1 Basic Topology

Why do we need topology?

- It defines the concept of *neighborhoods* and *continuity*, which are crucial to optimization problems in economics.

1.1 The Basics

First, the basic definitions:

- A set S is compact if every sequence in S has a subsequence that converges to an element again contained in S.

1.2 Preferences and Utility

Where does the utility function come from?

- We would like to have utility functions to represent one's preference because mathematical tools can be readily applied to such constructs.
- It turns out that with a few (relatively) acceptable axioms on individual preferences, one can then analyze such individual's choice *as if* they have a utility function to maximize.

1.3 Functions

Variations of continuous functions:

- Functions that only jump up are called upper-semicontinuous
- Functions that only jump down are called lower-semicontinuous

Why do we care so much about this?

- In most economic problems, we need at least semi-continuity of the objective to ensure that optimum exists.
- The existence of indifference may induce a discontinuity in the payoff (example: simple take-it-or-leave-it bargaining game)

1.4 Correspondence

The idea:

- For functions, for any x in the domain, there must be one and only one value $f(x)$ in the range being assigned.
- We can relax this requirement and allow multiple values in the range to be assigned to the same x.

Upper/Lower Hemicontinuity:

- Upper hemicontinuity makes sure that the correspondence does not explode. Intuitively, upper hemicontinuity at x says that a small perturbation at x does not cause the image set $\Gamma(x)$ to suddenly get large.
- Lower hemicontinuity makes sure that the correspondence does not implode. Intuitively, this demands that if $x' \in X$ is close enough to x, then its image $\Gamma(x')$ should not be far away from any point in $\Gamma(x)$.
- See **here** for further discussion and its relevance to game theory.

1.5 Theorem of Maximum

First, consider a very standard way of formulating an economic problem. Let X and Θ be metric spaces, and consider a function $f(x, \theta)$ and a correspondence $\Gamma(\theta)$.

- Interpretation here is that x is a choice variable, whereas θ is a parameter of the environment that the economic agent cannot affect.
- Economic agent seeks to maximize $f(x, \theta)$ where $x \in \Gamma(\theta)$.
- The value function:

$$V(\theta) := \sup_{x \in \Gamma(\theta)} f(x, \theta)$$

- The optimal choice (= correspondence):

$$X^*(\theta) := \{x \in \Gamma(\theta) | f(x, \theta) = V(\theta)\}$$

The above characterization is a framework, and it's useful to translate an existing problem using this new language.

- Consider, for example, an optimization problem:

$$\max_{x \in \mathbb{R}^n_+} \text{ s.t. } \mathbf{p}^T \mathbf{x} \leq m$$

- The budget constraint is the correspondence on $\mathbb{R}^n_+ \times (0, \infty)$
- Using θ to denote $(\mathbf{p}, m)$, the consumer's problem can be thus written as:

$$\max_{x \in \Gamma(\theta)} u(x)$$

Now to the Theorem:

- It says that if the function and correspondence are continuous enough, the value function and the solution set will also be continuous.
- The proof is worth following, so check the lecture notes.
- One useful result used in the proof is that if every subsequence of x_n has a further subsequence which converges to x, then the sequence x_n also converges to x.

An immediate extension is the Demand Continuity Lemma.

- How do we show that the demand is continuous?
  - You first assume quasi-concavity of the utility function. This means that the demand *correspondence* becomes a demand *function*. Then the upper-hemicontinuity of the demand *correspondence* now becomes the continuity of the demand *function*.
- Also called Marshallian Demand, which is derived from the utility maximization problem. A competing concept is the Hicksian Demand, which is derived from an expenditure minimizing problem.

1.6 Fixed Point Theorem

One important task in economics is finding an equilibrium.

- Equilibrium uses the idea that if we are in an equilibrium, nobody wants to change the behavior. "Forces" come in to bring stuff to balance.
- Contents thus far is knowing how to maximize things and squeeze a little more juice by looking at how the solution set changes when the environment changes.
- The remaining piece is to bring these pieces together. This is what the set of fixed point theorems do. They ensure that you find an equilibrium.

Definition of a fixed point:

- Put the point into a function and it gives you the point.

- Put the point into a correspondence and it gives you the set containing the point.

When does a fixed point exist?

- *Intermediate Value Theorem* is a type of a fixed point theorem. Let $g(x) = f(x) - x$.
- *Brouwer's Fixed Point Theorem* For any $n \in \mathbb{N}$, let $S \subseteq \mathbb{R}^n$ be (1) non-empty, (2) compact, and (3) convex. Define $f : S \to S$ which is a continuous function. Then f has a fixed point. The proof is crazy long, but the intuition isn't too bad.
  - This thoerem is **critical** in showing general equilibrium.
- *Kakutani Fixed Point Theorem* is an analogous theorem for correspondences. This time the correspondence must be an upper-hemicontinuous correspondence. Otherwise the result is the same.

This result can be applied to Nash Equilibria:

- Consider a finite game G and its mixed extension $\bar{G}$. Then $\bar{G}$ has a Nash equilibrium. See lecture notes for proof.

Another very useful fixed point theorem for macroeconomics is the following:

- A mapping $f : X \to X$ is a *contraction* if the distance between $f(x)$ and $f(y)$ will be smaller than (a multiple) of the distance between x and y.
- *Banach Contraction Principle:* Basically, given a contraction and a complete metric space, you can find a *unique* fixed point.
- This principle is used to derive the Bellman equation and also the existence of solutions to first-order differential equations.