1 Short Questions

Problem 1.1. (5 points) Define what it means for a preference relation $\succsim$ on $\mathbb{R}^n_+$ to be continuous.

Solution. The preference relation $\succsim$ is continuous if and only if

$$\succsim (x), \precsim (x)$$

are closed. ■

Problem 1.2. (5 points) Prove that if two utility functions u and v represent the same preference relation on $\mathbb{R}^n_+$, then there is a strictly increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(x) = f(v(x))$ for every $x \in \mathbb{R}$. You should assume that the range of both $u(\cdot)$ and $v(\cdot)$ is all of $\mathbb{R}_+$.

Solution. Define $f := v^{-1}(u(x))$ and argue that f is well-defined and f is strictly increasing. ■

Problem 1.3. (5 points) Provide an Edgeworth box showing a Walrasian equilibrium allocation that is in the interior of the box but that is not Pareto efficient even though preferences are continuous.

Solution. The standard Edgeworth box with a "thick" indifference curve. ■

Problem 1.4. (5 points) State the first welfare theorem for the exchange economy.

Solution. The FWT is that if u^i is strictly increasing, then every WEA is PO. ■

2 Exchange Economy

Consider an economy with four consumers:

$$u^1(x_1, x_2) = u^2(x_1, x_2) = x_1 x_2, \quad \mathbf{e}^1 = \mathbf{e}^2 = (18, 2)$$
$$u^3(x_1, x_2) = u^4(x_1, x_2) = x_1 x_2, \quad \mathbf{e}^3 = \mathbf{e}^4 = (2, 18)$$

Problem 2.1. (5 points) Show that the allocation $\mathbf{x}^1 = \mathbf{x}^3 = (9, 9)$ and $\mathbf{x}^2 = \mathbf{x}^4 = (11, 11)$ is Pareto efficient.

Solution. It suffices to show that $\{\mathbf{x}^i\}$ is a WEA by finding a price vector that can sustain this allocation.

- ▷ At prices $(p_1, p_2) = (1, 1)$, the MRS of each agent are the same so the allocation is indeed PO from the FWT.

■

Problem 2.2. (10 points) Show that the allocation in part (a) is not in the core. (Hint: Find a blocking pair)

Solution. It suffices to find a blocking pair. Take [1] and [3] to have $\mathbf{x}^1 = \mathbf{x}^3 = (10, 10)$. ■

Problem 2.3. (15 points) Show that the allocation $\mathbf{x}^1 = \mathbf{x}^2 = (9, 9)$ and $\mathbf{x}^3 = \mathbf{x}^4 = (11, 11)$ is in the core but that it is not a Walrasian equilibrium allocation. (If needed, you may use the fact that $29\sqrt{19/11} > 38$)

Solution. We will first show that the allocation is in the core.

- ▷ Blocking individuals: No individual would form a blocking coalition since no individual is worse off relative to their endowments.
- ▷ Blocking pairs: Given symmetry, it suffices to verify that pairs $\{1, 2\}, \{1, 3\}, \{3, 4\}$ do not form blocking coalitions.
- ▷ Blocking trios: Given symmetry, it suffices to verify that trios $\{1, 2, 3\}$ and $\{1, 3, 4\}$ do not form a blocking coalition.
- ▷ Pareto optimality: From part (a), it follows that any interior allocation with $x_1^i = x_2^i$ is PO.

Next, we show that the allocation is not a WEA. Given the utility function, the only possible WEA is $p_1 = p_2 > 0$, but clearly $\mathbf{x}^3$ and $\mathbf{x}^4$'s bundles are not feasible under these prices. Therefore, we conclude that this is not a WEA. ■

3 Production Economy

Problem 3.1. (5 points) Define the profit function, $\Pi(\mathbf{p})$, for a price-taking firm with production set $Y \subset \mathbb{R}^n$.

Solution. The profit function is defined as

$$\Pi(\mathbf{p}) := \max_{\mathbf{y} \in Y} \mathbf{p}'\mathbf{y}$$

■

Problem 3.2. (5 points) Suppose that $\mathbf{y}^0 \in Y$ is a profit-maximizing production plan at the price vector $\mathbf{p}^0 \gg 0$. Suppose that the price of good k increases and that all other prices remain fixed, leading to the new price vector $\mathbf{p}^1$. Prove that if $\mathbf{y}^1 \in Y$ is profit-maximizing at the price vector $\mathbf{p}^1$, then $y_k^1 \geq y_k^0$ i.e. the supply of good k does not fall when its price goes up. Do not use envelope theorem or Hotelling's lemma.

Solution. We are given that $\mathbf{y}^0$ is profit-maximizing at $\mathbf{p}^0$ and $\mathbf{y}^1$ is profit-maximizing at $\mathbf{p}^1$ where $p_k^1 > p_k^0$.

▷ Suppose

$$\mathbf{p}^1 = \mathbf{p}^0 + \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix}$$

▷ From the assumptions of profit maximization, we have:

$$\sum_i p_i^1 y_i^1 \geq \sum_i p_i^1 y_i^0$$
$$\sum_i p_i^0 y_i^0 \geq \sum_i p_i^0 y_i^1$$

▷ Combining the two equations, we have:

$$\sum_i p_i^1 y_i^0 \leq \sum_i p_i^1 y_i^1 = \sum_i p_i^0 y_i^1 + \lambda y_k^1 \leq \sum_i p_i^0 y_i^0 + \lambda y_k^1$$

which implies that

$$\sum_i y_i^0 \left(p_i^1 - p_i^0\right) \leq \lambda y_k^1$$

or

$$y_k^0 \leq y_k^1$$

■

Problem 3.3. (5 points) Let Y be the *aggregate* production set of an economy with $J > 1$ firms. That is,

$$Y = \left\{ y \in \mathbb{R}^n : y \in \sum_{j=1}^{J} y^j, y^j \in Y^j \right\}$$

Fix some $\mathbf{p}$ and suppose that $\hat{\mathbf{y}} \in Y$ solves $\max \mathbf{p} \cdot \mathbf{y}$ subject to $\mathbf{y} \in Y$. Prove that there exists $\hat{\mathbf{y}}^1 \in Y^1, ..., \hat{\mathbf{y}}^J \in Y^J$ such that for each $j \in \{1, ..., J\}$, $\hat{\mathbf{y}}^j$ solves $\max \mathbf{p} \cdot \mathbf{y}^j$ subject to $\mathbf{y}^j \in Y^j$.

Solution. By way of contradiction, suppose that there exists $\tilde{\mathbf{y}}^k \in Y^k$ such that

$$\mathbf{p} \cdot \tilde{\mathbf{y}}^k > \mathbf{p} \cdot \hat{\mathbf{y}}^k$$

Then summing across all firms,

$$\sum_{j \in J} \mathbf{p} \cdot \tilde{\mathbf{y}}^j > \sum_{j \in J} \mathbf{p} \cdot \hat{\mathbf{y}}^j = \mathbf{p} \cdot \sum_{j \in J} \hat{\mathbf{y}}^j = \mathbf{p} \cdot \hat{\mathbf{y}}$$

which contradicts the assumption that $\hat{\mathbf{y}}$ maximized aggregate profits. ■

4 Social Welfare

Consider a society with three individuals $(1, 2, 3)$ and three social alternative (x, y, z). Let f be a social welfare function that yields:

Individual 1	Individual 2	Individual 3	Society f
x	y	z	x
y	z	x	z
z	x	y	y

Problem 4.1. (5 points) Provide precise statements of Arrow's four axioms using notations from class.

Solution. The four axioms are: U, WP, IIA, and ND.

- ▷ U says that (1) the domain of f must include all possible combinations of individual preferences and that (2) induced social welfare function

$$R = f\left(R^1, ..., R^N\right)$$

must be a valid preference relation.

- ▷ WP says that if $xR^iy, \forall i$ then xRy.
- ▷ IIA says that if everyone individual ranks $x, y \in X$ under R^i and $\tilde{R}^i$, then R and $\tilde{R}$ must rank x and y in the same way.
- ▷ D says that there is no individual such that $xP^iy \Rightarrow xPy$.

■

Problem 4.2. (5 Points) State Arrow's theorem. Without changing any individuals' rankings in the table above, use Arrow's theorem to conclude that f must violate at least one of U, WP or IIA.

Solution. If $|N| \geq 2$ and $|X| \geq 3$, then there is no social welfare function f that simultaneously satisfies conditions U, WP, IIA, and D. ■

Problem 4.3. (5 Points) Suppose that f satisfies U. By changing the ranking(s) of one or more individuals in the table above, show that f cannot also simultaneously satisfy both WP and IIA. You must answer this question without appealing to Arrow's theorem.

Solution. Change y and x for individuals 1 and 3. Since we did not change the ordering of (x, z) and (y, z), by IIA it must still be that

$$xPz, \quad zPy$$

By transitivity, we have xPy but yPx for all i which would contradict WP. ■