1 Announcements

Next Friday will be a lecture by Prof. Hansen.

2 Permanent and Transitory Shocks

We illustrate them using the VAR example:

$$X_{t+1} = AX_t + BW_{t+1}$$
$$Y_{t+1} - Y_t = \nu + D^T X_t + F^T W_{t+1}$$

where A is stable and $X_0 = 0$. X_{t+1} is a $k \times 1$ vector; Y_t is a scalar.

▷ In class, we saw the following decomposition:

$$Y_t = Y_0 + \nu t + \underbrace{\sum_{j=1}^{t} \left(F^T + D^T (I-A)^{-1} B\right) W_j}_{\text{= martingale component}} - D^T (I-A)^{-1} X_t$$

▷ Using the lag operator, rewrite the process as

$$(I - AL) X_t = BW_t$$

and plug this back into the decomposition:

$$Y_t = Y_0 + \nu t + \sum_{j=1}^{t} \left(F^T + D^T (I-A)^{-1} B\right) W_j - D^T (I-A)^{-1} \underbrace{(I - AL)^{-1} BW_t}_{(1)}$$

Note that (1) is

$$\left(I + AL + A^2L^2 + \cdots\right) BW_t = BW_t + ABW_{t-1} + A^2BW_{t-2} + \cdots$$

so we have

$$Y_t = Y_0 + \nu t + \sum_{j=1}^{t} \left(F^T + D^T (I-A)^{-1} B\right) W_j - D^T (I-A)^{-1} \sum_{j=0}^{t-1} A^j BW_{t-j}$$

▷ Assume $W_1 = 0$ and $W_t = 0, \forall t \geq 2$. Then

$$Y_t = Y_0 + \nu t + \underbrace{\left(F^T + D^T (I-A)^{-1} B\right) W_1}_{\text{permanent part}} - \underbrace{D^T (I-A)^{-1} A^{t-1} BW_1}_{\text{transient part}}$$

The transiency comes from the decay that comes from A^{t-1}.

* Permanent part is just a linear combination of W_1.
* If W_1 is *orthogonal* to $F^T + D^T (I-A)^{-1} B$, we call it the *transitory* shock.
* If W_1 is *parallel* to $F^T + D^T (I-A)^{-1} B$, we call it the *permanent* shock.

▷ For a transitory impulse response, the impulse should just decay exponentially.

▷ For all shocks that are not transitory, the impulse response will converge to

▷ The permanent shock can be found by fixing the magnitude of the impulse response and finding the W_1 that gives the largest steady-state.

* Even if the shock is entirely parallel, you still have a convergence over time to the steady state.

3 Small Shock Approximation, Lombardo and Uhlig (2018)

3.1 Setup

Consider the following setup:

$$X_{t+1} = AX_t + BW_{t+1}$$
$$y_{t+1} - y_t = \nu + D^T X_t + F^T W_{t+1}$$

where $y_t := \log Y_t$. We will illustrate how the previous technique can be applied in a macroeconomic framework.

▷ Technology: transfer 1 unit time t good to $\exp(\rho)$ units of time $t+1$ good. The $\{Y_t\}$s are the fruits.

▷ The feasibility constraint:

$$K_{t+1} + C_t = \exp(\rho) K_t + Y_t$$

This is equivalent to Prof. Stokey's formulation:

$$K_{t+1} + C_t = A_t K_t + (1-\delta) K_t$$

▷ The RA maximizes

$$\mathbb{E}\left[\sum_{j=0}^{\infty} \exp(-\delta t) \log C_t\right]$$

▷ The Euler Equation:

$$U'(C_t) = \mathbb{E}\left[\exp(\rho)\exp(-\delta) u'(C_{t+1}) | X_t\right]$$

which yields

$$1 = \mathbb{E}\left[\exp(-\delta+\rho)\frac{C_t}{C_{t+1}} | X_t\right]$$

3.2 Solving for Response in Consumption

We want to linearize the $\mathbb{E}[\cdot]$.

▷ To solve this, define

$$\hat{K}_t = \frac{K_t}{Y_t}, \qquad \hat{C}_t = \log C_t - \log Y_t$$

and re-write the feasibility constraint:

$$\hat{K}_{t+1}\exp(\log Y_{t+1} - \log Y_t) + \exp\left(\hat{C}_t\right) - \exp(\rho)\hat{K}_t - 1 = 0$$

and the Euler equation:

$$\exp(-\delta+\rho)\mathbb{E}\left[\exp\left(-\left(\hat{C}_{t+1} - \hat{C}_t\right) - (\log Y_{t+1} - \log Y_t)\right) | X_t\right] - 1 = 0$$

Essentially, we are deterending to get the stationary distribution.

▷ Now consider perturbing the system by q i.e. changing the exposure of Y_ts to the stochastic component:

$$X_{t+1} = AX_t + BW_{t+1}$$
$$y_{t+1}(q) - y_t(q) = \nu + \left[D^T X_t + F^T W_{t+1}\right] q$$

This allows us to reformulate the previous variables as a function of q

$$[1] : \hat{K}_{t+1}(q) \exp(\log Y_{t+1}(q) - \log Y_t(q)) + \exp\left(\hat{C}_t(q)\right) - \exp(\rho) \hat{K}_t(q) - 1 = 0$$

$$[2] : \exp(-\delta + \rho) \mathbb{E}\left[\exp\left(-\left(\hat{C}_{t+1}(q) - \hat{C}_t(q)\right) - (\log Y_{t+1}(q) - \log Y_t(q))\right) | X_t\right] - 1 = 0$$

▷ Consider a Taylor expansion around 0:

$$\hat{C}_t(q) \approx \hat{C}_t(0) + \hat{C}_t'(q)$$
$$\hat{K}_{t+1}(q) \approx \hat{K}_{t+1}(0) + \hat{K}_{t+1}'(q)$$

The reason we do it around 0 is becaues $\hat{C}_t(0)$ results in a deterministic fruits process $\{Y_t\}$ and thus it is very easy to compute.

▷ Note that the individual components above are processes, not numbers. Similarly to the macro class, we want to do

$$\hat{C}_t = C\left(X_t, \hat{K}_t\right), \quad \hat{K}_{t+1} = K\left(X_t, \hat{K}_t\right)$$

where X_t is the exogenous state and K_t is the endogenous state.

Now define a new function

$$F_1\left(\hat{K}_{t+1}(q), \hat{C}_t(q), \hat{K}_t(q), \Delta y_{t+1}(q)\right) \equiv [1]$$

where $\Delta y_{t+1}(q) = y_{t+1}(q) - y_t(q)$

▷ Since F_1 is equal to zero for all q, we have

$$F_1(q) \approx F_1|_{q=0} + q\frac{\partial F_1}{\partial q}|_{q=0} \approx 0$$

as well. Similar argument holds for $F_2 = 0$:

$$F_2(q) \approx F_2|_{q=0} + q\frac{\partial F_2}{\partial q}|_{q=0} \approx 0$$

▷ Obtaining $F_1|_{q=0}$: Making the assumption that $\delta = \rho - \nu$, the economy has a steady state of:

$$\hat{C}_t(0) = 0, \qquad \hat{K}_{t+1}(0) = 0$$

which yields

$$\log C_t - \log Y_t = 0 \Rightarrow C_t = Y_t$$
$$\hat{K}_{t+1}(0) = \frac{K_{t+1}}{Y_{t+1}} = 0$$

so you consume fruit everyday and save nothing.

▷ Obtaining $\partial F_1/\partial q$:

$$\frac{\partial F_1}{\partial q}|_{q=0} = \hat{C}_t'(0)\frac{\partial F_1}{\partial \hat{C}_t}|_{q=0} + \hat{K}_{t+1}'(0)\frac{\partial F_1}{\partial \hat{K}_{t+1}}|_{q=0} + \hat{K}_t'(0)\frac{\partial F_1}{\partial \hat{K}_t}|_{q=0} + \Delta y_{t+1}'(0)\frac{\partial F_1}{\partial \Delta y_{t+1}}|_{q=0} = 0$$

* Note that we already know

$$\frac{\partial F_1}{\partial \hat{C}_t}\left(\hat{K}_{t+1}(q), \hat{C}_t(q), \hat{K}_t(q), \Delta y_{t+1}(q)\right)|_{q=0}$$

since we've computed the relevant quantities evaluated at zero in the previous step. The similar argument follows for the other derivatives.

* Deriving analogously for F_2, we have two linear equations of $\hat{K}_{t+1}'(0)$ and $\hat{C}_t'(0)$.
* The term with $\Delta y_{t+1}'$ is good since

$$\frac{\partial F_1}{\partial \Delta y_{t+1}}|_{q=0} = 0$$

and the term vanishes.

* The term with $\hat{K}_t'(0)$ is also good since the derivative is simply $\exp(\rho)$.

Going through a similar process with F_2, we obtain:

$$[3] : \hat{K}_{t+1}^1 \exp(\nu) + \hat{C}_t^1 - \exp(\rho)\hat{K}_t' = 0$$
$$[4] : \mathbb{E}\left[\left(\hat{C}_{t+1}^1 - \hat{C}_t^1\right) + \Delta y_{t+1}'\right] = 0$$

To solve this, guess and verify:

$$\hat{C}_t^1 := C_t'(q=0) = MX_t + \Gamma_K \hat{K}_t^1$$

▷ Plug this into [3] and express $\hat{K}_{t+1}$ as a function of $\hat{K}_t$ and $\hat{X}_t$.

▷ Plug this into [4] and replace X_{t+1} as a function of X_t and solve for M and Γ_K.

The resulting solution is

$$\hat{C}_t' = \lambda D'(I-\lambda A)^{-1} X_t + \{\exp(\rho)\}$$
$$\hat{K}_{t+1}' = \hat{K}_t' - \exp(-\nu)\lambda D'(I-\lambda A)^{-1} X_t$$

where $\lambda = \exp(v-\rho)$. We also obtain

$$\hat{C}_{t+1}^1 - \hat{C}_t^1 = -D^T X_T + \lambda D^T (I-\lambda A)^{-1} BW_{t+1}$$

This allows us to compute the log consumption progress:

$$\log C_{t+1} - \log C_t = -D^T X_t + \lambda D^T (1-\lambda A)^{-1} BW_{t+1} + \nu + D^T X_t + F^T W_{t+1}$$
$$= \left(\lambda D^T (I-\lambda A)^{-1} B + F^T\right) W_{t+1} + \nu$$

This is the function that Professor Hansen plotted in class. He provided two plots – one is the permanent shock to the $\log Y_t$ process and one is the transitory shock to the $\log Y_t$ process.

- ▷ If λ is really close to one, then the permanent shock which is parallel to

$$D^T \left(I - \lambda A\right)^{-1} B + F^T$$

will be almost parallel to

$$\lambda D^T \left(I - \lambda A\right)^{-1} B + F^T$$

in which case the permanent shock will have a big impact on consumption.

- ▷ There is no D which is why you see the impulse response as a constant. This is constrast with the general response with the long-term convergence (adjustment takes time due to the transitory part, which was $D\left[\cdots\right]$). If $D = 0$, then it will be just a straight line.

3.3 Comparison with Log-linearization

If you want to attain the second order in log-linearization, you have to deal with

$$\hat{X}_{t+1} = a\hat{X}_t + b\hat{X}_t^2 + \cdots$$

which fucks with stationarity. But for this methodology, even if you go to second order:

$$\hat{X}_{t+1} = \hat{X}_{t+1}\left(0\right) + \left[\cdots\right]\hat{X}_t{}'\left(0\right) + \left[\cdots\right]\hat{X}_t''\left(0\right)$$

you are taking the square of the derivative so it's okay.