Theory of Income I
Autumn 2018

(Fernando Alvarez)

Notes on Aggregation

by Takuma Habu

University of Chicago

Contents

For typos/comments, email me at takumahabu@uchicago.edu.

1 Recap

Let's recite the First and the Second Welfare Theorems, as well as the aggregation theorem.

Assumption 1. *(Assumption HH) Assume that* $\mathbf{X}^i$ *are convex for all* $i \in \mathbf{I}$ *and that* $u^i : \mathbf{X}^i \to \mathbb{R}$ *are continuous and strictly quasiconcave.*

Assumption 2. *(Assumption FF) Assume that the aggregate production sumset of the economy is convex; i.e.*

$$\mathbf{Y} := \left\{ \mathbf{y} \in \mathbf{L} : \mathbf{y} = \sum_{j=1}^{J} \mathbf{y}^j,\ \mathbf{y}^j \in \mathbf{Y}^j,\ \forall j \in \mathbf{J} \right\}.$$

Assumption 3. *(Assumption CC)* u^i *are concave for all* $i \in \mathbf{I}$.

Theorem 1.1. *(First Welfare Theorem) Suppose that* u^i *satisfies local non-satiation for all* $i \in \mathbf{I}$. *Let* $\{\mathbf{p}, \bar{\mathbf{x}}^i, \bar{\mathbf{y}}^j\}$ *be a competitive equilibrium. Then,* $\{\bar{\mathbf{x}}^i, \bar{\mathbf{y}}^j\}$ *is a Pareto optimal allocation.*

Theorem 1.2. *(Second Welfare Theorem) Assume that assumptions HH and FF are satisfied. Let* $\{\bar{\mathbf{x}}^i, \bar{\mathbf{y}}^j\}$ *be a Pareto optimal allocation. Then, there exists a price vector* $\mathbf{p}$ *such that:*

(i) all firms maximise profits such that, for all $j \in \mathbf{J}$,

$$\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y},\ \forall \mathbf{y} \in \mathbf{Y}^j;$$

(ii) given allocation $\{\bar{\mathbf{x}}^i\}$, *consumers minimise expenditure subject to attaining at least the same utility obtained by consuming* $\bar{\mathbf{x}}^i$; *i.e.*

$$\bar{\mathbf{x}}^i \in \underset{\mathbf{x} \in \mathbf{X}^i}{\arg\min}\ \mathbf{p} \cdot \mathbf{x} \quad s.t. \quad u^i(\mathbf{x}) \geq u^i\left(\bar{\mathbf{x}}^i\right).$$

Theorem 1.3. *Assume that* u^i *are strictly increasing, and that assumptions HH, CC and FF are satisfied. Then,* $\{\bar{\mathbf{x}}^i, \bar{\mathbf{y}}^i\}$ *is a Pareto optimal allocation if and only if there exists a vector* $\boldsymbol{\lambda} \in \mathbb{R}^l_+$ *such that* $\{\bar{\mathbf{x}}^i, \bar{\mathbf{y}}^i\}$ *solves the problem W:*

$$\begin{aligned} W := \max_{\{\mathbf{x}^i, \mathbf{y}^j\}} \quad & \sum_{i \in \mathbf{I}} \lambda_i u^i\left(\mathbf{x}^i\right) \\ s.t. \quad & \left\{\mathbf{x}^i, \mathbf{y}^j\right\} \text{ is a feasible allocation.} \end{aligned}$$

So how do these theorems allow us to analyse an economy using a representative agent? Let us consider a pure endowment economy so that we do not have to worry about the $\mathbf{y}^j$'s.

- ▷ By the First Welfare Theorem, we know that competitive equilibrium allocations are Pareto optimal. Then, Theorem 1.3 tells us that there exists $\boldsymbol{\lambda}$ that defines the corresponding "representative agent's" utility function:

$$u\left(\left\{\mathbf{x}^i\right\}_{i\in\mathbf{I}}\right) := \sum_{i\in\mathbf{I}} \lambda_i u^i\left(\mathbf{x}^i\right).$$

- ▷ Theorem 1.3 also tells us that maximising the "representative agent's" utility above subject to a feasibility constraint will give us a Pareto optimal allocation. Then, by the Second Welfare Theorem, we know that we can find prices that support the Pareto optimal allocation in a competitive equilibrium. This competitive equilibrium corresponds to one for an economy with one "representative agent" for some weight $\boldsymbol{\lambda}$.

Importantly, the "representative agent's" utility depends on the weights $\boldsymbol{\lambda}$, which reflects each agent's marginal utility of income.

2 Planner's vs Household's problems

Compare the planner's problem using representative agent's utility and the household's problem.

Planner's problem

$$\max_{\mathbf{x}^i\in\mathbf{X}^i} \sum_{i\in\mathbf{I}} \lambda_i u^i\left(\mathbf{x}^i\right) \;\; s.t. \;\; \sum_{i\in\mathbf{I}} \mathbf{x}^i \leq \overline{\mathbf{e}}$$

Lagrangian

$$\mathcal{L} = \sum_{i\in\mathbf{I}} \lambda_i u^i\left(\mathbf{x}^i\right) + \boldsymbol{\gamma}\cdot\left(\overline{\mathbf{e}} - \sum_{i\in\mathbf{I}} \mathbf{x}^i\right).$$

$\boldsymbol{\gamma}$: marginal value of aggregate endowment

FOCs

$$\lambda_i \frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)}{\partial x^i_\ell} = \gamma_\ell$$

MRS for each pair of goods:

$$\frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_\ell}{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_k} = \frac{\gamma_\ell}{\gamma_k} = \frac{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_\ell}{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_k}$$

Consumption of goods across agents:

$$\frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_\ell}{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_\ell} = \frac{\lambda_j}{\lambda_i}$$

Household's problem

$$\max_{\mathbf{x}^i\in\mathbf{X}^i} u^i\left(\mathbf{x}^i\right) \;\; s.t. \;\; \mathbf{p}\cdot\mathbf{x}^i \leq \mathbf{p}\cdot\overline{\mathbf{e}}^i$$

Lagrangian

$$\mathcal{L} = u^i\left(\mathbf{x}^i\right) + \mu_i\left(\mathbf{p}\cdot\overline{\mathbf{e}}^i - \mathbf{p}\cdot\mathbf{x}^i\right).$$

μ_i: marginal value of income

FOCs

$$\frac{1}{\mu_i}\frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)}{\partial x_\ell} = p_\ell$$

MRS for each pair of goods:

$$\frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_\ell}{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_k} = \frac{p_\ell}{p_k} = \frac{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_\ell}{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_k}$$

Consumption of goods across agents:

$$\frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x^i_\ell}{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x^j_\ell} = \frac{1/\mu_j}{1/\mu_i}$$

With this side-by-side comparison, I hope it's clear that:

$$\lambda_i \equiv \frac{1}{\mu_i}, \ \gamma_\ell \equiv p_\ell.$$

What does this mean?

- ▷ an agent is assigned a high λ_i-weight if his μ_i is low $\Leftrightarrow$ low marginal value of income $\Leftrightarrow$ those with high consumption (since u^i is concave).[1]
- ▷ a good is relatively expensive if its γ_ℓ is high $\Leftrightarrow$ high marginal value of aggregate endowment $\Leftrightarrow$ high marginal social value.

3 Demand aggregation

We've now seen that we can write the planner's problem using representative agent's utility function. So aren't we done with aggregation? Not quite.

Before we get into the details, let's go remember what macroeconomics is about. Macroeconomics, as its name suggests, is a study of aggregate variables. So, for example, we are interested in studying the aggregate demand curve to make welfare comparisons. Put differently, we want to be able to treat aggregate demand as if it is generated by a representative agent and study the change in the welfare of the representative agent as if it corresponds to changes in the welfare of a society as a whole (which is given by the sum of the individual welfare changes).

Of course for a representative agent approach to make sense, the aggregate demand must be a function of a price vector and sum aggregate of the individual endowments. But the problem is that aggregate demand is a sum of individual demand function which depends on individual endowments (as well as the price vector). I will first demonstrate this concretely below.

3.1 The Gorman Form

In class, we worked with pure endowment economy but in this section, I'm going to move away from that and think in terms of Marshallian demand. That is, I'm going to think about consumer's demand for good $\ell \in \mathbf{M}$, x^i_ℓ, as a function of the price vector, $\mathbf{p}$, and his wealth w^i.[2] Then, aggregate demand for good ℓ is given by summing the demand for good ℓ for each agent; i.e.

$$\widetilde{D}_\ell \left(\mathbf{p}, \{w^i\}_{i\in\mathbf{I}}\right) := \sum_{i\in\mathbf{I}} x^i_\ell \left(\mathbf{p}, w^i\right).$$

So we can see that aggregate demand for good ℓ depends on the prices of goods, $\mathbf{p}$, as well as *each individual's* wealth.

The question we want to ask ourselves is this: when can we write aggregate demand as a function of just $\mathbf{p}$ and the aggregate wealth (i.e. $\overline{w} \equiv \sum_{i\in\mathbf{I}} w^i$)? That is, we wish to find a conditions under

[1] To see the last one, look at the FOC. Since p_ℓ (or γ_ℓ) are common across agents, for the FOC to hold, agents with low marginal utilities must have low μ_i. Since u^i is concave, low marginal utilities correspond to those with high consumption.

[2] The added complexity with pure endowment economy comes from the fact that $w^i = \mathbf{p}\cdot\overline{\mathbf{e}}^i$. That is, whenever prices change, the individual's wealth also changes.

which there exists function D_ℓ such that

$$D_\ell(\mathbf{p}, \overline{w}) \equiv \widetilde{D}_\ell\left(\mathbf{p}, \left\{w^i\right\}_{i\in\mathbf{I}}\right).$$

We say that *demand is aggregated* when we can find such function D_ℓ. Before establishing an important result, let us first establish some terminology.

Definition 3.1. (*Indirect utility function*) An indirect utility function, $v^i\left(\mathbf{p}, w^i\right)$, is defined as

$$v^i\left(\mathbf{p}, w^i\right) \equiv u^i\left(\left\{x_\ell^i\left(\mathbf{p}, w^i\right)\right\}_{\ell\in\mathbf{M}}\right).$$

Definition 3.2. (*Gorman form*) An indirect utility function is said to be of the *Gorman Form* if it can be written in terms of functions $a^i(\mathbf{p})$ and $b(\mathbf{p})$ such that

$$v^i\left(\mathbf{p}, w^i\right) \equiv a^i(\mathbf{p}) + b(\mathbf{p})\, w^i.$$

Observe that a^i may depend on the specific agent while $b(p)$ does not.

We now state the result (MWG Proposition 4.B.1).

Fact 3.1. *Demand is aggregated if and only if each individual's utility function has an indirect utility function of the Gorman form.*

To see where why Gorman Form implies aggregation, remember Roy's identity (which you should know as part of your innate knowledge of economics!):[3]

$$x_\ell^i\left(\mathbf{p}, w^i\right) = -\frac{\frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial p_i}}{\frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial w^i}}.$$

Substituting the expression for the Gorman Form yields

$$x_\ell^i\left(\mathbf{p}, w^i\right) = -\frac{a_\ell^i(\mathbf{p}) + b_\ell(\mathbf{p})\, w^i}{b(\mathbf{p})}, \tag{3.1}$$

where the subscript ℓ on a and b denote the partial derivative. We can derive the expression for

[3]To see where this comes from, observe that

$$\begin{aligned} v^i\left(\mathbf{p}, w^i\right) &= \max_{\mathbf{x}^i}\ u\left(\mathbf{x}^i\right) \ \ s.t.\ \mathbf{p}\cdot\mathbf{x}^i = w^i \\ &= u\left(\overline{\mathbf{x}}^i\right) + \lambda\left(w^i - \mathbf{p}\cdot\overline{\mathbf{x}}^i\right), \end{aligned}$$

where λ is the Lagrange multiplier and $\overline{\mathbf{x}}^i$ denotes the demand function. By the Envelope Theorem,

$$\frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial p_\ell} = -\lambda\overline{x}_\ell^i,\ \ \frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial w^i} = \lambda.$$

Then,

$$-\frac{\frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial p_i}}{\frac{\partial v^i\left(\mathbf{p}, w^i\right)}{\partial w^i}} = -\frac{-\lambda\overline{x}_\ell^i}{\lambda} = \overline{x}_\ell^i.$$

aggregate demand:

$$\begin{aligned}\widetilde{D}_\ell\left(\mathbf{p},\{w^i\}_{i\in\mathbf{I}}\right) &= \sum_{i\in\mathbf{I}}\left[-\frac{a_\ell^i(\mathbf{p})+b_\ell(\mathbf{p})\,w^i}{b(\mathbf{p})}\right]\\ &= -\frac{1}{b(\mathbf{p})}\sum_{i\in\mathbf{I}}a_\ell^i(\mathbf{p})-\frac{b_\ell(\mathbf{p})}{b(\mathbf{p})}\sum_{i\in\mathbf{I}}w^i\\ &= -\frac{1}{b(\mathbf{p})}\sum_{i\in\mathbf{I}}a_\ell^i(\mathbf{p})-\frac{b_\ell(\mathbf{p})}{b(\mathbf{p})}\overline{w}.\end{aligned}$$

Observe that $\widetilde{D}_\ell$ depends only on the aggregate wealth level and not on how wealth is distributed among the agents. That is, demand aggregates!

To get an intuition as to the Gorman form, we can differentiate both sides of (3.1) with respect to w^i (we can do this because the equation is an identity) and obtain

$$\frac{\partial x_\ell^i\left(\mathbf{p},w^i\right)}{\partial w^i}=\frac{b_\ell(\mathbf{p})}{b(\mathbf{p})},\ \forall\ell\in\mathbf{M}.$$

You should notice two things from this expression:

- ▷ for a fixed price $\mathbf{p}$, the derivative does not depend on wealth; i.e. as wealth increases, demand increases at a constant rate (i.e. Engel curves are straight lines);
- ▷ the derivative does not depend on i; i.e. the Engel curves for different consumers are all parallel.

The implication of these two is that each agent increases demand at a constant rate when their wealth increases so that total change in demand can be expressed as the constant rate times the change in aggregate wealth!

3.2 Examples of utility functions with Gorman Form

We will consider two cases: homothetic and quasilinear utility functions.

Definition 3.3. *(Homotheticity)* A homothetic function is a monotonic transformation of a function which is homogenous; i.e. u^i is homothetic if

$$u^i(\mathbf{x})=g^i\left(h^i(\mathbf{x})\right),$$

where g^i is a strictly increasing concave function and where h is homogenous of degree one, i.e.

$$h^i(\eta\mathbf{x})=\eta h^i(\mathbf{x}),\ \forall\eta>0,\ \forall\mathbf{x},$$

and independent of i.

Proposition 3.1. *If u^i is homothetic, then the slope of the indifference curves remain constant along any ray from the origin.*

Proof. The slope of the indifference curve is given by the (negative of) the marginal rate of substitution:

$$\frac{\frac{\partial u^i(\mathbf{x})}{\partial x_\ell}}{\frac{\partial u^i(\mathbf{x})}{\partial x_k}} = \frac{\frac{\partial g^i(h(\mathbf{x}))}{\partial h(\mathbf{x})}\frac{\partial h^i(\mathbf{x})}{\partial x_\ell}}{\frac{\partial g^i(h(\mathbf{x}))}{\partial h(\mathbf{x})}\frac{\partial h^i(\mathbf{x})}{\partial x_k}} = \frac{\frac{\partial h^i(\mathbf{x})}{\partial x_\ell}}{\frac{\partial h^i(\mathbf{x})}{\partial x_k}} = \frac{h^i_\ell(\mathbf{x})}{h^i_k(\mathbf{x})}.$$

Remember the properties of CRS production from the TA class on OLG? We prove there that if a function exhibits CRS (i.e. homogenous of degree one), then its partial derivatives are homogenous of degree zero. Since h is a homogenous of degree one, h^i_ℓ and h^i_k are homogenous of degree zero. That is,

$$\frac{h^i_\ell(\mathbf{x})}{h^i_k(\mathbf{x})} = \frac{h^i_\ell(\alpha\mathbf{x})}{h^i_k(\alpha\mathbf{x})},\ \forall\alpha > 0$$

so that along any ray from the origin, the MRS are the same so that the slope of the indifference curves remain constant. ■

Proposition 3.2. *If u^i is homothetic and h^i is strictly increasing, then the demand functions are linear in endowment/income; i.e.*

$$x^i_\ell\left(\mathbf{p}, w^i\right) = B_\ell\left(\mathbf{p}\right) w^i,\ \forall\ell\in\mathbf{M}$$

for some function B.

Proof. We prove by contradiction. First, since u^i is a strictly increasing transformation h, we know that

$$\begin{aligned}&\arg\max_{\mathbf{x}^i\in\mathbf{X}^i}\ u^i\left(\mathbf{x}^i\right)\ \ s.t.\ \mathbf{p}\cdot\mathbf{x}^i = \mathbf{p}\cdot\overline{\mathbf{e}}^i\\ \equiv\ &\arg\max_{\mathbf{x}^i\in\mathbf{X}^i}\ h^i\left(\mathbf{x}^i\right)\ \ s.t.\ \mathbf{p}\cdot\mathbf{x}^i = \mathbf{p}\cdot\overline{\mathbf{e}}^i,\end{aligned}$$

where h is homogenous of degree one. Now, suppose, by way of contradiction, that the optimal demand $x^i_\ell\left(\mathbf{p},\overline{\mathbf{e}}^i\right)$ is not homogenous of degree one; i.e.

$$x^i_\ell\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right) \neq t x^i_\ell\left(\mathbf{p},\overline{\mathbf{e}}^i\right)$$

for some $t > 0$. Observe that both $x^i_\ell\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right)$ and $t x^i_\ell\left(\mathbf{p},\overline{\mathbf{e}}^i\right)$ are budget feasible on the budget $\mathbf{p}\cdot t\overline{\mathbf{e}}^i$. But since, $x^i_\ell\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right)$ is the optimal choice, it must be that

$$h^i\left(\mathbf{x}^i\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right)\right) > h^i\left(t\mathbf{x}^i\left(\mathbf{p},\overline{\mathbf{e}}^i\right)\right).$$

Dividing both sides by $1/t$ and using the fact that h^i is homogenous of degree one:

$$h^i\left(\frac{\mathbf{x}^i\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right)}{t}\right) > h^i\left(\mathbf{x}^i\left(\mathbf{p},\overline{\mathbf{e}}^i\right)\right).$$

Note that $\mathbf{x}^i\left(\mathbf{p},\overline{\mathbf{e}}^i\right)$ is the optimal choice given budget $\mathbf{p}\cdot\overline{\mathbf{e}}^i$ but above inequality says that choosing $\mathbf{x}^i\left(\mathbf{p}, t\overline{\mathbf{e}}^i\right)/t$ which is also budget feasible under $\mathbf{p}\cdot\overline{\mathbf{e}}^i$ yields strictly higher utility—a contradiction that $\mathbf{x}^i\left(\mathbf{p},\overline{\mathbf{e}}^i\right)$ is the optimal choice. ■

Corollary 3.1. *If u^i is homothetic and h^i is strictly increasing, then the indirect utility can be* *represented by*

$$v^i\left(\mathbf{p}, w^i\right) = h^i\left(\mathbf{B}\left(\mathbf{p}\right)\right) w^i.$$

Proof. From the Proposition above, we know that

$$\mathbf{x}^i\left(\mathbf{p}, w^i\right) = \mathbf{B}\left(\mathbf{p}\right) w^i$$

so that

$$\begin{aligned} v^i\left(\mathbf{p}, w^i\right) &= h^i\left(\mathbf{x}^i\left(\mathbf{p}, w^i\right)\right) \\ &= h^i\left(\mathbf{B}\left(\mathbf{p}\right) w^i\right) \\ &= h^i\left(\mathbf{B}\left(\mathbf{p}\right)\right) w^i, \end{aligned}$$

where, in the last line, we used the fact that h^i is homogenous of degree one. Since g^i is strictly increasing, $g^i\left(h^i\left(\cdot\right)\right)$ and $h^i\left(\cdot\right)$ represents the same preferences. ■

The Corollary tells us that, homothetic preferences has the Gorman form given by

$$\begin{aligned} b\left(\mathbf{p}\right) &:= h^i\left(\mathbf{B}\left(\mathbf{p}\right)\right), \\ a\left(\mathbf{p}\right) &:= 0. \end{aligned}$$

3.3 Translating into Fernando's language

We need two translations. First, the discussions above was about wealth not endowments. Second, the discussion above was about aggregate demand, and not about competitive equilibrium. Let's go through each in turn.

In a pure exchange economy, differences in wealth across agents arises because of differences in endowments (since everyone faces the same prices). So, we can think of the question of whether we can aggregate demand as where we can write the aggregate demand as function of total endowment $\overline{\mathbf{e}} = \sum_{i\in\mathbf{I}} \overline{\mathbf{e}}^i$.

We also saw that λ_i is determined by the marginal value of income, which will be affected by the vector of individual endowments $\left\{\mathbf{e}^i\right\}$. Thus, we can also think of the aggregation problem as whether aggregate demand depends on the $\boldsymbol{\lambda}$-weights.

Now let's think about the implication of the fact that aggregate demand may depend on distribution of wealth/endowments/$\boldsymbol{\lambda}$-weights. Remember that we the competitive equilibrium prices are determined by combining the demand functions and the market clearing condition— we find finding the prices that equates aggregate demand to aggregate supply. So, if aggregate demand depends on these distributions, it also means that equilibrium prices depends on the distributions. Therefore, we can think of the problem of aggregation as a question of whether equilibrium prices $\mathbf{p}$ depends on the distribution of endowments, or equivalently, the distribution of the $\boldsymbol{\lambda}$ weights.

4 Checking for aggregation

Steps for checking for aggregation:

- ▷ Write down the planner's problem (maximise weighted utility subject to feasibility) and derive the first-order conditions.
- ▷ Sum across agents the first-order condition to get an expression for γ_ℓ in terms of aggregate endowment, $\overline{e}_\ell$, and $\boldsymbol{\lambda}$-weights.
- ▷ Take the ratio of γ_ℓ's and see if it depends on $\boldsymbol{\lambda}$-weights (remember that $\gamma_\ell = p_\ell$ and that we can only determine relative prices in a competitive equilibrium).

4.1 2013/14 Midterm 1 Q2: Social Planner's Problem

Consider a pure endowment economy with I agents and m commodities. Let $\bar{e}_\ell$ denote the aggregate endowment of each commodity ℓ and let $u^i : \mathbb{R}^m \to \mathbb{R}$ be the utility function of each agent i given by

$$u^i\left(\mathbf{x}^i\right) := -\sum_{l=1}^{m} \beta_\ell \exp\left[-\frac{x_\ell^i}{\rho_i}\right],$$

where $\beta_\ell > 0$ are common parameters across the I agents for each good $\ell \in \{1, 2, \ldots, m\} =: \mathbf{M}$, and where the parameter $\rho_i > 0$ are potentially different across agents. We will consider a problem with weights $\lambda_i > 0$ for each agent $i = 1, 2, \ldots, I$. Without loss of generality, assume that the weights satisfy

$$\sum_{i=1}^{I} \rho_i \ln \lambda_i = 0. \tag{4.1}$$

Why is it without loss of generality? Note that we are free to normalise the $\boldsymbol{\lambda}$-weights. That is, we can divide each λ_i by some constant and the solution to the planner's problem will not change. So, given ρ_i's, we can always rescale λ_i's so that (4.1) holds.

Question 1 Using γ_ℓ for the Lagrange multiplier of the constraint on commodity ℓ, write the first-order conditions of the planning problem for agent i's consumption of commodity ℓ.

......

The planner's problem is given by the following

$$\max_{\{\mathbf{x}^i\}_{i\in\mathbf{I}}} \quad \sum_{i=1}^{I} \lambda_i \left(-\sum_{\ell=1}^{m} \beta_\ell \exp\left[-\frac{x_\ell^i}{\rho_i}\right]\right)$$
$$s.t. \quad \sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \overline{\mathbf{e}}^i.$$

The Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \left(-\sum_{l=1}^{m} \beta_\ell \exp\left[-\frac{x_\ell^i}{\rho_i}\right]\right) + \sum_{l=1}^{m} \gamma_\ell \left(\sum_{i=1}^{I} \overline{e}_\ell - \sum_{i=1}^{I} x_\ell^i\right).$$

The first-order conditions with respect to x_ℓ^i is then

$$\lambda_i \frac{\beta_\ell}{\rho_i} \exp\left[-\frac{x_\ell^i}{\rho_i}\right] = \gamma_\ell, \ \forall i \in \mathbf{I}, \ \forall \ell \in \mathbf{M}.$$

Question 2 Take logs of of the first-order conditions derived above, multiply by ρ_i and add across the I agents. Use the definition of aggregate endowment for good ℓ. Show an expression for γ_ℓ.

......

Taking logs and multiplying both sides by ρ_i yields

$$\begin{aligned}
\rho_i \ln\left(\lambda_i \frac{\beta_\ell}{\rho_i} \exp\left[-\frac{x_\ell^i}{\rho_i}\right]\right) &= \rho_i \ln \gamma_\ell \\
\Leftrightarrow \rho_i \ln\left(\lambda_i \frac{\beta_\ell}{\rho_i}\right) - x_\ell^i &= \rho_i \ln \gamma_\ell \\
\Rightarrow \sum_{i=1}^{I} \rho_i \ln\left(\frac{\beta_\ell}{\rho_i}\right) + \underbrace{\sum_{i=1}^{I} \rho_i \ln \lambda_i}_{=0} - \bar{e}_\ell &= \ln \gamma_\ell \left(\sum_{i=1}^{I} \rho_i\right) \\
\Leftrightarrow \ln \beta_\ell \left(\sum_{i=1}^{I} \rho_i\right) - \sum_{i=1}^{I} \rho_i \ln(\rho_i) - \bar{e}_\ell &= \ln \gamma_\ell \left(\sum_{i=1}^{I} \rho_i\right) \\
\Leftrightarrow -\frac{\sum_{i=1}^{I} \rho_i \ln(\rho_i) + \bar{e}_\ell}{\sum_{i=1}^{I} \rho_i} &= \ln \frac{\gamma_\ell}{\beta_\ell} \\
\Leftrightarrow \beta_\ell \exp\left[-\frac{\sum_{i=1}^{I} \rho_i \ln(\rho_i) + \bar{e}_\ell}{\sum_{i=1}^{I} \rho_i}\right] &= \gamma_\ell
\end{aligned}$$

Question 3 According to your answer above, do these preferences show a *strong* form of aggregation? Do equilibrium prices (or equivalently, the ratio of the $\boldsymbol{\gamma}$'s) depend on the distribution of wealth (or equivalently, on the distribution of the $\boldsymbol{\lambda}$'s)?

......

The ratio of γ_ℓ/γ_k is given by

$$\begin{aligned}
\frac{\gamma_\ell}{\gamma_k} &= \frac{\beta_\ell}{\beta_k} \exp\left[-\frac{\sum_{i=1}^{I} \rho_i \ln(\rho_i) + \bar{e}_\ell}{\sum_{i=1}^{I} \rho_i} + \frac{\sum_{i=1}^{I} \rho_i \ln(\rho_i) + \bar{e}_k}{\sum_{i=1}^{I} \rho_i}\right] \\
&= \frac{\beta_\ell}{\beta_k} \exp\left[\frac{\bar{e}_k - \bar{e}_\ell}{\sum_{i=1}^{I} \rho_i}\right].
\end{aligned}$$

Thus, equilibrium prices do not depend on $\boldsymbol{\lambda}$'s. Hence, distribution of wealth does matter for the ratio and we have aggregation.

Remark 4.1. You may wonder that (4.1) means that we may have to change ρ_i's whenever we change λ_i's. But, as per the explanation of why we can assume (4.1) without loss of generality, (4.1) can be made to hold by rescaling λ_i's while maintaining the same ρ_i's so that $\sum_{i=1}^{I} \rho_i$ remains constant.