2.4 TA Session (2019.01.11)

2.4.1 Sufficient Statistics

Example 2.6. (Sufficient statistics for normal distribution) We have a density $f_X(x)$ such that we can factorize it into

$$f_X(x) = \exp\left(g(\theta)K(x) + s(\theta) + h(x)\right)$$

then $K(x)$ is your sufficient statistic. Once you pin down the value of $K(x)$ then you don't have to care about the part where θ and x interact. For example, the sufficient statistics for

$$\begin{aligned} & x_1, ..., x_N \sim N(\mu, \sigma^2) \\ f_X(x_1, ..., x_N) &= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}\right) = \left(2\pi\sigma^2\right) \exp\left(-\sum_{i=1}^{n} \frac{x_i^2}{\sigma^2} + 2\sum_{i=1}^{n} \frac{\mu x_i}{\sigma^2} - \frac{n\mu^2}{\sigma^2}\right) \end{aligned}$$

is $T(X) = \left(\sum X_i, \sum X_i^2\right)$.

To see this, note that

$$f_{X|T}(x|t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{f_X(x) \cdot 1\{t, x \text{ are compatible}\}}{f_T(t)}$$

Example 2.7. (Sufficient statistics for Poisson) Given $x_1, ..., x_N \sim Poisson(\lambda)$, we have

$$f_X(x) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-n\lambda}\lambda^{\sum x_i} \prod_{i=1}^{n} \frac{1}{x_i!}$$

In this case, $\sum x_i$ is the sufficient statistic. To see this more explicitly,

$$f_{X,T}(x,t) = f_X(x) \cdot 1\{t, x \text{ are compatible}\}$$

Furthermore, $T = \sum x_i \sim Poisson(n\lambda)$ so the marginal density should be

$$f_T(t) = e^{-n\lambda}(n\lambda)^t \frac{1}{t!}$$

and the conditional density is then obtained as

$$f_{X|T}(x|t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{1}{n^t} t! \prod_{i=1}^{n} \frac{1}{x_i!}$$

which is $Multinomial\left(\frac{1}{n}, ..., \frac{1}{n}, N\right)$

2.4.2 Conjugate Prior

Example 2.8. Prior $\theta \sim Beta(\alpha, \beta)$ and conditional $x|\theta \sim Binomial(p, \theta)$ yields posterior

$$\theta|x \sim Beta(\alpha + x, \beta + n - x)$$

where the density of a Beta distribution is given as

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} 1\{0 \leq \theta \leq 1\}$$

and the density of a Binomial distribution is given as

$$\frac{n!}{x!(n-x)!}\theta^x(1-\theta)^{n-x}1\left\{x=0,1,...,n\right\}$$

Then

$$f_{\Theta|X}\left(\theta|x\right)=\frac{f_{X|\Theta}(x|\theta)f_{\Theta}(\theta)}{f_X(x)}$$

Note that $P(x=1)=P(x=0)=1/2$, so we have

$$f\left(\theta|x\right)=2\theta^x(1-\theta)^{1-x}1\left\{0\leq\theta\leq1\right\}\sim Beta(1+x,2-x)$$