Price Theory, III
Economics 33003

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Assignment 6
(Due Friday, May 24, prior to the start of the Review session)

Problem 1 Consider a natural (but regulated) monopolist with a constant marginal cost of production equal to c which is private information at the time of contracting and uniformly distributed on $[1, 2]$. The monopolist's profit for producing output q at marginal cost c with transfer t from the regulator is

$$u(q, c, t) \equiv t - cq.$$

Note that the firm does not get any revenue from selling q; all payments are through t. The firm's participation constraint is $\underline{U} = 0$.

The regulator's objective function is to maximize

$$\int_{\underline{c}}^{\overline{c}} [vq(c) - \frac{1}{2}q(c)^2 - t(c)]f(c)dc,$$

over the class of all DRM's which satisfy IC and IR constraints. Assume $v \geq 3$. Among other things, this implies the first-best, full-information output would be $q^{fb}(c) = v - c > 0$.

(a). What are the IC and IR constraints for this problem? State and prove a characterization theorem for the set of all IC mechanisms $\{q(\cdot), t(\cdot)\}$ in terms of a monotonicity condition and an integral condition.

(b). Solve for the regulator's optimal revelation mechanism, $\{q(\cdot), t(\cdot)\}$, and compare $q(c)$ to $q^{fb}(c)$. Explain the inequality that you find.

(c). Set $v = 3$. Compute an indirect tariff, $T(q)$, with the property that the regulator can offer the firm this schedule, $T(q)$, letting the firm freely choose its output q in exchange for $T(q)$, and the resulting choice and payments are equivalent to those from (b).

Problem 2 Suppose that an airline selling tickets to business customers (b) and tourists (t). Each category of customer draws its value from a different distribution: $F_b(\theta)$ and $F_t(\theta)$. For now, we don't make any assumptions about how these relate. At the time of purchase, the customer knows whether they are a business or tourist flyer, but the airline does not; the customer does not know his ex post type, θ. Business customers represent the proportion $\phi \in (0, 1)$ of potential consumers.

The airline has decided to restrict attention to a simple pricing scheme where a ticket costs p at the time of purchase, but also has a refund provision that allows the consumer to return the ticket for r after the consumer learns θ. Hence, a ticket is defined by its price and the amount that is refundable, (p, r). The airline wants to design a menu of tickets, $\{(p_b, r_b), (p_t, r_t)\}$, to maximize its profits. It has constant unit cost of serving either customer of c.

(a). Write down the airline's program, including the IC and IR constraints for both types.

(b). Argue that the IR constraint of the business customer can be ignored if either F_b first-order stochastically dominates F_t, or F_b is a mean-preserving spread of F_t.

(c). Consider the relaxed program in which the IC constraint for the t consumer is ignored. Show in the relaxed program that IR for t must bind and IC for b must bind.

(d). State and solve the relaxed program in which IR_t and IC_b bind (and IC_t is ignored). Describe the optimal refund policy for the two classes of tickets for the case when $F_b \geq_{FOSD} F_t$. [You may assume that IC_t is slack, though this can be proved.] Given your solution, why might this not be such a great model of airline pricing that we observe in the real world?

(e). Suppose instead that F_b is a mean-preserving spread of F_t. In particular, let's suppose that $F_b(\theta) = \theta$ on $[0, 1]$, but the tourists have a triangular distribution,

$$F_t(\theta) = \begin{cases} 2\theta^2 & \text{if } \theta \leq \frac{1}{2} \\ 4\theta - 2\theta^2 - 1 & \text{if } \theta > \frac{1}{2}. \end{cases}$$

Let $c = \frac{3}{8}$ and $\phi = \frac{1}{2}$. Describe the optimal refund policy for the two classes of tickets. [Hint: Look for the solution to the FOC for r_t where $r_t \leq \frac{1}{2}$.] Describe your solution. Does this do a better job at fitting actual airline pricing?

Problem 3 (From B. Szentes.) Consider a trade of a divisible good between a seller (principal) and a buyer (agent). The buyer's payoff is $vq - t$ where v is her valuation, q is the quantity traded, and t the payment to the seller. The seller's payoff is $t - q^2/2$, where $q^2/2$ is the cost of producing q units of the good.

Suppose that the buyer's valuation is

$$v = \lambda\theta + (1 - \lambda)\varepsilon,$$

where θ is the buyer's private information and ε is a publicly observable and contractible shock. Assume that both θ and ε are independently and uniformly distributed on $[0, 1]$. The seller can offer a contract to the buyer prior to the realization of ε, but θ is privately known by the buyer at the time of the contract. The buyer's outside option is zero.

(a). What is the set of contracts to which it is without loss of generality to restrict attention?

(b). Consider an incentive compatible contract and let $U(\theta)$ denote the equilibrium payoff of the buyer with type θ. What is $U(\theta) - U(0)$?

(c). Use your result in part (b) to express the buyer's ex-ante expected payoff, $E_\theta[U(\theta)]$.

(d). Express the seller's expected payoff as the difference between social surplus and the buyer's payoff.

(e). Use your result is part (d) to derive the optimal quantity produced as a function of θ and ε.

(f). Suppose that ε is not observable by the seller, but instead remains private information to the buyer. Otherwise, the timing is as before: the contract is offered after the buyer has observed θ but before the buyer observes ε. Using a result from class, argue that the same allocation in (e) will be implemented by the seller.

Problem 4 Consider an IPV auction environment with two bidders, one "strong" and one "weak". The strong bidder's type θ_s, is uniformly distributed on $[2, 3]$ and the weak bidder's type, θ_w, is uniformly distributed on $[0, 1]$.

(a). Compute the equilibrium bidding functions in the second-price auction. Compute the expected revenue to the seller.

(b). Compute the equilibrium bidding functions in the first-price auction for the equilibrium in which the weak player bids $\bar{b}_w(\theta_w) = \theta_w$. Compute the expected revenue to the seller.

(c). Compare the expected revenues. Explain why they are the same (i.e., explain how the revenue equivalence theorem applies to this setting), or explain why they are different (i.e., why the revenue equivalence theorem does not apply to this situation).

Problem 5 (JR, Exercise 9.8) In a first-price, all-pay auction, the bidders simultaneously submit sealed bids. The highest bid wins the object and every bidder pays the seller the amount of his bid. Consider the independent private values model with symmetric bidders whose values θ_i are each distributed according to the distribution function F, with density f.

(a). Find the unique symmetric equilibrium bidding function.

(b). Do bidders bid higher or lower than in a first-price auction?

(c). Find an expression for the seller's expected revenue.

(d). Both with and without using the revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

Problem 6 (JR, Exercise 9.9) Suppose there are just two bidders. In a second-price, all-pay auction, the two bidders simultaneously submit sealed bids. The highest bid wins the object and both bidders pay the second-highest bid.

(a). Find the unique symmetric equilibrium bidding function. Interpret

(b). Do bidders bid higher or lower than in a first-price, all-pay auction?

(c). Find an expression for the seller's expected revenue.

(d). Both with and without using the revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

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Answers to Assignment 6

1 (a). Incentive compatibility requires

$$U(c) \equiv t(c) - cq(c) \geq U(\hat{c}|c) \equiv t(\hat{c}) - cq(\hat{c}) \text{ for all } c, \hat{c} \in [1,2].$$

The individual rationality (or participation) constraints are

$$U(c) \geq 0 \text{ for all } c \in [1,2].$$

Result: $\{q(\cdot), t(\cdot)\}$ is incentive compatible if and only if

1. (monotonicity) $q(c)$ is nonincreasing,
2. (envelope/integral condition)

$$U(c) = U(\bar{c}) + \int_c^2 q(s)ds, \text{ for all } c \in [1,2].$$

The proof of this follows directly from the lecture notes, so I won't reproduce it here.

(b). First, we want an expression for $E[U(c)]$ that involves only $q(\cdot)$ and $U(2)$. I'm omitting the steps (these are in the notes for Baron and Myerson (1981)).

$$E[U(c)] = U(\bar{c}) + E\left[q(c)\frac{F(c)}{f(c)}\right].$$

Inserting this into the regulator's program, we have the regulator's simplified program:

$$\max_{q(\cdot)\downarrow,\, U(2)\geq 0} E\left[vq(c) - \frac{1}{2}q(c)^2 - \left(c + \frac{F(c)}{f(c)}\right)q(c)\right] - U(2).$$

Note that $U(2) \geq 0$ implies the IR constraints are satisfied for all types (because $u_c = -q \leq 0$).

The regulator will always choose $U(2) = 0$. Solving the relaxed program for $q(\cdot)$ (ignoring monotonicity of $q(\cdot)$), the pointwise optimal allocation satisfies

$$v - q(c) - \left(c + \frac{F(c)}{f(c)}\right) = 0,$$

or, using $F(c)/f(c) = c - 1$,

$$q(c) = (v - c) - (c - 1) = v - 2c + 1 > 0.$$

This is clearly decreasing in c, so the solution to the relaxed program also solves the full program. Computing the firm's indirect utility, we have

$$U(c) = \int_c^2 (v - 2s + 1)ds = (2 - c)(v - c - 1).$$

Substituting to find t, we have

$$t(c) = U(c) + cq(c) = 2v - c^2 - 2.$$

(c). Inverting $q(\cdot)$, (and using $v = 3$) we have

$$q^{-1}(q) \equiv \mathcal{C}(q) = \frac{1}{2}(4 - q).$$

Again, using $v = 3$,

$$t(c) = 4 - c^2.$$

Thus,

$$T(q) = t(\mathcal{C}(q)) = 4 - \left(\frac{1}{2}(4 - q)\right)^2 = \frac{1}{4}(8 - q)q.$$

2 [This is based on Courty and Li (*REStud.*,2000).]

(a). Airline chooses $\{(p_b, r_b), (p_t, r_t)\}$ to maximize

$$\phi(p_b - r_b F_b(r_b) - c(1 - F(r_b)) + (1 - \phi)(p_t - r_t F_t(r_t) - c(1 - F_t(r_t))),$$

subject to

$$\begin{aligned}
\int_{r_t}^{\overline{\theta}} \theta f_t(\theta) d\theta + r_t F_t(r_t) - p_t &\geq 0, \quad \text{IRt} \\
\int_{r_b}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_b F_b(r_b) - p_b &\geq 0, \quad \text{IRb} \\
\int_{r_b}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_b F_b(r_b) - p_b &\geq \int_{r_t}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_t F_b(r_t) - p_t \quad \text{ICb} \\
\int_{r_t}^{\overline{\theta}} \theta f_t(\theta) d\theta + r_t F_t(r_t) - p_t &\geq \int_{r_b}^{\overline{\theta}} \theta f_t(\theta) d\theta + r_b F_t(r_b) - p_b \quad \text{ICt.}
\end{aligned}$$

(b). We will show that ICb and IRt imply IRb.

$$\begin{aligned}
\int_{r_b}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_b F_b(r_b) - p_b \quad &\geq \textstyle\int_{r_t}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_t F_b(r_t) - p_t \qquad \text{from ICb} \\
&= \textstyle\int_{\underline{\theta}}^{\overline{\theta}} \max\{r_t, \theta\} f_b(\theta) d\theta - p_t \qquad \text{rearrangement} \\
&\geq \textstyle\int_{\underline{\theta}}^{\overline{\theta}} \max\{r_t, \theta\} f_t(\theta) d\theta - p_t \qquad \text{FOSD or MPS} \\
&= \textstyle\int_{r_t}^{\overline{\theta}} \theta f_t(\theta) d\theta + r_t F_t(r_t) - p_t \geq 0, \quad \text{IRt.}
\end{aligned}$$

To be clear, if $F_b \geq_{FOSD} F_t$, then the fact that $\max\{r_t, \theta\}$ is an increasing function implies the inequality in the third line above; if F_b is a mean-preserving spread, the fact that $\max\{r_t, \theta\}$ is a convex function of θ implies the inequality.

(c). In the relaxed program in which ICt is ignored, IRt must bind. If it were slack, one could increase p_t by a small amount, thereby increasing profits, while not violating ICb. IRb must also bind. If not, we can increase p_b, raising profits, without impacting IRt.

(d). The IRt and ICb constraints in the relaxed program bind. We use these constraints to substitute for p_t and p_b:

$$p_t = r_t F_t(r_t) + \int_{r_t}^{\overline{\theta}} \theta f_t(\theta) d\theta,$$

$$\begin{aligned} p_b &= r_b F_b(r_b) + \int_{r_b}^{\overline{\theta}} \theta f_b(\theta) d\theta + r_b F_b(r_b) + \int_{r_t}^{\overline{\theta}} \theta (f_t(\theta) - f_b(\theta)) d\theta + r_t (F_t(r_t) - F_b(r_t)), \\ &= r_b F_b(r_b) + \int_{r_b}^{\overline{\theta}} \theta f_b(\theta) d\theta - \int_{r_t}^{\overline{\theta}} (F_t(\theta) - F_b(\theta)) d\theta. \end{aligned}$$

The objective function can now be written as a function of r_b and r_t.

$$\phi \left(\int_{r_b}^{\overline{\theta}} (\theta - c) f_b(\theta) d\theta \right) + (1-\phi) \left(\int_{r_t}^{\overline{\theta}} (\theta - c) f_t(\theta) d\theta \right) - \phi \left(\int_{r_t}^{\overline{\theta}} (F_t(\theta) - F_b(\theta)) d\theta \right).$$

Taking first-order conditions, we obtain

$$-\phi (r_b - c) f_b(r_b) = 0,$$

$$-(1-\phi)(r_t - c) f_t(r_t) + \phi (F_t(r_t) - F_b(r_t)) = 0.$$

Thus,

$$r_b = c$$

$$r_t = c + \frac{\phi}{1-\phi} \frac{F_t(r_t) - F_b(r_t)}{f_t(r_t)} > c.$$

The effect is that the tourist customer is less likely to take a flight once they learn their ex post type. This outcome, that tourists fly less than business customers is not surprising given they have lower valuations on average.

What is peculiar in the context of airline pricing is that the tourists pay more for their ticket $p_t > p_b$ and get better refund terms. Business customers do not value refund terms as much since they are more certain they will fly, and hence they have a lower purchase price but a lower refund as well. In practice, we observe business customers buying more expensive tickets without refund penalties and tourists buying cheaper non-refundable tickets.

Aside: Note we have not yet checked that the IC constraint for the tourist is slack. (You were not asked to do so, but we'll include the argument here for completeness.) Because IC_b binds,

$$p_t - p_b = \int_{r_b}^{r_t} F_b(\theta) d\theta.$$

This, in turn, implies

$$-p_t + r_t F_t(r_t) + \int_{r_t}^{\overline{\theta}} \theta f_t(\theta) d\theta = -p_b + r_b F_t(r_b) + \int_{r_b}^{\overline{\theta}} \theta F_t(\theta) d\theta - \int_{r_b}^{r_t} (F_b(\theta) - F_t(\theta)) d\theta.$$

It follows that IC_t is satisfied iff

$$\int_{r_b}^{r_t} (F_b(\theta) - F_t(\theta))d\theta \leq 0.$$

Suppose this inequality is violated. Then consider the alternative offer of $\tilde{r}_t = \tilde{r}_b = c$. The surplus of the t type is higher in the alternative menu and the rent to the b type is smaller because

$$\int_{r_t}^{\overline{\theta}} (F_t(\theta) - F_b(\theta))d\theta > \int_{r_t}^{\overline{\theta}} (F_t(\theta) - F_b(\theta))d\theta + \int_{c}^{r_t} (F_t(\theta) - F_b(\theta))d\theta.$$

Hence, in the relaxed program, supposing that the solution violates the IC_t leads to a contradiction.

(e). The solution to the FOC for r_t is found by solving

$$r_t = c + \frac{\phi}{1-\phi}\frac{F_t(r_t) - F_b(r_t)}{f_t(r_t)},$$

which, given our functional forms and the hint, translates to

$$r_t = \frac{3}{8} + \frac{2(r_t)^2 - r_t}{4r_t},$$

or

$$r_t = \frac{1}{4} < c = \frac{3}{8}.$$

Thus, the refund policy is efficient for the business traveler, but encourages too much travel for the tourist (because the refund is less generous). The business traveler pays more for the better refund policy up front because she values the option to cancel her flight given the greater uncertainty in θ_b. This seems to be closer to the real world practice of selling expensive refundable tickets to business customers and cheap, nonrefundable tickets to tourists. Still, we have assumed that the means are the same, which is probably not realistic, so we would want to mix the two models (FOSD and something like SOSD into the same model.)

3 (a). By the revelation principle, it is without loss of generality to restrict attention to contracts $\{q(\cdot), t(\cdot)\}_{\theta \in [0,1]}$, where $q : [0,1]^2 \rightarrow \Re$ and $t : [0,1]^2 \rightarrow \Re$. $q(\theta, \varepsilon)$ is the quantity traded if θ is reported and ε is later observed and $t(\theta, \varepsilon)$ is the corresponding payment.

(b). Using the standard implementation result (specifically, the envelope theorem)

$$U(\theta) - U(0) = E_\varepsilon \left[\int_0^\theta u_\theta(q(\theta, \varepsilon), (\lambda s + (1-\lambda)\varepsilon))ds \right] = \lambda E_\varepsilon \left[\int_0^\theta q(s, \varepsilon)ds \right].$$

(c).

$$E_\theta[U(\theta)] = \lambda E_\varepsilon \int_0^1 q(\theta, \varepsilon)\frac{1 - F(\theta)}{f(\theta)} f(\theta)d\theta + U(0).$$

Thus,

$$E_theta[U(\theta)] = \lambda E_{\theta,\varepsilon} \left[q(\theta, \varepsilon)(1 - \theta) \right] + U(0).$$

(d).

$$E_{\theta,\varepsilon}\left[q(\theta,\varepsilon)(\lambda\theta+(1-\lambda)\varepsilon)-\lambda(1-\theta))-\frac{1}{2}q(\theta,\varepsilon)^2\right]-U(0).$$

(e). Maximize pointwise over (θ,ε).

$$q(\theta,\varepsilon)=\max\{2\lambda\theta+(1-\lambda)\varepsilon-\lambda,0\}.$$

(f). Recall from the lecture on dynamic screening, in the two-period consumption model, we know that if $G_{\theta_1}(\theta_2|\theta_1)=0$, then q_2 is set at first best. Here, because θ and ε are independently distributed (the analogue is θ_1 and θ_2 are independently distributed), we get the first best allocation. Effectively, the seller "pre-sells" the efficient quantity related to ε , and the buyer obtains no information rents from the private information arising from ε.

4 (a). In the second-price auction, it is still a dominate strategy to bid one's true type. Thus, $\bar{b}(\theta_i)=\theta_i$ for both the strong and weak players is an equilibrium. The good is always purchased by the strong player. The expected price is the expected type of the weak player, which is $\frac{1}{2}$. Thus, expected revenue is $ER=\frac{1}{2}$.

(b). Assume, at first, that the weak player bids his type, $\bar{b}_w(\theta)=\theta$. This is weakly optimal, given (as we will see) that the weak player will never win. Given this strategy by the weak player, the strong player solves

$$\max_{b_s[0,1]} F_w(b_s)(\theta_s-b_s)=b_s(\theta_s-b_s).$$

Differentiating this over b_s, we obtain

$$\theta_s-2b_s$$

which is positive for all $\theta_s\in[2,3]$ and $b_s\in[0,1]$. Hence, $b_s=1$ is optimal. Given this strategy, it is optimal for the weak player to bid his type. The expected revenue is 1.

Aside: There are other equilibria that generate even more revenue. The weak player could always bid 2. In this case, the strong bidder bids slightly more than 2 (assuming there is a minimum price increment above 2 or assuming that ties are broken in favor of the strong type.) This would generate revenue of 2.

(c). The revenues are different. The first-price auction raises more revenue. The revenue equivalence theorem cannot be applied because the expected utility of the worst type of strong buyer $\theta_s=2$ is not the same in the two auction formats. In the second-price auction, expected utility is $2-\frac{1}{2}=\frac{3}{2}$; in the first-price auction, expected utility is $2-1=1$.

Aside: In this example, the high type bidder always wins. If the supports overlap, however, this would not generally hold. The weak bidder might sometimes win with a lower type than the strong bidder. Consequently, the revenue equivalence theorem cannot be applied in that circumstance either. Interesting results comparing the two auction formats exist, however. See Maskin and Riley (*Ema*, 2000).

5 (a). Suppose that there is a symmetric equilibrium bidding function that is strictly increasing, and therefore the highest type bidder wins the auction.

There are two ways to find the equilibrium bidding function. Let's first use the envelope-theorem approach. In equilibrium, bidder i with type θ_i has expected utility

$$G(\theta_i)\theta_i - \bar{b}(\theta_i).$$

By the envelope theorem, $U_i'(\theta_i) = G(\theta)$, so

$$U_i(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} G(s)ds.$$

Consider the lowest type bidder, $\theta = \underline{\theta}$. The probability of this bidder winning is zero, and so the bidder will bid zero (because bids are paid regardless of winning). Hence, $U_i(\underline{\theta}) = 0$. Now we set the two statements of expected utility equal to each other:

$$G(\theta_i)\theta_i - \bar{b}(\theta_i) = \int_{\underline{\theta}}^{\theta} G(s)ds.$$

Solving for $\bar{b}(\theta_i)$, we obtain

$$\bar{b}(\theta_i) = G(\theta_i)\theta_i - \int_{\underline{\theta}}^{\theta} G(s)ds.$$

Now let's try the brute force method of differentiating the equilibrium bid function and solving a differential equation. Bidder i chooses b_i to solve

$$\max_{b_i} \; G(\bar{b}^{-1}(b_i))\theta_i - b_i.$$

The first-order condition is

$$g(\bar{b}^{-1}(b_i))\frac{d\bar{b}^{-1}(b_i)}{db_i}\theta_i - 1 = 0.$$

In a symmetric equilibrium, $b_i = \bar{b}(\theta_i)$, and hence we have

$$g(\theta_i)\theta_i = \bar{b}'(\theta_i).$$

Integrating, and using the fact that $b(\underline{\theta}) = 0$, we have

$$\bar{b}(\theta_i) = \int_0^{\theta_i} sg(s)ds.$$

We could stop there, but let's integrate by parts to confirm this is the same as our solution above:

$$\bar{b}(\theta_i) = \int_{\underline{\theta}}^{\theta_i} sg(s)ds = sG(s)|_0^{\theta_i} - \int_{\underline{\theta}}^{\theta_i} G(s)ds = \theta_i G(\theta_i) - \int_{\underline{\theta}}^{\theta_i} G(s)ds.$$

Now we have to verify that it is an equilibrium to use this bid function. Consider bidder i. If all other bidders are using the strategy $\bar{b}(\theta)$, then the highest bid is $\bar{b}(\bar{\theta})$. Hence, it is always optimal to restrict attention to bids in $[0, \overline{b}(\overline{\theta})]$. Because $\overline{b}$ is strictly increasing, we can think of bidder i as choosing a report, $\hat{\theta}$, instead of b_i (where the $\hat{\theta} = \overline{b}^{-1}(b_i)$). Thus, we want to show

$$\theta_i \in \arg\max_{\hat{\theta}} \; G(\hat{\theta})\theta_i - \overline{b}(\hat{\theta}).$$

Using our formula for $\overline{b}$, we have

$$U(\hat{\theta}|\theta_i) = G(\hat{\theta})\theta_i - G(\hat{\theta})\hat{\theta} + \int_{\underline{\theta}}^{\hat{\theta}} G(s)ds.$$

Simplifying,

$$U(\hat{\theta}|\theta_i) = G(\hat{\theta})(\theta_i - \hat{\theta}) + \int_{\underline{\theta}}^{\hat{\theta}} G(s)ds.$$

We want to show $U(\theta_i|\theta_i) - U(\hat{\theta}|\theta_i) \geq 0$. To this end,

$$\Delta U = U(\theta_i|\theta_i) - U(\hat{\theta}|\theta_i) = \int_{\underline{\theta}}^{\theta_i} G(s)ds - G(\hat{\theta})(\theta_i - \hat{\theta}) - \int_{\underline{\theta}}^{\hat{\theta}} G(s)ds.$$

$$\Delta U = \int_{\hat{\theta}}^{\theta_i} G(s)ds - G(\hat{\theta})(\theta_i - \hat{\theta}) = \int_{\hat{\theta}}^{\theta_i} (G(s) - G(\hat{\theta}))ds \geq 0.$$

(b). In the first-price auction, the equilibrium bid function is

$$\overline{b}^{fp}(\theta_i) = \int_{\underline{\theta}}^{\theta_i} \frac{sg(s)}{G(\theta_i)} ds.$$

Above, we established

$$\overline{b}(\theta_i) = \int_{\underline{\theta}}^{\theta_i} sg(s)ds.$$

Because $G(\theta_i) < 1$, bids are higher in the first-price auction compared to the all-pay auction.

(c). Expected revenue is

$$ER = nE\left[\int_{\underline{\theta}}^{\theta_i} sg(s)ds\right] = n\int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta_i} sg(s)ds\right) f(\theta)d\theta.$$

Integrating by parts,

$$ER = n\left(\left(\int_{\underline{\theta}}^{\theta} sg(s)ds\right)(F(\theta) - 1)\Big|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} \theta g(\theta)(F(\theta) - 1)d\theta.\right.$$

Thus,

$$ER = n\int_{\underline{\theta}}^{\overline{\theta}} \theta g(\theta)(1 - F(\theta))d\theta.$$

Substituting for $g(\theta) = (n-1)f(\theta)F(\theta)^{n-2}$, we have

$$ER = \int_{\underline{\theta}}^{\overline{\theta}} \theta n(n-1)f(\theta)F(\theta)^{n-2}(1 - F(\theta))d\theta.$$

But, as we noted in class,

$$n(n-1)f(\theta)F(\theta)^{n-2}(1-F(\theta))d\theta$$

is the density of the second-order statistic. Thus,

$$ER = E[\theta|\theta = \text{2nd order statistic}].$$

(d). We computed the expected revenue of the first-price auction in class and found

$$ER = E[\theta|\theta = \text{2nd order statistic}].$$

This is the same as in the all-pay auction.

Alternatively, we could apply the revenue–equivalence theorem and note that the highest type wins the object in both the all-pay and the first-price auction. Moreover, the lowest type bidder receives nothing in either auction. Thus, RET implies the expected revenues are equal.

6 [To be completed.]