1 Bayesian Inference

(Very similar to PS1 Q1 for 2018-19) We are given $x \sim N\left(\mu, \sigma^2\right)$ with the loss function

$$L\left(\mu, \delta\left(x\right)\right) = \left(\mu - \delta\left(x\right)\right)^2$$

where the decision rule is $\delta\left(x; \nu\right) = \nu x$.

Problem 1.1. Write as exponential family with as few sufficient statistics as possible.

Solution. Skipped.

Problem 1.2. Find the conjugate prior.

Solution. Skipped.

Problem 1.3. Rewrite the prior as a familiar density.

Solution. Skipped.

Problem 1.4. Compute the average loss $R\left(\theta; \delta\right)$.

Solution. The average loss is

$$R\left(\theta; \nu\right) = \int \left(\mu - \nu x\right)^2 f\left(x|\mu\right) d\mu = \nu^2\sigma^2 + \left(1 - \nu\right)^2 \mu^2$$

■

Problem 1.5. Find the ν that minimizes $R\left(\theta; \delta\right)$.

Solution. The minimizing ν is

$$\nu^* = \frac{\mu^2}{\mu^2 + \sigma^2}$$

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Problem 1.6. Compute the posterior expected loss, $\rho\left(\pi; \delta\right)$.

Solution. The posterior expected loss is:

$$\rho(\pi;\delta) = \int (\mu - \nu x)^2 \pi(\mu|x)\, d\mu = \tilde{\sigma}^2 + \tilde{\mu}^2 - 2\nu x\tilde{\mu} + \nu^2 x^2$$

where $\tilde{\sigma}^2$ and $\tilde{\mu}^2$ are the parameters of our prior density. ■

Problem 1.7. Restricting our mean for the prior to be zero, find the Bayes estimator that minimizes this.

Solution. The answer is $\nu^* = 0$. ■

2 Markov Chains

Consider a Markov transition matrix

$$A = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

which has a stationary distribution of $(1/2, 1/2)$.

Problem 2.1. Find all forms of A that satisfy detailed balance.

Solution. The forms are:

$$A = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

where $p \in [0, 1]$. ■

Problem 2.2. When is A not ergodic?

Solution. It is not ergodic if $p = 1$. ■

Problem 2.3. When is A^2 not ergodic?

Solution. It is not ergodic if $p = 1$ and $p = 0$. ■

3 MA(1)

Suppose we are given the following MA(1) process:

$$y_t = \epsilon_t - \alpha\epsilon_{t-1}$$

with $\alpha > 1, \sigma^2 = \mathbb{E}\left[\epsilon_t^2\right]$.

Problem 3.1. Express this using a lag operator.

Solution. The answer is

$$y_t = \epsilon_t \left(1 - \alpha L\right)$$

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Problem 3.2. Define the Blaschke factor and the fundamental representation.

Solution. The Blaschke factor is defined as:

$$B = \frac{z - \alpha}{1 - \alpha z}$$

Since $\alpha > 1$, the fundamental representation is:

$$y_t = \left(1 - \frac{1}{\alpha}L\right) u_t, Var\left(u_t\right) = \alpha^2\sigma^2$$

■

Problem 3.3. Express u_t as a function of all present and past values of ϵ_t.

Solution. From the fundamental representation:

$$y_t = \left(1 - \frac{1}{\alpha}L\right) u_t$$

Using the expression for y_t:

$$\epsilon_t \left(1 - \alpha L\right) = \left(1 - \frac{1}{\alpha}L\right) u_t$$

Rearranging:

$$\begin{aligned}
u_t &= \left(1 - \frac{1}{\alpha}L\right)(1-\alpha L)\,\epsilon_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{\alpha}L\right)^j (1-\alpha L)\,\epsilon_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{\alpha}L\right)^j \epsilon_t - \alpha\sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\right)^j L^{j+1}\epsilon_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j} - \alpha\sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j-1} \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j} - \alpha\sum_{j=1}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j} \\
&= \epsilon_t + \sum_{j=1}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j} - \alpha\sum_{j=1}^{\infty}\left(\frac{1}{\alpha}\right)^j \epsilon_{t-j}
\end{aligned}$$

which yields:

$$u_t = \epsilon_t + \sum_{j=1}^{\infty}\left(\frac{1}{\alpha}\right)^{j-1}\left(\frac{1}{\alpha} - \alpha\right)\epsilon_{t-j}$$

■

Problem 3.4. If we run an OLS regression of y_t on all past values of $y_{t-1}, \ldots,$, what coefficients will we get?

Solution. Since

$$y_t = \left(1 - \frac{1}{\alpha}L\right)u_t$$

Rearranging:

$$\begin{aligned}
u_t &= \left(1 - \frac{1}{\alpha}L\right)^{-1} y_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\right) y_{t-j}
\end{aligned}$$

which yields:

$$y_t = u_t - \sum_{j=1}^{\infty}\left(\frac{1}{\alpha}\right)^j y_{t-j}$$

■

4 AR(1)

Suppose we are given the following AR(1) process:

$$y_t = \theta y_{t-1} + \epsilon_t, \epsilon_t \sim \mathcal{N}(0,1)$$

with some data $x = (y_0, ..., y_T)$.

Problem 4.1. Find the log-likelihood.

Solution. Since

$$f(y_t|y_{t-1}) \sim \mathcal{N}(\theta y_{t-1}, 1)$$

we have:

$$\begin{aligned} L_n(\theta|y_1, ..., y_T) = f(y_1, ..., y_T|\theta) &= \prod_{t=1}^{T} f(y_t|y_{t-1}, y_0, \theta) \\ &= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_t - \theta y_{t-1})^2}{2}\right) \end{aligned}$$

Taking log on each side, we have

$$\ell_n(\theta|y_1, ..., y_T) = -\frac{T}{2}\log(2\pi) - \sum_{t=1}^{T} \frac{(y_t - \theta y_{t-1})^2}{2}$$

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Problem 4.2. Is $\exp(\ell(\theta|x))$ an exponential family? If so, what is the conjugate prior?

Solution. We can rewrite it as:

$$\begin{aligned} \exp(\ell(\theta|x)) &= \exp\left\{-\frac{T}{2}\log(2\pi) - \sum_{t=1}^{T} \frac{(y_t - \theta y_{t-1})^2}{2}\right\} \\ &= \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} y_t^2 - \theta \sum_{t=1}^{T} y_t y_{t-1} - \frac{\theta^2}{2}\sum_{t=1}^{T} y_{t-1}^2 - \frac{T}{2}\log(2\pi)\right\} \end{aligned}$$

so it is indeed an exponential family with

$$f(x|\theta) = \exp\left(\sum_{i=1}^{2} c_i(\theta) T_i(x) + d(\theta) + S(x)\right) 1_{\mathbb{A}}(y)$$

wheere

$$\begin{aligned}
c_1(\theta) &= -\theta, \quad T_1(x) = \sum_{t=1}^{T} y_t y_{t-1} \\
c_2(\theta) &= -\frac{\theta^2}{2} \quad T_2(x) = \sum_{t=1}^{T} y_{t-1}^2 \\
d(\theta) &= -\frac{T}{2}\log(2\pi) \\
S(x) &= -\frac{1}{2}\sum_{t=1}^{T} y_t^2
\end{aligned}$$

and the conjugate prior is

$$\pi(\theta; t_1, t_2, t_3) = \exp\left(\sum_{i=1}^{2} c_i(\theta)\, t_i + t_3\left(-\frac{T}{2}\log(2\pi)\right) - \log\omega\right)$$

■

Problem 4.3. Given the conjugate prior, what is the posterior?

Solution. The posterior is

$$\pi(x|\theta) = \exp\left(\sum_{i=1}^{2} c_i(\theta)\,(t_i + T_i(x)) + (t_3+1)\, d(\theta) - \log\omega'\right)$$

where $\log\omega'$ is a normalizing constant. ■

5 VAR(1)

We have a VAR(1):

$$y_t = By_{t-1} + u_t$$

with

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1/2 \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

Problem 5.1. Find the characteristic polynomial and its roots.

Solution. We have

$$p(\lambda) = (1 - \lambda)(0.5 - \lambda) = 0 \Rightarrow \lambda^* = 1, 0.5$$

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Problem 5.2. Find the error-correction representation.

Solution. The eigenvectors of B are:

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which yields the diagonalization of the form:

$$B = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Since 0.5 is the stable root, we have

$$\nu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \nu^* = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

which yields:

$$\alpha = \nu(1 - 0.5) = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \beta = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus the error correction representation is

$$\Delta y_t = -\alpha\beta' y_{t-1} + u_t$$

where the vectors are given as above. ■

Problem 5.3. Find the Cholesky decomposition of Σ.

Solution. This is easy:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

■

Problem 5.4. Characterize all impulse responses.

Solution. Given

$$y_t = \begin{bmatrix} \nu & \nu^* \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ \beta^* \end{bmatrix}$$

the impulse response is characterized as:

$$r_a(k) = \nu (0.5)^k \beta' a + \nu^* (\beta^*)' a$$

Denoting

$$a = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

we then have

$$r_a(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right)^k (2u_1 + u_2) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u_1$$

■

Problem 5.5. Find the Blanchard-Quar decomposition.

Solution. The answer is:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

■