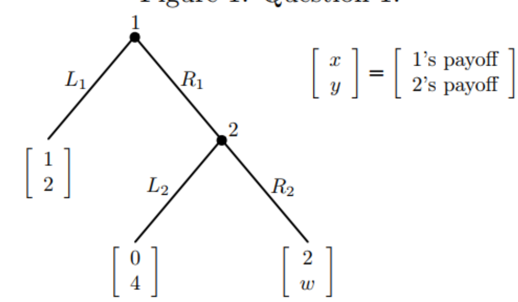
1 Game Theory

Consider the following game tree:

Figure 1: Question 1.



Problem 1.1. For what values of w in the game does there exist a pure strategy profile that is a Nash equilibrium but that is not a backward induction strategy profile?

Solution. Consider $\sigma_1 = [L_1]$ and $\sigma_2 = [R_2]$. Then this is a NE for $w \geq 4$. ■

Problem 1.2. For what values of w in the game does there exist a behavioral strategy profile that is a NE but that is not SPE?

Solution. Consider $\sigma_1 = [L_1]$ and $\sigma_2 = p\,[L_2] + (1-p)\,[R_2]$. For 1 to play $[L_1]$ instead of $[R_1]$, we require:

$$1 \geq 2\,(1-p) \Rightarrow p \geq 1/2$$

For this to be a NE but not SPE, it should be that $w \neq 4$. ■

2 Game Theory

Consider the strategic form game between players 1 and 2 in the following figure:

Figure 2: Question 2.

$1\backslash 2$	L	M	R
T	$1, 1-x$	$0, 0$	$-1, -3+x$
B	$-1, 7-x$	$0, 0$	$1, -9+x$

Problem 2.1. For what values of w does player 2 have a strictly dominant pure strategy?

Solution. Player 2 has such strategy for $x < 1, x > 9$. ■

Problem 2.2. For what values of x does L strictly dominate R for player 2?

Solution. This happens if $1 - x > -3 + x$ and $7 - x > -9 + x$, which happens when $x < 2$. ■

Problem 2.3. For what values of x does this game have no pure-strategy NE?

Solution. Looking at what player 1 chooses, there would be no pure-strategy equilibrium if player 2 chooses $[R]$ given 1 choosing $[T]$ and chooses $[L]$ given 1 choosing $[B]$. Mathematically:

$$\begin{aligned} -3 + x > 0, -3 + x > 1 - x \\ 7 - x > 0, 7 - x > -9 + x \end{aligned}$$

which yields:

$$3 < x < 7$$

■

3 Game Theory

Consider the game

Figure 2: Question 2.

$1\backslash 2$	L	M	R
T	$1, 1-x$	$0, 0$	$-1, -3+x$
B	$-1, 7-x$	$0, 0$	$1, -9+x$

but now suppose that x is known only to player 2 but is known to player 1. More precisely, suppose that Nature draws x uniformly from the interval $[0, 10]$ and that player 2 learns x before she must choose her action from $\{L, M, R\}$. Player 1 knows only that x is drawn uniformly from $[0, 10]$ and he must choose an action from $\{L, R\}$.

Problem 3.1. If player 1 chooses $[T]$, what types x of player 2 would choose L; what types x would choose M; and what types x would choose R?

Solution. Player 2 will choose L if $1 - x > 0$ and $1 - x > -3 + x$ or $x < 1$. Player 2 will choose R if $x > 3$. Players in $(1, 3)$ will choose $[M]$. ■

Problem 3.2. If player 1 chooses $[B]$, what types x of player 2 would choose $[L]$; what types x would choose $[M]$; and what types x would choose $[R]$?

Solution. Player 2 will choose $[L]$ if $x < 7$; $[M]$ if $x \in (7, 9)$; and $[R]$ if $x > 9$. ■

Problem 3.3. Based on your answers to parts (a) and (b), argue that this game has no Bayes-Nash equilibrium in which player 1 uses a pure strategy.

Solution. Suppose player 1 plays $[T]$ in equilibrium. Then player 2 plays according to the previous rule, in which case player 1 wants to play $[B]$ instead, so it cannot be maintained as an equilibrium. A symmetric argument holds for why player 1 does not play $[B]$ in equilibrium. ■

Problem 3.4. For player 1 to be indifferent between $[T]$ and $[B]$, what must be true about the fractions of player 2's types who choose $[L]$ and who chose $[R]$?

Solution. For player 1 to be indifferent the expected payoff must be the same; this implies that the fractions of player 2's types choosing $[L]$ and choosing $[R]$ must be equal. ■

Problem 3.5. If player 1 chooses $[T]$ with probability p, what types x of player 2 would choose $[L]$; what types x would choose $[M]$; and what types x would choose $[R]$?

Solution. Computing the expected payoff for each strategy, we have:

$$\begin{aligned}
[L] &: p\,(1-x) + (1-p)\,(7-x) = 7 - x - 6p \\
[M] &: 0 \\
[R] &: p\,(-3+x) + (1-p)\,(-9+x) = -9 + x + 6p
\end{aligned}$$

and so we have

$$\sigma_2 = \begin{cases} [L] & x \in [0, 7-6p] \\ [M] & x \in [7-6p, 9-6p] \\ [R] & x \in [9-6p, 10] \end{cases}$$

■

Problem 3.6. Find a Bayes-Nash equilibrium of this game.

Solution. Imposing the constraint that the fraction of people has to be the same, we have

$$7 - 6p = 10 - (9 - 6p) \Rightarrow p = \frac{1}{2}$$

Therefore, the BNE is

$$\sigma_1 = \frac{1}{2}\,[T] + \frac{1}{2}\,[B]$$

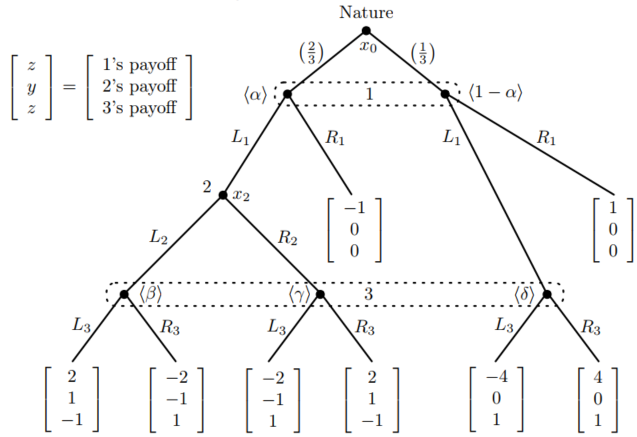
$$\sigma_2 = \begin{cases} [L] & x \in [0, 4] \\ [M] & x \in [4, 6] \\ [R] & x \in [4, 10] \end{cases}$$

■

4 Game Theory

Consider the extensive form game.

Figure 3: Question 4.



Problem 4.1. Find a pure strategy profile that is a NE but for which there is no system of belief that could ever make the resulting assessment sequentially rational.

Solution. Consider

$$\sigma_1 = [R_1], \quad \sigma_2 = [R_2], \quad \sigma_3 = [L_3]$$

We can verify that this is a NE since players do not have the incentive to deviate. To see that there is no system of belief that could ever make the resulting assessment sequentially rational, observe that player 2 has an incentive to deviate at his node. ■

Problem 4.2. Find a sequentially rational assessment whose strategy profile is pure but is not a NE.

Solution. We will cook up some "fucked up" beliefs so that it is optimal to play a pure strategy profile that is not NE. Consider

$$\sigma_1 = [R_1], \sigma_2 = [L_1], \sigma_3 = [L_3]$$

This can be sustained if $\beta = 0, \gamma = 1, \delta = 0$ (since 3 will always play L_3). Also, let $\alpha = 0$ so that player 1 finds it optimal to play $[R_1]$. But this is not NE since given the true Nature's probabilities, he will find it optimal to play $[L_1]$ instead. ■

Problem 4.3. Show that if $(\alpha, \beta, \gamma, \delta, q_1, q_2, q_3)$ is consistent, then $\delta = 1/3$.

Solution. First, note that consistency requires $\alpha = 2/3$. Consistency requires that for any $q_1^n, q_2^n \in (0,1)$ that converges to equilibrium strategies, we have

$$\delta^n \to \delta$$

The LHS can be written as

$$\delta^n = \frac{(1-\alpha)\, q_1^n}{\alpha q_1^n q_2^n + \alpha q_1^n (1-q_2^n) + (1-\alpha)\, q_1^n} = \frac{(1-\alpha)\, q_1^n}{\alpha q_1^n + (1-\alpha)\, q_1^n} = 1-\alpha = \frac{1}{3}$$

so we conclude that $\delta = 1/3$. ■

Problem 4.4. Find a sequential equilibrium assessment $(\alpha, \beta, \gamma, \delta, q_1, q_2, q_3)$ using your answer to the previous part.

Solution. We showed above that $\alpha = 2/3$ and $\delta = 1/3$. Furthermore, at the node where δ is formed, the payoff is equal for $[L_3]$ and $[R_3]$ which implies that 2 must be mixing. Furthermore, at the x_2 node, we have a matching pennies game between 2 and 3, so they must be mixing with equal probabilities, yielding:

$$q_2 = q_3 = \frac{1}{2}$$

Now 1's expected payoffs are

$$\begin{aligned} [L_1] &: \frac{2}{3}(0) + \frac{1}{3}\left(\frac{1}{2}(-4) + \frac{1}{2}(4)\right) = 0 \\ [R_1] &: -\frac{1}{3} \end{aligned}$$

so $q_1 = 1$. Bayes rule implies that $\beta = 1/3$ and $\gamma = 1/3$. There is no need to check explicitly for consistency since every information set is reached with positive probability. ■