Theory of Income, Fall 2009

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Problem Set 2

1 Homothetic but not identical preferences

Consider a set of agents $i = 1, 2, ..., I$ with preferences indexed by a vector β:

$$u^i\left(x^i\right) \equiv f\left(x^i; \beta^i\right) = \sum_{l=1}^{m} \beta_l^i \log\left(x_l^i\right),$$

where

$$\sum_{l=1}^{m} \beta_l^i = 1,\ \ \beta_l^i \geq 0 \text{ all } l.$$

We assume that different agents have different β^i vectors.

Given a vector of λ weights, the Social Planner's problem is:

$$\max_{\left\{x_l^i\right\}} \sum_{i=1}^{I} \lambda_i f\left(x^i; \beta^i\right),$$

subject to

$$\sum_{i=1}^{I} x_l^i = \bar{e}_l \text{ for } l = 1, 2, .., m.$$

1.1) Question. Write down the foc for x_l^i.

Ans:

$$\lambda_i \beta_l^i \frac{1}{x_l^i} = \gamma_l,$$

or

$$\lambda_i \beta_l^i = \gamma_l x_l^i.$$

1.2) Question. Use the foc for x_l^i for different agents i, and feasibility for the l^{th} commodity to obtain an expression for γ_l in terms of $\bar{e}_l$, $\{\lambda_i\}_{i=1}^I$ and $\left\{\beta_l^i\right\}_{i=1}^I$.

Ans: Adding across agents,

$$\sum_{i=1}^{I} \lambda_i \beta_l^i = \gamma_l \sum_{i=1}^{I} x_l^i = \gamma_l \bar{e}_l,$$

so

$$\gamma_l = \frac{\sum_{i=1}^{I} \lambda_i \beta_l^i}{\bar{e}_l}.$$

1.3) Question. Write down an expression for the relative shadow values for two commodities, γ_l/γ_k. This expression should depend on $(\bar{e}_k/\bar{e}_l)$ and the vectors $\{\lambda_i\}_{i=1}^{I}$, $\{\beta_l^i\}_{i=1}^{I}$ and $\{\beta_k^i\}_{i=1}^{I}$.

Ans: Dividing the expressions for γ_l and γ_k we have

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{e}_k}{\bar{e}_l}\right) \frac{\sum_{i=1}^{I} \lambda_i \beta_l^i}{\sum_{i=1}^{I} \lambda_i \beta_k^i}.$$

1.4) Question. Write down γ_l/γ_k for $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$. What do you conclude about the dependence of the relative shadow values on the vector of $\lambda' s$?

Ans:

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{e}_k}{\bar{e}_l}\right) \frac{\beta_l^i}{\beta_k^i},$$

so the relative shadow values depend on the particular $\lambda' s$ chosen. There is no aggregation, in the sense that marginal valuations are *not* independent of the $\lambda-$weights.

1.5) Question. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? [Hint: use your answer to the previous question].

Ans: Remember that in the corresponding competitive equilibrium, $p_l/p_k = \gamma_l/\gamma_k$ and $\mu_i = 1/\lambda_i$, where μ_i is the Lagrange multiplier on the budget constraint. Thus, in the competitive equilibrium, a higher λ_i is equivalent to a lower μ_i, or equivalently, to a wealthier agent. In other words, if the relative prices depend on the λ -weights, they will depend on the distribution of wealth. The last question shows that relative prices depend on the particular choice of $\lambda' s$. Thus, relative prices depend on the distribution of wealth and hence, there is no aggregation.

2 Aggregation without identical and homothetic preferences (first example)

Consider the case where preferences are indexed by two vectors $\beta^i = \left(\beta_l^i\right)_{l=1}^{m}$ and $\theta^i = \left(\theta_l^i\right)_{l=1}^{m}$ as follows:

$$u^i\left(x_1, x_2, ..., x_m\right) = \sum_{l=1}^{m} \beta_l^i \log\left(x_l^i - \theta_l^i\right).$$

The constants θ^i_j are interpreted as a subsistence level of consumption of good l for agent i. (The above preferences are known as Stone-Geary).

2.1) Question. Write down the foc for the social planner problem for x^i_l.

Ans:

$$\lambda_i \beta^i_l \frac{1}{x^i_l - \theta^i_l} = \gamma_l,$$

or

$$\lambda_i \beta^i_l = \gamma_l \left(x^i_l - \theta^i_l \right).$$

2.2) Question. Use the foc(s) for the social problem for x^i_l for all agents for commodity l and the market clearing condition for l to obtain an expression for γ_l. Your answer should be a function of $\left(\beta^i_l\right)^I_{i=1}$, $\bar{e}_l$, $(\lambda_i)^I_{i=1}$ and $\bar{\theta}_l$ where

$$\bar{\theta}_l = \sum_{i=1}^{I} \theta^i_l.$$

Ans: Adding across agents,

$$\sum_{i=1}^{I} \lambda_i \beta^i_l = \gamma_l \sum_{i=1}^{I} \left(x^i_l - \theta^i_l \right) = \gamma_l \left(\bar{e}_l - \bar{\theta}_l \right),$$

so

$$\gamma_l = \frac{1}{\left(\bar{e}_l - \bar{\theta}_l \right)} \sum_{i=1}^{I} \lambda_i \beta^i_l.$$

2.3) Question. Derive an expression for γ_l/γ_k. Does it depend on $(\lambda_i)^I_{i=1}$? Assuming that $\beta^i_l = \beta^j_l$ for all $j, i \in I$, $l = 1, .., m$, does γ_l/γ_k depend on $(\lambda_i)^I_{i=1}$? Are preferences in this case identical? Are they homothetic?

Ans:

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{e}_k - \bar{\theta}_k}{\bar{e}_l - \bar{\theta}_l} \right) \frac{\sum_{i=1}^{I} \lambda_i \beta^i_l}{\sum_{i=1}^{I} \lambda_i \beta^i_k}.$$

If $\beta^i_l = \beta^j_l$, then

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{e}_k - \bar{\theta}_k}{\bar{e}_l - \bar{\theta}_l} \right) \frac{\beta_l}{\beta_k}.$$

Preferences are neither identical nor homothetic.

2.4) Question. Assume that $\beta_l^i = \beta_l^j$ for all $j, i \in I$. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? [Hint: use your answer to the previous question].

Ans: Since $\gamma_l/\gamma_k\,(= p_l/p_k)$ is independent of the λ -weights, relative prices are independent of the distribution of wealth and there is aggregation.

3 Aggregation without identical and homothetic preferences (second example)

Consider the case where preferences are indexed by two vectors $\beta^i = \left(\beta_l^i\right)_{l=1}^m$ and $\theta^i = \left(\theta_l^i\right)_{l=1}^m$ as follows:

$$u^i\left(x_1, x_2, ..., x_m\right) = -\frac{1}{2}\sum_{l=1}^m \beta_l^i \left(x_l^i - \theta_l^i\right)^2.$$

In this case (quadratic utility), the constants θ_l^i are interpreted as the satiation point of that particular good, for if $x_l^i > \theta_l^i$ the marginal utility of an additional unit of x_l^i is negative.

3.1) Question. Write down the foc for the social planner problem for x_l^i.

Ans:

$$-\lambda_i \beta_l^i \left(x_l^i - \theta_l^i\right) = \gamma_l,$$

or

$$\theta_l^i - x_l^i = \frac{\gamma_l}{\lambda_i \beta_l^i}.$$

3.2) Question. Use the foc(s) for the social problem for x_l^i for all agents for commodity l and the market clearing condition for l to obtain an expression for γ_l. Your answer should be a function of $\left(\beta_l^i\right)_{i=1}^I$, $\bar{e}_l$, $\left(\lambda_i\right)_{i=1}^I$ and $\bar{\theta}_l$ where

$$\bar{\theta}_l = \sum_{i=1}^I \theta_l^i.$$

Ans: Adding across agents,

$$\begin{aligned} \sum_{i=1}^I \left(\theta_l^i - x_l^i\right) &= \sum_{i=1}^I \frac{\gamma_l}{\lambda_i \beta_l^i}, \\ \left(\bar{\theta}_l - \bar{e}_l\right) &= \gamma_l \sum_{i=1}^I \frac{1}{\lambda_i \beta_l^i}, \end{aligned}$$

so

$$\gamma_l = \left(\bar{\theta}_l - \bar{e}_l\right)\left[\sum_{i=1}^{I}\frac{1}{\lambda_i \beta_l^i}\right]^{-1}.$$

3.3) Question. Derive an expression for γ_l/γ_k. Does it depend on $(\lambda_i)_{i=1}^{I}$? Assuming that $\beta_l^i = \beta_l^j$ for all $j, i \in I$, $l = 1, .., m$, does γ_l/γ_k depend on $(\lambda_i)_{i=1}^{I}$? Are preferences in this case identical? Are they homothetic?

Ans:

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{\theta}_l - \bar{e}_l}{\bar{\theta}_k - \bar{e}_k}\right)\frac{\sum_{i=1}^{I}\left(1/\lambda_i \beta_k^i\right)}{\sum_{i=1}^{I}\left(1/\lambda_i \beta_l^i\right)}.$$

If $\beta_l^i = \beta_l^j$, then

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{\theta}_l - \bar{e}_l}{\bar{\theta}_k - \bar{e}_k}\right)\frac{\beta_l}{\beta_k}.$$

Preferences are neither identical nor homothetic.

3.4) Question. Assume that $\beta_l^i = \beta_l^j$ for all $j, i \in I$. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? [Hint: use your answer to the previous question].

Ans: Since $\gamma_l/\gamma_k \, (= p_l/p_k)$ is independent of the λ -weights, relative prices are independent of the distribution of wealth and there is aggregation.

4 Identical but not homothetic preferences: lack of aggregation

Assume $I = m = 2$ and that households have the following identical preferences:

$$u^i\left(x_1, x_2\right) = \frac{x_1^{1-\rho}}{1-\rho} + \frac{x_2^{1-\beta}}{1-\beta}; \; i = 1, 2.$$

4.1) Question. Write down the foc for the social planner problem for x_l^i, $l = 1, 2$.

Ans:

$$\lambda_i \left(x_1^i\right)^{-\rho} = \gamma_1; \quad i = 1, 2,$$

$$\lambda_i \left(x_2^i\right)^{-\beta} = \gamma_2; \quad i = 1, 2.$$

4.2) Question. Use the foc(s) for the social problem for x_l^i for all agents for commodity l and the market clearing condition for l to obtain an expression for γ_l. Your answer should be a function of $\bar{e}_l$, $(\lambda_i)_{i=1}^{I}, \rho$ and β.

Ans: Solving for x_1^i and x_2^i,

$$x_1^i = \left(\frac{\gamma_1}{\lambda_i}\right)^{-1/\rho},$$

and

$$x_2^i = \left(\frac{\gamma_2}{\lambda_i}\right)^{-1/\beta}.$$

Adding for all agents we obtain

$$\bar{e}_1 = (\gamma_1)^{-1/\rho}\left(\lambda_1^{1/\rho} + \lambda_2^{1/\rho}\right).$$

and

$$\bar{e}_2 = (\gamma_2)^{-1/\beta}\left(\lambda_1^{1/\beta} + \lambda_2^{1/\beta}\right).$$

Thus

$$\gamma_1 = \bar{e}_1^{-\rho}\left(\lambda_1^{1/\rho} + \lambda_2^{1/\rho}\right)^{\rho},$$

and

$$\gamma_2 = \bar{e}_2^{-\beta}\left(\lambda_1^{1/\beta} + \lambda_2^{1/\beta}\right)^{\beta}.$$

4.3) Question. Derive an expression for γ_1/γ_2. Does it depend on λ_1 and λ_2 if $\beta \neq \rho$?

Ans:

$$\frac{\gamma_1}{\gamma_2} = \left(\frac{\bar{e}_1^{-\rho}}{\bar{e}_2^{-\beta}}\right)\frac{\left(\lambda_1^{1/\rho} + \lambda_2^{1/\rho}\right)^{\rho}}{\left(\lambda_1^{1/\beta} + \lambda_2^{1/\beta}\right)^{\beta}}.$$

4.4) Question. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? [Hint: use your answer to the previous question].

Ans: Since $\gamma_l/\gamma_k\ (= p_l/p_k)$ depends on the particular choice of λ -weights, relative prices depend on the distribution of wealth and hence, there is no aggregation.

5 Aggregation without identical and homothetic preferences (third example)*

Consider the case where preferences are indexed by two vectors $\beta = (\beta_l)_{l=1}^m$ and $\theta^i = (\theta_l^i)_{l=1}^m$ and a parameter σ as follows:

$$u^i(x_1, x_2, ..., x_m) = \sum_{l=1}^m \beta_l \frac{\left(x_l^i - \theta_l^i\right)^{1-\sigma}}{1-\sigma}.$$

The constants θ_j^i are interpreted as a subsistence level of consumption of good l for agent i. (The above preferences are known as Stone-Geary).

5.1) Question. Write down the foc for the social planner problem for x_l^i.

Ans:

$$\lambda_i \beta_l \left(x_l^i - \theta_l^i\right)^{-\sigma} = \gamma_l,$$

or

$$\left(x_l^i - \theta_l^i\right) = \gamma_l^{-1/\sigma} \left(\lambda_i \beta_l\right)^{1/\sigma}.$$

5.2) Question. Use the foc(s) for the social problem for x_l^i for all agents for commodity l and the market clearing condition for l to obtain an expression for γ_l. Your answer should be a function of $\sigma,\ \bar{e}_l,\ \beta_l,\ (\lambda_i)_{i=1}^I$ and $\bar{\theta}_l$ where

$$\bar{\theta}_l = \sum_{i=1}^I \theta_l^i.$$

Ans: Adding across agents,

$$\begin{aligned} \sum_{i=1}^I \left(x_l^i - \theta_l^i\right) &= \gamma_l^{-1/\sigma} \beta_l^{1/\sigma} \sum_{i=1}^I \lambda_i^{1/\sigma}, \\ \left(\bar{e}_l - \bar{\theta}_l\right) &= \gamma_l^{-1/\sigma} \beta_l^{1/\sigma} \sum_{i=1}^I \lambda_i^{1/\sigma}, \end{aligned}$$

so

$$\gamma_l = \left(\bar{e}_l - \bar{\theta}_l\right)^{-\sigma} \beta_l \left[\sum_{i=1}^I \lambda_i^{1/\sigma}\right]^{\sigma}.$$

5.3) Question. Derive an expression for γ_l/γ_k. Does it depend on $(\lambda_i)_{i=1}^I$? Are preferences in this case identical? Are they homothetic?

Ans:

$$\frac{\gamma_l}{\gamma_k} = \left(\frac{\bar{e}_k - \bar{\theta}_k}{\bar{e}_l - \bar{\theta}_l}\right)^{\sigma} \frac{\beta_l}{\beta_k}.$$

Preferences are neither identical nor homothetic.

5.4) Question. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? [Hint: use your answer to the previous questions].

Ans: Since $\gamma_l/\gamma_k \left(= p_l/p_k\right)$ is independent of the λ -weights, relative prices are independent of the distribution of wealth and there is aggregation.

6 Concavity of the "representative agent" utility function*

Let $X^i \subseteq R^m$ be the consumption possibility set of agent i, and $u^i : X^i \to R$ the utility function of agent i.

Let $\lambda_i \geq 0$ be the weight of agent i in the social planner's problem. Define the representative agent utility function $u(x;\lambda) : X \to R$ where X is the set sum of the X^i and

$$u(x;\lambda) \doteq \max_{\{x^i\}} \sum_{i=1}^{I} \lambda_i u^i\left(x^i\right),$$

subject to

$$\begin{aligned} x^i &\in X^i, \\ \sum_{i=1}^{I} x^i &= x. \end{aligned}$$

6.1) Question. Show that if X^i is convex for each $i = 1, ..., I$, then X is convex.

Ans: X is defined as follows:

$$X = \left\{ x \in R^m : x = \sum_{i=1}^{I} x^i, \text{ all } x^i \in X^i, \text{ all } i \in I \right\}.$$

Start with any given $\hat{x}^i \in X^i$ and $\bar{x}^i \in X^i$, all $i \in I$. By definition of X, $\hat{x} \equiv \sum_{i=1}^{I} \hat{x}^i \in X$ and $\bar{x} \equiv \sum_{i=1}^{I} \bar{x}^i \in X$. Since X^i is convex for each $i \in I$, then $\alpha\hat{x}^i + (1-\alpha)\bar{x}^i \in X^i$, all $\alpha \in (0,1)$, $i \in I$. But then, by definition of X,

$$\sum_{i=1}^{I} \left(\alpha\hat{x}^i + (1-\alpha)\bar{x}^i\right) = \alpha\hat{x} + (1-\alpha)\bar{x} \in X,$$

which implies that the set X is convex, as was to be shown.

6.2) Question. Show that if X^i is convex and u^i is concave for each $i = 1, ..., I$, then $u(x;\lambda)$ is a concave function of x. You need to show that given any $\hat{x} \in X$, $\bar{x} \in X$ and $\rho \in (0,1)$,

$$u(\rho\hat{x} + (1-\rho)\bar{x};\lambda) \geq \rho u(\hat{x};\lambda) + (1-\rho)u(\bar{x};\lambda).$$

[Hint: use as a feasible consumption for each agent i when aggregate consumption is $\rho\hat{x}+(1-\rho)\bar{x}$ the convex combination of the consumption for each agent i used when aggregate consumption was $\bar{x}$ and $\hat{x}$].

Ans: Let

$$\left\{\hat{x}^i\right\} = \arg\max_{\{x^i\}} \sum_{i=1}^{I} \lambda_i u^i\left(x^i\right) \text{ s.t. } x^i \in X^i, \sum_{i=1}^{I} x^i = \hat{x},$$

and

$$\left\{\bar{x}^i\right\} = \arg\max_{\{x^i\}} \sum_{i=1}^{I} \lambda_i u^i\left(x^i\right) \text{ s.t. } x^i \in X^i, \sum_{i=1}^{I} x^i = \bar{x}.$$

Since X^i and X are convex, we know that $\rho\hat{x}^i + (1-\rho)\,\bar{x}^i \in X^i$ and $\rho\hat{x} + (1-\rho)\,\bar{x} \in X$ for any $\rho \in (0,1)$. Moreover, since u^i is concave,

$$u^i\left(\rho\hat{x}^i + (1-\rho)\,\bar{x}^i\right) \geq \rho u^i\left(\hat{x}^i\right) + (1-\rho)\,u^i\left(\bar{x}^i\right).$$

Multiplying each side of the inequality by λ_i and adding across agents we obtain

$$\begin{aligned}\sum_{i=1}^{I} \lambda_i u^i\left(\rho\hat{x}^i + (1-\rho)\,\bar{x}^i\right) &\geq \rho\sum_{i=1}^{I} \lambda_i u^i\left(\hat{x}^i\right) + (1-\rho)\sum_{i=1}^{I} \lambda_i u^i\left(\bar{x}^i\right) \\ &= \rho u\left(\hat{x};\lambda\right) + (1-\rho)\,u\left(\bar{x};\lambda\right).\end{aligned}$$

But by definition of the maximum, we must have that

$$u(\rho\hat{x} + (1-\rho)\,\bar{x};\lambda) \geq \sum_{i=1}^{I} \lambda_i u^i\left(\rho\hat{x}^i + (1-\rho)\,\bar{x}^i\right),$$

since $\rho\hat{x}^i + (1-\rho)\,\bar{x}^i$ is a feasible consumption for each agent i when aggregate consumption is $\rho\hat{x} + (1-\rho)\,\bar{x}$. Hence, combining the above two expressions we obtain

$$u\left(\rho\hat{x} + (1-\rho)\,\bar{x};\lambda\right) \geq \rho u\left(\hat{x};\lambda\right) + (1-\rho)\,u\left(\bar{x};\lambda\right),$$

for any $\hat{x} \in X,\ \bar{x} \in X$ and $\rho \in (0,1)$, which is the desired result.

6.3) Question. Assume that each u^i is strictly increasing, concave and differentiable, that $X^i = R^m$. Let $x \in R^m_+$. Show that

$$\frac{\partial u\left(x;\lambda\right)}{\partial x_l}$$

is decreasing as a function of x_l for any $l = 1, 2, ..., m$. What is the economic interpretation of this? (maximum two lines).

Ans: This follows immediately from the concavity of $u\left(x;\lambda\right)$. Recall that in equilibrium the marginal utility of good l is directly related to its marginal social value γ_l (or, equivalently, to its price p_l). Thus, this result tells us that the scarcity of good l is directly related to its marginal social value.

7 Adding production: PO and CE allocations, shadow values and prices*

Recall that in our abstract GE notation there are m goods in the commodity space ($L = R^m$). To simplify the introduction of production into our analysis of P.O. allocations and shadow values, and of C.E. allocations and prices, we split the m goods into types: final output and inputs. Recall that abstract GE notation allows a good to be either. We let the first r goods, $l = 1, 2, ..., r$ be the final goods, produced by the firms and consumed by households. We let the second $m-r$ goods, $l = r+1, r+2, ..., m$ be inputs supplied by the households and bought by firms.

Moreover we let the production possibility set of firm j, Y^j, be described by a production function. We also specialize the model so that each firm produces only one type of final goods using all the inputs. To further simplify the problem we also allow only one type of firm producing each final good (we return to this below). Hence there is one firm (type) for each final output produced, so $J = r$ and hence firm j produces final good $l = j$, $1 \leq j \leq r$. (The general case allows a firm to produce jointly many final goods).

The production function f^j describes the production possibility set Y^j. Recall that in the abstract GE notation, purchases from firms are negative numbers, and sales from firms are positive numbers. Thus, we specify the production of final good $l = j$ using quantities of inputs $-y_{r+1}$, $-y_{r+2}$, $..., -y_m$ be described by a production function $f^j : R_+^{m-r} \to R_+$,

$$y_j^j = f^j \left(-y_{r+1}, -y_{r+2}, ..., -y_m\right),$$

where the function $f(\cdot)$ is increasing and concave in its $m-r$ arguments. Take, for example, the Cobb-Douglas case:

$$f^j \left(n_1, n_2, ..., n_{m-r}\right) = A \ \ (n_1)^{\alpha_1} \ \ (n_2)^{\alpha_2} \ \ \cdots \ \ (n_{m-r})^{\alpha_{m-r}} .$$

Formally the production possibility set Y^j is described as follows: $y \in Y^j$ if and only if

i) firm j produces only final output of good j, i.e.

$$y_l^j = 0 \text{ for } 1 \leq l \leq r \text{ and } l \neq j \ .$$

ii) final good $l = j$ is produced with the production function f^j, i.e.

$$y_j^j \leq f^j \left(-y_{r+1}, -y_{r+2}, \ \cdots \ , -y_m\right) \ .$$

For the utility function we let $u^i : R_+^r \times R_-^{m-r} \to R$ be increasing and concave in x. The first r commodities are final goods consumed by the households, so that

$$x_l^i \geq 0 \text{ for } l = 1, ..., r,$$

and that the second $m - r$ commodities are inputs, supplied by the households, so that

$$x_l^i \leq 0 \text{ for } l = r + 1, ..., m.$$

For example, the $r + 1$ commodity may represent labor services, so $x_{r+1} < 0$ means that the household is selling these services, i.e. suppling labor. We may also allow for some of these inputs not to enter in u^i, i.e. to have $\partial u^i / \partial x_l \equiv 0$. An example of such a case is land. In such a case we should set $x_l = 0$ for this input, and let $e_l > 0$ be the endowment of land supplied by this household.

Summarizing, a feasible allocation $\{x^i, y^j\}$ must satisfy:

$$\begin{aligned} x_l^i &\geq 0, \text{ for } l = 1, 2, ..., r \\ \left(-x_{r+s}^i\right) &\geq 0, \text{ for } s = 1, 2, ..., m - r, \end{aligned}$$

for all $i = 1, ..., I$ and

$$f^l\left(-y_{r+1}^l, -y_{r+2}^l, ..., -y_m^l\right) - y_l^l \geq 0, \tag{1}$$

$$\left(-y_{r+s}^l\right) \geq 0, \text{ for } s = 1, 2, ..., m - r \ ,$$

for all $l = 1, 2, ..., r(= J)$, and market clearing:

$$\sum_{i=1}^{I} x_l^i = \sum_{j=1}^{J} y_l^j + \sum_{i=1}^{I} e_l^i, \tag{2}$$

for all $l = 1, 2, ..., m$.

We can obtain the PO allocations by solving

$$\max_{\{x^i, y^j\}} \sum_{i=1}^{I} \lambda_i u^i\left(x^i\right),$$

subject to $\{x^i, y^j\}$ being feasible.

7.1) Question. Write the Lagrangian for the planner's problem. Use γ_l as the multiplier for the market clearing constraint of good l (2), and ρ_l as the multiplier of the production function for l (1).

Ans:

$$\begin{aligned}
\mathcal{L} &= \max_{\{x^i,y^j\}} \sum_{i=1}^{I} \lambda_i u^i\left(x_1^i,...,x_r^i,x_{r+1}^i,...,x_m^i\right) + \sum_{l=1}^{m}\gamma_l\left(\sum_{j=1}^{J} y_l^j + \sum_{i=1}^{I} e_l^i - \sum_{i=1}^{I} x_l^i\right) \\
&\quad + \sum_{l=1}^{r}\rho_l\left(f^l\left(-y_{r+1}^l,...,-y_m^l\right) - y_l^l\right).
\end{aligned}$$

We say that an allocation is interior if:

a) $x_l^i > 0$ for all $i = 1,...,I$ and $l = 1,...,r$,

b) $-x_{r+s}^i > 0$ for all $i = 1,...,I$ and $s = 1,...,m-r$,

c) $y_l^l > 0$ for all $l = 1,...,r$,

d) $-y_{r+s}^l > 0$ for all $l = 1,...,r$ and $s = 1,...,m-r$.

7.2) Question. Let $\{\bar{x}^i, \bar{y}^j\}$ be a PO allocation. Derive the first order condition (foc) for the Lagrangian written in question 7.1. Assume, to simplify that the allocation is interior. Write the foc separately for the following cases:

a) foc with respect to (wrt) x_l^i for $i = 1,2,...,I$ and $l = 1,...,r$,

b) foc wrt x_l^i for $i = 1,2,...,I$ and $l = r+1,...,m$, assuming $x_l^i < 0$,

c) foc wrt y_j^j for $j = 1,2,...,r$ assuming $y_j^j > 0$, and

d) foc wrt y_l^j for $l = r+1, r+2,...,m$ assuming $y_l^j < 0$.

Ans: We will derive the FOCs without assuming that the allocations are interior:

$$\begin{aligned}
x_l^i &: \lambda_i \frac{\partial u^i}{\partial x_l^i}\left(\bar{x}^i\right) - \gamma_l \le 0, \quad \left(= 0 \text{ if } x_l^i > 0\right) \quad \text{for } i = 1,2,...,I \text{ and } l = 1,...,r \\
x_{r+s}^i &: \lambda_i \frac{\partial u^i}{\partial x_{r+s}^i}\left(\bar{x}^i\right) - \gamma_{r+s} \ge 0, \quad \left(= 0 \text{ if } x_{r+s}^i < 0\right) \quad \text{for } i = 1,2,...,I \text{ and } s = 1,...,m-r \\
y_l^l &: \gamma_l - \rho_l \le 0, \quad \left(= 0 \text{ if } y_l^l > 0\right) \quad \text{for } l = 1,2,...,r \\
y_{r+s}^l &: \gamma_{r+s} - \rho_l \frac{\partial f^l}{\partial y_{r+s}^l}\left(\bar{y}^l\right) \ge 0, \quad \left(= 0 \text{ if } y_{r+s}^l < 0\right) \quad \text{for } l = 1,2,...,r \text{ and } s = 1,...,m-r.
\end{aligned}$$

When the allocations are interior, we can rewrite the above FOCs as follows:

$$\begin{aligned}
x_l^i &: \lambda_i \frac{\partial u^i}{\partial x_l^i}\left(\bar{x}^i\right) = \gamma_l \quad \text{for } i = 1,2,...,I \text{ and } l = 1,...,r \\
x_{r+s}^i &: \lambda_i \frac{\partial u^i}{\partial x_{r+s}^i}\left(\bar{x}^i\right) = \gamma_{r+s} \quad \text{for } i = 1,2,...,I \text{ and } s = 1,...,m-r \\
y_l^l &: \gamma_l = \rho_l \quad \text{for } l = 1,2,...,r \\
y_{r+s}^l &: \gamma_{r+s} = \rho_l \frac{\partial f^l}{\partial y_{r+s}^l}\left(\bar{y}^l\right) \quad \text{for } l = 1,2,...,r \text{ and } s = 1,...,m-r.
\end{aligned}$$

7.3) Question. Let $\left\{\bar{x}^i, \bar{y}^j\right\}$ be an interior PO allocation. Show that for any two final goods, $l, k = 1, ..., r$:

$$\begin{aligned} &= \frac{\partial u^1\left(\bar{x}^1\right)/\partial x_l}{\partial u^1\left(\bar{x}^1\right)/\partial x_k} = \frac{\partial u^2\left(\bar{x}^2\right)/\partial x_l}{\partial u^2\left(\bar{x}^2\right)/\partial x_k} = \cdots = \frac{\partial u^I\left(\bar{x}^I\right)/\partial x_l}{\partial u^I\left(\bar{x}^I\right)/\partial x_k} \\ &= \frac{\partial f^k\left(\bar{y}^k\right)/\partial y_{r+1}}{\partial f^l\left(\bar{y}^l\right)/\partial y_{r+1}} = \frac{\partial f^k\left(\bar{y}^k\right)/\partial y_{r+2}}{\partial f^l\left(\bar{y}^l\right)/\partial y_{r+2}} = \cdots = \frac{\partial f^k\left(\bar{y}^k\right)/\partial y_m}{\partial f^l\left(\bar{y}^l\right)/\partial y_m}. \end{aligned}$$

Give an economically interpretable label to these equations (two lines maximum).

Ans: Combining the FOC for x_l^i for any two $l, k = 1, ..., r$ yields

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_k^i} = \frac{\gamma_l}{\gamma_k}.$$

Similarly, combining the FOC for y_{r+s}^l for any two $l, k = 1, ..., r$ yields

$$\frac{\partial f^k\left(\bar{y}^k\right)/\partial y_{r+s}^k}{\partial f^l\left(\bar{y}^l\right)/\partial y_{r+s}^l} = \frac{\rho_l}{\rho_k} = \frac{\gamma_l}{\gamma_k}.$$

Thus, for any two final goods $l, k = 1, ..., r$ we obtain

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_k^i} = \frac{\left(\partial f^l\left(\bar{y}^l\right)/\partial y_{r+s}^l\right)^{-1}}{\left(\partial f^k\left(\bar{y}^k\right)/\partial y_{r+s}^k\right)^{-1}} = \frac{\gamma_l}{\gamma_k}, \tag{3}$$

for $i = 1, 2, ..., I$ and $s = 1, 2, ..., m - r$, which is the desired result. The above equation says that the every consumer's marginal rate of substitution must equal every input's marginal rate of technological transformation for all pairs of goods (notice that $\left(\partial f^l\left(\bar{y}^l\right)/\partial y_{r+s}^l\right)^{-1}$ is the additional number of units of input y_{r+s} needed to increase the output of good l in one unit).

7.4) Question. Let $\left\{\bar{x}^i, \bar{y}^j\right\}$ be an interior PO allocation. Show that for any final good $l = 1, ..., r$, and any input $r + s$, $s = 1, 2, ..., m - r$:

$$\begin{aligned} &\frac{\partial f^l\left(\bar{y}^l\right)}{\partial y_{r+s}} \\ &= \frac{\partial u^1\left(\bar{x}^1\right)/\partial x_{r+s}}{\partial u^1\left(\bar{x}^1\right)/\partial x_l} = \frac{\partial u^2\left(\bar{x}^2\right)/\partial x_{r+s}}{\partial u^2\left(\bar{x}^2\right)/\partial x_l} = \cdots = \frac{\partial u^I\left(\bar{x}^I\right)/\partial x_{r+s}}{\partial u^I\left(\bar{x}^I\right)/\partial x_k}. \end{aligned}$$

Give an economically interpretable label to these equations (two lines maximum).

Ans: Combining the FOC for x_l^i for any $l = 1, ..., r$ and any $s = 1, 2, ..., m - r$ yields

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_{r+s}^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i} = \frac{\gamma_{r+s}}{\gamma_l}.$$

Now, the FOC for y_{r+s}^l can be written as

$$\frac{\partial f^l\left(\bar{y}^l\right)}{\partial y_{r+s}^l} = \frac{\gamma_{r+s}}{\rho_l} = \frac{\gamma_{r+s}}{\gamma_l}$$

Thus, for any final good $l = 1, ..., r$ and any input $r + s,\ \ s = 1, 2, ..., m - r$ we obtain

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_{r+s}^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i} = \frac{\partial f^l\left(\bar{y}^l\right)}{\partial y_{r+s}^l} = \frac{\gamma_{r+s}}{\gamma_l}, \tag{4}$$

for $i = 1, 2, ..., I$, which is the desired result. The above equation says that the every consumer's marginal rate of substitution must equal every firm's marginal rate of transformation for all input-ouput pairs.

7.5) Question. Show that if an interior allocation $\left\{\bar{x}^i, \bar{y}^j\right\}$ satisfies the equations in Question 7.3 and 7.4, and market clearing (2), then the allocation is PO. In particular, construct the Lagrange multipliers γ_l for $l = 1, 2, ..., m$, and ρ_k for $k = 1, 2, ..., r$, and the weights λ_i for $i = 1, 2, ..., I$ and check the (sufficient) foc derived in Question 7.2.

Ans: From the Lagrangian of the planner's problem it is evident that if we multiply λ_i, γ_l and ρ_k for all i, l and k by a constant, then the optimal equilibrium allocation will be unmodified. Thus, we are free to normalize these parameters by γ_1 so that $\gamma_1 = 1$. Now assume that the interior allocation $\left\{\bar{x}^i, \bar{y}^j\right\}$ satisfies equations (3), (4) and market clearing (2). Then, from (3) and (4) we can write

$$\gamma_l = \frac{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_1^i},$$

for $l = 2, 3, ..., m$, where we have used the fact that $\gamma_1 = 1$. Also, we will set

$$\rho_k = \gamma_k,$$

for $k = 1, 2, ..., r$. Finally, we will let

$$\lambda_i = \frac{1}{\partial u^i\left(\bar{x}^i\right)/\partial x_1^i},$$

for $i = 1, 2, ..., I$. It is now straightforward to show that, given the above $\lambda-$weights and

Lagrange multipliers, the feasible allocation $\left\{\bar{x}^i, \bar{y}^j\right\}$ satisfies the sufficient FOCs derived in Question 7.3. Thus, this allocation is PO.

We now turn to the analysis of the CE for this economy. We let $p = (p_1, p_2, ..., p_m)'$ denote the price vector.

The problem of firm $j = 1, 2, ..., J \ (= r)$

$$\pi^j = \max_{y^j \in Y^j} \ p \ y^j,$$

can be written as

$$\pi^j = \max_{\left(-y^j_{r+s}\right) \geq 0, \ s=1,...,m-r} \ p_j \ f^j \left(\left(-y^j_{r+1}\right), \ \left(-y^j_{r+2}\right), \ \cdots, \ \left(-y^j_m\right) \right) \ - \sum_{s=1}^{m-r} p_{r+s} \ \left(-y^j_{r+s}\right).$$

The problem of household $i \ = 1, 2, ..., I$

$$\max_{x \in X^i} u^i \left(x^i\right),$$

subject to

$$px^i \leq pe^i + \sum_{j=1}^{J} \theta^i_j \pi^j,$$

can be written as

$$\max u^i \left(x^i\right),$$

subject to

$$\begin{aligned} x^i_l &\geq 0 \text{ for } l = 1, 2, ..., r, \\ -x^i_{r+s} &\geq 0 \text{ for } s = 1, 2, ..., m-r, \end{aligned}$$

and

$$\sum_{l=1}^{r} p_l \ x^i_l \leq \sum_{l=1}^{r} p_l \ e^i_l + \sum_{s=1}^{m-r} p_{r+s} \ \left(-x^i_{r+s} + e^i_{r+s}\right) + \sum_{j=1}^{J} \theta^i_j \pi^j \ .$$

7.6) Question. Write the foc for the problem of firm j assuming that y^j is interior.

Ans:

$$y^l_{r+s} : -p_l \frac{\partial f^l}{\partial y^l_{r+s}} \left(\bar{y}^l\right) + p_{r+s} \geq 0, \quad \left(= 0 \text{ if } y^l_{r+s} < 0\right) \ \ \text{for } l = 1, 2, ..., r \text{ and } s = 1, ..., m-r.$$

When the allocations are interior, we can rewrite the above FOC as follows:

$$y_{r+s}^{l}: p_{r+s} = p_l \frac{\partial f^l}{\partial y_{r+s}^l}\left(\bar{y}^l\right) \text{ for } l = 1,2,...,r \text{ and } s = 1,...,m-r.$$

7.7) Question. Write the foc condition for the problem of household i assuming that x^i is interior. Use μ_i to denote the Lagrange multiplier on the budget constraint of household i.

Ans:

$$\begin{aligned}
x_l^i &: \frac{\partial u^i}{\partial x_l^i}\left(\bar{x}^i\right) - p_l\mu_i \leq 0, \quad \left(= 0 \text{ if } x_l^i > 0\right) \text{ for } i = 1,2,...,I \text{ and } l = 1,...,r \\
x_{r+s}^i &: \frac{\partial u^i}{\partial x_{r+s}^i}\left(\bar{x}^i\right) - p_{r+s}\mu_i \geq 0, \quad \left(= 0 \text{ if } x_{r+s}^i < 0\right) \text{ for } i = 1,2,...,I \text{ and } s = 1,...,m-r.
\end{aligned}$$

When the allocations are interior, we can rewrite the above FOCs as follows:

$$\begin{aligned}
x_l^i &: \frac{1}{\mu_i}\frac{\partial u^i}{\partial x_l^i}\left(\bar{x}^i\right) = p_l \text{ for } i = 1,2,...,I \text{ and } l = 1,...,r \\
x_{r+s}^i &: \frac{1}{\mu_i}\frac{\partial u^i}{\partial x_{r+s}^i}\left(\bar{x}^i\right) = p_{r+s} \text{ for } i = 1,2,...,I \text{ and } s = 1,...,m-r.
\end{aligned}$$

7.8) Question. Show that if $\left\{\bar{x}^i, \bar{y}^j, p\right\}$ is a CE with an interior allocation, then the equations in Questions 7.3 and 7.4 are satisfied.

Ans: Combining the FOC for x_l^i for any two $l, k = 1,...,r$ yields

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_k^i} = \frac{p_l}{p_k}.$$

Similarly, combining the FOC for y_{r+s}^l for any two $l, k = 1,...,r$ yields

$$\frac{\partial f^k\left(\bar{y}^k\right)/\partial y_{r+s}^k}{\partial f^l\left(\bar{y}^l\right)/\partial y_{r+s}^l} = \frac{p_l}{p_k}.$$

Thus, for any two final goods $l, k = 1,...,r$ we obtain

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_k^i} = \frac{\partial f^k\left(\bar{y}^k\right)/\partial y_{r+s}^k}{\partial f^l\left(\bar{y}^l\right)/\partial y_{r+s}^l} = \frac{p_l}{p_k}, \tag{5}$$

for $i=1,2,...,I$ and $s=1,2,...,m-r$, so the equations in Question 7.3 are satisfied. Now, combining the FOC for x_l^i for any $l=1,...,r$ and any $s=1,2,...,m-r$ yields

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_{r+s}^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}=\frac{p_{r+s}}{p_l}.$$

Now, the FOC for y_{r+s}^l can be written as

$$\frac{\partial f^l\left(\bar{y}^l\right)}{\partial y_{r+s}^l}=\frac{p_{r+s}}{p_l}$$

Thus, for any final good $l=1,...,r$ and any input $r+s,\ s=1,2,...,m-r$ we obtain

$$\frac{\partial u^i\left(\bar{x}^i\right)/\partial x_{r+s}^i}{\partial u^i\left(\bar{x}^i\right)/\partial x_l^i}=\frac{\partial f^l\left(\bar{y}^l\right)}{\partial y_{r+s}^l}=\frac{p_{r+s}}{p_l}, \tag{6}$$

for $i=1,2,...,I$, so the equations in Question 7.4 are satisfied.

7.9) Question. Assume that $\left\{\bar{x}^i,\bar{y}^j,p\right\}$ is a CE with an interior allocation. Use your answer to the previous question to show that $\left\{\bar{x}^i,\bar{y}^j\right\}$ is a Pareto Optimal allocation. What is the relationship between the price p and the Lagrange multiplier γ? What is the relationship between the Lagrange multipliers μ_i and the weights λ_i?

Ans: Comparing the FOCs of the social planner problem with those of the CE it is evident that we can make these FOCs identical by setting

$$\gamma_l=p_l,$$

for $l=1,2,...,m$, and

$$\lambda_i=\frac{1}{\mu_i},$$

for $i=1,2,...,I$. To pin down the $\lambda-$weights we will let $p_1=1$. Then, from the FOC for x_1^i we can write

$$\mu_i=\frac{\partial u^i}{\partial x_1^i}\left(\bar{x}^i\right),$$

for $i=1,2,...,I$. It is now straightforward to show that, given the above $\lambda-$weights and Lagrange multipliers, the feasible CE allocation $\left\{\bar{x}^i,\bar{y}^j\right\}$ satisfies the sufficient FOCs derived in Question 7.3. Thus, this allocation is PO.

As usual, goods with high marginal social value (high γ) have high prices in the CE allocation. Similarly, agents with high $\lambda-$weights are those with low marginal utility of income (low γ) in the CE; that is, agents that enjoy high levels of consumption.

7.10) Question. Assume that $\{\bar{x}^i, \bar{y}^j\}$ is an interior PO allocation. Use your answer to the previous questions (7.3, 7.4, 7.6, and 7.7) to find a price p so that $\{\bar{x}^i, \bar{y}^j, p\}$ is a CE for some ownership structure. Make sure you specify how the given allocation is used to construct the prices p, and to check that the allocation and prices indeed constitute a competitive equilibrium. Specify the ownership structure (i.e., the θ_j^i and the e^i).

Ans: Given the interior PO allocation $\{\bar{x}^i, \bar{y}^j\}$, we first use our answer for Question 7.5 to construct the $\lambda-$weights and Lagrange multipliers γ of the social planner's problem. Then, we use our previous answer to find the prices and Lagrange multipliers μ of the CE allocation. In particular, we set $p = \gamma$ and $\mu = 1/\lambda$. Given these prices and Lagrange multipliers, it is straightforward to check that the allocation $\{\bar{x}^i, \bar{y}^j, p\}$ satisfies the sufficient FOCs derived in Questions 7.6 and 7.7. Hence, this allocation is a CE. It only rests to specify the ownership structure. To do so, consider the budget constraint of agent i and the feasibility constraint for good l:

$$\sum_{l=1}^{m} p_l \ \bar{x}_l^i = \sum_{l=1}^{m} p_l \ e_l^i + \sum_{j=1}^{J} \theta_j^i \left(p \ \bar{y}^j\right) \ ,$$

$$\sum_{i=1}^{I} \bar{x}_l^i = \sum_{i=1}^{I} e_l^i + \sum_{j=1}^{J} \bar{y}_l^j.$$

We also have the constraint that each firm is entirely owned by households:

$$\sum_{i=1}^{I} \theta_j^i = 1.$$

Thus, we have $I + m + J$ equations (I budget constraints, m market clearing constraints, and $J\,(= r)$ adding-up restrictions) in $(m + J) \times I$ unknowns (e_l^i for I agents and m goods and θ_j^i for I agents and $J(= r)$ firms). This gives us $m\,(I - 1) + I\,(J - 1) - J$ free variables, which is a strictly positive number for $I \geq 2$ (of course, when $I = 1$ the unique agent would own all of the firms and endowments in the economy). Hence, there will typically be infinitely many ownership structures consistent with the CE allocation $\{\bar{x}^i, \bar{y}^j, p\}$. To pin down a particular ownership structure, we will assume that there are as many firms as there are agents, $I = J$, and that agent i only owns firm $j = i$:

$$\bar{\theta}_j^i = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the BC for agent i can be written as

$$\sum_{l=1}^{m} p_l \ \bar{x}_l^i = \sum_{l=1}^{m} p_l \ e_l^i + \sum_{l=1}^{m} p_l \ \bar{y}_l^i = \sum_{l=1}^{m} p_l \ \left(e_l^i + \bar{y}_l^j\right).$$

Hence, if we let

$$\bar{e}_l^i = \bar{x}_l^i - \bar{y}_l^j,$$

for $i = 1, ..., I$ and $l = 1, ..., m$, the BC for agent i will be satisfied by construction. Moreover, adding the above equation across agents we obtain

$$\sum_{i=1}^{I} \bar{x}_l^i = \sum_{i=1}^{I} \bar{e}_l^i + \sum_{i=1}^{I} \bar{y}_l^j,$$

which implies that the proposed ownership structure $\left(\bar{\theta}^i, \bar{e}^i\right)$ satisfies market clearing as well. Thus, we conclude that the allocation $\left\{\bar{x}^i, \bar{y}^j, p\right\}$ is a CE allocation for the ownership structure $\left(\bar{\theta}^i, \bar{e}^i\right)$, as was to be shown.

7.11) Question. Suppose that there are several firms that can produce the same final good j using inputs $l = r+1, r+2, ..., m$ but possibly using different technologies. In particular, let's denote two such firms by h and q with production functions f^h and f^q. Assume that $\left\{\bar{x}^i, \bar{y}^j, p\right\}$ is a CE. What is the relationship between $\partial f^h\left(\bar{y}^h\right)/\partial y_{r+s}$ and $\partial f^q\left(\bar{y}^q\right)/\partial y_{r+s}$?

Ans: Combining the FOC for y_{r+s}^l for any two $h, q = 1, ..., r$ yields

$$\frac{\partial f^h\left(\bar{y}^h\right)/\partial y_{r+s}^h}{\partial f^q\left(\bar{y}^q\right)/\partial y_{r+s}^q} = \frac{p_q}{p_h}.$$

Since both firms are producing the same good, we must have that $p_q = p_h$. Then,

$$\frac{\partial f^h\left(\bar{y}^h\right)}{\partial y_{r+s}^h} = \frac{\partial f^q\left(\bar{y}^q\right)}{\partial y_{r+s}^q},$$

for $s = 1, ..., m - r$. That is, on the margin, every input must be equally productive for all firms that produce the same good.

8 Exercise: "Donward Slopping Demand"

In this problem we examine the $\lambda-$weighted planner's problem for a pure-exchange economy. We want to relate the Lagrange multipliers of the endowment vector of the planner's problem to the prices in the corresponding competitive equilibrium (CE). In particular we want to find conditions under which the relative price of a good decreases as its endowment increases. We find that when a good is normal, then if the endowment of that good increases, its relative shadow value for the planner will decrease. The exercise is divided into 5 sections, and has a total of 185 points.

Part I. Social Planner's problem [35 points].

Let $L \equiv R^m$ be the commodity space (i.e. there are m goods), let $e \in L$ be the aggregate endowment. Let $u^i : R^m_+ \to R$ be the utility function of each agent $i = 1, 2, ..., I$. We assume that all u^i are strictly increasing and strictly concave. Let $\lambda \in R^I_{++}$ be the Pareto weight assigned to each agent. Consider the representative agent utility $U : R^m_+ \times R^I_+ \to R$ defined as

$$U(e; \lambda) = \max_{x^i \in R^n_+, \quad i=1,...,I} \sum \lambda_i \, u^i(x^i) \tag{7}$$

such that

$$\sum_{i=1}^{I} x^i = e. \tag{8}$$

I.1) [5 points] Form the Lagrangian of the problem denoting by $\gamma \in R^m$ the Lagrange multiplier of the constraint (8). Make sure to use the convention of the signs so that γ will be positive. Your answer should be one line.

Ans:

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \, u^i(x^i) + \gamma \left(e - \sum_{i=1}^{I} x^i \right). \tag{9}$$

I.2) [5 points] Write the foc of the problem w.r.t. x^i. Your answer should be one line.

Ans:

$$\lambda_i \, \frac{\partial u(x^i)}{\partial x} - \gamma = 0. \tag{10}$$

I.3) [5 points] Use the envelope theorem to find an expression for $\partial U(e; \lambda) / \partial e$. Your answer should be one line. Be carefull on writing all the argument of the functions involved.

Ans:

There are two ways that this can be done.

1) From (9)

$$\frac{\partial U(e; \lambda)}{\partial e} = \frac{\partial \mathcal{L}}{\partial e} = \gamma.$$

2) Alternative:

Totally differentiate $\mathcal{L}$ wrt e

$$\begin{aligned} \frac{d\mathcal{L}}{de} &= \sum_{i=1}^{I} \lambda_i \frac{\partial u^i(x^*)}{\partial x^i} \frac{\partial x^*}{\partial e} + \gamma \frac{\partial \left(e - \sum_{i=1}^{I} x^* \right)}{\partial x^i} \frac{\partial x^*}{\partial e} \\ &+ \left(e - \sum_{i=1}^{I} x^* \right) \frac{\partial \gamma}{\partial e} + \sum_{i=1}^{I} \lambda_i \frac{\partial u^i(x^*)}{\partial e} + \gamma \frac{\partial \left(e - \sum_{i=1}^{I} x^i \right)}{\partial e} \end{aligned}$$

We can rewrite this expression as

$$\begin{aligned}\frac{d\mathcal{L}}{de} &= \left(\sum_{i=1}^{I}\left[\lambda_i \frac{\partial u^i(x^*)}{\partial x^i} - \gamma\right]\right)\frac{\partial x^*}{\partial e} \\ &\quad + \left(e - \sum_{i=1}^{I} x^*\right)\frac{\partial \gamma}{\partial e} + \sum_{i=1}^{I}\lambda_i \frac{\partial u^i(x^*)}{\partial e} + \gamma \frac{\partial\left(e - \sum_{i=1}^{I} x^i\right)}{\partial e}\end{aligned}$$

Using (10) and that $e = \sum_{i=1}^{I} x^*$ or $\gamma = 0$ and thus $\frac{\partial\gamma}{\partial e} = 0$,

$$\frac{d\mathcal{L}}{d\gamma} = \sum_{i=1}^{I}\lambda_i \frac{\partial u^i(x^*)}{\partial e} + \gamma$$

Finally, given that $\frac{\partial u^i(x^*)}{\partial e} = 0$ (e affects u^i only through x^*),

$$\frac{d\mathcal{L}}{d\gamma} = \gamma$$

I.4) [10 points] Show that $U(e;\lambda)$, as defined in (7), is concave on e, i.e. that for all $\kappa \in (0,1)$ and $\hat{e},\ \tilde{e} \in R_+^m$:

$$U(\bar{e};\lambda) > \kappa U(\hat{e};\lambda) + (1-\kappa)U(\tilde{e};\lambda),$$

where

$$\bar{e} = \kappa\hat{e} + (1-\kappa)\tilde{e}.$$

[Hints: Let $x^i(\hat{e};\lambda)$ and $x^i(\tilde{e};\lambda)$ be the solution of each of the problems. Define $\bar{x}^i$ as

$$\bar{x}^i = \kappa x^i(\hat{e};\lambda) + (1-\kappa)x^i(\tilde{e};\lambda).$$

Show that $\bar{x}^i$ is feasible for $\bar{e}$ and use the concavity of each u^i to establish the result.]

Ans: To simplify the notation we suppress the argument λ. Notice that

$$\begin{aligned}\sum_i x^i(\hat{e}) &= \hat{e}, \\ \sum_i x^i(\tilde{e}) &= \tilde{e},\end{aligned}$$

and hence

$$\sum_i \bar{x}^i = \kappa\sum_i x^i(\hat{e}) + (1-\kappa)\sum_i x^i(\tilde{e}) = \kappa\hat{e} + (1-\kappa)\tilde{e} = \bar{e},$$

so $\bar{x}^i$ is feasible for $\bar{e}$. Thus:

$$U(\bar{e}) \geq \sum_{i=1}^{I} \lambda_i u^i\left(\bar{x}^i\right),$$

and by concavity of u^i

$$u^i\left(\bar{x}^i\right) \geq \kappa u^i\left(x^i(\hat{e})\right) + (1-\kappa) u^i\left(x^i(\tilde{e})\right).$$

Hence

$$\begin{aligned} U(\bar{e}) &\geq \sum_{i=1}^{I} \lambda_i u^i\left(\bar{x}^i\right) \geq \sum_{i=1}^{I} \lambda_i \left[\kappa u^i\left(x^i(\hat{e})\right) + (1-\kappa) u^i\left(x^i(\tilde{e})\right)\right], \\ &= \kappa \sum_{i=1}^{I} \lambda_i u^i\left(x^i(\hat{e})\right) + (1-\kappa) \sum_{i=1}^{I} \lambda_i u^i\left(x^i(\tilde{e})\right), \\ &= \kappa U(\hat{e}) + (1-\kappa) U(\tilde{e}). \end{aligned}$$

I.5) [10 points] Let $\lambda' = \alpha\,\lambda$ for an arbitrary scalar $\alpha > 0$, so that λ' are a rescaled version of the weights λ. Show that

$$U\left(e; \lambda'\right) = \alpha U(e; \lambda).$$

Letting $x^i(e,\lambda)$ be the optimal allocation and $\gamma(e,\lambda)$ the Lagrange multiplier of the feasibility constraint in the original problem, show that

$$\begin{aligned} x^i\left(e, \lambda'\right) &= x^i(e, \lambda), \\ \gamma\left(e, \lambda'\right) &= \alpha\,\gamma(e, \lambda). \end{aligned}$$

[Hint: Use the necessary foc for the problem with λ to verify that sufficient foc for the problem with λ'.]

Ans: FOCs for $U(e,\lambda)$:

$$\lambda_i \frac{\partial u^i\left(x^i\right)}{\partial x^i} = \gamma(e, \lambda) \tag{11}$$

$$\sum_i x^i = e \tag{12}$$

by the fact that $\lambda > 0$.

FOCs for $U\left(e,\lambda'\right)$:

$$\begin{aligned} \lambda_i' \frac{\partial u^i\left(x^i\right)}{\partial x^i} &= \gamma\left(e,\lambda'\right) && (13) \\ \sum_i x^i &= e && (14) \end{aligned}$$

From (13) and using (11)

$$\gamma\left(e,\lambda'\right) = \lambda_i' \frac{\partial u^i\left(x^i\right)}{\partial x^i} = \alpha \lambda_i \frac{\partial u^i\left(x^i\right)}{\partial x^i} = \alpha \gamma\left(e,\lambda\right)$$

So,

$$\gamma\left(e,\lambda'\right) = \alpha\gamma\left(e,\lambda\right)$$

This fact implies that actually (11) and (13) contain de same information (actually, they are the same!). This, together with the fact that the budget constraints are the same, implies that

$$x^i\left(e,\lambda'\right) = x^i\left(e,\lambda\right)$$

Finally,

$$\begin{aligned} U\left(e,\lambda'\right) &= \sum_{i=1}^{I} \lambda_i' u^i\left(x^i\left(e,\lambda'\right)\right), \\ &= \sum_{i=1}^{I} \lambda_i' u^i\left(x^i\left(e,\lambda\right)\right), \\ &= \alpha \sum_{i=1}^{I} \lambda_i u^i\left(x^i\left(e,\lambda\right)\right), \\ &= \alpha U\left(e,\lambda\right). \end{aligned}$$

Part II. Agents' problem in a Competitive Equilibrium. [60 points]

Given prices $p \in R_+^m$ and expenditure $y \in R_+$, define the indirect utility function of agent i as

$$V^i\left(y;p\right) = \max_{x \in R_+^m} \ u^i\left(x\right), \tag{15}$$

subject to:

$$p\ x = y. \tag{16}$$

A competitive equilibrium for an economy described by utilities and endowments $\left\{u^i, e^i\right\}$ is an allocation and price vector $\left\{x^i, p\right\}$ such that x^i solves problem (15) for $y^i = pe^i$ and markets clear,

$$\sum_{i=1}^{I} x^i = e, \tag{17}$$

where

$$e \equiv \sum_{i=1}^{I} e^I. \tag{18}$$

II.1) [5 points] Form the Lagrangian of the problem (15) letting μ_i be the Lagrange multiplier of the budget constraint (16). Make sure to use a convention for the signs so that μ_i will be positive. Your answer should be one line.

Ans:

$$\mathcal{L}^i \ = u^i(x) + \mu_i \ (y - px).$$

II.2) [5 points] Write the foc of the problem w.r.t. x^i. Your answer should be one line.

Ans:

$$\frac{\partial u^i(x)}{\partial x} - \mu_i p = 0$$

II.3) [5 points] Use the envelope theorem to find an expression for $\partial V_i(y,p)/\partial y$. Your answer should be one line. Be carefull on writing all the argument of the functions involved.

Ans:

$$\frac{\partial V_i(y,p)}{\partial y} = \mu_i.$$

II.4) [15 points] Let's consider the case where there are two commodities, i.e. $m = 2$. Find conditions on $u_1 \equiv \partial u^i/\partial x_1$, $u_2 \equiv \partial u^i/\partial x_2$, $u_{12} \equiv \partial^2 u^i/\partial x_1 \partial x_2$, and $u_{22} = \partial^2 u^i/\partial x_2 \partial x_2$ so that good 1 is a normal good:

$$\frac{\partial x_1^i(y,p)}{\partial y} > 0.$$

In particular differentiate the following foc's with respect to y:

$$\frac{u_1(x_1(y), [y - x_1(y) \ p_1]/p_2)}{u_2(x_1(y), [y - x_1(y) p_1]/p_2)} = \frac{p_1}{p_2},$$

where we suppress the argument p and the superindex i for clarity. Write this equation as

$$\phi(x_1(y), y) = \frac{p_1}{p_2},$$

where

$$\phi\left(x_1, y\right) \equiv \frac{u_1\left(x_1, \left[y - x_1 p_1\right]/p_2\right)}{u_2\left(x_1, \left[y - x_1 p_1\right]/p_2\right)},$$

and hence

$$\frac{\partial x_1}{\partial y} = \frac{\partial \phi}{\partial y} / \left[-\frac{\partial \phi}{\partial x_1}\right]$$

[Hint: if you obtain an expression such as

$$\left(u_2\right)^2 u_{11} - 2u_1 u_2\ u_{12} + \left(u_1\right)^2 u_{22},$$

notice that it equals

$$= \left(u_2\ u_1\right)\left[\frac{\partial^2 u}{\partial x \partial x}\right]\left(u_2\ u_1\right)',$$

which is negative since it is quadratic function of the second derivative of u.]

Ans:

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{u_2\left[u_{11} - \frac{p_1}{p_2} u_{12}\right] - u_1\left[u_{21} - \frac{p_1}{p_2} u_{22}\right]}{\left[u_2\right]^2}, \\
\frac{\partial \phi}{\partial y} &= \frac{1}{p_2} \frac{u_{12} u_2 - u_{22} u_1}{\left[u_2\right]^2},
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{u_2\left[u_{11} - \frac{p_1}{p_2} u_{12}\right] - u_1\left[u_{21} - \frac{p_1}{p_2} u_{22}\right]}{\left[u_2\right]^2}, \\
&= \frac{u_2\left[u_{11} - \frac{u_1}{u_2} u_{12}\right] - u_1\left[u_{21} - \frac{p_1}{p_2} u_{22}\right]}{\left[u_2\right]^2}, \\
&= \frac{\left(u_2\right)^2 u_{11}/u_2 - 2u_1 u_2\ u_{12}/u_2 + \left(u_1\right)^2 u_{22}/u_2}{\left[u_2\right]^2}, \\
&= \frac{\left(u_2\right)^2 u_{11} - 2u_1 u_2\ u_{12} + \left(u_1\right)^2 u_{22}}{\left[u_2\right]^3} < 0,
\end{aligned}$$

since

$$\left(u_2\right)^2 u_{11} - 2u_1 u_2\ u_{12} + \left(u_1\right)^2 u_{22} = \left(u_2\ u_1\right)\left[\frac{\partial^2 u}{\partial x \partial x}\right]\left(u_2\ u_1\right)',$$

a quadratic of the second derivative of u. Then

$$\frac{\partial x_1}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right) / \left(-\frac{\partial \phi}{\partial x_1}\right),$$

and thus

$$sign\left(\frac{\partial x_1}{\partial y}\right) = sign\left(\frac{\partial \phi}{\partial y}\right) = sign\left(u_{12}u_2 - u_{22}u_1\right).$$

II.5) [10 points] Define the marginal rate of substitution as:

$$m\left(x_1, x_2\right) = \frac{u_1\left(x_1, x_2\right)}{u_2\left(x_1, x_2\right)}.$$

Compute and expression for

$$\frac{\partial}{\partial x_2} m\left(x_1, x_2\right).$$

Ans:

$$\frac{\partial}{\partial x_2} m\left(x_1, x_2\right) = \frac{u_{12}u_2 - u_{22}u_1}{\left[u_2\right]^2}.$$

II.6) [5 points] Use your answers to II.4 and II.5 to show that $\partial x_1\left(p, y\right)/\partial y > 0$ iff $\partial m\left(x_1, x_2\right)/\partial x_2 > 0$.

Ans:

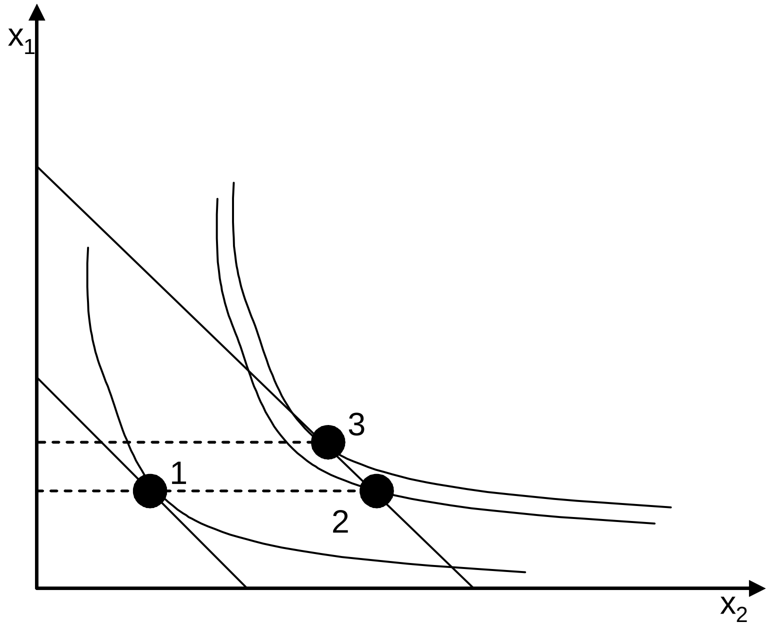
$$\begin{aligned} sign\left(\frac{\partial}{\partial x_2} m\left(x_1, x_2\right)\right) &= sign\left(u_{12}u_2 - u_{22}u_1\right), \\ sign\left(\frac{\partial x_1}{\partial y}\right) &= sign\left(u_{12}u_2 - u_{22}u_1\right). \end{aligned}$$

II.7) [15 points] Interpret your answer to II.5 using a graph with x_1 and x_2. You must produce 3 figures. In figure 1 good x_1 is a normal good. In figure 2 good one has zero income elasticity. In figure 3 good one has negative income elasticity. In each figure draw two budget lines that corresponds to the same relative prices p_1/p_2 but with different income. Graph x_2 in the horizontal axis of each figure. Include the two indifferent curves that correspond to the choices for the two different budget sets, one for low income and one for high income. Include also an indifference curve that intersects the budget line with the highest income and that keeps x_1 at the same level as the optimal choice for the low income.

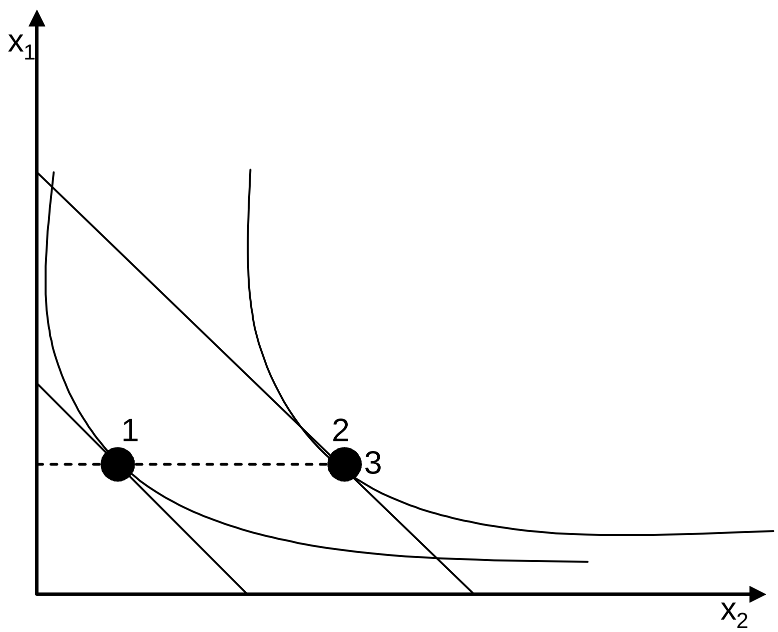
Ans:

For further reference: the point denoted by the number 1 is the original allocation, the point denoted by 2 is the allocation that keeps unchanged x_1 when the income is increased, and the point denoted by 3 is the optimal allocation under the new budget constraint.

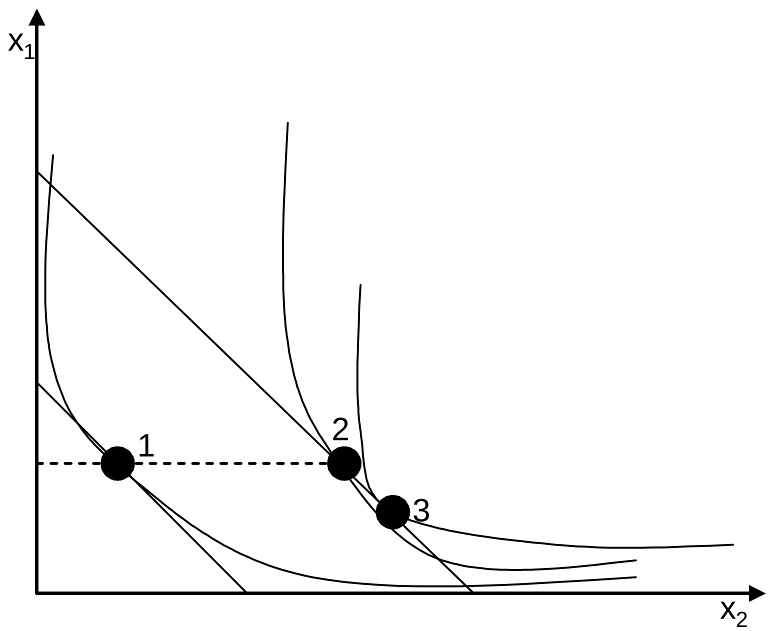
When x_1 is a normal good:



When x_1 has zero income elasticity:



When x_1 has negative income elasticity:



For future reference, we state without proof, that if u^i is strictly concave, then $V^i(y,p)$ is strictly concave in y. (the proof of this proposition follows the same logic than the one of question I.4)

Here is the proof for completeness: Let $y = \omega y_1 + (1-\omega) y_2$ for $\omega \in (0,1)$. Let $x(y_j;p)$ be the optimal solution. Then

$$\begin{aligned}
& p\left[\omega\ x(y_1,p) + (1-\omega)\ \ x(y_2,p)\right], \\
= & \ \omega\ px(y_1,p) + (1-\omega)\quad p\ x(y_2,p), \\
= & \ \omega\ y_1 + (1-\omega)\quad p\ y_2, \\
= & \ y,
\end{aligned}$$

and hence $\omega\ x(y_1,p) + (1-\omega)\ \ x(y_2,p)$ is budget feasible for y. Thus

$$\begin{aligned}
V^i(y,p) & \geq u^i\left(\omega\ x(y_1,p) + (1-\omega)\ \ x(y_2,p)\right), \\
& = \omega\ u^i\left[x(y_1,p)\right] + (1-\omega)\, u^i\left[\ x(y_2,p)\right], \\
& = \omega\ V^i(y_1,p) + (1-\omega)\, V^i(y_2,p).
\end{aligned}$$

III. Interpretation of prices in a CE as multipliers in the Pareto problem [15

points]

Consider the solution of the planner's problem (7) x^i and its associated lagrange multiplier vector γ for a vector of weights λ and aggregate endowment e.

Consider a CE for the economy with allocation $\bar{x}^i$, associated price vector p and lagrange multipliers μ^i for the constraint (16).

III.1) [10 points] Let's denote by $\left\{p, \bar{x}^i\right\}$ the price and allocation in a CE, and by $\{\mu_i\}$ the corresponding Lagrange mulitpliers of the budget constraints in tha agent problem for an economy with endowments $\left\{\bar{e}^i\right\}$. Let's denote by $\left\{x^i\right\}$ a Pareto Optimal allocation solving the planner's problem with weights λ in an economy with aggregate endowment e. Assume that $e = \sum_{i=1}^{I} \bar{e}^i$. If $\bar{x}^i = x^i$ what is the relationship between γ and p? What is the relationship between μ^i to λ_i ? (your answer should be two formulas)

Ans:

Recall that the FOC in a CE is

$$\frac{\partial u^i\left(x^i\right)}{\partial x^i} = p\mu_i$$

Then, it is trivial to set

$$\begin{aligned} p &= \gamma \\ \lambda_i &= 1/\mu_i \end{aligned}$$

III.2) [5 points] In a CE only relative prices are determined, i.e. the CE has the same allocation for the price vector p than for the vector κp for any $\kappa > 0$.. What is the analogous property for the planner's problem? Your answer can be done in one line. [Hint: It suffices to point out which previous question addressed this issue.]

Ans: The optimal allocation of the planner's problem is homogenous of degree zero in λ. This argument is analogous to think that we already found that the PO allocation remains unchanged for any $\gamma' = \alpha\gamma$, $\alpha > 0$.

IV. Relative prices p and "relative multipliers γ" [35 points]

Let's index the endowment of the n commodity by ε, i.e.:

$$e = (e_1, ..., e_n, ..., e_m) = (e_1, ..., \varepsilon, ..., e_m)$$

so that $e_n = \varepsilon > 0$.

IV.1) [5 points] Consider the derivative of the planner's problem defined in (7). Argue that $\partial U(e;\lambda)/\partial e_n$ is decreasing in ε. Your answer can be done in one line using properties of U previously obtained.

Ans: We have shown that $U(\cdot,\lambda)$ is concave, and a concave function has negative second derivatives.

The previous question has shown that if the endowment of a commodity n increases its associated shadow value γ_n in the social planner decreases. But we have also shown that the units of γ are meaningless, i.e. only relative values of γ matter. This leads us to analyze the following ratio:

$$\frac{\gamma_n(e)}{\gamma_j(e)} = \frac{\gamma_n(e_1,...,\varepsilon,...,e_m)}{\gamma_j(e_1,...,\varepsilon,...,e_m)} = \frac{\partial U(e;\lambda)/\partial e_n}{\partial U(e;\lambda)/\partial e_j} \tag{19}$$

We now show that $\gamma_n(e)/\gamma_j(e)$ is not necessarily decreasing in $e_n = \varepsilon$ for $j \neq n$. To see this, consider a simpler case where there is only one agent and two goods, so that trivially $x^1(e,\lambda) = e$ and $U(e,\lambda) = u(e)$, where $u(\cdot)$ is the utility function of agent $i=1$. We are asking whether

$$\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)} = \frac{\partial u(e_1,e_2)/\partial e_1}{\partial u(e_1,e_2)/\partial e_2},$$

is decreasing in e_1.

IV.2) [10 points] Consider the case of $I=1$ and $m=2$. Find an expression for

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right],$$

by differentiating with respect to e_1 the left hand side of the previous expression and find a condition in terms of the second derivatives u_{11}, u_{22} and u_{21} that determines the sign of $\partial[\gamma_1/\gamma_2]/\partial e_1$.

Ans:

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right] = \frac{u_{11}\ u_2 - u_1 u_{21}}{[u_2]^2}.$$

Thus,

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right] \geq 0 \text{ iff } u_{11}\ u_2 - u_1 u_{21} \geq 0.$$

IV.3) [5 points] Use your answer to II.4-II.7 to give and economic interpretation to the condition you obtained in IV.2. Your answer can be done in one line.

Ans: $\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right] \geq 0$ when good two is a normal good.

IV.4) [5 points] Since your previous answer involves second derivatives, write down the three inequalities that u_{11}, u_{22} and u_{21} must satisfies for u to be strictly concave.

Ans:

$$\begin{aligned} u_{11} &< 0, \\ u_{22} &< 0, \\ u_{11}\ u_{22} - [u_{12}]^2 &> 0. \end{aligned}$$

IV.5) [10 points] Show that it is possible to have

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right] < 0,$$

and u to be strictly concave. In particular, consider the following example of quadratic preferences:

$$u(x_1,x_2) = -\frac{1}{2}(x_1-2)^2 - \frac{1}{2}(x_2-2)^2 + b\ x_1x_2,$$

which are strictly concave if $b^2 < 1$ and strictly increasing in the range $x_2 < 2$ and $x_1 < 2$. Show that

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1,e_2)}{\gamma_2(e_1,e_2)}\right] < 0,$$

evaluated at

$$e_1 = e_2 = 1,$$

even though u is strictly concave.

Ans: We require

$$\begin{aligned} u_{11}\ u_2 - u_1 u_{21} &< 0, \\ u_{11}\ u_{22} - [u_{12}]^2 &> 0. \end{aligned}$$

Differentiating u :

$$\begin{aligned} u_1(x_1,x_2) &= 2 - x_1 + bx_2, \\ u_2(x_1,x_2) &= 2 - x_2 + bx_1, \end{aligned}$$

and hence u is strictly increasing if

$$\begin{aligned} 2 + bx_2 &> x_1, \\ 2 + bx_1 &> x_2. \end{aligned}$$

The second derivatives of u are:

$$\begin{aligned} u_{11} &= -1, \\ u_{22} &= -1, \\ u_{12} &= b. \end{aligned}$$

Thus for u to be strictly concave we require that $b^2 < 1$ or $|b| < 1$.

We have

$$\begin{aligned} \frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1, e_2)}{\gamma_2(e_1, e_2)}\right] &= \frac{u_{11}\ u_2 - u_1 u_{21}}{[u_2]^2}, \\ &= \frac{u_{11}\ - u_{21}(u_1/u_2)}{u_2} = \frac{-1\ - b(u_1/u_2)}{u_2}, \end{aligned}$$

and using the expression for u_1/u_2 evaluated at

$$x_1 = x_2 = 1,$$

we get:

$$\frac{u_1}{u_2} = 1,$$

so

$$\frac{\partial}{\partial e_1}\left[\frac{\gamma_1(e_1, e_2)}{\gamma_2(e_1, e_2)}\right] = -\frac{1+b}{u_2} < 0,$$

since $|b| < 1$ by concavity of u.

V. Representative Agent's problem in a CE. [45 points]

In this section we analyze the problem of the representative agent facing prices, i.e. we analyze the problem of a decision maker that has to allocate expenditure across I units in order to maximize the utility of the representative agent.

Let $V^i(\cdot)$ be the function defined in (15), let $p \in R^m$ be an arbitrary "price" vector and let $\bar{y} \in R$ be total expenditure. The following problem maximizes the weighted sum of utilities assigning expenditure y_i to each agent and letting them trade at prices p:

$$V(\bar{y}, p; \lambda) = \max_{y_i \in R_+, i=1,\dots,I} \sum_{i=1}^{I} \lambda_i V^i(y_i, p), \tag{20}$$

subject to:

$$\sum_{i=1}^{I} p\ y_i = \bar{y}. \tag{21}$$

Interpretation: we will show that $V(y, p; \lambda)$ is the indirect utility function that correspond to the utility function $U(x; \lambda)$ of the representative agent.

V.1) [5 points] Write the Lagrangian for problem (20) using $\theta \in R$ for the multiplier on constraint (21). Make sure to use a convention of the signs to that θ will be positive. Your answer should be one line.

Ans:

$$\mathcal{L}^i = \sum_{i=1}^{I} \lambda_i V^i(y_i, p) + \theta\left(\bar{y} - \sum_{i=1}^{I} y_i\right).$$

V.2) [5 points] Write the foc for the Lagrangian of the problem (20).

Ans:

$$\lambda_i \frac{\partial V^i}{\partial y}(y_i, p) - \theta = 0$$

V.3) [5 points] Use V.2) and the envelope to find an expression for $\partial V(\bar{y}, p; \lambda)/\partial \bar{y}$ in terms of $\partial V^i(y_i, p)/\partial y$, and the vectors λ_i .Your answer should be one line.

Ans:

$$\frac{\partial V(\bar{y}, p; \lambda)}{\partial \bar{y}} = \theta = \lambda_i \frac{\partial V^i}{\partial y}(y_i, p),$$

where he have used the condition founf in V.2.

For future reference, we state without proof that if all $V^i(y, p)$ are strictly concave in y then $V(\bar{y}, p; \lambda)$ is strictly concave in $\bar{y}$.

Here is the proof for completeness: Let $\bar{y} = \omega\ \bar{y}_1 + (1-\omega)\bar{y}_2$ for $\omega \in (0, 1)$. Let $y^i(y, p; \lambda)$ be the corresponding optimal policies. Then we have that

$$\begin{aligned} & p\left[\omega y^i(\bar{y}_1, p; \lambda) + (1-\omega) y^i(\bar{y}_2, p; \lambda)\right], \\ & \omega\ p y^i(\bar{y}_1, p; \lambda) + (1-\omega) p y^i(\bar{y}_2, p; \lambda), \\ = \ & \bar{y}, \end{aligned}$$

and hence

$$\omega y^i(\bar{y}_1, p; \lambda) + (1-\omega) y^i(\bar{y}_2, p; \lambda),$$

is feasible for $\bar{y}$, and thus

$$\begin{aligned} V(\bar{y}, p; \lambda) &\geq \omega \sum_i \lambda_i V^i \left(y^i (\bar{y}_1, p; \lambda), p\right) \\ &\quad + (1-\omega) \sum_i \lambda_i V^i \left(y^i (\bar{y}_2, p; \lambda), p\right), \\ &= \omega V(\bar{y}_1, p; \lambda) + (1-\omega) V(\bar{y}_2, p; \lambda). \end{aligned}$$

V.4) [5 points] Assume that all $V^i(y,p)$ are strictly concave in y. Let $y^i(\bar{y}, p; \lambda)$ be the optimal solution of problem (20). Show that it is strictly increasing in $\bar{y}$. Hints: Use the foc of the problem as well as the strict concavity of $V(\bar{y}, p; \lambda)$ on $\bar{y}$.

Ans: The foc equates

$$\lambda_i \frac{\partial V^i}{\partial y}\left(y^i, p\right) = \frac{\partial V}{\partial y}(\bar{y}, p; \lambda).$$

Hence differentiating both sides w.r.t. $\bar{y}$ we obtain

$$\lambda_i \frac{\partial^2 V^i}{\partial y^2}\left(y^i, p\right) \frac{\partial y}{\partial \bar{y}} = \frac{\partial^2 V}{\partial y^2}(\bar{y}, p; \lambda)$$

Solving for $\frac{\partial y}{\partial \bar{y}}$,

$$\frac{\partial y}{\partial \bar{y}} = \frac{\frac{\partial^2 V}{\partial y^2}(\bar{y}, p; \lambda)}{\lambda_i \frac{\partial^2 V^i}{\partial y^2}(y^i, p)} > 0$$

because V^i is strictly concave.

V.5) [15 points] Now we show that $V(\bar{y}, p; \lambda)$ is the indirect utility function that corresponds to the utility function of the representative agent implicitly defined by the planner's problem:

$$V(\bar{y}, p; \lambda) = \max_x U(x; \lambda), \tag{22}$$

subject to:

$$p\ x = \bar{y}.$$

V.5.a) [5 points] Write the foc for problem (22). Your answer should be one line. Be carefull on writing all the argument of the functions involved.

Ans:

$$\frac{\partial U(x, \lambda)}{\partial x} = \mu_i p.$$

V.5.b) [10 points] Let $x^i(e,\lambda)$ be the solution of the social planner's problem (7) and $\gamma(e,\lambda)$ be its associated multiplier for weights λ. Let e be the aggregate endowment and let

$$\begin{aligned} p &= \gamma(e,\lambda), \\ \bar{y} &= p\, e. \end{aligned}$$

Let $x^i(\bar{y};p,\lambda)$ be the solution of problem (20). Show that

$$x^i(\bar{y};p,\lambda) = x^i(e,\lambda).$$

[Hints: Construct the sufficient foc of problem (20) guessing that $\theta = 1$. For this, use the necessary foc of problem (7) and first construct a solution to the sufficient conditions for the foc of the (15) problem with $\mu_i = 1/\lambda_i$. Find an expression for the envelope $\partial V_i/\partial y$.]

Ans: The necessary foc of the planner's problem are:

$$\lambda_i \frac{\partial u^i(x)}{\partial x} - \gamma = 0$$

The sufficient foc and envelope of problem (15) are:

$$\begin{aligned} \frac{\partial u^i(x)}{\partial x} - \mu_i p &= 0 \\ \frac{\partial V_i(y,p)}{\partial y} &= \mu_i \end{aligned}$$

hence the foc for x^i at $x^i(e,\lambda)$ are satisfied by setting

$$\mu_i = 1/\lambda_i.$$

The envelope $\partial V_i(y,p)/\partial y$ then reads

$$\frac{\partial V_i(y,p)}{\partial y} = \frac{1}{\lambda_i}.$$

Using this expression for the envelope it is immediate that the foc for problem (20):

$$\lambda_i \frac{\partial V^i}{\partial y}(y_i,p) - \theta = 0,$$

for all i is satisfied setting $\theta = 1$. Then it is clear that

$$x^i(\bar{y};p,\lambda) = x^i(e,\lambda)$$

Notice that we have established that the optimal decision rule $x(\bar{y}, p, \lambda)$ can be written as

$$x(\bar{y}, p, \lambda) = \sum_{i=1}^{I} x^i \left(p, y^i(\bar{y}, p; \lambda)\right)$$

where $x^i(p, y)$ is the solution of the agent problem and where $y^i(\bar{y}, p; \lambda)$ is the solution to the problem (20).

V.6) [10 points] Show that if each agent utility $u^i(x)$ is such that good 1 is a normal good, then good one is normal in the utility function of the representative agent $U(x; \lambda)$.

[Hints: Normality of good 1 for the representative agent is equivalent to saying that the optimal choice $x_1(\bar{y}, p; \lambda)$ is strictly increasing in $\bar{y}$. Use that $y^i(\bar{y}, p; \lambda)$ is increasing in $\bar{y}$.]

Ans: Let $x^i(y, p)$ be the agent optimal choice. Since $y^i(\bar{y}, p; \lambda)$ is strictly increasing we have that the demand for good one is given by

$$x_1(\bar{y}, p, \lambda) = \sum_{i=1}^{I} x_1^i \left(p, y^i(\bar{y}, p; \lambda)\right),$$

and hence $x_1(\bar{y}, p, \lambda)$ is increasing in $\bar{y}$.