Price Theory III
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(Lars Stole)

Monopolistic Screening and Mechanism Design
by Takuma Habu

University of Chicago

Contents

For typos/comments, email me at takumahabu@uchicago.edu.

1 Introduction

Recall the signalling and screening models that we went over in the first half of the quarter (e.g. Spence, 1973; Rothschild and Stiglitz, 1976). They were models of incomplete information—i.e. there were informed and uninformed players—with *competition* among the uninformed players.

- ▷ Signalling: the informed chose a signal, then uninformed players responded; PBE/SE.
- ▷ Screening: uninformed players offers contracts; the informed picks a contract; SPNE.

Our focus now is on screening problems with *no* competition; i.e. informed buyer(s) vs a monopolistic seller. For whatever reason, the literature uses the following terminology

- ▷ single buyer: monopolistic screening
- ▷ multiple buyers: mechanism design

In both cases we will be applying BNE.

My goal here is to highlight the similarity between the single buyer and multiple buyers models.

2 Monopolistic screening with a single buyer

By the revelation principle, it is without loss (in finding the maximum revenue for the seller) to focus on truthful equilibria in a direct mechanism.

Seller's problem: version 1

$$\begin{aligned}\max_{(q(\theta),t(\theta))_{\theta\in\Theta}} \quad & \mathbb{E}_\theta\left[t\left(\theta\right)-C\left(q\left(\theta\right)\right)\right]\\ \text{s.t. (IC)} \quad & u\left(q\left(\theta\right),\theta\right)-t\left(\theta\right)\geq u\left(q\left(\hat{\theta}\right),\theta\right)-t\left(\hat{\theta}\right),\ \forall\theta,\hat{\theta}\in\Theta,\\ \text{(IR)} \quad & u\left(q\left(\theta\right),\theta\right)-t\left(\theta\right)\geq 0,\ \forall\theta\in\Theta.\end{aligned}$$

Remark 2.1. In static mechanism design problems, there are three substages that we can think about:

- ▷ ex ante: everyone is uninformed (i.e. neither the buyer nor the seller knows θ)
- ▷ interim: only the uninformed in uninformed (i.e. only the buyer knows θ and the seller does not)
- ▷ ex post: everyone is informed (i.e. both the buyer and the seller knows θ)

Since reporting happens at the interim stage, we usually care about IC and IR at the interim stage.

Remark 2.2. Incentive compatibility ensures that that it is optimal for each player to report truthfully conditional on all others reporting truthfully; i.e. it rules out unilateral deviations. This is different from (weak) dominance which means that it is optimal to report truthfully no matter what others do (e.g. second price auction).

Proposition. *A direct mechanism* $(q(\theta), t(\theta))_{\theta\in\Theta}$ *is incentive compatible if and only if*

(i) $t(\theta) = u(q(\theta), \theta) - u(q(\underline{\theta}), \underline{\theta}) + t(\underline{\theta}) - \int_{s=\underline{\theta}}^{\theta} u_\theta(q(s), s)\,\mathrm{d}s$, $\forall\theta \in \Theta$.

(ii) $q(\cdot)$ *is nondecreasing;*

Proof. See Lemma 1 and 2 from class notes. ∎

Remark 2.3. This proposition does not have anything to say about IR. Indeed, the expression for the transfer, $t(\theta)$, does not pin down the level of $t(\cdot)$ (e.g. add $k \in \mathbb{R}$ to both sides).

This also explains why we refer to this equation sometimes as *revenue equivalence.* Observe that any IC direct mechanisms that have the same $q(\cdot)$ and $t(\underline{\theta})$ must have the same $t(\cdot)$; since t is revenue for the seller, such mechanisms all raise the same revenue for the monopolist.

Remark 2.4. Note that we can rewrite the IC constraint as

$$\arg\max_{\hat\theta\in\Theta} u\left(q\left(\hat\theta\right), \theta\right) - t\left(\hat\theta\right).$$

Then, condition (i) corresponds to the first-order condition and (ii) corresponds to the second-order condition of the problem above. We will confirm this when we look at auctions and assume a more specific form of u.

The characterisation above allows us to substitute out $t(\theta)$ in the seller's problem. With $t(\theta)$ as given above, we can rewrite the IR constraint as

$$0 \le u(q(\underline{\theta}), \underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)\,\mathrm{d}s,\ \forall\theta \in \Theta.$$

Since $u_\theta > 0$, above holds if and only if the inequality above holds for $\theta = \underline{\theta}$. Moreover, since the seller's payoff is increasing in $t(\underline{\theta})$, for any given $q(\cdot)$, the seller maximises revenue by setting $t(\underline{\theta})$ as high as possible. In other words, we set

$$\begin{aligned} t(\underline{\theta}) &= u(q(\underline{\theta}), \underline{\theta}) \\ \Rightarrow t(\theta) &= u(q(\theta), \theta) - \int_{\underline{\theta}}^{\theta} u_\theta(q(s), s)\,\mathrm{d}s. \end{aligned}$$

Substituting this into the seller's problem gives us

Seller's problem: version 2

$$\begin{aligned} \max_{q(\cdot)} \quad & \mathbb{E}_{\tilde\theta}\left[u\left(q\left(\tilde\theta\right), \tilde\theta\right) - \int_{s=\underline{\theta}}^{\tilde\theta} u_\theta(q(s), s)\,\mathrm{d}s - C\left(q\left(\tilde\theta\right)\right)\right] \\ \text{s.t.} \quad & q(\cdot) \text{ is nondecreasing} \end{aligned}$$

Remember that t is a transfer from the buyer and so what we've done above is to rewrite the problem in terms of maximising the total surplus, which consists of: (i) utility that the buyer gets from buying q; (ii) inefficiency arising from monopoly (monopolist charging too much and therefore underselling); and (iii) cost of production.

Let $\theta \sim F(\theta)$, by exchanging integral (Sota showed you this trick!), we can rewrite the objective function by:

$$\begin{aligned}
&\int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left[u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - \int_{s=\underline{\theta}}^{\tilde{\theta}} u_\theta\left(q\left(s\right), s\right) \mathrm{d}s - C\left(q\left(\tilde{\theta}\right)\right) \right] f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} \\
&= \int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left[u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - C\left(q\left(\tilde{\theta}\right)\right) \right] f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} - \underbrace{\int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \int_{s=\underline{\theta}}^{\tilde{\theta}}}_{\underline{\theta} \le s \le \tilde{\theta} \le \overline{\theta}} u_\theta\left(q\left(s\right), s\right) \mathrm{d}s f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} \\
&= \int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left[u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - C\left(q\left(\tilde{\theta}\right)\right) \right] f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} - \underbrace{\int_{s=\underline{\theta}}^{\overline{\theta}} \int_{\tilde{\theta}=s}^{\overline{\theta}}}_{\underline{\theta} \le s \le \tilde{\theta} \le \overline{\theta}} f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} u_\theta\left(q\left(s\right), s\right) \mathrm{d}s \\
&= \int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left[u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - C\left(q\left(\tilde{\theta}\right)\right) \right] f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} - \int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left(1 - F\left(\tilde{\theta}\right)\right) u_\theta\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) \mathrm{d}\tilde{\theta} \\
&= \int_{\tilde{\theta}=\underline{\theta}}^{\overline{\theta}} \left(\underbrace{u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - \frac{1 - F\left(\tilde{\theta}\right)}{f\left(\tilde{\theta}\right)} u_\theta\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right)}_{:=\tilde{u}(q(\cdot),\theta)} - C\left(q\left(\tilde{\theta}\right)\right) \right) f\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta}.
\end{aligned}$$

We refer to

$$\tilde{u}\left(q\left(\theta\right), \theta\right) = u\left(q\left(\theta\right), \theta\right) - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} u_\theta\left(q\left(\theta\right), \theta\right)$$

as the *virtual utility*, where we interpret the second term as the buyer's information rent from having private information. It is the source of welfare loss relative to the first best.

We can now rewrite the seller's still as

Seller's problem: version 3

$$\begin{aligned}
\max_{q(\cdot)} \quad & \mathbb{E}_{\tilde{\theta}} \left[u\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - \frac{1 - F\left(\tilde{\theta}\right)}{f\left(\tilde{\theta}\right)} u_\theta\left(q\left(\tilde{\theta}\right), \tilde{\theta}\right) - C\left(q\left(\tilde{\theta}\right)\right) \right]. \\
\text{s.t.} \quad & q\left(\cdot\right) \text{ is nondecreasing}
\end{aligned}$$

Remark 2.5. Regularity condition(s) just ensures that objective function is (quasi)concave so that we can pointwise maximise $q(\cdot)$. Most of the time, we impose $u\left(q\left(\theta\right), \theta\right) = q\left(\theta\right)\theta$, convex $C\left(\cdot\right)$ and the monotone hazard rate condition on F—i.e. $\left(1 - F\left(\theta\right)\right) / f\left(\theta\right)$ is nondecreasing in θ.

Remark 2.6. We also call this problem nonlinear monopoly pricing problem—nonlinearity in the problem above can arise from two sources: one due to u and another due to c.

Assuming regularity condition, the first-order condition is given by

$$\tilde{u}_q\left(q\left(\theta\right), \theta\right) = u_q\left(q\left(\theta\right), \theta\right) - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} u_{\theta q}\left(q\left(\theta\right), \theta\right) = C_q\left(q\left(\theta\right)\right),$$

where we can interpret the left-hand side as the marginal revenue from selling to type θ and the right-hand side is the marginal cost.

Remark 2.7. Let $u(q(\theta),\theta) = q(\theta)\theta$ and $C(q(\theta)) := 0$ and replace θ with θ. Revenue equivalence is now

$$t(\theta) = u(q(\theta),\theta) - q(\underline{\theta})\underline{\theta} + t(\underline{\theta}) - \int_{s=\underline{\theta}}^{\theta} q(s)\,\mathrm{d}s,$$

and the seller's problem is now

$$\begin{aligned} \max_{q(\cdot)} \quad & \mathbb{E}_\theta\left[\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) q(\theta)\right]. \\ \text{s.t.} \quad & q(\cdot) \text{ is nondecreasing} \end{aligned}$$

Keep these in mind. Note that this is now a model of linear pricing!

3 Mechanism design: monopolistic screening with multiple buyers

Let us now imagine that the monopolist/seller wishes to sell a good to multiple buyers. Specifically, we will look at this problem in the independent private values environment (meaning to be clarified in a moment).

The seller has one unit of quantity to sold. There are N buyers indexed by i and each buyer has a value $\theta_i \sim F_i$ with common atomless, full support, say $\Theta := [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}_+$. Let $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. Payoff for each buyer is $u\left(q, t; \vec{\theta}\right) := q\theta_i - t$, where t is the payment to the seller. If the good being sold is indivisible, then q is to be interpreted as the probability that the buyer gets the good; if, instead, the good is divisible, we can think of q as the share of the good sold to buyer i. We can now talk about *independent* and *private* means.

- ▷ Independent: each buyer's values statistically independent of others; i.e. $\theta_i \perp\!\!\!\perp \theta_j$ for all $i \neq j$.
- ▷ Private: buyer i's utility does not depends on the valuation of others; i.e. $u(q, t; \theta_i, \theta_{-i}) = u(q, t; \theta_i, \theta'_{-i})$ for all θ'_{-i}.

Remark 3.1. Of course, you can imagine θ_i's to be correlated. Another environment you might hear about is the *common value* environment. There are two interpretations: (i) $\theta + \epsilon_i$ where ϵ_i is conditionally independent (of v) among buyers; (ii) *pure* common value; i.e. everyone has value θ.

Our goal is to maximise seller's ex ante revenue across all BNE. Revelation principle applies so that it is without loss to focus on truth telling equilibrium of a direct mechanism. In this context, a direct mechanism is

$$(q, t) := \left(q_i\left(\vec{\theta}\right), t_i\left(\vec{\theta}\right)\right)_{i=1,\ldots,N,\ \vec{\theta}\in\Theta^N},$$

where $q_i : \Theta^N \to [0,1]$ and $t_i : \Theta^N \to \mathbb{R}$ are the probability that buyer i is allocated the good (or the share of good allocated to buyer i) if buyers report $\vec{\theta}$ (q_i must be feasible; i.e. $\sum_{i=1}^N q_i(\theta) \leq 1$), and $t_i\left(\vec{\theta}\right)$ is the payment to buyer i makes to the seller, respectively.

Exercise. Write down the direct mechanism corresponding to: first-price, second-price, first-price-all-pay and second-price-all-pay auctions.

Given a direct mechanism (q, t), type θ_i's expected utility at the interim stage from reporting $\hat{\theta}_i$ is

$$
\begin{aligned}
&\mathbb{E}_{\theta_{-i}}\left[q_i\left(\hat{\theta}_i;\theta_{-i}\right)\theta_i - t_i\left(\hat{\theta}_i;\theta_{-i}\right)\right] \\
&= \left[\underbrace{\int_\Theta \cdots \int_\Theta}_{N-1} q_i\left(\hat{\theta}_i;\theta_{-i}\right)\prod_{j\neq i} f_j\left(\theta_j\right)\mathrm{d}\theta_j\right]\theta_i - \left[\underbrace{\int_\Theta \cdots \int_\Theta}_{N-1} t_i\left(\hat{\theta}_i;\theta_{-i}\right)\prod_{j\neq i} f_j\left(\theta_j\right)\mathrm{d}\theta_j\right] \\
&= \mathbb{E}_{\theta_{-i}}\left[q_i\left(\hat{\theta}_i;\theta_{-i}\right)\right]\theta_i - \mathbb{E}_{\theta_{-i}}\left[t_i\left(\hat{\theta}_i;\theta_{-i}\right)\right] \\
&\equiv \overline{q}_i\left(\hat{\theta}_i\right)\theta_i - \overline{t}_i\left(\hat{\theta}_i\right).
\end{aligned}
$$

Since we are looking for a truthful telling equilibrium, we need incentive compatibility. For the same reason as in monopolistic screening, we need IC to hold at the interim stage; i.e.

$$
\overline{q}_i\left(\theta_i\right)\theta_i - \overline{t}_i\left(\theta_i\right) \geq \overline{q}_i\left(\hat{\theta}_i\right)\theta_i - \overline{t}_i\left(\hat{\theta}_i\right),\ \forall\theta_i,\hat{\theta}_i \in \Theta,\ \forall i.
$$

The IR constraint is that $\overline{q}_i\left(\theta_i\right)\theta_i - \overline{t}_i\left(\theta_i\right) \geq 0$ for all i and $\theta_i \in \Theta$.

The seller's ex ante expected revenue is

$$
\mathbb{E}_{\vec{\theta}}\left[\sum_{i=1}^N t_i\left(\vec{\theta}\right)\right].
$$

Therefore, we can now write down the following:

Seller's problem: version 1*

$$
\begin{aligned}
\max_{(q,t)} \quad & \mathbb{E}_{\vec{\theta}}\left[\sum_{i=1}^N t_i\left(\vec{\theta}\right)\right] \\
\text{s.t. (IC)} \quad & \overline{q}_i\left(\theta_i\right)\theta_i - \overline{t}_i\left(\theta_i\right) \geq \overline{q}_i\left(\hat{\theta}_i\right)\theta_i - \overline{t}_i\left(\hat{\theta}_i\right),\ \forall\theta_i,\hat{\theta}_i \in \Theta,\ \forall i \\
\text{(IR)} \quad & \overline{q}_i\left(\theta_i\right)\theta_i - \overline{t}_i\left(\theta_i\right) \geq 0,\ \forall\theta_i \in \Theta
\end{aligned}
$$

Does this look familiar?

Proposition. *A direct mechanism (q, t) is incentive compatible if and only if*

(i) $\overline{t}_i\left(\theta_i\right) = \overline{q}_i\left(\theta_i\right)\theta_i - \overline{q}_i\left(\underline{\theta}\right)\underline{\theta} + \overline{t}_i\left(\underline{\theta}\right) - \int_{s=\underline{\theta}}^{\theta_i}\overline{q}_i(s)\,\mathrm{d}s,\ \forall\theta_i \in \Theta,\ \forall i.$

(ii) $\overline{q}_i(\cdot)$ *is nondecreasing for all i;*

Proof. See end of this section. ■

Deja vu?

We can use the above proposition to rewrite the seller's problem by replacing t_i. But as before, necessary and sufficient condition for IR to hold is for IC to hold and IR for the lowest type to hold; i.e.

$$
\overline{q}_i\left(\underline{\theta}\right)\underline{\theta} - \overline{t}_i\left(\underline{\theta}\right) \geq 0.
$$

Since the seller's revenue is increasing in $\bar{t}_i$ and changing the level of $\bar{t}_i$ does not affect IC, the seller maximises revenue by making the above constraint bind. So we now have

$$\bar{t}_i(\theta_i) = \bar{q}_i(\theta_i)\,\theta_i - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\,\mathrm{d}s.$$

Now, observe that

$$\begin{aligned}
\mathbb{E}_{\vec{\theta}}\left[\sum_{i=1}^{N} t_i\left(\vec{\theta}\right)\right] &= \sum_{i=1}^{N} \mathbb{E}_{\theta_i}\left[\int_{\Theta}\cdots\int_{\Theta} t_i(\theta_i;\theta_{-i}) \prod_{j\neq i} f_j(\theta_j)\,\mathrm{d}\theta_j\right] \\
&= \sum_{i=1}^{N} \mathbb{E}_{\theta_i}\left[\bar{t}_i(\theta_i)\right].
\end{aligned}$$

Thus, we can rewrite the seller's problem as

Seller's problem: version 2*

$$\begin{aligned}
\max_{q} \quad & \sum_{i=1}^{N} \mathbb{E}_{\theta_i}\left[\bar{q}_i(\theta_i)\,\theta_i - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\,\mathrm{d}s\right] \\
& \text{s.t. } q_i(\cdot) \text{ is nondecreasing } \forall i.
\end{aligned}$$

Let us now use the exchange of integral trick:

$$\begin{aligned}
&\mathbb{E}_{\theta_i}\left[\bar{q}_i(\theta_i)\,\theta_i - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\,\mathrm{d}s\right] \\
&= \int_{\theta_i=\underline{\theta}}^{\bar{\theta}} \bar{q}_i(\theta_i)\,\theta_i f_i(\theta_i)\,\mathrm{d}\theta_i - \underbrace{\int_{\theta_i=\underline{\theta}}^{\bar{\theta}}\int_{s=\underline{\theta}}^{\theta_i}}_{\underline{\theta}\leq s\leq\theta_i\leq\bar{\theta}} \bar{q}_i(s)\,\mathrm{d}s f_i(\theta_i)\,\mathrm{d}\theta_i \\
&= \int_{\theta_i=\underline{\theta}}^{\bar{\theta}} \bar{q}_i(\theta_i)\,\theta_i f_i(\theta_i)\,\mathrm{d}\theta_i - \underbrace{\int_{s=\underline{\theta}}^{\bar{\theta}}\int_{s=\theta_i}^{\bar{\theta}}}_{\underline{\theta}\leq s\leq\theta_i\leq\bar{\theta}} f_i(\theta_i)\,\mathrm{d}\theta_i \bar{q}_i(s)\,\mathrm{d}s \\
&= \int_{\theta_i=\underline{\theta}}^{\bar{\theta}} \bar{q}_i(\theta_i)\,\theta_i f_i(\theta_i)\,\mathrm{d}\theta_i - \int_{\theta_i=\underline{\theta}}^{\bar{\theta}} (1-F_i(\theta_i))\,\bar{q}_i(\theta_i)\,\mathrm{d}\theta_i \\
&= \int_{\theta_i=\underline{\theta}}^{\bar{\theta}} \left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)\bar{q}_i(\theta_i)\,f_i(\theta_i)\,\mathrm{d}\theta_i.
\end{aligned}$$

The term inside the bracket is called the *virtual value*! Just as in the monopoly screening case, the term $(1-F(\theta_i))/f(\theta_i)$ is the buyer's information rent (and cause of a welfare loss).

So we can rewrite the seller's problem now as

Seller's problem: version 3*

$$\begin{aligned}
\max_{q} \quad & \sum_{i=1}^{N} \mathbb{E}_{\theta_i}\left[\left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)\bar{q}_i(\theta_i)\right] \\
& \text{s.t. } \bar{q}_i(\cdot) \text{ is nondecreasing } \forall i.
\end{aligned}$$

So, by now, you should be convinced that monopolistic screening problems and auctions (at least in the IPV environment) are very similar!

3.1 Optimal direct mechanism

By substituting the expression for $\overline{q}_i(\theta_i) = \mathbb{E}_{\theta_{-i}}[q_i(\theta_i;\theta_{-i})]$ and moving the summation inside the expectation, the objective function becomes

$$\mathbb{E}_{\vec{\theta}}\left[\sum_{i=1}^{N}\left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right) q_i\left(\vec{\theta}\right)\right].$$

As before, let's impose a regularity condition—in this case, we just need to impose that F satisfies MHRC. This ensures that objective is concave, and we can once again, pointwise maximise; i.e. fix $\vec{\theta}$ and maximise with respect to $q_i\left(\vec{\theta}\right)$. Since the objective is linear in $q_i\left(\vec{\theta}\right)$, this just means that we should set $q_i\left(\vec{\theta}\right) = 1$ for i with the highest virtual utility. Not quite. Recall our feasibility constraint that

$$\sum_{i=1}^{N} q_i\left(\vec{\theta}\right) \leq 1$$

In other words, it's possible that with the optimal mechanism, the inequality above holds strictly. Why might this be the case? It could be that the virtual utility for everyone is negative, in which case the objective is maximised by setting $q_i\left(\vec{\theta}\right)$ to be zero for all i. Thus, the optimal allocation mechanism q is

$$q_i\left(\vec{\theta}\right) = \begin{cases} 1 & \text{if } \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} > \max\left\{0, \theta_j - \frac{1-F_j(\theta_j)}{f_j(\theta_j)}\right\}, \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

Since we assumed MHRC, we $\overline{q}_i(\theta_i)$ is nondecreasing.

Remark 3.2. Recall that we interpreted the virtual utility as marginal revenue in the monopoly screening problem. Recall the optimal $q(\theta)$ for each θ was characterised by the first-order condition that equated marginal revenue (derivative of the virtual utility) with the marginal cost. In the auction setting, we set $C(\cdot) := 0$ and $u(q(\theta), \theta) := q(\theta)\theta$ so that the problem is linear—so the first-order condition doesn't quite work.[1] But the inequality

$$\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} > \max\left\{0, \theta_j - \frac{1-F_j(\theta_j)}{f_j(\theta_j)}\right\}$$

can still be thought of as comparing the marginal revenue (the left-hand side) with the marginal cost. Here, marginal cost refers to the marginal *opportunity* cost. By selling the good to buyer i, the monopolist forgoes the opportunity to sell to other buyers j, as well as forgoing the opportunity to keep the good.

If you take Ben (Brook)'s class in the second year, you will see this in a much more rigorous way!

[1] The problem is now something like $\max_{q_1,q_2\in[0,1],q_1+q_2\leq 1} 3q_1 + 2q_2$.

Now, think back to our characterisation of IC direct mechanism. We now need to construct $t_i\left(\vec{\theta}\right)$ such that

$$\bar{t}_i\left(\theta_i\right) = \bar{q}_i\left(\theta_i\right)\theta_i - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i\left(s\right) \mathrm{d}s.$$

But note that we have quite a bit of flexibility here. Why? Well, the direct mechanism involves specifying ex post payment $t_i\left(\vec{\theta}\right)$ while the restriction relates to interim payment.

Exercise. Write down the $t_i\left(\vec{\theta}\right)$ corresponding to the optimal q_i we found above to: first-price, second-price, first-price-all-pay and second-price-all-pay auctions.

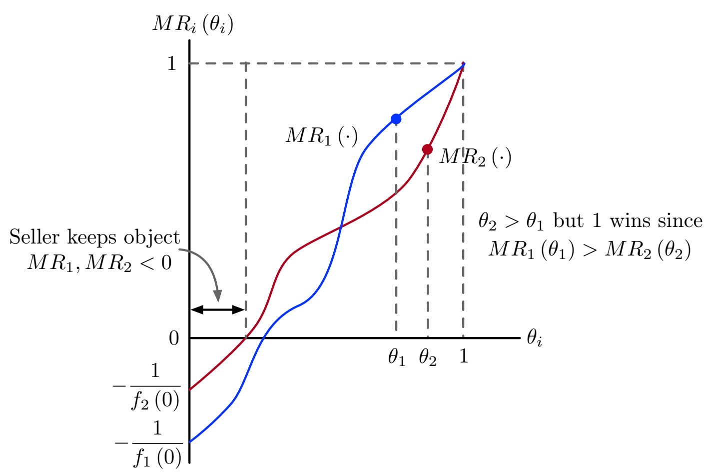
3.2 Inefficiency

Let us think about allocative (in)efficiency in the context of a selling mechanism.

▷ We implicitly assumed that the seller assigns zero value to the object, whereas buyer assign some positive value to the object. Hence, the seller should always sell the object to one of the buyers.

▷ In fact, the object should be assigned to the buyer with the highest valuation.

We already saw that the seller keeps the object if $MR_i\left(\theta_i\right)$ for each buyer is negative. To see how the second might arise, consider the following figure.



Note that: (i) when $\theta_i = 0$, $MR_i\left(0\right) = -1/f_i\left(0\right)$ since $F_i\left(0\right) = 0$; (ii) when $\theta_i = 1$, then $MR_i\left(1\right) = 1$; and (iii) MR_i is strictly increasing between these two points. The figure shows how there can be certain realisations of θ_i's such that the seller will keep the object, as well as an example of realisation in which the buyer with the highest value does not get the object. We can view the fact that sometimes the seller keeps the object as the seller exercising his monopoly power over the good.

Now suppose that buyers are symmetric so that

$$f \equiv f_1 \equiv f_2,\ F \equiv F_1 \equiv F_2.$$

In this case, the two curves in the figure are overlapping so that the second type of inefficiency cannot arise. The optimal selling mechanism is now that the buyer with the highest reported value wins and pays the seller θ_i^*, the largest value that he could have reported, given the other buyer's reported values, without winning the object. However, the first source of inefficiency still remains so that if there are no buyers with positive reported value of MR_i, then the seller keeps the object. This means that, in fact, the buyer with the highest value, say buyer i with value θ_i, wins as so long as $\theta_i > \rho^* \in [0,1]$, where ρ^* is implicitly defined as

$$\rho^* - \frac{1 - F(\rho^*)}{f(\rho^*)} = 0.$$

Buyer i does not get the object unless the reported value is strictly highest and strictly above ρ^*. So, the largest his report can be without receiving the object (i.e. the amount he has to pay) is the largest of the other buyers' values or ρ^*, whichever is larger. Thus, it is as if we've added another buyer whose value is ρ^* to the set up.

Remark 3.3. We can mimic this optimal direct selling mechanism by running a second-price auction with reserve price ρ^*.

3.3 "Missing" proofs

Note that the two propositions below are just corollaries of the IC proposition we proved in class. The integral condition of IC we derived in class (2), in tandem with the definition that $t(\theta) = q(\theta)\theta - U(\theta)$ implies (i) in the proposition. Similarly, the second proposition here is an immediate corollary to the IC proposition in class given the definition of $t(\theta) = q(\theta)\theta - U(\theta)$ and the IR requirement with an outside option 0.

Proposition. *A direct mechanism (q,t) is incentive compatible if and only if*

(i) $\bar{t}_i(\theta_i) = \bar{q}_i(\theta_i)\theta_i - \bar{q}_i(\underline{\theta})\underline{\theta} + \bar{t}_i(\underline{\theta}) - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\,\mathrm{d}s$, $\forall \theta_i \in \Theta$, $\forall i$.

(ii) $\bar{q}_i(\cdot)$ *is nondecreasing for all i;*

Proof. Necessity: IC $\Rightarrow$ (i) and (ii)) That (q,t) is incentive compatible means that $u_i(\hat{\theta}_i, \theta_i) := \bar{q}_i(\hat{\theta}_i)\theta_i - \bar{t}_i(\hat{\theta}_i)$ *is maximised in $\hat{\theta}_i$ when $\hat{\theta}_i = \theta_i$ for all i and θ_i* ; i.e.

$$\begin{aligned} 0 = \left.\frac{\partial u_i(\hat{\theta}_i, \theta_i)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}_i = \theta_i} &= \left.\frac{\partial\left[\bar{q}_i(\hat{\theta}_i)\theta_i - \bar{t}_i(\hat{\theta}_i)\right]}{\partial \hat{\theta}_i}\right|_{\hat{\theta}_i = \theta_i} \\ &= \bar{q}_i'(\hat{\theta}_i)\theta_i - \bar{t}_i'(\hat{\theta}_i)\big|_{\hat{\theta}_i = \theta_i} \\ &= \bar{q}_i'(\theta_i)\theta_i - \bar{t}_i'(\theta_i) \\ \Leftrightarrow \bar{t}_i'(\theta_i) &= \bar{q}_i'(\theta_i)\theta_i, \ \forall \theta_i \in [\underline{\theta}, \bar{\theta}]. \end{aligned} \tag{3.1}$$

Since the equation holds for all θ_i, we can integrate both sides with respect to θ_i over the interval

$[\underline{\theta}, \theta_i]$:

$$
\begin{aligned}
\bar{t}_i(\theta_i) - \bar{t}_i(\underline{\theta}) &= \int_0^{\theta_i} \bar{q}_i'(x)\, x dx \\
\Leftrightarrow \bar{t}_i(\theta_i) &= \bar{t}_i(\underline{\theta}) + [\bar{q}_i(x)\, x]_{\underline{\theta}}^{\theta_i} - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\, \mathrm{d}s \\
&= \bar{q}_i(\theta_i)\, \theta_i - \bar{q}_i(\underline{\theta})\, \underline{\theta} + \bar{t}_i(\underline{\theta}) - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i(s)\, \mathrm{d}s,
\end{aligned}
$$

where we used integration by parts in the second line. This gives us condition (ii).

To see (i), consider

$$
\begin{aligned}
\left.\frac{\partial^2 u_i(\hat{\theta}_i, \theta_i)}{\partial\hat{\theta}_i \partial\hat{\theta}_i}\right|_{\hat{\theta}_i=\theta_i} &= \bar{q}_i''(\hat{\theta}_i)\, \theta_i - \bar{t}_i''(\hat{\theta}_i)|_{\hat{\theta}_i=\theta_i} \\
&= \bar{q}_i''(\theta_i)\, \theta_i - \bar{t}_i''(\theta_i) \le 0,\ \forall \theta_i \in [0,1]\,,
\end{aligned} \tag{3.2}
$$

where the inequality follows from the second-order condition. Now since (3.1) holds for all θ_i, we can differentiate both sides,

$$
\begin{aligned}
\bar{t}_i''(\theta_i) &= \bar{q}_i''(\theta_i)\, \theta_i + \bar{q}_i'(\theta_i)\,,\ \forall \theta_i \in \Theta \\
\Leftrightarrow -\bar{q}_i'(\theta_i) &= \bar{q}_i''(\theta_i)\, \theta_i - \bar{t}_i''(\theta_i)\,,\ \forall \theta_i \in \Theta.
\end{aligned}
$$

Substituting this into (3.2) gives

$$
\bar{q}_i'(\theta_i) \ge 0,\ \forall \theta_i \in [0,1]\,.
$$

So it must be that $\bar{q}_i'$ is nonnegative; i.e. $\bar{q}_i(\theta_i)$ is nondecreasing in θ_i.

(Sufficiency: (i) and (ii) imply IC) Consider

$$
\frac{\partial u_i(\hat{\theta}_i, \theta_i)}{\partial \hat{\theta}_i} = \bar{q}_i'(\hat{\theta}_i)\, \theta_i - \bar{t}_i'(\hat{\theta}_i)\,. \tag{3.3}
$$

Since (ii) holds for all θ_i, we can differentiate it with respect to θ_i to obtain (and write θ_i as $\hat{\theta}_i$)

$$
\begin{aligned}
\bar{t}_i'(\hat{\theta}_i) &= \bar{q}_i'(\hat{\theta}_i)\, \hat{\theta}_i + \bar{q}_i(\hat{\theta}_i) - \bar{q}_i(\hat{\theta}_i) \\
\Leftrightarrow \bar{t}_i'(\hat{\theta}_i) &= \bar{q}_i'(\hat{\theta}_i)\, \hat{\theta}_i.
\end{aligned}
$$

Substituting this into (3.3),

$$
\frac{\partial u_i(\hat{\theta}_i, \theta_i)}{\partial \hat{\theta}_i} = \bar{q}_i'(\hat{\theta}_i)\, \theta_i - \bar{q}_i'(\hat{\theta}_i)\, \hat{\theta}_i = \bar{q}_i'(\hat{\theta}_i)\, (\theta_i - \hat{\theta}_i)\,.
$$

From (i), we know that $\bar{q}_i'(\hat{\theta}_i) \ge 0$ for all $\hat{\theta}_i$.

- ▷ Suppose first that $\bar{q}_i'(\hat{\theta}_i) > 0$ for all $\hat{\theta}_i \in \Theta$, then the slope is positive while if $\hat{\theta}_i > \theta_i$, and negative if $\hat{\theta}_i < \theta_i$. Hence, there exists a unique global maximum at $\hat{\theta}_i = \theta_i$, which implies that truth telling is a dominant strategy for player i.

- ▷ If, instead, $q_i'(\hat{\theta}_i) = 0$ for all $\hat{\theta}_i \in \Theta$, then the player is indifferent between reporting $\hat{\theta}_i = \theta_i$

and any other value (expected utility is constant with respect to $\hat{\theta}_i$ in this case). Hence, truth telling is a weakly dominant strategy.

- ▷ Finally, suppose that $q_i\left(\hat{\theta}_i\right) > 0$ for some values of $\hat{\theta}_i \in \Theta$ but $q_i\left(\tilde{r}_i\right) = 0$ some other values of $\tilde{r}_i \in [0,1]$. In this case, truth telling is a weakly dominant strategy.

Hence, we realise that $\hat{\theta}_i = \theta_i$ is a weakly dominant strategy for player i given (i) and (ii) so that we have our desired result. ■

Remark 3.4. Condition (i) ensures that the second-order condition of the maximisation of $u_i\left(\hat{\theta}_i, \theta_i\right)$ with respect to $\hat{\theta}_i$ (evaluated at $\hat{\theta}_i = \theta_i$) is satisfied, while condition (ii) ensures that the first-order condition is satisfied. That $\bar{q}_i\left(\theta_i\right)$ is nondecreasing means that there is incentive for bidder i to report a higher $\hat{\theta}_i$ than his true valuation since he can increase his probability of winning by reporting $\hat{\theta}_i > \theta_i$. We can then interpret the integral term in condition (ii) as putting a limit on the bidder's incentive to report higher $\hat{\theta}_i$.

Proposition. *An IC DM is individual rational if and only if*

$$\bar{t}_i\left(0\right) \leq 0,\ \forall i.$$

Proof. Recall that an IC DM satisfies condition (ii) of the previous proposition; i.e.

$$\bar{t}_i\left(\theta_i\right) = \bar{q}_i\left(\theta_i\right)\theta_i - \bar{q}_i\left(\underline{\theta}\right)\underline{\theta} + \bar{t}_i\left(\underline{\theta}\right) - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i\left(s\right) \mathrm{d}s,\ \forall \theta_i \in \Theta.$$

Substituting for $\bar{t}_i\left(\theta_i\right)$ into the individual rationality constraint yields that, for all $\theta_i \in \Theta$,

$$\begin{aligned} & \bar{q}_i\left(\theta_i\right)\theta_i \geq \bar{q}_i\left(\theta_i\right)\theta_i - \bar{q}_i\left(\underline{\theta}\right)\underline{\theta} + \bar{t}_i\left(\underline{\theta}\right) - \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i\left(s\right) \mathrm{d}s \\ \Leftrightarrow & \int_{s=\underline{\theta}}^{\theta_i} \bar{q}_i\left(s\right) \mathrm{d}s \geq \bar{t}_i\left(\underline{\theta}\right). \end{aligned}$$

The left-hand side is smallest when $\theta_i = \underline{\theta}$, in which case, we require

$$\bar{t}_i\left(\underline{\theta}\right) \leq 0.$$

Reversing the steps gives us the other direction. ■