Price Theory III
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(Lars Stole)

Some solutions to
Assignment 7
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Contents

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v1.0 Initial version

1 Problem 1: JR Exercise 9.4

There are n bidders participating in a first-price auction. Each bidder's value is independently drawn from $[0,1]$ according to the distribution function F, having continuous and strictly positive density f. If a bidder's value is θ and he wins the object with a bid of $b<\theta$, then his von Neumann-Morgenstern utility is $(\theta-b)^{\frac{1}{\alpha}}$, where $\alpha \geq 1$ is fixed and common to all bidders. Consequently, the bidders are risk averse when $\alpha>1$ and risk neutral when $\alpha=1$. Given the risk-aversion parameter α, let $\bar{b}_\alpha(\theta)$ denote the (symmetric) equilibrium bid of a bidder when his value is θ. The following parts will guide you toward finding $\bar{b}_\alpha(\theta)$ and uncovering some of its implications.

1.1 Part (a)

Let $U(\hat{\theta}|\theta)$ denote a bidder's expected utility from bidding $\bar{b}_\alpha(\hat{\theta})$, given that all other bidders employ $\bar{b}_\alpha(\cdot)$. Show that

$$U(\hat{\theta}|\theta)=F(\hat{\theta})^{n-1}(\theta-\bar{b}_\alpha(\hat{\theta}))^{\frac{1}{\alpha}}.$$

Why must $U(\hat{\theta}|\theta)$ be maximised in $\hat{\theta}$ when $\hat{\theta}=\theta$?

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In a first-price auction, the bidder with the highest bid wins and pays the winning bid. If the bidder does not win, then his utility is zero. The probability that bidder i who bids $\bar{b}_\alpha(\hat{\theta})$ wins is the probability that all other bidders bid less than $\bar{b}_\alpha(\hat{\theta})$. Let us assume for now that $\bar{b}_\alpha(\hat{\theta})$ is strictly increasing. Hence,

$$\bar{b}_\alpha\left(\hat{\theta}\right)>\bar{b}_\alpha\left(\theta_j\right),\ \forall j \Leftrightarrow \hat{\theta}>\theta_j,\ \forall j.$$

Given that values are drawn independently, the probability of winning with bid $\bar{b}_\alpha(\hat{\theta})$ is

$$\left[F\left(\hat{\theta}\right)\right]^{n-1},$$

where we implicitly assume that all other bidders bid according to $\bar{b}_\alpha$. In case the bidder wins, he must pay $\bar{b}_\alpha(\hat{\theta})$ so that his expected utility is given by

$$U\left(\hat{\theta}|\theta\right)=\left[F\left(\hat{\theta}\right)\right]^{n-1}\left(\theta-\bar{b}_\alpha\left(\hat{\theta}\right)\right)^{\frac{1}{\alpha}}.$$

We want to argue that the bidder maximises utility by acting truthfully, i.e. that bidding $\hat{b}_\alpha(\theta)$ when his value is θ is optimal. This is equivalent to requiring that $U(\hat{\theta}|\theta)$ is maximised in $\hat{\theta}$ when $\hat{\theta}=\theta$; i.e.

$$\theta=\underset{\hat{\theta}}{\arg\max}\ U\left(\hat{\theta}|\theta\right).$$

1.2 Part (b)

Use part (a) to argue that

$$U(\hat{\theta}|\theta)^{\alpha} = F(\hat{\theta})^{\alpha(n-1)}(\theta - \overline{b}(\hat{\theta})).$$

must be maximised in $\hat{\theta}$ when $\hat{\theta} = \theta$.

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From Part (a),

$$\left[U\left(\hat{\theta}|\theta\right)\right]^{\alpha} = \left[F\left(\hat{\theta}\right)\right]^{\alpha(n-1)}\left(\theta - \overline{b}_{\alpha}\left(\hat{\theta}\right)\right).$$

Since $\alpha \geq 1$ is a strictly increasing transformation of $U(\hat{\theta}|\theta)$, the point of maximum does not change; i.e. $[U(\hat{\theta}|\theta)]^{\alpha}$ is maximised in $\hat{\theta}$ at the same point as $U(\hat{\theta}|\theta)$, which is when $\hat{\theta} = \theta$; i.e.

$$\theta = \arg\max_{\hat{\theta}}\ U\left(\hat{\theta}|\theta\right) = \arg\max_{\hat{\theta}}\ \left[U\left(\hat{\theta}|\theta\right)\right]^{\alpha},\ \forall\alpha \geq 1.$$

1.3 Part (c)

Use part (b) to argue that a first-price auction with the n risk-averse bidders above whose values are each independently distributed according to $F(\theta)$, is equivalent to a first-price auction with n risk-neutral bidders whose values are each independently distributed according to $F(\theta)^{\alpha}$. Use the solution for the risk-neutral case to conclude that

$$\overline{b}_{\alpha}(\theta) = \theta - \int_0^{\theta}\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(n-1)} \mathrm{d}x.$$

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In a first-price auction with n *risk-neutral* bidders whose values are each independently distributed according to $G(\theta) = F(\theta)^{\alpha}$, the expected utility from bidding $\overline{b}(\hat{\theta})$ when value is θ (assuming symmetric bidding function) is

$$\begin{aligned}\tilde{U}\left(\hat{\theta}|\theta\right) &= \left[G\left(\hat{\theta}\right)\right]^{(N-1)}\left(\theta - \overline{b}\left(\hat{\theta}\right)\right)\\ &= \left[F\left(\hat{\theta}\right)\right]^{\alpha(N-1)}\left(\theta - \overline{b}\left(\hat{\theta}\right)\right)\\ &= \left[U\left(\hat{\theta}|\theta\right)\right]^{\alpha}.\end{aligned}$$

From parts (a) and (b), we know that $[U(\hat{\theta}|\theta)]^{\alpha}$ and $U(\hat{\theta}|\theta)$ are maximised when $\hat{\theta} = \theta$. Using the symmetric equilibrium bidding function from the risk-neutral case when values are each independently distributed according to $G(\theta) = F(\theta)^{\alpha}$ gives us that

$$\begin{aligned}\overline{b}\left(\theta\right) &= \theta - \int_0^{\theta}\left(\frac{G\left(x\right)}{G\left(\theta\right)}\right)^{N-1} \mathrm{d}x\\ &\equiv \theta - \int_0^{\theta}\left(\frac{\left[F\left(x\right)\right]^{\alpha}}{\left[F\left(\theta\right)\right]^{\alpha}}\right)^{N-1} \mathrm{d}x\\ &= \theta - \int_0^{\theta}\left(\frac{F\left(x\right)}{F\left(\theta\right)}\right)^{\alpha(N-1)} \mathrm{d}x \equiv \overline{b}_{\alpha}\left(\theta\right).\end{aligned}$$

1.4 Part (d)

Prove that $\bar{b}_\alpha(\theta)$ is strictly increasing in α. Does this make sense? Conclude that as bidders become more risk averse, the seller's revenue from a first-price auction increases.

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Consider

$$\begin{aligned}
\frac{\partial \bar{b}_\alpha(\theta)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(\theta - \int_0^\theta \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x\right) \\
&= -\frac{\partial}{\partial \alpha}\left(\int_0^\theta \exp\left[\alpha(N-1)\ln\left(\frac{F(x)}{F(\theta)}\right)\right] \mathrm{d}x\right) \\
&= -\int_0^\theta (N-1)\ln\left(\frac{F(x)}{F(\theta)}\right)\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x.
\end{aligned}$$

Note that

$$\begin{aligned}
F(x) &\leq F(v)\,,\ \forall 0 \leq x \leq \theta \\
\Rightarrow \ln\left(\frac{F(x)}{F(\theta)}\right) &\leq 0,\ \forall 0 \leq x \leq \theta.
\end{aligned}$$

Thus, so long as $\theta > 0$,

$$\frac{\partial \bar{b}_\alpha(\theta)}{\partial \alpha} > 0.$$

his tells us that bidders bid more as they become more risk averse reflecting the fact that they are more willing to pay to avoid uncertainty of not winning the object. Since the seller's revenue is the winning bid, it follows that the seller's revenue from a first-price auction increases as agents become more risk averse.

1.5 Part (e)

Use part (d) and the revenue equivalence result for the standard auctions in the risk-neutral case to argue that when bidders are risk averse as above, a first-price auction raises more revenue for the seller than a second-price auction. Hence, these two standard auctions no longer generate the same revenue when bidders are risk averse.

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We know that when agents are risk neutral, a first-price auction raises the same revenue for the seller as a second-price auction. In part (d), we argued that a first-price auction with risk-averse agents raises more revenue than a first-price auction with risk neutral agents. So it follows that a first-price auction with risk-averse agents raises more revenue than a second-price auction with risk-neutral agents. Now, to compare the seller's revenue from a first-price auction with risk-averse agents and a second-price auction with risk-averse agents, consider what the players would do in the latter case.

In a second-price auction, the bidder with the highest bid wins and pays an amount equal to the second highest bid. The (weakly) dominant strategy with risk-neutral agents was that the bidders will bid their true value. In fact, even if the agents are risk averse, bidding v remains a dominant strategy. To see this, suppose the bidder i bids $\hat{\theta}_i < \theta_i$.

- ▷ If $\hat{\theta}_i < \theta_i < \max\{\theta_j\}_{j \neq i}$, then the bidder does not win and obtains zero utility;
- ▷ If $\hat{\theta}_i < \max\{\theta_j\}_{j \neq i} < \theta_i$, then reporting θ_i gives the bidder utility of

$$\left(\theta_i - \max\{\theta_j\}_{j \neq i}\right)^{1/\alpha} > 0$$

  and reporting $\hat{\theta}_i$ gives zero utility.

- ▷ If $\max\{\theta_j\}_{j \neq i} < \hat{\theta}_i < \theta_i$, then the bidder wins whether he reports $\hat{\theta}_i$ or θ_i and the utility is given by

$$\left(\theta_i - \max\{\theta_j\}_{j \neq i}\right)^{1/\alpha}.$$

Hence, bidding θ_i is not dominated by bidding $\hat{\theta}_i < \theta_i$. Analogous argument tells us that bidding θ_i is also not dominated by bidding $\hat{\theta}_i > \theta_i$. It follows then that the revenue for the seller from a second-price auction with risk-averse agents are the same as the revenue from a second-price auction with risk-neutral agents.

Putting all of these together, we realise that the seller's revenue from a first-price auction with risk-averse agents must be greater than his revenue from a second-price auction with risk-averse agents.

1.6 Part (f)

What happens to the seller's revenue as the bidders become infinitely-risk averse (i.e., as $\alpha \to \infty$)?

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The seller's revenue is the winning bid. Let $\theta = \max\{\theta_i\}$, then his revenue is

$$\bar{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)} \mathrm{d}x.$$

Taking limit as $\alpha \to \infty$ while recalling that $0 \leq F(x)/F(\theta) \leq 1$ for all $x \in [0, \theta]$,

$$\lim_{\alpha\to\infty} \bar{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\lim_{\alpha\to\infty}\left(\frac{F(x)}{F(\theta)}\right)^{\alpha(N-1)}\right) \mathrm{d}x = \theta.$$

So the seller's revenue is the highest value among the bidders and the seller is able to extract the entire surplus from the bidders.

2 Problem 2: Auctioning procurement contracts

[This is closely related to Problem 1 from Problem Set 6. You may want to consult that question and solution before tackling this one.]

A monopsony buyer is interested in purchasing a large quantity of output from one of n possible suppliers. Each supplier i has a constant marginal cost of production equal to c_i, which is private information to the supplier and is uniformly distributed on $[1, 2]$. Supplier i's payoff from producing $q \in [0, Q]$ units of output for a transfer of t dollars is

$$t - c_i q.$$

Each supplier's outside option is 0. The buyer's payoff from purchasing q units of output at a total price of t dollars is

$$vq - \frac{1}{2}q^2 - t,$$

where we assume $v \geq 3$.

The buyer's objective is to design an optimal direct-revelation mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, where each component is a mapping from cost reports $c = (c_1, c_2, \ldots, c_n)$ to probabilities of selecting firm i, output for the selected firm, and transfers to each supplier, respectively, in order to maximise

$$\mathbb{E}_c \left[\sum_{i=1}^n \left(\phi_i(c) \left(vq_i(c) - \frac{1}{2}(q_i(c))^2 - c_i q_i(c) \right) - U_i(c_i) \right) \right].$$

2.1 Part (a)

State the two conditions that any incentive compatible mechanism must satisfy. [Hint: the monotonicity condition will now involve both ϕ_i and q_i.]

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In the usual set up (without q_i), we need ϕ_i to be nonincreasing and t_i to satisfy the envelope condition (the revenue equivalence formula). So we should expect similar conditions to be relevant here. The idea is to replicate the usual proof for the characterisation of incentive compatible mechanism.

First, let us think about the type-c_i supplier's payoff from reporting $\hat{c}_i$ when others report c_{-i}:

$$U_i(\hat{c}_i | c_i) := \mathbb{E}_{c_{-i}} \left[\phi_i(\hat{c}_i; c_{-i}) \left(t_i(\hat{c}_i; c_{-i}) - c_i q_i(\hat{c}_i; c_{-i}) \right) \right].$$

Now since the buyer has quasilinear preferences, only the expected transfer (i.e. interim and not the ex post transfer) is relevant for incentive compatibility. That is, for whatever ϕ_i we choose, we can the the transfer to satisfy

$$\bar{t}_i(\hat{c}_i) := \mathbb{E}_{c_{-i}} \left[\phi_i(\hat{c}_i; c_{-i}) t_i(\hat{c}_i; c_{-i}) \right].$$

Hence, we can write

$$\begin{aligned} U_i(\hat{c}_i|c_i) &= \bar{t}_i(\hat{c}_i) - \mathbb{E}_{c_{-i}}\left[\phi_i(\hat{c}_i;c_{-i})\, q_i(\hat{c}_i;c_{-i})\right] c_i \\ &\equiv \bar{t}_i(\hat{c}_i) - \overline{\Phi}_i(\hat{c}_i)\, c_i, \end{aligned}$$

where we defined $\overline{\Phi}_i(\cdot)$ in an obvious manner. As usual, define $U_i(c_i) \coloneqq U_i(c_i|c_i)$. From this point, the proof simply mimics the monopolistic screening case from PS5, where $\overline{\Phi}_i$ corresponds to q.

Lemma 2.1. *If a direct mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, is incentive compatible, then for each $i = 1, 2, \ldots, n$,*

(i) $\overline{\Phi}_i(\cdot)$ is nonincreasing; and

(ii) $U_i(c_i) = U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s)\,\mathrm{d}s$, $\forall c_i \in [1,2]$.

Proof. The IC constraints are that, for all i and for all c_i,

$$U_i(c_i) \geq U_i(\hat{c}_i|c_i), \ \forall \hat{c}_i.$$

Subtracting $U_i(\hat{c}_i) = \bar{t}_i(\hat{c}_i) - \overline{\Phi}_i(\hat{c}_i)\,\hat{c}_i$ from both sides yields

$$\begin{aligned} U_i(c_i) - U_i(\hat{c}_i) &\geq \left[\bar{t}_i(\hat{c}_i) - \overline{\Phi}_i(\hat{c}_i)\, c_i\right] - \left[\bar{t}_i(\hat{c}_i) - \overline{\Phi}_i(\hat{c}_i)\,\hat{c}_i\right] \\ &= \overline{\Phi}_i(\hat{c}_i)(\hat{c}_i - c_i). \end{aligned}$$

Reversing the roles of $\hat{c}_i$ and c_i, we can write

$$\begin{aligned} & U_i(\hat{c}_i) - U_i(c_i) \geq \overline{\Phi}_i(c_i)(c_i - \hat{c}_i) \\ \Leftrightarrow\ & \overline{\Phi}_i(c_i)(\hat{c}_i - c_i) \geq U_i(c_i) - U_i(\hat{c}_i). \end{aligned}$$

Combining the two inequalities, we obtain

$$\begin{aligned} & \overline{\Phi}_i(c_i)(\hat{c}_i - c_i) \geq \overline{\Phi}_i(\hat{c}_i)(\hat{c}_i - c_i) \\ \Leftrightarrow\ & \left[\overline{\Phi}_i(c_i) - \overline{\Phi}_i(\hat{c}_i)\right](\hat{c}_i - c_i) \geq 0. \end{aligned}$$

The inequality above is trivially satisfies if $\hat{c}_i \neq c_i$. However, it requires that: (i) if $\hat{c}_i > c_i$, then $\overline{\Phi}_i(c_i) \geq \overline{\Phi}_i(\hat{c}_i)$; (ii) if $c_i > \hat{c}_i$, then $\overline{\Phi}_i(c_i) \leq \overline{\Phi}_i(\hat{c}_i)$. In other words, we require $\overline{\Phi}_i(\cdot)$ to be nonincreasing—this is gives us the monotonicity condition (i).

We can write the IC constraint as

$$U_i(c_i) = \max_{\hat{c}_i \in [1,2]} \bar{t}_i(\hat{c}_i) - \overline{\Phi}_i(\hat{c}_i)\, c_i.$$

Since $\overline{\Phi}(\cdot)$ is bounded (it's an expectation of two bounded functions), the envelope theorem (corollary from the note "A useful envelope theorem") tells us that $U_i(c_i)$ is absolutely continuous so that

$$U_i'(c_i) = -\overline{\Phi}_i(c_i).$$

Integrating both sids from 0 to c_i yields

$$U_i\left(c_i\right) = -\int_0^{c_i} \overline{\Phi}_i\left(s\right)\mathrm{d}s + C.$$

To find out C, evaluate above at $c_i = 2$, which gives us that

$$C = U_i\left(2\right) + \int_0^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s.$$

Hence,

$$\begin{aligned} U_i\left(c_i\right) &= -\int_0^{c_i} \overline{\Phi}_i\left(s\right)\mathrm{d}s + U_i\left(2\right) + \int_0^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s \\ &= U_i\left(2\right) + \int_{c_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s, \end{aligned}$$

which is the relevant integral condition. ■

We now prove the converse.

Lemma 2.2. *Suppose, for all $i = 1, 2, \ldots, n$, $\overline{\Phi}_i\left(\cdot\right)$ is nonincreasing and $U_i\left(c_i\right) = U_i\left(2\right) + \int_{c_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s$ for all $c_i \in [1, 2]$. Then, the direct mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, is incentive compatible with any $t_i\left(c\right)$ such that $\overline{t}_i\left(c_i\right) = U_i\left(c_i\right) + \overline{\Phi}_i\left(c_i\right)c_i$.*

Proof. Suppose, for all $i = 1, 2, \ldots, n$, $\overline{\Phi}_i\left(\cdot\right)$ is nonincreasing and that $U_i\left(c_i\right) = U_i\left(2\right) + \int_{c_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s$ for all $c_i \in [1, 2]$. Let $t_i\left(c\right)$ be such that $\overline{t}_i\left(c_i\right) = U_i\left(c_i\right) + \overline{\Phi}_i\left(c_i\right)c_i$. It suffices to show that

$$U_i\left(c_i\right) \geq U_i\left(\hat{c}_i | c_i\right) = \overline{t}_i\left(\hat{c}_i\right) - \overline{\Phi}_i\left(\hat{c}_i\right)c_i, \ \forall c_i \in \left[1, 2\right].$$

Substituting the expressions for $\overline{t}_i$, we obtain, $\forall c_i$ and $\forall \hat{c}_i$,

$$\begin{aligned} & U_i\left(c_i\right) \geq U_i\left(\hat{c}_i\right) + \overline{\Phi}_i\left(\hat{c}_i\right)\hat{c}_i - \overline{\Phi}_i\left(\hat{c}_i\right)c_i \\ \Leftrightarrow\ & U_i\left(c_i\right) - U_i\left(\hat{c}_i\right) \geq \overline{\Phi}_i\left(\hat{c}_i\right)\left(\hat{c}_i - c_i\right) \\ \Leftrightarrow \int_{c_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s - \int_{\hat{c}_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s \geq\ & \overline{\Phi}_i\left(\hat{c}_i\right)\left(\hat{c}_i - c_i\right) = \int_{c_i}^{\hat{c}_i} \overline{\Phi}_i\left(\hat{c}_i\right)\mathrm{d}s. \end{aligned}$$

In other words, we want to show that

$$\int_{c_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s - \int_{\hat{c}_i}^2 \overline{\Phi}_i\left(s\right)\mathrm{d}s - \int_{c_i}^{\hat{c}_i} \overline{\Phi}_i\left(\hat{c}_i\right)\mathrm{d}s \geq 0.$$

Three cases:

(i) If $c_i = \hat{c}_i$, above inequality is trivial satisfied.

(ii) If $\hat{c}_i > c_i$, then, we may write the right-hand side as

$$\int_{c_i}^{\hat{c}_i} \overline{\Phi}_i\left(s\right)\mathrm{d}s - \int_{c_i}^{\hat{c}_i} \overline{\Phi}_i\left(\hat{c}_i\right)\mathrm{d}s = \int_{c_i}^{\hat{c}_i} \left(\overline{\Phi}_i\left(s\right) - \overline{\Phi}_i\left(\hat{c}_i\right)\right)\mathrm{d}s.$$

That $\overline{\Phi}_i(\cdot)$ is nonincreasing implies $\overline{\Phi}_i(s) \geq \overline{\Phi}_i(\hat{c}_i)$ for all $s \in [c_i, \hat{c}_i]$ so that above must be nonnegative.

(iii) If $c_i > \hat{c}_i$, then, we may write the right-hand side as

$$\begin{aligned}\int_{\hat{c}_i}^{c_i} \left(-\overline{\Phi}_i(s)\right) \mathrm{d}s - \int_{c_i}^{\hat{c}_i} \overline{\Phi}_i(\hat{c}_i) \, \mathrm{d}s &= \int_{\hat{c}_i}^{c_i} \left(-\overline{\Phi}_i(s)\right) \mathrm{d}s + \int_{\hat{c}_i}^{c_i} \overline{\Phi}_i(\hat{c}_i) \, \mathrm{d}s \\ &= \int_{\hat{c}_i}^{c_i} \left(\overline{\Phi}_i(\hat{c}_i) - \overline{\Phi}_i(s)\right) \mathrm{d}s.\end{aligned}$$

That $\overline{\Phi}_i(\cdot)$ is nonincreasing implies $\overline{\Phi}_i(\hat{c}_i) \geq \overline{\Phi}_i(s)$ for all $s \in [\hat{c}_i, c_i]$ so that above must be nonnegative.

This completes the proof that $\{\phi_i, q_i, t_i\}_{i=1}^n$, is incentive compatible. ■

Let us record what we found below.

Proposition 2.1. *[Characterisation of IC mechanism in procurement auctions] A direct mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, is incentive compatible if and only if, for each $i = 1, 2, \ldots, n$,*

(i) $\overline{\Phi}_i(\cdot)$ *is nonincreasing; and*

(ii) $\bar{t}_i(c_i) = \overline{\Phi}_i(c_i) c_i + U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s) \, \mathrm{d}s$, $\forall c_i \in [1, 2]$.

Proof. Follows immediately from the previous two lemmata and substituting $\bar{t}_i(c_i) = U_i(c_i) + \overline{\Phi}_i(c_i) c_i$ into the envelope characterisation. ■

2.2 Part (b)

Using your conditions in (a), find an expression of $\mathbb{E}_c[U_i(c_i)]$ that is entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

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Substituting the expected transfer from Proposition 2.1 into $U_i(c_i)$ yields

$$\begin{aligned}U_i(c_i) \equiv U_i(c_i|c_i) &\equiv \bar{t}_i(c_i) - \overline{\Phi}_i(c_i) c_i \\ &= \overline{\Phi}_i(c_i) c_i + U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s) \, \mathrm{d}s - \overline{\Phi}_i(c_i) c_i \\ &= U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s) \, \mathrm{d}s\end{aligned}$$

Taking expectations with respect to c yields

$$\mathbb{E}_c[U_i(c_i)] = \mathbb{E}_c\left[U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s) \, \mathrm{d}s\right] = U_i(2) + \int_1^2 \left(\int_{c_i}^2 \overline{\Phi}_i(s) \, \mathrm{d}s\right) f(c_i) \, \mathrm{d}c_i,$$

where $f(\cdot)$ in this case is the density of Uniform $[1,2]$. To simplify the expression, we exchange the integral, which gives

$$\begin{aligned}
\mathbb{E}_c\left[U_i(c_i)\right] &= U_i(2) + \underbrace{\int_1^2 \int_{c_i}^2}_{1 \le c_i \le s \le 2} \overline{\Phi}_i(s)\,\mathrm{d}s f(c_i)\,\mathrm{d}c_i \\
&= U_i(2) + \underbrace{\int_1^2 \int_1^s}_{1 \le c_i \le s \le 2} f(c_i)\,\mathrm{d}c_i \overline{\Phi}_i(s)\,\mathrm{d}s \\
&= U_i(2) + \int_1^2 F(c_i)\,\overline{\Phi}_i(c_i)\,\frac{f(c_i)}{f(c_i)}\mathrm{d}c_i \\
&= U_i(2) + \mathbb{E}_{c_i}\left[\overline{\Phi}_i(c_i)\,\frac{F(c_i)}{f(c_i)}\right] \\
&= U_i(2) + \mathbb{E}_{c_i}\left[\mathbb{E}_{c_{-i}}\left[\phi_i(c_i;c_{-i})\,q_i(c_i;c_{-i})\right]\frac{F(c_i)}{f(c_i)}\right] \\
&= U_i(2) + \mathbb{E}_c\left[\phi_i(c)\,q_i(c)\,\frac{F(c_i)}{f(c_i)}\right],
\end{aligned}$$

where $F(c_i)$ is the cumulative distribution function of Uniform $[1,2]$.

2.3 Part (c)

Using your result in (b), substitute into the buyer's objective function to obtain a maximisation program that is expressed entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

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Substituting the expression we obtained for $\mathbb{E}_c[U_i(c_i)]$ that we obtained into the buyer's objective function yields

$$\begin{aligned}
&\mathbb{E}_c\left[\sum_{i=1}^n\left(\phi_i(c)\left(vq_i(c) - \frac{1}{2}(q_i(c))^2 - c_iq_i(c)\right) - \left(U_i(2) + \mathbb{E}_c\left[\phi_i(c)\,q_i(c)\,\frac{F(c_i)}{f(c_i)}\right]\right)\right)\right] \\
&= \mathbb{E}_c\left[\sum_{i=1}^n \phi_i(c)\left(vq_i(c) - \frac{1}{2}(q_i(c))^2 - \left(c_i + \frac{F(c_i)}{f(c_i)}\right)q_i(c)\right)\right] - \sum_{i=1}^n U_i(2)\,.
\end{aligned}$$

Effectively, what we have done is to build the IC constraint into the objective function. Thus, the buyer's problem is to maximise above subject to IR and that $\overline{\Phi}_i(\cdot)$ is nonincreasing. Given that $\overline{\Phi}_i(\cdot)$ is nonincreasing, the IR constraints are equivalent to, for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
&0 \le U_i(c_i) = U_i(c_i|c_i) = U_i(2) + \int_{c_i}^2 \overline{\Phi}_i(s)\,\mathrm{d}s,\ \forall c_i \in [1,2] \\
\Leftrightarrow\ &0 \le U_i(2)\,.
\end{aligned}$$

2.4 Part (d)

Find the optimal $\phi_i(c)$ and $q_i(c)$ components of the optimal mechanism. Make whatever regularity assumptions you use to this end explicit.

......

Define the virtual cost function as

$$J_i(c_i) := c_i + \frac{F(c_i)}{f(c_i)} = 2c_i - 1,$$

where we used that F is uniform on $[1,2]$. Observe, in particular, that $J_i(\cdot)$ is increasing. Thus, we have the usual requirement that the virtual cost function is monotonic. Using $J_i(c_i)$, we may write the problem now as

$$\begin{aligned} \max_{(\phi_i, q_i)} \quad & \mathbb{E}_c\left[\sum_{i=1}^{n} \phi_i(c)\left(vq_i(c) - \frac{1}{2}(q_i(c))^2 - J_i(c_i)\, q_i(c)\right)\right] - \sum_{i=1}^{n} U_i(2) \\ \text{s.t.} \quad & U_i(2) \geq 0,\ \forall i = 1, 2, \ldots, n, \\ & \overline{\Phi}_i(\cdot) \ \text{is nonincreasing.} \end{aligned}$$

In the standard setting, the regularity assumptions allowed us to solve this problem via pointwise maximisation of the objective. So let's see what happens if we tried to do the same. Ignoring the monotonicity condition, pointwise maximisation (i.e. fixing c) yields that:

(i) $U_i(2) = 0$ for all $i = 1, 2, \ldots, n$;

(ii) for any seller who is selected (i.e. $\phi_i(c) > 0$), the optimal $q_i(c)$ solves

$$q_i(c) = \operatorname*{arg\,max}_{\tilde{q} \in [0,Q]} \ v\tilde{q} - \frac{1}{2}\tilde{q}^2 - J_i(c_i)\,\tilde{q}.$$

The first-order condition yields that

$$\begin{aligned} v - q_i(c) - J_i(c_i) = 0 &\Leftrightarrow v - q_i(c) - (2c_i - 1) = 0 \\ &\Leftrightarrow q_i(c) = v - 2c_i + 1. \end{aligned}$$

Notice that $q_i(c)$ is a function of only c_i and that it is decreasing in c_i. Moreover, given $c_i \in [1,2]$, and $v \geq 3$, we know that $q_i(c_i) \geq 0$ and the lower bound constraint on q does not bind. Define the virtual utility as

$$\begin{aligned} V_i^*(c_i) &:= \max_{\tilde{q} \in [0,Q]} \ v\tilde{q} - \frac{1}{2}\tilde{q}^2 - J_i(c_i)\,\tilde{q} \\ &= (v - J_i(c_i))\,\tilde{q} - \frac{1}{2}\tilde{q}^2 \\ &= (v - 2c_i + 1)^2 - \frac{1}{2}(v - 2c_i + 1)^2 \\ &= \frac{1}{2}(v - 2c_i + 1)^2 \geq 0. \end{aligned}$$

Te nonnegative implies that it is always optimal to award the supply contract to some supplier. Hence, $\sum_{i=1}^{n} \phi_i(c) = 1$ for all $c \in [1,2]^n$. The optimal pointwise solution is to award it to the highest $V_i^*(c_i)$, where:[1]

$$\phi_i(c) = \begin{cases} 1 & \text{if } V_i^*(c_i) > \max_{j \neq i} V_j^*(c_j), \\ 0 & \text{otherwise.} \end{cases}$$

It remains to verify the monotonicity condition; i.e. $\overline{\Phi}_i(c_i) = \mathbb{E}_{c_{-i}}[q_i(c)\,\phi_i(c)]$ is nonincreasing in c_i. Since $V_i^*(c_i)$ is strictly decreasing in c_i, $\phi_i(c)$ is weakly decreasing in c_i. As remarked previously, $q_i(c)$ is decreasing in c_i so that, together, we must have that $\phi_i(c)\,q_i(c)$ is weakly decreasing in c_i. Hence, it follows that $\overline{\Phi}_i(\cdot)$ is nonincreasing as we wanted.

Remark 2.1. See the official solution for how this question relates to the result from Laffont and Tirole (JPE, 1987).

[1] Tie breaks are measure zero events since c_i are continuously distributed.

3 Problem 3: Mineral-rights auctions

[See the official solutions.]

4 Problem 4: Auctions with adverse selection

The economics department is trying to procure teaching services from one of n potential external lecturers. Candidate i has an outside opportunity of $\theta_i \in [0,1]$ with distribution $F(\cdot)$. This opportunity is private information and can be thought of as the candidate's type. The department gets teaching value $v(\theta_i)$ from a lecturer with type θ_i; the function $v(\cdot)$ is increasing and differentiable.

Consider a direct revelation mechanism consisting of allocation function $\phi_i(\theta_1, \theta_2, \ldots, \theta_n) \in [0,1]$ for each lecturer i, and a transfer function $t_i(\theta_1, \theta_2, \ldots, \theta_n)$, which is the payment to each lecturer i. i's utility from reporting $\hat{\theta}_i$ when her true type is θ_i is

$$U_i\left(\hat{\theta}_i|\theta_i\right) = \mathbb{E}_{\theta_{-i}}\left[t_i\left(\hat{\theta}_i;\theta_{-i}\right) - \phi_i\left(\hat{\theta}_i;\theta_{-i}\right)\theta_i\right] = \bar{t}_i\left(\hat{\theta}_i\right) - \overline{\phi}_i\left(\hat{\theta}_i\right)\theta_i.$$

[Note that the lecturer utility is decreasing in θ_i and the single-crossing terms is negative.] The department's objective is to maximise

$$\Pi \coloneqq \mathbb{E}_\theta\left[\sum_{i=1}^{n}\left(\phi_i\left(\theta\right)v\left(\theta_i\right) - t_i\left(\theta\right)\right)\right].$$

Remark 4.1. Notice that $U_i(\hat{\theta}_i|\theta_i)$ already builds the lecturer's outside option into the utility. Thus, the IR constraint is simply that $U_i(\hat{\theta}_i|\theta_i) \geq 0$ as usual.

Remark 4.2. Normally, we assume, we assume $\tilde{\Pi} = \sum_{i=1}^{n}\left(v\phi\left(\theta_i\right) - t_i\left(\theta\right)\right)$ as the principal's utility. In this case, principal's utility is not directly affected by the type of agent hired. However, in this problem, v depends on θ_i so that there is selection. in particular the higher type brings more benefit for the principal but has a higher outside option; hence, we have adverse selection similar to the labour market case we looked at earlier in the quarter.

4.1 Part (a)

Characterise incentive compatibility in terms of an integral equation for the agent's utility, $U_i(\theta_i)$, and a monotonicity constraint.

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The proof here is the same as in Question 2 when we replace $\overline{\Phi}_i(\cdot)$ with $\overline{\phi}_i(\cdot)$ so I will not repeat it here. Given that the support of the distribution is now $[0,1]$ (instead of $[1,2]$), we arrive at the following result.

Proposition 4.1. *A direct-revelation mechanism, $\{\phi_i, t_i\}_{i=1}^n$, is incentive compatible if and only if, for all $i = 1, 2, \ldots, n$,*

(i) $\overline{\phi}_i(\theta_i)$ *is nonincreasing; and*

(ii) $U_i(\theta_i) = U_i(1) + \int_{\theta_i}^{1}\overline{\phi}_i(s)\,\mathrm{d}s$ *for all* $\theta_i \in [0,1]$

where $t_i(\theta_i)$ *satisfies* $\bar{t}_i(\theta_i) = U_i(\theta_i) + \overline{\phi}_i(\theta_i)\theta_i$ *for all* θ_i.

4.2 Part (b)

Using (a), what is the departments profit expressed in terms of $\phi_i(\cdot)$ and $U_i(1)$.

$\cdots\cdots$

Recall that $\bar{t}_i(\theta_i) = U_i(\theta_i) + \overline{\phi}_i(\theta_i)\theta_i$ so that we may write the envelope condition as

$$\mathbb{E}_{\theta_{-i}}[t_i(\theta)] = \bar{t}_i(\theta_i) = \overline{\phi}_i(\theta_i)\theta_i + U_i(1) + \int_{\theta_i}^1 \overline{\phi}_i(s)\,\mathrm{d}s.$$

Taking expectations with respect to θ_i and exchanging integrals, we obtain

$$\begin{aligned}
\mathbb{E}_\theta[t_i(\theta)] &= \mathbb{E}_{\theta_i}\left[\overline{\phi}_i(\theta_i)\theta_i + U_i(1) + \int_{\theta_i}^1 \overline{\phi}_i(s)\,\mathrm{d}s\right] \\
&= U_i(1) + \int_0^1 \overline{\phi}_i(\theta_i)\theta_i f(\theta_i)\,\mathrm{d}\theta_i + \underbrace{\int_0^1\int_{\theta_i}^1}_{0\le\theta_i\le s\le 1} \overline{\phi}_i(s)\,\mathrm{d}s f(\theta_i)\,\mathrm{d}\theta_i \\
&= U_i(1) + \int_0^1 \overline{\phi}_i(\theta_i)\theta_i f(\theta_i)\,\mathrm{d}\theta_i + \underbrace{\int_0^1\int_0^s}_{0\le\theta_i\le s\le 1} f(\theta_i)\,\mathrm{d}\theta_i\overline{\phi}_i(s)\,\mathrm{d}s \\
&= U_i(1) + \int_0^1 \overline{\phi}_i(\theta_i)\theta_i f(\theta_i)\,\mathrm{d}\theta_i + \int_0^1 F(\theta_i)\overline{\phi}_i(\theta_i)\,\mathrm{d}\theta_i \\
&= U_i(1) + \int_0^1 \left[\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)\overline{\phi}_i(\theta_i) f(\theta_i)\,\mathrm{d}\theta_i\right] \\
&= U_i(1) + \mathbb{E}_{\theta_i}\left[\overline{\phi}_i(\theta_i)\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)\right] \\
&= U_i(1) + \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\theta_{-i}}[\phi_i(\theta_i;\theta_{-i})]\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)\right] \\
&= U_i(1) + \mathbb{E}_\theta\left[\phi_i(\theta)\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)\right].
\end{aligned}$$

Using the linearity of $\mathbb{E}$ operator and the expression above, we may write Π as

$$\begin{aligned}
\Pi &= \sum_{i=1}^n \mathbb{E}_\theta[\phi_i(\theta)v(\theta_i)] - \sum_{i=1}^n \mathbb{E}_\theta[t_i(\theta)] \\
&= \sum_{i=1}^n \mathbb{E}_\theta[\phi_i(\theta)v(\theta_i)] - \sum_{i=1}^n\left(U_i(1) + \mathbb{E}_\theta\left[\phi_i(\theta)\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)\right]\right) \\
&= \mathbb{E}_\theta\left[\sum_{i=1}^n\left(\phi_i(\theta)\left(v(\theta_i) - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}\right) - U_i(1)\right)\right].
\end{aligned}$$

Remark 4.3. We could have let $t_i(\theta) = \phi_i(\theta)\theta_i + U_i(\theta)$ and followed the same way as in Question 2.

4.3 Part (c)

For the remainder of the question, assume that $F_i(\theta_i)/f_i(\theta)$ is increasing in θ_i.

If $v'(\theta_i) \le 1$, what is the department's optimal hiring policy, $\{\phi_i(\cdot)\}_{i=1}^n$?

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Define virtual value as

$$V_i(\theta_i) := v(\theta_i) - \theta_i - \frac{F(\theta_i)}{f(\theta_i)} = v(\theta_i) - 2\theta_i,$$

where we used that $F(\theta_i) = \theta_i$ and $f(\theta_i) = 1$ since $\theta_i \sim \text{Uniform}[0,1]$.

We can write the problem now as

$$\begin{aligned} \max_{\phi_i} \quad & \mathbb{E}_\theta \left[\sum_{i=1}^n \left(\phi_i(\theta) V_i(\theta_i) - U_i(1) \right) \right] \\ \text{s.t.} \quad & \overline{\phi}_i(\theta_i) \text{ is nonincreasing}, \ \forall i = 1, 2, \dots, n, \\ & U_i(1) \ge 0, \ \forall i = 1, 2, \dots, n, \end{aligned}$$

where the second constraint is the IR constraint (while assuming $\phi_i(\theta)$ is nonincreasing). As in Question 2, let us solve for the pointwise maximum while ignoring the monotonicity constraint.

(i) $U_i(1) = 0$ for all $i = 1, 2, \dots, m$.

(ii) since the objective is linear and $\phi_i \in [0,1]$, pointwise maximum is attained by setting $\phi_i(\theta)$ to i with the largest nonzero $V_i(\theta_i)$; i.e.

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } V_i(\theta_i) > \max\{\max_{j \ne i} V_i(\theta_j), 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, to verify that monotonicity condition holds, observe that

$$V_i'(\theta_i) = v'(\theta_i) - 2 < 0$$

so that $V_i(\theta_i)$ is strictly decreasing in θ_i. Hence, $\overline{\phi}_i(\theta_i)$ is weakly decreasing (i.e. nonincreasing) in θ_i.

4.4 Part (d)

Suppose instead that $v'(\theta_i) > 2$ and $\mathbb{E}[v(\theta_i)] \ge 1$. What is the department's optimal hiring policy? [Hint: If an unrelated program violates the required monotonicity at every point of θ_i, then the constrained-optimal solution must be constant.]

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With $v'(\theta_i) > 2$, we would now have that

$$V_i'(\theta_i) = v'(\theta_i) - 2 \ge 0;$$

i.e. $V_i(\theta_i)$ is strictly increasing. It follows then that $\overline{\phi}_i(\theta_i)$ is strictly increasing for all θ_i; i.e. the monotonicity constraint is violated everywhere. Applying the hint, we therefore conclude that the optimal allocation rule is constant. That is $\overline{\phi}_i \equiv \overline{\phi}_i(\theta)$ so that incentive compatibility requires that

$$\begin{aligned} t_i \equiv \bar{t}_i(\theta_i) &= \overline{\phi}_i(\theta_i)\,\theta_i + U_i(1) + \int_{\theta_i}^{1} \overline{\phi}_i(s)\,\mathrm{d}s \\ &= \overline{\phi}_i\theta_i + U_i(1) + \overline{\phi}_i(1-\theta_i) \\ &= U_i(1) + \overline{\phi}_i \\ &= \overline{\phi}_i, \end{aligned}$$

where the last line follows from individual rationality (i.e. $U_i(1) = 0$). Since $\mathbb{E}[v]$

$$\mathbb{E}_{\theta_i}[v(\theta_i) - 2\theta_i] = \mathbb{E}_{\theta_i}[v(\theta_i)] - 1 \geq 0,$$

it is always optimal for the department to hire a lecturer; i.e.

$$\sum_{i=1}^{n} \overline{\phi}_i = 1 \Leftrightarrow \overline{\phi}_i = \frac{1}{n}.$$

That is, it is optimal to choose a lecturer randomly among the n candidates and pay 1 to the chosen candidate (since this implies that $\bar{t}_i(\theta_i) = 1/n$). Intuitively, the department cares so much more about quality relative to costs that it gives up trying to reduce expenditures by screening among lecturers.

5 Problem 5: Auctions with endogenous entry

[See the official solutions.]

6 Problem 6: Reserve prices in IPV auctions

Consider an IPV setting with $n = 2$ and θ_i uniformly distributed on $[0, 1]$ for both bidders. Suppose that $\theta_0 = 0$ (the seller's opportunity cost of the objective is zero).

6.1 Part (a)

In the Myersonian optimal auction, what is the allocation function $\phi_i(\theta_i, \theta_{-i})$ and what is the reserve type, θ^* (i.e., what is the highest type buyer θ_i such that $\overline{\phi}_i(\theta_i) = 0$). In describing the optimal allocation function, you may ignore the situations with zero probability (i.e. ignore ties).

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I solved this in my TA session (TA class 7), for derivation, take a look at the notes that I uploaded (the notation I used there was slightly different: I used q_i instead of ϕ_i).

Letting $J_i(\theta_i)$ denote i's virtual value function, the optimal allocation is given by

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) \geq \max\left\{\max_{j \neq i} J_j(\theta_j), 0\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we need not worry about tie breaks here since values are drawn from a continuous distribution. In this case, $J_i(\theta_i)$ is given by

$$J_i(\theta_i) := \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} = \theta_i - \frac{1 - \theta_i}{1} = 2\theta_i - 1$$

so that $J_i(\theta_i^*) \geq 0 \Leftrightarrow \theta_i^* \geq \frac{1}{2}$. We can therefore write

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \max\left\{\max_{j \neq i} \theta_j, \frac{1}{2}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

The reserve type is $\theta^* = \frac{1}{2}$.

6.2 Part (b)

Show that a first-price auction with an appropriately chosen reserve price is also optimal. Argue that the optimal reserve price is set at $r^* = \theta^*$.

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By the revenue equivalence theorem, a first-price auction with reserve will be optimal if it implements the same ϕ_i's and $U_i(0) = 0$.

- ▷ Among the bidders who submit bids above the reserve price, the highest bidder wins. Because the distributions are symmetric, this corresponds to the highest virtual value type winning as in the optimal auction in a. Hence, ϕ_i's coincide.

▷ It remains to show that a reserve price of $r^* = \theta^*$, where $J_i(\theta^*) = 0$, implements the same threshold type, θ^*; i.e. $\theta_i < \theta^*$ do not bid, and $\theta_i \geq \theta^*$ bids. Notice that any bidder with value $\theta_i < r^*$ will not bid (if he bids, then there is some chance that he might win and earn a strictly negative payoff) and so earn $U(\theta_i) = 0$. Any bidder with type $\theta_i > r^*$ will find it optimal to bid and will choose a bid in $[r^*, \theta_i)$. Hence, type $\theta_i > r^*$ will win with some probability in the first-price auction. It follows that, by setting $r^* = \theta^*$, only $\theta \geq \theta^*$ will bid and have any probability of winning. Because $U_i(\theta_i) = 0$ for all $\theta_i \leq \theta^*$, the utility of the lowest type is $U_i(0) = 0$, just as in the optimal auction.

6.3 Part (c)

Compute the equilibrium bidding function in (b) give the optimal reserve. [Hint: it is *not* linear. Try using the envelope theorem to find $\bar{b}(\cdot)$.]

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Consider the expected utility of a bidder that faces a first-price auction with type θ^*. As argued above, since only types $\theta_i \geq \theta^*$ participates,

$$\begin{aligned} U_i\left(\hat{\theta}_i|\theta_i\right) &= \mathbf{1}_{\{\theta_i \geq \theta^*\}} F^{N-1}\left(\hat{\theta}_i\right)\left(\theta_i - \bar{b}\left(\hat{\theta}_i\right)\right) \\ \Rightarrow U_i(\theta_i) \coloneqq U_i(\theta_i|\theta_i) &= \mathbf{1}_{\{\theta_i \geq \theta^*\}} F^{N-1}(\theta_i)\left(\theta_i - \bar{b}(\theta_i)\right). \end{aligned}$$

Incentive compatibility requires that

$$\begin{aligned} U_i(\theta_i) &= U_i(0) + \int_0^{\theta_i} \bar{\phi}_i(s)\,\mathrm{d}s \\ &= 0 + \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[\mathbf{1}_{\{s \geq \max_{j\neq i}\{\theta_j, \theta^*\}\}}\right]\mathrm{d}s. \end{aligned}$$

Observe that

$$\mathbb{E}_{\theta_{-i}}\left[\mathbf{1}_{\{s \geq \max_{j\neq i}\{\theta_j, \theta^*\}\}}\right] = \mathbb{E}_{\theta_{-i}}\left[\mathbf{1}_{\{\max_{j\neq i}\theta_j \leq s\}}\right]\mathbf{1}_{\{s \geq \theta^*\}} = F^{N-1}(s)\,\mathbf{1}_{\{s \geq \theta^*\}}$$

so that

$$U_i(\theta_i) = \int_0^{\theta_i} F^{N-1}(s)\,\mathbf{1}_{\{s \geq \theta^*\}}\mathrm{d}s = \mathbf{1}_{\{\theta_i \geq \theta^*\}} \int_{\theta^*}^{\theta_i} F(s)\,\mathrm{d}s. \tag{6.1}$$

Thus,

$$\mathbf{1}_{\{\theta_i \geq \theta^*\}} \int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s = U_i(\theta_i) = \mathbf{1}_{\{\theta_i \geq \theta^*\}} F^{N-1}(\theta_i)\left(\theta_i - \bar{b}(\theta_i)\right).$$

Then, for $\theta_i \geq \theta^*$,

$$\begin{aligned} \int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s &= F^{N-1}(\theta_i)\left(\theta_i - \bar{b}(\theta_i)\right) \\ \Leftrightarrow \bar{b}(\theta_i) &= \theta_i - \frac{\int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s}{F^{N-1}(\theta_i)}. \end{aligned}$$

Integration by parts yields

$$\begin{aligned}\int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s &= \left[F^{N-1}(s)\,s\right]_{\theta^*}^{\theta_i} - \int_{\theta^*}^{\theta_i} (N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s \\ &= F^{N-1}(\theta_i)\,\theta_i - F^{N-1}(\theta^*)\,\theta^* - \int_{\theta^*}^{\theta_i} (N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s\end{aligned}$$

and substituting this into $\bar{b}(\theta_i)$ for $\theta_i \geq \theta^*$ yields

$$\begin{aligned}\bar{b}(\theta_i) &= \theta_i - \frac{F^{N-1}(\theta_i)\,\theta_i - F^{N-1}(\theta^*)\,\theta^* - \int_{\theta^*}^{\theta_i}(N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s}{F^{N-1}(\theta_i)} \\ &= \frac{F^{N-1}(\theta_i)\,\theta_i - F^{N-1}(\theta_i)\,\theta_i + F^{N-1}(\theta^*)\,\theta^*}{F^{N-1}(\theta_i)} + \frac{\int_{\theta^*}^{\theta_i}(N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s}{F^{N-1}(\theta_i)} \\ &= \frac{F^{N-1}(\theta^*)}{F^{N-1}(\theta_i)}\theta^* + \frac{1}{F^{N-1}(\theta_i)}\int_{\theta^*}^{\theta_i}(N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s.\end{aligned}$$

Thus, the bidding function is the bidding function without reservation price (with the lower limit on the integral changed) plus another term.[2] The bidding function is

$$\bar{b}(\theta_i) = \begin{cases} \frac{F^{N-1}(\theta^*)}{F^{N-1}(\theta_i)}\theta^* + \frac{1}{F^{N-1}(\theta_i)}\int_{\theta^*}^{\theta_i}(N-1)\,f(s)\,F^{N-2}(s)\,s\mathrm{d}s & \text{if } \theta_i \geq \theta^*, \\ 0 & \text{if } \theta_i > \theta^*. \end{cases}$$

Since $F^{N-1}(\theta) = \theta$ and $\theta^* = \frac{1}{2}$, we have that, for all $\theta_i \geq \frac{1}{2}$,

$$\begin{aligned}\bar{b}(\theta_i) = \theta_i - \frac{\int_{\theta^*}^{\theta_i} s\mathrm{d}s}{\theta_i} &= \theta_i - \frac{1}{2}\frac{\theta_i^2 - (\theta^*)^2}{\theta_i} \\ &= \frac{1}{2}\theta_i + \frac{1}{2}\frac{(\theta^*)^2}{\theta_i} \\ &= \frac{1}{2}\theta_i + \frac{1}{8}\frac{1}{\theta_i}.\end{aligned}$$

Observe that the bidding function is *not* a truncation of equilibrium bid function in the no-reserve case—it's not even linear anymore!

Remark 6.1. In the symmetric IPV environment, you show that the first-price, Dutch and English auctions are all optimal for the seller once an appropriate reserve price is chosen (see JR3, Exercise 9.21). Moreover, the optimal reserve value is the same for all four of the standard auctions (i.e. the aforementioned plus second-price auction). The point of this question was to highlight that, although the reserve value are the same, the bidding function are not the same and is not a simple transformation of the original equilibrium bidding function without a reserve price.

6.4 Part (d)

In order to implement the optimal auction allocation in a standard all-pay auction (i.e., highest bidder wins but everyone pays their bids), what is the optimal reserve price, r^*, that must be set. [Hint: for all-pay auction, the answer is not the same as in (b). The equilibrium bid function will

[2] You can check that this function is strictly increasing in θ_i^* so that it remains the case that the bidder with the highest value bids the highest and wins.

be of the form $\overline{b}(\theta) = 0$ for all $\theta < \theta^*$, a jump at $b(\theta^*) = r^*$, and $\overline{b}(\theta) > r^*$ for all $\theta > \theta^*$. Try using the envelope theorem.]

. . . . . .

By the same logic as in part (b), using the revenue equivalence theorem, it suffices to find a reserve price r^* such that all types $\theta \leq \theta^*$ chooses $\overline{b}(\theta) = 0$ and all types $\theta > \theta^*$ choose $\overline{b}(\theta) \geq r^*$ and win with positive probability.

The expected utility of a bidder that faces an all-pay auction with reservation type θ^* is

$$\begin{aligned} U_i\left(\hat{\theta}_i|\theta_i\right) &= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\left(F^{N-1}\left(\hat{\theta}_i\right)\theta_i - \overline{b}\left(\hat{\theta}_i\right)\right) \\ \Rightarrow U_i(\theta_i) \coloneqq U_i(\theta_i|\theta_i) &= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\left(F^{N-1}(\theta_i)\theta_i - \overline{b}(\theta_i)\right). \end{aligned}$$

Notice that (6.1) remains unchanged so that

$$\mathbf{1}_{\{\theta_i \geq \theta^*\}}\int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s = U_i(\theta_i) = \mathbf{1}_{\{\theta_i \geq \theta^*\}}\left(F^{N-1}(\theta_i)\theta_i - \overline{b}(\theta_i)\right).$$

Then, for $\theta_i \geq \theta^*$,

$$\begin{aligned} \int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s &= F^{N-1}(\theta_i)\theta_i - \overline{b}(\theta_i) \\ \Leftrightarrow \overline{b}(\theta_i) &= F^{N-1}(\theta_i)\theta_i - \int_{\theta^*}^{\theta_i} F^{N-1}(s)\,\mathrm{d}s \\ &= F^{N-1}(\theta_i)\theta_i - \left[F^{N-1}(\theta_i)\theta_i - F^{N-1}(\theta^*)\theta^* - \int_{\theta^*}^{\theta_i}(N-1)f(s)F^{N-2}(s)\,s\mathrm{d}s\right] \\ &= F^{N-1}(\theta^*)\theta^* + \int_{\theta^*}^{\theta_i}(N-1)f(s)F^{N-2}(s)\,s\mathrm{d}s \\ &= F^{N-1}(\theta^*)\theta^* + \int_{\theta^*}^{\theta_i} s\mathrm{d}F^{N-1}(s)\,s, \end{aligned}$$

Once again, the bidding function is the bidding function for all-pay first-price auction without reservation price (with the lower limit changed again) plus another term. The bidding function is

$$\overline{b}(\theta_i) = \begin{cases} F^{N-1}(\theta^*)\theta^* + \int_{\theta^*}^{\theta_i} s\mathrm{d}F^{N-1}(s)\,s & \text{if } \theta_i \geq \theta^*, \\ 0 & \text{if } \theta_i < \theta^*. \end{cases}$$

Since $F^{N-1}(\theta) = \theta$ and $\theta^* = \frac{1}{2}$, we have that, for all $\theta_i \geq \frac{1}{2}$,

$$\overline{b}(\theta_i) = (\theta^*)^2 + \int_{\theta^*}^{\theta_i} s\mathrm{d}s = \frac{1}{2}\theta_i^2 + \frac{1}{2}(\theta^*)^2 = \frac{1}{2}\theta_i^2 + \frac{1}{8}.$$

For $\theta_i = \theta^*$, we need $\overline{b}(\theta^*) = r^*$; i.e. for all $\theta_i \geq \frac{1}{2}$,

$$\overline{b}(\theta^*) = \frac{1}{2}(\theta_i^*)^2 + \frac{1}{8} = \frac{1}{4} \Rightarrow r^* = \frac{1}{4}.$$

To be complete, the equilibrium bidding function is

$$\bar{b}\left(\theta_i\right) = \begin{cases} \frac{1}{2}\theta_i^2 + \frac{1}{8} & \text{if } \theta_i \geq \frac{1}{2}, \\ 0 & \text{if } \theta_i < \frac{1}{2}. \end{cases}$$