1 PS6 Q1

Consider a natural (but regulated) monopolist with a constant marginal cost of production equal to c which is private information at the time of contracting and uniformly distributed on [1, 2]. The monopolistâs profit for producing output q at marginal cost c with transfer t from the regulator is

$$u(q, c, t) \equiv t - cq.$$

Note that the firm does not get any revenue from selling q; all payments are through t. The firmâs participation constraint is $\underline{U} = 0$. The regulatorâs objective function is to maximize

$$\int_{\underline{c}}^{\overline{c}} \left[vq(c) - \frac{1}{2}q(c)^2 - t(c) \right] f(c)dc$$

over the class of all DRMâs which satisfy IC and IR constraints. Assume $v \geq 3$. Among other things, this implies the first-best, full-information output would be

$$q^{fb}(c) = v - c > 0 \tag{1}$$

Problem 1.1. What are the IC and IR constraints for this problem? State and prove a characterization theorem for the set of all IC mechanisms $\{q(\cdot), t(\cdot)\}$ in terms of a monotonicity condition and an integral condition.

Solution. The IR constraint is

$$t(c) - cq(c) \geq \underline{U} = 0, \forall c \in [1, 2],$$

so that all types of firms participate.

The IC characterization theorem is: A mechanism is IC if and only if

1. $q(\cdot)$ is non-increasing
2. $\Pi(C) = \Pi(1) - \int_1^c q(s)ds, = \Pi(2) + \int_c^2 q(s)ds,$

where $\Pi(c) = \Pi(c|c)$, for $\Pi(\hat{c}|c) = t(\hat{c}) - cq(\hat{c})$.

Proof (Necessity). The necessary FOC for truth-telling is

$$0 = \Pi_1(c|c) = t'(c) - cq'(c).$$

Totally differentiating this expression yields

$$0 = \Pi_{11}(c|c) + \Pi_{12}(c|c) = t''(c) - cq''(c) - q'(c).$$

By the necessary SOC for truth-telling (i.e. $Pi()$ is concave), we have

$$\Pi_{11}(c|c) \leq 0,$$

which implies that necessarily

$$\begin{aligned} \Pi_{12}(c|c) = -q'(c) &\geq 0 \\ \leftrightarrow q'(c) &\leq 0 \end{aligned}$$

Thus, $q(\cdot)$ non-increasing is necessary for IC.

Now consider the firm's envelope theorem:

$$\Pi_2(c|c) = -q(c).$$

Then

$$\Pi'(c) = \Pi_1(c|c) + \Pi_2(c|c) = -q(c) \leq 0,$$

where the second inequality follows by the FOC for truth-telling. Note that $\Pi'(c) \leq 0$ implies that the highest cost type is worst off, so we need only bind his IR to ensure that the IR constraints for all other types are satisfied. Integrating both sides yields of the above equation (second equality) yields

$$\Pi(c) = \Pi(2) + \int_c^2 q(s)ds.$$

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Proof (Sufficiency). It suffices to show that

$$\Pi(c) \geq \Pi(\hat{c}|c) = t(c') - cq(c'), \forall c, c' \in [1, 2].$$

WLOG, assume $c \leq c'$. Then

$$\begin{aligned} q(c) &\geq q(c') \\ \leftrightarrow \int_c^{c'} q(s)ds &\geq \int_c^{c'} q(c')ds \\ \leftrightarrow \Pi(c) - \Pi(c') &\geq (c' - c)q(c')ds, \text{ using the integral condition} \\ \leftrightarrow \Pi(c) &\geq c'q(c') - cq(c') + \Pi(c')ds \\ \leftrightarrow \Pi(c) &\geq t(c') - cq(c'), \text{ since } \Pi(c') = t(c') - c'q(c') \end{aligned}$$

as desired.

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Problem 1.2. Solve for the regulatorâs optimal revelation mechanism, $\{q(\cdot), t(\cdot)\}$, and compare q(c) to $q^{fb}(c)$. Explain the inequality that you find.

Solution. As discussed in part a, the IR for the highest cost type must bind

$$\Pi(2) = 0,$$

and so

$$\Pi(c) = \int_c^2 q(s)ds.$$

Thus,

$$\begin{aligned}
\mathbb{E}[\Pi(c)] &= \int_1^2 \left(\int_c^2 q(s)ds\right) f(c)dc \\
&= \int_1^2 \left(\int_1^s f(c)dc\right) q(s)ds \\
&= \int_1^2 q(s)F(s)ds \\
&= \int_1^2 q(s)\frac{F(s)}{f(s)} f(s)ds \\
&= \mathbb{E}\left[q(c)\frac{F(c)}{f(c)}\right] \\
&= \mathbb{E}\left[q(c)(c-1)\right].
\end{aligned}$$

Plugging $t(c') = \Pi(c') + c'q(c')$ into the regulator's objective function yields

$$\max_{q(\cdot) \text{ non-increasing}} \mathbb{E}[vq(c) - .5q(c)^2 - \Pi(c) - cq(c)]$$

Substituting our above expression $\mathbb{E}[\Pi(c)]$ yields the following objective:

$$\mathbb{E}[vq(c) - .5q(c)^2 - q(c)(2c-1)].$$

Now let

$$\Lambda(q, c) = vq - .5q^2 - q(2c - 1).$$

Since $Lambda$ is clearly jointly concave in (q, c), the FOC is neccessary and sufficient. Taking the FOC of Λ wrt. q we get

$$v - q - (2c - 1) = 0,$$

and so $q(c) = v - 2c + 1$ is optimal. Note that

$$q(c) = v - 2c + 1 = v - c - (c - 1) \leq v - c = q^{fb}(c),$$

since $c \geq 1$.

Now we need to compute the optimal $t(c)$.

$$\begin{aligned}\Pi(c) &= \int_c^2 q(s)ds \\ &= \int_c^2 (v - 2s + 1)ds \\ &= (2-c)(v+1) - (4 - c^2) \\ &= (2-c)(v+1-2-c) \\ &= (2-c)(v-1-c).\end{aligned}$$

Thus,

$$\begin{aligned}q(c) &= v - 2c + 1 \\ t(c) &= \Pi(c) + cq(c) = 2v - 2 - c^2.\end{aligned}$$

Above we find that $q(c) \leq q^{fb}(c)$ because the regulator effectively sets price higher than marginal cost (so that $q(c)$ is less than its efficient value) to reduce the monopoly's information rent and transfer that surplus to customers.

Problem 1.3. Set v = 3. Compute an indirect tariff, $T(q)$, with the property that the regulator can offer the firm this schedule, $T(q)$, letting the firm freely choose its output q in exchange for $T(q)$, and the resulting choice and payments are equivalent to those from (b).

Solution. We seek $T(q)$ such that

$$\begin{aligned}q(c) &= v - 2c + 1 = 4 - 2c \\ T(q(c)) &= t(c) = 2v - 2 - c^2 = 4 - c^2.\end{aligned}$$

Note that the first equality implies

$$c = \frac{4 - q(c)}{2}.$$

Plugging this result into the second equation yields

$$T(q(c)) = t(c) = 4 - \left(\frac{4 - q(c)}{2}\right)^2 = 2q(c) - \frac{1}{4}q(c)^2.$$

So we conjecture that $T(q) = 2q - \frac{1}{4}q^2$. To verify that the firm with cost c will indeed pick $q(c)$ under this tariff, we consider the firm's problem

$$\max_q T(q) - cq$$

which has FOC wrt q:

$$T'(q) = c.$$

Then note that

$$T'(q(c)) = 2 - \frac{1}{2}q(c) = 2 - \frac{1}{2}(4 - 2c) = c,$$

as desired.

Thus, under the tariff $T(q) = 2q - \frac{1}{4}q^2$, the firm will pick the same quantity as in part b and, bu construction, the transfer payment to the firm is the same.

2 PS6 Q3 (From B. Szentes)

Consider a trade of a divisible good between a seller (principal) and a buyer (agent). The buyer's payoff is $vq - t$ where v is her valuation, q is the quantity traded, and t the payment to the seller. The seller's payoff is $t - \frac{q^2}{2}$ where $\frac{q^2}{2}$ is the cost of producing q units of the good.
Suppose the buyer's valuation is

$$v = \lambda\theta + (1-\lambda)\varepsilon,$$

where θ is the buyer's private information and ε is a publicly observable and contractible shock. Assume that both θ and ε are independently and uniformly distributed on $[0,1]$. The seller can offer a contract to the buyer prior to the realization of ε, but θ is privately known by the buyer at the time of the contract. The buyer's outside option is zero.

Problem 2.1. What is the set of contracts to which it is without loss of generality to restrict attention?

Solution. We restrict our attention to contracts that are optimal deterministic direct mechanisms. That is, $\{q(\cdot), t(\cdot)\}$ that solve

$$\begin{aligned}
&\max_{\{q(\cdot),t(\cdot)\}} E_\theta[t(\theta) - \frac{q(\theta)^2}{2}],\\
&s.t. \theta \in \arg\max_{\hat\theta\in\Theta}(\lambda\theta + (1-\lambda)\varepsilon)q(\hat\theta) - t(\hat\theta), \forall\theta\in\Theta, (IC)\\
&(\lambda\theta + (1-\lambda)\varepsilon)q(\theta) - t(\theta) \geq 0, \forall\theta\in\Theta, (IR).
\end{aligned}$$

Problem 2.2. Consider an incentive compatible contract and let $U(\theta)$ denote the equilibrium payoff of the buyer with type θ. What is $U(\theta) - U(0)$?

Solution. We have

$$U(\theta) = u(q(\theta),\theta) - t(\theta) = (\lambda\theta + (1-\lambda)\varepsilon)q(\theta) - t(\theta).$$

By Lemma 1 in the lecture notes, in an incentive compatible contract we have

$$U(\theta) - U(0) = \int_0^\theta u_\theta(q(s),s)ds = \lambda\int_0^\theta q(s)ds$$

Problem 2.3. Use your result in part (b) to express the buyer's ex-ante expected payoff, $E_\theta[U(\theta)]$.

Solution. Using (b), the buyer's ex-ante expected payoff is now

$$E_\theta[U(\theta)] = U(0) + \lambda E_\theta[\int_0^\theta q(s)ds].$$

Problem 2.4. Express the seller's expected payoff as the difference between social surplus and the buyer's payoff.

Solution. By the revenue principle we have

$$t(\theta) = (\lambda\theta + (1-\lambda)\varepsilon)q(\theta) - U(0) - \lambda E_\theta[\int_0^\theta q(s)ds].$$

Therefore the seller's expected payoff can be simplified as

$$E_\theta[t(\theta) - \frac{q(\theta)^2}{2}] = E_\theta[(\lambda\theta + (1-\lambda)\varepsilon)q(\theta) - \frac{q(\theta)^2}{2} - \lambda E_\theta[\int_0^\theta q(s)ds]] - U(0).$$

Problem 2.5. Use your result in part (d) to derive the optimal quantity produced as a function of θ and ε.

Solution. Setting $U(0)$ = 0 and simplifying the inside ntegral by integration by parts as we did in class, we get that the seller's problem is now

$$\max_{\{q(\cdot)\}} E_\theta[(\lambda\theta + (1-\lambda)\varepsilon)q(\theta,\varepsilon) - \frac{q(\theta)^2}{2} - \lambda(1-\theta)q(\theta,\varepsilon)],$$
$$\text{s.t. } q(\cdot) \text{ nondecreasing in } \theta.$$

So now we simply need to pointwise maximize for $q(\cdot)$. The FOC is

$$\begin{aligned} &(\lambda\theta + (1-\lambda)\varepsilon) - q(\theta,\varepsilon) - \lambda(1-\theta) = 0 \\ \implies &q(\theta,\varepsilon) = \lambda(2\theta - 1) + (1-\lambda)\varepsilon \end{aligned}$$

which is indeed nondecreasing since $q_\theta(\theta,\varepsilon) = 2\lambda > 0$ and $q_\varepsilon(\theta,\varepsilon) = 1-\lambda > 0$.

Problem 2.6. Suppose that ε is not observable by the seller, but instead remains private information to the buyer. Otherwise, the timing is as before: the contract is offered after the buyer has observed θ but before the buyer observes ε. Using a result from class, argue that the same allocation in (e) will be implemented by the seller.

Solution. We showed in class that as long as the program is regular, a random direct mechanism design cannot outperform a deterministic one. In this case, we convert the previous problem to a random direct mechanism by having the seller now offer $\{\phi(\cdot|\theta,\varepsilon)\}$ where the probability distributions depend on the random variables θ and ε and only places mass on the deterministic direct mechanism from (e). Thus, our solution to the problem will be the same. This is because although now ε is private information, the contract is still incentive compatible and the program is regular.

3 Q5 (JR 9.8)

In a first-price, all-pay auction, the bidders simultaneously submit sealed bids. The highest bid wins the object and every bidder pays the seller the amount of his bid. Consider the independent private values model with symmetric bidders whose values θ_i are each distributed according to the distribution function F, with density f.

Problem 3.1. Find the unique symmetric equilibrium bidding function.

Solution. Conjecture that the symmetric equilibrium bid function $b(\theta)$ is strictly increasing. Then the highest type bidder will win. Moreover, the lowest type $\underline{\theta}$ will definitely lose and so will bid 0, and have payoff of 0. Then the Revenue Equivalence Theorem (light) (RET) applies. We know from the proof of the RET that a bidder of type θ_i receives expected utility

$$U_i(\theta_i) = \int_{\underline{\theta}}^{\theta_i} F(s)^{n-1} ds.$$

We also know that

$$U_i(\theta_i) = F(\theta_i)^{n-1}\theta_i - b(\theta_i),$$

since $F(\theta_i)^{n-1}$ is the probability type θ_i wins (since $b()$ assumed to be strictly increasing in type), θ_i is the value he receives if he wins, and he always pays his bid $b(\theta_i)$. Thus the equilibrium symmetric bid function is

$$b^{FAPA}(\theta_i) = F(\theta_i)^{n-1}\theta_i - \int_{\underline{\theta}}^{\theta_i} F(s)^{n-1} ds = \int_{\underline{\theta}}^{\theta_i} s dF(s)^{n-1},$$

where the second equality follows by integration by parts. As is made clear by the second equality, $b^{FAPA}(\theta_i)$ is indeed strictly increasing, as conjectured.

Problem 3.2. Do bidders bid higher or lower than in a first-price auction.

Solution. We know that in a first-price auction, the symmetric equilibrium bid function ios

$$\begin{aligned} b^{FPA}(\theta_i) &= \theta_i - \int_{\underline{\theta}}^{\theta_i} \frac{F^{n-1}(s)}{F^{n-1}(\theta_i)} ds \\ &= \frac{1}{F^{n-1}(\theta_i)} b^{FPA}(\theta_i), \text{ by part a} \\ &\geq b^{FAPA}(\theta_i), \text{ since } F^{n-1}(\theta_i) \leq 1 \end{aligned}$$

where the last inequality is strict with probability one. Thus, bidders will bid more in a first-price auction because they know that they won't have to pay their bids if they lose.

Problem 3.3. Find an expression for the sellerâs expected revenue.

Solution.

$$\begin{aligned} ER^{FAPA} &= N\mathbb{E}[b^{FAPA}(\theta_i)], \text{ since all bidders pay their bids} \\ &= N\int_{\underline{\theta}}^{\overline{\theta}} b^{FAPA}(\theta_i)dF(\theta_i) \\ &= N\int_{\underline{\theta}}^{\overline{\theta}} \left(F(\theta_i)^{n-1}\theta_i - \int_{\underline{\theta}}^{\theta_i} F(s)^{n-1}ds\right)dF(\theta_i) \end{aligned}$$

Problem 3.4. Both with and without using the revenue equivalence theorem, show that the sellerâs expected revenue is the same as in a first-price auction.

Solution. As discussed in part a, for RET (light) to hold we need:

1. In both auctions, the highest type of bidder wins. We indeed satisfy this conditions since $b^{FAPA}()$ and $b^{FPA}()$ are both strictly increasing.

2. The lowest type gets payoff zero. We also satisfy this condition. In both auctions, since the symmetric equilibrium bid functions are strictly increasing, the lowest type definitely loses and gets value zero. In FAPA, the lowest type of bidder bids 0 so pays 0, while in FPA he pays 0 because he loses. In both cases, the lowest type gets a payoff of 0.

3. In the RET (light) from class we also need that all bidders have the same type distribution, which is indeed the case in both auctions here.

So by RET, we have $ER^{APA} = ER^{FPA}$. Now we check revenue equivalance directly:

$$\begin{aligned} ER^{FPA} &= N\mathbb{E}[F^{n-1}(\theta_i)b^{FPA}(\theta_i) + (1 - F^{n-1}(\theta_i)) \cdot 0], \text{ where the expectation is the expected payment per bidder} \\ &= N\int_{\underline{\theta}}^{\overline{\theta}} F^{n-1}(\theta_i)b^{FPA}(\theta_i)dF(\theta_i) \\ &= Nint_{\underline{\theta}}^{\overline{\theta}} b^{FAPA}(\theta_i)dF(\theta_i), \text{ by part b} \\ &= ER^{FAPA}, \end{aligned}$$

as desired.