Theory of Income, Fall 2008
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Problem Set 8

1 Fiscal Policy in the Neoclassical Growth Model

1.1 Steady states

Feasibility is given by

$$g_t + c_t + k_{t+1} = F(k_t, n_t) + (1-\delta) k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. There is an endowment of time of 1 per period, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v(c_t, 1-n_t).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t [c_t + x_t + \tau_t] = \sum_{t=0}^{\infty} p_t [(1-\tau_{lt}) w_t n_t + k_t v_t - k_t (v_t - \delta) \tau_{kt}],$$

and the law of motion of capital

$$k_{t+1} = x_t + (1-\delta) k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the before tax real wage and rental rate of capital in terms of consumption t goods. There are two tax rates, τ_{lt} is the tax rate on labor income and τ_{kt} is that tax rate on capital and a lump sum tax τ_t, denominated in time t units of the consumption good. Government purchases are denoted by g_t. Notice that the capital income taxes are levied on the net income of capital, i.e. there is an allowance for depreciation, thus the time t income from renting capital equals

$$v_t k_t - \tau_{kt} (v_t - \delta) k_t = v_t k_t (1-\tau_{kt}) + \tau_{kt} \delta k_t.$$

The firm's problem is

$$\max_{k_t, n_t} F(k_t, n_t) - w_t n_t - v_t k_t.$$

The government budget constraint is

$$\sum_{t=0}^{\infty} p_t [\tau_t + \tau_{lt} w_t n_t + k_t (v_t - \delta) \tau_{kt}] = \sum_{t=0}^{\infty} p_t g_t.$$

We will analyze the steady state as a function of the assumed constant g, τ_k and τ_l. We assume that the lump sum taxes τ adjust to satisfy the government budget constraint.

We will assume that v is C^2, strictly concave and strictly increasing in (c, l), and that satisfies the following Inada conditions:

$$\begin{aligned}
\lim_{c \to 0} \frac{v_c(c,l)}{v_l(c,l)} &= \infty \text{ for all } l \in (0,1). \\
\lim_{l \to 0} \frac{v_l(c,l)}{v_c(c,l)} &= \infty \text{ for all } c > 0. \\
\lim_{l \to 1} \frac{v_l(c,l)}{v_c(c,l)} &= 0 \text{ for all } c > 0.
\end{aligned}$$

so that we don't have to worry about corner solutions.

1. Using λ for the multiplier of the budget constraint of the agent, write down the FOC w.r.t. c_t and n_t. Use these FOCs to obtain an expression for the marginal rate of substitution of c_{t+1} and c_t and for the marginal rate of substitution of c_t and n_t.

Ans: The FOCs w.r.t. c_t and n_t are,

$$\beta^t v_c(c_t, 1-n_t) = \lambda p_t,$$

$$\beta^t v_l(c_t, 1-n_t) = \lambda p_t (1-\tau_{lt}) w_t.$$

The marginal rates of substitution are,

$$\beta \frac{v_c(c_{t+1}, 1-n_{t+1})}{v_c(c_t, 1-n_t)} (1+r_t) = 1, \tag{1}$$

$$\frac{v_l(c_t, 1-n_t)}{v_c(c_t, 1-n_t)} = (1-\tau_{lt}) w_t. \tag{2}$$

2. Write down the FOCs for the firm's maximization problem.

Ans: The FOCs w.r.t. k_t, and n_t are,

$$F_k(k_t, n_t) = v_t,$$

$$F_n(k_t, n_t) = w_t.$$

3. Using the households budget constraint, show that

$$v_{t+1} = \delta + \frac{r_t}{1-\tau_{kt+1}},$$

must hold if agents will find optimal to choose $0 < k_{t+1} < \infty$. [Hint: Collect the terms in the budget constraint with k_{t+1}

$$... - p_t k_{t+1} + ... + p_{t+1} k_{t+1} \left[(1-\delta) + v_{t+1} (1 - \tau_{kt+1}) + \tau_{kt+1} \delta \right] + ...,$$

use the definition of r_t, impose that the optimal k_{t+1} is interior, and go through the algebra].

Ans: Replacing x_t into the household budget constraint, we note that the latter becomes

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta) k_t + \tau_t \right] = \sum_{t=0}^{\infty} p_t \left[(1 - \tau_{lt}) w_t n_t + k_t v_t - k_t (v_t - \delta) \tau_{kt} \right].$$

Consider the terms involving only capital:

$$\begin{aligned}
& \sum_{t=0}^{\infty} p_t \left[k_{t+1} - (1-\delta+v_t) k_t + k_t (v_t - \delta) \tau_{kt} \right] \\
= & \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_t [(1-\delta+v_t) k_t - k_t (v_t - \delta) \tau_{kt}] \\
= & \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=1}^{\infty} p_t [(1-\delta+v_t) k_t - k_t (v_t - \delta) \tau_{kt}] - p_0 [(1-\delta+v_0) k_0 - k_0 (v_0 - \delta) \tau_{k0}] \\
= & \sum_{t=0}^{\infty} p_t k_{t+1} - \sum_{t=0}^{\infty} p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1}) k_{t+1} - p_0 (1-\delta+v_0 - (v_0-\delta) \tau_{k0}) k_0 \\
= & \sum_{t=0}^{\infty} [p_t - p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1})] k_{t+1} - p_0 (1-\delta+v_0 - (v_0-\delta) \tau_{k0}) k_0.
\end{aligned}$$

Thus, the budget constraint becomes

$$\begin{aligned}
& \sum_{t=0}^{\infty} p_t c_t + \sum_{t=0}^{\infty} [p_t - p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1})] k_{t+1} \\
= & \; p_0 (1-\delta+v_0 - (v_0-\delta) \tau_{k0}) k_0 + \sum_{t=0}^{\infty} p_t w_t n_t
\end{aligned}$$

Now, if the term in square brackets is strictly negative at some period t, i.e.,

$$p_t < p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1}),$$

then the agent will be able to increase arbitrarily her wealth by choosing an arbitrarily high value of k_{t+1}. Therefore, consumption can be made arbitrarily high without violating the budget constraint. Clearly, this cannot be a part of any equilibrium, so we conclude that in any equilibrium $p_t \geq p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1})$. On the other hand, if in some period t,

$$p_t > p_{t+1} (1-\delta+v_{t+1} - (v_{t+1}-\delta) \tau_{kt+1})$$

then the agent will set k_{t+1} as small as possible (either negative, if feasible, or zero). Again, this cannot be part of any equilibrium since with zero capital there is zero consumption. Therefore, the only configuration of parameters consistent with an equilibrium is when

$$p_t = p_{t+1}\left(1-\delta+v_{t+1}-\left(v_{t+1}-\delta\right)\tau_{kt+1}\right) \text{ for all } t.$$

This is equivalent to

$$1-\delta+v_{t+1}-\left(v_{t+1}-\delta\right)\tau_{kt+1} = \frac{p_t}{p_{t+1}} \equiv 1+r_t \text{ for all } t,$$

or

$$v_{t+1} = \delta + \frac{r_t}{1-\tau_{kt+1}}.$$

4. Assume that the tax rates are constant, i.e., $\tau_{kt} = \tau_k$ and $\tau_{lt} = \tau_l$ for all t. Show that in a steady state r, $\kappa \equiv k/n$, w, x, c, n and k solve the following system of 7 equations:

$$\begin{aligned}
r &= \rho, \\
F_k\left(\kappa, 1\right) &= \frac{\rho}{1-\tau_k} + \delta, \\
w &= F_n\left(\kappa, 1\right), \\
\frac{v_l\left(c, l\right)}{v_c\left(c, l\right)} &= w\left(1-\tau_l\right), \\
c &= \left(F\left(\kappa, 1\right) - \delta\kappa\right)\left(1-l\right) - g, \\
x &= \delta\kappa \ \left(1-l\right), \\
k &= \kappa \ \left(1-l\right),
\end{aligned}$$

where $\beta = 1/\left(1+\rho\right)$. [Hint: Modify the equations used for the steady state with no taxation in the class notes].

Ans: Using that at the steady state $v_c\left(c_t, l_t\right) = v_c\left(c_{t+1}, l_{t+1}\right)$, from (1) we obtain

$$\left(1+r\right)\beta = 1,$$

or

$$r = \rho. \tag{3}$$

Using $v = \delta + \rho/\left(1-\tau_k\right)$ and $v = F_k\left(k, n\right)$ we can then determine the capital to labor ratio, $\kappa = k/\left(1-l\right)$, as the solution to

$$\delta + \frac{\rho}{1-\tau_k} = F_k\left(\kappa, 1\right). \tag{4}$$

Taking the resulting κ and plugging it into the firm's FOC for n, we obtain w,

$$w = F_n\left(\kappa, 1\right).$$

Moreover, in steady state we know that $x = \delta k = \delta \kappa \ (1-l)$. Thus, from feasibility we obtain

$$c + \delta k + g = F(\kappa, 1)(1-l)$$

or,

$$c = (F(\kappa, 1) - \delta\kappa)(1-l) - g. \tag{5}$$

Finally, from the consumer's problem we know that,

$$\frac{v_l(c, 1-n)}{v_c(c, 1-n)} = (1-\tau_l) w,$$

or, using the firm's FOC w.r.t n,

$$\frac{v_l(c,l)}{v_c(c,l)} = (1-\tau_l) F_n(\kappa, 1). \tag{6}$$

Note that given κ, which is obtained from (4), (5) and (6) constitute a system of two equations in two unknowns, c and l, which can be used to find the steady state values of consumption and leisure. Moreover, under normality, c and l are unique (this is proved below). Then, the steady state level of capital k, can be recovered from $\kappa = k/(1-l)$.

The point of this problem is to realize that the introduction of distorting taxes precludes us from solving the consumer problem subject to the feasibility constraint (i.e., the Pareto problem), since in this case there is a wedge between the marginal rate of substitution and the marginal rate of transformation due to the presence of distortionary taxation.

5. i) Show that if c and l are normal goods, then

$$\begin{aligned} \frac{\partial}{\partial c}\frac{v_l(c,l)}{v_c(c,l)} &> 0 \\ \frac{\partial}{\partial l}\frac{v_l(c,l)}{v_c(c,l)} &< 0 \end{aligned}$$

respectively (i.e., the first condition is necessary for normality of leisure and the second for normality of consumption).

Ans: Consider the following (well behaved) utility function $v(c,l)$, where l and c are any pair of goods. Without loss of generality, normalize the price of good l to one and let p denote the price of good c in terms of good l. Let M denote the agent's income. We say that l and c are normal goods if $dl(p,M)/dM > 0$ and $dc(p,M)/dM > 0$ holding prices constant. The conditions that determine l and c are:

$$\frac{v_l(l,c)}{v_c(l,c)} = \frac{1}{p}, \tag{7}$$

and

$$pc + l = M.$$

Differentiating the budget constraint w.r.t. M we obtain

$$p\frac{dc}{dM}+\frac{dl}{dM}=1. \tag{8}$$

In turn, differentiating condition (7) w.r.t. M yields

$$\underbrace{\frac{\partial}{\partial c}\frac{v_l\left(l,c\right)}{v_c\left(l,c\right)}}_{\Psi_c}\frac{dc}{dM}+\underbrace{\frac{\partial}{\partial l}\frac{v_l\left(l,c\right)}{v_c\left(l,c\right)}}_{\Psi_l}\frac{dl}{dM}=0,$$

or, using (8),

$$\Psi_c\frac{dc}{dM}+\Psi_l\left(1-p\frac{dc}{dM}\right)=0,$$

or

$$\frac{dc}{dM}=-\frac{\Psi_l}{\Psi_c-p\Psi_l}.$$

Finally, using (8) again we obtain

$$\begin{aligned}\frac{dl}{dM} &= 1-p\frac{dc}{dM}\\ &= 1+\frac{p\Psi_l}{\Psi_c-p\Psi_l},\end{aligned}$$

or

$$\frac{dl}{dM}=\frac{\Psi_c}{\Psi_c-p\Psi_l}.$$

Now, note that

$$\Psi_l=\frac{v_{ll}v_c-v_lv_{lc}}{v_c^2}=\frac{v_{ll}}{v_c}-\frac{1}{p}\frac{v_{lc}}{v_c},$$

and

$$\Psi_c=\frac{v_{lc}v_c-v_lv_{cc}}{v_c^2}=\frac{v_{lc}}{v_c}-\frac{1}{p}\frac{v_{cc}}{v_c},$$

where we have used that $v_l=v_c/p$. Thus,

$$\begin{aligned}\Psi_c-p\Psi_l &= \frac{v_{lc}}{v_c}-\frac{1}{p}\frac{v_{cc}}{v_c}-p\frac{v_{ll}}{v_c}+\frac{v_{lc}}{v_c}\\ &= -\frac{1}{v_c}\left[pv_{ll}-2v_{lc}+\frac{1}{p}v_{cc}\right].\end{aligned}$$

The term in brackets is the quadratic expression

$$\left[\begin{array}{cc}-\sqrt{p} & 1/\sqrt{p}\end{array}\right]\left[\begin{array}{cc}v_{ll} & v_{lc}\\ v_{lc} & v_{cc}\end{array}\right]\left[\begin{array}{c}-\sqrt{p}\\ 1/\sqrt{p}\end{array}\right],$$

which is negative by concavity of v. Hence, $\Psi_c-p\Psi_l>0$, which implies that

$$\operatorname{sgn}\left[\frac{dl}{dM}\right]=\operatorname{sgn}\left[\Psi_c\right],$$

and

$$\operatorname{sgn}\left[\frac{dc}{dM}\right] = \operatorname{sgn}\left[-\Psi_l\right].$$

That is, normality of l requires that $\Psi_c > 0$ and normality of c requires that $\Psi_l < 0$, which is the desired result.

ii) Show that if c and l are normal goods, one can define a function $\phi(l,\omega)$ such that

$$\frac{v_l(\phi(l,\omega), l)}{v_c(\phi(l,\omega), l)} = \omega.$$

Moreover ϕ is increasing in ω and l.

Ans: By the implicit function theorem, there exists a function $\phi(l,\omega)$ such that

$$\frac{v_l(\phi(l,\omega), l)}{v_c(\phi(l,\omega), l)} = \omega. \tag{9}$$

This is so because the derivatives of the function $\phi(l,\omega)$ are well defined under the assumption that l is a normal good. To show this, differentiate the above expression to obtain

$$\Psi_c \frac{d\phi(l,\omega)}{dl} + \Psi_l = 0 \Rightarrow \frac{d\phi(l,\omega)}{dl} = -\frac{\Psi_l}{\Psi_c} > 0,$$

which is well-defined since l is normal, and positive since c is normal as well. Moreover,

$$\Psi_c \frac{d\phi(l,\omega)}{d\omega} = 1 \Rightarrow \frac{d\phi(l,\omega)}{d\omega} = \frac{1}{\Psi_c} > 0,$$

which is well-defined and positive since l is normal. Thus, ϕ is increasing in ω and l.

iii) Suppose that $v(c,l) = (1-\alpha)\log c + \alpha \log l$. Show that in this case the function $\phi(l,\omega)$ is given by

$$\phi(l,\omega) = \omega \frac{1-\alpha}{\alpha} l.$$

Ans: Using the proposed functional form in (7) we obtain

$$\frac{\alpha/l}{(1-\alpha)/c} = \omega,$$

or

$$c = \phi(l,\omega) = \omega \frac{1-\alpha}{\alpha} l.$$

6. Consider the system of two equations

$$\begin{aligned}\frac{v_l(c,l)}{v_c(c,l)} &= F_n(\kappa,1)(1-\tau_l),\\ c &= (F(\kappa,1)-\delta\kappa)(1-l)-g,\end{aligned}$$

in two unknowns, c and l, for given κ, g, and τ_l. Assume that $v(c,l)$ is such that c and l are both normal goods. Using the previous results write this system as

$$\begin{aligned}c &= \phi(l\ ,\ F_n(\kappa,1)(1-\tau_l)\ ),\\ c &= (F(\kappa,1)-\delta\kappa)\ (1-l)-g.\end{aligned}$$

i) Show that, given κ, g, and τ_l, there is a unique c and l that solve this equation. Define $C(\kappa,g,\tau_l)$ and $L(\kappa,g,\tau_l)$ as the solution. [Hint: Plot the RHS of the two expressions with c in the vertical axis and l in the horizontal axis. The intersection of the two functions define the solutions $C(\kappa,g,\tau_l)$ and $L(\kappa,g,\tau_l)$].

Ans: Given our previous results, the first equation is increasing in l. Moreover, the second equation is decreasing (an linear) in l. Thus, in a graph with c in the vertical axis and l in the horizontal axis, both equations will intersect once at most which implies that the steady state is unique. We let $l^* = L(\kappa,g,\tau_l)$ and $c^* = C(\kappa,g,\tau_l)$, denote such intersection.

ii) Show that $C(\kappa,g,\tau_l)$ is decreasing in τ_l and $L(\kappa,g,\tau_l)$ is increasing in τ_l. [Hint: Use the plot described above and shift the RHS of the top equation].

Ans: Given our previous results, the first equation is increasing in $\omega \equiv F_n(\kappa,1)(1-\tau_l)$ and, hence, decreasing in τ_l. Thus, the curve $c=\phi(l,\omega)$ shifts down in the (l,c) space. Moreover, the second equation does not depend on τ_l. Hence, a rise in τ_l increases consumption and reduces labor supply, i.e., $C(\kappa,g,\tau_l)$ is decreasing in τ_l and $L(\kappa,g,\tau_l)$ is increasing in τ_l.

iii) Show that $C(\kappa,g,\tau_l)$ is increasing in κ but $L(\kappa,g,\tau_l)$ can be decreasing or increasing in κ. [Hint: Use the plot described above and shift the RHS of the both equations].

Ans: The first equation is increasing in $\omega \equiv F_n(\kappa,1)(1-\tau_l)$ and, hence, increasing in κ (recall that $F_{kn} > 0$ under CRS). Thus, the curve $c=\phi(l,\omega)$ shifts up in the (l,c) space. Moreover, the second equation is increasing in κ as well. To see this, note that

$$\frac{d}{d\kappa}(F(\kappa,1)-\delta\kappa) = F_k - \delta = \frac{\rho}{1-\tau_k} > 0,$$

where we use that in steady state $F_k = \rho/(1-\tau_k)+\rho$, by equation (4). Thus, the line $c=(F(\kappa,1)-\delta\kappa)\ (1-l)-g$ shifts up in the (l,c) space as well. Hence, a rise in κ increases consumption but has ambiguous effects on labor supply, i.e., $C(\kappa, g, \tau_l)$ is increasing in κ and $L(\kappa, g, \tau_l)$ can be decreasing or increasing in κ.

iv) Show that $C(\kappa, g, \tau_l)$ is decreasing in g and $L(\kappa, g, \tau_l)$ is decreasing in g. [Hint: Use the plot described above and shift the RHS of the bottom equation].

Ans: The first equation is independent of g. Moreover, the second equation is decreasing in g. Thus, the line $c = (F(\kappa, 1) - \delta\kappa)\ (1-l) - g$ shifts down in the (l, c) space. Hence, a rise in g reduces both consumption and labor supply, i.e., $C(\kappa, g, \tau_l)$ and $L(\kappa, g, \tau_l)$ are decreasing in g.

v) Define $C^*(\tau_k, \tau_l, g)$ and $L^*(\tau_k, \tau_l, g)$ as

$$\begin{aligned} C^*(\tau_k, \tau_l, g) &= C(\kappa^*(\tau_k), g, \tau_l), \\ L^*(\tau_k, \tau_l, g) &= L(\kappa^*(\tau_k), g, \tau_l), \end{aligned}$$

where $\kappa^*(\tau_k)$ solves

$$F_k(\kappa^*(\tau_k), 1) = \delta + \frac{\rho}{1-\tau_k}.$$

7. Based on the previous results, complete the following table. In each entry write $+$, $-$, $=$ or ? if the quantity increases, decreases, stays constant, or its effect cannot be determined. That is, examine what happens with the steady states values C^*, L^*, κ^* as well as other steady state objects ($r,\ w,\ w(1-\tau_l),\ v,\ v-\delta,\ (v-\delta)(1-\tau_k),\ n,$ and x) as a function of the fiscal policy parameters τ_l, τ_k and g. Assume that c and l are normal goods. [Hint: Use the plot described above and shift the RHS of both equations, also taking into account the effect of τ_k in $\kappa^*(\tau_k)$].

Ans:

steady-state \ fiscal policy	τ_l	τ_k	g
interest rate, r	$=$	$=$	$=$
before tax wages, w	$=$	$-$	$=$
after-tax wages, $w(1-\tau_l)$	$-$	$-$	$=$
capital-labor ratio, κ	$=$	$-$	$=$
before-tax rental rate of capital, v	$=$	$+$	$=$
before-tax net rental rate of capital, $v-\delta$	$=$	$+$	$=$
after-tax net rental rate of capital, $(v-\delta)(1-\tau_k)$	$=$	$=$	$=$
consumption, c	$-$	$-$	$-$
labor supply, n	$-$	?	$+$
capital stock, k	$-$	?	$+$
investment, x	$-$	?	$+$

1.2 Productivity growth

Now we analyze the effect of labor augmenting labor productivity and an increase in steady state allocation. Let A be an index of labor augmenting productivity, so that the production function is written as

$$y = F(k, n\ A).$$

Notice that using κ for the capital-labor ratio (in natural, as opposed to efficiency, units), $\kappa = k/n$ we have:

$$\begin{aligned}
\frac{\partial}{\partial k} F(k, n\ A) &= F_k(k, nA) = F_k\left(\frac{\kappa}{A}, 1\right), \\
\frac{\partial}{\partial n} F(k, n\ A) &= A\ F_n(k, nA) = A\ F_k\left(\frac{\kappa}{A}, 1\right).
\end{aligned}$$

1. Show that steady states are given by the solution $\kappa^*(A) = k(A)/n(A)$ to:

$$F_k\left(\kappa \frac{1}{A}, 1\right) = \delta + \frac{\rho}{1-\tau_k},$$

so that

$$\kappa^*(A) = \kappa^*(1)\ A,$$

for all $A > 0$. Also show that

$$C^*(A, \tau_k, \tau_l, g) = C(\kappa, A, \tau_k, \tau_l, g),$$

and

$$L^*(A, \tau_k, \tau_l, g) = L(\kappa, A, \tau_k, \tau_l, g),$$

where $C(A, \kappa, \tau_k, \tau_l, g)$ and $L(\kappa, A, \tau_k, \tau_l, g)$ are the values of c and l that, given $\kappa, \tau_k, \tau_l, g$ solve:

$$\begin{aligned}
c &= \phi\left(l\ ,\ AF_n\left(\frac{\kappa}{A}, 1\right)(1-\tau_l)\ \right), \\
c &= A\left(F\left(\frac{\kappa}{A}, 1\right) - \delta\frac{\kappa}{A}\right)\ (1-l) - g.
\end{aligned}$$

Ans: Feasibility is now given by

$$g_t + c_t + k_{t+1} = F(k_t, An_t) + (1-\delta) k_t, \tag{10}$$

where for now we take A as a constant. The household's budget constraint is identical, thus

$$v_{t+1} = \delta + \frac{r_t}{1-\tau_{kt+1}}, \tag{11}$$

still holds. The household's optimality conditions are the same and the firm's FOCs now become

$$w_t = AF_n\left(k_t, An_t\right) = AF_n\left(\frac{k_t}{n_t}\frac{1}{A}, 1\right), \tag{12}$$

$$v_t = F_k\left(k_t, n_t A\right) = F_k\left(\frac{k_t}{n_t}\frac{1}{A}, 1\right), \tag{13}$$

where we are using that both partial derivatives are homogeneous of degree zero.

We now proceed to solve for the steady state values. From the household's intertemporal condition we still obtain

$$r = \rho.$$

Evaluating (11) and (13) at the steady state and letting $\kappa \equiv k/n$, we find

$$F_k\left(\kappa\frac{1}{A}, 1\right) = \delta + \frac{\rho}{1-\tau_k}, \tag{14}$$

which determines κ. Given the other parameters, let $\kappa^*(A)$ denote the steady state capital to labor ratio as a function of A. The previous equation shows that for all A, $\kappa^*(A)/A$ must be constant over time (if we let $h(a)$ denote the inverse function of F_k, so that $F_k(h(a), 1) = a$, then the constant is $h(\delta + \rho/(1-\tau_k))$). That is,

$$\kappa^*(A) = A \times \text{constant},$$

which implies that

$$\kappa^*(A) = A\kappa^*(1). \tag{15}$$

Given κ, we can obtain the steady state wage rate from (12):

$$w = F_n\left(\kappa\frac{1}{A}, 1\right).$$

Feasibility at the steady state is given by

$$\begin{aligned} g + c &= F(k, A(1-l)) - \delta k \\ &= (1-l)\, AF\left(\frac{k}{(1-l)}\frac{1}{A}, \right) - \delta\frac{\kappa}{A}A(1-l) \\ &= A\left[F\left(\frac{\kappa}{A}, 1\right) - \delta\frac{\kappa}{A}\right](1-l). \end{aligned}$$

Hence

$$c = A\left[F\left(\frac{\kappa}{A}, 1\right) - \delta\frac{\kappa}{A}\right](1-l) - g \tag{16}$$

Finally, the other equation used to solve for l and c is the intratemporal condition between consumption and leisure (using the steady state wage rate),

$$\frac{v_l(c,l)}{v_c(c,l)} = AF_n\left(\frac{\kappa}{A}, 1\right)(1-\tau_l),$$

which, under normality of consumption and leisure, can be written as

$$c = \phi\left(l, AF_n\left(\frac{\kappa}{A}, 1\right)(1-\tau_l)\right). \tag{17}$$

Now, in a steady state $\kappa = \kappa^*(A)$ solves (14). Hence (16) and (17) become

$$c = A\left[F\left(\kappa^*(1), 1\right) - \delta\kappa^*(1)\right](1-l) - g, \tag{18}$$

and

$$c = \phi\left(l, AF_n\left(\kappa^*(1), 1\right)(1-\tau_l)\right). \tag{19}$$

This is a system of two equations in two unknowns, which can be used to solve for the steady state levels of consumption and leisure $C^*(A, \tau_k, \tau_l, g)$ and $L^*(A, \tau_k, \tau_l, g)$.

2. Assume that $v(c,l)$ are normal goods. Show that $C^*(A)$ is increasing in A. Is $L^*(A)$ increasing in A? Discuss your answer in terms of income and substitution effects.

Ans: The idea is to use the same type of plots as in the previous exercise, with c in the vertical axis and l in the horizontal axis. For a given l, as A increases both equations (18) and (19) shift upwards. Thus, c increases and l can either increase or decrease. Since wealth increases, consumption increases. Regarding leisure, we have both, income and substitution effects: wealth increases but the relative price of leisure also increases.

3. Let

$$v(c,l) = \frac{c^{1-\sigma}h(l)}{1-\sigma},$$

for $\sigma > 0$, $\sigma \neq 1$. Assume that

$$\frac{h'(l)}{h(l)}\frac{1}{1-\sigma},$$

is a decreasing function of l. i) Show that c and l are normal goods. ii) Show that the function ϕ defined above takes the form

$$\phi(l,\omega) = \omega\ \varphi(l),$$

for an increasing function φ. iii) Assume that $g = A\hat{g}$, so government purchases are proportional to A. Use ii) to argue that in this case

$$\begin{aligned} C^*(A) &= A\ C^*(1), \\ L^*(A) &= L(1), \end{aligned}$$

for all $A > 0$.

Ans: Remember that for v to have positive marginal utility of consumption and leisure we need $h(l) > 0$ and $h'(l)/(1-\sigma) > 0$.The marginal rate of substitution is

$$\frac{v_l(c,l)}{v_c(c,l)} = c\left(\frac{h'(l)}{h(l)}\frac{1}{1-\sigma}\right).$$

i) c is normal:

$$\frac{\partial}{\partial l}\left[\frac{v_l(c,l)}{v_c(c,l)}\right] = c\frac{\partial}{\partial l}\left[\frac{h'(l)}{h(l)}\frac{1}{1-\sigma}\right] < 0,$$

where the inequality follows by assumption.

l is normal:

$$\frac{\partial}{\partial c}\left[\frac{v_l(c,l)}{v_c(c,l)}\right] = \frac{h'(l)}{h(l)}\frac{1}{1-\sigma} > 0,$$

where we used that $h'(l)/(1-\sigma) > 0$ and $h(l) > 0$.

ii) The function ϕ was defined as

$$\frac{v_l(\phi(l,\omega),l)}{v_c(\phi(l,\omega),l)} = \omega.$$

In our case

$$\phi(l,\omega)\frac{h'(l)}{h(l)}\frac{1}{1-\sigma} = \omega,$$

so that

$$\phi(l,\omega) \equiv \omega\varphi(l) = \omega\frac{h(l)}{h'(l)}(1-\sigma).$$

Thus

$$\varphi'(l) = \frac{\partial}{\partial l}\left[\frac{h(l)}{h'(l)}(1-\sigma)\right] > 0,$$

since by assumption

$$\frac{h'(l)}{h(l)}\frac{1}{1-\sigma},$$

is decreasing in l.

iii) If $\tilde{g} = Ag$, using ii) the system of equations (18) and (19) becomes

$$c = A\left[F(\kappa^*(1),1) - \delta\kappa^*(1)\right](1-l) - Ag,$$

and

$$c = \varphi(l)\,AF_n(\kappa^*(1),1)(1-\tau_l),$$

or dividing by A and letting $\tilde{c} \equiv c/A$,

$$\begin{aligned}\tilde{c} &= \left[F(\kappa^*(1),1) - \delta\kappa^*(1)\right](1-l) - g,\\ \tilde{c} &= \varphi(l)\,F_n(\kappa^*(1),1)(1-\tau_l).\end{aligned}$$

Thus, the pair $\tilde{c}$ and l which solves this system of equations is invariant to A. Hence, the solution satisfies

$$\begin{aligned} C^*(A) &= AC^*(1), \\ L^*(A) &= L^*(1). \end{aligned}$$

We have shown that L^* can be increasing or decreasing in A, depending on the income and substitution effects, while L^* is decreasing in τ_l. Why is that for the case of labor taxes the substitution effect dominates the income effect? The next exercise is designed to explain this difference. The key to it, is that, in equilibrium, for labor taxes there is NO income effect, since it is assumed that the extra revenue produced by the income taxes is rebated back to the households. To simplify the exposition in the next exercise we set $A = 1$, $g = 0$ and $\tau_k = 0$, and just consider the comparative static w.r.t. τ_l.

4. i) Show that a steady state is the solution of the following 6 equations in 6 unknowns:

$$\begin{aligned} \rho + \delta &= F_k(\kappa, 1), \\ \kappa &= k/n, \\ w &= F_n(\kappa, 1), \\ l &= 1 - n, \\ c + wl &= w + k\rho, \\ \frac{v_l(c,l)}{v_c(c,l)} &= w(1 - \tau_l). \end{aligned}$$

Why is it that there are no taxes in the equation $c + wl = w + k\rho$ (that is neither τ_l, τ_k or the lump sum tax τ show up there)?

Ans: Those are the same equations that we obtained in the previous exercise evaluated at $g = 0$ and $\tau_k = 0$, except, perhaps, $c + wl = w + k\rho$. So let's compute it. From feasibility at steady state we know that

$$c + \delta k = F(k, n).$$

Now, write F as

$$F(k, n) = F_k(\kappa, 1)\, k + F_n(\kappa, 1)(1 - l),$$

and use that

$$\begin{aligned} w &= F_n(\kappa, 1), \\ \rho + \delta &= F_k(\kappa, 1). \end{aligned} \tag{20}$$

Thus,

$$\begin{aligned} c + \delta k &= F(k, n) \\ &= (\rho + \delta)\, k + w(1 - l), \end{aligned}$$

or

$$c + wl = \rho k + w. \tag{21}$$

The last equation is *aggregate* feasibility at the steady state, thus no taxes should appear in there.

ii) Let κ^* be the solution of $\rho + \delta = F_k(\kappa^*, 1)$, let w^* be the solution of $w^* = F_n(\kappa^*, 1)$. Draw a diagram with c in the vertical axis and l in the horizontal axis. iia) Draw a piecewise linear set of points for which $c = w^*(1-l) + k\rho$ for l in $[0,1]$ for a given k. This is like a budget line for the points $l > 0$ and then has a vertical part for $l = 1$, so we will refer to it as the "economy-wide budget set". iib) Draw a set of indifference curves for $v(c,l)$ in the same graph. iic) Identify the pair $\left(\bar{c}(k), \bar{l}(k)\right)$ where the indifference curve is tangent to the economy-wide budget set. iid) Identify the pair $(C(k;\tau_l), L(k;\tau_l))$ on the economy-wide budget set whose slope is $(1-\tau_l)$ times the slope of the economy-wide budget set, i.e., the pair

$$(C(k;\tau_l), L(k;\tau_l)) = (c, l),$$

that solves:

$$\begin{aligned} c &= w^*(1-l) + k\rho, \\ \frac{v_l(c,l)}{v_c(c,l)} &= w^*(1-\tau_l). \end{aligned}$$

iie) Argue that $L(k;\tau_l) > \bar{l}(k)$. Explain why the relative strength of the income and substitution effects are not relevant to establish this inequality (i.e., why labor income taxes do not have, in equilibrium, an income effect). iif) Argue that the steady state l is given by $L^*(\tau_l) = L(k^*(\tau_l), \tau_l)$, where $k = k^*(\tau_l)$ solves

$$(1 - L(k;\tau_l)) = (1/\kappa^*)\ k.$$

iig) Use the function $N(k;\tau_l) \equiv 1 - L(k;\tau_l)$ and the equation

$$N(k^*(\tau_l);\tau_l) = (1/\kappa^*)\ k^*(\tau_l),$$

to argue that the steady state labor supply is decreasing in τ_l. [Hint: In a diagram with k in the horizontal axis and N in the vertical axis, use the the solution to iid)-iie) to compare the intersection of the functions $N(k;\tau_l')$ and $N(k;\tau_l)$ with $\tau_l' > \tau_l$ with the the function $(1/\kappa^*)\ k$].

Ans:
iia-d) See Figure 1.

iie) It is obvious from the figure that $L(k;\tau_l) > \bar{l}(k)$. Indeed, the marginal rate of substitution is lower than the marginal rate of transformation, which means that the equilibrium is in a point where the marginal rate of substitution is flatter than the marginal rate of transformation. All points where $\bar{l}(k) < L(k;\tau_l)$ have higher MRS than MRT. If total income were to remain constant, there would be offsetting income and substitution effects playing in the determination of leisure. However income does not remain constant. The labor tax receipts are rebated back to the households as lump-sum transfers. Thus, the income effect disappears and only the substitution effect remains, which means that leisure increases and consumption decreases, since the relative price of leisure decreases.

iif) The previous question took k as given, but in general k depends on τ_l. To obtain $k^*(\tau_l)$, the steady state of capital as a function of τ_l, note that in the previous question we obtained leisure as a function of k (i.e., $L(k;\tau_l)$) and we also know that (20) determines the capital to labor ratio κ^*. Thus $k^*(\tau_l)$ is the k that solves

$$\frac{k}{(1-L(k,\tau_l))} = \kappa^*,$$

or, equivalently

$$(1-L(k;\tau_l)) = (1/\kappa^*)\ \ k.$$

iig) Fix an arbitrary k. Assuming that leisure and consumption are normal and using that κ^* solves (20), the solution to c and l satisfies

$$\begin{aligned} c &= F_n(\kappa^*,1)(1-l)+k\rho, \\ c &= \phi(l, F_n(\kappa^*,1)(1-\tau_l)). \end{aligned}$$

It is easy to show that $L(k,\tau_l)$ is increasing in τ_l and increasing in k. We now construct the function $N(k;\tau_l) \equiv 1 - L(k;\tau_l)$, which is decreasing in τ_l and decreasing in k. Figure 2 shows the determination of $N^*(\tau_l)$ and $k^*(\tau_l)$. Both the steady state stock of capital and the steady state labor supply are decreasing in τ_l.

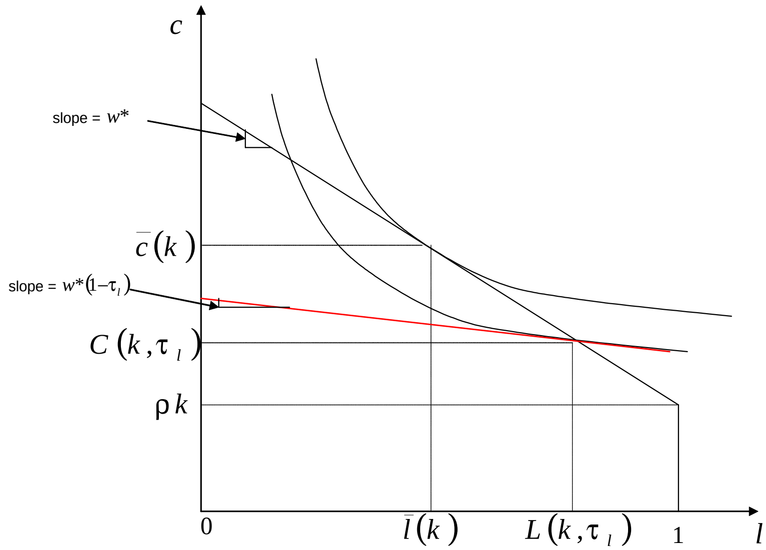

Figure 1

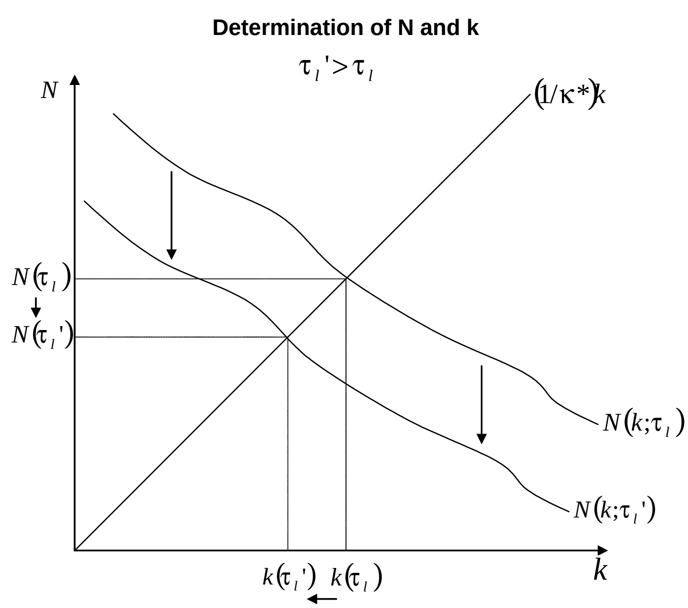

Figure 2

1.3 Population and productivity growth

Consider a version of the neoclassical growth model with population growth, at the rate of λ per year, and labor augmenting productivity, at the rate of π per year.

Assume that preferences are given by

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v\left(c_t, 1-n_t\right) = \sum_{t=0}^{\infty} \Lambda_t \beta^t v\left(c_t, l_t\right),$$

where c_t is per-capita consumption, and n_t is per-capita labor supply at time t. The size of the population at time t is $\Lambda_t = \Lambda_0 \left(1+\lambda\right)^t$. The time t total production if per-capita capital is k_t and labor per person n_t are used is

$$F\left(\Lambda_t k_t, \left(1+\pi\right)^t \Lambda_t n_t\right).$$

Thus, feasibility is given by

$$\Lambda_t \left[c_t + x_t + g_t\right] = F\left(\Lambda_t k_t, \Lambda_t \left(1+\pi\right)^t n_t\right),$$

where x_t and g_t denote per-capita investment and government purchases. The law of motion of capital is given by

$$\Lambda_{t+1} k_{t+1} = \Lambda_t x_t + \Lambda_t k_t \left(1-\delta\right).$$

We define the * allocation by letting

$$\begin{aligned} g_t^* &= \frac{g_t}{\left(1+\pi\right)^t}, \\ c_t^* &= \frac{c_t}{\left(1+\pi\right)^t}, \\ x_t^* &= \frac{x_t}{\left(1+\pi\right)^t}, \\ k_t^* &= \frac{k_t}{\left(1+\pi\right)^t}, \\ l_t^* &= l_t, \\ n_t^* &= n_t, \end{aligned}$$

and prices

$$\begin{aligned} r_t^* &= \frac{p_t}{p_{t+1}} - 1, \\ w_t^* &= \frac{w_t}{\left(1+\pi\right)^t}, \\ \tau_t^* &= \frac{\tau_t}{\left(1+\pi\right)^t}. \end{aligned}$$

5. Show that feasibility and law of motion equations are equivalent to

$$c_t^* + x_t^* + g_t^* = F\left(k_t^*, n_t^*\right),$$

and

$$(1+\lambda)(1+\pi)k_{t+1}^* = x_t^* + k_t^*(1-\delta).$$

Ans: Feasibility is given by,

$$\Lambda_t\left[c_t + x_t + g_t\right] = F\left(\Lambda_t k_t, \Lambda_t (1+\pi)^t n_t\right).$$

Since F is constant returns to scale, then

$$\Lambda_t\left[c_t + x_t + g_t\right] = \Lambda_t F\left(k_t, (1+\pi)^t n_t\right).$$

Dividing both sides of this expression by $(1+\pi)^t$ and using that F is CRS again we get

$$\frac{c_t}{(1+\pi)^t} + \frac{x_t}{(1+\pi)^t} + \frac{g_t}{(1+\pi)^t} = F\left(\frac{k_t}{(1+\pi)^t}, n_t\right),$$

or

$$c_t^* + x_t^* + g_t^* = F\left(k_t^*, n_t^*\right).$$

Rewrite the law of motion of capital as

$$\frac{\Lambda_{t+1}}{\Lambda_t}k_{t+1} = x_t + k_t(1-\delta),$$

or, using that $\Lambda_{t+1}/\Lambda_t = 1+\lambda$,

$$(1+\lambda)\frac{k_{t+1}}{(1+\pi)^{t+1}}(1+\pi) = \frac{x_t}{(1+\pi)^t} + \frac{k_t}{(1+\pi)^t}(1-\delta).$$

Thus,

$$(1+\lambda)(1+\pi)k_{t+1}^* = x_t^* + k_t^*(1-\delta).$$

6. Show that the following budget constraint for the family

$$\begin{aligned} &\sum_{t=0}^{\infty} p_t \Lambda_t\left[c_t + x_t + (1-\tau_{tl})w_t l_t + \tau_t\right] \\ = &\sum_{t=0}^{\infty} p_t \Lambda_t\left[w_t(1-\tau_{tl}) + k_t v_t (1-\tau_{kt}) + k_t \delta \tau_{kt}\right], \end{aligned}$$

is equivalent to

$$\begin{aligned} &\sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t\left[c_t^* + x_t^* + (1-\tau_{tl})w_t^* l_t^* + \tau_t^*\right] \\ = &\sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t\left[w_t^*(1-\tau_{tl}) + k_t^*\left(v_t(1-\tau_{kt}) + \delta\tau_{kt}\right)\right]. \end{aligned}$$

Ans: To get the result just multiply and divide each term in the sum by $(1+\pi)^t$ an apply the definition of the star variables.

7. Show, by an arbitrage argument, that collecting all the terms involving k^*_{t+1} for $t \geq 0$ in the budget constraint, such as

$$-p_t\Lambda_t (1+\lambda) + p_{t+1}\Lambda_{t+1}\left[(1-\delta) + (v_{t+1}(1-\tau_{kt+1}) + \delta\tau_{kt+1})\right],$$

then in equilibrium if $0 < k^*_{t+1} < \infty$ it must be that

$$v_{t+1} = \delta + \frac{r_t}{1-\tau_{kt+1}}.$$

Use this result to show that the budget constraint of the household, in equilibrium, must be

$$\begin{aligned}
& \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t \left[c^*_t + (1-\tau_{tl}) w^*_t l^*_t + \tau^*_t\right] \\
= \quad & \Lambda_0 k^*_0 (1 + (v_0 - \delta)(1-\tau_{k0})) + \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t \left[w^*_t (1-\tau_{tl})\right].
\end{aligned}$$

Ans: Introduce x^* into the budget constraint to get

$$\begin{aligned}
& \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t \left[c^*_t + (1+\lambda)(1+\pi) k^*_{t+1} - k^*_t (1-\delta) + (1-\tau_{tl}) w^*_t l^*_t + \tau^*_t\right] \\
= \quad & \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t \left[w^*_t (1-\tau_{tl}) + k^*_t \left[v_t (1-\tau_{kt}) + \delta\tau_{kt}\right]\right].
\end{aligned}$$

Consider the terms involving only capital:

$$\begin{aligned}
& \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t \left[(1+\lambda)(1+\pi) k^*_{t+1} - k^*_t (1-\delta)\right] - \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t k^*_t \left[v_t (1-\tau_{kt}) + \delta\tau_{kt}\right] \\
= \quad & \sum_{t=0}^{\infty} p_t\Lambda_t (1+\lambda)(1+\pi)^t (1+\pi) k^*_{t+1} - \sum_{t=0}^{\infty} p_t\Lambda_t (1+\pi)^t k^*_t \left[1 - \delta + v_t (1-\tau_{kt}) + \delta\tau_{kt}\right] \\
= \quad & \sum_{t=0}^{\infty} p_t\Lambda_{t+1} (1+\pi)^{t+1} k^*_{t+1} - \sum_{t=1}^{\infty} p_t\Lambda_t (1+\pi)^t k^*_t \left[1 + (v_t - \delta)(1-\tau_{kt})\right] \\
& -p_0\Lambda_0 k^*_0 \left[1 - \delta + v_0 - \tau_{k0}(v_t - \delta)\right] \\
= \quad & \sum_{t=0}^{\infty} p_t\Lambda_{t+1} (1+\pi)^{t+1} k^*_{t+1} - \sum_{t=0}^{\infty} p_{t+1}\Lambda_{t+1} (1+\pi)^{t+1} k^*_{t+1} \left[1 + (v_t - \delta)(1-\tau_{kt})\right] \\
& -p_0\Lambda_0 k^*_0 \left[1 + (v_0 - \delta)(1-\tau_{k0})\right] \\
= \quad & \sum_{t=0}^{\infty} p_t\Lambda_{t+1} (1+\pi)^{t+1} k^*_{t+1} \left\{1 - \frac{p_{t+1}}{p_t}\left[1 + (v_t - \delta)(1-\tau_{kt})\right]\right\} \\
& -p_0\Lambda_0 k^*_0 \left[1 + (v_0 - \delta)(1-\tau_{k0})\right].
\end{aligned}$$

Thus, the household's budget constraint becomes

$$\sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t \left[c_t^* + (1-\tau_{tl}) w_t^* l_t^* + \tau_t^* - w_t^* (1-\tau_{tl})\right] - p_0 \Lambda_0 k_0^* \left[1 + (v_0 - \delta)(1-\tau_{k0})\right]$$
$$+ \sum_{t=0}^{\infty} p_t \Lambda_{t+1} (1+\pi)^{t+1} k_{t+1}^* \left\{1 - \frac{p_{t+1}}{p_t}\left[1 + (v_t - \delta)(1-\tau_{kt})\right]\right\} = 0.$$

The same argument used before can be applied here to show that no-arbitrage implies:

$$1 = \frac{p_{t+1}}{p_t}\left[1 + (v_t - \delta)(1-\tau_{kt})\right],$$

or, using the definition $p_t/p_{t+1} \equiv 1 + r_t$,

$$v_t = \delta + \frac{r_t}{(1-\tau_{kt})}. \tag{22}$$

Moreover, the household's budget constraint becomes

$$\begin{aligned} & \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t \left[c_t^* + (1-\tau_{tl}) w_t^* l_t^* + \tau_t^* - w_t^* (1-\tau_{tl})\right] \\ = & \sum_{t=0}^{\infty} p_t \Lambda_t (1+\pi)^t w_t^* (1-\tau_{tl}) + p_0 \Lambda_0 k_0^* \left[1 + (v_0 - \delta)(1-\tau_{k0})\right]. \end{aligned}$$

8. Using preferences of the form

$$v(c,l) = c^{1-\gamma} h(l) / (1-\gamma),$$

for $\gamma > 0$, $\gamma \neq 1$, and

$$v(c,l) = \log c + h(l),$$

for $\gamma = 1$, show that

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, l_t) = \sum_{t=0}^{\infty} \Lambda_t \beta^{*t} v(c_t^*, l_t),$$

where

$$\beta^* = \beta (1+\pi)^{1-\gamma},$$

for $\gamma > 0$, $\gamma \neq 1$and,

$$\sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t, l_t) = \log(1+\pi) \sum_{t=0}^{\infty} \Lambda_t \beta^t t + \sum_{t=0}^{\infty} \Lambda_t \beta^t v(c_t^*, l_t),$$

for $\gamma = 1$.

Ans: Consider first the case $v(c,l) = c^{1-\gamma} h(l) / (1-\gamma)$:

$$
\begin{aligned}
\sum_{t=0}^{\infty} \Lambda_t \beta^t v\left(c_t, l_t\right) &= \sum_{t=0}^{\infty} \Lambda_t \beta^t c_t^{1-\gamma} h\left(l_t\right) / (1-\gamma) \\
&= \sum_{t=0}^{\infty} \Lambda_t \left(\beta (1+\pi)^{1-\gamma}\right)^t \left(\frac{c_t}{(1+\pi)^t}\right)^{1-\gamma} h\left(l_t\right) / (1-\gamma) \\
&= \sum_{t=0}^{\infty} \Lambda_t \beta^{*t} c_t^{*1-\gamma} h\left(l_t\right) / (1-\gamma) \\
&= \sum_{t=0}^{\infty} \Lambda_t \beta^{*t} v\left(c_t^*, l_t\right).
\end{aligned}
$$

Now, take the case $v(c,l) = \log c + h(l)$:

$$
\begin{aligned}
\sum_{t=0}^{\infty} \Lambda_t \beta^t v\left(c_t, l_t\right) &= \sum_{t=0}^{\infty} \Lambda_t \beta^t \left[\log c_t + h\left(l_t\right)\right] \\
&= \sum_{t=0}^{\infty} \Lambda_t \beta^t \left[\log c_t + h\left(l_t\right) + \log (1+\pi)^t - \log (1+\pi)^t\right] \\
&= \sum_{t=0}^{\infty} \Lambda_t \beta^t \log (1+\pi)^t + \sum_{t=0}^{\infty} \Lambda_t \beta^t \left(\log \frac{c_t}{(1+\pi)^t} + h\left(l_t\right)\right) \\
&= \log (1+\pi) \sum_{t=0}^{\infty} \Lambda_t \beta^t t + \sum_{t=0}^{\infty} \Lambda_t \beta^t \left[\log c_t^* + h\left(l_t\right)\right] \\
&= \log (1+\pi) \sum_{t=0}^{\infty} \Lambda_t \beta^t t + \sum_{t=0}^{\infty} \Lambda_t \beta^t v\left(c_t^*, l_t\right).
\end{aligned}
$$

Using the above preferences and the intertemporal budget constraint, the household's first order conditions are

$$\mu \Lambda_t p_t (1+\pi)^t = \Lambda_t \left(\beta^*\right)^t v_c\left(c_t^*, l_t^*\right),$$

$$\mu \Lambda_t p_t w_t^* \left(1-\tau_{tl}\right)(1+\pi)^t = \Lambda_t \left(\beta^*\right)^t v_l\left(c_t^*, l_t^*\right),$$

where μ is the Lagrange multiplier on the budget constraint. Thus, the marginal rate of substitutions are

$$\beta (1+\pi)^{-\gamma} \left(1+r_t\right) \frac{v_c\left(c_{t+1}^*, l_{t+1}^*\right)}{v_c\left(c_t^*, l_t^*\right)} = 1, \tag{23}$$

and

$$\frac{v_l\left(c_t^*, l_t^*\right)}{v_c\left(c_t^*, l_t^*\right)} = \left(1-\tau_{tl}\right) w_t^*. \tag{24}$$

9. Show that the vector $(r, w, v, x,\ k, l, \kappa, c)$ is a steady state, that is

$$\begin{aligned} g_t^* &= g, \\ c_t^* &= c, \\ x_t^* &= x, \\ k_t^* &= k, \\ l_t^* &= l, \\ n_t^* &= n, \end{aligned}$$

and

$$\begin{aligned} r_t^* &= r, \\ w_t^* &= w, \\ \tau_t^* &= \tau, \end{aligned}$$

if and only if it satisfies the following equations:

$$\begin{aligned} 1+r &= (1+\rho)(1+\pi)^{\gamma} \text{ or } r \cong \rho + \gamma\pi, \\ x &= [(1+\lambda)(1+\pi) - (1-\delta)]\, k \text{ or } x \cong (\lambda + \delta + \pi)\, k, \\ v &= \delta + \frac{r}{1-\tau_k}, \\ v &= F_k(\kappa, 1), \\ w &= F_l(\kappa, 1), \\ w(1-\tau_l) &= c\frac{h'(l)}{(1-\gamma)\, h(l)}, \\ g + c &= F(\kappa, 1)(1-l) - [(1+\lambda)(1+\pi) - (1-\delta)]\, \kappa (1-l) \text{ or} \\ g + c &\cong F(\kappa, 1)(1-l) - (\delta + \lambda + \pi)\, \kappa (1-l), \\ \kappa &= \frac{k}{1-l}, \end{aligned}$$

where the symbol $\cong$ is interpreted as a first order approximation around $\rho = \pi = \lambda = 0$ of the relevant expression.

Ans: The Euler equation (23) evaluated at the steady state becomes

$$\frac{(1+\pi)^{-\gamma}}{1+\rho}(1+r)\frac{v_c(c,l)}{v_c(c,l)} = 1.$$

Thus,

$$1 + r = (1+\rho)(1+\pi)^{\gamma}.$$

Taking logs of the previous expression,

$$\log(1+r) = \log(1+\rho) + \gamma \log(1+\pi),$$

or, using that $\log(1+x) \cong x$ for x small,

$$r \cong \rho + \gamma\pi.$$

At the steady state (22) becomes

$$v = \delta + \frac{r}{1-\tau_k}. \tag{25}$$

Moreover, from the firm's problem (and the homogeneity of degree zero of F_k),

$$v = F_k(\kappa, 1).$$

Thus, we can determine the capital to labor ratio $\kappa \equiv k/(1-l)$ as the solution to

$$\delta + \frac{r}{1-\tau_k} = F_k(\kappa, 1). \tag{26}$$

The κ that solves the previous equation can be used to find the steady state wage rate using the firm's FOC w.r.t. n:

$$w = F_n(\kappa, 1). \tag{27}$$

Also, in steady state $k_t^* = k_{t+1}^* = k$. Hence, from the definition of x,

$$\begin{aligned} x &= [(1+\lambda)(1+\pi) - (1-\delta)]\,k \\ &= [(1+\lambda)(1+\pi) - (1-\delta)]\,\kappa(1-l) \\ &\cong [\lambda + \pi + \delta]\,\kappa(1-l). \end{aligned}$$

Substituting the previous equation into the feasibility constraint we obtain

$$g + c = F(\kappa, 1)(1-l) - [(1+\lambda)(1+\pi) - (1-\delta)]\,\kappa(1-l), \tag{28}$$

or

$$g + c \cong F(\kappa, 1)(1-l) - [\lambda + \pi + \delta]\,\kappa(1-l).$$

Finally, the intratemporal condition (24) becomes

$$w(1-\tau_l) = c\frac{h'(l)}{(1-\gamma)h(l)}. \tag{29}$$

Given κ, which is obtained from (26) and (25), we can use (27) to obtain the steady state wage. Then, the system of equations given by (28) and (29) can be used to find the steady state values for consumption and leisure, c and l. Finally, k can be recovered from $\kappa = k/(1-l)$.

10. Assume that

$$\begin{aligned} v(c,l) &= \left(c^{1-\alpha}l^{\alpha}\right)^{1-\phi}/(1-\phi) = c^{(1-\alpha)(1-\phi)}l^{\alpha(1-\phi)}/(1-\phi), \\ F(k,n) &= k^{\theta}n^{1-\theta}, \end{aligned}$$

so that

$$\gamma = \alpha + \phi(1-\alpha).$$

Take the benchmark values

$$\begin{aligned}
\rho &= 0.01, \\
\gamma &= 2, \\
\alpha &= 0.7, \\
\pi &= 0.02, \\
\lambda &= 0.01, \\
\delta &= 0.05, \\
\theta &= 0.3, \\
g/y &= 0.14, \\
\tau_k &= 0.30, \\
\tau_l &= 0.20.
\end{aligned}$$

Then a steady state is given by

$$\begin{aligned}
r &\cong \rho + \gamma\pi = 0.01 + 2 \times 0.02 = 0.05, \\
v &\cong \delta + \frac{r}{1-\tau_k} = 0.05 + \frac{0.05}{0.7} = 0.05 + 0.07 = 0.12, \\
\frac{k}{y} &\cong \frac{\theta}{v} = \frac{0.3}{0.12} = 2.5, \\
\frac{x}{k} &\cong (\delta + \pi + \lambda) = 0.05 + 0.02 + 0.01 = 0.08, \\
\frac{x}{y} &\cong \frac{x}{k}\frac{k}{y} = 0.08 \times 2.5 = 0.2, \\
\frac{c}{y} &\cong 1 - \frac{g}{y} - \frac{x}{y} = 1 - 0.14 - 0.2 = 0.66, \\
l &\cong 1/\left(1 + \frac{(1-\theta)}{c/y}(1-\tau_l)\frac{1-\alpha}{\alpha}\right) = 1/\left(1 + \frac{0.7}{0.66}0.8\frac{0.3}{0.7}\right), \\
l &\cong 0.73, \ n \cong .27, \\
k &= \left[\frac{v}{\theta}\right]^{\frac{1}{\theta-1}} n = \left[\frac{0.12}{0.3}\right]^{\frac{1}{-0.7}} 0.27 = [.4]^{-\frac{1}{0.7}} 0.27.
\end{aligned}$$

Recompute the values for r, v, k/y, x/y, $c/y, l$ and k for
i) $\tau_k = 0.2$, $\tau_l = 0.2$ and $g/y = 0.14$,
ii) $\tau_k = 0.3$, $\tau_l = 0.3$ and $g/y = 0.14$,
iii) $\tau_k = 0.3$, $\tau_l = 0.2$ and $g/y = 0.16$, and
iv) $\tau_k = 0.2$, $\tau_l = 0.3$ and $g/y = 0.16$.

Ans:

Variable \ Parameters	$\tau_l = 0.20$ $\tau_k = 0.30$ $\frac{g}{y} = 0.14$	$\tau_l = 0.20$ $\tau_k = 0.20$ $\frac{g}{y} = 0.14$	$\tau_l = 0.30$ $\tau_k = 0.30$ $\frac{g}{y} = 0.14$	$\tau_l = 0.30$ $\tau_k = 0.20$ $\frac{g}{y} = 0.16$	$\tau_l = 0.20$ $\tau_k = 0.30$ $\frac{g}{y} = 0.16$
Interest rate, r	0.050	0.050	0.050	0.050	0.050
Before-tax rental rate, v	0.121	0.113	0.121	0.113	0.121
Capital-output ratio, k/y	2.471	2.667	2.471	2.667	2.471
Investme-outptut ratio, x/y	0.198	0.213	0.198	0.213	0.198
Consum-output ratio, c/y	0.662	0.647	0.662	0.627	0.642
Leisure, l	0.734	0.729	0.759	0.749	0.728
Labor, n	0.266	0.271	0.241	0.251	0.272
Capital stock, k	0.968	1.099	0.876	1.019	0.990

2 Ricardian Equivalence does not imply that the timing of DISTORTING taxes is immaterial

So far in class we considered either lump sum taxes, constant income taxes in the case with no variable labor supply, or taxation in steady sate. With lump sum taxes, the only important feature of the tax system, as we have seen, is the present value of taxes. In this case, the timing of taxation was immaterial. To avoid that you get the incorrect notion that this is the whole story, I include here some simple analysis of distorting taxation with variable labor supply. In this case there are powerful forces that indicates that it is better, from a welfare perspective, to keep the tax rates constant trough time. To simplify the point I have excluded capital accumulation (including capital accumulation so that we can talk about capital taxation is very important, but capital accumulation brings a different set of issues so it is better to treat that important case elsewhere). The analysis here follows closely Lucas and Stokey's JME paper.

2.1 Tax smoothing in the Ramsey model. Warming up with a simple case

Assume that all agents are identical and have utility function given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right),$$

where c_t is consumption and l_t is labor supply and $\beta \in (0,1)$. There is no capital in this economy, the production technology is linear in labor and each period the government purchases a variable amount $g_t \leq 1$ of consumption goods, which must be financed using a linear income tax of rate τ_t. Thus, the resource constraint is $c_t + g_t = l_t$ for $t \geq 0$. Competitive firms pay a wage equal to marginal productivity, 1, and households receive after-tax wages of $(1-\tau_t)$. The government has no other expenditures and starts in period $t=0$ with no assets or liabilities. The government chooses $\{\tau_t\}$ to maximize the representative agent utility, subject to being able to finance its purchases $\{g_t\}$ with taxes $\{\tau_t\}$. In this case the Arrow-Debreu budget constraint of a household and the government are

$$\sum_{t=0}^{\infty} p_t \left[c_t - l_t \left(1-\tau_t\right)\right] = 0, \text{ and } \sum_{t=0}^{\infty} p_t \left[l_t \tau_t - g_t\right] = 0,$$

where p_t is the price of a unit of consumption at time t in terms of time zero consumption goods.

a. Write down the FOCs for the household problem using θ for the Lagrange multiplier of the budget constraint.

Ans:

$$\begin{aligned} c_t &: \quad \beta^t u_c\left(c_t, l_t\right) = \theta p_t, \\ l_t &: \quad -\beta^t u_l\left(c_t, l_t\right) = \theta p_t\left(1-\tau_t\right). \end{aligned}$$

Consider the utility function

$$u\left(c, l\right) = c - \frac{1}{1+\phi} l^{\phi+1},$$

for $\phi > 0$.

b. Use a. to argue that if the solution of the household problem is interior, $p_t = \beta^t$, after-tax wages at time t satisfy $1-\tau_t = l_t^{\phi}$, and time t tax revenues satisfy $\tau_t l_t = l_t - l_t^{1+\phi}$.

Ans: With the assumed functional form for u the FOCs become

$$\begin{aligned} c_t &: \quad \beta^t = \theta p_t, \\ l_t &: \quad \beta^t l_t^{\phi} = \theta p_t\left(1-\tau_t\right). \end{aligned}$$

The first equation implies that $p_t/p_0 = \beta^t$, or normalizing p_0 to one (which implies that $\theta = 1$), $p_t = \beta^t$. Thus, from the second equation we obtain $l_t^{\phi} = 1-\tau_t$. Finally, tax revenues are $\tau_t l_t = \left(1-l_t^{\phi}\right) l_t = l_t - l_t^{1+\phi}$.

c. Replace the expressions obtained in b. for tax revenues and $\{p_t\}$ in the budget constraint of the government. Your answer should be one equation that depend on the sequences $\{g_t, l_t\}$ and the parameters β and ϕ.

Ans:

$$\sum_{t=0}^{\infty} \beta^t \left[l_t - l_t^{1+\phi} - g_t\right] = 0.$$

d. Replace c_t using the resource constraint in the agent's utility function. Your answer should be one expression that depends on the sequences $\{g_t, l_t\}$ and the parameters β and ϕ.

Ans: The agent's utility function is

$$\sum_{t=0}^{\infty} \beta^t \left[c_t - \frac{1}{1+\phi} l_t^{1+\phi} \right].$$

Thus,

$$\sum_{t=0}^{\infty} \beta^t \left[l_t - g_t - \frac{1}{1+\phi} l_t^{1+\phi} \right].$$

e. Using c. and d. consider the welfare consequences of two fiscal policies. Assume that the allocations are interior. In the first policy the budget is balanced period by period. In the second policy, tax rates are constant. Which one is better? Why?

Ans: Consider a benevolent government who wishes to find the sequence of optimal labor supply $\{l_t\}_{t=0}^{\infty}$ that maximizes the lifetime utility of the agent, such that this sequence allows the government to finance its expenditures. From c. and d., this problem consists of

$$\max_{\{l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[l_t - g_t - \frac{1}{1+\phi} l_t^{1+\phi} \right],$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \left[l_t - l_t^{1+\phi} - g_t \right] = 0.$$

Letting μ denote the multiplier associated with the budget constraint, the FOC associated with this problem can be written as

$$l_t : 1 - l_t^{\phi} + \mu \left[1 - (1+\phi) \, l_t^{\phi} \right] = 0.$$

Thus,

$$l_t = l^* = \left(\frac{1+\mu}{1+\mu\,(1+\phi)} \right)^{1/\phi},$$

where we have used that the RHS of the above expression is independent of time. Since $\tau_t = 1 - l_t^{\phi}$, we then see that the best tax policy is one where labor supply and tax rates are constant over time. This result is due to the fact that the marginal utility of consumption is constant, so that it is optimal to let consumption pick up any variations in government expenditures and, hence, keep labor supply constant over time. Of course, this implies that a policy of balanced budget is typically suboptimal (unless government expenditures are constant over time, in which case both policies would be equivalent).

Notice that when $\mu = 0$, so that the government has nothing to finance, labor coincides with its optimal undistorted level, $l^* = 1$. Moreover, as $\mu \to \infty$, so that the government must finance a large amount of expenditures, optimal labor supplies approaches $(1/(1+\phi))^{1/\phi}$ which, not surprisingly, happens to be the labor supply that maximizes total tax revenues, $l_t - l_t^{1+\phi}$.

d. What is the best policy among all tax policies (assume that the allocations are all interior)? How is labor supply l_t during time periods with different government purchases when the best policy is followed?

Ans: Above.

2.2 Tax smoothing in a Ramsey problem. A bit more general result

Consider the same model as in the previous question.

i. Write down (again!) the time zero Arrow-Debreu budget constraint of a household. Use p_t for price of a unit of consumption at time t in terms of zero consumption goods for arbitrary taxes τ_t and intertemporal prices p_t. Your answer should take 1 line.

Ans:

$$\sum_{t=0}^{\infty} p_t \left[c_t - l_t \left(1 - \tau_t\right)\right] = 0. \tag{30}$$

ii. Write down (again!) the time zero Arrow-Debreu budget constraint of the government. Your answer should take 1 line.

Ans:

$$\sum_{t=0}^{\infty} p_t \left[l_t \tau_t - g_t\right] = 0. \tag{31}$$

iii. Write down (again!) the first order conditions for c_t and l_t all t, for the household problem, for arbitrary taxes τ_t and intertemporal prices p_t. These are two equations for each t; use θ for the Lagrange multiplier of the budget constraint of the household.

Ans:

$$\begin{aligned} c_t &: \quad \beta^t u_c\left(c_t, l_t\right) = \theta p_t, \\ l_t &: \quad -\beta^t u_l\left(c_t, l_t\right) = \theta p_t \left(1 - \tau_t\right). \end{aligned}$$

iv. Define a competitive equilibrium with taxes for this economy (list all the objects and the conditions that they must satisfy, your answer should take no more than 3 lines).

Ans: A competitive equilibrium with taxes is a sequence $\{c_t, l_t, p_t, \tau_t, g_t\}_{t=0}^{\infty}$ such that

1. Given $\{p_t, \tau_t\}_{t=0}^{\infty}$, the sequence $\{c_t, l_t\}_{t=0}^{\infty}$ maximizes the agent's utility subject to (30);
2. The budget constraint of the government, equation (31), holds; and
3. The goods market clears: $c_t + g_t = l_t$, for all $t \geq 0$.

v. Use i., ii. and iii. to derive an equation that we will call "implementability constraint". The implementability constraint is the budget constraint of the agent where the prices p_t and after-tax wages $1 - \tau_t$ have been replaced by the marginal rate of substitution from the agent's problem. Thus, the implementability constraint is a function of $\{c_t, l_t\}$; no $p_t's$ or $\tau_t's$ should be in this expression. The implementability constraint is

$$\sum_{t=0}^{\infty} \beta^t \left[u_c\left(c_t, l_t\right) c_t + u_l\left(c_t, l_t\right) l_t \right] = 0.$$

[Hint: Use i. and iii. to derive this equation. Your answer should take no more than five lines].

Ans: From iii. we obtain

$$\begin{aligned} \frac{1}{\theta} \beta^t u_c\left(c_t, l_t\right) &= p_t, \\ -\frac{1}{\theta} \beta^t u_l\left(c_t, l_t\right) &= p_t\left(1 - \tau_t\right). \end{aligned}$$

Using this result in (30) it follows that

$$\sum_{t=0}^{\infty} \beta^t \left[u_c\left(c_t, l_t\right) c_t + u_l\left(c_t, l_t\right) l_t \right] = 0. \tag{32}$$

vi. Show that if an allocation $\{c_t, l_t\}_{t=0}^{\infty}$ is resource feasible (i.e., $c_t + g_t = l_t$ for $t \geq 0$) and satisfies the implementability constraint in v., it is a competitive equilibrium with taxes. [Hint: Use the given allocation, i. and iii. to find p_t, τ_t and θ]. Your answer should take no more than five lines.

Ans: Since the allocation $\{c_t, l_t\}_{t=0}^{\infty}$ is resource feasible, we only need to check that the first two conditions in iv. hold. Let

$$\begin{aligned} p_t &= \beta^t u_c(c_t, l_t), \\ 1-\tau_t &= -\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)}, \\ \theta &= u_c(c_0, l_0). \end{aligned}$$

Then, it is straightforward to check that the allocation $\{c_t, l_t\}_{t=0}^{\infty}$ satisfies the FOCs of the agent's problem as well as the budget constraint, so condition iv.1 is satisfied. Since the goods market clears, by Walras' law this implies that the government budget constraint holds as well, so condition iv.2 is also satisfied. Hence, $\{c_t, l_t\}_{t=0}^{\infty}$ is a competitive equilibrium with taxes.

vii. The Ramsey problem, as explained above, is to choose taxes $\{\tau_t\}$ such that the corresponding competitive equilibrium maximizes the utility of the representative agent. In view of the previous results, this problem is then

$$\max_{\{c_t, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

subject to the implementability constraint derived in v. and the resource constraint $c_t + g_t = l_t$ for all $t \geq 0$. Use iv. to derive the FOC for the Ramsey problem w.r.t. c_t and l_t. Use λ for the Lagrange multiplier of the implementability constraint and $\beta^t \mu_t$ for the Lagrange multiplier of resource constraint at time t.

Ans: The Lagrangian of the Ramsey problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, l_t) + \mu_t (l_t - c_t - g_t) + \lambda \left[u_c(c_t, l_t) c_t + u_l(c_t, l_t) l_t \right] \right\}.$$

FOCs:

$$\begin{aligned} c_t &: \beta^t \left\{ u_c(c_t, l_t) - \mu_t + \lambda \left[u_{cc}(c_t, l_t) c_t + u_c(c_t, l_t) + u_{cl}(c_t, l_t) l_t \right] \right\} = 0, \\ l_t &: \beta^t \left\{ u_l(c_t, l_t) + \mu_t + \lambda \left[u_{cl}(c_t, l_t) c_t + u_l(c_t, l_t) + u_{ll}(c_t, l_t) l_t \right] \right\} = 0, \end{aligned}$$

or

$$\begin{aligned} (1+\lambda) u_c(c_t, l_t) + \lambda \left[u_{cc}(c_t, l_t) c_t + u_{cl}(c_t, l_t) l_t \right] - \mu_t &= 0, \\ (1+\lambda) u_l(c_t, l_t) + \lambda \left[u_{cl}(c_t, l_t) c_t + u_{ll}(c_t, l_t) l_t \right] + \mu_t &= 0. \end{aligned}$$

viii. Consider the case where $g_t = 0$ for all $t \geq 0$, except for some date $s > 0$. Solve for the Ramsey allocation. Use the FOCs for the Ramsey problem derived in vii. to argue that this allocation has $c_t = \bar{c}$, $l_t = \bar{l}$ for all $t \neq s$.

Show that then $\tau_t = \bar{\tau}$ for all dates $t \neq s$. Finally argue that $\bar{\tau} > 0$ in spite of the fact that $g_t = 0$ for $t \neq s$. What is the intuition for this result? Why taxes are positive even in periods when government purchases are zero, that is, why not just tax on the period s where $g_s > 0$? [Hint: To formally show this result, take an arbitrary period $t \neq s$. Multiply the FOC of the Ramsey problem w.r.t. c_t by c_t, and the FOC w.r.t. l_t by l_t and add them up for each t. Use the resource constraint and the fact that $g_t = 0$ to eliminate the terms containing the multiplier μ_t. Notice that this sum has a term which is a quadratic form of u, and hence it can be signed as negative. Using that the multiplier of the implementability constraint, λ, is positive, this implies that other term, $u_c(t) + u_l(t)$ is positive. Use the FOC for the agent's problem to show that this implies that $\tau_t > 0$].

Ans: Adding the FOCs and using that $l_t = c_t + g_t$, yields

$$(1+\lambda)(u_c + u_l) + \lambda\left[(u_{cc} + u_{cl})c_t + (u_{cl} + u_{ll})(c_t + g_t)\right] = 0,$$

where the first and second derivatives of u are evaluated at $(c_t, c_t + g_t)$. This condition gives an equation for c_t for each t as a function of λ and g_t only. Since λ is time invariant, $g_t = 0$ for all $t \neq s$ and $g_s > 0$, this immediately implies that $c_t = \bar{c}$ for all $t \neq s$ and $c_s = \hat{c}$. By feasibility, it follows that $l_t = \bar{l}$ for all $t \neq s$ and $l_s = \hat{l}$. Finally, since $1 - \tau_t = -u_l(c_t, l_t)/u_c(c_t, l_t)$, we also have that $\tau_t = \bar{\tau}$ for all $t \neq s$ and $\tau_s = \hat{\tau}$.

Now, multiplying the first FOC by c_t and the second FOC by l_t and adding the result we obtain

$$(1+\lambda)(u_c c_t + u_l l_t) + \lambda\left[u_{cc}c_t^2 + 2u_{cl}c_t l_t + u_{ll}l_t^2\right] + \mu_t(l_t - c_t) = 0,$$

or, using that $l_t = c_t$ for all $t \neq s$,

$$(1+\lambda)(u_c + u_l)c_t + \lambda\left[u_{cc}c_t^2 + 2u_{cl}c_t l_t + u_{ll}l_t^2\right] = 0.$$

Since $\lambda > 0$ and the terms in brackets is strictly negative, it follows that $u_c + u_l > 0$. But from vi. we know that $\tau_t = (u_c + u_l)/u_c$, so we conclude that $\tau_t = \bar{\tau} > 0$ for all $t \neq s$ and $\tau_s = \hat{\tau}$.

We thus see that taxes are positive in periods when there is no government spending. This is due to the fact that the Ramsey planner wishes to minimize the deadweight losses associated with distortionary taxation. Due to the concavity of the utility function, this is best accomplished by setting small but positive and constant tax rates in all periods, as opposed to just one large tax rate when government purchases are carried out.

ix. Consider the utility function

$$u(c_t, l_t) = \frac{1}{1-\alpha}c^{1-\alpha} - \frac{1}{1+\phi}l_t^{\phi+1},$$

for $1 \neq \alpha > 0$ and $\phi > 0$. Show that if the allocation is interior, i.e. c_t and $l_t > 0$ all t, then the Ramsey allocation has *constant income taxes* τ_t for all t.

What is the intuition for this result? How do l_t and c_t move in periods where g_t is bigger? [Hint: To formally show this result, take an arbitrary period t. Add the FOC for c_t and l_t to eliminate μ_t, use the functional form of u to relate $u_{cc}c$ to u_c and $u_{ll}l$ to u_l. Finally compare the marginal rate of substitution between c_t and l_t to determine that it is the same for any period t and, hence, tax rates are the same for all time periods].

Ans: With the assumed functional form for u the FOCs become

$$\begin{aligned} (1+\lambda) u_c(c_t) + \lambda u_{cc}(c_t) c_t - \mu_t &= 0, \\ (1+\lambda) u_l(l_t) + \lambda u_{ll}(l_t) l_t + \mu_t &= 0, \end{aligned}$$

or, adding them up and using that $u_{cc}c = -\alpha u_c$ and $u_{ll}l = \phi u_l$,

$$(1+\lambda) u_c(c_t) - \alpha\lambda u_c(c_t) + (1+\lambda) u_l(l_t) + \phi\lambda u_l(l_t) = 0,$$

or

$$-\frac{u_l(l_t)}{u_c(c_t)} = \frac{1+\lambda(1-\alpha)}{1+\lambda(1+\phi)}.$$

Since the RHS is independent of time, it follows that the MRS between c_t and l_t is independent of time as well, so the Ramsey allocation has constant income taxes τ_t for all t. This is the classic uniform commodity taxation result, which here is due to the fact that the elasticity of substitution of labor supply is constant over time.

Finally, since $-u_l$ is strictly increasing in l while u_c is strictly decreasing in c, consumption and leisure move in opposite directions (if they moved in the same direction, the MRS between c_t and l_t would not be constant over time). In particular, labor supply rises and consumption falls in periods of high government spending.

3 Ramsey Taxation in the Lucas-Uzawa Endogenous Growth Model

We consider here a version of the Lucas-Uzawa model of section 1 where we introduce distorting taxation and leisure. Our aim is to analyze the optimal Ramsey taxation in this set-up in light of the result of zero capital taxation for the standard neoclassical growth model. In this model labor income is proportional to human capital, which itself is accumulated, so even labor income has features similar to capital in the standard neoclassical growth model.

Let preferences be given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, l_{1t} + l_{2t}), \tag{33}$$

where c_t is consumption, l_{1t} time working and l_{2t} time learning. We assume that U is strictly concave in its two arguments $U(c,l)$, strictly increasing in c, strictly decreasing in l, and satisfy standard Inada conditions.

The budget constraint of the household is

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1}\right] = \sum_{t=0}^{\infty} p_t \left[\left(1-\tau_t\right) w_t h_t l_{1t} + k_t R_{k,t}\right], \tag{34}$$

and

$$R_{kt} = 1 + \left(1-\theta_t\right)\left(r_t - \delta\right) \quad \text{for all } t \geq 0, \tag{35}$$

where p_t is the Arrow-Debreu price of a consumption good at time t, w_t the real wage, τ_t the tax rate on wages, h_t the human capital of the household, so $l_{1t}h_t$ gives the total labor services offered by the household, and $w_t\left(1-\tau_t\right)$ the after-tax wage for each unit of labor services. We use k_t for the physical capital, R_{kt} for the gross after-tax return on capital, r_t the rental rate of capital, δ the depreciation rate of capital, and θ_t the tax rate on the net rental rate of capital.

The human capital accumulation technology of the households is described by

$$h_{t+1} = G\left(l_{2t}\right) h_t + \left(1-\delta_h\right) h_t, \tag{36}$$

for all $t \geq 0$, so that $\left[G\left(l_{2t}\right) + 1 - \delta_h\right]$ gives the (gross) growth rate of human capital as a function of l_{2t}, the time devoted to learning, and δ_h, the depreciation rate of human capital. We assume that G is positive, and unless otherwise indicated, we assume that G is strictly increasing, strictly concave, and satisfies standard Inada conditions.

The household problem is to maximize (33) subject to (34), (35) and (36), given h_0 and k_0.

The firm's problem is

$$\max_{k_t, n_t} F\left(n_t, k_t\right) - k_t r_t - w_t n_t, \tag{37}$$

where n_t are the labor services demanded and k_t the capital services demanded. We assume that F has constant returns to scale and it is neoclassical (strictly quasi-concave, satisfying Inada conditions).

Feasibility is given by

$$g_t + c_t + k_{t+1} = F\left(l_{1t}h_t, k_t\right) + \left(1-\delta\right) k_t, \tag{38}$$

for all $t \geq 0$, where g_t are the government purchases at time t.

The government budget constraint is given by

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \left[\tau_t w_t h_t l_{1t} + \theta_t k_t \left(r_t - \delta\right)\right]. \tag{39}$$

Definition: A competitive equilibrium with taxes where the government finances the purchases $\{g_t\}$ is an allocation $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a price system $\{p_t, w_t, R_{kt}, r_t\}_{t=0}^{\infty}$, and taxes $\{\tau_t, \theta_t\}_{t=0}^{\infty}$, such that i) $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ is feasible for $\{g_t\}$ and initial conditions k_0, h_0, ii) $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ maximizes the household utility given prices $\{p_t, w_t, R_{kt}, r_t\}_{t=0}^{\infty}$, taxes $\{\tau_t, \theta_t\}_{t=0}^{\infty}$, and initial conditions k_0, h_0, iii) firms maximize profits given prices $\{w_t, r_t\}$, and iv) the government budget constraint holds.

A balanced-growth path is an equilibrium where l_{1t} and l_{2t} are constant and where h_t, c_t, k_t grow at a common constant rate, say γ. To allow the possibility of a balanced-growth path we specialize preferences to

$$\begin{aligned} U(c,l) &= \frac{[c\varphi(l)]^{1-\sigma}}{1-\sigma} \text{ for } \sigma \neq 1, \\ &= \log c + \varphi(l) \text{ for } \sigma = 1, \end{aligned} \tag{40}$$

for some function $\varphi(l)$ and where $\sigma > 0$.

1) Write down the FOCs for the household problem w.r.t. c_t, l_{1t}, l_{2t}, k_{t+1} and h_{t+1} for $t \geq 0$. Use λ_h for the multiplier on the household budget constraint (34)

$$\sum_{t=0}^{\infty} p_t \left[-c_t - k_{t+1} + (1-\tau_t) w_t h_t l_{1t} + k_t R_{k,t}\right] = 0,$$

and $\beta^t \phi_t$ for the multiplier on the household human capital accumulation technology (36)

$$-h_{t+1} + G(l_{2t}) h_t + (1-\delta_h) h_t = 0.$$

Ans: The Lagrangian of the household problem is

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t, l_{1t} + l_{2t}) + \phi_t \left[G(l_{2t}) h_t + (1-\delta_h) h_t - h_{t+1}\right] \right\} \\ & + \lambda_h \sum_{t=0}^{\infty} p_t \left[(1-\tau_t) w_t h_t l_{1t} + k_t R_{k,t} - c_t - k_{t+1}\right]. \end{aligned}$$

FOCs:

$$\begin{aligned} c_t &: \beta^t U_{ct} = \lambda_h p_t, \\ l_{1t} &: \beta^t U_{lt} = -\lambda_h p_t w_t (1-\tau_t) h_t, \\ l_{2t} &: \beta^t \left(U_{lt} + \phi_t G'(l_{2t}) h_t\right) = 0, \\ k_{t+1} &: \lambda_h \left(-p_t + p_{t+1} R_{k,t+1}\right) = 0, \\ h_{t+1} &: -\beta^t \phi_t + \beta^{t+1} \phi_{t+1} \left[G(l_{2t+1}) + (1-\delta_h)\right] + \lambda_h p_{t+1} w_{t+1} (1-\tau_{t+1}) l_{1t+1} = 0. \end{aligned}$$

2) Consider the following variational argument. Let $l'_{1t} = l_{1t} - \delta$, and $l'_{2t} = l_{2t} + \delta$, where the plan with $'$ is the variation on the original plan. In particular, analyze the effects of a small change on δ on the household budget constraint. Show that if in the original plan the household is maximizing utility, it must be that

$$p_t w_t (1-\tau_t) h_t = \frac{G'(l_{2t})}{\left[G(l_{2t}) + (1-\delta_h)\right]} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} l_{1t+s}, \tag{41}$$

for all $t \geq 0$. [Hint: Derive first an expression for dh_{t+s}/dl_{2t} using (36). Perturb l_{1t} and l_{2t} in the budget constraint (34) by $-\delta$ and δ, use that the original plan was optimal, and replace your expression for dh_{t+s}/dl_{2t}]. Give a two lines economic interpretation to each side of this equation. FYI: Lucas displays a continuous time version of this equation in his supply-side paper.

Ans: Solving (36) forward, we obtain

$$h_{t+s} = h_t \, \Pi_{r=0}^{s-1} \left[G\left(l_{2t+r}\right) + \left(1 - \delta_h\right) \right].$$

Then,

$$\begin{aligned} \frac{dh_{t+s}}{dl_{2t}} &= G'\left(l_{2t}\right) h_t \Pi_{r=1}^{s-1} \left[G\left(l_{2t+r}\right) + \left(1-\delta_h\right) \right] \\ &= \frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right) + \left(1-\delta_h\right) \right]} h_{t+s}. \end{aligned}$$

Now, since $l'_{1t} = l_{1t} - \delta$ and $l'_{2t} = l_{2t} + \delta$, then $dl_{1t}/d\delta + dl_{2t}/d\delta = 0$. Perturbing l_{1t} and l_{2t} in the budget constraint by $-\delta$ and δ respectively we obtain

$$p_t w_t \left(1 - \tau_t\right) h_t \frac{dl_{1t}}{d\delta} + \sum_{s=1}^{\infty} p_{t+s} \frac{dh_{t+s}}{dl_{2t}} \frac{dl_{2t}}{d\delta} w_{t+s} \left(1 - \tau_{t+s}\right) l_{1t+s} = 0,$$

where the equality follows because in an optimal solution the perturbation does not have any effect on total wealth (otherwise, we are not in an optimal point). Using $dl_{1t}/d\delta + dl_{2t}/d\delta = 0$ we get

$$p_t w_t \left(1 - \tau_t\right) h_t = \sum_{s=1}^{\infty} p_{t+s} \frac{dh_{t+s}}{dl_{2t}} w_{t+s} \left(1 - \tau_{t+s}\right) l_{1t+s},$$

or, using our expression for dh_{t+s}/dl_{2t}, we obtain

$$p_t w_t \left(1 - \tau_t\right) h_t = \frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right) + \left(1-\delta_h\right) \right]} \sum_{s=1}^{\infty} p_{t+s} h_{t+s} w_{t+s} \left(1 - \tau_{t+s}\right) l_{1t+s}.$$

3) Show that the FOCs of this problem imply equation (41). [Hints: write the FOC w.r.t. h_t as a first order linear difference equation with time varying coefficients, and solve $\beta^t \phi_t$ forward. Replace $\beta^t \phi_t$ from the FOCs w.r.t. l_{2t} and l_{1t}, use the human capital accumulation technology and rearrange].

Ans: Using the human capital accumulation technology, the FOC w.r.t. h_{t+1} can be written as

$$\beta^t \phi_t = \frac{h_{t+2}}{h_{t+1}} \beta^{t+1} \phi_{t+1} + A_{t+1},$$

where $A_{t+1} \equiv \lambda_h p_{t+1} w_{t+1} (1-\tau_{t+1}) l_{1t+1}$. Solving this equation forward we obtain

$$\begin{aligned}
\beta^t \phi_t &= \frac{h_{t+2}}{h_{t+1}} \left(\frac{h_{t+3}}{h_{t+2}} \beta^{t+2} \phi_{t+2} + A_{t+2} \right) + A_{t+1} \\
&= \frac{h_{t+3}}{h_{t+1}} \beta^{t+2} \phi_{t+2} + \frac{h_{t+2}}{h_{t+1}} A_{t+2} + A_{t+1} \\
&= \frac{h_{t+3}}{h_{t+1}} \left(\frac{h_{t+4}}{h_{t+3}} \beta^{t+3} \phi_{t+3} + A_{t+3} \right) + \frac{h_{t+2}}{h_{t+1}} A_{t+2} + A_{t+1} \\
&= \frac{h_{t+4}}{h_{t+1}} \beta^{t+3} \phi_{t+3} + \frac{h_{t+3}}{h_{t+1}} A_{t+3} + \frac{h_{t+2}}{h_{t+1}} A_{t+2} + A_{t+1},
\end{aligned}$$

so that

$$\beta^t \phi_t = \frac{\lambda_h}{h_{t+1}} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} l_{1t+s}.$$

Now, combining the FOCs w.r.t. l_{1t} and l_{2t} we obtain

$$\lambda_h p_t w_t (1-\tau_t) h_t = G'(l_{2t}) h_t \beta^t \phi_t.$$

Thus, using our expression for $\beta^t \phi_t$ in the above equation yields,

$$p_t w_t (1-\tau_t) h_t = G'(l_{2t}) \frac{h_t}{h_{t+1}} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} l_{1t+s},$$

or, using the human capital accumulation technology,

$$p_t w_t (1-\tau_t) h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} l_{1t+s}.$$

4) Show that the FOC of the household problem derived in 1), including the multipliers λ_h and $\beta^t \phi_t$, are equivalent to the following system,

$$\begin{aligned}
\frac{U_{lt}}{h_t G'(l_{2t})} &= \frac{\beta U_{lt+1}}{h_{t+1} G'(l_{2t+1})} [G(l_{2t+1}) + (1-\delta_h)] + \beta U_{lt+1} l_{1t+1}/h_{t+1}, && (42) \\
U_{ct} &= \beta U_{ct+1} [1 + (1-\theta_{t+1})(r_t - \delta)], && (43) \\
-\frac{U_{lt}}{U_{ct}} &= (1-\tau_t) w_t h_t, && (44)
\end{aligned}$$

for all $t \geq 0$. FYI: the Euler equation for human capital (42) is the one displayed by Atkeson, Chari and Kehoe.

Ans: Dividing the FOCs for c_t and l_{1t} we obtain

$$-\frac{U_{lt}}{U_{ct}} = w_t (1-\tau_t) h_t.$$

From the FOC w.r.t k_{t+1} we obtain

$$\frac{p_t}{p_{t+1}} = R_{kt+1}.$$

Dividing the FOC for c_t at periods t and $t+1$, using the previous result and the definition of R_{kt+1}, and rearranging we find

$$U_{ct} = \beta U_{ct+1}\left[1 + (1-\theta_{t+1})(r_{t+1} - \delta)\right].$$

Finally, introducing the FOCs for l_{1t} and l_{2t} into the FOC for h_{t+1}, and rearranging we get

$$\frac{U_{lt}}{G'(l_{2t})\,h_t} = \frac{\beta U_{lt+1}}{G'(l_{2t+1})\,h_{t+1}}\left[G(l_{2t+1}) + (1-\delta_h)\right] + \frac{\beta U_{lt+1} l_{1t+1}}{h_{t+1}}.$$

5) Show that if the implementability constraint

$$\sum_{t=0}^{\infty}\beta^t\left[U_{ct}c_t + U_{lt}l_{1t}\right] = U_c k_0\left(1 + (1-\theta_0)(F_{k0} - \delta)\right), \tag{45}$$

the Euler equation for human capital (42), the human capital accumulation technology (36), and feasibility condition (38) hold, then there is a competitive equilibrium with taxes where the government finances the government purchases. Conversely, show that in any competitive equilibrium with taxes where the government finances the government purchases, the implementability constraint (45) and the Euler equation for human capital (42) hold. To answer this question you should use the definition of equilibrium stated above.

Ans: *Necessity:* We will first prove that if $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ is a competitive equilibrium allocation, then (45), (42), (36) and (38) have to be satisfied. That (38), (36) and (42) hold in a CE allocation is obvious. It remains to show that (45) also holds. Rewrite the household's budget constraint as

$$\begin{aligned}
0 &= \sum_{t=0}^{\infty} p_t\left[(1-\tau_t)\,w_t h_t l_{1t} + k_t R_{k,t} - c_t - k_{t+1}\right] \\
&= \sum_{t=0}^{\infty} p_t\left[(1-\tau_t)\,w_t h_t l_{1t} - c_t\right] + \sum_{t=0}^{\infty} p_t k_t R_{k,t} - \sum_{t=0}^{\infty} p_t k_{t+1} \\
&= \sum_{t=0}^{\infty} p_t\left[(1-\tau_t)\,w_t h_t l_{1t} - c_t\right] + p_0 k_0 R_{k,0} + \sum_{t=0}^{\infty} k_{t+1}\left[p_{t+1}R_{k,t+1} - p_t\right].
\end{aligned}$$

Using that in any CE $p_{t+1}R_{k,t+1} = p_t$ and the definition of $R_{k,0}$, the household's budget constraint becomes

$$\sum_{t=0}^{\infty} p_t\left[(1-\tau_t)\,w_t h_t l_{1t} - c_t\right] + p_0 k_0\left[1 + (1-\theta_0)(F_{k0} - \delta)\right] = 0.$$

Introducing the FOC w.r.t c_t and c_0, i.e.,

$$\beta^t U_{ct} = \lambda_h p_t;\ U_{c0} = \lambda_h p_0,$$

in place of the prices p_t and dividing by λ_h we get,

$$\sum_{t=0}^{\infty} \beta^t U_{ct} \left[(1-\tau_t) w_t h_t l_{1t} - c_t \right] + U_{c0} k_0 \left[1 + (1-\theta_0)(F_{k0} - \delta) \right] = 0.$$

Finally, using (44) we obtain

$$\sum_{t=0}^{\infty} \beta^t U_{ct} \left[-\frac{U_{lt}}{U_{ct}} l_{1t} - c_t \right] + U_{c0} k_0 \left[1 + (1-\theta_0)(F_{k0} - \delta) \right] = 0$$

or, rearranging,

$$\sum_{t=0}^{\infty} \beta^t \left[U_{lt} l_{1t} + U_{ct} c_t \right] = U_{c0} k_0 \left[1 + (1-\theta_0)(F_{k0} - \delta) \right].$$

Sufficiency: Now will prove that if an allocation $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ satisfies (45), (42), (36) and (38), then such allocation belongs to a competitive equilibrium for some prices and government taxes. First, note that the solution to the household problem is summarized by equations (42), (43) and (44). It follows then that if (42) holds, one of the optimality conditions of the household's problem holds. Given the allocation $\{c_t, l_{1t}, l_{2t}, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, define the prices as

$$F_n(l_{1t} h_t, k_t) = w_t,$$
$$F_k(l_{1t} h_t, k_t) = r_t,$$

and

$$\frac{\beta U_{ct+1}}{U_{ct}} = \frac{p_{t+1}}{p_t}.$$

Then, from the definition of R_{kt} and the previous prices obtain the capital tax rate θ_t as

$$\frac{p_t}{p_{t+1}} = 1 + (1-\theta_t)\left(F_k(l_{1t} h_t, k_t) - \delta \right).$$

Finally, from (44) we can pin down the labor tax rate:

$$-\frac{U_{lt}}{U_{ct}} = (1-\tau_t) F_n(l_{1t} h_t, k_t) h_t.$$

Given those prices and tax instruments plus feasibility, households and firms are maximizing, and by Walras' law, the government budget constraint also holds. Therefore, the proposed allocation belongs to a competitive equilibrium for certain prices and certain tax instruments.

6) Equivalences. i) Show that the variational condition (41) and the FOC w.r.t. l_{1t} imply the following variational conditions purely in terms of allocations,

$$U_{lt} = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1}^{\infty} \beta^s U_{lt+s} l_{1t+s}, \tag{46}$$

for all $t \geq 0$. [Hint: use the FOC for l_{t+s} from 1)]. ii) Show that the Euler equation for human capital (43) also implies this variational conditions purely in terms of allocations (46). [Hint: Treat the Euler equation for human capital as a linear difference equation, solve $U_{lt}/h_t G(l_{2t})$ forward and use the human capital accumulation technology].

Ans: i) Substituting the FOC for l_{1t} in the variational condition (41) yields

$$\frac{\beta^t U_{lt}}{\lambda_h} = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1}^{\infty} \frac{\beta^{t+s} U_{lt+s}}{\lambda_h} l_{1t+s},$$

or

$$U_{lt} = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1}^{\infty} \beta^s U_{lt+s} l_{1t+s}.$$

ii) Using the human capital accumulation technology, the Euler equation for human capital (43) can be written as

$$B_t = \frac{h_{t+2}}{h_{t+1}} \beta B_{t+1} + \beta C_{t+1},$$

where $B_{t+1} \equiv U_{lt+1}/(h_{t+1} G'(l_{2t+1}))$ and $C_{t+1} = U_{lt+1} l_{1t+1}/h_{t+1}$. Solving this equation forward we obtain

$$\begin{aligned}
B_t &= \frac{h_{t+2}}{h_{t+1}} \beta \left(\frac{h_{t+3}}{h_{t+2}} \beta B_{t+2} + \beta C_{t+2} \right) + \beta C_{t+1} \\
&= \frac{h_{t+3}}{h_{t+1}} \beta^2 B_{t+2} + \frac{1}{h_{t+1}} \beta^2 C_{t+2} + \beta C_{t+1} \\
&= \frac{h_{t+3}}{h_{t+1}} \beta^2 \left(\frac{h_{t+4}}{h_{t+3}} \beta B_{t+3} + \beta C_{t+3} \right) + \frac{h_{t+2}}{h_{t+1}} \beta^2 C_{t+2} + \beta C_{t+1} \\
&= \frac{h_{t+4}}{h_{t+1}} \beta^3 B_{t+3} + \frac{h_{t+3}}{h_{t+1}} \beta^3 C_{t+3} + \frac{h_{t+2}}{h_{t+1}} \beta^2 C_{t+2} + \beta C_{t+1},
\end{aligned}$$

so that

$$\frac{U_{lt}}{h_t G'(l_{2t})} = \frac{1}{h_{t+1}} \sum_{s=1}^{\infty} \beta^s U_{lt+s} l_{1t+s},$$

or, using the human capital accumulation technology,

$$U_{lt} = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1}^{\infty} \beta^s U_{lt+s} l_{1t+s}.$$

7) Show that the implementability constraint (45) and the variational equations in terms of allocations (46) are equivalent to the system given by the implementability constraint

$$\sum_{t=0}^{\infty} \beta^t U_{ct} c_t = U_{c0} \left[1 + (1 - \theta_0)(F_{k0} - \delta)\right] - U_{l10} \left[l_{10} + \frac{G(l_{20}) + (1 - \delta_h)}{G'(l_{20})}\right], \tag{47}$$

and the variational equations in terms of allocations (46). [Hint: Use the variational equation in terms of allocations for $t = 0$ into the implementability constraint (45) and rearrange].

Ans: We show that (47) is equivalent to

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct} c_t + U_{lt} l_{1t}\right] = U_c k_0 \left(1 + (1 - \theta_0)(F_{k0} - \delta)\right),$$

and (46) for time $t = 0$. To see this, consider the equation (46) for period $t = 0$, namely,

$$U_{l0} \frac{G(l_{20}) + (1 - \delta_h)}{G'(l_{20})} = \sum_{t=1}^{\infty} \beta^t U_{lt} l_{1t}.$$

Then,

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \left[U_{ct} c_t + U_{lt} l_{1t}\right] &= \sum_{t=0}^{\infty} \beta^t U_{ct} c_t + U_{l10} l_{10} + \sum_{t=1}^{\infty} \beta^t U_{lt} l_{1t} \\
&= \sum_{t=0}^{\infty} \beta^t U_{ct} c_t + U_{l10} \left[l_{10} + \frac{G(l_{20}) + (1 - \delta_h)}{G'(l_{20})}\right],
\end{aligned}$$

so replacing this result into (45),

$$\sum_{t=0}^{\infty} \beta^t U_{ct} c_t = U_{c0} \left[1 + (1 - \theta_0)(F_{k0} - \delta)\right] - U_{l10} \left[l_{10} + \frac{G(l_{20}) + (1 - \delta_h)}{G'(l_{20})}\right].$$

The converse also holds.

8) Using the preferences described in (40) show that the Lagrangian of the Ramsey problem can be written as

$$\begin{aligned}
&\max_{\{c_t, l_{1t}, l_{2t}, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_{1t} + l_{2t}) \left[1 + \lambda(1 - \sigma)\right] \\
&-\lambda U_{c0} \left[1 + (1 - \theta_0)(F_{k0} - \delta)\right] + \lambda U_{l10} \left[l_{10} + \frac{G(l_{20}) + (1 - \delta_h)}{G'(l_{20})}\right],
\end{aligned}$$

subject to feasibility (38), the variational equation using allocations (46), the human capital accumulation technology (36), and the implementability constraint

(47) for some λ, the Lagrange multiplier of the implementability constraint (47). [Hint: Use the form of the preferences to collect $U_c c$ with U and use the answer to 7)].

Ans: Let λ be the Lagrange multiplier on the implementability (47). Then the Lagrangian of the Ramsey problem is

$$\max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t}+l_{2t}\right) + \lambda \sum_{t=0}^{\infty} \beta^t U_{ct}\left(c_t, l_{1t}+l_{2t}\right) c_t$$
$$-\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right]$$

subject to (38), (46) and (36). But with the above preferences, i.e.,

$$U(c,l) = \frac{c^{1-\sigma}}{1-\sigma}\varphi(l)^{1-\sigma},$$

we have that

$$U_c c = c^{1-\sigma}\varphi(l)^{1-\sigma} = (1-\sigma)\, U(c,l).$$

Thus, the Lagrangian becomes

$$\max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t}+l_{2t}\right)\left[1+\lambda(1-\sigma)\right]$$
$$-\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right]$$

subject to (38), (46) and (36).

9) Consider the problem described in 8). i) Derive the FOCs for k_{t+1} for $t \geq 1$. [Hint: Replace consumption in $U\left(c_t, l_{1t}+l_{2t}\right)$ using feasibility and differentiate w.r.t. k_{t+1}]. ii) Use i) and the Euler equation for consumers to establish whether capital taxes equal zero for all $t \geq 2$. Give a proof or a counter-example. Give a two lines intuitive explanation of your answer.

Ans: Using the hint, and attaching multipliers $\left\{\beta^t \xi_t\right\}$ and $\left\{\beta^t \eta_t\right\}$ to (46) and (36), we can rewrite the Lagrangian of the Ramsey problem as

$$\max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_{1t}+l_{2t}\right)\left[1+\lambda(1-\sigma)\right]$$
$$+\sum_{t=0}^{\infty} \beta^t \xi_t \left[U_l\left(c_t, l_{1t}+l_{2t}\right) - \frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right)+\left(1-\delta_h\right)\right]} \sum_{s=1}^{\infty} \beta^s U_l\left(c_{t+s}, l_{1t+s}+l_{2t+s}\right) l_{1t+s}\right]$$
$$+\sum_{t=0}^{\infty} \beta^t \eta_t \left[\left[G\left(l_{2t}\right)-\left(1-\delta_h\right)\right] h_t - h_{t+1}\right]$$
$$-\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right],$$

where

$$c_{t+s} = F\left(l_{1t+s}h_{t+s}, k_{t+s}\right) + (1-\delta)\,k_{t+s} - k_{t+s+1} - g_{t+s}.$$

i) The FOC w.r.t k_{t+1} for $t \geq 1$ is

$$\begin{gathered} -U_{ct}\left[1+\lambda\left(1-\sigma\right)\right] + \beta U_{ct+1}\left[1+F_{kt+1}-\delta\right] \\ +\xi_t\left[-U_{lct} - \frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right)+\left(1-\delta_h\right)\right]}\beta U_{lct+1}l_{1t+1}\right] \\ +\beta\xi_{t+1}U_{lct+1}\left[F_{kt+1}+1-\delta\right] = 0. \end{gathered}$$

ii) The Euler equation for consumers is

$$-U_{ct} + \beta U_{ct+1}\left[1+\left(1-\theta_{t+1}\right)\left(F_{kt+1}-\delta\right)\right] = 0.$$

Thus, in general, capital tax rates are not zero for an arbitrary $t \geq 2$.

10. Consider the problem described in 8). Assume that $\lambda > 0$, and that θ_0 is restricted to be on $[0,1]$. What will be the optimal value of θ_0? Give a two lines intuitive explanation of your answer.

Ans: The optimal θ_0 is 1. The intuition is that k_0 is given (i.e., capital at period zero is in perfect inelastic supply), hence it is optimal to tax it at the highest possible rate.

11) Consider the following relaxed version of problem described in 8):

$$\begin{gathered} \max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty}\beta^t U\left(c_t, l_{1t}+l_{2t}\right)\left[1+\lambda\left(1-\sigma\right)\right] \\ -\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right] + \lambda U_{l10}\left[l_{10} + \frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right], \end{gathered}$$

subject to feasibility (38), the human capital accumulation technology (36), and the implementability constraint (47) for some number λ. i) Show that if a balanced growth path exists for this relaxed problem, equation (46) is satisfied for the balanced growth path. ii) Use i) to show that if a balanced growth path exists for this problem, it is the same as the balanced growth path for the first best economy (the one that has access to lump sum taxes). iii) What are the implications of ii) for labor taxes τ_t in a balanced growth path of the Ramsey problem described in 8)? Give a two lines intuitive explanation of your answer to iii).

Ans: Our utility function implies

$$U_l\left(c,l\right) = c^{1-\sigma}\varphi\left(l\right)^{-\sigma}.$$

Then, (46) at period t becomes

$$c_t^{1-\sigma}\varphi\left(l_{1t}+l_{2t}\right)^{-\sigma}=\frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right)+\left(1-\delta_h\right)\right]}\sum_{s=1}^{\infty}\beta^s c_{t+s}^{1-\sigma}\varphi\left(l_{1t+s}+l_{2t+s}\right)^{-\sigma}l_{1t+s},$$

or

$$\varphi\left(l_{1t}+l_{2t}\right)^{-\sigma}=\frac{G'\left(l_{2t}\right)}{\left[G\left(l_{2t}\right)+\left(1-\delta_h\right)\right]}\sum_{s=1}^{\infty}\beta^s\left(\frac{c_{t+s}}{c_t}\right)^{1-\sigma}\varphi\left(l_{1t+s}+l_{2t+s}\right)^{-\sigma}l_{1t+s}.$$

But in a BGP we have that $l_{1t}=l_1$, $l_{2t}=l_2$ and $c_{t+1}/c_t=\gamma$ (so that $c_{t+s}/c_t=\gamma^s$). Thus, the previous equation becomes

$$\varphi\left(l_1+l_2\right)^{-\sigma}=\frac{G'\left(l_2\right)}{\left[G\left(l_2\right)+\left(1-\delta_h\right)\right]}\sum_{s=1}^{\infty}\beta^s\gamma^{s(1-\sigma)}\varphi\left(l_1+l_2\right)^{-\sigma}l_1,$$

or

$$1=\frac{G'\left(l_2\right)l_1}{\left[G\left(l_2\right)+\left(1-\delta_h\right)\right]}\left[\frac{\beta\gamma^{1-\sigma}}{1-\beta\gamma^{1-\sigma}}\right], \tag{48}$$

which is the same for all t. In other words, (46) is satisfied in a BGP.

ii) Thus, the Ramsey problem solves

$$\max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}}\sum_{t=0}^{\infty}\beta^t U\left(c_t,l_{1t}+l_{2t}\right)\left[1+\lambda\left(1-\sigma\right)\right]$$
$$-\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right]+\lambda U_{l10}\left[l_{10}+\frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right]$$

subject to the technological constraints (38) (i.e. feasibility) and (36) (i.e. the human capital accumulation technology). Since no other restrictions are added and $\left[1+\lambda\left(1-\sigma\right)\right]$ is a constant, the BGP of the Ramsey solution is the same as the BGP for the first-best economy. Further, if we take the FOCs and rearrange, we will notice that (48) is satisfied, so that the solution to the relaxed problem is indeed the solution to the Ramsey problem in a BGP.

iii) Not only there is no taxation of capital in the BGP but also there is no taxation of labor.

12) *Skilled and Unskilled labor* (Optional ***).

We now consider a version with two types of labor services: skilled and unskilled. The time t utility is $U\left(c,l_{1t}+l_{2t}+l_{3t}\right)$ where l_{3t} is time devoted to unskilled labor. Preferences $U\left(c,l_1+l_2+l_3\right)$ are restricted as in (40). The production function F is Cobb-Douglas and uses capital k_t, skilled labor $h_t l_{1t}$ and unskilled labor services l_{3t}:

$$F\left(k_t,\ l_{1t}h_t,\ l_{3t}\right)=k_t^{\alpha_1}\left(l_{1t}h_t\right)^{\alpha_2}\left(l_{3t}\right)^{1-\alpha_2-\alpha_3}.$$

The household budget constraint is

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1}\right] = \sum_{t=0}^{\infty} p_t \left[(1-\tau_t)\, w_t h_t l_{1t} + k_t R_{k,t} + (1-\omega_t)\, u_t l_{3t}\right],$$

where u_t is the wage for unskilled workers and ω_t is tax rate on unskilled labor services. The rest of the economy is as before (with the obvious changes in the government budget constraint and firm's problem).

i) Write down the FOCs for the household's problem. What is the difference with the answer to 1)?

Ans: Using the same multipliers as above, the FOCs are

$$\begin{aligned}
c_t &: \quad \beta^t U_{ct} = \lambda_h p_t, \\
l_{1t} &: \quad \beta^t U_{lt} = -\lambda_h p_t (1-\tau_t)\, w_t h_t, \\
l_{2t} &: \quad \beta^t \left(U_{lt} + \phi_t G'(l_{2t})\, h_t\right) = 0, \\
l_{3t} &: \quad \beta^t U_{lt} = -\lambda_h p_t (1-\omega_t)\, u_t, \\
k_{t+1} &: \quad \lambda_h \left(-p_t + p_{t+1} R_{k,t+1}\right) = 0, \\
h_{t+1} &: \quad -\beta^t \phi_t + \beta^{t+1} \phi_{t+1} \left[G(l_{2t+1}) + (1-\delta_h)\right] + \lambda_h p_{t+1} w_{t+1} (1-\tau_{t+1})\, l_{1t+1} = 0.
\end{aligned}$$

The difference is that now we have one more FOC, the one for unskilled labor services.

ii) Show that this economy is compatible with a balanced growth path where: a) all tax rates are constant, b) l_{1t}, l_{2t} and l_{3t} are constant, c) the constant gross growth rate of k_t and c_t is γ, d) the constant gross growth rate of h_t is γ_h, with $(\gamma)^{1-\alpha_1} = (\gamma_h)^{\alpha_2}$, d) wage u_t, w_t grow at a constant rate, and e) r_t is constant. [Hint: Just show that the FOCs, feasibility and human capital technology can be satisfied with the path described above].

Ans: Using the guess, the household's Euler equation becomes

$$c_t^{-\sigma} \varphi (l_1 + l_2 + l_3)^{1-\sigma} = \beta c_{t+1}^{-\sigma} \varphi (l_1 + l_2 + l_3)^{1-\sigma} \left[1 + (1-\theta)\left(\alpha_1 k_t^{\alpha_1 - 1} h_t^{\alpha_2} l_1^{\alpha_2} (l_3)^{1-\alpha_2-\alpha_3} - \delta\right)\right],$$

or, using $c_{t+1}/c_t = \gamma$,

$$\gamma^\sigma = \beta \left[1 + (1-\theta)\left(\alpha_1 k_t^{\alpha_1 - 1} h_t^{\alpha_2} l_1^{\alpha_2} (l_3)^{1-\alpha_2-\alpha_3} - \delta\right)\right].$$

Hence, if a BGP exists, then $k_t^{\alpha_1 - 1} h_t^{\alpha_2}$ has to be constant. That is,

$$\left(\frac{h_{t+1}}{h_t}\right)^{\alpha_2} = \left(\frac{k_{t+1}}{k_t}\right)^{1-\alpha_1},$$

i.e., $(\gamma)^{1-\alpha_1} = (\gamma_h)^{\alpha_2}$, where $\gamma_h = h_{t+1}/h_t$.

From feasibility we can write

$$\frac{c_t}{k_t} = k_t^{\alpha_1 - 1} (h_t)^{\alpha_2} l_t^{\alpha_2} (l_3)^{1-\alpha_2-\alpha_3} + (1-\delta) - \frac{k_{t+1}}{k_t} - \frac{g_t}{k_t},$$

or

$$\frac{c_t}{k_t} + \gamma - k_t^{\alpha_1 - 1} (h_t)^{\alpha_2} l_t^{\alpha_2} (l_3)^{1-\alpha_2-\alpha_3} - (1-\delta) = -\frac{g_t}{k_t}.$$

Since the LHS is constant, g_t/k_t has to be constant as well (i.e. $g_{t+1}/g_t = \gamma$).

Finally, the rental rates satisfy

$$\begin{aligned} w_t &= \alpha_2 k_t^{\alpha_1} (h_t)^{\alpha_2 - 1} l_1^{\alpha_2 - 1} (l_3)^{1-\alpha_2-\alpha_3}, \\ u_t &= (1-\alpha_2-\alpha_3) k_t^{\alpha_1} (h_t)^{\alpha_2} l_1^{\alpha_2} (l_3)^{-\alpha_2-\alpha_3}, \\ r_t &= \alpha_1 k_t^{\alpha_1 - 1} (l_{1t} h_t)^{\alpha_2} (l_{3t})^{1-\alpha_2-\alpha_3}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{w_{t+1}}{w_t} &= \left(\frac{k_{t+1}}{k_t}\right)^{\alpha_1} \left(\frac{h_{t+1}}{h_t}\right)^{\alpha_2 - 1} = \gamma/\gamma_h, \\ \frac{u_{t+1}}{u_t} &= \left(\frac{k_{t+1}}{k_t}\right)^{\alpha_1} \left(\frac{h_{t+1}}{h_t}\right)^{\alpha_2} = \gamma, \\ \frac{r_{t+1}}{r_t} &= \left(\frac{k_{t+1}}{k_t}\right)^{\alpha_1 - 1} \left(\frac{h_{t+1}}{h_t}\right)^{\alpha_2} = 1, \end{aligned}$$

where we used that $\left(\frac{k_{t+1}}{k_t}\right)^{\alpha_1} \left(\frac{h_{t+1}}{h_t}\right)^{\alpha_2} = \gamma$. Hence, the rental rates of labor grow at constant rates, while the rental rate of capital is constant.

iii) Show that the equivalent of the implementability constraint (45) is

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct} c_t + U_{lt} (l_{1t} + l_{3t})\right] = U_{c0} k_0 \left[1 + (1-\theta_0)(F_{k0} - \delta)\right].$$

Ans: Using the same trick we used before to get rid of k_{t+1}, we can rewrite the household's budget constraint as

$$\sum_{t=0}^{\infty} p_t \left[(1-\tau_t) w_t h_t l_{1t} + (1-\omega_t) u_t l_{3t} - c_t\right] + p_0 k_0 \left[1 + (1-\theta_0)(F_{k0} - \delta)\right] = 0.$$

Introducing the FOCs

$$\begin{aligned} \beta^t U_{ct} &= \lambda_h p_t, \\ \beta^t U_{lt} &= -\lambda_h p_t (1-\tau_t) w_t h_t, \\ \beta^t U_{lt} &= -\lambda_h p_t (1-\omega_t) u_t, \end{aligned}$$

into the previous equation and rearranging, we obtain

$$\sum_{t=0}^{\infty} \beta^t \left[U_{lt}\left(l_{1t}+l_{3t}\right)+U_{ct}c_t\right] = U_{c0}k_0\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right].$$

iv) Show that the variational condition (41), and its related expression (46) also hold.

Ans: Exactly the same algebra as before.

v) Adapt the argument in 7) replacing (47) for the following implementability:

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}c_t+U_{lt}l_{3t}\right] = U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right] - U_{l10}\left[l_{10}+\frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right].$$

Ans: In 7) we used (46) to replace for $\sum_{t=0}^{\infty}\beta^t U_{lt}l_{1t}$. Doing the same here we obtain the required equation.

vi) Adapt the argument in 8) using the following objective function

$$\max_{\{c_t,l_{1t},l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{U\left(c_t,l_{1t}+l_{2t}\right)\left[1+\lambda\left(1-\sigma\right)\right]+U_{lt}l_{3t}\right\}$$
$$-\lambda U_{c0}\left[1+\left(1-\theta_0\right)\left(F_{k0}-\delta\right)\right]+\lambda U_{l10}\left[l_{10}+\frac{G\left(l_{20}\right)+\left(1-\delta_h\right)}{G'\left(l_{20}\right)}\right].$$

Ans: Exactly the same algebra as in 8), but now the term $U_{lt}l_{3t}$ remains there.

vii) Adapt the argument in 11) to show that in a balanced growth path both capital and skilled labor taxes are zero, but unskilled labor taxes are, in general, not zero.

Ans: Unfortunately, this statement is not true. In fact, capital taxes can be shown to be zero but skilled labor taxes will, in general, be non-zero. Furthermore, it is incorrect to ignore (46) in this case.

13) *Inelastic labor supply.*

Consider a simplified version of the problem analyzed so far. Assume that $l_{1t}+l_{2t}=1$, but that labor does not enter in the utility function U. Use $l_{1t}=1-l_{2t}$ for all t.

i) Derive the FOCs of the household's problem.

Ans: The Lagrangian is

$$\max_{\{c_t, l_{2t}, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} - \lambda_h \sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\tau_t) w_t h_t (1-l_{2t}) + k_t R_{k,t}\right]$$
$$+ \sum_{t=0}^{\infty} \beta^t \phi_t \left[\left[G(l_{2t}) + (1-\delta_h)\right] h_t - h_{t+1}\right].$$

FOCs:

$$\begin{aligned}
c_t &: \quad \beta^t U_{ct} = \lambda_h p_t \\
l_{2t} &: \quad -\lambda_h p_t w_t (1-\tau_t) h_t + \beta^t \phi_t G'(l_{2t}) h_t = 0 \\
k_{t+1} &: \quad \lambda_h \left[-p_t + p_{t+1} R_{kt+1}\right] = 0 \\
h_{t+1} &: \quad -\beta^t \phi_t + \beta^{t+1} \phi_{t+1} \left[G(l_{2t+1}) + (1-\delta_h)\right] + \lambda_h p_{t+1} w_{t+1} (1-\tau_{t+1}) (1-l_{2t+1}) = 0
\end{aligned}$$

As before, from the FOCs w.r.t. c_t and k_{t+1}, and using $r_t = F_{kt}$, we obtain

$$U_{ct} = \beta U_{ct+1} \left[1 + (F_{kt+1} - \delta)(1-\theta_{t+1})\right].$$

ii) Since the variational argument used in 2 still applies, use the FOCs of the problem to show that it implies that

$$U_{ct} F_{lt} (1-\tau_t) h_t = \frac{G'(l_{2t})}{\left[G(l_{2t}) + (1-\delta_h)\right]} \sum_{s=1} \beta^s U_{ct+s} F_{lt+s} (1-\tau_{t+s}) h_{t+s} (1-l_{2t+s}), \tag{49}$$

for all $t \geq 0$.

Ans: Solving the FOC w.r.t h_{t+1} forward we find (see exercise 3)

$$\beta^t \phi_t = \frac{\lambda_h}{h_{t+1}} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} (1-l_{2t+s}).$$

Now, the FOC w.r.t. l_{2t} is

$$\lambda_h p_t w_t (1-\tau_t) h_t = G'(l_{2t}) h_t \beta^t \phi_t.$$

Thus, using our expression for $\beta^t \phi_t$ in the above equation yields,

$$p_t w_t (1-\tau_t) h_t = G'(l_{2t}) \frac{h_t}{h_{t+1}} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} (1-l_{2t+s}),$$

or, using the human capital accumulation technology,

$$p_t w_t (1-\tau_t) h_t = \frac{G'(l_{2t})}{\left[G(l_{2t}) + (1-\delta_h)\right]} \sum_{s=1}^{\infty} p_{t+s} w_{t+s} (1-\tau_{t+s}) h_{t+s} (1-l_{2t+s}).$$

Finally, using the FOC w.r.t. c_t and the fact that $w_t = F_{lt}$ in the above expression yields

$$\frac{\beta^t U_{ct}}{\lambda_h} F_{lt} (1 - \tau_t) h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1 - \delta_h)]} \sum_{s=1}^{\infty} \frac{\beta^t U_{ct}}{\lambda_h} F_{lt+s} (1 - \tau_{t+s}) h_{t+s} (1 - l_{2t+s}),$$

or

$$U_{ct} F_{lt} (1 - \tau_t) h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1 - \delta_h)]} \sum_{s=1}^{\infty} \beta^s U_{ct+s} F_{lt+s} (1 - \tau_{t+s}) h_{t+s} (1 - l_{2t+s}).$$

iii) Show that by replacing the FOCs of the household's problem, the household's budget constraint can be written as

$$\sum_{t \geq 0} \beta^t U_{ct} [c_t - (1 - \tau_t) F_{lt} (1 - l_{2t}) h_t] = U_{c0} k_0 (1 + (1 - \theta_0)(F_{k0} - \delta)). \quad (50)$$

Ans: Using the trick to eliminate capital and the fact that $R_{k0} = (1 + (1 - \theta_0)(F_{k0} - \delta))$, the budget constraint becomes

$$\sum_{t=0}^{\infty} p_t [c_t - (1 - \tau_t) w_t h_t (1 - l_{2t})] = p_0 k_0 (1 + (1 - \theta_0)(F_{k0} - \delta)),$$

or, using the FOC w.r.t. c_t and the fact that $w_t = F_{lt}$,

$$\sum_{t=0}^{\infty} \beta^t U_{ct} [c_t - (1 - \tau_t) F_{lt} (1 - l_{2t}) h_t] = U_{c0} k_0 (1 + (1 - \theta_0)(F_{k0} - \delta)).$$

iv) Show that if the implementability constraint (50), the variational condition (49), the human capital accumulation technology (36), and feasibility (38) hold, then there is a competitive equilibrium with taxes where the government finances the government purchases. Conversely, show that in any competitive equilibrium with taxes where the government finances the government purchases, the implementability constraint (50) and the variational equation (49) hold. Notice that both (49) and (50) include the sequence of labor taxes $\{\tau_t\}_{t=0}^{\infty}$.

Ans: Straightforward modification of exercise 5). Note, however, that the sequence of labor taxes $\{\tau_t\}$ remains there. This is so because we don't have the marginal condition w.r.t. leisure that we had before to eliminate labor taxes.

v) Show that in the solution of the Ramsey problem, $\theta_t = 0$ for all $t \geq 1$ and $\tau_t = \bar{\tau} > 0$ for all $t \geq 0$. Show that the resulting allocation coincides with the first best allocation, i.e., the one with lump sum taxes. [Hint: Simply check that the FOCs for the first best allocation and the budget constraint of the government can be simultaneously satisfied if taxes are set as stated above]. Give a two lines intuitive explanation of the difference between this case and the previous one.

Ans: The first best allocation solves

$$\max_{\{c_t,l_{2t},k_{t+1},h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \eta_t \{[G(l_{2t}) + 1 - \delta_h] h_t - h_{t+1}\}$$
$$- \sum_{t=0}^{\infty} \beta^t \gamma_t \{c_t + k_{t+1} - [F((1-l_{2t}) h_t, k_t) + (1-\delta) k_t]\}.$$

FOCs:

$$\begin{aligned}
c_t &: U_{ct} = \gamma_t, \\
l_{2t} &: \eta_t G'(l_{2t}) = \gamma_t F_l h_t, \\
k_{t+1} &: \beta \gamma_{t+1} [F_{kt+1} + 1 - \delta] = \gamma_t, \\
h_{t+1} &: -\eta_t + \beta \eta_{t+1} [G(l_{2t}) + 1 - \delta_h] + \beta \gamma_{t+1} F_{lt+1} (1 - l_{2t+1}) = 0.
\end{aligned}$$

Doing the same algebra as in 13.ii but now with the first best allocation we get

$$U_{ct} = \beta U_{ct+1} [F_{kt+1} + 1 - \delta],$$

and

$$U_{ct} F_{lt} h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1} \beta^s U_{ct+s} F_{lt+s} h_{t+s} (1 - l_{2t+s}).$$

The corresponding conditions for the household's problem are

$$U_{ct} = \beta U_{ct+1} [1 + (F_{kt+1} - \delta)(1 - \theta_{t+1})],$$

$$U_{ct} F_{lt} (1-\tau_t) h_t = \frac{G'(l_{2t})}{[G(l_{2t}) + (1-\delta_h)]} \sum_{s=1} \beta^s U_{ct+s} F_{lt+s} h_{t+s} (1 - \tau_{t+s})(1 - l_{2t+s}).$$

Thus, the solution of the Ramsey problem entails $\theta_{t+1} = 0$ for $t \geq 1$. Moreover, note that by setting $\tau_t = \bar{\tau}$, the household's condition for accumulation of human capital is the same as in the first best. That is, by setting $\bar{\tau}$ such that the government budget constraint holds, we are able to obtain the first best allocation. Comparing this result with the previous exercise we see that the key to this result is whether agents derive utility from leisure or not. If agents value leisure, labor taxes will distort the intertemporal and intratemporal leisure decisions.