The Maximum Likelihood Estimator (MLE)

Empirical Analysis II, Econ 311: Topic 1

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Outline

Measure Spaces

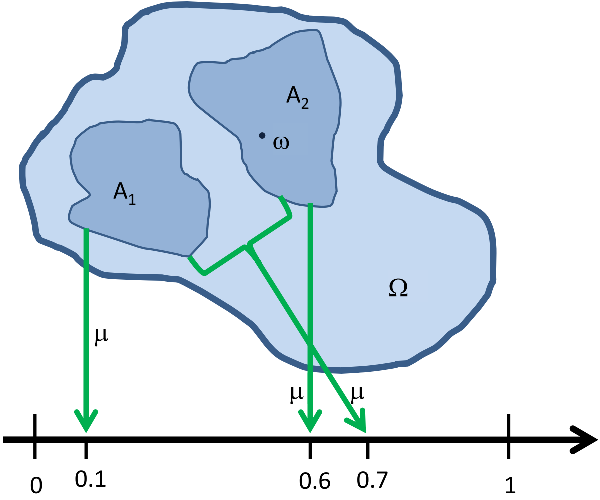
A measure space $(\Omega, \mathcal{F}, \mu)$ consists of

1. Ω: a set of points ("states of nature") ω.
2. $\mathcal{F}$: a set of subsets ("events") of Ω, which form a σ-algebra:
    1. $\Omega \in \mathcal{F}$.
    2. If $A \in \mathcal{F}$, then so is its complement $A^c = \Omega \backslash A \in \mathcal{F}$.
    3. $A_j \in \mathcal{F}, j = 1, 2, \ldots$ implies $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.
3. A measure μ, i.e. a mapping $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{\infty\}$ with
    1. Positivity: $\mu(A) \geq 0$.
    2. σ-additivity: if $A_j \in \mathcal{F}, j = 1, 2, \ldots$ are disjoint, then

    $$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

    3. $\mu(\emptyset) = 0$.
4. Probability space / probability measure: $\mu(\Omega) = 1$.

A measure space



Example 1: Rolling two dice

- $\Omega = \{\omega = (x, y) \mid x, y \in \{1, \ldots, 6\}\}$
- Three σ-algebras:
  - $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
  - $\mathcal{F}_1 = \{A_x \times \{1, \ldots, 6\} \mid A_x \subseteq \{1, \ldots, 6\}\}$
  - $\mathcal{F}_2 = \{A \subseteq \Omega\}$
- $\mu(A) = \sum_{\omega \in A} \frac{1}{36}$: probability measure.
- A filtration: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$.
- Dice roll at $t = 1, t = 2$. $\mathcal{F}_t$: events "known" at t. Information.

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Example 2: An infinite sequence

- $\Omega = \mathbb{N}$
- $\mathcal{F} = \{A \subseteq \Omega\}$.
- Let $\alpha_j = \mu(\{j\})$. Then $\mu(A) = \sum_{j \in A} \alpha_j$.

Example 3: the Lebesgue measure

- $\Omega = \mathbb{R}^m$.
- $\mathcal{F} = \mathcal{B}(\Omega)$: the Borel-$\sigma$-algebra, i.e. the smallest σ-algebra, which contains all open subsets of Ω.
- Let $I_j = [a_j, b_j], a_j \leq b_j \in \mathbb{R}$ be intervals. Define the box $B = I_1 \times \ldots \times I_n$. Define

$$\mu(B) = (b_1 - a_1)(b_2 - a_2) \ldots (b_m - a_m).$$

  Extend this to $\mathcal{F}$.
- For the mathematicians.
  - Extend to the Lebesgue-measurable sets $\bar{\mathcal{F}} = \{A \cup B \mid A \in \mathcal{F}, B \subseteq C \in \mathcal{F}, \mu(C) = 0\}$ per $\mu(A \cup B) = \mu(A)$.
  - Or this way. Outer measure:

$$\mu^*(A) = \inf\{\sum_j \mu(B_i) \mid A \subseteq \bigcup_i B_i, B_i \text{ is a box}\}$$

    A is Lebesgue measurable if $\mu^*(A) = \mu^*(A \cap C) + \mu^*(A \backslash C)$ for all $C \subseteq \mathbb{R}^n$. Define $\mu(A) = \mu^*(A)$.

Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

- A function $f : \Omega \to \mathbb{R}^k$ is called $\mathcal{F}$-measurable if $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \in \mathcal{B}(\mathbb{R}^k)$.
- Suppose $f = 1_A$ is an indicator function on a set $A \in \mathcal{F}$: $1_A(\omega) = 1$, if $\omega \in A$ and $1_A(\omega) = 0$, if $\omega \notin A$. Define the integral

$$\int f \, d\mu = \int f(\omega) \, \mu(d\omega) = \mu(A)$$

- Suppose f is a linear combination of indicator functions,

$$f(\omega) = \sum_{j=1}^{n} \psi_j 1_{A_j}, \; A_j \in \mathcal{F}$$

  Define the integral per linear extension,

$$\int f \, d\mu = \sum_{j=1}^{n} \psi_j \int 1_{A_j} d\mu = \sum_{j=1}^{n} \psi_j \, \mu(A_j)$$

Integration

- Extend to all positive measurable functions. Extend to all measurable functions f per

$$\int f d\mu = \int \max(f, 0) d\mu - \int \max(-f, 0) d\mu$$

provided at least one of the integrals is finite.

- For $A \in \mathcal{F}$, define

$$\int_A f d\mu = \int 1_A(\omega) f(\omega) \mu(d\omega).$$

- If μ is a probability measure, define the expectation $E[f] = \int f d\mu$.

The Radon-Nikodym Theorem

Theorem

Let $\mathcal{F}$ be a σ-algebra on Ω. Let μ and ν be two measures on $\mathcal{F}$. Suppose that $\mu(\Omega) < \infty$ and $\nu(\Omega) < \infty$. Suppose that ν is absolutely continuous with respect to μ, $\nu \ll \mu$, i.e. $\mu(A) = 0$ implies $\nu(A) = 0$ for $A \in \mathcal{F}$. Then there exists a positive measurable function g, called the Radon-Nikodym derivative,

$$g : \Omega \to \mathbb{R}_+ \text{ or } g = \frac{d\nu}{d\mu} \text{ with } \nu(A) = \int_A g\, d\mu = \int_A \frac{d\nu}{d\mu} d\mu$$

for all $A \in \mathcal{F}$.

Remark: This can be extended to signed measures $\nu : \Omega \to \mathbb{R}$: drop the requirement of "positiveness", but impose $-\infty < \nu(\Omega) < \infty$.

Example 1: Rolling two dice

- $\Omega = \{\omega = (x, y) \mid x, y \in \{1, \ldots, 6\}\}$.
- $\mathcal{F} = \mathcal{F}_j$, for one of $j = 0, 1, 2$.
- $\mu(A) = \sum_{\omega \in A} \frac{1}{36}$.
- Let $f : \Omega \to \mathbb{R}$ be measurable.

$$\int f \, d\mu = \sum_{\omega \in \Omega} \frac{f(\omega)}{36}$$

- Conditional expectation. Let $f : \Omega \to \mathbb{R}$ be $\mathcal{F}_2$-measurable. Find a $\mathcal{F}_1$-measurable function $g : \Omega \to \mathbb{R}$, $g = E[f \mid \mathcal{F}_1] = E_1[f]$ per the property

$$\int_A g \, d\mu = \int_A f \, d\mu, \quad \text{for all } A \in \mathcal{F}_1$$

- Existence of g: per the Radon-Nikodym-Theorem for signed measures. Here:

$$g(x, y) = E_1[f(x, y)] = E[f(x, y) \mid \mathcal{F}_1] = E[f(x, y) \mid x] = \sum_{j=1}^{6} \frac{1}{6} f(x, j)$$

Example 2: An infinite sequence

- $\Omega = \mathbb{N}$
- $\mathcal{F} = \{A \subseteq \Omega\}$.
- $\mu(A) = \sum_{j \in A} \alpha_j$.
- For $f : \Omega \to \mathbb{R}$,

$$\int f \, d\mu = \sum_{j=1}^{\infty} \alpha_j f(j)$$

  where $\alpha_j = \mu(\{j\})$.
- "Summation is integration".

Example 3: the Lebesgue measure

- Ω: an open subset of some $\mathbb{R}^n$.
- $\mathcal{F} = \mathcal{B}(\Omega)$: the Borel-$\sigma$-algebra.
- μ: the Lebesgue measure.
- For a measurable function $f : \Omega \to \mathbb{R}$, $\int f \, d\mu$ is what you expect it to be.
- Example: let $I_j = [a_j, b_j], a_j \leq b_j \in \mathbb{R}$ be intervals. Define the box $B = I_1 \times \ldots \times I_n$. Let $f(\omega) = \kappa 1_B$. Then,

$$\int f \, d\mu = \kappa(b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$$

Outline

The framework

- (Unknown) parameter $\theta \in \Theta$. Measure $\mu(d\theta)$. Often: $\Theta \subseteq \mathbb{R}^m$.
- Observation $y \in Y$. Measure $\nu(dy)$.
- Probability density $f(y \mid \theta)$ wrt ν. Thus,

$$\int f(y \mid \theta)\nu(dy) = 1, \text{ for all } \theta \in \Theta$$

- Likelihood function: $L(\theta \mid y) = f(y \mid \theta)$.
- Log-likelihood function: $\ell(\theta \mid y) = \ln L(\theta \mid y)$.
- Experiment on θ. Leads to an observation $y \sim f(y \mid \theta)$ for some known f, if it is carried out.
- Unconditional vs conditional likelihood. Suppose, draws of X do not depend on θ. Then

$$f(X, y \mid \theta) = f(y \mid \theta, X) f(X)$$

- Often: iid observations, $y = (y_1, \ldots, y_n)$,

$$f^{(n)}(y \mid \theta) = \prod^{n} f(y_j \mid \theta)$$

Example 1: linear regression

- $y \in \mathbb{R}^n, X \in \mathbb{R}^{n \times k}, \beta \in \mathbb{R}^k, \Sigma \in \mathbb{R}^{n \times n}$ pos.def.,

$$y = X\beta + \epsilon, \; \epsilon \sim \mathcal{N}(0, \Sigma)$$

- Solve for ϵ:

$$\epsilon = y - X\beta \sim \mathcal{N}(0, \Sigma)$$

- $\theta = \beta$. Or: $\theta = (\beta, \Sigma)$. Or: $\theta = \Sigma$. Or ...
- Assume: Distribution of X does not depend on θ.
- Conditional likelihood function: $L(\theta \mid y, X) = f(y \mid \theta, X)$,

$$L(\theta \mid y, X) = \frac{1}{(2\pi)^{n/2} \mid \Sigma \mid^{1/2}} \exp\left(-\frac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\right)$$

- Special case: iid assumption, $\Sigma = \sigma^2 I_n$.

$$L(\theta \mid y, X) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j - X_j\beta)^2\right)$$

Example 2: binomial distribution

"Biased coin"

- $y \in \{0, 1\}$.
- Likelihood function:

$$\begin{aligned} L(\theta \mid y = 1) &= f(y = 1 \mid \theta) = P(y = 1 \mid \theta) &= \theta \\ L(\theta \mid y = 0) &= f(y = 0 \mid \theta) = P(y = 0 \mid \theta) &= 1 - \theta \end{aligned}$$

- Observation: $y = 1$.
- "How likely is it to get the observed data $y = 1$?". Answer: $L(\theta \mid y = 1) = \theta$.
- $y = 1$ is 9 times more likely for $\theta = 0.9$ than $\theta = 0.1$.
- Likelihood ratio: 9. Log-Likelihood-Ratio: 2.2.

Example 3: binary choice. Probit and Logit.

- Example: choose to accept ($y = 1$) or reject ($y = 0$) a job, if $X\beta$ is larger than some random outside option, i.e.

$$y = 1, \text{ iff } \epsilon \leq X\beta, \qquad y = 0, \text{ iff } \epsilon > X\beta$$

- Data $y \in \{0, 1\}$, X. Cannot observe ϵ. $\theta = \beta$.
- CDF (=cum.distr.function) G for ϵ. $\theta = \beta$. Likelihood function:

$$L(\theta \mid y = 1) = G(X\beta), \;\; L(\theta \mid y = 0) = 1 - G(X\beta)$$

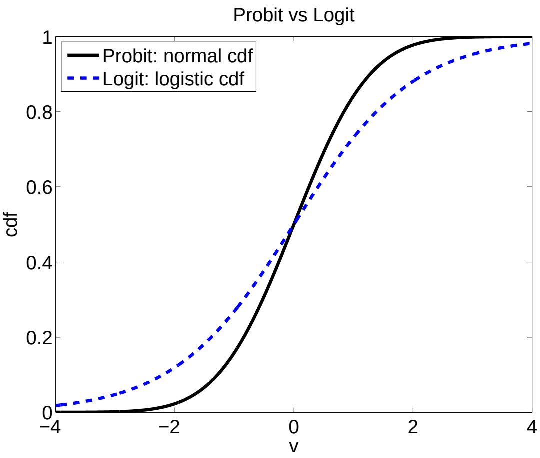
- Probit: standard normal,

$$G(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds$$

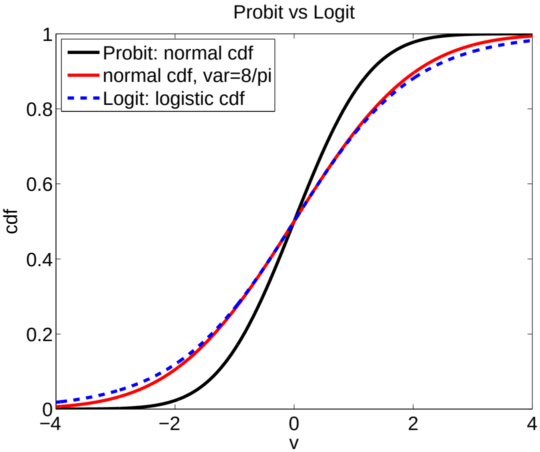
- Logit: logistic distribution,

$$G(v) = \frac{e^v}{1 + e^v}$$

Probit vs Logit



Probit vs Logit vs normal cdf with $\sigma = \sqrt{8/\pi} \approx 1.6$.



Example 4: censored regression. Tobit

- Example: hours worked depend on wage, provided the agent chooses to work at all. Data on hours available only then, on wage always.
- Tobit model:

$$y = \max\{y^*; 0\}, \text{ where } y^* = X\beta + \epsilon, \ \epsilon \sim \mathcal{N}(0, \sigma^2)$$

- Data: y, X. Not observed: y^*, ϵ.
- $\theta = (\beta, \sigma^2)$. Likelihood function:

$$L(\theta \mid y, X) = \begin{cases} \frac{1}{\sigma}\varphi\left(\frac{y - X\beta}{\sigma}\right) & \text{if } y > 0 \\ \Phi\left(\frac{-X\beta}{\sigma}\right) & \text{if } y = 0 \end{cases}$$

  where φ, Φ are the pdf and cdf of a standard normal.
- $L(\cdot \mid y, X)$: a Radon-Nikodym derivative wrt to which measure?

Outline

The Maximum Likelihood Estimator

$$\hat{\theta} = \text{argmax}_{\theta} L(\theta \mid y)$$

Remarks

- Maximizing the log-likelihood is the same:

$$\begin{aligned}\hat{\theta} &= \text{argmax}_\theta L(\theta \mid y) \\ &= \text{argmax}_\theta \ell(\theta \mid y)\end{aligned}$$

- Unconditional vs unconditional likelihood. Suppose, draws of X do not depend on θ, i.e.

$$f((X, y) \mid \theta) = f(y \mid \theta, X) f(X)$$

  Then, the MLE of the conditional and the unconditional likelihood function are the same,

$$\hat{\theta} = \text{argmax}_\theta f((X, y) \mid \theta) = \text{argmax}_\theta f(y \mid \theta, X)$$

The Score

From here on, $\theta \in \Theta \subseteq \mathbb{R}^m$, an open set.
The score is the first derivative of the log-likelihood function,

$$s(\theta) = s(\theta \mid y) = \frac{\partial \ell(\theta \mid y)}{\partial \theta}$$

Remark: the score is defined to be a column vector.

The Score

$$\boxed{E[s(\theta \mid y)] = 0}$$

Proof: Note: θ in arg. is also "truth". For emphasis: $E_\theta[s(\theta \mid y)] = 0$.

- For all θ, $\int f(y \mid \theta)\nu(dy) = 1$.
- Therefore $\int \frac{\partial f(y|\theta)}{\partial \theta}\nu(dy) = 0$.
- Rewrite:

$$\begin{aligned} 0 &= \int \frac{\partial f(y \mid \theta)}{\partial \theta}\nu(dy) \\ &= \int \frac{\frac{\partial f(y|\theta)}{\partial \theta}}{f(y \mid \theta)} f(y \mid \theta)\nu(dy) \\ &= \int \frac{\partial \ell(\theta \mid y)}{\partial \theta} f(y \mid \theta)\nu(dy) \\ &= E[s(\theta \mid y)] \end{aligned}$$

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&= \int \frac{\partial \ell(\theta \mid y)}{\partial \theta} f(y \mid \theta)\nu(dy) \\
&= E[s(\theta \mid y)]
\end{aligned}$$

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The Information Matrix

The Fisher information matrix is defined as

$$\mathcal{I}(\theta) = E\left[s(\theta \mid y)s(\theta \mid y)'\right]$$

The Information Matrix Equality

Theorem

$$\mathcal{I}(\theta) = E\left[s(\theta \mid y)s(\theta \mid y)'\right] = -E\left[\frac{\partial^2 \ell(\theta \mid y)}{\partial\theta\,\partial\theta'}\right]$$

Proof: θ is "truth", i.e. $\mathcal{I}(\theta) = E_\theta\left[s(\theta \mid y)s(\theta \mid y)'\right] = -E_\theta\left[\frac{\partial^2 \ell(\theta|y)}{\partial\theta\,\partial\theta'}\right]$.

- Recall: $0 = E[s(\theta \mid y)] = \int \frac{\partial \ell(\theta|y)}{\partial\theta} f(y \mid \theta)\nu(dy)$.
- Differentiate wrt θ',

$$\begin{aligned} 0 &= \int \frac{\partial^2 \ell(\theta \mid y)}{\partial\theta\,\partial\theta'} f(y \mid \theta)\nu(dy) \\ &\quad + \int \frac{\partial \ell(\theta \mid y)}{\partial\theta} \frac{\frac{\partial f(y|\theta)}{\partial\theta'}}{f(y \mid \theta)} f(y \mid \theta)\nu(dy) \\ &= E\left[\frac{\partial^2 \ell(\theta \mid y)}{\partial\theta\,\partial\theta'}\right] + \mathcal{I}(\theta) \end{aligned}$$

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&= E\left[\frac{\partial^2 \ell(\theta \mid y)}{\partial\theta\,\partial\theta'}\right] + \mathcal{I}(\theta)
\end{aligned}$$

First- and second order expansions around some θ

$$
\begin{aligned}
\ell(\tilde{\theta}) &\approx \ell(\theta) + s(\theta)'(\tilde{\theta} - \theta) + \frac{1}{2}(\tilde{\theta} - \theta)' \frac{\partial^2 \ell(\theta)}{\partial\theta\,\partial\theta'}(\tilde{\theta} - \theta) \\
s(\tilde{\theta}) &\approx s(\theta) + \frac{\partial^2 \ell(\theta)}{\partial\theta\,\partial\theta'}(\tilde{\theta} - \theta)
\end{aligned}
$$

Likewise, with θ as "truth":

$$
\begin{aligned}
E_\theta[\ell(\tilde{\theta})] &\approx E_\theta[\ell(\theta)] + E_\theta[s(\theta)]'(\tilde{\theta} - \theta) - \tfrac{1}{2}(\tilde{\theta} - \theta)'\mathcal{I}(\theta)(\tilde{\theta} - \theta) \\
&\approx E_\theta[\ell(\theta)] - \tfrac{1}{2}(\tilde{\theta} - \theta)'\mathcal{I}(\theta)(\tilde{\theta} - \theta)
\end{aligned}
$$

$$
\begin{aligned}
E_\theta[s(\tilde{\theta})] &\approx E_\theta[s(\theta)] - \mathcal{I}(\theta)(\tilde{\theta} - \theta) \\
&\approx - \mathcal{I}(\theta)(\tilde{\theta} - \theta)
\end{aligned}
$$

Outline

Asymptotic Theory

- iid sample, $y = (y_1, \ldots, y_n)$. Truth: $\theta = \theta_0$.
- Correct is: $\ell, s, \mathcal{I}$ depend on entire sample. For example, $\ell(\theta \mid y) = \ell(\theta \mid (y_1, \ldots, y_n))$.
- Now: slight abuse of notation. $\ell, s, \mathcal{I}$ for one obs., e.g. $\ell(\theta \mid y_j)$.
- Let

$$
\begin{aligned}
\ell_n(\theta) &= \tfrac{1}{n} \textstyle\sum_{j=1}^{n} \ell(\theta \mid y_j) &&= \frac{1}{n} \ell(\theta \mid y_1, \ldots, y_n) \\
s_n(\theta) = \frac{\partial \ell_n(\theta)}{\partial \theta} &= \tfrac{1}{n} \textstyle\sum_{j=1}^{n} \frac{\partial \ell(\theta|y_j)}{\partial \theta} &&\xrightarrow{P} E_{\theta_0}[s(\theta)] = 0 \text{ at } \theta = \theta_0 \\
H_n(\theta) = \frac{\partial^2 \ell_n(\theta)}{\partial \theta \partial \theta'} &= \tfrac{1}{n} \textstyle\sum_{j=1}^{n} \frac{\partial^2 \ell(\theta|y_j)}{\partial \theta \partial \theta'} &&\xrightarrow{P} -\mathcal{I}(\theta_0) \text{ at } \theta = \theta_0
\end{aligned}
$$

- Central limit theorem:

$$
\sqrt{n}\, s_n(\theta_0) \xrightarrow{d} \mathcal{N}(0, E[s(\theta_0)s(\theta_0)']) = \mathcal{N}(0, \mathcal{I}(\theta_0))
$$

Asymptotic Normality for the MLE: Delta method

- The MLE $\hat{\theta}_n$ solves $s_n(\hat{\theta}_n) = 0$.
- First-order expansion around θ_0:

$$0 = s_n(\hat{\theta}_n) \approx s_n(\theta_0) + H_n(\theta_0)(\hat{\theta}_n - \theta_0)$$

- Assume $\mathcal{I}(\theta)$ is invertible (hence: positive definite).

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \approx -\sqrt{n}\,\mathcal{I}(\theta_0)^{-1} H_n(\theta_0)(\hat{\theta}_n - \theta_0) \approx \sqrt{n}\,\mathcal{I}(\theta_0)^{-1} s_n(\theta_0)$$

- Take the limit.

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}\left(0, \mathcal{I}(\theta_0)^{-1}\mathcal{I}(\theta_0)\,\mathcal{I}(\theta_0)^{-1}\right) = \mathcal{N}\left(0, \mathcal{I}(\theta_0)^{-1}\right)$$

Asymptotic Normality for the MLE

Theorem

If $\mathcal{I}(\theta)$ is invertible at the true θ, then

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}\left(0, \mathcal{I}(\theta)^{-1}\right)$$

The inverse of the information matrix is the asymptotic variance of the MLE

Feasible estimation of $\mathcal{I}(\theta)$

- Recall:

$$\mathcal{I}(\theta) = E\left[s(\theta \mid y)s(\theta \mid y)'\right] = -E\left[\frac{\partial^2 \ell(\theta \mid y)}{\partial\theta\, \partial\theta'}\right]$$

  Typically, not known and needs to be estimated.

- As average of score products

$$\hat{\mathcal{I}}_n^{(1)}(\theta_n) = \frac{1}{n}\sum_{j=1}^{n} s(\theta_n \mid y_j)s(\theta_n \mid y_j)'$$

- As average of second derivatives (see $H_n(\theta)$):

$$\hat{\mathcal{I}}_n^{(2)}(\theta_n) = -\frac{1}{n}\sum_{j=1}^{n} \frac{\partial^2 \ell(\theta_n \mid y_j)}{\partial\theta_n\, \partial\theta_n'}$$

- If $\theta_n \xrightarrow{P} \theta$, then $\hat{\mathcal{I}}_n^{(j)}(\theta_n) \xrightarrow{P} \mathcal{I}(\theta)$ is a consistent estimator.
- Often: $\theta_n = \hat{\theta}_n$.

Example 1: linear regression

$y_i \in \mathbb{R}$, $X_i \in \mathbb{R}^{1\times k}$, $\theta = [\beta, \sigma^2]$, $\beta \in \mathbb{R}^k, \sigma^2 > 0$,

$$y_i = X_i\beta + \epsilon_i,\ \epsilon_i \sim \mathcal{N}(0, \sigma^2),\ i = 1, \ldots, n \text{ iid}$$

Let X be $n \times k$, with X_i as i-th row. Similarly, let y be $n \times 1$.

$$\ell_n(\theta \mid y, X) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{1}{2n\sigma^2}\sum_{j=1}^{n}(y_j - X_j\beta)^2$$

$$s_n(\theta \mid y, X) = \begin{bmatrix} \frac{1}{n\sigma^2}(X'y - X'X\beta) \\ -\frac{1}{2\sigma^2} + \frac{1}{2n\sigma^4}\sum_{j=1}^{n}(y_j - X_j\beta)^2 \end{bmatrix}$$

$$\hat{\theta}_n = \begin{bmatrix} \hat{\beta} \\ \widehat{\sigma^2} \end{bmatrix} = \begin{bmatrix} (X'X)^{-1}X'y \\ \frac{1}{n}\sum_{j=1}^{n}(y_j - X_j\hat{\beta})^2 \end{bmatrix}$$

$$-E\left[H_n(\theta) \mid X\right] = \begin{bmatrix} \frac{X'X}{n\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix} = \begin{bmatrix} \sigma^2\left(\frac{X'X}{n}\right)^{-1} & 0 \\ 0 & 2\sigma^4 \end{bmatrix}^{-1} = \mathcal{I}(\theta \mid X)$$

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\end{aligned}$$

Example 2: binomial distribution

$y_i \in \{0,1\}, i = 1, \ldots, n$, with $P(y_i = 1 \mid \theta) = \theta$ iid. Observe: k "ones".

$$
\begin{aligned}
L(\theta) &= \theta^k (1-\theta)^{n-k} \\
\ell_n(\theta) &= \frac{1}{n} \sum_{i=1}^{n} \left(y_i \ln \theta + (1-y_i) \ln(1-\theta) \right) \\
&= \frac{k}{n} \ln \theta + \frac{n-k}{n} \ln(1-\theta) \\
s_n(\theta) &= \frac{k}{n} \frac{1}{\theta} - \frac{n-k}{n} \frac{1}{1-\theta} = \frac{\frac{k}{n} - \theta}{\theta(1-\theta)} \\
\hat{\theta}_n &= \frac{k}{n} \\
H_n(\theta) &= -\frac{k}{n} \frac{1}{\theta^2} - \frac{n-k}{n} \frac{1}{(1-\theta)^2} \\
-E[H_n(\theta)] &= \frac{1}{\theta} + \frac{1}{1-\theta} = (\theta(1-\theta))^{-1} = \mathcal{I}(\theta)
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-E\left[H_n(\theta)\right] &= \frac{1}{\theta} + \frac{1}{1-\theta} = (\theta(1-\theta))^{-1} = \mathcal{I}(\theta)
\end{aligned}
$$

Example 3: binary choice.

Data $y_i \in \{0,1\}$, X_i. $P(y_i = 1 \mid X_i) = G(X\beta)$ iid. Density $g(v) = G'(v)$. $\theta = \beta$. Abbreviate $G_i = G(X_i\beta),\ g_i = g(X_i\beta)$. Define

Inverse Mills ratio or hazard: $\lambda(v) = \dfrac{g(v)}{1 - G(v)}$ (think: "$= \dfrac{P(V = v)}{P(V \geq v)}$")

$$
\begin{aligned}
\ell(\beta \mid y_i, X_i) &= y_i \ln G(X_i\beta) + (1 - y_i) \ln(1 - G(X_i\beta)) \\
&= y_i \ln G_i + (1 - y_i) \ln(1 - G(X_i\beta)) \\
s(\beta \mid X_i) &= \frac{y_i g_i X_i'}{G_i} - \frac{(1 - y_i) g_i X_i'}{1 - G_i} \\
&= \frac{(y_i - G_i) g_i X_i'}{G_i(1 - G_i)} \\
-E[H(\beta) \mid X_i] &= \frac{g_i^2 X_i' X_i}{G_i(1 - G_i)} = \left(\frac{G_i(1 - G_i)}{g_i^2 X_i' X_i} \right)^{-1} = \mathcal{I}(\beta \mid X_i) \\
&= \lambda(X_i\beta)\,\lambda(-X_i\beta)\,X_i' X_i \text{ ,if symmetry } g(v) = g(-v)
\end{aligned}
$$

Outline

Information Inequality

A statistic is a function $T : y \in Y \to \mathbb{R}^k$.

Theorem

Suppose $Var_\theta(T(y)) < \infty$ and $\mathcal{I}(\theta)$ is invertible. Let $\psi(\theta) = E_\theta[T(y)]$. Then,

$$Var_\theta(T(y)) \geq \left(\frac{\partial \psi(\theta)}{\partial \theta}\right) \mathcal{I}(\theta)^{-1} \left(\frac{\partial \psi(\theta)}{\partial \theta}\right)'$$

In particular:

Cramér-Rao lower bound:
Suppose that $E_\theta[T(y)] = \theta$,
i.e. $T(y)$ is an unbiased estimator of θ. Then
$Var_\theta(T(y)) \geq \mathcal{I}(\theta)^{-1}$

In words: MLE's are asymptotically efficient: they have minimal asymptotic variance among all asymptotically unbiased estimators.

Proof of the Information Inequality (scalar case)

Recall that

$$0 = E[s(\theta \mid y)] = \int \frac{\partial \ell(\theta \mid y)}{\partial \theta} f(y \mid \theta) \nu(dy) \tag{1}$$

Therefore

$$\begin{aligned}
\psi(\theta) &= \int T(y) f(y \mid \theta) \nu(dy) \\
\mid \psi'(\theta) \mid &= \mid \int T(y) \frac{\frac{\partial f(y \mid \theta)}{\partial \theta}}{f(y \mid \theta)} f(y \mid \theta) \nu(dy) \mid \\
per(1): &= \mid \int (T(y) - E_\theta[T(y)]) (s(\theta \mid y) - E[s(\theta \mid y)]) f(y \mid \theta) \nu(dy) \mid \\
&= \mid (Cov_\theta(T(y), s(\theta \mid y)) \mid \\
&\leq \sqrt{\text{Var}_\theta(T(y)) \, \text{Var}_\theta(s(\theta \mid y))}
\end{aligned}$$

or

$$\mid \psi'(\theta) \mid^2 \leq \text{Var}_\theta(T(y)) \mathcal{I}(\theta)$$

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Outline

Identification

- With enough data, will there eventually be a unique θ or will several θ work as well, even asymptotically? Or: is θ identified?
- Definition: θ is identified, if there is no other $\tilde{\theta}$ with $L(\theta \mid y) = L(\tilde{\theta} \mid y)$ for all y.
- Example for lack of identification:
  - Suppose, some function $\vartheta : x \in (a, b) \rightarrow \theta = \vartheta(x)$ results in constant log-likelihood $\ell(\vartheta(x) \mid y)$, for all y.
  - ... therefore, $0 = s(\vartheta(x) \mid y)v(x)$, where $v(x) = \frac{d\vartheta(x)}{dx}$.
  - ... therefore

$$0 = v(x)' \frac{\partial s(\vartheta(x) \mid y)}{\partial \theta'} v(x) + s(\vartheta(x) \mid y) \frac{d^2 \vartheta(x)}{dx^2}$$

  - Take expectations:

$$0 = v(x)' \mathcal{I}(\vartheta(x)) v(x)$$

Lack of identification

If $\mathcal{I}(\theta)$ is not invertible,
θ may not be identified.

Outline

A framework for testing

- Hypothesis: $\theta \in \Theta_0$. Alternative: $\theta \in \Theta_1$.
- Observations y.
- Test: A decision $\delta(y) \in 0, 1$. $\delta(y) = 0$: "accept". $\delta(y) = 1$: "reject".
- Power function: $\beta(\theta, \delta) = P_\theta(\delta(y) = 1)$. Probability of rejection, if θ is true.
- Error of Type I: reject, even though $\theta \in \Theta_0$.
- Size of the test: $\sup_{\theta \in \Theta_0} \beta(\theta, \delta)$. Maximal probability of rejecting true hypothesis, i.e. max. prob. of type-I error. "5 % significance".
- Error of Type II: accept, even though $\theta \in \Theta_1$.

Likelihood-ratio test: a special case

- Hypothesis $\theta = \theta_0$. Alternative: $\theta = \theta_1$.
- Likelihood ratio:

$$LR^*(\theta_1, \theta_0 \mid y) = \frac{L(\theta_1 \mid y)}{L(\theta_0 \mid y)}$$

- Reject, if likelihood ratio exceeds some threshold Ψ,

$$\delta_\Psi^{(LR)}(y) = \begin{cases} 1 & \text{if } LR^*(\theta_1, \theta_0 \mid y) \geq \Psi \\ 0 & \text{if } LR^*(\theta_1, \theta_0 \mid y) < \Psi \end{cases}$$

The Neyman-Pearson-Lemma

Theorem

Consider testing $\theta = \theta_0$ vs $\theta = \theta_1$. Given Ψ, let $\delta(y)$ be some test with size equal or smaller than the size of the test $\delta_\Psi^{(LR)}$. Then, the power at the alternative is smaller too,

$$\beta(\delta, \theta_0) \leq \beta(\delta_\Psi^{(LR)}, \theta_0) \textit{ implies } \beta(\delta, \theta_1) \leq \beta(\delta_\Psi^{(LR)}, \theta_1)$$

In words: test δ that are as least as "careful" in avoiding type-I errors as a likelihood-ratio test will make at least as many type-II errors as a likelihood-ratio test.

Likelihood ratio tests are nice!

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Likelihood ratio tests: general case

- Hypothesis $\theta \in \Theta_0$. Alternative: $\theta \in \Theta_1$.
- Likelihood ratio: typically written in terms of 2*log-likelihood

$$LR(\Theta_1, \Theta_0 \mid y) = 2(\sup_{\theta_1 \in \Theta_1} \ell(\theta_1 \mid y) - \sup_{\theta_0 \in \Theta_0} \ell(\theta_0 \mid y)) = 2\log(LR^*(\Theta_1, \Theta_0 \mid y))$$

- Reject, if likelihood ratio exceeds some threshold $\psi = 2\log(\Psi)$,

$$\delta_\Psi^{(LR)}(y) = \begin{cases} 1 & \text{if } LR(\Theta_1, \Theta_0 \mid y) \geq \psi \\ 0 & \text{if } LR(\Theta_1, \Theta_0 \mid y) < \psi \end{cases}$$

- Suppose $\sup L = \max L$ on Θ_0. Suppose $\sup_{\theta_1 \in \Theta_1} L = \max_{\theta \in \Theta} L$. Let $\hat{\theta}_c$ be the constrained MLE on Θ_0. Let $\hat{\theta}$ be unconstrained MLE. Then

$$\begin{aligned} LR(\Theta_1, \Theta_0 \mid y) &= 2(\ell(\hat{\theta} \mid y) - \ell(\hat{\theta}_c \mid y)) \\ &= 2\ln\left(\frac{L(\hat{\theta} \mid y)}{L(\hat{\theta}_c \mid y)}\right) \end{aligned}$$

Outline

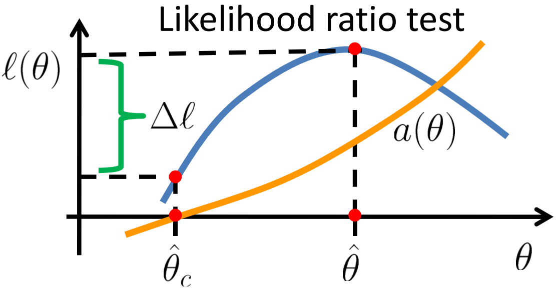
Testing a parametric constraint

- $\theta \in \mathbb{R}^m$ (or open subset)
- Constraint $a : \mathbb{R}^m \to \mathbb{R}^k$, differentiable. Derivative: rank k.
- Or: $a : \mathbb{R}^m \to \mathbb{R}^l$, differentiable, $l > k$, but derivative has rank k.
- Hypothesis: k constraints hold,

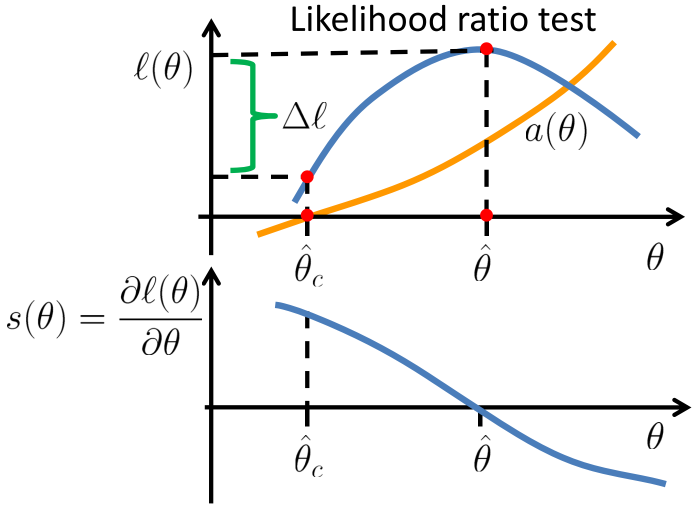
$$H_0 : a(\theta) = 0$$

- $\hat{\theta}$ or $\hat{\theta}_n$: unconstrained MLE.
- $\hat{\theta}_c$ or $\hat{\theta}_{c,n}$: constrained MLE, satisfies $a(\hat{\theta}_c) = 0$.
- Some material: Hayashi, Econometrics, Princeton Univ. Press (2000).

LR, Score and Wald



LR, Score and Wald



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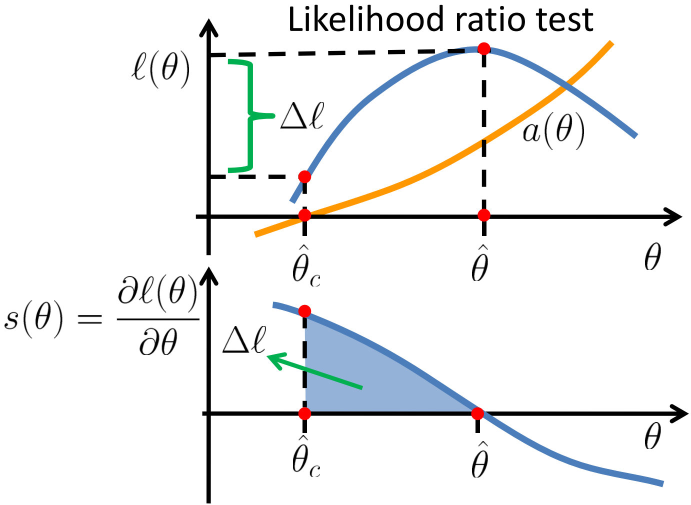

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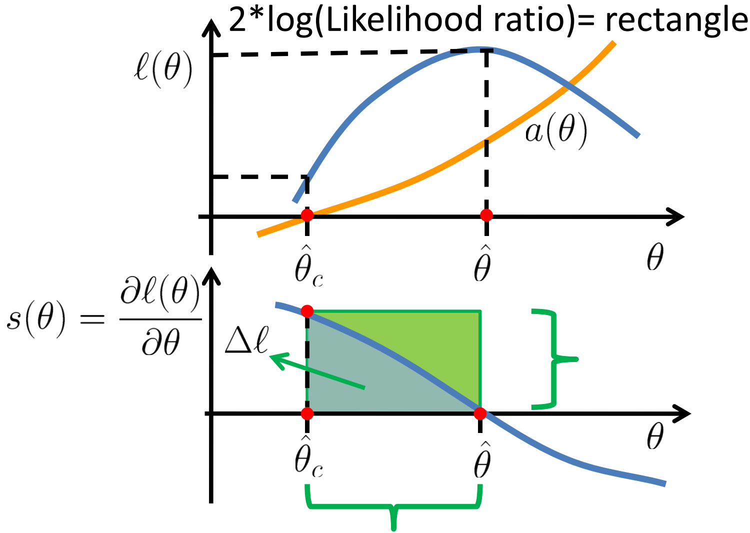

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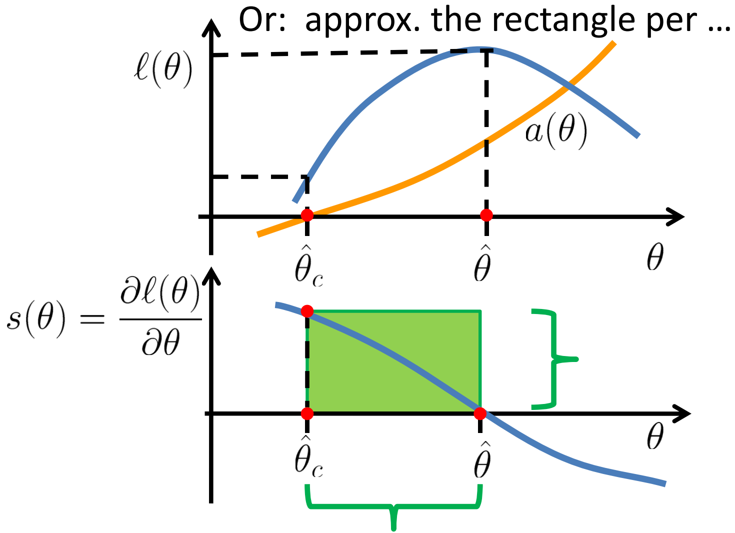

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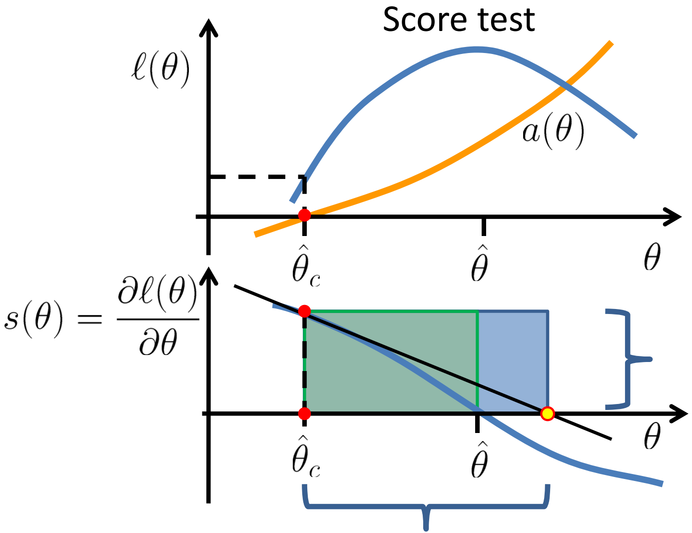
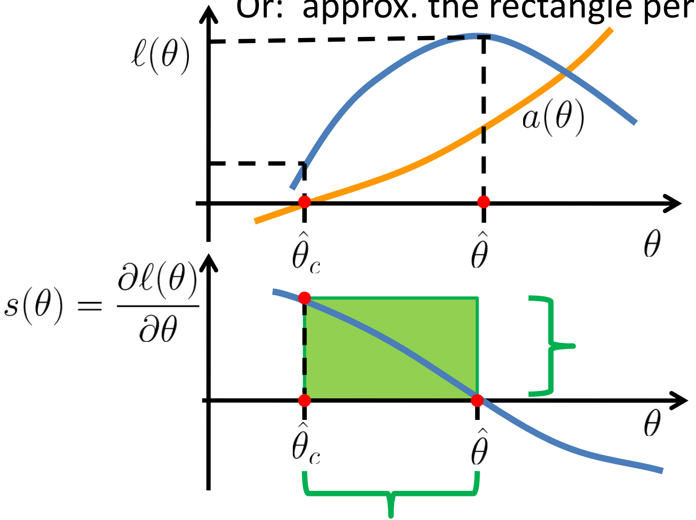


LR, Score and Wald

Or: approx. the rectangle per …



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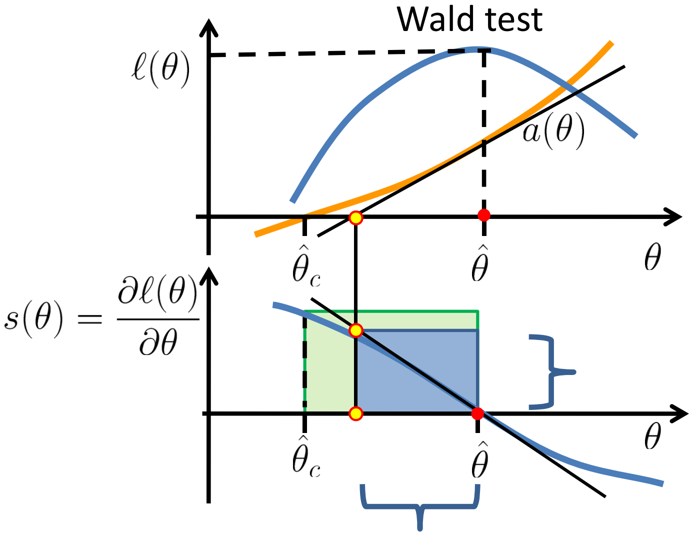

Three tests:

$$\textbf{rectangle} = s(\hat{\theta}_c)'(\hat{\theta} - \hat{\theta}_c)$$

1. Likelihood-ratio test:

$$\textbf{rectangle} \approx LR = 2 * (\ell(\hat{\theta}) - \ell(\hat{\theta}_c))$$

2. Score test or Lagrange multiplier test or Rao test:

$$\begin{aligned} \textbf{rectangle} &\approx s(\hat{\theta}_c)'\mathcal{I}(\hat{\theta}_c)^{-1}s(\hat{\theta}_c) \\ \text{per: } -s(\hat{\theta}_c) &= s(\hat{\theta}) - s(\hat{\theta}_c) \approx -\mathcal{I}(\hat{\theta}_c)(\hat{\theta} - \hat{\theta}_c) \end{aligned}$$

3. Wald test: [Remark: invertibility of $\partial a / \partial \theta$?]

$$\begin{aligned} \textbf{rectangle} &\approx a(\hat{\theta})' \left(\frac{\partial a(\hat{\theta})}{\partial \theta} \mathcal{I}(\hat{\theta})^{-1} \left(\frac{\partial a(\hat{\theta})}{\partial \theta} \right)' \right)^{-1} a(\hat{\theta}) \\ \text{per: } -a(\hat{\theta}) &= a(\hat{\theta}_c) - a(\hat{\theta}) \approx \frac{\partial a(\hat{\theta})}{\partial \theta}(\hat{\theta}_c - \hat{\theta}) \\ s(\hat{\theta}_c) &= s(\hat{\theta}_c) - s(\hat{\theta}) \approx -\mathcal{I}(\hat{\theta})(\hat{\theta}_c - \hat{\theta}) \end{aligned}$$

3. Wald statistic. Asymptotics.

- Truth: θ_0. Under H_0: $a(\theta_0) = 0$.
- Define

$$A(\theta) = \frac{\partial a(\theta)}{\partial \theta}, \ \ A = A(\theta_0), \ \ \mathcal{I} = \mathcal{I}(\theta_0)$$

- MLE asymptotics:

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}\left(0, \mathcal{I}^{-1}\right)$$

- Delta method:

$$\sqrt{n}(a(\hat{\theta}_n) - a(\theta_0)) \xrightarrow{d} \mathcal{N}\left(0, A\,\mathcal{I}^{-1} A'\right)$$

- Wald statistic:

$$W = n\, a(\hat{\theta}_n)' \left(A(\hat{\theta}_n)\,\hat{\mathcal{I}}_n^{-1} A(\hat{\theta}_n)'\right)^{-1} a(\hat{\theta}_n) \xrightarrow{d} \chi^2_k$$

2. Score test or Lagrange multiplier test. Asymptotics.

- Truth: θ_0. Maximize $\ell_n(\theta)$ s.t. $a(\theta) = 0$. Lagrangian. FONC:

$$s_n(\hat{\theta}_{c,n}) - A(\hat{\theta}_{c,n})' \lambda_n = 0$$

- Solve for $\lambda_n \in \mathbb{R}^k$: pre-multiply with some $B_n \xrightarrow{P} B$, size $k \times m$:

$$\begin{aligned} \sqrt{n}\,\lambda_n &= \sqrt{n}\left(B_n A(\hat{\theta}_{c,n})'\right)^{-1} B_n\, s_n(\hat{\theta}_{c,n}) \\ &\xrightarrow{d} \mathcal{N}\left(0, (B\,A')^{-1} B\,\mathcal{I}\,B'(A\,B')^{-1}\right) \end{aligned}$$

- A clever choice: $B = A\,\mathcal{I}^{-1}$ and $B_n = A(\hat{\theta}_{c,n})\hat{\mathcal{I}}_n^{-1}$. Then

$$\sqrt{n}\,\lambda_n \xrightarrow{d} \mathcal{N}\left(0, \left(A\,\mathcal{I}^{-1} A'\right)^{-1}\right)$$

- Lagrange multiplier statistic: [Remark: not $s_n' \hat{\mathcal{I}}_n^{-1} s_n \xrightarrow{d} \chi^2_m$]

$$LM = n\, s_n(\hat{\theta}_{c,n})' \,\hat{\mathcal{I}}_n^{-1}\, s_n(\hat{\theta}_{c,n}) = n\,\lambda_n' A(\hat{\theta}_{c,n})\,\hat{\mathcal{I}}_n^{-1} A(\hat{\theta}_{c,n})'\,\lambda_n \xrightarrow{d} \chi^2_k$$

2. Score test or Lagrange multiplier test. Asymptotics.

- Truth: θ_0. Maximize $\ell_n(\theta)$ s.t. $a(\theta) = 0$. Lagrangian. FONC:

$$s_n(\hat{\theta}_{c,n}) - A(\hat{\theta}_{c,n})'\lambda_n = 0$$

- Solve for $\lambda_n \in \mathbb{R}^k$: pre-multiply with some $B_n \xrightarrow{P} B$, size $k \times m$:

$$\begin{aligned} \sqrt{n}\,\lambda_n &= \sqrt{n}\left(B_n A(\hat{\theta}_{c,n})'\right)^{-1} B_n\, s_n(\hat{\theta}_{c,n}) \\ &\xrightarrow{d} \mathcal{N}\left(0, (B\,A')^{-1} B\,\mathcal{I}\,B'(A\,B')^{-1}\right) \end{aligned}$$

- A clever choice: $B = A\,\mathcal{I}^{-1}$ and $B_n = A(\hat{\theta}_{c,n})\hat{\mathcal{I}}_n^{-1}$. Then

$$\sqrt{n}\,\lambda_n \xrightarrow{d} \mathcal{N}\left(0, \left(A\,\mathcal{I}^{-1}A'\right)^{-1}\right)$$

- Lagrange multiplier statistic: [Remark: not $s_n'\hat{\mathcal{I}}_n^{-1}s_n \xrightarrow{d} \chi^2_m$]

$$LM = n\,s_n(\hat{\theta}_{c,n})'\,\hat{\mathcal{I}}_n^{-1}\,s_n(\hat{\theta}_{c,n}) = n\,\lambda_n'\,A(\hat{\theta}_{c,n})\,\hat{\mathcal{I}}_n^{-1}\,A(\hat{\theta}_{c,n})'\,\lambda_n \xrightarrow{d} \chi^2_k$$

1. Likelihood ratio test. Asymptotics.

- Truth: θ_0. Second-order approximation around $\hat{\theta}_n$:

$$\begin{aligned} LR &= 2n\,(\ell_n(\hat{\theta}_n) - \ell_n(\hat{\theta}_{c,n})) \\ &= n\,(\hat{\theta}_n - \hat{\theta}_{c,n})'\,\hat{\mathcal{I}}_n^{(2)}\,(\hat{\theta}_n - \hat{\theta}_{c,n}) + o_P \end{aligned}$$

- From calculation on score statistic plus first-order approx. of score:

$$A(\hat{\theta}_{c,n})'\,\lambda_n = s_n(\hat{\theta}_{c,n}) = \hat{\mathcal{I}}_n^{(2)}(\hat{\theta}_n - \hat{\theta}_{c,n}) + \frac{o_P}{n}$$

  Solve for $(\hat{\theta}_n - \hat{\theta}_{c,n})$.

- Therefore

$$LR = n\,\lambda_n' A(\hat{\theta}_{c,n}) \left(\hat{\mathcal{I}}_n^{(2)}\right)^{-1} A(\hat{\theta}_{c,n})'\,\lambda_n + o_P \xrightarrow{d} \chi_k^2$$

  per result about Lagrange multiplier statistic.