1 Recap

1.1 Motivation

Consider the transition matrix $\mathbb{P}$:

$$\mathbb{P} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $\mathbb{P}$ is not ergodic. Furthermore, recall that

▷ Λ is invariant if $\mathbb{S}^{-1}(\Lambda) = \Lambda$

▷ $\mathbb{S}$ is ergodic if $Pr(\Lambda) = 0$ or 1 where Λ is invariant.

1.2 Sufficient conditions for ergodicity

Define

$$\mathbb{T}f(x) = \mathbb{E}\left[f(X_{t+1})\,|X_t = x\right]$$

Then f is an eigenfunction with unit eigenvalue if $\mathbb{T}f = f$. This implies

$$X_t = x \Rightarrow \underbrace{\mathbb{E}\left[f(X_{t+1})\,|X_t = x\right]}_{=\mathbb{T}f(x)} = f(x) = f(X_t)$$

Proposition 1.1. *(3.4.1) When the solution to $\mathbb{T}f = f$ is a constant function (with Q meausre one), then we can find some $\mathbb{S}$ such that*

$$X_t(\omega) = X\left(S^t(\omega)\right), \quad \mathbb{S} \text{ is ergodic}$$

Proposition 1.2. *(3.4.2) Suppose that for any (i) $f \geq 0$ such that (ii) $\int f(x)Q(dx) > 0$, it is the case that $\mathbb{M}f > 0$ where $\mathbb{M}$ is the discounted sum of conditional expectations,*

$$\mathbb{M}f(x) = (1-\delta)\sum_{j=0}^{\infty}\delta^j\mathbb{T}^j f$$

Then any solution to $\mathbb{T}\tilde{f} = \tilde{f}$ is constant with respect to Q.

▷ TA's Proof:

* Consider an eigenfunction $\tilde{f}$. Then define f as the following:

$$f(x) = \phi\left(\tilde{f}(x)\right), \quad \phi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{B}$ is a Borel set. Note that this satisfies $\mathbb{T}f = f$ since

$$\begin{aligned}\mathbb{T}f &= \mathbb{E}\left[f(X_{t+1})\,|X_t = x\right] \\ &= \mathbb{E}\left[\phi\left(\tilde{f}(x)\right)|X_t = x\right] \\ &= \end{aligned}$$

* This also means $\mathbb{M}f = f$ since

$$\mathbb{M}f = (1-\delta)\sum_{j=0}^{\infty}\delta^j\mathbb{T}^j f = (1-\delta)\sum_{j=0}^{\infty}\delta^j f = f$$

* Suppose $\int f(x)\,Q(dx) > 0$. Then by the hypothesis of the proposition $\mathbb{M}f = f > 0$ with probability 1. Since ϕ only takes values of 1 and 0, it must be the case that $f(x) = 1$.
* Given this fact, suppose there is a number y that $\tilde{f}$ takes on, i.e.

$$\Pr\left\{\tilde{f}(x) = y\right\} > 0$$

Then take the set $\mathcal{B} = \{y\}$ which implies

$$f(x) = \phi\left(\tilde{f}(x)\right) = 1 \Rightarrow \tilde{f}(x) = y \text{ with probability 1}$$

▷ Taks' proof:

* Let $\tilde{f}$ be an arbitrary function such that $\mathbb{T}\tilde{f} = \tilde{f}$. Then the function $f_{\mathfrak{b}} = \phi_{\mathfrak{b}} \circ \tilde{f}$ also solves $\mathbb{T}f_{\mathfrak{b}} = f_{\mathfrak{b}}$, where ϕ is defined as

$$f_{\mathfrak{b}}(x) := \left(\phi_{\mathfrak{b}} \circ \tilde{f}\right)(x) = \begin{cases} 1 & \text{if } \tilde{f}(x) \in \mathfrak{b} \\ 0 & \text{if } \tilde{f}(x) \notin \mathfrak{b} \end{cases}.$$

for some Borel set $\mathfrak{b}$ in $\mathbb{R}$.

  - Since f can only take values zero or one, we have condition (i). If f does not meet condition (ii), then it must be that $f_{\mathfrak{b}} = 0$ with Q-measure one, which implies that we must have $\tilde{f} = 0$ with Q-measure 1 so that $\tilde{f}$ is a constant.

* Consider the case when $f_{\mathfrak{b}}$ meets condition (ii).
  - Then, by the hypothesis of the proposition, we must have

$$(\mathbb{M}f_{\mathfrak{b}})(x) > 0,\ \forall x \in \mathcal{X}$$

  with Q-measure one.
  - Since $f_{\mathfrak{b}}$ is also an eigenfunction of $\mathbb{T}$ associated with a unit vector, $f_{\mathfrak{b}} = \mathbb{M}f_{\mathfrak{b}}$ so that, together, we have

$$f_{\mathfrak{b}}(x) = (\mathbb{M}f_{\mathfrak{b}})(x) > 0,\ \forall x \in \mathcal{X}$$

  with Q-measure one. But because f only takes values zero or one, if it is strictly positive, it must be 1.
  - This implies that $\tilde{f}(x) \in \mathfrak{b}$ with Q-measure 1.
* If $\mathfrak{b}$ consists of two subsets $\mathfrak{b}'$ and $\mathfrak{b}''$ each of positive Q-measure, then we can define $f_{\mathfrak{b}'}$ and $f_{\mathfrak{b}''}$. These satisfy the conditions of the proposition—but they cannot both be strictly positive with Q-measure 1. Thus, it must be that $\mathfrak{b}$ is a singleton set. This means that $\tilde{f}(x) = \mathfrak{b}$ with Q-measure one; i.e. $\tilde{f}$ is constant.

1.3 More ergodicity

Reconsider the matrix

$$\mathbb{P} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the stationary distribution is

$$\mu = \begin{bmatrix} \alpha & \alpha & 1-2\alpha \end{bmatrix}^T, \forall \alpha \in \left[0, \frac{1}{2}\right]$$