1 Long-run Risk

Consider the following long-run-risk model to the macroeconomy specified as:

$$Y_{t+1} - Y_t = \alpha_y + \beta_y Z_t + \begin{bmatrix} \sigma_y & 0 \end{bmatrix} W_{t+1}$$
$$Z_{t+1} = \beta_z Z_t + \begin{bmatrix} 0 & \sigma_z \end{bmatrix} W_{t+1}$$

where $\{W_{t+1} : t \geq 0\}$ is an iid, bivariate normal with mean zero and covariance I. In this equation, the first difference in Y_{t+1} is the logarithm of the stochastic growth in the macroeconomy. Assume $0 \leq \beta_z < 1$. Note that if $\log X \sim \mathcal{N}\left(\mu, \sigma^2\right)$,then $\mathbb{E}\left[X\right] = \exp\left(\mu + 0.5\sigma^2\right)$.

Problem 1.1. Compute the stationary distribution for the growth rate process $\{Z_t : t \geq 0\}$. In particular, what is the mean of this process and what is the variance as a function of the underlying parameters?

Solution. Define $\mu_t := \mathbb{E}\left[Z_t\right]$ and $\Sigma_t := \mathrm{Var}\left[Z_t\right]$.

- ▷ The mean of the stationary distribution is given as

$$\mu = \beta_z \mu \Rightarrow \mu = 0$$

  which is well-defined since $\beta_z < 1$.

- ▷ The variance of the stationary distribution is given as

$$\Sigma = \beta_z^2 \Sigma + \sigma_z^2 \Rightarrow \Sigma = \frac{\sigma_z^2}{1 - \beta_z^2}$$

  which is also well-defined since $\beta_z < 1$.

Since Z_t is a sum of independent normal shocks, it follows that the stationary distribution is $\mathcal{N}\left(0, \frac{\sigma_z^2}{1-\beta_z^2}\right)$. ■

Problem 1.2. Suppose a decision maker uses the following objective specified recursively to assess the macro economy:

$$\log V_t = \left[1 - \exp\left(-\delta\right)\right] Y_t + \exp\left(-\delta\right) \mathbb{R}\left(\log V_{t+1} | \mathfrak{F}_t\right)$$

where

$$\mathbb{R}\left(\log V_{t+1} | \mathfrak{F}_t\right) = \frac{1}{1-\gamma} \log \mathbb{E}\left(\exp\left[\left(1-\gamma\right) \log V_{t+1}\right] | \mathfrak{F}_t\right)$$

and $\mathfrak{F}_t$ contains information revealed by shocks $W_1, \ldots, W_t$ and the initial condition $\left(Y_0, X_0\right)$. Produce a corresponding recursive equation for $v_t = \log V_t - Y_t$.

Solution. Rewriting the objective of the decision maker, we have:

$$\begin{aligned}\log V_t - Y_t &= \exp\left(-\delta\right) \mathbb{R}\left[\log V_{t+1} - Y_t | \mathfrak{F}_t\right] \\ &= \exp\left(-\delta\right) \mathbb{R}\left[\log V_{t+1} - Y_{t+1} + \left(Y_{t+1} - Y_t\right) | \mathfrak{F}_t\right]\end{aligned}$$

Plugging in the expression for $Y_{t+1} - Y_t$,we have

$$\underbrace{\log V_t - Y_t}_{\equiv v_t} = \exp(-\delta)\,\mathbb{R}\left[\underbrace{\log V_{t+1} - Y_{t+1}}_{\equiv v_{t+1}} + \left\{\alpha_y + \beta_y Z_t + \begin{bmatrix} \sigma_y & 0 \end{bmatrix} W_{t+1}\right\} | \mathfrak{F}_t\right]$$

■

Problem 1.3. Do you expect $\log V_t$ and Y_t to be cointegrated? If so, what is the cointegrating vector?

Solution. $\log V_t$ and Y_t are cointegrated if there exists a linear combination of them that is stationary. The cointegrating vector is $[1, -1]^\top$. ■

Problem 1.4. Next, do guess and verify to construct a formula for the coefficients of $v_t = \alpha_v + \beta_v Z_t$. Provide a formula for α_v and β_v.

Solution. We will guess that v_t has the form $\alpha_v + \beta_y Z_t$. Define $M_{t+1} := v_{t+1} + Y_{t+1} - Y_t$. Under this guess,

$$\begin{aligned}
M_{t+1} &:= v_{t+1} + Y_{t+1} - Y_t \\
&= (\alpha_v + \beta_v Z_{t+1}) + \left(\alpha_y + \beta_y Z_t + \begin{bmatrix} \sigma_y & 0 \end{bmatrix} W_{t+1}\right) \\
&= (\alpha_v + \alpha_v) + \beta_v\left(\beta_z Z_t + \begin{bmatrix} 0 & \sigma_z \end{bmatrix} W_{t+1}\right) + \beta_y Z_t + \begin{bmatrix} \sigma_y & 0 \end{bmatrix} W_{t+1} \\
&= (\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t + \beta_v \begin{bmatrix} 0 & \sigma_z \end{bmatrix} W_{t+1} + \begin{bmatrix} \sigma_y & 0 \end{bmatrix} W_{t+1}
\end{aligned}$$

This implies that

$$M_{t+1} | \mathfrak{F}_t \sim \mathcal{N}\left((\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t, \beta_v^2\sigma_z^2 + \sigma_y^2\right)$$

i.e. M_{t+1} is conditionally normally distributed with mean $(\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t$ and variance $\beta_z^2\sigma_z^2 + \sigma_y^2$.

▷ Then consider:

$$\mathbb{R}(M_{t+1}|\mathfrak{F}_t) = \frac{1}{1-\gamma}\log\mathbb{E}\left(\exp\left[(1-\gamma) M_{t+1}\right] | \mathfrak{F}_t\right)$$

Since

$$(1-\gamma) M_{t+1} | \mathfrak{F}_t \sim \mathcal{N}\left((1-\gamma)\left[(\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t\right], (1-\gamma)^2\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}\right)$$

it follows that

$$\log\mathbb{E}\left[\exp\left[(1-\gamma) M_{t+1}\right]\right] = (1-\gamma)\left[(\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t\right] + \frac{1}{2}(1-\gamma)^2\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}$$

which implies

$$\mathbb{R}(M_{t+1}|\mathfrak{F}_t) = \left[(\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t\right] + \frac{1}{2}(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}$$

▷ Going back to our original recursion, we have

$$v_t = \exp(-\delta)\,\mathbb{R}(M_{t+1}|\mathfrak{F}_t)$$

Plugging in our conjecture:

$$\begin{aligned}\alpha_v + \beta_v Z_t &= \exp(-\delta)\left[\left[(\alpha_v + \alpha_v) + (\beta_v\beta_z + \beta_y) Z_t\right] + \frac{1}{2}(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}\right] \\ &= \left[\exp(-\delta)(\alpha_v + \alpha_y) + \frac{1}{2}\exp(-\delta)(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}\right] + \left[\exp(-\delta)(\beta_v\beta_z + \beta_y)\right] Z_t\end{aligned}$$

which yields:

$$\begin{aligned}\alpha_v &= \exp(-\delta)(\alpha_v + \alpha_y) + \frac{1}{2}\exp(-\delta)(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\} \\ \beta_v &= \exp(-\delta)(\beta_v\beta_z + \beta_y)\end{aligned}$$

Rearranging:

$$\begin{aligned}\alpha_v &= \frac{\exp(-\delta)\alpha_y + \frac{1}{2}\exp(-\delta)(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}}{1-\exp(-\delta)} \\ \beta_v &= \frac{\exp(-\delta)\beta_y}{1-\beta_z\exp(-\delta)}\end{aligned}$$

■

Problem 1.5. Consider two processes for $\{Z_t : t \geq 0\}$. The processes differ in terms of the coefficients $\left(\beta_z^i, \sigma_z^i\right)$ with for process i and the remaining coefficients held the same across models. Presume that $\beta_z^1 > \beta_z^2$ and that σ_z^is are adjusted so that the stationary distribution for the resulting $\{Z_t : t \geq 0\}$ remains the same. Use your answer in part 1a to answer this. Set $\alpha_y = 0$ to facilitate a small δ limiting comparison between the two processes. Compare the coefficients α_v^i and β_v^i for the two processes in the $\delta = 0$ limit using your answer to 1(d).

Solution. Recall that the stationary distribution for Z_t is given as:

$$\mathcal{N}\left(0, \frac{\sigma_z^2}{1-\beta_z^2}\right)$$

- ▷ If we have $\beta_z^1 > \beta_z^2$, then $\sigma_z^1 < \sigma_z^2$ in order to preserve the original stationary distribution.
- ▷ Now set $\alpha_y = 0$ to the coefficients derived in the previous section:

$$\begin{aligned}\alpha_v &= \frac{\frac{1}{2}\exp(-\delta)(1-\gamma)\left\{\beta_v^2\sigma_z^2 + \sigma_y^2\right\}}{1-\exp(-\delta)} \\ \beta_v &= \frac{\exp(-\delta)\beta_y}{1-\beta_z\exp(-\delta)}\end{aligned}$$

so α_v is increasing in σ_z, which means it is decreasing in β_z. β_v is increasing in β_z.

We see that as $\delta \to \infty$, $\beta_v \to \beta_y$ i.e. the two processes with different $\beta_z^1 > \beta_z^2$ converges to the same value. ■

Problem 1.6. How does the decision maker view persistence in the growth rate process in this example? Please provide only a short answer that builds, if possible on your answer to part 1e.

Solution. We saw that higher persistence in the growth rate (β_z) implies a lower α_v but a higher β_v . So he prefers higher persistence if β_v is sufficiently higher than α_v. ■

2 Markov Chain

Suppose that a process $\{Z_t : t \geq 0\}$ evolves as a two-state Markov Chain with transition matrix:

$$\mathbb{P} = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}$$

where $0 < p_{ii} \leq 1, \forall i =\in \{1, 2\}$. Let realized values of Z_t be one of the two coordinate vectors. Suppose when state i is realized, the next period macroeconomic growth rate, $Y_{t+1} - Y_t$, is distributed as a normal with mean μ_i and standard deviation σ_i.

Problem 2.1. Suppose that p_{11} and p_{22} are both strictly less than 1. Provide a formula for the stationary distribution of $\{Z_t : z \geq 0\}$. Is this process ergodic under this stationary distribution?

Solution. The problem statement implies that

$$\begin{aligned} Y_{t+1} - Y_t &= \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} Z_t + \begin{bmatrix} \sigma_1 & \sigma_2 \end{bmatrix} W_{t+1} \\ \mathbb{E}\left[Z_{t+1} | \mathfrak{F}_t\right] &= \mathbb{P}' Z_t \end{aligned}$$

where

$$Z_t \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad W_{t+1} \sim \mathcal{N}(0, 1)$$

To find the stationary distribution, we can solve for

$$\begin{bmatrix} \pi & 1 - \pi \end{bmatrix} \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix} = \begin{bmatrix} \pi & 1 - \pi \end{bmatrix}$$

which yields:

$$\mathbb{E}\left[Z_t\right] = \begin{bmatrix} \pi \\ 1 - \pi \end{bmatrix}$$

where

$$\pi = \frac{1 - p_{22}}{2 - p_{11} - p_{22}}, \quad 1 - \pi = \frac{1 - p_{11}}{2 - p_{11} - p_{22}}$$

The process is indeed ergodic under this stationary distribution.

■

Problem 2.2. Provide a formula for the mean and the variance of $\{Y_{t+1} - Y_t : t \geq 0\}$ for the stationary distribution that you computed in part 2a. Will time-series estimators recover this mean and variance? If so, what estimators? Will the implied stationary distribution for the macro growth rate process be a normal distribution? Why or why not?

Solution. Recall that we had:

$$\begin{aligned} Y_{t+1} - Y_t &= \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} Z_t + \begin{bmatrix} \sigma_1 & \sigma_2 \end{bmatrix} W_{t+1} \\ \mathbb{E}\left[Z_{t+1} | \mathfrak{F}_t\right] &= \mathbb{P}' Z_t \end{aligned}$$

▷ The mean can be obtained by

$$\mathbb{E}\left[Y_{t+1} - Y_t\right] = \mu_1 \pi + \mu_2 \left(1 - \pi\right)$$

▷ To obtain the variance, first note that

$$\text{Var}\left[Z_t\right] = \begin{bmatrix} \pi - \pi^2 & -\pi\left(1-\pi\right) \\ -\pi\left(1-\pi\right) & \pi - \pi^2 \end{bmatrix}$$

* To see this:

$$\begin{aligned}
\text{Var}\left[Z_{t+1}\right] &= \mathbb{E}\left[\left(Z_{t+1} - \mathbb{E}\left[Z_{t+1}\right]\right)\left(Z_{t+1} - \mathbb{E}\left[Z_{t+1}\right]\right)^\top\right] = \mathbb{E}\left[Z_{t+1} Z_{t+1}^\top\right] - \mathbb{E}\left[Z_{t+1}\right]\mathbb{E}\left[Z_{t+1}\right]^\top \\
&= \mathbb{E}\left[\begin{pmatrix} \left(Z_{t+1}^1\right)^2 & Z_{t+1}^1 Z_{t+1}^2 \\ Z_{t+1}^1 Z_{t+1}^2 & \left(Z_{t+1}^2\right)^2 \end{pmatrix}\right] - \mathbb{E}\left[\begin{pmatrix} Z_{t+1}^1 \\ Z_{t+1}^2 \end{pmatrix}\right]\mathbb{E}\left[\begin{pmatrix} Z_{t+1}^1 & Z_{t+1}^2 \end{pmatrix}\right]^\top \\
(\because)\, Z_{t+1}^2 = Z_{t+1} &= \begin{bmatrix} \mathbb{E}\left[Z_{t+1}^1\right] - \mathbb{E}\left[Z_{t+1}^1\right]^2 & 0 - \mathbb{E}\left[Z_{t+1}^1\right]\mathbb{E}\left[Z_{t+1}^2\right] \\ 0 - \mathbb{E}\left[Z_{t+1}^1\right]\mathbb{E}\left[Z_{t+1}^2\right] & \mathbb{E}\left[Z_{t+1}^2\right] - \mathbb{E}\left[Z_{t+1}^2\right]^2 \end{bmatrix} \\
&= \begin{bmatrix} \pi - \pi^2 & -\pi\left(1-\pi\right) \\ -\pi\left(1-\pi\right) & \left(1-\pi\right) - \left(1-\pi\right)^2 \end{bmatrix}
\end{aligned}$$

* Simplifying yields:

$$\text{Var}\left[Z_t\right] = \begin{bmatrix} \pi - \pi^2 & -\pi\left(1-\pi\right) \\ -\pi\left(1-\pi\right) & \pi - \pi^2 \end{bmatrix}$$

* Now writing out the variance of $Y_{t+1} - Y_t$:

$$\begin{aligned}
\text{Var}\left[Y_{t+1} - Y_t\right] &= \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} \text{Var}\left[Z_t\right] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \left(\sigma_1^2 + \sigma_2^2\right) \\
&= \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} \pi - \pi^2 & -\pi\left(1-\pi\right) \\ -\pi\left(1-\pi\right) & \pi - \pi^2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \left(\sigma_1^2 + \sigma_2^2\right) \\
&= \begin{bmatrix} \mu_1\left(\pi - \pi^2\right) - \mu_2\left(\pi - \pi^2\right) & -\mu_1\left(\pi - \pi^2\right) + \mu_2\left(\pi - \pi^2\right) \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \left(\sigma_1^2 + \sigma_2^2\right) \\
&= \mu_1^2\left(\pi - \pi^2\right) - 2\mu_1\mu_2\left(\pi - \pi^2\right) + \mu_2^2\left(\pi - \pi^2\right) + \left(\sigma_1^2 + \sigma_2^2\right)
\end{aligned}$$

Note that the stationary distribution is not necessarily normal because Z_t is not necessarily normally distributed.

■

Problem 2.3. Now suppose that $p_{11} = p_{22} = 1$. What are the implied stationary distributions for $\{Z_t : t \geq 0\}$? Under which of these distributions the process be ergodic?

Solution. If $p_{11} = p_{22} = 1$, then the transition matrix is

$$\mathbb{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so any stationary distributions $\left[\alpha, 1-\alpha\right]^\top$can be sustained. However the process is ergodic only under the following stationary distributions:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

■

3 Markov Chain

For some GMM estimation problems, there is a special structure for the moment restrictions captured by

$$F(x,b) = \begin{bmatrix} F_1(x,b_1) \\ F_2(x,b_1,b_2) \end{bmatrix}$$

where

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and b_1 has k_1 entries, b_2 has k_2 entries, F_1 has $r_1 = k_1$ entries and F_2 has $r_2 > k_2$ entries. Suppose that

$$\mathbb{E}\left[F_1(X_t,\beta_1)\right] = 0$$

is used to identify and estimate β_1. Given the estimate of β_1,

$$\mathbb{E}\left[F_2(X_t,\beta_1,\beta_2)\right] = 0$$

is used to identify and estimate β_2 given an initial estimate of β_1. Recall the GMM approximation formulas:

$$\sqrt{N}(b_N-\beta) \approx -\left(A'D\right)^{-1} A' \frac{1}{\sqrt{N}} \sum_{t=1}^{N} F(X_t,\beta)$$

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} F(X_t,\beta) \approx \left[I - D\left(A'D\right)^{-1} A'\right] \frac{1}{\sqrt{N}} \sum_{t=1}^{N} F(X_t,\beta)$$

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} F(X_t,\beta) \Rightarrow \text{Normal}(0,V)$$

Problem 3.1. Show that the matrix D is lower block triangular:

$$D = \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}$$

where the partitioning is conformable with the partitioning of b and F.

Solution. Note that the D matrix is defined as

$$D = \mathbb{E}\left[\partial F(X_t,\beta)/\partial\beta\right]$$

Denote

$$\begin{aligned}
D_{11} &:= \mathbb{E}\left[\partial F_1(X_t,\beta_1)/\partial\beta_1\right] \\
D_{12} &:= \mathbb{E}\left[\partial F_1(X_t,\beta_1)/\partial\beta_2\right] \\
D_{21} &:= \mathbb{E}\left[\partial F_2(X_t,\beta_1,\beta_2)/\partial\beta_1\right] \\
D_{22} &:= \mathbb{E}\left[\partial F_2(X_t,\beta_1,\beta_2)/\partial\beta_2\right]
\end{aligned}$$

It is thus clear that $D_{12} = 0$ since F_1 is not a function of β_2. ■

Problem 3.2. The two-step nature of the estimation can be captured by a block diagonal selection matrix:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Explain.

Solution. Recall that our desired moment conditions are:

$$\mathbb{E}\left[F_1\left(X_t, \beta_1\right)\right] = 0$$
$$\mathbb{E}\left[F_2\left(X_t, \beta_1, \beta_2\right)\right] = 0$$

and the corresponding GMM estimators are given by solutions to

$$A'_{11} \frac{1}{N} \sum_{t=1}^{N} F_1\left(X_t, \beta_1\right) = 0$$
$$A'_{22} \frac{1}{N} \sum_{t=1}^{N} F_2\left(X_t, \beta_1, \beta_2\right) = 0$$

which can be expressed as

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}' \begin{bmatrix} \frac{1}{N}\sum_{t=1}^{N} F_1\left(X_t, \beta_1\right) & \frac{1}{N}\sum_{t=1}^{N} F_2\left(X_t, \beta_1, \beta_2\right) \\ \frac{1}{N}\sum_{t=1}^{N} F_1\left(X_t, \beta_1\right) & \frac{1}{N}\sum_{t=1}^{N} F_2\left(X_t, \beta_1, \beta_2\right) \end{bmatrix} = 0$$

■

Problem 3.3. The matrix $(A_{11})'$ can be set to $(D_{11})^{-1}$ for the purposes of deriving limiting properties provided that D_{11} is non-singular. Explain.

Solution. This is setting A_{11} to be the efficient selection matrix.

▷ To see this, recall that for a given A, the asymptotic covariance matrix for a GMM estimator is given as

$$\text{Cov}\left[A\right] = (AD)^{-1} AVA' \left(D'A'\right)^{-1}$$

▷ Now consider A that satisfies the above and any BA where B is non-singular. Plugging into the above formula yields:

$$\text{Cov}\left[BA\right] = \text{Cov}\left[A\right]$$

which means that without loss of generality, we may assume that

$$AD = 1$$

provided D is non-singular.

Since here we are focusing on β_1, we can set $A_{11}D_{11} = I$ and retain the same limiting properties. ■

Problem 3.4. Construct an approximation formula for the GMM estimator of β_2 as a function of the selection matrix A_{22} taking account of the initial stage estimation. You may use the following approximation formulas for a family of GMM estimators.

Solution. The estimation exercise of interest is

$$A_{22} g_N^{[2]} \left(b_N^1, \beta^{[2]} \right) = 0$$

To proceed, we use partitioning and the property

$$\sqrt{N} \left(b_N - \beta_0 \right) \approx - \left(AD \right)^{-1} A \sqrt{N} g_N \left(\beta_0 \right)$$

to obtain the limiting distribution for the estimator $b_N^{[2]}$:

$$\sqrt{N} \left(b_N^{[2]} - \beta_0^{[2]} \right) \approx - \left(A_{22} D_{22} \right)^{-1} A_{22} \left[\begin{array}{cc} -D_{21} \left(A_{11} D_{11} \right)^{-1} A_{11} & I \end{array} \right] \sqrt{N} g_N \left(\beta_0 \right)$$

▷ Recall that

$$A = \left[\begin{array}{cc} A_{11} & 0 \\ 0 & A_{22} \end{array} \right], \qquad D = \left[\begin{array}{cc} D_{11} & 0 \\ D_{21} & D_{22} \end{array} \right]$$

and the inverse formula for the block matrix:

$$\left[\begin{array}{cc} A & B \\ C & D \end{array} \right]^{-1} = \left[\begin{array}{cc} \left(A - BD^{-1} C \right)^{-1} & -A^{-1} B \left(D - CA^{-1} B \right)^{-1} \\ -D^{-1} C \left(A - BD^{-1} C \right)^{-1} & \left(D - CA^{-1} B \right)^{-1} \end{array} \right]$$

to obtain:

$$AD = \left[\begin{array}{cc} A_{11} D_{11} & 0 \\ A_{22} D_{21} & A_{22} D_{22} \end{array} \right] \Rightarrow \left(AD \right)^{-1} = \left[\begin{array}{cc} \left(A_{11} D_{11} \right)^{-1} & 0 \\ - \left(A_{22} D_{22} \right)^{-1} \left(A_{22} D_{21} \right) \left(A_{11} D_{11} \right)^{-1} & \left(A_{22} D_{22} \right)^{-1} \end{array} \right]$$

and thus:

$$\begin{aligned} \left(AD \right)^{-1} A &= \left[\begin{array}{cc} \left(A_{11} D_{11} \right)^{-1} & 0 \\ - \left(A_{22} D_{22} \right)^{-1} \left(A_{22} D_{21} \right) \left(A_{11} D_{11} \right)^{-1} & \left(A_{22} D_{22} \right)^{-1} \end{array} \right] \left[\begin{array}{cc} A_{11} & 0 \\ 0 & A_{22} \end{array} \right] \\ &= \left[\begin{array}{cc} \left(A_{11} D_{11} \right)^{-1} A_{11} & 0 \\ - \left(A_{22} D_{22} \right)^{-1} \left(A_{22} D_{21} \right) \left(A_{11} D_{11} \right)^{-1} A_{11} & \left(A_{22} D_{22} \right)^{-1} A_{22} \end{array} \right] \end{aligned}$$

▷ Looking at the second block, we thus have:

$$\left[\begin{array}{c} \sqrt{N} \left(b_N^{[1]} - \beta_0^{[1]} \right) \\ \sqrt{N} \left(b_N^{[2]} - \beta_0^{[2]} \right) \end{array} \right] \approx \left[\begin{array}{cc} \left(A_{11} D_{11} \right)^{-1} A_{11} & 0 \\ - \left(A_{22} D_{22} \right)^{-1} \left(A_{22} D_{21} \right) \left(A_{11} D_{11} \right)^{-1} A_{11} & \left(A_{22} D_{22} \right)^{-1} A_{22} \end{array} \right] \left[\begin{array}{c} \sqrt{N} g_N \left(\beta_0^{[1]} \right) \\ \sqrt{N} g_N \left(\beta_0^{[2]} \right) \end{array} \right]$$

Focusing on the second block, we have

$$\begin{aligned} \sqrt{N} \left(b_N^{[2]} - \beta_0^{[2]} \right) &\approx - \left(A_{22} D_{22} \right)^{-1} \left(A_{22} D_{21} \right) \left(A_{11} D_{11} \right)^{-1} A_{11} \left[\sqrt{N} g_N \left(\beta_0^{[1]} \right) \right] \\ &\quad + \left(A_{22} D_{22} \right)^{-1} A_{22} \left[\sqrt{N} g_N \left(\beta_0^{[2]} \right) \right] \\ &= - \left(A_{22} D_{22} \right)^{-1} A_{22} \left[\begin{array}{cc} -D_{21} \left(A_{11} D_{11} \right)^{-1} A_{11} & I \end{array} \right] \sqrt{N} g_N \left(\beta_0 \right) \end{aligned}$$

Therefore, we have the following desired approximation:

$$\sqrt{N} \left(b_N^{[2]} - \beta_0^{[2]} \right) \approx - \left(A_{22} D_{22} \right)^{-1} A_{22} \left[\begin{array}{cc} -D_{21} \left(A_{11} D_{11} \right)^{-1} A_{11} & I \end{array} \right] \sqrt{N} g_N \left(\beta_0 \right)$$

■