# 1 PS5 Q2

Problem 2 (MWG, Exercise 14.C.6) Reconsider the labor-market screening model in MWG Exercise 13.D.1 (Problem set 3, Problem 2) in which tasks are productive and a type-$\theta$ worker produces $\theta(1+t)$ in output for the firm. Now, however, suppose that there is a single employer. Characterize the solution to this firmâs screening problem (assume that both types of workers have a reservation utility level of 0). Compare the task levels in this solution with those in the equilibrium of the competitive screening model (assuming an equilibrium exists) that you derived in Exercise 13.D.l.

**Solution.** The firm's problem is

$$\max_{(w_H,t_H),(w_L,t_L)} \phi(\theta_H(1+t_H)-w_H)+(1-\phi)(\theta_L(1+t_L)-w_L)$$

$$\begin{aligned} s.t.\quad & w_L - c(t_L,\theta_L) \geq 0 \quad \text{(IR-L)} \\ & w_H - c(t_H,\theta_H) \geq 0 \quad \text{(IR-H)} \\ & w_L - c(t_L,\theta_L) \geq w_H - c(t_H,\theta_L) \quad \text{(IC-L)} \\ & w_H - c(t_H,\theta_H) \geq w_L - c(t_L,\theta_H) \quad \text{(IC-H)} \end{aligned}$$

Note that (IR-L), (IC-H), and $c_\theta < 0$ imply (IR-H):

$$\begin{aligned} 0 &\leq w_L - c(t_L,\theta_L), \text{ by IR-L} \\ &< w_L - c(t_L,\theta_H), \text{ by } c_\theta < 0 \\ &\leq w_H - c(t_H,\theta_H), \text{ by IC-H,} \end{aligned}$$

which implies that (IR-H) is slack.

Then note that (IC-H) must bind, because otherwise the firm could raise $t_H$ without breaking slack (IR-H) or (IC-L), which slackens if we increase $t_H$, thereby raising profits in a feasible manner.

Since (IC-H) binds, (IC-L) must be slack:

$$\begin{aligned} c(t_H,\theta_L) - c(t_L,\theta_L) &> c(t_H,\theta_H) - c(t_L,\theta_H), \text{ since } c_{t\theta} < 0 \\ &= w_H - w_L, \text{ since (IC-H) binds} \\ \leftrightarrow w_L - c(t_L,\theta_L) &> w_H - c(t_H,\theta_L), \end{aligned}$$

so (IC-L) is slack. Now note that (IR-L) must bind, because otherwise the firm could raise $t_L$ without breaking the slack (IC-L) or (IC-H), which slackens if we raise $t_L$.

Thus, (IR-L) and (IC-H) bind while (IR-H) and (IC-L) are slack. We thus rewrite the firm's problem as:

$$\max_{t_H,t_L} \phi[\theta_H(1+t_H) - c(t_H,\theta_H) - c(t_L,\theta_L) + c(t_L,\theta_H)] + +(1-\phi)[\theta_L(1+t_L) - c(t_L,\theta_L)].$$

The FOC with respect to $t_H$ is:

$$\theta_H = c_t(t_H^*, \theta_H),$$

which means that $t_H^*$ here matches the first-best value of $t_H$. The FOC with respect to $t_L$ is:

$$\theta_L = \frac{\phi}{1-\phi}[c_t(t_L^*, \theta_L) - c_t(t_L^*, \theta_H)] + c_t(t_L^*, \theta_L).$$

Note that

$$c_t(t_L^*, \theta_L) - c_t(t_L^*, \theta_H) > 0$$

because $c_{t\theta} < 0$. Furthermore, $\phi/(1-\phi) > 0$, clearly. Thus, since $c(t, \theta)$ is convex, $c_t(t, \theta)$ is increasing in $t$, and so we have $t_L^*$ is less than the first best value of $t_L$, under which

$$\theta_L = c_t(t_L^{FB}, \theta_L).$$

The EQ allocation in the competitive screening model yielded

$$t_L^C = t_L^{FB} > t_L^*$$
$$t_H^C \geq t_H^{FB} = t_H^*$$

## 2 PS5 Q4

(MWG, Exercise 14.C.9) Consider a risk-averse individual who is an expected utility maximizer with a Bernoulli utility function over wealth $u(\cdot)$. The individual has initial wealth $y$ and faces a probability $\pi$ of suffering a loss of size $L$, where $y > L > 0$. An insurance contract may be described by a pair $(c_1, c_2)$, where $c_1$ is the amount of wealth the individual has in the event of no loss and $c_2$ is the amount the individual has if a loss is suffered. That is, in the event no loss occurs the individual pays the insurance company an amount $y - c_1$, whereas if a loss occurs the individual receives a payment $c_2 - y + L$ from the company.

**Procblem 2.1.** Suppose that the individual's only source of insurance is a risk-neutral monopolist (i.e., the monopolist seeks to maximize its expected profits). Characterize the contract the monopolist will offer the individual in the case in which the individual's probability of loss, $\pi$, is observable.

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**Solution.** If the individual $i$'s type $\pi_i$ is observable, then the firm would maximize profits by offering the policy $(c_1, c_2)$ such that the individual is indifferent between the two scenarios with insurance, and between having and not having insurance, i.e.

$$c_1 = c_2 \text{ and } u(c_1) = u(c_2) = \pi_i u(y - L) + (1 - \pi_i) u(y)$$

So, define $\underline{u} \equiv \pi_i u(y - L) + (1 - \pi_i) u(y)$, and we have that the monopolist firm would offer the contract

$$c_1 = c_2 = u^{-1}(\underline{u})$$

Note that offering coverage earns strictly positive profits for the firm because customers are risk averse, so actuarially fair policy (zero profit for the firm) would give strictly positive surplus to customers, and now that customers have zero surplus, so it must be the case the firms are making strictly positive profits. So, for the firm, offering policy is better than offering nothing.

**Procblem 2.2.** Suppose, instead, that $\pi$ is not observable by the insurance company (only the individual knows $\pi$). The parameter $\pi$ can take one of two values $\pi_H > \pi_L > 0$ and Prob $[\pi = \pi_L] = \pi \in (0, 1)$. Characterize the optimal contract offers of the monopolist. Can one speak of one type of insured individual being "rationed" in his purchases of insurance (i.e., he would want to purchase more insurance if allowed to at fair odds)? Intuitively, why does this rationing occur? [Hint: It might be helpful to draw a picture in $(c_1, c_2)$-space. To do so, start by locating the individual's endowment point, that is, what he gets if he does not purchase any insurance.]

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**Solution.** The monopolist would offer an optimal contract in the form $\left\{\left(c_1^L, c_2^L\right), \left(c_1^H, c_2^H\right)\right\}$, which is the solution to

$$\max_{\left\{\left(c_1^L, c_2^L\right), \left(c_1^H, c_2^H\right)\right\}} \phi\left(\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)\right)+(1-\phi)\left(\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right)\right)$$

subject to

$$\begin{aligned}
\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right) &\geq \underline{u} && \text{IR 1}\\
\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right) &\geq \underline{u} && \text{IR 2}\\
\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right) &\geq \left(1-\pi_H\right) u\left(c_1^L\right)+\pi_H u\left(c_2^L\right) && \text{IC 1}\\
\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right) &\geq \left(1-\pi_L\right) u\left(c_1^H\right)+\pi_L u\left(c_2^H\right) && \text{IC 2}
\end{aligned}$$

We can show that IR1 + IC2 $\Rightarrow$ IR2.

*Proof.* Note that $\pi_H > \pi_L$, $c_1^H \geq c_2^H$ and $u(\cdot)' > 0$ together imply that

$$\left(1-\pi_L\right) u\left(c_1^H\right)+\pi_L u\left(c_2^H\right) \geq \left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)$$

Then, use this result in IC2 to have

$$\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right) \geq \left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)$$

By IR1 we know that

$$\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right) \geq \underline{u}$$

So, finally we have

$$\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right) > \underline{u}$$

■

So, IR2 is redundant. We can drop IC1 for now (and later check that IC1 satisfied) to consider the relaxed seller's problem:

$$\max_{\left\{\left(c_1^L, c_2^L\right), \left(c_1^H, c_2^H\right)\right\}} \phi\left(\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)\right)+(1-\phi)\left(\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right)\right)$$

subject to

$$\begin{aligned}
\left(1-\pi_H\right) u\left(c_1^H\right)+\pi_H u\left(c_2^H\right) &\geq \underline{u} && \text{IR 1}\\
\left(1-\pi_L\right) u\left(c_1^L\right)+\pi_L u\left(c_2^L\right) &\geq \left(1-\pi_L\right) u\left(c_1^H\right)+\pi_L u\left(c_2^H\right) && \text{IC 2}
\end{aligned}$$

First, let's prove that IC2 is binding in the relaxed problem.

*Proof.* Suppose not, then we have

$$(1-\pi_L)\,u\left(c_1^L\right)+\pi_L u\left(c_2^L\right)>(1-\pi_L)\,u\left(c_1^H\right)+\pi_L u\left(c_2^H\right)$$

In that case, we have a $\varepsilon>0$ s.t. IC2 still holds for $\tilde{c}_1^L=c_1^L-\varepsilon$ and $\tilde{c}_2^L=c_2^L-\varepsilon$, i.e.

$$(1-\pi_L)\,u\left(\tilde{c}_1^L\right)+\pi_L u\left(\tilde{c}_2^L\right)>(1-\pi_L)\,u\left(c_1^H\right)+\pi_L u\left(c_2^H\right)$$

And notice that $\left(c_1^L,c_2^L\right)$ only appears in IC2, so IR1 is not affected, therefore both constraints hold with $\left(\tilde{c}_1^L,\tilde{c}_2^L\right)$ . Since the seller makes more money with $\left(\tilde{c}_1^L,\tilde{c}_2^L\right)$ than$\left(c_1^L,c_2^L\right)$, we know that $\left(c_1^L,c_2^L\right)$ cannot be the solution, which implies that any solution must have IC2 binding. ■

Then, let's prove that IR1 is binding in the relaxed problem.

*Proof.* Suppose not, then we have

$$(1-\pi_H)\,u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)>\underline{u}$$

In that case, we have a $\varepsilon>0$ s.t. IR1 still holds for $\tilde{c}_1^H=c_1^H-\varepsilon$ and $\tilde{c}_2^H=c_2^H-\varepsilon$, i.e.

$$(1-\pi_H)\,u\left(\tilde{c}_1^H\right)+\pi_H u\left(\tilde{c}_2^H\right)>\underline{u}$$

And notice that in IC2 a lower $\left(c_1^H,c_2^H\right)$ makes IC2 hold more easily, therefore both constraints hold with $\left(\tilde{c}_1^H,\tilde{c}_2^H\right)$. Since the seller makes more money with $\left(\tilde{c}_1^H,\tilde{c}_2^H\right)$ than $\left(c_1^H,c_2^H\right)$, we know that $\left(c_1^H,c_2^H\right)$ cannot be the solution, which implies that any solution must have IR1 binding. ■

Based on the two results shown above, we have

$$\begin{aligned}
(1-\pi_H)\,u\left(c_1^H\right)+\pi_H u\left(c_2^H\right)&=\underline{u} && \text{IR 1}\\
\Rightarrow u\left(c_1^H\right)&=\frac{\underline{u}-\pi_H u\left(c_2^H\right)}{1-\pi_H}\\
(1-\pi_L)\,u\left(c_1^L\right)+\pi_L u\left(c_2^L\right)&=(1-\pi_L)\,u\left(c_1^H\right)+\pi_L u\left(c_2^H\right) && \text{IC 2}\\
u\left(c_1^L\right)&=\frac{(1-\pi_L)\,u\left(c_1^H\right)+\pi_L u\left(c_2^H\right)-\pi_L u\left(c_2^L\right)}{1-\pi_L}\\
&=\frac{(1-\pi_L)\,\frac{\underline{u}-\pi_H u\left(c_2^H\right)}{1-\pi_H}+\pi_L u\left(c_2^H\right)-\pi_L u\left(c_2^L\right)}{1-\pi_L}\\
&=\frac{\underline{u}-\pi_H u\left(c_2^H\right)}{1-\pi_H}+\frac{\pi_L u\left(c_2^H\right)-\pi_L u\left(c_2^L\right)}{1-\pi_L}
\end{aligned}$$

and we can rewrite the relaxed seller's problem as:

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\begin{aligned}
&\max_{\left\{c_2^H, c_2^L\right\}} \phi\left((1-\pi_H)\frac{\underline{u}-\pi_H u\left(c_2^H\right)}{1-\pi_H}+\pi_H u\left(c_2^H\right)\right) \\
&\qquad +(1-\phi)\left((1-\pi_L)\left(\frac{\underline{u}-\pi_H u\left(c_2^H\right)}{1-\pi_H}+\frac{\pi_L u\left(c_2^H\right)-\pi_L u\left(c_2^L\right)}{1-\pi_L}\right)+\pi_L u\left(c_2^L\right)\right) \\
\Leftrightarrow &\max_{\left\{c_2^H, c_2^L\right\}} \phi\underline{u}+(1-\phi)\left(\frac{(1-\pi_L)\left(\underline{u}-\pi_H u\left(c_2^H\right)\right)}{1-\pi_H}+\pi_L u\left(c_2^H\right)-\pi_L u\left(c_2^L\right)+\pi_L u\left(c_2^L\right)\right) \\
\Leftrightarrow &\max_{\left\{c_2^H\right\}} \phi\underline{u}+(1-\phi)\left(\frac{(1-\pi_L)\left(\underline{u}-\pi_H u\left(c_2^H\right)\right)}{1-\pi_H}+\pi_L u\left(c_2^H\right)\right)
\end{aligned}
$$

Since

$$\pi_L\left(1-\pi_H\right)<\left(1-\pi_L\right)\pi_H$$

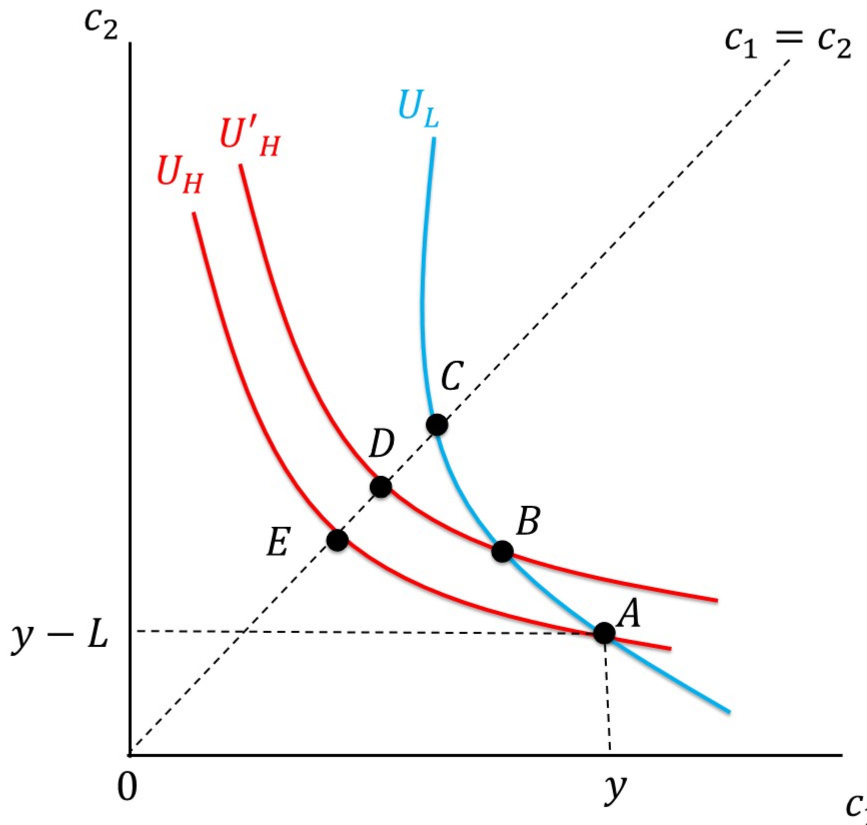
We have that

$$\left[-\frac{\left(1-\pi_L\right)\pi_H}{1-\pi_H}+\pi_L\right]u'\left(c_2^H\right)<0$$

which means that the objective function is decreasing in $c_2^H$, so the optimal $c_2^H$ is the lowest corner solution, which is characterized in (a), i.e.

$$c_1^H=c_2^H=u^{-1}\left(\underline{u}\right)$$

which is true because we would always have the optimal contract as $c_1^H=c_2^H$ because agent is risk averse while firm is risk neutral, so consumption smoothing enables the firm to even lower the consumption bundles and still make the agent indifferent. So, high type have the full coverage. See the graph below.




The point $A$ is the utility level without insurance for both agents. Since we have shown that high type guy will receive full insurance and stay indifferent between having and not having insurance, point $E$ is the contract offered to high type guy.

The monopolist would also like to offer policy $C$ to the low type guy, however, this is not incentive compatible for high type guy, because point $C$ is more favorable to him than point $E$. So, point $E$ and $C$, which are first best for both types, are not implementable.

One candidate contract for the low type guy such that high type won't have incentive to deviate from point $E$ is a point to the right of point $A$ below the red line. However, such point represents for the contract that awards money to customers when risk doesn't happen and take away money when risk does happen. Since the customer has very low marginal utility without risk happening and high marginal utility with risk happening, such policy implies that the firm need to award much more money in one scenario than it takes away from the customer in the other scenario, which is not profit maximizing for the firm.

So, any non-negative insurance offered to low type guy must raise the utility level of high type guy. For example, if the low type guy's fraction is high enough, then the firm may decide to offer the low type some insurance, then point $B$ would be offered to low type, and incentive compatibility implies that point $D$ is the contract offered to high type in that case.

In conclusion, the result is that high type would always have the first best (full coverage), and low type would be under-insured.

**Procblem 2.3.** Compare your solution in (b) with your what we know about the competitive screening outcome in the insurance-market model of JR, chapter 8.

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**Solution.** The difference is that in this monopolistic screening model, the firm gets the positive profits and customers (at least the low type) are left with indifference between insurance and no insurance (i.e. have reservation utility). However, in competitive screening case, firms have zero profits and customers have positive surplus.