PRICE THEORY III
SPRING 2019

(LARS STOLE)

SOLUTIONS TO
FINAL

UNIVERSITY OF CHICAGO

Contents

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v1.0 Initial version

1 Question 1 (30 points)

Consider a restaurant selling to a diner with unknown demand. The consumer desires at most a single meal, and has a payoff of $u(q,\theta) - p$ for a meal of quality $q \geq 0$ and price p, where

$$u(q,\theta) := \theta q.$$

The consumer's type is privately known to the consumer, but distributed uniformly on $[\underline{\theta}, \overline{\theta}]$.

The restaurant's costs are strictly increasing and convex in quality. In addition, consumers with higher θ are also more demand to serve and increase the restaurant's costs. Let costs of serving quality q to consumer with type θ be

$$C(q,\theta) = c(\theta) q + \frac{1}{2} q^2,$$

where $c(\theta)$ is a strictly increasing function.

1.1 Part (a)

What is the full-information, efficient allocation, $q^{fb}(\theta)$?

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Under full information, we may assume that the restaurant can observe the consumer's type perfectly. For each θ, the restaurant thus solves the following problem

$$\begin{aligned} \max_{p, q \geq 0} \quad & p - \left(c(\theta) q + \frac{1}{2} q^2 \right) \\ \text{s.t.} \quad & \theta q - p \geq 0, \end{aligned}$$

where the constraint is the participation constraint. At the optimum, since restaurant's objective function is strictly increasing in p, the constraint must bind. Hence, we can solve the following problem instead:

$$\max_{q \geq 0} \theta q - c(\theta) q - \frac{1}{2} q^2.$$

We are told that the cost function is strictly increasing and convex quality q so that the first-order condition from above is necessary and sufficient for a global maximum in case of interior solution. The full-information, efficient allocation is then

$$q^{fb}(\theta) = \max\{\theta - c(\theta), 0\}.$$

1.2 Part (b)

Assume in parts (b)–(e) that **the allocation in (a) is strictly increasing in θ**. The firm wishes to maximise profits by offering a menu (literally) of prices and qualities, indexed by the different types of consumer: $\{p(\cdot), q(\cdot)\}_{\theta\in[\underline{\theta},\overline{\theta}]}$. Define the type-$\theta$ consumer's indirect utility when choosing the menu item designed for θ as

$$U(\theta) := \theta q(\theta) - p(\theta).$$

Provide the standard representation of incentive-compatibility for $\{p(\cdot), q(\cdot)\}_{\theta\in[\underline{\theta},\overline{\theta}]}$ in terms of a condition for $q(\cdot)$ and an integral condition for the consumer's indirect utility, $U(\cdot)$.

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Since $\frac{\partial^2 u(q,\theta)}{\partial q \partial \theta} = 1 > 0$, the two conditions are

(i) $q(\cdot)$ nondecreasing; and

(ii) $U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\mathrm{d}s$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

1.3 Part (c)

Compute an expression for $\mathbb{E}[U(\theta)]$ and use your result to reduce the restaurant's objective function to an object that depends only on the choice variables $q(\cdot)$ and $U(\underline{\theta})$.

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Let $F(\cdot)$ and $f(\cdot)$ denote the cumulative distribution function and probability density function of Uniform$[\underline{\theta}, \overline{\theta}]$, respectively. Given the integral condition from part (b), we can compute $\mathbb{E}[U(\theta)]$:

$$\begin{aligned}
\mathbb{E}[U(\theta)] &= \mathbb{E}_\theta\left[U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,\mathrm{d}s\right] \\
&= U(\underline{\theta}) + \underbrace{\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\theta}}_{\underline{\theta}\le s\le\theta\le\overline{\theta}} q(s)\,\mathrm{d}s f(\theta)\,\mathrm{d}\theta = U(\underline{\theta}) + \underbrace{\int_{\underline{\theta}}^{\overline{\theta}}\int_{s}^{\overline{\theta}}}_{\underline{\theta}\le s\le\theta\le\overline{\theta}} f(\theta)\,\mathrm{d}\theta q(s)\,\mathrm{d}s \\
&= U(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} (1-F(s))\frac{f(s)}{f(s)} q(s)\,\mathrm{d}s = U(\underline{\theta}) + \mathbb{E}\left[\frac{1-F(\theta)}{f(\theta)} q(\theta)\right],
\end{aligned}$$

where we exchanged integral in the usual manner. Since $U(\theta) = \theta q(\theta) - p(\theta)$, we may rewrite above as

$$\begin{aligned}
\mathbb{E}[\theta q(\theta) - p(\theta)] &= U(\underline{\theta}) + \mathbb{E}\left[\frac{1-F(\theta)}{f(\theta)} q(\theta)\right] \\
\Leftrightarrow \mathbb{E}[p(\theta)] &= \mathbb{E}\left[q(\theta)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)\right] - U(\underline{\theta}).
\end{aligned}$$

We can substitute the expression into the restaurant's objective function:

$$\mathbb{E}[p(\theta) - C(q(\theta), \theta)] = \mathbb{E}\left[q(\theta)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - \left(c(\theta)q(\theta) + \frac{1}{2}(q(\theta))^2\right)\right] - U(\underline{\theta}).$$

Since $\theta \sim \text{Uniform}[\underline{\theta}, \overline{\theta}]$, $F(\theta) = (\theta - \underline{\theta}) / (\overline{\theta} - \underline{\theta})$ and $f(\theta) = 1/(\overline{\theta} - \underline{\theta})$; i.e.

$$\theta - \frac{1 - F(\theta)}{f(\theta)} = \theta - \frac{1 - \frac{\theta - \underline{\theta}}{\overline{\theta} - \underline{\theta}}}{\frac{1}{\overline{\theta} - \underline{\theta}}} = 2\theta - \overline{\theta}$$

so that the objective function further simplifies to

$$\mathbb{E}\left[\left(2\theta - \overline{\theta}\right) q(\theta) - \left(c(\theta) q(\theta) + \frac{1}{2}(q(\theta))^2\right)\right] - U(\underline{\theta}).$$

1.4 Part (d)

Characterise the optimal allocation $q(\cdot)$.

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The IR constraint is that

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s) \mathrm{d}s \geq 0, \ \forall \theta \in \left[\underline{\theta}, \overline{\theta}\right].$$

If $q(\theta)$ is nondecreasing, then above holds if and only if

$$U(\underline{\theta}) \geq 0.$$

Since the restaurant's objective is strictly decreasing in $U(\underline{\theta})$ and this terms appears additively separably, at the optimum, the constraint must bind. Thus, we can write the restaurant's program as:

$$\max_{q(\cdot) \geq 0} \quad \mathbb{E}\left[q(\theta)\left(2\theta - \overline{\theta}\right) - \left(c(\theta) q(\theta) + \frac{1}{2}(q(\theta))^2\right)\right] \text{ s.t. } q(\cdot) \text{ is nondecreasing.}$$

Consider the relaxed problem in which we ignore the monotonicity constraint. Since $q(\theta)$ is independent of $q(\theta')$, we may maximise pointwise to obtain that pointwise to obtain that

$$q(\theta) = \max\left\{\theta - c(\theta) + \left(\theta - \overline{\theta}\right), 0\right\}.$$

Let us verify that the solution to this relaxed problem solves the original problem with the monotonicity constraint: for all $q(\theta) > 0$,

$$q'(\theta) = 2 - c'(\theta).$$

Recall that we were told that $q(\theta)$ in part (a) is a strictly increasing function; i.e. $c'(\theta) < 1$. Hence, we realise that $q'(\theta) \geq 0$ as we wanted to show.

1.5 Part (e)

Does the price schedule which implements this allocation exhibit quality discounts or quality premia over the qualities that are implemented? [Hint: Don't solve for $P(q)$, but instead consider the nature of the consumer's utility function.]

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This is the Mussa and Rosen (1978) setting in which there is quality premia since the cost function is convex and utility is linear. Formally, since the utility function is linear in q, incentive compatibility requires that $P(q)$ is convex; if it were not, then $u(q, \theta) - P(q) = \theta q - P(q)$ would not be concave around the qualities that are implemented.

1.6 Part (f)

Now suppose that $c'(\theta) > 2$ which implies, among other things, that the allocation in (a) is strictly decreasing in θ. Also, suppose that $\overline{\theta} \geq c(\overline{\theta})$ so that it is efficient for type $\overline{\theta}$ to consume a meal in the full-information setting. How does this change your analysis in (b)–(d). Be specific. Then, explain briefly (without mathematical proof) how the optimal $q(\cdot)$ will depend upon θ.

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Parts (b) and (c) remain unchanged as we did not rely on the assumption about $c'(\theta)$. However, the optimal allocation $q(\cdot)$ that we derived in part (d) is now weakly decreasing. Since $\overline{\theta} \geq c(\overline{\theta})$, the relaxed solution is

$$q(\theta) = 2\theta - c(\theta) - \overline{\theta},$$

which is strictly decreasing over $[\underline{\theta}, \overline{\theta}]$. Hence, the solution to the unrelaxed program must have a binding monotonicity condition and there must be pooling of types consuming same equality in the optimal mechanism. Thus, $q(\theta)$ will be constant in θ. This is because of the adverse selection in $C(q, \theta)$. Notice that similarity between this question and PS7.

2 Question 2 (20 points)

There are n firms involves in an R&D race in which a single firm will win. The winning firm i will obtain a payoff of $\pi_i > 0$, which is independently and identically distributed across the firms according to continuous cumulative distribution $F(\cdot)$ on $[0,1]$. The return to winning is private information to firm i, although F is commonly known to all firms. The firms simultaneously choose their R&D expenditures, r_i. The firm with the highest expenditure wins the race.

Assume that there is a symmetric equilibrium in which each firm chooses expenditures according to $\bar{r} : [0,1] \to \mathbb{R}_+$ and $\bar{r}$ is strictly increasing in π_i.

Derive the equilibrium R&D expenditure function. (You may assume that $F(\cdot)$ has a pdf $f(\cdot)$, but that is not necessary.)

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The key is to notice that this R&D race is a first-price, all-pay auction, where the expenditure is the bid and π_i is bidder i's value in the usual setting. We can then apply the usual method to obtain the bidding function.

The expected utility truthfully "bidding" according to i's value, π_i, is

$$U_i(\pi_i) = F^{n-1}(\pi_i)\,\pi_i - \bar{r}(\pi_i).$$

The integral condition gives that

$$U_i(\pi_i) = U_i(0) + \int_0^{\pi_i} F^{n-1}(p)\,\mathrm{d}p,$$

where we implicitly ignored tie breaks since $F(\cdot)$ is continuous. We can equate the two to obtain while noting that $U(0) = 0$ (since no firm will spend any money if its payoff from winning is zero),

$$\begin{aligned} F^{n-1}(\pi_i)\,\pi_i - \bar{r}(\pi_i) &= \int_0^{\pi_i} F^{n-1}(\pi_i)\,\mathrm{d}p \\ \Leftrightarrow \bar{r}(\pi_i) &= F^{n-1}(\pi_i)\,\pi_i - \int_0^{\pi_i} F^{n-1}(p)\,\mathrm{d}p. \end{aligned}$$

Although not necessary, you can "simplify" this using integration by parts:

$$\begin{aligned} \bar{r}(\pi_i) &= F^{n-1}(\pi_i)\,\pi_i - \left(\left[F^{n-1}(p)\,p\right]_0^{\pi_i} - \int_0^{\pi_i}(n-1)\,f(p)\,F^{n-2}(p)\,p\mathrm{d}p\right) \\ &= (n-1)\int_0^{\pi_i} f(p)\,F^{n-2}(p)\,p\mathrm{d}p. \end{aligned}$$

3 Question 3 (30 points)

Consider a two-person, public goods setting. There is a single public good that can be implemented, $x \in \{0,1\}$; let $x = 1$ represent that the public good is chosen. Each person i values the public good at θ_i, which is uniformly (and independently) distributed on $[0,1]$. If the public good is chosen, however, each person must contribute personal labour to build the public good equal to a cost of $\frac{1}{2}$. Thus, player i's value of the public good is $\theta_1 - \frac{1}{2}$ and player 2's value of the public good is $\theta_2 - \frac{1}{2}$.

3.1 Part (a) (4 points)

Characterise an ex post efficient public good allocation rule $\hat{x}(\cdot)$.

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Ex post efficient public good allocation rule $\hat{x}(\cdot)$ is given by

$$\hat{x}(\theta_1,\theta_2) \in \underset{x(\theta)\in\{0,1\}}{\arg\max}\; x(\theta)\left\{\left(\theta_1 - \frac{1}{2}\right) + \left(\theta_2 - \frac{1}{2}\right)\right\}.$$

One solution is

$$\hat{x}(\theta_1,\theta_2) = \mathbf{1}_{\{\theta_1+\theta_2\geq 1\}}.$$

Note that we could have specified x to be zero if $\theta_1 + \theta_2 = 1$ (which is a measure-zero event).

> **Mark scheme:** 2 points if you simply wrote the definition and 4 points for writing down the expression for this question.

3.2 Part (b) (8 points)

Design a dominant-strategy incentive compatible (DSIC), direct-revelation mechanism, $\{\hat{x}, t_1, t_2\}$, that implements the ex post efficient allocation in (a) and has the property that each pays a transfer equal to their externality on the other play whenever they are pivotal in the public good decision. Verify that your mechanism is DSIC for each player i.

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You should recall from class that a VCG mechanism would satisfy the required properties. To construct a VCG mechanism, we first find the optimal public good decision absent player i, denoted $\hat{x}_j(\theta_j)$, where $j \neq i$. Absent player i, player j should choose the public good if and only if $\theta_j \geq \frac{1}{2}$; i.e.

$$\hat{x}_j(\theta_j) = \mathbf{1}_{\left\{\theta_j\geq\frac{1}{2}\right\}}.$$

The VCG transfers are therefore

$$t_i^{vcg}(\theta) = u_j\left(\hat{x}_j(\theta_j),\theta_j\right) - u_j\left(\hat{x}(\theta),\theta_j\right) = \left(\hat{x}_j(\theta_j) - \hat{x}(\theta)\right)\left(\theta_j - \frac{1}{2}\right)$$

for each $i \in \{1,2\}$. When the first term is nonzero, then i is pivotal:

- ▷ if $\hat{x}_j(\theta_j) - \hat{x}(\theta) = 1 \Leftrightarrow (\theta_j \geq \frac{1}{2}) \wedge (\theta_i < 1 - \theta_j) \Leftrightarrow \frac{1}{2} \leq \theta_j < 1 - \theta_i$; i.e. if i is pivotal in the sense that i prevents the public good from being chosen, then i pays $\theta_j - \frac{1}{2}$;
- ▷ if $\hat{x}_j(\theta_j) - \hat{x}(\theta) = -1 \Leftrightarrow (\theta_j < \frac{1}{2}) \wedge (\theta_i \geq 1 - \theta_j) \Leftrightarrow 1 - \theta_i \leq \theta_j \leq \frac{1}{2}$; i.e. if i is pivotal in the sense that i instigates the public good being chosen, then i receives $\theta_j - \frac{1}{2}$ (or, equivalent, pays $\frac{1}{2} - \theta_j$)
- ▷ i pays zero if i is not pivotal.

To verify that the VCG mechanism defined above is DSIC, player-i-type-θ_i's ex post payoff from reporting $\hat{\theta}_i$ in the mechanism $\{\hat{x}, t_1^{vcg}, t_2^{vcg}\}$ when player $j \neq i$ report $\hat{\theta}_j$ is

$$\begin{aligned}
&\hat{x}\left(\hat{\theta}_i;\hat{\theta}_j\right)\left(\theta_i - \frac{1}{2}\right) - t_i^{vcg}\left(\hat{\theta}_i;\hat{\theta}_j\right) \\
&= \hat{x}\left(\hat{\theta}_i;\hat{\theta}_j\right)\left(\theta_i - \frac{1}{2}\right) - \left[\left(\hat{x}_j\left(\hat{\theta}_j\right) - \hat{x}\left(\hat{\theta}_i;\hat{\theta}_j\right)\right)\left(\hat{\theta}_j - \frac{1}{2}\right)\right] \\
&= \hat{x}\left(\hat{\theta}_i;\hat{\theta}_j\right)\left(\theta_i + \hat{\theta}_j - 1\right) - \hat{x}_j\left(\hat{\theta}_j\right)\left(\hat{\theta}_j - \frac{1}{2}\right).
\end{aligned}$$

By construction of $\hat{x}(\cdot)$, the first term is maximised when $\hat{\theta}_i = \theta_i$. Observe that the second term is independent of i's report so that it has no impact on the choice of i's report. Thus, truthful reporting is optimal for all $\hat{\theta}_j$ so that the mechanism we constructed is indeed DSIC.

Mark scheme: 4 points for writing the mechanism and 4 points for verifying DSIC. For each part, half mark if you just wrote down the definitions, full marks for writing down the expression for this question.
Comments: Many of you missed that player i can be pivotal in two ways!

3.3 Part (c) (10 points)

Design a Bayesian-incentive compatible (BIC), direct-revelation mechanism, $\{\hat{x}, t_1, t_2\}$, that implements the ex post efficient allocation in (a) and has the properties that each player pays a transfer equal to their *expected externality* on the other play and the total transfers are balanced (i.e., $t_1(\theta_1, \theta_2) = -t_2(\theta_1, \theta_2)$ for all θ_1, θ_2). Verify that this mechanism is budged balanced and is BIC for each player i.

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We now construct an expected externality mechanism using the VCG transfers from (b). We first take interim expectations:

$$\begin{aligned}
\bar{t}_i^{vcg}(\theta_i) &:= \mathbb{E}_{\theta_j}\left[t_i^{vcg}(\theta_i;\theta_j)\right] \\
&= \mathbb{E}_{\theta_j}\left[\left(\hat{x}_j(\theta_j) - \hat{x}(\theta)\right)\left(\theta_j - \frac{1}{2}\right)\right] = \mathbb{E}_{\theta_j}\left[\left(\mathbf{1}_{\left\{\theta_j \geq \frac{1}{2}\right\}} - \mathbf{1}_{\left\{\theta_i+\theta_j \geq 1\right\}}\right)\left(\theta_j - \frac{1}{2}\right)\right] \\
&= \int_{\frac{1}{2}}^{1}\left(\theta_j - \frac{1}{2}\right)\mathrm{d}\theta_j - \int_{1-\theta_i}^{1}\left(\theta_j - \frac{1}{2}\right)\mathrm{d}\theta_j = \left[\frac{1}{2}(\theta_j)^2 - \frac{\theta_j}{2}\right]_{\frac{1}{2}}^{1} - \left[\frac{1}{2}(\theta_j)^2 - \frac{\theta_j}{2}\right]_{1-\theta_i}^{1} \\
&= -\left(\frac{1}{8} - \frac{1}{4}\right) + \left(\frac{1}{2}(1-\theta_i) - \frac{1}{2}\right)(1-\theta_i) = \frac{1}{8} - \frac{\theta_i}{2} + \frac{\theta_i^2}{2}.
\end{aligned}$$

Taking expectation again, we obtain

$$\begin{aligned}
\bar{t}_i^{vcg} &:= \mathbb{E}_{\theta_i}\left[\bar{t}_i^{vcg}(\theta_i)\right] \\
&= \int_0^1\left(\frac{1}{8} - \frac{\theta_i}{2} + \frac{\theta_i^2}{2}\right)\mathrm{d}\theta_i = \left[\frac{1}{8}\theta_i - \frac{\theta_i^2}{4} + \frac{\theta_i^3}{6}\right]_0^1 \\
&= \frac{1}{8} - \frac{1}{4} + \frac{1}{6} = \frac{1}{24}.
\end{aligned}$$

This gives us all the ingredients for the EE mechanism:

$$\begin{aligned}
t_i^{ee}(\theta_i;\theta_j) &= \bar{t}_i^{vcg}(\theta_i) + \left(\bar{t}_j^{vcg} - \bar{t}_j^{vcg}(\theta_j)\right) - \frac{1}{2}\left(\bar{t}_i^{vcg} + \bar{t}_j^{vcg}\right) \\
&= \frac{1}{8} - \frac{\theta_i}{2} + \frac{\theta_i^2}{2} + \left(\frac{1}{24} - \left(\frac{1}{8} - \frac{\theta_j}{2} + \frac{\theta_j^2}{2}\right)\right) - \frac{1}{24} \\
&= \frac{\theta_j(1+\theta_j) - \theta_i(1+\theta_i)}{2}.
\end{aligned}$$

for each $i \in \{1,2\}$.

To verify budget balance, i.e. $\sum_{i=1,2} t_i^{ee}(\theta_i;\theta_j) = 0$, note that

$$\sum_{i=1,2} t_i^{ee}(\theta_i;\theta_j) = \frac{\theta_2(1+\theta_2) - \theta_1(1+\theta_1)}{2} + \frac{\theta_1(1+\theta_1) - \theta_2(1+\theta_2)}{2} = 0$$

as we wanted.

To verify that the EE mechanism defined above is BIC, player-i-type-θ_i's interim payoff from reporting $\hat{\theta}_i$ in the mechanism $\{\hat{x}, t_1^{ee}, t_2^{ee}\}$ when player $j \neq i$ report $\hat{\theta}_j$ is

$$\begin{aligned}
U_i^{ee}\left(\hat{\theta}_i|\theta_i\right) &= \mathbb{E}_{\theta_j}\left[\hat{x}\left(\hat{\theta}_i;\theta_j\right)\left(\theta_i - \frac{1}{2}\right) - t_i^{ee}\left(\hat{\theta}_i;\theta_j\right)\right] \\
&= \mathbb{E}_{\theta_j}\left[\hat{x}\left(\hat{\theta}_i;\theta_j\right)\left(\theta_i - \frac{1}{2}\right)\right] - \mathbb{E}_{\theta_j}\left[\bar{t}_i^{vcg}\left(\hat{\theta}_i\right) + \left(\bar{t}_j^{vcg} - \bar{t}_j^{vcg}\left(\theta_j\right)\right) - \frac{1}{2}\left(\bar{t}_i^{vcg} + \bar{t}_j^{vcg}\right)\right] \\
&= \int_{1-\hat{\theta}_i}^{1}\left(\theta_i - \frac{1}{2}\right)\mathrm{d}\theta_j - \bar{t}_i^{vcg}\left(\hat{\theta}_i\right) + \frac{1}{2}\left(\bar{t}_i^{vcg} + \bar{t}_j^{vcg}\right) \\
&= \left(\theta_i - \frac{1}{2}\right)\hat{\theta}_i - \left(\frac{1}{8} - \frac{\hat{\theta}_i}{2} + \frac{1}{2}\left(\hat{\theta}_i\right)^2\right) + \frac{1}{24} \\
&= \theta_i - \frac{1}{2}\left(\hat{\theta}_i\right)^2 - \frac{1}{12}.
\end{aligned}$$

Observe that above is strictly concave in $\hat{\theta}_i$ and attains its maximum at

$$\hat{\theta}_i = \theta_i;$$

i.e. truthful telling is optimal so that BIC holds for each player i. Alternatively, you can note that $t_i^{vcg}(\theta)$ is DSIC and so $\bar{t}_i^{vcg}(\theta_i)$ is BIC. Since $t_i^{ee}(\theta)$ differs from $\bar{t}_i^{vcg}(\theta_i)$ via a constant (with respect to θ_i), $t_i^{ee}(\theta)$ must also be BIC.

> **Mark scheme:** 4 points for writing the mechanism, 2 points for verifying BB and 4 points for verifying DSIC. For each part, half marks if you just wrote down the definition, full markets for writing down the expression for this question. I didn't penalise you if you got the wrong expression in the previous part.
> **Comments:** Even if you're not sure about how to verify BIC, write down what condition you need to show so you can get some partial credit.

3.4 Part (d)

Suppose that the players have the ability to reduce to play in the mechanism. Importantly, if player i refuses, player i cannot be forced to contribute to the public good and thus $x = 0$. What are the smallest fixed payments that needed to be added to the transfers you found in (a) to make that mechanism individually rational?

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We need to compute the IR-VCG payments. To do so, we need to compute the expected indirect utility of each player in the VCG mechanism when both players are truthful:

$$\begin{aligned}
U_i^{vcg}\left(\theta_i\right) &= \mathbb{E}_{\theta_j}\left[\hat{x}\left(\theta_i;\theta_j\right)\left(\theta_i - \frac{1}{2}\right) - t_i^{vcg}\left(\theta_i;\theta_j\right)\right] \\
&= \left(\theta_i - \frac{1}{2}\right)\theta_i - \left(\frac{1}{8} - \frac{\hat{\theta}_i}{2} + \frac{1}{2}\left(\hat{\theta}_i\right)^2\right) \\
&= \frac{1}{2}\theta_i^2 - \frac{1}{8}.
\end{aligned}$$

The smallest payment that guarantees individual rationality for i is

$$\begin{aligned} U_i^{vcg}(\theta_i) + \psi_i &\geq \underline{U}_i(\theta_i) = 0, \ \forall \theta_i \in [0,1] \\ \Leftrightarrow \psi_i &\geq -U_i^{vcg}(\theta_i) = \frac{1}{8} - \frac{1}{2}\theta_i^2 \\ \Rightarrow \psi_i^* &= \max_{\theta_i \in [0,1]} \frac{1}{8} - \frac{1}{2}\theta_i^2 = \frac{1}{8}. \end{aligned}$$

Thus, the IR-VCG payments are

$$t_i^{ir}(\theta) = t_i^{vcg}(\theta) - \frac{1}{8}.$$

> **Mark scheme:** 2 points for writing down the definition, 3 points for writing down the condition for current context and 4 points if you got the correct minimum payments. I didn't penalise you if you got the wrong expression in the previous parts.

3.5 Part (e)

Using your answer in (d), is it possible to implement the ex post efficient allocation using a budget-balanced, BIC and individually rational mechanism? You may use any result from class other than Myerson-Satterthwaite (1983) to establish your result.

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Recall from class the theorem that there exists a BIC, BB, ex post efficient, IR revelation-mechanism if and only if there exists expected budget surplus in the IR-VCG mechanism. Compute

$$\begin{aligned} \bar{t}_i^{ir} &:= \mathbb{E}_\theta\left[t_i^{ir}(\theta)\right] \\ &= \mathbb{E}_{\theta_1,\theta_2}\left[t_i^{vcg}(\theta) - \frac{1}{8}\right] = \bar{t}_i^{vcg} - \frac{1}{8} \\ &= \frac{1}{24} - \frac{1}{8} = -\frac{1}{12} \\ \Rightarrow \bar{t}_1^{ir} + \bar{t}_2^{ir} &= -\frac{1}{6} < 0. \end{aligned}$$

Hence, we conclude that it is not possible to obtain a mechanism with the desired properties.

> **Mark scheme:** 2 points if you wrote down the theory, 3 points if you also showed expression for expected budget surplus for this context and 4 points if you showed that this was negative.
> **Comments:** Many of you wrote down that *if* the mechanism runs an expected budget surplus, then the answer is "yes"; since the answer turns out to be that the mechanism runs an expected budget deficit, you actually needed to have specified "if and only if"; anyway, I didn't penalise you for this.

4 Question 4 (40 points)

A seller has up to Q identical units for sale in an n-person auction. Each bidder i has private value of $\theta_i \in [0,1]$ for one unit of good (and no value for nay additional units). θ_i is independently distributed on $[0,1]$ according to the cumulative distribution function $F(\cdot)$. We assume the distribution satisfies the monoton-hazard-rate condition. The seller's cost of each of the Q units is 0; the seller cannot sell more than Q units in total.

4.1 Part (a)

Assume that $Q = 2$ and $n > 2$. Suppose that the seller uses a third-price auction: the two highest bidders win a unit and each pays the third-highest bid. Show that there is a dominant-strategy equilibrium in which each player i bids θ_i.

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Consider player i's incentive to deviate from bidding his true value θ_i. It's useful to denote as $b^{(2)}$ is the second-highest bid among the other bidders.

- ▷ $\theta_i > b^{(2)}$. By bidding $b_i = \theta_i$, i's payoff is $\theta_i - b^{(2)} > 0$. Bidding any higher will not change the outcome and the payoff remains the same so that i can do no better by bidding $b_i > \theta_i$ than bidding $b_i = \theta_i$. A bid of $b_i = b^{(2)}$ would result in either i getting the good (in which case, the same surplus) or i not getting the good so that i's payoff is zero, depending on the tie break rule. In any case, i cannot do any better by bidding $b_i = b^{(2)}$ over bidding $b_i = \theta_i$. Finally, bidding $b_i < b^{(2)}$ means i does not win so that i earns zero. Hence, in this case, it is a dominant strategy for i to bid $b_i = \theta_i$.
- ▷ $\theta_i \leq b^{(2)}$. By bidding $b_i = \theta_i$, i's payoff is zero (note that even if i gets the good when i bids $b_i = b^{(2)}$, i's payoff is still zero). By bidding $b_i < \theta_i$, i's payoff remains at zero. If i bids $b_i > b^{(2)}$, i wins the good but pays at least $b^{(2)}$ in which case i's payoff is $\theta_i - b^{(2)} \leq 0$ so that i can do no better this way. Hence, in this case also, it is a dominant strategy for i to bid $b_i = \theta_i$.

Comments: You could have solved this indirectly along the lines of the VCG mechanism being a generalisation of a second-price auction. In the third-price auction as given in the question, we can think about the externality that a winner imposes on the others. Whether you consider the highest or the second-highest bidder, the externality that each impose is the third highest bid. So, VCG mechanism in this case tells us that the winner should pay the third highest bid. Now, we "know" that it is a (weakly) dominant strategy for each player to report truthfully in a VCG mechanism, and here, the report coincides with the bid. Thus, we may conclude that there is a dominant-strategy equilibrium in which each player i bids θ_i.

4.2 Part (b)

Assume that $Q = 1$ and $n > 2$. Suppose as in (a) that the seller uses a third-price auction: the highest bidder wins the single unit and pays the third-highest bid. Does the result in (a) still apply? Explain.

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It is no longer the case that bidding θ_i is a dominant strategy. To see this, let $b^{(1)}$ and $b^{(2)}$ denote the highest and the second-highest bids among the other plays, and suppose that $b^{(1)} > \theta_i > b^{(2)}$. In this case, bidder i can bid $b^{(1)} + \varepsilon$ to win the auction and earn $\theta_i - b^{(2)} > 0$; hence bidding truthful cannot be a weakly dominant strategy.

Recall that DSIC auction requires bidder i pay the externality i imposes on the others when winning the good. In a third-price auction with two goods, the externality caused by bidder i winning is the lost value that the third-highest bidder would have obtained consuming the good. In contrast, with only one good, the externality is the value that the second-highest bidder would have obtained, not the third highest.

4.3 Part (c)

Let $\{\phi_i(\cdot), t_i(\cdot)\}_{i=1}^n$ represent a direct-revelation auction mechanism. Let $\overline{\phi}_i(\hat{\theta}_i)$ be the expected probability that player i obtains a unit of the good given report $\hat{\theta}_i$ and all other plays report truthfully. Similarly, let $\bar{t}_i(\hat{\theta}_i)$ represent the expected payment of player i to the seller given report $\hat{\theta}_i$ and all other players report truthfully.

State a condition on $\overline{\phi}_i(\cdot)$ and an integral condition characterising

$$U_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\,\theta_i - \bar{t}_i(\theta_i)$$

that are necessary and sufficient conditions for the mechanism to be incentive compatible.

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We require:

(i) $\overline{\phi}_i(\cdot)$ to be nondecreasing; and

(ii) $U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} \overline{\phi}_i(s)\,\mathrm{d}s$ for all $\theta_i \in [0,1]$.

4.4 Part (d)

Using your result in (c) and the fact that

$$\bar{t}_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\,\theta_i - U_i(\theta_i),$$

state and sketch the proof to to the Revenue Equivalence Theorem applied to this multi-unit auction setting.

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Rewrite above as

$$U_i(\theta_i) \equiv \overline{\phi}_i(\theta_i)\,\theta_i - \bar{t}_i(\theta_i).$$

Substituting to the integral condition yields:

$$\begin{aligned}\overline{\phi}_i(\theta_i)\,\theta_i - \bar{t}_i(\theta_i) &= U_i(0) + \int_0^{\theta_i} \overline{\phi}_i(s)\,\mathrm{d}s \\ \Leftrightarrow \bar{t}_i(\theta_i) &= \overline{\phi}_i(\theta_i)\,\theta_i + U_i(0) - \int_0^{\theta_i} \overline{\phi}_i(s)\,\mathrm{d}s, \qquad (4.1)\end{aligned}$$

where$U_i(0) = \bar{t}_i(0)$. Hence, revenue from any mechanism that has the same $\overline{\phi}_i$ (i.e. interim probability of allocation to i) and $U_i(0)$ (i.e. interim expected utility for the lowest type) yields the same revenue for the seller.

4.5 Part (e)

Using your result in (c), construct the seller's expected payoff as a function that depends only on the choice variables $\{\phi_i(\cdot)\}_{i=1}^n$ and $\{U_i(0)\}_{i=1}^n$.

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The seller's payoff function is

$$\begin{aligned}\mathbb{E}_\theta\left[\sum_{i=1}^n \bar{t}_i(\theta_i)\right] &= \mathbb{E}_\theta\left[\sum_{i=1}^n \left(\overline{\phi}_i(\theta_i)\,\theta_i + U_i(0) - \int_0^{\theta_i}\overline{\phi}_i(s)\,\mathrm{d}s\right)\right] \\ &= \sum_{i=1}^n \mathbb{E}_{\theta_i}\left(\overline{\phi}_i(\theta_i)\,\theta_i + U_i(0) - \int_0^{\theta_i}\overline{\phi}_i(s)\,\mathrm{d}s\right) \\ &= \sum_{i=1}^n \mathbb{E}_{\theta_i}\left(\overline{\phi}_i(\theta_i)\,\theta_i + U_i(0)\right) - \sum_{i=1}^n \int_0^1\int_0^{\theta_i}\overline{\phi}_i(s)\,\mathrm{d}s f(\theta_i)\,\mathrm{d}\theta_i \\ &= \sum_{i=1}^n \mathbb{E}_{\theta_i}\left(\overline{\phi}_i(\theta_i)\,\theta_i + U_i(0)\right) - \sum_{i=1}^n \mathbb{E}_{\theta_i}\left[\frac{1-F(\theta_i)}{f(\theta_i)}\overline{\phi}_i(\theta_i)\right] \\ &= \sum_{i=1}^n \mathbb{E}_{\theta_i}\left(\overline{\phi}_i(\theta_i)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right) - \sum_{i=1}^n U_i(0) \\ &= \mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \sum_{i=1}^n U_i(0)\end{aligned}$$

This is very similar to Q1 part (c).

4.6 Part (f)

Find the optimal allocation $\{\phi_i(\cdot)\}_{i=1}^n$. [You may write it in terms of the multiplier, $\lambda(\theta)$, on the capacity constraint.]

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Given the capacity constraint, we need that

$$\sum_{i=1}^{n} \phi_i(\theta) \le Q,\ \phi_i(\theta) \in [0,1],\ \forall i.$$

At the optimum, IR is satisfied by setting $U_i(0) = 0$. Hence, the seller's program is

$$\begin{aligned} \max_{\phi_i(\cdot)\in[0,1]} \quad & \mathbb{E}_\theta \left[\sum_{i=1}^{n} \phi_i(\theta) \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right] \\ \text{s.t.} \quad & \sum_{i=1}^{n} \phi_i(\theta) \le Q, \\ & \overline{\phi}_i(\cdot) \text{ is nondecreasing.} \end{aligned}$$

Since F satisfies the monotone-hazard rate condition, we know that the virtual value function is strictly increasing. We can solve the relaxed problem (i.e. ignoring the monotonicity condition) by pointwise maximisation of the Lagrangian

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{n} \phi_i(\theta) \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) + \lambda(\theta) \left(Q - \sum_{i=1}^{n} \phi_i(\theta) \right) \\ &= \sum_{i=1}^{n} \phi_i(\theta) \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) \right) + \lambda(\theta) Q. \end{aligned}$$

Then,

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

where we ignore the tie break cases. The Lagrange multiplier, $\lambda(\theta) \ge 0$, is given by the complementary slackness condition:

$$\lambda(\theta) \left(Q - \sum_{i=1}^{n} \phi_i(\theta) \right) = 0.$$

Lastly, note that $\phi_i(\theta)$ must be weakly increasing in θ_i. To see this, note that, if $\lambda(\theta) = 0$, then ϕ_i is increasing because $J(\theta_i) := \theta_i - (1 - F(\theta_i))/f(v_i)$ is increasing. If $\lambda(\theta) > 0$, then $\sum_j \phi_j(\theta) = Q$. In this case, an increase in θ_i increases $J(\theta_i)$ but has no impact on $J(\theta_j)$ for $j \ne i$. Because all ϕ_j share the same multiplier in their construction, $\lambda(\theta)$, ϕ_i must weakly increase (and $\sum_{j\ne i} \phi_j$ must weakly decrease) in θ_i. Therefore, $\overline{\phi}_i(\cdot)$ from the relaxed program is nondecreasing. That is, the solution we found for the relaxed program is indeed the solution to the original problem.

4.7 Part (g)

Suppose that $Q > n$. Is there a simple way to implement the optimal mechanism. Explain?

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If $Q > n$, the part (f) implies that we should allocate the good all types with positive virtual value; i.e. $\theta_i - (1 - F(\theta_i))/f(\theta_i) \ge 0$. Since F satisfies MHRC, the virtual value is strictly increasing in θ. Therefore, there exists θ^* that satisfies

$$\theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} = 0,$$

such that, for all $\theta \geq \theta^*$, virtual value is nonnegative. Then, we can mimic the optimal allocation by a posted price mechanism—in particular, by charing $p = \theta^*$ for each unit of the good. It remains to verify that charing θ^* is optimal. Recalling that type-θ buyer purchases if and only if $\theta \geq p$, the posted price mechanism is formally given by

$$\phi_i(\theta) = \mathbf{1}_{\{\theta_i \geq \theta^*\}}, \; t_i(\theta) = \phi_i(\theta)\,\theta^*.$$

We have to check that above satisfies (4.1):

$$\bar{t}_i(\theta_i) = \overline{\phi}_i(\theta)\,\theta^* = \mathbf{1}_{\{\theta_i \geq \theta^*\}}\theta^*$$

and

$$\begin{aligned}
\overline{\phi}_i(\theta_i)\,\theta_i - \int_0^{\theta_i} \overline{\phi}_i(s)\,\mathrm{d}s &= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\theta_i - \int_0^{\theta_i} \mathbf{1}_{\{s \geq \theta^*\}}\mathrm{d}s \\
&= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\theta_i - \begin{cases} 0 & \theta_i \leq \theta^* \\ \int_{\theta^*}^{\theta_i} \mathrm{d}s & \theta_i > \theta^* \end{cases} \\
&= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\theta_i - \mathbf{1}_{\{\theta_i \geq \theta^*\}}(\theta_i - \theta^*) \\
&= \mathbf{1}_{\{\theta_i \geq \theta^*\}}\theta^* = \bar{t}_i(\theta_i)\,.
\end{aligned}$$

That is, $\{\phi_i, t_i\}_{i=1}^n$ specified above satisfies the integral condition. Note that it is the IC constraint the implies that the seller cannot discriminate while ensuring that all those types above θ^* purchase.

4.8 Part (h)

Return to the case where $Q = 2$ and $n > 2$ as in (a). Is the third-price auction optimal? Why or why not?

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In general, without a reserve price, third-price auction is not optimal since the virtual utility of the winning bidders could be negative (in which case, it is optimal for the seller to keep the good(s)).