Price Theory III
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(Lars Stole)

Bayesian Games

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Contents

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1 Increasing differences and cut-off strategies in Bayesian games

[I have borrowed heavily from Myerson's lecture notes. All mistakes are of mine own.]

1.1 Definitions

Consider Bayesian games in which each player i first learns his type $\tilde{t}_i$, and then i chooses his action c_i. We assume that $\tilde{t}_i$ is drawn from some probability distribution p_i independently of all other players. Independence allow us to write the joint probability distribution of the players' types as

$$p\left(t_1, t_2, \ldots, t_n\right) = \prod_{i \in N} p_i\left(t_i\right),$$

where $p_i\left(t_i\right)$ denotes the probability that player i is of type $t_i \in T_i$.

The payoffs of each player i may depend on all players' types and actions according to some utility payoff function $u_i\left(c_1, c_2, \ldots, c_n, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n\right)$. We assume that players' types and actions are ordered as numbers; i.e. $c_i \in C_i \subseteq \mathbb{R}$ and $\tilde{t}_i \in T_i \subseteq \mathbb{R}$.

Consider a two-player Bayesian game ($N = \{1, 2\}$) where player 1 has two possible actions, $C_1 = \{T, B\}$. Suppose that players have several possible types (each type is denoted t_i for $i \in N$) and actions. The difference in player 1's payoff from switching from B to T is given by

$$u_1\left(T, c_2, t_1, t_2\right) - u_1\left(B, c_2, t_1, t_2\right),$$

where the difference depends on player 1's type, t_1, player 2's action, c_2, and player 2's type, t_2.

We say that player 1's payoffs satisfy (weakly or strictly) *increasing differences* if

$$u_1\left(T, c_2, t_1, t_2\right) - u_1\left(B, c_2, t_1, t_2\right)$$

is a (weakly or strictly) increasing function of t_1 for any $c_2 \in C_2$ and any $t_2 \in T_2$. That is, player 1's payoff satisfy weakly increasing differences if, for any $r_1, t_1 \in T_i$ such that $r_1 \geq t_1$,

$$u_1\left(T, c_2, r_1, t_2\right) - u_1\left(B, c_2, r_1, t_2\right) \geq u_1\left(T, c_2, t_1, t_2\right) - u_1\left(B, c_2, t_1, t_2\right), \forall c_2 \in C_2, \forall t_2 \in T_2.$$

Similarly, player 1's payoff satisfy strictly increasing differences if, for any $r_1, t_1 \in T_i$ such that $r_1 > t_1$,

$$u_1\left(T, c_2, r_1, t_2\right) - u_1\left(B, c_2, r_1, t_2\right) > u_1\left(T, c_2, t_1, t_2\right) - u_1\left(B, c_2, t_1, t_2\right), \forall c_2 \in C_2, \forall t_2 \in T_2.$$

With increasing differences, player 1's higher types find T relatively more attractive than player 1's lower types.

A *cut-off strategy* for player 1 is one in which there exists a cut-off/threshold θ such that

$$\sigma_1 = \begin{cases} [T] & \text{if } t_1 > \theta \\ p\,[T] + (1-p)\,[B]\,, p \in [0,1] & \text{if } t_1 = \theta \\ [B] & \text{if } t_1 < \theta \end{cases}.$$

Observe that the probability of player 1 choosing T decreases as t_1 increases. The following fact gives us a useful property when looking for equilibria with payoffs that satisfy increasing differences.

Fact 1.1. *If player 1's payoff satisfy increasing differences, then player 1 will always want to use a cut-off strategy, independent of player 2's strategy.*

We can also define the property of increasing differences when player 1's set of actions is a subset of $\mathbb{R}$.

Definition 1.1. *(Increasing differences)* Suppose $N = \{1, 2\}$ and that player 1's action is $c_1 \in C_1 \subseteq \mathbb{R}$. We say that player 1's payoff satisfy *(weakly or strictly) increasing differences* if, for every pair of possible actions $c_1, d_1 \in C_i$ such that $c_1 > d_1$, the difference

$$u_1\,(c_1, c_2, t_1, t_2) - u_1\,(d_1, c_2, t_1, t_2)$$

is a (weakly or strictly) increasing function of player 1's type, $t_1 \in T_1$, for all $c_2 \in C_2$ and for all $t_2 \in T_2$.
If u_1 is differentiable , then the condition for increasing differences is

$$\frac{\partial^2 u_1}{\partial c_1 \partial t_1} \geq (>)\, 0.$$

Fact 1.2. *If player 1's payoffs satisfy weakly increasing differences, then, against any strategy of player 2, player 1 will have* ***some*** *best-response strategy $s_1 : T_1 \to C_1$ that is weakly increasing; i.e. for any $r_1, t_1 \in T_1$ such that $r_1 \geq t_1$, $s_1\,(r_1) \geq s_1\,(t_1)$.*

Fact 1.3. *If player 1's payoff satisfy strictly increasing differences, then,* ***all*** *player 1's best-response strategies must be weakly increasing. Specifically, if (i) $r_1, t_1 \in T_1$ such that $r_1 > t_1$; (ii) against some strategy $\sigma_2 \in \Delta\,(C_2)$ for player 2, action $c_1 \in C_1$ is optimal for type t_1 and $d_1 \in C_1$ is optimal for type r_1; then $d_1 \geq c_1$.*
In other words, if , in equilibrium, type t_1 chooses c_1 with positive probability (i.e. $\sigma_1\,(c_1|t_1) > 0$), and type r_1 chooses d_1 with positive probability (i.e. $\sigma_1\,(d_1|r_1) > 0$), then $d_1 \geq c_1$.

Remark 1.1. Strictly increasing differences does not imply that there exists some best-response strategy that is strictly increasing.

1.2 Example

Suppose that $N = \{1, 2\}$, $C_1 = \{T, B\}$, $C_2 = \{L, R\}$, and that neither player has private information (i.e. T_1 and T_2 are singleton sets). The payoff matrix is as follows:

Player 1\2	L	R
T	$0, \mathbf{0}$	$\mathbf{0}, -1$
B	$\mathbf{1}, 0$	$-1, \mathbf{3}$

There is no pure-strategy Nash equilibrium for this game. The unique Nash equilibrium with randomised strategies is given by

$$\sigma = \left(\frac{3}{4} [T] + \frac{1}{4} [B], \frac{1}{2} [L] + \frac{1}{2} [R] \right).$$

Suppose now that the set of player 1's possible types are $T_1 = \{0, 0.1, 0.2, 0.3\}$ and $p_1 (t_1) = 1/4$ for all $t_1 \in T_1$. Player 2 still has no private information (so T_2 is a singleton set). Given player 1's type, suppose that the payoff matrix is now given the matrix below.

Player 1\2	L	R
T	$t_1, \mathbf{0}$	$\mathbf{t_1}, -1$
B	$\mathbf{1}, 0$	$-1, \mathbf{3}$

Observe that there is no pure-strategy Nash equilibrium again so that there must exist randomised equilibria. Since player 1 has four types, the number of player 1's pure strategies is $2^4 = 16$. The number of possible support is thus, $2^{16} = 65,536$. It might therefore seem difficult to find the equilibria.

However, notice that player 1's utility from switching from B to T is

$$u_1 (T, L, t_1) - U_1 (B, L, t_1) = t_1 - 1,$$
$$u_1 (T, R, t_1) - U_1 (B, R, t_1) = t_1 + 1.$$

Thus, the differences are strictly increasing in t_1; i.e. higher type t_1 will always find T relatively more attractive than the lower types. This means that we can restrict player 1's possible support to the following:

$$\sigma_1 = \begin{cases} [T] & \text{if } t_1 > \theta \\ p [T] + (1-p) [B], \ p \in [0,1] & \text{if } t_1 = \theta \\ [B] & \text{if } t_1 < \theta \end{cases}.$$

We can summarise the possible support as follows (the last column shows player 2's belief about the probability that player 1 plays T).

$\theta \backslash t_1$	$t_1 = 0$	$t_1 = 0.1$	$t_1 = 0.2$	$t_1 = 0.3$	$P(T)$
$\theta > 0.3$	B	B	B	B	0
$\theta = 0.3$	B	B	B	$p[T] + (1-p)[B]$	$[0, 1/4]$
$0.2 < \theta < 0.3$	B	B	B	T	$[1/4]$
$\theta = 0.2$	B	B	$p[T] + (1-p)[B]$	T	$[1/4, 1/2]$
$0.1 < \theta < 0.2$	B	B	T	T	$[1/2]$
$\theta = 0.1$	B	$p[T] + (1-p)[B]$	T	T	$[1/2, 3/4]$
$0 < \theta < 0.1$	B	T	T	T	$3/4$
$\theta = 0$	$p[T] + (1-p)[B]$	T	T	T	$[3/4, 1]$
$\theta < 0$	T	T	T	T	1

Thus, Fact 1.1 allows us to rule out supports such as (B, T, B, T) and (T, B, T, T) etc. Moreover, given that we have strictly increasing differences, we can rule out randomisation between, say $(BBBB)$ and (B, B, T, T). This is because, if type 0.2 is indifferent between T and B, strictly increasing differences mean that type 0.3 must strictly prefer T over B and hence it cannot be that both types 0.2 and 0.3 are randomising.

Suppose player 2's strategy is

$$\sigma_2 = q\,[L] + (1-q)\,[R]\,.$$

For the cut-off strategy to be optimal, it must be that

$$\begin{aligned} t_1 \geq \theta \Leftrightarrow U_1\left(T, \sigma_2, t_1\right) &\geq U_1\left(B, \sigma_2, t_1\right) \\ \Leftrightarrow q t_1 + (1-q)\,t_1 &\geq q\,(1) + (1-q)\,(-1) \\ \Leftrightarrow t_1 &\geq 2q - 1, \end{aligned}$$

since $t_1 \geq \theta$ implies that player 1 (weakly) prefers to play T over B. Thus, we can obtain the cut-off value by setting $t_1 = \theta$ so that

$$\theta = 2q - 1 \Leftrightarrow q = \frac{1+\theta}{2}. \tag{1.1}$$

Thus, the cut off θ is optimal when $q = (1+\theta)/2$.

Given the payoffs, there is no equilibrium in which player 2 chooses L or R for sure. (If player 2 plays L, then player 1's best response is to play B (whatever the type), but then player 2 can do better by playing R. If player 2 plays R, then player 1 plays B, but, again, player 2 can do better by playing L instead.)[1] To make player 2 wiling to randomise, we, of course, must have

$$\begin{aligned} \mathbb{E}u_2\,(L) &= \mathbb{E}u_2\,(R) \\ \Rightarrow 0 &= P\,(T)\,(-1) + (1 - P\,(T))\,3 \\ \Leftrightarrow P\,(T) &= \frac{3}{4}, \end{aligned}$$

where $P\,(T)$ denotes the unconditional probability of player 1 choosing T as assessed by player 2 (who does not know player 1's type). In equilibrium, this conditional probability must be consistent with the behavioural strategy for player 1 which specifies, for each $t_1 \in T_1$, the conditional probability $\sigma_1\,(T|t_1)$. Consistency here is in the sense of Bayes' rule so, we need

$$P\,(T) = \sum_{t_1 \in T_1} \sigma_1\,(T|t_1)\,p_1\,(t_1)\,.$$

[1] We can also argue this in terms of θ. If player 2 plays $[L]$, then it must be that $\theta > 0.3$ so that all types of player 1 choose B, which implies that $P\,(T) = 0$. But then this implies $\mathbb{E}u_2\,(R) = 3 > 0 = \mathbb{E}u_2\,(L)$ so that player 2, in fact, wants to play $[R]$. Similarly, if player 2 plays $[R]$, then it must be that $\theta < 0$ so that all types of player 1 choose T, so $P\,(T) = 1$. But this implies that $\mathbb{E}u_2\,(R) = -1 < 0 = \mathbb{E}u_2\,(L)$ so that player 2 would actually want to play $[L]$.

The cut-off strategy specifies

$$\sigma_1 (T|t_1) = \begin{cases} 1 & \text{if } t_1 > \theta \\ \sigma_1 (T|\theta) & \text{if } t_1 = \theta \\ 0 & \text{if } t_1 < \theta \end{cases},$$

which implies that

$$\begin{aligned} P(T) &= \sum_{t_1<\theta} (0)\, p_1 (t_1) + \sigma_1 (T|t_1)\, p_1 (t_1) + \sum_{t_1>\theta} (1)\, p_1 (t_1) \\ &= \sigma_1 (T|\theta)\, p_1 (\theta) + \sum_{t_1>\theta} p_1 (t_1) . \end{aligned}$$

We know from the table above that for $P(T) = 3/4$, we need $0 \leq \theta \leq 0.1$.

Let q denote the probability of player 2 choosing L. To make player 1's cut-off strategy optimal, player 2's randomised strategy $q[L] + (1-q)[R]$ must make player 1 prefer B when $t_1 = 0$ but make player 1 prefer T when $t_1 = 0.1$; i.e.

$$\begin{aligned} \mathbb{E}u_1 (T|t_1 = 0) &\leq \mathbb{E}u_1 (B|t_1 = 0) \\ \Leftrightarrow q(0) + (1-q)(0) &\leq q(1) + (1-q)(-1) \\ \Leftrightarrow \frac{1}{2} &\leq q \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}u_1 (T|t_1 = 0.1) &\geq \mathbb{E}u_1 (B|t_1 = 0.1) \\ \Leftrightarrow q(0.1) + (1-q)(0.1) &\geq q(1) + (1-q)(-1) \\ \Leftrightarrow q &\leq \frac{11}{20}. \end{aligned}$$

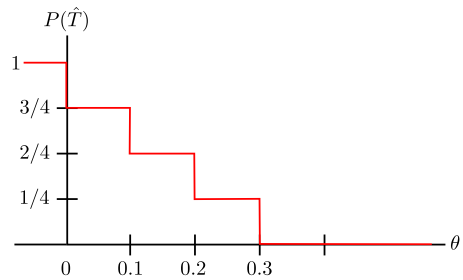
Thus, we must have

$$\frac{1}{2} \leq q = \frac{1+\theta}{2} \leq \frac{11}{20}.$$

Therefore, the equilibrium is given by

$$\left((\sigma_1 (t_1))_{t_1 \in T_1}, \sigma_2 \right) = \left(([B], [T], [T], [T]), (q[L] + (1-q)[R]) \right)$$

for any $q \in [1/2, 11/20]$. To see why we get a range for q, consider the following figure which shows player 2's belief about the probability that player 1 will play T, $P(T)$, as a function of the cut-off value.



Since $P(T) = 3/4$ in this case, notice that any value of $\theta \in [0, 0.1]$ would be consistent.

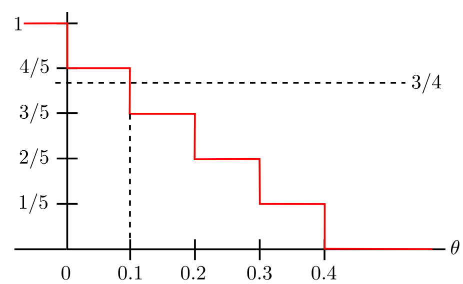
Now suppose we have one more type for player 1 so that

$$T_1 = \{0, 0.1, 0.2, 0.3, 0.4\}, \; p(t_1) = \frac{1}{5}, \forall t_1 \in T_1.$$

It remains the case that, in equilibrium, we must have

$$P(T) = \frac{3}{4}.$$

Given that player 1 has increasing strategies, this means that for $P(T) = 3/4$, it must be that $\theta = 0.1$. This is clear from the figure below.



This allows us to calculate the probability with which player 1 of type $\theta = 0.1$ plays T:

$$\begin{aligned} \frac{3}{4} = P(T) &= \sigma_1(T|\theta)\, p_1(\theta) + \sum_{t_1 > \theta} p_1(t_1) \\ &= \sigma_1(T|\theta)\frac{1}{5} + \frac{3}{5} \\ \Leftrightarrow \sigma_1(T|\theta) &= \frac{3}{4}. \end{aligned}$$

Using (1.1), we can then obtain q as

$$q = \frac{1+\theta}{2} = \frac{11}{20}.$$

2 Review problems

2.1 Problem 1

2.1.1 Part 1

Consider a Cournot duopoly model with linear demand, $p = a - bQ$, and constant unit costs, $C_i(q) = c_i q$. Both firms initially have unit costs equal to c, but with probability $\phi \in (0,1)$, firm 1 obtains a cost reduction equal to $\delta \in (0,c)$ before outputs are chosen. Thus, firm 1's unit cost is $c_{1h} = c$ with probability $1-\phi$ and $c_{1\ell} = c - \delta$ with probability ϕ; firm two's unit cost is $c_2 = c$.

(a) Complete information Assume that any cost reduction to firm 1 is publicly observable to both firms before they choose their outputs. Solve the Nash equilibrium outputs and firm 1's equilibrium profit following the two possible cost realisations. (Assume, throughout, that a is sufficiently high that each firm produces positive output in equilibrium.)

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Let $t_1 \in \{h, \ell\}$ and define the profit function for firm $i \in \{1,2\}$ as

$$\pi_i(q_i, q_j; t_1) = \left[a - b(q_i + q_j) - \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right)\right] q_i.$$

The problem we wish to solve is then

$$\max_{q_i} \ \pi_i(q_i, q_j; t_1).$$

The profit function is globally concave so that the first-order condition is both necessary and sufficient for a unique maximum:

$$\begin{aligned} 0 &= -bq_i^* + a - b(q_i^* + q_j) - \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right) \\ \Leftrightarrow q_i^* &= \frac{a - c - bq_j + \delta \mathbf{1}_{\{t_1=\ell,i=1\}}}{2b}. \end{aligned}$$

Case 1: $t_1 = h$. We find a fixed point of the best response. With slight abuse of notation, we get, for each $i \in \{1,2\}$,

$$\begin{aligned} q_i^* \equiv q_i^*\left(q_j^*(q_i^*); t_1 = h\right) &= \frac{a - c - b\left(\frac{a - bq_i^* - c}{2b}\right)}{2b} = \frac{a-c}{4b} + \frac{q_i^*}{4} \\ \Leftrightarrow q_i^* &= \frac{a-c}{3b}. \end{aligned}$$

The equilibrium profit is, for each $i \in \{1,2\}$,

$$\pi_i^*\left(q_i^*, q_j^*; t_1 = h\right) = \left(a - c - b\left(\frac{a-c}{3b} + \frac{a-c}{3b}\right)\right)\frac{a-c}{3b} = \frac{(a-c)^2}{9b}.$$

Case 2: $t_1 = \ell$. For firm 1, we have

$$
\begin{aligned}
q_1^* \equiv q_1^*\left(q_2^*\left(q_1^*\right); t_1 = \ell\right) &= \frac{a - c - b\left(\frac{a - bq_1^* - c}{2b}\right) + \delta}{2b} = \frac{a - c + 2\delta}{4b} + \frac{q_1^*}{4} \\
\Leftrightarrow q_1^* &= \frac{a - c + 2\delta}{3b}, \\
q_2^* &= \frac{a - c - b\left(\frac{a - c + 2\delta}{3b}\right)}{2b} = \frac{a - c - \delta}{3b}.
\end{aligned}
$$

The equilibrium profits are:

$$
\begin{aligned}
\pi_1\left(q_1^*, q_2^*; t_1 = \ell\right) &= \left(a - c + \delta - b\left(\frac{a - c + 2\delta}{3b} + \frac{a - c - \delta}{3b}\right)\right)\left(\frac{a - c + 2\delta}{3b}\right) \\
&= \frac{\left(a - c + 2\delta\right)^2}{9b}.
\end{aligned}
$$

(b) Incomplete information. Suppose instead that any cost reduction is private information to firm 1. Solve for the Bayes-Nash equilibrium outputs, $(q_{1h}^*, q_{1\ell}^*, q_2^*)$, and compute firm 1's profit for each possible cost.

......

Type-h and type-ℓ firm 1's best responses remain unchanged. Firm 2's maximisation problem is now

$$
\max_{q_2} \ \mathbb{E}_{t_1}\left[\pi_i\left(q_i, q_j; t_1\right)\right] \equiv \max_{q_2} \ a - b\left(q_2 + \mathbb{E}_{t_1}\left[q_1\right]\right) q_2,
$$

where we note that the profit function is linear in the expected quantity produced by firm 1. Then, it follows from what we already computed that

$$
\begin{aligned}
q_2^* &= \frac{a - c - b\mathbb{E}_{t_1}\left[q_1\right]}{2b} \\
&= \frac{a - c}{2b} - \frac{1}{2}\left[\phi q_1^*\left(q_2^*; t_1 = \ell\right) + (1 - \phi)\, q_1^*\left(q_2^*; t_1 = h\right)\right] \\
&= \frac{a - c}{2b} - \frac{1}{2}\left[\phi \frac{a - c - bq_2^* + \delta}{2b} + (1 - \phi)\, \frac{a - c - bq_2^*}{2b}\right] \\
&= \frac{a - c}{2b} - \frac{1}{2}\left(\frac{a - c}{2b} - \frac{q_2^*}{2} + \frac{\phi\delta}{2b}\right) \\
\Leftrightarrow q_2^* &= \frac{a - c - \phi\delta}{3b}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
q_{1h}^* &:= q_1^*\left(q_2^*; t_1 = h\right) = \frac{a - c - b\left(\frac{a - c - \phi\delta}{3b}\right)}{2b} = \frac{a - c}{3b} + \frac{\phi\delta}{6b}, \\
q_{1\ell}^* &:= q_1^*\left(q_2^*; t_1 = \ell\right) = \frac{a - c - b\left(\frac{a - c - \phi\delta}{3b}\right) + \delta}{2b} = \frac{a - c}{3b} + \frac{(3 + \phi)\,\delta}{6b}.
\end{aligned}
$$

Firm 1's profits are then

$$
\begin{aligned}
\pi_{1h} &\coloneqq \pi_1\left(q_{1h}^*, q_2^*; t_1 = h\right) \\
&= \left(a - c - b\left(\frac{a-c}{3b} + \frac{\phi\delta}{6b} + \frac{a-c-\phi\delta}{3b}\right)\right)\left(\frac{a-c}{3b} + \frac{\phi\delta}{6b}\right) \\
&= \left(\frac{a-c}{3} + \frac{\phi\delta}{6}\right)\left(\frac{a-c}{3b} + \frac{\phi\delta}{6b}\right) = \frac{\left(a-c+\frac{\delta\phi}{2}\right)^2}{9b}, \\
\pi_{1\ell} &\coloneqq \pi_1\left(q_{1\ell}^*, q_2^*; t_1 = \ell\right) \\
&= \left(a - c + \delta - b\left(\frac{a-c}{3b} + \frac{(3+\phi)\,\delta}{6b} + \frac{a-c-\phi\delta}{3b}\right)\right)\left(\frac{a-c}{3b} + \frac{(3+\phi)\,\delta}{6b}\right) \\
&= \left(\frac{a-c}{3} + \frac{(3+\phi)\,\delta}{6}\right)\left(\frac{a-c}{3b} + \frac{(3+\phi)\,\delta}{6b}\right) = \frac{\left(a-c+\frac{(3+\phi)\delta}{2}\right)^2}{9b}.
\end{aligned}
$$

(c) Incomplete information with the possibility of evidence disclosure. Sticking with the private information assumption in (b), assume that, if a cost reduction is achieved, firm 1 can choose to reveal evidence proving the reduction to firm 2 before outputs are chosen. To be precise, at date 0, nature chooses firm 1's cost type; at date 1, if firm 1's cost is low, it can reveal evidence proving this to firm 2; at date 2, outputs are simultaneously chosen by the firms. What is the equilibrium to this disclosure-output game (i.e., with what probability does the low-cost firm reveal evidence at stage 1; what are the corresponding outputs at stage 2)? [Hint: use your profit computations from (a) and (b).]

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If type-ℓ firm 1 reveals its cost, then it obtains the outcome as we found in part (a); i.e. it obtains a profit of

$$\frac{(a-c+2\delta)^2}{9b}.$$

On the other hand, if it does not reveal its cost, then it obtains the outcome as we found in part (b); .e. it obtains a profit of

$$\frac{\left(a-c+\frac{(3+\phi)\delta}{2}\right)^2}{9b}.$$

Thus, revealing its cost is optimal if and only if

$$
\begin{gathered}
\frac{(a-c+2\delta)^2}{9b} \geq \frac{\left(a-c+\frac{(3+\phi)\delta}{2}\right)^2}{9b} \\
\Leftrightarrow a-c+2\delta \geq a-c+\frac{(3+\phi)\,\delta}{2} \\
\Leftrightarrow \phi \leq 1.
\end{gathered}
$$

Since $\phi \in (0,1)$, it follows that type-ℓ firm 1 will always disclose its costs. It remains to check that type-ℓ firm 1 does not have an incentive to pretend to be type h.

So suppose that, in an equilibrium, the type-ℓ firm 1 reveals its cost. If it deviates and does not reveal information, firm 2 would react by assuming that firm 1 is of h type so that firm 2 would raise output from $q_2 = \frac{a-c-\delta}{3b}$ to $q_2 = \frac{a-c}{3b}$. This hurts firm 1 since it leads to lower prices and lower output.[2] Hence, the low-cost firm 1 will produce cost-reduction evidence in equilibrium and there will be full separation, leading to the outcomes in (a).

So why does this happen? Note that

$$q_{1h}^* < \mathbb{E}_{t_1}\left[q_1^*\right] < q_{1\ell}^*$$

and that firm 2's best response function is strictly decreasing in the quantity produced by firm 1. These mean that, when firm 1 does not disclose any information, firm 2's conjecture about firm 1's output, $\mathbb{E}_{t_1}\left[q_1^*\right]$, is strictly lower than $q_{1\ell}^*$ so that firm 2 produces greater output. But this leads to lower prices, and lower quantity for firm 1 (since firm 1's best response is also decreasing in firm 2's output), leading to lower profits.

2.1.2 Part 2

Consider a differentiated Bertrand duopoly game with linear demand:

$$\begin{aligned} q_1 &= \alpha - \beta p_1 + \gamma p_2, \\ q_2 &= \alpha - \beta p_2 + \gamma p_1, \end{aligned}$$

and constant unit costs as in Part 1: $c_{1h} = c$ with probability $1-\phi$ and $c_{1\ell} = c - \delta$ with probability ϕ, and $c_2 = c$.

(d) In the incomplete information game with cost-reduction evidence, will the low-cost firm reveal evidence to firm 2? Note that evidence only exists regarding cost reductions; if a cost reduction did not happen, firm 1 cannot prove that it did not happen (i.e. evidence is not symmetric). Explain. You do not need to explicitly prove your statement using profit computations as you did in Part 1, but you do need to give the intuition. [Hint: Drawing a set of reaction functions can be illuminating.]

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Let's first work through the complete information case. The profit function is now

$$\pi_i\left(p_i, p_j; t_1\right) = \left[p_i - \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right)\right]\left(\alpha - \beta p_i + \gamma p_j\right)$$

and the firms solve the following problem:

$$\max_{p_i}\ \left(p_i - \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right)\right)\left(\alpha - \beta p_i + \gamma p_j\right).$$

[2] To see that this leads to lower output,

$$q_1 = \frac{a - c - b\left(\frac{a-c}{3b}\right) + \delta}{2b} = \frac{a-c}{3b} + \frac{\delta}{2b} < q_{1\ell}^* = \frac{a-c}{3b} + \frac{(3+\phi)\,\delta}{6b} = \frac{a-c}{3b} + \frac{\delta}{2b} + \frac{\phi\delta}{6b}.$$

The first-order condition is, again, necessary and sufficient for a maximum and gives us that

$$\begin{aligned} 0 &= \alpha - \beta p_i + \gamma p_j - \beta \left(p_i - \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right)\right) \\ \Leftrightarrow p_i^* \left(p_j\right) &= \frac{\alpha + \beta \left(c - \delta \mathbf{1}_{\{t_1=\ell, i=1\}}\right) + \gamma p_j}{2\beta}. \end{aligned}$$

Now consider the incomplete information case. Firm 2's problem is now

$$\max_{p_2} \ (p_i - c)\left(\alpha - \beta p_2 + \gamma \mathbb{E}_{t_1}\left[p_1\right]\right),$$

which is, once again, linear in the expected price set by firm 1. Thus, the best response function is simply:

$$p_2^* = \frac{\alpha + \beta c + \gamma \mathbb{E}_{t_1}\left[p_1\right]}{2\beta}.$$

In this case, we have that

$$p_{1\ell}^* < \mathbb{E}_{t_1}\left[p_1^*\right] < p_{1h}^*$$

and that, unlike in the Cournot case, the best response functions are now increasing in the other firm's action. If type-ℓ firm 1 does not disclose, then firm 2 sets prices according to $\mathbb{E}_{t_1}\left[p_1^*\right] > p_{1\ell}^*$ and since firm 2's best response function is strictly increasing, this means that firm 2 sets a higher price in case firm 1 does not disclose. This is surely good since this means that firm 1 captures greater demand. Therefore, in contrast to the Cournot case, type-ℓ firm 1 prefers not to disclose its costs. That is, we get a pooling equilibrium.

Note that it's important here that the type-h firm 1 cannot disclose its costs—the same argument as above implies that type-h firm would profit from being able to do so, which would lead to a separating equilibrium.

2.2 Problem 2

Consider a public goods game with two players, $i \in \{1, 2\}$, and private information. Each player can choose to either contribute to the public good ($s_i = 1$) or not ($s_i = 0$); i.e. $S_i = \{0, 1\}$. If one or both players contribute, then the public good is produced and each player receives a benefit of 1. Player i's cost of contributing is $c_i \in [\underline{c}, \overline{c}]$, where $\underline{c} < 1 < \overline{c}$; c_i is independently distributed for both players according to the continuous distribution function $F\left(c_i\right)$. If player i contributes, her payoff is $1 - c_i$; if player i does not contribute, her payoff is 1 if player $j \neq i$ contributed, and 0 otherwise. Formally,

$$u_i\left(s_i, s_j, c_i\right) = \max\left\{s_1, s_2\right\} - c_i s_i.$$

A pure-strategy equilibrium of this game is a profile of contribution functions, $\left\{s_1^*\left(\cdot\right), s_2^*\left(\cdot\right)\right\}$, where $s_i^* : [\underline{c}, \overline{c}] \to \{0, 1\}$.

(a) Characterise the symmetric pure-strategy Bayesian-Nash equilibrium. Prove that it is unique and always exists.

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Observe first that u_i has strictly increasing differences with $s_i = 0$ being the "high" action: for any $c_i' > c_i$,

$$\begin{aligned} u_i(0,1,c_i') - u_i(1,1,c_i') &= (1) - (1 - c_i') = c_i' \\ &> c_i = u_i(0,1,c_i) - u_i(1,1,c_i), \\ u_i(0,0,c_i') - u_i(1,0,c_i') &= 0 - (1 - c_i') = c_i' - 1 \\ &> c_i - 1 = u_i(0,0,c_i) - u_i(1,0,c_i). \end{aligned}$$

Hence, it follows that each player must use a cut-off strategy in any equilibrium. Let the cut-off strategy be

$$s_i(c_i) = \begin{cases} [0] & \text{if } c_i > \theta_i, \\ p[0] + (1-p)[1] & \text{if } c_i = \theta_i, \\ [1] & \text{if } c_i < \theta_i. \end{cases}$$

Let $\alpha_j \coloneqq F(\theta_j)$ denote the probability that player j plays [1]. For this cut-off strategy to be optimal, it must be that

$$\begin{aligned} c_i \geq \theta_i &\Leftrightarrow [0] \succsim_i [1] \\ &\Leftrightarrow \alpha_j (\max\{0,1\} - c_i(0)) + (1-\alpha_j)(\max\{0,0\} - c_i(0)) \\ &\qquad \geq \alpha_j (\max\{1,1\} - c_i s_i) + (1-\alpha_j)(\max\{1,0\} - c_i s_i) \\ &\Leftrightarrow \alpha_j (1) + (1-\alpha_j)(0) \geq \alpha_j (1 - c_i) + (1-\alpha_j)(1-c_i) \\ &\Leftrightarrow \alpha_j \geq 1 - c_i. \end{aligned}$$

The cut-off θ_i, θ_i^*, is such that

$$\alpha_j = F(\theta_j) = 1 - \theta_i^*.$$

So in equilibrium, we need that

$$F\left(\theta_j^*\right) = 1 - \theta_i^*, \ \forall i, j \in \{1,2\}, i \neq j. \tag{2.1}$$

Notice that $c_i = \theta_i$ is a measure zero event given that F is continuous, hence we need not worry about this case.

We are looking for a symmetric equilibrium and so we need $\theta_1^* = \theta_2^* = \theta^*$, and θ^* must satisfy

$$F(\theta^*) = 1 - \theta^*.$$

We must prove that such θ^* exists and is unique. To do so, define the function $\Phi : [\underline{c}, \overline{c}] \to \mathbb{R}$

$$\Phi(c) \coloneqq F(c) - 1 + c$$

and observe that Φ is continuous and strictly increasing in θ (recall F must be nondecreasing since it is a CDF), and that

$$\begin{aligned} \Phi(\underline{c}) &= F(\underline{c}) - 1 + \underline{c} = \underline{c} - 1 < 0, \\ \Phi(\overline{c}) &= F(\overline{c}) - 1 + \overline{c} = \overline{c} > 1 \end{aligned}$$

Then, by the intermediate value theorem, there exists θ^* such that $\Phi(\theta^*) = 0$. Moreover, since Φ is strictly increasing, θ^* must be unique. In other words, the symmetric equilibrium, characterised by (2.1) exists and is unique.

(b) Characterise a pair of asymmetric pure-strategy equilibria in which one player never contributes. Give the condition on the distribution of costs that guarantees the existence of such equilibria.

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As we established above, since we have strictly increasing differences, every best response is a cut-off strategy and must satisfy (2.1). Suppose that $\theta_1^* = \underline{c}$ so that player 1 always fails to contribute; i.e. $\alpha_1 = 0$. Then,

$$[1] \succsim_2 [0] \Leftrightarrow 1 - c_2 \geq \alpha_1 = 0 \Leftrightarrow c_2 \leq 1.$$

Hence, $\theta_2^* = 1$. It remains to check that player 1 finds it optimal to never contribute, which is the case if

$$\begin{aligned} [0] \succsim_1 [1] &\Leftrightarrow 1 - c_1 \leq \alpha_2 = F(1), \forall c_1. \\ &\Leftrightarrow 1 - F(1) \leq \underline{c}. \end{aligned}$$

This gives the condition on the distribution of costs that guarantees the existence of this type of equilibria. Of course, our choice of player 1 as the "slacker" was arbitrary so the other equilibrium in the pair is the one in which player 2 always fails to contribute, player 1 contributes with probability $F(1)$ and the same condition as above must also hold.

2.3 Problem 3

Consider a first-price auction (without a reserve price) between two buyers. Valuations are $\theta_h > \theta_\ell > 0$ and the probability that $\theta = \theta_h$ is $\phi \in (0,1)$.

There is no pure-strategy equilibrium to the first-price auction with discrete types. Find the symmetric Bayesian-Nash equilibrium in which type θ_ℓ bids $b_\ell = \theta_\ell$, but type θ_h randomises her bid according to some equilibrium distribution $G(b_h)$ on $\left[\underline{b}, \overline{b}\right]$.

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Given the proposed equilibrium, type-θ_i player i's expected payoff from bidding $b \in \left[\underline{b}, \overline{b}\right]$ is

$$u_i(b, \theta_i) = (\theta_i - b)\left(\phi G(b) + (1-\phi)\mathbf{1}_{\{b > \theta_\ell\}} + \left(\frac{1-\phi}{2}\right)\mathbf{1}_{\{b = \theta_\ell\}}\right),$$

where I've assumed that if two players bid the same, we flip a fair coin to determine who wins (the result is not sensitive to the tie breaking rule). For type θ_ℓ, we need

$$\begin{aligned} & u_i(\theta_\ell,\theta_\ell) \geq u_i(b,\theta_\ell), \ \forall b \\ \Leftrightarrow (\theta_\ell-\theta_\ell)\left(\phi G(\theta_\ell)+(1-\phi)\mathbf{1}_{\{\theta_\ell>\theta_\ell\}}+\left(\frac{1-\phi}{2}\right)\mathbf{1}_{\{\theta_\ell=\theta_\ell\}}\right) & \geq (\theta_\ell-b)\left(\phi G(b)+(1-\phi)\mathbf{1}_{\{b>\theta_\ell\}}\right) \\ & \Leftrightarrow 0 \geq (\theta_\ell-b)\left(\phi G(b)+(1-\phi)\mathbf{1}_{\{b>\theta_\ell\}}\right). \end{aligned}$$

Since the right-hand side is positive, we require that $\underline{b}=\theta_\ell$ to ensure that the inequality above holds for all b.

For type θ_h to randomise, it must be that

$$\begin{aligned} u_i(b,\theta_h) &= k, \ \forall b\in\left[\theta_\ell,\overline{b}\right] \\ \Leftrightarrow (\theta_h-b)(\phi G(b)+(1-\phi)) &= k, \ \forall b\in\left[\theta_\ell,\overline{b}\right], \end{aligned} \tag{2.2}$$

where k is some constant.[3] Note that $G(\theta_\ell)=0$ so k is determined by

$$k=(\theta_h-\theta_\ell)(\phi G(\theta_\ell)+(1-\phi))=(\theta_h-\theta_\ell)(1-\phi).$$

Then,

$$\begin{aligned} (\theta_h-b)(\phi G(b)+(1-\phi)) = k &= (\theta_h-\theta_\ell)(1-\phi) \\ \Leftrightarrow G(b) &= \left(\frac{\theta_h-\theta_\ell}{\theta_h-b}-1\right)\frac{1-\phi}{\phi} \\ &= \frac{b-\theta_\ell}{\theta_h-b}\frac{1-\phi}{\phi}. \end{aligned}$$

It remains to determine $\overline{b}$. By definition, $G\left(\overline{b}\right)=1$ so that

$$\begin{aligned} G\left(\overline{b}\right)=1 &= \frac{\overline{b}-\theta_\ell}{\theta_h-\overline{b}}\frac{1-\phi}{\phi} \\ \Leftrightarrow \phi\left(\theta_h-\overline{b}\right) &= (1-\phi)\left(\overline{b}-\theta_\ell\right) \\ \Leftrightarrow \phi\theta_h+(1-\phi)\theta_\ell &= (1-\phi)\overline{b}+\phi\overline{b} \\ \Leftrightarrow \overline{b} &= \phi\theta_h+(1-\phi)\theta_\ell. \end{aligned}$$

Remark 2.1. You were not given that G was differentiable (i.e. that it has a density). If it were, then we could have differentiated (2.2) against b (why?) to obtain that

$$0=\phi G(b)+1-\phi+(\theta_h-b)\phi g(b),$$

which is a first-order differential equation. Then, using the hint would have given us the same function as $G(b)$.

Exercise 2.1. Why can there be no pure strategy equilibria?

[3] I assume G is atomless (which it is!), in particular, at $b=\theta_\ell$ so that the tie break rule does not matter.