Theory of Income III
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Contents

1 Real business cycle models

The goal here is to build a model that can describe the observed real business cycle behaviour in developed countries such as the US.

1.1 Canonical model

1.1.1 The planner's problem

We work in discrete time so $t = 0, 1, 2 \ldots$. Let

- ▷ $\mathbf{s}_t \in \mathbf{S}_t$ denote the period-t state which is only learnt at the beginning of the period. We assume $\mathbf{s}^0$ is a singleton and that $\mathbf{S}_t$ is finite.
- ▷ $\mathbf{s}^t = \{s_0, s_1 \ldots, s_t\} \in \mathbf{S}^t$ denote the history of states in period t.[1]
- ▷ $\Pi_t(\mathbf{s}^t)$ is the probability in time period t of realising history $\mathbf{s}^t$.

The planner's problem is given by

$$\begin{aligned}
\max_{\{C_t(\mathbf{s}^t), H_t(\mathbf{s}^t), K_{t+1}(\mathbf{s}^t)\}_{\forall t, \mathbf{s}^t}} \quad & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t \in \mathbf{S}^t} \beta^t \Pi_t(\mathbf{s}^t) U\left(C_t(\mathbf{s}^t), H_t(\mathbf{s}^t)\right) \\
s.t. \quad & F_t\left(K_t(\mathbf{s}^{t-1}), H_t(\mathbf{s}^t), \mathbf{s}^t\right) = C_t(\mathbf{s}^t) + K_{t+1}(\mathbf{s}^t), \forall t, \mathbf{s}^t, \\
& K_0(\mathbf{s}^{-1}) \coloneqq K_0 \text{ given.}
\end{aligned}$$

- ▷ The planner chooses a (time-consistency) contingent plan in period 0 for all future periods, and for all possible histories.
- ▷ The planner chooses consumption, $C_t(\mathbf{s}^t)$, labour, $H_t(\mathbf{s}^t)$, and capital, $K_{t+1}(\mathbf{s}^t)$, for each $t = 0, 1, \ldots$ and for each possible history $\mathbf{s}^t$ in each period $t = 0, 1, \ldots$. We embed the assumption that the decision about $t+1$-capital is made in period t (i.e. before the planner knows about the state in $t+1$).
- ▷ Implicit in the use of the inner summation is the assumption of expected utility. We also assume that utility is additively separable over time.
- ▷ The production function depends on the current state (e.g. productivity shock) and is *net* of depreciation. Hence, depreciation (i.e. $(1-\delta) K_t$) does not appear in the feasibility condition.
- ▷ Feasibility condition implies that we have no government in the model.
- ▷ To ensure that the first-order conditions are necessary and sufficient, we assume that: (i) the objective function is concave; (ii) the constraint set is convex; (iii) U and F_t are at least once differentiable. We also assume Inada conditions to ensure that the solution is interior.

1.1.2 Intra- and inter-temporal conditions

To solve the model, we use the Lagrangian method,[2] where the Lagrangian multiplier is defined as

$$\beta^t \Pi_t(\mathbf{s}^t) \lambda_t(\mathbf{s}^t).$$

[1] For the problem to be written in a recursive form, we would require s_t to depend only on s_{t-1}.

[2] We could write the problem using a value function and solve via dynamic programming. However, the value function would have to be indexed by both t and s_t so will not any simpler.

The Lagrangian is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t \in \mathbf{S}^t} \beta^t \Pi_t \left(\mathbf{s}^t\right) \left[U\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right) + \lambda_t\left(\mathbf{s}^t\right)\left(F_t\left(K_t\left(\mathbf{s}^{t-1}\right), H_t\left(\mathbf{s}^t\right), \mathbf{s}^t\right) - C_t\left(\mathbf{s}^t\right) - K_{t+1}\left(\mathbf{s}^t\right)\right)\right].$$

The first-order conditions are, for all t and for all $\mathbf{s}^t$,

$$\begin{aligned}
\left\{C_t\left(\mathbf{s}^t\right)\right\} \qquad & U_C\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right) = \lambda_t\left(\mathbf{s}^t\right), \\
\left\{H_t\left(\mathbf{s}^t\right)\right\} \qquad & -U_H\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right) = \lambda_t\left(\mathbf{s}^t\right) F_{H,t}\left(K_t\left(\mathbf{s}^{t-1}\right), H_t\left(\mathbf{s}^t\right), \mathbf{s}^t\right), \\
\left\{K_{t+1}\left(\mathbf{s}^t\right)\right\} \qquad & \beta^t \Pi_t\left(\mathbf{s}^t\right)\lambda_t\left(\mathbf{s}^t\right) = \beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right)\lambda_{t+1}\left(\mathbf{s}^{t+1}\right) \\
& \qquad \times F_{K,t+1}\left(K_{t+1}\left(\mathbf{s}^t\right), H_{t+1}\left(\mathbf{s}^{t+1}\right), \mathbf{s}^{t+1}\right),
\end{aligned}$$

where $\mathbf{s}^{t+1}$ is a *continuation* of $\mathbf{s}^t$; i.e.

$$\mathbf{s}^{t+1} \succ \mathbf{s}^t \Leftrightarrow \mathbf{s}^{t+1} \in \left\{\left\{\mathbf{s}^t, s_{t+1}\right\} : s_{t+1} \in S_{t+1}\right\}.$$

We define some short-hand notation:

$$\begin{aligned}
U_{C,t}\left(\mathbf{s}^t\right) &\coloneqq U_C\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right), \\
U_{H,t}\left(\mathbf{s}^t\right) &\coloneqq U_H\left(C_t\left(\mathbf{s}^t\right), H_t\left(\mathbf{s}^t\right)\right), \\
F_{i,t}\left(\mathbf{s}^t\right) &\coloneqq \frac{\partial}{\partial i} F_t\left(K_t\left(\mathbf{s}^{t-1}\right), H_t\left(\mathbf{s}^t\right), \mathbf{s}^t\right), \forall i \in \left\{K_t, H_t\right\}.
\end{aligned}$$

The *intertemporal decision* (between consumption and labour) is given by dividing $\{H_t(\mathbf{s}^t)\}$ by $\{C_t(\mathbf{s}^t)\}$:

$$-\frac{U_{H,t}\left(\mathbf{s}^t\right)}{U_{C,t}\left(\mathbf{s}^t\right)} = F_{H,t}\left(\mathbf{s}^t\right),\ \forall t, \mathbf{s}^t$$

This equates the marginal rate of substitution (the left-hand side) with the marginal product of labour (the right-hand side). The condition equates the gains from one more unit of labour supply (which produces $F_{H,t}$ more output to be consumed, which is worth $U_{C,t}F_{H,t}$ in utility terms) with the cost of supplying the additional labour ($-U_{H,t}$).

The *intertemporal decision* (consumption across periods) is obtained by eliminating $\lambda_t(\mathbf{s}^t)$ from $\{K_{t+1}(\mathbf{s}^t)\}$ using $\{C_t(\mathbf{s}^t)\}$.

$$\beta^t \Pi_t\left(\mathbf{s}^t\right) U_{C,t}\left(\mathbf{s}^t\right) = \beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) U_{C,t+1}\left(\mathbf{s}^{t+1}\right) F_{K,t+1}\left(\mathbf{s}^{t+1}\right),\ \forall t, \mathbf{s}^t.$$

Dividing through by $\beta^t \Pi_t(\mathbf{s}^t)$ yields the usual Euler equation:

$$U_{C,t}\left(\mathbf{s}^t\right) = \beta \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)} U_{C,t+1}\left(\mathbf{s}^{t+1}\right) F_{K,t+1}\left(\mathbf{s}^{t+1}\right),\ \forall t, \mathbf{s}^t.$$

Note that $\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)/\Pi_t\left(\mathbf{s}^t\right)$ is the probability that the next period state is $\mathbf{s}^{t+1} \succ \mathbf{s}^t$ conditional on the history of $\mathbf{s}^t$. To save notation, we often write

$$U_{C,t}\left(\mathbf{s}^t\right) = \beta \mathbb{E}_t\left[U_{C,t+1}\left(\mathbf{s}^{t+1}\right) F_{K,t+1}\left(\mathbf{s}^{t+1}\right)\right],\ \forall t, \mathbf{s}^t.$$

1.1.3 Conditions required for the existence of a balanced growth path

We now ask, if there are no shocks, under what conditions do we have a balanced-growth-path steady state? That is, a steady state in which C and K grows at a constant rate, say g, while H remains constant. (The motivation for the balanced growth path is based on observed long-run data.)[3]

To impose the assumption that there are no shocks, we can either:

- ▷ assume that all $\mathbf{s}^t$'s are singleton sets (so the path of history is unique; i.e. deterministic); or
- ▷ remove s_t from the production function since, in the given set up, s_t affects only the production function.[4]

Assume either of these, then C_t, H_t and K_t does not depend on $\mathbf{s}^t$ (or only trivially depends on $\mathbf{s}^t$).

Proposition 1.1. *Suppose $F_t(K_t, H_t) = \hat{F}\left(K_t, (1+g)^t H_t\right)$ and the production function has constant returns to scale. Then, the model permits a balanced growth path if and only if, for $\sigma > 0$,*

$$U(C, H) = \begin{cases} \frac{\left(Ce^{-v(H)}\right)^{1-\sigma} - 1}{1-\sigma} + a & \text{if } \sigma \neq 0 \\ \log C - v(H) + a & \text{if } \sigma = 1 \end{cases}$$

with a being some constant.

Proof. We only prove sufficiency where we assume that F_t is strictly concave and v is strictly convex, and they both satisfy the Inada conditions. We consider the case where $\sigma \neq 1$ and we will find that the same expression applies by setting $\sigma = 1$.

Given the functional form for U:

$$\begin{aligned} U_C &= e^{-v(H_t)} \left(C_t e^{-v(H_t)}\right)^{-\sigma}, \\ U_H &= -v'(H_t) C_t e^{-v(H_t)} \left(C e_t^{-v(H_t)}\right)^{-\sigma}. \end{aligned}$$

Hence, the intratemporal condition becomes

$$\begin{aligned} F_{H,t}(K_t, H_t) &= -\frac{-v'(H_t) C_t e^{-v(H_t)} \left(C e_t^{-v(H_t)}\right)^{-\sigma}}{e^{-v(H_t)} \left(C_t e^{-v(H_t)}\right)^{-\sigma}} \\ &= C_t v'(H_t). \end{aligned} \tag{1.1}$$

The Euler equation is now

$$\begin{aligned} U_{C,t} &= \beta U_{C,t+1} F_{K,t+1} \\ \Rightarrow e^{-v(H_t)} \left(C_t e^{-v(H_t)}\right)^{-\sigma} &= \beta e^{-v(H_{t+1})} \left(C_{t+1} e^{-v(H_{t+1})}\right)^{-\sigma} F_{K,t+1}(K_{t+1}, H_{t+1}) \\ \Leftrightarrow C_t^{-\sigma} \left(e^{-v(H_t)}\right)^{1-\sigma} &= \beta C_{t+1}^{-\sigma} \left(e^{-v(H_{t+1})}\right)^{1-\sigma} F_{K,t+1}(K_{t+1}, H_{t+1}). \end{aligned} \tag{1.2}$$

[3]That H is constant is more controversial than that K and C grow at the same, constant rate.
[4]For example, it's possible to set up the model so that U depends on $\mathbf{s}^t$—e.g. a preference shock.

The feasibility condition becomes

$$K_{t+1} = F_t\left(K_t, H_t\right) - C_t. \tag{1.3}$$

That the productivity shock is labour augmenting means that

$$F_t\left(K_t, H_t\right) = \hat{F}\left(K_t, (1+g)^t H_t\right).$$

Define detrended variables:

$$k_t := \frac{K_t}{(1+g)^t},\ c_t := \frac{C_t}{(1+g)^t}. \tag{1.4}$$

Then,

$$\begin{aligned} F_{H,t}\left(K_t, H_t\right) &= (1+g)^t \hat{F}_H\left(K_t, (1+g)^t H_t\right) \\ &= (1+g)^t \hat{F}_H\left(\frac{K_t}{(1+g)^t}, H_t\right) \\ &= (1+g)^t \hat{F}_H\left(k_t, H_t\right), \end{aligned}$$

where, in the second line, we used the fact that $\hat{F}_H$ is homogeneous of degree zero (since $\hat{F}$ is constant returns to scale). We also have

$$\begin{aligned} F_{K,t+1}\left(K_{t+1}, H_{t+1}\right) &= \hat{F}_K\left(K_{t+1}, (1+g)^{t+1} H_{t+1}\right) \\ &= \hat{F}_K\left(\frac{K_{t+1}}{(1+g)^{t+1}}, H_{t+1}\right) \\ &= \hat{F}_K\left(k_{t+1}, H_{t+1}\right), \end{aligned}$$

where we used the fact that $\hat{F}_K$ is homogeneous of degree zero.

Using this detrended notation, the intratemporal condition, (1.1), can be written as

$$\begin{aligned} (1+g)^t \hat{F}_H\left(k_t, H_t\right) &= C_t v'\left(H_t\right) \\ \Leftrightarrow \hat{F}_H\left(k_t, H_t\right) &= c_t v'\left(H_t\right). \end{aligned} \tag{1.5}$$

Similarly, the Euler equation, (1.2), can be written as

$$\begin{aligned} C_t^{-\sigma}\left(e^{-v(H_t)}\right)^{1-\sigma} &= \beta C_{t+1}^{-\sigma}\left(e^{-v(H_{t+1})}\right)^{1-\sigma} \hat{F}_K\left(k_{t+1}, H_{t+1}\right) \\ \Leftrightarrow \left(\frac{C_t}{(1+g)^t}\right)^{-\sigma}\left(e^{-v(H_t)}\right)^{1-\sigma} &= \beta\left(\frac{C_{t+1}}{(1+g)^{t+1}}(1+g)\right)^{-\sigma}\left(e^{-v(H_{t+1})}\right)^{1-\sigma} \hat{F}_K\left(k_{t+1}, H_{t+1}\right) \\ \Leftrightarrow c_t^{-\sigma}\left(e^{-v(H_t)}\right)^{1-\sigma}(1+g)^{\sigma} &= \beta c_{t+1}^{-\sigma}\left(e^{-v(H_{t+1})}\right)^{1-\sigma} \hat{F}_K\left(k_{t+1}, H_{t+1}\right). \end{aligned} \tag{1.6}$$

Finally, the feasibility condition, (1.3), can be written as

$$\begin{aligned}
\frac{K_{t+1}}{(1+g)^{t+1}}(1+g) &= \frac{1}{(1+g)^t}F_t(K_t,H_t) - \frac{C_t}{(1+g)^t} \\
\Leftrightarrow k_{t+1}(1+g) &= \frac{1}{(1+g)^t}\hat{F}\left(K_t,(1+g)^t H_t\right) - c_t \\
&= \hat{F}(k_t,H_t) - c_t. \qquad (1.7)
\end{aligned}$$

On the balanced growth path, in the steady state, $H_{t+1} = H_t = \bar{H}$, $k_{t+1} = k_t = \bar{k}$ and $c_{t+1} = c_t = \bar{c}$ (this is why we introduced the detrended variables). Thus, (1.5) simplifies to

$$\hat{F}_H\left(\bar{k},\bar{H}\right) = \bar{c}v'\left(\bar{H}\right), \qquad (1.8)$$

and (1.6) simplifies to

$$\begin{aligned}
\bar{c}^{-\sigma}\left(e^{-v(\bar{H})}\right)^{1-\sigma}(1+g)^\sigma &= \beta\bar{c}^{-\sigma}\left(e^{-v(\bar{H})}\right)^{1-\sigma}\hat{F}_K\left(\bar{k},\bar{H}\right) \\
\Leftrightarrow (1+g)^\sigma &= \beta\hat{F}_K\left(\bar{k},\bar{H}\right), \qquad (1.9)
\end{aligned}$$

and (1.7) simplifies to

$$\bar{k}(1+g) = \hat{F}\left(\bar{k},\bar{H}\right) - \bar{c}. \qquad (1.10)$$

So, we now have three equations, (1.8), (1.9) and (1.10), and three unknowns ($\bar{H}$, $\bar{k}$ and $\bar{c}$). It remains to show that there exists a (unique) solution to this system of equations.

Since $\hat{F}_K$ is homogeneous of degree zero, (1.9) can be written as

$$(1+g)^\sigma = \beta\hat{F}_K\left(\frac{\bar{k}}{\bar{H}},1\right). \qquad (1.11)$$

Assuming that F is strictly concave, then $\hat{F}_K$ is strictly decreasing in its first argument so there can be at most one value of $\bar{k}/\bar{H}$ that solves the equation. That the solution exists is ensured if we impose Inada conditions on F. Moreover, since the left-hand side is a constant, it must be that $\bar{k}/\bar{H}$ is a constant as values of $\bar{k}$ and/or $\bar{H}$ changes.

Eliminating $\bar{c}$ from (1.8) and (1.10) yields

$$\begin{aligned}
\hat{F}_H\left(\bar{k},\bar{H}\right) &= \left(\hat{F}\left(\bar{k},\bar{H}\right) - \bar{k}(1+g)\right)v'\left(\bar{H}\right) \\
\Leftrightarrow \hat{F}_H\left(\frac{\bar{k}}{\bar{H}},1\right) &= \left(\bar{H}\hat{F}\left(\frac{\bar{k}}{\bar{H}},1\right) - \bar{k}(1+g)\right)v'\left(\bar{H}\right) \\
\Leftrightarrow \frac{1}{v'\left(\bar{H}\right)\bar{H}} &= \frac{1}{\hat{F}_H\left(\frac{\bar{k}}{\bar{H}},1\right)}\left(\hat{F}\left(\frac{\bar{k}}{\bar{H}},1\right) - \frac{\bar{k}}{\bar{H}}(1+g)\right).
\end{aligned}$$

As argued before, $\bar{k}/\bar{H}$ is constant so that the right-hand side is constant. Assuming that v is strictly convex, v' is strictly increasing so that $v'\left(\bar{H}\right)\bar{H}$ is strictly increasing in $\bar{H}$; i.e. the left-hand side is strictly decreasing in $\bar{H}$. Thus, there is at most one value of $\bar{H}$ that satisfies the equation above. We can then assume Inada condition on v to ensure such a unique solution exists. This allows us to back out $\bar{k}$ from the fact we know $\bar{k}/\bar{H}$ from (1.11). Then, we can derive $\bar{c}$ using either (1.8) or (1.10). Hence, we conclude that there exists steady state in detrended notation exists. But this means that K_t and C_t both grow at rate g in the steady state, while H_t remains constant in the steady state. ■

Remark 1.1. For the proof in the other direction (i.e. necessity), see Fernando's Problem Set 4, Q1 (viii).

1.2 Calibration

1.2.1 Stylised long-run (US) facts

(i) Output and consumption per capita have been growing by around 2%; i.e. the ratio is constant.

(ii) Hours worked has no trend.

(iii) Share of income accrued to labour, $w_t H_t / Y_t$, has been broadly constant; i.e. wage has been growing at around 2% (since H_t has been constant and Y_t has been growing at 2%).

(iv) Rate of return on capital has no trend.

1.2.2 Specialisation

We specify functional forms on the production and utility functions. We maintain the assumption that there are no shocks.

Utility function We take the $\sigma = 1$ case for utility for simplicity (results hold for $\sigma \neq 1$):

$$U(C, H) = \log C - v(H).$$

Production function Production function is Cobb-Douglas with constant labour-augmenting technological progress (at rate g):

$$F_t(K_t, H_t) = K_t^{\alpha} \left[(1+g)^t H_t \right]^{1-\alpha} + (1-\delta) K_t. \tag{1.12}$$

Given that the production function is Cobb-Douglas, we can rewrite the production function in terms of neutral technological progress (remember, we defined production function net of depreciation):

$$F_t(K_t, H_t) = \left[\underbrace{(1+g)^{1-\alpha}}_{=1+\tilde{g}} \right]^t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t.$$

Thus, with Cobb-Douglas production function, we cannot distinguish whether technological progress is labour augmenting or neutral (but, for example, with CES production function, we can distinguish). We maintain the functional form as in (1.12) for convenience (since, on the balanced growth path, c and k grows at rate g, not $\tilde{g}$).

We implicitly assume that the level of technological progress in period 0 is 1 (since $(1+g)^0 = 1$), which is a normalisation of units of output. It means that with one unit of H and K, output produced is 1 in period 0.

1.2.3 Equilibrium conditions

With the assumed functional forms, the intratemporal condition, (1.1), the intertemporal condition (i.e. Euler equation), (1.2), and the feasibility condition, (1.3), are given respectively by

$$C_t v'(H_t) = F_{H,t}(K_t, H_t) = (1-\alpha)\left[(1+g)^t\right]^{1-\alpha} K_t^{\alpha} H_t^{-\alpha}, \tag{1.13}$$

$$\frac{1}{C_t} = \beta \frac{F_{K,t+1}(K_{t+1}, H_{t+1})}{C_{t+1}} = \beta \frac{\alpha K_{t+1}^{\alpha-1}\left[(1+g)^{t+1} H_{t+1}\right]^{1-\alpha} + 1 - \delta}{C_{t+1}}, \tag{1.14}$$

$$K_{t+1} = K_t^{\alpha}\left[(1+g)^t H_t\right]^{1-\alpha} + (1-\delta) K_t - C_t. \tag{1.15}$$

Using the detrended notation, (1.4), we can rewrite the conditions above as

$$\begin{aligned} c_t v'(H_t) &= (1-\alpha) k_t^{\alpha} H_t^{-\alpha}, \\ \frac{1+g}{c_t} &= \beta \frac{\alpha k_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}}, \\ k_{t+1}(1+g) &= k_t^{\alpha} H_t^{1-\alpha} + (1-\delta) k_t - c_t, \end{aligned}$$

where we: (i) applied the definition of detrended variables in (1.13); (ii) divided both sides by $(1+g)^{t+1}$ while applying the definition in (1.14); and (iii) divided both sides of (1.15) by $(1+g)^t$.

1.2.4 Balanced growth path

On the balanced growth path, $k_{t+1} = k_t = \bar{k}$, $c_{t+1} = c_t = \bar{c}_t$ and $H_{t+1} = H_t = \bar{H}$. We also know from Proposition 1.1 that there exists a unique solution to $\bar{k}$, $\bar{c}$ and $\bar{H}$.

To show this explicitly, we can evaluate the three conditions on the balanced growth path:

$$\bar{c} v'(\bar{H}) = (1-\alpha)\left(\frac{\bar{k}}{\bar{H}}\right)^{\alpha}, \tag{1.16}$$

$$1+g = \beta\left(\alpha\left(\frac{\bar{k}}{\bar{H}}\right)^{-(1-\alpha)} + 1 - \delta\right), \tag{1.17}$$

$$\bar{k}(1+g) = \bar{k}^{\alpha}\bar{H}^{1-\alpha} + (1-\delta)\bar{k} - \bar{c}. \tag{1.18}$$

The Euler equation (1.17) pins down $\bar{k}/\bar{H}$. Dividing (1.18) through by $\bar{k}$ gives the ratio $\bar{c}/\bar{k}$ since

$$\frac{\bar{c}}{\bar{k}} = \left(\frac{\bar{k}}{\bar{H}}\right)^{-(1-\alpha)} - (\delta + g).$$

Then, we can calculate $\bar{c}/\bar{H}$ as

$$\frac{\bar{c}}{\bar{H}} = \frac{\bar{k}}{\bar{H}}\frac{\bar{c}}{\bar{k}}.$$

We can then replace $\bar{c}$ in (1.16) to obtain:

$$\bar{H} v'(\bar{H}) = (1-\alpha)\left(\frac{\bar{k}}{\bar{H}}\right)^{\alpha-1}\left(\frac{\bar{c}}{\bar{k}}\right)^{-1}.$$

Since v is strictly convex (and strictly increasing), the left-hand side is strictly increasing in $\bar{H}$ so that there exists at most one solution. To guarantee existence, one sufficient condition is to impose

Inada condition on v. A weaker sufficient condition is

$$\lim_{H\downarrow 0} Hv'(H) = 0$$

since Inada condition on the production ensures that the right-hand side tends to infinity as $\bar{H} \downarrow 0$. Therefore, (given these assumptions) there is a unique solution to the system of equations for $\bar{c}$, $\bar{k}$ and $\bar{H}$.

1.2.5 Model implication vs stylised facts

We now show that the model is able to replicate stylised facts.

That $\bar{c}$, $\bar{k}$ and $\bar{H}$ are constant on the balanced growth path means that, in terms of the original variables,

- ▷ C_t and K_t grows at rate g;
- ▷ H_t is constant.

So the model is able to match the stylised facts (i) and (ii).

If we were to decentralise, we will have

$$C_t v'(H_t) = w_t = (1-\alpha)\left[(1+g)^t\right]^{1-\alpha} K_t^{\alpha} H_t^{-\alpha};$$

i.e. the marginal rate of substitution and the marginal product of labour both equal the wage rate, w_t.[5] From the first equality, since C_t grows at rate g while H_t is constant on the balanced growth path, w_t must be growing at rate g. Using the second equality,

$$\begin{aligned} w_t H_t &= (1-\alpha)\left[(1+g)^t\right]^{1-\alpha} K_t^{\alpha} H_t^{1-\alpha} \\ &= (1-\alpha) K_t^{\alpha}\left((1+g)^t H_t\right)^{1-\alpha} = (1-\alpha) Y_t \\ \Rightarrow \frac{w_t H_t}{Y_t} &= 1-\alpha. \end{aligned}$$

That is, the share of labour income is constant and is given by $1-\alpha$. Hence, the model matches the stylised fact (iii).

Rearranging (1.14),

$$\frac{C_{t+1}}{C_t} = \beta F_{K,t+1}(K_{t+1}, H_{t+1}) = \beta\alpha K_{t+1}^{\alpha-1}\left[(1+g)^{t+1} H_{t+1}\right]^{1-\alpha} + 1 - \delta.$$

On the balanced growth path $C_{t+1}/C_t = 1+g$ so

$$\frac{1+g}{\beta} = F_{K,t+1}(K_{t+1}, H_{t+1});$$

i.e. the marginal product of capital is constant. Since the marginal product of capital equals the interest rate/return on capital in the decentralised economy,[6] it follows that the return on capital is constant on the balanced growth path, which is consistent with stylised fact (iv).

[5] Recall that solving the firm's problem gives that $w_t = F_{H,t}$ as the profit maximisation condition.
[6] Recall that solving the firm's problem gives that $r_t = F_{K,t}$ as the profit maximisation condition.

1.2.6 Calibrating the parameters of the model

We need to obtain parameter values for

$$\alpha, \beta, g, \delta$$

and the function

$$v(H).$$

The idea is to use the long-run facts to calibrate the parameters. This ensures that when we use the calibrated model to study real business cycles (which is a short-run analysis), the model provides long-run implications that are consistent with the aforementioned long-run stylised facts.

In calibrating the model, we consider both annual and quarterly time period.

Capital share of output (α) We can obtain α as $1-$labour share. The long-run average capital share of output in the US has been around 0.4.

$$\alpha = 0.4.$$

Balanced path growth rate (g) As mentioned earlier, consumption, output and capital have been growing at around 2% per year on average, which is equivalent to a growth rate of approximately 0.5% per quarter.[7]

$$g = \begin{cases} 0.02 & \text{annual} \\ 0.005 & \text{quarterly} \end{cases}.$$

Depreciation rate (δ) One way is to take an weighted average of depreciation rates for goods in the economy. Alternatively, recall the law of motion for capital:

$$K_{t+1} = (1-\delta) K_t + X_t,$$

where X_t is investment. Dividing through by K_t and evaluating this on the balanced growth path,

$$1 + g = \frac{K_{t+1}}{K_t} = (1-\delta) + \frac{X_t}{K_t}.$$

We already know g, so if we can obtain a value of X_t/K_t (ratio of investment to capital), then we can back out δ. In the US, long-run average capital to output ratio is

$$\frac{K}{Y} = \begin{cases} 3.2 & \text{annual} \\ 12.8 & \text{quarterly} \end{cases}, \quad \frac{X}{Y} = 0.26.$$

Note that K is a stock while Y is a flow variable. This means that conversion between annual and quarterly involves multiplying by 4 (K is the "same" whether we consider annual or quarterly, while quarterly output c. 1/4 of annual output). In contrast, because investment and output are both flow variables, the value of X/Y is the same whether we consider annual or quarterly time periods. Using this, we obtain that

$$\delta = \frac{X}{Y}\frac{Y}{K} - g = \begin{cases} 0.06 & \text{annual} \\ 0.015 & \text{quarterly} \end{cases}.$$

[7]Strictly speaking, quarterly growth is given by $(1+2\%)^{1/4} - 1$.

Discount factor (β) Recall (1.17), which is the Euler equation evaluated on the balanced growth path:

$$
\begin{aligned}
1+g &= \beta\left(\alpha\left(\frac{\bar{k}}{\bar{H}}\right)^{-(1-\alpha)}+1-\delta\right) \\
&= \beta\left(\alpha\frac{\bar{k}^{\alpha}\bar{H}^{1-\alpha}}{\bar{k}}+1-\delta\right) = \beta\left(\alpha\frac{Y}{K}+1-\delta\right) \\
\Rightarrow \beta &= \frac{1+g}{\alpha\frac{Y}{K}+1-\delta}.
\end{aligned}
$$

Substituting the values we obtained already, we get

$$
\beta = \begin{cases} 0.958872 & \text{annual} \\ 0.989234 & \text{quarterly} \end{cases}.
$$

Disutility from working (v) The only equation in which v appears is in the intratemporal condition (1.16), and it appears as $v'\left(\bar{H}\right)$. On the balanced growth path, we can only observe the value of $v'(H)$ evaluated at $H=\bar{H}$, and so it is not possible to obtain the function v from long-run data.

One solution is to impose a one-parameter functional form for $v(H)$ so that knowing the value of the function at a particular point is sufficient to obtain the function in full. For example, Prescott imposes that

$$
v(H) = -\gamma\log(1-H),
$$

where γ describes the disutility of working.

A better way is to allow for more flexibility in the functional form. We will see later when we consider log-linearised equilibrium conditions that what matters is v' and v''. This motivates us to use the following functional form:[8]

$$
v(H) = \gamma\frac{\varepsilon}{1+\varepsilon}H^{\frac{1+\varepsilon}{\varepsilon}}, \tag{1.19}
$$

where ε is the Frisch elasticity of labour supply (i.e. percentage change in labour supply due to a one percentage change in income keeping the marginal utility of wealth fixed). To calibrate the value of ε, we need to look beyond the data on the balanced growth path.

Add notes on Frisch elasticity of labour supply.

The estimates of ε are wide ranging (anywhere from zero to "∞") but (Shimer says that) it is reasonable to use an estimate of around 1. The problem with estimating ε arises from the fact that, here, we are assuming a representative agent who has much larger margin of labour supply adjustment than an individual (e.g. extensive vs intensive margin). In any case, once we have an estimate of ε, we can back out the value of γ using (1.16).

Suppose we assume the Frisch elasticity of labour supply, ε, to be one; i.e. $\varepsilon=1$. As it turns out, γ, the disutility from working, only affects the level of the stationary point but not the dynamics. We therefore use γ to normalise the level/unit of $\bar{H}$. Since data suggests average weekly hours to be around 23 hours, given $\varepsilon=1$, this implies that $\gamma=0.00153$.

[8] With this functional form,

$$
v'(H) = \gamma H^{\frac{1}{\varepsilon}},\; v''(H) = \frac{1}{\varepsilon}\gamma H^{\frac{1}{\varepsilon}-1}
$$

so that v' and v'' are completely parameterised by γ and ε.

To see this, consider the intratemporal condition evaluated at the stationary point:

$$(1-\alpha)\,\bar{k}^{\alpha}\,(\bar{s})^{1-\alpha}\left(\bar{H}\right)^{-\alpha}=\gamma\bar{c}\left(\bar{H}\right)^{\frac{1}{\varepsilon}}.$$

If $\varepsilon = 1$, recalling that $\bar{s} = 1$, then

$$\begin{aligned}\gamma &= \frac{(1-\alpha)\,\bar{k}^{\alpha}\bar{H}^{-\alpha-1}}{\bar{c}}\\ &= \frac{(1-\alpha)\left(\frac{\bar{k}}{\bar{H}}\right)^{\alpha}}{\bar{c}\bar{H}}\end{aligned}$$

From the feasibility condition evaluated at the stationary point:

$$\begin{aligned}\bar{c} &= \bar{k}^{\alpha}\left(\bar{H}\right)^{1-\alpha}-(\delta+g)\,\bar{k}\\ &= \left[\frac{\bar{y}}{\bar{k}}-(\delta+g)\right]\frac{\bar{k}}{\bar{H}}\bar{H}.\end{aligned}$$

Note that

$$\left(\frac{\bar{k}}{\bar{H}}\right)^{-(1-\alpha)}=\frac{\bar{k}^{\alpha}\bar{H}^{1-\alpha}}{\bar{k}}=\frac{\bar{y}}{\bar{k}}\Leftrightarrow\frac{\bar{k}}{\bar{H}}=\left(\frac{\bar{k}}{\bar{y}}\right)^{\frac{1}{1-\alpha}}.$$

Since Y and K grow at the same rate on the balanced growth path:

$$\frac{\bar{k}}{\bar{y}}=\frac{K}{Y}=\begin{cases}3.2 & \text{annual}\\ 12.8 & \text{quarterly}\end{cases}.$$

Letting $\bar{H} = 23$, and given the other calibrated values,

$$\begin{aligned}\bar{c} &= \left[\frac{1}{12.8}-(0.015+0.005)\right](12.7)^{\frac{1}{1-0.4}}\times 23 = 93.1321,\\ \Rightarrow\gamma &= \frac{(1-0.4)\left(12.8^{\frac{1}{1-0.4}}\right)^{0.4}}{c^{*}\times 23} = 0.00153272\end{aligned}$$

Remark 1.2. See summary of reading for Stokey's class on labour supply elasticities.

1.3 Transition dynamics

We now consider how the economy behaves off of the balanced growth path; i.e. its transition dynamics. Recall that we have three equations that describe the equilibrium:

$$c_t v'(H_t) = (1-\alpha)\,k_t^{\alpha}H_t^{-\alpha}, \tag{1.20}$$

$$\frac{1+g}{c_t}=\beta\frac{\alpha k_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha}+1-\delta}{c_{t+1}}, \tag{1.21}$$

$$k_{t+1}(1+g)=k_t^{\alpha}H_t^{1-\alpha}+(1-\delta)\,k_t-c_t, \tag{1.22}$$

We also need a transversality condition:

$$\lim_{T\to\infty}\beta^{T}F_{K,T}(K_T,H_T)\,U_C(C_T,H_T)\,K_T=0. \tag{1.23}$$

To give some intuition about the transversality condition, consider the finite-horizon case. If the horizon is finite, to the extent capital has some value, agents would want to run down the capital stock in the last period. For the social planner's problem to have a solution in the infinite horizon, the same must be true, which is why we require the limit condition as above.

Observe that the intratemporal condition is a static equation. This means that we can do one of two things to reduce the number of equations from three to two:

- ▷ solve for C_t and plug into the Euler equation and the feasibility conditions. This means that we have a system of (two) difference equations in K_t and H_t.
- ▷ if v' is invertible, we can solve for H_t and plug into the into the Euler equation and the feasibility conditions. In this case, we have a system of (two) difference equations in K_t and C_t.

We will focus on the second option.

First, let us suppose the labour supply is inelastic; i.e. $\varepsilon = 0$, which means that $v(H) = 0$. That is, labour supply drops out of the utility function. We can assume that $H_t = H$, where H is usually normalised to one. In effect, this is assuming that the disutility of work is infinite if $H_t > H$. That labour supply is fixed also implies that the intratemporal condition (1.20) drops out of the system. Rearranging the remaining conditions gives

$$\frac{c_{t+1}}{c_t}(1+g) = \beta\left(\alpha k_{t+1}^{\alpha-1} H^{1-\alpha} + 1 - \delta\right), \tag{1.24}$$

$$\begin{aligned} k_{t+1} - k_t &= \frac{k_t^{\alpha} H^{1-\alpha} + (1-\delta) k_t - c_t}{1+g} - k_t \\ &= \frac{k_t^{\alpha} H^{1-\alpha} - (\delta + g) k_t - c_t}{1+g}. \end{aligned} \tag{1.25}$$

$c_{t+1} = c_t$ locus Now consider the case when $c_{t+1} = c_t$ (but not necessarily $k_{t+1} = k_t$). From the first equation, we realise that

$$k_{t+1}^{\alpha-1} = \frac{1}{\alpha H^{1-\alpha}}\left(\frac{1+g}{\beta} - (1-\delta)\right). \tag{1.26}$$

Since the right-hand side is a constant, whenever consumption is static, k_{t+1} takes on a constant value. Denote this value as $\bar{k}$. In (k_{t+1}, c_t) space, the $c_{t+1} = c_t$ locus is a vertical line.

$k_{t+1} = k_t$ locus Now consider the case when $k_{t+1} = k_t$ (but not necessarily $c_{t+1} = c_t$). From the second equation, this is true whenever the numerator on the right-hand side equal zero; i.e.

$$c_t = k_t^{\alpha} H^{1-\alpha} - (\delta + g) k_t. \tag{1.27}$$

In (k, c) space, this is a concave hump-shaped function with the following features:

- ▷ at $k = 0$, the slope is positive infinity since

$$\begin{aligned} \lim_{k\to 0} \frac{\partial}{\partial k}\left[k^{\alpha} H^{1-\alpha} - (\delta+g) k\right] &= \lim_{k\to 0} \alpha k^{\alpha-1} H^{1-\alpha} - (\delta + g) \\ &= \lim_{k\to 0} \alpha \left(\frac{H}{k}\right)^{1-\alpha} - (\delta+g) = \infty \end{aligned}$$

▷ maximum value is attained at

$$\begin{aligned}0 &= \frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right]\\ \Rightarrow \alpha\hat{k}^{\alpha-1}H^{1-\alpha} &= \delta+g\\ \Rightarrow \hat{k} &= \left(\frac{\delta+g}{\alpha}\right)^{\frac{1}{\alpha-1}}H.\end{aligned}$$

▷ at $k=\bar{k}$, the slope is positive:

$$\begin{aligned}\frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right]\Big|_{k=\bar{k}} &= \alpha k^{\alpha-1}H^{1-\alpha} - (\delta+g)\big|_{k_t=\bar{k}}\\ &= \alpha\frac{1}{\alpha H^{1-\alpha}}\left(\frac{1+g}{\beta} - (1-\delta)\right)H^{1-\alpha} - (\delta+g)\\ &= \frac{1+g}{\beta} - (1+g) > 0\end{aligned}$$

since $\beta \in (0,1)$. In particular, this means that the maximum k_t is obtained at $\hat{k} > \bar{k}$.

▷ as $k \to \infty$, the slope tends to $-(\delta+g)$:

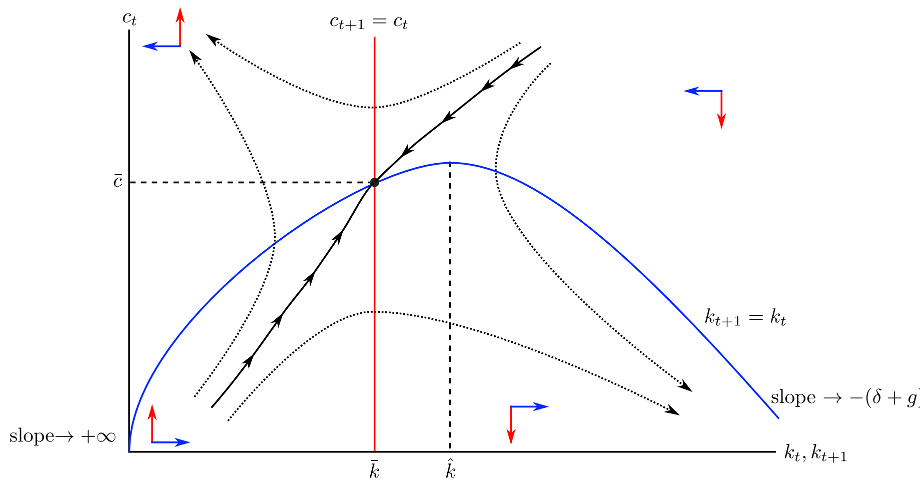
$$\begin{aligned}\lim_{k\to\infty}\frac{\partial}{\partial k}\left[k^{\alpha}H^{1-\alpha} - (\delta+g)\,k\right] &= \lim_{k\to\infty}\alpha\left(\frac{H}{k}\right)^{1-\alpha} - (\delta+g)\\ &= -(\delta+g)\,.\end{aligned}$$

Where the two loci, (1.26) and (1.27), intersect is the balanced growth path, $(\bar{k},\bar{c})$. To obtain the transition dynamics, observe the following

▷ If we start with k above this level, then the right-hand side of (1.24) is smaller so that $c_{t+1}/c_t < 1 \Leftrightarrow c_{t+1} < c_t$; i.e. consumption is falling. Hence, to the right of the locus, consumption is falling, and to the left, consumption is increasing.

▷ If we start with c above the $k_{t+1} = k_t$ locus, then the right-hand side of (1.25) is larger, which means that $k_{t+1} - k_t < 0$; i.e. capital is falling. In contrast, if we start with c below the locus, then capital is increasing.

The figure below gives the phase diagram for this model.



These transition dynamics mean that there is a one-dimensional set of points, called the saddle path, that converges to the balanced growth path $\left(\bar{k}, \bar{c}\right)$.

If we start off the saddle path, then one of two things happen:

▷ if we start above the saddle path, then $k_t \to 0$ in finite time;

▷ if we start below the saddle path, then $c_t \to 0$.

However, neither of these cases can be the solution to the social planner's problem. If $K_t = 0$, then output is zero (recall Inada condition says that $F(0, H) = 0$) so that consumption is zero, which implies $-\infty$ utility for the agent. Similarly, if $c_t \to 0$, then $U_C \to \infty$ while k_t tends to some positive number so that this violates the transversality condition (1.23).

Therefore, the solution to the problem must be on the saddle path.

1.3.1 Numerically solving for the saddle path

Inelastic labour supply ($\varepsilon = 0$) We fix the initial capital to be some value k_0. The idea is to guess the value of c_0 and iterate using the difference equation (1.21) and (1.22).

▷ If we guessed the correct value of c_0 (i.e. (k_0, c_0) is on the saddle path), then we know that $(k_t, c_t) \to \left(\bar{k}, \bar{c}\right)$.

▷ If we guessed too high, then we know that $c_t \not\to 0$ and $k_t \to 0$.

▷ If we guessed too low, then $c_t \to 0$ but $k_t \not\to 0$.

Remark 1.3. Since we will be solving this numerically, in reality, we will find that the economy may converge very close to $\left(\bar{k}, \bar{c}\right)$ and stay there for many periods, but then shoot off. We can't help that.

Remark 1.4. *(Reverse shooting)* Shooting algorithm is not very efficient (see Problem Set 1). What we can do is to reverse. Suppose we set the "initial" point to be $\left(\bar{k}, \bar{c} - \varepsilon\right)$, where ε is some small positive number. If we use the difference equations backwards, then this gives us the path that leads to the point $\left(\bar{k}, \bar{c} - \varepsilon\right)$. We can do this until we reach some desired level of initial capital (e.g. $k_0 = \bar{k}/2$), and read off the value of c_t at this point. This will not give us an exact value but by choosing ε small, we can be very close to the correct initial point on the saddle path.

Elastic labour supply ($\varepsilon \neq 0$) The method is essentially the same. However, we now need to "guess" H_0 as well as c_0 given k_0. But given our assumption about the invertibility of v', we can simply replace H_t and H_{t+1} in the Euler equation and the feasibility condition using the intratemporal condition. In effect, for any given c_0 and k_0, we must solve for H_t using the intratemporal condition. This leaves a system of difference equations on c_t and k_t and we can proceed as before.

Remark 1.5. See Problem Set 1.

1.4 Log linearisation

We can log linearise the system of equations (the intratemporal condition, the Euler equation, and the feasibility condition) to study the dynamics around the balanced growth path. In doing so, we also assume that there are no shocks in the future.

1.4.1 General case

Suppose that the system of equations is given by

$$\mathbf{G}\left(\mathbf{x}_t, \mathbf{x}_{t+1}\right) = \mathbf{0}, \tag{1.28}$$

where $\mathbf{x}_t = \{x_{1,t}, x_{2,t}, \ldots, x_{m,t}\}'$ are m variables consisting of both state and control variables and $\mathbf{G} = \{G_1, G_2, \ldots, G_m\}'$ are m explicit/implicit equations for the m variables. If the system has a stationary point, denoted $\mathbf{x}^*$, it satisfies

$$\mathbf{G}\left(\mathbf{x}^*, \mathbf{x}^*\right) = \mathbf{0}.$$

In the case of the model we have been studying (i.e. with a balanced growth path), $\mathbf{x}_t$ must be detrended.

Take the jth equation and write

$$0 = G_j\left(\mathbf{x}_t, \mathbf{x}_{t+1}\right) \equiv G_j\left(e^{\hat{\mathbf{x}}_t}\mathbf{x}^*, e^{\hat{\mathbf{x}}_{t+1}}\mathbf{x}^*\right),$$

where

$$\hat{\mathbf{x}}_t \coloneqq \log \mathbf{x}_t - \log \mathbf{x}^*$$

and $e^{\hat{\mathbf{x}}_t}$ and $\log \mathbf{x}_t$ means that we take exponent and logarithm of the vector $\hat{\mathbf{x}}_t/\mathbf{x}_t$ component by component. We can interpret the log deviation from the steady state, $\hat{\mathbf{x}}_t$, as percentage deviation of $\mathbf{x}_t$ from $\mathbf{x}^*$. To log linearise, we use first-order Taylor expansion around the stationary point,

$$\mathbf{x}_t = \mathbf{x}_{t+1} = \mathbf{x}^* \Leftrightarrow \hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t+1} = \mathbf{0}.$$

For the jth equation, Taylor expansion gives

$$
\begin{aligned}
0 &\simeq \underbrace{G_j\left(\mathbf{x}^*,\mathbf{x}^*\right)}_{=0} + \sum_{i=1}^{m}\left(\frac{\partial G_j\left(e^{\hat{\mathbf{x}}_t}\mathbf{x}^*, e^{\hat{\mathbf{x}}_{t+1}}\mathbf{x}^*\right)}{\partial x_{i,t}} e^{\hat{\mathbf{x}}_t} x_i^*\Bigg|_{\hat{\mathbf{x}}_t=\hat{\mathbf{x}}_{t+1}=0}\right)\left(\hat{x}_{i,t}-0\right) \\
&\quad + \sum_{i=1}^{m}\left(\frac{\partial G_j\left(e^{\hat{\mathbf{x}}_t}\mathbf{x}^*, e^{\hat{\mathbf{x}}_{t+1}}\mathbf{x}^*\right)}{\partial x_{i,t+1}} e^{\hat{\mathbf{x}}_{t+1}} x_i^*\Bigg|_{\hat{\mathbf{x}}_t=\hat{\mathbf{x}}_{t+1}=0}\right)\left(\hat{x}_{i,t+1}-0\right) \\
&= \sum_{i=1}^{m}\frac{\partial G_j\left(\mathbf{x}^*,\mathbf{x}^*\right)}{\partial x_{i,t}} x_i^*\hat{x}_{i,t} + \sum_{i=1}^{m}\frac{\partial G_j\left(\mathbf{x}^*,\mathbf{x}^*\right)}{\partial x_{i,t+1}} x_i^*\hat{x}_{i,t+1}.
\end{aligned}
$$

Define an $m \times m$ matrix $\mathbf{G}_t^*$ as

$$
\mathbf{G}_t^* := \left[\frac{\partial G_i\left(\mathbf{x}^*,\mathbf{x}^*\right)}{\partial x_{j,t}} x_j^*\right]_{i,j}, \ \forall t.
$$

Using this matrix notation, we can rewrite the log-linearised system of equations succinctly as

$$
\mathbf{G}_t^*\hat{\mathbf{x}}_t + \mathbf{G}_{t+1}^*\hat{\mathbf{x}}_{t+1} \simeq \mathbf{0}.
$$

We “approximate”; i.e. we assume that the equation above holds with equality.

If $\mathbf{G}_{t+1}^*$ is invertible, then we can write

$$
\hat{\mathbf{x}}_{t+1} = -\left(\mathbf{G}_{t+1}^*\right)^{-1}\mathbf{G}_t^*\hat{\mathbf{x}}_t = \mathbf{M}\hat{\mathbf{x}}_t.
$$

Notice that $\mathbf{M}$ does not depend on t since the partial derivatives are evaluated at the stationary point.

Remark 1.6. In the neoclassical model, $\mathbf{G}_{t+1}^*$ is *not* invertible. This is because of the intratemporal condition which only contains variables in period t, and not $t+1$. Therefore, the row corresponding to this equation in $\mathbf{G}_{t+1}^*$ are all zeros so that $\mathbf{G}_{t+1}^*$ is not invertible.

Backwards substitution yields

$$
\hat{\mathbf{x}}_t = \mathbf{M}\hat{\mathbf{x}}_{t-1} = \mathbf{M}^2\hat{\mathbf{x}}_{t-2} = \cdots = \mathbf{M}^t\hat{\mathbf{x}}_0. \tag{1.29}
$$

To solve the this difference equation, it is easier for us to diagonalise the matrix $\mathbf{M}$ (see my my notes for Fernando for details). Let $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ denote the m *distinct* and *real* eigenvalues of $\mathbf{M}$, and denote the corresponding $m \times m$ matrix of eigenvectors as $\mathbf{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$. By definition, $\mathbf{e}_i$ is an eigenvector of $\mathbf{M}$ if

$$
\mathbf{M}\mathbf{e}_i = \lambda_i\mathbf{e}_i,
$$

which means that we can write

$$
\mathbf{M}\mathbf{E} = \mathbf{E}\boldsymbol{\Lambda},
$$

where $\mathbf{\Lambda} \coloneqq \operatorname{diag}[\boldsymbol{\lambda}]$. Furthermore, if $\mathbf{E}$ is invertible, then,

$$\begin{aligned}
\mathbf{M} &= \mathbf{E\Lambda E}^{-1} \\
\hat{\mathbf{x}}_{t+1} &= \mathbf{E\Lambda E}^{-1}\hat{\mathbf{x}}_t \\
&= \mathbf{E\Lambda E}^{-1}\mathbf{E\Lambda} = \mathbf{E\Lambda}^2\mathbf{E}^{-1}\hat{\mathbf{x}}_{t-1} \\
&= \quad \vdots \\
&= \mathbf{E\Lambda}^{t+1}\mathbf{E}^{-1}\hat{\mathbf{x}}_0 \\
\Rightarrow \mathbf{E}^{-1}\hat{\mathbf{x}}_{t+1} &= \mathbf{\Lambda}^{t+1}\mathbf{E}^{-1}\hat{\mathbf{x}}_0
\end{aligned}$$

Define $\mathbf{z}_t \coloneqq \mathbf{E}^{-1}\hat{\mathbf{x}}_{t+1}$ so we can express above as

$$\mathbf{z}_{t+1} = \mathbf{\Lambda}^{t+1}\mathbf{z}_0.$$

But because $\mathbf{\Lambda}$ is a diagonal matrix, each of the m equations are independent; i.e. (rolling back one period)

$$z_{i,t} = \lambda_{i,t}^t z_{i,0}, \ i = 1, 2, \ldots, m.$$

Note that

$$\begin{aligned}
\hat{\mathbf{x}}_t &= \mathbf{E}\mathbf{z}_t = \mathbf{E\Lambda}^t\mathbf{z}_0 \\
&= \begin{bmatrix} e_{11} & e_{22} & \cdots & e_{m1} \\ e_{12} & e_{22} & \cdots & e_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1m} & e_{2m} & \cdots & e_{mm} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^t \end{bmatrix} \mathbf{E}^{-1}\hat{\mathbf{x}}_0 \\
&= \begin{bmatrix} \mathbf{e}_1\lambda_1^t & \mathbf{e}_2\lambda_2^t & \cdots & \mathbf{e}_m\lambda_m^t \end{bmatrix} \mathbf{E}^{-1}\hat{\mathbf{x}}_0 \\
\Rightarrow \hat{\mathbf{x}}_t' &= \underbrace{\hat{\mathbf{x}}_0'\mathbf{E}^{-1\prime}}_{=\boldsymbol{\mu}_{1\times m}} \begin{bmatrix} \lambda_1^t\mathbf{e}_1' \\ \lambda_2^t\mathbf{e}_2' \\ \vdots \\ \lambda_m^t\mathbf{e}_m' \end{bmatrix}_{m\times m} \\
&= \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix} \begin{bmatrix} \lambda_1^t e_{11} & \lambda_1^t e_{12} & \cdots & \lambda_1^t e_{1m} \\ \lambda_2^t e_{21} & \lambda_2^t e_{22} & \cdots & \lambda_2^t e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m^t e_{m1} & \lambda_m^t e_{m2} & \cdots & \lambda_m^t e_{mm} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^m \mu_j\lambda_j^t e_{j1} & \sum_{j=1}^m \mu_j\lambda_j^t e_{j2} & \cdots & \sum_{j=1}^m \mu_j\lambda_j^t e_{jm} \end{bmatrix} \\
\Rightarrow \hat{x}_{i,t} &= \sum_{j=1}^m \mu_j\lambda_j^t e_{ji}
\end{aligned}$$

$$\Rightarrow \begin{bmatrix} \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \vdots \\ \hat{x}_{m,t} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m \mu_j\lambda_j^t e_{j1} \\ \sum_{j=1}^m \mu_j\lambda_j^t e_{j2} \\ \vdots \\ \sum_{j=1}^m \mu_j\lambda_j^t e_{jm} \end{bmatrix} = \sum_{j=1}^m \begin{bmatrix} \mu_j\lambda_j^t \\ \mu_j\lambda_j^t \\ \vdots \\ \mu_j\lambda_j^t \end{bmatrix} \mathbf{e}_j = \sum_{i=1}^m \mu_i\lambda_i^t\mathbf{e}_i$$

That is,

$$\hat{\mathbf{x}}_t = \sum_{i=1}^{m} \mu_i \lambda_i^t \mathbf{e}_i.$$

Thus, we can express $\hat{\mathbf{x}}_t$ as a linear combination of the initial values of $\hat{\mathbf{x}}_0$ with weights given by the eigenvalues and the eigenvectors.

Since we assumed that the eigenvalues are distinct, we can pin down μ_i using the initial condition $\hat{\mathbf{x}}_0$ by solving

$$\hat{\mathbf{x}}_0 = \sum_{i=1}^{m} \mu_i \mathbf{e}_i.$$

Given that the system converges, it must be the case that, if $|\lambda_i| \geq 1$, then $\mu_i = 0$ (this is not clear if we were only looking at (1.29)). We say that the system is *saddle path stable* if not all eigenvalues are in the unit circle.

1.4.2 Log linearisation versus linearisation

There is no rule that says how one should choose between log-linearisation or linearisation. Of course, log-linearisation will not be appropriate if we expect the variables to be negative (e.g. the real interest rate, the inflation rate). However, in general, the approximation tends to be more accurate with log-linearisation if variables are positive. Another advantage of log-linearisation is that we need not worry about the units of the variable as we can interpret in terms of percentage changes with log linearisation.

If we were to linearise (1.28) around the stationary point $\mathbf{x}^*$, then we obtain

$$0 \simeq \sum_{i=1}^{m} \frac{\partial G_j\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial x_{i,t}} \left(x_{i,t} - x_i^*\right) + \sum_{i=1}^{m} \frac{\partial G_j\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial x_{i,t+1}} \left(x_{i,t+1} - x_i^*\right).$$

Define

$$\tilde{\mathbf{G}}_t^* := \left[\frac{\partial G_j\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial x_{i,t}}\right]_{i,j}, \ \forall t,$$

then

$$\tilde{\mathbf{G}}_t^* \left(\mathbf{x}_t - \mathbf{x}^*\right) + \tilde{\mathbf{G}}_{t+1}^* \left(\mathbf{x}_{t+1} - \mathbf{x}^*\right) \simeq \mathbf{0}.$$

So, if $\tilde{\mathbf{G}}_{t+1}^*$ is invertible,

$$\begin{aligned} \mathbf{x}_{t+1} - \mathbf{x}^* &= \left(\tilde{\mathbf{G}}_{t+1}^*\right)^{-1} \tilde{\mathbf{G}}_t^* \left(\mathbf{x}_t - \mathbf{x}^*\right) \\ &= \tilde{\mathbf{M}} \left(\mathbf{x}_t - \mathbf{x}^*\right). \end{aligned}$$

1.4.3 Log linearising the Neoclassical Growth Model

Suppose that

$$\begin{aligned} u\left(C, H\right) &= \log C - \frac{\gamma \varepsilon}{1+\varepsilon} H^{\frac{1+\varepsilon}{\varepsilon}}, \\ F_t\left(K, H, \mathbf{s}^t\right) &= K^\alpha \left(\left(1+g\right)^t s_t H\right)^{1-\alpha} + \left(1-\delta\right) K, \\ \log s_{t+1} &= \rho \log s_t + \upsilon_{t+1}, \end{aligned}$$

where $|\rho| < 1$ and υ is iid with mean zero. With this functional form,

$$\begin{aligned}
v\left(H_t\right) &= \frac{\gamma\varepsilon}{1+\varepsilon} H_t^{\frac{1+\varepsilon}{\varepsilon}} \Rightarrow v'\left(H_t\right) = \gamma H_t^{\frac{1}{\varepsilon}}, \\
F_{K,t}\left(K_t, H_t\right) &= \alpha K_t^{-(1-\alpha)} \left(\left(1+g\right)^t s_t H_t\right)^{1-\alpha} + 1 - \delta \\
&= \alpha k_t^{-(1-\alpha)} s_t^{1-\alpha} H_t^{1-\alpha} + 1 - \delta, \\
F_{H,t}\left(K_t, H_t\right) &= \left(1-\alpha\right)\left(1+g\right)^t s_t K_t^{\alpha} \left(\left(1+g\right)^t s_t H_t\right)^{-\alpha} \\
&= \left(1-\alpha\right)\left(1+g\right)^t k_t^{\alpha} s_t^{1-\alpha} H_t^{-\alpha},
\end{aligned}$$

where lower case letters mean detrended (i.e. divide by $(1+g)^t$). Then the intratemporal condition, (1.5) is now

$$\begin{aligned}
\left(1-\alpha\right)\left(1+g\right)^t k_t^{\alpha} s_t^{1-\alpha} H_t^{-\alpha} &= \gamma C_t H_t^{\frac{1}{\varepsilon}} \\
\Leftrightarrow \left(1-\alpha\right) k_t^{\alpha} s_t^{1-\alpha} H_t^{-\alpha} &= \gamma c_t H_t^{\frac{1}{\varepsilon}}
\end{aligned} \tag{1.30}$$

The Euler equation, (1.6), becomes (set $\sigma = 1$)

$$\frac{1+g}{c_t} = \beta \mathbb{E}_t \left[\frac{\alpha k_{t+1}^{-(1-\alpha)} s_{t+1}^{1-\alpha} H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}} \right],$$

where we added the expectations operator. The feasibility condition, (1.7), is

$$k_{t+1}\left(1+g\right) = k_t^{\alpha}\left(s_t H_t\right)^{1-\alpha} + \left(1-\delta\right) k_t - c_t.$$

Since t does not appear explicitly in the equations, we have

$$\begin{aligned}
k_t &\to k^*, \\
H_t &\to H^*, \\
c_t &\to c^*, \\
s_t &\to s^* = 1,
\end{aligned}$$

where the last equality follows since the unconditional (i.e. long-run) mean of $\log s_{t+1}$ is zero given $|\rho| < 1$.

We now assume shutdown the shocks; i.e. $\upsilon_{t+1} = 0$ for all $t \geq 0$. This means that we can remove the expectation term from the Euler equation. The law of motion for the state is now

$$\log s_{t+1} = \rho \log s_t.$$

Rewrite the set of equations to equal zero then log-linearise.

$$
\begin{aligned}
&\{\mathbf{G}_1(\mathbf{x}_t, \mathbf{x}_{t+1})\} \quad 0 = (1-\alpha) k_t^{\alpha} s_t^{1-\alpha} H_t^{-\alpha} - \gamma c_t H_t^{\frac{1}{\varepsilon}},\\
&\{\mathbf{G}_2(\mathbf{x}_t, \mathbf{x}_{t+1})\} \quad 0 = \frac{1+g}{c_t} - \beta \frac{\alpha k_{t+1}^{-(1-\alpha)} s_{t+1}^{1-\alpha} H_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}},\\
&\{\mathbf{G}_3(\mathbf{x}_t, \mathbf{x}_{t+1})\} \quad 0 = k_{t+1}(1+g) - k_t^{\alpha}(s_t H_t)^{1-\alpha} - (1-\delta) k_t + c_t,\\
&\{\mathbf{G}_4(\mathbf{x}_t, \mathbf{x}_{t+1})\} \quad 0 = \log s_{t+1} - \rho \log s_t.
\end{aligned}
$$

where $\mathbf{x}_t = (k_t, c_t, s_t, H_t)$. We can change the variable and compute the derivatives or we can use the expression we obtained for the general case.

Take $\mathbf{G}_1$, the intratemporal condition, first.

$$
\begin{aligned}
\frac{\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*)}{\partial k_t} k^* &= \alpha(1-\alpha)(k^*)^{\alpha}(s^*)^{1-\alpha}(H^*)^{-\alpha},\\
\frac{\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*)}{\partial c_t} c^* &= -\gamma c^* (H^*)^{\frac{1}{\varepsilon}},\\
\frac{\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*)}{\partial s_t} s^* &= (1-\alpha)^2 (k^*)^{\alpha}(s^*)^{1-\alpha}(H^*)^{-\alpha},\\
\frac{\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*)}{\partial H_t} H^* &= -\alpha(1-\alpha)(k^*)^{\alpha}(s^*)^{1-\alpha}(H^*)^{-\alpha} - \frac{1}{\varepsilon}\gamma c^* (H^*)^{\frac{1}{\varepsilon}},
\end{aligned}
$$

and $\partial \mathbf{G}_1 / \partial \mathbf{x}_{t+1} = \mathbf{0}$. Notice that $(\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*) / \partial c_t) c^*$ contains $v'(H^*)$ and the second term in $(\partial \mathbf{G}_1(\mathbf{x}^*, \mathbf{x}^*) / \partial H_t) H^*$ contains $v''(H^*)$ so that, as mentioned before, log-linearisation requires specification of v' as well as v''.

The log-linearised approximation of the intratemporal condition is

$$
0 = \left(\frac{\partial \mathbf{G}_1}{\partial k_t} k^*\right)\hat{k}_t + \left(\frac{\partial \mathbf{G}_1}{\partial c_t} c^*\right)\hat{c}_t + \left(\frac{\partial \mathbf{G}_1}{\partial s_t} s^*\right)\hat{s}_t + \left(\frac{\partial \mathbf{G}_1}{\partial H_t} H^*\right)\hat{H}_t \tag{1.31}
$$

Next, consider the Euler equation, $\mathbf{G}_2$.

$$
\begin{aligned}
&\frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial k_t} k^* = \frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial s_t} s^* = \frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial H_t} H^* = 0,\\
&\frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial c_t} c^* = -\frac{1+g}{c^*},\\
&\frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial k_{t+1}} k^* = (1-\alpha)\beta \frac{\alpha (k^*)^{-(1-\alpha)}(s^*)^{1-\alpha}(H^*)^{1-\alpha}}{c^*},\\
&\frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial c_{t+1}} c^* = \beta \frac{\alpha (k^*)^{-(1-\alpha)}(s^*)^{1-\alpha}(H^*)^{1-\alpha} + 1 - \delta}{c^*},\\
&\frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial s_{t+1}} s^* = \frac{\partial \mathbf{G}_2(\mathbf{x}^*, \mathbf{x}^*)}{\partial H_{t+1}} H^* = -(1-\alpha)\beta \frac{\alpha (k^*)^{-(1-\alpha)}(s^*)^{1-\alpha}(H^*)^{1-\alpha}}{c^*}.
\end{aligned}
$$

So the log-linearised approximation of the Euler equation is

$$
\begin{aligned}
0 = &\left(\frac{\partial \mathbf{G}_2}{\partial c_t} c^*\right)\hat{c}_t + \left(\frac{\partial \mathbf{G}_2}{\partial k_{t+1}} k^*\right)\hat{k}_{t+1} + \left(\frac{\partial \mathbf{G}_2}{\partial c_{t+1}} c^*\right)\hat{c}_{t+1}\\
&+ \left(\frac{\partial \mathbf{G}_2}{\partial s_{t+1}} s^*\right)\hat{s}_{t+1} + \left(\frac{\partial \mathbf{G}_2}{\partial H_{t+1}} H^*\right)\hat{H}_{t+1}
\end{aligned}
$$

Consider the feasibility condition, $\mathbf{G}_3$.

$$\begin{aligned}
\frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial k_t} k^* &= -\alpha \left(k^*\right)^\alpha \left(s^* H^*\right)^{1-\alpha} - (1-\delta) k^*, \\
\frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial c_t} c^* &= c^*, \\
\frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial s_t} s^* &= \frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial H_t} H^* = -(1-\alpha)\left(k^*\right)^\alpha \left(s^* H^*\right)^{1-\alpha}, \\
\frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial k_{t+1}} k^* &= (1+g) k^*, \\
\frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial c_{t+1}} c^* &= \frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial s_{t+1}} s^* = \frac{\partial \mathbf{G}_3\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial H_{t+1}} H^* = 0.
\end{aligned}$$

The log-linearised approximation of the feasibility condition is

$$\begin{aligned}
0 = &\left(\frac{\partial \mathbf{G}_3}{\partial k_t} k^*\right) \hat{k}_t + \left(\frac{\partial \mathbf{G}_3}{\partial c_t} c^*\right) \hat{c}_t + \left(\frac{\partial \mathbf{G}_3}{\partial s_t} s^*\right) \hat{s}_t + \left(\frac{\partial \mathbf{G}_3}{\partial H_t} H^*\right) \hat{H}_t \\
&+ \left(\frac{\partial \mathbf{G}_3}{\partial k_{t+1}} k^*\right) \hat{k}_{t+1}
\end{aligned}$$

Finally, consider the law of motion for productivity,

$$\begin{aligned}
\frac{\partial \mathbf{G}_4\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial s_t} s^* &= -\rho, \\
\frac{\partial \mathbf{G}_4\left(\mathbf{x}^*, \mathbf{x}^*\right)}{\partial s_{t+1}} s^* &= 1,
\end{aligned}$$

and the log-linearised approximation is given by

$$0 = \left(\frac{\partial \mathbf{G}_4}{\partial s_{t+1}} s^*\right) \hat{s}_{t+1} + \left(\frac{\partial \mathbf{G}_4}{\partial s_t} s^*\right) \hat{s}_t.$$

We can now write $\mathbf{G}_t^*$ and $\mathbf{G}_{t+1}^*$ as

$$\mathbf{G}_t^* = \begin{bmatrix}
\frac{\partial \mathbf{G}_1}{\partial k_t} k^* & \frac{\partial \mathbf{G}_1}{\partial c_t} c^* & \frac{\partial \mathbf{G}_1}{\partial s_t} s^* & \frac{\partial \mathbf{G}_1}{\partial H_t} H^* \\
\frac{\partial \mathbf{G}_2}{\partial k_t} k^* & \frac{\partial \mathbf{G}_2}{\partial c_t} c^* & \frac{\partial \mathbf{G}_2}{\partial s_t} s^* & \frac{\partial \mathbf{G}_3}{\partial H_t} H^* \\
\frac{\partial \mathbf{G}_3}{\partial k_t} k^* & \frac{\partial \mathbf{G}_3}{\partial c_t} c^* & \frac{\partial \mathbf{G}_4}{\partial s_t} s^* & \frac{\partial \mathbf{G}_3}{\partial H_t} H^* \\
\frac{\partial \mathbf{G}_4}{\partial k_t} k^* & \frac{\partial \mathbf{G}_2}{\partial c_t} c^* & \frac{\partial \mathbf{G}_4}{\partial s_t} s^* & \frac{\partial \mathbf{G}_4}{\partial H_t} H^*
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \mathbf{G}_1}{\partial k_t} k^* & \frac{\partial \mathbf{G}_1}{\partial c_t} c^* & \frac{\partial \mathbf{G}_1}{\partial s_t} s^* & \frac{\partial \mathbf{G}_1}{\partial H_t} H^* \\
\frac{\partial \mathbf{G}_2}{\partial k_t} k^* & \frac{\partial \mathbf{G}_2}{\partial c_t} c^* & 0 & 0 \\
\frac{\partial \mathbf{G}_3}{\partial k_t} k^* & \frac{\partial \mathbf{G}_3}{\partial c_t} c^* & \frac{\partial \mathbf{G}_3}{\partial s_t} s^* & \frac{\partial \mathbf{G}_3}{\partial H_t} H^* \\
0 & 0 & 0 & \frac{\partial \mathbf{G}_4}{\partial s_t} s^*
\end{bmatrix},$$

$$\mathbf{G}_{t+1}^* = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{\partial \mathbf{G}_2}{\partial k_{t+1}} k^* & \frac{\partial \mathbf{G}_2}{\partial c_{t+1}} c^* & \frac{\partial \mathbf{G}_2}{\partial s_{t+1}} s^* & \frac{\partial \mathbf{G}_2}{\partial H_{t+1}} H^* \\
\frac{\partial \mathbf{G}_3}{\partial k_{t+1}} k^* & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial \mathbf{G}_4}{\partial s_{t+1}} s^*
\end{bmatrix}.$$

Observe that $\mathbf{G}_{t+1}^*$ is singular so that it is not invertible. The problem is that the intratemporal condition does not depend on the $t+1$ variables so that the first row of $\mathbf{G}_{t+1}^*$ are all zeros. However, we can reduce the dimension of the system by substituting $\hat{H}_t$ in terms of $\hat{c}_t$, $\hat{k}_t$ and $\hat{s}_t$.

Taking log of (1.30),

$$\begin{aligned}
\log H_t &= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\log k_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\log s_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log c_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log\frac{\gamma}{1-\alpha} \qquad (1.32)\\
\Rightarrow \hat{H}_t &= \log H_t - \log H^* \\
&= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\log k_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\log s_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log c_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log\frac{\gamma}{1-\alpha} - \log H^* \\
&= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log\frac{\gamma}{1-\alpha} - \log H^* \\
&\quad + \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\log k^* + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\log s^* - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log c^* \\
&\equiv \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_t + \Gamma,
\end{aligned}$$

where we grouped the constants as Γ. But, evaluating (1.32) at the stationary point, we have

$$\begin{aligned}
\log H^* &= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\log k^* + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\log s^* - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log c^* - \frac{1}{\alpha+\frac{1}{\varepsilon}}\log\frac{\gamma}{1-\alpha} \\
&\Rightarrow \Gamma = 0.
\end{aligned}$$

That is,

$$\hat{H}_t = \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_t. \qquad (1.33)$$

Remark 1.7. Observe that γ does not enter the equation above, nor does it enter into the other log-linearised equations. This is the reason why we say that γ does not affect the business cycles.

Substituting (1.33) into the log-linearised Euler equation:

$$\begin{aligned}
0 &= \left(\frac{\partial \mathbf{G}_2}{\partial c_t}c^*\right)\hat{c}_t + \left(\frac{\partial \mathbf{G}_2}{\partial k_{t+1}}k^*\right)\hat{k}_{t+1} + \left(\frac{\partial \mathbf{G}_2}{\partial c_{t+1}}c^*\right)\hat{c}_{t+1} + \left(\frac{\partial \mathbf{G}_2}{\partial s_{t+1}}s^*\right)\hat{s}_{t+1} \\
&\quad + \left(\frac{\partial \mathbf{G}_2}{\partial H_{t+1}}H^*\right)\left(\frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_{t+1} + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_{t+1} - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_{t+1}\right) \\
&= \left(\frac{\partial \mathbf{G}_2}{\partial c_t}c^*\right)\hat{c}_t + \left(\frac{\partial \mathbf{G}_2}{\partial k_{t+1}}k^* + \left(\frac{\partial \mathbf{G}_2}{\partial H_{t+1}}H^*\right)\frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\right)\hat{k}_{t+1} \\
&\quad + \left(\frac{\partial \mathbf{G}_2}{\partial c_{t+1}}c^* - \left(\frac{\partial \mathbf{G}_2}{\partial H_{t+1}}H^*\right)\frac{1}{\alpha+\frac{1}{\varepsilon}}\right)\hat{c}_{t+1} + \left(\frac{\partial \mathbf{G}_2}{\partial s_{t+1}}s^* + \left(\frac{\partial \mathbf{G}_2}{\partial H_{t+1}}H^*\right)\frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\right)\hat{s}_{t+1} \\
&\equiv \tilde{G}_{2c_t}\hat{c}_t + \tilde{G}_{2k_{t+1}}\hat{k}_{t+1} + \tilde{G}_{2c_{t+1}}\hat{c}_{t+1} + \tilde{G}_{2s_{t+1}}\hat{s}_{t+1}.
\end{aligned}$$

Similarly, substituting (1.33) into the feasibility condition yields

$$\begin{aligned}
0 &= \left(\frac{\partial \mathbf{G}_3}{\partial k_t}k^*\right)\hat{k}_t + \left(\frac{\partial \mathbf{G}_3}{\partial c_t}c^*\right)\hat{c}_t + \left(\frac{\partial \mathbf{G}_3}{\partial s_t}s^*\right)\hat{s}_t \\
&\quad + \left(\frac{\partial \mathbf{G}_3}{\partial H_t}H^*\right)\left(\frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_t\right) + \left(\frac{\partial \mathbf{G}_3}{\partial k_{t+1}}k^*\right)\hat{k}_{t+1} \\
&= \left(\frac{\partial \mathbf{G}_3}{\partial k_t}k^* + \left(\frac{\partial \mathbf{G}_3}{\partial H_t}H^*\right)\frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\right)\hat{k}_t + \left(\frac{\partial \mathbf{G}_3}{\partial c_t}c^* - \left(\frac{\partial \mathbf{G}_3}{\partial H_t}H^*\right)\frac{1}{\alpha+\frac{1}{\varepsilon}}\right)\hat{c}_t \\
&\quad + \left(\frac{\partial \mathbf{G}_3}{\partial s_t}s^* + \left(\frac{\partial \mathbf{G}_3}{\partial H_t}H^*\right)\frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\right)\hat{s}_t + \left(\frac{\partial \mathbf{G}_3}{\partial k_{t+1}}k^*\right)\hat{k}_{t+1} \\
&\equiv \tilde{G}_{3k_t}\hat{k}_t + \tilde{G}_{3c_t}\hat{c}_t + \tilde{G}_{3s_t}\hat{s}_t + \tilde{G}_{3k_{t+1}}\hat{k}_{t+1}.
\end{aligned}$$

The law of motion for productivity shock remains unchanged. We can now write the system as

$$\begin{aligned}
\tilde{\mathbf{G}}_t^* &= \begin{bmatrix} 0 & \tilde{G}_{2c_t} & 0 \\ \tilde{G}_{3k_t} & \tilde{G}_{3c_t} & \tilde{G}_{3s_t} \\ 0 & 0 & \frac{\partial \mathbf{G}_4}{\partial s_t}s^* \end{bmatrix}, \\
\mathbf{G}_{t+1}^* &= \begin{bmatrix} \tilde{G}_{2k_{t+1}} & \tilde{G}_{2c_{t+1}} & \tilde{G}_{2s_{t+1}} \\ \tilde{G}_{3k_{t+1}} & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{G}_4}{\partial s_{t+1}}s^* \end{bmatrix}.
\end{aligned}$$

Now the $\mathbf{G}_{t+1}^*$ could be invertible.

To make headway, we can substitute the calibrated values from before. Assuming quarterly time period,[9]

$$\begin{aligned}
\alpha &= 0.4, & g &= 0.005, \\
\delta &= 0.015, & \beta &= 0.989, \\
\rho &= 0.95, & \gamma &= 0.00153.
\end{aligned}$$

The stationary values are

$$\mathbf{x}^* = \begin{bmatrix} k^* \\ c^* \\ s^* \\ H^* \end{bmatrix} = \begin{bmatrix} 1610.93 \\ 93.1321 \\ 1 \\ 23 \end{bmatrix}.$$

Using these values, we can obtain that

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} = \begin{bmatrix} 1.024 & -0.091 & 0.067 \\ -0.013 & 0.988 & 0.024 \\ 0 & 0 & 0.95 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{s}_t \end{bmatrix}. \tag{1.34}$$

[9] We can estimate ρ as the total factor productivity (also called the Solow residual). Specifically, we use the feasibility condition, with calibrated values for g, α and δ, and data on k_t, k_{t+1}, H_t and c_t, to obtain a prediction of output. We then compare the predicted output against the actual output. The residual is the total factor productivity. We can then compute the autocorrelation within the residuals to obtain ρ. Empirically, the productivity factor, s_t, is found to be persistent with $\rho \simeq 1$. If $\rho \geq 1$, we know that the law of motion for productivity would not converge, so, here, we assume $\rho = 0.95$.

The eigenvalues of the $\mathbf{M}$ matrix are

$$\begin{aligned}
\lambda_1 &= 0.950, \\
\lambda_2 &= 0.967, \\
\lambda_3 &= 1.045.
\end{aligned}$$

Then

$$\begin{aligned}
\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{s}_t \end{bmatrix} &= \mu_1 \lambda_1^t \mathbf{e}_1 + \mu_2 \lambda_2^t \mathbf{e}_2 \\
&= \mu_1 \lambda_1^t \begin{bmatrix} -0.834 \\ -0.471 \\ 0.287 \end{bmatrix} + \mu_2 \lambda_2^t \begin{bmatrix} 0.846 \\ 0.533 \\ 0 \end{bmatrix},
\end{aligned}$$

where the vectors on the right-hand side are the eigenvectors corresponding to the eigenvalues. Then,

$$\begin{aligned}
\hat{s}_t &= \mu_1 \lambda_1^t e_{13} \\
\Leftrightarrow \mu_1 \lambda_1^t &= \frac{1}{e_{13}} \hat{s}_t,
\end{aligned}$$

which allows us to write

$$\begin{aligned}
\hat{k}_t &= \mu_1 \lambda_1^t e_{11} + \mu_2 \lambda_2^t e_{21} \\
&= \frac{e_{11}}{e_{13}} \hat{s}_t + \mu_2 \lambda_2^t e_{21} \\
\Leftrightarrow \mu_2 \lambda_2^t &= \frac{1}{e_{21}} \hat{k}_t - \frac{e_{11}}{e_{21} e_{13}} \hat{s}_t.
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{c}_t &= \mu_1 \lambda_1^t e_{12} + \mu_2 \lambda_2^t e_{22} \\
&= \frac{e_{12}}{e_{13}} \hat{s}_t + \left(\frac{1}{e_{21}} \hat{k}_t - \frac{e_{11}}{e_{21} e_{13}} \hat{s}_t \right) e_{22} \\
&= \frac{e_{22}}{e_{21}} \hat{k}_t + \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \hat{s}_t.
\end{aligned}$$

Let m_{ij} denote the (i, j)th element of $\mathbf{M}$, then

$$\begin{aligned}
\hat{k}_{t+1} &= m_{11} \hat{k}_t + m_{12} \hat{c}_t + m_{13} \hat{s}_t \\
&= m_{11} \hat{k}_t + m_{12} \left(\frac{e_{22}}{e_{21}} \hat{k}_t + \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \hat{s}_t \right) + m_{13} \hat{s}_t \\
&= \left(m_{11} + m_{12} \frac{e_{22}}{e_{21}} \right) \hat{k}_t + \left(m_{13} + m_{12} \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \right) \hat{s}_t.
\end{aligned}$$

We also have from (1.33) that

$$\begin{aligned}
\hat{H}_t &= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\hat{c}_t \\
&= \frac{\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{k}_t + \frac{1-\alpha}{\alpha+\frac{1}{\varepsilon}}\hat{s}_t - \frac{1}{\alpha+\frac{1}{\varepsilon}}\left(\frac{e_{22}}{e_{21}}\hat{k}_t + \frac{e_{22}}{e_{21}}\left(1-\frac{e_{11}}{e_{13}}\right)\hat{s}_t\right) \\
&= \frac{1}{\alpha+\frac{1}{\varepsilon}}\left(\alpha - \frac{e_{22}}{e_{21}}\right)\hat{k}_t + \frac{1}{\alpha+\frac{1}{\varepsilon}}\left(1-\alpha-\frac{e_{22}}{e_{21}}\left(1-\frac{e_{11}}{e_{13}}\right)\right)\hat{s}_t.
\end{aligned}$$

Substituting the calibrated values, we obtain

$$\begin{aligned}
\hat{c}_t &= 0.630\hat{k}_t + 0.190\hat{s}_t, \\
\hat{k}_{t+1} &= 0.967\hat{k}_t + 0.049\hat{s}_t \\
\hat{H}_t &= -0.164\hat{k}_t + 0.293\hat{s}_t.
\end{aligned}$$

If we consider $t=0$, given $\hat{k}_0$ and $\hat{s}_0$, we have three equations and three unknowns ($\hat{c}_0$, μ_1 and μ_2). So we can solve for the unknowns to give

$$\begin{aligned}
\mu_1 &= 3.481\hat{s}_0, \\
\mu_2 &= 1.182\hat{k}_0 + 3.432\hat{s}_0.
\end{aligned}$$

Note that

- ▷ when $\hat{k}_t$ is higher (i.e. capital stock is above trend), then consumption increases. In contrast, this results in a lower $\hat{H}_0$ so that hours worked is below trend.
- ▷ when $\hat{s}_0$ is higher (i.e. productivity shock is positive), then consumption today increases, but savings (i.e. $\hat{k}_1$) also increases. Hours worked also increases which means that the income and substitution effects is such that the substitution effect is larger than the income effect.

Remark 1.8. Sensitivity of the dynamics on the parameters. The choice of ε, ρ and α (especially close to 1) affects the dynamics significantly. However, g, β and δ have relatively small impact. γ has no impact on the dynamics.

Investment We can derive an expression for the detrended investment, i_t, from the law of motion for capital:

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\Leftrightarrow \frac{K_{t+1}}{(1+g)^{t+1}}(1+g) &= (1-\delta)\frac{K_t}{(1+g)^t} + \frac{I_t}{(1+g)^t} \\
\Leftrightarrow i_t &= k_{t+1}(1+g) - (1-\delta)k_t.
\end{aligned}$$

The stationary value of investment is given by

$$i^* = k^*(1+g) - (1-\delta)k^* = k^*(g+\delta) = 32.72.$$

To log linearise the investment equation, rewrite as

$$e^{\hat{i}_t} i^* = e^{\hat{k}_{t+1}} k^* (1+g) - (1-\delta) e^{\hat{k}_t} k^*.$$

Taylor expansion around the steady state yields

$$\hat{i}_t = \frac{k^* (1+g)}{i^*} \hat{k}_{t+1} - \frac{(1-\delta) k^*}{i^*} \hat{k}_t.$$

Substituting the expressions for $\hat{k}_{t+1}$, we obtain

$$\begin{aligned}
\hat{i}_t &= \frac{k^* (1+g)}{i^*} \left(\left(m_{11} + m_{12} \frac{e_{22}}{e_{21}} \right) \hat{k}_t + \left(m_{13} + m_{12} \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \right) \hat{s}_t \right) - \frac{(1-\delta) k^*}{i^*} \hat{k}_t \\
&= \left(\frac{k^* (1+g)}{i^*} \left(m_{11} + m_{12} \frac{e_{22}}{e_{21}} \right) - \frac{(1-\delta) k^*}{i^*} \right) \hat{k}_t + \frac{k^* (1+g)}{i^*} \left(m_{13} + m_{12} \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \right) \hat{s}_t.
\end{aligned}$$

Substituting the calibrated values, we obtain

$$\hat{i}_t = -0.632 \hat{k}_t + 2.444 \hat{s}_t.$$

Output We can also derive an expression for the detrended output from the production function:

$$\begin{aligned}
Y_t &= K_t^\alpha \left[(1+g)^t H_t \right]^{1-\alpha} \\
\Leftrightarrow y_t &= k_t^\alpha H_t^{1-\alpha}.
\end{aligned}$$

The stationary value is given by

$$y^* = (k^*)^\alpha (H^*)^{1-\alpha} = 125.85.$$

To log-linearise, we can take the log of both sides to obtain

$$\log y_t = \alpha \log k_t + (1-\alpha) \log H_t.$$

Subtracting from this the equation evaluated at the stationary point yields

$$\begin{aligned}
\log y_t - \log y^* &= \alpha (\log k_t - \log k^*) + (1-\alpha)(\log H_t - \log H^*) \\
\Leftrightarrow \hat{y}_t &= \alpha \hat{k}_t + (1-\alpha) \hat{H}_t.
\end{aligned}$$

Using the expression for $\hat{H}_t$, we obtain

$$\begin{aligned}
\hat{y}_t &= \alpha \hat{k}_t + (1-\alpha) \hat{H}_t \\
&= \alpha \hat{k}_t + (1-\alpha) \left(\frac{1}{\alpha + \frac{1}{\varepsilon}} \left(\alpha - \frac{e_{22}}{e_{21}} \right) \hat{k}_t + \frac{1}{\alpha + \frac{1}{\varepsilon}} \left(1 - \alpha - \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \right) \hat{s}_t \right) \\
&= \left(\alpha + \frac{1-\alpha}{\alpha + \frac{1}{\varepsilon}} \left(\alpha - \frac{e_{22}}{e_{21}} \right) \right) \hat{k}_t + \frac{1-\alpha}{\alpha + \frac{1}{\varepsilon}} \left(1 - \alpha - \frac{e_{22}}{e_{21}} \left(1 - \frac{e_{11}}{e_{13}} \right) \right) \hat{s}_t.
\end{aligned}$$

Substituting the calibrated values, we obtain

$$\hat{y}_t = 0.302 \hat{k}_t + 0.176 \hat{s}_t.$$

Summary: Log-linearised equations Putting together all the expressions we derived gives the following.

$$
\begin{aligned}
\hat{c}_t &= 0.630\hat{k}_t + 0.190\hat{s}_t, \\
\hat{k}_{t+1} &= 0.967\hat{k}_t + 0.049\hat{s}_t, \\
\hat{H}_t &= -0.164\hat{k}_t + 0.293\hat{s}_t, \\
\hat{i}_t &= -0.632\hat{k}_t + 2.444\hat{s}_t, \\
\hat{y}_t &= 0.302\hat{k}_t + 0.176\hat{s}_t, \\
\hat{s}_{t+1} &= 0.95\hat{s}_t + v_{t+1}
\end{aligned}
$$

1.5 Impulse response

Suppose we are on the balanced growth path in period $t = 0$. What would happen if the economy experienced a productivity shock in period $t = 0$, say $\hat{s}_0 = 1$ (with $\hat{k}_0 = 0$)?

We can see how this "shock" propagate through the system via the log-linearised equation above. We assume that $v_t = 0$ for all $t > 0$, and first calculate a sequence of

$$\left\{\hat{k}_{t+1}, \hat{s}_{t+1}\right\}.$$

Given this sequence, we can then compute $\hat{c}_t$, $\hat{H}_t$, $\hat{i}_t$, and $\hat{y}_t$ using the equations we obtained before.

Remark 1.9. Since the log-linearised system is linear by construction, if we were to set $\hat{s}_0 = \alpha \neq 0$, then we would find that the impact scales linearly (if $\alpha < 0$, then the impact "flips" in sign).

Remark 1.10. In general, the dynamic response from the log-linearised system and the original system would differ. The non-linearity in the original system means that the expected value is not given when $v_t = 0$. In order to calculate the expected value, we would have to calculate all possible paths of shocks and then taken expectation. As it turns out, RBC models are close enough to be linear so that log-linearisation gives good a approximation.

Figure below plots the impulse response both in log deviation (z_t is the trend component equal to $(1+g)^t$). Observe that:

- ▷ productivity shock leads to a build up capital initially, which leads to higher consumption consumption initially;
- ▷ hours worked increases too, initially, but as capital is built up, the wealth effect (the negative coefficient on $\hat{k}_t$ in $\hat{H}_t$) dominates so that labour decreases.

Figure 1.1: Impulse response in log deviations.

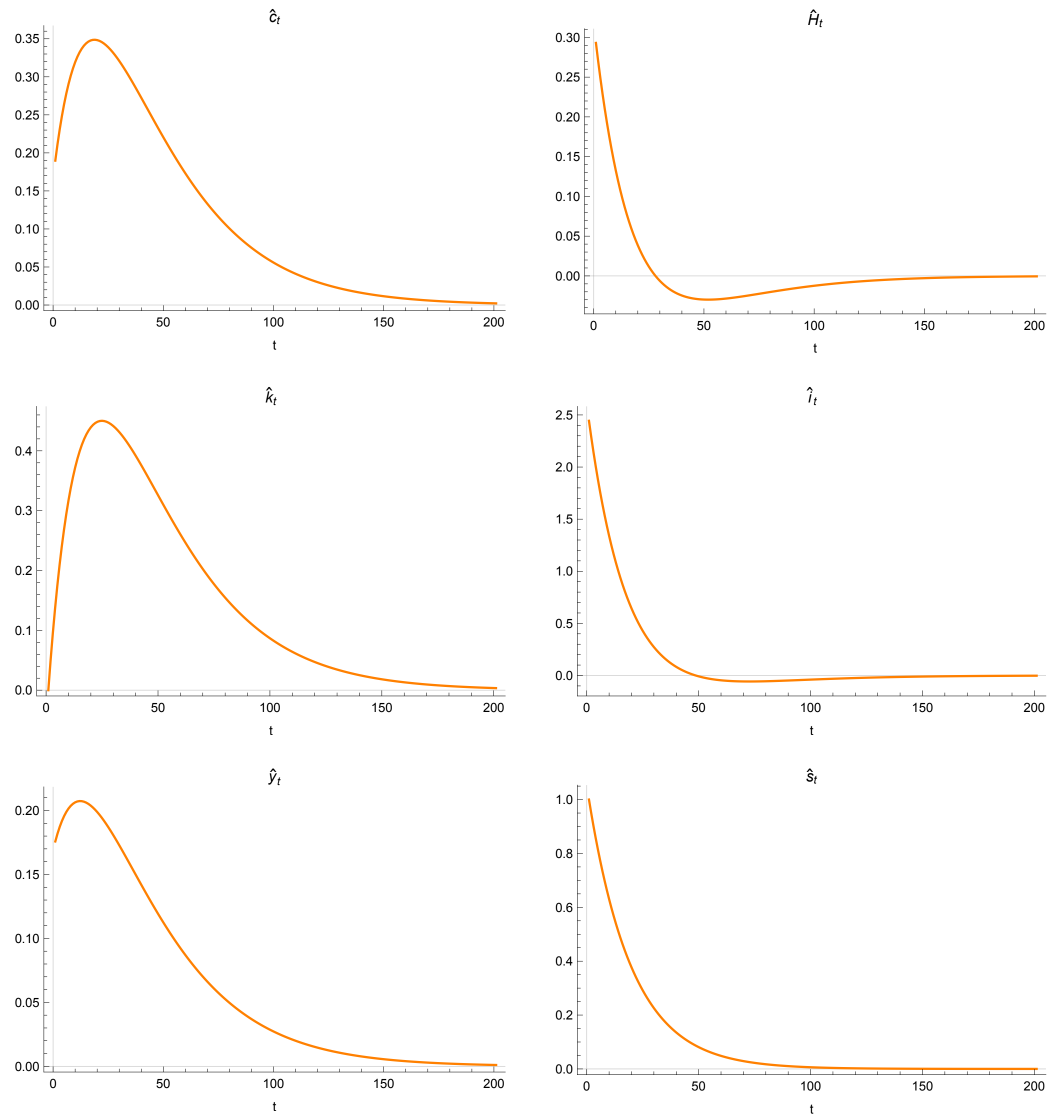
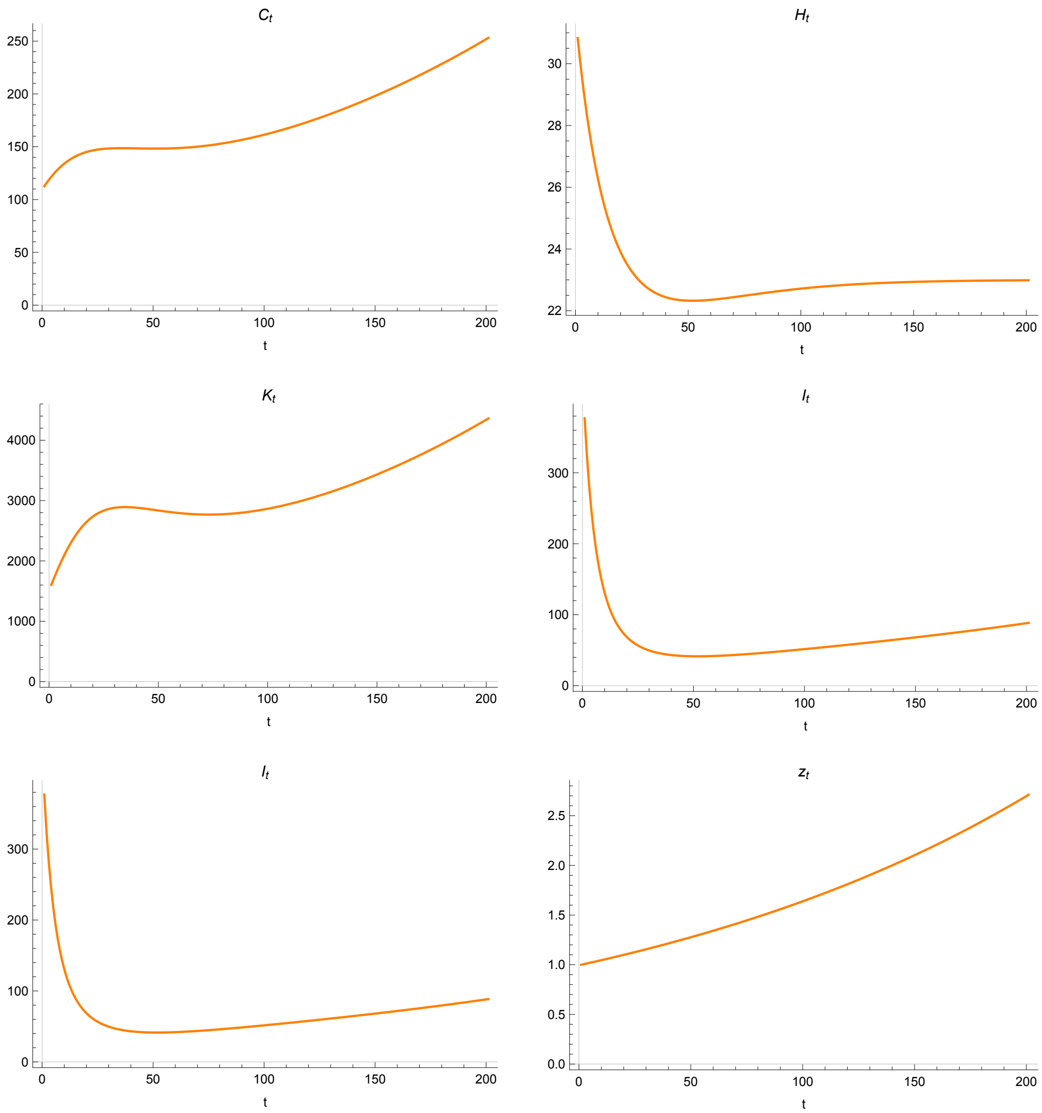


Figure 1.2: Impulse response in levels.



1.6 Testing the model

How can we test whether the model fits the data? We have three approaches.

1.6.1 Isolating productivity shocks

One way is to isolate the productivity shocks from the data. If we can isolate the productivity shock in a particular period, and the effects of that shock in subsequent periods, we can compare the impulse response predicted by the model against the data. This is usually done through VAR. In practice, however, the number of observations are too few for this to work well.

1.6.2 Analytical approach

Suppose we take many draws of υ_{t+1} and simulate the model, we can then compare the co-movements in the variables against the co-movements we observe in the data. We can do this analytically by deriving an expression for the variance-covariance matrix. Let $\boldsymbol{\xi}_t$ be the a column vector consisting of n number of variables of interest (e.g. $\xi_t = \{k_t, c_t, \ldots\}$). Using (1.34), we can write the system as

$$\boldsymbol{\xi}_{t+1} = \mathbb{M}\boldsymbol{\xi}_t + \mathbf{d}\upsilon_{t+1},$$

where $\mathbb{M}$ is an $n \times n$ matrix and $\mathbf{d}$ is a $n \times 1$ vector. Note that $\mathbb{M}$ is not the same as the $\mathbf{M}$ matrix before. Recall that we have

$$\begin{aligned}
\hat{c}_{t+1} &= 0.630\hat{k}_{t+1} + 0.190\hat{s}_{t+1} \\
\hat{k}_{t+1} &= 0.967\hat{k}_t + 0.049\hat{s}_t, \\
\hat{s}_{t+1} &= 0.95\hat{s}_t + \upsilon_{t+1} \\
\Rightarrow \hat{c}_{t+1} &= 0.630\left(0.967\hat{k}_t + 0.049\hat{s}_t\right) + 0.190\left(0.95\hat{s}_t + \upsilon_{t+1}\right) \\
&= 0.630 \times 0.967\hat{k}_t + \left(0.630 \times 0.049 + 0.190 \times 0.95\right)\hat{s}_t \\
&\quad + 0.190\upsilon_{t+1}.
\end{aligned}$$

This ensures that the $\mathbb{M}$ matrix is stable (remember that $\mathbf{M}$ was not a stable matrix!).

Let $\boldsymbol{\Sigma}_t \coloneqq \mathrm{Var}[\boldsymbol{\xi}_t]$ and $\sigma_\upsilon^2 \coloneqq \mathbb{E}[\upsilon_{t+1}^2]$, then

$$\begin{aligned}
\boldsymbol{\Sigma}_{t+1} &= \mathbb{M}\boldsymbol{\Sigma}_t\mathbb{M}' + \mathbf{d}\sigma_\upsilon^2\mathbf{d}' + 2\underbrace{\mathrm{Cov}\left[\mathbb{M}\boldsymbol{\xi}_t, \mathbf{d}\upsilon_{t+1}\right]}_{=0} \\
&= \mathbb{M}\boldsymbol{\Sigma}_t\mathbb{M}' + \mathbf{d}\mathbf{d}'\sigma_\upsilon^2.
\end{aligned}$$

Repeated substitution yields

$$\begin{aligned}
\boldsymbol{\Sigma}_{t+1} &= \mathbb{M}\left(\mathbb{M}\boldsymbol{\Sigma}_{t-1}\mathbb{M}' + \mathbf{d}\mathbf{d}'\sigma_\upsilon^2\right)\mathbb{M}' + \mathbf{d}\mathbf{d}'\sigma_\upsilon^2 \\
&= \mathbb{M}^2\boldsymbol{\Sigma}_{t-1}\left(\mathbb{M}'\right)^2 + \mathbb{M}\mathbf{d}\mathbf{d}'\mathbb{M}'\sigma_\upsilon^2 + \mathbf{d}\mathbf{d}'\sigma_\upsilon^2 \\
&= \sigma_\upsilon^2\sum_{j=1}^{t}\mathbb{M}^{j-1}\mathbf{d}\mathbf{d}\left(\mathbb{M}'\right)^{j-1} + \mathbb{M}^t\boldsymbol{\Sigma}_0\left(\mathbb{M}'\right)^t.
\end{aligned}$$

Taking limits as $t \to \infty$,

$$\boldsymbol{\Sigma}_\infty \coloneqq \lim_{t\to\infty}\boldsymbol{\Sigma}_{t+1} = \sigma_\upsilon^2\sum_{j=1}^{\infty}\mathbb{M}^{j-1}\mathbf{d}\mathbf{d}\left(\mathbb{M}'\right)^{j-1} + \lim_{t\to\infty}\mathbb{M}^t\boldsymbol{\Sigma}_0\left(\mathbb{M}'\right)^t.$$

The deterministic model we are considering is stationary; i.e. $\mathbb{M}$ is a stable matrix (absolute values of the eigenvalues are strictly less than one), so the last term is equal to zero. This gives us that

$$\begin{aligned}
\boldsymbol{\Sigma}_\infty &= \sigma_\upsilon^2\sum_{j=1}^{\infty}\mathbb{M}^{j-1}\mathbf{d}\mathbf{d}\left(\mathbb{M}'\right)^{j-1} \\
&= \sigma_\upsilon^2\sum_{j=0}^{\infty}\mathbb{M}^{j}\mathbf{d}\mathbf{d}\left(\mathbb{M}'\right)^{j}
\end{aligned}$$

We can verify that that this expression is, in fact, the variance of the stationary distribution of $\boldsymbol{\xi}_{t+1}$. To verify, consider

$$\begin{aligned}
\mathbb{M}\boldsymbol{\Sigma}_\infty\mathbb{M}' + \mathbf{dd}'\sigma_\upsilon^2 &= \mathbb{M}\left(\sigma_\upsilon^2 \sum_{j=0}^{\infty} \mathbb{M}^j \mathbf{dd}\left(\mathbb{M}'\right)^j\right)\mathbb{M}' + \mathbf{dd}'\sigma_\upsilon^2 \\
&= \sigma_\upsilon^2 \sum_{j=0}^{\infty} \mathbb{M}^{j+1}\mathbf{dd}\left(\mathbb{M}'\right)^{j+1} + \mathbf{dd}'\sigma_\upsilon^2 \\
&= \sigma_\upsilon^2 \left[\mathbf{dd}' + \mathbb{M}\mathbf{dd}\left(\mathbb{M}'\right) + \mathbb{M}^2\mathbf{dd}\left(\mathbb{M}'\right)^2 + \cdots\right] \\
&= \sigma_\upsilon^2 \sum_{j=0}^{\infty} \mathbb{M}^j \mathbf{dd}\left(\mathbb{M}'\right)^j = \boldsymbol{\Sigma}_\infty.
\end{aligned}$$

We can then compare $\boldsymbol{\Sigma}_\infty$ implies by the model with that from the data.

Remark 1.11. If $\rho = 0.9999$ (i.e. very close to one), the finite sample property is bad.

Remark 1.12. What about omitted variables? For example, the US economy has experienced changes in fertility or demographics (female labour force participation) over the last 50 years but our model is silent about such factors. In contrast, the data contains the effect of such unmodelled factors. We can hope that these factors affect the economy at frequencies that are different (longer) than the business cycle (around eight years) and choose to ignore it (filtering at the frequency corresponding to the business cycle would remove the effect of such factors). Alternatively, we could model such factors explicitly but, of course, the model will become more cumbersome as a result.

1.6.3 Monte Carlo simulation

We can also compare comovements implied by the model against the data using Monte Carlo simulation. Suppose we have 50 years worth of quarterly data (i.e. 200 observations). Then, we would proceed as follows.

(i) Specify the distribution of the shocks, υ_{t+1}. Take 200 draws of υ_{t+1}.

(ii) Feed the shocks into the model and compute the variables of interest for 200 periods.

(iii) Detrend the data (more on this below).

(iv) Calculate the variance-covariance matrix.

(v) Repeat a "bunch" of times.

See Problem Set 2 for an implementation of this.

1.7 Detrending

The aggregate variables in the data generally has both trend and cyclical part. The idea of detrending is to isolate the cyclical part (i.e. the business cycle) from the trend part. A widely used filter is the Hodrick-Prescott filter.

Let y_t denote a time series data of a variable. We want to decompose this into two parts: the trend, g_t, and the cyclical part, d_t. We suppose that

$$y_t = g_t + d_t, \forall t.$$

The Hodrick-Prescott filtering solves the following problem.

$$\min_{\{g_t, d_t\}_{t=1}^T} \left\{ \sum_{t=1}^{T} (d_t)^2 + \lambda \sum_{t=2}^{T-1} \left[(g_{t+1} - g_t) - (g_t - g_{t-1}) \right]^2 \right\}.$$

The first component of the objective function penalises deviations from the trend. To interpret the second, note that $(g_{t+1} - g_t)$ and $(g_t - g_{t-1})$ are slopes of the trend. Thus, the second part of the objective function penalises changes in slope of the trend, and λ is a parameter that specifies the relative weights between the two "losses". Note that

$$\lambda = 0 \Rightarrow \begin{cases} d_t = 0 \\ g_t = y_t \end{cases},$$

$$\lambda \to \infty \Rightarrow \tau_t = \text{a linear trend}.$$

The customary values of λ are:

Data frequency	Yearly	Quarterly	Monthly
λ	100	1,600	14,400

Remark 1.13. If we think that variables are growing exponentially over time, then we would detrend the log of the series. If a variable grows linear, there is no need. (We can only detrend the log of the series if the variable is always nonnegative (e.g. we cannot take logs of inflation or the interest rate!).

1.7.1 Solving the HP filter

Note that for any sequence $\{x_t\}_{t=1}^T$,

$$\sum_{t=1}^{T} x_t^2 = \begin{bmatrix} x_1 & x_2 & \cdots & x_T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix}^\intercal \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix}.$$

Using this, we can write

$$\sum_{t=1}^{T}(d_t)^2 = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_T \end{bmatrix}^{\intercal} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_T \end{bmatrix},$$

$$\sum_{t=2}^{T-1}(g_{t-1} - 2g_t + g_{t+1})^2 = \left(\begin{bmatrix} g_1 - 2g_2 + g_3 \\ g_2 - 2g_3 + g_4 \\ \vdots \\ g_{T-2} - 2g_{T-1} + g_T \end{bmatrix}^{\intercal}\right)_{1\times(T-2)} \begin{bmatrix} g_1 - 2g_2 + g_3 \\ g_2 - 2g_3 + g_4 \\ \vdots \\ g_{T-2} - 2g_{T-1} + g_T \end{bmatrix}_{(T-2)\times 1},$$

where

$$\begin{bmatrix} g_1 - 2g_2 + g_3 \\ g_2 - 2g_3 + g_4 \\ \vdots \\ g_{T-2} - 2g_{T-1} + g_T \end{bmatrix}_{(T-2)\times 1} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}_{(T-2)\times T} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_T \end{bmatrix}_{T\times 1}.$$

Denote the $(T-2)\times T$ matrix above by $\mathbf{A}$, the column vector of g_t's by $\mathbf{g}$, and the column vector of d_t's by $\mathbf{d}$, the objective function can be written as

$$\mathbf{d}^{\intercal}\mathbf{d} + \lambda(\mathbf{A}\mathbf{g})^{\intercal}(\mathbf{A}\mathbf{g}).$$

Finally, since $d_t = y_t - g_t$ for all t, and denoting the column vector of y_t's as $\mathbf{y}$, we can write the minimisation problem as

$$\min_{\mathbf{d}} \quad \mathbf{d}^{\intercal}\mathbf{d} + \lambda(\mathbf{A}(\mathbf{y}-\mathbf{d}))^{\intercal}(\mathbf{A}(\mathbf{y}-\mathbf{d})).$$

The objective function is clearly strictly convex so we know that the first-order conditions are both necessary and sufficient. Recall that, given a matrix $\mathbf{A}$ and a vector $\mathbf{x}$,

$$\frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}^{\intercal}, \frac{\partial \mathbf{A}\mathbf{x}^{\intercal}}{\partial \mathbf{x}} = \mathbf{A}.$$

Then,

$$\begin{aligned}
\frac{\partial \mathbf{d}^{\intercal}\mathbf{d}}{\partial \mathbf{d}} &= 2\mathbf{d}, \\
\frac{\partial(\mathbf{A}(\mathbf{y}-\mathbf{d}))^{\intercal}(\mathbf{A}(\mathbf{y}-\mathbf{d}))}{\partial \mathbf{d}} &= \frac{\partial(\mathbf{y}^{\intercal}\mathbf{A}^{\intercal} - \mathbf{d}^{\intercal}\mathbf{A}^{\intercal})(\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{d})}{\partial \mathbf{d}} \\
&= \frac{\partial(\mathbf{y}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{y} - \mathbf{d}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{y} - \mathbf{y}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d} + \mathbf{d}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d})}{\partial \mathbf{d}} \\
&= -\frac{\partial(\mathbf{d}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{y})}{\partial \mathbf{d}} - \frac{\partial \mathbf{y}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d}}{\partial \mathbf{d}} + \frac{\partial \mathbf{d}^{\intercal}\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d}}{\partial \mathbf{d}} \\
&= -\mathbf{A}^{\intercal}\mathbf{A}\mathbf{y} - \mathbf{A}^{\intercal}\mathbf{A}\mathbf{y} + (\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d} + \mathbf{A}^{\intercal}\mathbf{A}\mathbf{d}) \\
&= -2\mathbf{A}^{\intercal}\mathbf{A}\mathbf{y} + 2\mathbf{A}^{\intercal}\mathbf{A}\mathbf{d}
\end{aligned}$$

so that the first-order condition is given by

$$\begin{aligned} 2\mathbf{d}^* + 2\lambda\left(-\mathbf{A}^\intercal\mathbf{A}\mathbf{y} + \mathbf{A}^\intercal\mathbf{A}\mathbf{d}^*\right) = 0 \\ \Leftrightarrow \left(\mathbb{I} + \lambda\mathbf{A}^\intercal\mathbf{A}\right)\mathbf{d}^* = \lambda\mathbf{A}^\intercal\mathbf{A}\mathbf{y} \\ \Rightarrow \mathbf{d}^* = \lambda\left(\mathbb{I} + \lambda\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\mathbf{A}\mathbf{y}, \end{aligned}$$

where $\mathbb{I}$ is the $T \times T$ identity matrix.

Remark 1.14. Suppose we wanted to detrend n series of the same length, say T. In computing the detrended series, we only need to compute the matrix $\lambda\left(\mathbb{I} + \lambda\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\mathbf{A}$ once and then multiply by this by $T \times n$ matrix of the data to detrend (each column is a series of data).

1.8 Wedge accounting

1.8.1 US data vs Model output

The table below gives some statistics on logged output (y), consumption (c), investment (x) and hours worked (h). Since housing can be thought of as a capital good (i.e. it provides services to the owner over time), it is common to include house purchases in investment. Sometimes, other durable goods (e.g. cars) are also included in investment for the same reason. Hours worked does not include vacation.

Table 1: Historical US data vs model output.

	US data				Model output				
	y	c	x	h	y	c	x	h	z
Standard deviation[10]	1	0.563	3.684	1.017	1	0.293	0.181	0.378	1.284
Autocorrelation	0.992	0.935	0.912	0.917	0.911	0.933	0.909	0.909	0.910
Cross correlation (log)									
y	1	0.827	0.864	0.849	1	0.876	0.992	0.982	0.997
c	.	1	0.756	0.759	.	1	0.807	0.771	0.837
x	.	.	1	0.699	.	.	1	0.998	0.999
h	.	.	.	1	.	.	.	1	0.994
z	n.a.				.	.	.	.	1

Comparing the data with the output, we can see that:

- ▷ consumption, investment and hours worked are all too "smooth" in the model;
- ▷ cross-correlations tend to be too high in the model.

With respect to hours worked, recall that we had set the Frisch elasticity of labour, ε, to be one. We can introduce more volatility in the hours worked in the model by increasing this parameter; however, for reasonable values (up to around 4), we cannot reproduce the volatility observed in the real data (see Problem Set 3). To fix other aspects, we may wish extend the RBC model by adding government and/or export/imports into the model.

Alternatively, we could look at the three optimality conditions to see which of them are violated in the data. This is what is referred to as *wedge accounting*.

[10]Normalised by the standard deviation of y.

1.8.2 Benchmark prototype economy

We first define a competitive equilibrium in an RBC model. We assume that households own the firms, and that firms own capital. Given history $\mathbf{s}^t$ in period t, let $\tau_{h,t}(\mathbf{s}^t)$ denote the tax on labour income, $\tau_{x,t}(\mathbf{s}^t)$ denote the tax on investment. $G_t(\mathbf{s}^t)$ represents (exogenous) government purchases, while $T_t(\mathbf{s}^t)$ denotes the lump-sum tax transfers to households from the government. $w_t(\mathbf{s}^t)$ denotes the wages and $q_0^t(\mathbf{s}^t)$ denotes the (Arrow-Debreu) prices which represents the price in period 0 of a claim of one unit of consumption good in period t given history $\mathbf{s}^t$.

Definition 1.1. *(Competitive equilibrium)* A competitive equilibrium is a sequence of allocations, $\{C_t(\mathbf{s}^t), H_t(\mathbf{s}^t), X_t(\mathbf{s}^t), Y_t(\mathbf{s}^t), K_{t+1}(\mathbf{s}^t)\}$, and a sequence of prices, $\{w_t(\mathbf{s}^t), q_0^t(\mathbf{s}^t)\}$, given initial values $\{a_0, v_0, K_0(\mathbf{s}^{-1})\}$, policy variables $\{T_t(\mathbf{s}^t), \tau_{h,t}(\mathbf{s}^t), \tau_{x,t}(\mathbf{s}^t), G_t(\mathbf{s}^t)\}$ and shocks $\{z_t(\mathbf{s}^t)\}$ such that

- ▷ given $\{a_0, w_t(\mathbf{s}^t), q_0(\mathbf{s}^t), T_t(\mathbf{s}^t), \tau_{h,t}(\mathbf{s}^t)\}$, the household chooses $\{C_t(\mathbf{s}^t), H_t(\mathbf{s}^t)\}$ that solves the following problem

$$\max_{\{C_t(\mathbf{s}^t), H_t(\mathbf{s}^t)\}} \quad \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t} \beta^t \Pi_t(\mathbf{s}^t)\, U\left(C_t(\mathbf{s}^t), H_t(\mathbf{s}^t)\right)$$
$$s.t. \quad a_0 \geq \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t} q_0^t(\mathbf{s}^t)\left[C_t(\mathbf{s}^t) - \left(1-\tau_{h,t}(\mathbf{s}^t)\right) w_t(\mathbf{s}^t) H_t(\mathbf{s}^t) - T_t(\mathbf{s}^t)\right];$$

- ▷ given $\{K_0(\mathbf{s}^{-1}), w_t(\mathbf{s}^t), q_0(\mathbf{s}^t), z_t(\mathbf{s}^t)\}$, firms choose $\{H_t(\mathbf{s}^t), K_{t+1}(\mathbf{s}^t), X_t(\mathbf{s}^t), Y_t(\mathbf{s}^t)\}$ that solves the following problem

$$v_0 = \max_{\{H_t(\mathbf{s}^t), K_{t+1}(\mathbf{s}^t), X_t(\mathbf{s}^t), Y_t(\mathbf{s}^t)\}} \quad \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t} q_0^t(\mathbf{s}^t)\left(Y_t(\mathbf{s}^t) - w_t(\mathbf{s}^t) H_t(\mathbf{s}^t) - \left(1+\tau_{x,t}(\mathbf{s}^t)\right) X_t(\mathbf{s}^t)\right)$$
$$s.t. \quad Y_t(\mathbf{s}^t) = \left(K_t(\mathbf{s}^{t-1})\right)^{\alpha}\left(z_t(\mathbf{s}^t) H_t(\mathbf{s}^t)\right)^{1-\alpha}, \tag{1.35}$$
$$X_t(\mathbf{s}^t) = K_{t+1}(\mathbf{s}^t) - (1-\delta) K_t(\mathbf{s}^{t-1});$$

- ▷ government budget constraint holds:

$$\tau_{h,t}(\mathbf{s}^t) w_t(\mathbf{s}^t) H_t(\mathbf{s}^t) + \tau_{x,t}(\mathbf{s}^t) X_t(\mathbf{s}^t) = T_t(\mathbf{s}^t) + G_t(\mathbf{s}^t), \forall t, \mathbf{s}^t;$$

- ▷ goods market clears:

$$Y_t(\mathbf{s}^t) = C_t(\mathbf{s}^t) + X_t(\mathbf{s}^t) + G_t(\mathbf{s}^t), \forall t, \mathbf{s}^t; \tag{1.36}$$

- ▷ financial market clears:

$$a_0 = v_0.$$

Note:

- ▷ the A-D price $q_0(\mathbf{s}^t)$ embeds the probability of $\mathbf{s}^t$, $\Pi_t(\mathbf{s}^t)$; i.e. it reflects the probability of state $\mathbf{s}^t$ from the point of view of period 0;
- ▷ since $T_t(\mathbf{s}^t)$ can be negative or positive (so long as it satisfies the government budget constraint), government can borrow in this economy;

- ▷ we should think of government purchases as including net exports too.
- ▷ government purchases, $G_t(\mathbf{s}^t)$, have no role in the economy. Alternatively, we could add this to the budget constraint or utility. In doing so, we may wish to add them in an additively separable manner if we want to avoid altering the optimality conditions.

Household's problem Write the Lagrangian as

$$\begin{aligned}\mathcal{L} = & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)U\left(C_t\left(\mathbf{s}^t\right),H_t\left(\mathbf{s}^t\right)\right)\\ & +\lambda\left(a_0-\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}q_0^t\left(\mathbf{s}^t\right)\left[C_t\left(\mathbf{s}^t\right)-\left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)w_t\left(\mathbf{s}^t\right)H_t\left(\mathbf{s}^t\right)-T_t\left(\mathbf{s}^t\right)\right]\right).\end{aligned}$$

Letting $\beta^t\Pi_t\left(\mathbf{s}^t\right)\lambda_t\left(\mathbf{s}^t\right)$ denote the Lagrange multiplier on the constraint, the first-order conditions are

$$\begin{aligned}\left\{C_t\left(\mathbf{s}^t\right)\right\} \qquad & \beta^t\Pi_t\left(\mathbf{s}^t\right)U_{C,t}\left(\mathbf{s}^t\right)=\lambda q_0^t\left(\mathbf{s}^t\right),\\ \left\{H_t\left(\mathbf{s}^t\right)\right\} \qquad & -\beta^t\Pi_t\left(\mathbf{s}^t\right)U_{H,t}\left(\mathbf{s}^t\right)=\lambda q_0^t\left(\mathbf{s}^t\right)\left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)w_t\left(\mathbf{s}^t\right).\end{aligned} \tag{1.37}$$

Dividing the second by the first gives us the intratemporal condition that equate marginal rate of substitution between consumption and labour with the after-tax wage.

$$\frac{-U_{H,t}\left(\mathbf{s}^t\right)}{U_{C,t}\left(\mathbf{s}^t\right)}=\left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)w_t\left(\mathbf{s}^t\right). \tag{1.38}$$

Consider the first-order condition with respect to $C_t\left(\mathbf{s}^t\right)$ in period t. Since $\mathbf{s}^0$ is a singleton, $\Pi_0\left(\mathbf{s}^0\right)=1$. We normalise $q_0^0\left(\mathbf{s}^0\right)=1$ (i.e. the unit is in marginal utility of consumption in period 0), then

$$\lambda=U_{C,0}\left(\mathbf{s}^0\right).$$

Firm's problem Eliminating $X_t\left(\mathbf{s}^t\right)$ and $Y_t\left(\mathbf{s}^t\right)$ using the constraints, we can write the firm's problem as

$$\begin{aligned}v_0=\max_{\left\{H_t\left(\mathbf{s}^t\right),K_{t+1}\left(\mathbf{s}^t\right)\right\}}\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}q_0^t\left(\mathbf{s}^t\right)\Big[&\left(K_t\left(\mathbf{s}^{t-1}\right)\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}\\ &-w_t\left(\mathbf{s}^t\right)H_t\left(\mathbf{s}^t\right)-\left(1+\tau_{x,t}\left(\mathbf{s}^t\right)\right)\left(K_{t+1}\left(\mathbf{s}^t\right)-\left(1-\delta\right)K_t\left(\mathbf{s}^{t-1}\right)\right)\Big].\end{aligned}$$

The first-order conditions are:

$$\begin{aligned}\left\{H_t\left(\mathbf{s}^t\right)\right\} \qquad & \left(1-\alpha\right)\frac{Y_t\left(\mathbf{s}^t\right)}{H_t\left(\mathbf{s}^t\right)}=w_t\left(\mathbf{s}^t\right),\\ \left\{K_{t+1}\left(\mathbf{s}^t\right)\right\} \qquad & \sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}q_0^{t+1}\left(\mathbf{s}^{t+1}\right)\left(\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+\left(1-\delta\right)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)=\left(1+\tau_{x,t}\left(\mathbf{s}^t\right)\right)q_0^t\left(\mathbf{s}^t\right).\end{aligned} \tag{1.39}$$

Substituting the first-order condition with respect to $C_t(\mathbf{s}^t)$ from the household's problem, (1.37), into the firms' first-order condition with respect to $K_{t+1}(\mathbf{s}^t)$, (1.39), yields

$$\begin{aligned}
\frac{(1+\tau_{x,t}(\mathbf{s}^t))\,\beta^t\Pi_t(\mathbf{s}^t)\,U_{C,t}(\mathbf{s}^t)}{\lambda} &= \sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\frac{\beta^{t+1}\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)U_{C_{t+1}(\mathbf{s}^{t+1})}}{\lambda}\left(\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+(1-\delta)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)\\
\Leftrightarrow \left(1+\tau_{x,t}\left(\mathbf{s}^t\right)\right)U_{C,t}\left(\mathbf{s}^t\right) &= \sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t}\beta\frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)}U_{C_{t+1}(\mathbf{s}^{t+1})}\left(\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+(1-\delta)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)\\
&= \beta\mathbb{E}_t\left[U_{C_{t+1}(\mathbf{s}^{t+1})}\left(\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+(1-\delta)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)\right]
\end{aligned} \tag{1.40}$$

This condition says that, on the margin, consuming today (the left-hand side) should give the same benefit as foregoing consumption today and investing today instead.

Specialisation Suppose that

$$u(C,H) \coloneqq \log C - \gamma\frac{\varepsilon}{1+\varepsilon}H^{\frac{1+\varepsilon}{\varepsilon}}. \tag{1.41}$$

Then, the intratemporal condition, (1.38), becomes

$$\begin{aligned}
\gamma C_t\left(\mathbf{s}^t\right)H_t\left(\mathbf{s}^t\right)^{\frac{1}{\varepsilon}} &= \left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)w_t\left(\mathbf{s}^t\right)\\
&= \left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)(1-\alpha)\frac{Y_t\left(\mathbf{s}^t\right)}{H_t\left(\mathbf{s}^t\right)},
\end{aligned} \tag{1.42}$$

where in the second line, we used the optimality condition for the firm.

Using (1.41), the Euler equation, (1.40), becomes

$$\frac{1+\tau_{x,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)} = \beta\mathbb{E}_t\left[\frac{\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+(1-\delta)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right]. \tag{1.43}$$

1.8.3 The four wedges

Labour wedge Recall the specialised intratemporal condition, (1.42),

$$\gamma C_t(s)^t H_t\left(\mathbf{s}^t\right)^{\frac{1}{\varepsilon}} = \left(1-\tau_{h,t}\left(\mathbf{s}^t\right)\right)(1-\alpha)\frac{Y_t\left(\mathbf{s}^t\right)}{H_t\left(\mathbf{s}^t\right)}.$$

The labour wedge is defined as the term $(1-\tau_{h,t}(\mathbf{s}^t))$.

Since this condition must hold in every period, given γ and ε, we can use data on α, C_t, H_t and Y_t to back out the implied labour wedge, $1-\tau_{h,t}$. If we find that the wedge does not equal one, then we see that there is a wedge in this optimality condition—although the source of the wedge may not be labour tax, we can think of the sources of this wedge as something that acts *as if* it is a labour tax.

Investment wedge Recall the specialised Euler equation, (1.43),

$$\frac{1+\tau_{x,t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)}=\beta\mathbb{E}_t\left[\frac{\alpha\frac{Y_{t+1}\left(\mathbf{s}^{t+1}\right)}{K_{t+1}\left(\mathbf{s}^{t+1}\right)}+(1-\delta)\left(1+\tau_{x,t+1}\left(\mathbf{s}^{t+1}\right)\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right].$$

The investment wedge is the term $1+\tau_{x,t}\left(\mathbf{s}^t\right)$. We can conduct a similar exercise as for the labour wedges using data/values on/for C_t, Y_t, K_t, β and δ, to obtain the investment wedge, $1+\tau_{x,t}\left(\mathbf{s}^t\right)$.

However, there are two complications here. First, we have the expectations operator on the right-hand side—of course, in the data, we only observed the realisation of possible outcomes. Second, we have a dynamic equation in $\tau_{x,t}\left(\mathbf{s}^t\right)$, which requires us to pin down $\tau_{x,0}\left(\mathbf{s}^0\right)$. Chari, Kehoe and McGrattan (2007) ignores the expectations operator and uses historical average of investment tax as $\tau_{x,0}$. In ignoring the expectations operator, we are implicitly assuming that no forecast error is made by the agents in the economy; in other words, we ignore the possibility that the investment wedge we calculate could simply be a result of expectation not aligning with actual realisation.

Efficiency wedge We can rearrange the production function, (1.35), and write

$$\begin{aligned}\left(z_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}&=\frac{Y_t\left(\mathbf{s}^t\right)}{\left(K_t\left(\mathbf{s}^{t-1}\right)\right)^{\alpha}\left(H_t\left(\mathbf{s}^t\right)\right)^{1-\alpha}}\\ \Rightarrow\log z_t\left(\mathbf{s}^t\right)&=\frac{1}{1-\alpha}\log Y_t\left(\mathbf{s}^t\right)-\frac{\alpha}{1-\alpha}\log K_t\left(\mathbf{s}^{t-1}\right)-\log H_t\left(\mathbf{s}^t\right).\end{aligned}$$

Using data/values on/for Y_t, K_t, H_t and α, we can back out z_t. Chari, Kehoe and McGrattan (2007) refers to this as the efficiency wedge, although the literature also refers to this as total factor productivity (TFP).

Government wedge Using the feasibility condition, (1.36), we can write

$$G_t\left(\mathbf{s}^t\right)=Y_t\left(\mathbf{s}^t\right)-C_t\left(\mathbf{s}^t\right)-X_t\left(\mathbf{s}^t\right).$$

We refer to $G_t\left(\mathbf{s}^t\right)$ as the government wedge (which includes government purchases as well as net exports).

1.8.4 Which wedges matter?

One simple interpretation of the wedges is that they represent taxes, TFP and government spending that exists in real life. Chari, Kehoe and McGrattan (2007) interprets the wedges to be misspecification of the model. Specifically, they back out the time series for each wedge. Since the time series may contain interaction among the wedges, they run a VAR to back out the "pure" effect of each wedge. They then plug this back into the model, one wedge at at a time, and see which ones matter the most in replicating the observed data (note that with all four wedges "in action", the model is able to fully replicate the data by construction).

Chari, Kehoe and McGrattan (2007) conclude that the efficiency and labour wedges are the most important out of the four. This suggests that extensions to the model that provides way to affect the intratemporal condition and the Euler equations might lead to better performance against the observed data.

1.8.5 Some mechanisms that affect the labour wedge

- ▷ Better modelling of stochastic process for productivity will not help explain the labour wedge since the productivity shock does not appear in the intratemporal condition. That is, for any stochastic process for the productivity shock, the intratemporal condition must hold. The same applies for government spending.

- ▷ The data suggests that the financial market crisis in 2008 was driven mainly by the shock to the labour wedge. This might be surprising since the cause of the crisis was the financial market. However, this observation simply implies that whatever the source of shock affected the intratemporal condition, and not, for example, the Euler equation.

- ▷ (Moll (2015) AEJ) Suppose there are heterogeneity among entrepreneurs in the economy (i.e. there are good ones and bad ones). The assumption that the production function is constant returns to scale means that resources should be rented to the best entrepreneur.[11] However, inefficiencies in the capital markets might mean that such resource transfers might not occur, so that bad entrepreneurs are still producing output; i.e. there will be misallocation in the economy. This will show up as low TFP.

- ▷ (Hsieh-Kelnow (2009) QJE) Suppose that government prefers some firms over others (assume firms are heterogeneous). We again will have a misallocation problem which will appear as low TFP.

- ▷ Another source of labour wedge might be that we are not very good at measuring "output". Suppose, for example, that during slow periods, workers invest their time in "business development" activities. Such activities has no measurable output but might be important once things get busy again. So what might appear as a fall in output might in fact be leading to higher output in the future.

- ▷ Firing and hiring workers are costly. This might lead to "labour hoarding" that appear as a labour wedge in the model. Such behaviour can be explained if we assume that firms/workers solve a dynamic problem. For firms, it might be the case that, since hiring/firing is costly, it is optimal for them to maintain the workers in a (short) downturn so as to be able to produce output once things get busy again.

1.8.6 Labour wedge

There are two sides to the labour wedge: household's and firm's, given respectively by

$$MRS_{H,C} = \gamma C_t H_t^{\frac{1}{\varepsilon}} = (1 - \tau_{h,t})\, w_t = \text{after-tax wage}, \tag{1.44}$$

$$MPL = (1 - \alpha)\, \frac{Y_t}{H_t} = w_t = \text{wage}. \tag{1.45}$$

Is the labour wedge being caused by the violating of the household's condition or the firm's condition, or both? (Gali Gertler Lopez-Salido (2007) REStat).

[11] In reality, there is likely to be diminishing returns to scale if you gave one person everything! But we are abstracting away from such an issue for now. We're also ignoring the possibility that the best entrepreneur may abuse his position if given all the resources in the economy.

There are some empirical problems in trying to answer this question. For example, there are many measures of wages and it is not clear which one we should use (wages should include benefits that workers receive in the form of pension, health insurance, etc. as well as salary). There are also issues in how to convert nominal wages observed in the data to real wages (e.g. which deflation measure to use?) However, in general, wages are thought to be "mildly procyclical". We also know that output and hours workers have similar standard deviations and the correlation is closed to one; i.e. labour productivity, Y_t/H_t, is also mildly procyclical. These suggest that (1.45) is not violated significantly.

However, recall that we observe in the data that consumption is smooth, wages are not volatile, but hours worked is volatile. This would imply violation of (1.44). This does not seem too unrealistic. For example, suppose you're on a long-term contract (Shimer gave himself as an example). Then your wages are unlikely to be volatile, while your hours worked might be vary quite a lot. However, you would probably not worry about violating the intratemporal condition in some periods since you know that you're on a long-term contract, and that you'll be paid a certain amount (at least for the duration of the contract). In this circumstance, it seems reasonable to expect violation of (1.44). To explain this rationale, we might wish to make (1.44) dynamic so that it depends variables in periods other than t.

There could be other (less plausible, according to Shimer) explanations. For example, labour wedges could be caused by uncompetitive markets (e.g. unions, monopsonies, monopolies). However, it's important that any source of labour wedge that we incorporate in the model must have a *cyclical component* for the extended model to be able to explain data (how realistic is it to assume that degree of competition is higher/lower during recessions than during booms?).

2 Search model

2.1 Pissarides

2.1.1 The setup

We work in continuous time. Let u denote number of those unemployed, and v be the number of vacancies. We denote the matching function—a reduced-form representation of how firms find workers and how workers find jobs—as

$$m(u,v).$$

The matching function gives the rate of matches per unit of time (in discrete time, m would be the number of job matches). If there are no frictions (e.g. geographical, skill constraints) in the economy, then

$$m(u,v)=\min\{u,v\}.$$

However, we assume instead there are frictions and impose the following properties on the matching function:

- ▷ m is increasing;
- ▷ m exhibits constant returns to scale.

The CRS assumption means that, with respect to any one factor (i.e. u or v), there is diminishing returns. That is, if you double the number of job vacancies holding fixed the number of unemployed workers, then job matches less than double; i.e. the rate of matches falls. This reflects the fact that it is harder for each firm to hire workers when more firms are trying to recruit workers.

Define

$$\theta := \frac{v}{u}.$$

We think of θ as measuring *market tightness.* The situation where $\theta > 1 \Leftrightarrow v > u$ is preferred by the worker whereas $\theta < 1 \Leftrightarrow v < u$ is preferred by the firm; i.e. if market is tight for one side of the market, then it is relaxed for the other side of the market.

The rate at which the unemployed finds a job is

$$\frac{m(v,u)}{u} = m\left(1,\frac{v}{u}\right) = m(1,\theta) := f(\theta).$$

This shows an important deviation from the neoclassical models—the rate at which a worker finds a job depends on how many other workers are seeking to find a job (through θ). That is, there is a congestion externality in the model. Given the property of m, f is increasing in θ (in discrete time, $m(v,u)/u$ is the number of job matches divided by the number of unemployed; i.e. the share of unemployed who finds a job).

The rate at which vacancies are filled can be expressed analogously as

$$\frac{m(v,u)}{v} = m\left(\frac{1}{\theta},1\right) := \mu(\theta),$$

where μ is decreasing in θ. There is a congestion externality on this side of the market too—the rate at which a firm fills a vacancy depends on how many other firms are seeking to fill positions.

We assume that workers are risk neutral and has a discount rate $r > 0$. We assume also that employed workers lose their job at an exogenous rate x, which we referred to as the *job separation* *rate*. If workers do not lose jobs, then everyone would be employed in the long run; i.e. full employment will be the absorbing state, and the steady state would not involve unemployment. Workers earn a wage w when employed (this will be an endogenous variable), and γ when unemployed (think of this as the unemployment benefit or disutility of work).

Firms have CRS production technology that uses only labour and we assume that the marginal product of labour is constant at z (i.e. a linear production function). Firms' cost of posting a job vacancy is $c > 0$ (per vacancy). To ensure that the model is economically interesting, we assume $z > \gamma$ (i.e. since firms will never pay above z, if $z < \gamma$, then no one will look for a job). We would therefore expect that, in equilibrium,

$$z > w^* > \gamma,$$

where w^* is the equilibrium wage.

2.1.2 Value functions

[See technical appendix for derivation of the continuous-time value functions with Poisson arrival rates.]

Workers Workers have two "states": employed and unemployed. We define a value function for each, denoted V^e and V^u respectively.

If the worker is unemployed, then he receives γ and, at rate $f(\theta)$, he is matched with a job and earns some $w = w^*$, where w^* denotes the equilibrium wage rate. The value function for an unemployed worker is therefore

$$rV^u = \gamma + f(\theta)\left(V^e(w^*) - V^u\right). \tag{2.1}$$

The term $V^e(w^*) - V^u$ is the net benefit from changing the state from being unemployed to employed. $-V^u$ represents the opportunity cost of becoming employed.

If, instead, the worker is employed at wage rate w, then he receives this wage, w, and at rate x, he loses the job. The "gain" from losing a job is given by $V^u - V^e(w)$. Thus, the value function for an employed worker is

$$rV^e(w) = w + x\left(V^u - V^e(w)\right), \tag{2.2}$$

where $-V^e(w)$ represents the opportunity cost of becoming unemployed.

Firms Given the assumption that firms have CRS technology, we can analyse the firm's decision to hire job-by-job. Hence, there are two "states" for the firms: with one vacancy (v) or with the vacancy filled (f). Let V^v and V^f be the value function in each state respectively. Unlike workers, firms do not have an opportunity cost of hiring. This is because firms have no upper bound on how many workers to hire given the assumption of constant returns to scale/free entry.

If the firm has its vacancy filled and has a worker at wage rate w, then its profit is given by $z - w$. At rate x, the worker is fired (remember, firing happens for exogenous reasons in the model), in which case the firm's value is reduced by $V^f(w)$. Thus, the value function of a firm with no vacancy is given by:[12]

$$rV^f(w) = z - w - xV^f(w). \tag{2.3}$$

[12] Given free entry of vacancies, this expression is mathematically equivalent to

$$rV^f(w) = z - w + x\left(V^v - V^f(w)\right).$$

If, instead, the firm has vacancy, then it does not earn z but must pay a cost of c to post a vacancy. At rate $\mu(\theta)$, the firm finds a worker, which gives an additional value of $V^f(w^*)$. Hence, the value function of a firm with vacancy is

$$rV^v = -c + \mu(\theta) V^f(w). \tag{2.4}$$

We assume *free entry of vacancies.* This means that, if $V^v > 0$, then firms will continue to post vacancies. Thus, in equilibrium, it must be that:[13]

$$V^v = 0.$$

This implies that

$$c = \mu(\theta) V^f(w).$$

2.1.3 Pinning down w^*

To make the problem nontrivial, we assume that $z > \gamma$. For any wage level $w \in (z, \gamma)$, the worker is willing to work, and the firm is willing to hire; i.e. the value functions above cannot pin down the wage level. This is a consequence of having search costs in the labour market so that the labour market is no longer competitive. Here, both workers and firms have some bargaining power—for firms, if workers quit, then it may take several periods and incurs cost in filing the vacancy, and similarly, for workers, if they are fired, it may take several periods and loss in income to find a new job.

To pin down the equilibrium wage, w^*, we adopt the Nash bargaining solution so that w^* is given by

$$w^* \in \arg\max_w \left(V^e(w) - V^u\right)^\phi \left(V^f(w)\right)^{1-\phi}.$$

Note that

- ▷ $V^e(w) - V^u$ is the net surplus for the worker from working (relative to his threat point or outside option);
- ▷ $V^f(w)$ is the firm's net surplus from hiring a worker (relative to its threat point of zero profits);
- ▷ $\phi \in (0,1)$ is a parameter that relates to the worker's relative bargaining power over firms. If $\phi = 1/2$, then firms an workers have equal bargaining power. If $\phi \to 1$, then the solution would involve setting $V^e(w) - V^u$ as high as possible while setting $V^f(w)$ small.

We are assuming that the bargaining happens between an individual worker and an individual firm when there are many workers and firms in the market. This means that the outcome of the bargaining does not affect the market wages w^*. So, when solving the bargaining problem, we can take V^u as a constant with respect to w. (But since every worker-firm pair is the same, the market wages w^* coincides with the Nash bargaining solution of work-firm pair.)

However, this expression implies that, when firms lose a worker, then they automatically post a vacancy, which may not be true in general. Therefore, V^f does not involve V^v.

[13] We implicitly assume that firms always have the choice to not hire so that the value function is bounded below at zero. Similarly, for workers, we assume that they can always choose not work.

Before solving for w^*, we use the Bellman equations to simplify the objective function (we use the equations without $f(\theta)$ and $\mu(\theta)$). Subtracting rV^u from both sides of (2.2) gives

$$\begin{aligned} rV^e(w) - rV^u &= w + x\left(V^u - V^e(w)\right) - rV^u \\ \Leftrightarrow (r+x)\left(V^e(w) - V^u\right) &= w - rV^u \\ \Leftrightarrow V^e(w) - V^u &= \frac{w - rV^u}{r+x}. \end{aligned}$$

Rearranging (2.3) yields

$$V^f(w) = \frac{z-w}{r+x}.$$

Thus, we can write the Nash bargaining problem as

$$\begin{aligned} &\arg\max_w \left(\frac{w - rV^u}{r+x}\right)^\phi \left(\frac{z-w}{r+x}\right)^{1-\phi} \\ \equiv &\arg\max_w \phi \ln\left(w - rV^u\right) + (1-\phi)\ln(z-w) - \ln(r+x). \end{aligned}$$

The first-order condition implies

$$\begin{aligned} \frac{\phi}{w^* - rV^u} &= \frac{1-\phi}{z-w^*} \\ \Leftrightarrow \phi\left(z - w^*\right) &= (1-\phi)\left(w^* - rV^u\right) \\ \Leftrightarrow \left(\phi + (1-\phi)\right)w^* &= (1-\phi)rV^u + \phi z \\ \Leftrightarrow w^* &= \phi z + (1-\phi)rV^u. \end{aligned} \tag{2.5}$$

The payoff to workers and firms are then

$$\begin{aligned} V^e(w^*) - V^u &= \frac{w^* - rV^u}{r+x} = \frac{\phi z + (1-\phi)rV^u - rV^u}{r+x} \\ &= \phi\frac{z - rV^u}{r+x}, \qquad (2.6) \\ V^f(w^*) &= \frac{z - \left(\phi z + (1-\phi)rV^u\right)}{r+x} \\ &= (1-\phi)\frac{z - rV^u}{r+x}. \qquad (2.7) \end{aligned}$$

Observe that

$$\frac{z - rV^u}{r+x} = \frac{w - rV^u}{r+x} + \frac{z-w}{r+x} = V^e(w) - V^u + V^f(w)$$

is the total net surplus from successful bargaining. It is a characteristic of the Nash bargaining solution that each side of the bargaining receives a share of the total net surplus equal to ϕ that parameterises the relative bargaining power.

We now have four equations, (2.1), (2.4) (the two Bellman equations with $f(\theta)$ and $\mu(\theta)$), (2.6) and (2.7) and four unknowns, $V^e(w^*)$, V^u, $V^f(w^*)$ and θ:

$$rV^u = \gamma + f(\theta)\left(V^e(w^*) - V^u\right), \tag{2.8}$$

$$rV^v = 0 = -c + \mu(\theta) V^f(w^*), \tag{2.9}$$

$$V^e(w^*) - V^u = \phi \frac{z - rV^u}{r+x}, \tag{2.10}$$

$$V^f(w^*) = (1-\phi)\frac{z - rV^u}{r+x}. \tag{2.11}$$

We want to pin down the equilibrium θ, denoted θ^*. From (2.9), we have

$$V^f(w^*) = \frac{c}{\mu(\theta^*)}.$$

This allows us to write (2.11) as

$$\frac{c}{\mu(\theta^*)(1-\phi)} = \frac{z - rV^u}{r+x}, \tag{2.12}$$

where the right-hand side is equal to the net surplus from bargaining, $V^e(w) - V^u + V^f(w)$. Then, using (2.10),

$$V^e(w^*) - V^u = \frac{\phi}{1-\phi}\frac{c}{\mu(\theta^*)}.$$

Substituting this into (2.8) gives

$$rV^u = \gamma + f(\theta)\frac{\phi}{1-\phi}\frac{c}{\mu(\theta^*)}.$$

We can now eliminate rV^u from (2.12) to obtain

$$\frac{c(r+x)}{\mu(\theta^*)(1-\phi)} = z - \left(\gamma + \underbrace{\frac{f(\theta^*)}{\mu(\theta^*)}}_{=\theta^*}\frac{\phi}{1-\phi}c\right)$$

$$\Leftrightarrow \frac{r+x}{\mu(\theta^*)} = (1-\phi)\frac{z-\gamma}{c} - \phi\theta^*. \tag{2.13}$$

Since r, x, ϕ, γ, z and c are constant, it follows that θ^* is constant at any point in time in the model (in equilibrium). In other words, the ratio between v and u, which measures the labour market tightness) is always constant in the model, even if u (or v) might be changing over time.

Since $\mu(\theta)$ is decreasing in θ, the left-hand side is strictly increasing in θ. In contrast, the right-hand side is strictly decreasing in θ. Hence, there can be at most one value of θ^* that satisfies (2.17). To ensure existence, we can put boundary conditions (recall $\mu(\theta)$ is decreasing and $\mu(\theta) = m(u,v)/v \geq 0$ and $\theta = v/u \geq 0$):

$$\lim_{\theta\to 0}\mu(\theta) = m\left(\frac{1}{\theta}, 1\right) = \infty, \ \lim_{\theta\to\infty}\mu(\theta) = m\left(\frac{1}{\theta}, 1\right) = 0.$$

2.1.4 Unemployment process

We normalise population to be one so that u represents the unemployment rate. Let u_0 denote the initial unemployment rate. Then the law of motion for unemployment is given by

$$\dot{u}(t) = x(1 - u(t)) - f(\theta)u(t).$$

The first term represents the change from worker being employed to unemployed (x is the rate at which workers lose jobs and $1 - u(t)$ is the employment rate), and the second term represents the change from worker being unemployed to employed ($f(\theta)$ is the rate at which a vacancy is filled and $u(t)$ is the unemployment rate). Note, in particular, that the law of motion holds for any θ, in particular, it holds even if $\theta \neq \theta^*$.

Rewrite the law of motion as

$$\dot{u}(t) = x - (x + f(\theta))u(t), \tag{2.14}$$

which is a first-order differential equation with boundary condition $u(0) = u_0$. To solve for $u(t)$, guess that the solution is of the form

$$u(t) = \exp[-(x + f(\theta))t]h(t),$$

where $h(t)$ is the constant of variation. Then,

$$\begin{aligned}\dot{u}(t) &= -(x + f(\theta))\exp[-(x + f(\theta))t]h(t) + \exp[-(x + f(\theta))t]\dot{h}(t)\\ &= -(x + f(\theta))u(t) + \exp[-(x + f(\theta))t]\dot{h}(t).\end{aligned}$$

Substituting this into (2.14) yields

$$\begin{aligned}-(x + f(\theta))u(t) + \exp[-(x + f(\theta))t]\dot{h}(t) &= x - (x + f(\theta))u(t)\\ \Leftrightarrow \dot{h}(t) &= x\exp[(x + f(\theta))t]\\ \Rightarrow \int \dot{h}(t)\,dt &= \int x\exp[(x + f(\theta))t]\,dt\\ \Rightarrow h(t) &= \frac{x}{x + f(\theta)}\exp[(x + f(\theta))t] + C,\end{aligned}$$

where C is some constant of integration. Then,

$$\begin{aligned}u(t) &= \exp[-(x + f(\theta))t]\left(\frac{x}{x + f(\theta)}\exp[(x + f(\theta))t] + C\right)\\ &= C\exp[-(x + f(\theta))t] + \frac{x}{x + f(\theta)}.\end{aligned}$$

Using the boundary condition,

$$u_0 = C + \frac{x}{x + f(\theta)} \Leftrightarrow C = u_0 - \frac{x}{x + f(\theta)}.$$

Hence,

$$
\begin{aligned}
u(t) &= \left(u_0 - \frac{x}{x+f(\theta)}\right)\exp\left[-\left(x+f(\theta)\right)t\right] + \frac{x}{x+f(\theta)} \\
&= u_0 \exp\left[-\left(x+f(\theta)\right)t\right] + \frac{x}{x+f(\theta)}\left(1-\exp\left[-\left(x+f(\theta)\right)t\right]\right). \qquad (2.15)
\end{aligned}
$$

In the steady state (in equilibrium), $\dot{u}=0$, and $\theta=\theta^*$, which implies that

$$
x = \left(x+f(\theta^*)\right)u^* \Leftrightarrow u^* = \frac{x}{x+f(\theta^*)}. \qquad (2.16)
$$

Therefore, in (2.15), we see that $u(t)$ is a weighted average between the initial unemployment rate u_0 and the steady-state unemployment rate. As t becomes large, $\exp\left[-\left(x+f(\theta)\right)t\right] \to 0$ so that $u(t) \to u^*$.

2.1.5 Dynamics

Suppose that $u_0 = 2u^*$; i.e. the initial unemployment rate is double that of the steady-state unemployment rate. Then what does the dynamics look like?

To understand the dynamics, we can draw a phase diagram. From (2.13), we know that, at any point in time, in equilibrium, the ratio θ^* is constant. In the steady state, unemployment rate is unchanging over time; i.e. $\dot{u}=0$. The steady state is given by the intersection, where $v/u=\theta^*$ and $\dot{u}=0$.

In the (u,v) space, the condition that $v/u=\theta^*$ is represented by a straight line given by

$$
v = \theta^* u.
$$

Since $\theta^* > 0$, this is an upward sloping line.

The other locus of relevance is the $\dot{u}=0$ locus. Recalling (2.16) without imposing $\theta=\theta^*$ gives

$$
\begin{aligned}
x &= \left(x+f(\theta)\right)u = \left(x + \frac{m(v,u)}{u}\right)u \\
&= xu + m(v,u).
\end{aligned}
$$

Differentiating with respect to u gives

$$
\begin{aligned}
0 &= x + m_v \frac{\partial v}{\partial u} + m_u \qquad (2.17) \\
\Leftrightarrow \frac{\partial v}{\partial u} &= -\frac{x+m_u}{m_v} < 0,
\end{aligned}
$$

where the inequality follows since the matching function is strictly increasing. Differentiating (2.17) with respect to u again, we obtain

$$\begin{aligned}
0 &= m_v \frac{\partial^2 v}{\partial u^2} + \left(m_{vu} + m_{vv}\frac{\partial v}{\partial u}\right)\frac{\partial v}{\partial u} + m_{uu} + m_{uv}\frac{\partial v}{\partial u} \\
&= m_v \frac{\partial^2 v}{\partial u^2} + \left(m_{vu} + m_{vu} + m_{vv}\frac{\partial v}{\partial u}\right)\frac{\partial v}{\partial u} + m_{uu} \\
\Leftrightarrow \frac{\partial^2 v}{\partial u^2} &= -\frac{\left(2\overbrace{m_{vu}}^{>0} + \overbrace{m_{vv}\frac{\partial v}{\partial u}}^{>0}\right)\overbrace{\frac{\partial v}{\partial u}}^{<0} + \overbrace{m_{uu}}^{<0}}{\underbrace{m_v}_{>0}} > 0,
\end{aligned}$$

where the inequality follows since m exhibits constant returns to scale, which implies $m_{vu} > 0$ and $m_{uu}, m_{vv} < 0$.[14] This means that the $\dot{u}$ locus is strictly convex in (u, v) space.

Off-steady state dynamics. Now consider what happens when we are above or below that $\dot{u} = 0$ locus. That $v/u = \theta^*$ always in equilibrium means that we must still be on the locus $v = \theta^* u$ even when we are not at the steady state. Rearranging (2.14) gives

$$\begin{aligned}
\dot{u} &= x - (x + f(\theta))\,u = x - \left(x + \frac{m(u,v)}{u}\right)u \\
&= x - xu - m(u,v).
\end{aligned}$$

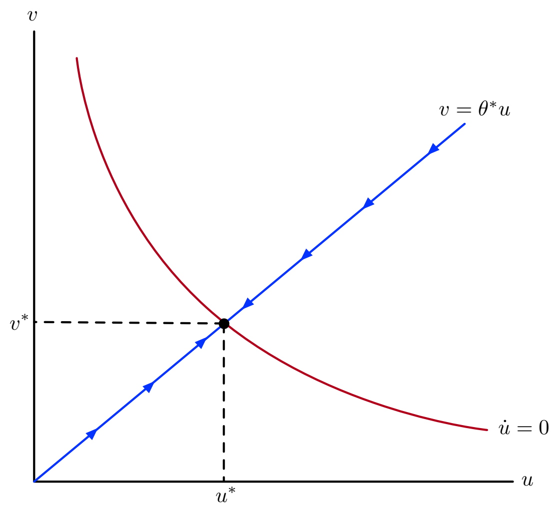
Since m is strictly increasing, we realise that, if we are above (below) the $\dot{u} = 0$ locus, then $\dot{u} < 0$ ($\dot{u} > 0$). Thus, we conclude that the saddle path is given by the locus $v = \theta^* u$. If we start with $u_0 = 2u^*$, then we immediately jump to $v = \theta^* 2u^*$, and the move down along line and converge to the steady state.

The phase diagram looks as follows.

[14] Think Cobb-Douglas. Let

$$\begin{aligned}
f(K, L) &= K^{\alpha} L^{1-\alpha}, \\
\Rightarrow f_K(K, L) &= \alpha K^{\alpha-1} L^{1-\alpha} > 0 \\
\Rightarrow f_{KL}(K, L) &= \alpha(1-\alpha) K^{\alpha-1} L^{-\alpha} > 0.
\end{aligned}$$

The intuition is that capital becomes more productive (i.e. marginal product of capital is higher) when more labour input is used.



2.1.6 MIT shock

Consider an one-time unanticipated increase in labour productivity, z (i.e. MIT shock). We can see how this affects θ^* via (2.13). Differentiating this with respect to z gives

$$-\frac{r+x}{\left(\mu\left(\theta^*\right)\right)^2}\mu'\left(\theta^*\right)\frac{\partial\theta^*}{\partial z} = \frac{1-\phi}{c} - \frac{\partial\theta^*}{\partial z}\phi$$

$$\Leftrightarrow \frac{\partial\theta^*}{\partial z} = \frac{\frac{1-\phi}{c}}{\phi - \frac{r+x}{\left(\mu\left(\theta^*\right)\right)^2}\mu'\left(\theta^*\right)} = \frac{1-\phi}{c\left(\phi - \frac{r+x}{\left(\mu\left(\theta^*\right)\right)^2}\mu'\left(\theta^*\right)\right)} > 0.$$

Thus, we see that a positive shock to z leads to a higher value of θ^* so that there are relatively more vacancies than unemployment (labour market tightness moves in favour of workers). Intuitively, a higher productivity of workers mean that the value of filled jobs increase initially (i.e. holding fixed wages and θ^*) so that firms are willing to hire more workers. This means that θ^* increases so that it becomes harder for firms to fill jobs. Wages also adjust.

We can see from (2.5) that higher productivity leads directly to higher wages, which pushes down the value of filled jobs (recall (2.7)). There is also an upward pressure on w^* from V^u. Higher equilibrium wages, and the faster rate at which workers are finding jobs means that the value of being unemployed, V^u is higher (since it is easier to find employment now). To see this, combine (2.1) and (2.6):

$$rV^u = \gamma + f\left(\theta^*\right)\phi\frac{z - rV^u}{r+x}.$$

Differentiating this with respect to z yields

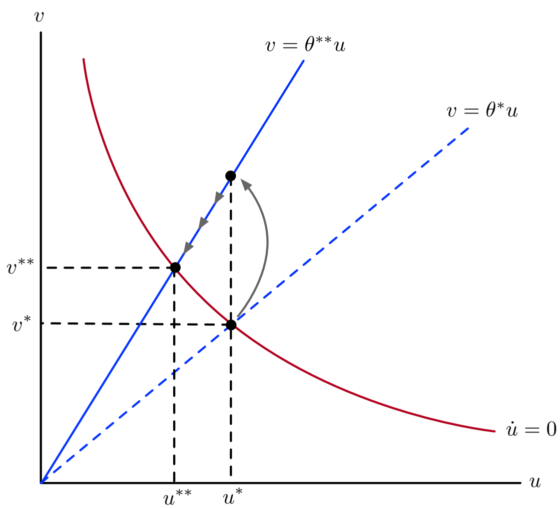
$$r\frac{\partial V^u}{\partial z} = f'\left(\theta\right)\frac{\partial\theta^*}{\partial z}\left(\phi\frac{z-rV^u}{r+x}\right) + f\left(\theta^*\right)\frac{\phi}{r+x}\left(1 - r\frac{\partial V^u}{\partial z}\right)$$

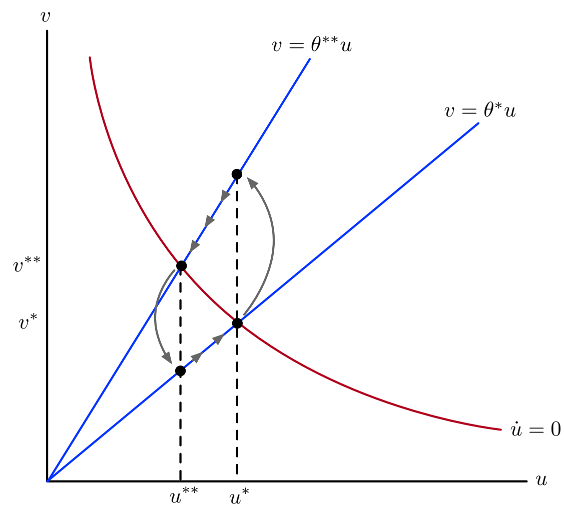
$$\left(1 + f\left(\theta^*\right)\frac{\phi}{r+x}\right)r\frac{\partial V^u}{\partial z} = f'\left(\theta\right)\frac{\partial\theta^*}{\partial z}\left(\phi\frac{z-rV^u}{r+x}\right) + f\left(\theta^*\right)\frac{\phi}{r+x}$$

$$\frac{\partial V^u}{\partial z} = \frac{\overbrace{f'\left(\theta^*\right)}^{>0}\overbrace{\frac{\partial\theta^*}{\partial z}}^{>0}\left(\phi\frac{z-rV^u}{r+x}\right) + f\left(\theta^*\right)\frac{\phi}{r+x}}{\left(1 + f\left(\theta^*\right)\frac{\phi}{r+x}\right)r} > 0.$$

The change in the value of θ^* means that the $v = \theta^* u$ locus rotates anticlockwise. The $\dot{u} = 0$ locus remains the same. Since u is a state variable that cannot "jump", following the MIT shock, v must jump up to the new $v = \theta^{**} u$ $(\theta^{**} > \theta^*)$ locus. Then, we gradually converge to the new steady state $u^{**} < u^*$ and $v^{**} > v^*$ as we see in the figure on the left below.

If we have another MIT shock back such that θ^{**} falls back to θ^*, then we get the dynamics in the opposite direction, and we see that (u, v) moves in an anticlockwise direction. Pissarides refers to this as *counterclockwise loops*.





2.1.7 Speed of convergence

Recall the law of motion for unemployment, (2.15),

$$u(t) = u_0 \exp\left[-\left(x + f(\theta)\right)t\right] + \frac{x}{x + f(\theta)}\left(1 - \exp\left[-\left(x + f(\theta)\right)t\right]\right).$$

The speed of convergence is determined by $x + f(\theta)$. What is the magnitude of $x + f(\theta)$?

Recall that $f(\theta)$ is the rate at which vacancies are filled. To get some data on this, we can consider $1/f(\theta)$, which is the average duration of unemployment. Data suggests that, in the US (faster than continental Europe), $1/f(\theta)$ is around 2–3 months, and so we generally take

$$f(\theta) \simeq 0.4$$

(on a monthly basis). To obtain a value of x, we can consider the steady-state level of unemployment

$$u^* = \frac{x}{x + f(\theta)} \Rightarrow x = \frac{u^* f(\theta)}{1 - u}.$$

Long-run (US) average gives the value of u^* to be around 5%, and so

$$x = \frac{5\% \times 0.4}{1 - 5\%} \simeq 0.02.$$

To see what these values imply, suppose that at $u(0) = 10\%$ and that $u^* = u(\infty) = 5\%$. We can think about how long it takes for the unemployment rate to fall by a half from 10%; i.e. the half life. Recalling (2.15), we find t such that

$$\begin{aligned}
\frac{1}{2}\left(u_0 - \underbrace{\frac{x}{x+f(\theta^*)}}_{=u^*}\right) &= u(0) - u(t) \\
&= u_0 - \left[\left(u_0 - \frac{x}{x+f(\theta^*)}\right)\exp\left[-\left(x+f(\theta^*)\right)t\right] + \frac{x}{x+f(\theta^*)}\right] \\
&= \left(u_0 - \frac{x}{x+f(\theta^*)}\right)\exp\left[-\left(x+f(\theta^*)\right)t\right] \\
\Leftrightarrow \exp\left[-\left(x+f(\theta^*)\right)t\right] &= \frac{1}{2} \\
\Leftrightarrow t &= \frac{\ln 2}{x+f(\theta^*)} \\
&\simeq \frac{\ln 2}{0.02+0.4} = 1.65,
\end{aligned}$$

where the unit of t is months. If, instead, $f(\theta) = 0.1$, then the half-life would be 5.8 months. Thus, the model implies a very fast adjustment, which does not accord with the data. The message is that, in order to assume protracted convergence, we must assume x and $f(\theta)$ that are very small.

Suppose we are hit continuously by MIT shocks and they are shifting θ around (up and down), then we will remain close to $\dot{u} = 0$. We are unlikely to see large movements along the $\dot{u} = 0$ locus because the adjustment mechanism is fast. We therefore would not expect to see prolonged period in which movement in u and v in the same direction.

See Shimer (2005) AER, in which Markov Poisson shocks is added to the model z. The implication from this model is similar to what we saw above, but the model will now have forward-looking elements in the counterpart to condition (2.13). It remains the case that if high z implies high z in the future (in the sense of first-order stochastic dominance), then θ^* increases with z and the convergence is fast. The model with shocks generates quantitatively small movements in vacancies and unemployment as long as

$$\frac{z}{\gamma} \gg 1 \Rightarrow \frac{\mathrm{Var}\left[\theta(z)\right]}{\mathrm{Var}\left[z\right]} \text{ is small.}$$

Alternatively, if z/γ is close to 1 (e.g. 1.05), then the movements are large. If we knew worker's bargaining power, then using data on wages and z, we can pin down the value of γ (or if we know γ, we can obtain the worker's bargaining power). However, we do not know either so to assess whether the model is a good fit of reality, the question comes down to whether the ratio z/γ is much larger than 1 or not. Arguments in favour of each case:

- ▷ $z \gg \gamma$ (Shimer's choice). z is the value of what worker can do in the market and γ is what the worker can do at "home". If we think about a world in which there is no market production, we would expect people would be worse off. This would suggest that z should be much larger than γ.

- ▷ $z \simeq \gamma$ (Hagedorn and Manovskii (2008) AER). If we take this model as a statement about a locally linear approximation around the neighbourhood of what we observe, then z represents the marginal product of labour, and γ represents the marginal rate of substitution between consumption and leisure. In a competitive environment, the two are equal. In this light, we might think that search friction should not drive a large wedge between the two on the margin.

The rationale for the latter case suggests that we might consider a model in which we have non-linear technology and preferences.

Remark 2.1. The constant returns to scale assumption in the model means that there is no distinction between average and margin—the two are the same.

2.2 Planner's problem

Consider the following social planner's problem:

$$\begin{aligned} W(u_0) = \max_{v(t)} \quad & \int_0^\infty e^{-\rho t}\left[u(t)\gamma + (1-u(t))z - v(t)c\right]dt \\ s.t. \quad & \dot{u}(t) = x(1-u(t)) - m(u(t), v(t)) \\ & u(0) = u_0. \end{aligned}$$

Since agents are risk neutral, we take their utility function to be linear. $u(t)\gamma$ is the "output" from the unemployed, $(1-u(t))z$ is the output from the employed. $v(t)c$ is the cost firms pay to fill vacancies.

We will solve this using Bellman equation,[15] which is given by

$$rW(u) = \max_{v\in[0,1]}\left\{u\gamma + (1-u)z - vc + W'(u)(x(1-u) - m(u,v))\right\}.$$

We claim, without proof, that (i) there is a unique solution that solves this problem for all $u \in (0,1)$; and (ii) the value function is linear so that $W'(u) = k$ for some constant k. The first-order condition is

$$-c - W'(u)m_v(u,v) = 0 \Rightarrow c = -km_v(u,v).$$

[15] To see this, for small $\varepsilon > 0$ (see Stokey notes on option pricing in deterministic setting),

$$\begin{aligned} W(u(0)) &= \int_0^\varepsilon e^{-\rho t}\left[u(t)\gamma + (1-u(t))z - v(t)c\right]dt + e^{-\rho\varepsilon}W(u(\varepsilon)) \\ &\approx \left[u(0)\gamma + (1-u(0))z - v(0)c\right]\varepsilon + \frac{1}{1+\rho\varepsilon}W(u(\varepsilon)), \end{aligned}$$

where we used that, when ε is small, the integral is approximately equal to the term inside the square bracket (with $t=0$) multiplied by ε, and $e^{-\rho\varepsilon} \approx (1+\rho\varepsilon)^{-1}$. Taylor expansion of $W(u(\varepsilon))$ around $\varepsilon = 0$ gives

$$W(u(\varepsilon)) \approx W(u(0)) + W'(u(0))\dot{u}(0)\varepsilon.$$

Substituting yields

$$\begin{aligned} W(u(0)) &\approx \left[u(0)\gamma + (1-u(0))z - v(0)c\right]\varepsilon + \frac{1}{1+\rho\varepsilon}\left[W(u(0)) + W'(u(0))\dot{u}(0)\varepsilon\right] \\ \Rightarrow (1+\rho\varepsilon)W(u(0)) &\approx \left[u(0)\gamma + (1-u(0))z - v(0)c\right]\varepsilon(1+\rho\varepsilon) + W(u(0)) + W'(u(0))\dot{u}(0)\varepsilon \\ \Rightarrow \rho\varepsilon W(u(0)) &\approx \left[u(0)\gamma + (1-u(0))z - v(0)c\right]\varepsilon + W'(u(0))\dot{u}(0)\varepsilon, \end{aligned}$$

where we assume ε^2 term is (approximately) zero. Dividing through by ε and writing $u(0) = u$ and $v(0) = v$, we obtain

$$\begin{aligned} \rho W(u) &= u\gamma + (1-u)z - vc + W'(u)\dot{u}(0) \\ &= u\gamma + (1-u)z - vc + W'(u)\left[x(1-u) - m(u,v)\right]. \end{aligned}$$

The envelope condition is

$$\begin{aligned} rW'(u) &= \gamma - z + W''(u)[\cdot] + W'(u)[-x - m_u(u,v)] \\ \Rightarrow rk &= \gamma - z - k[x + m_u(u,v)] \\ \Rightarrow k &= \frac{\gamma - z}{r + x + m_u(u,v)}, \end{aligned} \tag{2.18}$$

where we abbreviate the coefficient on W'' since $W'' = 0$ (since we assumed W is linear). Combining the two conditions, we obtain

$$\begin{aligned} c &= -\frac{\gamma - z}{r + x + m_u(u,v)} m_v(u,v) \\ \Leftrightarrow \frac{r+x}{m_v(u,v)} + \frac{m_u(u,v)}{m_v(u,v)} &= \frac{z - \gamma}{c} \\ \Leftrightarrow \frac{r+x}{m_v(u,v)} &= \frac{z-\gamma}{c} - \frac{m_u(u,v)}{m_v(u,v)}, \end{aligned} \tag{2.19}$$

which is very similar to the condition we found previously, (2.13). Since m exhibits constant returns to scale, its partial derivatives are homogenous of degree zero. This allows us to write

$$\frac{r+x}{m_v(1,\theta)} = \frac{z-\gamma}{c} - \frac{m_u(1,\theta)}{m_v(1,\theta)}$$

so that social planner's problem amounts to choosing θ (given u).

Remark 2.2. To verify that our guess was correct, we need to show that k is a constant. k is given by

$$k = \frac{\gamma - z}{r + x + m_u(u,v)}.$$

Using again the fact that m exhibits constant returns to scale, we can write

$$k = \frac{\gamma - z}{r + x + m_u(1,\theta)}.$$

We already proved that θ is constant so that we can conclude that k is a constant; i.e. our guess that the value function is linear was correct.

Exercise 2.1. Assume the following functional form for the matching function:

$$m(u,v) := \bar{m} u^{\eta} v^{1-\eta}, \ \eta \in (0,1).$$

We are thus assuming a Cobb-Douglas matching function. Given the functional form,

$$\begin{aligned} m_u(u,v) &= \eta \bar{m} u^{\eta-1} v^{1-\eta} = \eta \bar{m} \left(\frac{v}{u}\right)^{1-\eta} = \eta \bar{m} \theta^{1-\eta} \\ m_v(u,v) &= (1-\eta) \bar{m} u^{\eta} v^{-\eta} = (1-\eta) \bar{m} \left(\frac{v}{u}\right)^{-\eta} = (1-\eta) \bar{m} \theta^{-\eta} \end{aligned}$$

Then we can write (2.19) as

$$\frac{r+x}{(1-\eta)\,\bar{m}\theta^{-\eta}} = \frac{z-\gamma}{c} - \frac{\eta\bar{m}\theta^{1-\eta}}{(1-\eta)\,\bar{m}\theta^{-\eta}}$$
$$= \frac{z-\gamma}{c} - \frac{\eta}{1-\eta}\theta.$$

Note that

$$\mu(\theta) = \frac{m(u,v)}{v} = \frac{\bar{m}u^{\eta}v^{1-\eta}}{v} = \bar{m}\theta^{-\eta}.$$

So, in fact, we have

$$\frac{r+x}{\mu(\theta)} = (1-\eta)\frac{z-\gamma}{c} - \eta\theta,$$

which is exactly (2.13) with $\eta = \phi$. In other words, the equilibrium solves the planner's problem if and only if $\eta = \phi$. This is referred to as the *(Mortensen-) Hosios condition.*

Consider what happens if the bargaining power of the worker, ϕ, increases. Then more of the net surplus accrues to the worker from the bargaining so that w^* increases. Consequently, firms will reduce vacancies so that the probability of filling the vacancy increases to restore the zero profit condition, and the equilibrium θ^* increases. Observe that ϕ does not play a role in the planner's problem since the planner has access to lump-sum transfers. Mortensen (1982, AER) provides some intuition for the result. Let us consider the two extremes, $\eta = 1$ and $\eta = 0$.

- ▷ If $\eta = 0$, then $m(u,v) = \bar{m}v$ so that the matching function depends only on the number of vacancies and not on the number of unemployed. We can think of this situation as one in which the employers are making all the effort to fill vacancies—unemployed workers simply wait for the "phone call" from firms offering them jobs. In this case, if the worker obtains some surplus from job matching, then firm has less incentive to hire; i.e. there will be loss in welfare from a hold-up problem. Thus, for Pareto optimality, we require all of the net surplus from matching to go to the firms. In other words, $\phi = 0$.
- ▷ If $\eta = 1$, then $m(u,v) = \bar{m}u$. Now, it is the worker that makes all the effort in the matching process. This means that there will be vacancies and the rate at which vacancies are filled $\mu(\theta)$ is infinite.
- ▷ For intermediate values of η, elasticity tells you the importance of each's role in the matching process.

The Mortensen-Hosios condition holds more generally. Mortensen-Pissarides extends the model to the case in which there are heterogeneity in productivity among workers. Specifically, they assume that new workers start with high productivity and subsequently experiences idiosyncratic (i.e. worker specific) shocks. Thus, in their model, workers must decide when to end the job so that the separation rate, x, becomes endogenous. Workers then have two decision margins (i.e. whether to take the job, and when to leave the job) (note that when workers wishes to end the match, so do the firms). The same result holds that workers act in the same way as the social planner if and only if $\phi = \eta$. In the steady state, the model with endogenous x gives the same result as the case of exogenous x but off the steady state, the two models differ.

An adverse shock results in end of (some) jobs so that unemployment increases. The model suggests that the increase in unemployment increases because workers leave jobs. However, the data suggests that the increase in unemployment during times of recessions are driven mainly by the fact that the unemployed stays unemployed longer. That is, it is the lower probability of finding a job that increases unemployment and not the increase in separation rate that matters.

2.3 Incorporating search friction into the Neoclassical growth model

As mentioned before when we discussed the dynamics, whether the model fits the data well depends upon the ratio z/γ being close to one or not. The problem was that the linearity in the model meant that there was no distinction between marginal and average effects. Our goal here is to introduce search friction into an RBC model (which is non-linear) and to study whether this extension means that the model fits the data better. We first solve the planner's problem. (If we were to decentralise this, then we would need to specify the worker's bargaining power, ϕ.)

2.3.1 The setup

Households choose consumption and whether to work or not; i.e. the work decision is binary so that there is no intensive decision regarding how much to work.[16] We assume that the disutility from not working is zero, while γ is the disutility of work conditional on working. This means that the total disutility of work in the economy is given by γ times the number of individuals working, N_t. Workers lose jobs with constant probability χ.

Firms (representative) choose capital and labour. In particular, past hiring decisions of the firm affects the availability of labour that can be used in production, L_t, in period t. In this way, firms *invest* in hiring labour, and doing so requires some labour, which we denote as V_t (we can think of this as firms having an HR/recruitment department).

(Aggregate) matching technology The matching technology, $m(u_t, V_t)$, depends on the number of unemployed, u_t, as well as the amount of labour in the HR department, V_t. So V_t plays the role of vacancies in the Pissarides model. As before, we assume that m is increasing in both elements and exhibits constant returns to scale.

Planner's problem We assume that the planner has the same utility as the household and 2 laws of motions (one for capital and one for labour). The planner maximises household utility subject to these law of motions, as well as the aggregate feasibility constraint for goods and labour. The planner solves the following maximisation problem with respect to $\{K_{t+1}, N_{t+1}, C_t, V_t, L_t, u_{t+1}\}$:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \sum_{\mathbf{s}^t} \beta^t \Pi_t\left(\mathbf{s}^t\right)\left(\log C_t\left(\mathbf{s}^t\right) - \gamma N_t\left(\mathbf{s}^{t-1}\right)\right) \\
s.t. \quad & K_{t+1}\left(\mathbf{s}^t\right) = K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right) L_t\left(\mathbf{s}^t\right)\right)^{1-\alpha} + (1-\delta) K_t\left(\mathbf{s}^{t-1}\right) - C_t\left(\mathbf{s}^t\right) \\
& N_{t+1}\left(\mathbf{s}^t\right) = (1-\chi) N_t\left(\mathbf{s}^{t-1}\right) + m\left(u_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right) \\
& u_t\left(\mathbf{s}^{t-1}\right) + N_t\left(\mathbf{s}^{t-1}\right) = 1, \\
& V_t\left(\mathbf{s}^t\right) + L_t\left(\mathbf{s}^t\right) = N_t\left(\mathbf{s}^{t-1}\right) \\
& K_0\left(\mathbf{s}^{-1}\right) = K_0, \ N_0\left(\mathbf{s}^{-1}\right) = N_0.
\end{aligned}$$

[16]The data suggests about about 1/3 of the variation in aggregate hours worked is due to the intensive margin, and 2/3 is due to the extensive margin.

Observe that;

- ▷ $\gamma N_t\left(\mathbf{s}^t\right)$ represents the fact that decision to work involves only the extensive margin;
- ▷ the first constraint is the law of motion for capital, which is the standard one except that we now denote labour (used in production of goods) as L_t;
- ▷ the second constraint is the law of motion for labour. The number of total workers in the next period is the remaining workers from the previous period (since worker lose job with probability χ) plus the number of job matches given by the matching function $m\left(u_t, V_t\right)$;
- ▷ the third constraint just says that there is a unit measure of individuals, and each individual is either employed or unemployed;
- ▷ the fourth constraint says that total number of workers are determined one period before the decision as to whether workers work on producing output, L_t, or in the HR department, V_t;
- ▷ the problem has two state variables, K_{t+1} and N_{t+1}.

2.3.2 Optimality conditions

We eliminate L_t and u_t from the problem using the constraints. The problem is then

$$\begin{aligned}
\max_{\{K_{t+1}, N_{t+1}, C_t, V_t\}} \quad & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)\left(\log C_t\left(\mathbf{s}^t\right)-\gamma N_t\left(\mathbf{s}^{t-1}\right)\right)\\
s.t. \quad & K_{t+1}\left(\mathbf{s}^t\right)=K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)\left(N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)\right)\right)^{1-\alpha}\\
& \qquad +(1-\delta)K_t\left(\mathbf{s}^{t-1}\right)-C_t\left(\mathbf{s}^t\right)\\
& N_{t+1}\left(\mathbf{s}^t\right)=(1-\chi)N_t\left(\mathbf{s}^{t-1}\right)+m\left(1-N_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right)\\
& K_0\left(\mathbf{s}^{-1}\right)=K_0,\ N_0\left(\mathbf{s}^{-1}\right)=N_0.
\end{aligned}$$

As before, let the Lagrangian multipliers be $\beta^t\Pi_t\left(\mathbf{s}^t\right)\lambda_t\left(\mathbf{s}^t\right)$ for the law of motion for capital and $\beta^t\Pi_t\left(\mathbf{s}^t\right)\mu_t\left(\mathbf{s}^t\right)$ for the law of motion for labour. The Lagrangian is given by

$$\begin{aligned}
\mathcal{L}=\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)\Big[&\log C_t\left(\mathbf{s}^t\right)-\gamma N_t\left(\mathbf{s}^{t-1}\right)\\
&+\lambda_t\left(\mathbf{s}^t\right)\Big(K_{t+1}\left(\mathbf{s}^t\right)-K_t\left(\mathbf{s}^{t-1}\right)^{\alpha}\left(z_t\left(\mathbf{s}^t\right)\left(N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)\right)\right)^{1-\alpha}\\
&\qquad -(1-\delta)K_t\left(\mathbf{s}^{t-1}\right)-C_t\left(\mathbf{s}^t\right)\Big)\\
&+\mu_t\left(\mathbf{s}^t\right)\left(N_{t+1}\left(\mathbf{s}^t\right)-(1-\chi)N_t\left(\mathbf{s}^{t-1}\right)-m\left(1-N_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right)\right)\Big].
\end{aligned}$$

Then, the first-order conditions are

$$
\begin{aligned}
\left\{C_t\left(\mathbf{s}^t\right)\right\} \quad & 0=\beta^t \Pi_t\left(\mathbf{s}^t\right)\left[\frac{1}{C_t\left(\mathbf{s}^t\right)}-\lambda_t\left(\mathbf{s}^t\right)\right], \\
\left\{V_t\left(\mathbf{s}^t\right)\right\} \quad & 0=\lambda_t\left(\mathbf{s}^t\right)(1-\alpha) K_t\left(\mathbf{s}^{t-1}\right)^\alpha z_t\left(\mathbf{s}^t\right)^{1-\alpha}\left(N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)\right)^{-\alpha} \\
& \quad-\mu_t\left(\mathbf{s}^t\right) m_v\left(1-N_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right), \\
\left\{K_{t+1}\left(\mathbf{s}^t\right)\right\} \quad & 0=\beta^t \Pi_t\left(\mathbf{s}^t\right) \lambda_t\left(\mathbf{s}^t\right)-\beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) \lambda_t\left(\mathbf{s}^{t+1}\right) \times \\
& \qquad \left[\alpha K_{t+1}\left(\mathbf{s}^t\right)^{\alpha-1}\left(z_{t+1}\left(\mathbf{s}^{t+1}\right)\left(N_{t+1}\left(\mathbf{s}^t\right)-V_{t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)^{1-\alpha}-(1-\delta)\right], \\
\left\{N_{t+1}\left(\mathbf{s}^t\right)\right\} \quad & 0=-\beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) \gamma \\
& \quad+\beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) \lambda_t\left(\mathbf{s}^{t+1}\right)(1-\alpha) K_{t+1}\left(\mathbf{s}^t\right)^\alpha z_{t+1}\left(\mathbf{s}^{t+1}\right)^{1-\alpha}\left(N_{t+1}\left(\mathbf{s}^t\right)-V_{t+1}\left(\mathbf{s}^{t+1}\right)\right)^{-\alpha} \\
& \quad+\beta^t \Pi_t\left(\mathbf{s}^t\right) \mu_t\left(\mathbf{s}^t\right) \\
& \quad-\beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) \mu_{t+1}\left(\mathbf{s}^{t+1}\right)\left[(1-\chi)-m_u\left(1-N_{t+1}\left(\mathbf{s}^t\right), V_{t+1}\left(\mathbf{s}^t\right)\right)\right].
\end{aligned}
$$

From $\{C_t(\mathbf{s}^t)\}$,

$$
\lambda_t\left(\mathbf{s}^t\right)=\frac{1}{C_t\left(\mathbf{s}^t\right)}.
$$

Substituting this into $\{K_{t+1}(\mathbf{s}^t)\}$ gives the usual Euler equation:

$$
\begin{aligned}
\frac{\beta^t \Pi_t\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right)} & =\beta^{t+1} \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \Pi_{t+1}\left(\mathbf{s}^{t+1}\right) \frac{F_{K, t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)} \\
\Leftrightarrow \frac{1}{C_t\left(\mathbf{s}^t\right)} & =\beta \sum_{\mathbf{s}^{t+1} \succ \mathbf{s}^t} \frac{\Pi_{t+1}\left(\mathbf{s}^{t+1}\right)}{\Pi_t\left(\mathbf{s}^t\right)} \frac{F_{K, t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)} \\
& =\beta \mathbb{E}_t\left[\frac{F_{K, t+1}\left(\mathbf{s}^{t+1}\right)}{C_{t+1}\left(\mathbf{s}^{t+1}\right)}\right],
\end{aligned}
$$

As where (as before production function is net of depreciation):

$$
\begin{aligned}
F_{K, t+1}\left(\mathbf{s}^{t+1}\right) \coloneqq & \alpha K_{t+1}\left(\mathbf{s}^t\right)^{\alpha-1}\left(z_{t+1}\left(\mathbf{s}^{t+1}\right)\left(N_{t+1}\left(\mathbf{s}^t\right)-V_{t+1}\left(\mathbf{s}^{t+1}\right)\right)\right)^{1-\alpha} \\
& -(1-\delta)
\end{aligned}
$$

From $\{V_t(\mathbf{s}^t)\}$:

$$
\mu_t\left(\mathbf{s}^t\right)=\frac{\lambda_t\left(\mathbf{s}^t\right) F_{\ell, t}\left(\mathbf{s}^t\right)}{m_{v, t}\left(\mathbf{s}^t\right)}=\frac{F_{\ell, t}\left(\mathbf{s}^t\right)}{C_t\left(\mathbf{s}^t\right) m_{v, t}\left(\mathbf{s}^t\right)},
$$

where

$$
\begin{aligned}
F_{\ell, t}\left(\mathbf{s}^t\right) & \coloneqq(1-\alpha) K_t\left(\mathbf{s}^{t-1}\right)^\alpha z_t\left(\mathbf{s}^t\right)^{1-\alpha}\left(\underbrace{N_t\left(\mathbf{s}^{t-1}\right)-V_t\left(\mathbf{s}^t\right)}_{=L_t\left(\mathbf{s}^t\right)}\right)^{-\alpha} \\
m_{v, t}\left(\mathbf{s}^t\right) & \coloneqq m_v\left(1-N_t\left(\mathbf{s}^{t-1}\right), V_t\left(\mathbf{s}^t\right)\right).
\end{aligned}
$$

Together with the expression for $\lambda_t(\mathbf{s}^t)$, we can write $\{N_{t+1}(\mathbf{s}^t)\}$ as

$$\begin{aligned}\frac{F_{\ell,t}(\mathbf{s}^t)}{C_t(\mathbf{s}^t)\,m_{v,t}(\mathbf{s}^t)} &= \beta \sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t} \frac{\Pi_{t+1}(\mathbf{s}^{t+1})}{\Pi_t(\mathbf{s}^t)} \left(\frac{F_{\ell,t+1}(\mathbf{s}^{t+1})}{C_{t+1}(\mathbf{s}^{t+1})} - \gamma\right) \\ &\quad + \beta \sum_{\mathbf{s}^{t+1}\succ\mathbf{s}^t} \frac{\Pi_{t+1}(\mathbf{s}^{t+1})}{\Pi_t(\mathbf{s}^t)} \frac{F_{\ell,t+1}(\mathbf{s}^{t+1})}{C_{t+1}(\mathbf{s}^{t+1})\,m_{v,t+1}(\mathbf{s}^{t+1})}\left[1-\chi-m_{u,t+1}(\mathbf{s}^{t+1})\right] \\ \Leftrightarrow \frac{F_{\ell,t}(\mathbf{s}^t)}{C_t(\mathbf{s}^t)} &= \beta m_{v,t}(\mathbf{s}^t)\,\mathbb{E}_t\left[\frac{F_{\ell,t+1}(\mathbf{s}^{t+1})}{C_{t+1}(\mathbf{s}^{t+1})}\left(1+\frac{1-\chi-m_{u,t+1}(\mathbf{s}^{t+1})}{m_{v,t+1}(\mathbf{s}^{t+1})}\right)-\gamma\right].\end{aligned}$$

To summarise, we have the following consumption Euler equation:

$$\frac{1}{C_t(\mathbf{s}^t)} = \beta\mathbb{E}_t\left[\frac{F_{K,t+1}(\mathbf{s}^{t+1})}{C_{t+1}(\mathbf{s}^{t+1})}\right]$$

which tells us that, on the margin, consuming today (the left-hand side) should give the same benefit as foregoing consumption today and investing today. Investing today gives additional output of $F_{k,t+1}$ tomorrow which can be eaten (we are implicitly assuming K_{t+1} remains the same), which is worth $F_{k,t+1}/C_{t+1}$ in utils, and we discount this back to today by β. Thus, the consumption Euler equation tells us the optimal trade off between consuming today and investing in capital today.

The labour Euler equation is

$$\frac{F_{\ell,t}(\mathbf{s}^t)}{C_t(\mathbf{s}^t)} = \beta m_{v,t}(\mathbf{s}^t)\,\mathbb{E}_t\left[\frac{F_{\ell,t+1}(\mathbf{s}^{t+1})}{C_{t+1}(\mathbf{s}^{t+1})}\left(1+\underbrace{\frac{1-\chi-m_{u,t+1}(\mathbf{s}^{t+1})}{m_{v,t+1}(\mathbf{s}^{t+1})}}_{*}\right)-\gamma\right].$$

This tells us the optimal trade off between working for production today and working in recruitment/HR. The left-hand side gives the marginal utility from working in production today (working in production today gives additional output of $F_{\ell,t}$ which is worth $F_{\ell,t}/C_t$ in utils). If we instead work in recruiting, then this increases the number of matches by $m_{v,t}$. The return from this is the additional production worker we get the next period worth $F_{\ell,t+1}/C_{t+1}$ in utils. The $*$ term plays the same role as $1-\delta$ in the consumption Euler equation. It reflects the fact that the additional workers are likely to be around in $t+2$ which allows the planner to move more workers into production in $t+1$. Of course, having additional workers mean that there is an associated disutility of work γ.

In addition to the consumption and Euler equations, we also have the two feasibility constraints/laws of motion that gives a system of four equations. (We also have two transversality conditions for each of the state variables capital and labour).

Remark 2.3. Recall that in the standard RBC model, the optimal conditions were: (i) intratemporal condition that equated the marginal rate of substitution with marginal product of labour; (ii) consumption Euler equation; and (iii) feasibility condition (goods market). In the RBC model with search friction, we no longer have the intratemporal condition—this is replaced by the labour Euler equation.

2.3.3 Detrended optimality conditions

Let us assume the deterministic growth path case, so

$$z_t\left(\mathbf{s}^t\right) = (1+g)^t s_t.$$

We normalise variables by dividing by $(1+g)^t$; i.e.

$$k_t := \frac{K_t}{(1+g)^t},\ c_t := \frac{C_t}{(1+g)^t}.$$

Then,

$$\begin{aligned} F_{K,t+1}\left(\mathbf{s}^{t+1}\right) &:= \alpha K_{t+1}^{\alpha-1}\left((1+g)^{t+1} s_{t+1}\left(N_{t+1}-V_{t+1}\right)\right)^{1-\alpha} - (1-\delta) \\ &= \alpha k_{t+1}^{\alpha-1}\left(s_{t+1}\left(N_{t+1}-V_{t+1}\right)\right)^{1-\alpha} - (1-\delta), \end{aligned}$$

so that the consumption Euler equation becomes

$$\frac{1+g}{c_t} = \beta \mathbb{E}_t\left[\frac{\alpha k_{t+1}^{\alpha-1}\left(s_{t+1}\left(N_{t+1}-V_{t+1}\right)\right)^{1-\alpha} - (1-\delta)}{c_{t+1}}\right].$$

Similarly,

$$\begin{aligned} F_{\ell,t}\left(\mathbf{s}^t\right) &:= (1-\alpha) K_t{}^{\alpha}\left((1+g)^t s_t\right)^{1-\alpha}\left(N_t-V_t\right)^{-\alpha} \\ &= (1-\alpha) k_t^{\alpha}(1+g)^t\left(s_t\right)^{1-\alpha}\left(N_t-V_t\right)^{-\alpha} \end{aligned}$$

and so the labour Euler equation is now

$$\begin{aligned} &\frac{(1-\alpha) k_t^{\alpha}\left(s_t\right)^{1-\alpha}\left(N_t-V_t\right)^{-\alpha}}{c_t} \\ =&\beta m_{v,t}\mathbb{E}_t\left[\frac{(1-\alpha) k_{t+1}^{\alpha}(1+g)^{t+1}\left(s_{t+1}\right)^{1-\alpha}\left(N_{t+1}-V_{t+1}\right)^{-\alpha}}{c_{t+1}(1+g)^{t+1}}\left(1+\frac{1-\chi-m_{u,t+1}}{m_{v,t+1}}\right)-\gamma\right] \\ =&\beta m_{v,t}\mathbb{E}_t\left[\frac{(1-\alpha) k_{t+1}^{\alpha}\left(s_{t+1}\right)^{1-\alpha}\left(N_{t+1}-V_{t+1}\right)^{-\alpha}}{c_{t+1}}\left(1+\frac{1-\chi-m_{u,t+1}}{m_{v,t+1}}\right)-\gamma\right]. \end{aligned}$$

Dividing the goods market clearing condition through by $(1+g)^t$ gives

$$\begin{aligned} &\frac{K_{t+1}}{(1+g)^t} = \frac{K_t^{\alpha}\left((1+g)^t s_t L_t\right)^{1-\alpha}}{(1+g)^t} + (1-\delta)\frac{K_t}{(1+g)^t} - \frac{C_t}{(1+g)^t} \\ \Leftrightarrow\ &k_{t+1}(1+g) = k_t^{\alpha}\left(s_t L_t\right)^{1-\alpha} + (1-\delta) k_t - c_t \end{aligned}$$

2.3.4 Balanced growth path

On the balanced growth path

$$
\begin{aligned}
k_{t+1} &= k_t = \bar{k}, \\
c_{t+1} &= c_t = \bar{c}, \\
N_{t+1} &= N_t = \bar{N}, \\
V_{t+1} &= V_t = \bar{V}, \\
L_{t+1} &= L_t = \bar{L} \\
s_{t+1} &= s_t = 1, \\
m_{u,t+1} &= m_{u,t} = \bar{m}_u, \\
m_{v,t+1} &= m_{v,t} = \bar{m}_v.
\end{aligned}
$$

Imposing these conditions on the consumption Euler equation gives

$$
\begin{aligned}
\frac{1+g}{\bar{c}} &= \beta \mathbb{E}_t \left[\frac{\alpha \bar{k}^{\alpha-1} \left(\bar{N} - \bar{V}\right)^{1-\alpha} - (1-\delta)}{\bar{c}} \right] \\
\Leftrightarrow \frac{1}{\alpha} \left(\frac{1+g}{\beta} + (1-\delta) \right) &= \left(\frac{\bar{k}}{\bar{N} - \bar{V}} \right)^{\alpha-1}.
\end{aligned}
$$

Since the left-hand side is a constant, this tells us that the ratio $\bar{k}/\left(\bar{N} - \bar{V}\right) = \bar{k}/\bar{L}$ must be constant, and is pinned downed by this expression.

For labour market feasibility, we have

$$
\begin{aligned}
\bar{N} &= (1-\chi)\bar{N} + m\left(1 - \bar{N}, \bar{V}\right) \\
\Leftrightarrow \chi &= \frac{m\left(1 - \bar{N}, \bar{V}\right)}{\bar{N}} \qquad (2.20)
\end{aligned}
$$

Observe that the right-hand side is strictly decreasing in $\bar{N}$ (matching function is strictly increasing in u and V). Give some Inada conditions on m, for any given $\bar{V}$, we can find $\bar{N}$ that satisfies the equation above.

From the labour Euler equation:

$$
\begin{aligned}
\frac{(1-\alpha)\bar{k}^{\alpha}\left(\bar{N} - \bar{V}\right)^{-\alpha}}{\bar{c}} &= \beta \bar{m}_v \mathbb{E}_t \left[\frac{(1-\alpha)\bar{k}^{\alpha}\left(\bar{N} - \bar{V}\right)^{-\alpha}}{\bar{c}} \left(1 + \frac{1-\chi-\bar{m}_u}{\bar{m}_v}\right) - \gamma \right] \\
\Leftrightarrow \beta \bar{m}_v \gamma &= \frac{(1-\alpha)}{\bar{c}} \left(\frac{\bar{k}}{\bar{N} - \bar{V}} \right)^{\alpha} \left[\beta \bar{m}_v \left(1 + \frac{1-\chi-\bar{m}_u}{\bar{m}_v}\right) - 1 \right] \\
\Leftrightarrow \frac{\beta\gamma}{1-\alpha}\bar{c} &= \left(\frac{\bar{k}}{\bar{N} - \bar{V}} \right)^{\alpha} \left[\beta \left(1 + \frac{1-\chi-\bar{m}_u}{\bar{m}_v}\right) - \frac{1}{\bar{m}_v} \right],
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{m}_v &= m_v\left(1 - \bar{N}, \bar{V}\right) = m_v\left(1, \frac{\bar{V}}{1 - \bar{N}}\right), \\
m_u &= m_u\left(1 - \bar{N}, \bar{V}\right) = m_u\left(1, \frac{\bar{V}}{1 - \bar{N}}\right),
\end{aligned}
$$

Feasibility condition on the balanced growth path is

$$\begin{aligned}
\bar{k}(1+g) &= \bar{k}^{\alpha}\left(\bar{N}-\bar{V}\right)^{1-\alpha} + (1-\delta)\,\bar{k} - \bar{c} \\
\Leftrightarrow \bar{c} &= \bar{k}^{\alpha}\left(\bar{N}-\bar{V}\right)^{1-\alpha} - (\delta+g)\,\bar{k} \\
\Leftrightarrow \frac{\bar{c}}{\bar{k}} &= \left(\frac{\bar{k}}{\bar{N}-\bar{V}}\right)^{-(1-\alpha)} - (\delta+g)\,.
\end{aligned}$$

Observe that $\bar{c}/\bar{k}$ is a constant. This also implies that

$$\frac{\bar{c}}{\bar{N}-\bar{V}} = \frac{\bar{k}}{\bar{N}-\bar{V}}\frac{\bar{c}}{\bar{k}}$$

is a constant. Then,

Hence,

$$\begin{aligned}
\frac{\beta\gamma}{1-\alpha}\frac{\bar{c}}{\bar{N}-\bar{V}}\left(\frac{\bar{k}}{\bar{N}-\bar{V}}\right)^{-\alpha} &= \frac{1}{\bar{N}-\bar{V}}\left[\beta\left(1+\frac{1-\chi-\bar{m}_u}{\bar{m}_v}\right)-\frac{1}{\bar{m}_v}\right] \\
&= \frac{1}{\bar{N}-\bar{V}}\frac{1}{\bar{m}_v}\left[\beta\left(1-\chi+\bar{m}_v-\bar{m}_u\right)-1\right]. \qquad (2.21)
\end{aligned}$$

where the left-hand side is a constant. We now have two equations—(2.20) and (2.21)—in two unknowns, $\bar{V}$ and $\bar{N}$.

What's the condition for the solution to exist?

2.3.5 Calibration

We will now calibrate the model on a monthly basis.

We first need to define N, the size of the labour force. The calibrated valued of N depends on whether we let N represent the labour force or the entire population. Suppose we take the former option.[17] Since the long-run US average unemployment rate is around 5%, then

$$\begin{aligned}
u &= 0.05, \\
N &= 0.95.
\end{aligned}$$

We now have some new parameters to calibrate: the job separation rate χ. Although this varies across countries, in the US, the average rate is around 3.4% (on a monthly basis):

$$\chi = 0.034.$$

With respect to V, the literature suggests that V accounts for around 0.4% of the labour force:

$$\frac{V}{N} = 0.004.$$

and that the cost of recruiting is around 1/8th of a worker's wage. We assume that the matching function is Cobb-Douglas with equal weights:

What's the point of this?

[17] Alternatively, if we were to define N as the population, then since around 2/3 of the population is in the labour force, $N \simeq 0.6$ (and $u = 0.4$). Although the choice how we measure N affects the balanced growth path, it does not affect the cyclical movements; i.e. both will give the same log-linear approximation.

$$m(u,v) = \bar{m}\sqrt{uv}.$$

Recall that α represents the capital share of income. Unlike in the standard NCG/RBC model, there are now two types of labour: workers can be involved in production or recruitment. This means that α, the capital share of income, is based on L_t and not on N_t as before. Since the proportion of those working in recruitment is small (0.4%), α does not differ much from before:

$$\alpha = [???].$$

The other parameters on a monthly basis are:

$$\begin{aligned} g &= 0.00167, \\ \delta &= 0.00510, \\ \beta &= 0.99637, \\ \epsilon &= \infty, \\ \rho &= 0.98. \end{aligned}$$

2.3.6 Model implication

Let

$$\mathbf{x}_t = [k_t, c_t, s_t, N_t, V_t]'$$

and

$$0 = \mathbf{G}(\mathbf{x}_t, \mathbf{x}_{t+1}) = \begin{bmatrix} \text{Consumption Euler} \\ \text{Labour Euler} \\ \text{Goods market feasiblity} \\ \text{Labour market feasiblity} \\ \text{Law of motion for } s_t \end{bmatrix}.$$

Then

$$\mathbf{G}_t^* := \left[\frac{\partial G_i(\mathbf{x}^*, \mathbf{x}^*)}{\partial x_{j,t}} x_j^*\right]_{i.j},$$

which is a 5×5 matrix. Log linearising this gives us that

$$\begin{aligned} \mathbf{0} &= \mathbf{G}_t^* \hat{\mathbf{x}}_t + \mathbf{G}_{t+1}^* \hat{\mathbf{x}}_{t+1} \\ \Leftrightarrow \hat{\mathbf{x}}_{t+1} &= -\left(\mathbf{G}_{t+1}^*\right)^{-1} \mathbf{G}_t^* \hat{\mathbf{x}}_t \\ &\equiv \mathbf{M}\hat{\mathbf{x}}_t, \end{aligned}$$

where $\mathbf{M}$ is a 5×5 matrix with five eigenvalues (and five associated eigenvectors). Since the system is saddle-path stable and there are 3 state variables, k_t, N_t and s_t, we will have three eigenvalues inside the unit circle. With the calibrated model, we obtain

$$\lambda_1 = 0.99,\ \lambda_2 = 0.98,\ \lambda_3 = 0.31.$$

As before $\lambda_2 = \rho$. λ_1 relates to capital and that its value is close to one reflects the fact that capital adjusts slowly in this model. λ_3 relates to labour and its relatively low value reflects the fact that labour is a fast-moving state variable. We had a similar result in the linear search model, where we found that unemployment adjusts to the steady state quickly.

Letting e_i denote the eigenvectors associated with eigenvalue λ_i, recall that we can write

$$\hat{\mathbf{x}}_t = \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{s}_t \\ \hat{N}_t \\ \hat{V}_t \end{bmatrix} = \mu_1 \lambda_1^t e_1 + \mu_2 \lambda_2^t e_2 + \mu_3 \lambda_3^t e_3.$$

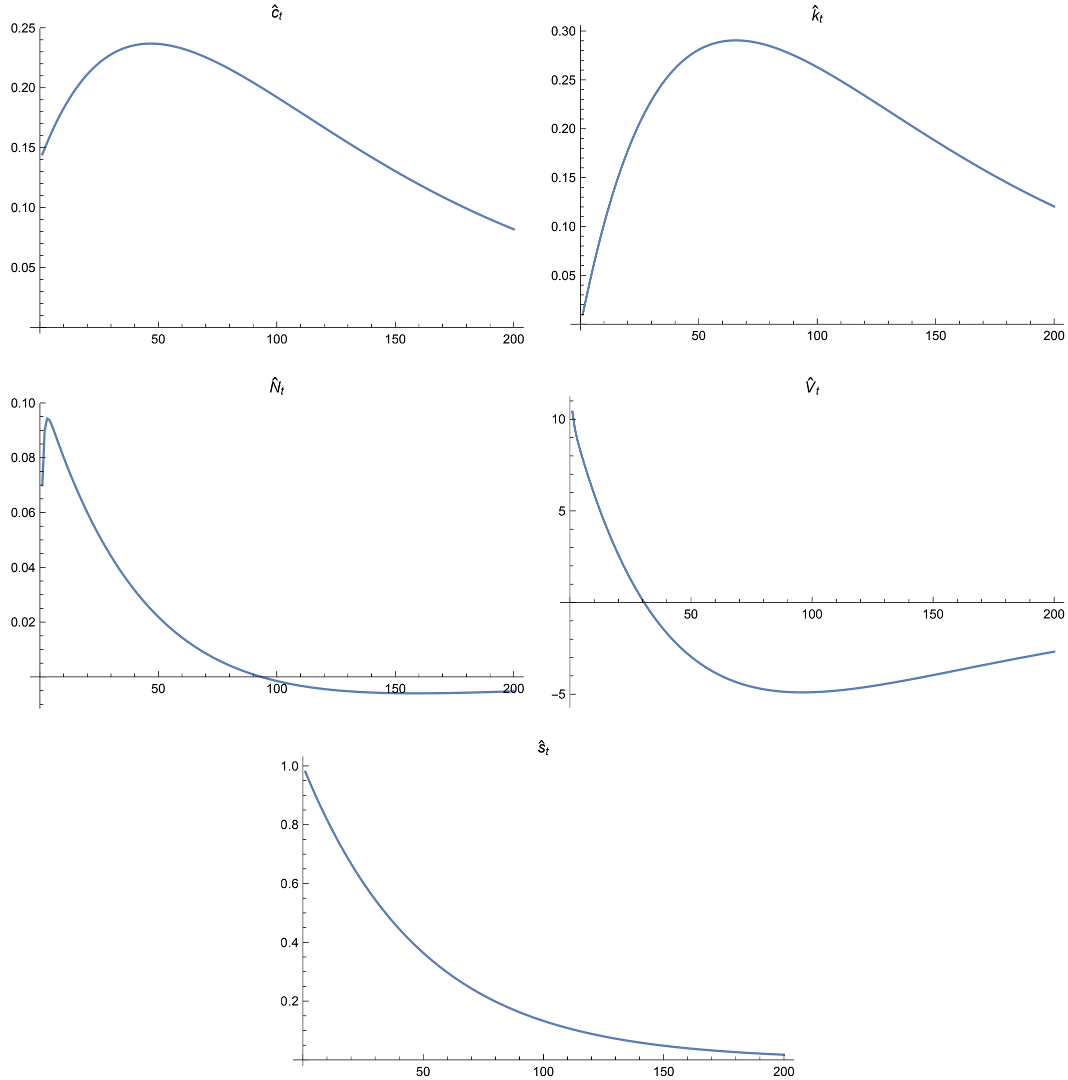
Since λ_3 is small, as $t \to \infty$, the dynamics will be driven predominantly by λ_1 and λ_2 (i.e. by capital and productivity shock). When we solve for μ_1, μ_2 and μ_3 given initial conditions, we obtain that

$$\begin{bmatrix} \hat{c}_t \\ \hat{V}_t \\ \hat{s}_{t+1} \\ \hat{k}_{t+1} \\ \hat{N}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.66 \\ -2.49 \\ 0 \\ 0.99 \\ -0.04 \end{bmatrix} \hat{k}_t + \begin{bmatrix} 0.01 \\ -19.5 \\ 0 \\ 0.02 \\ 0.31 \end{bmatrix} \hat{N}_t + \begin{bmatrix} 0.14 \\ 12.27 \\ 0.98 \\ 0.01 \\ 0.07 \end{bmatrix} \hat{s}_t. \tag{2.22}$$

From (2.22), we can see that the qualitatively implications from this model is similar to that of the RBC model.

Consider an unit increase in $\hat{s}_0$. The higher productivity leads to a build up of capital initially, which leads to higher consumption. Labour increases too, at least initially, since productivity is high, also contributing to higher consumption. However, as capital increases, eventually the wealth effect (the negative coefficient on $\hat{k}_t$ in $\hat{N}_{t+1}$) dominates and labour decreases. Recruitment labour initially increases following the shock, however, higher employment means less recruitment is required, and together with the wealth effect, $\hat{V}_t$ decreases over time.

Figure 2.1: Impulse response to $\hat{s}_0 = 1$.



The quantitative implications of the RBC model with search friction, however, is different. We can think of search friction as an adjustment cost on employment. Recruiting more people implies that both unemployment and V increases, which reduces the marginal product of labour from recruiting. Thus, search friction dampens volatility in employment.

Remark 2.4. Observe that the coefficient on $\hat{k}_t$ in $\hat{k}_{t+1}$ and the the coefficient on $\hat{N}_t$ in $\hat{N}_{t+1}$ is approximately equal to the eigenvalues λ_1 and λ_2 respectively. To see why, suppose we "kill" productivity shocks, then the last two rows gives the following system:

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{N}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.99 & 0.02 \\ -0.04 & 0.31 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{N}_t \end{bmatrix}.$$

Recall that, with a 2×2 matrix, eigenvalues are given by

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \mathbb{I} \right| = (a - \lambda)(d - \lambda) - bc = 0.$$

Hence, if $bc \simeq 0$, then $\lambda_1 \simeq a$ and $\lambda_2 \simeq d$.

2.3.7 Does incorporating search frictions help?

We can compare the results with a version of RBC model with indivisible labour , which we can think of as the limit of RBC model as we let the Frisch elasticity of labour supply to infinity. (We take Hansen (1985) and Rogerson (1988).) Although individuals makes labour decision on the extensive margin (i.e. to work or not), the social planner has an intensive margin—he can choose how many people should work. Calibrating this model on a monthly basis yields

$$\lambda_1 = 0.99, \ \lambda_2 = 0.98$$

and

$$\begin{bmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{s}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.59 \\ -0.49 \\ 0 \\ 0.99 \end{bmatrix} \hat{k}_t + \begin{bmatrix} 0.21 \\ 0.98 \\ 0.98 \\ 0.03 \end{bmatrix} \hat{s}_t, \tag{2.23}$$

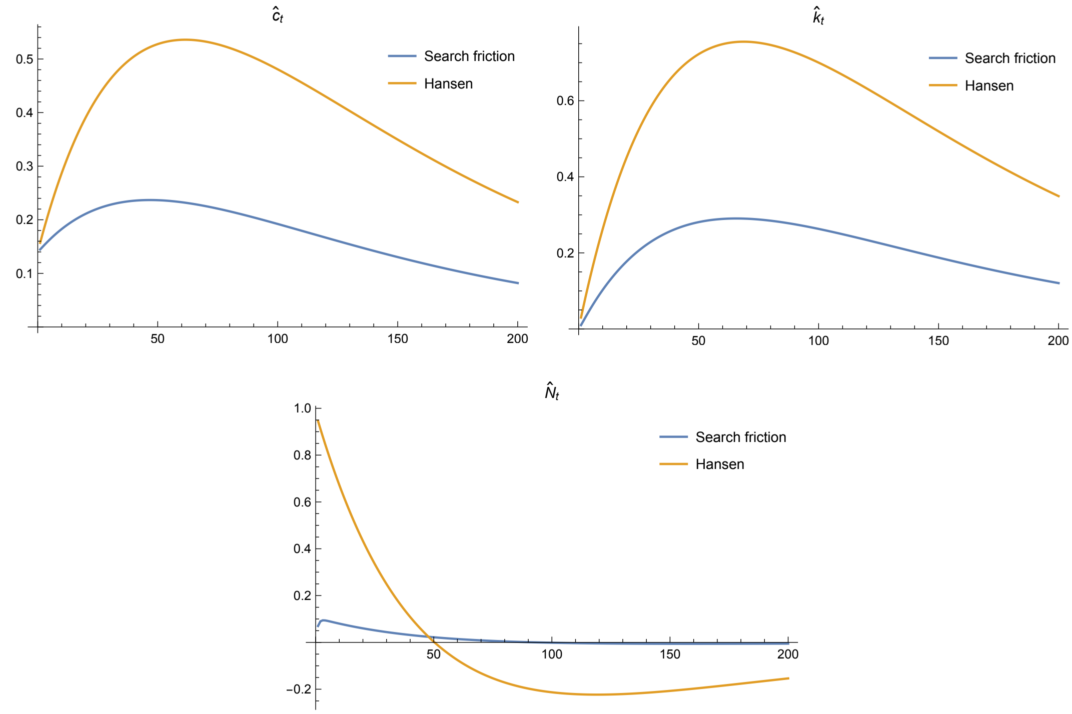
where we note that N_t is not a state variable in the RBC model.

Comparing (2.22) and (2.23), we see that $\hat{n}_t$ reacts more to the state variables. There are two reasons for this.

- ▷ In the search model, labour is chosen a period before the RBC model ($N_{t+1}(\mathbf{s}^t)$ is chosen not $N_t(\mathbf{s}^t)$). This is not a "big deal".
- ▷ More importantly, search model has a matching function which means that it is costly to adjust employment.

We can also see that the wealth effect (the coefficient on $\hat{k}_t$ in $\hat{n}_t$) is also dampened in the search model.

Figure 2.2: Impulse response to $\hat{s}_0 = 1$.



The implication of the difference means that RBC model with search frictions predicts lower volatility in employment than the RBC model. Recall that one of the failures of the RBC model was that it predicted employment to be too smooth. Thus, incorporating search frictions, in and of itself, does help us better match the stylised facts.

We can also measure the labour wedge a la Chari, Kehoe and McGrattan (2007). By running Monte Carlo simulations, we can generate data from the RBC model with search frictions. We can then measure the labour wedge using RBC model a la Hansen/Rogerson. What we would find is that when employment is high (i.e. a boom time), then the labour wedge is high. Since labour wedge is $1 - \tau_{h,t}$, this means that, in times of economic booms, it is as if there is a reduction in labour on tax. This is the opposite of what we would expect.

So is this the end of the road for search frictions? Not quite.

2.4 Decentralisation

We want to decentralise the planner's problem which would allow us to study equilibrium wage determination.

Recall that in the linear search model, the Bellman equations did not allow us to pin down the equilibrium wage. We obtained the equilibrium wage by imposing Nash bargaining between firms and workers. We will find that in a decentralised version of the RBC model with search friction, there is also some freedom in determining equilibrium wages. By choosing an appropriate wage determination method, perhaps we can make the model better match the data?

2.4.1 Complete market assumption

We assume in the RBC model with search frictions that individuals can work or not work. We can think of this as individuals experiencing idiosyncratic shock. However, the optimal consumption was constant across all individuals—in effect, the social planner provided insurance for individuals so as to ensure that consumption is the same whether an individual is working or not. This means that, in order for our decentralisation to replicate the planner's solution, we must insure agents against shocks.

One way is to introduce securities that allow individuals to insure against idiosyncratic shocks. But this requires, in principle, Arrow-Debreu securities for every individuals for every possible history; i.e. many many securities. We would rather keep the problem simple.

The way we proceed is to assume that each individuals belongs to a household, and that the household provides insurance for its members. As Shimer puts it, each household is lead by a "grandpa" that tells members what to do and what to consume. Within each household, some members could be working or not working, but the insurance means that all members consume the same amount. We assume that there is a measure one of households and each household has a measure one of members.

2.4.2 Competitive equilibrium

Define

$$\theta_t\left(\mathbf{s}^t\right)=\frac{V_t\left(\mathbf{s}^t\right)}{U_t\left(\mathbf{s}^t\right)}=\frac{V_t\left(\mathbf{s}^t\right)}{1-N_t\left(\mathbf{s}^t\right)}.$$

The matching function is constant returns to scale:

$$\begin{aligned} m\left(1-N,V\right) &\equiv f\left(\theta\right)\left(1-N\right) \\ &\equiv \mu\left(\theta\right)V, \end{aligned}$$

where f is increasing in θ while μ is decreasing in θ.

Definition 2.1. A competitive equilibrium in this economy is the sequence of allocations

$$\left\{C_t\left(\mathbf{s}^t\right),L_t\left(\mathbf{s}^t\right),V_t\left(\mathbf{s}^t\right),N_{t+1}\left(\mathbf{s}^t\right),K_{t+1}\left(\mathbf{s}^t\right),\theta_t\left(\mathbf{s}^t\right)\right\}_{t=1,\dots,\infty,\forall\mathbf{s}^t}$$

and sequence of prices

$$\left\{w_t\left(\mathbf{s}^t\right),q_0^t\left(\mathbf{s}^t\right)\right\}_{t=1,\dots,\infty,\forall\mathbf{s}^t}$$

such that

(i) given $\{w_t\left(\mathbf{s}^t\right),q_0^t\left(\mathbf{s}^t\right),\theta_t\left(\mathbf{s}^t\right)\}$, a_0 and $N_0\left(\mathbf{s}^{t-1}\right)=N_0$, the sequence $\{C_t\left(\mathbf{s}^t\right),N_t\left(\mathbf{s}^t\right)\}$ solves the household's problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}\beta^t\Pi_t\left(\mathbf{s}^t\right)\left(\log C_t\left(\mathbf{s}^t\right)-\gamma N_t\left(\mathbf{s}^t\right)\right) \\ s.t. \quad & a_0\geq\sum_{t=0}^{\infty}\sum_{\mathbf{s}^t}q_0^t\left(\mathbf{s}^t\right)\left(C_t\left(\mathbf{s}^t\right)-w_t\left(\mathbf{s}^t\right)N_t\left(\mathbf{s}^t\right)\right) \\ & N_{t+1}\left(\mathbf{s}^t\right)=\left(1-\chi\right)N_t\left(\mathbf{s}^{t-1}\right)+f\left(\theta_t\left(\mathbf{s}^t\right)\right)\left(1-N_t\left(\mathbf{s}^{t-1}\right)\right),\ \forall t,\mathbf{s}^t; \end{aligned}$$

(ii) given $\{w_t(\mathbf{s}^t), q_0^t(\mathbf{s}^t), \theta_t(\mathbf{s}^t)\}$, $K_0(\mathbf{s}^{-1}) = K_0$ and $N_0(\mathbf{s}^{-1}) = N_0$, the sequence $\{L_t(\mathbf{s}^t)$, $V_t(\mathbf{s}^t)$, $K_{t+1}(\mathbf{s}^t)$, $N_{t+1}(\mathbf{s}^t)\}$ solves the firm's problem:

$$\begin{aligned} v_0 = \max \quad & \sum_{t=0}^{\infty}\sum_{\mathbf{s}^t} q_0^t(\mathbf{s}^t)\Big[\left(K_t(\mathbf{s}^{t-1})\right)^{\alpha}\left(z_t(\mathbf{s}^t)L_t(\mathbf{s}^t)\right)^{1-\alpha} \\ & \qquad -\left(K_{t+1}(\mathbf{s}^t)-(1-\delta)K_t(\mathbf{s}^{t-1})\right) \\ & \qquad -w_t(\mathbf{s}^t)N_t(\mathbf{s}^{t-1})\Big] \\ s.t. \quad & N_{t+1}(\mathbf{s}^t) = (1-\chi)N_t(\mathbf{s}^{t-1}) + \mu(\theta_t(\mathbf{s}^t))V_t(\mathbf{s}^t),\ \forall t,\mathbf{s}^t, \\ & N_t(\mathbf{s}^{t-1}) = V_t(\mathbf{s}^t) + L_t(\mathbf{s}^t),\ \forall t,\mathbf{s}^t; \end{aligned}$$

(iii) the goods market clear:

$$K_{t+1}(\mathbf{s}^t) = \left(K_t(\mathbf{s}^{t-1})\right)^{\alpha}\left(z_t(\mathbf{s}^t)L_t(\mathbf{s}^t)\right)^{1-\alpha} - (1-\delta)K_t(\mathbf{s}^{t-1}) - C_t(\mathbf{s}^t),\ \forall t,\mathbf{s}^t;$$

(iv) consistency condition holds:

$$\theta_t(\mathbf{s}^t) = \frac{V_t(\mathbf{s}^t)}{1-N_t(\mathbf{s}^t)},\ \forall t,\mathbf{s}^t;$$

(v) given $\left\{K_t(\mathbf{s}^{t-1}), L_t(\mathbf{s}^t), \theta_t(\mathbf{s}^t), C_t(\mathbf{s}^t)\right\}$, $\{w_t(\mathbf{s}^t)\}$ solves the Nash bargaining problem

$$\begin{aligned} w_t(\mathbf{s}^t) = {} & \phi(1-\alpha)\left(K_t(\mathbf{s}^{t-1})\right)^{\alpha}\left(z_t(\mathbf{s}^t)\right)^{1-\alpha}\left(L_t(\mathbf{s}^t)\right)^{1-\alpha}\left(1+\theta_t(\mathbf{s}^t)\right) \\ & + (1-\phi)\gamma C_t(\mathbf{s}^t),\ \forall t,\mathbf{s}^t. \end{aligned}$$

Observe the following.

▷ We assume that each household/firm is too small to affect $\theta_t(\mathbf{s}^t)$.

▷ $q_0^t(\mathbf{s}^t)$ depends on the (aggregate) history of productivity shocks but not on idiosyncratic shocks for the workers.

▷ Given initial condition N_0, and the law of motion, $N_t(\mathbf{s}^t)$ is completely exogenous for the household.

▷ Unlike households, firms can choose $V_t(\mathbf{s}^t)$ (and $L_t(\mathbf{s}^t)$).

▷ The Nash bargaining problem has similar structure to before; i.e. a geometric average of the net benefit of match for workers and the net benefit for firms. Notice that if $\alpha = 0$, then we have the same coefficient on ϕ in the wages equation, except for the $(1+\theta_t(\mathbf{s}^t))$ term. Unlike in the linear model case, γ is the marginal rate of substitution between consumption and leisure (previously, this was the disutility of work/unemployment benefit).

Fact 2.1. *The competitive equilibrium solves the planner's problem (i.e. it is Pareto optimal) if and only if*

$$\phi = m_u(u,v)\frac{u}{m(u,v)};$$

i.e. competitive equilibrium is efficient if and only if the worker's relative bargaining power is equal to the elasticity of matching function with respect to unemployment.

The benefit of the competitive equilibrium is that household and the firm's problems do not pin down wages; i.e. we can alter condition (iii). Take the extreme case where

$$w_t\left(\mathbf{s}^t\right) = (1+g)^t w_0.$$

If we calibrate this model choosing w_0 to be the same as in case of Nash bargaining, then we obtain

$$\lambda_1 = 0.999,\ \lambda_2 = 0.98,\ \lambda_3 = 0.27$$

and

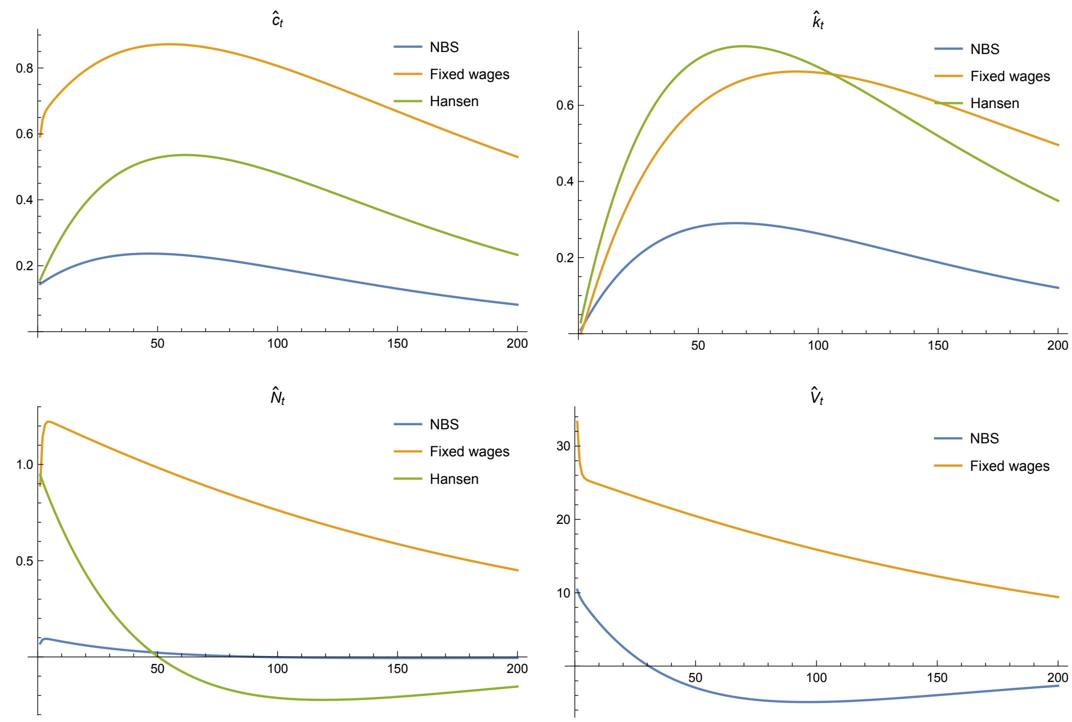
$$\begin{bmatrix} \hat{c}_t \\ \hat{V}_t \\ \hat{s}_{t+1} \\ \hat{k}_{t+1} \\ \hat{N}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.89 \\ 35.4 \\ 0 \\ 0.98 \\ 0.60 \end{bmatrix} \hat{k}_t + \begin{bmatrix} 0.18 \\ -20.1 \\ 0 \\ 0.017 \\ 0.30 \end{bmatrix} \hat{N}_t + \begin{bmatrix} 0.44 \\ 52.2 \\ 0.00 \\ 0.89 \end{bmatrix} \hat{s}_t. \tag{2.24}$$

Crucially, notice that the coefficient on $\hat{k}_t$ in $\hat{V}_t$ and $\hat{N}_{t+1}$ is now positive, whereas they were negative previously.

Suppose again that $\hat{s}_0 = 1$. Then, since productivity is high, employment increases and so does $\hat{V}$. With wages determined by Nash bargaining, higher productivity implies higher wages which dampens the firm's incentive to hire. However, when wages are fixed there is no such dampening effect. Thus, employment reacts much greater to productivity shocks. The increase employment also leads to higher consumption and capital, at least initially.

We can see from the impulse response below that fixed wages lead to greater volatility in employment. In fact, we will find that volatility in employment and consumption will be similar to output (as observed in the data).

Figure 2.3: Impulse response to $\hat{s}_0 = 1$.



2.5 Technical appendix

2.5.1 Poisson process

Definition 2.2. (*Counting process*) $\{N(t)\}_{t=0}^{\infty}$ is a countering process if:

- $\triangleright$ $N(t) \geq 0$;
- $\triangleright$ $N(t) \in \mathbb{Z}$;
- $\triangleright$ $t \geq s \Rightarrow N(t) \geq N(s)$;

For any $t > s$, then $N(t) - N(s)$ is the number of events/arrivals that occurred during the interval $[s, t]$.

Definition 2.3. (*Poisson process*) A counting process $\{N(t)\}_{t=0}^{\infty}$ is a *Poisson counting process* with rate $\lambda > 0$ if

- $\triangleright$ $N(0) = 0$;
- $\triangleright$ has independent increments;—i.e. the number of arrivals in the future interval $[s, t]$ does not depend on the number of arrivals to date: $N(t) \perp\!\!\!\perp N(t+s) - N(t)$;
- $\triangleright$ the number of events in any interval of length Δt is a Poisson random variable with parameter $\lambda \Delta t$—i.e. $N(s + \Delta t) - N(s) \sim \text{Poisson}(\lambda \Delta t)$.

The probability that n number of arrivals occur in an interval of length t (i.e. from period zero to period t) is given by

$$\mathbb{P}\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

The probability that at least one arrival occurs in a time period of duration t is

$$\lambda t + o(t).$$

The probability of an arrival occurring strictly more than once in an interval $[0,t]$ is given by

$$\mathbb{P}(N(t) \geq n) = o(t), \; n > 1.$$

As $t \to 0$, then this probability tends to zero.

Suppose we start from $t = 0$ and we wish to know the pdf for the time of occurrence of the first arrival. We define X to be the waiting time random variable. To obtain its pdf, we think of

$$\begin{aligned} f(x)\,dx &\equiv \mathbb{P}\{\text{first arrival occurs in the interval } x \text{ to } x+dx\} \\ &= \mathbb{P}\{\text{exactly 0 arrivas in the interval } [0,x] \text{ and exactly one arrival in } [x,x+dx]\} \\ &= \frac{(\lambda x)^0}{0!} e^{-\lambda t} (\lambda dx + o(dx)) \\ &= \lambda e^{-\lambda t} dx \\ \Rightarrow f(x) &= \lambda e^{-\lambda t}, \end{aligned}$$

where we ignore the $o(dx)$ term since it is small. Therefore, we realise that waiting time between arrivals of a Poisson counting process is exponentially distributed with mean $1/\lambda$.

2.5.2 Multiple independent Poisson processes

Suppose we have two independent Poisson processes with rate λ_1 and λ_2, and $N_1(t)$ and $N_2(t)$ are the respective numbers of events from period zero to t. Then, the combined process has a cumulative number of events equal to $N(t) = N_1(t) + N_2(t)$, and the combined process is also a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.

Since the processes are independent, the joint density is given by

$$\begin{aligned} f_{x_1,x_2}(x_1,x_2) &= f_{x_1}(x_1) f_{x_2}(x_2) \\ &= \left(\lambda_1 e^{-\lambda_1 x_1}\right)\left(\lambda_2 e^{-\lambda_2 x_2}\right) \\ &= \lambda_1 \lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2}. \end{aligned}$$

Hence, the probability that either an event from process 1 or 2 occurs is given by setting $x_1 = x_2 = x$,

$$f_{x_1,x_2}(x,x) = \lambda_1 \lambda_2 e^{-(\lambda_1+\lambda_2)x};$$

i.e. it has exponential distribution with mean $1/(\lambda_1 + \lambda_2)$.

Suppose we wish to know the probability that an event from process 1 occurs, conditional on at least one event from either of the processes. In other words, given two independently operating Poisson process with rate parameters λ_1 and λ_2, what is the probability that an arrival from process 1 occurs *before* an arrival from process 2? That is, we want to compute $\mathbb{P}(X_1, X_2)$,where X_1 and

X_2 denote the waiting time for the two processes. Then,

$$\begin{aligned}
\mathbb{P}\{X_1 < X_2\} &= \int_0^\infty dx_1 \int_{x_1}^\infty dx_2 \lambda_1 \lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2} \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} dx_1 \int_{x_1}^\infty \lambda_2 e^{-\lambda_2 x_2} dx_2 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} dx_1 \left[-e^{-\lambda_2 x_2}\right]_{x_1}^\infty \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} e^{-\lambda_2 x_1} dx_1 \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2) x_1} dx_1 \\
&= \left[-\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)x_1}\right]_0^\infty \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

2.5.3 Limiting approach for HJB equation

Suppose there are two states $s \in \{i, j\}$ and that we are currently in state i. The rate with which we move from state i to state j is given by λ. The probability that we move from state i to j in the interval Δt is then $\lambda \Delta t$ and the probability that we remain in state i is $1 - \lambda \Delta t$. Let c_i denote the payoff rate. The HJB equation is

$$v_{\Delta t}^i = c_i \Delta t + \frac{\lambda \Delta t}{1+\rho \Delta t} v_{\Delta t}^j + \frac{1-\lambda \Delta t}{1+\rho\Delta t} v_{\Delta t}^i + o(\Delta t).$$

Multiplying both sides by $1 + \rho\Delta t$ yields

$$\begin{aligned}
(1+\rho\Delta t) v_{\Delta t}^i &= (1+\rho\Delta t) c_i \Delta t + \lambda \Delta t v_{\Delta t}^j + (1-\lambda\Delta t) v_{\Delta t}^i + (1+\rho\Delta t) o(\Delta t) \\
\Rightarrow \rho \Delta t v_{\Delta t}^i &= (1+\rho\Delta t) c_i \Delta t + \lambda \Delta t v_{\Delta t}^j - \lambda \Delta t v_{\Delta t}^i + (1+\rho\Delta t) o(\Delta t).
\end{aligned}$$

Dividing through by Δt gives

$$\rho v_{\Delta t}^i = (1+\rho\Delta t) c_i + \lambda v_{\Delta t}^j - \lambda v_{\Delta t}^i + (1+\rho\Delta t)\frac{o(\Delta t)}{\Delta t}.$$

Then, letting $\Delta t \to 0$, we obtain the continuous-time version of the HJB:

$$\begin{aligned}
\rho v^i &= c_i + \lambda v^j - \lambda v^i \\
&= c_i + \lambda \left(v^j - v^i\right).
\end{aligned}$$

More generally, let the current state be i and that there are $j = 1, 2, \ldots, n$ states that we could move to, with the rate of transition given by λ_j. Then the continuous-time HJB is given by

$$\rho v^i = c_i + \sum_{j \neq i} \lambda_j \left(v^j - v_i\right).$$

Let $\tilde{v}^i$ be the expected value at $t = 0$ given $s(0) = i$, where the instantaneous payoff is denoted

c^i . Let

$$f(s(t)) = \begin{cases} c^i & \text{if } s(t) = i \\ c^j & \text{if } s(t) = j \end{cases}.$$

Then,

$$\tilde{v}^i = \mathbb{E}\left[\int_0^\infty e^{-\rho t} f(s(t))\, dt | s(0) = i\right].$$

We wish to show that $\tilde{v}^i$ satisfies the HJB equation. As before, let X be the waiting time for the first change of state from i to j. By law of iterated expectations

$$\begin{aligned}\tilde{v}^i &= \mathbb{E}\left[\mathbb{E}\left[\int_0^\infty e^{-\rho t} f(s(t))\, dt | X = \tilde{t}\right]\right] \\ &= \mathbb{E}\left[\int_0^{\tilde{t}} e^{-\rho t} c^i dt + \mathbb{E}\left[\int_{\tilde{t}}^\infty e^{-\rho t} f\left(s\left(t - \tilde{t}\right)\right) dt | X = \tilde{t}\right]\right].\end{aligned}$$

Note that, by construction, $s\left(\tilde{t}\right) = j$. We also have $t - \tilde{t}$ since we reset the initial period to $\tilde{t}$ (instead of 0) (we are assuming an underlying Poisson process so that probability of switching state is only dependent on the time elapsed from the initial period). We now apply the change of formula on the second part. Let

$$\begin{aligned} u &= t - \tilde{t}, \\ \Rightarrow t &= u + \tilde{t} \\ du &= dt. \end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}_{\tilde{t}}\left[\int_{\tilde{t}}^\infty e^{-\rho t} f\left(s\left(t - \tilde{t}\right)\right) dt | s\left(\tilde{t}\right) = j\right] &= \mathbb{E}_{\tilde{t}}\left[\int_0^\infty e^{-\rho\left(u + \tilde{t}\right)} f(s(u))\, du | X = \tilde{t}\right] \\ &= \mathbb{E}_{\tilde{t}}\left[e^{-\rho\tilde{t}} \int_0^\infty e^{-\rho u} f(s(u))\, du | X = \tilde{t}\right] \\ &= \mathbb{E}_{\tilde{t}}\left[e^{-\rho\tilde{t}} \tilde{v}^j | s\left(\tilde{t}\right) = j | X = \tilde{t}\right].\end{aligned}$$

We assume that the transition of state is given by a Poisson process with rate λ. Thus,

$$
\begin{aligned}
\tilde{v}^i &= \mathbb{E}\left[\int_0^{\tilde{t}} e^{-\rho t} c^i dt + \mathbb{E}\left[e^{-\rho\tilde{t}}\tilde{v}^j | X = \tilde{t}\right]\right] \\
&= \mathbb{E}\left[\int_0^{\tilde{t}} e^{-\rho t} c^i dt + e^{-\rho\tilde{t}}\tilde{v}^j\right] \\
&= \int_0^{\infty}\left[\left(\int_0^{\tilde{t}} e^{-\rho t} c^i dt\right) + e^{-\rho\tilde{t}}\tilde{v}^j\right]\lambda e^{-\lambda\tilde{t}} d\tilde{t} \\
&= \int_0^{\infty} c^i \left(\int_0^{\tilde{t}} e^{-\rho t} dt\right)\lambda e^{-\lambda\tilde{t}} d\tilde{t} + \int_0^{\infty} e^{-\rho\tilde{t}}\tilde{v}^j \lambda e^{-\lambda\tilde{t}} d\tilde{t} \\
&= \int_0^{\infty} c^i \left[-\frac{1}{\rho}e^{-\rho t}\right]_0^{\tilde{t}} \lambda e^{-\lambda\tilde{t}} d\tilde{t} + \tilde{v}^j\lambda \int_0^{\infty} e^{-(\rho+\lambda)\tilde{t}} d\tilde{t} \\
&= \int_0^{\infty} c^i \left(-\frac{1}{\rho}e^{-\rho\tilde{t}} + \frac{1}{\rho}\right)\lambda e^{-\lambda\tilde{t}} d\tilde{t} + \tilde{v}^j\lambda\left[-\frac{1}{\rho+\lambda}e^{-(\rho+\lambda)\tilde{t}}\right]_0^{\infty} \\
&= \int_0^{\infty} \frac{c^i}{\rho}\left(1 - e^{-\rho\tilde{t}}\right)\lambda e^{-\lambda\tilde{t}} d\tilde{t} + \frac{\lambda}{\rho+\lambda}\tilde{v}^j \\
&= \frac{c^i}{\rho}\underbrace{\int_0^{\infty} \lambda e^{-\lambda\tilde{t}} d\tilde{t}}_{=1} - \frac{c^i}{\rho}\lambda \int_0^{\infty} e^{-(\rho+\lambda)\tilde{t}} d\tilde{t} + \frac{\lambda}{\rho+\lambda}\tilde{v}^j \\
&= \frac{c_i}{\rho} - \frac{c_i}{\rho}\frac{\lambda}{\rho+\lambda} + \frac{\lambda}{\rho+\lambda}\tilde{v}^j = \frac{c_i}{\rho}\left(1 - \frac{\lambda}{\rho+\lambda}\right) + \frac{\lambda}{\rho+\lambda}\tilde{v}^j \\
&= \frac{c_i}{\rho}\left(\frac{\rho}{\rho+\lambda}\right) + \frac{\lambda}{\rho+\lambda}\tilde{v}^j \\
&= \frac{c_i}{\rho+\lambda} + \frac{\lambda}{\rho+\lambda}\tilde{v}^j.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\rho+\lambda)\,\tilde{v}^i &= c_i + \lambda\tilde{v}^j \\
\Leftrightarrow \rho\tilde{v}^i &= c_i + \lambda\left(\tilde{v}^j - \tilde{v}^i\right).
\end{aligned}
$$

Suppose now that there are three states and value functions denotes v^1, v^2 and v^3. From state 1, we move to state 2 and 3 according to independent Poisson processes with rates λ_2 and λ_3

respectively. Consider

$$\begin{aligned}
v^1 &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} f\left(s\left(t\right)\right) dt | s\left(0\right) = 1\right] \\
&= \mathbb{E}\left[\int_0^{\tilde{t}} e^{-\rho t} c^1 dt + \frac{\lambda_2}{\lambda_2+\lambda_3}\mathbb{E}\left[\int_{\tilde{t}}^\infty e^{-\rho t} f\left(s\left(t-\tilde{t}\right)\right) dt | s\left(\tilde{t}\right) = 2\right]\right. \\
&\quad \left. + \frac{\lambda_3}{\lambda_2+\lambda_3}\mathbb{E}\left[\int_{\tilde{t}}^\infty e^{-\rho t} f\left(s\left(t-\tilde{t}\right)\right) dt | s\left(\tilde{t}\right) = 3\right]\right] \\
&= \frac{c^1}{\rho+\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_2+\lambda_3}\left[\int_0^\infty e^{-(\lambda_2+\lambda_3)\tilde{t}} e^{-\rho\tilde{t}} v^2 \left(\lambda_2+\lambda_3\right) d\tilde{t}\right] \\
&\quad + \frac{\lambda_3}{\lambda_2+\lambda_3}\left[\int_0^\infty e^{-(\lambda_2+\lambda_3)\tilde{t}} e^{-\rho\tilde{t}} v^3 \left(\lambda_2+\lambda_3\right) d\tilde{t}\right] \\
&= \frac{c^1}{\rho+\lambda_2+\lambda_3} + \lambda_2 \frac{v^2}{\rho+\lambda_2+\lambda_3} + \lambda_3 \frac{v^3}{\rho+\lambda_2+\lambda_3}.
\end{aligned}$$

Rearranging gives

$$\rho v^1 = c^1 + \lambda_2\left(v^2 - v^1\right) + \lambda_3\left(v^3 - v^1\right).$$

3 New Keynesian Model

New Keynesian model introduces price stickiness and monopolistic competition into the RBC Model and, unlike in the Neoclassical Growth Model, features what can be interpreted as aggregate demand. We will first consider the fully flexible prices model before introducing sticky prices a la Calvo in the model.

3.1 Flexible prices

3.1.1 The set up

We assume representative household that consumes differentiated goods produced by measure one of heterogeneous firms, indexed by $j \in [0,1]$, that produces a continuum of goods $c_{j,t}$ in period t. Monopolistic competition among the firms imply that they charge a price, $p_{j,t}$, with a mark up over their marginal cost to the household. We assume that there is no capital and production occurs with labour as the only input. We also begin by assuming that firms are unrestricted in their ability to change prices; i.e. flexible prices. Although we do not introduce money explicitly, we consider the model in nominal terms.

3.1.2 The household problem

The infinitely lived household seeks to solve the following problem:

$$\begin{aligned} \max_{\{C_t, c_{j,t}, H_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t U\left(C_t, H_t\right) \\ s.t. \quad A_0 &= \sum_{t=0}^{\infty} Q_0^t \left(\int_0^1 p_{j,t} c_{j,t} dj - W_t H_t \right) \\ & A_0, \left\{Q_0^t, W_t, p_{j,t}\right\}_{t=0}^{\infty} \text{ given,} \end{aligned} \tag{3.1}$$

where

- $\triangleright$ $c_{j,t}$ denote household's consumption of goods produced by firm j in period t;
- $\triangleright$ C_t is the aggregate consumption index given by

$$C_t := \left(\int_0^1 \left(c_{j,t}\right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}, \tag{3.2}$$

where $\eta > 1$ is the elasticity of substitution between goods.[18] Since a higher η means that goods are more substitutable, as $\eta \to \infty$ is the case in which goods are perfect substitutes. On the other hand, as $\eta \to 1$, consumers would want to diversify their consumption as much

[18]To see this, first note that

$$\begin{aligned} U_{c_j} := \frac{\partial U\left(C,H\right)}{\partial c_j} &= U_C\left(C,H\right) \frac{\partial C}{\partial c_j} \\ &= U_C\left(C,H\right) \left(\int_0^1 \left(c_{j,t}\right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{1}{\eta-1}} c_j^{-\frac{1}{\eta}} = U_C\left(C,H\right) C_t^{\frac{1}{\eta}} c_j^{-\frac{1}{\eta}}. \end{aligned}$$

as possible.[19]

- ▷ H_t is the hours worked by the household;
- ▷ Q_0^t gives the period-0 worth of a dollar in period t;
- ▷ W_t is the nominal wage.

In solving its problem, the household takes as given their initial level of wealth, A_0, prices of goos in each period, $\{p_{j,t}\}_{j\in[0,1],t=0,\ldots,\infty}$, nominal wages $\{W_t\}_{t=0}^{\infty}$ as well as Arrow-Debreu prices $\{Q_0^t\}_{t=0}^{\infty}$.

We can substitute the definition of the aggregate consumption index into the objective function and write the Lagrangian as

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U\left(\left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}}\, dj\right)^{\frac{\eta}{\eta-1}}, H_t\right) \\ + \lambda\left(A_0 - \sum_{t=0}^{\infty} Q_0^t\left(\int_0^1 p_{j,t}c_{j,t}dj - W_t H_t\right)\right).\end{aligned}$$

The first-order condition with respect to $c_{j,t}$ is given by

$$\begin{aligned}\beta^t U_C\left(C_t, H_t\right)\left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}}\, dj\right)^{\frac{1}{\eta-1}} c_{j,t}^{-\frac{1}{\eta}} &= \lambda Q_0^t p_{j,t} \\ \Leftrightarrow \beta^t U_C\left(C_t, H_t\right) C_t^{\frac{1}{\eta}} c_{j,t}^{-\frac{1}{\eta}} &= \lambda Q_0^t p_{j,t}.\end{aligned} \tag{3.3}$$

Let $j = 0$ be the numeraire, then dividing the first-order condition with respect to $c_{j,t}$ by that with respect to $c_{0,t}$ yields

$$\begin{aligned}\frac{c_{j,t}^{-\frac{1}{\eta}}}{c_{0,t}^{-\frac{1}{\eta}}} = \frac{p_{j,t}}{p_{0,t}} \Leftrightarrow c_{0,t}^{\frac{1}{\eta}} c_{j,t}^{-\frac{1}{\eta}} &= \frac{p_{j,t}}{p_{0,t}} \\ \Leftrightarrow c_{j,t} = \left(\frac{p_{j,t}}{p_{0,t}} c_{0,t}^{-\frac{1}{\eta}}\right)^{-\eta} &= c_{0,t} p_{0,t}^{\eta} p_{j,t}^{-\eta}.\end{aligned} \tag{3.4}$$

The elasticity of substitution between good j and k is given by

$$\begin{aligned}\frac{d\ln\left(c_j/c_k\right)}{d\ln\left(MRS_{k,j}\right)} = \frac{d\ln\left(c_j/c_k\right)}{d\ln\left(U_{c_k}/U_{c_j}\right)} &= \frac{d\ln\left(c_j/c_k\right)}{d\ln\left(c_k^{-1/\eta}/c_j^{-1/\eta}\right)} \\ = \frac{d\ln\left(c_j/c_k\right)}{d\ln\left(\left(c_j/c_k\right)^{1/\eta}\right)} &= \frac{d\ln\left(c_j/c_k\right)}{\frac{1}{\eta} d\ln\left(c_j/c_k\right)} = \eta.\end{aligned}$$

[19] Write:

$$\begin{aligned}C_t &= \exp\left[\frac{\eta}{\eta-1}\ln\left(\int_0^1 (c_{j,t})^{\frac{\eta-1}{\eta}}\, dj\right)\right] \\ &= \exp\left[\frac{\eta}{\eta-1}\ln\left(\int_0^1 \exp\left[\frac{\eta-1}{\eta}\ln\left(c_{j,t}\right)\right] dj\right)\right]\end{aligned}$$

Define, $t := \eta - 1$ so that $\eta \to 1 \Leftrightarrow t \to 0$, and

$$C_t = \exp\left[\frac{1+t}{t}\ln\left(\int_0^1 \exp\left[\frac{t}{1+t}\ln\left(c_{j,t}\right)\right] dj\right)\right].$$

Consider

$$\lim_{t\to 0}\exp\left[\frac{t}{1+t}\ln\left(c_{j,t}\right)\right] =$$

??

Substituting this into (3.2), we can write the aggregate consumption index as

$$C_t = \left(\int_0^1 \left(p_{j,t}^{-\eta} p_{0,t}^{\eta} c_{0,t} \right)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}$$
$$= \left(\int_0^1 p_{j,t}^{-(\eta-1)} p_{0,t}^{\eta-1} c_{0,t}^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}.$$

But since $p_{0,t}$ and $c_{0,t}$ does not depend on j, we can take it outside the integral so that

$$C_t = c_{0,t} p_{0,t}^{\eta} \left(\int_0^1 p_{j,t}^{1-\eta} dj \right)^{\frac{\eta}{\eta-1}}. \tag{3.5}$$

Now consider the household's expenditure on consumption goods. Substituting out $c_{j,t}$ using (3.4) gives

$$\int_0^1 p_{j,t} c_{j,t} dj = \int_0^1 p_{j,t} c_{0,t} p_{0,t}^{\eta} p_{j,t}^{-\eta} dj = c_{0,t} p_{0,t}^{\eta} \int_0^1 p_{j,t}^{1-\eta} dj. \tag{3.6}$$

Define the *ideal price index* as

$$P_t := \left(\int_0^1 p_{j,t}^{1-\eta} dj \right)^{\frac{1}{1-\eta}}. \tag{3.7}$$

To see the reasoning for naming this an ideal price index, using P_t, we can write (3.5) and (3.6) as

$$C_t = c_{0,t} p_{0,t}^{\eta} P_t^{-\eta} \Leftrightarrow C_t P_t^{\eta} = c_{0,t} p_{0,t}^{\eta} \tag{3.8}$$
$$\int_0^1 p_{j,t} c_{j,t} dj = c_{0,t} p_{0,t}^{\eta} P_t^{1-\eta}.$$

Combining the two, we realise that

$$\int_0^1 p_{j,t} c_{j,t} dj = C_t P_t^{\eta} P_t^{1-\eta} = C_t P_t. \tag{3.9}$$

That is, total expenditure on goods by the household (i.e. the left-hand side) equals the product of aggregate consumption index by the ideal price index.

Recall that the choice of good c_0 was arbitrary, so (3.8), in fact, holds for all $j \in [0,1]$; i.e.

$$c_{j,t} = C_t \left(\frac{P_t}{p_{j,t}} \right)^{\eta}, \ \forall j \in [0,1]. \tag{3.10}$$

This gives us the demand function for each good j.

We can also simplify the first-order condition (3.3) using (3.10). First, rewrite (3.3) as

$$\beta^t U_C \left(C_t, H_t \right) C_t^{\eta} = \lambda Q_0^t \left(c_{j,t} p_{j,t}^{\eta} \right)^{\frac{1}{\eta}},$$

then substituting in (3.10) yields

$$\beta^t U_C \left(C_t, H_t \right) C_t^{\frac{1}{\eta}} = \lambda Q_0^t \left(C_t P_t^{\eta} \right)^{\frac{1}{\eta}}$$
$$\Leftrightarrow \beta^t U_C \left(C_t, H_t \right) = \lambda Q_0^t P_t.$$

This expression should be familiar from the RBC model (decentralised) in which we had $q_0^t \equiv Q_0^t P_t$.

Dividing the first-order condition with respect to $c_{j,t}$ by that respect to $c_{j,t+1}$ yields

$$\frac{\beta^t U_C(C_t, H_t)}{\beta^{t+1} U_C(C_{t+1}, H_{t+1})} = \frac{Q_0^t P_t}{Q_0^{t+1} P_t^{t+1}}.$$

Rearranging this gives the familiar Euler equation:

$$U_C(C_t, H_t) = \beta \frac{Q_0^t P_t}{Q_0^{t+1} P_t^{t+1}} U_C(C_{t+1}, H_{t+1}).$$

The first-order condition with respect to H_t is

$$-\beta^t U_H(C_t, H_t) = \lambda Q_0^t W_t$$

and we can obtain the usual intratemporal condition by dividing above by the first-order condition with respect to $c_{j,t}$:

$$\frac{-U_H(C_t, H_t)}{U_C(C_t, H_t)} = \frac{W_t}{P_t}.$$

The condition equates the marginal rate of substitution between consumption and leisure (i.e. the left-hand side) with the real wage (i.e. the right-hand side).

HH problem

To summarise, we have obtained the following optimality conditions in aggregate variables:

$$U_C(C_t, H_t) = \beta \frac{Q_0^t P_t}{Q_0^{t+1} P_{t+1}} U_C(C_{t+1}, H_{t+1}), \tag{3.11}$$

$$\frac{-U_H(C_t, H_t)}{U_C(C_t, H_t)} = \frac{W_t}{P_t}, \tag{3.12}$$

and the demand curve

$$c_{j,t} = C_t \left(\frac{P_t}{p_{j,t}}\right)^{\eta}, \ \forall j \in [0,1]. \tag{3.13}$$

Remark 3.1. That we can obtain optimality conditions in aggregate variables is clear if we substitute (3.9) into the household problem.

3.1.3 The firm problem

We assume that there is a continuum of firms such that each firm $j \in [0,1]$ takes as given P_t when choosing $p_{j,t}$. Note that we are implicitly assuming that the set of products are the same over time. The firm's period-t problem is statistic (there is no capital here) and is given by

$$\begin{aligned} v_t^j := \max_{\{p_{j,t} y_{j,t} c_{j,t} h_{j,t}\}} \quad & p_{j,t} y_{j,t} - W_t h_{j,t} \\ s.t. \quad & y_{j,t} = z_t h_{j,t} \\ & y_{j,t} = c_{j,t} = \frac{C_t P_t^{\eta}}{p_{j,t}^{\eta}} \\ & \{W_t, P_t, C_t\} \text{ given.} \end{aligned} \tag{3.14}$$

The first constraint is the production function where we assume aggregate productivity, z_t (i.e. has no j subscript), and the first equality in the second constraint is the market clearing condition. Using the constraints, we can write the objective function in terms of $p_{j,t}$'s (and C_t, P_t) only:

$$\begin{aligned} p_{j,t}y_{j,t} - W_t \frac{y_{j,t}}{z_t} &= p_{j,t} \frac{C_t P_t^\eta}{p_{j,t}^\eta} - \frac{W_t}{z_t} \frac{C_t P_t^\eta}{p_{j,t}^\eta} \\ &= C_t P_t^\eta \left(p_{j,t}^{1-\eta} - \frac{W_t}{z_t} p_{j,t}^{-\eta} \right). \end{aligned} \tag{3.15}$$

The first-order condition is given by[20]

$$\begin{aligned} 0 &= (1-\eta)\, p_{j,t}^{-\eta} + \eta \frac{W_t}{z_t} p_{j,t}^{-\eta-1} \\ &= (1-\eta) + \eta \frac{W_t}{z_t} p_{j,t}^{-1} \\ \Leftrightarrow p_{j,t} &= \frac{\eta}{\eta-1} \frac{W_t}{z_t}. \end{aligned} \tag{3.16}$$

Note that W_t/z_t is the marginal cost of production and $\eta/(\eta-1)$ is the mark-up over the marginal cost. We see that as $\eta \to \infty$, $p_{j,t} \to W_t/z_t$ so that we get the perfect competition outcome. In contrast as $\eta \downarrow 1$, we see that the firms would charge arbitrarily high prices.

Since the right-hand side of (3.16) does not depend on j, we realise that all firms charge the same price. This implies that

$$P_t = \left(\int_0^1 p_{j,t}^{1-\eta} dj \right)^{\frac{1}{1-\eta}} = p_{j,t}$$

and so

$$P_t \coloneqq \frac{\eta}{\eta-1} \frac{W_t}{z_t}, \ \forall j \in [0,1].$$

Then we can write the real wage as

$$\frac{W_t}{P_t} = z_t \frac{\eta-1}{\eta} < z_t.$$

Thus, we see that there is a wedge, $(\eta-1)/\eta$, between the real wage and the marginal product of labour.

Labour market clearing condition is:

$$H_t = \int_0^1 h_{j,t} dj.$$

[20] The second derivative of the objective function is

$$\begin{aligned} &C_t P_t \left(\eta(\eta-1) p_{j,t}^{-\eta-1} - (\eta+1)\eta \frac{W_t}{z_t} p_{j,t}^{-\eta-2} \right) \\ =&C_t P_t p_{j,t}^{-\eta-1} \eta \left((\eta-1) - (\eta+1) \frac{W_t}{z_t} p_{j,t}^{-1} \right). \end{aligned}$$

Evaluating this at the optimum,

$$C_t P_t p_{j,t}^{-\eta-1} \eta \left((\eta-1) - \frac{(\eta+1)(\eta-1)}{\eta} \right) = C_t P_t p_{j,t}^{-\eta-1} \eta (\eta-1) \left(\frac{-1}{\eta} \right) \leq 0.$$

Hence, we conclude that (3.16) is a global maximum (since we are maximising with respect to a single variable).

Using the production function and the consumer's demand function, we can write

$$H_t = \int_0^1 \frac{C_t P_t^\eta}{p_{j,t}^\eta z_t} dj = \int_0^1 \frac{C_t P_t^\eta}{P_t^\eta z_t} dj = \frac{C_t}{z_t}.$$

Rearranging this gives the aggregate production in the economy:

$$C_t = z_t H_t.$$

FF problem

To summarise, we have obtained the following optimality conditions in aggregate variables:

$$\frac{W_t}{P_t} = z_t \frac{\eta - 1}{\eta}, \tag{3.17}$$

$$P_t = p_{j,y}, \forall j \in [0,1]. \tag{3.18}$$

Labour market clearing

$$C_t = z_t H_t. \tag{3.19}$$

3.1.4 Equilibrium

An equilibrium here is a sequence of allocations

$$\left\{C_t, H_t, \{c_{t,j}, h_{j,t}, y_{j,t}\}_{j\in[0,1]}\right\}_{t=0}^{\infty}$$

and a sequence of prices

$$\left\{P_t, Q_0^t, W_t, \{p_{t,j}\}_{j\in\{0,1\}}\right\}$$

and initial conditions A_0 such that: (i) household solves the problem (3.1); (ii) firms solve the problem (3.14) in all periods; and (iii) markets clear:

$$H_t = \int_0^1 h_{j,t} dj, \ \forall t = 1, \dots, \infty,$$

$$y_{j,t} = c_{j,t}, \ \forall j \in [0,1], t = 1, \dots, \infty,$$

$$A_0 = \sum_{t=0}^{\infty} Q_0^t \int_0^1 v_t^j dj.$$

3.1.5 Labour wedge

Let us assume that

$$U(C,H) \coloneqq \log C - v(H). \tag{3.20}$$

Using (3.12), (3.17) and (3.19), we can write

$$z_t H_t v'(H_t) = C_t v'(H_t) = \frac{W_t}{P_t} = z_t \frac{\eta - 1}{\eta},$$

which simplifies to

$$H_t v'(H_t) = \frac{\eta - 1}{\eta}.$$

In the social planner's problem, the left-hand side would equal one (which coincides with the $\eta \to \infty$ case) so we can interpret $(\eta - 1)/\eta$ as a labour wedge. In this set up, however, since η is a fixed parameter, this labour wage does not affect cyclicality. For η (the elasticity of demand) or equivalently $(\eta - 1)/\eta$ (mark up over the marginal cost charges by firms) to affect business cycles, they must vary cyclically. One rationale is that, in times of recession, due to some firms exiting, market power of remaining firms may increase (and the opposite in times of booms).

3.1.6 Classical dichotomy and money neutrality

An economy exhibits the classical dichotomy if real variables such as output and real interest rates can be completely analysed without considering what is happening to their nominal counterparts, the money value of output and the interest rate. In particular, this means that real GDP and other real variables can be determined without knowing the level of the nominal money supply or the rate of inflation. An economy exhibits the classical dichotomy if money is neutral, affecting only the price level, not real variables.

If we have H_t, C_t and Q_0^t, the model gives us $P_{t+1}/P_t = \pi_t$ (inflation). Thus, everything is identified up to a a level of prices (to pin down price level, we would need an initial condition for the price level). In other words, price level itself has no affect on the optimality conditions and thus "money" is neutral in the model.

3.2 Calvo model

We now introduce price stickiness into the model by adding some friction into the firm's ability to change prices. There are two main methods:

Time dependent each firm's ability to change prices depends only upon time;

State dependent each firm's ability to change prices depends only upon time as well as the relative price between its price and market price.

We focus on the former. Specifically, we assume that, in each period, each firm has probability $1-\theta$ of being able to readjust its prices, and with probability θ, the firm must maintain the price from the previous period.

In closing the model, we would need initial conditions that tells us the distribution of prices at the beginning; however, since there are infinitely many firms in the model, and we would need initial conditions for each firm, we would have infinitely many initial conditions. Since prices are persistent, we would also have to have infinitely many state variables.

To overcome this problem with infinite dimensions, we explore the Calvo model of price stickiness.

3.2.1 Firm's problem

We first note that introducing Calvo pricing does not alter the household's problem. Thus, the demand function (3.13) remains the same as before. Firms that are not graced with the "Calvo fairy" are unable to change prices. So we only consider the lucky firms' problem in period t, which is given by

$$\begin{aligned}
\max_{\{\{c_{j,t'},h_{j,t'}\}_{t'=t}^{\infty},p_{j,t}\}} \quad & \sum_{t'=t}^{\infty}\frac{Q_0^{t'}}{Q_0^t}\theta^{t'-t}\left(p_{j,t}c_{j,t'}-W_{t'}h_{j,t'}\right) \\
s.t. \quad & c_{j,t'}=C_{t'}\left(\frac{P_{t'}}{p_{j,t}}\right),\ \forall t'\geq t, \\
& c_{j,t'}=z_{t'}h_{j,t'},\ \forall t'\geq t, \\
& \left\{P_{t'},Q_0^{t'},C_{t'},W_{t'}\right\}_{t=t'}^{\infty}\ \text{given.}
\end{aligned}$$

Recall that θ is the probability that the firm is unable to change its price in any period. Thus, if the firm changes its price in period t, with probability θ, in period $t+1$, the firm must maintain the same price; i.e. $p_{j,t+1}=p_{j,t}$ with probability θ. Now, in period $t+1$, with probability $1-\theta$ is able to change its price and thus there is a continuation value associated with this event. However, in our set up, since $p_{j,t}$ only affects demand today, we do not need to include the continuation value in the objective function. In other words, the problem is the same whenever the firm is able to change its price. We can substitute the constraint, which is the demand functions in periods $t'=t,t+2,\ldots$, and the production function to write the objective function as

$$\begin{aligned}
\sum_{t'=t}^{\infty}\frac{Q_0^{t'}}{Q_0^t}\theta^{t'-t}\left(p_{j,t}c_{j,t'}-W_{t'}h_{j,t'}\right) &= \sum_{t'=t}^{\infty}\frac{Q_0^{t'}}{Q_0^t}\theta^{t'-t}\left(p_{j,t}C_{t'}\left(\frac{P_{t'}}{p_{j,t}}\right)-\frac{W_{t'}}{z_{t'}}C_{t'}\left(\frac{P_{t'}}{p_{j,t}}\right)\right) \\
&= \sum_{t'=t}^{\infty}\frac{Q_0^{t'}}{Q_0^t}\theta^{t'-t}C_{t'}P_{t'}^{\eta}\left(p_{j,t}^{1-\eta}-\frac{W_{t'}}{z_{t'}}p_{j,t}^{-\eta}\right).
\end{aligned}$$

Observe that as $\theta\to 0$, since $0^1,0^2,\ldots=0$ and $0^0=1$, the objective function tends to static problem (3.15).

The first-order condition is given by:[21]

$$\begin{aligned}
0 &= \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \left[(1-\eta) p_{j,t}^{-\eta} + \eta \frac{W_{t'}}{z_{t'}} p_{j,t}^{-\eta-1} \right] \\
&= \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \left[(1-\eta) + \eta \frac{W_{t'}}{z_{t'}} p_{j,t}^{-1} \right] \\
&= (1-\eta) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} + \frac{\eta}{p_{j,t}} \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}} \\
p_{j,t} &= \frac{\eta}{\eta-1} \frac{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}}}{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta}}. \qquad (3.21)
\end{aligned}$$

Since the right-hand side does not feature j, we realise that every firm that changes its price in period t changes to the same price. Define

$$P_t^* := \frac{\eta}{\eta-1} \frac{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}}}{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta}}.$$

3.2.2 Law of motion for the ideal price index

Recall (3.7), which can be written as

$$P_{t+1}^{1-\eta} = \int_0^1 p_{j,t+1}^{1-\eta} dj = \mathbb{E}\left[p_{j,t+1}\right]$$

We know that $1-\theta$ proportion of the firms will change their price to P_{t+1}^*, and θ proportion of firms will keep their price the same as in period t. Since the firms that are allowed to change prices are chosen randomly, the distribution of prices among those that can change prices is the same as the distribution of overall prices in the previous period; i.e. the average price among the θ proportion of firms that must keep their price has the same average price as the average price in the previous period.

[21] The second derivative of the objective function is

$$\begin{aligned}
&\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \eta \left[(\eta-1) p_{j,t}^{-\eta-1} - (\eta+1) \frac{W_{t'}}{z_{t'}} p_{j,t}^{-\eta-2} \right] \\
=&\eta p_{j,t}^{-\eta-1} \left((\eta-1) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} - p_{j,t}^{-1} (\eta+1) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}} \right)
\end{aligned}$$

Evaluating this at the optimum,

$$\begin{aligned}
&\eta p_{j,t}^{-\eta-1} \left((\eta-1) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} - \frac{(\eta+1) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}}}{\frac{\eta}{\eta-1} \left(\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_t P_t^{\eta} \frac{W_{t'}}{z_{t'}} \right) / \left(\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_t P_t^{\eta} \right)} \right) \\
=&\eta p_{j,t}^{-\eta-1} \left((\eta-1) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} - \frac{\eta(\eta+1)}{\eta-1} \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_t P_t^{\eta} \right) \\
=&\eta(\eta-1) p_{j,t}^{-\eta-1} \left(\frac{-1}{\eta} \right) \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_t P_t^{\eta} \leq 0.
\end{aligned}$$

Hence, we conclude that (3.21) is a global maximum (since we are maximising with respect to a single variable).

Hence, we can write the law of motion for the ideal price (to the power of $1-\eta$) as:

$$\begin{aligned} P_{t+1}^{1-\eta} &= \mathbb{E}\left[p_{j,t+1}^{1-\eta}\right] \\ &= \mathbb{E}\left[p_{j,t+1}^{1-\eta}|\text{price change}\right]\mathbb{P}\left(\text{price change}\right) \\ &\quad + \mathbb{E}\left[p_{j,t+1}^{1-\eta}|\text{no price change}\right]\mathbb{P}\left(\text{no price change}\right) \\ &= \left(P_{t+1}^{*}\right)^{1-\eta}\left(1-\theta\right) + P_t^{1-\eta}\theta. \end{aligned}$$

This tells us that we must know the $1-\eta$th moment of the prices, i.e. P_t, as one of the state variables.

3.2.3 Law of motion for the production price index

Recall the labour market clearing condition:

$$H_t = \int_0^1 h_{j,t}dj.$$

We can replace $h_{j,t}$ using the production function, goods market clearing condition, and the consumer's demand function to write above as

$$H_t = \int_0^1 \frac{C_t P_t^{\eta}}{z_t p_{j,t}^{\eta}}dj = \frac{C_t P_t^{\eta}}{z_t}\int_0^1 p_{j,t}^{-\eta}dj. \tag{3.22}$$

We define the *production price index*,[22] $\tilde{P}_t$, so as to satisfy

$$\tilde{P}_t^{-\eta} \equiv \int_0^1 p_{j,t}^{-\eta}dj.$$

By the same argument as in the case for the ideal price index, we can write the law of motion as

$$\tilde{P}_{t+1}^{-\eta} = \left(P_{t+1}^{*}\right)^{-\eta}\left(1-\theta\right) + \tilde{P}_t^{-\eta}\theta,$$

which tells us that, in addition to the $1-\eta$th moment, we also need to know the $-\eta$th moment of the prices, $\tilde{P}_t$, as another state variable.

We can then write (3.22) as

$$H_t = \frac{C_t}{z_t}\left(\frac{P_t}{\tilde{P}_t}\right)^{\eta}.$$

We can see from this that if $P_t = \tilde{P}_t$, then the aggregate labour supply multiplied by labour productivity, $H_t z_t$, would equal aggregate demand. But as the claim below shows, in general the two are not equal such that aggregate demand is less than what could potentially be produced ($H_t z_t$). We can think of the difference as the cost of inefficiencies arising from price stickiness since production is being distorted towards goods with lower prices.

Claim 3.1. $P_t \geq \tilde{P}_t$ and $P_t = \tilde{P}_t$ if and only if the price distribution is degenerate.

Proof. Define

$$\Phi\left(p\right) = p^{\frac{\eta}{\eta-1}}.$$

[22] Unlike in the case of the ideal price index, there is no deep rationale for calling $\tilde{P}_t$ the production price index, except that it appears on the production side.

Since $\eta > 1$, $\eta/(\eta-1) > 1$ so that Φ is a strictly convex function. Using Φ , we can write

$$\tilde{P}_t^{-\eta} = \int_0^1 p_{j,t}^{-\eta} dj = \int_0^1 \Phi\left(p_{j,t}^{1-\eta}\right) dj = \mathbb{E}\left[\Phi\left(p_{j,t}^{1-\eta}\right)\right].$$

By Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\left[\Phi\left(p_{j,t}^{1-\eta}\right)\right] &\geq \Phi\left(\mathbb{E}\left[p_{j,t}^{1-\eta}\right]\right) \\
&= \Phi\left(\int_0^1 p_{j,t}^{1-\eta} dj\right) = \Phi\left(P_t^{1-\eta}\right) = P_t^{-\eta}.
\end{aligned}$$

Since Φ is strictly convex, the inequality above holds with equality if and only if $p_{j,t}$ has a degenerate distribution. ■

3.2.4 Closing the model

Let us assume the utility function is given by (3.20). We now have the following six equations:

▷ Intratemporal condition for the household:

$$C_t v'(H_t) = \frac{W_t}{P_t}. \tag{3.23}$$

▷ Euler equation for the household:

$$\frac{1}{C_t} = \beta \frac{Q_0^t P_t}{Q_0^{t+1} P_t^{t+1}} \frac{1}{C_{t+1}}. \tag{3.24}$$

▷ Optimal repricing for firms:

$$P_t^* := \frac{\eta}{\eta-1} \frac{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}}}{\sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta}}. \tag{3.25}$$

▷ Law of motion for the ideal price index:

$$P_{t+1}^{1-\eta} = \left(P_{t+1}^*\right)^{1-\eta}(1-\theta) + P_t^{1-\eta}\theta \tag{3.26}$$

▷ Law of motion for the production price index:

$$\tilde{P}_{t+1}^{-\eta} = \left(P_{t+1}^*\right)^{-\eta}(1-\theta) + \tilde{P}_t^{-\eta}\theta \tag{3.27}$$

▷ Labour market clearing:

$$H_t = \frac{C_t}{z_t}\left(\frac{P_t}{\tilde{P}_t}\right)^{\eta}. \tag{3.28}$$

The state variables are P_t and $\tilde{P}_t$, and the unknowns are:

$$\left\{C_t, H_t, W_t, P_t, \tilde{P}_t, P_t^*, Q_0^t\right\}_{t=0}^{\infty},$$

and we assume $\{Z_t\}_{t=0}^{\infty}$ is given. Hence, we have one too many unknowns—what we are missing is a monetary policy rule. This means, for example, that we can achieve some sequence of desired price level for given $\{Z_t\}_{t=0}^{\infty}$ by choosing $\{Q_0^t\}_{t=0}^{\infty}$ appropriately.

Constant price level Suppose we want to make the price level constant over time and we set

$$P_{-1} = \tilde{P}_{-1} = P. \tag{3.29}$$

(Initial condition is for period $t = -1$ since we have aggregate variables from period 0, so we need period -1 value to obtain P_0 etc.). From (3.25), for P_t^* to be constant, we need the marginal cost of labour, W_t/z_t, to remain constant over time. Assuming that it is, then

$$P_t^* = \frac{\eta}{\eta - 1}\frac{W_t}{z_t}.$$

For P_t to be constant, from (3.26), it must be that $P_t^* = P$. Thus,

$$\frac{W_t}{P} = \frac{\eta - 1}{\eta} z_t.$$

Substituting this into the intratemporal condition (3.23) gives

$$C_t v'(H_t) = \frac{W_t}{P} = \frac{\eta - 1}{\eta} z_t.$$

Given (3.23), we realise that $\tilde{P}_t = P$ so that (3.26) gives us that $C_t = H_t z_t$, which we can use to eliminate C_t from the expression above:

$$H_t v'(H_t) = \frac{\eta - 1}{\eta} z_t.$$

This equation pins down H_t. Then using the Euler equation, (3.28), we can pin down:

$$\frac{Q_0^t}{Q_0^{t+1}} = \frac{1}{\beta}\frac{H_{t+1} z_{t+1}}{H_t z_t},$$

where the left-hand side is the period-t gross interest rate.

Notice that we can follow the steps in reverse so that, given $\{Q_0^t\}_{t=0}^{\infty}$ (and $\{z_t\}_{t=0}^{\infty}$), we can obtain

$$\left\{C_t, H_t, W_t, P_t, \tilde{P}_t, P_t^*\right\}_{t=0}^{\infty}.$$

In this model, money is neutral—e.g. if we double the price level, C_t and H_t does not change. In this sense, it is only the relative price that matters.

3.2.5 Recursive form

Notice that, except for the optimal pricing condition, (3.25), all equilibrium conditions are recursive. To work with (3.25) we introduce the following notation:

$$P_t^* = \frac{\eta}{\eta - 1}\frac{m_t}{d_t} P_t.$$

To find expressions for m_t and d_t, divide both the numerator and the denominator by $C_t P_t^{\eta+1}$, and multiply the numerator by $P_{t'}/P_{t'}$ so that

$$\begin{aligned} P_t^* &:= \frac{\eta}{\eta-1} \frac{\frac{1}{C_t P_t^{\eta+1}} \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta} \frac{W_{t'}}{z_{t'}} \frac{P_{t'}}{P_{t'}}}{\frac{1}{C_t P_t^{\eta+1}} \sum_{t'=t}^{\infty} \frac{Q_0^{t'}}{Q_0^t} \theta^{t'-t} C_{t'} P_{t'}^{\eta}} \\ &= \frac{\eta}{\eta-1} \frac{\sum_{t'=t}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} \frac{W_{t'}}{z_{t'} P_{t'}}}{\sum_{t'=t}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta}}{Q_0^t C_t P_t^{\eta}}} P_t \end{aligned}$$

Define the summation in the numerator as

$$m_t := \sum_{t'=t}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} \frac{W_{t'}}{z_{t'} P_{t'}}.$$

We can interpret $W_{t'}/Z_{t'}P_{t'}$ as the real marginal cost and m_t as the discounted present value of the real marginal cost of production. Expanding the summation, we can write m_t recursively as

$$\begin{aligned} m_t &= \frac{W_t}{z_t P_t} + \sum_{t'=t+1}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} \frac{W_{t'}}{z_{t'} P_{t'}} \\ &= \frac{W_t}{z_t P_t} + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} \sum_{t'=t+1}^{\infty} \theta^{t'-(t+1)} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta+1}}{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta+1}} \frac{W_{t'}}{z_{t'} P_{t'}} \\ &= \frac{W_t}{z_t P_t} + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} m_{t+1}. \end{aligned}$$

Define the summation in the denominator as

$$d_t := \sum_{t'=t}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta}}{Q_0^t C_t P_t^{\eta}}.$$

We can write this recursively as

$$\begin{aligned} d_t &= 1 + \sum_{t'=t+1}^{\infty} \theta^{t'-t} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta}}{Q_0^t C_t P_t^{\eta}} \\ &= 1 + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta}}{Q_0^t C_t P_t^{\eta}} \sum_{t'=t+1}^{\infty} \theta^{t'-(t+1)} \frac{Q_0^{t'} C_{t'} P_{t'}^{\eta}}{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta}} \\ &= 1 + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta}}{Q_0^t C_t P_t^{\eta}} d_{t+1}. \end{aligned}$$

We can then replace P_t^* in our optimality conditions to eliminate P_t^*.

$$C_t v'(H_t) = \frac{W_t}{P_t}, \tag{3.30}$$

$$\frac{1}{C_t} = \beta \frac{Q_0^t P_t}{Q_0^{t+1} P_{t+1}} \frac{1}{C_{t+1}}, \tag{3.31}$$

$$m_t = \frac{W_t}{z_t P_t} + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} m_{t+1}, \tag{3.32}$$

$$d_t = 1 + \theta \frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta}}{Q_0^t C_t P_t^{\eta}} d_{t+1}, \tag{3.33}$$

$$P_{t+1}^{1-\eta} = \left(\frac{\eta}{\eta-1} \frac{m_{t+1}}{d_{t+1}} P_{t+1} \right)^{1-\eta} (1-\theta) + P_t^{1-\eta}\theta, \tag{3.34}$$

$$\tilde{P}_{t+1}^{-\eta} = \left(\frac{\eta}{\eta-1} \frac{m_{t+1}}{d_{t+1}} P_{t+1} \right)^{-\eta} (1-\theta) + \tilde{P}_t^{-\eta}\theta, \tag{3.35}$$

$$H_t = \frac{C_t}{z_t} \left(\frac{P_t}{\tilde{P}_t} \right)^{\eta}. \tag{3.36}$$

3.2.6 Homogeneity and further simplification the system of equations

If we, for example, double initial the ideal price index P_0 and the production price index $\tilde{P}_0$, then we would construct a new equilibrium in which wages are double, and sequence of prices doubles while real variables (consumption, hours and output) remains the same. This means that we can further reduce the system of rewriting the equation in terms of relative prices. Define

$$\Pi_t := \frac{P_t}{P_{t-1}},\ X_t := \frac{\tilde{P}_t}{P_t},\ i_t := \frac{Q_0^t}{Q_0^{t+1}},$$

where Π_t is the (gross) inflation, X_t is a measure of price dispersion, and i_t is the nominal interest rate. The first step is to reduce the dimension of the system of equations.

Dividing (3.34) by $P_t^{1-\eta}$ yields

$$\left(\frac{P_{t+1}}{P_t} \right)^{1-\eta} = \left(\frac{\eta}{\eta-1} \frac{m_{t+1}}{d_{t+1}} \frac{P_{t+1}}{P_t} \right)^{1-\eta} (1-\theta) + \frac{P_t^{1-\eta}}{P_t^{1-\eta}}\theta$$
$$\Leftrightarrow \Pi_{t+1}^{1-\eta} = \left(\frac{\eta}{\eta-1} \frac{m_{t+1}}{d_{t+1}} \Pi_{t+1} \right)^{1-\eta} (1-\theta) + \theta.$$

We can rearrange this to obtain an expression for d_{t+1} in terms of m_{t+1} and Π_t:

$$\Leftrightarrow \frac{\eta}{\eta-1} \frac{m_{t+1}}{d_{t+1}} \Pi_{t+1} = \left(\frac{\Pi_{t+1}^{1-\eta} - \theta}{1-\theta} \right)^{\frac{1}{1-\eta}}$$
$$\Leftrightarrow d_{t+1} = \frac{\eta}{\eta-1} \Pi_{t+1} m_{t+1} \left(\frac{\Pi_{t+1}^{1-\eta} - \theta}{1-\theta} \right)^{\frac{1}{\eta-1}}. \tag{3.37}$$

Using this, we can eliminate d_t from (3.33):

$$\begin{aligned}\frac{\eta}{\eta-1}\Pi_t m_t\left(\frac{\Pi_t^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} &= 1+\theta\frac{Q_0^{t+1}C_{t+1}P_{t+1}^{\eta}}{Q_0^tC_tP_t^{\eta}}\frac{\eta}{\eta-1}\Pi_{t+1}m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}\\ \Leftrightarrow \Pi_t m_t\left(\frac{\Pi_t^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} &= \frac{\eta-1}{\eta}+\theta\frac{Q_0^{t+1}C_{t+1}P_{t+1}^{\eta}}{Q_0^tC_tP_t^{\eta}}\Pi_{t+1}m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}. \qquad (3.38)\end{aligned}$$

Notice that the Euler equation, (3.31), gives us that

$$\beta=\frac{Q_0^{t+1}C_{t+1}P_{t+1}}{Q_0^tC_tP_t}\Rightarrow\frac{Q_0^{t+1}C_{t+1}P_{t+1}^{\eta}}{Q_0^tC_tP_t^{\eta}}=\beta\left(\frac{P_{t+1}}{P_t}\right)^{\eta-1}=\beta\Pi_{t+1}^{\eta-1} \qquad (3.39)$$

Hence, we can simplify (3.38) as

$$\Pi_t m_t\left(\frac{\Pi_t^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}=\frac{\eta-1}{\eta}+\theta\beta\Pi_{t+1}^{\eta}m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}.$$

So, taking stock, we've used (3.34) and (3.33) as well as (3.31), and we've eliminated d_t from the system of equations.

Divide (3.35) by $P_t^{-\eta}$ to obtain

$$\begin{aligned}\left(\frac{\tilde{P}_{t+1}}{P_t}\right)^{-\eta}&=\left(\frac{\eta}{\eta-1}\frac{m_{t+1}}{d_{t+1}}\frac{P_{t+1}}{P_t}\right)^{-\eta}(1-\theta)+\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\theta\\ \Leftrightarrow\left(\frac{\tilde{P}_{t+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right)^{-\eta}&=\left(\frac{\eta}{\eta-1}\frac{m_{t+1}}{d_{t+1}}\Pi_{t+1}\right)^{-\eta}(1-\theta)+X_t^{-\eta}\theta\\ \Leftrightarrow(X_{t+1}\Pi_t)^{-\eta}&=\left(\frac{\eta}{\eta-1}\frac{m_{t+1}}{d_{t+1}}\Pi_{t+1}\right)^{-\eta}(1-\theta)+X_t^{-\eta}\theta.\end{aligned}$$

Substituting d_{t+1} using (3.37) gives

$$\begin{aligned}(X_{t+1}\Pi_{t+1})^{-\eta}&=\left(\frac{\eta}{\eta-1}m_{t+1}\Pi_{t+1}\left(\frac{\eta}{\eta-1}\Pi_{t+1}m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}\right)^{-1}\right)^{-\eta}(1-\theta)+X_t^{-\eta}\theta\\ &=\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{\eta}{\eta-1}}(1-\theta)+X_t^{-\eta}\theta.\end{aligned}$$

We want to eliminate W_t from the system. First, notice that we can use (3.36) to write

$$C_t=z_tH_t\left(\frac{P_t}{\tilde{P}_t}\right)^{-\eta}=z_tH_tX_t^{\eta}. \qquad (3.40)$$

Substituting this into (3.30) gives

$$\frac{W_t}{z_tP_t}=H_tX_t^{\eta}v'(H_t).$$

We can then substitute this expression into (3.32) so that

$$m_t = H_t X_t^{\eta} v'(H_t) + \theta\beta\Pi_{t+1}^{\eta} m_{t+1},$$

where we also used (3.39) to write

$$\frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\eta+1}}{Q_0^t C_t P_t^{\eta+1}} = \beta\Pi_{t+1}^{\eta}.$$

Finally, we can write i_t using (3.31) as

$$i_t \coloneqq \frac{Q_0^t}{Q_0^{t+1}} = \frac{1}{\beta}\frac{C_{t+1}P_{t+1}}{C_t P_t} = \frac{1}{\beta}\frac{C_{t+1}}{C_t}\Pi_{t+1}$$

We can use (3.40) to eliminate C_t and C_{t+1}:

$$i_t = \frac{1}{\beta}\frac{z_{t+1}H_{t+1}X_{t+1}^{\eta}}{z_t H_t X_t^{\eta}}\Pi_{t+1}.$$

So, we now have the following system of equations:

$$\Pi_t m_t \left(\frac{\Pi_t^{1-\eta} - \theta}{1-\theta}\right)^{\frac{1}{\eta-1}} = \frac{\eta-1}{\eta} + \theta\beta\Pi_{t+1}^{\eta} m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta} - \theta}{1-\theta}\right)^{\frac{1}{\eta-1}}, \tag{3.41}$$

$$(X_{t+1}\Pi_{t+1})^{-\eta} = \left(\frac{\Pi_{t+1}^{1-\eta} - \theta}{1-\theta}\right)^{\frac{\eta}{\eta-1}}(1-\theta) + X_t^{-\eta}\theta, \tag{3.42}$$

$$m_t = H_t X_t^{\eta} v'(H_t) + \theta\beta\Pi_{t+1}^{\eta} m_{t+1}, \tag{3.43}$$

$$i_t = \frac{1}{\beta}\frac{z_{t+1}H_{t+1}X_{t+1}^{\eta}}{z_t H_t X_t^{\eta}}\Pi_{t+1}. \tag{3.44}$$

The first equation is a combination of (3.34) and (3.33) as well as (3.31). The second, in addition, used (3.35). The third, in addition, used (3.32), (3.36) and (3.30). Finally, the last used the definition of i_t and (3.36). The unknowns are:

$$\{\Pi_t, m_t, X_t, H_t, i_t\}.$$

Which are the state variables?

3.2.7 Steady state

Let us evaluate the four equations we obtained at a steady state. Let

$$\Pi_{t-1} = \Pi_t = \Pi^*,$$
$$m_{t+1} = m_t = m^*,$$
$$X_{t+1} = X_t = X^*,$$
$$H_{t+1} = H_t = H^*,$$
$$z_{t+1} = z_t = 1.$$

Then, (3.41) becomes

$$\begin{aligned}
\Pi^* m^* \left(\frac{(\Pi^*)^{1-\eta} - \theta}{1-\theta} \right)^{\frac{1}{\eta-1}} &= \frac{\eta - 1}{\eta} + \theta\beta \left(\Pi^*\right)^{\eta} m \left(\frac{(\Pi^*)^{1-\eta} - \theta}{1-\theta} \right)^{\frac{1}{\eta-1}} \\
\Leftrightarrow \frac{\eta-1}{\eta} &= m^* \left(\frac{(\Pi^*)^{1-\eta} - \theta}{1-\theta} \right)^{\frac{1}{\eta-1}} \Pi^* \left(1 - \theta\beta \left(\Pi^*\right)^{\eta-1}\right) \\
m^* &= \frac{\eta-1}{\eta} \frac{1}{\Pi^* \left(1 - \theta\beta \left(\Pi^*\right)^{1-\eta}\right)} \left(\frac{1-\theta}{(\Pi^*)^{1-\eta} - \theta} \right)^{\frac{1}{\eta-1}}.
\end{aligned}$$

Now consider (3.42),

$$\begin{aligned}
\left(X^* \Pi^*\right)^{-\eta} &= \left(\frac{(\Pi^*)^{1-\eta} - \theta}{1-\theta} \right)^{\frac{\eta}{\eta-1}} (1-\theta) + \left(X^*\right)^{-\eta} \theta \\
\Leftrightarrow \left(X^*\right)^{-\eta} &= \left(\frac{(\Pi^*)^{1-\eta} - \theta}{1-\theta} \right)^{\frac{\eta}{\eta-1}} \frac{1-\theta}{(\Pi^*)^{-\eta} - \theta},
\end{aligned}$$

and (3.43),

$$\begin{aligned}
m^* &= H^* \left(X^*\right)^{\eta} v'\left(H^*\right) + \theta\beta \left(\Pi^*\right)^{\eta} m^* \\
\Leftrightarrow H^* v'\left(H^*\right) &= m^* \left(1 - \theta\beta \left(\Pi^*\right)^{\eta}\right) \left(X^*\right)^{-\eta}.
\end{aligned}$$

We want to consider a steady state around zero inflation. Since Π^* is the gross inflation, we set

$$\Pi^* = 1.$$

This implies that

$$\begin{aligned}
\left(X^*\right)^{-\eta} &= \left(\frac{1-\theta}{1-\theta} \right)^{\frac{\eta}{\eta-1}} \frac{1-\theta}{1-\theta} \\
\Rightarrow X^* &= 1,
\end{aligned}$$

and

$$m^* = \frac{\eta-1}{\eta} \frac{1}{1\left(1-\theta\beta\right)} \left(\frac{1-\theta}{1-\theta} \right)^{\frac{1}{\eta-1}} = \frac{\eta-1}{\eta} \frac{1}{1-\theta\beta}.$$

We also have

$$H^* v'\left(H^*\right) = \frac{\eta-1}{\eta} \frac{1}{1-\theta\beta} \left(1-\theta\beta\right) 1 = \frac{\eta-1}{\eta}.$$

3.2.8 Log-linearisation

We differentiate with respect to the variable of interest, evaluate the derivative at the steady state, and multiply by the variable of interest. This gives us the coefficient on the log deviation term in the log-linearised equation.

Take (3.42) first

$$0 = \left(\frac{\Pi_{t+1}^{1-\eta} - \theta}{1-\theta}\right)^{\frac{\eta}{\eta-1}} (1-\theta) + X_t^{-\eta}\theta - (X_{t+1}\Pi_{t+1})^{-\eta}.$$

The derivatives are

$$\begin{aligned}
\left.\frac{\partial RHS}{\partial \Pi_{t+1}}\right|_{SS} \Pi^* &= \frac{\eta}{\eta-1}\frac{(1-\eta)(\Pi^*)^{1-\eta}}{1-\theta}\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{\eta}{\eta-1}-1}(1-\theta) + \eta (X^*\Pi^*)^{-\eta} \\
&= -\eta (\Pi^*)^{1-\eta}\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} + \eta (X^*\Pi^*)^{-\eta} \\
&= -\eta + \eta = 0, \\
\left.\frac{\partial RHS}{\partial X_t}\right|_{SS} X^* &= -\eta\theta (X^*)^{-\eta} = -\eta\theta, \\
\left.\frac{\partial RHS}{\partial X_{t+1}}\right|_{SS} X^* &= \eta (X^*\Pi^*)^{-\eta} = \eta.
\end{aligned}$$

Hence, the log-linearised equation is

$$\begin{aligned}
0 &= -\eta\theta\hat{X}_t + \eta\hat{X}_{t+1} \\
\Leftrightarrow \hat{X}_{t+1} &= \theta\hat{X}_t. \qquad (3.45)
\end{aligned}$$

This gives the law of motion for the state variable.

Now take (3.43):

$$0 = H_t X_t^{\eta} v'(H_t) + \theta\beta\Pi_{t+1}^{\eta} m_{t+1} - m_t.$$

The derivatives are

$$\begin{aligned}
\left.\frac{\partial RHS}{\partial m_t}\right|_{SS} m^* &= -m^* = -\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta} \\
\left.\frac{\partial RHS}{\partial H_t}\right|_{SS} H^* &= H^*(X^*)^{\eta} v'(H^*) + (H^*)^2 (X^*)^{\eta} v''(H^*) \\
&= (H^* v''(H^*) + v'(H^*)) H^* \\
\left.\frac{\partial RHS}{\partial X_t}\right|_{SS} X^* &= \eta H^* (X^*)^{\eta} v'(H^*) = \eta\frac{\eta-1}{\eta} = \eta - 1, \\
\left.\frac{\partial RHS}{\partial \Pi_{t+1}}\right|_{SS} \Pi^* &= \theta\beta\eta (\Pi^*)^{\eta} m^* = \theta\beta\eta\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta} = \frac{\theta\beta}{1-\theta\beta}(\eta-1), \\
\left.\frac{\partial RHS}{\partial m_{t+1}}\right|_{SS} m^* &= \beta\theta (\Pi^*)^{\eta} m^* = \theta\beta\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta} = \frac{\theta\beta}{1-\theta\beta}\frac{\eta-1}{\eta}.
\end{aligned}$$

Now, recall that Frisch elasticity is given by

$$\varepsilon := \frac{v'(H)}{Hv''(H)} \Rightarrow Hv''(H) = \frac{v'(H)}{\varepsilon}$$

and so

$$H^* v''(H)^* H^* + v'(H^*) H^* = \left(\frac{1}{\varepsilon}+1\right) v'(H^*) H^* = \frac{\eta-1}{\eta}\left(1+\frac{1}{\varepsilon}\right).$$

Hence, the log-linearised equation is

$$
\begin{aligned}
0 &= -\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta}\hat{m}_t + \frac{\eta-1}{\eta}\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + (\eta-1)\hat{X}_t \\
&\quad + \frac{\theta\beta}{1-\theta\beta}(\eta-1)\hat{\Pi}_{t+1} + \frac{\theta\beta}{1-\theta\beta}\frac{\eta-1}{\eta}\hat{m}_{t+1} \\
\Leftrightarrow \frac{\hat{m}_t}{\eta(1-\theta\beta)} &= \frac{1}{\eta}\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \hat{X}_t + \frac{\theta\beta}{1-\theta\beta}\hat{\Pi}_{t+1} + \frac{\theta\beta}{1-\theta\beta}\frac{1}{\eta}\hat{m}_{t+1} \\
\Leftrightarrow \hat{m}_t &= (1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \eta(1-\theta\beta)\hat{X}_t + \eta\theta\beta\hat{\Pi}_t + \theta\beta\hat{m}_{t+1}.
\end{aligned}
$$

Recall (3.45), since $\theta \in (0,1)$, give any initial value $\hat{X}_0$, $\hat{X}_t$ converges to zero in the long run; i.e. there is no price dispersion in the long run. In the log-linearisation, we assume that $\hat{X}_t = 0$. In a zero inflation steady state, small deviation in the price has only a second-order cost in terms of efficiencies of production since the starting points coincides with the planner's problem (envelope theorem). This is why we can ignore the state variable in the log-linearised system of equations. If we instead log-linearised around a positive inflation steady state, then we cannot eliminate $\hat{X}_t$ from the system. This is because small deviations now result in first-order cost in terms of efficiency.

We can now write

$$
\hat{m}_t = (1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \eta\theta\beta\hat{\Pi}_{t+1} + \theta\beta\hat{m}_{t+1}. \tag{3.46}
$$

Now take (3.41).

$$
0 = \frac{\eta-1}{\eta} + \theta\beta\Pi_{t+1}^{\eta}m_{t+1}\left(\frac{\Pi_{t+1}^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} - \Pi_t m_t\left(\frac{\Pi_t^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}}.
$$

Then,

$$
\begin{aligned}
\left.\frac{\partial RHS}{\partial m_t}\right|_{SS} m^* &= -\Pi^* m^*\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} \\
&= -\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta} = -\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta}, \\
\left.\frac{\partial RHS}{\partial \Pi_t}\right|_{SS} \Pi^* &= -m^*\Pi^*\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} - \Pi^* m^*\frac{1}{\eta-1}\frac{1-\eta}{1-\theta}(\Pi^*)^{-\eta}\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}-1} \\
&= m^*\left(\frac{1}{1-\theta}-1\right) = \frac{\eta-1}{\eta}\frac{1}{1-\theta\beta}\frac{\theta}{1-\theta}, \\
\left.\frac{\partial RHS}{\partial m_{t+1}}\right|_{SS} m^* &= \theta\beta(\Pi^*)^{\eta}m^*\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} = \frac{\eta-1}{\eta}\frac{\theta\beta}{1-\theta\beta}, \\
\left.\frac{\partial RHS}{\partial \Pi_{t+1}}\right|_{SS} \Pi^* &= \eta\theta\beta(\Pi^*)^{\eta-1}m^*\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}} + \theta\beta(\Pi^*)^{\eta}m^*\frac{1}{\eta-1}\frac{1-\eta}{1-\theta}(\Pi^*)^{-\eta}\left(\frac{(\Pi^*)^{1-\eta}-\theta}{1-\theta}\right)^{\frac{1}{\eta-1}-1} \\
&= \theta\beta m^*\left(\eta-\frac{1}{1-\theta}\right) = \frac{\eta-1}{\eta}\frac{\theta\beta}{1-\theta\beta}\left(\eta-\frac{1}{1-\theta}\right).
\end{aligned}
$$

The log-linearised equation is

$$\begin{aligned}
\frac{\eta-1}{\eta}\frac{1}{1-\theta\beta}\hat{m}_t &= \frac{\eta-1}{\eta}\frac{1}{1-\theta\beta}\frac{\theta}{1-\theta}\hat{\Pi}_t + \frac{\eta-1}{\eta}\frac{\theta\beta}{1-\theta\beta}\hat{m}_{t+1} + \frac{\eta-1}{\eta}\frac{\theta\beta}{1-\theta\beta}\left(\eta-\frac{1}{1-\theta}\right)\hat{\Pi}_{t+1} \\
\Leftrightarrow \hat{m}_t &= \frac{\theta}{1-\theta}\hat{\Pi}_t + \theta\beta\hat{m}_{t+1} + \theta\beta\left(\eta-\frac{1}{1-\theta}\right)\hat{\Pi}_{t+1}.
\end{aligned}$$

Substituting (3.46), we have

$$\begin{aligned}
(1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \eta\theta\beta\hat{\Pi}_{t+1} + \theta\beta\hat{m}_{t+1} &= \frac{\theta}{1-\theta}\hat{\Pi}_t + \theta\beta\hat{m}_{t+1} + \theta\beta\left(\eta-\frac{1}{1-\theta}\right)\hat{\Pi}_{t+1} \\
\Leftrightarrow (1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t &= \frac{\theta}{1-\theta}\hat{\Pi}_t - \theta\beta\frac{1}{1-\theta}\hat{\Pi}_{t+1} \\
\Leftrightarrow \frac{\theta}{1-\theta}\hat{\Pi}_t &= (1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \theta\beta\frac{1}{1-\theta}\hat{\Pi}_{t+1} \\
\Leftrightarrow \hat{\Pi}_t &= \frac{1-\theta}{\theta}(1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\hat{H}_t + \beta\hat{\Pi}_{t+1}.
\end{aligned}$$

Now recall the production function:

$$\begin{aligned}
y_{j,t} = c_{j,t} = z_t h_{j,t} &\Rightarrow \int_0^1 y_{j,t}dj = z_t\int_0^1 h_{j,t}di \\
&\Rightarrow Y_t = z_t H_t \\
&\Rightarrow \hat{Y}_t = \hat{z}_t + \hat{H}_t \\
&\Leftrightarrow \hat{H}_t = \hat{Y}_t - \hat{z}_t,
\end{aligned}$$

where we assume that $\hat{X}_t = 0$ (i.e. no efficiency in the long run). Hence, we can write

$$\hat{\Pi}_t = \frac{1-\theta}{\theta}(1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)\left(\hat{Y}_t-\hat{z}_t\right) + \beta\hat{\Pi}_{t+1}.$$

We refer to this equation as the New Keynesian Phillips curve since it relates inflation to output.

Finally, take (3.44). To log-linearise, take log of both sides

$$\begin{aligned}
&\log i_t = -\log\beta + \log z_{t+1} - \log z_t + \log H_{t+1} - \log H_t + \eta\left(\log X_{t+1} - \log X_T\right) + \log\Pi_{t+1} \\
&\Rightarrow \hat{i}_t = \hat{\Pi}_{t+1} + \left(\hat{z}_{t+1}+\hat{H}_{t+1}\right) - \left(\hat{z}_t + \hat{H}_t\right) \\
&\qquad = \hat{\Pi}_{t+1} + \hat{Y}_{t+1} - \hat{Y}_t = \hat{\Pi}_{t+1} + \Delta\hat{Y}_{t+1}.
\end{aligned}$$

where we again assumed that $\hat{X}_t = \hat{X}_{t+1} = 0$. This equation is called the *dynamic IS curve*.

3.2.9 The "Three" equations

New Keynesian Phillips Curve (NKPC)

$$\hat{\Pi}_t = \kappa\hat{H}_t + \beta\hat{\Pi}_{t+1},$$

where $\kappa \coloneqq (1-\theta)(1-\theta\beta)\left(1+\frac{1}{\varepsilon}\right)/\theta > 0$. This says that we have inflation if the firm who can reprice wants to set a high price. One reason for them to set a high price is if the firm "expects" (note, we're technically working with a deterministic model here) next period inflation to be high because they anticipate that there is a chance that they will not be able to change their price next period. They also set a high price when employment is high, which is the case when wages are high; i.e. when marginal cost of production is high which induces firms to set a higher price.

Dynamic IS curve (DIS)

$$\hat{H}_t = (\hat{z}_{t+1} - \hat{z}_t) - \underbrace{\left(\hat{i}_{t+1} - \hat{\Pi}_{t+1}\right)}_{\text{real interest rate}} + \hat{H}_{t+1}.$$

From the production function (recalling that price dispersion is zero), we have

$$\hat{C}_t = \hat{Y}_t = \hat{z}_t + \hat{H}_t.$$

Hence, the dynamic IS curve can be written also as

$$\hat{C}_t = -\left(\hat{i}_{t+1} - \hat{\Pi}_{t+1}\right) + \hat{C}_{t+1};$$

i.e. consumption today is negatively related to the real interest rate in period $t+1$ and positively related to consumption in period $t+1$. So the dynamic IS curve is, in fact, the consumption Euler equation. We can interpret the equation as saying that if real interest rate is high, then people are wanting to have high consumption growth:

$$\underbrace{\hat{i}_{t+1} - \hat{\Pi}_{t+1}}_{\text{real interest rate}} = \underbrace{\hat{C}_{t+1} - \hat{C}_t}_{\text{consumption growth}}.$$

Taylor rule The NKPC and DIS would represent a pair of difference equations for $\hat{H}_t$ and $\hat{\Pi}$ if we knew $\hat{z}_t$ and if we knew the path of $\hat{i}_t$. The path of $\hat{i}_t$ is a specification of the monetary policy. We will assume that nominal interest rate for between periods t and $t+1$, i_{t+1}, is set in period t according to:[23]

$$\hat{i}_{t+1} = \phi_H \hat{H}_t + \phi_\Pi \hat{\Pi}_t + \upsilon_t, \ \phi_H, \phi_\Pi > 0. \tag{3.47}$$

This is based on an empirical paper by John Taylor which found that historical relationship between nominal interest, hours and inflation can be described well by the structure above with coefficients that are "relatively stable" over time. We often interpret this as a prescription rule; e.g. if employment is high or when inflation is high, the central bank increases interest rates.

υ_t is called the monetary policy shock. Thinking of the Taylor rule as a implementation rule, we can think of υ as capturing some objective by the monetary policy that is not completely captured by the first two terms. Alternatively, we can think of it as representing the unpredictability in monetary policy (e.g. committee members could change over time).

[23] In a stochastic setting, i_{t+1} must be set using only information available in period t.

3.2.10 Solving the system

Since $\hat{i}_t$ appears in DIS and the Taylor rule, we can combine the two by eliminating $\hat{i}_t$:

$$\begin{aligned}\hat{H}_t &= (\hat{z}_{t+1} - \hat{z}_t) - \left(\left(\phi_H \hat{H}_t + \phi_\Pi \hat{\Pi}_t + \upsilon_t\right) - \hat{\Pi}_{t+1}\right) + \hat{H}_{t+1} \\ \Leftrightarrow (1+\phi_H)\,\hat{H}_t &= \hat{z}_{t+1} - \hat{z}_t + \hat{\Pi}_{t+1} - \phi_\Pi \hat{\Pi}_t + \hat{H}_{t+1} - \upsilon_t.\end{aligned}$$

This, together with the NKPC gives us a pair of dynamic equations for $\hat{H}_t$ and $\hat{\Pi}_t$.

No shocks Let us shutdown the exogenous shocks in the model and set $\hat{z}_t = \upsilon_t = 0$ for all t.

$$\begin{aligned}\hat{\Pi}_t &= \kappa \hat{H}_t + \beta \hat{\Pi}_{t+1}, \\ (1+\phi_H)\,\hat{H}_t &= \hat{\Pi}_{t+1} - \phi_\Pi \hat{\Pi}_t + \hat{H}_{t+1}.\end{aligned}$$

One solution to the pair of equation is:

$$\hat{\Pi}_t = \hat{H}_t = 0,\ \forall t. \tag{3.48}$$

In general, however, there are many other solutions to this system of difference equations. Some solutions may be explosive. Note that we cannot use the log-linearised equations to rule out explosive solutions (no transversality conditions as in the NCG model). The "dirty" method we use to rule out explosive solutions is the following. We suppose that the model is correct until some period T, and after period T, we assume that there will be flexible prices/money neutrality holds. Since we solved the flexible prices model already, we know what the economy will look like in period T, and we use the difference equations above to solve backwards from period T. If the system is globally unstable, then the only solution that would converge to the flexible price economy is (3.48).

The condition for the system to be globally unstable is:

$$(\phi_\Pi - 1)\,\kappa + \phi_H\,(1-\beta) > 0. \tag{3.49}$$

If this condition is true, then the unique bounded solution is (3.48). If not, there are other bounded solutions. Note that since $\beta \in (0,1)$ and $\phi_\Pi, \phi_H, \kappa > 0$, a sufficient condition for global instability is

$$\phi_\Pi > 1.$$

In other words, looking back at the Taylor rule, (3.47), monetary policy has to react strongly to inflation. This is called the *Taylor Principle.* If (3.49) does not hold, then there will be a continuum of equilibria that converges to the flexible price economy by period T.

Remark 3.2. Taylor argues that when US inflation was high, $\phi_\Pi \approx 0.5$ and in the Volcker period, inflation was brought down with $\phi_\Pi \approx 1.5$.

Let us assume that υ_t and z_t follows first-order Markov process.

If we add shocks, then the system of equation becomes:

$$\hat{\Pi}_t = \kappa \hat{H}_t + \beta \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right], \tag{3.50}$$

$$(1+\phi_H)\,\hat{H}_t = \mathbb{E}_t\left[\hat{z}_{t+1}\right] - \hat{z}_t + \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right] - \phi_\Pi \hat{\Pi}_t + \mathbb{E}_t\left[\hat{H}_{t+1}\right] - \upsilon_t. \tag{3.51}$$

With monetary policy shocks, υ_t We assume that $\phi_\Pi > 1$ and that

$$\mathbb{E}_t\left[\upsilon_{t+1}\right] = \rho_\upsilon \upsilon_t,\ \hat{z}_t = 0,\ \forall t.$$

Then

$$\hat{\Pi}_t = \kappa \hat{H}_t + \beta \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right], \tag{3.52}$$

$$\left(1+\phi_H\right)\hat{H}_t = \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right] - \phi_\Pi \hat{\Pi}_t + \mathbb{E}_t\left[\hat{H}_{t+1}\right] - \upsilon_t. \tag{3.53}$$

We can write the system as

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \beta & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \hat{\Pi}_{t+1} \\ \hat{H}_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & \kappa \\ -\phi_\Pi & -\left(1+\phi_H\right) \end{bmatrix}\begin{bmatrix} \hat{\Pi}_t \\ \hat{H}_t \end{bmatrix} \\
&\equiv \mathbf{G}^*_{t+1}\hat{\mathbf{x}}_{t+1} + \mathbf{G}^*_t \hat{\mathbf{x}}_t \\
\Rightarrow \hat{\mathbf{x}}_{t+1} &= -\left(\mathbf{G}^*_{t+1}\right)^{-1}\mathbf{G}^*_t\hat{\mathbf{x}}_t \\
&\equiv \mathbf{M}\hat{\mathbf{x}}_t \\
\begin{bmatrix} \hat{\Pi}_{t+1} \\ \hat{H}_{t+1} \end{bmatrix} &= \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\beta}+\phi_\Pi & 1+\frac{\kappa}{\beta}+\phi_H \end{bmatrix}\begin{bmatrix} \hat{\Pi}_t \\ \hat{H}_t \end{bmatrix}.
\end{aligned}$$

The characteristic polynomial is then

$$\begin{aligned}
0 &= \left(\frac{1}{\beta}-\lambda\right)\left(1+\frac{\kappa}{\beta}+\phi_H-\lambda\right) - \left(-\frac{\kappa}{\beta}\right)\left(-\frac{1}{\beta}+\phi_\Pi\right) \\
&= \left(1-\beta\lambda\right)\left(1+\frac{\kappa}{\beta}+\phi_H-\lambda\right) + \left(\kappa\right)\left(\phi_\Pi - \frac{1}{\beta}\right) \\
&= 1+\frac{\kappa}{\beta}+\phi_H-\lambda-\beta\lambda-\kappa\lambda-\beta\lambda\phi_H+\beta\lambda^2+\kappa\phi_\Pi-\frac{\kappa}{\beta} \\
&= \beta\lambda^2 - \left(1+\beta\left(1+\phi_H\right)+\kappa\right)\lambda + \left(1+\phi_H+\kappa\phi_\Pi\right).
\end{aligned}$$

This give us the two eigenvalues λ_1 and λ_2. Define

$$Q\left(\lambda\right) \coloneqq \beta\lambda^2 - \left(1+\beta\left(1+\phi_H\right)+\kappa\right)\lambda + \left(1+\phi_H+\kappa\phi_\Pi\right).$$

Then,

$$\begin{aligned}
Q\left(0\right) &= 1+\phi_H+\kappa\phi_\Pi > 0, \\
Q'\left(\lambda\right) &= 2\beta\lambda - \left(1+\beta\left(1+\phi_H\right)+\kappa\right) \\
\Rightarrow Q'\left(0\right) &= -\left(1+\beta\left(1+\phi_H\right)+\kappa\right) < 0 \\
Q'\left(1\right) &= \left(\beta-1\right)-\beta\phi_H-\kappa < 0, \\
Q''\left(\lambda\right) &= 2\beta > 0.
\end{aligned}$$

Since Q is strictly convex, positive at $\lambda = 0$ and is downward sloping at $\lambda = 0$ and $\lambda = 1$, it must be that the two roots/eigenvalues are both greater than one (draw the figure).

Suppose we guess that the solution is linear:

$$\hat{H}_t = A\upsilon_t,\ \hat{\Pi}_t = B\upsilon_t.$$

Substituting into (3.50)

$$\begin{aligned} B\upsilon_t &= \kappa A\upsilon_t + \beta B\rho_\upsilon \upsilon_t \\ \Leftrightarrow B &= \frac{\kappa A}{1-\beta\rho_\upsilon}; \end{aligned}$$

and into

$$\begin{aligned} (1+\phi_H) A\upsilon_t &= B\rho_\upsilon\upsilon_t - \phi_\Pi B\upsilon_t + A\rho_\upsilon\upsilon_t - \upsilon_t \\ \Leftrightarrow B &= \frac{(1+\phi_H-\rho_\upsilon) A + 1}{\rho_\upsilon - \phi_\Pi}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\kappa A}{1-\beta\rho_\upsilon} &= \frac{(1+\phi_H-\rho_\upsilon) A + 1}{\rho_\upsilon - \phi_\Pi} \\ \Leftrightarrow \kappa A(\rho_\upsilon - \phi_\Pi) &= (1-\beta\rho_\upsilon)(1+\phi_H-\rho_\upsilon) A + 1 - \beta\rho_\upsilon \\ \Leftrightarrow A &= \frac{1-\beta\rho_\upsilon}{\kappa(\rho_\upsilon-\phi_\Pi) - (1-\beta\rho_\upsilon)(1+\phi_H-\rho_\upsilon)} < 0, \end{aligned}$$

where the sign follows since $\rho_\upsilon \in (0,1)$ while $\phi_\Pi > 1$. This, in turn implies that

$$B = \frac{\kappa}{\kappa(\rho_\upsilon-\phi_\Pi) - (1-\beta\rho_\upsilon)(1+\phi_H-\rho_\upsilon)} < 0$$

So we can see that our guess was correct and that the solution is given by

$$\begin{aligned} \hat{H}_t &= \frac{1-\beta\rho_\upsilon}{\kappa(\rho_\upsilon-\phi_\Pi) - (1-\beta\rho_\upsilon)(1+\phi_H-\rho_\upsilon)}\upsilon_t \\ \hat{\Pi}_t &= \frac{\kappa}{\kappa(\rho_\upsilon-\phi_\Pi) - (1-\beta\rho_\upsilon)(1+\phi_H-\rho_\upsilon)}\upsilon_t, \end{aligned}$$

where both coefficients are negative. This means that following a positive monetary policy shock, both employment and inflation falls. That $\hat{H}_t$ is affected also implies that monetary policy has a real effect on the economy (which directly affects $\hat{Y}_t$).

Why does monetary policy have a real effect? Suppose that we think that after some period T, there will be no monetary shocks; i.e. $\upsilon_t = 0$ for all $t > T$. We think of solving this model backwards. The impact of monetary shock, υ_t affects nominal interest and it ends up affecting real interest rate. High value of υ_t (contractionary monetary policy), raises nominal interest and raises real interest. After period T, we must be in the flexible price case and so $\hat{H}_t = 0$ and $H = H^*$.

So when each consumer sees that interest rate will be higher from today to tomorrow, the consumption Euler equation tells us that consumers would want steeply upward consumption profile. But since the end point cannot move, in order to achieve this, consumers must cut back consumption today and "save". Here, since there is no capital, saving is only possible if consumers demand less today. This means that less labour is required and consumers feel poor. As hours worked falls, wages decreases until the intertemporal condition is satisfied.

Falling wages imply that cost of production is low and so firms set prices low; i.e. inflation is low.

Remark 3.3. If ρ_υ small, then high υ_t implies high $\hat{i}_{t+1}$; but if ρ_υ close to one, then high υ_t implies low $\hat{i}_{t+1}$.

With productivity shocks, z_t We assume that $\phi_\Pi > 1$ and that

$$\mathbb{E}_t\left[\hat{z}_{t+1}\right] = \rho_z \hat{z}_t,\ \upsilon_t = 0,\ \forall t.$$

Then

$$\hat{\Pi}_t = \kappa \hat{H}_t + \beta \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right], \tag{3.54}$$

$$\left(1+\phi_H\right)\hat{H}_t = \mathbb{E}_t\left[\hat{z}_{t+1}\right] - \hat{z}_t + \mathbb{E}_t\left[\hat{\Pi}_{t+1}\right] - \phi_\Pi \hat{\Pi}_t + \mathbb{E}_t\left[\hat{H}_{t+1}\right]. \tag{3.55}$$

Let us guess again that the solutions are linear:

$$\hat{H}_t = C\hat{z}_t,\ \hat{\Pi}_t = D\hat{z}_t.$$

Then, (3.54) becomes

$$\begin{aligned} D\upsilon_t &= \kappa C \hat{z}_t + \beta D \rho_z \hat{z}_t \\ \Leftrightarrow D &= \frac{\kappa C}{1-\beta\rho_z}; \end{aligned}$$

and (3.55) becomes

$$\begin{aligned} \left(1+\phi_H\right) C\hat{z}_t &= \rho_z \hat{z}_t - \hat{z}_t + \rho_z D \hat{z}_t - \phi_\Pi D \hat{z}_t + \rho_z C \hat{z}_t \\ D &= \frac{\left(\rho_z - 1\right) + \left(\rho_z - \left(1+\phi_H\right)\right) C}{\phi_\Pi - \rho_z}. \end{aligned}$$

Combining, we get

$$\begin{aligned} \frac{\kappa C}{1-\beta\rho_z} &= \frac{\left(\rho_z - 1\right) + \left(\rho_z - \left(1+\phi_H\right)\right) C}{\phi_\Pi - \rho_z} \\ \Leftrightarrow \left(\phi_\Pi - \rho_z\right)\kappa C &= \left(1-\beta\rho_z\right)\left(\rho_z - 1\right) + \left(1-\beta\rho_z\right)\left(\rho_z - \left(1+\phi_H\right)\right) C \\ \Leftrightarrow C &= \frac{\left(1-\beta\rho_z\right)\left(1-\rho_z\right)}{\kappa\left(\rho_z - \phi_\Pi\right) - \left(1-\beta\rho_z\right)\left(1+\phi_H-\rho_z\right)} < 0 \end{aligned}$$

and so

$$D = \frac{\kappa\left(1-\rho_z\right)}{\kappa\left(\rho_z - \phi_\Pi\right) - \left(1-\beta\rho_z\right)\left(1+\phi_H-\rho_z\right)} < 0.$$

Hence, the solution is given by

$$\begin{aligned} \hat{H}_t &= \frac{\left(1-\beta\rho_z\right)\left(1-\rho_z\right)}{\left(1-\beta\rho_z\right)\left(\rho_z - \left(1+\phi_H\right)\right) - \left(\phi_\Pi - \rho_z\right)\kappa}\hat{z}_t, \\ \hat{\Pi}_t &= \frac{\kappa\left(1-\rho_z\right)}{\left(1-\beta\rho_z\right)\left(\rho_z - \left(1+\phi_H\right)\right) - \left(\phi_\Pi - \rho_z\right)\kappa}\hat{z}_t. \end{aligned}$$

Since both coefficients are negative, this tells us that, in the New Keynesian model, an increase in productivity shock leads to a reduction in inflation as well as hours worked. The latter is the contrary to what we found with the real business cycle model (with balanced growth path preferences). This is a controversial conclusion from the New Keynesian model

Recall that in the RBC, a temporary increase in productivity leads people to work more to take and build up the capital stock (hours worked eventually falls, however). But in the NK model we are looking at, there is no capital. So what happens in the RBC model without capital? Without capital (and balanced growth preferences, in particular, log utility for consumption), income (higher productivity means greater income from working a given amount so incentive to work less) and substitution effects (higher productivity means greater opportunity cost of leisure so work more) due to the productivity shock exactly offset each other. So in the RBC model without capital, hours worked will not change in response to productivity shocks.

In the NK model, firms set prices before shocks are realised. With higher productivity, it becomes cheaper to produce output but firms cannot cut their price. Thus, firms respond by hiring fewer workers, since each worker is now more productive. Lower hours worked then results in lower wages and lower inflation. Note that positive productivity shock does lead to an increase in aggregate consumption,[24] however, the effect of firms reducing labour input is greater.

Statistic Phillips curve Observe that if ρ_v and ρ_z are the same, then the implicit relationship between $\hat{H}_t$ and $\hat{\Pi}_t$ is the same when we have (i) just the monetary policy shock; and (ii) just the productivity shock. This statistic relationship is the more convention to think of Phillips curve trade off. Temporarily high (low) hours worked is occurs when inflation is relatively high (low).

Remark 3.4. With log-linearised equations, if we have both shocks in the model, and if they are uncorrelated, then the effects are additive.

3.2.11 Relationship between productivity shock and hours worked

As we saw in the previous section, the New Keynesian model (with log utility) predicts that when the economy experiences a positive productivity shock, hours worked declines; i.e. there is a negative correlation between productivity shocks and hours worked. The intuition was that since firms are unable to cut prices, they instead hire fewer workers following a positive productivity shock.

Gali (1999) uses VAR and finds that, in the short run, productivity and hours worked/employment are negatively correlated while in the long run, he finds that the only shock that affects labour productivity are productivity shocks. However, VAR analysis with small data can be biased signifinantly.

24

$$
\begin{aligned}
\hat{C}_t = \hat{z}_t + \hat{H}_t = \hat{z}_t + \frac{(1-\beta\rho_z)(1-\rho_z)}{(1-\beta\rho_z)(\rho_z-(1+\phi_H))-(\phi_\Pi-\rho_z)\kappa}\hat{z}_t \\
= \frac{-(1-\beta\rho_z)(1-\rho_z+\phi_H)-(\phi_\Pi-\rho_z)\kappa+(1-\beta\rho_z)(1-\rho_z)}{(1-\beta\rho_z)(\rho_z-(1+\phi_H))-(\phi_\Pi-\rho_z)\kappa}\hat{z}_t \\
= \frac{\overbrace{-(1-\beta\rho_z)\phi_H-(\phi_\Pi-\rho_z)\kappa}^{<0}}{\underbrace{(1-\beta\rho_z)(\rho_z-(1+\phi_H))-(\phi_\Pi-\rho_z)\kappa}_{<0}}\hat{z}_t.
\end{aligned}
$$

4 Adjustment cost in investment

4.1 Impulse vs propagation mechanisms

The most volatile component of GDP at the business cycle frequency is investment. Recall that RBC model is able to generate volatile investment. Now if we add adjustment cost to the model, this would dampen the fluctuations in investment. So why do we want to introduce adjustment costs?

The real business cycle is about impulse. It is about persistent technology shocks that lead to some short-lived dynamics of investment and employment, and then there are some propagation that work in the model through the long-lasting impact on capital stock. The reason why we get persistent effect on investment is because the shock is not really a one-time shock—it is due to persistent technology shock (recall AR coefficient was around 0.95). Something we might want to get out of the model is a persistent dynamics on investment following a one-time (i.e. not persistent) shock; we want a propagation mechanism is generated internally in the model, rather than specified via the persistence in shocks.

RBC models do have some propagation mechanism via behaviour of capital stock but the effect on investment (following a positive productivity shock) is positive initially that turns negative. What we hope adjustment cost in investment would do is to generate persistent positive investment behaviour following a one-time shock. Adjustment costs provide a microfoundation for why the desired level of capital might take a while to show up as actual level of capital and a prolonged period of high investment.

The basic intuition is that with convex adjustment cost, investment is cheaper when done in steps rather than in one giant leap. This is the way in which it produces propagation mechanism.

4.2 Convex cost: Lucas & Prescott (1971)

4.2.1 Firm problem

When we look at investment behaviour by firms, fixed cost adjustment models (which predicts no investment until certain thresholds are reached) do not fit the data well. We therefore consider convex adjustment costs that help to explain small investment adjustments by firms that we observe in the data.

The firm's problem is the following:

$$
\begin{aligned}
v^*\left(k_0\right) := \max_{\left\{k_{t+1}\left(\mathbf{s}^t,k_0\right),h_t\left(\mathbf{s}^t,k_0\right)\right\}_{t,\mathbf{s}^t}} \sum_{t=0}^{\infty} \beta^t \sum_{\mathbf{s}^t} \Pi_t\left(\mathbf{s}^t\right) \Big[& p_t\left(\mathbf{s}^t\right) f\left(k_t\left(\mathbf{s}^{t-1},k_0\right),h_t\left(\mathbf{s}^t,k_0\right)\right) \\
& -w_t\left(\mathbf{s}^t\right) h_t\left(\mathbf{s}^t,k_0\right) - c\left(\frac{k_{t+1}\left(\mathbf{s}^t,k_0\right)}{k_t\left(\mathbf{s}^{t-1},k_0\right)}\right) k_t\left(\mathbf{s}^{t-1},k_0\right)\Big],
\end{aligned}
$$

where the each firm takes as given the sequence $\left\{p_t\left(\mathbf{s}^t\right), w_t\left(\mathbf{s}^t\right)\right\}$ (i.e. there are many firms competing against each other). Note:

- ▷ $v^*\left(k_0\right)$ is the value of the firm (expected present discounted value of future profits);
- ▷ k_0 is the initial level of capital;
- ▷ f is an increasing, concave, constant returns to scale production function;

- ▷ c represents the adjustment cost that is increasing, convex, constant returns to scale function (continuously differentiable). The CRS assumption implies that the adjustment cost is proportional to different level of k_0.
- ▷ If $k_{t+1} = k_t$, then adjustment cost if equal to $c(1)\,k_t$. We can think of this as depreciation.

Claim 4.1. The value function is linear in k_0; i.e.

$$v^*(k_0) = v_0^* k_0. \tag{4.1}$$

Proof. Let $k_{t+1}^*(\mathbf{s}^t, k_0)$ and $h_t^*(\mathbf{s}^t, k_0)$ denote the optimal choice of capital and labour for a firm with initial capital stock of k_0. Then consider another firm with initial level of capital k_0' and the following policy for capital and labour for the firm with k_0': for all $\mathbf{s}^t$ and t,

$$\begin{aligned} k_{t+1}\left(\mathbf{s}^t, k_0'\right) &:= k_{t+1}^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0}, \\ h_t\left(\mathbf{s}^t, k_0'\right) &:= h_t^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0}. \end{aligned}$$

For any sequence of $\{k_{t+1}(\mathbf{s}^t, k_0), h_t(\mathbf{s}^t, k_0)\}_{t,\mathbf{s}^t}$, define

$$\tilde{v}(k_0) = \sum_{t=0}^{\infty}\beta^t\sum_{\mathbf{s}^t}\Pi_t\left(\mathbf{s}^t\right)\tilde{v}_t(k_0),$$

where

$$\tilde{v}_t(k_0) := p_t\left(\mathbf{s}^t\right) f\left(k_t\left(\mathbf{s}^{t-1}, k_0\right), h_t\left(\mathbf{s}^t, k_0\right)\right) - w_t\left(\mathbf{s}^t\right) h_t\left(\mathbf{s}^t, k_0\right) - c\left(\frac{k_{t+1}\left(\mathbf{s}^t, k_0\right)}{k_t\left(\mathbf{s}^{t-1}, k_0\right)}\right) k_t\left(\mathbf{s}^{t-1}, k_0\right).$$

Now, consider $\tilde{v}_t(k_0')$:

$$\begin{aligned} \tilde{v}_t\left(k_0'\right) &= p_t\left(\mathbf{s}^t\right) f\left(k_{t+1}\left(\mathbf{s}^{t-1}, k_0'\right), h_t\left(\mathbf{s}^t, k_0'\right)\right) - w_t\left(\mathbf{s}^t\right) h_t\left(\mathbf{s}^t, k_0'\right) \\ &\qquad - c\left(\frac{k_{t+1}\left(\mathbf{s}^t, k_0'\right)}{k_t\left(\mathbf{s}^{t-1}, k_0'\right)}\right) k_t\left(\mathbf{s}^{t-1}, k_0'\right) \\ &= p_t\left(\mathbf{s}^t\right) f\left(k_{t+1}^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0}, h_t^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0}\right) - w_t\left(\mathbf{s}^t\right) h_t^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0} \\ &\qquad - c\left(\frac{k_{t+1}^*\left(\mathbf{s}^t, k_0\right)\frac{k_0'}{k_0}}{k_t^*\left(\mathbf{s}^{t-1}, k_0\right)\frac{k_0'}{k_0}}\right) k_t^*\left(\mathbf{s}^{t-1}, k_0\right)\frac{k_0'}{k_0} \\ &= \frac{k_0'}{k_0} p_t\left(\mathbf{s}^t\right) f\left(k_{t+1}^*\left(\mathbf{s}^t, k_0\right), h_t^*\left(\mathbf{s}^t, k_0\right)\right) - \frac{k_0'}{k_0} w_t\left(\mathbf{s}^t\right) h_t^*\left(\mathbf{s}^t, k_0\right) \\ &\qquad - \frac{k_0'}{k_0} c\left(\frac{k_{t+1}^*\left(\mathbf{s}^t, k_0\right)}{k_t^*\left(\mathbf{s}^{t-1}, k_0\right)}\right) k_t^*\left(\mathbf{s}^{t-1}, k_0\right) \\ &= \frac{k_0'}{k_0} v_t^*(k_0), \ \forall t, \mathbf{s}^t, \end{aligned}$$

where

$$v_t^*(k_0) := p_t\left(\mathbf{s}^t\right) f\left(k_t^*\left(\mathbf{s}^{t-1}, k_0\right), h_t^*\left(\mathbf{s}^t, k_0\right)\right) - w_t\left(\mathbf{s}^t\right) h_t^*\left(\mathbf{s}^t, k_0\right) - c\left(\frac{k_{t+1}^*\left(\mathbf{s}^t, k_0\right)}{k_t^*\left(\mathbf{s}^{t-1}, k_0\right)}\right) k_t^*\left(\mathbf{s}^{t-1}, k_0\right).$$

Hence,

$$\tilde{v}\left(k_0'\right) = \frac{k_0'}{k_0}\sum_{t=0}^{\infty}\beta^t\sum_{\mathbf{s}^t}\Pi_t\left(\mathbf{s}^t\right)v_t^*\left(k_0\right) = \frac{k_0'}{k_0}v^*\left(k_0\right)$$

Since $v^*\left(k_0'\right)$ solves the firm's problem, it follows that

$$v^*\left(k_0'\right) \geq \tilde{v}\left(k_0'\right) = \frac{k_0'}{k_0}v^*\left(k_0\right). \tag{4.2}$$

We now reverse the role of k_0 and k_0'. Define

$$\begin{aligned} k_{t+1}\left(\mathbf{s}^t, k_0\right) &\coloneqq k_{t+1}^*\left(\mathbf{s}_t, k_0'\right)\frac{k_0}{k_0'}, \\ h_t\left(\mathbf{s}^t, k_0\right) &\coloneqq h_t^*\left(\mathbf{s}_t, k_0'\right)\frac{k_0}{k_0'}. \end{aligned}$$

Then, following the same logic,

$$\begin{aligned} \tilde{v}_t\left(k_0\right) &= \frac{k_0}{k_0'}v_t^*\left(k_0'\right),\ \forall t, \mathbf{s}^t \\ \Rightarrow \tilde{v}\left(k_0\right) &= \frac{k_0}{k_0'}v^*\left(k_0'\right) \\ \Rightarrow v^*\left(k_0\right) \geq \tilde{v}\left(k_0\right) &= \frac{k_0}{k_0'}v^*\left(k_0'\right) \\ \Leftrightarrow v^*\left(k_0'\right) &\leq \frac{k_0'}{k_0}v^*\left(k_0\right). \end{aligned} \tag{4.3}$$

Then, (4.2) and (4.3) together imply that

$$\begin{aligned} v^*\left(k_0'\right) &= \frac{k_0'}{k_0}v^*\left(k_0\right) \\ \Leftrightarrow v^*\left(k_0\right) &= v_0^*k_0. \end{aligned} \qquad \blacksquare$$

Corollary 4.1. *For any nonzero k_0 and k_0',*

$$\frac{k_{t+1}^*\left(\mathbf{s}^t, k_0\right)}{k_0} = \frac{k_{t+1}^*\left(\mathbf{s}^t, k_0'\right)}{k_0'},\ \frac{h_t^*\left(\mathbf{s}^t, k_0\right)}{k_0} = \frac{h_t^*\left(\mathbf{s}^t, k_0'\right)}{k_0'}.$$

We can interpret v_0^* as the optimum investment rate.

To pin down v_0^*, given (4.1), we know that

$$v_0^* = \frac{\partial v^*\left(k_0\right)}{\partial k_0}.$$

Since $v^*\left(k_0\right)$ is maximised, by the Envelope Theorem, we do not need to consider how k^* and h^*

change with k_0; i.e.

$$\begin{aligned}
v_0^* &= p_0\left(\mathbf{s}^0\right) f_k\left(k_0, h_0\left(\mathbf{s}^0, k_0\right)\right) - c\left(\frac{k_1\left(\mathbf{s}^0, k_0\right)}{k_0}\right) \\
&\quad - c'\left(\frac{k_1\left(\mathbf{s}^0, k_0\right)}{k_0}\right) \frac{k_1\left(\mathbf{s}^0, k_0\right)}{-k_0^2} k_0 \\
&= p_0\left(\mathbf{s}^0\right) f_k\left(1, \frac{h_0\left(\mathbf{s}^0, k_0\right)}{k_0}\right) - c\left(\gamma_0\right) + c'\left(\gamma_0\right)\gamma_0.
\end{aligned}$$

The first-order condition with respect to $h_t\left(\mathbf{s}^t, k_0\right)$ gives us that, in any period t with history $\mathbf{s}^t$,

$$\begin{aligned}
\frac{w_t\left(\mathbf{s}^t\right)}{p_t\left(\mathbf{s}^t\right)} &= f_h\left(k_t\left(\mathbf{s}^{t-1}, k_0\right), h_t\left(\mathbf{s}^t, k_0\right)\right) \\
\Rightarrow g\left(\frac{h_t\left(\mathbf{s}^t, k_0\right)}{k_t\left(\mathbf{s}^{t-1}, k_0\right)}\right) &= \frac{w_t\left(\mathbf{s}^t\right)}{p_t\left(\mathbf{s}^t\right)}.
\end{aligned}$$

This allows us to write

$$v_0^* = \left[p_0 \tilde{g}\left(\frac{w_0}{p_0}\right)\right] + \left[-c\left(\gamma_0\right) + c'\left(\gamma_0\right)\gamma_0\right],$$

where $\gamma_0 \coloneqq k_1\left(\mathbf{s}^0, k_0\right)/k_0$ and $\tilde{g}\left(x\right) \coloneqq f_k\left(1, g^{-1}\left(x\right)\right)$. We can interpret the first term in square brackets as the marginal revenue of the firm from a small increase in k_0, and the second square bracket as the change in the investment rate of the firm. Note that γ_0 is the growth rate of capital between period 0 and 1.

Suppose we observe v_0^*; i.e. the value of the firm. Since p_0 and w_0 are also given:

$$v_0^* - p_0 \tilde{g}\left(\frac{w_0}{p_0}\right) = -c\left(\gamma_0\right) + c'\left(\gamma_0\right)\gamma_0. \tag{4.4}$$

Now, notice that

$$\frac{d\left(-c\left(\gamma_0\right) + c'\left(\gamma_0\right)\gamma_0\right)}{d\gamma_0} = -c'\left(\gamma_0\right) + c''\left(\gamma_0\right)\gamma_0 + c'\left(\gamma_0\right) = c''\left(\gamma_0\right)\gamma_0 > 0$$

since we assume c is strictly convex. Thus, the right-hand side is strictly increasing in γ_0 and the left-hand side is a constant with respect to γ_0. Thus, with appropriate Inada conditions on c, there exists a unique solution for γ_0 in (4.4); i.e. we can back out the the optimal growth rate in capital—in other words, optimal investment rate.

This means that the expected value of the firm v_0^* is sufficient to determine the optimal investment rate for the firm. In other words, investment is only a function of average value of a unit of capital—called the Tobin's q:

$$\begin{aligned}
\frac{v\left(k_0\right)}{k_0} &\coloneqq \text{Tobin's average } q, \\
v'\left(k_0\right) &\coloneqq \text{Tobin's marginal } q.
\end{aligned}$$

Of course, investment decision is decided on the marginal (rather than average). However, as we can see here, under our set of assumptions (perfect competition and constant returns to scale production and investment cost functions), $v\left(k_0\right)$ is linear such that average and marginal Tobin's q are equal.

Since $v(k_0)$ is linear, as it is under our set of assumptions (perfect competition and constant returns to scale production and investment cost functions), the two are equal. This helps us empirically since average q is easier to measure than marginal q.

If we did not have constant returns to scale, then there will be a wedge between average and marginal q. Hayashi (1982) constructs this wedge that also includes the effect of taxes. He then regresses aggregate investment rate on this tax-adjusted measure of marginal q and finds that it explains around 46% of the variation (R^2). However, this represented the high point of q-theory. Subsequent work has found that q-theory does not perform well on micro data although one might expect it to work better (since micro data suffers less from the effect of aggregating over heterogenous firms). They find that other things explains investment behaviour, despite Tobin's q being a sufficient statistic theoretically. For example, current cash flow of the firms, even conditional on tax adjusted Tobin's q. (This did not appear in our model since we assumed perfect access to capital markets.)

One possibility of failure is that we are not measuring q accurately. This might be due to the fact that what we observe in the data is the average q and so we may be misspecifying how we adjust the average q to obtain estimates of marginal q. Another possibility is that we may have the whole adjustment cost specification wrong. The nature of adjustment cost often seems non-convex and exhibits irreversibility of investment (e.g. a wedge between a buy and sell price of capital). It turns out that if we can calculate marginal q in models with non-convex costs, marginal q is still a sufficient statistic for investment. But the relationship between marginal q and investment becomes highly non-linear—we can obtain inaction regions—which would be difficult to pick up in regressions.

Example 4.1. Suppose

$$c(\gamma) := \frac{\gamma\delta}{1+\delta(1-\gamma)},\ 0 \leq \gamma < 1 + \frac{1}{\delta},$$

where δ has the interpretation as the deprecation rate since $c(1) = \delta$.Then,

$$\begin{aligned}
c'(\gamma) &= \frac{\delta(1+\delta(1-\gamma)) + \delta^2\gamma}{(1+\delta(1-\gamma))^2} = \frac{\delta(1+\delta)}{(1+\delta(1-\gamma))^2} \\
\Rightarrow c'(\gamma)\gamma &= \frac{1+\delta}{1+\delta(1-\gamma)}c(\gamma) \\
\Rightarrow -c(\gamma) + c'(\gamma)\gamma &= -c(\gamma) + \frac{1+\delta}{1+\delta(1-\gamma)}c(\gamma) = \frac{-c(\gamma) - c(\gamma)\delta(1-\gamma) + (1+\delta)c(\gamma)}{1+\delta(1-\gamma)} \\
&= c(\gamma)\frac{\gamma}{1+\delta(1-\gamma)} = c(\gamma)^2.
\end{aligned}$$

Define $z := v^{*\prime}(k_0) - p_0\tilde{g}(w_0/p_0)$, so that

$$
\begin{aligned}
z &= \left(\frac{\gamma_0\delta}{1+\delta(1-\gamma_0)}\right)^2 \\
\Leftrightarrow \gamma_0^2\delta^2 &= z\left(1+2\delta(1-\gamma_0)+\delta^2(1-\gamma_0)^2\right) \\
0 &= z\left(1+2\delta-2\gamma_0\delta+\delta^2-2\gamma_0\delta^2+\gamma_0^2\delta^2\right)-\gamma_0^2\delta^2 \\
&= \gamma_0^2\left(\delta^2(z-1)\right)+\gamma_0\left(-2\delta z(1+\delta)\right)+z\left(1+2\delta+\delta^2\right) \\
&= \gamma^2\left(\delta^2(z-1)\right)+\gamma_0\left(-2\delta z(1+\delta)\right)+z(1+\delta)^2 \\
\Leftrightarrow \gamma &= \frac{2\delta z(1+\delta)\pm\sqrt{(2\delta z(1+\delta))^2-4\delta^2(z-1)z(1+\delta)^2}}{2\delta^2(z-1)} \\
&= \frac{1+\delta}{\delta}\frac{z\pm\sqrt{z}}{z-1}.
\end{aligned}
$$

4.2.2 Industry equilibrium

Lucas and Prescott (1971) assumes large number of price-taking firms who, as an industry, faces a downward-sloping demand curve (and/or an upward-sloping labour supply curve). We showed that the growth rate of the firm (for given price and wage) was independent of its size (i.e. v_0^* was independent of k_0), it follows that all firms would want to grow at the same rate. Thus, to determine the future growth rate of the industry, it suffices to know the total capital stock of the industry. Thus, we need not think about the distribution of the sizes of the firms in the industry.

Solving for the industry equilibrium involves first solving for the firms optimal policy given wages and prices. Then, ensuring that wages and prices are consistent with the policy decisions of the firms. So it appears to be involve solving a fixed-point problem. However, Lucas and Prescott (1971) show that the industry equilibrium can be characterised by solving a social planner's problem in which consumer surplus is maximised. This is how they prove existence and uniqueness.

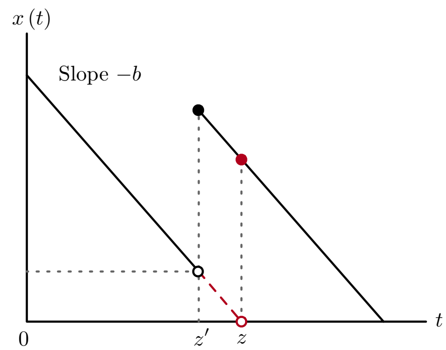
4.3 Non-convex adjustment costs

We start with a simple inventory problem.

4.3.1 Inventory management problem with discounting (Baumol-Tobin model of money)

For convenience, we work in continuous time. Consider a firm that sells goods at a deterministic rate, $\dot{x}(t) = -b$, from its inventory, $x(t) \geq 0$. As it runs out of inventory, it must restock its goods. There are both costs associated with restocking, c (independent of the how much they restock), as well as holding inventory, $ax(t)$. We assume that firm has discount rate $\rho > 0$.

When would you restock? Suppose that the firm decides to restock at time $t = z'$ such that $x(z') > 0$ to some positive level. Now, consider an alternative policy in which the firm restocks at time $t = z$ such that $x(z) = 0$ to the level equivalent to the original policy at time t (i.e. to $x(z') - (z - z')b$). This means that after $t > z$, the two policies are exactly equivalent. But the second policy is better for two reasons: (i) inventory level is lower at every point in time so that inventory cost is lower; and (ii) the second policy involves paying the restocking cost at a later point in time; i.e. discounted cost of restocking is lower. Hence, the optimal policy involves waiting until the current stock falls to zero before restocking.



We wish to study how much the firm should restock. Since the model is stationary, the firm would restock to the same level at every time stock level reaches zero. Now, let $v(x)$ denote the value function where x denotes the current level of stock. Since stock level declines at rate $-b$, it takes x/b "periods" for the stock to be ran down. During this time, the firm has to pay the discounted holding cost at rate $e^{-\rho t}a$. Once the stock is ran down to zero after x/b periods, the firm restocks, and we denote the value (cost) at that point in time as v_0. Hence, the value function is given by

$$v(x) = \int_0^{x/b} e^{-\rho t} a (x - bt)\, dt + e^{-\rho x/b} v_0,$$

and v_0 is such that the firm chooses the restock level y that minimises the future cost

$$v_0 = \min_y \{c + v(y)\}. \tag{4.5}$$

Using integration by parts, we can write $v(x)$ as

$$\begin{aligned} v(x) &= a \int_0^{x/b} e^{-\rho t} (x - bt)\, dt + e^{-\rho x/b} v_0 \\ &= a \left(\left[-\frac{1}{\rho} e^{-\rho t} (x - bt) \right]_0^{x/b} - \int_0^{x/b} -\frac{1}{\rho} e^{-\rho t} (-b)\, dt \right) + e^{-\rho x/b} v_0 \\ &= a \left(\frac{x}{\rho} - \frac{b}{\rho} \left[-\frac{1}{\rho} e^{-\rho t} \right]_0^{x/b} \right) + e^{-\rho x/b} v_0 \\ &= a \left(\frac{x}{\rho} - \frac{b}{\rho^2} \left(1 - e^{-\rho x/b} \right) \right) + e^{-\rho x/b} v_0 \\ &= \frac{a}{\rho^2} \left(x\rho - b \left(1 - e^{-\rho x/b} \right) \right) + e^{-\rho x/b} v_0. \end{aligned}$$

We can solve this by substituting in for $v(y)$ using above into (4.5) (while ignoring the min):

$$\begin{aligned} v_0 &= c + \frac{a}{\rho^2}\left(x\rho - b\left(1 - e^{-\rho x/b}\right)\right) + e^{-\rho x/b} v_0 \\ \Leftrightarrow \left(1 - e^{-\rho x/b}\right) v_0 &= c + \frac{a}{\rho^2}\left(x\rho - b\left(1 - e^{-\rho x/b}\right)\right) \\ \Leftrightarrow v_0 &= \frac{c + \frac{a}{\rho^2}\left(x\rho - b\left(1 - e^{-\rho x/b}\right)\right)}{1 - e^{-\rho x/b}} \\ &= \frac{c + ax/\rho}{1 - e^{-\rho x/b}} - \frac{ab}{\rho^2}. \end{aligned}$$

We can then solve

$$\min_x \ \frac{c + ax/\rho}{1 - e^{-\rho x/b}} - \frac{ab}{\rho^2}.$$

The first-order condition gives that

$$\begin{aligned} 0 &= \frac{\frac{a}{\rho}\left(1 - e^{-\rho\bar{x}/b}\right) - \left(c + \frac{a\bar{x}}{\rho}\right)\left(\frac{\rho}{b} e^{-\rho\bar{x}/b}\right)}{\left(1 - e^{-\rho\bar{x}/b}\right)^2} \\ \Leftrightarrow c\frac{\rho}{b} e^{-\rho\bar{x}/b} &= \frac{a}{\rho}\left(1 - e^{-\rho\bar{x}/b}\right) - \frac{a\bar{x}}{b} e^{-\rho\bar{x}/b} \\ \Leftrightarrow c &= \frac{ab}{\rho^2}\left(e^{\rho\bar{x}/b} - 1\right) - \frac{a\bar{x}}{\rho} \\ &= \frac{a\left[b\left(e^{\rho\bar{x}/b} - 1\right) - \rho\bar{x}\right]}{\rho^2}, \end{aligned}$$

which implicitly defines $\bar{x}$. Let us consider what happens as $\rho \to 0$. Using L'hôpital's rule twice:

$$\begin{aligned} \lim_{\rho\to 0} c &= \lim_{\rho\to 0} \frac{\frac{\partial}{\partial\rho} a\left[b\left(e^{\rho\bar{x}/b} - 1\right) - \rho\bar{x}\right]}{2\rho} = \lim_{\rho\to 0} \frac{a\left(\bar{x}e^{\rho\bar{x}/b} - \bar{x}\right)}{2\rho} \\ &= \lim_{\rho\to 0} \frac{\frac{\partial}{\partial\rho} a\left(\bar{x}e^{\rho\bar{x}/b} - \bar{x}\right)}{2} = \lim_{\rho\to 0} \frac{a\bar{x}^2}{2b} e^{\rho\bar{x}/b} \\ &= \frac{a\bar{x}^2}{2b}. \\ \Rightarrow \bar{x} &\simeq \sqrt{\frac{2cb}{a}}. \end{aligned}$$

For values of c close to zero (restocking cost), the restocking level $\bar{x}$ is large so that the inaction region (the range of x for which no action is taken) will be large.

Remark 4.1. (Baumol-Tobin interpretation) $\bar{x}$, which is how much to withdraw from the bank, is a function of: (i) c, the cost of going to the bank; b, the rate at which you spend money; and (ii) a, the cost per unit of holding money (foregone interest/risk of theft).

Remark 4.2. We can solve for $\bar{x}$ another way. Since $v(x)$ is the cost of starting from a particular level of stock, we can consider x that minimises this cost. In doing so, we will use the fact that v_0 is not a function of x in the equation for $v(x)$.

$$\begin{aligned} &\min_x \int_0^{x/b} e^{-\rho t} a(x - bt)\, dt + e^{-\rho x/b} v_0 \\ \equiv &\min_x \ \frac{a}{\rho^2}\left(x\rho - b\left(1 - e^{-\rho x/b}\right)\right) + e^{-\rho x/b} v_0. \end{aligned}$$

The first-order condition gives

$$\begin{aligned}\frac{a}{\rho}\left(1-e^{-\rho\bar{x}/b}\right)&=\frac{\rho}{b}e^{-\rho\bar{x}/b}v_0\\ \Leftrightarrow v_0&=\frac{ab}{\rho^2}\left(e^{\rho\bar{x}/b}-1\right)\end{aligned}$$

Substituting this into (4.5) yields the same expression for $\bar{x}$:

$$\begin{aligned}v_0&=\min_y\left\{c+v\left(y\right)\right\}\\ &=c+\frac{a}{\rho^2}\left(\bar{x}\rho-b\left(1-e^{-\rho\bar{x}/b}\right)\right)+e^{-\rho\bar{x}/b}v_0\\ \Leftrightarrow c&=\left(1-e^{-\rho x/b}\right)v_0-\frac{a}{\rho^2}\left(\bar{x}\rho-b\left(1-e^{-\rho\bar{x}/b}\right)\right)\\ &=\left(1-e^{-\rho\bar{x}/b}\right)\frac{ab}{\rho^2}\left(e^{\rho\bar{x}/b}-1\right)-\frac{a}{\rho^2}\left(\bar{x}\rho-b\left(1-e^{-\rho\bar{x}/b}\right)\right)\\ &=\frac{a}{\rho^2}\left[\left(1-e^{-\rho\bar{x}/b}\right)\left(e^{\rho\bar{x}/b}-1\right)b-\rho\bar{x}+b\left(1-e^{-\rho\bar{x}/b}\right)\right]\\ &=\frac{a}{\rho^2}\left[\left(1-e^{-\rho\bar{x}/b}\right)\left(\left(e^{\rho\bar{x}/b}-1\right)b+b\right)-\rho\bar{x}\right]\\ &=\frac{a\left[b\left(e^{\rho\bar{x}/b}-1\right)-\rho\bar{x}\right]}{\rho^2}.\end{aligned}$$

4.3.2 Inventory management problem with no discounting

Let us now consider how we can solve the inventory management problem without discounting. Often discounting rate is small relative to other parts of the model and we can get good intuition by considering an undercounted problem.

We cannot use the previous setup since when $\rho=0$, $v_0=\infty$. But, we know from the discounted problem that there is a constant threshold, $\bar{x}$, such that when you restock, you will always order amount $\bar{x}$. This reflects the stationary nature of the problem. Let's think about choosing an arbitrary $\bar{x}$. This affects the average inventory holding cost and it's going to affect the frequency with which to restock. The average inventory holding cost is given by

$$\int_0^{\bar{x}/b}a\left(\bar{x}-bt\right)dt=a\left[\bar{x}t-\frac{1}{2}bt^2\right]_0^{\bar{x}/b}=a\left(\frac{\bar{x}^2}{b}-\frac{\bar{x}^2}{2b}\right)=\frac{a\bar{x}^2}{2b}.$$

The average cost for each cycle is given by the sum of the average inventory holding cost and the restocking cost, c; i.e.

$$\frac{a\bar{x}^2}{2b}+c.$$

We then want to divide this by the length of the inventory cycle, which is given by $\bar{x}/b$, to obtain the average cost for each cycle per unit of time:

$$\frac{1}{\bar{x}/b}\left(\frac{a\bar{x}^2}{2b}+c\right)=\frac{a\bar{x}}{2}+\frac{bc}{\bar{x}}.$$

Then, we want to choose $\bar{x}$ to minimise this cost:

$$\bar{x}=\arg\min_x\ \frac{ax}{2}+\frac{bc}{x}.$$

The first-order condition gives us that

$$\frac{a}{2} = \frac{bc}{\bar{x}^2} \Leftrightarrow \bar{x} = \sqrt{\frac{2bc}{a}}.$$

Thus, we obtain the same solution as in the discounted problem as $\rho \to 0$.

4.3.3 Fixed cost of adjustment with no discounting

Single firm's problem We now think of a firm that chooses a capital stock at every point in time (with no discounting) to maximise its profits. We assume that there exists some shock that affects its desired level of capital stock, k_t^*, in each period (think of this as some wage/productivity shock). Specifically,

$$\log k_t^* = \begin{cases} \log k_{t-1}^* - \Delta & \text{w.p. } p \\ \log k_{t-1}^* & \text{w.p. } 1-2p \\ \log k_{t-1}^* + \Delta & \text{w.p. } p \end{cases},$$

where $\Delta > 0$ is the percentage change in the desired level of capital stock, and $p \in (0, 1/2)$. The firm faces a fixed cost of adjustment, given by

$$a\left(\log k_t - \log k_t^*\right)^2 + x_t, \tag{4.6}$$

where $\log k_t$ is the actual capital stock, and x_t is investment given by

$$x_t = \begin{cases} 0 & \text{if } k_t = k_{t-1} \\ \lambda & \text{if } k_t \neq k_{t-1} \end{cases},$$

where $\lambda > 0$. The firm therefore faces two types of costs: (i) (symmetric) loss from operating at a suboptimal level of capital stock; and (ii) a "restocking" cost (as before, this does not depend on the size of adjustment of capital stock).

The existence of adjustment cost means that, most of the time, the firm will keep its capital stock level fixed but if the level strays too far from the desired level, the firm will incur the adjustment cost and adjust its actual level of capital stock.

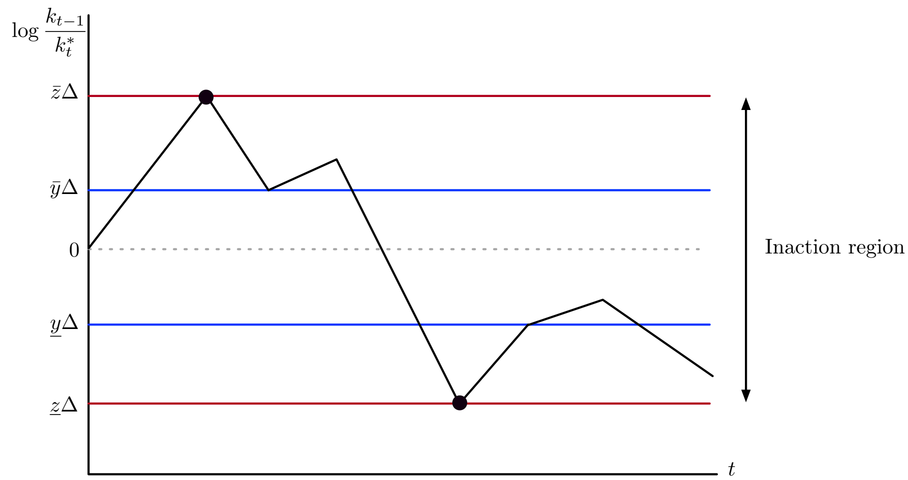
Profit of the firm is:[25]

$$\pi - \text{cost}.$$

What is the problem for the firm?

Since π is constant, profit maximising is the same as minimising the cost. Now, the firm "wakes up" each morning with some capital stock carried over, $\log k_{t-1}$, and the desired level of capital, $\log k_t^*$. Given the adjustment cost, the firm would only invest if the two are sufficiently different. There will be some threshold $\bar{z}$ such that if $\log k_{t-1} - \log k_t^* \geq \bar{z}\Delta > 0$, then the firm adjusts to a *return point*, say $\bar{y}\Delta$. Similarly, there will be some threshold $\underline{z}$ such that if $\log k_{t-1} - \log k_t^* \leq \underline{z}\Delta < 0$, then the firm adjusts to another return point, $\underline{y}\Delta$. The figure below shows how the $\log k_{t-1} - \log k_t^*$ (% deviation of actual capital stock from the desired level in each period) might look over time.

[25] Alternatively, we can account for the scale of the firm by writing the profit of the firm as $(\pi - \text{cost})\, k_t^*$. But the algebra becomes more complicated!



Given the symmetry (of the loss function and the shocks) we will find the following:

▷ $-\underline{y} = \overline{y}$ and $-\underline{z} = \bar{z}$.

▷ $\underline{y} = \overline{y} = 0$.

That is, the inaction region is symmetric around zero and that when the threshold is reached, the firm readjusts to the static optimal level of capital. This also implies that the optimal policy is characterised by $\bar{z}$.

In a similar way to how we solved the inventory management problem with no discounting, we will think about the fraction of time that the firm spends in each possible states of the economy. Let the number of "steps" aways from the ideal capital stock level as y_t

$$y_t \coloneqq \frac{\log k_t - \log k_t^*}{\Delta}$$

and the optimal policy takes the form

$$\{\bar{z}, \underline{z}, 0, 0\}.$$

Observe that the possible values of y_t are

$$\{-\bar{z}+1, -\bar{z}+2, \ldots, 0, \ldots, \bar{z}-2, \bar{z}-1\} \tag{4.7}$$

so it can take one of $2\bar{z}-1$ possible values. Let $\phi(y)$ denote the density of the ergodic distribution where the support of this distribution is given by (4.7). How do we derive this distribution?

By definition, ergodic distribution is such that $\phi(y)$ remains constant over time. For this to hold, it must be that the probability of "leaving" a particular state is the same as the probability of "arriving" from other states (i.e inflow equals out flow). If the current state is $y \neq 0$ ($y = 0$ is special since you can arrive from non-adjacent states!), then the probability of leaving is $2p$. The probability of arrival is: (i) with probability p, arrives from $y+1$; (ii) with probability p arrives from $y-1$. Hence,[26]

$$\begin{aligned} 2p\phi(y) &= p\phi(y+1) + p\phi(y-1), \ \forall y \neq 0 \\ \Leftrightarrow 2\phi(y) &= \phi(y+1) + \phi(y-1), \ \forall y \neq 0 \end{aligned}$$

where we define the boundary conditions as

$$\phi(-\bar{z}), \phi(\bar{z}) \coloneqq 0.$$

This is a second-order difference equation. Note that $\phi(y)$ does not depend on p. We can solve this difference equation. Let us rewrite as

$$0 = \phi(y+1) - 2\phi(y) + \phi(y-1)$$

and so the characteristic polynomial is given by

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

This has a single real root $\lambda = 1$ so that the solution to the difference equation has the form:[27]

$$\phi(y) = (A + By)\lambda^t = A + By.$$

Given that $y \neq 0$, we must consider two cases. First, when $y > 0$, the relevant boundary condition gives us that

$$\begin{aligned} \phi^+(\bar{z}) &= \bar{A} + \bar{B}\bar{z} = 0 \\ \Leftrightarrow \bar{z} &= -\frac{\bar{A}}{\bar{B}}. \end{aligned}$$

So

$$\begin{aligned} \phi^+(y) &= \bar{A} + \bar{B}y = -\bar{B}\left(-\frac{\bar{A}}{\bar{B}} - y\right) \\ &= -\bar{B}(\bar{z} - y) = -\bar{B}\bar{z}\left(1 - \frac{y}{\bar{z}}\right) \\ &\equiv \bar{D}\left(1 - \frac{y}{\bar{z}}\right). \end{aligned}$$

Second, when $y < 0$,

$$\begin{aligned} \phi^-(-\bar{z}) &= \underline{A} - \underline{B}\bar{z} = 0 \\ \Leftrightarrow \bar{z} &= \frac{\underline{A}}{\underline{B}}. \end{aligned}$$

[26] Let $\phi(y;t)$ denote the density for state y in period t. We can obtain the same expression by noting that

$$\phi(y;t) = (1-2p)\phi(y;t-1) + p\phi(y-1;t-1) + p\phi(y+1;t-1)$$

and imposing that stationary condition that $\phi(y;t) = \phi(y;t-1) = \phi(y)$:

$$2p\phi(y) = p\phi(y-1) + p\phi(y+1).$$

[27] With two distinct real roots, solution takes the form $A\lambda_1^t + B\lambda_2^t$.

Hence,

$$\begin{aligned}\phi^-(y) &= \underline{A} - \underline{B}y = -\underline{B}\left(-\frac{\underline{A}}{\underline{B}} + y\right)\\ &= -\underline{B}(-\bar{z}+y) = \underline{B}\bar{z}\left(1+\frac{y}{\bar{z}}\right)\\ &\equiv \underline{D}\left(1+\frac{y}{\bar{z}}\right).\end{aligned}$$

We can summarise the result as

$$\phi(y) = \begin{cases}\bar{D}\left(1-\frac{y}{\bar{z}}\right) & \text{if } y>0\\ \underline{D}\left(1+\frac{y}{\bar{z}}\right) & \text{if } y<0\end{cases}.$$

Let us now consider the case when $y=0$. The leaving rate from $y=0$ is $2p\phi(0)$. The arrival rate is the sum of: (i) probability of arrival from $y=1$, p; (ii) probability of arrival from $y=-1$, p; (iii) probability of arrival from $y=-\bar{z}+1$, p; (iv) probability of arrival from $y=\bar{z}-1$. Hence,

$$\begin{aligned}2p\phi(0) &= p\phi(1)+p\phi(-1)+p\phi(-\bar{z}+1)+p\phi(\bar{z}-1)\\ \Leftrightarrow 2\phi(0) &= \phi(1)+\phi(-1)+\phi(-\bar{z}+1)+\phi(\bar{z}-1).\end{aligned}$$

Substituting $\phi(y)$ that we obtained for $y\neq 0$, we obtain

$$\begin{aligned}2\phi(0) &= \bar{D}\left(1-\frac{1}{\bar{z}}\right)+\underline{D}\left(1-\frac{1}{\bar{z}}\right)+\underline{D}\left(1-\frac{\bar{z}-1}{\bar{z}}\right)+\bar{D}\left(1-\frac{\bar{z}-1}{\bar{z}}\right)\\ &= \left(\bar{D}+\underline{D}\right)\left(1-\frac{1}{\bar{z}}\right)+\left(\bar{D}+\underline{D}\right)\left(1-\frac{\bar{z}-1}{\bar{z}}\right)\\ &= \left(\bar{D}+\underline{D}\right)\left(1-\frac{1}{\bar{z}}\right)+\left(\bar{D}+\underline{D}\right)\left(\frac{1}{\bar{z}}\right)\\ \Leftrightarrow \phi(0) &= \frac{\bar{D}+\underline{D}}{2}.\end{aligned}$$

This is the boundary condition which we can use for ϕ^+ and ϕ^-.

$$\begin{aligned}\phi^+(0) &= \bar{A} = \frac{\bar{D}+\underline{D}}{2},\\ \phi^-(0) &= \underline{A} = \frac{\bar{D}+\underline{D}}{2},\\ &\Rightarrow \bar{A} = \underline{A}\\ &\Rightarrow \bar{B} = -\underline{B}.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{D} &= -\bar{B}\bar{z}\\ &= \underline{B}\bar{z} = \underline{D}.\end{aligned}$$

That is, the two constants are equal, which, in turn, implies that

$$\phi(0) = \bar{D}.$$

Since $\phi(y)$ is a probability, $\phi(y)$'s must sum to one. We use this to back out the value of $\bar{D}$. In doing so, remember that y can take $2\bar{z}-1$ distinct values.

$$\begin{aligned}
1 &= \sum_{y=-\bar{z}+1}^{\bar{z}-1} \phi(y) = \sum_{y=-\bar{z}+1}^{-1} \phi(y) + \phi(0) + \sum_{y=1}^{\bar{z}-1} \phi(y) \\
&= \bar{D} \sum_{y=-\bar{z}+1}^{-1} \left(1+\frac{y}{\bar{z}}\right) + \bar{D} + \bar{D}\sum_{y=1}^{\bar{z}-1}\left(1-\frac{y}{\bar{z}}\right) \\
&= \bar{D}\left(1+\left(2\sum_{y=1}^{\bar{z}-1} 1\right) + \frac{2}{\bar{z}}\sum_{y=1}^{\bar{z}-1} y\right) \\
&= \bar{D}\left(1+2(\bar{z}-1)+\frac{2}{\bar{z}}(-1-2-\cdots-(\bar{z}-2)-(\bar{z}-1))\right) \\
&= \bar{D}\left(1+2(\bar{z}-1)-\frac{2}{\bar{z}}(1+2+\cdots+(\bar{z}-2)+(\bar{z}-1))\right) \\
&= \bar{D}\left(1+2(\bar{z}-1)+\frac{2}{\bar{z}}(\bar{z}-1)\frac{1+(\bar{z}-1)}{2}\right) = \bar{D}(1+2(\bar{z}-1)-(\bar{z}-1)) \\
\Leftrightarrow \bar{D} &= \frac{1}{\bar{z}}.
\end{aligned}$$

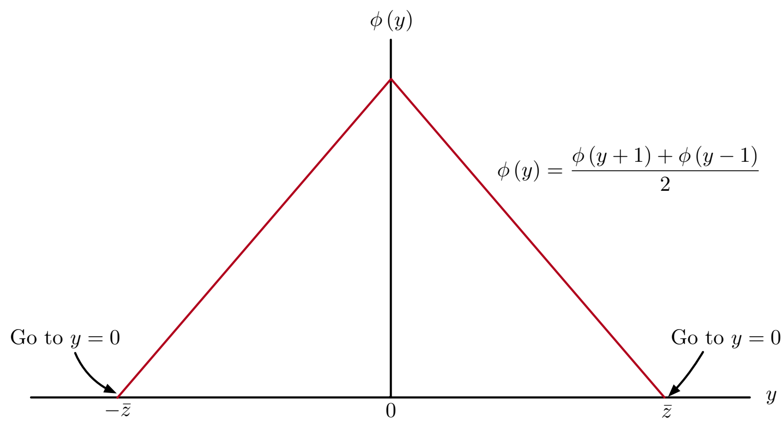
We can now write the unique stationary solution as

$$\begin{aligned}
\phi(y) &= \begin{cases} \frac{1}{\bar{z}}\left(1-\frac{y}{\bar{z}}\right) & \text{if } y>0 \\ \frac{1}{\bar{z}} & \text{if } y=0 \\ \frac{1}{\bar{z}}\left(1+\frac{y}{\bar{z}}\right) & \text{if } y<0 \end{cases} \\
&= \frac{1}{\bar{z}}\left(1-\frac{|y|}{\bar{z}}\right) \\
&= \frac{1-|y|/\bar{z}}{\bar{z}},\ \forall -\bar{z} \le y \le \bar{z}.
\end{aligned}$$

The density $\phi(y)$ is triangular. The linearity comes from the fact that the difference equation tells us that $\phi(y)$ is an average of densities at the adjacent states for all $-\bar{z} < y < \bar{z}$:

$$\phi(y) = \frac{\phi(y+1)+\phi(y-1)}{2}.$$

The shape of the density reflects the symmetric structure we imposed on the model. More generally, the density are asymmetric and, for example, can be convex and concave.



We can now compute the average cost of a particular policy, $\{\bar{z}, -\bar{z}, 0, 0\}$ ($\phi(y)$ gives the "time spent" in each state and can be used as the weight in calculating the average cost). Recall (4.6). Since, in state y, $\log k_t - \log k_t^* = \Delta y$, the expected (or average) cost is given by

$$\text{cost} = \sum_{y=-\bar{z}}^{\bar{z}} a(\Delta y)^2 \phi(y) + \lambda p (\phi(-\bar{z}+1) + \phi(\bar{z}-1)),$$

where the second term follows because, with probability p, in states $y = -\bar{z}+1$ and $y = \bar{z}-1$, the state moves to $y=0$, so the firm incurs the restocking cost λ. Substituting the functional forms for ϕ's yields

$$\begin{aligned}
\text{cost} &= \sum_{y=-\bar{z}}^{\bar{z}} a(\Delta y)^2 \phi(y) + \lambda p \left(\frac{1 - |-\bar{z}+1|/\bar{z}}{\bar{z}} + \frac{1 - |\bar{z}-1|/\bar{z}}{\bar{z}} \right) \\
&= \sum_{y=-\bar{z}}^{\bar{z}} a(\Delta y)^2 \phi(y) + 2\lambda p \frac{1 - \frac{\bar{z}-1}{\bar{z}}}{\bar{z}} \\
&= \sum_{y=-\bar{z}}^{\bar{z}} a(\Delta y)^2 \phi(y) + \frac{2\lambda p}{\bar{z}^2}.
\end{aligned}$$

Now, let us focus on the summation term. Since $\phi(y)$ is symmetric, and the term is zero when $y=0$,

$$\begin{aligned}
\sum_{y=-\bar{z}}^{\bar{z}} a(\Delta y)^2 \phi(y) &= 2a\Delta^2 \sum_{y=1}^{\bar{z}} y^2 \phi(y) \\
&= 2a\Delta^2 \sum_{y=1}^{\bar{z}} y^2 \frac{1 - |y|/\bar{z}}{\bar{z}} \\
&= \frac{2a\Delta^2}{\bar{z}^2} \left(\bar{z} \sum_{y=1}^{\bar{z}} y^2 - \sum_{y=1}^{\bar{z}} y^3 \right).
\end{aligned}$$

Faulhaber's formula gives us that

$$\begin{aligned}\sum_{y=1}^{\bar{z}} y^2 &= \frac{\bar{z}\left(\bar{z}+1\right)\left(2\bar{z}+1\right)}{6},\\ \sum_{y=1}^{\bar{z}} y^3 &= \frac{\bar{z}^2\left(\bar{z}+1\right)^2}{4}.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{z}\sum_{y=1}^{\bar{z}} y^2 - \sum_{y=1}^{\bar{z}} y^3 &= \frac{2\bar{z}^2\left(\bar{z}+1\right)\left(2\bar{z}+1\right) - 3\bar{z}^2\left(\bar{z}+1\right)^2}{12}\\ &= \frac{\bar{z}^2\left(\bar{z}+1\right)\left(4\bar{z}+2-3\bar{z}-3\right)}{12}\\ &= \frac{\bar{z}^2\left(\bar{z}+1\right)\left(\bar{z}-1\right)}{12} = \frac{\bar{z}^2\left(\bar{z}^2-1\right)}{12}.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{cost} &= \frac{2a\Delta^2}{\bar{z}^2}\frac{\bar{z}^2\left(\bar{z}^2-1\right)}{12} + \frac{2\lambda p}{\bar{z}^2}\\ &= \frac{a\Delta^2}{6}\left(\bar{z}^2-1\right) + \frac{2\lambda p}{\bar{z}^2}.\end{aligned}$$

The optimal $\bar{z}$, then minimises this cost; i.e.

$$\bar{z} = \arg\max_{z}\ \frac{a\Delta^2}{6}\left(z^2-1\right) + \frac{2\lambda p}{z^2}.$$

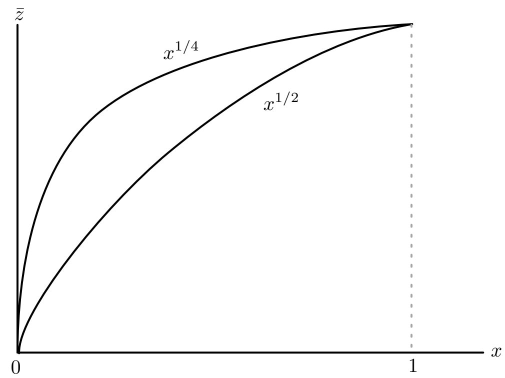
The first-order condition gives

$$\begin{aligned}\frac{a\Delta^2}{3}\bar{z} &= \frac{4\lambda p}{z^3}\\ \Leftrightarrow \bar{z} &= \left(\frac{12\lambda p}{a\Delta^2}\right)^{\frac{1}{4}}.\end{aligned}$$

Observe that:

- ▷ when a is large, missing the desired level of capital is costly, so that $\bar{z}$ is small;
- ▷ if Δ is large, steps are large, so that each shock can move you far away, so $\bar{z}$ is small;
- ▷ if p is small, once you move away from the desired point, you will stay there a for a long time, and that makes you more willing to pay to adjust ($\bar{z}$ larger); in contrast, as p becomes larger, there is option value of waiting, so you are more willing to wait.
- ▷ if the fixed cost of adjustment λ is larger, you adjust less frequently; i.e. $\bar{z}$ is higher.
- ▷ the power on the right-hand side is $1/4$, rather than $1/2$ as was the case in the Baumol-Tobin model. This means that for small fixed cost, λ, $\bar{z}$ is higher in this model (see figure below) so that this model has a larger inaction region than the Baumol-Tobin model. The difference in the power is driven by two difference between this stochastic model and the Baumol-Tobin

model. One is that there is a stochastic change in the desired capital level in this model whereas in the Baumol-Tobin model, there is a deterministic running down of inventory. The other is that there is now optimal thresholds on both sides.



Remark 4.3. As time periods become shorter, the model converges to a Brownian motion.

Industry problem with aggregate shocks Let us assume that there are many firms that experience independent idiosyncratic shocks to their desired level of capital (shocks are also independent over time). Thus, in the industry, a fraction p experiences a positive shock, another fraction p experiences a negative shock, and remaining fraction $1 - 2p$ does not experience a shock. We now want to think about industry investment level. In the absence of aggregate shocks, a fraction $p\phi(\bar{z} - 1)$ and $p\phi(-\bar{z} + 1)$ makes investment and others do not.

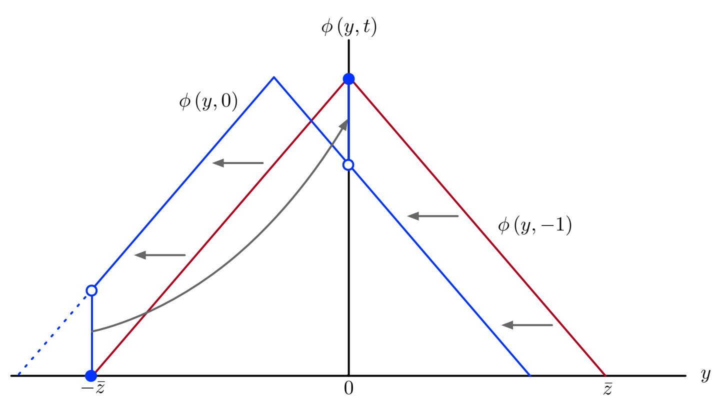
Let $\phi(y; t)$ denote the fraction of firms in the industry in state y in period t. We assume that in period $t = -1$, we begin with the ergodic distribution; i.e.

$$\phi(y; -1) = \frac{1 - |y| / \bar{z}}{\bar{z}}.$$

Consider the impact of a one-time unanticipated (i.e. MIT) shock in period 0:

$$\log k_0^* = \log k_{-1}^* + \Delta.$$

Since this is a one-time shock, in particular, $\bar{z}$ remains unchanged.



As the figure shows, this positive MIT shock in period 0 shifts the distribution to the left. This is because a higher k_t^* reduces y_t, which is defined as $y_t := (\log k_t - \log k_t^*)/\Delta$. However, since firms invest at $y = -\bar{z}$, those that would have moved to state $y < -\bar{z}$ would, in fact, adjust to $y = 0$. Hence, the new distribution has zero mass at $y = -\bar{z}$ (as usual) and has a "spike" for state $y = 0$. The shock causes firms in state $-\bar{z}+1$ at $t = -1$ to invest in period $t = 0$ following the shock. That is, a fraction

$$\phi(-\bar{z}+1) = \frac{1 - |-\bar{z}+1|/\bar{z}}{\bar{z}} = \frac{1-(\bar{z}-1)/\bar{z}}{\bar{z}} = \frac{1}{\bar{z}^2}$$

of firms invest. When they invest, they invest such that their state will be 0; i.e. they invest $\bar{z}\Delta$. Thus, net investment in period 0, x_0, is given

$$x_0 = \frac{1}{\bar{z}^2}\bar{z}\Delta = \frac{\Delta}{\bar{z}}.$$

We now want to see how the distribution changes over time in subsequent periods. We know that, in the long-run, all firms wish to invest Δ step more to move back to the ergodic distribution. So the sum of net investment from period $t = 1$ onwards must equal

$$\sum_{t=1}^{\infty} x_t = \Delta\left(1 - \frac{1}{\bar{z}}\right),$$

where we subtracted investment already made in period 0, $x_0 = \Delta/\bar{z}$.

Now, let us take the simplest nontrivial case when $\bar{z} = 2$ (if $\bar{z} = 1$, then firms would always be at the desired level of capital stock) so $y \in \{-1, 0, 1\}$. This gives us three difference equations for $\phi(-1;t)$, $\phi(0;t)$ and $\phi(1;t)$. But one of these equation is redundant since ϕ's all has to sum to one. So let us consider $\phi(-1;t)$ and $\phi(0;t)$.

Now in period $t \geq 1$, the fraction of firms in state -1 is: (i) $1-2p$ proportion of firms who were in state -1 in the previous period and received no shock; and (ii) p proportion of firms who were in state 0 in the previous period and received a positive shock; i.e.

$$\phi(-1;t) = (1-2p)\,\phi(-1;t-1) + p\phi(0;t-1), \ \forall t \geq 1. \tag{4.8}$$

The fraction of firms in state 0 is: (i) $1-2p$ proportion of firms who were already in state 0 and did not receive a shock; (ii) p proportion of firms who were in state -1 and receives a positive shock so that they invest to move to state 0; (iii) p proportion of firms who were in state -1 and receives a negative shock and moves to state 0 (without any investment); (iv) same as (ii) but for those who were in state 1; and (v) same as (iii) but for those who were in state 1:

$$\phi(0;t) = (1-2p)\,\phi(0;t-1) + 2p\,(\phi(-1;t-1) + \phi(1;t-1)). \tag{4.9}$$

Recall that the ergodic distribution is such that

$$\phi(0) = \frac{1}{\bar{z}} = \frac{1}{2},$$
$$\phi(1) = \phi(-1) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

Hence, in period $t = 0$, since the distribution is shifted one step to the left:

- ▷ those in state 1 in the previous period moves to state 0;

▷ those in state 0 in the previous period moves to state -1;

▷ those in state -1 in the previous period moves to state 0.

So the effect of the MIT shock is:

$$\begin{aligned}\phi(1;0) &= 0,\\ \phi(0;0) &= \phi(1;-1)+\phi(-1;-1)=\frac{1}{2},\\ \phi(-1;0) &= \phi(0;-1)=\frac{1}{2}.\end{aligned}$$

We can then establish the following which is useful in solving the problem when $\bar{z}=2$.

Claim 4.2. If $\phi(0;-1)=1/2$, then

$$\phi(0;t)=\frac{1}{2},\ \forall t.$$

Proof. Since there are only three states

$$\phi(-1;t)+\phi(1;t)=1-\phi(0;t),\ \forall t.$$

Substituting into (4.9) gives

$$\begin{aligned}\phi(0;t) &= (1-2p)\,\phi(0;t-1)+2p\,(\phi(-1;t-1)+\phi(1;t-1))\\ &= (1-2p)\,\phi(0;t-1)+2p\,(1-\phi(0;t-1))\end{aligned}$$

Now, since $\phi(0;t-1)=1/2$, we have

$$\begin{aligned}\phi(0;1) &= (1-2p)\,\frac{1}{2}+2p\frac{1}{2}=\frac{1}{2},\\ \Rightarrow \phi(0;t) &= \frac{1}{2},\ \forall t.\end{aligned}$$

■

We can then write (4.8) as

$$\phi(-1;t)=(1-2p)\,\phi(-1;t-1)+\frac{p}{2},$$

which is a first-order (partial) difference equation. The steady state (which is well-defined since $p\in(0,0.5)$) is

$$\phi(-1)=(1-2p)\,\phi(-1)+\frac{p}{2}\Leftrightarrow\phi(-1)=\frac{1}{4}$$

as we had seen before. We can write the difference equation as deviation from the steady state:

$$\begin{aligned}
\phi(-1;t)-\phi(-1) &= (1-2p)\,\phi(-1;t-1)+\frac{p}{2}-\left((1-2p)\,\phi(-1)+\frac{p}{2}\right)\\
&= (1-2p)\left(\phi(-1;t-1)-\phi(-1)\right)\\
&= (1-2p)^t\left(\phi(-1;0)-\phi(-1)\right)\\
\Leftrightarrow \phi(-1;t) &= (1-2p)^t\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{4}\\
&= \frac{1}{4}\left(1+(1-2p)^t\right).
\end{aligned}$$

Now consider net investment in each period. In period t, a fraction p of firms who were in state -1 in period $t-1$ is hit by a positive shock so that they adjust the capital stock. In doing so, they invest 2 units (since $\bar{z}=2$). A fraction p of firms who were in state 1 in period $t-1$ is hit by a negative shock and disinvest 2 units. Thus, net investment is given by

$$x_t = 2p\left(\phi(-1;t-1)-\phi(1;t-1)\right).$$

Substituting the functional form we found already gives

$$\begin{aligned}
x_t &= 2p\left(\phi(-1;t-1)-\left(1-\phi(0;t-1)-\phi(-1;t-1)\right)\right)\\
&= 2p\left(-1+\frac{1}{2}+2\phi(-1;t-1)\right)\\
&= 2p\left(2\frac{1}{4}\left(1+(1-2p)^{t-1}\right)-\frac{1}{2}\right)\\
&= p\,(1-2p)^{t-1}
\end{aligned}$$

Hence, we see that investment exponentially declines over time. Thus, we see that there is propagation mechanism for investment following an investment shock which we saw with convex adjustment cost. The difference, however, is that with non-convex cost, although there is no such propagation mechanism at the firm level, we see that such a mechanism exists at the aggregate level.

In class, Shimer wrote down $p\,(1-2p)^{t-1}/2$...

4.3.4 Fixed cost of adjustment with discounting

Firm chooses a sequence k_t that maximises

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t\left(\pi-c_t\right)\right],$$

subject to

$$c_t = a\left(\log k_t-\log k_t^*\right)^2+\lambda\mathbf{1}\left\{k_t\neq k_{t+1}\right\}$$

and we assume the same stochastic process for $\log k_t^*$. Since π plays no role in this model, the firm is just minimising the discounted cost. We can write the value function as

$$\begin{aligned} v(k,k^*) &= \min\{\text{don't adjust}, \text{adjust}\} \\ &= \min\left\{w(k,k^*), \min_{k'}\{w(k',k^*)+\lambda\}\right\}, \\ w(k,k^*) &= a(\log k - \log k^*)^2 + \beta\left(pv\left(k,k^*e^{\Delta}\right) + pv\left(k,k^*e^{-\Delta}\right) + (1-2p)v(k,k^*)\right). \end{aligned}$$

To solve this analytically, we want to translate the problem into choosing a sequence of y_t's, where

$$y \coloneqq \frac{\log k - \log k^*}{\Delta}.$$

Note that we can write c_t as

$$a\Delta^2 y_t^2 + \lambda \mathbf{1}\{k_{t+1} \neq k_t\}.$$

Observe that

$$\begin{aligned} \mathbf{1}\{k_{t+1} \neq k_t\} &= \mathbf{1}\left\{\frac{k_t}{k_t^*} \neq \frac{k_{t+1}}{k_{t+1}^*}\frac{k_{t+1}^*}{k_t^*}\right\} \\ &= \mathbf{1}\left\{\frac{1}{\Delta}\log\frac{k_t}{k_t^*} \neq \frac{1}{\Delta}\log\frac{k_{t+1}}{k_{t+1}^*} + \frac{1}{\Delta}\log\frac{k_{t+1}^*}{k_t^*}\right\} \\ &= \mathbf{1}\{y_t \neq y_{t+1} + \varepsilon_{t+1}\}, \end{aligned}$$

where

$$\varepsilon_{t+1} = \frac{\log k_{t+1}^* - \log k_t}{\Delta} = \begin{cases} -1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-2p \\ 1 & \text{w.p. } p \end{cases}.$$

Thus, we can write the sequence problem equivalently as

$$\max_{\{y_t\}} -\mathbb{E}\left[\sum_{t=1}^{\infty} \beta^t a\Delta^2 y_t^2 + \lambda \mathbf{1}\{y_t \neq y_{t+1} + \varepsilon_{t+1}\}\right].$$

The value function is then

$$\begin{aligned} v(y) &= \min\left\{w(y), \min_z\{w(z)+\lambda\}\right\}, \\ w(y) &= a\Delta^2 y^2 + \beta(pv(y-1) + pv(y+1) + (1-2p)v(y)). \end{aligned} \tag{4.10}$$

This appears similar to the difference equations for ϕ; however, there are two main differences: (i) we have the cost term, $a\Delta^2 y^2$; and (ii) we have discounting.

The optimal policy is that the firm there will be an inaction region y lies in the range $[\underline{z}, \bar{z}]$. That is, the firm would only adjust its capital stock when the state reaches some threshold. As before, we let $\bar{y}$ denote the level that the firm adjusts to when y reaches $\bar{z}$, and $\underline{y}$ denote the level that the firm adjusts to when y reaches $\underline{z}$. Here, $\bar{y} = \underline{y} = y$ since the cost of adjustment does not depend on which state you come from, and where y solves the problem above.

In the inaction region, $\underline{z} < y < \bar{z}$, "don't adjust" must be optimal; i.e.

$$\begin{aligned} v(y) &= w(y), \ \forall \underline{z} < y < \bar{z} \\ &= a\Delta^2 y^2 + \beta\left(pv(y-1) + pv(y+1) + (1-2p)\,v(y)\right). \end{aligned} \tag{4.11}$$

We see that this is a second-order difference equation in v. We first solve the homogenous equation

$$\begin{aligned} v(y) &= \beta\left(pv(y-1) + pv(y+1) + (1-2p)\,v(y)\right) \\ \Leftrightarrow 0 &= \beta pv(y-1) + \beta pv(y+1) + \beta(1-2p)\,v(y) - v(y) \\ &= \beta pv(y+1) + (\beta(1-2p)-1)\,v(y) + \beta pv(y-1). \end{aligned}$$

The characteristic polynomial is

$$\beta p\lambda^2 + (\beta(1-2p)-1)\,\lambda + \beta p$$

which has roots given by

$$\begin{aligned} \lambda &= \frac{1-\beta(1-2p) \pm \sqrt{(1-\beta(1-2p))^2 - 4\beta^2 p^2}}{2\beta p} \\ &= \frac{1-\beta(1-2p) \pm \sqrt{1-2\beta(1-2p) + \beta^2\left((1-2p)^2 - 4p^2\right)}}{2\beta p} \\ &= \frac{1-\beta(1-2p) \pm \sqrt{(1-2\beta(1-2p)) + \beta^2(1-4p)}}{2\beta p}. \end{aligned}$$

So we see that at at least one root is positive. Now, let λ_1 denote the positive root. Then, it must be such that

$$\beta p\lambda_1^2 + (\beta(1-2p)-1)\,\lambda_1 + \beta p = 0. \tag{4.12}$$

Now consider $\lambda_2 = 1/\lambda_1$. Substituting in for λ_1 in the expression above yields

$$\begin{aligned} &\beta p\lambda_2^2 + (\beta(1-2p)-1)\,\lambda_2 + \beta p \\ &= \beta p\left(\frac{1}{\lambda_1}\right)^2 + (\beta(1-2p)-1)\,\frac{1}{\lambda_1} + \beta p \\ &= \lambda_1^2\underbrace{\left(\beta p + (\beta(1-2p)-1)\,\lambda_1 + \beta p\lambda_1^2\right)}_{=0}. \end{aligned}$$

That is, the roots come in "almost reciprocal pairs" (remember Fernando's class material). Thus, we have that

$$\lambda_1 = \frac{1}{\lambda_2} > 0.$$

We can also show that $\lambda_1 \in (0,1)$. To see this, define

$$Q(\lambda) := \lambda^2 + \frac{\beta(1-2p)-1}{\beta p}\lambda + 1$$

so that $Q(\lambda) = 0$ if and only if λ solves the characteristic polynomial. Observe then that

$$\begin{aligned} Q(0) &= 1, \\ Q(1) &= \frac{\beta(1-2p)-1}{\beta p} + 2 \\ &= \frac{\beta(1-2p)-1+2\beta p}{\beta p} = \frac{\beta-1}{\beta p} < 0. \end{aligned}$$

Since $Q(\lambda)$ is U-shaped, these two imply that one of the root, say λ_1, lies in the interval $[0,1]$ (draw the usual figure from Fernando's class).

The solution to the homogenous equation is given by

$$v_1 \lambda_1^y + v_2 \lambda_2^y = v_1 \lambda_1^y + v_2 \lambda_1^{-y}.$$

We "guess" that the particular solution is of the form:

$$v(y) = v_1 \lambda_1^y + v_2 \lambda_1^{-y} + C_1 y^2 + C_2.$$

Substituting this into (4.11), we get

$$\begin{aligned} v_1 \lambda_1^y + v_2 \lambda_1^{-y} + C_1 y^2 + C_2 &= a\Delta^2 y^2 + \beta p \left(v_1 \lambda_1^{y-1} + v_2 \lambda_1^{-y+1} + C_1 (y-1)^2 + C_2 \right) \\ &\quad + \beta p \left(v_1 \lambda_1^{y+1} + v_2 \lambda_1^{-y-1} + C_1 (y+1)^2 + C_2 \right) \\ &\quad + \beta (1-2p) \left(v_1 \lambda_1^y + v_2 \lambda_1^{-y} + C_1 y^2 + C_2 \right) \\ &= a\Delta^2 y^2 + \beta p \left(v_1 \lambda_1^{y-1} + v_2 \lambda_1^{-y+1} + C_1 \left(y^2 - 2y + 1 \right) + C_2 \right) \\ &\quad + \beta p \left(v_1 \lambda_1^{y+1} + v_2 \lambda_1^{-y-1} + C_1 \left(y^2 + 2y + 1 \right) + C_2 \right) \\ &\quad + \beta (1-2p) \left(v_1 \lambda_1^y + v_2 \lambda_1^{-y} + C_1 y^2 + C_2 \right) \end{aligned}$$

Collecting the terms with y^2,

$$\begin{aligned} C_1 &= a\Delta^2 + 2\beta p C_1 + \beta(1-2p) C_1 \\ &= \frac{a\Delta^2}{1 - 2\beta p - \beta(1-2p)} = \frac{a\Delta^2}{1-\beta}. \end{aligned}$$

Collecting the constant terms,

$$\begin{aligned} C_2 &= 2\beta p (C_1 + C_2) + \beta(1-2p) C_2 \\ &= \frac{2\beta p a \Delta^2}{1-\beta} \frac{1}{1 - 2\beta p - \beta(1-2p)} = \frac{2\beta p a\Delta^2}{(1-\beta)^2}. \end{aligned}$$

Observe that the coefficient on y cancel. Hence, the solution is

$$v(y) = \frac{a\Delta^2}{1-\beta} + \frac{2\beta p a \Delta^2}{(1-\beta)^2} y^2 + v_1 \lambda_1^y + v_2 \lambda_1^{-y},$$

where v_1 and v_2 are constants to be determined.[28]

[28] For completeness, we should check that the coefficients on λ_1's equate. Ignoring the constant and y^2 terms, we

At this stage, we have six unknowns:

$$\left\{\bar{z}, \underline{z}, \bar{y}, \underline{y}, v_1, v_2\right\}.$$

How do we pin these down? We use *value-matching* and *smooth-pasting* conditions.

Value-matching conditions are

$$\begin{aligned}
v\left(\underline{z}\right) &= v\left(\underline{y}\right) + \lambda, \\
v\left(\bar{z}\right) &= v\left(\bar{y}\right) + \lambda.
\end{aligned}$$

These say that, at the time the firm adjusts its capital stock, it must be that value of not adjusting (the left-hand side) is equal to the value of adjusting (the right-hand side). These conditions give us v_1 and v_2 as functions of $\boldsymbol{\theta} := \left\{\bar{z}, \underline{z}, \bar{y}, \underline{y}\right\}$, which denote as $v_1\left(\boldsymbol{\theta}\right)$ and $v_2\left(\boldsymbol{\theta}\right)$.

So far, we have computed the value function for an arbitrary policy. In particular, we have not done any optimisation. The optimal $\bar{y}$ and $\underline{y}$ minimises the cost for the firm so that they are given by

$$\bar{y} = \underline{y} = \arg\min_{y} \; v\left(y\right).$$

Notice that the policy variables, $\boldsymbol{\theta}$, appear in the value function $v\left(y\right)$ as part of v_1 and v_2. It turns out that we can find $\boldsymbol{\theta}^*$ that simultaneously both v_1 and v_2 (i.e. the same $\boldsymbol{\theta}^*$ minimises both v_1 and v_2 on their own). To understand why, since v_1 and v_2 appear in $v\left(y\right)$ as $v_1\lambda_1^y$ and $v_2\lambda_1^{-y}$ respectively

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have

$$\begin{aligned}
v_1\lambda_1^y + v_2\lambda_1^{-y} &= \beta p\left(v_1\lambda_1^{y-1} + v_2\lambda_1^{-y+1}\right) \\
&\quad + \beta p\left(v_1\lambda_1^{y+1} + v_2\lambda_1^{-y-1}\right) \\
&\quad + \beta\left(1-2p\right)\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right) \\
&= \beta p\left(v_1\lambda_1^{y-1} + v_2\lambda_1^{-y+1} + v_1\lambda_1^{y+1} + v_2\lambda_1^{-y-1}\right) \\
&\quad + \beta\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right) - \beta p\left(2v_1\lambda_1^{y} + 2v_2\lambda_1^{-y}\right) \\
&= \beta p\left(v_1\lambda_1^{y-1}\left(\lambda_1^2 - 2\lambda_1 + 1\right) + v_2\lambda_1^{-y-1}\left(\lambda_1^2 - 2\lambda_1 + 1\right)\right) \\
&\quad + \beta\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right) \\
&= \beta p\left(v_1\lambda_1^{y-1} + v_2\lambda_1^{-y-1}\right)\left(\lambda_1 - 1\right)^2 + \beta\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right) \\
&= \beta p\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right)\frac{\left(\lambda_1-1\right)^2}{\lambda_1} + \beta\left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right) \\
&= \left(v_1\lambda_1^{y} + v_2\lambda_1^{-y}\right)\beta\left(p\frac{\left(\lambda_1-1\right)^2}{\lambda_1} + 1\right)
\end{aligned}$$

For the left-hand side and the right-hand side to equal, it must be that

$$\begin{aligned}
\beta\left(p\frac{\left(\lambda_1-1\right)^2}{\lambda_1} + 1\right) &= 1 \\
\Leftrightarrow \left(\lambda_1 - 1\right)^2 &= \lambda_1\frac{1}{p}\left(\frac{1}{\beta} - 1\right) \\
\Leftrightarrow \beta p\lambda_1^2 - 2p\beta\lambda_1 + \beta p &= \lambda_1\left(1-\beta\right) \\
\Leftrightarrow \beta p\lambda_1^2 + \left(\beta - 1 - 2p\beta\right)\lambda_1 + \beta p &= 0 \\
\Leftrightarrow \beta p\lambda_1^2 + \left(\left(1-2p\right)\beta - 1\right)\lambda_1 + \beta p &= 0,
\end{aligned}$$

and we know that λ_1 satisfies the expression above from (4.11).

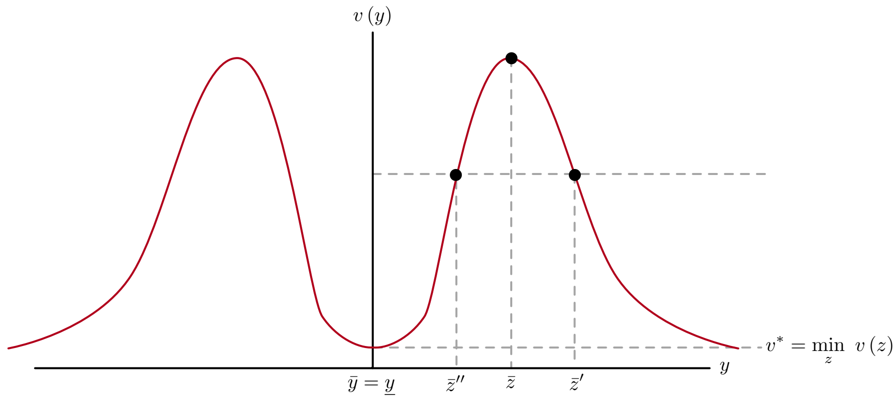
(and recall $\lambda_1 \in (0,1)$), if the current value of y is large, then one would care more about minimising v_2 and if y is small, then one would care about minimising v_1 (remember that $y \in \mathbb{Z}$). However, in the set up of the problem, the choice as to where to adjust to does not depend on the current state (as mentioned, adjustment cost is independent of the magnitude of adjustment). This gives us the economic rational for why the same $\boldsymbol{\theta}$ that minimises v_1 also minimises v_2. Solving this would give that

Check: Shimer said the other way around

$$\bar{y} = \underline{y} = 0.$$

We still need to determine $\bar{z}$ and $\underline{z}$. To this end, consider the following plot of the value function, $v(y)$, below (symmetric and has a minimum at $\bar{y} = \underline{y}$),[29] where

$$v^* = \min_y \; v(y).$$



Then given the value function (4.12), the firm will adjust its capital stock if $v(y) - v^* \geq \lambda$; i.e. if the gain from adjusting is greater than the fixed cost of adjustment. Let us suppose that the threshold is at $y = \bar{z}'$ in the figure. But since $v(y)$ is "continuous", before reaching $\bar{z}'$, there were some times during which the difference $v(y) - v^*$ was greater than λ; i.e. the firm would have adjusted before getting to this point. Hence, $\bar{z}'$ cannot be optimal.

What about $\bar{z}''$? This will not be optimal either due to "option value". Suppose that the firm's policy is to adjust at $\bar{z}''$. Now, consider an alternative policy that involves waiting one more period—until the next shock is realised—and then to decide whether to adjust. If the next period shock is a "good shock" (that moves you to $y = \bar{y} = \underline{y}$), then the firm would not adjust. If, instead, the next period shock is a "bad shock", the firm can still choose to adjust—importantly, with the same cost (since distance between y and $\bar{y} = \underline{y}$ does not affect the adjustment cost). Thus, waiting one more has a positive option value. The only point where this argument would not work, which is the point $\bar{z}$ in the figure, is when the firm is already at the maximum possible difference (which equals λ because of the value-matching conditions). This tells us that

$$\bar{z}, \underline{z} \in \arg\max_z \; v(z).$$

[29] What we plot in the figure the expression for $v(y)$ we found for all y. However, the value function itself has a restricted domain, in $[\underline{z}, \bar{z}]$. Alternatively, for $y < \underline{z}$ and $y > \bar{z}$, we can think of $v(y)$ as being flat.

The conditions are called *smooth-pasting* conditions. The symmetry of the problem gives us that

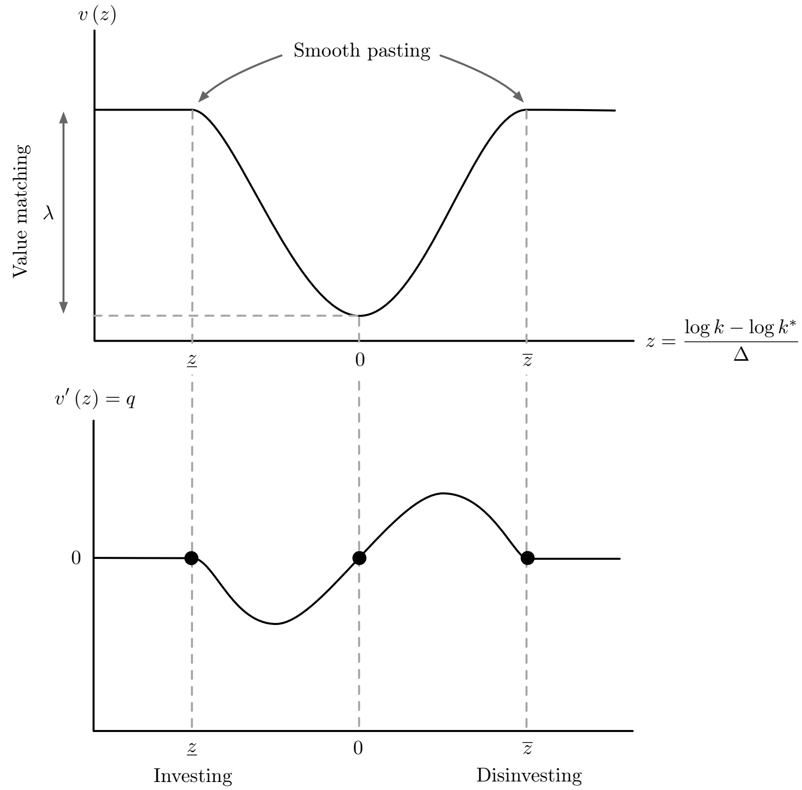
$$\underline{z} = -\bar{z}.$$

If we take the optimal $\bar{z}$, it would not look like the smooth-pasting conditions. However, if we take the expression and plug into the value function, we can verify that the it satisfied the first-order condition (since $\bar{z}$ is integer, the first-order condition is, in fact, that the value function at the two neighbouring points of $\bar{z}$ is equal to each other) for smooth pasting. Do the maths

4.3.5 Takeaways

Technical We can study non-convex costs without appealing to stochastic calculus/Itoh's lemma.

Marginal q With convex adjustment costs, we found that marginal q was a sufficient statistic for investment. With non-convex adjustment costs, marginal q is still informative of investment. This comes out from the smooth-pasting conditions—marginal value of unit to capital has to be the cost of unit of capital (in the model we considered is zero) exactly at the two adjustment threshold. Marginal q is a non-monotone function of the level of capital to relative to the desired level of capital, z. The figure below shows the value function (which represents cost here), $v(z)$. Value matching implies that the vertical difference between $v(\underline{z}) = v(\bar{z})$ and $v(0)$ is equal to λ. Smooth pasting ensures that $v'(\underline{z}) = v'(\bar{z}) = 0$. Recall that marginal q is given by $v'(z)$, plotted below also. We can see that marginal q is non-monotone/non-linear and it equals zero (i.e. the price of capital) at three points—at the two threshold where there is investment and disinvestment, and at $z = 0$ when there is neither investment nor disinvestment. Marginal q in this model is not sufficient since knowing that marginal q equals zero does not tell us which of the three points we are at (of course, we would know if purchase/sale price of capital were positive and they differed).



Effect of shocks to the level of uncertainty: Bloom (2009, Econometrica) Shocks to the level of uncertainty are important in the aggregate economy—we can think of this as an increase in variance. With log-linearised system, variance (which is a second moment) does not affect the behaviour of the model. Even if we study the RBC model in a non-linear way so that second moment can impact the model, we would find that not much will change.

With non-convex adjustment costs, however, there is potential that changes in variance/uncertainty can have a larger impact because the models feature the option value of delay. If the firm decides to wait, instead of acting now, there is a chance that, in the next period, things improve so that they would no longer want to act (if things do not improve, then the firm can act in then next period). We can think of high variance as meaning greater speed with which uncertainty is resolved, so the option value of delay is higher so that firms do not act. Bloom (2009) argues that this is a significant source of aggregate fluctuations.

To see this, consider a model in which firms are investing on average. In our set up, we can model this as the desired capital level is increasing over time, i.e. has a drift, by, for example, having a greater probability that k^* moves up than down. We can then construct an optimal policy for the firm and we will find that the structure of the policy will be the same (i.e. two thresholds with optimal return points). In this set up, an increase in the variance for the k^* process leads firms to widen the inaction region (i.e. widen the thresholds). Thus, immediately following a shock that widens the threshold, there will be no firms at the thresholds so that there will be zero investment in aggregate; i.e. investment collapses. Thus, with non-convex adjustment costs, second-moment shocks can now have a possibility of delivering first moment changes in the quantity of investment (unlike in RBC or convex adjustment cost models).

Financial constraints: Hubbar (1998, JEL) Hubbar (1998, JEL) finds that cash flows matter for firms' investment behaviour. In the model we considered so far, we implicitly assumed that firms have perfect access to capital markets. However, for cash flows to matter, we would need some kind of violations of the Modigliani-Miller theorem. We explore some ideas below.

- ▷ Bernanke Gertler (1989, AER): Costly state verification. Existence of moral hazard in financing projects creates a wedge (if a Kickstarter project fails, could it be because the developer decided to buy a new kitchen with the money from the funders?) In case the project fails, the investor would have to undertake costly verification of the "effort" exerted by the firm. The problem is the same in case the funding comes from banks—the cost of verification (and perhaps also for monitoring) would be part of the interest rate. These imply that projects financed by external funding would have to have a higher expected return than if the firm had cash on hand.

- ▷ Kiyotaki Moore (1997, JPE); Brunnermeier Sannikov (2014, AER): The amount that the banks would lend to firms depends on the amount of collateral that the firm can post. Requiring firms to post collateral incentives the firm to exert appropriate effort (think Myerson's model of moral hazard). We can also think about endogenising the value of collateral. For example, if collateral value collapses, then firms would not be able to borrow and the collateral may be used to finance less efficient uses. Self-fulfilling equilibrium in Kiyotaki Moore (1997), and large responses to small shocks in Brunnermeier Sannikov (2014).

5 Bubbles

We gives ourselves a working definition of a bubble:

Definition 5.1. A bubble is a situation in which the price of an asset exceeds the discounted present value of its future dividends.

Remark 5.1. This implies that all fiat money are bubbles since it pays no dividends. In NCG, there can be no bubbles.

We will find that rational bubbles and failure of the First Welfare Theorem are closely related. We first study the failure of FWT in an OLG-like setting. However, to emphasis the fact that the failure of the FWT is unrelated to time, we work with a static model.

5.1 Failure of the First Welfare Theorem (FWT)

Suppose there are infinitely many individuals indexed by $i \in \mathbb{Z}$ (i can be positive/negative integers) and infinitely many goods indexed by $j \in \mathbb{Z}$. We assume that each individual has the same measure. Let e_{ij} denote individual i's endowment of good j. We assume that individual i has endowments of just good $j = i$ and $j = i + 1$; i.e.

$$\begin{aligned} e_{ii} &\coloneqq e_1 > 0, \\ e_{ii+1} &\coloneqq e_2 > 0, \\ e_{ij} &\coloneqq 0, \ \forall j \neq i, i+1. \end{aligned}$$

Let c_{ij} denote individual i's consumption of good j. We assume that i's utility is given by

$$\log c_{ii} + \log c_{ii+1}$$

so that i only values goods $j = i$ and $j = i + 1$. Let q_i^{i+1} denote the price of good $i + 1$ in terms of good i.

The set up here is similar to OLG but differs in that there is no concept of "initial period" and i (which corresponds to t in the OLG) extends is not one directional. Moreover, any individual can trade with another individual (in OLG, you cannot trade with a "dead" generation).

Definition 5.2. *(Competitive equilibrium)* A competitive equilibrium is a nonnegative sequence $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ and $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$ such that

(i) utility is maximised; i.e. for all $i \in \mathbb{Z}$, $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ solves

$$\begin{aligned} \max_{\{c_{ii}, c_{ii+1}\}} \quad & \log c_{ii} + \log c_{ii+1} \\ s.t. \quad & c_{ii} + q_i^{i+1} c_{ii+1} \leq e_{ii} + q_i^{i+1} e_{ii+1} = e_1 + q_i^{i+1} e_2 \end{aligned} \tag{5.1}$$

(ii) market clears; i.e.

$$c_{ii} + c_{i-1i} = e_{ii} + e_{i-1i} = e_1 + e_2, \ \forall i \in \mathbb{Z}.$$

Let λ_i denote the Lagrange multiplier on the budget constraint in the individual's utility maximisation problem. The first-order conditions are

$$\frac{1}{c_{ii}} = \lambda_i, \ \frac{1}{c_{ii+1}} = q_i^{i+1}\lambda_i.$$

These imply that

$$c_{ii} = q_i^{i+1} c_{ii+1}.$$

Substituting into the budget constraint (which binds with equality) gives

$$c_{ii} = \frac{e_1 + q_i^{i+1} e_2}{2}, \ c_{ii+1} = \frac{e_1 + q_i^{i+1} e_2}{2q_i^{i+1}}. \tag{5.2}$$

Then, market clearing implies that

$$\begin{aligned} e_1 + e_2 &= c_{ii} + c_{i-1i} \qquad (5.3)\\ &= \frac{e_1 + q_i^{i+1} e_2}{2} + \frac{e_1 + q_{i-1}^{i} e_2}{2q_{i-1}^{i}} \\ \Leftrightarrow q_i^{i+1} e_2 &= 2\left(e_1 + e_2\right) - e_1 - \frac{e_1 + q_{i-1}^{i} e_2}{q_{i-1}^{i}} \\ &= e_1 - \frac{e_1}{q_{i-1}^{i}} + 2e_2 - e_2 \\ \Leftrightarrow q_i^{i+1} &= 1 + \frac{e_1}{e_2}\frac{q_{i-1}^{i} - 1}{q_{i-1}^{i}}. \end{aligned}$$

This is a first-order difference equation.

Claim 5.1. There exists a competitive equilibrium for any strictly positive sequence of $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$ that satisfies (5.3).

Proof. Given $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$, we can construct $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ using (5.2). By construction, $\{c_{ij}\}_{i,j\in\mathbb{Z}}$ and $\left\{q_i^{i+1}\right\}_{i\in\mathbb{Z}}$ maximises utility and satisfies market clearing. ■

There are three cases to consider: $e_1 = e_2$, $e_1 > e_2$ and $e_2 < e_1$. The first case is a knife-edge case and so we do not focus on it. Given the set up, the last two are symmetric (relabel all i to $-i$). In what follows, we focus on $e_2 > e_1$ case to save notation.

Claim 5.2. The difference equation, (5.3), has two fixed points given by

$$q_{i-1}^{i} = 1, \frac{e_1}{e_2}.$$

Proof. We first verify that the two values are indeed fixed points:

$$\begin{aligned} 1 + \frac{e_1}{e_2}\frac{1-1}{1} &= 1, \\ 1 + \frac{e_1}{e_2}\frac{\frac{e_1}{e_2} - 1}{\frac{e_1}{e_2}} &= \frac{e_1}{e_2}. \end{aligned}$$

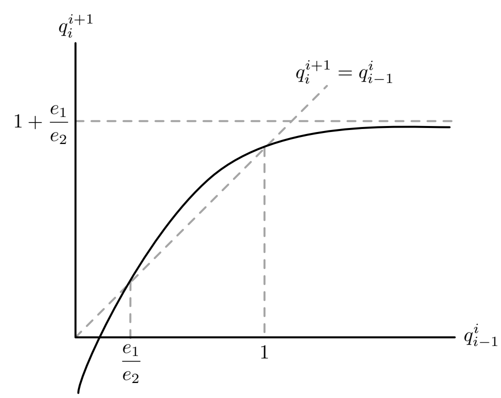
To show that these are the only fixed points, we note that the right-hand side of (5.3) is strictly increasing and concave.

$$\frac{\partial RHS}{\partial q_{i-1}^{i}} = \frac{e_1}{e_2}\frac{q_{i-1}^{i} - \left(q_{i-1}^{i} - 1\right)}{\left(q_{i-1}^{i}\right)^2} = \frac{e_1}{e_2}\frac{1}{\left(q_{i-1}^{i}\right)^2} > 0,$$
$$\frac{\partial^2 RHS}{\partial \left(q_{i-1}^{i}\right)^2} = -2\frac{e_1}{e_2}\frac{1}{\left(q_{i-1}^{i}\right)^3} < 0.$$

Note

$$\lim_{q_{t-1}^{i}\to\infty} 1 + \frac{e_1}{e_2}\frac{q_{i-1}^{i} - 1}{q_{i-1}^{i}} = 1 + \frac{e_1}{e_2},$$
$$\lim_{q_{t-1}^{i}\to 0} 1 + \frac{e_1}{e_2}\frac{q_{i-1}^{i} - 1}{q_{i-1}^{i}} = \lim_{q_{t-1}^{i}\to 0} 1 + \frac{e_1}{e_2}\frac{1 - \frac{1}{q_{i-1}^{i}}}{1} = -\infty.$$

We can then plot the function as below.



It is clear from the figure that there can be at most two fixed points, and the knife edge case is when $e_1 = e_2$. ■

Remark 5.2. We can just compute the fixed points normally. Setting $q_{i-1}^{i} = q_i^{i+1} = q$, we can obtain the quadratic expression

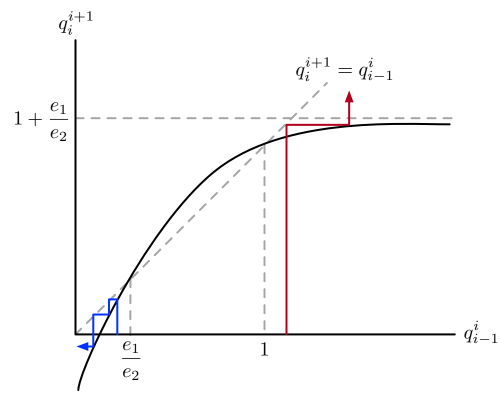
$$q^2 - \left(1 + \frac{e_1}{e_2}\right) q + \frac{e_1}{e_2} = 0.$$

Then, using the usual formula,

$$\frac{\frac{e_2+e_1}{e_2} \pm \sqrt{\left(\frac{e_2+e_1}{e_2}\right)^2 - 4\frac{e_1}{e_2}}}{2} = \frac{\frac{e_2+e_1}{e_2} \pm \sqrt{\frac{1}{e_2^2}\left((e_2+e_1)^2 - 4e_1e_2\right)}}{2}$$
$$= \frac{\frac{e_2+e_1}{e_2} \pm \sqrt{\frac{1}{e_2^2}(e_2^2 - 2e_1e_2 + e_1^2)}}{2}$$
$$= \frac{\frac{e_2+e_1}{e_2} \pm \sqrt{\frac{1}{e_2^2}(e_2-e_1)^2}}{2}$$
$$= \frac{\frac{e_2+e_1}{e_2} \pm \frac{|e_2-e_1|}{e_2}}{2} = \frac{e_2+e_1 \pm |e_2-e_1|}{2e_2}$$
$$= 1 \text{ or } \frac{e_1}{e_2},$$

where we assume $e_2 > e_1$.

Remark 5.3. So we found at least two equilibria. One in which $q_i^{i+1} = 1$ for all i, and another with $q_i^{i+1} = e_1/e_2$ for all i. There are others. To see this, first note that for any $q_{i-1}^i < e_1/e_2$, if we solve the difference equation forward, we will eventually find that for some $i' > i$, $q_{i'-1}^{i'} < 0$ (blue arrow in the figure below). Similarly, for any $q_{i-1}^i > 1$, there exists $i'' > i'$ such that $q_{i''-1}^{i''}$ does not exist (red arrow in the figure below).



Now, for any $q_{i-1}^i \in (e_1/e_2, 1)$, we converge to either of the two fixed points so that the difference equation is satisfied for such sequence. So, in fact, there is a continuum of equilibria. We now claim that the equilibria are Pareto rankable. In particular,

▷ the worst equilibrium is $q_{i-1}^i = q_i^{i+1} = e_1/e_2$;

▷ the best equilibrium is $q_{i-1}^i = q_i^{i+1} = 1$.

To get some intuition on why this is the case, notice that with relative prices given by e_1/e_2 , consumption for i is, in fact, the endowment points:

$$c_{ii} = \frac{e_1 + \frac{e_1}{e_2}e_2}{2} = e_1,$$
$$c_{ii+1} = \frac{e_1 + \frac{e_1}{e_2}e_2}{2\frac{e_1}{e_2}} = e_2.$$

Since we know that individuals are utility maximising, and consuming the endowment is always possible for any relative prices, for individuals to pick another bundle, it must be that the new bundle gives him a higher utilities (this is a revealed preference argument). Consider the derivative of the utility with respect to q_i^{i+1} when the individual is consuming the optimal bundle.

$$\begin{aligned}\frac{\partial}{\partial q_i^{i+1}}\left(\log\frac{e_1+q_i^{i+1}e_2}{2}+\log\frac{e_1+q_i^{i+1}e_2}{2q_i^{i+1}}\right) &= \frac{\partial}{\partial q_i^{i+1}}\left(\log\frac{e_1+q_i^{i+1}e_2}{2}+\log\frac{\frac{e_1}{q_i^{i+1}}+e_2}{2}\right)\\ &= \frac{e_2/2}{\frac{e_1+q_i^{i+1}e_2}{2}}+\frac{-\frac{e_1}{2\left(q_i^{i+1}\right)^2}}{\frac{e_1+q_i^{i+1}e_2}{2q_i^{i+1}}}\\ &= \frac{1}{e_1+q_i^{i+1}e_2}\left(e_2-\frac{e_1}{q_i^{i+1}}\right).\end{aligned}$$

The derivative is smallest when q_i^{i+1} is lowest, which is e_1/e_2, the value of the derivative when $q_i^{i+1}=e_1/e_2$ is

$$\frac{1}{e_1+q_i^{i+1}e_2}\left(e_2-\frac{e_1}{e_1/e_2}\right)=0.$$

Hence, we realise that the higher the q_i^{i+1}, the higher is the individual's utility from consuming the optimum bundle (since the derivative is strictly positive when $q_i^{i+1}>e_1/e_2$). This tells us that that $q_{i-1}^i=q_i^{i+1}=1$ is the best equilibrium (in any equilibrium sequence of prices, the relative prices cannot be greater than one).

So we conclude that not all competitive equilibria are Pareto optimal; i.e. the First Welfare Theorem fails.

5.2 First Welfare Theorem

Suppose that there measure ω_i of type i individuals. Let I denote the set of types and $|I|<\infty$. Let J denote the set of all goods. $\{c_{ij}\}_{j\in J}$ denotes the consumption bundle for type i. The utility from consuming this bundle is

$$u_i\left(\{c_{ij}\}_{j\in J}\right).$$

We assume that u_i satisfies local non-satiation (which implies $p_j>0$ in equilibrium for all j). $\{e_{ij}\}_{j\in J}$ denotes the bundle of endowments for type i and p_j the price of good j. The budget constraint is then given by

$$\sum_{j\in J}p_jc_{ij}\le\sum_{j\in J}p_je_{ij},\ \forall i\in I.$$

For the problem to be "well-posed", in case when $|J|=\infty$, we require the right-hand side to be finite.

We also have resource constraints for each good:

$$\sum_{i\in I}\omega_ic_{ij}\le\sum_{i\in I}\omega_ie_{ij},\ \forall j\in J.$$

Definition 5.3. A competitive equilibrium is a sequence of $\{p_j\}_{j\in J}$, $\{c_{ij}\}_{i\in I,j\in J}$ such that each individual maximises his utility subject to the budget constraint, and resource constraints hold.

Definition 5.4. An allocation $\{c_{ij}\}_{i\in I,j\in J}$ is Pareto optimal if, for all $\left\{c'_{ij}\right\}_{i\in I,j\in J}$ that satisfy the resource constraints, there exists i such that

$$u_i\left(\{c_{ij}\}\right) > u_i\left(\left\{c'_{ij}\right\}\right) \text{ or } u_i\left(\{c_{ij}\}\right) = u_i\left(c'_{ij}\right), \ \forall i \in I.$$

Theorem 5.1. *(First Welfare Theorem). Any competitive equilibrium is Pareto optimal.*

Proof. Let $\left\{\{c_{ij}\}_{i\in I}\{p_j\}\right\}_{j\in J}$ be a competitive equilibrium. Since u_i satisfies LNS, we know that

$$\sum_{j\in J} p_j c_{ij} = \sum_{j\in J} p_j e_{ij}, \ \forall i \in I. \tag{5.4}$$

Suppose, by way of contradiction, that there exists $\left\{c'_{ij}\right\}_{i\in I, j\in J}$ that Pareto dominates $\{c_{ij}\}_{i\in I, j\in J}$ and also satisfies the resource constraints. Since $\{c_{ij}\}$ was chosen instead of $\left\{c'_{ij}\right\}$, then it must be that the bundle $\left\{c'_{ij}\right\}$ was not affordable; i.e.

$$\sum_{j\in J} p_j c_{ij} \leq \sum_{j\in J} p_j c'_{ij}, \ \forall i \in I$$

and that this inequality holds strictly for some i. Summing across i and using (5.4),

$$\begin{aligned} &\sum_{i\in I}\omega_i \sum_{j\in J} p_j c_{ij} < \sum_{i\in I}\omega_i \sum_{j\in J} p_j c'_{ij} \\ \Leftrightarrow &\sum_{i\in I}\omega_i \sum_{j\in J} p_j e_{ij} < \sum_{i\in I}\omega_i \sum_{j\in J} p_j c'_{ij}. \end{aligned} \tag{5.5}$$

Since $p_j > 0$ for all $j \in J$ (LNS), above implies that for $j \in J$,

$$\sum_{i\in I}\omega_i p_j e_{ij} < \sum_{i\in I}\omega_i p_j c'_{ij};$$

i.e. the resource constraint is violated—a contradiction. ■

What might fail if $|I| = \infty$? In this case, the left-hand side of (5.5), which is the value of aggregate endowment, could be infinite. In this, we cannot apply the proof as we stated above. In other words, if $|I|$ is infinite, then we need to add an additional condition that the value of endowment is finite. Note that this is only a sufficient additional assumption for the First Welfare Theorem to hold when $|I| = \infty$.

Recall that, in our example, we found one equilibrium to be $q_i^{i+1} = 1$ for all $i \in I$. In this case,

$$c_{ii} = \frac{e_1 + q_i^{i+1} e_2}{2} = \frac{e_1 + e_2}{2} = \frac{e_1 + q_i^{i+1} e_2}{2 q_i^{i+1}} = c_{ii+1}.$$

Since q_i^{i+1} represents the relative price of good $i+1$ in terms of i, if all

$$q_i^{i+1} = 1, \ \forall i \in I \Rightarrow p_j = p, \ \forall j \in J,$$

where p is some strictly positive constant. The value of the endowment in this case is

$$\sum_{i\in I}\sum_{j\in J}\omega_i p_j e_{ij} = p \sum_{i=-\infty}^{\infty} (e_1 + e_2) = \infty.$$

Note that

$$\begin{aligned} c_{ii} &= c_{ii+1} \\ \Leftrightarrow \frac{e_1 + q_i^{i+1} e_2}{2} &= \frac{e_1 + q_i^{i+1} e_2}{2 q_i^{i+1}} \\ \Leftrightarrow e_1 + q_i^{i+1} e_2 &= \frac{e_1}{q_i^{i+1}} + e_2 \\ \Leftrightarrow \left(1 - \frac{1}{q_i^{i+1}}\right) e_1 &= e_2 \left(1 - q_i^{i+1}\right). \end{aligned}$$

For any $e_1 \neq e_2$, for this to hold, it must be that $q_i^{i+1} = 1$. If $e_1 = e_2$, then we must have

$$1 - \frac{1}{q_i^{i+1}} = 1 - q_i^{i+1} \Leftrightarrow -1 = q_i^{i+1}$$

but this cannot be since all prices has to be strictly positive in equilibrium. Hence, we conclude that, in any equilibrium in which $c_{ii} \neq c_{ii+1}$, it must be that

$$q_i^{i+1} \neq 1.$$

But note that, given the symmetry, it must be that

$$q_i^{i+1} = q$$

for some constant q. Then

$$p_j = p_0 q^j$$

but this would again imply that the value of aggregate endowment is infinite.

Why would it be a constant? There are only two fixed points to the difference equation...

5.3 Bubbles in an OLG model

For each cohort born at $t \in \{0, 1, \ldots\}$, suppose they solve the following problem:

$$\begin{aligned} \max_{c_{1,t} c_{2,t}} \quad & \log c_{1,t} + \log c_{2,t+1} \\ s.t. \quad & c_{1,t} + q_t^{t+1} c_{2,t+1} \leq e_1 + q_t^{t+1} e_2. \end{aligned}$$

We assume a unit measure of initial old whose preference is simply

$$\begin{aligned} \max_{c_{2,0}} \quad & \log c_{2,0} \\ s.t. \quad & c_{2,0} \leq e_2. \end{aligned}$$

Definition 5.5. Competitive equilibrium is $\left\{c_{1,t}, c_{2,t}, q_t^{t+1}\right\}_{t=0}^{\infty}$ such that every cohort maximises their utility and resource constraint holds:

$$c_{1,t} + c_{2,t} = e_1 + e_2, \ \forall t.$$

5.3.1 Competitive equilibrium

Observe that for those born at $t = 0, 1, \dots$, the problem is the same as in (5.1) so that optimality and market clearing require

$$q_i^{i+1} = 1 + \frac{e_1}{e_2}\frac{q_{i-1}^i - 1}{q_{i-1}^i},\ \forall t \geq 1. \tag{5.6}$$

For period $t = 0$, we know that the initial old will consume all his endowment so that $c_{2,0} = e_2$. By market clearing,

$$c_{2,0} = e_2 \Rightarrow c_{1,0} = e_1.$$

For this to be optimal for the cohort born in period 0,

$$c_{1,0} = \frac{e_1 + q_0^1 e_2}{2} = e_1 \Rightarrow q_0^1 = \frac{e_1}{e_2}.$$

This gives us a boundary condition for the difference equation (5.6). But since e_1/e_2 is a fixed point of the difference equation, it follows that

$$q_t^{t+1} = \frac{e_1}{e_2},\ \forall t \geq 0.$$

This, in turn, implies that

$$c_{1,t} = e_1,\ c_{2,t} = e_2,\ \forall t \geq 0,$$

which gives us the (unique) competitive equilibrium.

5.3.2 Violation of the First Welfare Theorem

Consider the allocation

$$c'_{1,t} = c'_{2,t} = \frac{e_1 + e_2}{2}. \tag{5.7}$$

First, observe that this is resource feasible. Since the utility function is concave, if $e_1 \neq e_2$,

$$\log c_{1,t} + \log c_{2,t} < \log c'_{1,t} + \log c'_{2,t},\ \forall t \geq 0.$$

For the initial old,

$$e_1 > e_2 \Leftrightarrow e_2 = \log c_{2,0} < \log c'_{2,0} = \frac{e_1 + e_2}{2}.$$

Hence, if $e_1 > e_2$, the competitive equilibrium is not Pareto optimal— we can make everyone at least as well off by changing consumption bundles to (5.7). In fact, it can be shown that violation of First Welfare Theorem occurs if and only if $e_1 > e_2$ in this model.

If $e_1 > e_2$, in the competitive equilibrium

$$q_t^{t+1} = \frac{e_1}{e_2} > 1.$$

Here, we think of q_t^{t+1} as the price of goods in the next period in terms of goods this period; i.e.

$$q_t^{t+1} = \frac{1}{1+r},$$

where r is the interest rate. Hence, we realise that, in the competitive equilibrium,

$$\frac{1}{1+r} > 1 \Leftrightarrow r < 0.$$

We interpret this to mean that the interest rate is less than the growth rate of the economy, which is zero in this case. Moreover, consider the value of endowment in this case:

$$\sum_{t=0}^{\infty} \left(\frac{e_1}{e_2}\right)^t (e_1 + e_2) = \infty.$$

However, if, instead $e_1/e_2 < 1$, then the sum if finite (note that unlike in the case with $i \in \mathbb{Z}$, here, t is one directional so that $e_1/e_2 < 1$ implies sum is finite).

5.3.3 Bubbles

We now "give" to the initial old a piece of paper that lasts forever (and no more is produced). Let p_t denote the price of the paper. We will show that there exists a competitive equilibrium in which $p_t = 0$ for all t but, if $e_1/e_2 > 1$, then there exists many equilibria in which $p_t \neq 0$ for all t; i.e. bubbles can exist in competitive equilibria.

Definition 5.6. Competitive equilibrium is $\{c_{1,t}, c_{2,t}, a_t, p_t\}_{t=0}^{\infty}$ such that: (i) for those born in $t = 0, 2, \ldots,$

$$\begin{aligned} \max_{\{c_{1,t}, c_{2,t}, a_t\}} \quad & \log c_{1,t} + \log c_{2,t+1} \\ s.t. \quad & c_{1,t} + p_t a_t \leq e_1, \\ & c_{2,t+1} \leq e_2 + p_{t+1} a_t; \end{aligned}$$

(ii) for the initial old:

$$\begin{aligned} \max_{\{c_{2,0}\}} \quad & \log c_{2,0} \\ s.t. \quad & c_{2,0} \leq e_2 + p_0; \end{aligned}$$

(iii) resource constraints hold:

$$\begin{aligned} a_t &= 1, \\ c_{1,t} + c_{2,t} &= e_1 + e_2, \end{aligned}$$

for all t.

No bubble equilibrium Suppose $p_t = 0$, $c_{1,t} = e_1$ and $c_{2,t} = e_2$ for all t. In particular, consider the case in which the individual faces $p_{t+1} = 0$. Since purchasing asset in period t does not yield any benefit in period $t+1$ (since $p_{t+1} = 0$), the optimal consumption profile is for the individual to consume their endowment in each period. Moreover, in order for the markets to clear, i.e. $a_t = 1$, we must have $p_t = 0$ since, if $p_{t+1} = 0$, the demand for asset is is zero for any positive p_t, and arbitrary if p_t is zero. This tells us that $p_{t+1} = 0$ implies $p_t = 0$, which, in turn, implies that $p_t = 0$

for all t.

Equilibrium with a bubble Now suppose $p_t > 0$ for all t. Notice that prices in all periods must be strictly positive in the bubble equilibrium. To see this, suppose $p_t = 0$ for any t. Then, by the same argument as before, in period $t-1$, it must be that $p_{t-1} = 0$ for markets to clear. This, in turn, would mean $p_{t-2} = 0$ etc.

Since prices are positive and the budget constraints hold with equality, we can combine the two budget constraints by eliminating a_t:

$$\begin{aligned} c_{1,t} + p_t\left(\frac{c_{2,t+1} - e_2}{p_{t+1}}\right) &= e_1 \\ \Leftrightarrow c_{1,t} + \frac{p_t}{p_{t+1}} c_{2,t+1} &= e_1 + \frac{p_t}{p_{t+1}} e_2. \end{aligned}$$

If we let $p_t/p_{t+1} = q_t^{t+1}$, then the problem is isomorphic to the OLG model we considered previously. Hence, optimality and market clearing requires

$$\begin{aligned} \frac{p_t}{p_{t+1}} &= 1 + \frac{e_1}{e_2}\frac{\frac{p_{t-1}}{p_t} - 1}{\frac{p_{t-1}}{p_t}}, \ \forall t \geq 1 \\ \Leftrightarrow \frac{p_t}{p_{t+1}} &= 1 + \frac{e_1}{e_2}\frac{p_{t-1} - p_t}{p_{t-1}} \\ &= \frac{e_2 p_{t-1} + e_1\left(p_{t-1} - p_t\right)}{e_2 p_{t-1}}, \ \forall t \geq 1 \\ \Leftrightarrow p_{t+1} &= \frac{e_2 p_{t-1} p_t}{\left(e_2 + e_1\right) p_{t-1} - e_1 p_t}, \ \forall t \geq 1. \end{aligned} \tag{5.8}$$

For the initial old,

$$c_{2,0} = e_2 + p_0,$$

market clearing requires that

$$c_{1,0} = e_1 + e_2 - e_2 - p_0 = e_1 - p_0. \tag{5.9}$$

For this to be optimal for cohort zero, it must be that

$$\begin{aligned} c_{1,0} = \frac{e_1 + \frac{p_0}{p_1} e_2}{2} &= e_1 - p_0 \\ \Leftrightarrow e_1 + \frac{p_0}{p_1} e_2 &= 2e_1 - 2p_0 \\ \Leftrightarrow p_1 &= \frac{e_2 p_0}{e_1 - 2p_0}. \end{aligned} \tag{5.10}$$

This gives us the boundary condition for the (second-order) difference equation (5.8). For any given p_0, (5.10) gives us p_1, and we can use (5.8) to obtain a sequence of prices p_3, $p_4, \ldots$. However, not all sequence generated in this way is a competitive equilibrium. We have to ensure that

$$0 \leq p_t < e_1, \ \forall t.$$

The second inequality ensures that the young, who has an endowment of e_1, is able to purchase one unit of the asset in every period (if the price is larger than e_1, then the asset market would not clear.)

The following proposition gives the condition for the existence of a bubble equilibrium.

Proposition 5.1. *There exists a bubble equilibrium if and only if $e_1 > e_2$. In other words, the unique competitive equilibrium if $e_1 < e_2$ is the no bubble equilibrium in which $p_t = 0$ for all t.*

Proof. Admitted. ■

Assuming $e_1 > e_2$, an example of a sequence of prices that form a bubble equilibrium is

$$p_t = \frac{e_1 - e_2}{2},\ \forall t.$$

This implies that

$$\frac{p_t}{p_{t+1}} = 1 \Rightarrow c_{1,t} = c_{2,t} = \frac{e_1 + e_2}{2},\ \forall t.$$

There is a continuum of bubble equilibrium for all $p_0 \in (0, (e_1 - e_2)/2)$ that are Pareto rankable. For such initial price p_0, p_t is a decreasing sequence that converges to zero. The Pareto dominant bubble equilibrium is the example given above.

Observe that the bubble equilibrium represents a Pareto improvement over the no bubble (autarky) equilibrium. That is, in the present model, a bubble leads to a better allocation.

5.3.4 Incomplete markets

Can competitive equilibrium be inefficient even with finite types? We look at the case when the market is incomplete. We will find that in this case also, each individual can have a present value of endowment of infinity and the competitive equilibrium can be inefficient, and a bubble can lead to a Pareto dominant allocation.

One example of an incomplete market model is to assume that the agents cannot borrow or lend. Typically, in this environment, we consider per-period budget constraints rather than a single lifetime budget constraint which allows us to keep track of the agent's wealth at every point in time (and impose a nonnegative constraint). When we do not have a lifetime budget constraint, then we can have another source of failure of the First Welfare Theorem—when we try to add up the individual's per-period budget constraints, we can get that the present value of the endowment is infinite from the individual's perspective.

To illustrate, suppose there are odd and even *types* in the economy. An odd (even) type receives high (low) endowment in odd periods and low (high) endowments in even periods. Let the two possible possible values of endowment be e_1 and e_2, where

$$\beta e_1 > e_2.$$

The two types have a common discount rate factor of $\beta \in (0, 1)$. Crucially, we assume that neither types can go into debt. This means that both types must consume their endowment.

Each type solves the following problem

$$\max_{\{c_t^i\}} \quad \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$s.t. \quad e_t^i + a_t^i = c_t^i + q_t^{t+i} a_{t+1}^i$$
$$a_t^i \geq 0,$$

where the second constraint reflects the fact that individuals cannot go into debt. Market clearing condition for the asset market is

$$a_t^{\text{odd}} + a_t^{\text{even}} = 0,$$

which implies that there is zero net supply of assets in the economy. Then, it is immediate that, in a competitive equilibrium,

$$a_t^i = 0, \ \forall t$$
$$e_t^i = c_t^i.$$

To show that this is Pareto inefficient, we think of transferring m from high-endowment type to low-endowment type in any period t. With such transfer scheme in place, the lifetime utility of those that start with high endowment initially is

$$\begin{aligned}
&\log(e_1 - m) + \beta \log(e_2 + m) + \beta^2 \log(e_1 - m) + \beta^3 \log(e_2 + m) + \cdots \\
&= \left(\sum_{t=0}^{\infty} \beta^{2t} \log(e_1 - m) \right) + \beta \left(\sum_{t=0}^{\infty} \beta^{2t} \log(e_2 + m) \right) \\
&= \frac{\log(e_1 - m) + \beta \log(e_2 + m)}{1 - \beta^2}.
\end{aligned}$$

Maximising above with respect to m gives the first-order condition:

$$\begin{aligned}
\frac{1}{e_1 - m} &= \beta \frac{1}{e_2 + m} \\
\Leftrightarrow e_2 + m &= \beta e_1 - \beta m \\
\Leftrightarrow m &= \frac{\beta e_1 - e_2}{1 + \beta} > 0. \qquad (5.11)
\end{aligned}$$

Since $m \neq 0$, it follows that the initial high-endowment type is better off under this scheme. To verify that the initial-low endowment type is also better off:

$$\begin{aligned}
&\log(e_2 + m) + \beta \log(e_1 - m) + \beta^2 \log(e_2 + m) + \beta^3 \log(e_1 - m) + \cdots \\
&= \left(\sum_{t=0}^{\infty} \beta^{2t} \log(e_2 + m) \right) + \beta \left(\sum_{t=0}^{\infty} \beta^{2t} \log(e_1 - m) \right) \\
&= \frac{\log(e_2 + m) + \beta \log(e_1 - m)}{1 - \beta^2} = \frac{\log(e_2 + m)(e_1 - m)^{\beta}}{1 - \beta^2}.
\end{aligned}$$

If $m = 0$, then above equals

$$\frac{\log e_2 e_1^{\beta}}{1 - \beta^2}.$$

If m is as given by (5.11), then

$$\frac{\log\left(e_2 + \frac{\beta e_1 - e_2}{1+\beta}\right)\left(e_1 - \frac{\beta e_1 - e_2}{1+\beta}\right)^{\beta}}{1-\beta^2} = \frac{\log\left(\frac{\beta}{1+\beta}\left(e_1 + e_2\right)\right)\left(\frac{1}{1+\beta}\left(e_1 + e_2\right)\right)^{\beta}}{1-\beta^2}.$$

So consider

$$\begin{aligned}
&\log\left(\frac{\beta}{1+\beta}\left(e_1+e_2\right)\right) + \beta\log\left(\frac{1}{1+\beta}\left(e_1+e_2\right)\right) - \log e_2 - \beta\log e_1 \\
&= \log\left(\frac{\beta}{1+\beta}\frac{e_1+e_2}{e_2}\right) + \beta\log\left(\frac{1}{1+\beta}\frac{e_1+e_2}{e_1}\right) \geq 0.
\end{aligned}$$

Hence, the initial-low endowment type is also better off.

Suppose each initial low-endowment type receives a piece of paper. We can find a bubble equilibrium with $p_t = m$. Of course, there is also a no-bubble equilibrium in which $p_t = 0$.

5.3.5 What's wrong with bubbles?

What we have seen above is that bubbles can lead to a better allocation in certain situations. So are bubbles always good?

First, it is worth noting that, in models we see above, we can infer that bubbles bursting is "bad"—people are made worse off. But another concern with bubbles is its redistributive effects. Suppose that some people hold money and some people do not and hold assets instead (in the models above, individuals are indifferent. between holding one or the other). However, if the value of money changes differently to that of the value of assets, then there could be redistributive consequences—if value of money rises faster, then those with greater holding of money would be made relatively better off.

5.4 Milgrom & Stokey (1982): No trade theorem

We now consider bubbles from a different perspective—with heterogeneous beliefs.

Suppose that you hold an asset that you think is no good. You might believe that if you hold it for long enough, some "sucker" would come by and purchase the asset from you. To model this, we can think of individuals as having signals about the future dividends of the assets in question. If you believe that there are others that have a signal that is wrong in a way that makes them pay more for the asset. In this case, you might be willing to pay more for the asset than what you believe is its worth expecting it to sell the asset to the one with the incorrect signal.

Milgrom & Stokey (1982) shows that, in a rational expectations equilibrium, these types of trades motivated by differences in information cannot exist—the intuition is that one is able to infer (enough about) the other's signal from the fact that he is willing to trade at that price.

We consider a model with heterogeneous expectations model by Harrison and Kreps—we can think of such individual as very stubborn Bayesians who are unwilling to update their priors. In this environment, Harrison and Kreps show that we can have assets that trade, not just above the value that an individual thinks it is worth, but also above what everyone in the economy think it is worth. Such prices come about because individuals believe that they will be able to sell the asset onto other individuals with "stupid" beliefs in the future, and that person believes the same with respect to (the same or) another individual.

We assume each individual $i \in I$ is risk neutral, discounts the future at rate β and has a belief about the transition probabilities for the future dividends d' conditional on the current dividends d as as $\pi^i(d'|d)$. We assume that individual's holding of assets has to be nonnegative (without the nonnegative constraints on the asset holding, there will be infinite supply of the asset). In this case, the post dividend price of the asset is given by

$$p(d) = \max_{i \in I} \; \beta \sum_{d'} \pi^i(d'|d)(d' + p(d')).$$

That is, the price of the asset is determined by the individual with the highest valuation of the next period's dividends based on their beliefs.

In contrast, each individual i's valuation of the asset (assuming that i holds the asset forever) is

$$q^i(d) = \beta \sum_{d'} \pi^i(d'|d)\left(d' + q^i(d')\right).$$

The highest valuation of the asset among the I individual is

$$q(d) = \max_{i \in I} \; q^i(d).$$

Definition 5.7. We say that there is a *speculative behaviour* if and only if

$$p(d) > q(d). \tag{5.12}$$

That is, there is a speculative behaviour if each individual believes that it is not worth holding the asset at that price, but someone is willing to purchase at that price. In such situations, it must be that the one willing to purchase believes that he will be willing to sell it for a higher price at a latter point. We can also think of (5.9) as saying that the price of the asset is greater than the buy-and-hold value/price of the asset.

As an example in which we can get a speculative behaviour, suppose there are two types of investors A and B, and two dividend levels, High (H) and Low (L). A thinks that dividends payments are persistent while B thinks that dividends payments are iid over time. If the current dividend is high, A will put a high valuation on the asset since they think high dividend payments will persist. In contrast, if current dividend is low, A thinks that it will stay low for a long time. But A believes there are "suckers" (B) who believe that the states are transitory (so they think dividends will recover to H quickly). Then, A will be wiling to pay more for the asset today (since it has an option value) when dividends are high, because they anticipate that they can sell it at to B if the dividends are low. In contrast, B are willing to buy when dividends are low, because they believe that shocks are transitory so that dividends will recover to H quickly. When it does recover, B think that they will be able to sell it to the "suckers" A who think that high dividends will persist.

Is speculative behaviour a bubble? People sometimes feel uncomfortable calling (5.9) a bubble. Consider the case of a car—part of the value of a new car is the value of selling it as used. If reselling is not allowed, however, the price that you are wiling to pay for the car would be lower. This appears to the case in which (5.9) holds: $q(d)$ is the the price if reselling is not allowed, and $p(d)$ is the value of the new car. But we would not want to call this a bubble.

One way to convince ourselves that this is not a bubble is to imagine that, in addition to the market for the ownership of the car, there is also a (frictionless) rental market for cars; i.e. even if you do not/cannot resell the car, you could rent it out to someone else, and you would receive the proceeds. In this world, there is no advantage to having a resale market and so $q(d) = p(d)$. So whether something is a bubble in the sense of (5.9) becomes whether d is tried to the owner of the asset or not. We would say that there is a bubble if we can observe (5.9) even with the rental market.