Lecture 5, Theory Income, Fall 2018

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Euler Equations and Transversality Conditions for Dynamic Problems, CRTS case

- This notes introduces the elements of discrete time dynamic optimization problems.
- Conditions under which Euler equations and Transversality conditions are necessary and sufficient for a path to be optimal are discussed.
- Most of this note follows RMED, “Recursive Methods in Economic Dynamics", by Stokey and Lucas with Prescott, Chapter 6.
- We use this material for different examples, including those of CRTS or Homogenous returns and growth models.

Set up in discrete time

The elements of a Dynamic Programming problem are $[X, \Gamma, F, \beta]$. X is the set of states x. We typically let x be the current state and y the next period state. $\Gamma : X \to X$, is the correspondance describing the feasibility constraints. That is for each $x \in X$, $\Gamma(x)$ is the set of feasible values for the state variable next period if the current state is x, with its graph given by

$$Gr(\Gamma) \equiv \{(y, x) : x \in X,\ y \in \Gamma(x)\}.$$

The period return function $F(x, y)$ is defined on $F : Gr(\Gamma) \to R$. Finally a discount factor $\beta \in (0, 1)$.
The sequence problem is

$$V^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma(x_t) \text{ for all } t \geq 0$$

with x_0 given.

Recall:

- $i + c = G(K, 1)$
- $K_{t+1} = i + K_t(1-\delta)$

Example: Neoclassical growth model

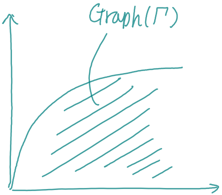
$$V^*(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U \underbrace{(f(k_t) - k_{t+1})}_{c_t}$$

subject to

$$0 \leq k_{t+1} \leq f(k_t)$$

k_0 given. This fits in the general notation by letting

$$\underbrace{\begin{aligned} F(x, y) &= U(f(x) - y) \\ \Gamma(x) &= [0, f(x)] \end{aligned}}_{\text{non-negative consumption \& capital}}$$



and

$$f(k) = G(k, 1) + (1 - \delta) k$$

where $G(\cdot, \cdot)$ is a neoclassical constant return production function and δ the depreciation rate.

A related notation distinguishes between controls, u_t, and states, x_t. In this notation the sequence problem is described by $[X, U, h, g, \beta]$. Where U is the set of feasible controls, h is the period return function and g is the law of motion of the state. The sequence problem is defined as:

know how to go back and forth

$$V^*(x_0) = \max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t h(x_t, u_t)$$

subject to the law of motion of the state

$$x_{t+1} = g(x_t, u_t) \text{ and } u_t \in U$$

for x_0 given.
To see that the control-state notation is equivalent to the previous one takes

$$\begin{aligned} F(x,y) &= \max_u \{h(x,u) : u \in U,\ y = g(x,u)\} \\ \Gamma(x) &= \{y : \text{ there is } u \in U, \text{ s.t. } y = g(x,u)\} \end{aligned}$$

Exercise

- Consider the neoclassical growth model as describe above (in terms of F and Γ).
- Describe it in terms of functions h and g. What is the controls and which is the state? Hint: the list of variables are capital, consumption and/or investment.

Exercise

- states : k_t
- controls : $k_{t+1} = f(k_t, n_t) - c_t$ so $\begin{bmatrix} n_t \\ c_t \end{bmatrix}$

- Consider the neoclassical growth model with variable labor supply.
- Denote hours work by n and leisure buy ℓ. Let the period utility function depend on (c, ℓ) consumption and leisure, where we assume that there an endowment one of time, so that $\ell + n = 1$.
- Let the production function be, again, $G(k, n)$ a function of capital k and labor n.
- Describe the problem in terms of functions h and g. What are the controls and which are (is) the state? Hint: the list of variables are capital, consumption, labor and investment.
- Describe the problem in terms of the period return function F and the feasible correspondence Γ.

Euler Equations (EE) and Transversality conditions (TC).

Assume that $X \in R^m$, F is C^1, $\beta \in (0,1)$.
Def. The path $\{x_{t+1}\}_{t=0}^{\infty}$ satisfies *EE* if * x, y are vectors

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0 \text{ for } t \geq 0$$

2nd-order implicit difference equation ✓

Def. The path $\{x_{t+1}\}_{t=0}^{\infty}$ satisfies *TC* if

$$\lim_{t \to \infty} \beta^t F_x(x_t, x_{t+1}) \cdot x_t = 0.$$

boundary condition ✓

Exercise. Write the Euler equations and TC for the neoclassical growth model.

"**General Principle:**" **EE and TC are necessary and sufficient for** the path **$\{x_{t+1}\}_{t=0}^{\infty}$ to be optimal.**

Sufficiency of EE and TC. We now show that the EE and TC are sufficiency for optimality, if the problem is convex. Assume that F is concave in (x, y), that $F_x\left(x_t^*, x_{t+1}^*\right) \geq 0$, and $X = R_+^m$. Then if $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ satisfies EE and TC, the path $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is optimal.

Proof. We use the fact that $f(x) \leq f(x^0) + f'(x^0)(x - x^0)$ for all x, if f is concave.
Take an arbitrary $\{x_{t+1}\}_{t=0}^{\infty}$ with $x_0 = x_0^*$ and $x_{t+1} \geq 0$ for all t.

$$
\begin{aligned}
& \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*) \right] \\
\leq \quad & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F_x(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_y(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*) \right]
\end{aligned}
$$

where the inequality follows by concavity.
Developing the summation in the right side:

$$
\begin{aligned}
= \quad & \lim_{T\to\infty} \{ \; F_x\,(x_0^*, x_1^*)\,(x_0 - x_0^*) + F_y\,(x_0^*, x_1^*)\,(x_1 - x_1^*) + \\
& +\beta \;\; [F_x\,(x_1^*, x_2^*)\,(x_1 - x_1^*) + F_y\,(x_1^*, x_2^*)\,(x_2 - x_2^*)] + \\
& + \ldots + \\
& +\beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)(x_t - x_t^*) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right] \\
& +\beta^{t+1} \left[F_x\left(x_{t+1}^*, x_{t+2}^*\right)\left(x_{t+1} - x_{t+1}^*\right) + F_y\left(x_{t+1}^*, x_{t+2}^*\right)\left(x_{t+2} - x_{t+2}^*\right)\right] \\
& + \ldots + \\
& \beta^T \left[F_x\left(x_T^*, x_{T+1}^*\right)(x_T - x_T^*) + F_y\left(x_T^*, x_{T+1}^*\right)\left(x_{T+1} - x_{T+1}^*\right)\right] \; \}
\end{aligned}
$$

using $x_0 = x_0^*$

this is essentially integration by parts!
(we will re-visit this later in continuous-time)

$$
\begin{aligned}
= \quad & \lim_{T\to\infty} \{ [F_y(x_0^*, x_1^*) + \beta F_x(x_1^*, x_2^*)] (x_1 - x_1^*) \\
& + \beta [F_y(x_1^*, x_2^*) + \beta F_x(x_1^*, x_2^*)] (x_2 - x_2^*) \\
& + \ldots + \\
& + \beta^t \left[F_y\left(x_t^*, x_{t+1}^*\right) + \beta F_x\left(x_{t+1}^*, x_{t+2}^*\right)\right] \left(x_{t+1} - x_{t+1}^*\right) \\
& + \beta^{t+1} \left[F_y\left(x_{t+1}^*, x_{t+2}^*\right) + \beta F_x\left(x_{t+2}^*, x_{t+3}^*\right)\right] \left(x_{t+2} - x_{t+2}^*\right) \\
& + \ldots + \\
& \beta^T F_y\left(x_T^*, x_{T+1}^*\right) \left(x_{T+1} - x_{T+1}^*\right) \}
\end{aligned}
$$

Using EE:

$$
\begin{aligned}
= \quad & \lim_{T\to\infty} \beta^T F_y\left(x_T^*, x_{T+1}^*\right) \left(x_{T+1} - x_{T+1}^*\right) \\
= \quad & -\lim_{T\to\infty} \beta^{T+1} F_x\left(x_{T+1}^*, x_{T+2}^*\right) \left(x_{T+1} - x_{T+1}^*\right)
\end{aligned}
$$

using $x_{T+1} \geq 0,\ F_x\left(x^*_{T+1}, x^*_{T+2}\right) \geq 0,$

$$
\begin{aligned}
&= -\lim_{T\to\infty} \beta^{T+1} F_x\left(x^*_{T+1}, x^*_{T+2}\right) x_{T+1} + \lim_{T\to\infty} \beta^{T+1} F_x\left(x^*_{T+1}, x^*_{T+2}\right) x^*_{T+1} \\
&\leq \lim_{T\to\infty} \beta^{T+1} F_x\left(x^*_{T+1}, x^*_{T+2}\right) x^*_{T+1}
\end{aligned}
$$

thus, if the TC holds:

$$
\begin{aligned}
&\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F\left(x_t, x_{t+1}\right) - F\left(x^*_t, x^*_{t+1}\right)\right] \\
\leq\ &\lim_{T\to\infty} \beta^T F_x\left(x^*_T, x^*_{T+1}\right) x^*_T = 0
\end{aligned}
$$

which finishes the proof.

concavity $\xrightarrow{X}$ does not imply unique steady state

"permanent income" $\longrightarrow$ multiple steady state!

we need concavity!

Def. Let $\bar{x}$ be a steady state, i.e. a solution to

$$F_y(\bar{x}, \bar{x}) + \beta F_x(\bar{x}, \bar{x}) = 0.$$

Exercise. For what kind of problems does $x_{t+1} = \bar{x}$ for $t \geq 0$ is optimal if $x_0 = \bar{x}$?
Exercise. Find the steady state(s) for the neoclassical growth model. Assume that G, the production function, satisfies Inada conditions.

Necessity of EE and TC. Assume that F is C^1. We will consider adding a variation around the optimal path $\{x\}$, denoted by ε. Let

$$x_t(\alpha, \varepsilon) = x_t + \alpha \varepsilon_t$$

for $\alpha \in R$ and $\varepsilon = \{\varepsilon_t\}_{t=0}^{\infty}$ with $\varepsilon_t \in R^m$ and $\varepsilon_0 = 0$. Then

$$\begin{aligned} V^*(x_0) &= v(0) \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t(0,\varepsilon), x_{t+1}(0,\varepsilon)) \\ &\geq v(\alpha) \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t(\alpha,\varepsilon), x_{t+1}(\alpha,\varepsilon)) \end{aligned}$$

for any α, ε such that $x_{t+1}(\alpha, \varepsilon) \in \Gamma(x_t(\alpha, \varepsilon))$ for all $t \geq 0$.

Since $\alpha = 0$ maximizes v, if v is differentiable, it must be that

$$\frac{\partial v(0)}{\partial \alpha} = 0.$$

Assuming that the limits involved in the derivative (with respect to α) and in the summation (with respect to T) can be exchanged we obtained:

$$\begin{aligned}
\frac{\partial v(0)}{\partial \alpha} &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F_x(x_t, x_{t+1})\varepsilon_t + F_y(x_t, x_{t+1})\varepsilon_{t+1}\right] \\
&= \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t \left[F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2})\right]\varepsilon_{t+1} \\
&\quad + \lim_{T\to\infty} \beta^T F_y(x_T, x_{T+1})\varepsilon_{T+1}
\end{aligned}$$

Exercise: Show the second equality above, i.e. fill the intermediate steps (Hint: imitate the sufficiency case).

Necessity of the EE. Consider the case where $\varepsilon_s = 0$ all s, except at time $t + 1$. In this case $x_{t+1}(\alpha, \varepsilon)$ will be feasible if $(x_{t+1}, x_t) \in Int\,(Gr\,(\Gamma))$. Also assume that, v is differentiable and the limits can be interchanged. Then, direct computation gives

$$\frac{\partial v\,(0)}{\partial \alpha} = \left[F_y\,(x_t, x_{t+1}) + \beta F_x\,(x_{t+1}, x_{t+2})\right] \varepsilon_{t+1} = 0,$$

so if ε_{t+1} can be anywhere in a neighborhood of 0, we get EE

$$F_y\,(x_t, x_{t+1}) + \beta F_x\,(x_{t+1}, x_{t+2}) = 0.$$

Necessity of TC. As explained above, assuming that v is differentiable, that interchanging the limits is valid, and using the EE:

$$\begin{aligned}\frac{\partial v(0)}{\partial \alpha} &= \lim_{T\to\infty} \beta^T F_y(x_T, x_{T+1})\,\varepsilon_{T+1} \\ &= -\lim_{T\to\infty} \beta^{T+1} F_x(x_{T+1}, x_{T+2})\,\varepsilon_{T+1}\end{aligned}$$

if $\varepsilon_{T+1} = -x_{T+1}$ is feasible

$$0 = \frac{\partial v(0)}{\partial \alpha} = \lim_{T\to\infty} \beta^T F_x(x_T, x_{T+1})\,x_T$$

i.e. TC must hold.

Uses of EE and TC.

Notice that EE can be regarded as a second order difference equation, i.e. define $x_{t+2} = \psi\left(x_{t+1}, x_t\right)$

$$F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_{t+1}, \psi\left(x_{t+1}, x_t\right)\right) = 0$$

There is an initial condition, x_0, and a boundary condition, namely TC.

Exercise. Assume that F is C^2. What condition will suffice to uniquely define ψ?

Exercise. Write down convexity conditions on $X,\ F, \Gamma$ so that the dynamic problem has, at most, one solution.
Shooting algorithm. This algorithm is described as follows. Given x_0 select x_1 arbitrarily. Generate a sequence $\{x\}$ using $x_{t+2} = \psi\left(x_{t+1}, x_t\right)$ for all $t \geq 2$. Compute if the limit of this sequence satisfies the TC for the arbitrary choice of x_1. If not, try a different one.

Exercise. For what type of problems does the shooting algorithm works? Why does it work?

Exercise: Transversality

- Consider a problem with

$$F(x, y) = U(w + x(1 + r) - y) \text{ with } (1 + r)\,\beta = 1$$

- This is a saving problem with constant income w and interest rate r.
- Question: Is the solution of this problem unique?
- Question: How many steady states does this problem has?
- Solution: $c_t^* = w + r\,x_t$ and $x_{t+1}^* = x_t^* = x_0$. Interpret it.
- Check that the proposed policy satisfied EE and Transversality.
- Give an interpretation to the EE.

Exercise: Transversality (cont)

- Consider and alternative policy $\tilde{c}_t = \tilde{c}_0 < c_0^* = c_t^*$ for all $t \geq 0$. We are keeping the same initial condition.
- Can this policy be optimal?
- Does the path satisfies EE? Interpret it.
- Compute the implied sequence of x_t for this policy.
- Does the implied path satisfies Transversality?
- Use the result so far to give an interpretation to the Transversality condition.

Takeaway: Reaching steady-state depends on:
(1) Curvature of the utility function
(2) The rate at which MPK drops

Exercise. Linear utility in the neoclassical growth model.
maximize PV of dividends = invest until 1 = discounted MPK

- Let $U(c) = c$ and
$$f(k) = G(k, 1) + (1 - \delta) k$$
where G is a neoclassical production function: strictly increasing and strictly concave in k, satisfying Inada conditions.

EE (doesn't need linear utility)

- Show that, as long as k_0 is such that $f(k_0) - \bar{k} \geq 0$ for $\beta f'(\bar{k}) = 1$, then capital converges to steady state $\bar{k}$ in one period, i.e $\bar{k} = g(x_t)$ were g denotes the optimal policy.

  (Hint: use the sufficiency of EE and Transversality). unique steady state

- If consumption is non-negative and $f(k_0) < \bar{k}$ what will be the optimal policy? Hint: trickier question, since you have to consider corners.

- cross-derivative is zero (like the case when we didn't have dynamic problems)
- so there's only one solution.
- polar opposite of permanent income (where you stayed where you began; now you go to the steady state right away)

Exercise: Adjustment cost model

- Let the adjustment cost model be:

$$
\begin{aligned}
F(x,y) &= -\frac{a}{2}y^2 - \frac{b}{2}(y-x)^2 \\
\Gamma(x) &= R
\end{aligned}
$$

- What is the interpretation of $b/a \geq 0$.
- Suppose that $x_0 = 0$. What is the optimal path after that initial condition?
- Write the EE and evaluate them at the steady state. What is that value?
- What is the optimal policy if $a = 0$?
- Show that the optimal policy is $x_{t+1} = g(x_t) = \gamma x_t$ for some $0 < \gamma < 1$. Characterize γ in terms of b/a and β. You should obtain a quadratic equation for γ in terms of the parameters.
- Give an economic interpretation of the results.

Exercise: constant saving rate.

- Consider the Neoclassical growth model with log utility, Cobb-Douglas production function and 100% depreciation: i.e.

$$\begin{aligned} F(x,y) &= \log\left(x^{\alpha}-y\right) \\ \Gamma(x) &= \left[0, x^{\alpha}\right] \end{aligned}$$

- Show that the optimal policy is of the form

$$k_{t+1} = g(k_t) = s\, x^{\alpha} \,.$$

- Find an expression for s in terms of α and β.

  Hints: Use EE and replace the optimal policy for consumption.

Exercise: constant savings rate.

- Consider the Neoclassical growth model with 100% depreciation,

$$f(k) = \left[\alpha k^{1-\frac{1}{\rho}} + (1-\alpha)^{1-\frac{1}{\rho}}\right]^{1/(1-1/\rho)}$$
$$U(c) = \left(c^{1-1/\sigma} - 1\right) / (1 - 1/\sigma)$$

- Look for the relationship between parameters ρ and σ such that the optimal policy is to have a constant savings rate:

$$k_{t+1} = g(k_t) = s\, f(k_t)$$

  for some number $s \in (0, 1)$.

- Hint: The previous exercise is an special case of this. In the previous case the elasticity of substitution of capital is one, and the intertemporal elasticity of substitution σ is also 1.

Homogeneous of degree 1 case (CRTS)

- Assume X is a cone, $x \in X \implies \lambda x \in X$ for all scalar $\lambda > 0$.
- $y \in \Gamma(x) \implies \lambda y \in \Gamma(\lambda x)$ for all scalar $\lambda > 0$.
- $F(\lambda x, \lambda y) = \lambda F(x, y)$ for all scalar λ and $(x, y) \in Graph(\Gamma)$
- Result: Optimal policy homogeneous of degree one, $y = g(x) \implies \lambda y = g(\lambda x)$.
- We will specialize on one dimensional case, so $y = g(x) = \bar{g}\, x$ for some constant $\bar{g}$.

Homogeneous of degree 1 case (Exercise)

- Using homogeneity on Euler Equation:

$$\begin{aligned} 0 &= F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_{t+1}, x_{t+2}\right) \\ &= F_y\left(1, \frac{x_{t+1}}{x_t}\right) + \beta F_x\left(1, \frac{x_{t+2}}{x_{t+1}}\right) \end{aligned}$$

- Guessing $\bar{g} = \frac{x_{t+1}}{x_t} = \frac{x_{t+2}}{x_{t+1}}$

$$0 = F_y\left(1, \bar{g}\right) + \beta F_x\left(1, \bar{g}\right)$$

- Write Transversality using homogeneity & form of decision rule.
- We require that $\beta|\bar{g}| < 1$. Why?
- Why is it OK to guess? What result are we using?

Homogeneous of degree 1 case (Exercise)

- Suppose that F is strictly quasi-concave, can the Euler equation be satisfied for multiple values of $\bar{g}$?
- Use concavity implies $F_{xx} < 0, F_{yy} < 0$ and jointly with homogeneity of degree 1, $F_{xy}^2 = F_{xx}F_{yy} > 0$.
- Differentiate Euler equation w.r.t. $\bar{g}$ and use $F_{xx}F_{yy} = F_{xy}^2$:

$$F_{yy}(1,\bar{g}) + \beta F_{xy}(1,\bar{g}) = F_{yy}(1,\bar{g}) + \beta\sqrt{F_{xx}(1,\bar{g})\,F_{yy}(1,\bar{g})}$$

$$= |F_{yy}(1,\bar{g})| \left[-1 + \beta\sqrt{\frac{F_{xx}(1,\bar{g})}{F_{yy}(1,\bar{g})}}\right]$$

- So, for β small enough the derivative is negative.
- It is also negative if $F_{xx}(1,\bar{g})/F_{yy}(1,\bar{g}) < 1$ for all $\bar{g}$ since $\beta < 1$.

General Homogeneity, Exercise

- Result extend to the case of homogeneity of degree $1-\gamma$:

$$\frac{F(\lambda x, \lambda y)}{1-\gamma} = \lambda^{1-\gamma} \frac{F(x,y)}{1-\gamma} \text{ for all} x, y \text{ and } \lambda > 0$$

  with the same assumptions on X and Γ.

- Alternatively $F(x,y) = H(x,y)^{1-\gamma}/(1-\gamma)$ for H homog. of degree one.

- In this case we also have $g(x)$ homogeneous of degree one, i.e.: $y = g(x) \implies y\lambda = g(x\lambda)$.

- Strict concavity requires $\gamma > 0$. The case of $\gamma = 1$ is the log case.

  This case is used a lot in growth theory. Simple example is Ak model: $c_t + i_t = A\,k_t$, standard l.o.m. for capital, and $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$.

  Transversality and Euler are a bit different. Left as **exercise** for the one dimensional case. Must use properties of derivatives of homogeneous of degree $1-\gamma$ function.

General Homogeneity, solutions

- $F(x, y)$ are homogeneous of degree $-\gamma$.
- Differentiate $\lambda^{1-\gamma} F(x, y) = F(\lambda x, \lambda y)$ with respect to x:

  $F_x(x, y) = \lambda^{\gamma} F_x(\lambda x, \lambda y)$ and $F_y(x, y) = \lambda^{\gamma} F_y(\lambda x, \lambda y)$

- Apply to Euler Equations:

$$F_x(x_t, x_{t+1}) = \left(\frac{1}{x_t}\right)^{\gamma} F_x\left(1, \frac{x_{t+1}}{x_t}\right)$$

$$F_y(x_{t+1}, x_{t+2}) = \left(\frac{1}{x_{t+1}}\right)^{\gamma} F_y\left(1, \frac{x_{t+2}}{x_{t+1}}\right)$$

- Use $x_{t+1} = g\, x_t$ and $x_{t+2} = g\, x_{t+1}$:

$$0 = \left(\frac{1}{x_t}\right)^{\gamma} F_x(1, g) + \beta \left(\frac{1}{x_{t+1}}\right)^{\gamma} F_y(1, g)$$

$$0 = F_x(1, g) + \beta\, g^{-\gamma} F_y(1, g)$$

$A\,k$, solutions

- Use $F_y(x,y) = -U(f(x)-y)$ and $F_x(x,y) = U'(f(x)-y)f'(x)$
- Specialize to $x=1, y=g$, $U'(c)=c^{-\gamma}$ and $f'(x)=A$:

$$0 = -(A-g)^{-\gamma} + \beta g^{-\gamma}(A-g)^{-\gamma} \text{ or } 1 = g^{-\gamma}\beta A$$

- Solution: $g = (\beta A)^{1/\gamma}$
- Taking logs, recall $\log(1+x) \approx x$:

$$\log g = \frac{1}{\gamma} log\,(\beta A)$$

- Higher value of γ, more curvature, reduces growth given $\beta A > 1$.

  What is the economic intuition for this result?
- Higher value of βA, increases growth, given γ.

  What is the economic intuition for this result?

Adjustment cost and Investment

- Maximize discounted profit net of investment expenditures.
- Problem of a firm, or for economy with $u(c) = c$.
- Production function $f(k)$.
  - Case $f(k)$ strictly concave.
  - Case $f(k)$ linear.
- Capital Law of motion $k_{t+1} = i_t + (1-\delta)k_t$
- Case w/additional cost of installing capital, in terms of final goods $\phi(i/k)k$ for some function ϕ.
- Problem: $\max_{\{i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[f(k_t) - i_t - \phi\left(\frac{i_t}{k_t}\right) k_t \right]$

  subject to law of motion capital.

Concave f and NO Adj. Costs ("old news")

- Consider the case of $f(k)$ strictly concave and satisfies Inada conditions.
- There is NO adjustment cost $\phi(\cdot) = 0$.
- Assume that investment can be positive or negative.
- Write $F(x, y)$ for this case.
- Write $F(x, y)$ the Euler Equation for this case.
- Show that steady state is achieve immediately.
- Can f be linear instead of strictly concave in this case?

Linear f w/Adj. Costs.

- Let $f(k) = A\,k$
- Use l.o.m. capital and define cost of adjustment function a as:

$$a\left(\frac{k_{t+1}}{k_t}\right) \equiv \phi\left(\frac{k_{t+1} - k_t(1-\delta)}{k_t}\right)$$

- Write $F(x, y)$ in using the constant A and the function a.
- We will assume that:
  - a is positive (so any change implies cost) and strictly concave (so cost are increasing in size of change)
  - $a'(1) = a(1) = 0$ both marginal and per unit cost are zero if capital stays constant.
  - $a(1/\beta) < A$, i.e. large changes are costly.

Linear f w/Adj. Costs. (Exercise)

1. Compute F_x and F_y in terms of A and $a(\cdot)$. Make sure your expressions depend only on the ration y/x.

2. Write the Euler equation for this model. Use that the optimal policy is homogeneous of degree one (Why?) and denote $y = \bar{g}\,x$. Your expression should be a function of A, $a(\cdot)$, $a'(\cdot)$, β and $\bar{g}$.

3. Differentiate the Euler equation with respect to $\bar{g}$. What is the sign of this expression for values $g < 1/\beta$?

4. Plot the constant βA against $\beta a(g) + a'(g)(1 - \beta g)$, with g in the horizontal axis. Indicate in your graph the value of $\bar{g}$, where both curves intersect. How is $\bar{g}$ compared with $1/\beta$? How is $\bar{g}$ compared with 1?

5. What happen with $\bar{g}$ if A increases?

6. What happens if we replace the function a but another one, say $\tilde{a}$, with higher cost, i.e. $\tilde{a}(g) > a(g)$ for all $g \neq 1$, also with $\tilde{a}(1) = \tilde{a}'(1) = 0$ and $\tilde{a}(1/\beta) > A$.

Linear f w/Adj. Costs. (answers)

1. $F(x,y) = Ax - a(y/x)x$, $F_x(x,y) = A - a(y/x) + a'(y/x)(y/x)$ and $F_y(x,y) = -a'(y/x)$

2. Euler: $0 = -a'(\bar{g}) + \beta\,[A - a(\bar{g}) + a'(\bar{g})\bar{g}]$.

3. $-a''(g) + \beta\,[-a(g) + a'(g) + a''(g)g] = -a''(g)(1-\beta g)$.

4. Differentiating $\beta a(g) + a'(g)(1-\beta g)$ we get $\beta a' + a''(1-\beta) - a'\beta = a''(1-\beta g)$ so it is strictly increasing until $g = 1/\beta$. At $g = 1$, we have $\beta A > \beta a(1) + a'(1)(1-\beta) = 0$. At $g = 1/beta$ we have $\beta A < \beta a(1/beta) + a'(g)(1-\beta/beta) = \beta a(1/beta)$.

5. If A increases, the value of $\bar{g}$ increases.

6. Higher adjustment cost implies lower $\bar{g}$.

First order condition, again

- Rewrite $a'(\bar{g}) = \beta\,[A - a(\bar{g}) - a'(\bar{g})\bar{g}]$ as $a'(\bar{g})\,[1 - \beta\bar{g}] = \beta\,[A - a(\bar{g})]$
- Can be expressed as $a'(\bar{g}) = \beta\,\frac{[A - a(\bar{g})]}{1 - \beta\bar{g}}$
- What is the interpretation of $A - a(\bar{g})$?
- What is the interpretation of $1/(1 - \beta\bar{g})$?
- Redefine net growth rate $\gamma = \bar{g} - 1$ and net interest $1 + r = 1/\beta$, rewrite

$$a'(1+\gamma) = \frac{A - a(1+\gamma)}{1/\beta - (1+\gamma)} = \frac{A - a(1+\gamma)}{r - \gamma} \text{ or}$$
$$A = a(1+\gamma) + a'(1+\gamma)\,[r - \gamma]$$

- What happens with γ if r increases?

Tobin's q

- Since growth is constant at $\bar{g}$, we can write expected discounted profits of firms with k, or its total market value as:

$$V(k) = k\,[A - a(\bar{g})] + \beta\, V(k\bar{g})$$

- The function V is also homogenous of degree one.(Why?), so $\lambda V(k) = V(\lambda k)$ for all k and λ, thus

$$V(k) = [A - a(\bar{g})]\, k + \beta \bar{g} V(k) \text{ or}$$
$$V(k) = V(1)k = \frac{A - a(\bar{g})}{1 - \beta\bar{g}} k$$

- Note that Market to Book value is $V(k)/k = V(1)$ with:

$$q \equiv \frac{V(k)}{k} = V(1) = \frac{A - a(\bar{g})}{1 - \beta\bar{g}}$$

Tobin's q (cont.)

- From above:

$$q \equiv \frac{V(k)}{k} = V(1) = \frac{A - a(\bar{g})}{1 - \beta\bar{g}}$$

- Compare with Euler Equation:

$$\beta\left[A - a(\bar{g})\right] = a'(\bar{g})(1 - \beta\bar{g}) \implies q \equiv V(1) = \frac{A - a(\bar{g})}{1 - \beta\bar{g}} = \frac{1}{\beta} a'(\bar{g})$$

$$q = (1 + r)\, a'(1 + \gamma)$$

  Tobin's q equals derivative of adjustment cost $a'(\cdot)$, evaluated at the optimal growth rate $\bar{g}$.

- Since in principle, Tobin's q is observable, as market capitalization divided by book value of capital, to forecast growth rate of investment $\bar{g}$.