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Part III: Monopoly Screening Contracts and Hidden Information

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1 General framework: revelation principle, incentive compatibility, optimality

In the following lectures, we explore the model of optimal (monopolistic) screening contracts. Because we assume a monopoly principal rather than a market of competitive firms offering screening contracts, we avoid many of the issues in competitive screening games (e.g., a unique equilibrium contract will be generally exist in the monopolistic screening environment). We also continue with our assumption of a single agent. We will later apply and extend the techniques from monopolistic screening to multi-agent settings (e.g., public goods games, bilateral trading mechanisms and auctions). When we do so, it is common to refer to the program as optimal mechanism design (rather than optimal screening contracts).

2 Monopoly nonlinear pricing

We begin is a workhouse of optimal screening theory: a single firm (seller, principal) wishes to sell its product to a single consumer (buyer, agent) who has unknown preferences for the seller's good. The buyer (he) knows his preferences for the good, characterized by $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$, but the seller (she) knows only the probability distribution for the buyer's type, characterized by the continuous CDF, $F(\theta)$, with corresponding density $f(\theta) > 0$ on Θ.

Payoffs: Both buyer and seller are risk neutral.

- The buyer's value of consuming q units of good at total price, t, is

  $$U = u(q, \theta) - t.$$

  We assume that u is increasing and concave in $q \in \mathcal{Q} = [\underline{q}, \overline{q}]$, u is increasing in θ, and a single-crossing property holds for (q, θ):

  $$u_{q\theta}(q, \theta) > 0 \text{ for all } \theta \in \Theta, q > 0.$$

Furthermore, for technical reasons we will find it useful to assume u is twice continuously differentiable and u_θ is bounded on Θ, a compact subset of $\mathbb{R}$.[1] A simple example to keep in mind is $u(q,\theta)=\theta q$ and $q \in \mathcal{Q}=\{0,1\}$; in this case, we can think of the monopolist selling a single unit and the buyer's type, θ, is his unit value of consumption.[2]

- The seller's profit from a sale of q at price t is

$$V = t - C(q),$$

where $C(q)$ is increasing and convex. Building again on our simple example, we can let $C(q)=cq$ and c denote the unit cost of producing output.

- **Full-information (efficient) benchmark.** We will denote the full-information, efficient q for each type θ as the function $q^{fb}(\theta)$ which is defined by

$$q^{fb}(\theta) = \arg\max_{q\in\mathcal{Q}} \; u(q,\theta) - C(q).$$

- **Demand curves.** Given our assumptions of quasi-linear utility, the demand curve for a consumer with preferences $u(q,\theta)$ facing a unit price of p is defined by the relationship

$$p = u_q(q,\theta).$$

Notice that our assumption that $u_{q\theta}>0$ implies that the family of demand curves indexed by $\theta\in\Theta$ is nested: the demand curve for for the type-θ consumer lies above the demand curve for type θ' if $\theta>\theta'$. In our special case of unit demands, $q\in\{0,1\}$, each consumer's demand curve is a "box" with height θ and width 1.

Timing:

1. Nature chooses the buyer's type according to $F(\theta)$ and the information, θ, is privately revealed to the buyer;
2. the seller offers the buyer a mechanism Γ (i.e., an extensive form game) to play; the seller has *full commitment power* in the design of the mechanism and can fully commit to how the seller will play the game (i.e., the seller's strategy can also be committed to in advance); if there are multiple equilibria in Γ, the seller chooses the equilibrium as well as the mechanism;

[1] The fact that u_θ is bounded on Θ implies that u is Lipschitz continuous on Θ. Recall that, u is Lipschitz-continuous on Θ if there exists a number K such that $|u(q,\theta)-u(q,\theta')|\le K|\theta-\theta'|$ for all $\theta,\theta'\in\Theta$.

[2] In contracting environments such procurement and regulation in which the principal is a buyer or regulator and the agent is a producer, it is sometimes more natural to think of the agent's underlying type as a representation of unit cost. As such, the corresponding assumptions are usually reversed: $u_\theta<0$ and $u_{q\theta}\le 0$. As you will see, the relevant inverse hazard rate will also change. Instead of $\frac{1-F(\theta)}{f(\theta)}$, the relevant ratio will be $\frac{F(\theta)}{f(\theta)}$.

3. the buyer either accepts or rejects the mechanism; if acceptable, the buyer plays the extensive form game Γ with the seller, resulting in a final (possibly random) equilibrium allocation of (q, t).

Remark about timing and information asymmetry at the time of contracting. Note that we assumed Nature moves first, assigning the agent's type θ, prior to the principal's contract offer. This is critical for generating distortions and second-best results. Because the principal and agent are risk neutral, if the principal could contract with the agent prior to the agent learning his type, the principal could effectively sell the agent the enterprise for its expected value. For example, a consumer with unit demand who has not yet learned the demand parameter, θ, would be willing to buy an option contract for future purchase at $p = c$ at for

$$\pi = E_\theta[\max\{\theta - c, 0\}].$$

For more general demands,

$$\pi = E_\theta[\max_{q \in \mathcal{Q}} \ u(q, \theta) - C(q)].$$

This would leave the consumer with zero expected surplus and the firm would obtain the full-information profit, π. This result is very general. With risk neutrality and no restrictions on transfers, the principal can achieve the full-information outcome if she can contract with the agent before nature chooses the agent's type. Indeed, this intuition carries over to dynamic settings as well. (See Eso and Szventes (*TE*, 2017), for example.) Notice that the full-information result requires that the agent is risk neutral and able to suffer negative outcomes. If the agent is infinitely-risk averse or liquidity constrained, or if the agent is allowed to undo the contract after observing θ, then for all practical purposes, then there is no advantage to signing the contract before nature chooses the agent's type. Indeed, if there is an ex post participation constraint, then contracting before nature chooses θ has no value.

Mechanisms: Although it is natural to think of the monopolist offering the consumer a deterministic pricing schedule, $P(q) : \mathcal{Q} \to \mathbb{R}_+$, or even a single price, $P(q) = pq$, we want to begin more generally (and abstractly) by allowing the seller to offer something even more general which could include a complex negotiation game, some randomization, etc. Specifically, we will allow the seller to design and offer the buyer an extensive form game, Γ, to play and a suggested equilibrium strategy for the buyer to play. (We assume that when the buyer is indifferent between strategies, the buyer is willing to play the equilibrium suggested by the seller.)

Throughout we assume that the seller can fully commit to her strategy in the game in advance, σ_s^*, at every decision node where the seller is called upon to act. As such, from the point of view of the agent, the resulting mechanism is nothing more than a single-person decision problem – a decision tree to be solved. When we think about multi-agent settings in later lectures, this will no longer be the case because a mechanism will generate a game played by the different agents. For these initial lectures on single-agent screening contracts, however, "optimal mechanism design" is really about "optimal contract design" because the seller is not designing an n-person game. Still, we will find we can export many of the techniques developed in the one-agent setting to more complex multi-agent settings.

Notice that the seller's set of available mechanisms (or contracts) includes the case of the seller committing to a posted unit price, p, or a nonlinear pricing schedule, $P(q)$. It may also include mechanisms that are more profitable than simple prices, which is the point of introducing the generality. Unfortunately, the space of available mechanisms (i.e., single-person, extensive-form decision problems) is huge and the seller's domain of optimization looks impossibly complicated at the outset. Fortunately, while the space of mechanisms is huge, the space of allocation functions associated with the mechanisms is easily characterized and presents a domain over which the principal can optimize.

To be specific, consider a mechanism (i.e., an extensive-form game between the principal and the agent) Γ. Let σ_s^* be the seller's committed strategy, and let σ_b^* be a buyer's equilibrium strategy in this game. The buyer's strategy in Γ will be a mapping from type, $\theta \in \Theta$, to some set of allowable actions in Γ at each decision node. The equilibrium outcome will therefore be a distribution over the final allocation, $y = (q, t) \in \mathcal{Y} \equiv \mathcal{Q} \times \mathbb{R}$, that is conditional on θ. (Often times, the outcome will be deterministic, but we allow for mixed strategy equilibria at this stage of generality.) Thus, for any Γ and equilibrium strategy $\{\sigma_b^*, \sigma_s^*\}$, there exists an equilibrium (conditional) distribution: $\phi^*(q, t|\theta)$ over final allocations, $y \in \mathcal{Y} = \mathcal{Q} \times \mathbb{R}$. Notice that while the space of Γ mechanisms is huge (each requiring a description of the entire game and its strategies), the space of equilibrium distributions is a simpler notion. It is the space of conditional distributions on $\mathcal{Q} \times \mathbb{R}$ indexed by $\theta \in \Theta$.

We will find it a great simplification to work with this space of allocations (mappings from types to outcomes). This projection into a simpler space is known as the revelation principle. We need a few definitions before we can present and prove the result. The first of which is a subset of mechanisms which the agent reports his type, $\hat{\theta}$, (possibly with dishonesty) and the seller commits to implementing some conditional allocation.

Definition 1. *A* **direct mechanism** *is a game in which the agent reports a type, $\hat{\theta} \in \Theta$, and the outcome $y = (q, t)$ is allocated (possibly randomly) as a function of report. Thus, a direct mechanism is conditional distribution, $\phi(y|\theta)$, that maps from the agent's type to probability distributions over $\mathcal{Y} = \mathcal{Q} \times \mathbb{R}$. A* **deterministic direct mechanism** *is an allocation, $y(\cdot) = \{q(\cdot), t(\cdot)\}$, that maps from the agent's type to $\mathcal{Y} = \mathcal{Q} \times \mathbb{R}$:*

$$q(\cdot) : \Theta \to \mathcal{Q}, \ \ \textit{and} \ \ t(\cdot) : \Theta \to \mathbb{R}.$$

In most of our applications, we will restrict attention to deterministic mechanisms and then verify that this is without loss of generality. When we do so, we will refer to $q(\cdot)$ as the *decision rule* and $t(\cdot)$ as the *transfer rule*.

Examples: A simple example of a deterministic, but not direct, mechanism is a posted price, p. In the extensive form, Γ, at date 1 the seller posts a price, p, at which the buyer is allowed to purchase one unit of output; at date 2, the buyer can choose to purchase the good at that price, or not. As an aside, note that if $u(q,\theta) = \theta q$, it is optimal for buyers with $\theta \geq p$ to purchase and for those with $\theta < p$ to not. Thus, in this mechanism, the equilibrium allocation is given by

$$q^*(\theta) = \begin{cases} 1 & \text{if } \theta \geq p, \\ 0 & \text{if } \theta < p \end{cases}$$

$$t^*(\theta) = \begin{cases} p & \text{if } \theta \geq p, \\ 0 & \text{if } \theta < p. \end{cases}$$

Compare the above indirect mechanism Γ (a posted price game) and its corresponding equilibrium allocation, $\{q^*(\cdot), t^*(\cdot)\}$, to a *direct* mechanism, $\tilde{\Gamma}$, in which the seller offers at date 1 the following:

$$q(\hat{\theta}) = \begin{cases} 1 & \text{if } \hat{\theta} \geq p, \\ 0 & \text{if } \hat{\theta} < p \end{cases}$$

$$t(\hat{\theta}) = \begin{cases} p & \text{if } \hat{\theta} \geq p, \\ 0 & \text{if } \hat{\theta} < p. \end{cases}$$

At date 2, the agent decides what type to report, $\hat{\theta} \in \Theta$. (We will generally use the $\hat{\theta}$ notation to reflect the agent's (possibly dishonest) report into the direct mechanism.) It is required that $\hat{\theta} \in \Theta$, but the agent could otherwise lie about his type if the seller doesn't take precautions against dishonesty in the design of the mechanism.

There are two interesting features to note in this pair of mechanisms when $u(q,\theta) = \theta q$. First, note that in the direct mechanism, the agent will find reporting truthfully is optimal. Second, notice also that the direct mechanism implements the same allocation (i.e., the same mapping from θ to (q,t)) as the posted price (indirect mechanism) game. This is an example of the revelation principle in operation.

We will state a general version of the revelation principle (allowing for randomization over $\mathcal{Y}$), which specializes to the more useful case of deterministic mechanisms, but for clarity we first present the special case as a separate result:

Proposition 1. (Deterministic) Revelation Principle. *For every mechanism Γ and every equilibrium buyer strategy σ_b^*, with deterministic equilibrium allocation $\{q^*(\theta), t^*(\theta)\}_{\theta\in\Theta}$ in Γ, there is a direct mechanism $\tilde{\Gamma}$ and an equilibrium buyer strategy $\tilde{\sigma}_b^*$, with deterministic equilibrium allocation $\{\tilde{q}^*(\theta), \tilde{t}^*(\theta)\}_{\theta\in\Theta}$ in $\tilde{\Gamma}$, such that*

1. *($\tilde{\sigma}_b^*$ exhibits truth-telling):*

$$\tilde{\sigma}_b^*(\theta) = \theta \;\; \textit{for all } \theta \in \Theta,$$

2. *($\tilde{\sigma}_b^*$ induces an identical equilibrium allocation):*

$$\tilde{q}^*(\theta) = q^*(\theta) \;\; \textit{and} \;\; \tilde{t}^*(\theta) = t^*(\theta) \;\; \textit{for all } \theta \in \Theta.$$

Proof: For any Γ and any equilibrium allocation $\{q^*(\theta), t^*(\theta)\}_{\theta\in\Theta}$, the seller can commit to offering the same equilibrium allocation as a function of reported types rather than actual types. [Recall our seller has full commitment power.] Given that $\{q^*(\cdot), t^*(\cdot)\}$ is an equilibrium distribution in Γ, it must be that buyer type θ does better playing the strategy $\sigma_b^*(\cdot|\theta)$ and obtaining $\{q^*(\theta), t^*(\theta)\}$ than playing the optimal strategy for type $\hat{\theta}$ and obtaining $\{q^*(\hat{\theta}), t^*(\hat{\theta})\}$. Thus,

$$u(q^*(\theta), \theta) - t^*(\theta) \geq u(q^*(\hat{\theta}), \theta) - t^*(\hat{\theta}), \quad \text{for all } \theta, \hat{\theta} \in \Theta.$$

Consequently, it is an equilibrium in $\tilde{\Gamma}$ to report truthfully, $\tilde{\sigma}_b^*(\theta) = \theta$. Having reported truthfully, the same allocation arises in the truth-telling equilibrium of $\tilde{\Gamma}$ as in the direct mechanism as in Γ. □

A revelation principle for the larger class of non-deterministic equilibrium allocations also holds using the same logic.

Proposition 2. Revelation Principle. *For every mechanism Γ and every equilibrium buyer strategy σ_b^*, with equilibrium allocation distribution $\phi^*(y|\theta)$ in Γ, there is a direct mechanism $\tilde{\Gamma}$ and an equilibrium buyer strategy $\tilde{\sigma}_b^*$, with equilibrium allocation distribution $\tilde{\phi}^*(y|\theta)$ in $\tilde{\Gamma}$, such that*

1. *($\tilde{\sigma}_b^*$ exhibits truth-telling):*

$$\tilde{\sigma}_b^*(\theta) = \theta \;\; \textit{for all } \theta \in \Theta,$$

2. *($\tilde{\sigma}_b^*$ induces an identical equilibrium allocation):*

$$\tilde{\phi}^*(y|\theta) = \phi^*(y|\theta), \;\; \textit{for all } y \in \mathcal{Y} \textit{ and } \theta \in \Theta.$$

Proof: For any Γ and equilibrium distribution $\phi^*(y|\theta)$, the seller commits designs a direct mechanism that offers the same equilibrium distribution, $\phi^*(y|\hat{\theta})$, as a function of reported types rather than actual types. Given that $\phi^*(y|\theta)$ is an equilibrium distribution in Γ, it must be that buyer type θ does better playing the strategy $\sigma_b^*(\cdot|\theta)$ than the optimal strategy for type $\hat{\theta}$, $\sigma_b^*(\cdot|\hat{\theta})$. Thus,

$$E_{\phi^*(\cdot|\theta)}[u(q,\theta)-t] \geq E_{\phi^*(\cdot|\hat{\theta})}[u(q,\theta)-t], \quad \text{for all } \theta, \hat{\theta} \in \Theta.$$

Consequently, it is an equilibrium in $\tilde{\Gamma}$ to report truthfully, $\tilde{\sigma}_b^*(\theta) = \theta$. Having reported truthfully, the same distribution over allocations arises in the truth-telling equilibrium of $\tilde{\Gamma}$ as in the direct mechanism as in Γ. □

Remarks:

- When do we use the deterministic revelation principle and when do we use the general revelation principle? Often times it is not natural to restrict attention to deterministic mechanisms and we will want to consider the larger class. Consider the simple example where $u(q,\theta) = \theta q$ and $q \in \{0,1\}$. Because the buyer is risk neutral, distributions over (q,t) can be reduced to distributions over q; that is, t can be made deterministic. Thus, the space of relevant allocations are distributions over $\{0,1\}$ which can be thought of as a probability $\phi(\theta) \in [0,1]$ of getting the good. We are therefore interested in direct mechanisms of the form $\phi : \Theta \to [0,1]$ and $t : \Theta \to \mathbb{R}$. By finding the optimal selling mechanism among the class of direct mechanisms, we will have solved the seemingly impossible problem of finding the best extensive-form selling game for the seller. We will also see that the optimal mechanism in this class is deterministic and the seller can do no better than committing to an optimal posted price.
- Sometimes, optimizing over the larger class of random mechanisms is difficult so we restrict attention to deterministic mechanisms. Consider, for example, the case where $u(q,\theta)$ is a more general functional form and $q \in \mathcal{Q} = [0,\overline{q}]$. Because the buyer is risk neutral, it is again without loss of generality to make t a deterministic function of type. Thus, the space of relevant allocations are distributions $\phi(\cdot|\theta)$ over $\mathcal{Q}$. This is a far more complicated domain for our optimization program than the previous example. Instead, we proceed by restricting attention to deterministic direct mechanisms of the form $q : \Theta \to \mathcal{Q}$ and $t : \Theta \to \mathbb{R}$, and then determine ex post if random mechanisms can improve the profit of the seller. Under some very reasonable conditions, we will see, the deterministic mechanisms will be optimal within the larger class.
- More general formulations of the revelation principle by Myerson allow agents to take actions as well. That is, y has some components which are under the agents' control and some components which are under the principal's control. A revelation principle still holds in which the principal implements y by choosing the components it controls and making recommendations to the agents as to which actions they should take that are under their control. Truthful revelation occurs in equilib-

rium and suggestions are followed. Myerson refers to this as "truth telling" and "obedience," respectively.

2.1 The optimal deterministic mechanism

We begin by finding the seller's optimal deterministic mechanism. Using the revelation principle, the seller can restrict her attention to the space of deterministic direct mechanisms in which truth telling is induced in equilibrium. Consequently, we can write the seller's program for the optimal deterministic mechanism, $\{q(\cdot), t(\cdot)\}$ as follows:

Program 1.

$$\max_{\{q(\cdot),t(\cdot)\}} E_\theta \left[t(\theta) - C(q(\theta))\right],$$

subject to

$$\theta \in \arg\max_{\hat{\theta}\in\Theta} \; u(q(\hat{\theta}), \theta) - t(\hat{\theta}), \; \forall \theta \in \Theta, \quad \textit{(IC)}$$

$$u(q(\theta), \theta) - t(\theta) \geq 0, \; \forall \theta \in \Theta, \quad \textit{(interim IR).}$$

Remark on IR constraint: Notice that if the agent has an outside option, say $\underline{U}(\theta)$, such that the agent's IR constraint is

$$u(q(\theta), \theta) - t(\theta) \geq \underline{U}(\theta),$$

we could define a new utility function $\tilde{u}(q, \theta) = u(q, \theta) - \underline{U}(\theta)$ and embed the (IR) constraint into this net utility function and the IR constraint can be written

$$\tilde{u}(q(\theta), \theta) - t(\theta) \geq 0.$$

Consequently, using an outside option of 0 is without loss of generality. Be careful, however, with the assumption that $u_\theta > 0$. If the outside option is type dependent, this simple assumption may no longer be obvious. In our present context, we assume that the agent's outside option is type independent, and so $u_\theta > 0$ is a natural assumption.

2.1.1 Incentive compatibility and implementable allocations

To be clear about the incentive compatibility and the related notion of implementability, we will make a few definitions.

Definition 2. *For any direct mechanism $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$, define the agent's* **indirect utility function** *from truth telling as*

$$U(\theta) \equiv u(q(\theta), \theta) - t(\theta).$$

Definition 3. *We say that a direct mechanism, $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$, is* **incentive compatible (IC)** *iff for all $\theta \in \Theta$*

$$U(\theta) \geq U(\hat{\theta}|\theta) \equiv u(q(\hat{\theta}), \theta) - t(\hat{\theta}), \; \forall \theta, \hat{\theta} \in \Theta.$$

Definition 4. *We say that an output-utility allocation, $\{q(\cdot), U(\cdot)\}_{\theta\in\Theta}$ is* **implementable** *if for transfer function*

$$t(\theta) = u(q(\theta), \theta) - U(\theta),$$

the direct mechanism $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$ is incentive compatible.

Remark: Note that for a given direct mechanism, $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$, the associated indirect utility function is immediately determined; similarly, for any implementable allocation $\{q(\cdot), U(\cdot)\}_{\theta\in\Theta}$, the associated transfer function is immediately determined. Thus, we can maximize over the set of incentive-compatible direct mechanisms, or we can change our domain and maximize over the set of implementable allocations. We will do the latter shortly.

Incentive compatibility (and implementability) constraints are challenging to work with in their current form, so we seek an alternative representation. First, we look for a set of necessary conditions for $\{q, U\}$ to be an implementable allocation. We then establish that these are sufficient.

Lemma 1. *If a direct mechanism $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$ is incentive compatible, then $q(\cdot)$ is nondecreasing and*

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)ds. \tag{1}$$

Proof: Incentive compatibility requires for any θ and θ' that

$$U(\theta) \geq u(q(\theta'), \theta) - t(\theta') = U(\theta') + u(q(\theta'), \theta) - u(q(\theta'), \theta').$$

Simplifying,

$$U(\theta) - U(\theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta').$$

Similarly, incentive compatibility requires

$$U(\theta') \geq u(q(\theta), \theta') - t(\theta) = U(\theta) + u(q(\theta), \theta') - u(q(\theta), \theta),$$

which after simplification yields

$$u(q(\theta), \theta) - u(q(\theta), \theta') \geq U(\theta) - U(\theta').$$

Putting these inequalities together, we have

$$u(q(\theta), \theta) - u(q(\theta), \theta') \geq U(\theta) - U(\theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta').$$

By the assumption that $u_{q\theta} > 0$, the fact that

$$u(q(\theta), \theta) - u(q(\theta), \theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta')$$

implies that $q(\theta) \geq q(\theta')$ for all $\theta > \theta'$. Hence, q is nondecreasing.

Without loss of generality, take $\theta > \theta'$ and divide by $(\theta - \theta')$. We now have

$$\frac{u(q(\theta),\theta) - u(q(\theta),\theta')}{\theta - \theta'} \geq \frac{U(\theta) - U(\theta')}{\theta - \theta'} \geq \frac{u(q(\theta'),\theta) - u(q(\theta'),\theta')}{\theta - \theta'}.$$

Because u is twice differentiable, the left and righthand side limits exist as $\theta \to \theta'$, and thus U has well-defined left and right derivatives. At any point where $q(\theta)$ is continuous, these derivatives coincide and the derivative of $U(\theta)$ is $u_\theta(q(\theta),\theta)$.

Lastly, note that

$$U(\theta) \equiv \max_{\hat{\theta}\in\Theta} u(q(\hat{\theta}),\theta) - t(\hat{\theta})$$

will be Lipschitz continuous given that $u(q,\theta)$ is Lipschitz continuous on Θ.[3] Lipschitz continuity implies absolute continuity. Because U is absolutely continuous, the fundamental theorem of calculus is valid and we can write

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s),s)ds.$$

□

Remarks:

- Lemma 1 gives the necessary condition as an integral equation. An equivalent statement (given that $U(\theta)$ is absolutely continuous) is

  $$U'(\theta) = u_\theta(q(\theta),\theta), \text{ for a.e. } \theta \in \Theta.$$

  In this form it is perhaps clearer that the property is an envelope condition. That is,

  $$U'(\theta) = \frac{d}{d\theta}u(q(\theta),\theta) - t(\theta) = u_\theta(q(\theta),\theta) + u_q(q(\theta),\theta)q'(\theta) - t'(\theta) = u_\theta(q(\theta),\theta),$$

[3] To see this, note that because $U(\hat{\theta}) \leq U(\theta|\hat{\theta})$, we have

$$U(\theta) - U(\hat{\theta}) \leq U(\theta) - U(\theta|\hat{\theta}).$$

Thus,

$$U(\theta) - U(\hat{\theta}) \leq u(q(\theta),\theta) - u(q(\theta),\hat{\theta}).$$

Because u is Lipschitz continuous on Θ, there exists a K such that for all $\theta, \hat{\theta}$,

$$U(\theta) - U(\hat{\theta}) \leq u(q(\theta),\theta) - u(q(\theta),\hat{\theta}) \leq |u(q(\theta),\theta) - u(q(\theta),\hat{\theta})| \leq K|\theta - \hat{\theta}|.$$

Using a similar argument,

$$U(\hat{\theta}) - U(\theta) \leq u(q(\hat{\theta}),\hat{\theta}) - u(q(\hat{\theta}),\theta) \leq |u(q(\hat{\theta}),\hat{\theta}) - u(q(\hat{\theta}),\theta)| \leq K|\theta - \hat{\theta}|.$$

Together

$$|U(\hat{\theta}) - U(\theta)| \leq K|\theta - \hat{\theta}|.$$

Hence, U is Lipschitz continuous on Θ.

where the final inequality is due to the fact that incentive compatibility implies the second and third terms sum to zero almost everywhere. Hence, the condition is akin to a first-order condition for incentive compatibility.

- The monotonicity condition can be thought of as a second-order condition for incentive compatibility.

- Here's a sloppy, but general, way to focus on the correct incentive compatibility conditions. Suppose that the direct deterministic mechanism, more generally, is some $y(\cdot)$ which may be a vector of transfer, actions, etc. Suppose that the utility function is simply $v(y, \theta)$, and that includes all aspects of y, including transfers. Then we can write the agent's utility as a function of reported $\hat{\theta}$ and true θ as simply

$$U(\hat{\theta}|\theta) = v(y(\hat{\theta}), \theta).$$

  Assuming $U(\hat{\theta}|\theta)$ is differentiable (this is the sloppy part), the first-order condition for incentive compatibility is

$$U_1(\theta|\theta) = 0.$$

  Thus,

$$\frac{d}{d\theta}U(\theta|\theta) = U_1(\theta|\theta) + U_2(\theta|\theta) = U_2(\theta|\theta) = v_\theta(y(\theta), \theta).$$

  What about the second-order condition? Differentiate $U_1(\hat{\theta}|\theta) = 0$ totally with respect to θ:

$$U_{11}(\theta|\theta) + U_{12}(\theta|\theta) = 0.$$

  If $U_{12}(\theta|\theta) \geq 0$, then $U_{11}(\theta|\theta) \leq 0$, which is the local second-order condition. Hence, the local second-order condition can be stated as

$$U_{12}(\theta|\theta) = \frac{\partial}{\partial\hat{\theta}}v_\theta(y(\hat{\theta}), \theta)\bigg|_{\hat{\theta}=\theta} \geq 0.$$

  In the case of $v(q, t, \theta) = u(q, \theta) - t$, we have $U_{12}(\theta|\theta) = u_{q\theta}(q(\theta), \theta)q'(\theta) \geq 0$. Hence, the single-crossing property and monotonicity in q guarantees the local second-order condition is satisfied.

- In some important settings such as auctions, we will assume $u(q, \theta) = \theta q$. In this special case, $U(\theta)$ is a convex function. This can be seen by noting that

$$U(\theta) = \max_{\hat{\theta}} \ \theta q(\hat{\theta}) - t(\hat{\theta}),$$

  which is the maximum of a family of affine functions (indexed by $\hat{\theta}$). The upper envelope of a family of affine functions is convex. In general, a convex function is differentiable almost everywhere; in the present case, at all points of differentiability, $U(\theta)$, has derivative $q(\theta)$. Because U is convex, $q(\cdot)$ is necessarily nondecreasing. In these circumstances, the necessary conditions in Lemma 1 are often written as (i) $U(\theta)$ is convex and (ii) $q(\theta) \in \partial U(\theta)$, where the latter denotes that q is in the subdifferential of $U(\theta)$ (i.e., it is a tangent).

In multi-dimensional settings, bilinearity between $q \in \mathcal{Q} \subseteq \mathbb{R}^k$ and $\theta \in \Theta \subseteq \mathbb{R}^k$ also gives rise to convexity:

$$U(\theta) = \max_{\hat{\theta} \in \Theta} \; q(\hat{\theta})' \cdot \theta - t(\hat{\theta}),$$

is a convex function. In this setting, a decision rule is implementable if and only if there exists a convex function $U(\theta)$ such that $q(\theta)$ is a subgradient at θ. (See Chone and Rochet (1998) or Borgers (2015), Proposition 5.3.) An example of such a setting is where there are k alternatives, each component of θ gives the value of alternative i, and q is a probability distribution over the alternatives (i.e., $\mathcal{Q} = [0,1]^k$).

Revenue equivalence. Because $t(\theta) = u(q(\theta), \theta) - U(\theta)$, an immediate consequence of the necessary conditions in Lemma 1 is that the seller's revenue, $t(\theta)$, in any incentive compatible mechanism, is entirely determined by the the allocation $q(\cdot)$ and the utility of the lowest type agent, $U(\underline{\theta})$. This is known as the *revenue equivalence theorem* for the single-agent setting. The more famous version of this result will appear when we study optimal auctions.

Corollary 1. Revenue equivalence. *For any direct mechanism, $\{q(\theta), t(\theta)\}_{\theta \in \Theta}$, that is incentive compatible,*

$$t(\theta) = u(q(\theta), \theta) - U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s) ds.$$

We now find, rather remarkably, that the two necessary conditions in Lemma 1 are also sufficient.

Lemma 2. *If $q(\cdot)$ is nondecreasing and*

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s) ds, \;\; \textit{for all } \theta \in \Theta,$$

then $\{q(\theta), U(\theta)\}_{\theta \in \Theta}$ is implementable with transfer function $t(\theta) = u(q(\theta), \theta) - U(\theta)$.

Proof: It is sufficient to show that $U(\theta) \geq u(q(\theta'), \theta) - t(\theta')$ for all $\theta, \theta' \in \Theta$, using $t(\theta) = u(q(\theta), \theta) - U(\theta)$:

$$\begin{aligned} U(\theta) &\geq u(q(\theta'), \theta) - t(\theta'), \;\forall \theta, \theta' \in \Theta \\ U(\theta) &\geq u(q(\theta'), \theta) - u(q(\theta'), \theta') + U(\theta'), \;\forall \theta, \theta' \in \Theta \\ \iff U(\theta) - U(\theta') &\geq u(q(\theta'), \theta) - u(q(\theta'), \theta'), \;\forall \theta, \theta' \in \Theta \\ \int_{\theta'}^{\theta} u_\theta(q(s), s) ds &\geq \int_{\theta'}^{\theta} u_\theta(q(\theta'), s) ds. \end{aligned}$$

Suppose $\theta > \theta'$. Then q nondecreasing implies $q(s) \geq q(\theta')$ for all $s \in [\theta', \theta]$. Consequently, $u_{q\theta} > 0$ implies $u_\theta(q(s), s) \geq u_\theta(q(\theta'), s)$ for all $s \in [\theta', \theta]$ and the inequality is satisfied. A symmetric argument holds for $\theta' > \theta$. □

Combining our previous two lemmata, we have our important characterization result.

Proposition 3. *(Incentive compatible mechanisms and implementable allocations.)*

- *A direct mechanism $\{q(\cdot), t(\cdot)\}_{\theta\in\Theta}$ is incentive compatible if and only if $q(\cdot)$ is nondecreasing and*
$$t(\theta) = u(q(\theta), \theta) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)ds.$$
- *An output-utility allocation $\{q(\cdot), U(\cdot)\}_{\theta\in\Theta}$ is implementable if and only if $q(\cdot)$ is nondecreasing and*
$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)ds.$$

2.1.2 Seller's program with revised IC constraints

Given our result in Proposition 3, we can simplify our seller's program. We will do this by also writing the program in terms of output-utility allocations, $\{q(\cdot), U(\cdot)\}$, instead of $\{q(\cdot), t(\cdot)\}$. Upon substitution, the seller's program can be restated:

$$\max_{\{q(\cdot), U(\cdot)\}} E_\theta \left[u(q(\theta), \theta) - C(q(\theta)) - U(\theta)\right],$$

subject to

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)ds,\ \forall\theta\in\Theta,$$
$$q(\cdot) \text{ nondecreasing in } \theta$$
$$U(\theta) \geq 0,\ \forall\theta\in\Theta,\quad \text{(IR)}.$$

Remarks:

- Notice that we have converted profits into total surplus minus consumer surplus. It is the same thing, of course.
- The first step to solving the program is to note is that $u_\theta > 0$ implies that (IR) is satisfied if $U(\underline{\theta}) \geq 0$. Looking at the above constraints, it is clear that setting $U(\underline{\theta}) = 0$ is optimal for the seller (it maximizes profit while satisfying IR).

Substituting $U(\underline{\theta}) = 0$ and replacing $U(\theta)$ in the objective function with its defining integral, we have the following program:

$$\max_{\{q(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left(u(q(\theta), \theta) - C(q(\theta) - \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)ds\right) f(\theta)d\theta,$$

subject to $q(\cdot)$ nondecreasing in θ.

As last trick is to simplify the inside integral. To be clear, notice that

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} u_\theta(q(s), s) ds \right) f(\theta) d\theta$$

$$= \left(\int_{\underline{\theta}}^{\theta} u_\theta(q(s), s) ds \right) (F(\theta) - 1) \Big|_{\theta=\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} u_\theta(q(\theta), \theta)(F(\theta) - 1) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} u_\theta(q(\theta), \theta)(1 - F(\theta)) d\theta.$$

Thus, the seller's program becomes

Program 2.

$$\max_{\{q(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left(u(q(\theta), \theta) - C(q(\theta) - \frac{1 - F(\theta)}{f(\theta)} u_\theta(q(\theta), \theta \right) f(\theta) d\theta,$$

subject to $q(\cdot)$ nondecreasing in θ.

To solve this program, we could use control-theory techniques to impose monotonicity on q. (We will return to this approach later.) Instead, we introduce a set of assumptions which guarantees that relaxed program which ignores monotonicity will produce an allocation $q(\cdot)$ that is nondecreasing.

Remark on "virtual utility": Myerson coined the phrase **virtual utility** as a way of thinking about the constrained-optimal optimization program. It is "as if" the agent's utility is

$$\tilde{u}(q, \theta) = u(q, \theta) - \frac{1 - F(\theta)}{f(\theta)} u_\theta(q(\theta),$$

and the seller's optimal program is to maximize bilateral surplus. Hence, $\tilde{u}$ is known as the virtual utility of the agent and $\frac{1-F(\theta)}{f(\theta)} u_\theta(q(\theta)$ is often referred to as the agent's (expected) **information rent**. In the simple setting where the agent's preferences are $u(q, \theta) = \theta q$ and $q \in \{0, 1\}$, then we have

$$\tilde{u}(q, \theta) = \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) q,$$

in which case $\theta - \frac{1-F(\theta)}{f(\theta)}$ is often referred to as the agent's **virtual type**.

Definition 5. *We say that a screening problem defined by $\{u, C, F\}$ is* **regular** *if the function*

$$\Lambda(q, \theta) \equiv u(q(\theta), \theta) - C(q(\theta)) - \frac{1 - F(\theta)}{f(\theta)} u_\theta(q(\theta)$$

is twice continuously differentiable, strictly quasi-concave over $q \in \mathcal{Q}$, and $\Lambda_{q\theta}(q, \theta) \geq 0$.

Remarks:

- "Regularity" ensures that the pointwise optimum over q defined by

$$\Lambda_q(q(\theta), \theta) = 0, \ \forall \theta \in \Theta,$$

is a pointwise global maximum of Λ and the corresponding function $q(\cdot)$ is nondecreasing:

$$q'(\theta) = -\frac{\Lambda_{q\theta}(q(\theta), \theta)}{\Lambda_{qq}(q(\theta), \theta)} \geq 0.$$

- In the literature, the assumption of regularity is often made by a series of assumptions, first on u and second on F. For example, suppose that u is quadratic and the inverse hazard rate

$$H(\theta) \equiv \frac{1 - F(\theta)}{f(\theta)}$$

is decreasing in θ. Then the program is regular.

- The condition that the inverse hazard rate is monotone is known as the *monotone hazard rate condition*.

Definition 6. *A distribution F satisfies the* **monotone-hazard rate condition (MHRC)** *iff*

$$H(\theta) = \frac{1 - F(\theta)}{f(\theta)} \ \text{ is nonincreasing in } \theta.$$

In some models (e.g., models in which cost is the private information), it is natural to assume $u_{q\theta} < 0$ and $u_\theta < 0$. In such models, the relevant hazard-rate condition is reversed and we typically assume $\frac{F(\theta)}{f(\theta)}$ is nondecreasing. This variation is also referred to as the monotone hazard rate condition, so some care must be exercised when interpreting the phrase. For both variations, the monotone-hazard-rate condition is not very restrictive and is satisfied for a large number of distributions, including normal, exponential and uniform.

We can now state our main result.

Proposition 4. *Suppose that the screening problem is regular and let $q(\theta)$ be the pointwise solution to*

$$u_q(q(\theta), \theta) - C_q(q(\theta)) = \frac{1 - F(\theta)}{f(\theta)} u_{q\theta}(q(\theta)), \ \forall \theta \in \Theta.$$

If $q(\theta) \in \mathcal{Q}$ for all $\theta \in \Theta$, then $q(\cdot)$ is a solution to the seller's program. The corresponding utility for the agent is

$$U(\theta) = \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s) ds,$$

and the direct transfer function is

$$t(\theta) - u(q(\theta), \theta) - U(\theta).$$

Remarks:

- If the solution to $\Lambda_q(q(\theta), \theta)$ violates the required range, $q : \Theta \to \mathcal{Q}$, then $q(\theta) = \underline{q}$ for lower bound violations and $q(\theta) = \overline{q}$ for upper bound violations.
- Existence. Providing the program is regular, the pointwise solution will exist. Recall that in the moral hazard setting, the source of the existence problem was that the choice of $w(x)$ was a pointwise optimum for a given pair of multipliers (λ, μ), but the multipliers were solutions to *global* complementary slackness conditions. In the present case, there are no global constraints other than the monotonicity constraint on q, and regularity assumption guarantees that this constraint is pointwise slack.
- Interpretation. It is insightful to write the first-order condition as

  $$[u_q(q(\theta), \theta) - C_q(q(\theta))]\, f(\theta) = [1 - F(\theta)] u_{q\theta}(q(\theta)).$$

  The lefthand side represents the social benefit from a marginal increase in q for type θ. The benefit from such a variation is weighted by $f(\theta)$. On the righthand side, however, is the cost of such an increase in the efficiency of type-θ's consumption - all higher types must be given a similar marginal increase in consumption; $u_{q\theta}$ indicates the marginal surplus increase to types above θ, which have mass $1 - F(\theta)$. Thus, inefficiencies in consumption are evaluated against increased rents to consumers.
- Notice that for $\theta = \overline{\theta}$, there is no distortion in consumption: $q(\theta) = q^{fb}(\theta)$, where we define $q^{fb}(\theta)$ as the solution to $u_q(q^{fb}(\theta), \theta) = C_q(q^{fb}(\theta))$. This is frequently referred to as the "no distortion at the top" screening result.
- Simple monopoly analogue. Return to our setting where $u(q, \theta) = \theta q$ and $q \in \{0, 1\}$, and suppose that the monopolist seller is using a posted price p (rather than a more complex mechanism) to sell the item. The demand from the consumer is given by $D(p) = 1 - F(p)$ (i.e., types $\theta \geq p$ buy at price p). The monopolist will choose p to maximize $(1 - F(p))(p - c)$. The first order condition can be written

  $$(p - c) f(p) = 1 - F(p).$$

  This is exactly the same tradeoff as in the general mechanism design problem. Indeed, we will shortly demonstrate that a posted price, in this setting, is the optimal mechanism.

2.1.3 Optimal pricing for unit demands

We now return to our simplest setting, where $u(q, \theta) = \theta q$, $C(q) = cq$, and $q \in \mathcal{Q} = \{0, 1\}$. First, note that we can define $\phi \in [0, 1]$ to be the probability of getting the good and hence our space of mechanisms allows for randomization. That is, we do *not* restrict attention to deterministic mechanisms. In this case, the seller chooses $\phi \in [0, 1]$ to maximize

$$\Lambda(\phi, \theta) = \phi \left(\theta - c - \frac{1 - F(\theta)}{f(\theta)} \right).$$

Notice that if the distribution F satisfies the monotone-hazard rate condition, then the program is regular. We can apply Proposition 4 to conclude that the optimal $\phi(\cdot)$ is

$$\phi(\theta) = \begin{cases} 1 & \text{if } \theta - c \geq \frac{1-F(\theta)}{f(\theta)} \\ 0 & \text{if } \theta - c < \frac{1-F(\theta)}{f(\theta)}. \end{cases}$$

Hence, the deterministic mechanism is optimal and we can use $q(\theta) = \phi(\theta) \in \{0, 1\}$!

Define θ^* as the unique type for which

$$\theta^* - c = \frac{1 - F(\theta^*)}{f(\theta^*)}.$$

We can now write the corresponding indirect utility function as

$$U(\theta) = \int_{\underline{\theta}}^{\theta} q(s)ds = \begin{cases} \theta - \theta^* & \text{if } \theta \geq \theta^* \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding optimal transfer function $t(\theta) = \theta q(\theta) - U(\theta)$ is

$$t(\theta) = \begin{cases} \theta^* & \text{if } \theta \geq \theta^* \\ 0 & \text{otherwise.} \end{cases}$$

Now that we have found the optimal direct mechanism which solves the seller's program, an obvious question is what indirect mechanism that we see in the real world accomplishes the same thing. Here, it is easy: the monopolist posts the price $p = \theta^*$ and the buyer purchases a unit if and only if $\theta > p$. So at the end of a lot of work, we are back to our simple monopoly program of picking a price. That said, we have also proven in this setting that the seller can do nothing better (e.g., randomization cannot improve profits).

2.1.4 Optimal nonlinear pricing for general preferences, $u(q, \theta)$

We now return to the more general monopoly setting in which $u(q, \theta)$ is not bilinear in q and θ and $\mathcal{Q} = [\underline{q}, \overline{q}]$. Assuming that the program is regular and the unconstrained solution to $\Lambda_q(q(\theta), \theta) = 0$ lies within the range $\mathcal{Q}$, Proposition 4 provide a full characterization of $q(\cdot)$:

$$(u_q(q(\theta), \theta) - C_q(q(\theta)))f(\theta) = (1 - F(\theta))u_{q\theta}(q(\theta)).$$

As before, we can use the resulting function $q(\cdot)$ to determine $U(\cdot)$, and then we can construct $t(\theta) = u(q(\theta), \theta) - U(\theta)$ and obtain the profit-maximizing, incentive-compatible deterministic direct mechanism, $\{q(\cdot), t(\cdot)\}$. Is there an indirect mechanism that is something we see in the real world that implements the same allocation?

To this end, we seek to derive a nonlinear price schedule, $P : \mathcal{Q} \to \mathbb{R}_+$, that is the generalization to the previous posted-price result. The idea is that the monopolist offers a schedule or menu of prices, $P(\cdot)$, and the consumer decides which q (if any) to choose. We want to create the price schedule so that it implements the same output-utility allocation, $\{q(\cdot), U(\cdot)\}$, as in the direct mechanism. The fact that such a menu can always be constructed is known as the **Taxation principle**, and it is really just the inverse of the revelation principle.

Proposition 5. Taxation principle. *For any incentive compatible (deterministic) direct mechanism, $\{q(\cdot), t(\cdot)\}$, that implements $U(\cdot)$, there exists a nonlinear price function, $P : \mathcal{Q} \to \mathbb{R}$ such that*

$$U(\theta) = \max_{q \in \mathcal{Q}} u(q, \theta) - P(q),$$

$$q(\theta) = \arg\max_{q \in \mathcal{Q}} u(q, \theta) - P(q),$$

Proof: By construction. If $\{q(\cdot), t(\cdot)\}$ is an incentive compatible direct mechanism, then define

$$P(q) = \begin{cases} t(\theta) & \text{if } q = q(\theta) \\ \infty & \text{otherwise.} \end{cases}$$

The buyer has the same set of price-output pairs available under $P(q)$ as with the direct mechanism $\{q(\cdot), t(\cdot)\}$. Given the direct mechanism is incentive compatible, then type θ buys $q(\theta)$. $\square$

A similar taxation principle applies to random direct mechanisms, except that now $P(\phi)$ with be a nonlinear price function over lotteries on $\mathcal{Q}$.

Given functions for $q(\cdot)$ and $t(\cdot)$, it is straightforward to construct $P(q)$ by inverting $q(\cdot)$. Note that $q(\cdot)$ is nondecreasing, and therefore it is invertible almost everywhere. Define the function

$$\vartheta(q) = \max_{\theta}\{\theta \in \Theta \mid q(\theta) = q\}.$$

At any output where $q(\theta)$ is strictly increasing, $\{\theta \in \Theta \mid q(\theta) = q\}$ is a single type. At any q that is chosen by more than one type, $\vartheta(q)$ selects the highest type. Note, however, that all types that are allocated the same q will also pay the same t (otherwise the direct mechanism would not be incentive compatible). Now it is straightforward to construct $P(q)$:

$$P(q) = t(\vartheta(q)).$$

Worked Example: Let's try a numerical example to illustrate each approach. Suppose that $\mathcal{Q} = [0, \overline{q}]$, $C(q) = q$, $u(q, \theta) = \theta q - \frac{1}{2}q^2$ and θ is distributed uniformly on $[1,2]$. Assuming that $\overline{q} > 1$, Proposition 4 implies

$$q(\theta) = \max\{0, 2\theta - 3\}.$$

The buyer's indirect utility function is

$$U(\theta) = \int_1^\theta q(s)ds = \begin{cases} \frac{9}{4} - (3-\theta)\theta & \text{if } \theta \geq \frac{3}{2} \\ 0 & \text{if } \theta < \frac{3}{2} \end{cases} = \max\left\{0, \frac{9}{4} - (3-\theta)\theta\right\}$$

The corresponding direct-mechanism transfer function is

$$t(\theta) = u(q(\theta),\theta) - U(\theta) = \theta\max\{0, 2\theta-3\} - \frac{1}{2}\left(\max\{0, 2\theta-3\}\right)^2 - \max\left\{0, \frac{9}{4} - (3-\theta)\theta\right\}.$$

After much simplification

$$t(\theta) = \begin{cases} (6-\theta)\theta - \frac{27}{4} & \text{if } \theta \geq \frac{3}{2} \\ 0 & \text{if } \theta < \frac{3}{2}. \end{cases}$$

To find $P(q)$, we invert $q(\cdot)$ and obtain, for any $q \geq 0$,

$$\vartheta(q) = \frac{q+3}{2}.$$

Hence,

$$P(q) = \left(6 - \frac{q+3}{2}\right)\left(\frac{q+3}{2}\right) - \frac{27}{4} = \frac{1}{4}(6-q)q.$$

A (marginal) monopoly price-elasticity rule: Suppose that the agent's utility function is linear in θ: $u(q,\theta) = \theta v(q)$. Then we can rearrange the optimality condition

$$(u_q(q(\theta),\theta) - C_q(q(\theta)))f(\theta) = (1 - F(\theta))u_{q\theta}(q(\theta)),$$

and use $u_q(q(\theta),\theta) = P'(q(\theta))$ to obtain

$$\frac{P'(q(\theta)) - C_q(q(\theta))}{P'(q(\theta))} = \frac{1 - F(\theta)}{\theta f(\theta)},$$

which is a marginal variation of the familiar monopoly pricing condition and we can think of the righthand side term as the marginal inverse elasticity of demand, where

$$\varepsilon(q) = \frac{\vartheta(q)f(\vartheta(q))}{1 - F(\vartheta(q))}.$$

Quantity discounts? The nonlinear pricing model above was first presented by Mussa and Rosen (*JET*, 1978) with the assumption that $u(q,\theta) = \theta q$ (and $C(q)$ strictly convex), and was later extended by Maskin and Riley (*RAND Journal*, 1984). One novel question in Maskin and Riley (1984) was whether or not $P(q)$ would exhibit quantity discounts (i.e.,

$P''(q) < 0$). The answer can be found by differentiating our derived expression for $P(q)$ twice. Using procedure 1, $P(q) = t(\vartheta(q)) = u(q, \vartheta(q)) - U(\vartheta(q))$, we have

$$P'(q) = u_q(q, \vartheta(q)) + (u_\theta(q, \vartheta(q)) - U'(\vartheta(q)))\, \vartheta' q = u_q(q, \vartheta(q)),$$

where we have used $U'(\theta) = u_\theta(q(\theta), \theta)$ to conclude that the second term is zero. Differentiating again,

$$P''(q) = u_{qq}(q, \vartheta(q)) + u_{q\theta}(q, \vartheta(q))\vartheta'(q).$$

Substituting for $\vartheta'(q)$ using the defining equation $\Lambda_q(q(\theta), \theta) = 0$, we have

$$P''(q) = u_{qq}(q, \vartheta(q)) + u_{q\theta}(q, \vartheta(q)) \frac{-\Lambda_{qq}(q, \vartheta(q))}{\Lambda_{q\theta}(q, \vartheta(q))}.$$

The sign of this can be determined with additional assumptions on u, C and F. As an example, suppose that $u_{qq\theta}(q, \theta) \leq 0$ and $u_{q\theta\theta}(q, \theta) \leq 0$. (Obviously this holds for quadratic preferences.) Suppose also that F satisfies the monotone hazard rate condition and $C(q) = cq$. Define the inverse hazard rate as $H(\theta) \equiv \frac{1-F(\theta)}{f(\theta)}$ for notational simplicity. With our assumptions,

$$\Lambda_{qq}(q, \theta) = u_{qq}(q, \theta) - H(\theta) u_{qq\theta}(q, \theta),$$

and

$$\Lambda_{q\theta}(q, \theta) = u_{q\theta}(q, \theta)\,(1 - H'(\theta)) - H(\theta) u_{q\theta\theta}(q, \theta).$$

All together now,

$$\begin{aligned} P''(q) &= u_{qq}(q, \vartheta(q)) + u_{q\theta}(q, \vartheta(q))\vartheta'(q) \\ &= u_{qq}(q, \vartheta(q)) + u_{q\theta}(q, \vartheta(q)) \frac{-u_{qq}(q, \vartheta(q)) + H(\vartheta(q)) u_{qq\theta}(q, \vartheta(q))}{u_{q\theta}(q, \vartheta(q))\,(1 - H'(\vartheta(q))) - H(\theta) u_{q\theta\theta}(q, \vartheta(q))} \\ &< 0. \end{aligned}$$

Hence, with quadratic u, constant marginal costs, and a monotone hazard rate, quantity discounts are optimal. Note, however, that in Mussa and Rosen (1978), where $u(q, \theta) = \theta q$ and $C(q)$ is strictly convex, the optimal nonlinear price $P(q)$ is convex!

2.1.5 Two-type models

Sometimes it is convenient to focus on simple models with only two types. I'll present here our nonlinear-pricing model with two types, $\theta \in \{\underline{\theta}, \overline{\theta}\}$, to illustrate how the marginal conditions in the continuously-distributed type case appear as simply inequalities. Later in these notes we generalize the setting to n types to provide a clearer analogue to the continuous case.

Suppose that there are two types of consumers, θ_2 and θ_1, with $\theta_2 > \theta_1$ and a proportion of $\phi \in (0, 1)$ of type θ_2 and $1 - \phi$ of θ_1. For this example, we assume the consumer has a unit demand, $u(q, \theta) = \theta q$, but we interpret q as the quality of the good which costs $c(q)$ per unit to the firm. (This is the setting of Mussa and Rosen (1978).) We will also assume that p is sufficiently small that the firm prefers to sell to both types rather than focus only on the θ_2-type customer.

By the revelation principle, the firm can restrict attention to contracts of the form $\{(q_2, t_2), (q_1, t_1)\}$ such that type θ_2 consumers find it optimal to choose the first contract pair and θ_1 consumers choose the second pair. Thus, we can write the firm's optimization program as:

$$\max_{\{(q_1,t_1),(q_2,t_2)\}} \phi[\theta_2 - c(q_2)] + (1-\phi)[\theta_1 - c(q_1)],$$

subject to

$$\begin{array}{rcll}
\theta_2 q_2 - t_2 & \geq & \theta_2 q_1 - t_1 & (\overline{IC}), \\
\theta_1 q_1 - t_1 & \geq & \theta_1 q_2 - t_2 & (\underline{IC}), \\
\theta_2 q_2 - t_2 & \geq & 0 & (\overline{IR}), \\
\theta_1 h q_1 - t_1 & \geq & 0 & (\underline{IR}),
\end{array}$$

where (IC) refers to an incentive compatibility constraint to choose the relevant contract and (IR) refers to an individual rationality constraint to choose some contract rather than no purchase at all.

To simplify the maximization program facing the firm, we consider the four constraints to determine which – if any – will be binding.

1. The two IC constraints imply

$$\theta_2 \Delta q \geq \Delta t \geq \theta_1 \Delta q,$$

   which further implies that $q_2 \geq q_1$. Hence, the IC constraints imply monotonicity.

2. Note that $\overline{IC}$ and $\underline{IR}$ imply that $\overline{IR}$ is slack. Hence, it will never be binding and we can ignore it.

3. A simple argument establishes that $\overline{IC}$ must always bind. Suppose otherwise and it was slack at the optimal contract offering. In such a case, t_2 could be raised slightly without disturbing this constraint or $\overline{IR}$, thereby increasing profits. Moreover, this increase only eases the $\underline{IC}$ constraint. Hence, the contract cannot be optimal. $\overline{IC}$ binds.

4. If $\overline{IC}$ binds, $\underline{IC}$ must be slack if $q_2 - q_1 \geq 0$ because IC implies

$$\theta_2 \Delta q = \Delta t \geq \theta_1 \Delta q.$$

   Hence, we can ignore $\underline{IC}$ if we impose $q_2 - q_1 \geq 0$.

5. We conclude that we can maximize subject to $\overline{IC}$ and $\underline{IR}$ as equalities and subject to monotonicity; we may ignore $\overline{IR}$ and $\underline{IC}$.

Because we have two constraints satisfied with equalities, we can use them to solve for t_2 and t_1 as functions of q_2 and q_1:

$$\begin{aligned} t_1 &= \theta_1 q_1, \\ t_2 &= t_1 + \theta_2 \Delta q \\ &= \theta_2 q_2 - \Delta\theta q_1. \end{aligned}$$

These t's are necessary and sufficient for all four constraints to be satisfied if $q_2 - q_1 \geq 0$. Substituting for the t's in the firm's objective program, the firm's maximization program becomes simply

$$\max_{\{(q_1,t_1),(q_2,t_2)\}} \phi[\theta_2 q_2 - c(q_2) - \Delta q_1] + (1-\phi)[\theta_1 q_1 - c(q_1)],$$

subject to $q_2 \geq q_1$. Ignoring the monotonicity constraint, the first-order conditions to this relaxed program imply

$$\theta_2 = c'(q_2),$$

$$\theta_1 = c'(q_1) + \frac{\phi}{1-\phi}\Delta\theta.$$

Hence, q_2 is set at the first-best efficient levels of consumption but q_1 is set at sub-optimal levels of consumption. This distortion also implies that $q_2 > q_1$, and hence our monotonicity constraint does not bind.

Having determined the optimal q_2 and q_1, the firm can easily determine the appropriate prices using the conditions for θ_2 and θ_1 above.

3 Technical amendments

3.1 When are deterministic mechanisms optimal?

The final question we must address in the nonlinear pricing context is whether or not the seller gave up expected profit by restricting attention to deterministic mechanisms. That is, if the seller offered a direct-mechanism of distributions, $\{\phi(\cdot|\theta), t(\theta)\}_{\theta \in \Theta}$, where $\phi(\cdot|\theta)$ is a probability distribution over $\mathcal{Q}$, could the seller increase profits? The textbook treatments of these conditions is not very clear and arguably incorrect in places. A clever paper by Strausz (*JET*, 2006) clarified much of the previous confusion and settled the question once and for all in the context of a discrete-type model (i.e., $\Theta = \{\theta_1, \ldots, \theta_n\}$) with quasi-linear preferences. He found that if the screening environment is regular so that the relaxed program delivers a nondecreasing solution, $q(\cdot)$, then a random scheme cannot improve upon a deterministic one. The key insight is that in a quasi-linear world with risk neutral principal and agent, a random scheme can only be valuable if it helps relax a binding monotonicity constraint. (As an aside, random mechanisms are able to implement allocations that are not monotonic in expected terms, hence their potential value. Strausz (2006) gives an example of this value to randomization.)

Note that we need to be a little careful exporting Strausz's claim for a discrete-type environment to our continuous-type setting. Here I will try to extend the result, but I will be a little sloppy with the measure theory. (Please use the following at your own risk.)

We are going to compare the value of four programs, two of which we have already seen. Let V^D be the value of the seller's program when restricted to deterministic mechanisms. Let V^D_{rel} be the value of the seller's program when restricted to deterministic mechanisms and ignoring the monotonicity condition on $q(\cdot)$. In general, $V^D_{rel} \geq V^D$, but in regular environments, $V^D_{rel} = V^D$. Now consider the unrelaxed, general mechanism design program:

$$\max_{\{\phi(\cdot|\cdot),t(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{q}}^{\overline{q}} (t(\theta) - C(q))\, \phi(q|\theta) dq f(\theta) d\theta$$

subject to

$$\int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi(q|\theta)dq - t(\theta) \geq \int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi(q|\theta')dq - t(\theta'), \quad \text{for all } \theta, \theta' \in \Theta, \quad \text{(IC)},$$

$$\int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi(q|\theta)dq - t(\theta) \geq 0, \quad \text{(IR)}.$$

Let the value of this program be V^ϕ. Given the deterministic mechanism is a degenerate case of the general mechanism, $V^\phi \geq V^D$. The question we are asking, of course, is whether or not this inequality is strict.

Lastly, consider a relaxed version of the general program in which only local IC constraints are imposed:

$$\max_{\{\phi(\cdot|\cdot),t(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{q}}^{\overline{q}} (t(\theta) - C(q))\, \phi(q|\theta) dq f(\theta) d\theta$$

subject to

$$\int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi_\theta(q|\theta)dq - t'(\theta) = 0, \quad \text{for almost all } \theta, \in \Theta, \quad \text{(local-IC)},$$

$$\int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi(q|\theta)dq - t(\theta) \geq 0, \quad \text{(IR)}.$$

Denote the value of this program as V^ϕ_{rel}. It follows that $V^\phi_{rel} \geq V^\phi$. Notice that the local IC constraint above is equivalent to requiring

$$U'(\theta) = \int_{\underline{q}}^{\overline{q}} u_\theta(q,\theta)\phi(q|\theta)dq.$$

Thus, an equivalent statement of this relaxed program is

$$\max_{\{\phi(\cdot|\cdot),U(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{q}}^{\overline{q}} (u(q,\theta) - C(q) - U(\theta))\, \phi(q|\theta) dq f(\theta) d\theta$$

subject to

$$U'(\theta) = \int_{\underline{q}}^{\overline{q}} u(q,\theta)\phi_\theta(q|\theta)dq, \quad \text{for almost all } \theta, \in \Theta, \quad \text{(local-IC)},$$

$$U(\theta) \geq 0, \quad \text{(IR)}.$$

Let's take stock. Of the four programs, we have the following relationships: $V^{\phi}_{rel} \geq V^{\phi} \geq V^D$ and $V^{\phi}_{rel} \geq V^D_{rel} \geq V^D$. Suppose that the program is regular so that the relaxed deterministic program delivers a monotone allocation and therefore $V^D_{rel} = V^D$. If we can also show that $V^{\phi}_{rel} = V^D_{rel}$, then it follows that $V^{\phi}_{rel} = V^D$. Because $V^{\phi}_{rel} \geq V^{\phi} \geq V^D$, we could conclude that $V^D = V^{\phi}$.

We now show that $V^{\phi}_{rel} = V^D_{rel}$. Recall that

$$U'(\theta) = \int_{\underline{q}}^{\overline{q}} u_\theta(q,\theta)\phi(q|\theta)dq, \quad \text{for almost all } \theta, \in \Theta, \quad \text{(local-IC)},$$

so it follows via integration by parts that

$$\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)f(\theta)d\theta = U(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{q}}^{\overline{q}} u_\theta(q,\theta)\phi(q|\theta)dq \right) \frac{1-F(\theta)}{f(\theta)} f(\theta)d\theta.$$

Substituting this into the seller's ϕ-relaxed program yields

$$\max_{\{\phi(\cdot|\cdot),U(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{q}}^{\overline{q}} \left(u(q,\theta) - C(q) - u_\theta(q,\theta)\frac{1-F(\theta)}{f(\theta)} \right) \phi(q|\theta)f(\theta)dqd\theta.$$

This can be solved pointwise in θ by choosing ϕ to have an atom of unit mass on the unique q for which

$$u(q,\theta) - C(q) - u_\theta(q,\theta)\frac{1-F(\theta)}{f(\theta)}$$

is maximized. But this is the deterministic solution in Proposition 4. Hence $V^{\phi}_{rel} = V^D$. $\square$

3.2 Dealing with non-regularity

Before considering problems of non-regularity, it is worth noting how one can set up the optimal contract problem as a standard program in optimal control. Indeed, many papers use control theory to solve the problem rather than integration by parts and pointwise maximization (e.g., all of the Laffont and Tirole papers on optimal regulation take a control-theoretic approach). The cost of this approach is that the standard (off the shelf) sufficient conditions for an optimal solution in control theory are stronger than what are typically needed for the direct approach (integration by parts and pointwise maximization). The benefits are that sometimes the direct approach cannot be used, as is the case when the monotonicity condition is binding. Control theory is far more powerful and general than our simple integration by parts trick. So for complicated problems, it is something which can be useful.

The basic idea is to treat θ as the state variable just as engineers would treat time. The control variable is q and the co-state variable is indirect utility, U. The Hamiltonian becomes

$$\mathcal{H}(q, U, \theta) \equiv (u(q, \theta) - C(q) - U) f(\theta) + \lambda(\theta) u_\theta(q, \theta).$$

Roughly speaking, providing $q(\cdot)$ is piecewise-C^1 and $\mathcal{H}$ is concave in (q, U) for any λ, the following conditions are necessary and sufficient for an optimum:

$$\mathcal{H}_q(q(\theta), U(\theta), \theta) = 0,$$

$$-\mathcal{H}_U(q, U, \theta) = \lambda'(\theta),$$

$$\lambda(1) = 0.$$

Solving these equations yields the same solution as we achieved with pointwise maximization. Weaker versions of the concavity conditions are available; see for example Seierstad and Sydsaeter's (1987) control theory book for details.

If it is unreasonable to assume regularity for the economic problem under study, one must maximize the unrelaxed program including a constraint for the monotonicity condition. This is done by choosing the derivative of q as the control variable, and letting q become a second co-state variable. Typically, the optimal solution will exhibit an "ironing out" of the decision function found in the relaxed program. Specifically, the solution to the optimal control problem will indicate optimal intervals over which q is made constant, but otherwise following the relaxed choice of q. The result is that there will be regions of pooling in the optimal contract, but in non-pooling regions, it will be the same as before. Typically (although not always) there will be no distortion at the top and downward distorted q for all other types.

See Fudenberg and Tirole (1991), chapter 7 (pages 303-306), for a summary of the approach.

3.3 Discrete distributions of θ with n types

Here, we will generalize our results for the 2-type setting. Suppose that there are now n types, $\theta_1 < \theta_2, \ldots, \theta_{n-1} < \theta_n$ with probability $\phi_i \in (0, 1)$ for each type and distribution function $\Phi_i \equiv \sum_{j=1}^{i} \phi_j$. Thus, $\Phi_1 = \phi_1$ and $\Phi_n = 1$. We state the principal's program as choosing an output-utility allocation, $\{q_i, U_i\}_{i=1}^n$ where $U_i = u(q_i, \theta_i) - t_i$:

$$\max_{\{(q_i, U_i)\}_{i=1}^n} \sum_{i=1}^{n} \phi_i \left\{ u(q_i, \theta_i) - C(q_i) - U_i \right\},$$

subject to, $\forall\ i, j$,,

$$U_i \geq u(q_j, \theta_i) - t_j, \qquad \text{(IC(i,j))}$$

and

$$U_i \geq \underline{U}. \tag{IR(i)}$$

Fortunately, we can eliminate many of these constraints and focus on local incentive compatibility.

Lemma 3. If $u_{q\theta} \geq 0$, then the local constraints

$$U_i \geq u(q_{i-1}, \theta_i) - t_{i-1} \tag{DLIC(i)}$$

and

$$U_i \geq u(q_{i+1}, \theta_i) - t_{i+1} \tag{ULIC(i)}$$

satisfied for all i are necessary and sufficient for global incentive compatibility.

Proof: Necessity is direct. Sufficiency is proven by induction. First, note that the local constraints imply that $q_i \geq q_{i-1}$. Specifically, the local constraints imply

$$U_i - (u(q_{i-1}, \theta_i) - t_{i-1}) \geq 0 \geq (u(q_i, \theta_{i-1}) - t_i) - U_{i-1}, \quad \forall\, i.$$

Rearranging, we have

$$u(q_i, \theta_i) - u(q_{i-1}, \theta_i) \geq u(q_i, \theta_{i-1}) - u(q_{i-1}, \theta_{i-1}).$$

Combining this inequality with the single-crossing property implies monotonicity.

Consider DLIC for type i and $i-1$. Restated in direct utility terms, these conditions are

$$u(q_i, \theta_i) - u(q_{i-1}, \theta_i) \geq t_i - t_{i-1},$$
$$u(q_{i-1}, \theta_{i-1}) - u(q_{i-2}, \theta_{i-1}) \geq t_{i-1} - t_{i-2}.$$

Adding the conditions imply

$$u(q_i, \theta_i) - u(q_{i-1}, \theta_i) + u(q_{i-1}, \theta_{i-1}) - u(q_{i-2}, \theta_{i-1}) \geq t_i - t_{i-2}.$$

By the single-crossing condition and monotonicity, the LHS is smaller than

$$u(q_i, \theta_i) - u(q_{i-1}, \theta_i) + u(q_{i-1}, \theta_i) - u(q_{i-2}, \theta_i) = u(q_i, \theta_i) - u(q_{i-2}, \theta_i),$$

and so IC(i,i-2) is satisfied:

$$u(q_i, \theta_i) - u(q_{i-2}, \theta_i) \geq t_i - t_{i-2}.$$

Thus, DLIC(i) and DLIC(i-1) imply IC(i,i-2). One can show that IC(i,i-1) and DLIC(i-2) imply IC(i,i-3), etc. Therefore, starting at $i = n$ and proceeding inductively, DLIC implies IC(i,j) holds for all $i \geq j$. A similar argument in the reverse direction establishes that ULIC implies IC(i,j) for $i \leq j$. $\square$

The basic idea of the lemma is that the local upward constraints imply global upward constraints, and likewise for the downward constraints. We have reduced our necessary and sufficient IC constraints from $n(n-1)$ to $2(n-1)$ constraints. We can now optimize using Kuhn-Tucker's theorem. If possible, however, it is better to check to see if we can simplify things a bit more.

We can still do better for our *particular* problem. Consider the following **relaxed program**.

$$\max_{y_i} \sum_{i=1}^{n} \phi_i \left\{ u(q_i, \theta_i) - C(q_i) - U(\theta_i) \right\},$$

subject to DLIC(i) for every i, IR(1), and q_i nondecreasing in i.

We will demonstrate that

Lemma 4. The solution to the unrelaxed program is equivalent to the solution of the relaxed program.

Proof: The proof proceeds in 3 steps.

Step 1: The constraints of the unrelaxed program imply those of the relaxed program. It is easy to see that IC(i,j) imply DLIC(i) and IR(i) imply IR(1). Take $i > j$. By IC(i,j) and IC(j,i) we have

$$u(q_i, \theta_i) - t_i \geq u(q_j, \theta_i) - t_j,$$
$$u(q_j, \theta_j) - t_j \geq u(q_i, \theta_j) - t_i.$$

Adding and rearranging,

$$[u(q_i, \theta_i) - u(q_j, \theta_i)] - [u(q_i, \theta_j) - u(q_j, \theta_j)] \geq 0.$$

By the single-crossing condition, if $\theta_i > \theta_j$, then $q_i \geq q_j$.

Step 2: At the solution of the relaxed program, DLIC(i) is binding for all i. Suppose not. Take i and ε such that $[u(q_i, \theta_i) - t_i] - [u(q_{i-1}, \theta_i) - t_{i-1}] > \varepsilon > 0$. Now for all $j \geq i$, raise transfers to $t_j + \varepsilon$. No IC constraints will be violated and profit is raised by $(1 - \Phi_{i-1})\varepsilon$, which contradicts $\{q, t\}$ being a solution to the relaxed program.

Step 3: The solution of the relaxed program satisfies the constraints for the unrelaxed program. Because DLIC(i) is binding, we have

$$u(q_i, \theta_i) - u(q_{i-1}, \theta_i) = t_i - t_{i-1}.$$

By monotonicity and the sorting condition,

$$u(q_i, \theta_{i-1}) - u(q_{i-1}, \theta_{i-1}) \leq t_i - t_{i-1}.$$

But this latter condition is ULIC(i-1). Hence, DLIC and ULIC are satisfied. By Theorem 3, this is sufficient for global incentive compatibility. Finally, it is straightforward to show that IC(i,j) and IR(1) implies IR(i). □

Optimal Contracts

We now solve the simpler relaxed program. Note that there are now only n constraints, all of which are binding. We solve

$$\max_{q_i,U_i} \mathcal{L} = \sum_{i=1}^{n} \phi_i \left\{ u(q_i,\theta_i) - C(q_i) - U_i \right\} + \sum_{i=2}^{n} \lambda_i (U_i - U_{i-1} - u(q_{i-1},\theta_i) + u(q_{i-1},\theta_{i-1})) + \lambda_1 (U_1 - \underline{U}),$$

ignoring the monotonicity constraint for now. [Using indirect utilities, the DLIC constraints become $U_i - U_{i-1} - u(q_{i-1},\theta_i) + u(q_{i-1},\theta_{i-1}) \geq 0$.] There are $2n$ necessary first-order conditions:

$$\phi_i \left(u_q(q_i,\theta_i) - C'(q_i) \right) = \lambda_{i+1} [u_q(q_i,\theta_{i+1}) - u_q(q_i,\theta_i)],\ i = 1,\ldots,n-1,$$

$$\phi_n \left(u_q(q_n,\theta_n) - C'(q_n) \right) = 0,$$

$$-\phi_i + \lambda_i - \lambda_{i+1} = 0,\ i = 1,\ldots,n-1,$$

$$-\phi_n + \lambda_n = 0.$$

Combining the last two sets of equations, we have a first-order difference equation which we can solve uniquely for $\lambda_i = \sum_{j=i}^{n} p_j$. Thus, assuming discrete analogues of our regularity assumption, we have the following result.

Proposition 6. In the case of a finite distribution of types, in the optimal mechanism q_i satisfies

$$\phi_i \left(u_q(q_i,\theta_i) - C'(q_i) \right) = [1 - \Phi_i](u_q(q_i,\theta_{i+1}) - u_q(q_i,\theta_i)),\ i = 1,...,n,$$

t_i is chosen as the unique solution to the first-order difference equation,

$$U_i - U_{i-1} = u(q_{i-1},\theta_i) + u(q_{i-1},\theta_{i-1}),$$

with initial condition, $U_1 = \underline{U}$.

Remarks:

- As before, we have no distortion at the top and a suboptimal level of activity for all lower types.
- Step 2 of the proof of Lemma 4 is sometimes ignored in papers, but be careful because DLIC does not bind in all economic environments. Moreover, it is incorrect to assume DLIC binds (i.e., impose DLIC's with equalities in the relaxed program)

and then check that the other constraints are satisfied in the solution to the relaxed program. This does not guarantee an optimum!!! Either you must show that the constraints are binding or you must use Kuhn-Tucker analysis which allows the constraints to bind or be slack. This is important. In many papers, it is not the case that the DLIC constraints bind. See, Hart, [1983] for example. In fact, in Rochet and Stole (2002), there is an example of a price discrimination problem where the upward local IC constraints bind rather than the downward ones. This is generated by adding noise to the reservation utility of consumers. As a consequence, you may want to leave rents to some types to increase the chances that they will visit your store, but then the logic of step 2 does not work and in fact the upward constraints may bind.

- Discrete models sometimes generate results which are quite different from those which emerge from the continuous-type settings. For example, in Rochet and Stole (2002), the discrete setting with random IR constraints exhibits a downward distortion for the lowest type which is absent in the continuous setting.

- It is frequently convenient to use two-type models to get the feel for an economic problem before going on to tackle the n-type or continuous-type case. While this is usually helpful, some economic phenomena will only be visible in $n \geq 3$ environments. (E.g., common agency models with complements generate the first-best as an outcome with two types but not with three or more.) Fundamentally, the IC constraints in the discrete setting are much more rigid with two types than with a continuum. Nonetheless, much can be learned from two-type models, although it would certainly be better if the result could be generalized to more.

- I encourage you to take a look at the excellent textbook by Martimort and Laffont (*Theory of Incentives*, 2002) in which they are able to capture most of the interesting economic phenomena behind optimal screening contracts in simple 2-type models!

3.4 Type-dependent participation constraints

Throughout we have assumed that $u_\theta(q, \theta)$ has a constant sign (e.g., $u_\theta(q, \theta) \geq 0$ for the case of nonlinear pricing). Recall that we have also normalized the agent's outside option to be zero, arguing that we can always embed $\underline{U}(\theta)$ directly into the agent's payoff function, $u(q, \theta)$. In some settings, however, the relevant outside option may be increasing in θ so that once we incorporate the participation constraint, we may be left with a situation where u_θ changes sign endogenously, as a function of $q(\cdot)$! This complication was first explored by Lewis and Sappington in a pair of papers (*JET*, 1989; *AER*, 1989), and they coined the phrase "countervailing incentives" to describe the situation.

Here is a simple example of what can happen. Suppose that the monopolist in our setting is selling to a consumer with payoff $\theta v(q) - t$ for consuming q units at price t with type θ, but now assume that the consumer also has the option to buy a market substitute with fixed output q_0 at price p_0.[4] Thus,

$$\underline{U}(\theta) = \max\{0, \theta v(q_0) - p_0\}.$$

For simplicity, assume that $\underline{\theta} v(q_0) \geq p_0$, so a non-participating agent will always buy the inferior substitute. Embedding this outside option into the agent's (net) utility function, we have

$$u(q, \theta) = \theta(v(q) - v(q_0)) - p_0.$$

Notice that u_θ changes signs at $q = q_0$. Thus, while it is still the case that any incentive compatible mechanism will satisfy

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q(s))ds,$$

setting $U(\underline{\theta}) = 0$ will possibly violate the IR constraint for a higher type.

Note that in the example we've chosen, the agent's utility is linear in θ. It therefore follows that $U(\theta)$ will be convex. Hence, the IR constraint will bind over an interval (possibly degenerate), $[\theta_1, \theta_2] \subseteq [\underline{\theta}, \overline{\theta}]$. It must be the case that over the binding interval that $U'(\theta) = 0$, so $q(\theta) = q_0$ and $t(\theta) = p_0$. If we knew θ_1 and θ_2, then we are left with two standard problems. Over $[\theta_2, \overline{\theta}]$, we have $u_\theta \geq 0$ and thus we can find $q(\cdot)$ so solve

$$\Lambda_q(q(\theta), \theta) = 0,$$

as before. To the left of the binding interval, $[\underline{\theta}, \theta_1]$, we have $u_\theta(q, \theta) \leq 0$, and hence we set $U(\theta_1) = 0$ to satisfy the IR constraint. To the left, the problem looks very much like the Baron-Myerson setting of regulating a firm with unknown marginal cost (see below), and we can use the integration by parts trick to show

$$\int_{\underline{\theta}}^{\theta_1} U(\theta) f(\theta) d\theta = F(\theta_1)U(\theta_1) - \int_{\underline{\theta}}^{\theta_1} (v(q(\theta)) - v(q_0)) \frac{F(\theta)}{f(\theta)} f(\theta) d\theta.$$

The pointwise-optimal $q(\cdot)$ over that region will maximize

$$\int_{\underline{\theta}}^{\theta_1} \left(\theta v(q(\theta) - v(q_0)) - C(q(\theta)) + \frac{F(\theta)}{f(\theta)} (v(q(\theta)) - v(q_0)) - U(\theta_1) \right) f(\theta) d\theta.$$

Thus, $q(\cdot)$ satisfies

$$\theta v'(q(\theta)) - C'(q(\theta)) = -\frac{F(\theta)}{f(\theta)} v'(q(\theta)) < 0.$$

Having solved both the left and right problems, the firm now chooses θ_1 and θ_2 optimally. The result is a kind of smooth-pasting property which ensures that the three components of $q(\cdot)$ come together continuously (hence U is smooth). The result is over provision of output relative to the first best for low types, under provision of output relative to the first best for high types, and intermediate types bunch or pool at $q = q_0$. Several papers and applications have followed the original Lewis and Sappington papers. See Jullien (*JET*, 2000) for a very nice general methodology using control theory techniques.

[4] We assume that the firm is more efficient and can supply this option at lower cost $C(q_0) < p_0$.

4 Additional Applications

4.1 Regulating a monopolist with unknown marginal cost

There is a large body of work on regulating firms with private information that was initiated by Baron and Myerson (1982) and brought to fruition by a number of interesting papers written in the 1980s and 1990s by Laffont, Tirole, Lewis, Sappington, McAfee and McMillan. Here, I present a simple version of Baron and Myerson (1982) to illustrate how the methods used in nonlinear pricing can easily be adapted to settings with private information over costs. In another section, we'll consider an important extension by Laffont and Tirole (1986) that combines cost observation and moral hazard.

Consider the case where a regulator must set the price for a natural monopolist with private information about marginal cost. Specifically, the firm's costs of production is

$$C(q) = \theta q + K,$$

where K is a fixed capital cost, and $\theta \sim F(\theta)$ on $[\underline{\theta}, \overline{\theta}]$.

The regulator sets a price at which consumers purchase output, p, a transfer to the firm t (in addition to the revenues it earns form the consumers), requiring that the monopolist satisfy all demand at the set price. Because the demand function for the consumer is well known, setting a price p is equivalent to setting a quantity q. We will therefore think of the regulator as designing a mechanism in transfers and output,$\{q(\cdot), t(\cdot)\}$, with the understanding that the regulated consumer price is determined by the demand condition, $p(\theta) = P(q\theta)$, where here we abuse notation and use $P(q)$ as a representation of market demand.

Payoffs: For a given q and t, the monopolist earns the consumer revenue, $R(q)$, associated with selling q units and the transfer t, but must pay cost $\theta q + K$. Hence, we can write the monopolist's profits as

$$\Pi = R(q) - \theta q - K + t.$$

The regulator cares only about consumer surplus minus the payments made to the monopolist. Hence, the regulator maximizes

$$CS(q) - t = CS(q) + R(q) - \theta q - K - \Pi.$$

As a benchmark, note that the full-information output $q^{fb}(\theta)$ satisfies

$$CS'(q^{fb}(\theta)) + R'(q^{fb}(\theta)) - \theta = 0.$$

A direct mechanism is a pair, $\{q(\cdot), t(\cdot)\}$, and the corresponding profit is

$$\Pi(\theta) = R(q(\theta)) + t(\theta) - \theta q(\theta) - K.$$

Define the type-θ monopolist's payoff when reporting $\hat{\theta}$ as

$$\Pi(\hat{\theta}|\theta) = R(q(\hat{\theta})) + t(\hat{\theta}) - \theta q(\hat{\theta}) - K.$$

The first step to solving for the optimal regulation is to characterize the set of implementable $\{q(\cdot), \Pi(\cdot)\}$. Following the same arguments as in the case of nonlinear pricing, we have the following result.

Proposition 7. *In the regulation setting, an output-profit allocation $\{q(\cdot), \Pi(\cdot)\}_{\hat{\theta}\in\Theta}$ is implementable if and only if*

1. *$q(\cdot)$ is nonincreasing,*
2. *$\Pi(\theta) = \Pi(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} q(s)ds$.*

Because we are in an environment where $\Pi_\theta = -q(\theta) < 0$, the participation constraint will bind for the highest-cost firm. As such, we know it will be optimal to set $\Pi(\overline{\theta}) = 0$, but we have no knowledge (ex ante) about the value of $\Pi(\underline{\theta})$. For this reason, it is easier to work with an equivalent form of the envelope condition for implementability,

$$\Pi(\theta) = \Pi(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q(s)ds.$$

We will want an expression for $E[\Pi(\theta)]$ that is a function of $\Pi(\overline{\theta})$, so we will use this alternative expression to determine $E[\Pi]$.

$$\begin{aligned} E[\Pi(\theta)] &= \Pi(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\theta}^{\overline{\theta}} q(s)ds \right) f(\theta)d\theta \\ &= \Pi(\overline{\theta}) + \left(\int_{\theta}^{\overline{\theta}} q(s)ds \right) F(\theta) \Bigg|_{\underline{\theta}}^{\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} q(\theta)\frac{F(\theta)}{f(\theta)} f(\theta)d\theta \\ &= \Pi(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} q(\theta)\frac{F(\theta)}{f(\theta)} f(\theta)d\theta. \end{aligned}$$

Returning to the regulator's program, after substitution we have

$$\max_{q(\cdot)} E\left[CS(q(\theta)) + R(q(\theta)) - \theta q(\theta) - K - q(\theta)\frac{F(\theta)}{f(\theta)} - \Pi(\overline{\theta}) \right],$$

subject to $q(\cdot)$ nonincreasing. If we assume that $W(q,\theta) \equiv CS(q) + R(q) - \theta q$ is strictly concave and F/f is nondecreasing (satisfies a form of MHRC), then this program is regular. In this case, the pointwise-optimum will be nonincreasing and $q(\cdot)$ satisfies

$$CS'(q(\theta)) + R'(q(\theta)) - \theta = \frac{F(\theta)}{f(\theta)}.$$

Note that output is distorted below the full-information solution (for all $\theta > \underline{\theta}$) in an attempt to reduce monopoly rents due to private information.

Given $q(\cdot)$ has been determined, we can use the envelope condition to determine $\Pi(\theta)$, which, in turn, can be used to compute $t(\theta)$. With $\{q(\cdot), t(\cdot)\}$ we can also apply the taxation principle and compute a nonlinear subsidy schedule, $T(q)$, which is an alternative to the direct mechanism.

4.2 Hybrid models of hidden-information/hidden-action

Myerson (1982) introduces moral hazard into hidden information models and extends the revelation principle to include a notion of *obedience*. As an example, suppose that the agent has private information θ, but also has a private action e (effort) which stochastically impacts the principal's returns, x, captured in a conditional density function, $f(x|e)$, as we studied in the moral hazard framework. Myerson's general revelation principle states that the principal can restrict attention to contracts of the following form:

1. After θ is realized, the agent reports $\hat{\theta} \in \Theta$.
2. The mechanism announces an effort recommendation, $e(\hat{\theta})$, and a compensation scheme $w(x|\hat{\theta})$.
3. For every θ, the agent is (i) *truthful* (i.e., announces $\hat{\theta} = \theta$) and (ii) *obedient* (i.e., the agent finds it optimal to choose effort $e(\theta)$).

The recommendation part may seem a bit unnecessary, since in most single-agent settings, the principal has no information that the agent doesn't possess. In multi-agent settings, however, the information of agent j may be useful for the choice of agent i, and hence the principal will possibly benefit by making a recommendation to agent i with the appropriate information.

In some important papers, moral hazard and hidden information combine to generate a model that is essentially one of only hidden information. This typically arises because some aspect aspect of performance can be contracted upon which allows effort to be imputed, conditional on type. A good example of this is Laffont and Tirole (1986), which we will review here.

Laffont and Tirole (*JPE*, 1986). A regulated firm has private information about its costs, $\theta \in [\underline{\theta}, \overline{\theta}]$, distributed according to F, which we will assume satisfies the MHRC. The firm exerts an effort level, e, which has the effect of reducing the firm's marginal cost of production. The total cost of production is $C(q) = (\theta - e)q$. This effort, however, is costly; the firm's costs of effort are given by $\psi(e)$, which is increasing, strictly convex, and $\psi'''(e) \geq 0$. It is assumed that the regulator can observe costs, and so without loss of generality, we assume that the regulator pays the observed costs of production rather than the firm. Hence, the firm's utility is given by

$$U = t - \psi(e).$$

Laffont and Tirole assume a novel contracting structure in which the regulator can observe costs, but must determine how much of the costs are attributable to effort and how much are attributed to inherent luck (i.e., type). The regulator cannot observe e but can observe and contract upon total production costs C and output q. For any given q, the regulator can perfectly determine the firm's marginal cost $c = \theta - e$. Thus, the regulator can ask the firm to report its type and assign the firm a marginal cost target of $c(\hat{\theta})$ and an output level of $q(\hat{\theta})$ in exchange for compensation equal to $t(\hat{\theta})$. A firm with type θ that wishes to make the marginal cost target of $c(\hat{\theta})$ must expend effort equal to $e = \theta - c(\hat{\theta})$. With such a contract, the firm's indirect utility function becomes

$$U(\hat{\theta}|\theta) \equiv t(\hat{\theta}) - \psi(\theta - c(\hat{\theta})).$$

Our implementation result implies that incentive compatible contracts, $\{c(\cdot), t(\cdot)\}$, are equivalent to requiring that

$$U(\theta) = U(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} \psi'(\theta - c(\theta))d\theta,$$

and that $c(\theta)$ be nondecreasing. Note that this expression is independent of $q(\theta)$. As we will see, $q(\cdot)$ is chosen independently, conditional on the optimal screening variable $c(\cdot)$.

To solve for the optimal contract, we need to state the regulator's objectives. Let's suppose that the regulator wishes to maximize a weighted average of strictly concave consumer surplus, $CS(q)$, (less costs and transfers) and producer surplus, U, with less weight afforded to the latter.

$$V = E_\theta[CS(q(\theta)) - c(\theta)q(\theta) - t(\theta) + \gamma U(\theta)],$$

or substituting out the transfer function,

$$V = E_\theta[CS(q(\theta)) - c(\theta)q(\theta) - \psi(\theta - c(\theta)) - (1-\gamma)U(\theta)],$$

where $0 \leq \gamma < 1$. Note that Baron and Myerson (1982) consider the case where $\gamma = 0$.

Remarks:

1. In our above development, we could instead write $S = CS - cq - \psi$, and then the regulator maximizes $E[S - (1-\gamma)U]$. If $\gamma = 0$, we are in the same situation as our initial framework where the principal doesn't directly value the agent's utility.

2. L&T motivate the cost of leaving rents to the firm as arising from the shadow cost of raising public funds. In that case, if $1 + \lambda$ is the cost of public funds, the regulator's objective function (after simplification) is $E[CS - (1+\lambda)(cq + \psi) - \lambda U]$. Except for the optimal choice of q, this yields identical results as using $1 - \gamma \equiv \frac{\lambda}{1+\lambda}$.

Our regulator solves the following program:

$$\max_{q,c,t} E_\theta[CS(q(\theta)) - c(\theta)q(\theta) - \psi(\theta - c(\theta)) - (1-\gamma)U(\theta)],$$

subject to

$$U(\theta) = U(\overline{\theta}) + \int_\theta^{\overline{\theta}} \psi'(\theta - c(\theta))d\theta,$$

$c(\theta)$ nondecreasing, and the firm making nonnegative profits (i.e., $U(\theta) \geq 0$). Integrating $E_\theta[U(\theta)]$ by parts (using our previously developed techniques), we can substitute out U from the objective function (thereby eliminating transfers from the program). We then have

$$\max_{q,c} E_\theta \left[CS(q(\theta)) - c(\theta)q(\theta) - \psi(\theta - c(\theta)) - (1-\gamma)\frac{F(\theta)}{f(\theta)}\psi'(\theta - c(\theta)) - (1-\gamma)U(\overline{\theta}) \right],$$

subject to $c(\theta)$ nondecreasing, and $U(\overline{\theta}) \geq 0$. We obtain the following results.

Results:

1. The choice of effort, $e(\theta)$, satisfies

$$q(\theta) - \psi'(e(\theta)) = (1-\gamma)\frac{F(\theta)}{f(\theta)}\psi''(e(\theta)) \geq 0.$$

   Note that the first-best level of effort, conditional on q, is $q = \psi'(e)$. As a consequence, suboptimal effort is provided for every type except the lowest. We always have no distortion on the bottom. Only if $\gamma = 1$, so the regulator places equal weight on the firm's surplus, do we have no distortion anywhere. In the L&T setting, this condition translates to $\lambda = 0$; i.e., no excess cost of public funds.

2. The optimal q is the full-information efficient production level conditional on the marginal cost $c(\theta) \equiv \theta - e(\theta)$:

$$CS'(q) = c(\theta).$$

   This is not the result of L&T. They find that because public funds are costly, $CS'(q) = (1+\lambda)c(\theta)$, where the RHS represents the effective marginal cost (taking into account public funds). Nonetheless, this solution still corresponds to the choice of q under full-information for a given marginal cost, because the cost of public funds is independent of informational issues. Hence, there is a dichotomy between screening θ (choosing $c(\theta)$) and production.

3. L&T also show that the optimal nonlinear contract can be implemented using a realistic menu of two-part tariffs (i.e., cost-sharing contracts). Let $\overline{C}(q)$ be a total cost target. The firm chooses an output and a corresponding cost target, $\overline{C}(q)$, and then is compensated according to a reimbursement rule that is linear in observed cost, C:

$$T(q, C) = z(q) + \alpha(q)[\overline{C}(q) - C],$$

where C is observed ex post cost. $z(q)$ is interpreted as a fixed payment and α is a cost-sharing parameter. If $(\overline{\theta} - \underline{\theta})$ goes to zero, α goes to one, implying a fixed-price contract in which the firm absorbs any cost overruns. Of course, this framework doesn't make much sense with no other uncertainty in the model (i.e., there would never be cost overruns which we would observe). But if some noise is introduced on the ex post cost observation, i.e. $\tilde{C} = C(q) + \varepsilon$, the mechanism is still optimal. It is robust to linear noise in observed contract variables because the firm is risk neutral and the implemented contract is linear in the observation noise.

5 Dynamic screening

Here, we briefly discuss the important extension of the static model to dynamic screening.

Consider again our monopoly model, with the simple setting of linear utility but with two periods. The timing is as follows:

0: Nature chooses $\theta_1 \sim F(\theta_1)$ for the buyer;

1: Seller offers buyer a direct mechanism (without loss of generality),

$$\{t(\hat{\theta}_1, \hat{\theta}_2), q_1(\hat{\theta}_1), q_2(\hat{\theta}_1, \hat{\theta}_2)\}_{(\hat{\theta}_1, \hat{\theta}_2)};$$

if buyer accepts, buyer reports $\hat{\theta}_1 \in \Theta$ and $q_1(\hat{\theta})$ is consumed;

1.5: Nature chooses $\theta_2 \sim G(\theta_2|\theta_1)$ for the buyer;

2: Buyer reports $\hat{\theta}_2$ and obtains $q_2(\hat{\theta})$ and total cost $t(\hat{\theta}_1, \hat{\theta}_2)$.

For simplicity, there is no discounting. The buyer's utility from consuming q_1 at date 1 and q_2 at date 2 for a total transfer of t is

$$\theta_1 q_1 + \theta_2 q_2 - t.$$

The seller earns

$$t - C(q_1) - C(q_2).$$

Static benchmark: Recall that if there is only one period and F satisfies MHRC, then in the optimal mechanism, $q^*(\theta)$ satisfies[5]

$$\theta - C'(q^*(\theta)) = \frac{1 - F(\theta)}{f(\theta)}.$$

Case 1: Constant θ. Suppose that $\theta_1 = \theta_2 = \theta$. Then Baron and Besanko (1984) show the following:

[5]For simplicity, we are assuming that the solution of $\Lambda_q(q^*(\theta), \theta) = 0$ is nonnegative, $q^*(\theta) \geq 0$.

Proposition 8. *If $\theta_1 = \theta_2$, then the optimal optimal dynamic screening allocation is the replication of the optimal one-period allocaiton: $q_1(\theta) = q_2(\theta) = q^*(\theta)$.*

Proof: First note that repeating the static mechanism twice is incentive compatible and individually rational. Second, suppose that the optimal IC and IR mechanism is $q_1(\theta) \neq q_2(\theta)$. Consider forming a new mechanism which implements the same output $\bar{q}(\theta) = \frac{1}{2}(q_1(\theta) + q_2(\theta))$ each period with the original transfer function. This new mechanism is also IC and IR. In addition, it has higher profits because costs are strictly convex (and the principal's program is strictly concave). Hence, the optimal solution must be $q_1(\theta) = q_2(\theta)$. The firm's optimal program, given this restriction, is identical to the static program and generates the same solution. □

Note that this result is only true under full commitment. After observing the agent's report in the first period, the principal would like to change the mechanism (or renegotiate to mutually beneficial new mechanism which reduces consumption distortions). Without full commitment, both buyer and seller would anticipate future renegotiations which would destroy the incentive compatibility of the original full-commitment mechanism.

Case 2: Independent θ_1 and θ_2: $G(\theta_2|\theta_1) = F(\theta_2)$. If the types are independent, the firm is in a position to implement the first best on q_2 by selling an option contract to the buyer (i.e., buy as much as you like in period 2 for $C(q)$) and charging a first-period price of $p = E[\max_q \ \theta_2 q - C(q)]$. The determination of q_1, however, follows the standard monopoly screening problem because the agent is privately informed about θ_1 at the time of contracting.

What is interesting here (and quite general), is that information which has not yet been revealed at the contracting stage can be acquired at no cost in information rents. Because θ_1 contains no information about θ_2, the second-period allocation can be made efficient and all of the buyer's surplus over q_2 consumption can be captured.

We now turn to the interesting case where θ_1 contains imperfect information about θ_2. The seminal paper is by Courty and Li (2000, *REStud*).

First, we need to be clear about the dynamic version of the revelation principle. (This was not an issue when $\theta_1 = \theta_2$, or when θ_1 and θ_2 are independently distributed.)

Proposition 9. (Dynamic revelation principle.) *For every dynamic mechanism Γ and every buyer optimal strategy σ^* in Γ, there is a direct mechanism, $\tilde{\Gamma}$ and an optimal buyer strategy $\tilde{\sigma}^* = (\tilde{\sigma}_1^*, \tilde{\sigma}_2^*)$ such that*

1. $\tilde{\sigma}_1^*(\theta_1) = \theta_1, \ \forall \theta_1 \in \Theta$
2. $\tilde{\sigma}_2^*(\theta_1, \theta_2, \hat{\theta}_1)_{\hat{\theta}_1 = \theta_1} = \theta_2, \ \forall \theta_1, \theta_2 \in \Theta,$

3. *for every* (θ_1, θ_2), $\tilde{q}^*(\theta_1, \theta_2) = q^*(\theta_1, \theta_2)$ *and* $\tilde{t}^*(\theta_1, \theta_2) = t^*(\theta_1, \theta_2)$.

The proof is straightforward. Please consult Krähmer and Strausz (in Borgers, ch. 11) for the details. Notice that the truth-telling constraints require that $\hat{\theta}_1 = \theta_1$ and that the agent is truthful about θ_2 (assuming she has been truthful about θ_1). They do not require that second-period report is truthful given the first-period report was dishonest. That said, second-period IC requires (conditional on $\hat{\theta}_1 = \theta_1$)

$$\theta_2 q_2(\theta_1, \theta_2) - t(\theta_1, \theta_2) \geq \theta_2 q_2(\theta_1, \hat{\theta}_2) - t(\theta_1, \hat{\theta}_2), \quad \text{for all } \theta_1, \theta_2, \hat{\theta}_2 \in \Theta.$$

Because this must hold for all θ_1, implicitly it must hold for $\hat{\theta}_1 \neq \theta_1$. That is, whether or not the agent was dishonest in period 1 has no impact to incentive compatibility in period 2 because the utility from period 1 consumption, $\theta_1 q_1$, is no longer relevant in period 2. Thus, we may proceed by looking at direct mechanisms, $q_1(\hat{\theta}_1)$, $q_2(\hat{\theta}_1, \hat{\theta}_2)$ and $t(\hat{\theta}_1, \hat{\theta}_2)$ which are incentive compatible with respect to both θ_1 and θ_2.

Second-period incentive compatibility: Define $U^2(\hat{\theta}_2|\theta_2, \hat{\theta}_1)$ as the second-period utility (ignoring q_1 consumption) of agent of type θ_2 who reported $\hat{\theta}_1$ in the first period:

$$U^2(\hat{\theta}_2|\hat{\theta}_1, \theta_2) = \theta_2 q_2(\hat{\theta}_1, \hat{\theta}_2) - t(\hat{\theta}_1, \hat{\theta}_2).$$

Also define the period 2 truthful utility, $\hat{\theta}_2 = \theta_2$, given a first period report $\hat{\theta}_1$ (whether truthful or not) as

$$U^2(\theta_2|\hat{\theta}_1, \theta_2) = \theta_2 q_2(\hat{\theta}_1, \theta_2) - t(\hat{\theta}_1, \theta_2).$$

Second-period incentive compatibility requires

$$U^2(\hat{\theta}_1, \theta_2) \geq U^2(\theta_2|\hat{\theta}_1, \theta_2).$$

Following the arguments of our standard implementation result, we have the equivalence between second-period incentive compatibility and (i) $q_2(\hat{\theta}_1, \theta_2)$ is nondecreasing in θ_2 and (ii)

$$U^2(\hat{\theta}_1, \theta_2) = U^2(\hat{\theta}_1, \underline{\theta}) + \int_{\underline{\theta}}^{\theta_2} q_2(\hat{\theta}_1, s)ds.$$

We want an expression for $E_{\theta_2}[U_2(\cdot)|\theta_1]$, and so we integrate by parts:

$$E_{\theta_2}\left[U^2(\hat{\theta}_1, \theta_2)|\theta_1\right] = U^2(\hat{\theta}_1, \underline{\theta}) + E\left[q_2(\hat{\theta}_1, \theta_2)\frac{1 - G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)} \mid \theta_1\right].$$

First-period incentive compatibility: Given our result for the second period, the expected utility for θ_1 when reporting $\hat{\theta}_1$ is

$$U(\hat{\theta}_1|\theta_1) = \theta_1 q_1(\hat{\theta}_1) + U^2(\hat{\theta}_1, \underline{\theta}) + E_{\theta_2}\left[q_2(\hat{\theta}_1, \theta_2)\frac{1 - G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)} \mid \theta_1\right].$$

A familiar envelope result emerges for first-period utility,

$$U'(\theta) = q_1(\theta_1) - E_{\theta_2}\left[q_2(\theta_1, \theta_2)\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)} \mid \theta_1\right].$$

Thus,

$$E[U(\theta_1)] = E_{\theta_1}\left[E_{\theta_2|\theta_1}\left[q_1(\theta_1) - q_2(\theta_1,\theta_2)\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right]\frac{1-F(\theta_1)}{f(\theta_1)}\right].$$

Remark: This integral condition is a necessary condition for first-period incentive compatibility. Unfortunately, we do not find that both $q_1(\theta_1)$ and $q_2(\theta_1,\theta_2)$ must be nondecreasing in θ_1. Intuitively, the local second-order condition on first-period utility, $U_{12}(\hat{\theta}_1|\theta_1) \geq 0$, is equivalent to

$$U_{12}(\hat{\theta}_1|\theta_1) = \frac{dq_1(\theta_1)}{d\theta_1} - \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial q_2(\theta_1,\theta_2)}{\partial\theta_1} G_{\theta_1}(\theta_2|\theta_1)d\theta_1 \geq 0.$$

What can be shown, however, is that if $q_1(\theta_1)$ and $q_2(\theta_1,\theta_2)$ are nondecreasing in θ_1 (and q_2 is also nondecreasing in θ_2), then there exists a transfer function which implements $\{q_1(\cdot), q_2(\cdot,\cdot)\}$. Hence, we will proceed by solving the relaxed program, and then determining regularity conditions which imply the solution is optimal.

All together now, in the relaxed program the principal chooses q_1 and q_2 to maximize (pointwise)

$$E\left[\theta_1 q_1(\theta_1) + \theta_2 q_2(\theta_1,\theta_2) - C(q_1(\theta_1)) - C(q_2(\theta_1,\theta_2)) - \left(q_1(\theta_1) - q_2(\theta_1,\theta_2)\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right)\frac{1-F(\theta_1)}{f(\theta_1)} - U(\underline{\theta})\right].$$

The optimal quantity allocations satisfy

$$\theta_1 - C'(q_1(\theta_1)) = \frac{1-F(\theta_1)}{f(\theta_1)}.$$

$$\theta_2 - C'(q_2(\theta_1,\theta_2)) = -\left(\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right)\frac{1-F(\theta_1)}{f(\theta_1)}.$$

Notice that $q_1(\theta_1)$ is increasing in θ_1 providing the F satisfies the MHRC.

To verify that q_2 satisfies our desired monotonicity properties, we make the following assumption.

Assumption 1. (Dynamic regularity.) *The function*

$$J(\theta_1,\theta_2) = \theta_2 + \left(\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right)\left(\frac{1-F(\theta_1)}{f(\theta_1)}\right)$$

is nondecreasing in θ_1 and θ_2.

Notice that if θ_1 shifts $G(\theta_2|\theta_1)$ in a first-order stochastically dominating way

$$\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1) < 0,$$

and $J(\theta_1, \theta_2)$ is increasing in θ_1 as required. The requirement that $J(\theta_1, \theta_2)$ is increasing in θ_2 is harder to motivate, but will be true if the *informativeness* term, $\frac{G_{\theta_1}(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}$ is sufficiently small in magnitude. As a more general example, suppose that $\theta_2 = \lambda\theta_1 + (1-\lambda)\varepsilon$, where $\varepsilon \sim H(\varepsilon)$ on $\mathbb{R}$. In this case,

$$G(\theta_2|\theta_1) = \text{Prob}\left[\varepsilon \leq \frac{\theta_2 - \lambda\theta_1}{1-\lambda}\right] = H\left(\frac{\theta_2 - \lambda\theta_1}{1-\lambda}\right).$$

In this case, G satisfies FOSD and

$$J(\theta_1, \theta_2) = \theta_2 - \lambda\frac{1-F(\theta_1)}{f(\theta_1)},$$

has the desired monotonicity properties.

Proposition 10. *Assume F and G satisfy dynamic regularity. Then the optimal consumption profile, $\{q_1(\theta_1), q_2(\theta_1, \theta_2)\}$ satisfies*

$$\theta_1 - C'(q_1(\theta_1)) = \frac{1-F(\theta_1)}{f(\theta_1)}.$$

$$\theta_2 - C'(q_2(\theta_1, \theta_2)) = -\left(\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right)\frac{1-F(\theta_1)}{f(\theta_1)}.$$

Remarks:

1. Providing dynamic regularity is satisfied, the second period quantity is distorted downwards as a function of the informativeness of the first-period signal. One can immediately see the special cases emerge. If θ_1 and θ_2 are distributed independently, then

$$\left(\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right) = 0,$$

and we have the first-best outcome in period two. If the $\theta_1 = \theta_2$, however, then

$$\left(\frac{\frac{\partial}{\partial\theta_1}G(\theta_2|\theta_1)}{g(\theta_2|\theta_1)}\right) = 1,$$

and we have a repetition of the static allocation as in Baron and Besanko (1984).

2. The model of these notes has consumption in period 1 mainly to provide a device for discussing the full-commitment result in Baron and Besanko (1984). The model presented could easily be reduced to a single period of consumption (at date 2), and the role of θ_1 relegated to merely a signal regarding the future value, θ_2. That is, we can fix $q_1(\theta_1) = 1$ for all θ_1. This single-consumption model is the underlying framework in Courty and Li (2000).

3. The above model focused on the case where θ_1 shifts the distribution of θ_2 is a FOSD manner. Courty and Li (2000), who are motivated by airline pricing refund policies, also considered the case where an increase in θ_1 causes a mean-preserving spread in the distribution of θ_2 with a common intersection in the CDF's at $E[\theta_2]$. Thus, one can think of θ_1 as a measure of variance of θ_2. Mapping this to the real world, it is easy to imagiene that business travelers have high θ_1 (high variance about their travel needs) while tourists have low θ_1 (low variance). Now reconsider the first-order condition for q_2. For below mean realizations of θ_2, it will be the case that $G_{\theta_1} > 0$. Thus, lower θ_1 types will be induced to consume an excess amount of q_2 (relative to the first-best). This corresponds to offering tourists a cheaper ticket with a no (or low) refund policy. If the refund is lower than the airline's marginal cost, the tourist will take the flight, even when it is inefficient to do so.