Theory of Income, Fall 2017
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Problem Set 4

1 Neoclassical Growth Model: Exogenous Growth

Let the exogenous time augmenting productivity growth gross rate be $\lambda > 1$. Feasibility is given by

$$c_t + k_{t+1} = F\left(k_t, \lambda^t n_t\right) - (1-\delta)\,k_t,$$

where F is a neoclassical constant returns to scale production function, k_t is capital, c_t is consumption, n_t is labor, and δ is the depreciation rate of capital. The endowment of time per period is normalized to 1, so that leisure is $l_t = 1 - n_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t v\left(c_t, 1-n_t\right).$$

The consumer has a budget constraint given by

$$\sum_{t=0}^{\infty} p_t\left[c_t + x_t\right] = \sum_{t=0}^{\infty} p_t\left[w_t n_t + k_t v_t\right],$$

and the law of motion of capital is

$$k_{t+1} = x_t + (1-\delta)\,k_t,$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and v_t are the real wage and rental rate of capital in terms of consumption goods at period t.

The firm's problem is

$$\max_{k_t, n_t}\; F\left(k_t, \lambda^t n_t\right) - w_t n_t - v_t k_t.$$

Exercise 1. Let r_t be the time t interest rate, i.e. $p_t/p_{t+1} = 1 + r_t$. Use the budget constraint of the household, and the law of motion of capital to show that, as long as $x_t > 0$,

$$v_{t+1} = r_t + \delta.$$

[Hint: Consider an investment of 1 at t, renting it on $t+1$ and consuming the undepreciated capital at $t+1$].

Exercise 2. Write down the first order condition w.r.t. c_t, n_t and k_{t+1}. Use μ for the multiplier on the budget constraint [Hint: Replace x_t in the household budget constraint using the law of motion of capital]. Combine the FOC for c_t for two consecutive periods to obtain a relationship between the marginal rate of substitution between c_t and c_{t+1} and r_t. Combine the FOC with respect to c_t and n_t to obtain a relationship between the marginal rate of substitution between n_t and c_t and w_t.

Exercise 3. Use the expression for the rental rate of capital v_t for $t \geq 1$ and the law of motion of capital for $t \geq 0$ (solving for x_t) to show that the household's budget constraint can be written as

$$\sum_{t=0}^{\infty} p_t c_t = p_0 k_0 (v_0 + 1 - \delta) + \sum_{t=0}^{\infty} p_t w_t n_t.$$

Definition: A *balanced growth path* is given by an initial capital k_0 and λ such that it is optimal to set

$$\begin{aligned} c_t &= c_0 \lambda^t, \\ k_t &= k_0 \lambda^t, \\ n_t &= n_0, \\ w_t &= \lambda^t w_0, \\ r_t &= r_0, \end{aligned}$$

for all $t \geq 0$.

Exercise 4. Write down the FOC for the household imposing a balanced growth path. Use the FOC for the household and firm's problem as well as feasibility to write down the system of equations in c_0, n_0, k_0, w_0, r_0 that a balanced growth path must satisfy.

Exercise 5. Show that if preferences are of the form

$$v(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} h(1-n), \tag{1}$$

for $\gamma \neq 1$, or

$$v(c, 1-n) = \log c + h(1-n),$$

then there is a balanced growth path.

Exercise 6. Assume that $v(c, l)$ is strictly concave and increasing in (c, l) and have the form described in (1). Consider first the case where $\gamma \in (0, 1)$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex? Next, consider the case where $\gamma = 1$. What are the properties of $h(l)$? i.e. is it increasing or decreasing, concave or convex? Finally, consider the case where $\gamma > 1$. What are the properties of $h(l)$? i.e. is it positive or negative, increasing or decreasing, concave or convex?

Exercise 7. Let $v(c, 1-n)$ be

$$v(c, 1-n) = g(c - n^{\sigma}/\sigma),$$

for $\sigma > 1$ and g strictly increasing and concave. What is the income elasticity of the labor supply for this utility function? Show that this preferences are inconsistent with a balanced growth path. [Hint: In a BGP we must have

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = w_0\lambda^t,$$

but

$$\frac{v_l(c_0\lambda^t, 1-n_0)}{v_c(c_0\lambda^t, 1-n_0)} = \frac{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)}{g'(c_0\lambda^t - n_0^{\sigma}/\sigma)} n_0^{\sigma-1} = n_0^{\sigma-1} = w_0\lambda^t,$$

so compare the LHS and RHS of the last equality].

Exercise 8. Show that if the economy admits a balanced growth path for an open set of parameters β, λ and δ, preferences must be of the form in (1). [Hint: Write down an Euler equation-like expression relating the marginal rate of substitution of consumption between t and $t+1$ with r. Impose the balanced growth condition on this, noticing that this expression must be satisfied for all t. Differentiate this expression with respect to t to obtain a differential equation, whose solution implies that v is of the form $B(1-n) + c^{1-\gamma(1-n)} h(1-n)$ or $B(1-n) + \log c + h(1-n)$ where γ is a function of $(1-n)$. Use this, and the condition that marginal rates of substitution equal relative prices on a balanced growth path to establish the required result].

2 Deriving the Euler Equation in Continuous Time

In this question we obtain the continuous time Euler Equation by taking limits of the discrete time Euler equation. The point of this is to realize that although the expression for the continuous time counterpart is less intuitive that the one for the discrete time -which has the natural interpretation of equating marginal cost to marginal benefit- they are really the same.

The idea is to consider a sequence of discrete time dynamic programing problems. In each problem the length of time between periods where the state is decided is denoted by Δ. Decisions are taken at times $0,\ \Delta,\ 2\Delta,\ 3\Delta, ...$ The sequence of states to be chosen is

$$\left\{x_{\Delta(t+1)}\right\}_{t=0}^{\infty} = \{x_{\Delta}, x_{2\Delta}, x_{3\Delta}, ...\},$$

where x_0 is given. Setting $\Delta = 1$ we obtain the standard problem analyzed in the class notes.

We adjust the discount factor for each problem accordingly letting

$$\beta = \frac{1}{1+\Delta\rho},$$

so that ρ has the interpretation of a discount rate.

For each Δ we write the period return function during the interval of time of length Δ as F, and the corresponding return function per unit of time as $\hat{F}$. They satisfy:

$$F\left(x_t, x_{t+\Delta}\right) \equiv \Delta\ \hat{F}\left(x_t, \frac{1}{\Delta}\left(x_{t+\Delta} - x_t\right)\right),$$

where $t = i\Delta$ for some integer i. It helps to write these return functions as

$$F\left(x, y\right) \equiv \Delta\ \hat{F}\left(x, \frac{1}{\Delta}\left(y - x\right)\right),$$

or

$$F\left(x, \dot{x}\Delta + x\right) = \Delta\ \hat{F}\left(x, \dot{x}\right),$$

where we define $\dot{x}$ as the change per unit of time on the state:

$$\dot{x} \equiv \frac{y-x}{\Delta},$$

or using time subscripts:

$$\dot{x}_t = \frac{x_{t+\Delta} - x_t}{\Delta},$$

for $t = i\Delta$ and any integer i.

Likewise we can define the feasible correspondence $\hat{\Gamma}$ for the change per unit of time $\dot{x}$ in terms of the feasible correspondence for levels Γ as

$$\hat{\Gamma}\left(x\right) = \left\{\dot{x} : y \in \Gamma\left(x\right),\ y = \dot{x}\Delta + x\right\}.$$

Thus for each Δ we consider the problem

$$\max_{\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^t\ F\left(x_{t\Delta},\ x_{(t+1)\Delta}\right),$$

subject to

$$x_{(t+1)\Delta} \in \Gamma\left(x_{t\Delta}\right)\ ,$$

for all $t \geq 0$, where x_0 given.

Equivalently, we can write this problem as a choice of the sequence of discrete time changes

$\{\dot{x}_{t\Delta}\}_{t=0}^{\infty}$:

$$\max_{\{\dot{x}_{t\Delta}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\Delta\rho}\right)^{t} \Delta \; \hat{F}\left(x_{t\Delta}, \; \dot{x}_{t\Delta}\right),$$

subject to

$$\dot{x}_{t\Delta} \in \hat{\Gamma}\left(x_{t\Delta}\right),$$

$$x_{t\Delta+\Delta} = x_{t\Delta} + \dot{x}_{t\Delta} \; \Delta,$$

for all $t \geq 0$, and for given x_0.

We emphasize that the optimal sequence $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$ that solves the problem with interval of length Δ is a function of Δ.

For future reference, we also introduce the notation for the changes per unit of time on the change per unit of time of the state, denoting it by $\ddot{x}_t$:

$$\ddot{x}_t \equiv \frac{1}{\Delta}\left(\dot{x}_{t+\Delta} - \dot{x}_t\right),$$

for all $t = i\Delta$ and an integer i.

Exercise 1. Derive a formula for F_y and F_x in terms of $\partial\hat{F}/\partial x$ and $\partial\hat{F}/\partial\dot{x}$. In particular use the relationship between F and $\hat{F}$ to show that

$$F_y\left(x, \dot{x}\Delta + x\right) = \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right),$$

$$F_x\left(x, \dot{x}\Delta + x\right) = \Delta\frac{\partial}{\partial x}\hat{F}\left(x, \dot{x}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x, \dot{x}\right).$$

Exercise 2. Write the Euler Equations for the problem where we chose the sequence of levels of the state: $\left\{x_{(t+1)\Delta}\right\}_{t=0}^{\infty}$. Your Euler equation should involve $F_y, \; F_x, \; \Delta, \; \rho$ and be evaluated at $x_t, \; x_{t+\Delta}$ and $x_{t+2\Delta}$. [Hint: This is the standard problem].

Exercise 3. Rewrite the Euler equation obtained in 2 replacing the $x_{t+\Delta}$ in F_y in terms of Δ , x_t and $\dot{x}_t$, and replacing the $x_{t+2\Delta}$ in F_x in terms of Δ , $x_{t+\Delta}$ and $\dot{x}_{t+\Delta}$.

Exercise 4. Use the relationship between the derivatives of F and $\hat{F}$ found in 1 into your expression for the Euler equation found in exercise 3.

Exercise 5. Show that by rearranging the terms in the expression found in 4, the Euler equation can be written as:

$$\frac{1}{\Delta}\left[\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right) - \frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t, \dot{x}_t\right)\right] = \rho\frac{\partial}{\partial\dot{x}}\hat{F}\left(x_t, \dot{x}_t\right) + \frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta}, \dot{x}_{t+\Delta}\right).$$

Assumptions. The next steps consists on taking the limit of the above expression as $\Delta \to 0$. For this we will assume that as we take the limit as $\Delta \to 0$, the solutions are such that the resulting path $x\left(t\right)$ is twice differentiable with respect to time, so that the following limits are well defined and given by the corresponding expressions:

$$\begin{aligned}
\lim_{\Delta\to 0} x_{t+\Delta} &= x_t, \\
\lim_{\Delta\to 0} \frac{x_{t+\Delta}-x_t}{\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \dot{x}_{t+\Delta} &= \dot{x}_t, \\
\lim_{\Delta\to 0} \frac{\dot{x}_{t+\Delta}-\dot{x}_t}{\Delta} &= \ddot{x},
\end{aligned}$$

for all t.

Exercise 6. Use the Assumptions to show that the limit of the RHS of the expression in 5 is

$$\lim_{\Delta\to 0}\left[\rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)\right]=\rho\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)+\frac{\partial}{\partial x}\hat{F}\left(x_t,\dot{x}_t\right).$$

Exercise 7. Taking the limit of the LHS of the expression in 5 is more subtle. Use the expressions for the limits as $\Delta \to 0$ in the Assumptions to show that the limit as $\Delta \to 0$ of the LHS of the EE derived in 5

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right],$$

requires the use of L'Hôpital's rule for its evaluation.

Exercise 8. We now apply L'Hôpital's rule to evaluate the limit as $\Delta \to 0$ of the LHS of the EE derived in 5. To do so use the definitions

$$\begin{aligned}
x_{t+\Delta} &= x_t+\dot{x}_t\Delta, \\
\dot{x}_{t+\Delta} &= \dot{x}_t+\ \ddot{x}_t\Delta,
\end{aligned}$$

so that

$$\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)=\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t+\dot{x}_t\Delta,\ \dot{x}_t+\ddot{x}_t\Delta\right),$$

in computing the derivative

$$\frac{\partial}{\partial \Delta}\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right).$$

Show that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_{t+\Delta},\dot{x}_{t+\Delta}\right)-\frac{\partial}{\partial \dot{x}}\hat{F}\left(x_t,\dot{x}_t\right)\right]=\frac{\partial^2}{\partial x\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ \dot{x}_t+\frac{\partial^2}{\partial \dot{x}\partial \dot{x}}\hat{F}\left(x_t,\ \dot{x}_t\right)\ddot{x}_t.$$

[Hint: Use the assumptions to take the limit].

Exercise 9. Use your answers to question 5, 6 and 8 to obtain the continuous time Euler equation:

$$\frac{\partial^2}{\partial \dot{x} \partial x} \hat{F}(x_t, \dot{x}_t) \; \dot{x}_t + \frac{\partial^2}{\partial \dot{x} \partial \dot{x}} \hat{F}(x_t, \dot{x}_t) \, \ddot{x}_t = \rho \frac{\partial}{\partial \dot{x}} \hat{F}(x_t, \dot{x}_t) + \frac{\partial}{\partial x} \hat{F}(x_t, \dot{x}_t).$$

3 Continuous time agent's problem

Exercise 1. Let the discrete time budget constraint for a problem with length of time period Δ be

$$a_{t+\Delta} + \Delta c_t + \Delta \tau_t = \Delta w_t (1 - \bar{\tau}_t) + (1 + \Delta r_t (1 - \bar{\tau}_t)) \, a_t,$$

where a_t are assets, w_t wages, τ_t lump sum taxes, $\bar{\tau}_t$ income tax rate, and r_t the interest rate. Show that as Δ goes to zero this gives the following asset accumulation equation:

$$\dot{a}(t) + c(t) + \tau(t) = (1 - \bar{\tau}(t)) [w(t) + r(t) \, a(t)].$$

[Hint: Rearrange the discrete time expression, divide by Δ, and take limits].

Exercise 2. Show that the following present value budget constraint

$$\int_t^\infty [c(s) + \tau(s) - w(s)(1 - \bar{\tau}(s))] \, e^{-\int_t^s r(u)(1-\bar{\tau}(u))du} ds = a(t),$$

is a solution of the previous asset accumulation equation. [Hint: Differentiate this expression with respect to time].

Exercise 3. Formulate the problem of an agent with utility

$$\int_0^\infty e^{-\rho t} U(c(t)) \, dt,$$

of choosing consumption subject to the present value budget constraint (at time $t = 0$) obtained in the previous exercise. Write the Lagrangian using λ for the multiplier of the present value budget constraint. Show that the FOC with respect to $c(t)$ is:

$$e^{-\rho t} U'(c(t)) = \lambda e^{-\int_0^t r(s)(1-\bar{\tau}(s))ds}.$$

Show that this equation implies

$$\frac{\dot{c}(t)}{c(t)} = [(1 - \bar{\tau}(t)) \, r(t) - \rho] \;\; / \;\; \left[-c(t) \frac{U''(c(t))}{U'(c(t))} \right].$$

[Hint: Differentiate both sides of the FOC with respect to time].

Exercise 4. Consider the budget constraint of the government with purchases g_t, lump sum taxes τ_t and income taxes at rate $\bar{\tau}(t)$, and government assets (i.e. minus government debt) b_t :

$$b_{t+\Delta} + \Delta g_t = \bar{\tau}_t (\Delta w_t + \Delta r_t a_t) + \Delta \tau_t + b_t (1 + \Delta r_t).$$

Show that, as Δ goes to zero it implies:

$$\dot{b}(t) + g(t) = \bar{\tau}(t)(w(t) + r(t) a(t)) + \tau(t) + b(t) r(t),$$

and that it corresponds to the following present value budget constraint:

$$b(t) + \int_t^\infty [\tau(s) + \bar{\tau}(s)(w(s) + r(s) a(s)) - g(s)] e^{-\int_t^s r(u)du} ds = 0.$$

Exercise 5. Walras' law. Show that if i) $a, c, \tau, \bar{\tau}, w, r$ satisfy the asset accumulation equation for the households, ii) $b, g, \tau, \bar{\tau}, w, a, r$ satisfy the asset accumulation equation for the government, iii) there is equilibrium in the asset market, i.e.

$$a(t) + b(t) = k(t),$$

for all $t \geq 0$, and iv) firms maximize profits, so that:

$$\begin{aligned} r(t) &= f'(k(t)), \\ w(t) &= f(k(t)) - f'(k(t)) k(t), \end{aligned}$$

for all $t \geq 0$. Then the allocation is feasible, i.e.

$$\dot{k}(t) + c(t) + g(t) = f(k(t)),$$

holds for all $t \geq 0$.

Exercise 6. Ricardian Equivalence. Let a, b, τ, g, r, w, k be an equilibrium with lump sum taxes, so $\bar{\tau}(t) = 0$ all t. Consider the following fiscal policies with lump sum taxes τ' and debt b' satisfying:

$$\int_0^\infty \tau'(t) e^{-\int_0^t r(s)ds} dt = \int_0^\infty \tau(t) e^{-\int_0^t r(s)ds} dt,$$

and $b'(0) = b(0)$. Show that $a', b', \tau', g, r, w, k$ is also an equilibrium with lump sum taxes for some path of assets a' such that $a'(0) = a(0)$. [Hint: You must show that agents still maximize with the same choices c given τ', r, w, for some path of assets a' with $a'(0) = a(0)$ given, that firms maximize their profits, and that the government budget constraint also holds].

4 Convergence after a permanent productivity shock in the Neoclassical Growth Model

Let $f(k) = \varepsilon\ k^\alpha$ be a neoclassical production function with productivity parameter $\varepsilon > 0$ and capital share parameter $\alpha \in (0,1)$. The law of motion of capital is given by $dk/dt = f(k) - c$ (we set depreciation rate to zero). Preferences are $\int_0^\infty e^{-\rho t} U(c(t))\, dt$ for $U(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma > 0$.

Exercise 1. Continuous time Euler Equation . Write down the differential equation that c must satisfy in an optimal path. Your solution should contain, $\rho,\ f'(k),\ dc/dt,\ c$ and γ.

Exercise 2. Steady states . Write down two equations in two unknowns (c^* and k^*) that determine the steady state values of consumption and capital.

Exercise 3. Steady state consumption-capital ratio . Write an expression for the steady state value of consumption over capital, i.e. c^*/k^* as a function of ρ and α.

Exercise 4. Slope of the saddle path or optimal consumption function . Let $c(k)$ be the value of consumption that belong to the saddle path for a given capital k. In this exercise you need to find an expression for the slope of the saddle path evaluated at (c^*, k^*), i.e. you need to find $c'(k^*)$. Notice that $c(k)$ is also the optimal decision rule for consumption for a given k.

4.1) Derive the following quadratic equation:

$$c'(k^*)\left[\rho - c'(k^*)\right] = -\frac{\rho^2}{\gamma}\frac{1-\alpha}{\alpha},$$

whose positive solution is $c'(k^*)$.

4.2) Give an intuitive explanation of why $c'(k^*)$ is decreasing in γ. Your explanation should contain, *at most,* three lines.

[Hint for 4.1: To obtain the quadratic equation, evaluate

$$\frac{dc(k^*)}{dk} = \lim_{k \to k^*} \frac{dc/dt}{dk/dt},$$

where dc/dt is the expression obtained in 1) and dk/dt is the law of motion of capital, and where c is written as a function of k, i.e. $c = c(k)$ in both expressions. You need to use L'Hôpital's rule to find an expression for the right hand side, since at the steady state $dc/dt = dk/dt = 0$. To use L'Hôpital, differentiate the denominator and the numerator with respect to k, and evaluate them at the steady state value k^*].

Exercise 5. Impact effect on consumption of a permanent change in productivity . Suppose that at time $t = 0$ the economy is in a steady state corresponding to productivity ε. Suppose that at $t = 0$ we learn that productivity will immediately and permanently increase by a very small amount from ε to $\varepsilon' (> \varepsilon)$. We let $c^*(\cdot)$ and $k^*(\cdot)$ be the steady state consumption and capital levels and $c(k, \cdot)$ be the optimal consumption decision rules -as a function of capital- that corresponding to each productivity level ε and ε'.

It turns out that consumption at time $t = 0$ may increase, decrease or stay the same value relative to the old steady state, i.e. that $c(k^*(\varepsilon), \varepsilon') \geq (\leq)\ c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$.

5.1. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c(k^*(\varepsilon), \varepsilon') > c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$. Indicate $c(k^*(\varepsilon), \varepsilon')$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.2. Draw a phase diagram, labelling the saddle paths and steady states for both values of productivity (ε and ε') that is consistent with $c(k^*(\varepsilon), \varepsilon') < c^*(\varepsilon) = c(k^*(\varepsilon), \varepsilon)$. Indicate $c(k^*(\varepsilon), \varepsilon')$ in your diagram. Make sure that your phase diagram respects the qualitative properties shown in 3) and 4) for the consumption-capital ratios and the slopes of the saddle paths. Draw the saddle path as if it has constant slope.

5.3. Explain why it may be the case that consumption does not increase in impact (i.e. explain case 5.2). Make sure to mention the income effect and intertemporal substitution effects of the increase in productivity in your explanation.

5.4. Given your previous explanation, for which values of γ do you think that case 5.2 will occur ? Explain.

Hints. Use 3), 4) to argue that $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$. Use a phase diagram to argue that whether $c(k^*(\varepsilon), \varepsilon') \geq (\leq)\ c^*(\varepsilon)$ depend on whether $c'(k^*(\varepsilon), \varepsilon) = c'(k^*(\varepsilon'), \varepsilon')$ is smaller (higher) than $c^*(\varepsilon)/k^*(\varepsilon) = c^*(\varepsilon')/k^*(\varepsilon')$.

5 Investment in the Neoclassical Growth Model and Business Cycles

This exercise examines the dynamic behavior of gross investment in the neoclassical growth model. Specifically, it considers the adjustment path to the steady state, starting with a capital stock below its steady state value. Since capital is the state variable and since the dynamics of the neoclassical growth model are stable, net investment (i.e. the change in the stock of capital) must be decreasing in its adjustment path. Nevertheless, the behavior of gross investment (i.e. the change in capital plus the value of depreciation) can be different. In particular, below it is shown that gross investment could either increase or decrease in the adjustment path to steady state. Moreover, the investment to GDP ratio could be increasing.

This question is interesting because the neoclassical growth model is used as a simple business cycles model where unanticipated permanent productivity shocks are the source of fluctuations. In particular, we will like to see if the adjustment path after a permanent unexpected increase in productivity is consistent with the increase in investment typical of an expansion.

The planner's problem is to maximize

$$\int_0^\infty e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt,$$

subject to

$$\dot{k} = Ak^\alpha - c - \delta k.$$

That is, the production function is Cobb-Douglas, preferences are of the CRRA form with relative risk aversion γ, ρ is the discount rate and δ is the depreciation rate of capital. The solution to this problem is given by the Euler Equation and the law of motion of capital:

$$\begin{aligned} \dot{c} &= \frac{c}{\gamma}\left(\alpha A k^{\alpha-1} - (\rho+\delta)\right), \\ \dot{k} &= Ak^\alpha - c - \delta k. \end{aligned}$$

Exercise 1. Letting x be gross investment, i.e. $x = \dot{k} + \delta k$, show that the linear approximation of $x(t)$ around the steady state satisfies

$$x(t) - x^* = (k(0) - k^*)\exp(\lambda_1 t)[\lambda_1 + \delta],$$

for all $t \geq 0$,where λ_1 solves

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2},$$

and where x^* and k^* are steady state values. [**Hint**: the following steps will be useful:

Step 1 : Differentiate the equation for $\dot{k}$ with respect to t and insert the $\dot{c}$ equation into the result you obtain. This gives a second order differential equation in k (the Euler equation!). Linearize the resulting equation, that is, obtain

$$\ddot{k} = g\left(\dot{k}, k\right) \cong g(0, k^*) + g_{\dot{k}}(0, k^*)\dot{k} + g_k(0, k^*)(k - k^*),$$

and compute the coefficients. You should obtain $\ddot{k} = \rho\dot{k} - f''(k^*)(c^*/\gamma)(k - k^*)$. Moreover, denoting $z(t) = k(t) - k^*$ the last equation becomes $\ddot{z} = \rho\dot{z} - f''(k^*)(c^*/\gamma)z$

Step 2 : Argue that the solution to the last differential equation is $z(t) = B\exp(\lambda t)$ where λ solves the quadratic equation $\lambda^2 - \rho\lambda + (c^*/\gamma)f''(k^*) = 0$. Pick the stable solution (i.e. the negative root of λ), which satisfies

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4f''(k^*)(c^*/\gamma)}}{2} < 0$$

Step 3 : Show that for the functional forms we are using:

$$f''(k^*)(c^*/\gamma) = (\alpha - 1)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho + \delta(1-\alpha)}{\alpha}\right)$$

which implies

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)\frac{(\rho+\delta)}{\gamma}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)}}{2}$$

Step 4 : Use

$$\begin{aligned} x(t) &= \dot{k}(t) + \delta k(t) \\ &= \dot{z}(t) + \delta(z(t) + k^*) \end{aligned}$$

and replace the linearized solution to obtain the result].

Exercise 2. Let $\lambda_1(p)$ be the solution of λ_1 as a function of the parameters $p = (\rho, \delta, \gamma, \alpha)$. Show that

$$\lambda_1(p) + \delta \geq 0,$$

if and only if

$$\gamma \geq (1-\alpha)\left[\frac{\rho/\delta + (1-\alpha)}{\alpha}\right].$$

What does this imply for the dynamics of investment x close to the steady state if:

a) γ is large enough so that $\gamma > (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

b) if δ is small enough so that $\gamma < (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this?

c) if α is small enough so that $\gamma < (1-\alpha)\left[\frac{\rho/\delta+(1-\alpha)}{\alpha}\right]$, what is the intuition for this? [Hint: how does the marginal productivity of capital vary with α *close* to the steady state?]

[Hint: Let $F(p) \equiv 2(\lambda_1(p) + \delta)$, and use the explicit expression for λ_1 derived above. Show that

$$F(p) = 2\delta + \rho - \sqrt{(2\delta+\rho)^2 + \left(4(\rho+\delta)\frac{\delta}{\gamma}\left[(1-\alpha)\left(\frac{\rho/\delta + (1-\alpha)}{\alpha}\right) - \gamma\right]\right)}$$

].

Exercise 3. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

Compute $\lambda_1 + \delta$ for these two set of parameters.

Now we examine the behavior of the ratio x/y, where y denotes GDP. This ratio is a natural business cycles indicator, high in the expansions and low in the recessions. It has the added advantage of being independent of common "trends" in x and y.

We focus in the following quantity:

$$\mu(k) = \left(\frac{k}{x(k)/y(k)} d\frac{x(k)/y(k)}{dk} \right)$$

where $x(k)$, and $y(k)$ are investment and GDP as function of capita (i.e. $\mu(k)$ is the elasticity of the investment to GDP ratio with respect to capital). We are interested in this quantity evaluated at $k = k^*$ which we simply refer to as μ. This can be written as

$$\begin{aligned} \mu(k(t)) &= \left(\frac{k(t)}{x(k(t))/y(k(t))} d\frac{x(k(t))/y(k(t))}{dk} \right) \\ &= \left(\frac{1}{x(t)/y(t)} d\frac{x(t)/y(t)}{dt} \right) \bigg/ \left(\frac{1}{k(t)} \frac{dk(t)}{dt} \right) \end{aligned}$$

Then we study

$$\upsilon(t) = \frac{1}{x(t)/y(t)} d\frac{x(t)/y(t)}{dt}$$

for the linear approximation to the solution of the model.

Exercise 4. Show that

$$\mu = \frac{\lambda_1}{\delta} + 1 - \alpha$$

[Hint: Start by finding the linearized approximation of $\mu(k(t))$. To that end, note that $\upsilon(t) = \frac{1}{x(t)}\frac{dx(t)}{dt} - \frac{1}{y(t)}\frac{dy(t)}{dt}$. Then use the linear approximations

$$\begin{aligned} x(t) - x^* &= (\lambda_1 + \delta)(k(t) - k^*), \\ y(t) - y^* &= (\rho + \delta)(k(t) - k^*), \\ k(t) - k^* &= (k(0) - k^*)\exp(\lambda_1 t), \\ dk(t)/dt &= (k(0) - k^*)\exp(\lambda_1 t)\lambda_1 \end{aligned}$$

to obtain the derivatives. Replace them into your expression for $\mu(k(t))$. Lastly, take the limit as t goes to infinity and use the equations relating x^*, k^* and y^* to obtain μ].

Exercise 5. Show that

$$\lim_{\rho/\delta \to 0} \mu = (1-\alpha)\left[1 - \frac{1}{\sqrt{\gamma\alpha}}\right].$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$, ρ/δ is small, and $\gamma\alpha > 1$? Does this setting of parameters made investment/GDP in the model procyclical?

Exercise 6. Show that

$$\lim_{\rho/\delta \to \infty} \mu = -\infty.$$

Is the path of $x(t)/y(t)$ increasing or decreasing through time if $k(0) < k^*$ and ρ/δ is very large? Is this consistent with the interpretation of the adjustment to steady state as an expansion?

Exercise 7. Consider the following set of parameters. The first set is

$$\alpha = 0.3,\ A = 1,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$$

and the second is

$$\alpha = 0.4,\ A = 1,\ \gamma = 2,\ \rho = 0.05,\ \delta = 0.10$$

For each setting of parameters compute $\lambda_1/\delta + (1-\alpha)$ and comment if they are consistent with the procyclicality of x/y.

Exercise 8. Suppose that the economy is in steady state with $A = 1$. Use the first set of parameters ($\alpha = 0.3,\ \gamma = 2,\ \rho = 0.075,\ \delta = 0.075$) for all the calculations.

i. What is the steady state investment to GDP ratio x^*/y^*?

Assume that A unexpectedly changes to $A' = (1+\varepsilon)^{1-\alpha}$, or approximately $(1-\alpha)\varepsilon/100$ %, for small ε. Assume that A' will stay at that level forever.

ii. What is the % change in the steady state capital? Denote this capital by k^{**}. Does the steady state value x/y depend on A?

iii. Use the definition of μ above to compute the change in the investment/GDP ratio x/y on impact. Your answer should be a function of μ, (x^*/y^*) and ε. [Hint. Let $z(k) = x(k)/y(k)$, use a first order approximation for z around k^{**} and evaluate it at $k = k^*$].

iv. Using the reference numerical values for all parameters, compute the new value of x/y just after the change in productivity if $\varepsilon = 0.1$ (10%).

6 Investment Specific Technological Progress

Consider the following version of the neoclassical growth model

$$\max \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt$$

$$\begin{aligned} \text{s.t.} \quad p_k x\left(t\right) + c\left(t\right) &= F\left[k(t), e^{\gamma t}\right], \\ \dot{k}\left(t\right) &= e^{\eta t} x\left(t\right) - \delta k\left(t\right), \quad \text{all } t \geq 0, \\ \text{given } k\left(0\right) &= k_0 > 0, \end{aligned}$$

where F has constant returns to scale, and (inelastically supplied) labor is normalized at unity. Labor-augmenting technical change occurs at the rate $\gamma \geq 0$, and investment-specific technical change at the rate η. The constant p_k corresponds to the initial (time $t = 0$) price of investment in terms of consumption.

Exercise 1. Formulate the Hamiltonian using c as the only control variable and write the first order conditions for an optimum.

Exercise 2. Reduce the system obtained in 1 to a pair of differential equations in (c, k).

Use the following functional form for the remaining of this question:

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \\ F\left(k, e^{\gamma t}\right) &= A\ k^\alpha e^{(1-\alpha)\gamma t}, \quad 0 < \alpha < 1. \end{aligned}$$

Balanced Growth Path. Suppose $\gamma, \eta > 0$. Consider the growth rates of capital and consumption in the long run. Conjecture that in the long run the ratio of consumption to capital falls at the rate η. That is, conjecture that along the balanced growth path $c(t)/k(t) = x^* e^{-\eta t}$, where x^* is a constant that must be determined.

Exercise 3. Calculate the long run growth rates for capital (g_k), consumption (g_c) and output as function of the parameters $\alpha, \sigma, \delta, \gamma$ and η.

Exercise 4. How would you measure the contributions to long term growth of labor-augmenting technical change and investment-specific technical change for the US economy? [Sketch only, 10 lines maximum. Hint: A good answer should mention measuring TFP (total factor productivity) being careful on the units on which GDP and factors are measured, and the rate at which the investment deflator and the consumption deflator grow. Notice also that labor-augmenting technical change is closely related to TFP when the production function is Cobb-Douglas].

For the rest of the question consider the case where $\eta = \gamma = 0$.

Exercise 5. Draw the phase diagram corresponding to the system in b. Have c in the vertical axis and k in the horizontal axis. Indicate the $\dot{k} = 0$ and $\dot{c} = 0$ locus, the steady state values k^*, c^*. Include arrows indicating the direction of movement in each relevant quadrant, display the saddle path clearly, and indicate typical paths of trajectories that start close but not on the saddle path.

Exercise 6. Find an expression for the steady state capital stock k^* as a function of p_k, α, ρ and δ. Find an expression for the value of capital relative to GDP, i.e. $p_k k^*/y^*$ where y^* is the steady state output.

Exercise 7. Find an expression for the steady state consumption c^* as a function of p_k, α, ρ and δ. Find an expression for the value of consumption relative to GDP, i.e. c^*/y^* where y^* is the steady state output.

For the remaining of this question, consider the case of no depreciation, i.e. $\delta = 0$.

Exercise 8. What is the steady state value of investment x^*? What is the steady state value of investment $p_k x^*/y^*$ relative to GDP.

Exercise 9. Assume that $k(0)$ is smaller than the its steady state value k^*. Draw a figure with time t in the horizontal axis, and output $y(t)$, investment $x(t)$, and consumption $c(t)$ in the vertical axis. Label the steady state values for output, investment and consumption, y^*, x^* and c^* in the vertical axis. Make sure $y(t) = c(t) + p_k x(t)$ and that x^* is as in exercise 8.

Exercise 10. Assume that $k(0) = k^*$ is the steady state value of capital for the price p_k. Assume that at time $t = 0$, the price of capital decreases to p'_k with $p'_k < p_k$ and will remain there forever. Denote the new steady state values as y^{**}, c^{**} and x^{**}. Draw a figure with time t in the horizontal axis, and output $y(t)$, investment $x(t)$ and consumption $c(t)$ in the vertical axis. Make sure $y(t) = c(t) + p'_k x(t)$ and that x^* and x^{**} are as in exercise 8. Label the steady states values for output y^* and y^{**}, investment x^* and x^{**} and consumption c^{**} and c^* in the vertical axis. Show whether $c^* < c(0)$ or $c^* \geq c(0)$ and whether $y^* < y(0)$ or $y^* \geq y(0)$.

What do you learn about the "cyclicality" of consumption? That is, would the transition caused by a permanent decrease in p_k starting from a steady state capital look like an economic boom? (recall that in a boom consumption and GDP both increase together). How do you think your answer will change for a higher (positive) value of the depreciation δ?

7 Habit Formation model

Assume that agents select $c(t)$ and $l(t)$ for $t \geq 0$ to maximize

$$\int_0^{\infty} e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - Al(t) \right] dt,$$

subject to

$$\begin{aligned} c(t) &= wl(t), \\ \dot{s}(t) &= c(t) - \delta s(t), \end{aligned}$$

for all $t \geq 0$, for given $s(0)$.

In this problem c is consumption, l labor, w the real wage, ρ the discount factor, s the undepreciated stock of past consumption or the habit level, δ the depreciation rate of consumption in the habit level, and $u(c - \delta s) - Al$ the period utility function.

As it is clear from the expression above the agent has no access to a savings account or a storage technology. At each point in time the agent must decide how much to work (or consume). The problem is not static because we assume that preferences depend on labor l, and consumption c *relative to the depreciated stock of consumption* s.

We can write this problem as

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \left[u\left(c(t) - \eta s(t)\right) - \frac{A}{w} c(t) \right] dt,$$

subject to

$$\dot{s}(t) = c(t) - \delta s(t),$$

for all $t \geq 0$ given $s(0)$. We will ignore the non-negativity conditions for $l(t)$.

We will assume that $0 < \eta < \delta$, $\delta > 0, \rho > 0$ and $w > 0$. At some points we will also assume that u has constant relative risk aversion γ, i.e.

$$u(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma},$$

for $\gamma > 0$.

Exercise 1. Write the Hamiltonian $H(c, s, \lambda)$. Which is the control and which is the state?

Exercise 2. Write the first order conditions (these are two equations $H_c = 0$ and $\dot{\lambda} = ...$). Your answer should be in terms of the parameters ρ, δ, η, w and the function u.

Exercise 3. Use the previous equations, including the one for $\dot{s}$, to solve for steady state c as a function of the parameters of the model.

Exercise 4. Show that the long run elasticity of consumption and labor supply is

$$\frac{w}{c^*}\frac{dc^*}{dw} = \left[\frac{u'}{-cu''}\left(c^*\left(1-\frac{\eta}{\delta}\right)\right)\right] \;/\; \left[1-\frac{\eta}{\delta}\right],$$

i.e. this is the percentage change in steady state consumption for a 1% change in wages.

Exercise 5. To be able to draw the phase diagram for this model we want to eliminate λ from the system of three equations (one for $\dot{\lambda}$, one for $\dot{s}$ and $H_c = 0$). This is analogous to what we did in the neoclassical growth model. You must show that you can solve for λ and express the system of two differential equations in $\dot{s}$ and $\dot{c}$ as

$$\begin{aligned}
\dot{s} &= c - \delta s, \\
\dot{c} &= \frac{u'}{-u''}\left[\eta - (\rho+\delta)\right] + \left[\frac{1}{-u''}\right]\frac{A}{w}(\rho+\delta) + \eta\,(c - \delta s).
\end{aligned}$$

Exercise 6. Define the function $\theta(s)$ as giving the combinations of $(c, s) = (\theta(s), s)$ such that $\dot{c} = 0$, where $\dot{c}$ is given in question 5. Show that, at the steady state level s^*

$$\theta'(s^*) = \frac{\eta\left[(\rho+2\delta)-\eta\right]}{(\rho+\delta)}.$$

Exercise 7. Show that, since $\eta < \delta$, then

$$0 < \theta'(s^*) < \delta.$$

From now on assume that the utility function u has CRRA γ.

Exercise 8. Show that $d\dot{c}/dc > 0$, where $\dot{c}$ is the equation found in exercise 5 evaluated at steady state values c^* and s^*. [Hint. Using that u has CRRA it is easy to show that $d(\dot{c}/c)/dc > 0$. This, of course, implies that $d\dot{c}/dc > 0$].

Exercise 9. Draw the phase diagram. To simplify draw the phase diagram as if both the $\dot{s} = 0$ and the $\dot{c} = 0$ loci were linear. Put s in the x-axis and c in the y-axis. Make sure to draw arrows with the directions in which s and c will move in all relevant quadrants. Make sure that you include some arrows that intersect the $\dot{s} = 0$ and $\dot{c} = 0$ loci at points different from the steady state, and that the slope of these arrows are consistent with the $\dot{c}$ and $\dot{s}$ equations (either they cross vertically or horizontally). Make sure your graph is readable, the points you'll obtain depend on it! Plot the saddle path, make sure that it crosses the right quadrants, and that it has the right slope (i.e. that it is stepper or flatter than the relevant $\dot{c} = 0$ or $\dot{s} = 0$ loci).

Exercise 10. Assume that at time $t = 0$, the habit levels $s(0)$ is given by the steady state value s^*. Then, "unexpectedly", agents learn at time $t = 0$ that w will be higher, say $\bar{w} > w$

for T periods, and then it will return to the steady state value w. That is, $w(t) = \bar{w}$ for $t = (0, T)$, and $w(t) = w$ for $t \geq T$. You have to analyze the path for optimal consumption and habit for the case of this transitory increase in wages. In this page you must draw a phase diagram with both saddle path corresponding to w and $\bar{w}$, but the flow (arrows) should correspond to $\bar{w}$. Remember that s cannot "jump" at time $t = 0$ but consumption can. Remember also that at time $t = T$ the system must land continually (with respect to time) in the saddle path corresponding to w. In this phase diagram you must draw the trajectory in the c-s space that the agent will chose. Make sure that this trajectory is clearly labeled, that includes arrows showing the direction of movement, that it starts at the right height (i.e. the qualitatively correct level of c) and that if it crosses any of the $\dot{c} = 0$ or $\dot{s} = 0$ it does so with the right slope. In the next page you must also draw two figures with time t in the horizontal axis and with the optimal path of c (one figure) and s in the vertical axis. Start these figures at some time $t < 0$ and clearly label the time periods $t = 0$, $t = T$ and include horizontal lines for the steady state values for consumption $c^*$and $\bar{c}^*$ and habit $s^*, \bar{s}^*$ that correspond to the values of w and $\bar{w}$, respectively. You should obtain the qualitative features of these time trajectories from your previous figure. Make sure your graphs are readable, the points you'll obtain depend on it!