1 Q1:

Reconsider the two-type signaling model with $r(\theta_l) = r(\theta_h) = 0$, assuming that a worker's productivity is $\theta(1+e)$. As before, $\theta_h > \theta_l > 0$, the probability of θ_h is $\phi \in (0,1)$, and the worker's cost of education is $c(e,\theta)$, where $c(e,\theta)$ is increasing and convex in e, decreasing in θ, $c(0,\theta) = 0$, and $c_{e\theta}(e,\theta) < 0$ for $e > 0$.

Procblem 1.1. Assume that the worker's productivity is observable and contractible by the labor market. Characterize the competitive equilibrium wages and education levels for each type of worker in this complete information game.

Solution. In complete information game, the competitive equilibrium wage and education levels are as follows.

▷ Education level:

* Interior solution case:
  - type θ_h: competitive equilibrium education level is $e_h^\star$ s.t. $c_e(e_h^\star, \theta_h) = \theta_h$
  - type θ_l: competitive equilibrium education level is $e_l^\star$ s.t. $c_e(e_l^\star, \theta_l) = \theta_l$
* Corner solution case
  - In the corner solution case, if for type $i \in \{l, h\}$ $c_e(0, \theta_i) > \theta_i$, then the optimal education level is $e_i^\star = 0$.

▷ Wage:

* type θ_h: competitive equilibrium wage is $w_h^\star = \theta_h(1 + e_h^\star)$
* type θ_l: competitive equilibrium wage is $w_l^\star = \theta_l(1 + e_l^\star)$

To see why this is true, let's prove by way of contradiction. Suppose wages are not set in this way, then it must be lower than this level since firms cannot lose money in equilibrium. WLOG, suppose the equilibrium wage for type θ_h is $w_h < \theta_h(1 + e_h^\star)$. Then there will be a third firm who can set the wage as $\tilde{w}_h = w_h + \varepsilon$ for $\varepsilon \in (0, \theta_h(1 + e_h^\star) - w_h)$, then this third firm can steal all the type θ_h workers and earn strictly positive profits. So, wages have to be set in the way described above.

Now, let's look at the education level. Workers are payoff maximizers, so they will take the f.o.c. to decide the optimal level of education. Given that firms offer wages equal to productivities in equilibrium, workers would have incentive to deviate if they don't put effort in this way.

Procblem 1.2. Demonstrate that for some preferences, the outcome in (a) may arise in a separating equilibrium in the incomplete information game.

Solution. See the figure below.

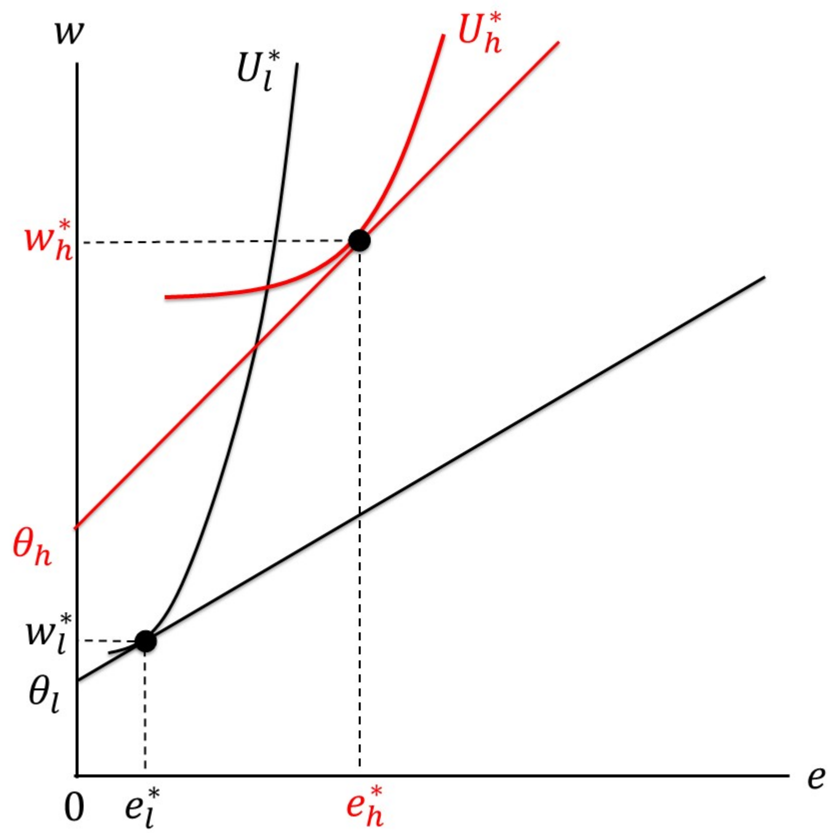

Figure 1: Separating Equilibrium with Same Outcome as in (a)

In this figure, the preferences are such that the low type won't have incentive to behave as high type and vice versa, which can also be characterized by

$$c\left(e_l^\star, \theta_h\right) \geq c\left(e_h^\star, \theta_h\right) + \theta_l\left(1 + e_l^\star\right) - \theta_h\left(1 + e_h^\star\right)$$

$$c\left(e_h^\star, \theta_l\right) \geq c\left(e_l^\star, \theta_l\right) + \theta_h\left(1 + e_h^\star\right) - \theta_l\left(1 + e_l^\star\right)$$

where we focus on the interior solution cases.

The result, as indicated in the figure, high type will put effort $e_h^\star$ s.t. $c_e\left(e_h^\star, \theta_h\right) = \theta_h$, low type will put effort $e_l^\star$ s.t. $c_e\left(e_l^\star, \theta_l\right) = \theta_l$. The wage for high type is $w_h^\star = \theta_h\left(1 + e_h^\star\right)$, and the wage for low type is $w_l^\star = \theta_l\left(1 + e_l^\star\right)$.

Procblem 1.3. Assume that the underlying preferences are such that full-information competitive allocation in (a) does not arise in any separating equilibrium to the incomplete information game. Indicate the range of education levels that can be supported in a PBE separating equilibrium and the range of outputs that can be supported in a PBE pooling equilibrium.

Solution. See the figure below.

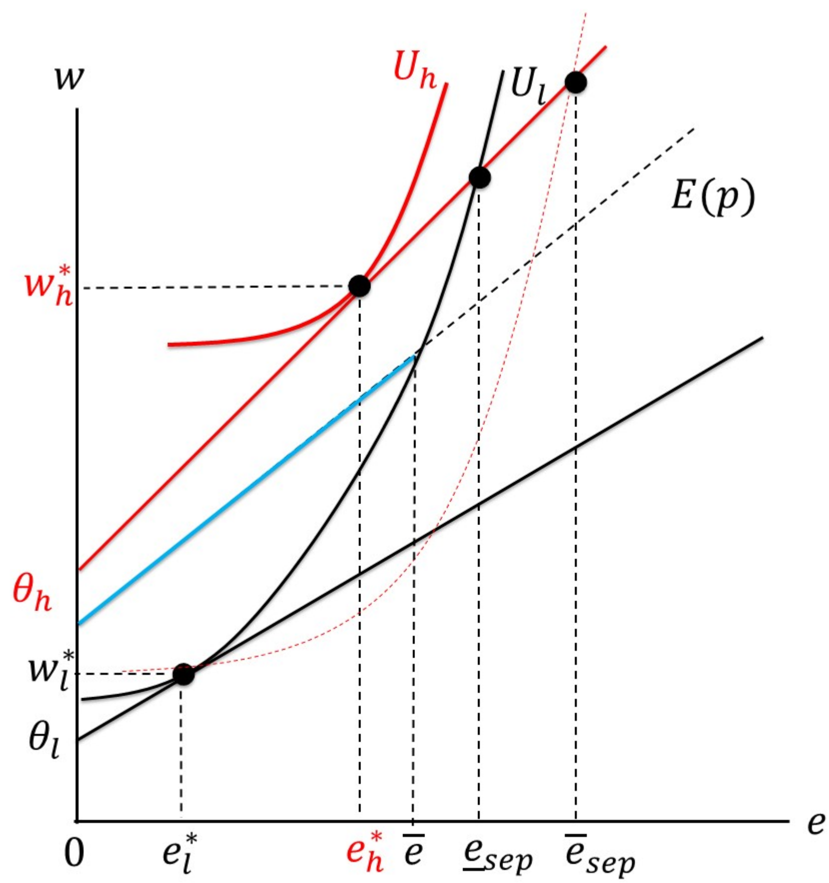

Figure 2: 1 (c)

As indicated in the graph, the wage offer

$$w(e) = \begin{cases} \theta_h (1+e) & \text{if } e \in \left[\underline{e}_{sep}, \overline{e}_{sep}\right] \\ \theta_l (1+e) & \text{otherwise} \end{cases}$$

supports separating equilibrium in incomplete information game, because $\underline{e}_{sep}$ is the lowest level of effort such that low type don't have incentive to pretend to be high type, and $\overline{e}_{sep}$ is the highest level of effort such that high type don't have incentive to pretend to be low type.

For pooling equilibrium ($E\left(p\right)$ in the figure is the pooling average productivity line), the blue line segment consists the set of offers of wage and effort levels that supports pooling equilibrium. Those wages are the average output of the pooling workers, and the education level that supports pooling is in the range $[0, \overline{e}]$.

To see why, first notice that the firm will earn zero profit in competitive equilibrium, so the offer must be on the $E\left(p\right)$ line. Then, to make sure that both types won't deviate from pooling, the offer cannot be to the right of the blue segment, otherwise the low type would deviate to play $(w_l^\star, e_l^\star)$.

2 Q3:

Consider a principal-agent model with moral hazard in which the principal is risk neutral and the agent is risk averse with $u(w) = \sqrt{100+w}$. There are two efforts, $e_h > e_l$, with personal cost to the agent of $\psi(e_h) = 1$ and $\psi(e_l) = 0$. The agent's expected utility from wage contract $w(\cdot)$ and effort $e$is $\mathbb{E}\left[\sqrt{100+w(x)} \mid e\right] - \psi(e)$.

There are two outcomes. The high-output outcome is $x_2 = 200$ and the low-outcome output is $x_1 = 100$. When the agent exerts low effort, each output is equally likely. When the agent exerts high effort, the probability of high output is $\frac{3}{4}$ (and the probability of low output is $\frac{1}{4}$). The agent's reservation utility is $\underline{\mathrm{U}} = u(0) = 10$.

Procblem 2.1. Does this distribution satisfy MLRP?

Solution. If $f(x \mid e)$ is differentiable in e, then we say that the distribution satisfied the monotone-likelihood ratio property (MLRP) iff for any e,

$$\frac{f_e(x \mid e)}{f(x \mid e)} \text{ is increasing in } x$$

If f is not differentible, (e.g. e can take on only two types), then we modify our definition to accommodate differences in place of derivatibes. In the 2-effort case, we say f satisfies MLRP iff

$$\frac{f_H(x) - f_L(x)}{f_H(x)} \text{ is increasing in } x$$

We can use the second definition to show that this distribution satisfies MLRP. In particular, notice the following

$$\frac{f_H(x_1) - f_L(x_1)}{f_H(x_1)} = \frac{\frac{1}{4} - \frac{1}{2}}{\frac{1}{4}} = -\frac{\frac{1}{4}}{\frac{1}{4}} = -1$$

and

$$\frac{f_H(x_2) - f_L(x_2)}{f_H(x_2)} = \frac{\frac{3}{4} - \frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

and so MLRP is satisfied as $\frac{1}{3} > -1$.

Procblem 2.2. Solve for the optimal output-contingent wage contract, $\{w_1^\star, w_2^\star\}$

Solution. Consider the following program

$$\max_{e,w} \int_X (x - w(x)) f(x \mid e)\, dx$$

subject to

$$\int_X u(w(x)) f(x \mid e)\, dx - \psi(x) \geq \underline{\mathrm{U}} \quad \text{(IR)}$$

$$e \in \max_e \int_X u(w(x)) f(x \mid e)\, dx - \psi(e) \qquad \text{(IC)}$$

the solution method is to find wages s.t. the IR and IC constraints bind, and then use these to solve for the equilibrium wage. First, let's consider the IC constraint

1. High effort: if high effort, then we have the following IR constraint:

$$\begin{aligned} \frac{1}{4}\sqrt{100 + w_1^\star} + \frac{3}{4}\sqrt{100 + w_2^\star} - 1 &\geq 10 \\ \sqrt{100 + w_1^\star} + 3\sqrt{100 + w_2^\star} &\geq 44 \end{aligned}$$

   and we also have the IC constraint

$$\begin{aligned} \frac{1}{4}\sqrt{100 + w_1^\star} + \frac{3}{4}\sqrt{100 + w_2^\star} - 1 &\geq \frac{1}{2}\sqrt{100 + w_1^\star} + \frac{1}{2}\sqrt{100 + w_2^\star} \\ -\sqrt{100 + w_1^\star} + \sqrt{100 + w_2^\star} &\geq 4 \end{aligned}$$

   we can add these two conditions together to get

$$\begin{aligned} 4\sqrt{100 + w_2^\star} &\geq 48 \\ \sqrt{100 + w_2^\star} &\geq 12 \\ w_2^\star &\geq 44 \end{aligned}$$

   which will hold with equality, so $w_2^\star = 44$. Then we have from the IR constraint that

$$\begin{aligned} 3 \times \sqrt{100 + w_1^\star} + \sqrt{144} &\geq 44 \\ \sqrt{100 + w_1^\star} &\geq 8 \\ 100 + w_1^\star &\geq 64 \\ w_1^\star &\geq -36 \end{aligned}$$

   and then from the IC constrant that

$$\begin{aligned} -\sqrt{100 + w_1^\star} + \sqrt{100 + 44} &\geq 4 \\ -\sqrt{100 + w_1^\star} + 12 &\geq 4 \\ 8 &\geq \sqrt{100 + w_1^\star} \\ 64 &\geq 100 + w_1^\star \\ -36 &\geq w_1^\star \end{aligned}$$

and so the cost-minimizing wage schedule that satisfies the IR and IC constraints is $(w_1^\star, w_2^\star) = (-36, 44)$.

2. Next let's try to characterize the low effort equilibrium. In particular, we can now specialize the problem to the following:

$$\begin{aligned}\frac{1}{2}\sqrt{100+w_1^\star}+\frac{1}{2}\sqrt{100+w_2^\star} &\geq 10\\ \sqrt{100+w_1^\star}+\sqrt{100+w_2^\star} &\geq 20\end{aligned}$$

and we also have the IC constraint

$$\begin{aligned}\frac{1}{2}\sqrt{100+w_1^\star}+\frac{1}{2}\sqrt{100+w_2^\star} \geq& \frac{1}{4}\sqrt{100+w_1^\star}+\frac{3}{4}\sqrt{100+w_2^\star}-1\\ \frac{1}{4}\sqrt{100+w_1^\star}-\frac{1}{4}\sqrt{100+w_2^\star} \geq& -1\\ \sqrt{100+w_1^\star}-\sqrt{100+w_2^\star} \geq& -4\end{aligned}$$

Notice that the wage contract $(w_1^\star, w_2^\star) = (0,0)$ is sufficient in this case to induce the agent to exert zero effort. Further, any further decrease in the wage $w_i^\star$ will induce a greater than one-to-one increase in the wage $w_{-i}^\star$ in order for the IR constraint to be satisfied and so this is the cost-minimizing wage schedule.

Now notice that the principal does better with the contract that induces effort, i.e.

$$\begin{aligned}\frac{3}{4}(200-44)+\frac{1}{4}(100+36) &\geq 150\\ 151 &\geq 150\end{aligned}$$

and so the optimal wage contract is $(w_1^\star, w_2^\star) = (-36, 44)$.

3 Q5:

Consider a simple moral hazard where the principal and the agent are risk neutral (i.e., $v'(\cdot) = u'(\cdot) = 1$), there is no uncertainty over output, but now output depends upon the actions of the agent and the principal. Specifically, the agent chooses $e_1 \in [0, 2]$, the principal chooses $e_2 \in [0, 2]$ and the output is deterministic:

$$x = e_1 + e_2 \in \mathcal{X} = [0, 4]$$

Assume that the cost of effort for each player is $\frac{1}{2}e_i^2$, so that first-best production requires $e_i^{fb} = 1$ and $x^{fb} = 2$.

Procblem 3.1. Assume that the principal can offer a contract to the agent which promises $s(x)$ in payment for the outcome x, and the residual profit, $x - s(x)$, is kept by the principal. After the agent accepts the contract, both the principal and agent simultaneously choose their individual efforts, e_i. Once x is revealed, payments are shared as promised with $s(x)$ going to the agent and $x - s(x)$ going to the principal.

If $s(x)$ is required to be continuously differentible on $\mathcal{X}$, show that the first best cannot be implemented by the principal.

Solution. Suppose that the principal could write down a contract that incentivized the first best effort level. We will show that, if the agent works at the first best effort level, the principal will have no incentive to work. Consider our original program:

$$\max_{e_2, s(x)} e_1 + e_2 - \frac{1}{2}e_2^2 - s(e_1 + e_2)$$

s.t.

$$s(e_1 + e_2) - \frac{1}{2}e_1^2 \geq 0 \tag{IR}$$

$$s(e_2 + 1) - \frac{1}{2} \geq s(e_2 + \tilde{e}_1) - \frac{1}{2}\tilde{e}_1^2 \quad \forall\, \tilde{e}_1 \tag{IC}$$

The agent's problem is

$$\max_{e_1} s(e_1 + e_2) - \frac{1}{2}e_1^2$$

and so we get the first-order condition

$$s'(e_1 + e_2) - e_1 = 0$$

but because of the IC constrant, we know that $e_1 = 1$ in this equation and so we get that

$$s'(e_1 + e_2) = e_1$$
$$s'(e_2 + 1) = 1$$

notice that the agent's IR constraint will hold for this effort level if $s(x) \geq \frac{1}{2}$, which is feasible since the output of the agent alone is strictly greater than $\frac{1}{2}$. So there exists a contract that satisfies both the agent's IR and IC constraints and where the agent exerts the first-best effort level. Now reconsider the principal's problem taking this contract as fixed. In particular, we have

$$\max_{e_2} e_2 + 1 - \frac{1}{2}e_2^2 - s(e_2 + 1)$$

with the associated FOC

$$\begin{aligned} 1 - e_2 - \underbrace{s'(e_2+1)}_{=1 \text{ from above}} &= 0 \\ e_2 &= 0 \end{aligned}$$

and so the first-best effort level cannot be implemented as now $e_2 = 0 \neq 1 = e_2^{fb}$.

Procblem 3.2. Suppose that instead of designing a sharing rule, $\{s(x), x - s(x)\}$, the principal can commit to giving some of the output to a third party for some values of x. I.e., a principal can offer $s(x)$ to the agent, and can commit to giving $z(x)$ to a third party, keeping $x - s(x) - z(x)$ for herself. Show that the first best can now be implemented as a NE between the principal and agent and that nothing is given to the third part along the equilibrium path. For this part, you may construct the implementing functions using discountinuous s and z functions.

Solution. Suppose that the we have

$$z(x) = \begin{cases} x & x \neq 2 \\ 0 & x = 2 \end{cases}$$

and let

$$s(x) = \begin{cases} 1 & x = 2 \\ 0 & x \neq 2 \end{cases}$$

Now consider the candidate Nash Equilibrium where both agents exert the first best effort level, $e_i = e_i^{fb} = 1$. Notice that neither agent has an incentive to deviate as if either agent unilaterally deviates then that agent will receive at most 0, which is strictly less than the payoff of $\frac{1}{2}$ that the agent receives from exerting effort level 1.

Procblem 3.3. Assume that we are back to setting (a) without a third party, and thus the share contracts are limited to $\{s(x), x - s(x)\}$. Now, however, assume that the sets of feasible actions are discrete $e_1 \in \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$ and $e_2 \in \left\{\frac{2}{3}, 1, \frac{4}{3}\right\}$. Can the first best $e_1^\star = e_2^\star = 1$ be implemented?

Solution. Yes, in particular consider a discrete contract that specifies

$$s(x) = \begin{cases} 1 & x = 2 \\ x & x \in \left\{\frac{5}{3}, \frac{7}{3}\right\} \\ 0 & \text{otherwise} \end{cases}$$

Now consider the incentives of the agent and principal, respectively,

1. The agent will exert effort 1. To see this, notice that if the agent exerts effort 1he is guaranteed a positve payoff, i.e. his minimum payoff when he exerts effort $e_1 = 1$ is

   $$1 - \frac{1}{2} > 0 - \frac{1}{2}\tilde{e}_1^2$$

   for any $\tilde{e}_1 \neq 1$. Where the payoff of any other effort level is zero because the sum of effort level $\tilde{e}_1 \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$ and any element $e_2 \in \left\{\frac{2}{3}, 1, \frac{4}{3}\right\}$ will not lie in the set $\left\{\frac{5}{3}, 2, \frac{7}{3}\right\}$.

2. The principal will exert effort 1 for the same reason. I.e., the minium payoff when the principal exerts effort $e_2 = 1$ is

   $$1 - \frac{1}{2} > 0 - \frac{1}{2}\tilde{e}_2^2$$

   as for any effort $\tilde{e}_2 \neq 1$ when the agent exerts effort one (as we have already shown he will do) the agent will receive all of the proceeds of the project.

So we have found a contract that implements the first best effort level.