Theory of Income III: Solutions Final Exam

Agustin Gutierrez[*] Hyejin Park [†]

June 16, 2019

[*]E-mail: agusting@uchicago.edu
[†]E-mail: hjpark0619@uchicago.edu

1 Capital Market in Industry Equilibrium

Firms in a competitive industry produce output using a decreasing returns to scale technology that is subject to an idiosyncratic technology shock. A typical firm $i \in [0,1]$ attempts to maximize:

$$\sum_{t=0}^{\infty} \beta^t \sum_{z_i^t} \Pi_{i,t}\left(z_i^t\right) \left[z_{i,t} k_{i,t}^{\alpha} - \bar{p}_t \max\left\{k_{i,t} - k_{i,t-1}, 0\right\} + \underline{p}_t \max\left\{k_{i,t-1} - k_{i,t}, 0\right\}\right].$$

Here $\Pi_{i,t}\left(z_i^t\right)$ is the probability that firm i has an idiosyncratic history of productivity shocks z_i^t; $z_{i,t} k_{it}^{\alpha}$ with $\alpha \in (0,1)$ is revenue in period t , and the last two terms are the cost of buying capital (at price $\bar{p}_t$) and revenue from selling capital (at price $\underline{p}_t < \bar{p}_t$) if the firm adjusts its capital stock. The discount factor is a constant $\beta \in (0,1)$. Firms take $\bar{p}_t$ and $\underline{p}_t$ as well as their initial capital stock $k_{i,-1}$ as given when solving this problem.

There is also a competitive industry that intermediates trade in capital. Firms in this industry can buy capital in a typical period t and then resell it in period $t+1$. Thus the intermediaries maximize profit by buying capital at price $\underline{p}_t$ and then reselling it in the following period at price $\bar{p}_{t+1}$. Profits are:

$$\sum_{t=0}^{\infty} \beta^t \left(\bar{p}_t K_t^s - \underline{p}_t K^B\right),$$

but they face a constraint that at the start of period $t+1$ their stock of capital is $K_{t+1} = K_t + K^b - K^s$, and they can only trade capital that they hold at the start of the period, $K_t^s \leq K_t$. Again the capital intermediaries take prices $\underline{p}_t$ and $\bar{p}_t$ as well as their initial capital stock K_0 as given.

In an industry equilibrium, the total quantity of capital bought by the producers is equal to the total quantity sold by the intermediaries, $\int_0^1 \max\left\{k_{i,t} - k_{i,t-1}, 0\right\} di = K_t^s$, and similarly $\int_0^1 \max\left\{k_{i,t-1} - k_{i,t}, 0\right\} di = K_t^b$. Finally, the total stock of capital is fixed, $\bar{K} = K_t + \int_0^1 k_{i,t} di$.

1.1 Question 1

Define an industry equilibrium.

. . . . . .

An industry equilibrium is a sequence of buying and selling prices $\left\{\bar{p}_t, \underline{p}_t\right\}$ and a sequence of quantities $\left\{K_t, K_t^s, K_t^b, \{k_{i,t}\}_{i\in[0,1]}\right\}$ such that:

1. Given prices the producer i solve:

$$v_i\left(k_{i,-1}, z_{i0}\right) \equiv \max_{\{k_{i,t}\}} \sum_{t=0}^{\infty} \beta^t \Pi_{i,t}\left(z_i^t\right)\left[z_{i,t}k_{i,t}^{\alpha} - \bar{p}_t \max\left\{k_{i,t} - k_{i,t-1}, 0\right\} + \underline{p}_t \max\left\{k_{i,t-1} - k_{i,t}, 0\right\}\right]$$
$$s.t. k_{i,-1}, z_{i,0} \text{ given.}$$

2. Given prices the intermediaries solve:

$$w\left(K_0\right) \equiv \max_{\left\{K_t^b, K_t^s, K_{t+1}\right\}} \sum_{t=0}^{\infty} \beta^t\left[\bar{p}_t K_t^s - \underline{p_t} K_t^b\right]$$
$$s.t.\ K_{t+1} = K_t + K_t^b - K_t^s,$$
$$K_t^s \leq K_t.$$

3. Market clears:

$$K_t^s = \int_0^1 \max\left\{k_{i,t} - k_{i,t-1}, 0\right\} di$$
$$K_t^b = \int_0^1 \max\left\{k_{i,t-1} - k_{i,t}, 0\right\} di$$
$$\bar{K} = K_t + \int_0^1 k_{i,t} di.$$

1.2 Question 2

Assume $z_{i,t}$ follows a two-state first order Markov process taking values $\bar{z} > \underline{z}$. Also assume the process is symmetric, so the probability that $z_{it} = z_{i,t-1}$ is $\rho \in (0,1)$. Assuming $\bar{p}_t = \bar{p}$ and $\underline{p_t} = \underline{p}$ are constant over time, find conditions on the two prices under which all firms in the high productivity state find it strictly optimal to use capital $\bar{k}$ and all firms in the low productivity state find it strictly optimal to use capital $\underline{k}$ for some $\underline{k} < \bar{k}$.

......

The BE for the producer given state (k, z) is:

$$v(k, z) = \max_{k'} zk'^{\alpha} - \bar{p}\max\{k' - k, 0\} + \underline{p}\max\{k - k', 0\} + \beta\{\rho v(k', z) + (1-\rho) v(k', z')\}.$$

We look for condition such that if $z = \bar{z}$ then I choose $k' = \bar{k}$ and if $z = \underline{z}$ the producer chooses $k = \underline{k}$. This mean that at the solution, the state can take four different combinations $\left\{\left(\bar{k}, \bar{z}\right), \left(\bar{k}, \underline{z}\right), (\underline{k}, \bar{z}), (\underline{k}, \underline{z})\right\}$. We study each case bellow.

1. If we start with $\left(\bar{k}, \underline{z}\right)$, then it has to be optimal to sell capital:

$$\begin{aligned}
v\left(\bar{k}, \underline{z}\right) &= \underline{z}\underline{k}^{\alpha} + \underline{p}\left(\bar{k} - \underline{k}\right) + \beta\{\rho v(\underline{k}, \underline{z}) + (1-\rho) v(\underline{k}, \bar{z})\} \\
\alpha\underline{z}\underline{k}^{\alpha-1} &- \underline{p} + \beta\{\rho v_1(\underline{k}, \underline{z}) + (1-\rho) v_1(\underline{k}, \bar{z})\} = 0 \\
v_1\left(\bar{k}, \underline{z}\right) &= \underline{p}
\end{aligned}$$

2. If we start with $(\underline{k}, \bar{z})$, then it has to be optimal to buy capital:

$$\begin{aligned}
v(\underline{k}, \bar{z}) &= \bar{z}\bar{k}^{\alpha} - \bar{p}\left(\bar{k} - \underline{k}\right) + \beta\left\{\rho v\left(\bar{k}, \bar{z}\right) + (1-\rho) v\left(\bar{k}, \underline{z}\right)\right\} \\
\alpha\bar{z}\bar{k}^{\alpha-1} &- \bar{p} + \beta\left\{\rho v_1\left(\bar{k}, \bar{z}\right) + (1-\rho) v_1\left(\bar{k}, \underline{z}\right)\right\} = 0 \\
v_1(\underline{k}, \bar{z}) &= \bar{p}.
\end{aligned}$$

3. If we start with $\left(\bar{k},\bar{z}\right)$, then it has to be optimal to keep capital:

$$v\left(\bar{k},\bar{z}\right)=\bar{z}\bar{k}^{\alpha}+\beta\left\{\rho v\left(\bar{k},\bar{z}\right)+(1-\rho)\,v\left(\bar{k},\underline{z}\right)\right\}$$
$$\lim_{k\uparrow\bar{k}}v_1\left(k,\bar{z}\right)=v_1\left(\underline{k},\bar{z}\right).$$

where the second equation is an smooth pasting condition for our value function.

4. If we start with $(\underline{k},\underline{z})$, then it has to be optimal to keep capital:

$$v\left(\underline{k},\underline{z}\right)=\underline{z}\underline{k}^{\alpha}+\beta\left\{\rho v\left(\underline{k},\underline{z}\right)+(1-\rho)\,v\left(\underline{k},\bar{z}\right)\right\}$$
$$\lim_{k\downarrow\underline{k}}v_1\left(k,\underline{z}\right)=v_1\left(\bar{k},\underline{z}\right).$$

Using these information it follows that $\bar{p}$ and $\underline{p}$ need to satisfy:

$$\alpha\underline{z}\underline{k}^{\alpha-1}+\beta\left\{\rho\underline{p}+(1-\rho)\,\bar{p}\right\}=\underline{p}$$
$$\alpha\bar{z}\bar{k}^{\alpha-1}+\beta\left\{\rho\bar{p}+(1-\rho)\,\underline{p}\right\}=\bar{p}.$$

1.3 Question 3

Still assume $\bar{p}_t=\bar{p}$ and $\underline{p}_t=\underline{p}$. Find conditions on the two prices under which intermediaries' profit maximization problem has a solution which involves buying capital and then selling it in the following period. Prove that in that solution, the constraint $K_t^s\leq K_t$ always binds.

. . . . . .

The intermediaries solve:

$$w(K) = \max_{\left\{K^b, K^s, K'\right\}} \bar{p}K^s - \underline{p}K^b + \beta w\left(K + K^b - K^s\right)$$
$$s.t.\ K^s \leq K.$$

The FOC:

$$\begin{aligned}
\{K^s\} :& \bar{p} = \beta w'(K') + \lambda \\
\left\{K^b\right\} :& \underline{p} = \beta w'(K') \\
\{EC\} :& w'(K) = \beta w'(K') + \lambda
\end{aligned}$$

Since $\bar{p} > \underline{p}$ it follows that $\lambda > 0$ so the constraint binds. This implies $K^s = K$. Hence,

$$\begin{aligned}
\bar{p} &= w'(K) \\
\underline{p} &= \beta w'\left(K^b\right)
\end{aligned}$$

Finally the stationarity of the solution we are trying to derive requires $K^b = K^s$ to keep the stock of capital constant. Prices then satisfy:

$$\bar{p}\beta = \underline{p}.$$

1.4 Question 4

Assume that in every period , half of all firms are in the high productivity state and half are in the low productivity state. Use the capital market clearing condition $\bar{K} = K_t + \int_0^1 k_{i,t} di$ to find the equilibrium prices. You should look for a steady state equilibrium with initial conditions chosen to ensure there is a steady state.

......

Market clearing for capital implies:

$$\begin{aligned}\bar{K} &= K + \frac{1}{2}\underline{k} + \frac{1}{2}\bar{k} \\ &= K_0 + \frac{1}{2}\underline{k} + \frac{1}{2}\bar{k}.\end{aligned}$$

From question (2) we can write the individual levels of capital as function of prices (use the previous result to get only one price):

$$\begin{aligned}\underline{k} &= \left[\frac{\alpha \underline{z}}{\bar{p}\rho\beta\left[1-\beta\right]}\right]^{\frac{1}{1-\alpha}} \\ \bar{k} &= \left[\frac{\alpha \bar{z}}{\bar{p}\left[1-\rho\beta^2+\beta^2-\beta\rho\right]}\right]^{\frac{1}{1-\alpha}}.\end{aligned}$$

Finally we need to figure out the condition for K_0. It follows from the assumption of the equilibrium we are constructing that:

$$\begin{aligned}K_0 &= K^S \\ &= \int_0^1 \max\left\{k_{i,0} - k_{i,-1}, 0\right\} di \\ &= \frac{1}{2}\left(1-\rho\right)\left\{\bar{k} - \underline{k}\right\},\end{aligned}$$

since half of the firm had $\underline{k}$ and from those a proportion $1-\rho$ gets the high productivity shock and needs to buy capital.

Putting everything together:

$$
\begin{aligned}
2\bar{K} &= \left[(2-\rho)\,\bar{k} + \rho\underline{k}\right] \\
\bar{p} &= \left(\frac{(2-\rho)}{2\bar{K}} \left[\frac{\alpha\bar{z}}{[1-\rho\beta^2+\beta^2-\beta\rho]} \right]^{\frac{1}{1-\alpha}} + \frac{\rho}{2\bar{K}} \left[\frac{\alpha\underline{z}}{\rho\beta\,[1-\beta]} \right]^{\frac{1}{1-\alpha}} \right)^{1-\alpha}.
\end{aligned}
$$

1.5 Question 5

Does an equilibrium of this form always exists? If not, describe another type of equilibrium.

......

Alternative we can construct an equilibrium where $\underline{p} > \bar{p}$ in which case everyone decides to keep their capital stock. Important capital does not depreciate.

2 Unions in a Neoclassical Growth Model

A representative firm uses capital K_t and a variety of different types of labor $h_{i,t}$ to produce output. It maximizes:

$$
\sum_{t=0}^{\infty} q_0^t \left(K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)\,K_t - K_{t+1} - \int_0^1 w_{i,t} h_{i,t}\, di \right),
$$

where the labor bundle H_t satisfies:

$$
H_t = \left(\int_0^1 h_{i,t}^{\frac{\eta-1}{\eta}}\, di \right)^{\frac{\eta}{\eta-1}}.
$$

The firm takes wages for each type of worker, $w_{i,t}$, as well as intertemporal prices q_0^t and its initial capital stock K_0, as given in solving this problem.

A typical household $i \in [0,1]$ has preferences over consumption $C_{i,t}$ and work $h_{i,t}$ given by

$$\sum_{t=0}^{\infty} \beta^t \left(\log C_{i,t} - v\left(h_{i,t}\right)\right).$$

The household faces a single lifetime budget constraint,

$$a_0 = \sum_{t=0}^{\infty} q_0^t \left(C_{i,t} - w_{i,t} h_{i,t}\right),$$

where a_0 is its initial wealth, q_0^t is the intertemporal price, and $w_{i,t}$ is the wage rate. In contrast to the usual household problem, the household does not take the wage as given, but rather sets the wage optimally to exploit its market power. That is, the household acts as a monopoly union in providing its specialized service.

2.1 Question 1

From the firm problem, find an expression for $h_{i,t}$ as a function of $H_t, w_{i,t}$ and a wage index W_t.

……

The objective function of the representative firm is

$$\sum_{t=0}^{\infty} q_0^t \left(K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t - K_{t+1} - \int_0^{\infty} w_{i,t} h_{i,t} di\right)$$

The first order condition with respect to $h_{i,t}$ is

$$(1-\alpha) K_t^{\alpha} H_t^{-\alpha} \left(\int_0^1 h_{i,t}^{\frac{\eta-1}{\eta}} di\right)^{\frac{1}{\eta-1}} h_{i,t}^{-\frac{1}{\eta}} = w_{i,t}$$

$$\Rightarrow \quad \left(\frac{h_{i,t}}{h_{0,t}}\right)^{-\frac{1}{\eta}} = \frac{w_{i,t}}{w_{0,t}}$$

Putting this expression in the definition of H_t,

$$
\begin{aligned}
H_t &= h_{0,t} w_{0,t}^{\eta} \left(\int_0^1 w_{i,t}^{1-\eta} di \right)^{\frac{-\eta}{1-\eta}} \\
&= h_{0,t} w_{0,t}^{\eta} W_t^{-\eta}
\end{aligned}
$$

where $W_t = \left(\int_0^1 w_{i,t}^{1-\eta} di \right)^{\frac{1}{1-\eta}}$ Then,

$$
h_{i,t} = H_t \left(\frac{w_{i,t}}{W_t} \right)^{-\eta}
$$

2.2 Question 2

Using the previous result, find the wage that each household sets.

......

Household maximization problem is given as

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\log C_{i,t} - v(h_{i,t}) \right) + \lambda \left(a_0 - q_0^t \left(C_{i,t} - w_{i,t} h_{i,t} \right) \right)
$$

subject to $h_{i,t} = H_t \left(\frac{w_{i,t}}{W_t} \right)^{-\eta}$. The first order conditions are

$$
\begin{aligned}
\beta^t \frac{1}{C_{i,t}} &= \lambda q_0^t \\
\beta^t v'(h_{i,t}) &= \lambda q_0^t H_t^{\frac{1}{\eta}} h_{i,t}^{-\frac{1}{\eta}} \frac{\eta - 1}{\eta} W_t \\
&= \lambda \frac{\eta - 1}{\eta} w_{i,t}
\end{aligned}
$$

Combining the two gives the wage expression

$$
w_{i,t} = \frac{\eta}{\eta - 1} v'(h_{i,t}) C_{i,t}
$$

2.3 Question 3

Solve the rest of the firm problem.

. . . . . .

The first order condition of the firm problem with respect to K_{t+1} is

$$q_0^t = q_0^{t+1}\left(\alpha K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha} + 1 - \delta\right)$$

Combined with the demand for $h_{i,t}$, this equaation characterizes the optimal policy of the firm.

2.4 Question 4

Solve the rest of the household problem.

. . . . . .

The consumption Euler equation is given as

$$\frac{1}{C_{i,t}} = \beta\frac{q_0^t}{q_0^{t+1}}\frac{1}{C_{i,t+1}}$$

2.5 Question 5

How does the presence of unions affect the Euler equation? The relationship between the marginal rate of substitution between consumption and leisure and the real wage?

. . . . . .

Combining the consumption Euler equation and the firm side capital equation,

$$\frac{1}{C_{i,t}} = \beta \frac{1}{C_{i,t+1}} \left(\alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta \right)$$

This is same as the Euler equation from the standard Neoclassical model. Next, after imposing the symmetric equilibrium,

$$v'(H_t)C_t = \frac{\eta - 1}{\eta} W_t$$

As a result of labor union, we have now labor wedge, which is constant over time.