Price Theory I
Problem Set 5, Question 2

Rafael Jiménez*

Fall 2018

For this question it is important to have in mind that the firm's problem can be broken down in two problems (Notes, p.93). First, cost minimization, taking as given quantity y to be produced which provides us with the conditional factor demands $(x(w, y))$ and cost function $(c(w, y))$. Second, we solve profit maximization in which producers choose the output they want to supply.

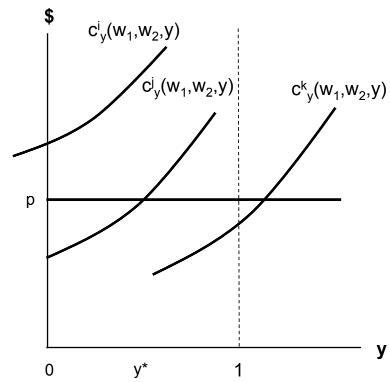
a.

Consider a set I of suppliers, each with a different production function and hence each with a different cost function $c^i(w_1, w_2, y)$. In an interior solution, we know that the efficiency condition is $p = c^i_y(w_1, w_2, y^i)$. However, in this problem corner solutions matter; producers for which $p < c^i(w_1, w_2, 0)$ will not produce—and hence will not demand any inputs—and producers such that $p > c^i(w_1, w_2, 1)$ will be constrained by the production capacity limit. These three cases are pictured in Figure 1.

Consider first the relationship between an individual firm's factor demand and output price. Panel a) of Figure 2 pictures a (marginal) increase in price from p to p' for the three cases outlined above. Note that, for the two constrained cases, the optimal output remains the same; hence, factor demands are perfectly inelastic in these cases. In an interior solution, a higher price increases the optimal output, which has an uncertain effect on factor demands unless we impose additional assumptions on the cost/production function. Recall from the class notes that conditional factor demands satisfy $x^i_k(w_1, w_2, y) = \frac{\partial c^i(w_1, w_2, y)}{\partial w_k}$.

*These answers benefited from discussion with Casey Mulligan, Ivan Kirov and Francisco del Villar. All errors are my own.

Figure 1: **Marginal costs for different producers**



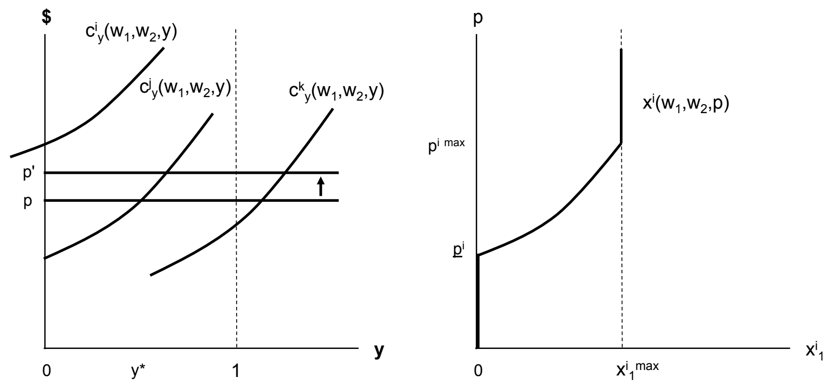
Pictured here are examples of marginal costs for three different producers. The dotted vertical line at $y = 1$ denotes the production capacity limit. Producer i's marginal cost at 0 is greater than the output price and hence chooses not to produce at all. Producer j is an interior solution so her marginal cost equals the output price at $y^* \in [0, 1]$. Producer k is constrained by the capacity limit; she would like to produce a higher quantity at the current price.

If we totally differentiate this expression with respect to output price we get: $\partial x_k^i(w_1, w_2, y)/\partial p = \frac{\partial^2 c^i(w_1,w_2,y)}{\partial w_k \partial y} \frac{dy}{dp}$. We have seen that $\frac{dy}{dp} \geq 0$; however, the other term could be positive or negative depending on whether the input is normal or inferior, as mentioned in the Notes (p. 100). The typical example of an inferior input is when the price of shovels goes sufficiently high so firms buy excavator machines and end up producing more. I will assume that inputs are normal since inferior inputs are rare, but it is something to have on the back of your mind. This gives rise to the relationship between factor demand $x_k^i(w_1, w_2, p)$, $k \in \{1, 2\}$, and p pictured in panel b) of Figure 2.

Now consider the relationship between factor demand $x_k^i(w_1, w_2, p)$ and factor price w_k. Panel a) of Figure 3 pictures a marginal increase in w_k. Again, we cannot guarantee that the marginal cost will increase (because of the possibility of having an inferior input). We know, however, that the demand is locally perfectly inelastic when firms are not producing—and hence when they demand zero input; it is downward sloping—with a slope equal to $\partial^2 c^i/\partial w_k^2$, since y is fixed at 1—when firms are constrained by the production capacity and downward sloping with a slope equal to $\partial^2 c^i/\partial w_k^2 - [(\partial^2 c^i/\partial w_k \partial y)^2/\partial^2 c^i/\partial y^2]$ (Slutsky equation) in an interior solution.[1] Panel b) of Figure 3 pictures the

[1]Recall that the total derivative of the conditional factor demand with respect to input

Figure 2: **Individual factor demand and output price**



(a) Marginal costs and price increase (b) Factor demand and output price

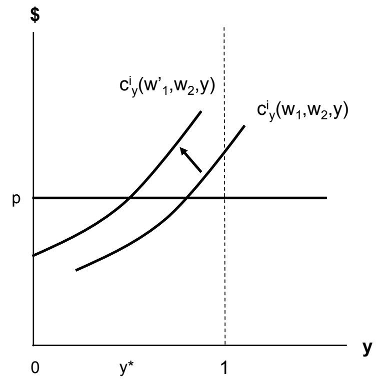
The price $\underline{p}^i$ is the reservation price of supplier i. Price $p^{i\ \max}$ is defined as: $p^{i\ \max} = c^i_y(w_1, w_2, 1)$. Quantity $x^{i\ \max}$ is an element of the argmin of the cost function when $y = 1$.

input demand in the normal case.

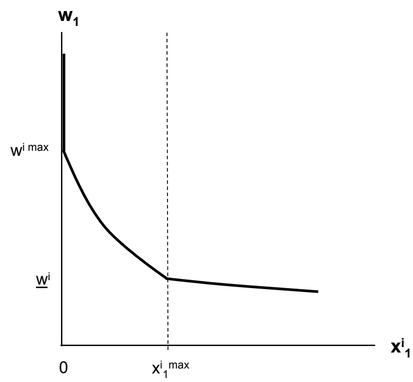
Finally, consider the relationship between factor demand $x^i_k(w_1, w_2, p)$ and cross factor price w_m. Again, it is not clear whether c_y shifts upwards or downwards with the change in price. Demand is again perfectly inelastic when firms are not producing. When firms are constrained, factor demand has a slope with respect to the cross input price given by $\partial^2 c^i/\partial w_k \partial w_j$, which constitutes the substitution effect—the scale effect, as above, is zero. However, this time the concavity of the cost function does not say anything about the sign of the substitution effect; it might be positive or negative. When firms are unconstrained, the slope of the factor demand has an extra term, the scale effect, given by: $-(\partial^2 c^i/\partial w_k \partial Y \times \partial^2 c^i/\partial Y \partial w_j)/(\partial^2 c^i/\partial Y^2)$. If both products are normal or inferior, the scale effect will be positive. The case when both inputs are complements looks very similar to panel b) of Figure 3 and there is no point in repeating the graph. Figure 4 plots the case of substitute inputs. Notice that when w_j is very low, there are no incentives to acquire good k. As w_j increases, assuming the producer is constrained by the production capacity limit, the substitution effect dictates the slope of demand, which is positive in this case. When w_j increases enough to be in an interior solution—remember that marginal cost

price is $\partial X_k/\partial w_k = \partial^2 c/\partial w_k^2 + \partial^2 C/\partial w_k \times \partial y dY/dw_k$. This last term is zero for constrained firms.

Figure 3: **Individual factor demand and own input price**



(a) Marginal costs and price increase

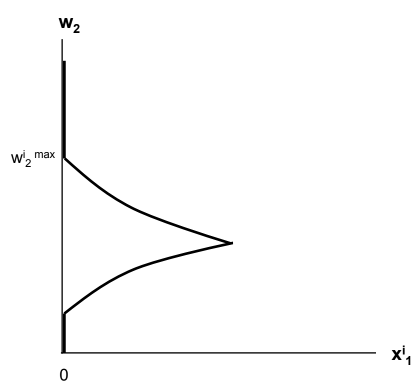


(b) Factor demand and input price

The price $\underline{w}^i$ is the price that makes the capacity constraint binding. Price $w^{i\ \max}$ is the reservation factor price. Note that the slope of demand to the right of $x^{i\ \max}$ is less steep that to the left; this is because when the capacity constraint becomes binding, the second term of the Slutsky equation becomes zero.

shifts left with a normal input—the scale effect comes into play, which leads to the negative slope we observe for high prices of w_j (i.e., assuming the scale effect dominates the substitution effect). Once the factor price increases enough above the reservation price of the supplier, demand stays at zero.

Figure 4: **Individual factor demand and cross input price**



Price $w_2^{i\ \max}$ is the reservation factor price.

b.

Supplier i's production function is: $\min\{\alpha^i x_1^i, x_2^i\}$, where α dictates the proportion of x_2^i relative to x_1^i needed to produce. From the cost minimization problem we get that $c^i(w_1, w_2, y) = y[w_1/\alpha^i + w_2]$.[2] This means that the marginal cost is constant for all producers and equal to: $c_y^i(w_1, w_2, y) = w_1/\alpha^i + w_2$. Producers such that $p > c_y^i$ will produce up to their capacity constraint, producers such that $p < c_y^i$ will not produce, and those such that $p = c_y^i$ will be indifferent between producing any amount in $[0, 1]$ (suppose they produce 1). Suppose there is a continuum of producers indexed by their α_i which is uniformly distributed in $(0, 1]$. Then, the marginal producer is given by $\alpha_M \equiv \frac{w_1}{p - w_2}$, which is increasing in w_k and decreasing in p: as the price of inputs increase, fewer suppliers produce positive amounts, but the reverse is true for the output price. Producers with higher alphas will produce 1—and hence demand 1 unit of x_2 and $1/\alpha^i$ units of x_1— and those with lower alphas will not produce—and thus demand zero input. Individual factor demands are summarized as:

$$x_1^i(w_1, w_2, p) = \begin{cases} \frac{1}{\alpha_i} & \text{if } \alpha^i \geq \frac{w_1}{p - w_2} \\ 0 & \text{otherwise} \end{cases}, \quad x_2^i(w_1, w_2, p) = \begin{cases} 1 & \text{if } \alpha^i \geq \frac{w_1}{p - w_2} \\ 0 & \text{otherwise} \end{cases}$$

The industry's factor demand is given by $X_k(w_1, w_2, p) \equiv \int_0^1 x_k^i(w_1, w_2, p) di$. Solving this integral gives us, for $p \geq w_1 + w_2$:[3]

$$\begin{aligned} X_1(w_1, w_2, p) &= \int_{\frac{w_1}{p - w_2}}^1 \frac{1}{\alpha_i} di = \ln(p - w_2) - \ln w_1 \\ X_2(w_1, w_2, p) &= 1 - \frac{w_1}{p - w_2} \end{aligned}$$

When $p < w_1 + w_2$, factor demands are zero for all suppliers. When $p \geq w_1 + w_2$, the industry's factor demands are increasing in p and decreasing in factor prices. This makes sense, since inputs are perfect complements. Moreover, the impact of prices on industry's demands is via the extensive margin (increase in the number of producers demanding positive amounts) with zero intensive margin effects (quantity demanded by producer); individual demands are perfectly inelastic, with a discrete jump at reservation prices.

[2] To see why, note that it is optimal to set $\alpha^i x_1^i = x_2^i$ (if $\alpha^i x_1^i > x_2^i$, the producer can lower down its expenditure on x_1^i without sacrificing production, and the same for the other case.)

[3] Minor detail: I exchange the di for $d\alpha_i$ since producers are indexed by their α_i.

c.

The cost function is $C(w_1, w_2, Y)$, where aggregate output is $Y = X_2(w_1, w_2, p)$. There are two ways to obtain the cost function in this case. One way is to integrate the individual cost functions and notice that this integral is a function of Y; then $C(w_1, w_2, Y) = \int_{\alpha_M}^{1} (w_1/\alpha^i + w_2) d\alpha_i$. The other way is to note that, in order for the industry to product Y units of the good we need exactly a mass of Y producers above the marginal producer, since each produces one unit. In other words, we need $\int_{\alpha_M}^{1} 1 di = Y$, or $\alpha_M = 1 - Y$. Then $C(w_1, w_2, Y) = \int_{1-Y}^{1} (w_1/\alpha^i + w_2) d\alpha_i$. In any case, the cost function for the industry is:

$$C(w_1, w_2, Y) = w_1 \ln \left(\frac{1}{1-Y} \right) + w_2 Y$$

You can verify that this cost function is increasing in factor prices and quantity and concave.

d.

For part a), now Panel b) of Figures 2 and 3 does not have the kink at $x_1^{i\,\max}$. Figure 4 could still have an increasing part, when the positive substitution effect dominates the scale effect, although the slope would look less positive in the range that previously corresponded to the the capacity limit constraint.

For part b), individual factor demand would now be perfectly elastic for those producers above the marginal one, $\alpha^i \geq \alpha_M$, and zero otherwise. This makes sense: the individual production function has constant returns to scale, so the quantity is not pinned down. The industry's factor demands are perfectly elastic for all prices $p \geq w_1 + w_2$ and zero otherwise.

To see part c), fix any price such that $p \geq w_1 + w_2$. All producers above α_M will want to produce an undetermined amount of output, since their profits increase linearly with y. Without further assumptions, the cost function in this industry is thus not pinned down; there is an infinite way for this industry to produce Y units. We can assume, however, that the first firm to produce will be the most efficient one; the one with $\alpha^i = 1$. In this case, the cost function of this firm—and hence the industry— will be simply $Y(w_1 + w_2)$.

e.

Part a) already covered the general case so to answer this question we need to make extra assumptions. First, assume individual production functions are CRS—which covers some functional forms such as the typical CES.[4] Under this assumption, we know that $c^i(w_1, w_2, y) = yc^i(w_1, w_2, 1)$, and thus the marginal cost is constant and equal to $c^i(w_1, w_2, 1)$. As above, this means that producers with costs below the marginal $c^M(w_1, w_2, 1) = p$ will produce up to their capacity constraint and those above the marginal will not produce. From the Slutsky equation, this means that only the substitution channel (not the scale channel) will be active when considering changes in factor prices—which is negative for own price and positive for cross-price. We cannot guarantee as in b) that individual factor demands will be perfectly inelastic such that the aggregate demand depends on factor prices via extensive margin only. However, individual demands will still be inelastic with respect to output price above and below the reservation price.[5]

You do not gain any additional insight (compared to assuming CRS) from assuming CES or Cobb-Douglas. When you assume a perfect-substitutes function such as $\alpha^i l + k$ there are many cases to consider;[6] however, the interesting case is when $w < r < p$ such that everyone produces, but there is a fraction of the population with $\alpha^i < \frac{w}{r}$ that specializes in using capital. Individual capital demand for these producers is perfectly inelastic at 1, but aggregate capital demand is $\frac{w}{r}$ (depends positively on w and negatively on r). The other fraction of producers demand $1/\alpha^i$ units of labor, which generates an aggregate demand of $\ln r - \ln w$.

[4]I'm not saying all CES functions are CRS (e.g., generalized CES), but CES is typically assumed to have CRS.

[5]To see why, notice that labor and capital demands, for those suppliers that produce positive amounts, are completely determined by the equations: $1 = F(K, L)$ and $\frac{w}{r} = \frac{F_L}{F_K}$. Output price only determines who the marginal producer is.

[6]Case 1: $w > r$, no one produces if $p < r$, everyone produces otherwise demanding zero labor and 1 unit of capital, and aggregate demands are equal to individual ones. Case 2: $w < p < r$, and only a fraction $\alpha \geq w/p$ produces, using demanding only $1/\alpha^i$ units of labor with an aggregate demand of $\ln p - \ln w$. Case 3 is described in the text.