4.2 Bounded Returns

In this section we study functional equations of the form

$$v(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta v(y)], \tag{1}$$

under the assumption that the function F is bounded and the discount factor β is strictly less than one.

As above, let X be the set of possible values for the state variable; let Γ: $X \rightarrow X$ be the correspondence describing the feasibility constraints; let

$A = \{(x, y) \in X \times X: y \in \Gamma(x)\}$ be the graph of Γ; let $F: A \to \mathbf{R}$ be the return function; and let $\beta \geq 0$ be the discount factor. Throughout this section, we will impose the following two assumptions on X, Γ, F, and β.

ASSUMPTION **4.3** *X is a convex subset of $\mathbf{R}^l$, and the correspondence Γ: $X \to X$ is nonempty, compact-valued, and continuous.*

ASSUMPTION **4.4** *The function $F: A \to \mathbf{R}$ is bounded and continuous, and $0 < \beta < 1$.*

It is clear that under Assumptions 4.3–4.4, Assumptions 4.1–4.2 hold, so the sequence problem corresponding to (1) is well defined. Moreover, Theorems 4.2–4.5 imply that under these assumptions solutions to (1) coincide exactly—in terms of both values and optimal plans—to solutions of the sequence problem.

The requirement that X be a subset of a finite-dimensional Euclidean space could be relaxed in much of what follows, but at the expense of a substantial additional investment in terminology and notation. (Recall that the definitions of u.h.c. and l.h.c. provided in Chapter 3 applied only to correspondences from one Euclidean space to another.) However, most of the arguments in this section apply much more broadly. Also note that the assumption that X is convex is not needed for Theorems 4.6 and 4.7.

If B is a bound for $|F(x, y)|$, then the supremum function v^* satisfies $|v^*(x)| \leq B/(1 - \beta)$, all $x \in X$. In this case it is natural to seek solutions to (1) in the space $C(X)$ of bounded continuous functions $f: X \to \mathbf{R}$, with the sup norm: $\|f\| = \sup_{x \in X}|f(x)|$. Clearly, any solution to (1) in $C(X)$ satisfies the hypothesis of Theorem 4.3 and hence is the supremum function. Moreover, given a solution $v \in C(X)$ to (1), we can define the policy correspondence $G: X \to X$ by

$$G(x) = \{y \in \Gamma(x): v(x) = F(x, y) + \beta v(y)\}, \tag{2}$$

and Theorems 4.4 and 4.5 imply that for any $x_0 \in X$, a sequence $\{x_t^*\}$ attains the supremum in the sequence problem if and only if it is generated by G.

The rest of the section proceeds as follows. Define the operator T on $C(X)$ by

$$(Tf)(x) = \max_{y \in \Gamma(x)}[F(x, y) + \beta f(y)], \tag{3}$$

so (1) becomes $v = Tv$. First, if we use only the boundedness and continuity restrictions in Assumptions 4.3 and 4.4, Theorem 4.6 establishes that $T: C(X) \to C(X)$, that T has a unique fixed point in $C(X)$, and that the policy correspondence G defined in (2) is nonempty and u.h.c. Theorem 4.7 establishes that under additional monotonicity restrictions on F and Γ, v is strictly increasing. Theorem 4.8 establishes that under additional concavity restrictions on F and convexity restrictions on Γ, v is strictly concave and G is a continuous (single-valued) function. Theorem 4.9 shows that if $\{v_n\}$ is a sequence of approximations defined by $v_n = T^n v_0$, with v_0 appropriately chosen, then the sequence of associated policy functions $\{g_n\}$ converges uniformly to the optimal policy function g given by (2). Finally, Theorem 4.11 establishes that if F is continuously differentiable, then v is, too.

THEOREM 4.6 *Let X, Γ, F, and β satisfy Assumptions 4.3 and 4.4, and let $C(X)$ be the space of bounded continuous functions $f: X \to \mathbf{R}$, with the sup norm. Then the operator T maps $C(X)$ into itself, $T: C(X) \to C(X)$; T has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$,*

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 0, 1, 2, \ldots. \tag{4}$$

Moreover, given v, the optimal policy correspondence $G: X \to X$ defined by (2) is compact-valued and u.h.c.

Proof. Under Assumptions 4.3 and 4.4, for each $f \in C(X)$ and $x \in X$, the problem in (3) is to maximize the continuous function $[F(x, \cdot) + \beta f(\cdot)]$ over the compact set $\Gamma(x)$. Hence the maximum is attained. Since both F and f are bounded, clearly Tf is also bounded; and since F and f are continuous, and Γ is compact-valued and continuous, it follows from the Theorem of the Maximum (Theorem 3.6) that Tf is continuous. Hence $T: C(X) \to C(X)$.

It is then immediate that T satisfies the hypotheses of Blackwell's sufficient conditions for a contraction (Theorem 3.3). Since $C(X)$ is a Banach space (Theorem 3.1), it then follows from the Contraction Mapping Theorem (Theorem 3.2), that T has a unique fixed point $v \in C(X)$, and (4) holds. The stated properties of G then follow from the Theorem of the Maximum, applied to (1). ■

It follows immediately from Theorem 4.3 that under the hypotheses of Theorem 4.6, the unique bounded continuous function v satisfying

(1) is the supremum function for the associated sequence problem. That is, Theorems 4.3 and 4.6 together establish that under Assumptions 4.3–4.4 the supremum function is bounded and continuous. Moreover, it then follows from Theorems 4.5 and 4.6 that there exists at least one optimal plan: any plan generated by the (nonempty) correspondence G is optimal.

To characterize v and G more sharply, we need more information about F and Γ. The next two results show how Corollary 1 to the Contraction Mapping Theorem can be used to obtain more precise characterizations of v and G.

ASSUMPTION 4.5 *For each y, $F(\cdot, y)$ is strictly increasing in each of its first l arguments.*

ASSUMPTION 4.6 *Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.*

THEOREM 4.7 *Let X, Γ, F, and β satisfy Assumptions 4.3–4.6, and let v be the unique solution to (1). Then v is strictly increasing.*

Proof. Let $C'(X) \subset C(X)$ be the set of bounded, continuous, nondecreasing functions on X, and let $C''(X) \subset C'(X)$ be the set of strictly increasing functions. Since $C'(X)$ is a closed subset of the complete metric space $C(X)$, by Theorem 4.6 and Corollary 1 to the Contraction Mapping Theorem (Theorem 3.2), it is sufficient to show that $T[C'(X)] \subseteq C''(X)$. Assumptions 4.5 and 4.6 ensure that this is so. ∎

ASSUMPTION 4.7 *F is strictly concave; that is,*

$$F[\theta(x, y) + (1 - \theta)(x', y')] \geq \theta F(x, y) + (1 - \theta)F(x', y'),$$

$$\text{all } (x, y), (x', y') \in A, \quad \text{and all } \theta \in (0, 1),$$

and the inequality is strict if $x \neq x'$.

ASSUMPTION 4.8 *Γ is convex in the sense that for any $0 \leq \theta \leq 1$, and $x, x' \in X$,*

$$y \in \Gamma(x) \quad \text{and} \quad y' \in \Gamma(x') \quad \text{implies}$$

$$\theta y + (1 - \theta)y' \in \Gamma[\theta x + (1 - \theta)x'].$$

Assumption 4.8 implies that for each $x \in X$, the set $\Gamma(x)$ is convex and there are no "increasing returns." Note that since X is convex, Assumption 4.8 is equivalent to assuming that the graph of Γ (the set A) is convex.

THEOREM **4.8** *Let X, Γ, F, and β satisfy Assumptions 4.3–4.4 and 4.7–4.8; let v satisfy (1); and let G satisfy (2). Then v is strictly concave and G is a continuous, single-valued function.*

Proof. Let $C'(X) \subset C(X)$ be the set of bounded, continuous, weakly concave functions on X, and let $C''(X) \subset C'(X)$ be the set of strictly concave functions. Since $C'(X)$ is a closed subset of the complete metric space $C(X)$, by Theorem 4.6 and Corollary 1 to the Contraction Mapping Theorem (Theorem 3.2), it is sufficient to show that $T[C'(X)] \subseteq C''(X)$.

To verify that this is so, let $f \in C'(X)$ and let

$$x_0 \neq x_1, \quad \theta \in (0, 1), \quad \text{and} \quad x_\theta = \theta x_0 + (1 - \theta)x_1$$

Let $y_i \in \Gamma(x_i)$ attain $(Tf)(x_i)$, for $i = 0, 1$. Then by Assumption 4.8, $y_\theta = \theta y_0 + (1 - \theta)y_1 \in \Gamma(x_\theta)$. It follows that

$$\begin{aligned}(Tf)(x_\theta) &\geq F(x_\theta, y_\theta) + \beta f(y_\theta)\\ &> \theta[F(x_0, y_0) + \beta f(y_0)] + (1 - \theta)[F(x_1, y_1) + \beta f(y_1)]\\ &= \theta(Tf)(x_0) + (1 - \theta)(Tf)(x_1),\end{aligned}$$

where the first line uses (3) and the fact that $y_\theta \in \Gamma(x_\theta)$; the second uses the hypothesis that f is concave and the concavity restriction on F in Assumption 4.7; and the last follows from the way y_0 and y_1 were selected. Since x_0 and x_1 were arbitrary, it follows that Tf is strictly concave, and since f was arbitrary, that $T[C'(X)] \subseteq C''(X)$.

Hence the unique fixed point v is strictly concave. Since F is also concave (Assumption 4.7) and, for each $x \in X$, $\Gamma(x)$ is convex (Assumption 4.8), it follows that the maximum in (3) is attained at a unique y value. Hence G is a single-valued function. The continuity of G then follows from the fact that it is u.h.c. (Exercise 3.11). ∎

Theorems 4.7 and 4.8 characterize the value function by using the fact that the operator T preserves certain properties. Thus if v_0 has property

P and if P is preserved by T, then we can conclude that each function in the sequence $\{T^n v_0\}$ has property P. Then, if P is preserved under uniform convergence, we can conclude that v also has property P. The same general idea can be used to establish facts about the policy function g, but we need to establish the sense in which the approximate policy functions—the functions g_n that attain $T^n v_0$—converge to g. The next result draws on Theorem 3.8 to address this issue.

THEOREM **4.9** (*Convergence of the policy functions*) *Let X, Γ, F, and β satisfy Assumptions 4.3–4.4 and 4.7–4.8, and let v and g satisfy (1) and (2). Let $C'(X)$ be the set of bounded, continuous, concave functions $f: X \to \mathbf{R}$, and let $v_0 \in C'(X)$. Let $\{(v_n, g_n)\}$ be defined by*

$$v_{n+1} = Tv_n, \quad n = 0, 1, 2, \ldots, \quad \text{and}$$

$$g_n(x) = \operatorname*{argmax}_{y \in \Gamma(x)} [F(x, y) + \beta v_n(y)], \quad n = 0, 1, 2,$$

Then $g_n \to g$ pointwise. If X is compact, then the convergence is uniform.

Proof. Let $C''(X) \subset C'(X)$ be the set of strictly concave functions $f: X \to \mathbf{R}$. As shown in Theorem 4.8, $v \in C''(X)$. Moreover, as shown in the proof of that theorem, $T[C'(X)] \subseteq C''(X)$. Since $v_0 \in C'(X)$, it then follows that every function v_n, $n = 1, 2, \ldots$, is strictly concave. Define the functions $\{f_n\}$ and f by

$$f_n(x, y) = F(x, y) + \beta v_n(y), \quad n = 1, 2, \ldots, \quad \text{and}$$

$$f(x, y) = F(x, y) + \beta v(y).$$

Since F satisfies Assumption 4.7, it follows that each function f_n, $n = 1, 2, \ldots$, is strictly concave, as is f. Hence Theorem 3.8 applies and the desired results are proved. ■

The next exercise deals with the case where the state space X is finite or countable, as it is in computational applications.

Exercise 4.4 Let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set; let the correspondence $\Gamma: X \to X$ be nonempty and finite-valued; let $A =$

$\{(x, y) \in X \times X: y \in \Gamma(x)\}$; let $F: A \to \mathbf{R}$ be a bounded function; and let $0 < \beta < 1$. Let $B(X)$ be the set of bounded functions $f: X \to \mathbf{R}$, with the sup norm. Define the operator T by (3).

a. Show that $T: B(X) \to B(X)$; that T has a unique fixed point $v \in B(X)$; that (4) holds for all $v_0 \in B(X)$; and that the optimal policy correspondence $G: X \to X$ defined by (2) is nonempty.

Let H be the set of functions $h: X \to X$ such that $h(x) \in \Gamma(x)$, all $x \in X$. For any $h \in H$, define the operator T_h on $B(X)$ by $(T_h f)(x) = F[x, h(x)] + \beta f[h(x)]$.

b. Show that for any $h \in H$, $T_h: B(X) \to B(X)$, and T_h has a unique fixed point $w \in B(X)$.

Let $h_0 \in H$ be given, and consider the following algorithm. Given h_n, let w_n be the unique fixed point of T_{h_n}. Given w_n, choose h_{n+1} so that $h_{n+1}(x) \in \operatorname{argmax}_{y \in \Gamma(x)} [F(x, y) + \beta w_n(y)]$.

c. Show that the sequence of functions $\{w_n\}$ converges to v, the unique fixed point of T. [*Hint.* Show that $w_0 \leq Tw_0 \leq w_1 \leq Tw_1 \leq \ldots$.]

An algorithm based on Exercise 4.4 involves applying the operators T_{h_n}—operators that require no maximization—repeatedly and applying T only infrequently. Since maximization is usually the expensive step in these computations, the savings can be considerable.

Once the existence of a unique solution $v \in C(X)$ to the functional equation (1) has been established, we would like to treat the maximum problem in that equation as an ordinary programming problem and use the standard methods of calculus to characterize the policy function g. For example, consider the functional equation for the one-sector growth model:

$$v(x) = \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\}.$$

If we knew that v was differentiable (and that the solution to the maximum problem in (1) was always interior), then the policy function g would be given implicitly by the first-order condition

$$U'[f(x) - g(x)] - \beta v'[g(x)] = 0. \tag{5}$$

Moreover, if we knew that v was twice differentiable, the monotonicity of g could be established by differentiating (5) with respect to x and examining the resulting expression for g'. However, the legitimacy of these methods depends upon the differentiability of the functions U, f, v, and g. We are free to make whatever differentiability assumptions we choose for U and f, but the properties of v and g must be established. We turn next to what is known about this issue.

It has been shown by Benveniste and Scheinkman (1979) that under fairly general conditions the value function v is *once* differentiable. That is, (5) is valid under quite broad conditions. However, known conditions ensuring that v is *twice* differentiable (and hence that g is once differentiable) are extremely strong (see Araujo and Scheinkman 1981). Thus differentiating (5) is seldom useful as a way of establishing properties of g. However, in cases where g is monotone, it is usually possible to establish that fact by a direct argument involving a first-order condition like (5).

We begin with the theorem proved by Benveniste and Scheinkman.

THEOREM 4.10 (*Benveniste and Scheinkman*) *Let $X \subseteq \mathbf{R}^l$ be a convex set, let $V: X \rightarrow \mathbf{R}$ be concave, let $x_0 \in \text{int } X$, and let D be a neighborhood of x_0. If there is a concave, differentiable function $W: D \rightarrow \mathbf{R}$, with $W(x_0) = V(x_0)$ and with $W(x) \leq V(x)$ for all $x \in D$, then V is differentiable at x_0, and*

$$V_i(x_0) = W_i(x_0), \quad i = 1, 2, \quad l.$$

Proof. Any subgradient p of V at x_0 must satisfy

$$p \cdot (x - x_0) \geq V(x) - V(x_0) \geq W(x) - W(x_0), \quad \text{all } x \in D,$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $W(x) \leq V(x)$, with equality at x_0. Since W is differentiable at x_0, p is unique, and any concave function with a unique subgradient at an interior point x_0 is differentiable at x_0 (cf. Rockafellar 1970, Theorem 25.1, p. 242). ■

Figure 4.1 illustrates the idea behind this result.

Applying this result to dynamic programs is straightforward, given the following additional restriction.

ASSUMPTION 4.9 *F is continuously differentiable on the interior of A.*

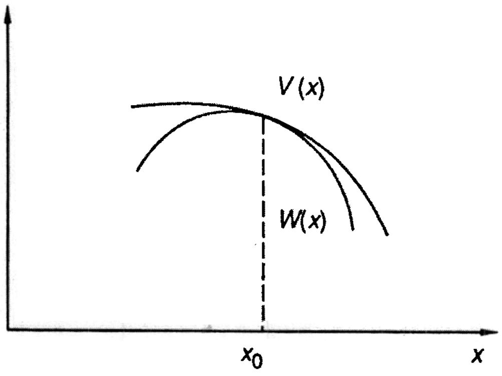

Figure 4.1

THEOREM **4.11** (*Differentiability of the value function*) *Let X, Γ, F, and β satisfy Assumptions 4.3–4.4 and 4.7–4.9, and let v and g satisfy (1) and (2). If $x_0 \in int\, X$ and $g(x_0) \in int\, \Gamma(x_0)$, then v is continuously differentiable at x_0, with derivatives given by*

$$v_i(x_0) = F_i[x_0, g(x_0)], \quad i = 1, 2, \ldots, l.$$

Proof. Since $g(x_0) \in \operatorname{int} \Gamma(x_0)$ and Γ is continuous, it follows that $g(x_0) \in \operatorname{int} \Gamma(x)$, for all x in some neighborhood D of x_0. Define W on D by

$$W(x) = F[x, g(x_0)] + \beta v[g(x_0)].$$

Since F is concave (Assumption 4.7) and differentiable (Assumption 4.9), it follows that W is concave and differentiable. Moreover, since $g(x_0) \in \Gamma(x)$ for all $x \in D$, it follows that

$$W(x) \leq \max_{y \in \Gamma(x)} [F(x, y) + \beta v(y)] = v(x), \quad \text{all } x \in D,$$

with equality at x_0. Hence v and W satisfy the hypotheses of Theorem 4.10, and the desired results follow immediately. ∎

Note that the proof requires only that F be differentiable in its first l arguments.

With differentiability of the value function established, it is often straightforward to show that the optimal policy function g is monotone, and to bound its slope.

Exercise 4.5 Consider the first-order condition (5). Assume that U, f, and v are strictly increasing, strictly concave, and once continuously differentiable, and that $0 < g(x) < f(x)$, all x. Use (5) to show that g is strictly increasing and has slope less than the slope of f. That is,

$$0 < g(x') - g(x) < f(x') - f(x), \quad \text{if } x' > x.$$

[*Hint.* Refer to Figure 4.2.]

In specific applications it is often possible to obtain much sharper characterizations of v or of G or of both than those provided by the theorems above. It is useful to keep in mind that once the existence and uniqueness of the solution to (1) has been established, the right side of that equation can be treated as an ordinary maximization problem. Thus whatever tools can be brought to bear on that problem should be exploited. But such arguments usually rely on properties of F or of Γ or of both that are specific to the application at hand. The problems in Chapter 5 provide a variety of illustrations of specific arguments of this type.

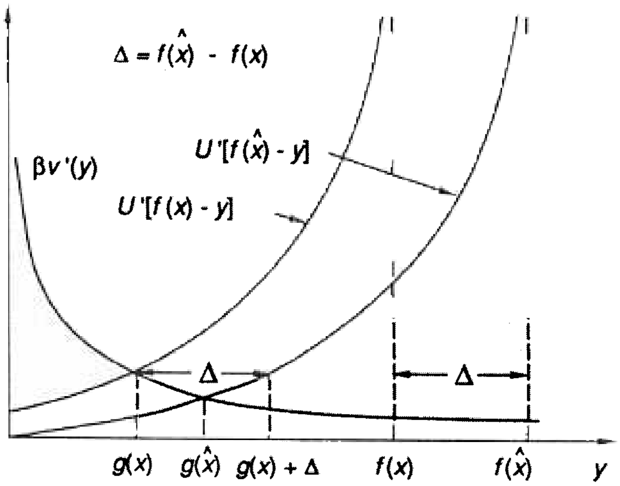

Figure 4.2

It should also be emphasized that even in cases that do not quite fit the assumptions of this section, arguments similar to the ones above can often be used. In this sense the results above should be viewed as suggestive, not (by any means) definitive. Sections 5.12 and 5.15 illustrate this point, as do many other applications in the literature. One particularly good illustration is the case of dynamic programming problems that exhibit constant returns to scale, to which we turn next.