Class Note 7: Theory of Income, Introduction to Dynamic Programming

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Principle of Optimality and Dynamic Programming

Bellman's Principle of Optimality provides conditions under which a programming problem expressed in sequence form is equivalent (in a precisely defined way described below) to a two period recursive programming problem (called the Bellman Equation).

This is to be developed fully in other quarters.

The end of the note focuses on continuous time Dynamic Programming and shows how it relates to the Maximum Principle.

Sequence Problem:

The sequence problem is to find a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ given an initial x_0, such that:

$$V^*(x_0) = \max_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{1}$$

subject to

$$x_{t+1} \in \Gamma(x_t) \text{ for all } t \geq 0$$

x_0 given.

We can write this problem differently, by defining

$$\begin{aligned} x^t &\equiv (x_0, x_1, ..., x_t), \\ \Pi^t(x_0) &\equiv \{(x_0, x_1, ..., x_t) : x_{s+1} \in \Gamma(x_s), s = 0, ..., t-1\} \\ u(\{x_{t+1}\}) &\equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}). \end{aligned}$$

Then (1) can be written as

$$V^*(x_0) = \max_{x^\infty \in \Pi^\infty(x_0)} u(\{x_{t+1}\})$$

Recursive Problem: Bellman Equation

The recursive problem is to find a function $V : X \to R$ such that

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \tag{2}$$

Notice that this is a functional equation, its solution is a function V, the same on both sides of (2) that must satisfy this equality for all $x \in X$.

Moreover the maximizer of the RHS of (2) is maximized by $g(x)$, which then satisfies

$$V(x) = F(x, g(x)) + \beta V(g(x))$$

Note that if the function V were known, or if we know properties of it, then (2) is a two period problem!

Principle of Optimality:

The principle of optimality states that

$$V^*(x) = V(x)$$

for all x.

Why is this progress? Because it says that the two period problem in (2) is equivalent to the infinitely dimension problem in (1)! Notice that once we have V and g, we have the solution of (1) for ANY initial condition $x_0 \in X$.

RMED has a superb treatment of the relation between these two problems.

We will only sketch the basic reasoning behind this principle.

Take the case where F is bounded so $|F(x,y)| \leq B < \infty$ for all $(y,x) \in Gr(\Gamma)$. Notice that in this case, for any $x^\infty \in \Pi^\infty(x_0)$:

$$|u(\{x_{t+1}\})| \leq \frac{B}{1-\beta}$$

$$u(\{x_{t+1}\}) = \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T \left[\sum_{t=T}^{\infty} \beta^{t-T} F(x_t, x_{t+1}) \right]$$

and

$$\left| u(\{x_{t+1}\}) - \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) \right| = \beta^T \left| \sum_{t=T}^{\infty} \beta^t F(x_t, x_{t+1}) \right| \leq \beta^T \frac{B}{1-\beta}$$

thus the value of a plan can be approximated arbitrary well by

$$\sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1})$$

for sufficiently large T.

Using that $|F| \leq B$ we also obtain that

$$|V(x)| \leq B/(1-\beta).$$

Thus for any $x^T \in \Pi^T(x_0)$

$$\lim_{T\to\infty} \beta^T V(x_T) = 0.$$

By definition (2) means

$$V(x_0) \geq F(x_0, x_1) + \beta V(x_1), \text{ all } (x_0, x_1) \in \Pi^1(x_0)$$

and

$$V(x_0) = F(x_0, x_1) + \beta V(x_1)$$

for some $(x_0, x_1) \in \Pi^1(x_0)$. Substituting $V(x_1)$ we have:

$$V(x_0) \geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2), \text{ all } (x_0, x_1, x_2) \in \Pi^2(x_0)$$

and

$$V(x_0) = F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2)$$

for some $(x_0, x_1, x_2) \in \Pi^2(x_0)$.

Continuing this way we get

$$\begin{aligned} V(x_0) &\geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T V(x_T) \text{ for all } x^T \in \Pi^T(x_0) \\ &= \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T V(x_T) \text{ for some } x^T \in \Pi^T(x_0) \end{aligned}$$

Since $\beta^T V(x_T)$ goes to zero for large T, then the value of $V(x_0)$ can be approximated arbitrarily well by

$$\sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}).$$

Thus we “conclude” that:

$$V(x_0) = V^*(x_0).$$

This is not a proof, but it contains the main idea behind a rigorous proof.

Envelope and first order conditions
The first order conditions and envelope are:

$$\begin{aligned} 0 &= F_y\left(x, g\left(x\right)\right) + \beta V'\left(g\left(x\right)\right) \\ V'\left(x\right) &= F_x\left(x, g\left(x\right)\right) \end{aligned}$$

for all x such that $\left(g\left(x\right), x\right) \in int\left(Gr\left(\Gamma\right)\right)$. Notice that combining them, we obtain the Euler Equations:

$$F_y\left(x, g\left(x\right)\right) + \beta F_x\left(g\left(x\right), g\left(g\left(x\right)\right)\right) = 0.$$

Example. For the neoclassical growth model we obtain:

$$\begin{aligned} U'\left(f\left(k\right)-g\left(k\right)\right) &= \beta V'\left(g\left(k\right)\right) \\ V'\left(k\right) &= U'\left(f\left(k\right)-g\left(k\right)\right)f'\left(k\right) \end{aligned}$$

Exercise. Show that g, the decision rule for the neoclassical growth model, is increasing in k. Graph $U(f(k) - y)$ vs $\beta V'(y)$ as functions of y, using the result that V is strictly concave. The interception of the two functions gives $y = g(k)$. Increase the value of k, say to $k' > k$, and argue that the new interception occurs at a larger value for y, so that g is strictly increasing, i.e. $g(k') > g(k)$. Use this graph to show that the slope of g is bounded by the slope of f. Show that the optimal consumption function $c(k) \equiv f(k) - g(k)$ is increasing in k.

Exercise. Assume that $X = R_+$, $\Gamma(x) = R$ and F be C^2 , $F_x \geq 0$, $F_y \leq 0$, and strictly concave. Use a similar argument than in the previous exercise to conclude that g is increasing in x if and only if $F_{xy} \geq 0$.

Advantages of Dynamic Programming Approach

- Much easier in models with shocks
- Techniques to solve value function directly.
- Guess and verify
- Computational ability
- Discrete choice problems, non-convex problems
- Easier to interpret

Continuous time Bellman Equation: set-up

Consider the following discrete time Bellman Equation

$$V(x_t) = \max_{u_t \in U} \left\{ \Delta h(x_t, u_t) + \frac{1}{1+\Delta\rho} V(x_{t+\Delta}) \right\}$$

subject to

$$x_{t+\Delta} = x_t + \Delta g(x_t, u_t)$$

We will analyze the continuous time Bellman Equation as a limit of the previous one.

Exercise. Take the limit as Δ goes to zero in the Bellman equation. Do you obtain a useful expression?

In view of the previous exercise we follow a different route. Use the Taylor's theorem to write

$$\begin{aligned} & V\left(x_{t+\Delta}\right) \\ = \quad & V\left(x_t + \Delta g\left(x_t, u_t\right)\right) = V\left(x_t\right) + V'\left(x_t\right) \Delta g\left(x_t, u_t\right) + o\left(\Delta g\left(x_t, u_t\right)\right) \end{aligned}$$

where $d\left(z\right) = o\left(z\right)$ means $\lim_{z \to 0} d\left(z\right)/z = 0$. Then the Bellman equation gives

$$\begin{aligned} & V\left(x_t\right) \\ = \quad & \max_{u_t \in U} \left\{ \Delta h\left(x_t, u_t\right) + \frac{1}{1+\Delta\rho} \left[V\left(x_t\right) + V'\left(x_t\right) \Delta g\left(x_t, u_t\right) + o\left(\Delta g\left(x_t, u_t\right)\right) \right] \right\} \end{aligned}$$

or multiplying by the positive constant $1 + \Delta\rho$ both sides

$$
\begin{aligned}
& (1+\Delta\rho)\, V(x_t) \\
= \quad & \max_{u_t \in U} \left\{ (1+\Delta\rho)\, \Delta h(x_t, u_t) + V(x_t) + V'(x_t)\, \Delta g(x_t, u_t) + o(\Delta g(x_t, u_t)) \right\}
\end{aligned}
$$

since the terms that do not involve u_t can be taken out of the maximum

$$
\begin{aligned}
& \Delta\rho V(x_t) \\
= \quad & \max_{u_t \in U} \left\{ (1+\Delta\rho)\, \Delta h(x_t, u_t) + V'(x_t)\, \Delta g(x_t, u_t) + o(\Delta g(x_t, u_t)) \right\}
\end{aligned}
$$

Dividing by Δ

$$
\begin{aligned}
& \rho V(x_t) \\
= \quad & \max_{u_t \in U} \left\{ (1+\Delta\rho)\, h(x_t, u_t) + V'(x_t)\, g(x_t, u_t) + o(\Delta g(x_t, u_t)) / \Delta \right\}
\end{aligned}
$$

Taking the limit as Δ goes to zero:

$$\rho V(x_t) = \max_{u_t \in U} \{h(x_t, u_t) + V'(x_t)\, g(x_t, u_t)\}$$

where we implicitly assumed that the limit w.r.t. to Δ of the max w.r.t. to u_t is the same as the max w.r.t. to u_t of the limit w.r.t. Δ. Getting rid of unnecessary time indexes we obtain the Bellman equation for continuous time:

$$\rho V(x) = \max_{u \in U} \{h(x, u) + V'(x)\, g(x, u)\} \tag{3}$$

Under regularity conditions the max of the RHS can be characterized using the following FOC for u :

$$0 = h_u (x, u^* (x)) + V' (x) \, g_u (x, u^* (x))$$

which define the optimal decision rule $u^* (x)$. Thus the following two equations summarize the dynamic programming problem:

$$\rho V (x) = h (x, u^* (x)) + V' (x) \, g (x, u^* (x)) \tag{4}$$
$$0 = h_u (x, u^* (x)) + V' (x) \, g_u (x, u^* (x)) \tag{5}$$

for all $x \in X$. Notice that these are two functional equations, i.e. equations whose solutions are functions. The functions are V and u^*, both functions of x.

Bellman equation and Maximum Principle.

We now show the sense where the Bellman equation and FOC above imply the equations for the Maximum Principle derived in the last class notes. First notice that from (5) we see that

$$\lambda(t) = V'(x(t))$$

i.e. that the co-state in the calculus of variations approach is the derivative of the value function. This is consistent with the interpretation for the discounted value of the co-state variables offered before: the marginal value of an extra unit of the state.

Second, using that $\lambda = V'(x)$ we can see that the FOC (5) is equivalent to the condition

$$H_u(x, u, \lambda) = 0$$

used in the Maximum Principle.
Third, differentiating (4) with respect to time:

$$\rho V' \dot{x} = h_x \dot{x} + h_u (u^*)' \dot{x} + \left[V'' g + V' g_x + V' g_u (u^*)' \right] \dot{x},$$

moreover using (5) we can simplify the terms containing $u^{*\prime}$ obtaining

$$\rho V' \dot{x} = h_x \ \dot{x} + V'' g \ \dot{x} + V' g_x \ \dot{x}$$

Which, outside steady state, i.e. for $\dot{x} \neq 0$, gives

$$\rho V' = h_x + \ V'' g \ + V' g_x$$

Differentiating $\lambda(t) = V'(x(t))$ with respect to time and using that $\dot{x}(t) = g(x(t), u(t))$ we get

$$\dot{\lambda} = V'' \, \dot{x} = \; V'' g$$

and using that $V'(x) = \lambda$:

$$\dot{\lambda} = \rho\lambda - h_x - \lambda g_x$$

which is equivalent to

$$\dot{\lambda} = \rho\lambda - H_x(x, u, \lambda),$$

the law of motion of the co-state variable obtained using the Maximum Principle.