Economics 331 Prof. Stokey

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# A BRIEF GUIDE TO DYNAMIC PROGRAMMING

## PART I. DETERMINISTIC PROBLEMS

### Overview

Dynamic programming is a method for solving (single-agent) optimization problems. We will use discrete time throughout, and for the most part will look at problems with an infinite horizon. These notes provide an introduction, taking a very 'cookbook' approach. They describe the basic steps in setting up and analyzing problems, and illustrate those steps with a series of examples. The examples have been chosen to display various limitations: to show what can go wrong and why, as well as what happens when everything goes right.

The main steps are:

1. Write the optimization problem as a sequence problem (SP), and define the value function $v$ in terms of this problem. Call solutions to (SP) *optimal plans*. (Plans are *sequences of actions*.) This step requires choosing the state variable(s) $x$ and state space $X$. We will focus on problems where $X \subseteq R^{\ell}$, although the techniques work fine for other spaces. It also involves defining the correspondence $\Gamma(x)$ describing the set of feasible choices $x'$ for next period, given today's state $x$; a one-period return function $F(x, x')$; and a one-period discount factor $\beta \in (0, 1)$.

2. Check some basic conditions: that

—the feasible set $\Gamma(x)$ is nonempty, for all $x \in X$;

—some limiting conditions hold that involve discounting.

3. Formulate the Bellman equation (BE), which involves an (as-yet-unknown) value function $\hat{v}$. Define the *optimal policy correspondence* $G(x) \subseteq \Gamma(x)$, the set of optimal choices for $x'$ for next period, given the current state $x$. The conditions in Step 2 insure that $v$ defined by the Sequence Problem satisfies the BE, and it is the only solution.

4. Check some additional condition for $X$, $\Gamma$, $F$ to insure that CMT applies.

5. Properties of $v, G$: check conditions on the return function $F$ and correspondence $\Gamma$ to insure $v$ is

—monotone

—concave (and the associated policy *function* $g$ is single valued)

—once differentiable

—twice differentiable (and the policy function is once differentiable).

6. Characterize steady state(s).

7. Euler equations: Suppose $v$ is concave and differentiable, so $g$ is single valued. Calculate the linear approximation to transitional dynamics near the SS. Use the resulting Euler equations to develop properties of $g$ and to study the transitional dynamics.

8. Study global properties of the transition path.

9. Ask how $v, g$ vary with exogenous parameters, such as

—the discount factor $\beta$,

—parameters of return function $F$

—parameters in the feasibility constraint(s) $\Gamma$.

**Step 1: Write the sequence problem**

Write the sequence problem (SP): given an initial state $x_0 \in X$, define $v(x_0)$ as

$$\begin{aligned} v(x_0) \quad &\equiv \quad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad \text{(SP)} \qquad (1) \\ \text{s.t.} \quad x_{t+1} \quad &\in \quad \Gamma(x_t), \qquad \text{all } t. \end{aligned}$$

Primitives:

a. The state variable(s) $x$ and state space $X \subseteq R^{\ell}$.

b. The correspondence $\Gamma$: $X \rightarrow X$ describing the feasibility constraint(s).

If the current state is $x$, then $\Gamma(x)$ is the feasible set for next period's state $y$.

If $X \subseteq R$, then $\Gamma$ may take the form $\Gamma(x) = [m(x), M(x)]$, all $x \in X$,

c. The return function $F(x, y)$. Make sure $F$ is defined on the set

$$A = \{(x, y) : x \in X, \quad y \in \Gamma(x)\}. \qquad (2)$$

d. The discount factor $\beta \in (0, 1)$.

**Step 2: Check some basic conditions**

For each $x \in X$, the feasible set is $\Gamma(x)$ is nonempty.

Some limiting conditions hold as $t \rightarrow \infty$ (enough discounting). These are always satisfied if $F$ is bounded and $\beta < 1$. They also hold more generally, but in cases where $F$ is not bounded, some conditions are needed to insure that growth is not "too fast" (relative to $\beta$).

**Step 3: Formulate Bellman equation**

Formulate the Bellman equation (BE)

$$\hat{v}(x) = \sup_{y \in \Gamma(x)} \left[ F(x, y) + \beta \hat{v}(y) \right], \quad \text{all } x \in X. \qquad \text{(BE)} \tag{3}$$

Note that (BE) is a *functional equation* in the (as yet) *unknown* function $\hat{v}$.

The conditions in Step 2 insure that $v$ defined in Step 1 is the unique solution to BE. That is,

$v$ defined in (SP) satisfies this equation *and* there are no other solutions.

**Step 4: Check properties of $X, F, \Gamma$**

**Sufficient conditions to insure DP methods apply.—**

a. State space is a convex subset $X \subseteq R^\ell$, OR a countable set $X = \{x_1, x_2, ....\}$.

b. For each $x \in X$, the feasible set is $\Gamma(x)$ is nonempty and compact.

In addition, $\Gamma(x)$ is continuous as a correspondence: the feasible set

$\Gamma(x)$ varies in a continuous way with $x$.

If $\Gamma(x) = [m(x), M(x)]$, this requires that $m(x)$ and $M(x)$ are finite,

with $m(x) \leq M(x)$, all $x$, and that both are continuous functions.

c. The return function $F$: $A \to R$ is continuous and is bounded on the set $A$ defined in (2). (It suffices if $F$ is bounded OR $X$ is closed and bounded). In addition $0 < \beta < 1$.

So far everything has been in terms of 'sup' rather than 'max'. But if $F$ is bounded and continuous, and $\Gamma(x)$ is compact-valued, then the set $G(x)$ is non-empty and we can replace 'sup' with 'max.' In this case the conditions in Step 2 insure:

every optimal plan can be generated by using $G$,

that every sequence generated by $G$ is an optimal plan.

**Basic conclusions.—**

(i) It then follows that $v$ in (1) is bounded:

$$|F(x,y)| \leq B, \quad \text{all } (x,y) \in A, \qquad \Longrightarrow \qquad |v(x)| \leq \frac{B}{1-\beta}, \quad \text{all } x \in X.$$

(ii) Suppose $\hat{v}(y)$ on the RHS of (3) is a continuous function. Since $F(x, \cdot)$ is continuous in $y$, the fact that $\Gamma(x)$ is compact (for example, that $[m(x), M(x)] \subset \mathbf{R}$ is a closed, bounded interval) insures the maximum in (3) is attained.

(iii) Since $F$ is continuous and $\Gamma$ is a continuous correspondence, the Theorem of the Maximum insures that the maximized RHS in (3) varies continuously with $x$.

(iv) Let $C(X)$ be the space of bounded, continuous functions $f : X \to \mathbf{R}$. Define the operator $T : C(X) \to C(X)$ by

$$Tf(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta f(y)\right].$$

Why are we sure that $Tf(x)$ is in $C(X)$? From (i) - (iii).

(v) The mapping $T$ is a contraction of modulus $\beta$. (Why? It satisfies Blackwell's sufficient conditions.) Hence $v$ defined in (SP) is the **UNIQUE** solution to (BE)

(vi) The set of maximizers $G(x)$ is nonempty, compact-valued, and (from the Theorem of the Maximum) u.h.c. (It can 'explode' but not 'implode.')

(vii) $v$ can be computed by successive approximations: for any continuous, bounded $v_0$, the sequence $\{v_n\}$ defined by $v_{n+1} = Tv_n$ has the property that $v_n \to v$, and the rate of convergence is $\beta$.

**Step 5: Properties of $v, G$**

**a. Monotonicity of $v$.—**

Suppose:

—$F(\cdot, y)$ is (strictly) increasing in its first argument;

—$\Gamma$ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.

Then $v$ is (strictly) increasing.

Proof 1: Let $\hat{C} \subset C$ be the space of weakly monotone functions. Apply the operator $T$ on $\hat{C}$.

Proof 2: Note that if $F, \Gamma$ are as above, and $x \leq x'$, then for *any* $f \in C$,

$$\begin{aligned} Tf(x) &= \max_{y \in \Gamma(x)} [F(x, y) + \beta v(f(y))] \\ &\leq \max_{y \in \Gamma(x')} [F(x', y) + \beta v(f(y))] \\ &= Tf(x'), \end{aligned}$$

Hence $T$: $C \rightarrow \hat{C}$. Moreover, the inequality is strict if $x < x'$ and $F$ is strictly increasing in $x$, so in this case $T$: $C \rightarrow \text{int}(\hat{C})$.

**b. Concavity of $v$, single-valued $g$.—**

For any

$$x, x', \ y, y' \in X, \qquad \theta \in (0,1),$$

let

$$x_\theta = \theta x + (1-\theta)\, x', \qquad y_\theta = \theta y + (1-\theta)\, y'.$$

Suppose:

—$\Gamma$ is convex in the sense that

$$y \in \Gamma(x), \ y' \in \Gamma(x') \quad \Longrightarrow \quad y_\theta \in \Gamma(x_\theta).$$

—$F$ is concave in the sense that

$$F(x_\theta, y_\theta) \geq \theta F(x,y) + (1-\theta)\, F(x', y'),$$

and the inequality is strict if $x \neq x'$.

Then:

a. $v$ is strictly concave,

b. the optimal policy correspondence is a (continuous) single-valued function $g$;

c. the sequence of approximate policy functions $\{g_n\}$ defined by

$$\begin{aligned} v_{n+1} &= T v_n(x) \\ g_n &= \arg\max_{y \in \Gamma(x)} \left[ F(x,y) + \beta v_n(y) \right], \end{aligned}$$

converges pointwise.

**c. Differentiability of $v$.—**

Argument of Benveniste and Scheinkman. Suppose:

—$F$ is continuously differentiable in its first argument

—$v$ is concave

—$\hat{x} \in \text{int } X$ and $g(\hat{x}) \in \text{int } \Gamma(\hat{x})$.

Then $v$ is differentiable at $\hat{x}$, and

$$v_i(\hat{x}) = F_i(\hat{x}, g(\hat{x})), \qquad i = 1, ..., \ell.$$

Proof: (picture + think of envelop theorem)

Key idea is that for $x$ near $\hat{x}$, $g(\hat{x}) \in \Gamma(x)$. This is true because

the correspondence $\Gamma$ is continuous—i.e. the set $\Gamma(x)$ changes continuous way with $x$, and $g(\hat{x})$ is in the *interior* of $\Gamma(\hat{x})$ (by hypothesis).

Hence for $x$ near $\hat{x}$,

$$\begin{aligned} v(x) &= F(x, g(x)) + \beta v(g(x)) \\ &\geq F(x, g(\hat{x})) + \beta v(g(\hat{x})). \end{aligned}$$

Since $F$ is differentiable in its first argument at $\hat{x}$, the function $v$ is also differentiable at $\hat{x}$, and its derivative is

$$v'(\hat{x}) = F_x(\hat{x}, g(\hat{x})).$$

When does differentiability fail? Suppose $g(x)$ is on the boundary of the feasible set, and the boundary has a kink at $\hat{x}$. Then the optimal policy $g(x)$ must 'change direction' at $\hat{x}$ to maintain feasibility. For example, suppose $X = [0, 2]$, $\Gamma(x) = [0, M(x)]$, where $M(x) \equiv \min\{x, 1\}$, and $g(x) = M(x)$. Then $v(x) = F(x, M(x)) + \beta v(M(x))$. Since $M(x)$ is not differentiable at $\hat{x} = 1$, one can construct $F$ so that $v$ has a kink at that point.

**d. Twice differentiability of $v$, differentiablity of $g$.**—

For twice differentiability see

Santos, Manuel S. 1991. Smoothness of the policy function in discrete time economic models," *Econometrica*, 59(5): 1365-82.

**Step 6: Characterize steady states**

Suppose $v$ is concave, so the optimal policy function $g(x)$ is single valued.

$\overline{x}$ is a steady state if and only if $g(\overline{x}) = \overline{x}$.

**Step 7: Euler equations**

Suppose $v$ is concave and differentiable.

Use (BE) to get the first order condition (FOC) and the envelop condition (EC)

$$\begin{aligned} 0 &= F_y[x, g(x)] + \beta v'(g(x)), \qquad \text{(FOC)} \qquad (4) \\ v'(x) &= F_x(x, g(x)). \qquad \text{(EC)} \qquad (5) \end{aligned}$$

Combine these to get the Euler equation (EE)

$$0 = F_y[x, g(x)] + \beta F_x[g(x), g(g(x))], \quad \text{all } x. \qquad \text{(EE)} \qquad (6)$$

Using

$$x = x_t, \quad g(x) = x_{t+1}, \quad g(g(x)) = x_{t+2},$$

gives the traditional Euler equation

$$0 = F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_{t+1}, x_{t+2}\right), \quad \text{all } t. \qquad \text{(EEt)} \qquad (7)$$

Note that $F_x, F_y$ and $v'$ are all vectors of length $\ell$, where $X \subset R^\ell$ is the state space.

**Step 8: Linearized dynamics**

Suppose $v$ is concave and differentiable.

An interior steady state $\overline{x}$ requires

$$0 = F_y(\overline{x}, \overline{x}) + \beta F_x(\overline{x}, \overline{x}).$$

Suppose $x_t$ is near $\overline{x}$. Let $z_t = x_t - \overline{x}$ and use a first-order Taylor series approximation to get

$$\begin{aligned} F_y(x_t, x_{t+1}) &\approx F_y + F'_{xy} z_t + F_{yy} z_{t+1}, \\ F_x\left(x_{t+1}, x_{t+2}\right) &\approx F_x + F_{xx} z_{t+1} + F_{xy} z_{t+2}, \end{aligned}$$

where $F_y, F_{xy}$, etc. are all evaluated at $\left(\overline{x}, \overline{x}\right)$, and $F'_{xy} = F_{yx}$. Note that $F_{xx}, F_{xy}$ etc. are $\ell \times \ell$ matrices. Using these approximations in (7) gives

$$\begin{aligned} 0 &= F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) \\ &\approx F_y + F'_{xy} z_t + F_{yy} z_{t+1} + \beta \left[F_x + F_{xx} z_{t+1} + F_{xy} z_{t+2}\right] \\ &= F'_{xy} z_t + \left(F_{yy} + \beta F_{xx}\right) z_{t+1} + \beta F_{xy} z_{t+2}. \end{aligned}$$

This second-order difference equation can be used to study the transitional dynamics. Write the 'stacked' difference equation

$$\begin{pmatrix} z_{t+1} \\ z_t \end{pmatrix} = \begin{pmatrix} -\beta^{-1} F_{xy}^{-1} \left(F_{yy} + \beta F_{xx}\right) & -\beta^{-1} F_{xy}^{-1} F'_{xy} \\ I & 0 \end{pmatrix} \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix}, \tag{8}$$

where each of the sub-matrices is $\ell \times \ell$.

Fact: If $\lambda$ is a characteristic root, then so is $\left(\beta\lambda\right)^{-1}$. Hence for $\ell = 1$, AT MOST one root is less than one in absolute value. Typically exactly one is.

In a problem with $\ell \geq 1$ state variables, there is a system of $\ell$ second-order difference equations. There are $2\ell$ roots, and typically HALF of the roots are less than one in absolute value.

Suppose that this is the case. The roots that are less than one in absolute value are the useful ones. Let $\lambda_1, ..., \lambda_\ell$ denote these roots, and let $V_1, ..., V_\ell$ be the associated eigenvectors. Note that each eigenvector has length $2\ell$. Write each eigenvector as $V_i = (V_i^c, V_i^p)$ (for current and past), where each component has length $\ell$.

Given any initial condition $x_0$ for the state, define $z_0 = x_0 - \overline{x}$. Write $z_0$ in terms of the $V_i^p$ components of the eigenvectors,

$$z_0 = \sum_{i=1}^{\ell} w_i V_i^p.$$

This is a system of $\ell$ equations in the $\ell$ unknown weights $w_i$. Suppose that it has a unique solution, and define $z_1$ by

$$z_1 = \sum_{i=1}^{\ell} w_i V_i^c.$$

Then use (8) to generate the infinite sequence $\{z_t\}_{t=2}^{\infty}$, and let $x_t = \overline{x} + z_t$, $t = 1, 2, ....$ For the initial condition $x_0$, the sequence $\{x_t\}_{t=1}^{\infty}$ is the unique solution to the planner's problem. It converges asymptotically to the steady state $\overline{x}$.

Exercise: Write the solution $\{z_t\}_{t=2}^{\infty}$ using $\lambda_1, ..., \lambda_\ell$ and $V_1, ..., V_\ell$ instead of (8).

**Step 9: Effect of parameter changes**

The analysis here is problem-specific.

## A (WELL-BEHAVED) PARETO PROBLEM

### Step 1

(a) Consider an economy with two agents, $i = 1, 2$, with identical preferences over consumption goods. Let $c_i = \{c_{it}\}_{t=0}^{\infty}$, $i = 1, 2$, denote the sequence of $i$'s consumption of the single non-storable good. Fix $\theta \in (0, 1)$ and consider the problem of choosing $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$ to maximize a weighted sum of the lifetime utilities of the two agents. In particular, define

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t \left[\theta u(c_{1t}) + (1-\theta) u(c_{2t})\right], \quad i = 1, 2,$$

subject to the constraints

$$\begin{aligned} c_{1t} + c_{2t} + k_{t+1} &\leq f(k_t) + (1-\delta)\, k_t, \qquad (1) \\ c_{1t}, c_{2t} &\geq 0, \quad k_{t+1} \geq (1-\delta)\, k_t, \qquad \text{all } t \geq 0, \end{aligned}$$

given $k_0$, where $0 < \delta \leq 1$ is the depreciation rate. Note that investment is irreversible: we require $k_{t+1} \geq (1-\delta)\, k_t$. Assume $u$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies $u(0) = 0$, $u'(0) = \infty$, and $\beta \in [0, 1)$. Assume $f$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies $f(0) = 0$, $f'(0) = \infty$, and $\lim_{k \to \infty} f'(k) = 0$.

a. For the state space, for now we require only $K \subseteq \mathbf{R}_+$ is a convex set.

More on this in Step 2.

b. Define $\Gamma$ by

$$\Gamma(k) \equiv \left[(1-\delta)\, k, (1-\delta)\, k + f(k)\right]. \qquad (9)$$

c. Define

$$U(c) \equiv \left[\theta u(c_{1t}) + (1-\theta) u(c_{2t})\right] \qquad \text{s.t. } c_1 + c_2 \leq c.$$

Note that $U$ inherits all of the properties of $u$. The define

$$F(k, y) \equiv U\left[(1-\delta)\, k + f(k) - y\right]. \qquad (10)$$

d. $\beta \in (0,1)$ is given.

**Step 2**

For each $k \in K$, $G(k)$ is nonempty.

To insure that $F$ is bounded, find an upper bound for the state space.

Here, we will assume

$$\lim_{k \to \infty} f'(k) < \delta,$$

and define $\overline{k}$ by $\delta \overline{k} = f(\overline{k})$. Then let $K = [0, \overline{k}]$. The

$$0 \leq F \leq U\left[(1-\delta)\,\overline{k} + f(\overline{k})\right] \equiv B,$$

so $F$ is bounded on the set $A$.

**Step 3**

The BE is

$$\hat{v}(k) = \max_{y \in \Gamma(k)} U\left[(1-\delta)\,k + f(k) - y\right] + \beta \hat{v}(y). \tag{11}$$

Under the assumptions in Step 2, we know that $v$ defined in Step 1 is the unique solution to this functional equation $\hat{v}$.

**Step 4**

Clearly the set $\Gamma(k)$ is compact-valued, for each $k \in K$, , and $\Gamma$ is continuous as a correspondence.

Let $G(k) \subseteq \Gamma(k)$, denote the set of maximizers in (11). The correspondence $G$ is non-empty and compact-valued, and $G$ is u.h.c.

Define the Bellman operator $T$: $C(K) \to C(K)$ by

$$T\hat{v}(k) = \max_{y \in \Gamma(k)} \left\{U\left[(1-\delta)\,k + f(k) - y\right] + \beta \hat{v}(y)\right\}. \tag{12}$$

**Step 5**

**a. Monotonicity** Clearly the RHS is increasing in $k$, but $\Gamma(k)$ is not increasing the required sense. Can we nevertheless show that $\hat{v}$ increasing implies $T\hat{v}$ increasing? Choose $k_1 < k_2$. Suppose $y_1$ is a maximizing value for $k_1$. Define

$$y_1 = (1-\delta)\,k_1 + f(k_1) - c_1.$$

as the corresponding consumption value. Then define

$$y_2 = (1-\delta)\,k_2 + f(k_2) - c_1.$$

Note that $y_2 \in \Gamma(k_2)$, and $y_2 > y_1$. Then since by assumption $\hat{v}$ is weakly monotone,

$$\begin{aligned} T\hat{v}(k_2) &\geq U\left[(1-\delta)\,k_2 + f(k_2) - y_2\right] + \beta\hat{v}(y_2) \\ &\geq U\left[(1-\delta)\,k_1 + f(k_1) - y_1\right] + \beta\hat{v}(y_1) \\ &= T\hat{v}(k_1), \end{aligned}$$

so $T\hat{v}$ is also weakly monotone.

But we should be able to prove that $\hat{v}$ is strictly increasing. Use the idea that for $k_2 > k_1$, both net investment and consumption can be higher. So try defining

$$\begin{aligned} I_1 &= y_1 - (1-\delta)\,k_1 \geq 0, \\ c_1 &= f(k_1) - I_1 \geq 0, \end{aligned}$$

and let

$$\begin{aligned} c_2 &= f(k_2) - I_1 > c_1, \\ y_2 &= (1-\delta)\,k_2 + I_1 > y_1. \end{aligned}$$

**b. Concavity** Clearly $\Gamma$ is convex as a correspondence. And it is pretty easy to show $F(k,y)$ defined in (10) is strictly concave in $(k,y)$. Hence

$v(k)$ is strictly concave;

the optimal policy function $g(k)$ is single valued and continuous;

the optimal policy function can be computed by iterating with $T$, in the sense that $\{g_n\}$ converges pointwise.

**c. Differentiability** Clearly $F(k, y)$ is continuously differentiable in $k$, and we just showed $v$ is concave.

Is it true that $g(k)$ is always interior, that

$$g(k) \in \text{int } \Gamma(k) = (\ (1-\delta)\, k,\ (1-\delta)\, k + f(k)\ )\,.$$

Use the Inada condition to argue that $c \neq 0$. Could $c = f(k)$. Yes, perhaps it could, if $k$ was large. Over any interval where this is the case,

$$v(k) = U(f(k)) + \beta v\left[(1-\delta)\, k\right],$$

so $v$ is differentiable at $k$ if it is differentiable at $(1-\delta)\, k$.

**Steps 6, 7**

Use (11) to get the FOC and EC,

$$\begin{aligned} U'\left[(1-\delta)\, k + f(k) - g(k)\right] &\geq \beta v'(g(k)), \qquad \text{w.e.i. } g(k) > (1-\delta)\, k, && (13) \\ v'(k) &= \left[(1-\delta) + f'(k)\right] U'\left[(1-\delta)\, k + f(k) - g(k)\right]. && (14) \end{aligned}$$

Combine to get the EE

$$U'\left[(1-\delta)\, k_t + f(k_t) - k_{t+1}\right] \geq \beta\left[(1-\delta) + f'(k_{t+1})\right] U'\left[(1-\delta)\, k_{t+1} + f(k_{t+1}) - k_{t+2}\right]. \tag{15}$$

At an interior steady state, the $U'$ terms cancel and we get

$$\beta^{-1} = 1 + \rho = (1-\delta) + f'(k^{ss}),$$

or

$$\rho = f'(k^{ss}) - \delta. \tag{16}$$

Since there is no growth, the steady state interest rate is equal to the rate of time preference. Thus, (16) says that the marginal product of capital, net of depreciation, is equal to the interest rate.

### Step 8

Linearize the Euler equation.

$$U'' \cdot [(1-\delta)\, z_t + f' z_t - z_{t+1}] = \beta f'' z_{t+1} U' + U'' \cdot [(1-\delta)\, z_{t+1} + f' z_{t+1} - z_{t+2}]$$

### Step 9

Which parameters affect the steady state? Which affect the speed of convergence?

**Features that lead to 'bad' behavior**

1. An unbounded return function can lead to extraneous solutions to BE and/or extraneous policies—policies that 'look' optimal but are not.
2. If the feasible set $\Gamma(x)$ is not closed and bounded, the max may not be attained.
3. If $\Gamma$ is not continuous $v$ may not be continuous.
4. If there are fixed costs of control, the feasible set $\Gamma(x)$ is not convex, and $v$ may not be concave.
5. Inequality constraints (e.g., a nonnegativity constraint) can produce a kink in the value function, a point of nondifferentiability.

**Badly behaved consumption-savings problems**

Primitives $u(\cdot), \beta, r, D$

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \\ \text{s.t.} \quad x_{t+1} &= (1+r)\left(x_t - c_t\right), \\ x_{t+1} &\geq -D, \quad \text{all } t. \end{aligned}$$

1. linear utility, $u(c) = c$

   discount rate = interest rate, $\beta = 1/\left(1+r\right)$,

   no borrowing limit, $-D = -\infty$

   Then $v(x) = \infty$.

   But BE has an extraneous solution, $v(x) = x$

   with an associated policy that does *not* attain the optimum.

2. Like (1) but with $D = 0$: no borrowing

   Then $v(x) = x$ is the only solution to the BE.

   Optimal policies are $\left(x_0, 0, 0, ...\right), \left(x_0, \beta^{-1}x_0, 0, 0, ...\right), \left(x_0, \beta^{-1}x_0, \beta^{-2}x_0, 0, 0, ...\right)$, etc. and all convex combinations thereof.

But the policy $x_{t+1} = \beta^{-1} x_t$, all $t$, (never consume) is **NOT** optimal

3. Upper bound on assets $a < B$.

   $\Gamma$ an open set: the maximum may not be attained,

   (there may be no optimal policies)

4. Two rates of return: $r^\ell < r^h$

   a higher rate of return for large asset holdings, if $x \geq A$

   $\Gamma$ is not continuous, so $v$ may not be continuous.

   [Maybe it is, but an argument specific to the problem is needed.]

5. Suppose the consumer gets welfare benefits if his wealth is below a threshold

$$y_t = \begin{cases} y, & \text{if } x_t \leq B, \\ 0, & \text{if } x_t > B. \end{cases}$$

   Then $\Gamma$ not continuous, leading to the same issue as in (4).

## INVESTMENT PROBLEMS (FIRM OR INDUSTRY)

Primitives $\pi(x), C(x,I), \delta, \beta = 1/(1+r)$

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[\pi\left(x_t\right) - C(x_t, I_t)\right],$$

$$\begin{aligned} \text{s.t.} \quad x_{t+1} &= (1-\delta)\, x_t + I_t, \\ x_{t+1} &\geq 0, \quad \text{all } t. \end{aligned}$$

Possibilities for $I, C$

1. convex costs: $C(x,I) = \phi(I), \quad C(x,I) = x\phi(I/x)$ and no constraints on $I$.

   The problem should be well behaved.

2. nonnegativity constraint $I \geq 0$, i.e., $y \geq (1-\delta)\, x$.

   Look out for a point of non-differentiability, a kink.

3. a fixed cost

$$C(I) = \begin{cases} 0, & \text{if } I = 0, \\ c_0 + c_1 I, & \text{if } I > 0. \end{cases}$$

   Expect $v$ to be non-concave.

## PART II. STOCHASTIC PROBLEMS

### Step 0: Define shock process

To maintain stationarity (recursivity), assume the shocks follow a **first-order Markov process with stationary transition function.**

Note that any Markov process of finite order can be written as a first-order process by expanding the state space. If the distribution of $z_{t+1}$ depends on $z_t, z_{t-1}, ..., z_{t-n}$ define the vector $\hat{z}_t = (z_t, z_{t-1}, ..., z_{t-n})$ and use $\hat{z}$.

There are two basic cases.

1. $Z = \{z_1, z_2, ...\}$ is a countable set and $Q = [q_{ij}]$ is a transition matrix:

$$q_{ij} \geq 0, \quad \text{all } i, j, \quad \sum_j q_{ij} = 1, \quad \text{all } i.$$

2. $Z \subseteq R^k$ is a closed rectangle (the product of intervals), and for each $z \in Z$, there is a continuous density $q(z, \cdot) > 0$ describing the distribution for the shock next period. Thus,

$$q\left(z, z'\right) \geq 0, \quad \text{all } z, z' \in Z, \qquad \int q(z, z')dz' = 1, \quad \text{all } z \in Z.$$

For case 2 assume in addition that for each $z' \in Z$, $q(\cdot, z')$ is continuous in its first argument. This assumption insures that the *Feller property* holds: That is, if $h(z)$ is continuous in $z$, the function $Mh$ defined by

$$\left(Mh\right)(z) = \mathrm{E}\left[h(z') \mid z\right] = \int h(z')q(z, z')dz' \tag{17}$$

is also continuous in $z$.

Note: if $Z$ is a discrete set, the Feller property holds, since 'continuity' holds vacuously.

The state variable and state space are $(x, z) = s \in S \equiv X \times Z$.

**Step 1: Write SP**

Write the sequence problem: given $x_0, z_0$, a **feasible plan** $\pi$ is a point

$$\pi_0 \in \Gamma(x_0, z_0),$$

and a sequence of functions

$$\pi_t\colon Z^t \to X, \quad \text{all } t,$$

such that

$$\pi_t(z^t) \in \Gamma\left[\pi_{t-1}(z^{t-1}), z_t\right], \quad \text{all } t,$$

where $z^t = (z_1, z_2, ..., z_t) \in Z^t$. Then $x_{t+1} = \pi_t(z^t)$ is chosen in period $t$.

As before, define the value function

$$v(x_0, z_0) \equiv \sup_{\pi} \left\{ F(x_0, \pi_0, z_0) + \mathrm{E}\left[ \sum_{t=1}^{\infty} \beta^t F(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t) \mid z_0 \right] \right\}. \quad \text{(SP)} \tag{18}$$

Define $A$ as before.

**Step 2: Formulate BE**

$$\hat{v}(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int \hat{v}(y, z') q(z, z') dz' \right\}. \qquad \text{(BE)} \tag{19}$$

The following Lemma is the basic result that allow us to apply the results from the deterministic case.

Lemma: If $f\colon\ X \times Z \to R$ is bounded and continuous, and $Q$ has the Feller property, then

$$\left(Mf\right)(y, z) = \int f(y, z') q(z, z') dz' \tag{20}$$

is also bounded and continuous. That is, $M\colon C(S) \to C(S)$.

In addition, $Mf$ inherits the following properties from $f$:

if $f$ is (strictly) increasing in $y$, so is $Mf$,

if $f$ is (strictly) concave in $y$, then so is $Mf$.

**Steps 3 - 5: Conditions on $F, \Gamma$ and results**

All the main requirements and results are as before. Assume $F$ is bounded and continuous, $\Gamma$ is compact-valued and continuous, and $q$ has the Feller property.

Define the operator $T$ in $C(S)$ by

$$\begin{aligned} Tf(x,z) &= \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \int f(y,z')q(z,z')dz' \right\} \\ &= \sup_{y\in\Gamma(x,z)} \left[ F(x,y,z) + \beta \left( Mf \right)(y,z) \right], \end{aligned} \tag{21}$$

where $Mf$ is defined in (20). The argument that $Tf$ is bounded and continuous. Hence $T$: $C(S) \to C(S)$. Apply the CMT as before.

The arguments for

monotonicity in $x$,

concavity in $x$, and single-valued, continuous policy function $g$,

differentiability in $x$

are all as before.

Suppose that in addition,

a. $F(x,y,\cdot)$ is (strictly) increasing in $z$, for all $(x,y)$;

b. $z \leq z'$ implies $\Gamma(x,z) \subseteq \Gamma(x,z')$, all $x$;

c. the transition function $Q$ is monotone in the sense that if $f$ is nondecreasing, then $Mf$ is also nondecreasing.

Then $v$ is (strictly) increasing in $z$.

Proof: Let $\hat{C}(S) \subset C(S)$ be the (closed) subset consisting of functions that are nondecreasing in $z$. Suppose $f \in \hat{C}(S)$, Then (c) implies that $Mf \in \hat{C}$. Then (a) and (b) insure that $Tf \in \hat{C}(S)$. Hence $T$: $\hat{C}(S) \to \hat{C}(S)$. Apply the CMT to $\hat{C}(S)$.

**Steps 3 - 5: Alternative BE**

Suppose that the decision maker cannot choose next period's state directly. For example, suppose in the consumption savings problem that the interest rate is stochastic. Then the agent chooses consumption $c_t$ this period, and next period's wealth is

$$x_{t+1} = (1 + r_{t+1}) [x_t + w - c_t],$$

which depends on $z_{t+1}$.

In this case, the DM chooses an action $y$ and next period's state is

$$x' = \phi(x, y, z').$$

Assume the function $\phi$ is continuous.

Write the Bellman operator as

$$Tf(x, z) = \max_{y \in \Gamma(x,z)} \left[ F(x, y, z) + \beta \int f [\phi (x, y, z'),\ z'] \, q(z, z') dz' \right]. \tag{22}$$

If $f$ and $\phi$ are continuous, and $q$ has the Feller property, then

$$h(x, y, x) \equiv \int f [\phi (x, y, z'),\ z'] \, q(z, z') dz'$$

is also continuous. The previous results can then be adapted to the present case.

Conclude

1. Same assumptions as before on $F, \Gamma, q$. Then $v$ defined by (SP) is the unique solution of (BE),

   and $v$ is bounded and continuous.

2. For monotonicity of $v$ in $x$, assume $\phi$ is nondecreasing in $x$.
3. For concavity of $v$ in $x$, assume that for each $z$, $\phi(\cdot, \cdot, z)$ is concave in $(x, y)$.

   Then the policy function $g$ is single valued and continuous.

4. For differentiability of $v$, assume $\phi (y, z)$ does not depend on $x$.
5. For monotonicity of $v$ in $z$, no additional restrictions are needed.

## 6. Euler equations

One can write the stochastic Euler equations, using the same method as before. (Use the FOC and the EC.) Then the FOC for an optimum is

$$0 = F_y\left[\pi_{t-1}, \pi_t, z_t\right] + \beta \int F_x\left[\pi_t, \pi_{t+1}, z_{t+1}\right] q(z_{t+1}, z_t) dz_{t+1},$$

where $x_{t+1} = \pi_t(z^t)$.

A stochastic system does not have any point that is a steady state. But what is often useful for simulations is to fix the shock(s) at their mean values, compute the steady state of the resulting deterministic system, and linearize the stochastic Euler equations around that point.

## 7. Dynamics

Alternatively, suppose that the problem is concave, so that there is a (single valued and continuous) policy function. That function, together with the transition function for the shocks, defines a transition function on the state space $S = X \times Z$.

**Lucas and Prescott: Industry investment under uncertainty**

Consider a perfectly competitive industry in which:

i. All firms have the same CRS production technology: capital is the only input, $Q_i = X_i$. (Or, capital and labor are used in fixed proportions, and wage costs are netted out.)

ii. All firms have the same CRS investment technology, and cost of investment depends on current capital

$$C(X_i', X_i) = X_i c(X_i'/X_i)$$

where

$$c\,(1-\delta) = 0,$$

and $c(\cdot)$ is strictly increasing and strictly convex on $[1-\delta, +\infty)$.

iii. Industry demand is stochastic, and $z$ is the demand shock. For each $z$, $D(\cdot, z)$ is a downward-sloping demand curve, and $U$ is the area under it,

$$U(q, z) = \int_0^q D(\nu, z)d\nu.$$

Hence for each $z$, $U$ is strictly increasing and strictly concave in $q$.

Let $x = \sum_i X_i$ be the (industry) capital stock. Then

$$q = \sum Q_i = \sum X_i = x$$

is total output.

$xc(x'/x)$ is the industry cost of increasing the capital stock from $x$ to $x'$

The Invisible Hand insures that a competitive industry maximizes total (consumers' plus producers') surplus, leading to problem

$$\sup \mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} (1+r)^{-t} \left[ U\left(x_t, z_t\right) - x_t c\left(x_{t+1}/x_t\right) \right] \right\}.$$

Choose $M$ large, and let $x \in X = (0, M]$ be the state space.

The Bellman equation is

$$v(x,z) = \max_{y \geq (1-\delta)x} \left[ U(x,z) - xc\left(y/x\right) + \frac{1}{1+r} \int v(y,z')q(z,z')dz' \right]$$

Note: $U$ is concave and $c$ is convex.

$v$ is continuous, strictly increasing, strictly concave, and differentiable in $x$

The optimal policy $y^* = g(x,z)$ is unique.

For any fixed $z$, $g(x,z)$ is continuous and strictly increasing in $x$.

Suppose $D(\nu, z)$ is strictly increasing in $z$. I.e., higher $z$ shifts demand (upward) outward.

Then $z$ is nondecreasing in $z$, and is strictly increasing if $g(x,z) > (1-\delta)\,x$

Note: $z$ is **useful** to the decision maker in choosing $y$ only because **it provides information about future demand.**

Current output is determined by past investments

So if $z$ is i.i.d., $g(x,z) = \hat{g}(x)$, and the capital stock converges deterministically to a SS, from any initial condition.