Theory of Income I: Euler Equation and TVC

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[*] All the mistakes in here are mine, and probably Tak's because I asked him to read this document. If you find something that sounds wrong let my know.
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[†] Based on Theory of Income I lecture note number 5 by Fernando Alvarez.

1 Introduction

In this note we are going to do a brief recap of the elements of dynamic programming and some of the propositions that Fernando discussed during class. The main focus of the note is to work with the examples at the end of Lecture 5, Neoclassical Growth Model (NGM) with homogeneous functions and adjustment cost models.

The note is organize as follow, section 2 is a review of what you've seen with Fernando (feel free to skip that part). Section 3 discuss examples of Dynamic programming problems where we made specific assumptions on the utility/production function (this are some of the exercises in Fernando's slides). Section 4 discuss a particular case of our sequential model in which we impose restrictive assumptions on the period return function. Finally in section 5 we discuss models with adjustment cost of insvestment, models where investment decision are carried by the firm and in which investing in new capital units involve an extra cost in the margin.

2 Recap

The elements of a Dynamic Programming problem are$[X, \Gamma, F, \beta]$:

- X is the set of states x.
- $\Gamma : X \to X$ is the feasibility correspondence, i.e. given $x \in X$, $\Gamma(x)$ is the set of feasible values for the state variable next period.
- $F(x, y)$ is the period return function defined as $F : Gr(\Gamma) \to \mathbb{R}$, where

$$Gr(\Gamma) = \{(x, y) : x \in X, y \in \Gamma(x)\}$$

- $\beta \in (0, 1)$ is a discount factor

Keep this formulation in mind, next quarter in Theory of Income II it will be the starting point when we try to get the Bellman representation of the problem. For now we'll work with the sequence problem given by (1)

$$V^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{1}$$

$$s.t. \quad x_{t+1} \in \Gamma(x_t), \quad \forall t \geq 0$$

$$x_0 \quad given$$

The main goal here is to characterize the dynamics of our state variable x_t, its evolution in time and its limit distribution. Here we focus only on non-stochastic problems so the limiting distribution is just the steady state. However we can consider extensions of the model that incorporates uncertainty in future periods. In this case our "steady state" is the stationary distribution (if it exists) of the state variable x_t.

Why we care about the state variable only? We actually don't! We also care about the dynamics of other variables in the model, those that we referred to as *control variables* (for example consumption in NGM). This variables, their policy functions are functions of the state. So once we understood the dynamics of the state, given the relationship between the controls and the state, we can back up their dynamics as well.

Remark 1. Fernando present during class an alternative formulation for this type of problem that distinguish between control and state variables. Of course the solution to each formulation is the same, which one we should use then depends entirely on the problem we are trying to solve.

Example 1. Neoclassical Growth model: Familiar notation to Sequence problem

Representative consumer model. Preferences over consumption stream, utility is time separable. Production technology is given by a neoclassical production function (meaning CRS plus Inada conditions). The planner for this economy solves

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{2}$$

$$\begin{aligned} s.t. \quad & c_t + I_t = G\left(k_t, 1\right), \qquad \forall t \\ & k_{t+1} = I_t + \left(1-\delta\right) k_t \\ & c_t \geq 0 \\ & k_{t+1} \geq 0, \qquad k_0 \quad given \end{aligned}$$

where the first constraint is feasibility in the good markets, and the second constraint is the production function for capital or law of motion of capital.

We want to express this problem in terms of the elements of DP described above. We start by substituting the second constraint into the first one, and writing

$$\begin{aligned} k_{t+1} &= G(k_t, 1) + (1-\delta) k_t - c_t, \\ c_t &\geq 0 \\ k_{t+1} &\geq 0 \end{aligned} \tag{3}$$

and,

$$c_t = G(k_t, 1) + (1-\delta) k_t - k_{t+1} \tag{4}$$

With equations (2)-(4) we can define $[X, \Gamma, F, \beta]$. We start by noticing that $X = \mathbb{R}_+$. To construct the feasibility correspondence we use (3). Given a level of capital at time t, k_t, the lowest value that tomorrow capital k_{t+1}can take is given by the constraint $k_{t+1} \geq 0$. On the other hand, the highest value that k_{t+1} could take is given by feasibility in the goods market. Note that this value depends on consumption at time t, however we can drive consumption to zero in order to increase our capital tomorrow. So given k_t we have,

$$\Gamma(k_t) = [0, G(k_t, 1) + (1-\delta) k_t] \tag{5}$$

To get the period return function, we combine the period utility with equation (4),

$$F(k_t, k_{t+1}) = u(G(k_t, 1) + (1-\delta) k_t - k_{t+1}) \tag{6}$$

Finally the discount factor in this case is trivially β.

2.1 Characterizing the solution

In what follows we assume the following on $[X, \Gamma, F, \beta]$.

Assumption 1. *We assume that $X \in \mathbb{R}^m$, $F \in \mathbb{C}^1$ and $\beta \in (0,1)$.*

We say that the sequence $\{x_{t+1}\}_{t=0}^{\infty}$ satisfies the *Euler Equation (EE)* if

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0, \quad \forall t \geq 0 \tag{7}$$

Furthermore, we say that the sequence $\{x_{t+1}\}_{t=0}^{\infty}$ satisfies the *Transversality Condition (TVC)* if

$$\lim_{t\to\infty} \beta^t F_x(x_t, x_{t+1}) \cdot x_t = 0 \tag{8}$$

Proposition 1. *If the period return function $F(\cdot,\cdot)$ is increasing and concave, a sequence $\{x_{t+1}\}$ satisfying the EE and TVC is an optimal sequence.*

Proof. See Lecture notes. □

Proposition 1 tell us that we only need to worry about sequence of states that satisfy EE and TVC in order to understand its dynamics. We also use the EE to understand its limiting value.

Definition 1. Let $\bar{x}$ be a steady state, i.e a solution to

$$F_y(\bar{x},\bar{x}) + \beta F_x(\bar{x},\bar{x}) = 0 \tag{9}$$

2.2 Uniqueness of the solution

We are not going to prove this here (you will spend most of your winter quarter working on this) but the solution of the dynamic problem is unique under certain assumptions of X, Γ, and F.

Proposition 2. *Let X be a convex set, F be a continuous and concave function on both arguments (x, y) and Γ be a convex correspondence describing a compact set for y. Then the solution to our DP exist and it is unique.*

Intuitively, the compactness of Γ and continuity of F ensures the existence of a solution to the problem. The concavity of F plus the convexity of Γ give us the uniqueness of that solution.

3 Exercises

3.1 Exercise 1: Transversality

Consider a problem with

$$F(x,y) = U(w + x(1+r) - y) \tag{10}$$

$$(1+r)\beta = 1$$

This is a saving problem with a constant income w and an interest rate r.

1. Is the solution to this problem unique?

Under regular assumptions on the utility function, increasing, concave plus Inada conditions. We'll we have that the solution is unique, since the feasibility constraint $\Gamma(x) = [0, w + x(1+r)]$ is compact and convex.

2. How many steady states does this problem has?

To characterize the steady states firs we need to compute the EE. Using equation (7) and (10) we have,

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0$$
$$-U'(w + (1+r)x_t - x_{t+1}) + \underbrace{(1+r)\beta}_{=1} U'(w + (1+r)x_{t+1} - x_{t+2}) = 0$$

Hence,

$$U'(w + (1+r)x_t - x_{t+1}) = U'(w + (1+r)x_{t+1} - x_{t+2})$$
$$U'(c_t) = U'(c_{t+1})$$
$$c_t = c_{t+1}, \quad \forall t \tag{11}$$

or

$$x_{t+2} - x_{t+1} = (1+r)[x_{t+1} - x_t] \tag{12}$$

Define $z_{t+1} = x_{t+1} - x_t$. Then,

$$z_{t+2} = (1+r)z_{t+1} \tag{13}$$

In steady state, $x_t = x_{t+1} = \bar{x}$, so $\bar{z} = 0$ which satisfies equation (13). Note that this is the only solution to that equation, any other value will generate a divergent sequence for z_t since $(1+r) > 1$. Furthermore, it has to be the case that $z_1 = \bar{z}$ otherwise we will never get to the point $\bar{z} = 0$. So in $t = 1$ we'll have,

$$z_1 = 0 \implies x_1 = x_0 = \bar{x} \tag{14}$$

3. Verify that the sequence $c_t^* = w + rx_t$, $x_{t+1}^* = x_t^* = x_0$ is the solution to the problem.

We argue before that at any time t we must have $z_t = 0$, so $x_t = x_{t-1}$ for any t. We have then,

$$x_t^* = x_{t+1}^* = x_0 \tag{15}$$

Replacing (15) into the equation describing consumption yields,

$$c_t^* = w + (1+r)x_t - x_{t+1} = w + rx_t \tag{16}$$

So the solution satisfies the EE. We can also check that it satisfies TVC,

$$\lim_{t\to\infty}\beta^t U'\left(w+(1+r)x_t-x_{t+1}\right)x_t = U'\left(w+rx_0\right)x_0 \underbrace{\lim_{t\to\infty}\beta^t}_{=0,\beta\in(0,1)} = 0$$

4. Consider and alternative policy $\tilde{c}_t = \tilde{c}_0 < c_0^* = c_t^*$ for all t. Can this policy be optimal? Does the path satisfies EE? Compute the implied sequence of x_t for this policy. Does the implied path satisfies TVC?

The proposed sequence cannot be a solution. We argued at the beginning of the exercise that the solution is unique and the proposed solution differ from the one we found. However, if we look at equation (11) we can immediately see that the sequence $\{\tilde{c}_t\}$ does satisfies the EE, since it implies $\tilde{c}_t = \tilde{c}_{t+1}$ for all t. It must be the case then, that the implied sequence of x_t violates TVC.

To see this we start computing the implied x_t. From the period constraint of the individual, evaluated at $\tilde{c}_t = \tilde{c}_0$ we have the following difference equation,

$$\tilde{x}_{t+1} = (w-\tilde{c}_0) + (1+r)\tilde{x}_t, \quad \tilde{x_0} = x_0 \tag{17}$$

Using equation (17) at $t+1$ and $t+2$ we have

$$\begin{aligned}\tilde{x}_{t+1} - (1+r)\tilde{x}_t &= \tilde{x}_{t+2} - (1+r)\tilde{x}_{t+1} \\ \tilde{z}_{t+1} &= (1+r)\tilde{z}_t \end{aligned} \tag{18}$$

where $\tilde{z}_t \equiv \tilde{x}_{t+1} - \tilde{x}_t$. So we ended up with the following difference equation,

$$\begin{aligned}\tilde{z}_{t+1} &= (1+r)\tilde{z}_t \\ \tilde{z}_0 &= \tilde{x}_1 - x_0 > 0\end{aligned}$$

where the second equation follows from (17) at $t=0$ and the assumption on the consumption path[1].The solution to this difference equation is given by,

$$\tilde{z}_t = \tilde{z}_0\left(1+r\right)^t$$

[1]Note that the optimal solution has $x_1 = x_0$. Then,

$$\tilde{x}_1 = w + (1+r)x_0 - \tilde{c}_0 > w + (1+r)x_0 - c_0 = x_1$$

Hence,

$$\tilde{x}_1 - x_0 > 0$$

or

$$
\begin{aligned}
\tilde{x}_{t+1} &= (w + r x_0 - \tilde{c}_0)(1+r)^t + \tilde{x}_t \\
&= (c_0^* - \tilde{c}_0)(1+r)^t + \tilde{x}_t
\end{aligned}
$$

Note that savings goes to infinity as time goes to infinity.

We can check that the TVC does not hold,

$$
\begin{aligned}
\lim_{t\to\infty} \beta^t U'(\tilde{c}_t)\,\tilde{x}_t &= U'(\tilde{c}_0) \lim_{t\to\infty} \beta^t \tilde{x}_t \\
&= U'(\tilde{c}_0) \lim_{t\to\infty} \beta^t \left[(c_0^* - \tilde{c}_0)(1+r)^{t-1} + \tilde{x}_{t-1} \right] \\
&= U'(\tilde{c}_0) \lim_{t\to\infty} (c_0^* - \tilde{c}_0) \frac{[\beta(1+r)]^t}{1+r} + \beta^t \tilde{x}_{t-1} \\
&= U'(\tilde{c}_0) \lim_{t\to\infty} (c_0^* - \tilde{c}_0) \frac{1}{1+r} + \beta^t \tilde{x}_{t-1} \\
&= \underbrace{U'(\tilde{c}_0)(c_0^* - \tilde{c}_0)\frac{1}{1+r}}_{>0} + \underbrace{U'(\tilde{c}_0) \lim_{t\to\infty} \beta^t \tilde{x}_{t-1}}_{\geq 0} \\
&> 0
\end{aligned}
$$

3.2 Exercise 2: Linear utility in the neoclassical growth model.

Let $U(c) = c$ and $f(k) = G(k,1) + (1-\delta)k$ where G is a neoclassical production function: strictly increasing and strictly concave in k, satisfying Inada conditions.

1. Show that, as long as k_0 is such that $f(k_0) - \bar{k} \geq 0$ for $\beta f'(\bar{k}) = 1$, then capital converges to steady state $\bar{k}$ in one period, i.e. $\bar{k} = g(x_t)$ where g denotes the optimal policy. Hint: use the sufficiency of EE and TVC).

We start by characterizing the steady state $\bar{k}$. We assume that the solution is interior, and we use the EE to characterize the solution to the problem. For this particular example we have,

$$
F(k_t, k_{t+1}) = f(k_t) - k_{t+1}
$$

Then, EE is given by

$$
-1 + \beta f'(k_{t+1}) = 0 \tag{19}
$$

Equation (19) pins down the value of k_{t+1},

$$f'\left(\bar{k}\right) = \left(\frac{1}{\beta}\right)$$

Note that this value does not depend on t. Now from feasibility

$$c_t = f\left(k_t\right) - \bar{k}$$

For $t = 0$, we have

$$c_0 = f\left(k_0\right) - \bar{k}.$$

Under our restriction that $f\left(k_0\right) - \bar{k} \geq 0$ for $\beta f'\left(\bar{k}\right) = 1$ we don't violate the constraint of consumption being non-negative, so we found our solution.

Note that this solution involves choosing $k_1 = \bar{k}$ and all subsequent values of k as well, that is we jump to the steady state in one period.

2. If consumption is non-negative and $f\left(k_0\right) < \bar{k}$ what will be the optimal policy? Hint: trickier question, since you have to consider corners.

From the previous analysis we can see that at time $t = 0$ we cannot choose $k_1 = \bar{k}$ since this implies negative consumption. That means that the non-negativity constraint is binding hence at time $t = 1$ we have,

$$c_0 = 0, \quad k_1 = f\left(\bar{k}\right)$$

For the following periods will have a similar situation. Now we need to compare $f\left(f\left(k_0\right)\right) - \bar{k}$, if it is positive that means that we jump to the steady state in two periods otherwise consumption is zero again and production is used to build more capital.

3.3 Exercise 3: A toy example of adjustment cost models

Let the adjustment cost model be:

$$F\left(x, y\right) = -\frac{a}{2} y^2 - \frac{b}{2}\left(y - x\right)^2$$
$$\Gamma\left(x\right) = \mathbb{R}$$

Note that this model has two opposing forces, on one side given x we want to drive down y in order to maximize the objective. However, if we move y to much (measure as the distance to x) then we incur in an extra cost for adjusting. So we want to keep y low and we do that in small steps.

1. Suppose that $x_0 = 0$. What is the optimal path after that initial condition?

If $x_0 = 0$, then the optimal path is to set $y = 0$ in every period. That yields the higher utility per period, zero.

2. Write the EE and evaluate them at the steady state. What is that value?

We use the period return function to write the Euler Equation,

$$-ag(x) - b(g(x) - x) + \beta b(g(g(x)) - g(x)) = 0$$

where we use g to denote the optimal policy for y. In steady state $g(x) = x$ which implies that $\bar{x} = 0$.

3. What is the optimal policy if $a = 0$?

If $a = 0$ then the first term in the period return function vanish. In order to maximize the utility we can choose $y = x$ in every period and get the highest utility of zero.

4. Show that the optimal policy is $x_{t+1} = g(x_t) = \gamma x_t$ for some $0 < \gamma < 1$. Characterize γ in terms of b/a and β.

We use our guess in our Euler Equation (guess and verify),

$$\begin{aligned} -ag(x) - b(g(x) - x) + \beta b(g(g(x)) - g(x)) &= 0 \\ -a\gamma x - b(\gamma x - x) + \beta b\left(\gamma^2 x - \gamma x\right) &= 0 \\ \left[-a\gamma - b(\gamma - 1) + \beta b\left(\gamma^2 - \gamma\right)\right] x &= 0 \end{aligned} \tag{20}$$

Equation (20) holds for any x, hence we must have

$$-a\gamma - b(\gamma - 1) + \beta b\left(\gamma^2 - \gamma\right) = 0$$

or

$$b\beta\gamma^2 - (a + b + \beta b)\gamma + b = 0$$

This has solutions

$$\gamma_{1,2} = \frac{(a + b + \beta b) \pm \sqrt{[(a + \beta b) + b]^2 - 4b}}{2b\beta}$$

you can show that one solution is positive and the other one is bigger than one. We ruled out the solution that is bigger than one since it implies and explosive behavior for x. We conclude the solution is

$$g(x_t) = \gamma x_t$$

3.4 Exercise 4: example of adjustment cost models II

I recommend you to solve the second Midterm of 2017/2018, it has an adjustment cost model similar to the one in Exercise 3.

4 Homogeneous of degree 1 case (CRS)

From now on we assume that

- X is a cone, i.e $x \in X$ implies $\lambda x \in X$ for all scalar $\lambda > 0$.
- $y \in \Gamma(x) \implies \lambda y \in \Gamma(\lambda x)$ for all scalar $\lambda > 0$.
- $F(x, y)$ is homogeneous of degree one, i.e. $F(\lambda x, \lambda y) = \lambda F(x, y)$ for all scalar λ and $(x, y) \in Gr(\Gamma)$.

Proposition 3. *Under the previous assumption the Optimal policy is homogeneous of degree one, that is*

$$y = g(x) \implies \lambda y = g(\lambda x)$$

Proof. Consider the sequential problem describe in (1)

$$V^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{21}$$

$$s.t. \quad x_{t+1} \in \Gamma(x_t), \quad \forall t \geq 0$$

$$x_0 \quad given$$

We denote with $y = g(x)$ the policy function that solves problem (21). Let $\lambda > 0$, $\tilde{x}_0 = \lambda x_0$, we want to show then that problem (21) with initial condition $\tilde{x}_0$ yield as solution $y = \frac{1}{\lambda} g(x\lambda)$.

We know that EE and TVC are sufficient conditions for optimality. The EE is the same across problems, the initial condition only change the solution to the difference equation (in principle). The solution for our tilde economy satisfies

$$F_y(\tilde{x}_t, \tilde{x}_{t+1}) + \beta F_x(\tilde{x}_{t+1}, \tilde{x}_{t+2}) = 0, \quad \forall t$$

Consider now time $t = 0$,

$$F_y\left(\lambda x_0, \tilde{x}_1\right) + \beta F_x\left(\tilde{x}_1, \tilde{x}_2\right) = 0$$

Since F is homogeneous of degree 1, their partial derivatives are homogeneous of degree zero, hence

$$F_y\left(x_0, \frac{1}{\lambda}\tilde{x}_1\right) + \beta F_x\left(\frac{1}{\lambda}\tilde{x}_1, \frac{1}{\lambda}\tilde{x}_2\right) = 0$$

But we know the solution for this type of EE, the solution is $\frac{1}{\lambda}\tilde{x}_{t+1} = x_{t+1}$. Finally note that choosing $\tilde{x}_{t+1} = \lambda x_{t+1}$ is always feasible because if

$$x_{t+1} \in \Gamma\left(x_t\right) \implies \tilde{x}_{t+1} \in \Gamma\left(\lambda x_t\right) \forall t$$

Also our propose solution satisfies the TVC since

$$\begin{aligned}
\lim_{t\to\infty} \beta^t F_x\left(\tilde{x}_t, \tilde{x}_{t+1}\right)\tilde{x}_t &= \lim_{t\to\infty} \beta^t F_x\left(\lambda x_t, \lambda x_{t+1}\right)\lambda x_t \\
&= \lambda \lim_{t\to\infty} \beta^t F_x\left(x_t, x_{t+1}\right) x_t \\
&= \lambda \times 0 \\
&= 0
\end{aligned}$$

So to summarize we have that, taking period $t = 0$,

$$g\left(\lambda x_0\right) = \tilde{x}_1 = \lambda x_1 = g\left(x_0\right)\lambda$$

and the same is true for $t > 0$. □

We will specialize to the one dimensional case, i.e. $X = \mathbb{R}$. In this particular case, proposition 3 implies that the policy function is linear in the state. This lead us to the following result.

Proposition 4. *In the one dimensional case, under the assumptions described above, the policy rule is homogeneous of degree one on the state, and can be written as*

$$y = \bar{g}x \tag{22}$$

where $\beta|\bar{g}|< 1$.

Proof. The first part of the proposition was proven before. Linearity of the policy function follows from x being one dimensional. We need to show then that $\beta|\bar{g}|< 1$.

Since (22) is a solution to our problem it satisfies TVC. Hence,

$$
\begin{aligned}
0 &= \lim_{t\to\infty} \beta^t F_x\left(x_t, g\left(x_t\right)\right) x_t \\
0 &= \lim_{t\to\infty} \beta^t F_x\left(x_t, \bar{g}x_t\right) x_t \qquad (23)
\end{aligned}
$$

Now use (22) to write x_t as a function of t and x_0 (you can either substitute backwards or solve the difference equation),

$$
x_t = \bar{g}^t x_0 \qquad (24)
$$

Then, combining (23) and (24),

$$
0 = \lim_{t\to\infty} \beta^t F_x\left(\bar{g}^t x_0, \bar{g}^{t+1} x_0\right) \bar{g}^t x_0
$$

By Homogeneity of degree zero of F_x,

$$
\begin{aligned}
0 &= \lim_{t\to\infty} \beta^t F_x\left(x_0, \bar{g}x_0\right) \bar{g}^t x_0 \\
0 &= F_x\left(x_0, \bar{g}x_0\right) x_0 \lim_{t\to\infty} \left(\beta\bar{g}\right)^t
\end{aligned}
$$

Since $F_x\left(x_0, \bar{g}x_0\right) x_0 > 0$, TVC impose that $\beta\bar{g} \in (-1, 1)$ or $\beta\left|\bar{g}\right| < 1$. □

In order to pin down the value $\bar{g}$, we work with the Euler Equation,

$$
F_y\left(x_t, x_{t+1}\right) + \beta F_x\left(x_{t+1}, x_{t+2}\right) = 0, \quad \forall t
$$

Substituting in the policy function yields

$$
F_y\left(x_t, \bar{g}x_t\right) + \beta F_x\left(\bar{g}x_t, \left(\bar{g}\right)^2 x_t\right) = 0, \quad \forall t
$$

Using the homogeneity of degree zero of both F_y and F_x this is equivalent to

$$
F_y\left(1, \bar{g}\right) + \beta F_x\left(1, \bar{g}\right) = 0 \qquad (25)
$$

So $\bar{g}$ is the solution to equation (25).

Exercise 1. Suppose that F is strictly quasi-concave. Can the EE be satisfied for multiple values of $\bar{g}$? HINT: use that concavity implies $F_{xx} < 0, F_{yy} < 0$ and jointly with homogeneity of degree 1, $F_{xy}^2 = F_{xx}F_{yy} > 0$.

Think of the LHS of equation (25) as a function of g,

$$H(g) = F_y(1, g) + \beta F_x(1, g)$$

We look for the roots of $H(g)$, that is the value(s) $\bar{g}$ such that $H(\bar{g}) = 0$. First we compute the slope of $H(\cdot)$ at any particular value,

$$H'(g) = F_{yy}(1, g) + \beta F_{xy}(1, g)$$

Using the hint we can write this as

$$\begin{aligned} H'(g) &= F_{yy}(1, g) + \beta\sqrt{F_{xx}(1, g) F_{yy}(1, g)} \\ &= |F_{yy}(1, g)| \left[-1 + \beta\sqrt{\frac{F_{xx}(1, g)}{F_{yy}(1, g)}}\right] \end{aligned}$$

Hence, the slope of the function $H(\cdot)$ depends on the sing of the terms in brackets, if $\left[-1 + \beta\sqrt{\frac{F_{xx}(1,g)}{F_{yy}(1,g)}}\right] \lesseqgtr 0$ for all values of g then $H(g)$ cross zero at most once so the solution is unique. If the term in brackets change sing for different values of g then we could have more than one value $\bar{g}$. Remember that all possible solutions that we find from (25) need to satisfy the constraint $\beta|\bar{g}| < 1$.

4.1 General Homogeneity

Suppose we keep our assumptions on X and $\Gamma(\cdot, \cdot)$, but instead we assume that the function $F(\cdot, \cdot)$ is homogenous of degree $1 - \gamma$,

$$F(x, y) = \frac{H(x, y)^{1-\gamma}}{1 - \gamma},$$

with $H(\cdot, \cdot)$ homogeneous of degree 1.

In this case we also have that the policy function $g(x)$ is homogeneous of degree one. The proof is left as an exercise [Try mimicking the logic used above].

5 Adjustment cost and investment

Consider an agent that maximize discounted profits net of investment expenditures. We can think of firm making investment decision or an economy with a representative agent with period utility function given by $u(c) = c$. We consider two possible production technologies $f(k)$,

- **Case 1:** $f(k)$ is strictly concave
- **Case 2:** $f(k)$ is linear

We use Fernando's notation here, keep in mind that now $f(k)$ does not include the un-depreciated capital.

The law of motion for capital is as usual,

$$k_{t+1} = i_t + (1-\delta) k_t \tag{26}$$

We introduce an extra margin to the problem. Installing capital has an additional cost in terms of the final good $\phi\left(\frac{i_t}{k_t}\right) k_t$.

The problem of the firms is given by (27)

$$\begin{aligned} \max_{\{i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[f(k_t) - i_t - \phi\left(\frac{i_t}{k_t}\right) k_t \right] \quad (27) \\ s.t \quad k_{t+1} = i_t + (1-\delta) k_t \\ k_{t+1} \geq 0 \\ k_0 \quad given \end{aligned}$$

5.1 Concave f and no adjustment cost

We consider first the case with no adjustment cost, $\phi\left(\frac{i_t}{k_t}\right) = 0$. Assume that $f(\cdot)$ is strictly concave and satisfies Inada conditions. Also assume that investment can be positive or negative.

The sequential problem under this set up is given by,

$$\begin{aligned} X &= \mathbb{R}_+ \\ i_t &\in [-(1-\delta) k_t, f(k_t)] \implies \Gamma(k_t) = [0, f(k_t) + (1-\delta) k_t] \\ F(k_t, k_{t+1}) &= f(k_t) + (1-\delta) k_t - k_{t+1} \end{aligned}$$

The EE is then,

$$\begin{aligned} F_y(k_t, k_{t+1}) + \beta F_x(k_{t+1}, k_{t+2}) &= 0 \\ -1 + \beta [f'(k_{t+1}) + 1 - \delta] &= 0 \quad (28) \end{aligned}$$

Note that from equation (28) follows that $\bar{k}$ is defined as

$$f'(k_{t+1}) = \frac{1}{\beta} - (1-\delta)$$

So we jump to the steady state immediately in time $t = 0$.

Exercise 2. Can the function $f(\cdot)$ be linear instead of strictly concave in this case?

Suppose the production function is given by $f(k) = Ak$. Then, the sequence problem reduces to

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\{A + (1-\delta)\} k_t - k_{t+1}]$$
$$s.t \quad k_{t+1} \geq 0$$
$$k_0 \quad given$$

Note that the objective function is equivalent to,

$$\begin{aligned}\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [\{A + (1-\delta)\} k_t - k_{t+1}] &= \lim_{T\to\infty} \{A + (1-\delta)\} k_0 - k_1 + \beta [\{A + (1-\delta)\} k_1 - k_2] + \cdots + \\ &\quad \beta^T [\{A + (1-\delta)\} k_{T-1} - k_T] \\ &= \lim_{T\to\infty} \{A + (1-\delta)\} k_0 + \sum_{t=1}^{T-1} \beta^t \left\{A + (1-\delta) - \frac{1}{\beta}\right\} k_t - \beta^T k_T\end{aligned}$$

So we want to solve

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \{A + (1-\delta)\} k_0 + \lim_{T\to\infty} \sum_{t=1}^{T-1} \{A + (1-\delta) - 1\} k_t - k_T$$
$$s.t \quad k_{t+1} \geq 0$$
$$k_0 \quad given$$

We have two cases. If $\{A + (1-\delta) - 1\} > 0$ the firm chooses $k_{t+1} = "\infty"$ hence the problem has no solution. On the other hand if $\{A + (1-\delta) - 1\} < 0$ then the optimal policy is to set $k_{t+1} = 0$, the firm operates for one period only.

5.2 Linear f and adjustment costs

We work with a particular example. Let $f(k) = Ak$. For the adjustment cost we define,

$$a\left(\frac{k_{t+1}}{k_t}\right) \equiv \phi\left(\frac{k_{t+1} - (1-\delta)k_t}{k_t}\right) \tag{29}$$

We assume that $a(\cdot)$ is positive (so any change implies cost) and **strictly convex** (so cost are increasing in size of change). We also assume that $a(1) = a'(1) = 0$, that is both marginal and per unit cost are zero if capital stays constant. Finally, $a\left(\frac{1}{\beta}\right) < A$, hence large changes are costly.

Be careful in L5 there is a typo it says concave. Fernando assumed $a\left(\frac{1}{\beta}\right) > A$ Fernando wrote in his notes a different equation, check slide 32 and 36 investment is missing in his expression.

In this case the period return function is given by

$$F(k_t, k_{t+1}) = Ak_t + (1-\delta)k_t - a\left(\frac{k_{t+1}}{k_t}\right)k_t - k_{t+1} \tag{30}$$

Note that the period return function is homogeneous of degree one.

1. Compute F_x and F_y in terms of A and $a(\cdot)$. Make sure your expressions depend only on the ratio y/x.

From equation (30) we have,

$$F_x(x,y) = A + (1-\delta) - a\left(\frac{k_{t+1}}{k_t}\right) + a'\left(\frac{k_{t+1}}{k_t}\right)\frac{k_{t+1}}{k_t} \tag{31}$$
$$F_y(x,y) = -a'\left(\frac{k_{t+1}}{k_t}\right) - 1$$

2. Write the Euler equation for this model. Use that the optimal policy is homogeneous of degree one (Why?) and denote $y = g(x) = \bar{g}x$. Your expression should be a function of A, $a(\cdot)$,$a'(\cdot)$, β and $\bar{g}$.

The Euler equation is given by equation (32),

$$-a'\left(\frac{k_{t+1}}{k_t}\right) - 1 + \beta\left\{A + (1-\delta) - a\left(\frac{k_{t+2}}{k_{t+1}}\right) + a'\left(\frac{k_{t+2}}{k_{t+1}}\right)\frac{k_{t+2}}{k_{t+1}}\right\} = 0 \tag{32}$$

Since the period return function is homogeneous of degree one for what we've seen before the optimal policy would be homogeneous of degree 1 as well, and since the state uni-dimensional k_{t+1} is a linear function of k_t. Hence, we have,

$$k_{t+1} = \bar{g}k_t \tag{33}$$

Combining equations (32) and (33), we get the expression defining $\bar{g}$,

$$-a'(\bar{g}) - 1 + \beta\{A + (1-\delta) - a(\bar{g}) + a'(\bar{g})\bar{g}\} = 0 \tag{34}$$

3. Differentiate the Euler equation with respect to $\bar{g}$. What is the sign of this expression for values $g < 1/\beta$?

Define the function $h(g)$ as,

$$h(g) \equiv a'(g) + \beta a(g) - \beta a'(g) g$$

Note that $h(\bar{g}) = \beta[A + 1 - \delta] - 1 > 0$ [2].

Taking the derivative with respect to g yields,

$$h'(g) = a''(g)[1 - \beta g]$$

Since $a(\cdot)$ is a **convex** function we have that for $g < \frac{1}{\beta}$, $h'(g) > 0$. So $h(\cdot)$ is strictly increasing until $g = \frac{1}{\beta}$ where it achieves its maximum.

4. Plot the constant $\beta[A + 1 - \delta] - 1$ against $\tilde{h}(g)$ with g in the horizontal axis. Indicate in your graph the value of $\bar{g}$ where both curves intersect. How is $\bar{g}$ compared with $\frac{1}{\beta}$? How is $\bar{g}$ compared with 1?

Take the function $h(\cdot)$, and evaluate it at $g = 1$ and $g = \frac{1}{\beta}$,

$$h(1) = a'(1) + \beta a(1) - \beta a'(1)1 = 0 < \beta[A + 1 - \delta] - 1$$
$$h\left(\frac{1}{\beta}\right) = \beta a\left(\frac{1}{\beta}\right)$$

Now since,

$$\beta a\left(\frac{1}{\beta}\right) > \beta A \implies \beta a\left(\frac{1}{\beta}\right) > \beta A - [1 - (1-\delta)\beta]$$

So the solution $\bar{g} \in \left(1, \frac{1}{\beta}\right)$.

Remark 2. The function decreases for $g > \frac{1}{\beta}$ so we have multiple solutions. If we check TVC we can ruled out the case where $\bar{g} > 1/\beta$.

5. What happen with $\bar{g}$ if A increases?

The function $h(\cdot)$ does not depend on A, so if A increases the constant $\beta A - [1 - (1-\delta)\beta]$

[2] We assumed the inequality to be true, we have no restrictions on A so we can choose a high value of it in order to satisfy the inequality.

increases so now $\bar{g}$ is higher.