Reference Sheet - Willy Chen (Latest Version)

Common Distributions

Normal $X \sim N(\mu, \sigma^2)$

$$PDF : \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2})$$
$$MGF : exp(\mu t + \frac{\sigma^2 t^2}{2}$$

Lognormal $X \sim Lognormal(\mu, \sigma^2)$

$$PDF : \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{1}{2}\frac{(log(x)-\mu)^2}{\sigma^2}),\ x > 0$$
$$E[X] = exp(\mu + \frac{\sigma^2}{2}),\ Var(X) = [exp(\sigma^2) - 1]E[X]^2$$

Note: A lognormally distributed r.v. is an r.v. whose logged version is normally distributed.

Chi-Square $X \sim \chi^2_n$

Let $Z \sim Normal(0, I_n)$, $Z'Z = \sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$

$$E[X] = n,\ Var(X) = 2n$$

t Distribution with $df = n$

Let $Z \sim Normal(0,1)$, $X \sim \chi^2_n$. Define $T \equiv \frac{Z}{\sqrt{X/n}}$.

Then $T \sim \mathcal{T}_n$. As $\lim_{n\to\infty} \mathcal{T}_n \to Normal(0,1)$

F Distribution with $df = n$

Let $X_1 \sim \chi^2_{k_1}$, $X_2 \sim \chi^2_{k_2}$. Define $W \equiv \frac{X_1/k_1}{X_2/k_2} \sim \mathcal{F}_{k_1,k_2}$

Gamma $X \sim Gamma(\alpha, \beta)$

$$PDF : \frac{1}{\beta\Gamma(\alpha)} x^{\alpha-1},\ x > 0$$
$$MGF : (1-\beta t)^{-\alpha},\ t < \frac{1}{\beta}$$
$$E[X] = \alpha\beta,\ Var(X) = \alpha\beta^2$$

When $\alpha = 1$, this is equivalent to $Exponential(\frac{1}{\beta})$.
When $\alpha = \frac{n}{2}$, $\beta = 2$, this is equivalent to the chi-square distribution with $df = n$.
α represents the time waiting and β represents the scale of the event (e.g. $\frac{1}{\beta}$ customers come in every α hours, $\lambda = \frac{\beta}{\alpha}$ for exponential).
Note: This distribution is typically used to model a continuous time until an event. However, generally, the **gamma distribution is NOT memoryless** unless it is the case of an exponential distribution. In a general quesiton, try to use exponential instead (reducing α to 1. See problem $\star$ in selected problems for variations.

Exponential $X \sim Expnential(\lambda)$

$$PDF : \lambda e^{-\lambda x},\ \lambda > 0$$
$$CDF : 1 - e^{-\lambda x}$$
$$MGF : \frac{\lambda}{\lambda - t},\ t < \lambda$$
$$E[X] = \frac{1}{\lambda},\ Var(X) = \frac{1}{\lambda^2}$$

Note: This distribution is typically used to model a continuous time until an event. For an example, see problem $\star$ in selected problems.

Binomial $X \sim Binomial(n, p)$

$$PMF : \binom{n}{k} p^k (1-p)^{n-k}$$
$$MGF : (1 - p + pe^t)^n$$
$$E[X] = np,\ Var(X) = np(1-p)$$

Negative Binomial $X \sim NegBin(\mu, \alpha)$

$$\Gamma(r) = \int_0^\infty exp(-u)u^{r-1}du,\ r > 0$$
$$\Gamma(k) = (k-1)!,\ k \in \mathbb{Z}_{++}$$
$$PMF : \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)x!}\left(\frac{\alpha}{\alpha+\mu}\right)^\alpha \left(\frac{\mu}{\alpha+\mu}\right)^x,\ x \in \mathbb{Z}_+$$
$$MGF : \left(1 + \frac{\mu}{\alpha}[1 - exp(t)]\right)^{-\alpha},\ t < -ln\left(\frac{\mu}{\alpha+\mu}\right)$$
$$E[X] = \mu,\ Var(X) = \mu + \frac{\mu^2}{\alpha}$$

When $\alpha = 1$, this is the *geometric* distribution
As $\alpha \to \infty$, NB converges to $Poisson(\mu)$

Poisson $X\ Poisson(\lambda)$

$$PMF : \frac{exp(-\theta)\theta^x}{x!},\ x \in \mathbb{N} \cup \{0\}$$
$$CDF : exp(-\theta)\sum_{x=0}^{t}\frac{\theta^x}{x!}$$
$$MGF : exp[\theta(exp(t) - 1)]$$
$$E[X] = \theta,\ Var(X) = \theta$$

Note: This distribution is typically used to model the probability of an event happening given a specific time period. λ is the frequency of the event in said time period. **Poisson is memoryless**.

Geometric $X \sim Geometric(p)$

k total trials ($k \in \mathbb{N}$)

$$PMF : (1-p)^{k-1}p$$
$$CDF : 1 - (1-p)^{\lfloor k \rfloor}$$
$$MGF : \frac{pe^t}{1 - (1-p)e^t},\ t < -ln(1-p)$$
$$E[X] = \frac{1}{p},\ Var(X) = \frac{1-p}{p^2}$$

k failures before success ($k \in \mathbb{N} \cup \{0\}$)

$$PMF : (1-p)^k p$$
$$CDF : 1 - (1-p)^{\lfloor k \rfloor + 1}$$
$$MGF : \frac{p}{1 - (1-p)e^t},\ t < -ln(1-p)$$
$$E[X] = \frac{1-p}{p},\ Var(X) = \frac{1-p}{p^2}$$

Important Properties

$\sigma-$algebra

Let Ω be the outcome space and $\mathcal{B}$ be the $\sigma-$algebra generated by $\mathcal{B}$. Then $\mathcal{B}$ must satisfy:

1. $\Omega \in \mathcal{B}$
2. $\forall A \in \mathcal{B},\ A^c \in \mathcal{B}$
3. $\forall i \in \mathbb{N}, A_i \in \mathcal{B},\ \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

Probability of Random Draws

	Without Replacement	With Replacement
Ordered	$P^n_k = \frac{n!}{(n-k)!}$	n^k
Unordered	$C^n_k = \frac{n!}{(n-k)!k!}$	$C^{n+k-1}_k = \frac{(n+k-1)!}{k!(n-1)!}$

Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$P(A|B)P(B) = P(B|A)P(A)$$
$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Probability as Expectation

Define the indicator function $I\{statement\}$ to be

$$I\{statement\} \equiv \begin{cases} 1 & \text{Statement is TRUE} \\ 0 & \text{Statement is False} \end{cases}$$

Then the probability of an event is the expectation of the indicator function of the event happening:

$$P(A) = E[I\{A\}]$$

Markov's Inequality

$$P(h(X) \geq b) \leq \frac{E[h(X)]}{b}$$

Chebyshev's Inequality

For $c > 0$, $a > 0$, $E[X^2] < \infty$

$$P(|X-\mu| \geq c) \leq \frac{\sigma_X^2}{c^2}$$
$$P(|X-\mu| \geq a\sigma) \leq \frac{1}{a^2}$$

Cauchy-Schwartz Inequality

$$|E[XY]| \leq E[|XY|] \leq [E[X^2]]^{\frac{1}{2}}[E[Y^2]]^{\frac{1}{2}}$$

Jensen's Inequality

Let $\mathcal{X} = supp(X)$, if $g : \mathcal{X} \to \mathbb{R}$ is **convex**, then

$$g(E[X]) \leq E[g(X)]$$

Holder's Inequality

$\forall p, q \in [1, \infty)\, s.t.\, \frac{1}{p} + \frac{1}{q} = 1$

$$\|fg\|_1 = \|f\|_p \|g\|_q$$

Minkowski's Inequality

$\forall p \in [1, \infty)$,

$$E[|X+Y|^p]^{\frac{1}{p}} \leq E[|X|^p]^{\frac{1}{p}} + E[|Y|^p]^{\frac{1}{p}}$$
$$E[|X+Y|] \leq E[|X|] + E[|Y|]$$
$$SD(X+Y) \leq SD(X) + SD(Y)$$

Law of Iterated Expectations

$$E_Y[Y] = E_X[E_Y[Y|X]] = E_X[E_Z[E_Y[Y|X,Z]|X]]$$

Law of Total Variance

$$Var(Y) = E[V[Y|X]] + V[E[Y|X]]$$

Conditional/Joint PDFs

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \iff X \perp\!\!\!\perp Y$$
$$f_X(x) = \int_{supp(Y)} f_{XY}(x,y)dy$$
$$\begin{aligned} f_{Y|X} &= \frac{f_{XY}}{f_Y} = \frac{\int_{supp(Z)} f_{XYZ} dz}{f_Y} \\ &= \int_{supp(Z)} \frac{f_{XYZ}(x,y,z)}{f_Y(y)} \cdot \frac{f_{XY}(x,y)}{f_{XY}(x,y)} dz \\ &= \int_{supp(Z)} f_{Z|X,Y} \cdot f_{X|Y} dz \end{aligned}$$

Moreoever,

$$\begin{aligned} f_{Y,X|Z} &= \frac{f_{YXZ}(y,x,z)}{f_Z(z)} = \frac{f_{Y|X,Z}(y|x,z) f_{X,Z}(x,z)}{f_Z(z)} \\ &= f_{Y|X,Z}(y|x,z) \cdot \frac{f_{X,Z}(x,z)}{f_Z(z)} = f_{Y|X,Z}(y|x,z) f_{X|Z}(x|z) \end{aligned}$$

Multivariate Normal Distribution

Conditional Normal

Consider random vectors $X_{m\times 1}, Y_{n\times 1}$ that are jointly normally distributed:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim Normal\left(\begin{pmatrix} \mu_X \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix} \right)$$

where

$$\Sigma_{XY} = Cov(X,Y)_{m\times n} = \sum_{YX}'$$

Then,

$$\begin{aligned} Y|X &\sim Normal(\alpha + B'X, \Sigma_{Y|X}) \\ B &= \Sigma_{XX}^{-1}\Sigma_{XY} \\ \alpha &= \mu_Y - B'\mu_X \\ \Sigma_{Y|X} &= \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY} \end{aligned}$$

Diagonalization of the Variance Matrix

A real, symmetric matrix Σ (which we assume variance matrices are), $\Sigma = QDQ'$ where Q is an orthonormal matrix ($QQ' = Q'Q = I$) and D is a diagonal matrix of eigenvalues. If we further assume that A is **positive definite**, then we can define $\Sigma^{-\frac{1}{2}} = QD^{-\frac{1}{2}}Q'$ where λ_i's are the eigenvalues and Q is made of corresponding eigenvectors.

$$D^{-\frac{1}{2}} = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-\frac{1}{2}} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & \lambda_n^{-\frac{1}{2}} \end{pmatrix}$$

If matrix B is symmetric and idempotent ($B^n = B$), then $X'BX = X'B'BX = (BX)'BX$.
If matrix $B_{n\times n}$ is symmetric, idempotent, and real with rank m ($\leq n$), it is diagonlizable with $B = QDQ'$ where D is a diagonal matrix with a total of m 1's in the diagonal.
$X \sum N(0, I_n) \Rightarrow X'_{1\times n} B_{n\times n} X_{n\times 1} \sim \chi_m^2$

Selected Problems

Find $f_Y(y)$ where $Y = e^X$ and $f_X = \frac{1}{\sigma^2} x \cdot exp(-\frac{x^2}{2\sigma^2})$
Sol: Since e^X is strictly monotonic, we can use the formula

$$f_Y(y) = [|\frac{dx(y)}{dy}|] f_x(g^{-1}(y)) =$$
$$\frac{1}{y}\frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)^2/2},\ y \in (1, \infty)$$

Find $f_Y(y)$ where $Y = \frac{4}{3}X - X^2$ and $X \sim Uniform[0,1]$
Sol:

$$\begin{aligned} F_Y(y) &= P(\frac{4}{3}X - X^2 \leq y) = P((X - \frac{2}{3})^2 \geq \frac{4}{9} - y) \\ &= 1 - P((X - \frac{2}{3})^2 \leq \frac{4}{9} - y) \\ &= 1 - P(\frac{2}{3} - (\frac{4}{9} - y)^{\frac{1}{2}} \leq X \leq \frac{2}{3} + (\frac{4}{9} - y)^{\frac{1}{2}}) \\ &= 1 - [F_X(\frac{2}{3} + (\frac{4}{9} - y)^{\frac{1}{2}}) - F_X(\frac{2}{3} - (\frac{4}{9} - y)^{\frac{1}{2}}] \end{aligned}$$

Notice that at $y \leq \frac{3}{9} = \frac{1}{3}$, $F_X(\frac{2}{3} + (\frac{4}{9} - y)^{\frac{1}{2}}) = 1$ since $x \in [0,1]$. Hence we have the CDF:

$$F_Y(y) = \begin{cases} \frac{2}{3} - (\frac{4}{9} - y)^{\frac{1}{2}} & , y \leq \frac{1}{3} \\ 1 - 2(\frac{4}{9} - y)^{\frac{1}{2}} & , \frac{1}{3} < y \leq \frac{4}{9} \\ 0 & \text{, otherwise} \end{cases}$$

and hence we have the PDF of Y as:

$$f_Y(y) = \begin{cases} \frac{1}{2}(\frac{4}{9} - y)^{-\frac{1}{2}} & , y \leq \frac{1}{3} \\ (\frac{4}{9} - y)^{-\frac{1}{2}} & , \frac{1}{3} < y \leq \frac{4}{9} \\ 0 & \text{, otherwise} \end{cases}$$

$X \sim Gamma(\alpha, \beta)$, show that $P(X \geq 2\alpha\beta) \leq (2/e)^\alpha$.
Sol: Using Markov's Inequality, we can bound the probability by:

$$\begin{aligned} P(X \geq 2\alpha\beta) &= P(e^{tX} \geq e^{t2\alpha\beta}) \leq \frac{E[e^{tX}]}{e^{t2\alpha\beta}} \\ &= \underbrace{\frac{(1-\beta t)^{-\alpha}}{e^{t2\alpha\beta}}}_{\text{using } t = \frac{1}{2\beta} < \frac{1}{\beta}} = \frac{(\frac{1}{2})^{-\alpha}}{e^\alpha} = (\frac{2}{e})^\alpha \end{aligned}$$

$X \sim Normal(\mu, \sigma^2)$, show $E[|X - \mu|] = \sigma\sqrt{2/\pi}$
Sol: Notice that $N(\mu, \sigma^2)$ is symmetric about $x = \mu$, so

$$\begin{aligned} E[|X-\mu|] &= 2 \cdot \int_\mu^\infty \frac{x-\mu}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= 2(-\frac{2\sigma}{2\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} |_\mu^\infty = 2(0 - (-\frac{\sigma}{\sqrt{2\pi}} e^0)) \\ &= \frac{2\sigma}{\sqrt{2\pi}} = \sigma\frac{\sqrt{2}}{\sqrt{\pi}} = \sigma\sqrt{\frac{2}{\pi}} \end{aligned}$$

Find the moment generating function for $f(x) = \frac{1}{4} exp\left(-\frac{|x-a|}{2}\right)$, $x, \alpha \in \mathbb{R}$

Sol:

$$\begin{aligned}
\psi_X(t) &= \int_{-\infty}^{\infty} \frac{1}{4} e^{-\frac{|x-\alpha|}{2}} e^{tx} dx \\
&= \int_{-\infty}^{\alpha} \frac{1}{4} e^{\frac{x-\alpha}{2}} e^{tx} dx + \int_{\alpha}^{\infty} \frac{1}{4} e^{\frac{\alpha-x}{2}} e^{tx} dx \\
&= \frac{1}{4} e^{-\frac{\alpha}{2}} \int_{-\infty}^{\alpha} e^{\frac{2t+1}{2}x} dx + \frac{1}{4} e^{\frac{\alpha}{2}} \int_{\alpha}^{\infty} e^{\frac{2t-1}{2}x} dx \\
&= \frac{1}{4} e^{-\frac{\alpha}{2}} \frac{2}{2t+1} e^{\frac{2t+1}{2}x}|_{-\infty}^{\alpha} + \frac{1}{4} e^{\frac{\alpha}{2}} \frac{2}{2t-1} e^{\frac{2t-1}{2}x}|_{\alpha}^{\infty} \\
&= \frac{2}{4(2t+1)} e^{\alpha t} + \frac{2}{4(2t-1)} e^{\alpha t} \\
&= \frac{2}{(2t+1)(2t-1)} e^{\alpha t}
\end{aligned}$$

Find the moment generating function for $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x \in \mathbb{N} \cup \{0\}$, $\lambda > 0$

Sol: Since X is a discrete random variable following the Poisson(λ) distribution, its MGF is:

$$\begin{aligned}
\psi(t) &= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} e^{tx} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x e^{tx}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= \underbrace{e^{-\lambda} e^{\lambda e^t}}_{\text{Using the power series expansion for exponential function}} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}$$

($\star$) Suppose in a shop on average ten customers come in per hour. What is the probability when you enter that you would have to wait more than twenty minutes for the next customer to come in?

Sol: The number of minutes we, on average, have to wait follows the **continuous** distribution $exponential(\frac{1}{6})$, so

$$\begin{aligned}
P(X \geq 20) &= 1 - P(X \leq 20) \\
&= 1 - (1 - e^{-\frac{1}{6} \cdot 20}) = \frac{1}{e^{\frac{10}{3}}} = 0.0357
\end{aligned}$$

Notice that there are several ways to specify the distribution for this problem. The following specifications are equivalent:

$$H \sim Gamma(1, \beta) \qquad \beta = \frac{1}{\lambda} = \frac{1}{10} \tag{1}$$

$$M \sim Gamma(1, \beta) \qquad \beta = \frac{1}{\lambda} = 6 \tag{2}$$

$$H \sim Exponential(\lambda) \qquad \lambda = \frac{1}{10} \tag{3}$$

$$M \sim Exponential(\lambda) \qquad \lambda = 6 \tag{4}$$

where H is the random variable representing the hours before an event and M is the random variable representing the minutes before an event. For each case, the heuristic description of the distribution is:

H is distributed such that per hour ($\alpha = 1$), there are 10 customers ($\lambda = \frac{1}{\beta} = 10$).

M is distributed such that per minute($\alpha = 1$), there are $\frac{1}{6}$ customers ($\lambda = \frac{1}{\beta} = \frac{1}{6}$).

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{2y + 4x}{1 + 4x} \\
f_X(x) &= \frac{1 + 4x}{3}
\end{aligned}$$

Find $f_{X|Y}$. **Sol:**

$$\begin{aligned}
f_{X|y} &= \frac{f_{Y|x} \cdot f_X}{f_Y} = \frac{f_{Y|x} \cdot f_X}{\int_0^1 f_{Y|x} \cdot f_X dx} \\
&= \frac{f_{Y|x} \cdot f_X}{\int_0^1 \frac{2y+4x}{3} dx} = \frac{f_{Y|x} \cdot f_X}{\int_0^1 \frac{2y+4x}{3} dx} = \frac{\frac{2y+4x}{3}}{\frac{2y+2}{3}} \\
&= \frac{y + 2x}{y + 1}, \ 0 < x < 1, 0 < y < 1
\end{aligned}$$

conditional density is 0 otherwise

$$\begin{aligned}
Y &= X + Z - 2XZ + U \\
E[U|X, Z] &= 0 \\
E[Z|X] &= 3 + 4X
\end{aligned}$$

Find $E[Y|X, Z]$ and $E[Y|X]$. **Sol:**

$$\begin{aligned}
E[Y|X, Z] &= E[X + Z - 2X \cdot Z + U|X, Z] \\
&= X + Z - 2X \cdot Z \\
E[Y|X] &= E[E[Y|X, Z]|X] = E[X + Z - 2X \cdot Z|X] \\
&= X + E[Z|X] - 2XE[Z|X] \\
&= X + (3 + 4X) - 2X(3 + 4X) \\
&= -8X^2 - X + 3
\end{aligned}$$