Theory of Income, Fall 2018

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Class Note 4

Uncertainty

Examples of Economies: Uncertainty

- By carefully choosing the commodity space, set of agent, etc., several interesting economic issues can be analyzed as a CE.
- In this note we analyze how uncertainty affects the extent of risk sharing and the price of risky securities.

- Assume that the state of the economy s can take m_1 different values.
- To simplify, we will assume that there are m_2 physically different goods, indexed by r. We will index a commodity both by its physical attributes as well as by the state, so that there are $m = m_1 \times m_2$ goods, or $L = R^m$.
- Thus the interpretation of vector x is the consumption of each of the physically different goods in each of the states.
- We write x_{sr} for the good in state s of physical characteristic r.
- Thus the utility function u^i is a function of a vector on R^m. The endowment of agent i, denoted by e^i are also indexed by state s and physical characteristic r.
- The production possibilities of the economy can also depend on the state, thus the production possibility sets of each firm are Y^j, subsets of R^m.

Expected Utility

The general specification of the utility function allows as a special case expected utility. In this case u^i is given by

additively separable
(more than weakly separable)

$$u^i(x) = \sum_{s=1}^{m_1} v^i(x_{s1}, x_{s2}, ..., x_{sm_2})\ \pi_s^i$$

for some function $v^i : R^{m_2} \to R$ and some vector of (subjective) probabilities $\pi^i \in R_+^{m_1}$, with $\sum_{s=1}^{m_1} \pi_s^i = 1$.

The subutility function $v^i(x_{s1}, x_{s2}, ..., x_{sm_2})$ has the interpretation of the utility that an agent will enjoy if she consumes the m_2 physically different goods $x_{s1}, x_{s2}, ..., x_{sm_2}$ in state s.

In this case, it is convenient to regard the vector $x_{s1}, x_{s2}, ..., x_{sm_2}$ as a random variable with m_1 possible realizations.

weak separability: u(x1,...,xm)=u(v(x1,...,xn),g(xn+1,...,xm))

To analyze attitudes toward risk, let's simplify and consider the case of only one physically different good so that $m_1 = m$. In this case we have

$$u^i(x) = \sum_{s=1}^{m} v^i(x_s)\,\pi^i_s$$

Definition. We say that u is risk averse if

$$v^i\left(\sum_{s=1}^{m} x_s \pi^i_s\right) > \sum_{s=1}^{m} v^i(x_s)\,\pi^i_s \tag{1}$$

for any random variable x. Notice that if $m = 2$, this coincides with the definition of v being strictly concave and the assumption that x, as a random variable, is not degenerate.

Question. Show that if v^i is concave then u^i is concave.

Jensen's Inequality and Concavity

To understand how concavity implies the Jensen's inequality (**??**), we will consider the case where v is concave and where the random variable can take 3 values so that $m = 3$. In this case we will show that

$$v\left(\sum_{s=1}^{3} x_s \eta_s\right) > \sum_{s=1}^{3} v(x_s)\,\eta_s$$

for any positive weights η_s.

Proof. Notice that we can write expected utility as:

$$\sum_{s=1}^{3} v(x_s)\,\eta_s = \left(\sum_{s'=1}^{2} \eta_{s'}\right)\left(\sum_{s=1}^{2} v(x_s)\,\frac{\eta_s}{\sum_{s'=1}^{2}\eta_{s'}}\right) + v(x_3)\,\eta_3$$

and expected consumption as

$$\sum_{s=1}^{3} x_s\eta_s = \left(\sum_{s'=1}^{2} \eta_{s'}\right)\sum_{s=1}^{2} x_s\frac{\eta_s}{\sum_{s'=1}^{2}\eta_{s'}} + \eta_3 x_3$$

where

$$\frac{\eta_s}{\sum_{s'=1}^{2}\eta_{s'}}$$

has the interpretation of the conditional probability of s being 1 or 2. Using the definition of concavity:

$$\sum_{s=1}^{2} v(x_s)\,\frac{\eta_s}{\sum_{s'=1}^{2}\eta_{s'}} < v\left(\sum_{s=1}^{2} x_s\frac{\eta_s}{\sum_{s'=1}^{2}\eta_{s'}}\right).$$

and multiplying it by $\sum_{s'=1}^{2}\eta_{s'}$ and adding $v(x_3)\,\eta_3$ we get,

$$\sum_{s=1}^{3} v(x_s)\,\eta_s$$

Equilibrium Risk Sharing

We will study the risk sharing implications for equilibrium allocations and Pareto Optimal allocations in the context of a pure endowment economy, with one good, m states of nature, and preferences given by expected utility displaying risk aversion and using the same probabilities π.

Under these hypothesis we will show that the consumption of each agent depends solely on the realization for the aggregate endowment, and that all the individual consumptions move together with the aggregate endowment.

- Let $\bar{e} \in R^m$ be the aggregate endowment, $\pi_s \in R^m_+$ the common probability of state s, and $v^i : R \to R$ the sub-utility function of agent i.
- Let $e^i \in R^m$ be the endowment of the agent i.
- We assume that v^i are differentiable, strictly increasing and strictly concave.
- We regard the vectors x^i, e^i and $\bar{e}$ as random variables.

Theorem. Fix an arbitrary vector of $\lambda-$weights.

The corresponding Pareto optimal allocation can be described by a set of *strictly increasing functions* g^i of the aggregate endowment, i.e. the optimal allocation can be written as

$$x^i_s = g^i\left(\bar{e}_s\right) \text{ for all } i \in I$$

Proof. Using the assumption of expected utility and the same subjective probabilities we can write the objective function of the Pareto problem as

$$\begin{aligned}
\sum_{i \in I} \lambda_i u^i\left(x^i\right) &= \sum_{i \in I} \lambda_i \sum_{s=1}^{m} v^i\left(x^i_s\right)\pi^i_s = \sum_{s=1}^{m}\sum_{i \in I} \lambda_i v^i\left(x^i_s\right)\pi^i_s \\
&= \sum_{s=1}^{m}\left[\sum_{i \in I} \lambda_i v^i\left(x^i_s\right)\right]\pi_s
\end{aligned}$$

subject to

$$\sum_{i \in I} x^i_s = \bar{e}_s$$

for $s = 1, ..., m$.

EXAM HERE

Thus the Pareto problem is solved by solving a different sub-problem state s by state s, namely:

$$\max_{\{x_s^i\}_{i\in I}} \sum_{i\in I} \lambda_i v^i\left(x_s^i\right)$$

subject to

$$\sum_{i\in I} x_s^i = \bar{e}_s.$$

Notice that the probability π_s does not enter in the state s problem. The only difference between the sub-problems for different states s is on the aggregate endowment $\bar{e}_s$. This establishes that the solution of the state s problem is given by

$$x_s^i = g^i\left(\bar{e}_s\right).$$

the last conclusion is from the concavity of the utility functions?
1) if larger aggregate endowment, you have to give one guy more
2) the FOCs still need to hold, so you end up giving more to each of the guys

Now we show that the g^i functions are strictly increasing. Consider two states s and s' with

$$\bar{e}_s > \bar{e}_{s'}$$

It must be that for at least some i,

$$x_s^i > x_{s'}^i.$$

Otherwise, the resource constraint will not hold with equality in state s. Now consider any other agent j. Using the foc for the sub-problem we obtain

$$\begin{aligned}
\lambda_i \frac{\partial v^i\left(x_s^i\right)}{\partial x} &= \lambda_j \frac{\partial v^j\left(x_s^j\right)}{\partial x} \\
\lambda_i \frac{\partial v^i\left(x_{s'}^i\right)}{\partial x} &= \lambda_j \frac{\partial v^j\left(x_{s'}^j\right)}{\partial x}
\end{aligned}$$

since $x_s^i > x_{s'}^i$, by the strict concavity of v^i,

$$\frac{\partial v^i\left(x_s^i\right)}{\partial x} < \frac{\partial v^i\left(x_{s'}^i\right)}{\partial x}$$

hence,

$$\frac{\partial v^j\left(x_s^j\right)}{\partial x} < \frac{\partial v^j\left(x_{s'}^j\right)}{\partial x}$$

and using the strict concavity of v^j, we obtain that

$$x_s^j > x_{s'}^j$$

thus g^j is strictly increasing. QED.

Remark: CE and Risk Sharing.

- Since the welfare theorems hold for this economy, the previous theorem implies that in any CE the consumption allocations depend only on the aggregate endowment $\bar{e}_s$ and NOT on the individual realization of e_s^i.
- Moreover, the consumption of all agents is higher if the realization s of the aggregate endowment is higher. This is the sense in which there is complete risk sharing.

Remark: CE and State Prices. In a CE the budget constraint of agent i is

$$\sum_{s=1}^{m} p_s x_s^i = \sum_{s=1}^{m} p_s e_s^i$$

where p_s are also referred to as state prices, the price of a security that pays one unit of the numeraire in state s and zero otherwise. Thus, agents can buy consumption contingent on the state, and they finance that by selling their endowment contingent on the state.

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Using the foc of agent i we obtain that in an equilibrium (or in its corresponding Pareto problem)

$$p_s = \frac{\partial v^i\left(x_s^i\right)}{\partial x} \pi_s \;/\; \mu_i = \lambda_i \frac{\partial v^i\left(g^i\left(\bar{e}_s\right)\right)}{\partial x} \pi_s$$

so that the state prices reflect the probability that the state s be realized as well as the scarcity of the aggregate endowment in state s.

The state prices are lower if the probability is small or if the aggregate endowment in that state is large, so that goods are relatively plentiful, and hence its marginal value relatively smaller.

$$p_s = \frac{\partial v^i\left(x_s^i\right)}{\partial x}\,\pi_s \;/\; \mu_i = \lambda_i \frac{\partial v^i\left(g^i\left(\bar{e}_s\right)\right)}{\partial x}\,\pi_s$$

if people are risk-neutral, prices only depend on probabilities.

Security Markets

- We will study an economy where trade does not occur in contingent markets but in securities markets.
- Each security pays different dividends in different states.
- Let d_{ks} be the payoff of security k in state s.
- There are K such securities. Let q_k be the price of the security k, h_k^i the purchases of this security by agent i, and θ_k^i the endowment of this security by agent i.
- In this case the budget constraint is written in two (set of) equations.
- Assume that before the state s is realized agents trade in competitive markets where they buy and sell securities $k = 1, ..., K$.

The first equation is given by

$$\sum_{k=1}^{K} h_k^i \; q_k = \sum_{k=1}^{K} \theta_k^i \; q_k \tag{2}$$

This says that the value of purchases is limited by the value of the sales of the securities.

no consumption today; only risk

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This says that the value of purchases is limited by the value of the sales of the securities.

When the state s is realized, consumption of agent i is given by her endowment of goods in that state $\hat{e}_s^i$ and by the payoffs of the securities purchased.

The second equation, indeed one for each state s, is given by

$$x_s^i = \sum_{k=1}^{K} h_k^i \; d_{ks} + \hat{e}_s^i \tag{3}$$

for each $s = 1, ..., m$.

Definition. Let D be the matrix with the payoffs of the K securities in the m states

$$D = \{d_{ks}\}_{k=1,\ldots,\, K, s=1,\ldots,m}$$

We will now compare two market structures.

The first has A-D markets (or contingent markets) with A-D prices (or state prices or contingent prices) as described in the previous section.

The agent's budget constraint is

$$\sum_{s=1}^{m} p_s x_s^i = \sum_{s=1}^{m} p_s e_s^i \tag{4}$$

In the second economy, which we refer to as the security market one, budget constraints are given by the two equations (**??**) and (**??**):

$$\begin{aligned} \sum_{k=1}^{K} h_k^i \, q_k &= \sum_{k=1}^{K} \theta_k^i \, q_k \qquad \text{you trade today...} \\ x_s^i &= \sum_{k=1}^{K} h_k^i \, d_{ks} + \hat{e}_s^i \text{ for each } s = 1, ..., m \end{aligned}$$

...and you have cash flows

Market clearing in the security market economy is given by the market clearing conditions for securities and goods, respectively:

$$\begin{aligned} \sum_{i=1}^{I} h_k^i &= \sum_{i=1}^{I} \theta_k^i \text{ for each } k = 1, 2, ...K \qquad \text{security-by-security} \\ \sum_{i=1}^{I} x_s^i &= \sum_{i=1}^{I} \left[\hat{e}_s^i + \left(\sum_{k=1}^{K} d_{ks} \, \theta_k^i \right) \right] \text{ for each } s = 1, 2...., m \end{aligned}$$

The next definition links the security prices and state prices in an obvious way.

Definition. We will say that security prices q and payoffs D are consistent with state price p, if

$$q_k = \sum_{s=1}^{m} p_s d_{ks} \qquad \text{for all securities } k = 1, ..., K. \tag{5}$$

Question: What is the economic interpretation of this equation?

Likewise, to relate the two economies we relate the goods endowment e^i in the AD economy with the goods endowments $\hat{e}^i$ and the endowment of securities.

Definition. The endowment e^i and $\left(\hat{e}^i, \theta^i\right)$ are equivalent if

$$\hat{e}^i_s + \sum_{k=1}^{K} d_{ks}\, \theta^i_k = e^i_s \tag{6}$$

for all states $s = 1, 2...., m$.

Proposition

Assume that prices q and payoffs D are consistent with state prices p, as in (**??**). Assume that the endowments are equivalent as in (**??**). Then:

1) If (x, h) is budget feasible in the security market economy, then x is budget feasible in the A-D economy. 1) you have the money to buy
2) you can actually buy

2) If x is budget feasible in the A-D economy, then it must be budget feasible in the security market economy, provided that D has full rank.

* the rank condition is not needed for 1) since the dimension of the budget constraint is smaller

Proof. We first characterize budget feasibility in both economies. x, h are budget feasible in the security market economy if x, h satisfy the following two equations

$$\begin{aligned} x^i &= D^T\ h^i + \hat{e}^i \\ q^T\ h^i &= q^T\ \theta^i \end{aligned}$$

or writing them in detail

$$\begin{bmatrix} x_1^i \\ x_2^i \\ \ldots \\ x_m^i \end{bmatrix} = \begin{bmatrix} d_{11} & d_{21} & d_{31} & \ldots & d_{K1} \\ d_{12} & d_{22} & d_{32} & \ldots & d_{K2} \\ d_{13} & d_{23} & d_{33} & \ldots & d_{K3} \\ \\ d_{1m} & d_{2m} & \ldots & & d_{Km} \end{bmatrix} \begin{bmatrix} h_1^i \\ h_2^i \\ h_3^i \\ \ldots \\ h_K^i \end{bmatrix} + \begin{bmatrix} \hat{e}_1^i \\ \hat{e}_2^i \\ \ldots \\ \hat{e}_m^i \end{bmatrix}$$

and

$$\begin{bmatrix} q_1 & q_2 & q_3 & \ldots & q_K \end{bmatrix} \begin{bmatrix} h_1^i \\ h_2^i \\ h_3^i \\ \vdots \\ h_k^i \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 & \ldots & q_K \end{bmatrix} \begin{bmatrix} \theta_1^i \\ \theta_2^i \\ \theta_3^i \\ \vdots \\ \theta_k^i \end{bmatrix}$$

Likewise, x is budget feasible in the A-D economy if it satisfies the following equation

$$p^T\ x^i = p^T\ e^i$$

or writing it in detail

$$\begin{bmatrix} p_1 & p_2 & \ldots & p_m \end{bmatrix} \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_m^i \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \ldots & p_m \end{bmatrix} \begin{bmatrix} e_1^i \\ e_2^i \\ \vdots \\ e_m^i \end{bmatrix}$$

We also have that we can write that the prices q and p are equivalent as in (**??**) as

$$q = D\ p$$

or writing it in detail

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \\ q_K \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ d_{31} & d_{32} & \ldots & d_{3m} \\ \\ d_{K1} & d_{K2} & \ldots & d_{Km} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \ldots \\ p_m \end{bmatrix}$$

Finally, we can write that the endowments e^i and $\hat{e}^i$ are equivalent as in (**??**) as

$$e^i = \hat{e}^i + D^T \, \theta^i$$

or writing it in detail

$$\begin{bmatrix} e_1^i \\ e_2^i \\ \dots \\ e_m^i \end{bmatrix} = \begin{bmatrix} \hat{e}_1^i \\ \hat{e}_2^i \\ \dots \\ \hat{e}_m^i \end{bmatrix} + \begin{bmatrix} d_{11} & d_{21} & d_{31} & \dots & d_{K1} \\ d_{12} & d_{22} & d_{32} & \dots & d_{K2} \\ d_{13} & d_{23} & d_{33} & \dots & d_{K3} \\ \dots & \dots & \dots & \dots & \dots \\ d_{1K} & d_{2K} & \dots & \dots & d_{mK} \end{bmatrix} \begin{bmatrix} \theta_1^i \\ \theta_2^i \\ \theta_3^i \\ \dots \\ \theta_k^i \end{bmatrix}$$

To show 1), assume that (x, h) is such that

$$x^i = D^T \, h^i + \hat{e}^i.$$

Multiply both sides by p^T

$$p^T \, x^i = p^T \, D^T \, h^i + p^T \, \hat{e}^i$$

use that the endowment are equivalent to obtain

$$\begin{aligned} p^T \, x^i &= p^T \, D^T \, h^i + p^T \, \left(e^i - D^T \theta^i\right) \\ p^T \, x^i &= p^T \, D^T \, \left(h^i - \theta^i\right) + p^T \, e^i \end{aligned}$$

use that p and q are equivalent and hence

$$p^T \, x^i = q^T \left(h^i - \theta^i\right) + p^T \, e^i$$

and finally use the second line of the security market budget constraint to obtain

$$p^T \, x^i = p^T \, e^i.$$

To show 2) we first notice that we can always find an h so that

$$x^i = D^T \; h^i + \hat{e}^i$$

by setting

$$h^i = \left(D^T\right)^{-1} \left(x^i - \hat{e}^i\right) \; .$$

We now show that such h is affordable. Start with

$$p^T \; x^i = p^T \; e^i$$

use that p and q are equivalent so p can be written as

$$p = D^{-1} q$$

so that

$$q^T \; \left(D^{-1}\right)^T \; x^i = q^T \; \left(D^{-1}\right)^T \; e^i$$

using that the endowments are equivalent we have

$$\begin{aligned} q^T \; \left(D^{-1}\right)^T \; x^i &= q^T \; \left(D^{-1}\right)^T \; \left(\hat{e}^i + D^T \; \theta^i\right) \\ &= q^T \; \left(D^{-1}\right)^T \; \hat{e}^i + q^T \; \theta^i \end{aligned}$$

Using that

$$x^i = D^T \; h^i + \hat{e}^i$$

$$\begin{aligned} q^T \; \left(D^{-1}\right)^T \; \left(D^T \; h^i + \hat{e}^i\right) &= q^T \; \left(D^{-1}\right)^T \; \hat{e}^i + q^T \; \theta^i \\ q^T \; h^i + q^T \; \left(D^{-1}\right)^T \hat{e}^i &= q^T \; \left(D^{-1}\right)^T \; \hat{e}^i + q^T \; \theta^i \end{aligned}$$

or

$$q^T \; h^i = q^T \; \theta^i$$

QED.

Question. Why do we need for 2) that D has full rank?.

If D does not have full rank, we refer to this economy as one with incomplete markets. If D has full rank we refer to as having complete markets.

Question. What is the minimum number of securities needed to have complete markets?

Question. How are the budget sets in the AD and security market economies with prices related by (**??**) if markets are incomplete? Which one is a larger set?.

Proposition.

Assume that the endowments e^i and $\left(\hat{e}^i, \theta^i\right)$ are equivalent, then

1) If $\left(x^i, h^i\right)$ clears the markets in the security market economy, then $\left(x^i\right)$ clears the markets in the A-D economy.

2) Assume also that D has full rank. If $\left(x^i\right)$ clears the markets in the A-D economy, the $\left(x^i, h^i\right)$ clears the markets in the security market economy.

Proof. Suppose that (x^i, h^i) clears the markets in the security market economy in the goods market. Then, using that the endowments are equivalent, we have

$$\sum_{i \in I} x^i = \sum_{i \in I} \left(\hat{e}^i + D^T \theta^i \right) = \sum_{i \in I} e^i$$

so that the goods market clears in the A-D economy.
Now suppose that (x^i) clears the market in the A-D economy. Then, using that the endowments are equivalent, we have

$$\sum_{i=1}^{I} x^i = \sum_{i=1}^{I} e^i = \sum_{i=1}^{I} \left[\hat{e}^i + D^T \ \theta^i \right]$$

so that the goods market clears in the security market economy.

Since h^i must solve

$$x^i = D^T \ h^i + \hat{e}^i$$

we have that h^i is given by

$$h^i = \left(D^T\right)^{-1} \left(x^i - \hat{e}^i\right)$$

using that the endowments are equivalent

$$h^i = \left(D^T\right)^{-1} \left(x^i - e^i + D^T \theta^i\right) = \left(D^T\right)^{-1} \left(x^i - e^i\right) + \theta^i$$

adding across consumers

$$\sum_{i \in I} h^i = \left(D^T\right)^{-1} \sum_{i \in I} \left(x^i - e^i\right) + \sum_{i \in I} \theta^i$$

and using that by hypothesis

$$\sum_{i \in I} \left(x^i - e^i\right) = 0$$

we have that the security market clears, i.e.

$$\sum_{i \in I} h^i = \sum_{i \in I} \theta^i. \qquad \text{QED.}$$

Question. Are the equilibrium allocations of a security market economy with complete market necessarily Pareto Optimal?

Question. Can the equilibrium allocations of a security market economy be Pareto Optimal if the security markets are not complete?

The Tilde Economy

Consider the following alternative analysis of the security market economy. We will use objects with tildes to denote the A-D economy that corresponds to the security market economy. In this economy we define the utility as a function of the portfolio shares, so that $\tilde{L} = R^K$. The utility is given by

$$\tilde{u}^i\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_K\right) \equiv$$

$$u^i(\underbrace{\sum_{k=1}^{K} d_{k1}\ \tilde{x}_k \ + e_1^i}_{x_1} \ , ..., \underbrace{\sum_{k=1}^{K} d_{ks}\ \tilde{x}_k \ + e_s^i}_{x_s} \ , ..., \underbrace{\sum_{k=1}^{K} d_{km}\ \tilde{x}_k \ + e_m^i}_{x_m} \ )$$

where now $\tilde{x}_k^i$ has the interpretation of the number of shares bought or sold of security k by agent i The budget constraint is

$$\sum_{k=1}^{K} \tilde{p}_k\ \tilde{x}_k^i = \sum_{k=1}^{K} \tilde{p}_k\ \tilde{e}_k^i.$$

Finally the endowment are in terms of shares of the securities,

$$\tilde{e}^i = \theta^i.$$

Clearly the equilibrium price $\tilde{p}$ will equal the equilibrium prices q analyzed above.
In this economy feasible allocations are defined relative to the rearrangements of securities across agents.

Question. Show that if u^i are strictly increasing, the first welfare theorem hold for this economy.

Question. Show that if u^i are strictly quasi-concave, and $\tilde{e}^i > 0$, the second welfare theorem holds too.

Question. Assume that D does not have full rank. Show that there could be PO allocations in the corresponding AD economy that are not equilibrium on this economy (consider the case, for example, where there are only one assets, two agents, no aggregate uncertainty, identical strictly concave utilities u^i , two states with equal probabilities and symmetric endowments). How is this possible in view of the previous results about incomplete markets? (Hint: are the set of feasible allocations in the tilde economy with K goods the same as in the original A-D economy with m goods?)

Asset Prices and “Equity Premium".

Consider an economy with one good, m states, and complete markets. Assume that all u^i are given by expected utility with strictly concave v^i and where all agents use the same probabilities π_s for all states. We are interested in understanding the price of two securities. Security $k = 1$ is a risk-less bond, i.e it pays

$$d_{ks} = 1$$

in all state of nature. Security $k = 2$ is similar to a stock, it pays

$$d_{ks} = \bar{e}_s$$

so that it pays a dividend equal to the aggregate endowment. We refer to this security as simple as the aggregate stock. We are interesting in the risk premium, the difference between the expected return of the aggregate stock and the return of the risk-less bond. The expected gross return of any security k is denoted by r_k and defined as

$$1 + r_k = \frac{\sum_{s=1}^{m} d_{ks}\ \pi_s}{q_k}$$

i.e. it is given by the expected payoffs divided by its price.

The risk premium is given by $(1 + r_k) / (1 + r_1)$.

Question. Let the equilibrium allocation be described by $x_s^i = g^i(\bar{e}_s)$ for g^i increasing (Why? Which past results are we using?). Show that the prices of securities 1, 2 and generic k security are given by

$$\begin{aligned} q_1 &= \frac{1}{\mu_i}\sum_{s=1}^{m}\frac{\partial v^i(g_i(\bar{e}_s))}{\partial x}\,\pi_s \\ q_2 &= \frac{1}{\mu_i}\sum_{s=1}^{m}\frac{\partial v^i(g_i(\bar{e}_s))}{\partial x}\bar{e}_s\,\pi_s \\ q_k &= \frac{1}{\mu_i}\sum_{s=1}^{m}\frac{\partial v^i(g_i(\bar{e}_s))}{\partial x}d_{ks}\,\pi_s \end{aligned}$$

for a positive number μ_i (what does μ_i represent).

Question. As a preliminary step, show that for two random variables Z and Y

$$E[Y\,Z] = E[Y]\;E[Z] + Cov[Y, Z]$$

Question. Show that

$$\begin{aligned} \frac{1+r_k}{1+r_1} &= \frac{E\left[\frac{\partial v^i\left(g^i(\bar{e})\right)}{\partial x}\right]E[d_k]}{E\left[\frac{\partial v^i(g^i(\bar{e}))}{\partial x}\,d_k\right]} \\ &= \frac{E\left[\frac{\partial v^i\left(g^i(\bar{e})\right)}{\partial x}\right]E[d_k]}{E\left[\frac{\partial v^i(g^i(\bar{e}))}{\partial x}\right]E[d_k] + Cov\left[\frac{\partial v^i(g^i(\bar{e}))}{\partial x},\;d_k\right]} \end{aligned}$$

so that

$$r_k - r_1 \geq (<)\,0 \iff Cov\left[\frac{\partial v^i(g^i(\bar{e}))}{\partial x},\;d_k\right] \leq (>)\,0$$

Give an intuitive explanation for this finding.

Question. Let f be a positive function that is strictly decreasing in X. Assume that X has strictly positive variance. Show that

$$Cov\,[X\ f\,(X)] < 0$$

(Hint: write

$$\begin{aligned} Cov\,[X\ f\,(X)] &= \int_{-\infty}^{E[X]} (X - E\,[X])\, f\,(X)\, F\,(dX) \\ &+ \int_{E[X]}^{\infty} (X - E\,[X])\, f\,(X)\, F\,(dX) \end{aligned}$$

where F is the cdf of X, use that $f > 0$, and that it is decreasing.)

Question. Use the previous result to show that if $\bar{e}$ has positive variance, then the premium in the aggregate stock is positive, i.e.

$$\frac{1 + r_2}{1 + r_1} > 1$$

Question. Assume that all $v^i\,(x) = x^{1-\gamma}/\,(1-\gamma)$ and $v^i\,(x) = \log x$ for $\gamma = 1$, then show that

$$x_s^i = \delta^i\ \bar{e}_s$$

for all s, for some fractions $\delta^i > 0$, $\sum_i \delta^i = 1$ (so that the functions g^i are linear, with no intercept). Show that in this case

$$\frac{1 + r_2}{1 + r_1} = \frac{E\,[\bar{e}^{-\gamma}]\, E\,[\bar{e}]}{E\,[\bar{e}^{-\gamma}\bar{e}]}$$

Question. Show, as an intermediate step, that if X is lognormally distributed, i.e.

$$\log X \sim N\left(\bar{\mu}, \sigma^2\right)$$

then

$$E\left[X^{-\gamma}\right] = \exp\left(-\gamma\bar{\mu} + \gamma^2 \frac{1}{2}\sigma^2\right)$$

(Hint: write down the integral using the form for the density of a normal distribution $x = \log X$ as

$$\begin{aligned} E\left[X^{-\gamma}\right] &= E\left[\exp\left(\log\left(X^{-\gamma}\right)\right)\right] = E\left[\exp\left(-\gamma x\right)\right] \\ &= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} \exp\left(-\lambda x\right) \exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right) dx \end{aligned}$$

Take common factor of the expressions with $\exp\left(X\right)$. Divide and multiply inside the integral by a constant as to get

$$\begin{aligned} & E\left[X^{-\gamma}\right] \\ = \; & \exp\left(-\gamma\bar{\mu} + \gamma^2 \frac{1}{2}\sigma^2\right) \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\left(x - \left(\mu - \lambda\sigma^2\right)\right)^2}{2\sigma^2}\right) dx \end{aligned}$$

so that you can use that the integral of a density of a normal with mean $\left(\mu - \lambda\sigma^2\right)$ and variance σ^2 is one, i.e.

$$1 = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\left(x - \left(\mu - \lambda\sigma^2\right)\right)^2}{2\sigma^2}\right) dx$$

and obtain the desired result.)

Question.

Show that if in addition to have $v^i\left(x\right) = x^{1-\gamma}/\left(1-\gamma\right)$ and $v^i\left(x\right) = \log x$ for $\gamma = 1$, the aggregate endowment is log-normally distributed (instead of having m states with probabilities π_s), i.e. if

$$\log \bar{e} \sim N\left(\bar{\mu}, \sigma^2\right)$$

then, the premium for the aggregate equity is

$$\frac{1 + r_2}{1 + r_1} = \exp\left(\gamma\sigma^2\right)$$

and hence, for small $\gamma\sigma^2$, the risk premium is

$$r_2 - r_1 \cong \gamma\sigma^2.$$

(Hint, use that log-normality implies

$$\begin{aligned} E(\bar{e}) &= \exp\left(\bar{\mu} + \frac{1}{2}\sigma^2\right) \\ E\left(\bar{e}^{-\gamma}\right) &= \exp\left(-\gamma\bar{\mu} + \frac{1}{2}\gamma^2\sigma^2\right) \\ E\left(\bar{e}^{1-\gamma}\right) &= \exp\left((1-\gamma)\bar{\mu} + \frac{1}{2}(1-\gamma)^2\sigma^2\right) \end{aligned}$$

solve for

$$\frac{1+r_2}{1+r_1} = \frac{\exp\left(\bar{\mu} + \frac{1}{2}\sigma^2\right)\exp\left(-\gamma\bar{\mu} + \frac{1}{2}\gamma^2\sigma^2\right)}{\exp\left((1-\gamma)\bar{\mu} + \frac{1}{2}(1-\gamma)^2\sigma^2\right)}$$

and develop the square $(1-\gamma)^2$.)
To help interpreting the previous answer, define as w the proportional insurance premium, i.e. for a random variable X, and utility function v it is defined as the solution to

$$v((1-w)E[X]) = E[v(X)]$$

the interpretation is that the consumer is equally happy with a fraction $[1-w]$ of the expected value of X, than with the random variable X.

Question. Show that if $\log X$ is normal with mean μ and variance σ^2 and if $v(x) = x^{1-\gamma}/(1-\gamma)$ then

$$1 - w = \exp\left(-\frac{\gamma\sigma^2}{2}\right)$$

or for small $\gamma\sigma^2$,

$$w \cong \frac{\gamma\sigma^2}{2}$$

(Hint: using one of the previous results

$$
\begin{aligned}
E\left[v\left(X\right)\right] &= \frac{1}{1-\gamma} E\left[X^{1-\gamma}\right] = \frac{\exp\left(\left(1-\gamma\right)\mu + \left(1-\gamma\right)^2 \frac{\sigma^2}{2}\right)}{1-\gamma}, \\
v\left(\left(1-w\right) E\left[X\right]\right) &= \frac{w^{1-\gamma}\left(\exp\left(\mu + \frac{1}{2}\sigma^2\right)\right)^{1-\gamma}}{1-\gamma}
\end{aligned}
$$

then equating these expressions and solving for w we obtain the desired result.)