Price Theory, III | **Professor Lars Stole**
Economics 33003 | **Spring 2019**

Assignment 7
(Due Friday, May 31, prior to the start of the Review session)

Problem 1 (JR Exercise 9.4) There are n bidders participating in a first-price auction. Each bidder's value is independently drawn from $[0, 1]$ according to the distribution function F, having continuous and strictly positive density f. If a bidder's value is θ_i and he wins the object with a bid of $b_i < \theta_i$, then his von Neumann-Morgenstern utility is $(\theta_i - b_i)^{\frac{1}{\alpha}}$, where $\alpha \geq 1$ is fixed and common to all bidders. Consequently, the bidders are risk averse when $\alpha > 1$ and risk neutral when $\alpha = 1$. Given the risk-aversion parameter α, let $\overline{b}_\alpha(\theta)$ denote the (symmetric) equilibrium bid of a bidder when his value is θ. The following parts will guide you toward finding $\overline{b}_\alpha(\theta)$ and uncovering some of its implications.

(a). Let $U(\hat{\theta}|\theta)$ denote a bidder's expected utility from bidding $\overline{b}_\alpha(\hat{\theta})$, given that all other bidders employ $\overline{b}_\alpha(\cdot)$. Show that

$$U(\hat{\theta}|\theta) = F(\hat{\theta})^{n-1}(\theta - \overline{b}_\alpha(\hat{\theta}))^{\frac{1}{\alpha}}.$$

Why must $U(\hat{\theta}|\theta)$ be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$?

(b). Use part (a) to argue that

$$U(\hat{\theta}|\theta)^\alpha = F(\hat{\theta})^{\alpha(n-1)}(\theta - \overline{b}(\hat{\theta})).$$

must be maximized in $\hat{\theta}$ when $\hat{\theta} = \theta$.

(c). Use part (b) to argue that a first-price auction with the n risk-averse bidders above whose values are each independently distributed according to $F(\theta)$, is equivalent to a first-price auction with n risk-neutral bidders whose values are each independently distributed according to $F(\theta)^\alpha$. Use the solution for the risk-neutral case to conclude that

$$\overline{b}_\alpha(\theta) = \theta - \int_0^\theta \left(\frac{F(x)}{F(\theta)}\right)^{\alpha(n-1)} dx.$$

(d). Prove that $\overline{b}_\alpha(\theta)$ is strictly increasing in α. Does this make sense? Conclude that as bidders become more risk averse, the seller's revenue from a first-price auction increases.

(e). Use part (d) and the revenue equivalence result for the standard auctions in the risk-neutral case to argue that when bidders are risk averse as above, a first-price auction raises more revenue for the seller than a second-price auction. Hence, these two standard auctions no longer generate the same revenue when bidders are risk averse.

(f) What happens to the seller's revenue as the bidders become infinitely-risk averse (i.e., as $\alpha \to \infty$)?

Problem 2 (Auctioning procurement contracts.) (This is closely related to Problem 1 from last week's Problem Set 6. You may want to consult that question and solution before tackling this one.)

A monopsony buyer is interested in purchasing a large quantity of output from one of n possible suppliers. Each supplier i has a constant marginal cost of production equal to c_i which is private information to the supplier and is uniformly distributed on $[1, 2]$. Supplier i's payoff from producing $q \in [0, Q]$ units of output for a transfer of t dollars is

$$t - c_i q.$$

Each supplier's outside option is 0. The buyer's payoff from purchasing q units of output at a total price of t dollars is

$$vq - \frac{1}{2}q^2 - t,$$

where we assume $v \geq 3$.

The buyer's objective is to design an optimal direct-revelation mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, where each component is a mapping from cost reports $c = (c_1, \ldots, c_n)$ to probabilities of selecting firm i, output for the selected firm, and transfers to each supplier, respectively, in order to maximize

$$E_c \left[\sum_{i=1}^{n} \phi_i(c) \left(vq_i(c) - \frac{1}{2}q_i(c)^2 - c_i q_i(c) \right) - U_i(c_i) \right].$$

(a). State the two conditions that any incentive compatible mechanism must satisfy. [Hint: the monotonicity condition will now involve both ϕ_i and q_i.]

(b). Using your conditions in (a), find an expression of $E_c[U_i(c_i)]$ that is entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

(c). Using your result in (b), substitute into the buyer's objective function to obtain a maximization program that is expressed entirely in terms $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$

(d). Find the optimal $\phi_i(c)$ and $q_i(c)$ components of the optimal mechanism. Make whatever regularity assumptions you use to this end explicit.

Problem 3 (Mineral-rights auctions). Consider a common-value auction with n bidders. Each bidder privately learns a signal, θ_i, independently distributed according to $F(\theta_i)$ on $[0, 1]$ (with positive density, $f(\theta_i)$). For now, notice we are allowing distributions to differ across bidders and the signals are independently distributed. Each bidder i has a common value of the good given by

$$v(\theta) = \frac{1}{n} \sum_{j=1} \theta_j.$$

The seller's value for the good is $\theta_0 = 0$, and does not depend upon the signals of the bidders.

We first want to compute the optimal auction using Myerson's framework.

(a). Write down the two conditions which characterize an incentive-compatible direct mechanism, $\{\phi_i(\cdot), t_i(\cdot)\}_i$.

(b). Using the conditions in (a), find an expression for $E_\theta[U_i(\theta_i)]$.

(c). Using (b), write the seller's objective function expressed only in terms of $\phi_i(\cdot)$ and $U(0)$.

(d). Using (c), find the optimal $\phi_i(\cdot)$ which maximizes the seller's expected profit and find an expression determining the reservation type for each i. Make any regularity assumption(s) that you use explicit.

(e). Now further assume that the signal distributions are symmetric across bidders and uniform on $[0,1]$. Suppose that the seller uses a first-price auction without reserve price to sell the good. Find the symmetric equilibrium bid function $\bar{b}(\cdot)$. [Hint: it is linear.]

(f). Is the auction in (e) optimal? Why or why not?

(g). Show that for the equilibrium in (e), if $n > 4$, the bid function declines as the number of bidders increases. Explain.

Problem 4 (Auctions with adverse selection.) The economics department is trying to procure teaching services from one of n potential external lecturers. Candidate i has an outside opportunity of $\theta_i \in [0,1]$ with distribution $F(\cdot)$. This opportunity is private information and can be thought of as the candidate's type. The department gets teaching value $v(\theta_i)$ from a lecturer with type θ_i; the function $v(\cdot)$ is increasing and differentiable.

Consider a direct revelation mechanism consisting of an allocation function $\phi_i(\theta_1, \ldots, \theta_n) \in [0,1]$ for each lecturer i, and a transfer function $t_i(\theta_1, \ldots, \theta_n)$ which is the payment to each lecturer i. i's utility from reporting $\hat{\theta}_i$ when her true type is θ_i is

$$U_i(\hat{\theta}|\theta) = E_{\theta_{-i}}\left[t_i(\hat{\theta}_i, \theta_{-i}) - \phi_i(\hat{\theta}_i, \theta_{-i})\theta_i\right] = \bar{t}_i(\theta_i) - \bar{\phi}_i(\theta_i)\theta_i.$$

[Note that the lecturer utility is decreasing in θ_i and the single-crossing term is negative.] The department's objective is to maximize

$$\Pi = E_\theta\left[\sum_{i=1}^{n} \phi_i(\theta)v(\theta_i) - t_i(\theta)\right].$$

(a) Characterize incentive compatibility in terms of an integral equation for the agent's utility, $U_i(\theta_i)$ and a monotonicity constraint.

(b). Using (a), what is the department's profit expressed in terms of $\phi_i(\cdot)$ and $U_i(1)$?

For the rest of the question, assume that $F(\theta_i)/f(\theta_i)$ is increasing in θ_i.

(c). If $v'(\theta_i) \leq 1$, what is the department's optimal hiring policy, $\{\phi_i(\cdot)\}_i$?

(d). Suppose instead that $v'(\theta_i) > 2$ and $E[v(\theta_i)] \geq 1$. What is the department's optimal hiring policy? [Hint: if an unrelaxed program violates the required monotonicity at every point of θ_i, then the constrained-optimal solution must be constant.]

Problem 5 (Auctions with endogenous entry.) Consider an IPV model of auctions with a large number of bidders, each of whom must pay k in order to learn their type before they can participate in the auction. The timing is as follows.

1. Seller offers an auction mechanism and offers invitations to bid to some subset of the population of bidders; each bidder sees how many bidders in total, n, are invited to the auction.

2. bidders with invitations to attend the auction decide whether or not to spend k to learn their type and participate in the auction; those that decide to do so learn their type θ_i which is distributed i.i.d. according to $F(\cdot)$ on $[\underline{\theta}, \overline{\theta}]$. The seller's cost of the good is θ_0.

3. bidders report their types into the mechanism and the winner and transfers are determined accordingly.

(a). Characterize incentive compatibility for those who decide to participate in the mechanism. As usual, you should have a monotonicity condition and an integral condition for the bidder's indirect utility function, $U_i(\theta)$.

(b). What is the relevant individual rationality constraint for the participating bidders that ensures all the invited bidders are willing to spend k to learn their type? Note that the bidders decide whether or not to participate *before* they learn their type, so this will involve the bidder's expected utility, $E[U_i(\theta)]$. In other words, the seller must satisfy ex ante IR constraints and not interim IR constraints for the bidders.

(c). Suppose that it is optimal for the seller to invite n bidders to the auction. Using (a) and (b), write the seller's program in terms of $\phi_i(\cdot)$ and $U_i(\underline{\theta})$.

(d). What is the optimal allocation for the seller? What is the optimal reserve type, θ^*? How does your answer differ from the standard optimal auction of Myerson? [Hint: to solve for the optimal auction, use a Lagrange multiplier, λ, for the participation constraint in (b), incorporate it into the seller's objective function, and prove that at the optimum the multiplier is $\lambda = 1$.]

(e). Using you answer from (d), show that the value function of the seller (as a function of n) is simply

$$\Pi(n) = E\left[\max\{\theta_1, \ldots, \theta_n, \theta_0\} - nk\right].$$

Conclude that the seller will want to invite the socially optimal number of bidders to the auction when there is a cost to entry.

Problem 6 (Reserve prices in IPV auctions.) Consider an IPV setting with $n = 2$ and θ_i uniformly distributed on $[0, 1]$ for both bidders. Suppose that $\theta_0 = 0$ (the seller's opportunity cost of the object is zero).

(a). In the Myerson optimal auction, what is the allocation function $\phi_i(\theta_i, \theta_{-i})$ and what is the reserve type, θ^* (i.e., what is the highest type buyer θ_i such that $\overline{\phi}_i(\theta_i) = 0$). In describing the optimal allocation function, you may ignore situations with zero probability (i.e., ignore ties).

(b). Show that a first-price auction with an appropriately chosen reserve price is also optimal. Argue that the optimal reserve price is set at $r^* = \theta^*$.

(c). Compute the equilibrium bidding function in (b) given the optimal reserve. [Hint: it is *not* linear. Try using the envelope theorem to find $\overline{b}(\cdot)$.]

(d). In order to implement the optimal auction allocation in a standard all-pay auction (i.e., highest bidder wins but everyone pays their bids), what is the optimal reserve price, r^*, that must be set. [Hint: for the all-pay auction, the answer is not the same as in (b). The equilibrium bid function will be of the form $\overline{b}(\theta) = 0$ for all $\theta < \theta^*$, a jump at $b(\theta^*) = r^*$, and $\overline{b}(\theta) > r^*$ for all $\theta > \theta^*$. Try using the envelope theorem.]