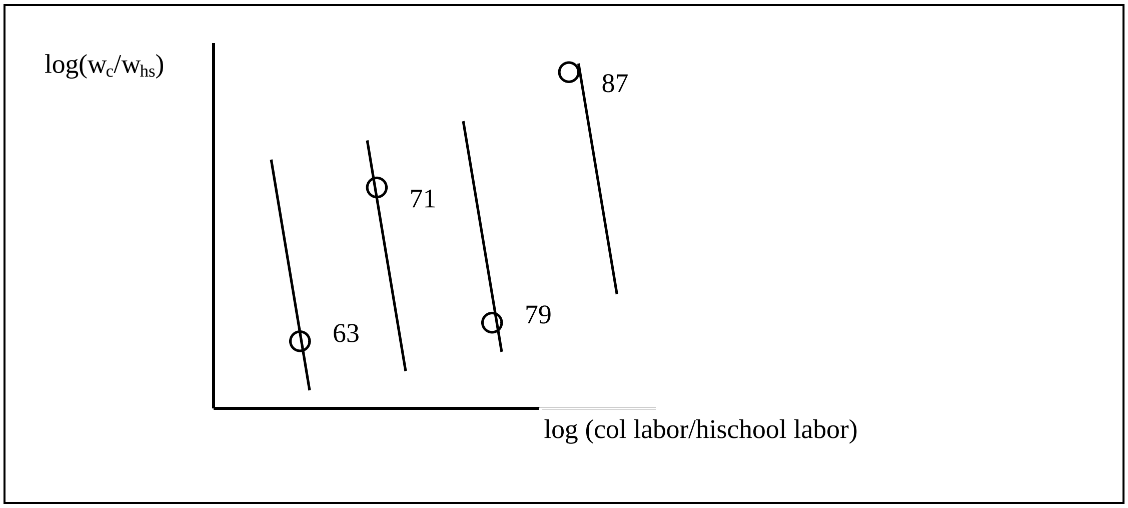


Figure 12.3 Supply drives most of the changes, demand shifts at a constant rate

Hypotheses 2. The relative demand for skilled labor first increases, then stays constant, and then increases again

If the elasticity of substitution is very high, then demand for college versus high school graduates must be fluctuating in order to explain this pattern. Since the two are good substitutes, a pure change in supply is unable, alone, to explain the pattern in the relative demand for college educated versus high school graduate labor.

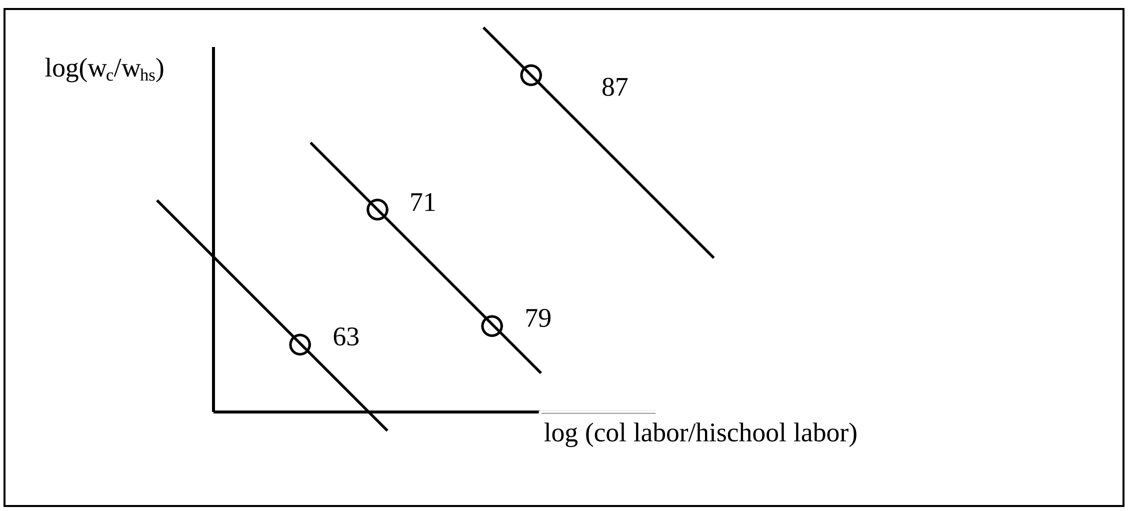


Figure 12.3 Demand drives most of the changes

Thus in order to identify which of the two stories explains the data we need to know the elasticity of substitution between college educated and high school graduates.

Lecture 13. Capital Market Dynamics and Investment

1. Special characteristics of the capital market

In what follows we will analyze the dynamics of the capital market, given the fact that the firm faces a rising supply price for investment. There are two main reasons why the firm may face a raising price of investment: the scarcity of resources, which makes their price go up us their demand increases (external adjustment costs) and the increasing costs to the firm of installing new investment and adjusting to it (internal adjustment costs). We will consider these reasons in detail and their consequences for the shape of the investment function in Lecture 14.

In this lecture, we will study the implications of the rising cost of investment for the dynamic behavior of the capital market. The main consequence is that it is not optimal for the firm to adjust instantaneously the capital stock.

Consider an industry with production function F (L,K). Assume the instantaneous product demand function

(13.1) $P(t) = P\,(Y(t), D\,(t)),$

where D(t) is a vector of demand shifters, and the instantaneous production function

(13.2) $Y(t) = F\,(K(t), L(t))$

The fact that the capital asset is durable means that, instead of one price and one quantity, we now have two prices and two quantities. The prices are the capital price, which is the price that must be paid in order to use the asset for ever, and the rental price, which is the cost of using the asset for one period. The two quantities that we will consider are the stock of capital, and the flow of new capital or investment.

We will call w(t) be the wage at paid at time t, q (t) the market price of capital, R(t) the rental price of capital, r the interest rate and δ the rate of depreciation of capital.

Using a constant rate of depreciation simplifies things considerably. It means that depreciation is only going to depend on total capital. Another, more complicated, assumption would be history dependent depreciation.

2. Rental price and capital price

The rental price is the cost of using the asset for one period. There are two ways to obtain a machine for ever. One is to pay the rent in every period. The other is to buy it. Arbitrage will make these two values equal, i.e.,

(13.3) $q(t) = R(t) + R(t+1)(1-\delta) / (1+r) + R(t+2)(1-\delta)^2 / (1+r)^2 + ...$

Or, in continuous time,

(13.4) $q(t) = \int_\tau^\infty e^{-(r+\delta)(\tau-t)} R(\tau)\, d\tau$

Sometimes, the rental market does not exist. In order to obtain the rental price, we can use the capital price. For example, in discrete time,

(13.5) $R(t) = q(t) - q(t+1)(1-\delta)/(1+r)$.

Rearranging,

(13.6) $(1+r) R(t) = rq(t) + [q(t) - (1-\delta)q(t+1)]$

This expression has a useful interpretation. In tomorrow's dollars, the rental cost has two components: interest cost and depreciation, which, in turn is due to physical depreciation and market change of the capital value of the asset. The market price of an asset includes both components of depreciation together. For example, the price of a 1986 car includes the change in the market price of the asset and the physical depreciation.

In the continuous time case, taking the derivative of (4) with respect to time,

(13.7) $dq / dt = - R(t) + (r+\delta) q(t)$

(13.8) $R(t) = rq(t) + [\delta q(t) - dq/dt]$

Which is the expression equivalent to (6).

Yet another interpretation can be obtained by rearranging,

(13.9) $rq(t) = R(t) - \delta q(t) + dq/dt$

We can read (13.9) as stating that in the optimum the opportunity cost of capital must be equal to the rental price plus the price appreciation minus the depreciation.

The usefulness of these formulas is that they allow us to read off the market expectation about the evolution of the asset price from the rental price and the capital price provided we know something about the physical depreciation. Rearranging (13.9) again,

(13.10) $(dq/dt)/q(t) = r + \delta - R(t)/q(t)$

Thus the relation between rental price and capital price allows us to infer the market expectation about future prices.

3. A model of the market for capital goods

The equilibrium in the capital market is like a normal demand and supply problem, except that the demand for capital services is a function of the stock, whereas the supply is a flow (investment). Apart from the demand and supply equations, we need an equation relating stock and flows and an equation relating the flow prices (rent) with the stock prices (capital price).

The first equation is the usual factor demand equation, given by the condition that, at the optimum, the rental price is equal to the value of the marginal product of capital.

(13.11) $R(t) = P(Y(t), D(t))\, F_K (K(t),L(t))$

To obtain a demand curve, we need to optimize L out of the problem. Then we will obtain

(13.12) $R(t) = VMPK\, (K\, (t), D(t),w(t))$

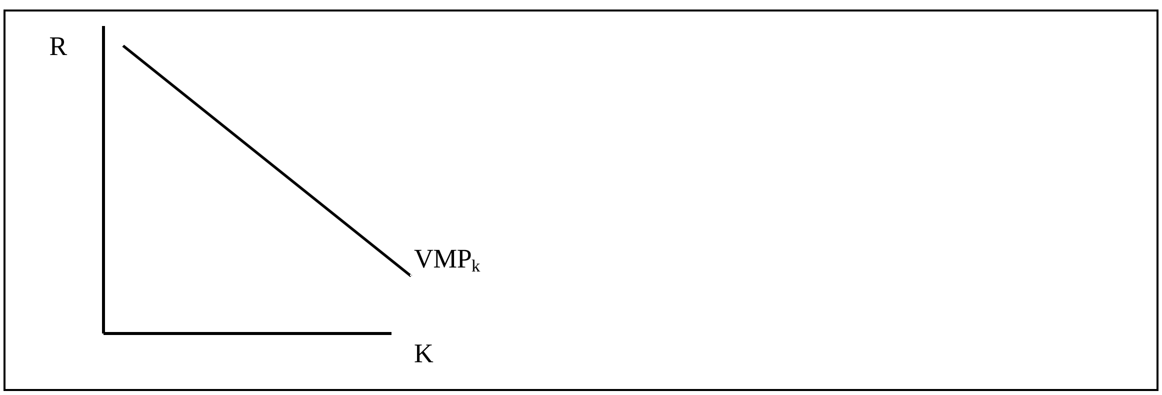


Figure 13.1. Demand for capital

The supply side will give us the supply for investment. As the amount of investment increases, so does its cost, due to internal and external adjustment cost. This is what originates the dynamic aspects of the problem. If the supply curve was horizontal, and the firm could obtain all the capital needed at no extra cost (neither internal nor external), this problem would be exactly like the usual one and the dynamic aspects would be irrelevant. We will discuss this function is Lecture 14. For the purposes of this lecture, it will be sufficient to express this relation as

(13.13) $I = S(q(t))$

Graphically,

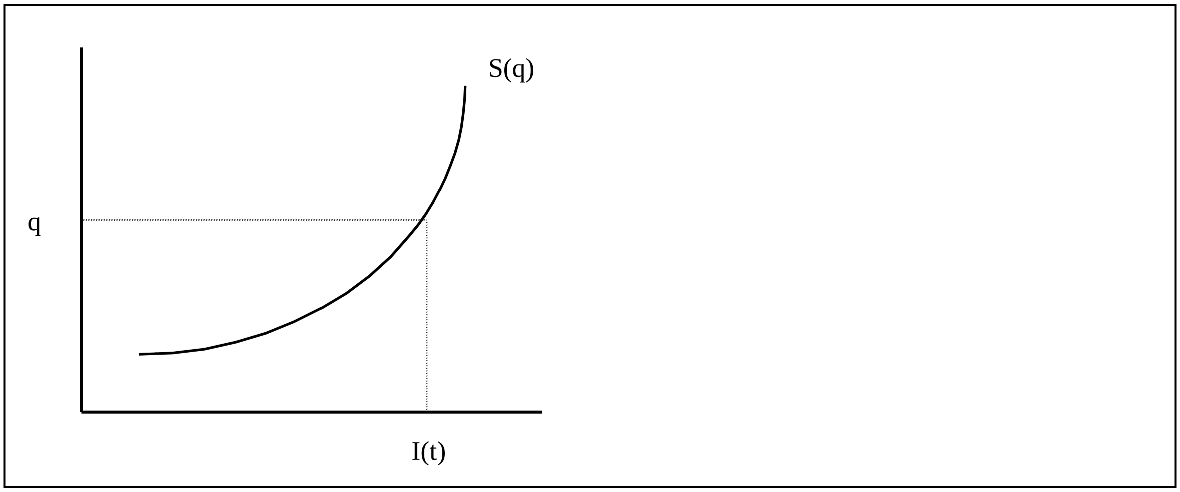


Figure 13.2 Supply of Investment

The two unusual equations are one relating stocks and flows and one relating capital prices and rental prices. The law of motion for capital is

(13.14) $dK/dt = I(t) - \delta K$

and the capital pricing equation is

(13.15) $R(t) = (r+\delta)q - dq/dt$

This is a standard dynamic model, with two state variables, q and K and two flow variables, I and R. We can think of it intuitively in the following way: K is a summary of the past, it is a consequence of past market conditions; q is the opposite, it is a summary of the future market conditions. K and q are linked trough time because capital is durable. Thus all of the variables in the system today are influenced by past and future supply and demand conditions.

4. The Steady State Equilibrium

In order to be able to analyze this problem, it is useful to think of a steady state equilibrium, in which none of the magnitudes is changing. This is interesting for two reasons. First, because this is the long run equilibrium. It will also allow us to investigate the dynamics in terms of deviations from the long run equilibrium.

By definition, in the steady state,

(13.16) $dK/dt = dq/dt = dR/dt = dI/dt = 0$

Then,

(13.17) $\underline{I} = \delta\ \underline{K}$

(13.18) $\underline{R} = (r+\delta)\underline{q}$

(13.19) $\underline{R} = VMPK\ (\underline{K})$

(13.20) $\underline{I} = S(\underline{q})$

(where the underlined values are steady state values)

We can simplify these equations into two,

(13.21) $\delta\underline{K} = S\ (\underline{q})$

(13.22) $VMPK\ (\underline{K}) = (r+\delta)\underline{q}$

Equation 22 says that, in the steady state, the marginal product of capital covers interest and depreciation. It gives us the demand side of our model: if the value of the marginal product of capital is higher than its cost (interest and depreciation) the firm will want to increase the amount of capital employed. Equation 21 is the supply side: the supply of capital is a function of the capital price and, in steady state, it must be equal to the loss of value of the current capital stock. We can represent these two equations graphically as in figure 13.3.

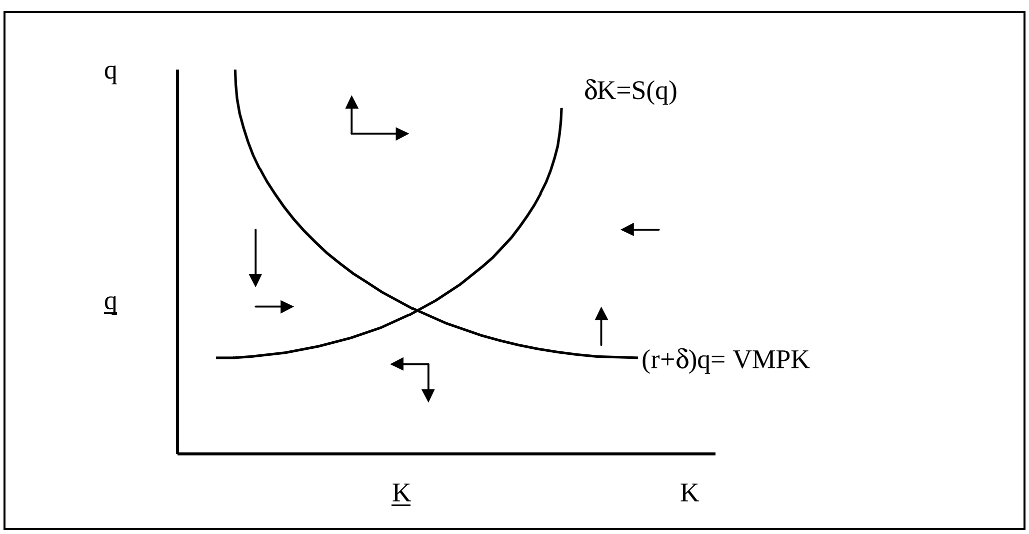


Figure 13.3 Steady State Equilibrium

How does the equilibrium react to perturbations? For values of q and K higher than those delimited by the equation dq/dt = 0 , (r+δ) q > VMPK so that dq/dt>0, and price is increasing. We represent this by an arrow pointing north. Similarly, on the right of dK/dt = 0, S(q(t)) < δK(t), and dK/dt < 0.

From this graphical representation, it is easy to do comparative statics on the solution. For example, how does a change in the discount rate affect the stationary steady state values of the price and the discount rate?

5. Dynamics out of the Steady State

Analytically,

(13.23) $dq/dt = (r+\delta)q - VMPK(K_t,D_t)$

(13.24) $dK/dt = S(q(t)) - \delta K$

This equations can be represented in the following way,

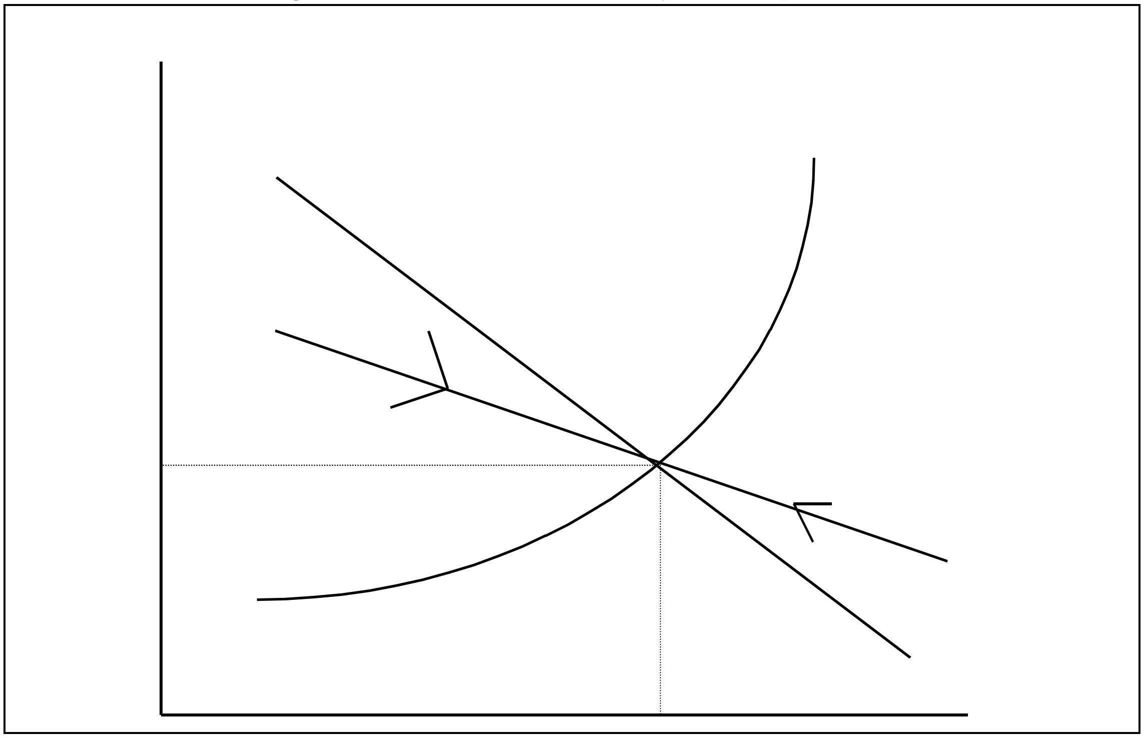
(13.25) $\begin{pmatrix} dk/dt \\ dq/dt \end{pmatrix} = G\begin{pmatrix} K(t) \\ q(t) \end{pmatrix}$

The steady state is the solution to G(t) = 0. In order to analyze the evolution of a perturbation of the steady state equilibrium, we can linearize this function around the steady state equilibrium values of K and q. Since, at these values, G(t) is 0, the approximation will be,

(13.26) $\begin{pmatrix} dk/dt \\ dq/dt \end{pmatrix} = \begin{pmatrix} -d & S' \\ -VMPK & (r+d) \end{pmatrix}\begin{pmatrix} K(t)-K \\ q(t)-q \end{pmatrix}$

The eigenvalues of the matrix in 13.26 are real and of opposite sign. The path to the stationary equilibrium is illustrated in figure 13.4. This means that there is only one line through which the system will converge towards the equilibrium. What makes us believe then that the system will converge?.

Figure 13.4. Path to the steady state equilibrium



The key is that q is cannot just take any value; q is a solution due to the forward looking behavior of the agents. For each value of K there is only one value of q that can be derived from optimization, and that is precisely the one that insures convergence towards the steady state equilibrium.

6. An application: the housing market

Let K be the stock of housing, R the rental rate for housing services, q the price of housing, and I the new housing construction (not the housing sales!).

Since this asset is very durable, the market for new construction is very volatile. First, the intertemporal substitution in consumption of housing is very low. Second, 2.5% of the stock every year is built every year. So if there is a drop in demand of 2%, 80% of new construction gets hit.

Imagine initially that the supply of new construction is perfectly elastic. Then the adjustment would take place in one period. Clearly, it is the upward sloping supply of new housing what makes it optimal to spread the new investment.

Assume, for example, that a tornado destroys part of the housing stock. Then the VMPK goes up the rental price also will increase and q increases as well. The increase in R will be larger than the increase in the capital price, since q is the present value of all future rents, and they will not increase so much.

How would the dynamics look? We can represent the time path of the four main variables in this problem after part of the capital stock has been wiped out,

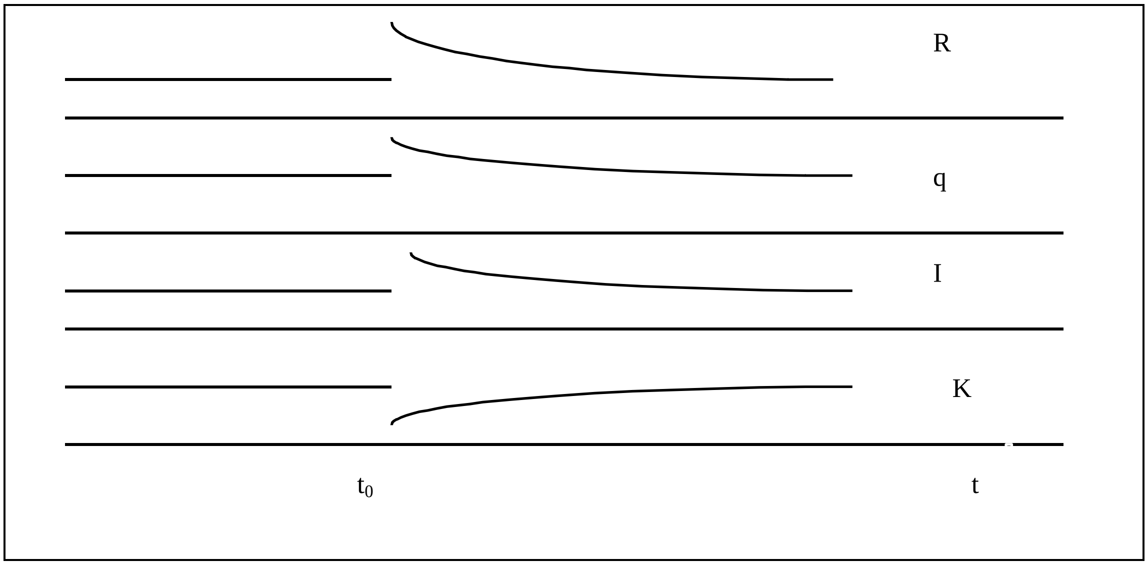


Figure 13.5 Time Path back to steady state equilibrium

What determines the time to get to steady state equilibrium?

- The rate of depreciation δ determines the memory of the problem. The larger δ the faster the rate of convergence.
- The elasticity of the demand of capital. The more inelastic the demand, the larger the change in price when there is a drop in the quantity, thus the larger will be the reaction of the investment, and the adjustment will be faster.
- The elasticity of the supply of investment. The higher the elasticity of supply, the higher will be the convergence, since a given change in price will lead to a large change in the amount of investment.

Lecture 14. The Investment Decision of the Firm

In Lecture 13 we took the equation describing the investment decision of the firm I=S(q) as given and used it, together with the stock flow accounting identity, the no arbitrage condition in the rental market, and the demand for capital, to analyze the equilibrium in the capital market. In this lecture we will derive them as a solution to the firm's profit maximization in the case in which there are adjustment costs. We will use the procedure of setting up a Hamiltonian, since this procedure allows us to be explicit about the marginal value of the constraint and about the possibility of corner solutions.

1. Setting up the problem as a simple profit maximization

Let V(K) be the total value to the firm of K units of capital, and C(I) the cost of investment. We can assume the firm has only one input, since otherwise we can optimize the other inputs out. Again, C(I) is shorthand for the external or internal adjustment costs the firm faces, which we will analyze in the next paragraph. As investment rises, the unit cost of investment increases.

We can obtain the investment decision of the firm as a solution to the following problem

(14.1) $\max \int_0^\infty e^{-rt}\{V(K(t)) - C(I(t))\}dt$

such that $I=dK/dt + \delta K(t)$, and $K(0) = K_0 \geq 0$, $I(t) \geq 0$ for every t.

The current value Hamiltonian is

(14.2) $H = V(K(t)) - C(I(t)) + q(t)\,(I - \delta K(t))$

The interior first order conditions will be

(14.3) $\partial H/\partial I$: $C' = q(t)$

This equation equalizes the marginal value of a unit of investment to its marginal cost. The first order condition for the present value multiplier will be[1]

[1]To obtain the present value multiplier note that, if the problem was solved by setting the normal Hamiltonian,
$H2 = e^{-rt}(V(K(t)) - C(I(t)) + \lambda(t)\,(I - \delta K(t))$,
The first order condition with respect to λ would be
$\lambda' = - H2_K$
But
$q = \lambda e^{rt}$, and $q' = r\lambda + \lambda' e^{rt} = rm - H_k$,
where H is the current value Hamiltonian of equation (14.2).

(14.4) $q'(t) = (r+\delta)q(t) - V'$

(14.5) $dK/dt = I - \delta K(t)$

Substitute 3 into 5 to get

(14.6) $dK/dt = C'^{-1}(q) - \delta K(t) = S(q) - \delta K(t)$

The present value multiplier is the marginal value of the resource constraint. This is the price of the capital stock. Thus, equations (14.6) and (14.4) give us the dynamics of the capital stock and the price. They are analogous to equations 13.23 and 13.24, and can be analyzed in an identical way.

We have obtained the four equations that we set up in Lecture 13 to describe the dynamics of the capital market from the firm's profit maximization when there is an adjustment cost of investment.

Equations 14.3 and 14.4 can also be expressed in terms of the usual marginal cost equal marginal value condition. Solving the differential equation in 14.4,

(14.7) $q'-(r+\delta)q = -V' \Rightarrow e^{-(r+\delta)t}[q'-(r+\delta)q] = -e^{-(r+\delta)t}V' \Rightarrow$

$e^{-(r+\delta)t} q = \int_t^\infty (-e^{-(r+\delta)\tau}V'd\tau$

(14.8) $q = \int_t^\infty (-e^{-(r+\delta)(\tau - t)}V'd\tau$

(14.9) $C'(I) = \int_t^\infty (-e^{-(r+\delta)(\tau-t)}V'(K(\tau))d\tau$

So, at the optimum, the marginal cost of investment is equal to its marginal value. We can interpret equation 8 as saying that, at the optimal, the price of the capital must be equal to the present discounted value of it return.

2. Internal and External Adjustment Costs

We have obtained the supply curve for investment as the inverse of the marginal adjustment cost. There are two main reasons that make us consider this adjustment cost. Depending on which one of them we are modeling, the equation describing adjustment costs will be slightly different.

$H_k = V' - \delta q$, and substituting in for q, we obtain expression (14.4).

The first reason for the upward sloping supply of capital investment is that when the firm tries to satisfy its investment demand immediately, investment price is driven up. The firm faces an upward sloping market supply of capital. This cost, due to the effect on the market price of capital of the firm's demand, is called external adjustment cost.

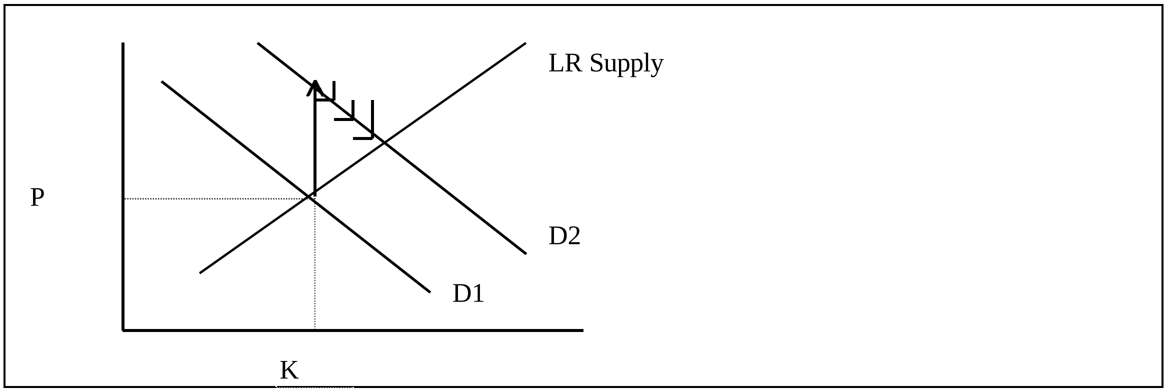


Figure 14.1. External Adjustment Cost

This model is useful in industries which have an specialized capital stock. For some industries this is not the case, and their own demand of capital has no effect on the market price of capital. Would we be back in this case to the instantaneous dynamics case?

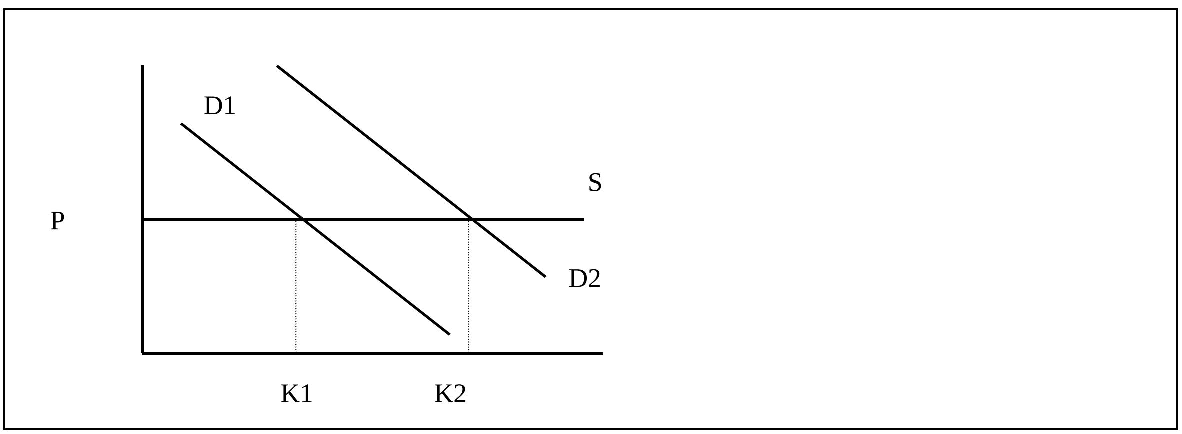


Figure 14.2. No External Adjustment Cost

In this case, the firm still faces a rising supply price to its investment. The reason is the adjustment cost that the firm must endure in order to adapt the new capital, to train its labor force to use it, etc. This force also pushes the firm to smooth its investment flow[2].

[2] We can also think of adjustment costs as a two step adjustment process. For example, why is there a rising long term supply price of investment in the housing market? The reason is that there is a second order adjustment cost: the capital to build new capital is limited, and it needs to be expanded first in order to expand the stock of houses. Introducing second order adjustment costs, we would get an overshooting behavior. We will not pursue this approach here, since most of the main characteristics of the process are well captured by our simpler adjustment cost model.

We can formalize both the idea of internal and of external adjustment costs, in order to obtain the implications of these costs for the behavior of the investment and the capital stock of the firm.

In the case of internal adjustment costs, since they are a consequence of the firms need to adjust to new capital, we can model them as a function of the net investment that the firm undertakes. We can then write the problem in 14.1 in a more explicit fashion as

(14.10) $$\max \int_0^\infty \{e^{-rt} [V(F(K(t))) - PI(t) - C(I(t) - \delta K(t))\} dt$$

such that $I(t) = dK/dt + \delta K(t)$.

Thus we have separated the cost function in 14.1 into two parts. The first one is a linear function of investment, which reflects the fact that, in this specification, the firm faces a fixed, given, external capital price. The second part is an adjustment cost function that depends only on the net change in capital stock of the firm, which is the reason for the need to adjust.

A common specification for the internal adjustment costs is

(14.11) $$C(dK/dt) = a(dK/dt)^2$$

If the firm's demand for capital does affect the market price for capital then the linear term in 14.7 is redundant, and we will write

(14.12) $$\max \int_0^\infty \{e^{-rt} [V(F(K(t))) - C ((dK/dt) + \delta K(t))\} dt$$

Here, according with the rationale expressed, the adjustment cost will be a function of the gross flow of investment, since the reason for the adjustment costs is that buying more capital drives up its supply price.

Thus the distinction between internal and external adjustment cost is somewhat trivial. The firm faces a rising cost of investment, and it does not make much difference why. In fact, depending in the problem we are considering, there are situations in which the internal adjustment costs will depend on gross, not net investment.

3. Solving the firms problem through Euler Equation methods

An equivalent way to solve the problem is, instead of setting up the constraint maximization and then the Hamiltonian, directly substituting in the constraint into the objective function and obtaining the Euler equation. Assuming that only internal adjustment costs matter, and that the firm faces a constant external price of investment, the firms profit function is

(14.13) $\max \int_0^\infty \{e^{-rt} [V(F(K(t))) - P(t)[dK/dt + \delta K(t)] - C(dK/dt)\}dt$

The formula that gives us the Euler Equation is

(14.14) $\frac{dF_{dk/dt}}{dt} = F_k$

Which is, in this case,

(14.15) $F_{dk/dt} = e^{-rt} [-P -C']$

(14.16) $F_k = e^{-rt} [V' - P\delta]$

(14.17) $F'_{dk/dt} = \{ e^{-rt} [-P -C']\}' = -re^{-rt} [-P-C'] + e^{-rt} [-P' - C'' d^2k/dt] =$

$= e^{-rt} [r(P+C') - (P' + C'' d^2K/dt)]$

(14.18) $V' - P\delta = r(P+C') - (P' + C'' d^2K/dt)$

Rearranging 14.15 we obtain an optimal rule which can be easily interpreted

(14.19) $V' = (\delta+r)P - P' + rC' - C'' (d^2K/dt)$

The left hand side is the marginal product of capital. The right hand side is the total user cost of a unit of capital. It has several terms, reflecting the opportunity cost of the money invested in capital (rP), the loss due to depreciation (δP), the capital gain or loss (P'), the opportunity cost of the investment made (rC') and the capital gain or loss on the investment made. Thus this equation tell us that we can obtain the optimal level of capital by making the marginal value of an extra unit of capital equal to the total cost to the user of an extra unit of capital.

By solving this equation we can obtain the investment function for the firm, the temporal path of the capital stock, and, substituting back in 14.10, the evolution of the firms profits and their present value.

4. Irreversible Investment

An irreversible investment is one which lasts forever. We can formalize this idea by writing

(14.20) $\delta = 0$

Given the constraint $I \geq 0$ in the usual investment set-up, this restriction implies that today's investment decision will be irreversible.

In order to appreciate the consequences of this restriction for the investment decision, we can analyze the case of a company which faces either a high or a low demand, with a known probability. This implies that there are two states concerning the level of the marginal productivity schedule of its capital, high and low.

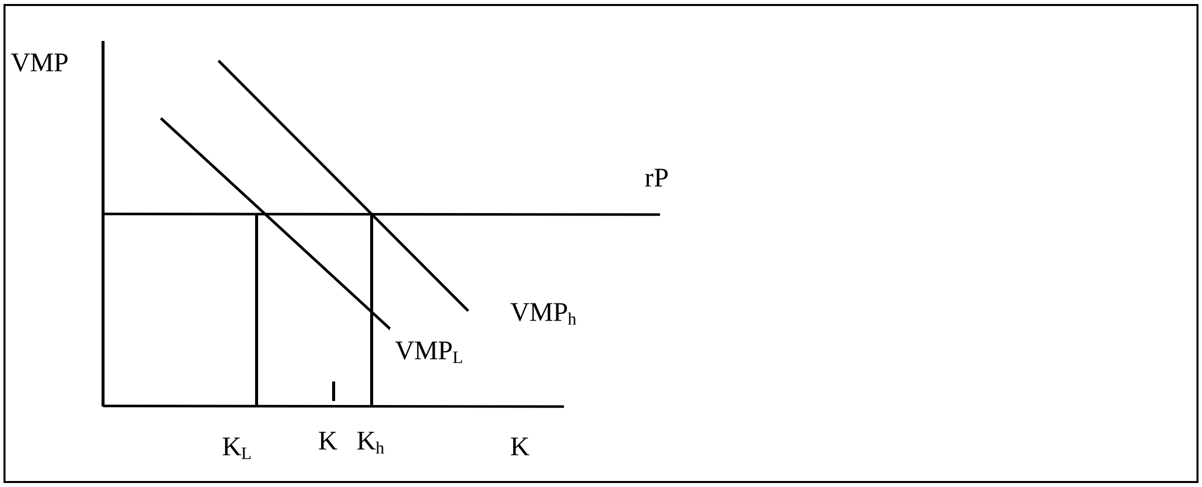


Figure 14.3 Demand for Capital in Two States

Assume that λ_1 is the continuous time probability of moving from state high to low and λ_2 is the probability of moving from state low to high.

If $\lambda_1 = 0$, the optimal level of capital is K_h, and if $\lambda_2 = 0$, the optimal level is K_L.

Suppose both λ_i are small, so that both states are very persistent. The firm in the high state will invest less than it would if it was going to be there for ever. It will obtain a level of capital $K^* \leq K_h$. The firm on the low state will invest K_L, since it knows that whenever it reaches the high state it can always invest the amount to get up to K_h. Thus the overall accumulation of capital in this economy will be lower than in a situation with depreciation.

In the long run, some of the firms which started in a low state will shift up to the high state and acquire K*. However, those in the high state are stuck with the level of capital accumulation they had regardless of whether they shift to the low state . Thus the capital stock will grow until the whole economy has K*.

Thus in the long run, the fact that investment is irreversible means that everybody ends up with more investment than they want. This is a 'high watermark' model. Your capital stock is given by what it was at its highest level.

The important consequence of this model is that, in order to think of a rental market the idea of reversibility is important

5. Tree Cutting Problems

1. Basic Problem

Assume the amount of wood in a tree is w(t). When is the optimal time to cut the tree given a fixed interest rate and price of wood? The problem is to maximize the present value of the tree by choice of the cutting time t*.

(14.21) $V = \max e^{-rt} p\, w(t)$

The first order condition is

(15.22) $-re^{-rt} w(t) + e^{-rt} p\, w'(t) = 0$

or

(14.23) $w'(t) / w(t) = r$

So that the tree should not be cut if it grows faster than the interest rate. If it grows slower, the opportunity cost of money is larger than the profit from the investment, and cutting it is the optimal strategy..

2. Cost to Cut

If we assume there is a cost to cut the trees K,

(14.24) $V = \max e^{-rt} [\, p\, w(t) - K]$

The new first order condition will be

(14.25) $-re^{-rt} [p\, w(t) - K] + e^{-rt} p\, w'(t) = 0$

or

(14.26) $pw'(t) = r\, [\, p\, w(t) - K]$

If the gain from waiting another period exceed the opportunity cost, the best strategy is to wait. The opportunity cost of the funds is the value of the net investment that could be carried out somewhere else. Thus K is deduced from the current value of the tree to get the net opportunity cost.

3. Replanting

If replanting is possible, we can set-up the problem recursively,

(14.27) $V = \max e^{-rt} [p\, w(t) - K + V]$

The reduced form of this equation is

(14.28) $V = \max e^{-rt} [p\, w(t) - K] / [1 - e^{-rt}]$

The solution of this problem will be to plant sooner than before, since now I do not only get the value of the asset but also the possibility to plant another tree in the same place. The first order conditions will be

(14.29) $\{-re^{-rt} [p\, w(t) - K] + e^{-rt} p\, w'(t)\}(1-e^{-rt}) -re^{-rt} \{ e^{-rt} [p\, w(t) - K]\} = 0$

or

(14.30) $p\, w'(t) = r\, e^{-rt} [p\, w(t) - K] / (1-e^{-rt}) + r [p\, w(t) - K]$

Now the opportunity cost of holding on to the tree has two components: the net opportunity cost of the money invested plus the present value of the alternative tree that could be planted after planting the tree currently growing.

4. Cutting cost not constant

If K = c(t), the set up is almost identical to the one in 2. The only change is that the relevant price know is not the gross but the net price.

(14.31) $V = \max e^{-rt} [p\, w(t) - c(t)]$

The first order condition will be

(14.32) $-re^{-rt} [p\, w(t) - c(t)] + e^{-rt} [p\, w'(t) - c'] = 0$

(14.33) $-r [p\, w(t) - c(t)] = [p\, w'(t) - c']$

5. The land has alternative uses

This set up is almost identical to 4,

(14.34) $V = \max e^{-rt} [pw + q(t)]$

(14.35) $p\, w'(t) = r\, p\, w(t) + r\, q(t) -q'(t)$

The rental price of the land is the opportunity cost of the money invested minus the appreciation of the value of the land. Thus we can call

$$R(t) = r\, q(t) - q'(t)$$

and, substituting in,

$$p\, w'(t) = r\, p\, w(t) + R(t)$$

Thus at the optimum, the growth in the value of the tree must be equal to the total opportunity cost, which is the cost of capital inverted plus the rental price of the land.

Lecture 15. The Neoclassical Growth Model

The neoclassical growth model tries to understand the factors that influence long run economic performance. The general set-up of the problem is based on a representative consumer who maximizes his welfare subject to an aggregate production function,

(15.1) $\max \int_0^\infty e^{-\rho t} C(t)\, dt$

subject to $C(t) = [\, F(K(t)) - \delta K(t) - dK/dt \,]$

In this model there are adjustment costs at economy-wide level: in order to produce more in the future we need to reduce present consumption. We can rewrite the problem as:

(15.2) $\max \int_0^\infty e^{-\rho t} [\, F(K(t)) - \delta K(t) - dK/dt \,]\, dt$

We will use this basic set-up with different assumptions about technological progress to in order to analyze the implications of these assumptions.

1. Exogenous technological progress

The aggregate production function that is most commonly used has Harrod neutral (labor augmenting) technological progress.

(15.3) $F(K(t), L(t)A(t))$

With this kind of technological progress, if the level of capital does not increase, there would be a declining ratio of capital to effective labor.

The usual way to specify this problem in order to obtain a closed form solution is to assume CES preferences:

(15.4) $U(c) = C^{\ +1} / \ +1$

With these preferences, the rate of growth of consumption also determines the interest rate of growth.

This kind of preferences and technological progress give a similar path for consumption and capital as the Solow growth model. This paths reflect a number of stylized facts about growth that have been found to characterize many economies.

- Labor share is constant in the long run.
- Output per worker and capital per worker grow at the same rate.
- The rental price of capital is constant.

- Wages grow at the same rate as technology.

We can set up the model in discrete time and obtain the first order conditions,

(15.5) $\max \Sigma \beta^t U(c_t)$

subject to $K_{t+1} = (1-\delta) K_t + F(K_t, L_t^*) - C_t$

Where L^* is the number of units of effective labor, $L^* = LA$. Substituting in the constraint,

(15.6) $\max \Sigma \beta^t U((1-\delta) K_t + F(K_t, L_t^*) - K_{t+1})$

And the first order condition for the capital level will be

(15.7) $1/\beta\ U'(C_{t-1})/ U'(C_t) = (1- \delta) + F_k (K_t, L_t^*)$

Which equals the marginal value of bringing a unit of consumption from today to tomorrow in consumption and production.

A two sector growth model which differentiates between a technology to produce consumption goods and a technology to produce production goods has sometimes been used. We can set up this model in the following way:

(15.8) $Y = F(L_1 , K_1)$

(15.9) $K_{1,t+1} = (1-\delta) K_t + G (L_2, K_2)$

The empirical evidence, however, seems to favor a one sector model, since the capital intensity of the two sectors is not substantially different.

2. Endogenous technological progress

Recent research has gone in models in which there is endogenous accumulation. They are based in a constant returns to scale technology without diminishing returns to capital.

(15.10) $Y_t = AK_t$

Using the first order conditions in (15.7), with this production function,

(15.11) $1/\beta\ U'(C_{t-1})/ U'(C_t) = (1- \delta) + A$

What does this model say a about the steady state level of capital? Assume we have a constant level of consumption in steady state, $C_t = C_{t+1}$;

Then if

(15.11) $A + (1-\delta) \geq 1/\beta$,

there would always be investment, and capital will grow for ever. Thus in this model there is, in steady state equilibrium, consumption growth without changing the parameter A.

For example, in the case of a CES technology,

(15.11) $1/\beta \; C_{t-1} / C_t = A + (1-\delta)$

Let $C_{t-1}/C_t = 1/(1+g)$. Then

(15.12) $g = [\beta(A+(1-\delta)]^{-1/}$ -1 is the growth rate of consumption in this model.

Thus consumption grows in this model at a constant rate forever. If is very close to 0, meaning that there is a very good intertemporal substitutability in consumption (people do not mind to consume now or later), the economy will have to grow at a very fast rate in order to drive up sufficiently interest rates.

Note that the marginal productivity of capital does not change in this model, even as more and more capital is accumulated. This model does not fit with some of the stylized facts noted above.

3. Endogenous Growth with Human Capital

We can take into account the existence of human capital by writing a production function which depends of both human capital and physical capital. Also human capital and physical capital can be used to produce human capital itself,

(15.12) $C_t = F(K_{ct},H_{ct}) + (1-\delta)\, K_t - K_{t+1}$

(15.13) $H_t = G\,(K_{ht},H_{ht})$

(15.14) $K_{ct} + K_{ht} = K_t$

(15.15) $H_{ct} + H_{ht} = H_t$

These models do not run into diminishing returns because in steady state both human capital and physical capital are growing. If a third argument L is added to the production function in (15.12) the model economy would run into diminishing returns.

For example, assume the following simple human capital production function

(15.15) $H_{t+1} = A^{\gamma} H_{ht}$

(15.17) $C_t = F(K_t, H_t - A^{-\gamma} H_{t+1}) + (1-\delta) K_t - K_{t+1}$

(15.18) $C_t = H_t \{F(K_t /H_t , 1 - A^{-\gamma} H_{t+1} /H_t) + (1-\delta) (K_t /H_t)- (K_{t+1} /K_t) (K_t /H_t)\}$

The solution of this problem can be found by choosing the human capital to physical capital rate such that the growth rate in human capital is the same as the growth rate in physical capital.

(15.19) Find K_t /H_t such that $H_{t+1} /H_t = K_{t+1} /K_t$

Thus in the steady state equilibrium growth will imply a path along which

- Human capital and physical capital are growing at the same rate
- The ratio of physical capital to human capital is constant

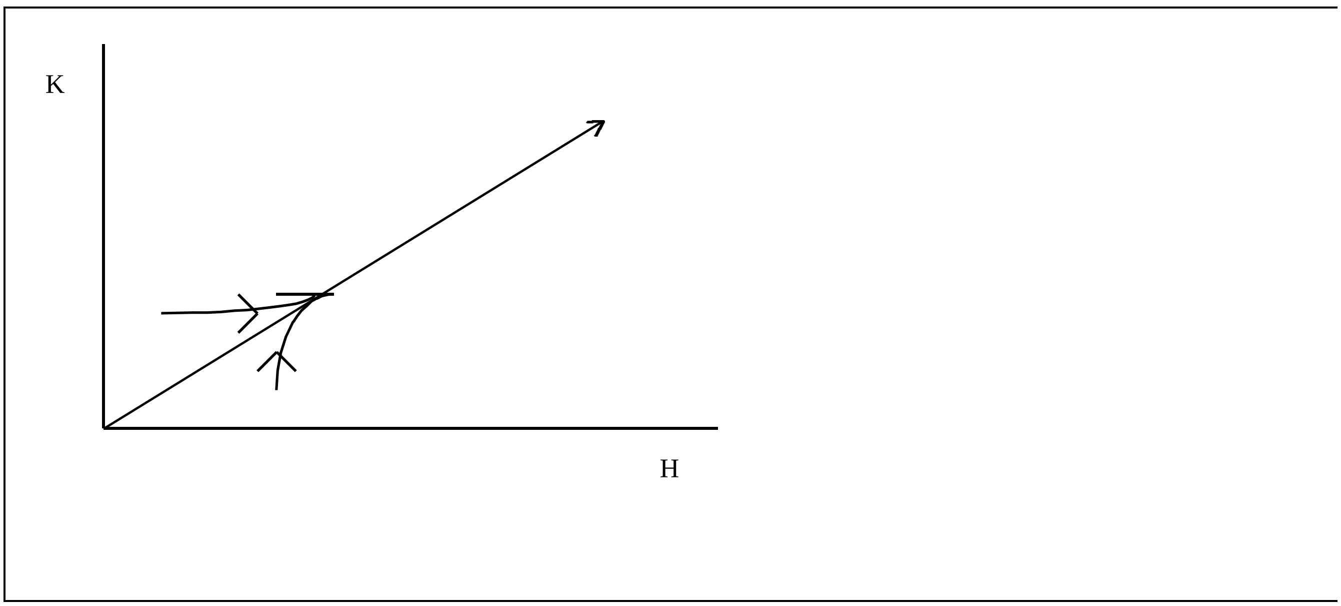


Figure 15.1. Steady State Growth of Human and Physical Capital

Lecture 16. Introduction to General Equilibrium

Most of the problems that we have dealt with so far were *partial equilibrium problems,* i.e. problems that concern a market at a time. We will now deal with a class of problems in which we must find the set of prices of all inputs and outputs and the set of quantities that satisfy all agents (firms and consumers) and clear all markets. We will deal here with two types of general equilibrium problems: representative agent problems and problems with heterogeneous agents.

In all of the problems that follow we are concerned with finding the labor supply of each consumer L^i, the supply of other inputs K^i, the consumer demand of each good Y^i, the set of market prices w, r, $P_1,...,P_k$ such that

1 All markets clear:

(16.1) $$\Sigma_i L^i = \Sigma_j L^j$$

$$\Sigma_i K^i = \Sigma_j K^j$$

$$\Sigma_i Y^{ki} = \Sigma_j Y^{kj}$$

Where i is the individual and j the firms.

2. Each consumer maximizes his utility subject to his budget constraint.

From this statement we will derive the supply of labor and other inputs such as capital, and the demand of final goods.

3. Firms maximize profits.

From this condition we can obtain the demand for inputs in the production process and the supply of final goods.

1. Representative agent models

1.1 Endowment Economy

In an endowment economy we only need to solve the consumer side. Y, L and K are given, so we only need to find the prices that make this the optimal allocation.

1.2 Perfect Substitutes case

This assumption allows us to pin down the prices: the ratio at which goods can be substituted is the price ratio.

1.3. General Case

We need to find both the production and the consumption vector apart from the prices. There will be K + 2 equations with K+2 unknowns as stated in (16.1), In fact, this is a singular system, and this problem has an infinite number of solutions. The reason is that in all demand and supply equations only relative prices matter, so you can eliminate one equation. The way to solve this problem is to pick one good as the nummeraire and express all of the solutions in terms of that good. This is called Walras law: if K-1 markets clear, market K clears as well.

The choice of the nummeraire can make it more or less difficult to solve the problem. In most problems the most convenient nummeraire is the wage, so that prices are the inverse of the marginal product of labor.

2. Heterogeneous agents

The biggest drawback of representative agent models is that there is no exchange in these economies. A model with heterogeneous agents will generate trade. The heterogeneity of the agents can be due to their having different endowments or preferences.

The simplest exchange economy is formed by two agents who are endowed with different quantities of goods A and B. The equilibrium is then going to be a set of prices such that what A wants to sell is what B want to buy. Since we must satisfy both budget constraints, we only need to think in terms of one good (Walras law).

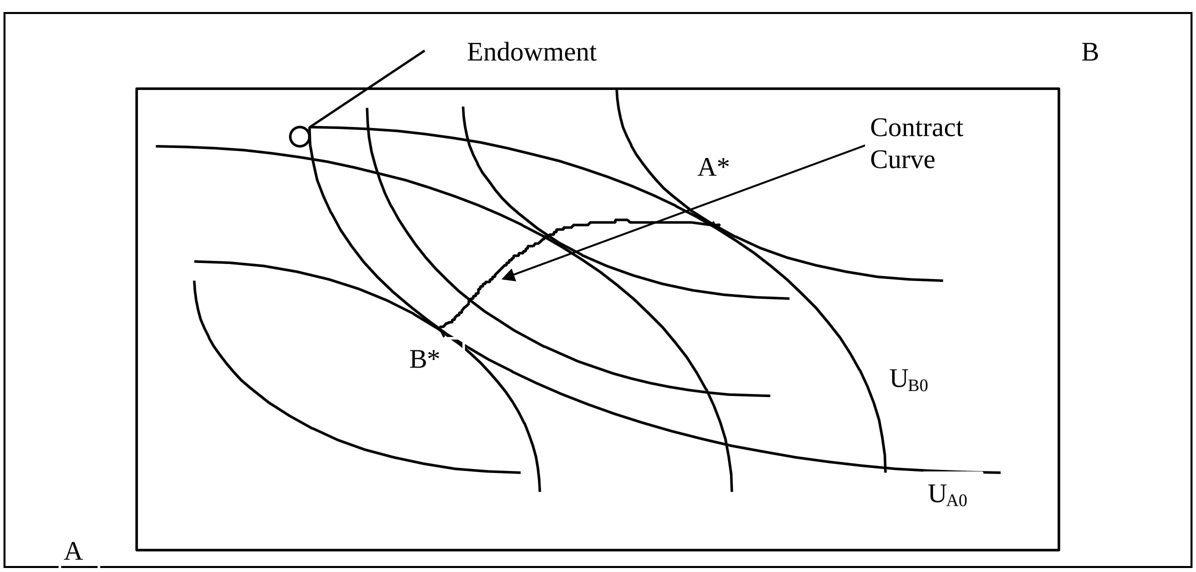


Figure 16.1 Edgeworth Box

We can represent this economy in an Edgeworth box. The utility of A increases from the South West corner of the box and the utility of B increase from the North East corner.

The Edgeworth box allows us to represent the whole set of feasible exchanges, and the desirability of them to each agent. Given the endowment that each agent has, U_{A0} and U_{B0} are the minimum levels of utility that each agent can achieve without trading. If they trade they must be at least as well off, so that A* and B* are the maximum achievable quantities for A and B. Anything between those two points is preferred by the two agents to the initial endowment.

The bargaining solution cannot be determined exactly. But any bargaining solution will be along the *contract curve* which is the line linking A* and B* along which the indifference curves of A and B are tangent.

The Walrasian equilibrium needs to satisfy one extra condition: there must be a price vector passing both through the Walrasian equilibrium and through the endowment point such that the indifferent curves of both agents are tangent to it. This means that there exists a set of prices such that agents will voluntarily choose the equilibrium point.

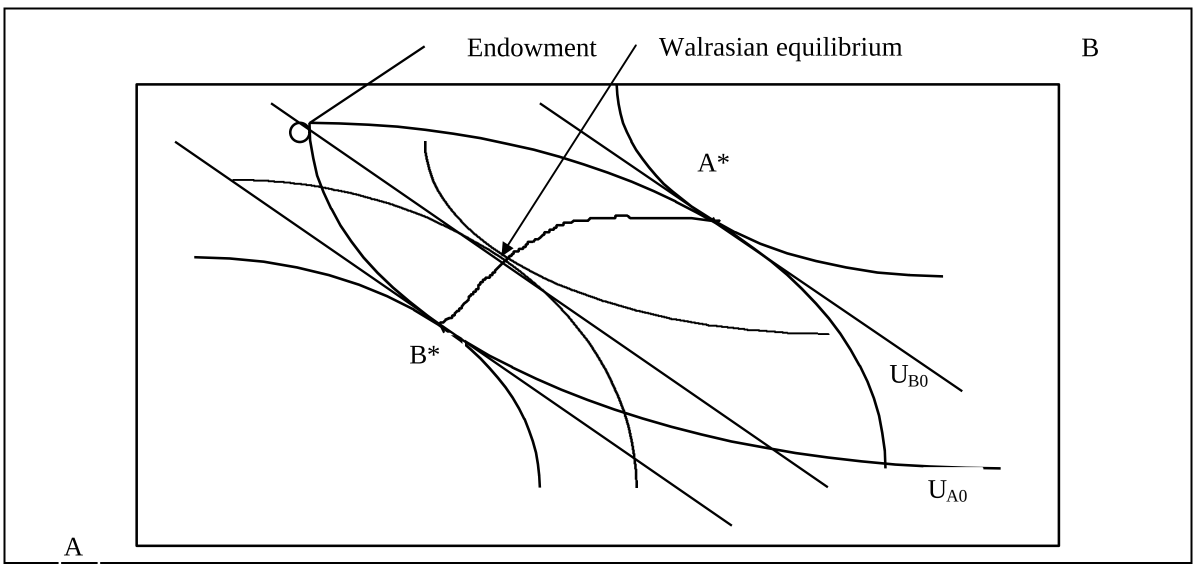


Figure 16.2 Walrasian Equilibrium

Are we sure a Walrasian equilibrium exists? The line through B* is to the left of E; the one through A* is to the right, so there must be one which goes through E. Is it unique? In any case there is an odd number of Walrasian equilibria, since according to the argument used above, if the price line passes E and then goes back through it, it still must pass once again.

The biggest difficulty with endowment economies and with general equilibrium in general is that income is endogenous, it adjusts as the price vector changes. As a consequence, there can be a large number of equilibria.