Theory of Income, Fall 2018

Fernando Alvarez, UofC

Class Note 6

Continuous Time Euler Equations and Transversality Conditions for Dynamic Problems, Hamiltonean

- This notes introduces the elements of continuous time dynamic optimization problems.
- We introduce and discuss the Maximum Principle and its relation to the classical variational approach used to derive the Euler Equations in countinuous time.
- The last part of the note shows how to obtain the continuous time version of the neoclassical growth model from its discrete time version counterpart.

Continuous time case

In this case the problem is to chose the derivative of the state with respect to time, $\dot{x}(t)$, in each time period. The elements of a Dynamic Programming problem are: the set of states X, the feasible correspondence $\Gamma : X \rightarrow X$, the period return function, $F(x, \dot{x})$, defined on $F : Gr(\Gamma) \rightarrow R$, and a discount rate ρ.
The problem is

$$V^*(x_0) = \max_{\{\dot{x}(t)\}_{t=0}^{\infty}} \lim_{T\rightarrow\infty} \int_0^T e^{-\rho t} F(x(t), \dot{x}(t))\, dt$$

subject to

$$\dot{x}(t) \in \Gamma(x(t))$$

for $t \geq 0$, x_0 given.

In the continuous time case the Euler Equations and Transversality Conditions are, respectively, as follows:

$$\begin{aligned} & F_x(x(t), \dot{x}(t)) + \rho F_{\dot{x}}(x(t), \dot{x}(t)) \\ = \; & F_{\dot{x}x}(x(t), \dot{x}(t))\ \dot{x}(t) + F_{\dot{x}\dot{x}}(x(t), \dot{x}(t))\, \ddot{x}(t), \end{aligned}$$

for all $t \geq 0$, and

$$0 = \lim_{T\rightarrow\infty} e^{-\rho T} F_{\dot{x}}(x(T), \dot{x}(T))\, x(T)\ .$$

Variational analysis. Assume that

$$(x(t), \dot{x}(t)) \in Int(Gr(\Gamma))$$

for $t \geq 0$. Consider the variational path $x(\alpha, \varepsilon) = x(t) + \alpha\varepsilon(t)$ where $\alpha \in R$ and $\varepsilon(t)$ is a differentiable function from R_+ to R^m, with $\varepsilon(0) = 0$. Define the value of the variational path as

$$\begin{aligned} v(\alpha) &= \lim_{T\to\infty} \int_0^T e^{-\rho t} F(x(\alpha,\varepsilon)(t), \dot{x}(\alpha,\varepsilon)(t))\, dt \\ &= \lim_{T\to\infty} \int_0^T e^{-\rho t} F(x(t) + \alpha\varepsilon(t), \dot{x}(t) + \alpha\dot{\varepsilon}(t))\, dt \end{aligned}$$

If the variational path is feasible, i.e. $\dot{x}(\alpha,\varepsilon)(t) \in \Gamma(x(\alpha,\varepsilon)(t))$ for all $t \geq 0$,

$$v(0) \geq v(\alpha).$$

Assuming that v is differentiable and that we can interchange derivative of the integral as the integral of the derivative:

$$\begin{aligned} \frac{\partial v(0)}{\partial \alpha} &= \lim_{T\to\infty} \int_0^T e^{-\rho t} F_x(x(t), \dot{x}(t))\, \varepsilon(t)\, dt \\ &\quad + \lim_{T\to\infty} \int_0^T e^{-\rho t} F_{\dot{x}}(x(t), \dot{x}(t))\, \dot{\varepsilon}(t)\, dt \end{aligned}$$

To be able to find the impact of the variational path ε for each t we use integration by parts. Let

$$\int_0^T p(t)\,\frac{d}{dt}\varepsilon(t)\,dt = p(t)\ \times \varepsilon(t)\,|_0^T - \int_0^T \frac{d}{dt}p(t)\,\varepsilon(t)\,dt$$

where

$$\begin{aligned}
p(t) &= e^{-\rho t}F_{\dot{x}}(x(t),\dot{x}(t)) \\
\frac{d}{dt}p(t) &= -\rho e^{-\rho t}F_{\dot{x}}(x(t),\dot{x}(t)) \\
&\quad + e^{-\rho t}F_{\dot{x}x}(x(t),\dot{x}(t))\ \dot{x}(t) + e^{-\rho t}F_{\dot{x}\dot{x}}(x(t),\dot{x}(t))\,\ddot{x}(t)
\end{aligned}$$

Thus, assuming that $x(t)$ is C^2,

$$\begin{aligned}
0 &= \frac{\partial v(0)}{\partial \alpha} \\
&= \lim_{T\to\infty}\int_0^T e^{-\rho t}F_x(x(t),\dot{x}(t))\ \varepsilon(t)\,dt \\
&\quad + \lim_{T\to\infty} e^{-\rho T}F_{\dot{x}}(x(T),\dot{x}(T))\ \varepsilon(T) \\
&\quad - \lim_{T\to\infty}\int_0^T e^{-\rho t}[-\rho F_{\dot{x}}(x(t),\dot{x}(t)) + \\
&\quad + F_{\dot{x}x}(x(t),\dot{x}(t))\ \dot{x}(t) + F_{\dot{x}\dot{x}}(x(t),\dot{x}(t))\,\ddot{x}(t)\ ]\,\varepsilon(t)\,dt
\end{aligned}$$

or omitting arguments

$$
\begin{aligned}
0 \quad = \quad & \frac{\partial v(0)}{\partial \alpha} = \int_0^\infty e^{-\rho t} F_x \varepsilon \, dt + \lim_{T \to \infty} e^{-\rho T} F_{\dot{x}}(T) \varepsilon(T) \\
& - \int_0^\infty e^{-\rho t} [-\rho F_{\dot{x}} + F_{\dot{x}x} \, \dot{x} + F_{\dot{x}\dot{x}} \, \ddot{x}] \, \varepsilon dt
\end{aligned}
$$

Necessity of the Euler Equations:
Take ε such that $\varepsilon(t) \neq 0$ and zero otherwise. Then $v'(0) = 0$ requires

$$
\begin{aligned}
& F_x(x(t), \dot{x}(t)) + \rho F_{\dot{x}}(x(t), \dot{x}(t)) \\
= \quad & F_{\dot{x}x}(x(t), \dot{x}(t)) \;\; \dot{x}(t) + F_{\dot{x}\dot{x}}(x(t), \dot{x}(t)) \, \ddot{x}(t)
\end{aligned}
$$

This is heuristic since before we assumed that ε is differentiable, but for the Euler equations we use a function ε that was discontinuous.
A rigorous proof needs to approximate this discontinuous case using a smooth function.

Necessity of the TC condition. Take a sequence x that satisfies the EE equations. Then,

$$0 = \frac{\partial v(0)}{\partial \alpha} = \lim_{T\to\infty} e^{-\rho T} F_{\dot{x}}\left(x(T), \dot{x}(T)\right)\ \varepsilon(T)$$

and if $\varepsilon(T) = -x(T)$ is feasible, we obtain the TC condition

$$0 = \lim_{T\to\infty} e^{-\rho T} F_{\dot{x}}\left(x(T), \dot{x}(T)\right)\ x(T)$$

Exercise. Adapt the proof of sufficiency of the Euler Equations and Transversality conditions for the optimal path in the discrete time case to the continuous time case. Hint: use integration by parts in a similar way that in the proof of necessity. Assume that $F(x, \dot{x})$ is concave, let $Gr(\Gamma)$ be convex set, and the optimal path $x^*(t)$ be interior.

Exercise. Write down the Euler equation for the Neoclassical growth model, i.e. for F given by:

$$\begin{aligned} F\left(k, \dot{k}\right) &= U\left(G(k,1) - \delta k - \dot{k}\right) \\ \Gamma(k) &= R \end{aligned}$$

Exercise. Use the Euler Equation,

$$\begin{aligned} & F_x\left(x(t), \dot{x}(t)\right) + \rho F_{\dot{x}}\left(x(t), \dot{x}(t)\right) \\ = \;& F_{\dot{x}x}\left(x(t), \dot{x}(t)\right)\ \dot{x}(t) + F_{\dot{x}\dot{x}}\left(x(t), \dot{x}(t)\right) \ddot{x}(t) \end{aligned}$$

setting $\dot{x}(t) = \ddot{x}(t) = 0$ to find the equation that the steady state $\bar{x}$ must solve.

Exercise. Use the answer to the previous two exercises to find an expression of the steady state $\bar{k}$ for the neoclassical growth model.

The Maximum Principle. Hamiltonian.

We use the control-state formulation. For this we have the instantaneous return function h depending on the state vector $x \in X \subseteq R^m$ and a control vector $u \in U \subseteq R^n$. The problem is

$$V^* (x_0) = \max_{u(t)_{t=0}^{\infty}} \lim_{T \to \infty} \int_0^T e^{-\rho t} h (x (t), u (t)) \, dt$$

subject to the law of motion for the state

$$\dot{x} (t) = g (x (t), u (t)) \text{ and } u (t) \in U$$

for $t \geq 0$, x_0 given.

The following is a procedure to obtain necessary and sufficient conditions for an optimum.
Of course, this requires, as the previous cases, some regularity conditions.
Additionally the sufficiency part will require convexity assumptions.
Let λ be a vector on R^m of co-state variables. The interpretation of the co-state is that $e^{-\rho t} \lambda (t)$ is the marginal value at time zero of an infinitesimal increase in the state x at time t.

Let H be the current-value Hamiltonian function, defined as

$$H (x, u, \lambda) = h (x, u) + \lambda g (x, u) .$$

Then the following conditions are necessary (under regularity assumptions) and sufficient (under regularity and convexity assumptions) for the path of x, u to be optimal:

$$H_u(x(t), u(t), \lambda(t)) = 0$$

$$\dot{\lambda}(t) = \rho\lambda(t) - H_x(x(t), u(t), \lambda(t))$$

and

$$\dot{x}(t) = g(x(t), u(t))$$

for all $t \geq 0$. The state variable(s) x has an initial value of x_0 and the co-state variable(s) $\lambda(t)$ have a boundary condition, the Transversality condition, given by

$$0 = \lim_{T\to\infty} e^{-\rho T}\lambda(T)\; x(T).$$

The initial value of the co-state variable $\lambda(0)$ is not predetermined, it has to be solved as part of the system.

Here is an heuristic proof of this characterization.
Form the Lagrangian, using $e^{-\rho t}\lambda(t)$ for the multiplier of $\dot{x}(t) = g(x(t), u(t))$,

$$\begin{aligned} L(x, u, \lambda) &= \lim_{T\to\infty}\int_0^T e^{-\rho t} h(x(t), u(t))\, dt \\ &+ \int_0^T e^{-\rho t}\lambda(t)\left[g(x(t), u(t)) - \dot{x}(t)\right] dt \end{aligned}$$

We need to maximize L with respect to x and u, and minimize it with respect to λ.
We can rewrite this expression using by part integration as follows

$$\begin{aligned} &\int_0^T e^{-\rho t}\lambda(t)\,\dot{x}(t)\, dt \\ = \;& e^{-\rho t}\lambda(t)\, x(t)\,|_0^T - \int_0^T \left[-\rho e^{-\rho t}\lambda(t) + e^{-\rho t}\dot{\lambda}(t)\right] x(t)\, dt \end{aligned}$$

Thus

$$
\begin{aligned}
L(x,u,\lambda) &= \lim_{T\to\infty}\int_0^T e^{-\rho t}h(x(t),u(t))\,dt \\
&+ \lim_{T\to\infty}\int_0^T e^{-\rho t}\left[\lambda(t)\,g(x(t),u(t)) - \rho\lambda(t)\,x(t) + \dot{\lambda}(t)\,x(t)\right]dt \\
&- \lim_{T\to\infty} e^{-\rho t}\lambda(t)\,x(t)\,|_0^T
\end{aligned}
$$

Since $L(x,\lambda)$ has to be maximized by x the first order conditions with respect to $x(t)$ gives

$$
\frac{dL(x,u,\lambda)}{dx(t)} = e^{-\rho t}\left[h_x(x(t),u(t)) + \lambda(t)\,g_x(x(t),u(t)) - \rho\lambda(t) + \dot{\lambda}(t)\right]
$$

or setting $dL/dx(t) = 0$,

$$
\dot{\lambda}(t) = \rho\lambda(t) - h_x(x(t),u(t)) - \lambda(t)\,g_x(x(t),u(t))
$$

obtaining the sought result that

$$
\dot{\lambda}(t) = \rho\lambda(t) - H_x(x(t),u(t),\lambda(t))
$$

and with respect to $u(t)$:

$$
\frac{dL}{du(t)} = e^{-\rho t}\left[h_u(x(t),u(t)) + \lambda(t)\,g_u(x(t),u(t))\right]
$$

or setting $dL/du(t) = 0$,

$$
h_u(x(t),u(t)) + \lambda(t)\,g_u(x(t),u(t)) = 0,
$$

obtaining the sought result that

$$
H_u(x(t),u(t),\lambda(t)) = 0.
$$

Exercise. Write down the system for λ and k for the neoclassical growth model, i.e. use k for state, c for control and

$$\begin{aligned} h(k,c) &= u(c), \\ g(k,c) &= f(k) - c. \end{aligned}$$

To see the relation with the classical Euler equations analysis consider the following special case:

$$\begin{aligned} u &= \dot{x} \\ F(x,u) &= h(x,u), \\ g(x,u) &= u, \end{aligned}$$

In this case we get

$$\begin{aligned} \lambda(t) &= -F_{\dot{x}}(x(t), \dot{x}(t)) \\ \dot{\lambda}(t) &= \rho\lambda(t) - F_x(x(t), \dot{x}(t)) \end{aligned}$$

or, combining both

$$\dot{\lambda}(t) = -\rho F_{\dot{x}}(x(t), \dot{x}(t)) - F_x(x(t), \dot{x}(t))$$

Finally, we can compute $\dot{\lambda}$ by differentiating w.r.t. to time,

$$\dot{\lambda}(t) = -\frac{d}{dt} F_{\dot{x}}(x(t), \dot{x}(t)) = -F_{\dot{x}x}(x(t), \dot{x}(t))\ \dot{x}(t) - F_{\dot{x}\dot{x}}(x(t), \dot{x}(t))\ x^{\cdot\cdot}(t)$$

replacing in the law of motion for λ

$$\begin{aligned} & F_{\dot{x}x}(x(t), \dot{x}(t))\ \dot{x}(t) + F_{\dot{x}\dot{x}}(x(t), \dot{x}(t))\ x^{\cdot\cdot}(t) \\ = \quad & \rho F_{\dot{x}}(x(t), \dot{x}(t)) + F_x(x(t), \dot{x}(t)) \end{aligned}$$

which is the Euler Equation obtained above.

Going from discrete to continuous time in the Neoclassical Growth model.

Consider the following version of the neoclassical growth model.

$$\max_{\{c_t, i_t\}} \sum_{t=0}^{\infty} \left(\frac{1}{1 + \Delta\rho} \right)^t \Delta U(c_{t\Delta})$$

and

$$\begin{aligned} \Delta c_t + \Delta i_t &= \Delta G(k_t, 1), \\ k_{t+\Delta} &= \Delta i_t + k_t(1 - \Delta\delta) \end{aligned}$$

for $t \geq 0$ where Δ is a positive parameter denoting the length of the time period.

Exercise. Write the Euler Equation and Law of motion of capital for this model for a fixed Δ.

Exercise. Take the limit of the Euler equation and law of motion of capital as Δ goes to zero. Show that you obtain the same equations than using the Hamiltonian. Hint: show that you can arrange the expression to get

$$\begin{aligned} \frac{k_{t+\Delta} - k_t}{\Delta} &= G(k_t, 1) - \delta k_t - c_t \\ (1 + \rho\Delta) U'(c_t) &= U'(c_{t+\Delta}) [\Delta G_k (k_{t+\Delta}, 1) + 1 - \Delta\delta] \end{aligned}$$

Take the limit on the first one. For the second use a first order approximation replacing $U'(c_{t+\Delta})$ with $U'(c_t) + U''(c_t)(c_{t+\Delta} - c_t) + o\left(\Delta \frac{c_{t+\Delta} - c_t}{\Delta}\right)$, further rearrange the terms, canceling the $u'(c_t)$ terms on both sides, divide by Δ and take limits when Δ goes to zero. Assume that c_t and k_t are differentiable functions of t.]

Euler Equations and speed of convergence: Cts time

- This section complements the derivation of the optimal decision rules for a continuous time dynamic problem.
- We focus in the 1 dimensional case, using the classical variational analysis, i.e. Euler equations. We seek to characterize the optimal decision rule $\dot{k} = g(k)$, determining the rate of change of the state as a function of the level of the state.
- We obtain a differential equation for the function g. We stress that we want to consider the state representation, i.e. a differential equation w.r.t. to the state k, as opposed to the standard representation of the Euler equation as a (second order) differential equation on the state at its derivatives w.r.t. to time.
- We use g to study the local dynamics of k, i.e. the dynamics of the state k close to the steady state value $\bar{k}$, which we summarize by $g'(\bar{k})$. We find an algebraic (quadratic) equation for $g'(\bar{k})$ and study sufficient conditions for local stability, i.e. for $g'(\bar{k}) < 0$.
- We use the neoclassical growth model as an illustration of the general principle.

Continuous time set up

We will work with the continuous time model define before: chose the derivative of the state with respect to time, $\dot{x}(t)$, in each time period to maximize:

$$V^*(x_0) = \max_{\{\dot{x}(t)\}_{t=0}^{\infty}} \lim_{T\to\infty} \int_0^T e^{-\rho t} F(x(t), \dot{x}(t))\, dt$$

subject to

$$\dot{x}(t) \in \Gamma(x(t))$$

for $t \geq 0$, x_0 given.

In the continuous time case the Euler Equations and Transversality Conditions are, respectively, as follows:

$$\begin{aligned} & F_x(x(t), \dot{x}(t)) + \rho F_{\dot{x}}(x(t), \dot{x}(t)) \\ = \;& F_{\dot{x}x}(x(t), \dot{x}(t))\ \dot{x}(t) + F_{\dot{x}\dot{x}}(x(t), \dot{x}(t))\, \ddot{x}(t), \end{aligned}$$

for all $t \geq 0$, and

$$0 = \lim_{T\to\infty} e^{-\rho T} F_{\dot{x}}(x(T), \dot{x}(T))\, x(T)\ .$$

We will write Euler Equation as:

$$H\left(\ddot{x}(t),\ \dot{x}(t),\ x(t)\right) = 0$$

for all $t \geq 0$, where the function $H : R^3 \to R$ is defined as:

$$H\left(\ddot{x}, \dot{x},\ x\right) = F_x(x, \dot{x}) + \rho F_{\dot{x}}(x, \dot{x}) - F_{\dot{x}x}(x, \dot{x})\ \dot{x} - F_{\dot{x}\dot{x}}(x, \dot{x})\, \ddot{x}$$

In the case of the neoclassical growth model we have:

$$\begin{aligned} F\left(k,\dot{k}\right) &= U\left(f(k)-\delta k-\dot{k}\right) \\ \Gamma(k) &= R \end{aligned}$$

then

$$\begin{aligned} F_k\left(k,\dot{k}\right) &= U'\left(f(k)-\delta k-\dot{k}\right)\left[f'(k)-\delta\right] \\ F_{\dot{k}}\left(k,\dot{k}\right) &= -U'\left(f(k)-\delta k-\dot{k}\right) \\ F_{\dot{k}k}\left(k,\dot{k}\right) &= -U''\left(f(k)-\delta k-\dot{k}\right)\left[f'(k)-\delta\right] \\ F_{\dot{k}\dot{k}}\left(k,\dot{k}\right) &= U''\left(f(k)-\delta k-\dot{k}\right) \end{aligned}$$

Steady states are given by

$$F_k\left(\bar{k},0\right)+\rho F_{\dot{k}}\left(\bar{k},0\right)=0$$

$$\begin{aligned} 0 &= U'\left(f\left(\bar{k}\right)-\delta\bar{k}\right)\left[f'\left(\bar{k}\right)-\delta\right]-\rho U'\left(f\left(\bar{k}\right)-\delta\bar{k}\right) \\ &\implies f'\left(\bar{k}\right)=\rho+\delta \end{aligned}$$

Note on notation: From now and on, we will use k for the state as opposed to x.

Time domain

Let's first review the analysis of the problem in the time domain. We are looking for a **path of** $k(t)$ **, i.e. capital as a function of time** that:
1) Starts at the initial conditions:

$$k(0)=k_o$$

2) Satisfies the Euler equation:

$$H\left(\ddot{k}(t),\ \dot{k}(t),\ k(t)\right)=0 \text{ for all } t>0$$

3) Goes to the unique steady state (and hence satisfies transversality)

$$k(t)\to\bar{k} \text{ as } t\to\infty$$

State domain

We are looking for **a function** $\dot{k} = g(k)$, i.e. **the rate of change of capital as a function of the level of capital** that:
1) Solves the Euler equation:

$$H(g'(k)\,g(k),\ g(k),\ k) = 0 \text{ for all } k$$

all value of k.
2) Goes to steady state (and hence satisfies TC)

$$g(k) > 0 \text{ if } k < \bar{k} \text{ and } g(k) < 0 \text{ if } k > \bar{k}$$

Condition 2) can be extended to the m dimensional case, for instance as:

$$\|g(k)\| \text{ is decreasing in } \|k - \bar{k}\|$$

or more generally, that there is a function $L : R^m \to R_+$ with $L(k) = 0$ iff $k = 0$, and $L(g(k))$

$$L(g(k))\,g'(k) < 0 \text{ all } k$$

Let's verify that:

$$H(\ g'(k)\,g(k),\ g(k),\ k) = 0 \text{ for all } k$$

is equivalent to:

$$H\left(\ \ddot{k}(t),\ k(t),\ k(t)\right) = 0 \text{ for all } t > 0$$

Notice that differentiating $\dot{k}(t) = g(k(t))$ w.r.t. t we have:

$$\ddot{k}(t)\ =\ g'(k(t))\ \dot{k}(t)\ = g'(k(t))\ g(k(t))$$

and replacing $\dot{k} = g(k)$ and $\ddot{k}\ = g'(k)\,g(k)$ we obtained the desired expression.

Now we focus on values of k close to the steady state $\bar{k}$.
In this case the condition

$$g(k) > 0 \text{ if } k < \bar{k} \text{ and } g(k) < 0 \text{ if } k > \bar{k}$$

is required only on a neighborhood of $\bar{k}$ (i.e. locally). This is equivalent to:

$$g'\left(\bar{k}\right) \equiv \frac{\partial g\left(\bar{k}\right)}{\partial k} < 0$$

This conditions ensure that the system defined by g is locally stable. The size of $\left|g'\left(\bar{k}\right)\right|$ describes the speed of convergence.

To see this write the (non-linear) law of motion of capital

$$\dot{k}(t) = g(k(t))$$

as

$$\dot{k} = g(k) = g\left(\bar{k}\right) + g'\left(\bar{k}\right)\left(k - \bar{k}\right) + o\left(k - \bar{k}\right)$$

or neglecting terms of order smaller than $k - \bar{k}$ obtaining the differential equation:

$$\dot{k} = g'\left(\bar{k}\right)\left(k - \bar{k}\right)$$

which has the solution:

$$k(t) = \bar{k} + \left[k(0) - \bar{k}\right] \exp\left(g'\left(\bar{k}\right) t\right)$$

which can be verified by differentiating this expression w.r.t. to t :

$$\dot{k}(t) = g'\left(\bar{k}\right)\left[k(0) - \bar{k}\right] \exp\left(g'\left(\bar{k}\right) t\right) = g'\left(\bar{k}\right)\left(k(t) - \bar{k}\right)$$

For the linearized version of the law of motion we have:
1) $k(t) \to \bar{k}$ iff $g'(\bar{k}) < 0$
2) the half life τ is defined as the time τ that it takes so that the system has close "half" of the difference its steady state value:

$$\left[k(\tau) - \bar{k}\right] = \frac{1}{2}\left[k(0) - \bar{k}\right]$$

or using the (linearized) law of motion:

$$\frac{1}{2}\left[k(0) - \bar{k}\right] = \left[k(0) - \bar{k}\right] \exp\left(g'\left(\bar{k}\right)\ \tau\right)$$

and solving for τ :

$$\tau = -\log(2) / g'\left(\bar{k}\right)$$

$\boxed{\text{Obtaining an expression of } g'\left(\bar{k}\right)}$

To obtain an expression of $g'\left(\bar{k}\right)$ use that

$$H(\ g'(k)\, g(k),\ g(k),\ k) = 0$$

holds for all k, and differentiate it w.r.t. to k to obtain:

$$\begin{aligned} 0 \quad = \quad & H_{\ddot{k}}(\ g'(k)\, g(k),\ g(k),\ k)\ \frac{\partial}{\partial k} g'(k)\, g(k) \\ & H_{\dot{k}}(\ g'(k)\, g(k),\ g(k),\ k)\, \frac{\partial}{\partial k} g(k) + H_k(\ g'(k)\, g(k),\ g(k),\ k) \end{aligned}$$

using

$$\frac{\partial}{\partial k} g'(k)\, g(k) = g''(k)\, g(k) + \left[g'(k)\right]^2$$

$$\begin{aligned} 0 \quad = \quad & H_{\ddot{k}}(\ g'(k)\, g(k),\ g(k),\ k)\left[g''(k)\, g(k) + \left(g'(k)\right)^2\right] \\ & H_{\dot{k}}(\ g'(k)\, g(k),\ g(k),\ k)\, g'(k) + H_k(\ g'(k)\, g(k),\ g(k),\ k) \end{aligned}$$

Evaluating this expression at the steady state $k = \bar{k}$:

$$g(\bar{k}) = 0 \implies \frac{\partial}{\partial k} g'(k)\, g(k) = g''(k)\, g(k) + [g'(k)]^2 = \left[g'(\bar{k})\right]^2$$

or

$$0 = H_{\ddot{k}}\left(0,\ 0,\ \bar{k}\right)\ g'\left(\bar{k}\right)^2 + H_{\dot{k}}\left(0,\ 0,\ \bar{k}\right) g'\left(\bar{k}\right) + H_k\left(0,\ 0,\ \bar{k}\right)$$

This a quadratic equation in $g'\left(\bar{k}\right)$, with coefficients given by $H_{\ddot{k}}\left(0,\ 0,\ \bar{k}\right)$, $H_{\dot{k}}\left(0,\ 0,\ \bar{k}\right)$ and $H_k\left(0,\ 0,\ \bar{k}\right)$. We next turn to the analysis of these coefficients.

Differentiating

$$H\left(\ddot{k}, \dot{k},\ k\right) = F_k\left(k, \dot{k}\right) + \rho F_{\dot{k}}\left(k, \dot{k}\right) - F_{\dot{k}k}\left(k, \dot{k}\right)\ \dot{k} - F_{\dot{k}\dot{k}}\left(k, \dot{k}\right) \ddot{k}\ = 0$$

with respect to its 3 arguments and evaluating it at s.s.:

$$H_{\ddot{k}}\ \left(0,\ 0,\ \bar{k}\right) = -F_{\dot{k}\dot{k}}\left(\bar{k}, 0\right),$$

$$H_{\dot{k}}\left(0,\ 0,\ \bar{k}\right) = F_{k\dot{k}}\left(\bar{k}, 0\right) + \rho F_{\dot{k}\dot{k}}\left(\bar{k}, 0\right) - F_{\dot{k}k}\left(\bar{k}, 0\right) = \rho F_{\dot{k}\dot{k}}\left(\bar{k}, 0\right)$$

using that $\dot{k} = \ \ddot{k}\ = 0$, and

$$H_k\left(0,\ 0,\ \bar{k}\right) = F_{kk}\left(\bar{k}, 0\right) + \rho F_{\dot{k}k}\left(\bar{k}, 0\right)$$

Then the quadratic equation is:

$$0 = H_{\ddot{k}}\ \left(0,\ 0,\ \bar{k}\right)\ g'\left(\bar{k}\right)^2 + H_{\dot{k}}\left(0,\ 0,\ \bar{k}\right) g'\left(\bar{k}\right) + H_k\left(0,\ 0,\ \bar{k}\right)$$

or

$$Q(\lambda) \equiv -F_{\dot{k}\dot{k}}\ [\lambda]^2 + \left[\rho F_{\dot{k}\dot{k}}\right]\ \lambda + \left[F_{kk} + \rho F_{\dot{k}k}\right]$$

Theorem:

Roots of Q comes in (almost) reciprocal pairs.
Let:

$$Q(\lambda) \equiv -F_{\dot{k}\dot{k}} \ [\lambda]^2 + \left[\rho F_{\dot{k}\dot{k}}\right] \ \lambda + \left[F_{kk} + \rho F_{k\dot{k}}\right]$$

If λ_1 solves $Q(\lambda_1) = 0$, so does $\lambda_2 = -\lambda_1 + \rho$.

Proof of the Theorem. Assume that λ_1 solves $Q(\lambda_1) = 0$ or

$$Q(\lambda_1) = -F_{\dot{k}\dot{k}} \ [\lambda_1]^2 + \left[\rho F_{\dot{k}\dot{k}}\right] \ \lambda_1 + \left[F_{kk} + \rho F_{k\dot{k}}\right] = 0$$

We have:

$$\begin{aligned} Q(-\lambda_1 + \rho) &= -F_{\dot{k}\dot{k}} \ [-\lambda_1 + \rho]^2 + \left[\rho F_{k\dot{k}}\right] \ (-\lambda_1 + \rho) + \left[F_{kk} + \rho F_{k\dot{k}}\right] \\ &= -F_{\dot{k}\dot{k}} \ [\lambda_1]^2 + 2F_{\dot{k}\dot{k}} \ \lambda_1 \rho - F_{\dot{k}\dot{k}} \ \rho^2 - \left[\rho F_{k\dot{k}}\right] \ \lambda_1 + \rho^2 F_{k\dot{k}} + \left[F_{kk} + \right. \end{aligned}$$

cancelling $F_{\dot{k}\dot{k}} \ \rho^2, \ F_{\dot{k}\dot{k}} \ \lambda_1 \rho, \ \rho F_{kk}$ we have:

$$\begin{aligned} Q(-\lambda_1 + \rho) &= -F_{\dot{k}\dot{k}} \ [\lambda_1]^2 + \left[F_{\dot{k}\dot{k}} \ \rho\right] \ \lambda_1 + \left[\rho F_{k\dot{k}} + F_{kk}\right] \\ &= Q(\lambda_1) = 0 \ . \end{aligned}$$

Interpretation of (almost) reciprocal pairs Thm.
- If $\lambda_1 < 0$ then $\lambda_2 > 0$. Hence if we find a solution

$$Q\left(g'\left(\bar{k}\right)\right) = 0 \text{ with } g'\left(\bar{k}\right) < 0$$

it is THE solution: it satisfies the Euler Eqn. and transversality, since it converges to the steady state. Also, the fact that it converges to steady state, justify the use of the approximation (i.e. linearization) of the law of motion, since k stays in a neighborhood of $\bar{k}$.
- There is AT MOST, one stable solution. This is reassuring, since a convex problem should have at least one solution.
- If $\lambda_1 > 0$ and $\lambda_2 > 0$ the system is NOT locally stable. In this case local arguments alone do not suffice to identify which one of the roots of Q is the solution for $g'\left(\bar{k}\right)$, but we know that one of them gives the value of $g'\left(\bar{k}\right)$.

The previous theorem also holds for higher dimensions.
If the states is of dimension m there will be $2m$ roots satisfying:

$$\lambda_{m+i} = -\lambda_i + \rho \text{ for } i = 1, ..., m.$$

In the m dimensional case the roots are NOT simply $\partial g_i / \partial k_i$. The roots are the eigenvalues of the matrix of the derivatives of g.
In the m dimensional case we have:

$$\dot{k}_i = g_i(k)$$

for $i = 1, ..., m$ or in vector notation:

$$\dot{k} = \left(\dot{k}_1, \dot{k}_2 ..., \dot{k}_m\right)$$

$$\dot{k} = g(k_1, ..k_m)$$

and

$$g(k_1, ..k_m) = (g_1(k_1, ..k_m), g_2(k_1, ..k_m), \cdots, g_m(k_1, ..k_m))$$

In this case $g'(k)$ is the matrix

$$\frac{\partial g(k)}{\partial k} = \begin{bmatrix} \frac{\partial g_1(\bar{k})}{\partial k_1} & \frac{\partial g_1(\bar{k})}{\partial k_2} & \cdots & \frac{\partial g_1(\bar{k})}{\partial k_m} \\ \frac{\partial g_2(\bar{k})}{\partial k_1} & \frac{\partial g_2(\bar{k})}{\partial k_2} & \cdots & \frac{\partial g_2(\bar{k})}{\partial k_m} \\ & & \cdots & \\ & & & \frac{\partial g_m(\bar{k})}{\partial k_m} \end{bmatrix}$$

The eigenvalues of the matrix $g'(k)$ control the (local) behavior of k around $\bar{k}$.

Analysis of the quadratic equation for the Neoclassical growth model

From the definition of

$$F\left(k, \dot{k}\right) = U\left(f(k) - \delta k - \dot{k}\right)$$

we have:

$$\begin{aligned}
F_k\left(k, \dot{k}\right) &= U'\left(f(k) - \delta k - \dot{k}\right)\left[f'(k) - \delta\right] \\
F_{\dot{k}}\left(k, \dot{k}\right) &= -U'\left(f(k) - \delta k - \dot{k}\right) \\
F_{\dot{k}k}\left(k, \dot{k}\right) &= -U''\left(f(k) - \delta k - \dot{k}\right)\left[f'(k) - \delta\right] \\
F_{\dot{k}\dot{k}}\left(k, \dot{k}\right) &= U''\left(f(k) - \delta k - \dot{k}\right) \\
F_{kk}\left(k, \dot{k}\right) &= U''\left(f(k) - \delta k - \dot{k}\right)\left[f'(k) - \delta\right] + U''\left(f(k) - \delta k - \dot{k}\right)\left[f'(k) - \delta\right]
\end{aligned}$$

which evaluated at the steady state $\left(\bar{c}, \bar{k}\right)$ give:

$$
\begin{aligned}
F_k\left(\bar{k}, 0\right) &= U'\left(\bar{c}\right)\left[f'\left(\bar{k}\right) - \delta\right] = U'\left(\bar{c}\right)\rho \\
F_{\dot{k}}\left(k, \dot{k}\right) &= -U'\left(\bar{c}\right) \\
F_{\dot{k}k}\left(k, \dot{k}\right) &= -U''\left(\bar{c}\right)\rho \\
F_{\dot{k}\dot{k}}\left(k, \dot{k}\right) &= U''\left(\bar{c}\right) \\
F_{kk}\left(\bar{k}, 0\right) &= U'\left(\bar{c}\right) f''\left(\bar{k}\right) + U''\left(\bar{c}\right)\rho^2
\end{aligned}
$$

Then the quadratic equation

$$
Q\left(\lambda\right) \equiv -F_{\dot{k}\dot{k}}\,\left[\lambda\right]^2 + \left[\rho F_{\dot{k}\dot{k}}\right]\,\lambda + \left[F_{kk} + \rho F_{\dot{k}k}\right]
$$

whose roots that determine g' becomes

$$
Q\left(\lambda\right) = -U''\lambda^2 + \rho U''\,\lambda + \left[U'f'' + U''\rho^2 - \rho^2 U''\right]
$$

which after simplifications can be written as

$$
\begin{aligned}
Q\left(\lambda\right) &= -U''\lambda^2 + \rho U''\,\lambda + \left[U'f''\right] \\
&= \left[-U''\right]\left\{\lambda^2 - \rho\,\lambda - \left(\frac{cf'}{k}\right)\frac{-k\,f''/f'}{-cU''/U'}\right\}
\end{aligned}
$$

The term

$$
\left(\frac{cf'}{k}\right)\frac{\left(-k\,f''/f'\right)}{\left(-cU''/U'\right)}
$$

contains the ratio of two elasticities: $\left(-k\,f''/f'\right)$, the elasticity of the marginal productivity, and $\left(-cU''/U'\right)$ the elasticity of the marginal utility, the reciprocal of the intertemporal elasticity of substitution.

Now we check that one root is negative, say λ_1, and the other is then positive and larger than ρ. To do this, notice that:

$$\begin{aligned} Q(0) &= [-U''] \left\{ -\left(\frac{cf'}{k}\right) \frac{-k\, f''/f'}{-cU''/U'} \right\} < 0 \\ Q(\rho) &= Q(0) \\ \frac{\partial^2 Q(\lambda)}{\partial \lambda^2} &= -2U'' > 0 \end{aligned}$$

so we can plot Q and verified that $Q(\lambda_1) = 0$ with $\lambda_1 < 0$ and $Q(\lambda_2) > \rho$.

We can indeed solve the roots explicitly obtaining:

$$\lambda = \frac{\rho \pm \left(\rho^2 + 4\left[\left(\frac{cf'}{k}\right) \frac{-k\, f''/f'}{-cU''/U'}\right]\right)^{1/2}}{2}$$

so the negative root is

$$g'\left(\bar{k}\right) = \frac{\rho - \left(\rho^2 + 4\left[\left(\frac{cf'}{k}\right) \frac{-k\, f''/f'}{-cU''/U'}\right]\right)^{1/2}}{2}$$

Example for the Neoclassical Growth Model:
Consider the following functional forms for the production function and the period utility:

$$\begin{aligned} f(k) &= A\, k^{\alpha} \\ u(c) &= \left(c^{1-\gamma} - 1\right)/(1-\gamma) \end{aligned}$$

The steady state equations are:

$$\rho + \delta = \alpha A \bar{k}^{\alpha-1} \text{ , and } \bar{c} = A\bar{k}^{\alpha} - \delta \bar{k}$$

or

$$\bar{c} = \frac{\bar{k}}{\alpha} \alpha A \bar{k}^{\alpha-1} - \delta \bar{k} = \bar{k} \frac{\rho + (1-\alpha)\delta}{\alpha}$$

and the elasticities used in Q are:

$$-kf''/k = (1-\alpha) \text{ and } -cU''(c)/c = \gamma$$

so

$$\left(\frac{cf'}{k}\right) \frac{(-k\, f''/f')}{(-cU''/U')} = \left[\frac{\rho + (1-\alpha)\delta}{\alpha}\right] (\rho + \delta)\; \frac{(1-\alpha)}{\gamma}$$

Now we return to the general problem. We analyze under what circumstances local stability, i.e. $g'\left(\bar{k}\right) < 0$, is obtained.
In the general case we have:

$$Q(\lambda) \equiv -F_{\dot{k}\dot{k}}\ [\lambda]^2 + \left[\rho F_{\dot{k}\dot{k}}\right]\ \lambda + \left[F_{kk} + \rho F_{k\dot{k}}\right]$$

which we can write as:

$$Q(\lambda) \equiv -F_{\dot{k}\dot{k}}\ \left\{[\lambda]^2 - \rho\lambda - \left[\frac{F_{kk} + \rho F_{\dot{k}k}}{F_{\dot{k}\dot{k}}}\right]\right\}$$

so we have:

$$\begin{aligned} Q(0) &= \left[-F_{\dot{k}\dot{k}}\right]\left[-\frac{F_{kk} + \rho F_{\dot{k}k}}{F_{\dot{k}\dot{k}}}\right] \\ Q(\rho) &= Q(0) \\ \frac{\partial^2 Q(\lambda)}{\partial \lambda^2} &= -F_{\dot{k}\dot{k}} \end{aligned}$$

The solutions are

$$\lambda = \frac{\rho \pm \sqrt{\rho^2 + 4\left[-\left(F_{kk} + \rho F_{\dot{k}k}\right)/\left(-F_{\dot{k}\dot{k}}\right)\right]}}{2}$$

Under the assumption that the period objective function $F\left(k, \dot{k}\right)$ is strictly concave, we have that:

$$F_{kk} < 0,\ F_{\dot{k}\dot{k}} < 0 \text{ and } F_{kk}\ F_{\dot{k}\dot{k}} - \left[F_{\dot{k}k}\right]^2 > 0$$

which implies $\quad \left[F_{\dot{k}k}\right]^2 \leq F_{kk}\ F_{\dot{k}\dot{k}}$ or $\left|F_{\dot{k}k}\right| \leq \sqrt{F_{kk}\ F_{\dot{k}\dot{k}}}$

Using this information we have the following cases:
1) If $-\left(F_{kk}+\rho F_{\dot{k}k}\right)/\left(-F_{\dot{k}\dot{k}}\right)>0$,

$$\lambda_1<0<\rho<\lambda_2$$

and hence:

$$g'\left(\bar{k}\right)=\lambda_1$$

and the steady state is locally stable.
2) If $-\left(F_{kk}+\rho F_{\dot{k}k}\right)/\left(-F_{\dot{k}\dot{k}}\right)<0$,

$$0<\lambda_1<\rho<\lambda_2$$

then the steady state is not locally stable.

Sufficient Conditions for Local Stability

I) First we consider a straightforward, but powerful result. Notice that as $\rho\to 0$:

$$\lambda=\frac{0\pm\sqrt{4\left[-F_{kk}/\left(-F_{\dot{k}\dot{k}}\right)\right]}}{2}=\pm\sqrt{F_{kk}/F_{\dot{k}\dot{k}}}$$

$$\text{so}\qquad \lim_{\rho\to 0}g'\left(\bar{k}\right)=-\sqrt{F_{kk}/F_{\dot{k}\dot{k}}}<0.$$

Thus, as long as F is strictly concave, for small ρ the steady state is globally stable.

II) Now we extend this result by finding a lower bound on ρ for which $g'\left(\bar{k}\right) < 0$. The smallest root λ_1 is negative if either

$$F_{\dot{k}k} \leq 0 \text{ or if } F_{k\dot{k}} > 0 \text{ and } \rho \leq \frac{-F_{kk}}{F_{k\dot{k}}}.$$

[see the argument below for a proof of the lower bound]. In either of these cases $g'\left(\bar{k}\right) = \lambda_1 < 0$. The lower bound for ρ can also be written as

$$\rho \leq \frac{-F_{kk}/F_k}{-F_{\dot{k}k} / \left(-F_{\dot{k}}\right)}$$

This condition compares the curvature of F_k to $F_{\dot{k}}$ with the discount rate. It is satisfied if the marginal cost $-F_{\dot{k}}$ of accumulating capital has a high curvature.

Notice also that the condition for local stability:

$$F_{kk} + \rho F_{\dot{k}k} < 0$$

is related to the sign of the comparative static of steady state capital with respect to ρ. Define $\bar{k}(\rho)$ as the solution to

$$F_k\left(\bar{k}(\rho), 0\right) + \rho F_{\dot{k}}\left(\bar{k}(\rho), 0\right) = 0$$

Differentiating w.r.t. to ρ :

$$\bar{k}_\rho = \left[F_{kk} + \rho F_{\dot{k}k}\right]^{-1}\left(-F_{\dot{k}k}\right)$$

Thus if $F_{\dot{k}k} < 0$ then $\bar{k}_\rho < 0$. This is the intuitive case: higher discount rate (i.e. more impatience) leads to lower capital accumulation in steady sate. If $F_{k\dot{k}} > 0$ but $F_{kk} + \rho F_{\dot{k}k} < 0$, the other case where the steady state is locally stable, we have the "counter-intutive" result that $\bar{k}_\rho > 0$.

Proof of the lower bound for ρ for local stability.
To see how we obtain this condition for the case where $F_{k\dot{k}} > 0$ notice that we can write:

$$\rho \leq \frac{-F_{kk}}{F_{\dot{k}k}} = \frac{-F_{kk}}{|F_{\dot{k}k}|}$$

and using concavity of F :

$$\rho \leq \frac{-F_{kk}}{\sqrt{F_{kk}\ F_{\dot{k}\dot{k}}}} = \frac{\sqrt{-F_{kk}}\sqrt{-F_{kk}}}{\sqrt{-F_{kk}}\ \sqrt{-F_{\dot{k}\dot{k}}}} = \sqrt{\frac{F_{kk}}{F_{\dot{k}\dot{k}}}}$$

or completing elasticities and using the steady state equations:

$$F_k + \rho F_{\dot{k}} = 0 \ \text{ or } F_k = -\rho F_{\dot{k}}$$

we obtain:

$$\rho \leq \sqrt{\frac{F_{kk}/F_k}{F_{\dot{k}\dot{k}}/F_{\dot{k}}}\frac{F_k}{F_{\dot{k}}}} = \sqrt{\frac{F_{kk}/F_k}{F_{\dot{k}\dot{k}}/\left(-F_{\dot{k}}\right)}\rho}$$

or

$$\rho \leq \frac{-F_{kk}/F_k}{-F_{\dot{k}\dot{k}}\ /\left(-F_{\dot{k}}\right)}.$$