Numerical approximation of DSGE models

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Introduction

- This presentation shows, step by step, how to solve and approximate numerically DSGE models
  - Example: Plain Vanilla RBC
- We'll discuss how to approximate the non-linear system of equilibrium conditions using linear approx. around the non-stochastic steady state.

RBC model

- Closed economy. Representative consumer. No Market failures $\Longrightarrow$ First and Second Welfare Theorem hold!
- Planner that chooses to maximize household's welfare.

Steps to solve a model

General procedure to solve a DSGE model:

1. Find equilibrium conditions of the model.
2. Find the steady state and calibrate the parameters of the model.
3. Log-linearize the equilibrium conditions around the steady state.
4. Write the linearized system of difference equations in a format consistent with:
$$\mathbf{A}\mathbb{E}\left[\mathbf{z}_{t+1}\right] = \mathbf{B}\mathbf{z}_t$$
where $\mathbf{z}_t$ is a vector with all the variables ordered in a particular way, and $\mathbf{A}$ and $\mathbf{B}$ are square matrices.
5. Use some routine to find approximate policy functions (see below).
6. Compute impulse responses, simulations, and so forth.

Planner's problem

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right]$$

subject to

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t$$

$$k_0, A_0 \quad \textit{given}$$

Lagrangian

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t - \eta \frac{l_t^{1+\frac{1}{v}}}{1+\frac{1}{v}} - \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta) k_t \right] \right\}$$

First order conditions:

$$\begin{aligned}
\frac{1}{c_t} &= \lambda_t \\
\eta\, l_t^{\frac{1}{v}} &= \lambda_t (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} \\
\lambda_t &= \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \\
c_t + k_{t+1} &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t
\end{aligned}$$

Transversality condition:

$$\lim_{T \to \infty} \mathbb{E}_0 \left[\beta^T \lambda_T k_{T+1} \right] = 0$$

Log of TFP follows an AR(1) process:

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1}$$

where ε_{t+1} is iid normal with mean 0 and variance σ_{ε}^2 and $|\rho| < 1$.

Equilibrium conditions

- Solution in the form of time-invariant "policy functions".
- As written:
  - Control variables: c_t, l_t and $\lambda_t \rightarrow y_t \equiv (c_t, l_t, \lambda_t)'$.
  - State variables: k_t and $A_t \rightarrow x_t \equiv (k_t, A_t)'$.
- We look for policy functions of the form:

$$c_t = c(x_t), \; l_t = l(x_t), \; \lambda_t = \lambda(x_t), \; k_{t+1} = k(x_t).$$

Equilibrium conditions

- But we are also interested in output and investment.
- Output is:

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}.$$

- Investment is:

$$i_t = k_{t+1} - (1-\delta) k_t.$$

- Remember, marginal product of capital and labor:

$$(1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} = (1-\alpha) \frac{y_t}{l_t}$$

$$\alpha A_t k_t^{\alpha-1} l_t^{1-\alpha} = \alpha \frac{y_t}{k_t}.$$

Equilibrium conditions

$$\frac{1}{c_t} = \lambda_t \tag{1}$$

$$\eta \, l_t^{\frac{1}{\nu}} = \lambda_t \left(1 - \alpha\right) \frac{y_t}{l_t} \tag{2}$$

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right] \tag{3}$$

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha} \tag{4}$$

$$c_t + i_t = y_t \tag{5}$$

$$i_t = k_{t+1} - \left(1 - \delta\right) k_t \tag{6}$$

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1} \tag{7}$$

Equilibrium conditions

Note that we can write the equilibrium conditions of the model as a system of equation of the form:

$$\mathbb{E}_t \left[f \left(x_{t+1}, y_{t+1}, x_t, y_t \right) \right] = \bar{0},$$

where $\bar{0}$ is a vector of zeros, and f is given by

$$f \left(x_{t+1}, y_{t+1}, x_t, y_t \right) = \begin{bmatrix} \frac{1}{c_t} - \lambda_t \\ \eta l_t^{\frac{1}{\nu}} - \lambda_t \left(1 - \alpha \right) \frac{y_t}{l_t} \\ \beta \lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) - \lambda_t \\ y_t - A_t k_t^{\alpha} l_t^{1-\alpha} \\ c_t + i_t - y_t \\ i_t - k_{t+1} + \left(1 - \delta \right) k_t \\ \log A_{t+1} - \rho \log A_t - \varepsilon_{t+1} \end{bmatrix}.$$

Hence, the equilibrium allocation solves a nonlinear system of stochastic difference equations.

We will solve the model using a first order perturbation around the non-stochastic steady state.

Non-Stochastic Steady state

Set $\varepsilon_t = 0$ for all t. The steady state satisfies $f(\bar{x}, \bar{y}, \bar{x}, \bar{y}) = 0$. Equations (1)-(7) become:

$$\frac{1}{\bar{c}} = \bar{\lambda} \tag{8}$$

$$\eta \bar{l}^{\frac{1}{v}} = \bar{\lambda}(1-\alpha)\frac{\bar{y}}{\bar{l}} \tag{9}$$

$$1 = \beta\left[\left(\alpha\frac{\bar{y}}{\bar{k}} + 1 - \delta\right)\right] \tag{10}$$

$$\bar{y} = \bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha} \tag{11}$$

$$\bar{c} + \bar{i} = \bar{y} \tag{12}$$

$$\bar{i} = \delta\bar{k} \tag{13}$$

$$\bar{A} = 1 \tag{14}$$

Calibration and steady state

- Set parameter values to match certain features observed in the data.
- α is the capital share in output. NIPA accounts for the U.S. imply a value of α of about $1/3 \implies \alpha = 1/3$.
- Set an average real interest rate of $\bar{R} = 0.01$ (1% per quarter). In the model, the (gross) steady state real interest rate is:

$$\alpha \frac{\bar{y}}{\bar{k}} + 1 - \delta = \bar{R}. \tag{15}$$

  Given α, this is a restriction between $\frac{\bar{k}}{\bar{y}}$ and δ.
- Equation (10) implies that β must satisfy:

$$\frac{1}{\beta} = \bar{R} \implies \beta = \frac{1}{1.01} \approx 0.99.$$

Calibration and steady state

Now match the average (long-run) investment rate $\bar{i}/\bar{y}$. Write (13) as:

$$\frac{\bar{i}}{\bar{y}} = \delta \frac{\bar{k}}{\bar{y}}.$$

But using (15) we can write

$$\frac{\bar{i}}{\bar{y}} = \delta \frac{\alpha}{\bar{R} - (1-\delta)}.$$

Solve for δ:

$$\delta = \frac{\left(\bar{R} - 1\right)\left(\bar{i}/\bar{y}\right)}{\alpha - \left(\bar{i}/\bar{y}\right)}.$$

Given a target value $\bar{i}/\bar{y} = 0.21$, and the calibrated values $\bar{R} = 1.01$ and $\alpha = 1/3$ we obtain:

$$\delta = \frac{0.01 \times 0.21}{0.33 - 0.21} \approx 0.017.$$

Calibration and steady state

- Calibrate the model so that the steady state labor input is $\bar{l} = 1/3$, roughly the fraction of total weekly hours that workers spend working.
- Using (14), we can write (11) as:

$$\bar{y} = \bar{k}^{\alpha} \bar{l}^{1-\alpha}.$$

  Dividing by $\bar{k}$

$$\frac{\bar{y}}{\bar{k}} = \left(\frac{\bar{l}}{\bar{k}} \right)^{1-\alpha}.$$

- Using condition (15) we can write:

$$\bar{k} = \bar{l} \left(\frac{\alpha}{\bar{R} - (1-\delta)} \right)^{\frac{1}{1-\alpha}}. \tag{16}$$

Calibration and steady state

Given $\bar{l} = 1/3$ and the other calibrated parameters, this equation delivers $\bar{k}$:

$$\bar{k} = \frac{1}{3}\left(\frac{1/3}{1.01-(1-0.017)}\right)^{\frac{1}{1-\frac{1}{3}}} \approx 14.46.$$

The steady state level of output is thus:

$$\bar{y} = \bar{k}^{\alpha}\bar{l}^{1-\alpha} \approx 1.17.$$

The steady state consumption $\bar{c}$ follows from feasibility (12):

$$\bar{c} = \bar{y} - \bar{i} = \bar{y}\left(1 - \frac{\bar{i}}{\bar{y}}\right) = 1.17(1-0.21) \approx 0.93.$$

Calibration and steady state

- It remains to calibrate η and v.
- Write condition (9) as

$$\eta \bar{l}^{1+\frac{1}{v}} = (1-\alpha)\frac{\bar{y}}{\bar{c}}.$$

  In this equation we know $\bar{l}, \bar{c}, \bar{y}$ and α.
- We have one equation and two parameters: η and v.
- Set the elasticity $v = 1$ (Some controversy here, remember our discussion last quarter).
- Then we recover η.

Calibration and steady state

- Calibration of the parameters of the stochastic process ρ and σ_ε^2?
- Two possibilities:
  1. Run a first order autoregression on estimated Solow residuals to estimate ρ and σ_ε^2.
  2. Set ρ to some number and then choose σ_ε^2 to match the volatility of output in the data.

Log-linearization

- We now approximate the policy functions around the steady state.
- Rather than linearizing, most economists choose to log-linearize their models.
  - Log-linear equation often describes the data better.
  - Nice interpretation as percentage deviation from steady state.
- Define for variable z_t, its log-deviation from the steady state

$$\hat{z}_t = \log\left(z_t/\bar{z}\right).$$

Note that z_t can then be written as

$$z_t = \bar{z}e^{\hat{z}_t}.$$

Log-linearization

- We start from our equilibrium conditions and write them in terms of *hat-variables:*

$$\mathbb{E}_t \left[f \left(\bar{x} e^{\hat{x}_{t+1}}, \bar{y} e^{\hat{y}_{t+1}}, \bar{x} e^{\hat{x}_t}, \bar{y} e^{\hat{y}_t} \right) \right] = 0. \tag{17}$$

- We linearize the system around $\hat{z}_t = 0$ for all variables z_t.
- Consider row j of matrix $f(\cdot)$. Then,

$$\mathbb{E}_t \left[\bar{f}_{j,1} \hat{x}_{t+1} + \bar{f}_{j,2} \hat{y}_{t+1} + \bar{f}_{j,3} \hat{x}_t + \bar{f}_{j,4} \hat{y}_t \right] \approx 0, \ \forall j$$

  where $f_{j,k}(\bar{x}, \bar{y}, \bar{x}, \bar{y})$ is the partial derivative of $f_j(\cdot)$ with respect to its k argument.
- Equivalently:

$$\begin{bmatrix} \bar{f}_{j,1} & \bar{f}_{j,2} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = - \begin{bmatrix} \bar{f}_{j,3} & \bar{f}_{j,4} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}$$

Log-linearization

Equivalently:

$$\begin{bmatrix} \bar{f}_{j,1} & \bar{f}_{j,2} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = - \begin{bmatrix} \bar{f}_{j,3} & \bar{f}_{j,4} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}.$$

Stacking over j

$$\underbrace{\begin{bmatrix} \bar{f}_{1,1} & \bar{f}_{1,2} \\ \vdots & \vdots \\ \bar{f}_{J,1} & \bar{f}_{J,2} \end{bmatrix}}_{\equiv \mathbf{A}} \mathbb{E}_t \underbrace{\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix}}_{\equiv \mathbf{z}_{t+1}} = - \underbrace{\begin{bmatrix} \bar{f}_{1,3} & \bar{f}_{1,4} \\ \vdots & \vdots \\ \bar{f}_{J,3} & \bar{f}_{J,4} \end{bmatrix}}_{\equiv \mathbf{B}} \underbrace{\begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}}_{\mathbf{z}_t}. \tag{18}$$

In what follows we construct the matrices **A** and **B** for the RBC model, and discuss how to solve equation (18) for the policy function.

Equation (1):

Write equation (1) as:

$$0 = \frac{1}{c_t} - \lambda_t$$
$$0 = \frac{1}{\bar{c}} e^{-\hat{c}_t} - \bar{\lambda} e^{\hat{\lambda}_t}.$$

First order Taylor expansion around $\left(\hat{c}_t, \hat{\lambda}_t\right) = (0,0)$,

$$0 \approx \frac{1}{\bar{c}} - \bar{\lambda} - \frac{1}{\bar{c}} \hat{c}_t - \bar{\lambda} \hat{\lambda}_t.$$

Using that in steady state $\frac{1}{\bar{c}} = \bar{\lambda}$ gives

$$0 \approx \hat{c}_t + \hat{\lambda}_t. \tag{19}$$

Equation (2):

$$
\begin{aligned}
0 &= \eta\, l_t^{\frac{1}{v}} - \lambda_t \left(1-\alpha\right) \frac{y_t}{l_t} \\
&= \eta\, \bar{l} e^{\frac{1}{v}\hat{l}_t} - \bar{\lambda} \left(1-\alpha\right) \frac{\bar{y}}{\bar{l}} e^{\hat{\lambda}_t + \hat{y}_t - \hat{l}_t}.
\end{aligned}
$$

Linearize around $\left(\hat{l}_t, \hat{\lambda}_t, \hat{y}_t\right) = (0,0,0)$,

$$
0 \approx \eta\, \bar{l}^{\frac{1}{v}} \frac{1}{v} \hat{l}_t - \bar{\lambda} \left(1-\alpha\right) \frac{\bar{y}}{\bar{l}} \left[\hat{\lambda}_t + \hat{y}_t - \hat{l}_t\right].
$$

In steady state $\eta\, \bar{l}^{\frac{1}{v}} = \bar{\lambda} \left(1-\alpha\right) \frac{\bar{y}}{\bar{l}}$, then

$$
0 \approx \left(1 + \frac{1}{v}\right) \hat{l}_t - \hat{\lambda}_t - \hat{y}_t \tag{20}
$$

Equation (3):

Disregard the expectation operator and write:

$$
\begin{aligned}
0 &= \beta\left[\lambda_{t+1}\left(\alpha\frac{y_{t+1}}{k_{t+1}}+1-\delta\right)\right]-\lambda_t \\
&= \beta\bar{\lambda}e^{\hat{\lambda}_{t+1}}\left(\alpha\frac{\bar{y}}{\bar{k}}e^{\hat{y}_{t+1}-\hat{k}_{t+1}}+1-\delta\right)-\bar{\lambda}e^{\hat{\lambda}_t}.
\end{aligned}
$$

Linearize around $\left(\hat{\lambda}_{t+1},\hat{y}_{t+1},\hat{k}_{t+1},\hat{\lambda}_t\right)=(0,0,0,0)$,

$$
0 \approx \beta\bar{\lambda}\left(\alpha\frac{\bar{y}}{\bar{k}}+1-\delta\right)\hat{\lambda}_{t+1}+\beta\bar{\lambda}\alpha\frac{\bar{y}}{\bar{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)-\bar{\lambda}\hat{\lambda}_t.
$$

Dividing by $\bar{\lambda}$ and using that in steady state $\beta\left(\alpha\frac{\bar{y}}{\bar{k}}+1-\delta\right)$,

$$
0 \approx \hat{\lambda}_{t+1}+\beta\alpha\frac{\bar{y}}{\bar{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)-\hat{\lambda}_t.
$$

Putting back the expectation operator gives

$$
0 \approx \mathbb{E}_t\left[\hat{\lambda}_{t+1}+\beta\alpha\frac{\bar{y}}{\bar{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)-\hat{\lambda}_t\right]. \tag{21}
$$

Equation (4):

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}$$

Already log-linear:

$$\log y_t = \log A_t + \alpha \log k_t + (1-\alpha) \log l_t.$$

Subtracting the same equation at the steady state gives

$$0 = \hat{y}_t - \hat{A}_t - \alpha \hat{k}_t - (1-\alpha)\hat{l}_t. \tag{22}$$

Equation (5):

$$
\begin{aligned}
0 &= y_t - c_t - i_t \\
&= \bar{y} e^{\hat{y}_t} - \bar{c} e^{\hat{c}_t} - \bar{i} e^{\hat{i}_t}
\end{aligned}
$$

Linearizing around $\left(\hat{y}_t, \hat{c}_t, \hat{i}_t\right) = (0, 0, 0)$ gives

$$
0 \approx \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t. \tag{23}
$$

Equation (6):

$$
\begin{aligned}
0 &= k_{t+1} - (1-\delta)\, k_t - i_t \\
&= \bar{k} e^{\hat{k}_{t+1}} - (1-\delta)\, \bar{k} e^{\hat{k}_t} - \bar{i} e^{\hat{x}_t}
\end{aligned}
$$

Linearizing this equation gives

$$0 \approx \bar{k} \hat{k}_{t+1} - (1-\delta)\, \bar{k} \hat{k}_t - \bar{i} \hat{i}_t .$$

But in steady stat $\bar{i} = \delta \bar{k}$ which implies

$$0 \approx \hat{k}_{t+1} - (1-\delta)\, \hat{k}_t - \delta \hat{i}_t . \tag{24}$$

Equation (7):

TFP equation is already log-linear

$$0 = \log A_{t+1} - \rho \log A_t - \varepsilon_{t+1}.$$

Subtracting the same equation at the steady state

$$0 = \hat{A}_{t+1} - \rho \hat{A}_t - \varepsilon_{t+1}.$$

Taking the conditional expectation as of time t then gives

$$0 = \mathbb{E}_t \hat{A}_{t+1} - \rho \hat{A}_t. \tag{25}$$

Summary of log-linear system of equations

$$
\begin{aligned}
0 &= \hat{c}_t + \hat{\lambda}_t \\
0 &= \left(1 + \frac{1}{v}\right)\hat{l}_t - \hat{\lambda}_t - \hat{y}_t \\
0 &= \hat{y}_t - \hat{A}_t - \alpha\hat{k}_t - (1-\alpha)\hat{l}_t \\
0 &= \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t \\
\mathbb{E}_t\left[\hat{k}_{t+1}\right] &= (1-\delta)\hat{k}_t + \delta\hat{i}_t \\
\mathbb{E}_t\left[\hat{\lambda}_{t+1} + \beta\alpha\frac{\bar{y}}{\bar{k}}\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right)\right] &= \hat{\lambda}_t \\
\mathbb{E}_t\hat{A}_{t+1} &= \rho\hat{A}_t.
\end{aligned}
$$

Note that I wrote $\mathbb{E}_t\left[\hat{k}_{t+1}\right]$ even though $\hat{k}_{t+1}$ is chosen (and therefore already known) at time t.

Log-linear system: matrix form

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta\alpha\frac{\bar{y}}{\bar{k}} & \beta\alpha\frac{\bar{y}}{\bar{k}} & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & \left(1+\frac{1}{\nu}\right) & 0 & -1 \\ -\alpha & -1 & 1 & 0 & -(1-\alpha) & 0 & 0 \\ 0 & 0 & \bar{y} & -\bar{c} & 0 & -\bar{i} & 0 \\ 1-\delta & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Numerical solution of the model

Three different ways to obtain numerical solutions using Matlab:

- **Uhlig's toolkit:** requires to log-linearize the system manually (what we did so far). Output includes simulation, detrending, impulse response, etc.
- **Dynare:** does not require to log-linearize. Input $=$ equilibrium conditions. Output as before and more.
- **Solab.m** by Paul Klein: compute approx policy function, requires to enter **A** and **B** matrix manually (the math above was presented with this method in mind).

Numerical solution of the model: solab.m

- We must tell the program how many of the variables in $\mathbf{z}_t$ are state variables. In our case, two: $\hat{k}_t$ and $\hat{A}_t$.
- $\mathbf{A}$ and $\mathbf{B}$ are 7×7 matrices described above.
- Let $\kappa_t = \left[\hat{k}_t, \hat{A}_t\right]'$ be the state variables and $\mathbf{u}_t = \left[\hat{y}_t, \hat{c}_t, \hat{l}_t, \hat{i}_t, \hat{\lambda}_t\right]$ the jump variables.
- The solver delivers the equilibrium of the 'certainty equivalent' model in the form

$$\mathbf{u}_t = \mathbf{F}\kappa_t$$
$$\kappa_{t+1} = \mathbf{P}\kappa_t.$$

- The stochastic solution of the model is obtained by replacing the second equation above with

$$\kappa_{t+1} = \mathbf{P}\kappa_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}$$

Numerical solution of the model: solab.m

Using the calibrated parameter values, the model delivers

$$\mathbf{F} = \begin{bmatrix} 0.22 & 1.33 \\ 0.57 & 0.34 \\ -0.17 & 0.5 \\ -1.1 & 5.07 \\ -0.57 & -0.34 \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} 0.96 & 0.09 \\ 0 & 0.95 \end{bmatrix}.$$

Equivalently, the policy functions are:

$$\begin{aligned} \hat{y}_t &= 0.22\hat{k}_t + 1.33\hat{A}_t \\ \hat{c}_t &= 0.57\hat{k}_t + 0.34\hat{A}_t \\ \hat{l}_t &= -0.17\hat{k}_t + 0.5\hat{A}_t \\ \hat{i}_t &= -1.1\hat{k}_t + 5.07\hat{A}_t \\ \hat{k}_{t+1} &= 0.96\hat{k}_t + 0.09\hat{A}_t \\ \hat{A}_{t+1} &= 0.95\hat{A}_t + \varepsilon_{t+1}. \end{aligned}$$

Once we have the solution, we can compute impulse responses, simulations, etc.