III. Game Theory (30 points)

In the extensive form game below, find

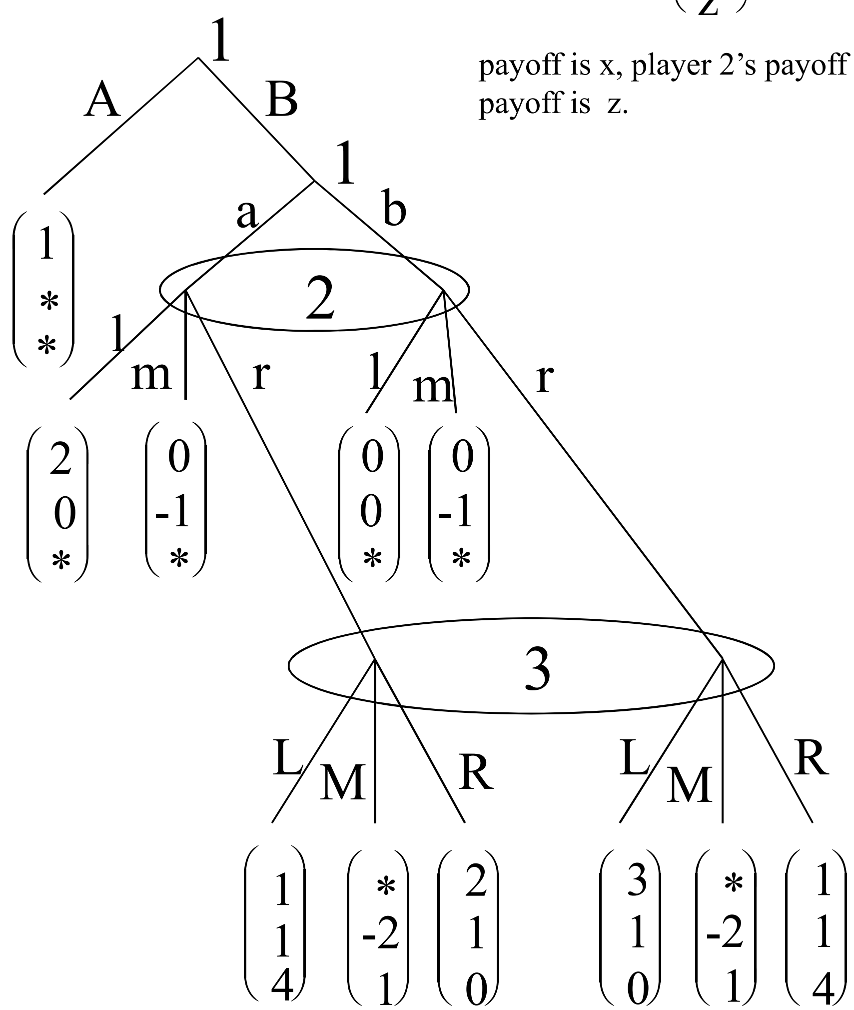
(a) (10 points) A pure strategy Nash equilibrium that is not subgame perfect. Why is it not subgame perfect?

(b) (10 points) A pure strategy subgame perfect equilibrium that is not sequential. Why is it not sequential?

(c) (10 points) A sequential equilibrium.

An asterisk in the figure below indicates that the particular payoff in that entry is not relevant.

The notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ indicates that player 1's payoff is x, player 2's payoff is y and player 3's payoff is z.



3.1 Part (a)

Find a pure strategy Nash equilibrium that is not subgame perfect. Why is it not subgame perfect.

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There are two subgames in the extensive-form game. The whole game and the game starting at the second decision node for player 1. Consider a pure strategy:

$$(A, b), m, R.$$

The strategies imply that the game ends after the first decision node, and player 1 obtains payoff 1. If he deviates, then he obtains payoff of zero. So player 1 has no incentive to deviate at this node. In his second decision node, if he deviates, he again obtains zero, so that he has no incentive to deviate.

For players and 3, deviating does not alter their payoffs since their decision node is off the equilibrium path. Hence, above is a Nash equilibrium.

To show that this strategy is not subgame perfect, we need to show that the proposed strategy is not a Nash equilibrium in the subgame starting from player 1's second decision node. But notice that playing m is strictly dominated by playing l for player 2. Hence, m is not a best response once the game reaches this second node. Thus, we have shown that the proposed strategy is not subgame perfect.

3.2 Part (b)

Find a pure strategy subgame perfect equilibrium that is not sequential. Why is it not sequential?

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Consider the following strategy:

$$(B, a), l, M.$$

In the subgame, the payoff from following the proposed strategy is 2 for player 1. If he deviates and plays b, then he obtains 0. So player 1 has no incentive to deviate. For player 2, playing m gives him -1 utility instead of zero, so he has no incentive to deviate to m. If player 2 instead deviates to r, then he receives -2 as player 3 plays M. So player 2 has no incentive to deviate.

For player 3, since his decision node is off the equilibrium path, deviating from M does not yield strictly better payoff so that he has no incentive to deviate.

To show that this is not sequentially rational, suppose that player 3 ever had the chance to make his decision. Then, observe that the strategy M is strictly dominated by a randomisation between L and R for player 3. Hence, M will not be played if player 3 ever has a chance to decide on the action to take.

3.3 Part (c)

Find a sequential equilibrium.

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As argued above, M will not be played by player 3. So we can eliminate it from the game. This, in turn, implies that, for player 2, m is strictly dominated by both l and r. So m will never be played in equilibrium.

Since m is never played by player 2, comparing l and r for player 2, we realise that playing r is the dominant strategy since he obtains 1 whether player 3 plays L or R and zero if player 2 plays l. Hence, player 2 plays r. We can then consider a two player subgame between players 1 and 3 that can be represented as

$P1\backslash P3$	L	R
a	$1, \mathbf{4}$	$\mathbf{2}, 0$
b	$\mathbf{3}, 0$	$1, \mathbf{4}$

The Nash equilibrium of this game is given by

$$\sigma_{1.2} = \frac{1}{2}[a] + \frac{1}{2}[b],$$
$$\sigma_{3.1} = \frac{1}{3}[L] + \frac{2}{3}[R].$$

Expected payoff for player 1 is then

$$\frac{1}{3}(1) + \frac{2}{3}(2) = \frac{5}{3} > 1.$$

For player 3 to randomise between L and R, his belief at $\alpha = \mu(B.a.r|3.1)$ must satisfy Bayes' consistency,

$$\begin{aligned}\alpha &= \frac{\sigma_{1.1}(B)\sigma_{1.2}(a)}{\sigma_{1.1}(B)\sigma_{1.2}(a) + \sigma_{1.1}(B)(1-\sigma_{1.2}(a))} \\ &= \frac{\sigma_{1.1}(B)0.5}{\sigma_{1.1}(B)0.5 + \sigma_{1.1}(B)0.5} = \frac{1}{2}.\end{aligned}$$

Bayes' consistency for $\beta = \mu(B.a|2.1) \in [0,1]$ requires

$$\beta = \frac{\sigma_{1.1}(B)\sigma_{1.2}(a)}{\sigma_{1.1}(B)\sigma_{1.2}(a) + \sigma_{1.1}(B)\sigma_{1.2}(b)} = \frac{1}{2}.$$

So a Nash equilibrium in sequentially rational strategy is given by

$$\begin{aligned}&((\alpha,\beta),(\sigma_{1.1},\sigma_{1.2},\sigma_{2.1},\sigma_{3.1})) \\ &= \left(\left(\frac{1}{2},\frac{1}{2}\right),\left([B],\frac{1}{2}[a]+\frac{1}{2}[b],[r],\frac{1}{3}[L]+\frac{2}{3}[R]\right)\right).\end{aligned}$$

Since every non-singleton information state is reached with positive probability, the belief-strategy pair above satisfies full consistency so that the pair is also sequentially rational.