Price Theory, III | **Professor Lars Stole**
Economics 33003 | **Spring 2019**

Assignment 2
(Due Friday, April 19, prior to the start of the Review session)

Problem 1 (JR, Exercise 8.9 - major variation) Analyze the insurance signaling game in JR (ch8) when the coverage B is restricted to being equal to L. Assume that the low risk consumer strictly prefers full insurance at the high-risk competitive price to no insurance. For simplicity, assume that there is a single firm.

(a). Characterize the unique pure-strategy sequential equilibrium price(s) when attention is restricted to those equilibria in which the insurance company earns zero profits.

(b). Show that there are pooling sequential equilibria in which the insurance company earns positive profits.

(c). Characterize a mixed-strategy sequential equilibrium strategy profile in which (1) the insurance company earns zero profits and (2) consumers fully separate. [Hint: assume that there are two offers in equilibrium, and the firm randomizes between "Accept" and "Reject" for one of them. If the firm "rejects", the game is over and the consumer is uninsured.]

Problem 2 Consider a simple monopoly setting in which a firm sells to a a consumer whose unit valuation is distributed uniformly on $[0, 1]$. The firm's unit costs are zero.

(a). What is the optimal price for the monopolist? What is the expected consumer surplus?

For the remainder of this problem, suppose that before the monopolist sets the price, the consumer can credibly reveal any closed subset, $M \subseteq [0, 1]$ which contains their valuation. E.g., a consumer of type $\theta = 0.6$ can reveal the *simple* message $\{0.6\}$ to the monopolist, or she could reveal a coarser signal such as "my value is in $[0.5, 0.7] \cup [.8, .9]$", but she cannot lie and say her type is in a set when it is not. A consumer can also send the message $M = [0, 1]$ which conveys no information. Revealing credible information does not have any direct cost to the consumer.

(b). Show that the original monopoly outcome is an equilibrium outcome in this certified disclosure game. Indicate the equilibrium messages and monopolist's beliefs (including out-of-equilibrium beliefs) which sustain the equilibrium.

(c). Show that full revelation with simple messages, followed by prices set to equal valuations, is also an equilibrium and in this equilibrium the monopolist extracts all of the consumer's surplus. Again, indicate the monopolist's beliefs that you are using to sustain this equilibrium.

(d). Construct an equilibrium in which the consumer does better (from an *ex ante* perspective) than the standard monopoly outcome in (a). [Hint: look for a simple 2-step partition, $M_1 = [0, a)$ and $M_2 = [a, 1]$.]

Problem 3 Consider the quadratic Cheap-talk model of Crawford and Sobel (1982), but with θ uniformly distributed on $\Theta = [0, 2]$, with message space $\mathcal{M} = [0, 2]$, and with $b = \frac{1}{8}$. Find the most informative equilibrium (the one with the most steps), fully characterizing the equilibrium partition.

Problem 4 Consider a setting of career concerns where a manager has one of two types, $1 > \theta_h > \theta_l > 0$, with prior probabilities ϕ and $(1-\phi)$, respectively. Let $E[\theta] = \phi\theta_h + (1-\phi)\theta_l$ be the players' prior expectation on θ. Neither the market nor the manager knows the manager's type.

In period 1, the manager privately learns of an investment project with some probability. An investment project, i, if available, gives a risky return paying y_i with probability θ and 0 with probability $1-\theta$. A safe asset is always available and pays $r > 0$. Assume that some projects are *ex ante* profitable (i.e., $E[\theta]y_i > r$ for some i and $E[\theta]y_i < r$ for others).

The firm cannot write contracts on investments or outcomes. Rather, it delegates the investment choice to the manager if a project (known only to the manager) becomes available. In period 1, the manager may either choose an investment project, or choose the safe risk-free project. The manager is paid w_1 in the current period and will be paid $w_2 = \text{Prob}[\theta = \theta_h|\cdot]$ after the market observes whether or not she invests and, if she invests, the outcome of the project (success, S, or failure, F). There is no discounting (i.e., $\delta = 1$).

(a). Compute the market's inference of the manager's type if she invests in the safe asset?

(b). Compute the market's inference of the manager's type if she invests in a risky project and it is a success? Compute the market's inference of the manager's type if she invests in a risky project and it is a failure?

(c). Show that a risk-neutral manager is indifferent between all projects (and the safe asset). Formally, show that the market's expected inference of the manager's type, conditional on investing, is the same as if she invests in the safe asset.

(d). Argue that if the manager is only slightly risk averse, she will not choose positive NPV projects.

Problem 5 Consider a setting of career concerns where a manager has one of two types, "talented" and "not" with prior probabilities ϕ and $(1-\phi)$, respectively. Neither the market nor the manager knows the manager's type. The manager is risk neutral and there is no discounting (i.e., $\delta = 1$).

In period 1, the manager learns of a random investment project that gives a risky return paying either 1 or -1 and a privately observed parameter $\gamma \in (0,1)$ associated with the project, where γ is distributed uniformly on $[0,1]$. If the manager is talented, the project pays 1 with probability $\gamma \in (0,1)$. If the manager is not talented, then the project pays 1 with probability $\frac{1}{2}$ (i.e., the expected value of the project is zero, independent of γ).

Timing: (1) the manager privately observes the project and it's γ (i.e., γ is private information); (2) the manager decides whether or not to invest in the project; (3) the market observes whether or not investment has taken place and, if an investment has been made, the outcome of the project (success or failure); (4) the manager receives a payment equal to the market's posterior probability that the manager is talented.

Note that the firm cannot write contracts on investments or outcomes. Rather, it delegates the investment choice to the manager who may decide whether or not to proceed with the project. The manager's incentives are entirely determined by the market's posterior of her talent going forward.

(a). Briefly explain why the firm would like the manager to follow the threshold strategy of invest if and only if $\gamma \geq \frac{1}{2}$.

(b). Suppose that the market believes that the manager follows a simple threshold strategy of investing if and only if $\gamma \geq \bar{\gamma}$. What is the market's posterior on the probability of talent as a function of $\overline{\gamma}$, conditional on success? What is the market posterior conditional on failure?

(c). If the market believes the manager is following the threshold strategy $\overline{\gamma}$, write an expression for the value of $\hat{\gamma}$ that makes the manager indifferent to investing. Show that there is a fixed point $\overline{\gamma} = \hat{\gamma} = \overline{\gamma}^*$ with $\overline{\gamma}^* = 1$. Show that the market's posterior that the manager is talented following investment and failure is 0.

(d). Is there another equilibrium in which all projects (independent of γ) are pursued and the market makes no inference about the manager following investment success or failure? Why or why not?

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Answers to Assignment 2

1 (a). The consumers have a single choice variable, p. If there were a separating pure-strategy equilibrium with $p_l \neq p_h$, then the consumer type with the higher price would deviate to mimic the consumer with the lower-price. Thus, full separation cannot happen in any pure-strategy equilibrium.

Consider next pooling equilibria. In any pure-strategy pooling sequential equilibrium with zero profits, it must be that

$$p^* = (\phi\pi_l + (1-\phi)\pi_h)L.$$

Therefore, if we can demonstrate that this outcome is part of a sequential equilibrium, we are done.

At this price, both consumer types would prefer to buy full insurance rather than to buy nothing. Suppose that the firm responds to any lower price by rejecting the offer and responds to any higher price by accepting. With these responses, it is optimal for both consumer types to offer p^* in equilibrium. For the firm's responses to be sequentially rational, we need a belief system that supports these actions. One such way to do it is to assign the following beliefs:

$$\mu(\pi_h) = \begin{cases} (1-\phi) & \text{if } p \geq p^* \\ 1 & \text{if } p < p^*. \end{cases}$$

Hence, $p^* = E[\pi]L$ is the unique pure-strategy equilibrium price in an equilibrium with zero profits.

(b). It is straightforward to change the pooling equilibrium in (a) to a higher price $\tilde{p} \in (p^*, \pi_h L)$. Because we assumed that the low-risk consumer *strictly* prefers to buy full insurance at $p = \pi_h L$, then she will certainly want to buy it at $\tilde{p}$. Hence, both consumers will choose $\tilde{p}$ versus the alternative of no insurance. We want to construct beliefs such that the firms believe that any price below $\tilde{p}$ must have come from the high-risk consumer. In this case, the firms will reject any offer with a lower price. The following beliefs will suffice:

$$\mu(\pi_h) = \begin{cases} (1-\phi) & \text{if } p \geq \tilde{p} \\ 1 & \text{if } p < \tilde{p}. \end{cases}$$

In this equilibrium, the firms make positive expected profits equal to $\tilde{p} - p^*$ per consumer.

(c). We construct a separating equilibrium in which the offered separating prices are zero-profit for the firm, $p_l = \pi_l L$, $p_h = \pi_h L$, and the firm randomizes between accepting and rejecting a low offer, (L, p_l), with probability σ and $(1-\sigma)$, respectively, where σ satisfies

$$u_h(L, \pi_h L) = \sigma u_h(L, \pi_l L) + (1-\sigma) u_h(0,0).$$

Notice that the firm is willing to randomize because all contracts (and no contract) yield zero profits. Given the above equality, with this randomization, σ, the high-risk type is willing to choose the higher price, p_h, to obtain full coverage with certainty. The low-risk type (because of the single-crossing property) strictly prefers the p_l offer, even though it is rejected $(1-\sigma)$ fraction of the time. We can induce the firm to reject any offers $p \in (p_l, p_h) = (\pi_l L, \pi_h L)$ by assuming $\mu(\pi_h | p \in (p_l, p_h)) = 1$. The firm will accept any offers $p \geq p_h$ with arbitrary beliefs. The firm will reject any offers below p_l with arbitrary beliefs.

2 Note: this question is based upon results in Ali, Lewis and Vasserman, "Voluntary disclosure and personalized pricing," (2018).

(a). The firm maximizes $(p-c)(1-F(p))$. Given $c=0$ and $F(p)=p$, the firm maximizes $p(1-p)$. The profit-maximizing price is $p=\frac{1}{2}$. At this price, $CS = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \frac{1}{2} \cdot \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{8}$.

(b). Suppose that only one message is sent in equilibrium: $[0,1]$. Given this message, the firm's optimal response is the same as in (a). [Assume that no message is taken to be equivalent to the message $M=[0,1]$.] Beliefs for out-of-equilibrium messages, $M \in \mathcal{M}$ that will support this equilibrium are the following *skeptical beliefs:*

$$\mu(\theta|M) = \begin{cases} 1 & \theta = \min M \\ 0 & \text{otherwise.} \end{cases}$$

With these beliefs, the firm responds to $M=[0,1]$ by setting $p=\frac{1}{2}$. Any deviating consumer type earns zero consumer surplus, either because $\min M = p < \theta$ and no purchase is made or $\min M = p = \theta$ and the purchase generates zero consumer surplus. Types $\theta \geq \frac{1}{2}$ strictly prefer to send the equilibrium message $M=[0,1]$; types $\theta \leq \frac{1}{2}$ can do no better than sending the equilibrium message $[0,1]$.

(c). In equilibrium, each consumer type sends the simple message $\{\theta\}$ and the firm responds with $p=\theta$. Consumers earn zero consumer surplus. For any out-of-equilibrium message (we take no message as equivalent to the message $M=[0,1]$), the firm holds skeptical beliefs

$$\mu(\theta|M) = \begin{cases} 1 & \theta = \min M \\ 0 & \text{otherwise.} \end{cases}$$

A deviating consumer also earns zero consumer surplus either because no trade takes place ($\min M = p < \theta$) or trade takes place but there is no surplus ($\min M = p = \theta$).

(d). There are an infinite number of equilibria that consumers prefer from an ex ante point of view. Here is a simple one with two partitions. Consumers send one of the following messages in equilibrium: $M_1 = [0, \frac{1}{2}], M_2 = [\frac{1}{2}, 1]$, with $\theta = \frac{1}{2}$ sending M_2. Any other message is met with skeptical beliefs by the firm and zero-consumer surplus. Given M_1, the optimal price is $p_1 = \frac{1}{4}$. Given M_2, the optimal price is $p_2 = \frac{1}{2}$. (Note that $F(\theta|M_2) = 2\theta - 1$.) Thus, consumers with $\theta \geq \frac{1}{2}$ and $\theta \leq \frac{1}{4}$ earn the same consumer surplus as in (a), but consumers $\theta \in (\frac{1}{4}, \frac{1}{2})$ now each strictly positive consumer surplus relative to zero under (a). Note that CS under (a) is $(\frac{1}{2})^3 = \frac{1}{8}$, while $CS = \frac{1}{8} + \frac{1}{32} = \frac{5}{32}$ in the constructed equilibrium.

Aside: More generally, Ali, Lewis and Vasserman (2018) show that the best possible consumer equilibrium is the infinite partition defined by the boundary points $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$. I.e., the messages sent in equilibrium are $\{[\frac{1}{2}, 1], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{8}, \frac{1}{4}], \dots\}$. The maximum consumer surplus that is achieved is $\frac{1}{6}$. See their paper for the explanation for why this partition represents the upper bound.

3 Given that θ is distributed on $[0,2]$ instead of $[0,1]$, we need to revise the relevant equations from the original $[0,1]$ model. Because preferences are still quadratic, we still obtain the same fundamental indifference relation which implies the same difference equation characterizes the indifferent types:

$$x_i + b = \frac{1}{2}\left(\frac{x_i - x_{i-1}}{2} + \frac{x_i + x_{i+1}}{2}\right),$$

implies

$$(x_{i+1} - x_i) = (x_i - x_{i-1}) + 4b.$$

The requirement that in any n-step equilibrium, we must have the steps exactly covering the support of types must be modified from Crawford and Soble (1982) because the type support has length 2 instead of 1:

$$x_1 + (x_1 + 4b) + \cdots + (x_1 + (n-1)4b) = 2.$$

Using the fact that $1 + 2 + \cdots + (n-1) = n(n-1)/2$, we have

$$nx_1 + n(n-1)2b = 2.$$

Given $n(n-1)2b < 2$, there exists a value of $x_1 \in (0, 2)$ that solves this equation. The inequality is satisfied if n is an integer less than

$$\overline{n} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{b}}\right).$$

The problem assumes $b = \frac{1}{8}$. In this case, we have

$$\overline{n}\left(\frac{1}{8}\right) = \frac{1}{2}(1 + \sqrt{33})$$

which is about 3.37, so the most informative equilibrium has three messages. (You could have solved this directly by plugging in $n = 1$, $n = 2$, etc. in the expression $n(n-1)2b < 2$ until you found a violation at $n = 4$.)

Now that we know there are at most three steps, we need to find the equilibrium partition of $[0, 2]$: x_1^* and x_2^*. Using the equation for x_1 and $n = 3$,

$$x_1 + (x_1 + 4b) + (x_1 + 8b) = 2,$$

and given $b = \frac{1}{8}$, we have

$$3x_1 = \frac{1}{2},$$

so $x_1^* = \frac{1}{6}$. The middle step must be $4b$ more in length than the lower step, so

$$x_2 - x_1^* = x_1^* + 4b,$$

or $x_2^* = 2x_1^* + 4b = \frac{5}{6}$. (As a check on our work, we can verify that the third step is $4b$ more in length than the second step, $2 - x_2^* = x_2^* - x_1^* + 4b$ or $x_2^* = 1 + \frac{1}{2}x_1^* - 2b = 1 + \frac{1}{12} - \frac{1}{4} = \frac{5}{6}$.)

We conclude, the maximum equilibrium partition is 3 steps with

$$\left[0, \frac{1}{6}\right), \left[\frac{1}{6}, \frac{5}{6}\right), \left[\frac{5}{6}, 2\right].$$

4 (This question is based on Holmstrom's 1982/99 paper, section 3.1.)

(a). If the manager does not invest, then the market learns nothing and Prob[θ_h|no invest] $= \phi$. The decision not to invest reveals no information because the manager does not know anything that the market does not know. Given that sometimes projects are not revealed or they are $NPV < 0$, not investing is not a negative signal.

(b). If the manager invests, we use Bayes' Rule to compute posterior probabilities:

$$\text{Prob}[\theta = \theta_h|\text{Invest \& Success}] = \frac{\phi\theta_h}{\phi\theta_h + (1-\phi)\theta_l} = \frac{\phi\theta_h}{E[\theta]},$$

$$\text{Prob}[\theta = \theta_h|\text{Invest \& Fail}] = \frac{\phi(1-\theta_h)}{\phi(1-\theta_h) + (1-\phi)(1-\theta_l)} = \frac{\phi(1-\theta_h)}{1-E[\theta]}.$$

(c). Given our results in (a) and (b), it is easy to see that investing generates a martingale with respect to market beliefs. That is, the expected market belief is the same as the prior, so the expected wage is the same regardless of project choice:

$$\text{Prob}[\theta_h|\text{I \& S}]E[\theta] + \text{Prob}[\theta_h|\text{I \& F}](1-E[\theta]) = \phi.$$

(d). Note that the posterior from not investing is ϕ while the distribution of posteriors from investing is random with mean ϕ. Because the distribution of posteriors is more disperse when a project is undertaken, a project that has positive expected value, $E[\theta]y_i > r$, generates wage risk. Consequently, a manager that is even slightly risk averse will prefer the safe asset even though the firm prefers her to invest in the project. Such investments introduce risk into her income without changing her mean wage.

5 (This question is based on Holmstrom's 1982/99 paper, section 3.2.)

(a). If the manager is not talented, investment has zero expected profit and the decision is irrelevant. If the manager is talented, however, the expected value of investment is

$$\gamma 1 + (1-\gamma)(-1) = 2\gamma - 1,$$

which is nonnegative iff $\gamma \geq \frac{1}{2}$.

(b). Given the threshold strategy, the expected probability of success by a talented manager is $\frac{\overline{\gamma}+1}{2}$. Bayes' Rule gives us

$$\text{Prob}[\text{talent}|I\&S] = \frac{\phi\left(\frac{\overline{\gamma}+1}{2}\right)}{\phi\left(\frac{\overline{\gamma}+1}{2}\right) + (1-\phi)\frac{1}{2}} = \frac{(1+\overline{\gamma})\phi}{1+\overline{\gamma}\phi},$$

$$\text{Prob}[\text{talent}|I\&F] = \frac{\phi\left(\frac{1-\overline{\gamma}}{2}\right)}{\phi\left(\frac{1-\overline{\gamma}}{2}\right) + (1-\phi)\frac{1}{2}} = \frac{(1-\overline{\gamma})\phi}{1-\overline{\gamma}\phi}.$$

(c). Given the market believes the manager is using threshold $\overline{\gamma}$, the expected manager's value from investing in a $\hat{\gamma}$-type project is equal to the value of not investing iff

$$\phi\left(\hat{\gamma}\left(\frac{(1+\overline{\gamma})\phi}{1+\overline{\gamma}\phi}\right)+(1-\hat{\gamma})\left(\frac{(1-\overline{\gamma})\phi}{1-\overline{\gamma}\phi}\right)\right)+(1-\phi)\left(\frac{1}{2}\left(\frac{(1+\overline{\gamma})\phi}{1+\overline{\gamma}\phi}\right)+\frac{1}{2}\left(\frac{(1-\overline{\gamma})\phi}{1-\overline{\gamma}\phi}\right)\right)=\phi.$$

Setting $\hat{\gamma}=\overline{\gamma}=\overline{\gamma}^*$, yields

$$\phi\left(\gamma^*\left(\frac{(1+\gamma^*)\phi}{1+\gamma^*\phi}\right)+(1-\gamma^*)\left(\frac{(1-\gamma^*)\phi}{1-\gamma^*\phi}\right)\right)+(1-\phi)\left(\frac{1}{2}\left(\frac{(1+\gamma^*)\phi}{1+\gamma^*\phi}\right)+\frac{1}{2}\left(\frac{(1-\gamma^*)\phi}{1-\gamma^*\phi}\right)\right)=\phi.$$

Solving this, one obtains

$$\frac{(1-\overline{\gamma}^*)\overline{\gamma}^*(1-\phi)\phi^2}{(\overline{\gamma}^*)^2\phi^2-1}=0,$$

and so $\overline{\gamma}^*=0$ or $\overline{\gamma}^*=1$ are the only solutions.

In the $\overline{\gamma}^*=1$ solution, no projects are undertaken by managers because the market assumes any project undertaken has a $\gamma=1$. As a consequence of this selection threshold, a manager is thought to be untalented with probability 1 following a failure:

$$\text{Prob}[\text{talent}|I\&F]=\frac{(1-\overline{\gamma}^*)\phi}{1-\overline{\gamma}^*\phi}=0.$$

In the event of a success, the manager does improve the posterior above ϕ, but given $\gamma<1$, this is insufficient to overcome the negative assessment from possible failure for any manger with $\gamma<1$. In short, in this equilibrium if the manager's investment information is not observed by the market, the manager becomes more conservative and the investment-choice decision unravels. Even a risk neutral manager acts as if she is risk averse in this example.

(d). Note that the above calculations in (c) show that there is a second equilibrium in which all projects, independent of γ, are chosen by the manager. In this case, the probability of success for a talented manager, conditional on investment, is $\frac{1}{2}$. This is the same as the probability of success for the untalented manager. Hence, the market does not update its posterior from its prior. Of course, too much investment is undertaken in equilibrium. [For those of you who read Holmstrom's original paper, you will have noticed this problem is from section 3.2. The always-invest equilibrium, however, was overlooked in that paper. You can check that $\overline{\gamma}^*=0$ (which is equivalent to $\overline{z}=-1$ in Holmstrom's notation), is also a solution to equation (34).]