

Classical Discrete Choice Theory

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- Classical regression model:

$$y = x\beta + \varepsilon$$

$$0 = E(\varepsilon|x)$$

$$\varepsilon \sim N(0, \sigma^2 I)$$

Model untenable if $y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- Unless we invoke the linear probability model, which is discussed below and has some unusual properties.

- To model discrete choices, need to think of the ingredients that give rise to choices.
- For example, suppose we want to forecast demand for a new good. We observe consumption data on old goods $x_1 \dots x_I$. (Each good could represent a transportation mode, for example, or an occupation choice.)
- Assume people choose a good that yields highest utility. When we have a new good, we need a way of putting it on a basis with the old.
- Earliest literature on discrete choice was developed in psychometrics where researchers were concerned with modeling choice behavior (Thurstone).
- These are also models of counterfactual utilities.

Two dominant modeling approaches

- i Luce Model (1953) \iff McFadden Conditional logit model
- ii Thurstone-Quandt Model (1929, 1930s). (Multivariate probit/normal model)

Other approaches

i GEV models

① Luce-McFadden Model

- widely used in economics
- easy to compute
- identifiability of parameters understood
- very restrictive substitution possibilities among goods
- restrictive heterogeneity
- imposes arbitrary preference shocks

② Quandt-Thurstone Model

- very general substitution possibilities
- allows for more general forms of heterogeneity
- more difficult to compute
- identifiability less easily established
- does not necessarily rely on preference shocks

- References: Manski and McFadden, Chapter 5, Yellot paper
- Notation:
 - X : universe of objects of choice
 - S : universe of attributes of persons
 - B : feasible choice set ($x \in B \subseteq X$)

- Behavior rule mapping attributes into choices: h

$$h(B, S) = x$$

- We might assume that there is a distribution of choice rules.
- h might be random because
 - a in observation we lose some information governing choices (unobserved characteristics of choice and person)
 - b there can be random variation in choices due to unmeasured psychological factors
- Define $P(x|S, B) = \Pr \{h \in H \ni h(S, B) = x\}$
- Probability that an individual drawn randomly from the population with attributes S and alternative set B chooses x .

- Maintain some restrictions on $P(x|S, B)$ and derive implications for the functional form of P .
- Axiom #1: “Independence of Irrelevant Alternatives”

$$\begin{array}{c} x, y \in B \quad s \in S \\ \frac{P(x|s, \{xy\})}{P(y|s, \{xy\})} = \frac{P(x|s, B)}{P(y|s, B)} \end{array} \quad B = \text{larger choice set}$$

- Example: Suppose choice is career decision and individual is choosing to be
 - an economist (E)
 - a fireman (F)
 - a policeman (P)

$$\frac{\Pr(E|s, \{EF\})}{\Pr(F|s, \{EF\})} = \frac{\Pr(E|s, \{EFP\})}{\Pr(F|s, \{EFP\})} \leftarrow$$

would think that introducing
3rd alternative might increase
ratio

- Another example: Red bus-Blue bus
- Choices:
 - take car C
 - red bus RB
 - blue bus BB
- Axiom #2

$\Pr(y|s, B) > 0 \quad \forall y \in B$ (i.e. eliminate 0 probability choices)

Implications of above axioms

- Define $P_{xy} = P(x|s, \{xy\})$
- Assume $P_{xx} = \frac{1}{2}$

$$P(y|s, B) = \frac{P_{yx}}{P_{xy}} P(x|s, B) \text{ by IIA axiom}$$

$$\sum_{y \in B} P(y|s, B) = 1 \implies P(x|s, B) = \frac{1}{\sum_{y \in B} \frac{P_{yx}}{P_{xy}}}$$

Implications of above axioms

- Furthermore,

$$P(y|s, B) = \frac{P_{yz}}{P_{zy}} P(z|s, B)$$

$$P(x|s, B) = \frac{P_{xz}}{P_{zx}} P(z|s, B)$$

$$P(y|s, B) = \frac{P_{yx}}{P_{xy}} P(x|s, B)$$

$$\frac{P_{yx}}{P_{xy}} = \frac{P(y|s, B)}{P(x|s, B)} = \frac{\frac{P_{yz}}{P_{zy}}}{\frac{P_{xz}}{P_{zx}}}$$

Implications of above axioms

- Define

$$\begin{aligned}\tilde{v}(s, x, z) &= \ln \left(\frac{P_{xz}}{P_{zx}} \right) & \tilde{v}(s, y, z) &= \ln \left(\frac{P_{yz}}{P_{zy}} \right) \\ \implies \frac{P_{yx}}{P_{xy}} &= \frac{e^{\tilde{v}(s, y, z)}}{e^{\tilde{v}(s, x, z)}}\end{aligned}$$

- Axiom #3: Separability Assumption

$$\tilde{v}(s, x, z) = v(s, x) - v(s, z) \quad \leftarrow \quad \begin{array}{l} v(s, z) \text{ can be interpreted as a} \\ \text{utility indicator of} \\ \text{representative tastes} \end{array}$$

- Then

$$P(x|s, B) = \frac{1}{\sum_{y \in B} \frac{P_{yx}}{P_{xy}}} = \frac{1}{\sum_{y \in B} \frac{e^{v(s,y) - v(s,z)}}{e^{v(s,x) - v(s,z)}}}$$

$$P(x|s, B) = \frac{e^{v(s,x)}}{\sum_{y \in B} e^{v(s,y)}} \quad \leftarrow \quad \begin{array}{l} \text{Get logistic from} \\ \text{from Luce Axioms} \end{array}$$

- Now link model to familiar models in economics.
- Marshak (1959) established link between Luce Model and random utility models (Rum's).

- Assume utility from choosing alternative j is

$$u_j = v(s, x_j) + \varepsilon(s, x_j)$$

- $v(s, x_j)$ is a nonstochastic function and $\varepsilon(s, x_j)$ is stochastic, reflecting idiosyncratic tastes.

- $\Pr(j \text{ is maximal in set } B) = \Pr(u(s, x_j) \geq u(s, x_l)) \quad \forall l \neq j$
 $= \Pr(v(s, x_j) + \varepsilon(s, x_j) \geq v(s, x_l) + \varepsilon(s, x_l)) \quad \forall l \neq j$
 $= \Pr(v(s, x_j) - v(s, x_l) \geq \varepsilon(s, x_l) - \varepsilon(s, x_j)) \quad \forall l \neq j$

- Specify a cdf $F(\varepsilon_1, \dots, \varepsilon_N)$
- Then

$$\begin{aligned}
 \Pr(v_j - v_l &\geq \varepsilon_l - \varepsilon_j \quad \forall l \neq j) \\
 &= \Pr(v_j - v_l + \varepsilon_j \geq \varepsilon_l \quad \forall l \neq j) \\
 &= \int_{-\infty}^{\infty} F_j(v_j - v_1 + \varepsilon_j, \dots, v_j - v_{j-1} + \varepsilon_j, \varepsilon_j, \dots, v_j - v_J + \varepsilon_j) d\varepsilon_j
 \end{aligned}$$

- If ε is iid, then

$$F(\varepsilon_1, \dots, \varepsilon_n) = \prod_{i=1}^n F_i(\varepsilon_i)$$

- So $\Pr(v_j - v_l \geq \varepsilon_l - \varepsilon_j \quad \forall l \neq j)$

$$\int_{-\infty}^{\infty} \left[\prod_{\substack{i=1 \\ i \neq j}}^n F_i(v_j - v_i + \varepsilon_j) \right] f_j(\varepsilon_j) d\varepsilon_j$$

Binary Example ($N = 2$)

$$P(1 \mid s, B) = \int_{-\infty}^{\infty} \int_{-\infty}^{v_1 - v_2 + \varepsilon_1} f_1(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2$$

- If $\varepsilon_1, \varepsilon_2$ are normal then $\varepsilon_1 - \varepsilon_2$ is normal, so $\Pr(v_1 - v_2 \geq \varepsilon_1 - \varepsilon_2)$ is normal.
- If $\varepsilon_1, \varepsilon_2$ are Weibull then $\varepsilon_1 - \varepsilon_2$ is logistic

$$\varepsilon \sim \text{Weibull} \implies \Pr(\varepsilon < c) = e^{-e^{-c+\alpha}}$$

- Also called double exponential” or “Type I extreme value”
- ε Weibull

$$\begin{aligned}\Pr(v_1 + \varepsilon_1 > v_2 + \varepsilon_2) &= \Pr(v_1 - v_2 > \varepsilon_2 - \varepsilon_1) \\ &= \Omega(v_1 - v_2) = \frac{e^{v_1 - v_2}}{1 + e^{v_1 - v_2}} = \frac{e^{v_1}}{e^{v_1} + e^{v_2}}\end{aligned}$$

- Result: Assuming that the errors follow a Weibull distribution yields same logit model derived from the Luce Axioms.
- This link was established by Marshak (1959)
- Turns out that Weibull is sufficient but not necessary.
- Some other distributions for ε generate a logit.
- Yellot (1977) showed that if we require “*invariance under uniform expansions of the choice set*” then only double exponential gives logit.
- Example: Suppose choice set is {coffee, tea, mild}, then “invariance” requires that probabilities stay the same if we double the choice set (i.e., 2 coffees, 2 teas, 2 milks).

Some Important Properties of the Weibull

- Developed 1928 (Fisher & Tippet showed it's one of 3 possible limiting distributions for the maximum of a sequence of random variables)
- Closed under maximization (i.e. max of n Weibulls is a Weibull)

$$\Pr(\max_i \varepsilon_i \leq c) = \prod_i e^{-e^{-(c+\alpha_i)}} = e^{-\sum_i e^{-c} e^{\alpha_i}} = e^{-e^{-c} \sum_i e^{\alpha_i}} = e^{-e^{-c + \ln \sum_i e^{\alpha_i}}}$$

- Difference between two Weibulls is a logit
- Under Luce axioms (on R.U.M. with Weibull assumption)

$$\Pr(j \mid s, B) = \frac{e^{v(s, x_j)}}{\sum_{l=1}^N e^{v(s, x_l)}}$$

- Now reconsider *the forecasting problem*:
- Let x_j = set of characteristics associated with choice j
- Usually, it is assumed that $v(s, x_j) = \theta(s)'x_j$
- Dependence of θ on s reflects fact that individuals differ in their evaluation of characteristics.

- Get

$$\Pr(j \mid s, B) = \frac{e^{\theta(s)'x_j}}{\sum_{l=1}^N e^{\theta(s)'x_l}}$$

- Solve by MLE

- Solve by MLE

$$\max_{\theta(s)} \prod_{i=1}^N \left[\frac{e^{\theta(s)x_1}}{\sum_{l=1}^N e^{\theta(s)x_l}} \right]^{D_{1i}} \cdots \left[\frac{e^{\theta(s)x_N}}{\sum_{l=1}^N e^{\theta(s)x_l}} \right]^{D_{Ni}}$$

- If a new good has same characteristics, get probabilities by

$$B' = \{B, N + 1\}$$
$$P(N + 1 \mid B', s) = \frac{e^{\theta(s)'x_{N+1}}}{\sum_{l=1}^{N+1} e^{\theta(s)'x_l}}$$

Debreu (1960) criticism of Luce Model

- “Red Bus - Blue Bus Problem”
- Suppose $N + 1^{th}$ alternative is identical to the first

$$\Pr(\text{choose 1 or } N + 1 \mid s, B') = \frac{2e^{\theta(s)'x_{N+1}}}{\sum_{l=1}^{N+1} e^{\theta(s)'x_l}}$$

- \implies Introduction of identical good changes probability of riding a bus.
 - not an attractive result
 - comes from need to make iid assumption on new alternative

Debreu (1960) criticism of Luce Model: Some Alternative Assumptions

- 1 Could let $v_i = \ln(\theta(s)'x_i)$

$$\Pr(j \mid s, B) = \frac{\theta(s)'x_j}{\sum_{l=1}^{N+1} \theta(s)'x_l}$$

If we also imposed $\sum_{l=1}^N \theta(s)'x_l = 1$, we would get linear probability model but this could violate IIA.

- 2 Could consider model of form

$$\Pr(j \mid s, B) = \frac{e^{\theta^j(s)x^j}}{\sum_{l=1}^N e^{\theta^l(s)x^l}}$$

but here we have lost our forecasting ability (cannot predict demand for a new good).

- 3 Universal Logit Model

$$\Pr(i \mid s, x_1, \dots, x_N) = \frac{e^{\varphi_i(x_1, \dots, x_N)\beta(s)}}{\sum_{l=1}^N e^{\varphi_l(x_1, \dots, x_N)\beta(s)}}$$

- ① Goal: We want a probabilistic choice model that
 - ① has a flexible functional form
 - ② is computationally practical
 - ③ allows for flexibility in representing substitution patterns among choices
 - ④ is consistent with a random utility model (RUM) \implies has a structural interpretation

How do you verify that a candidate PCS is consistent with a RUM?

1 Goal:

- a Either start with a R.U.M.

$$u^i = v(s, x^i) + \varepsilon(s, x^i)$$

and solve integral for

$$\Pr(u^i > u^l, \forall l \neq i) = \Pr(i = \arg \max_l (v^l + \varepsilon^l))$$

or

- b start with a candidate PCS and verify that it is consistent with a R.U.M. (easier)

2 McFadden provides sufficient conditions

3 See discussion of Daley-Zachary-Williams theorem

Link to Airum Models

Daly-Zachary-Williams Theorem

- Daly-Zachary (1976) and Williams (1977) provide a set of conditions that makes it easy to derive a PCS from a RUM with a class of models (“generalized extreme value” (GEV) models)
- Define $G : G(Y_1, \dots, Y_J)$
- If G satisfies the following
 - ① nonnegative defined on $Y_1, \dots, Y_J \geq 0$
 - ② homogeneous degree one in its arguments
 - ③ $\lim_{Y_i \rightarrow \infty} G(Y_1, \dots, Y_i, \dots, Y_J) \rightarrow \infty, \forall i = 1, \dots, J$

$$\frac{\partial^k G}{\partial Y_1 \cdots \partial Y_k} \quad \text{is} \quad \begin{array}{l} \text{nonnegative if } k \text{ odd} \\ \text{nonpositive if even} \end{array} \quad (1)$$

- Then for a R.U.M. with $u_i = v_i + \varepsilon_i$ and

$$F(\varepsilon_1, \dots, \varepsilon_J) = \exp \left\{ -G(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_J}) \right\}$$

- This cdf has Weibull marginals but allows for more dependence among ε 's.
- The PCS is given by

$$P_i = \frac{\partial \ln G}{\partial v_i} = \frac{e^{v_i} G_i(e^{v_1}, \dots, e^{v_J})}{G(e^{v_1}, \dots, e^{v_J})}$$

- Note: McFadden shows that under certain conditions on the form of the indirect utility function (satisfies AIRUM form), the DZW result can be seen as a form of Roy's identity.

- Let's apply this result
- Multinomial logit model (MNL)

$$\begin{aligned} \text{cdf } F(\varepsilon_1, \dots, \varepsilon_J) &= e^{-e^{-\varepsilon_1}} \dots e^{-e^{-\varepsilon_J}} \leftarrow \text{product of iid} \\ &\quad \text{Weibulls} \\ &= e^{-\sum_{j=1}^J e^{-\varepsilon_j}} \end{aligned}$$

- Can verify that $G(e^{\nu_1}, \dots, e^{\nu_J}) = \sum_{j=1}^J e^{\nu_j}$ satisfies DZW conditions

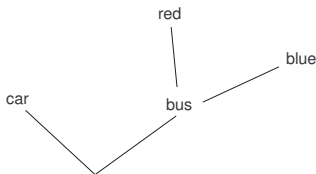
$$P(j) = \frac{\partial \ln G}{\partial \nu_j} = \frac{e^{\nu_j}}{\sum_{l=1}^J e^{\nu_l}} = \text{MNL model}$$

- Another GEV model
- Nested logit model (addresses to a limited extent the IIA criticism)
- Let

$$G(e^{v_1}, \dots, e^{v_J}) = \sum_{m=1}^M a_m \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{1-\sigma_m}$$

- σ_m like an elasticity of substitution.

- Idea: divide goods into branches
- First choose branch, then good within branch



- Will allow for correlation between errors (this is role of σ)
-

$$B_m \subseteq \{1, \dots, J\}$$
$$\bigcup_{m=1} B_m = B$$

is a single branch—need not have all choices on all branches

- Note: if $\sigma = 0$, get usual MNL form
- Calculate equation for

$$\begin{aligned}
 p_i &= \frac{\partial \ln G}{\partial v_i} = \frac{\partial \ln \left[\sum_{m=1}^m a_m \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{1-\sigma_m} \right]}{\partial v_i} \\
 &= \frac{\sum_{m \ni i \in B_m} a_m \left(e^{\frac{v_i}{1-\sigma_m}} \right) \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{-\sigma_m} \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{-1} \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]}{\sum_{m=1}^m a_m \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{1-\sigma_m}} \\
 &= \sum_{m=1}^m P(i \mid B_m) P(B_m)
 \end{aligned}$$

- Where

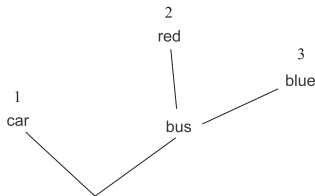
$$P(i \mid B_m) = \frac{e^{\frac{v_i}{1-\sigma_m}}}{\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}}} \text{ if } i \in B_m, 0 \text{ otherwise}$$

$$P(B_m) = \frac{a_m \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{1-\sigma_m}}{\sum_{m=1}^M a_m \left[\sum_{i \in B_m} e^{\frac{v_i}{1-\sigma_m}} \right]^{1-\sigma_m}}$$

- Note: If $P(B_m) = 1$ get logit form
- Nested logit requires that analyst make choices about nesting structure

- How does nested logit solve red bus/blue bus problem?
- Suppose

$$G = Y_1 + \left[Y_2^{\frac{1}{1-\sigma}} + Y_3^{\frac{1}{1-\sigma}} \right]^{1-\sigma} \quad Y_i = e^{v_i}$$



$$P(1 \mid \{123\}) = \frac{\partial \ln G}{\partial v_i} = \frac{e^{v_1}}{e^{v_1} + \left[e^{\frac{v_2}{1-\sigma}} + e^{\frac{v_3}{1-\sigma}} \right]^{1-\sigma}}$$

$$P(2 \mid \{123\}) = \frac{\partial \ln G}{\partial v_i} = \frac{e^{\frac{v_2}{1-\sigma}} \left[e^{\frac{v_2}{1-\sigma}} + e^{\frac{v_3}{1-\sigma}} \right]^{-\sigma}}{e^{v_1} + \left[e^{\frac{v_2}{1-\sigma}} + e^{\frac{v_3}{1-\sigma}} \right]^{1-\sigma}}$$

- As $v_3 \rightarrow -\infty$

$$P(1 \mid \{123\}) = \frac{e^{v_1}}{e^{v_1} + e^{v_2}} \quad (\text{get logistic})$$

What Role Does σ Play?

- σ is the degree of substitutability parameter
- Recall

$$F(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \exp\{-G(e^{-\varepsilon_1}, e^{-\varepsilon_2}, e^{-\varepsilon_3})\}$$

- Here

$$\sigma = \frac{\text{cov}(\varepsilon_2, \varepsilon_3)}{\sqrt{\text{var } \varepsilon_2} \sqrt{\text{var } \varepsilon_3}} = \text{correlation coefficient}$$

- Thus we require $-1 \leq \sigma \leq 1$, but turns out we also need to require $\sigma > 0$ for DZW conditions to be satisfied. This is unfortunate because it does not allow ε 's to be negatively correlated.
- Can show that

$$\lim_{\sigma \rightarrow 1} P(1 \mid \{123\}) = \frac{e^{v_1}}{e^{v_1} + \max(e^{v_2}, e^{v_3})} \quad (\text{L'Hôpital's Rule})$$

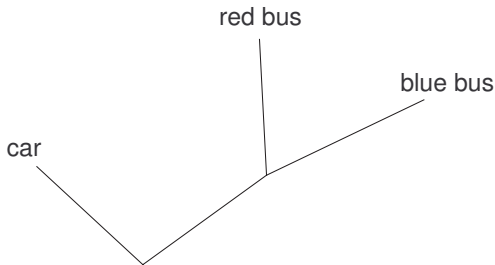
- If $v_2 = v_3$, then

$$\begin{aligned}
 P(2 \mid \{123\}) &= \frac{e^{\frac{v_2}{1-\sigma}} \left[2e^{\frac{v_2}{1-\sigma}} \right]^{-\sigma}}{e^{v_1} + \left[2e^{\frac{v_2}{1-\sigma}} \right]^{1-\sigma}} \\
 &= 2^{-\sigma} \frac{e^{v_2}}{e^{v_1} + (e^{v_2}) (2^{1-\sigma})}
 \end{aligned}$$

$$\lim_{\sigma \rightarrow 1} = 2^{-1} \frac{e^{v_2}}{e^{v_1} + e^{v_2}} \text{ when } v_1 = v_2$$

↗ introduce 3rd identical alternative and cut the probability of choosing 2 in half

- Solves red-bus/blue-bus problem
- Probability cut in half with two identical alternatives



- σ is a measure of similarity between red and blue bus.
- When σ close to one, the conditional choice probability selects with high probability the alternative.

We Can Expand Logit to Accommodate Multiple Nesting Levels

$$G = \sum_{q=1}^Q a_q \left\{ \sum_{m \in Q_q} a_m \left[\sum_{i \in B_m} y_i^{\frac{1}{1-\sigma_m}} \right]^{1-\sigma_m} \right\} \text{ 3 levels}$$

- Example: Two Choices

- ① Neighborhood (m)
- ② Transportation mode (t)
- ③ $P(m)$: choice of neighborhood
- ④ $P(i \mid B_m)$: probability of choosing i^{th} mode, given neighborhood m

① Not all modes available in all neighborhoods

$$P_{m,t} = \frac{e^{\frac{v(m,t)}{1-\sigma_m}} \left[\sum_{t=1}^{T_m} e^{\frac{v(m,t)}{1-\sigma_m}} \right]^{-\sigma_m}}{\sum_{j=1}^m \left[\sum_{t=1}^{T_j} e^{\frac{v(m,t)}{1-\sigma_m}} \right]^{1-\sigma_m}}$$

$$P_{t|m} = \frac{e^{\frac{v(m,t)}{1-\sigma_m}}}{\sum_{t=1}^{T_m} e^{\frac{v(m,t)}{1-\sigma_m}}}$$
$$P_m = \frac{\left[\sum_{t=1}^{T_m} e^{\frac{v(m,t)}{1-\sigma_m}} \right]^{1-\sigma_m}}{\sum_{j=1}^m \left[\sum_{t=1}^{T_j} e^{\frac{v(m,t)}{1-\sigma_m}} \right]^{1-\sigma_m}} = P(B_m)$$

- Standard type of utility function that people might use

$$v(m, t) = z'_t \gamma + x'_{mt} \beta + y'_m \alpha$$

- z'_t is transportation mode characteristics, x'_{mt} is interactions and y'_m is neighborhood characteristics.
- Then

$$P_{t|m} = \frac{e^{\frac{(z'_t\gamma + x'_{mt}\beta)}{1-\sigma_m}}}{\left[\sum_{t=1}^{T_m} e^{\frac{(z'_t\gamma + x'_{mt}\beta)}{1-\sigma_m}} \right]}$$

$$P_m = \frac{e^{y'_m\alpha} \left[\sum_{t=1}^{T_m} e^{\frac{(z'_t\gamma + x'_{mt}\beta)}{1-\sigma_m}} \right]^{1-\sigma_m}}{\sum_{j=1}^m e^{y'_m\alpha} \left[\sum_{t=1}^{T_m} e^{\frac{(z'_t\gamma + x'_{mt}\beta)}{1-\sigma_j}} \right]^{1-\sigma_j}}$$

- Estimation (in two steps) (see Amemiya, Chapter 9)
- Let

$$l_m = \sum_{t=1}^{T_m} e^{\frac{(z_t' \gamma + x_{mt}' \beta)}{1 - \sigma_m}}$$

- ① Within each neighborhood, get $\frac{\hat{\gamma}}{1-\sigma_m}$ and $\frac{\hat{\beta}}{1-\sigma_m}$ by logit
- ② Form \hat{I}_m
- ③ Then estimate by MLE

$$\frac{e^{y'_m \alpha + (1-\sigma_m) \ln \hat{I}_m}}{\sum_{j=1}^m e^{y'_m \alpha + (1-\sigma_j) \ln \hat{I}_j}} \quad \text{get } \hat{\alpha}, \hat{\sigma}_m$$

- Assume $\sigma_m = \sigma_j \forall j, m$ or at least need some restrictions across multiple neighborhoods?
- Note: \hat{I}_m is an estimated regressor (“Durbin problem”)
- Need to correct standard errors

Multinomial Probit Models

① Also known as:

- ① Thurstone Model V (1929; 1930)
- ② Thurstone-Quandt Model
- ③ Developed by Domencich-McFadden (1978) (on reading list)

$$u_i = v_i + \eta_i \quad i = 1, \dots, J$$

$$v_i = Z_i \beta \quad (\text{linear in parameters form})$$

$$u_i = Z_i \beta + \eta_i$$

MNL

(i) β fixed

(ii) η_i iid

MNP

(i) β random coefficient $\beta \sim N(\bar{\beta}, \Sigma_\beta)$

(ii) β independent of η $\eta \sim (0, \Sigma_\eta)$,

- Allow gen. forms of correlation between errors

$$u_i = Z_i \bar{\beta} + Z_i (\beta - \bar{\beta}) + \eta_i$$

- $(\beta - \bar{\beta}) = \varepsilon$ and $Z_i (\beta - \bar{\beta}) + \eta_i$ is a composite heteroskedastic error term.
- β random = taste heterogeneity,
- η_i can interpret as unobserved attributes of goods
- Main advantage of MNP over MNL is that it allows for general error covariance structure.
- Note: To make computation easier, users sometimes set $\Sigma_{\beta} = 0$ (fixed coefficient version)
 - allowing for β random
 - permits random taste variation
- allows for possibility that different persons value 2 characteristics differently

Problem of Identification and Normalization in the MNP Model

- Reference: David Bunch (1979), “Estimability In the Multinomial Probit Model” in *Transportation Research*
- Domencich and McFadden
- Let

$$Z\bar{\beta} = \begin{pmatrix} Z_1 \cdot \bar{\beta} \\ \vdots \\ Z_J \cdot \bar{\beta} \end{pmatrix} \quad \tilde{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_J \end{pmatrix} \quad \begin{array}{l} J \text{ alternatives} \\ K \text{ characteristics} \\ \beta \text{ random} \quad \beta \sim N(\beta, \Sigma_\beta) \end{array} \quad (2)$$

Problem of Identification and Normalization in the MNP Model

- $\Pr(\text{alternative } j \text{ selected})$:

$$= \Pr(u_j > u_i) \quad \forall i \neq j$$

$$= \int_{u_j=-\infty}^{\infty} \int_{u_i=-\infty}^{u_j} \int_{u_j=-\infty}^{u_j} \Phi(u \mid V_\mu, \Sigma_\mu) du_J du_I du_j$$

where $\Phi(u \mid V_\mu, \Sigma_\mu)$ is *pdf*

(Φ is J -dimensional MVN density with mean V_μ, Σ_μ)

- Note: Unlike the MVL, no closed form expression for the integral.
- The integrals often evaluated using simulation methods (we will work an example).

How many parameters are there?

- $\bar{\beta}$: K parameters
- Σ_{β} : $K \times K$ symmetric matrix $\frac{K^2-K}{2} + K = \frac{K(K+1)}{2}$
- Σ_{η} : $\frac{J(J+1)}{2}$
- Note: When a person chooses j , all we know is relative utility, not absolute utility.
- This suggests that not all parameters in the model will be identified.
- Requires normalizations.

- What does it mean to say a parameter is not identified in a model?
- Model with one parameterization is observationally equivalent to another model with a different parameterization

- Example: Binary Probit Model (fixed β)

$$\begin{aligned}\Pr(D = 1 \mid Z) &= \Pr(v_1 + \varepsilon_1 > v_2 + \varepsilon_2) \\ &= \Pr(x\beta + \varepsilon_1 > x_2\beta + \varepsilon_2) \\ &= \Pr((x_1 - x_2)\beta > \varepsilon_2 - \varepsilon_1) \\ &= \Pr\left(\frac{(x_1 - x_2)\beta}{\sigma} > \frac{\varepsilon_2 - \varepsilon_1}{\sigma}\right) \\ &= \Phi\left(\frac{\tilde{x}\beta}{\sigma}\right) \quad \bar{x} = x_1 - x_2\end{aligned}$$

- $\Phi\left(\frac{\tilde{x}\beta}{\sigma}\right)$ is observationally equivalent to $\Phi\left(\frac{\tilde{x}\beta^*}{\sigma^*}\right)$ for $\frac{\beta}{\sigma} = \frac{\beta^*}{\sigma^*}$.

- β not separably identified relative to σ but ratio is identified:

$$\begin{aligned}\Phi\left(\frac{\tilde{x}\beta}{\sigma}\right) &= \Phi\left(\frac{\tilde{x}\beta^*}{\sigma^*}\right) \\ \Phi^{-1} \cdot \Phi\left(\frac{\tilde{x}\beta}{\sigma}\right) &= \Phi^{-1}\Phi\left(\frac{\tilde{x}\beta^*}{\sigma^*}\right) \\ &\Rightarrow \frac{\beta}{\sigma} = \frac{\beta^*}{\sigma^*}\end{aligned}$$

- Set $\{b : b = \beta \cdot \delta, \delta \text{ any positive scalar}\}$ is identified (say “ β is identified up to scale and sign is identified”).

Identification in the MVP model

$$\Pr(j \text{ selected} \mid V_\mu, \Sigma_\mu) = \Pr(u_i - u_j < 0 \quad \forall i \neq j)$$

Define $\Delta_j = \begin{pmatrix} 1 & 0 & \dots & -1 & \dots & 0 \\ 0 & 1 & \dots & -1 & \dots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \dots & \dots & -1 & 0 & 1 \end{pmatrix}_{(J-1) \times J}$ (contrast matrix)

$$\Delta_j \tilde{u} = \begin{pmatrix} u^i - u^j \\ \vdots \\ u^J - u^j \end{pmatrix}$$

$$\begin{aligned}\Pr(j \text{ selected} \mid V_\mu, \Sigma_\mu) &= \Pr(\Delta_j \tilde{u} < 0 \mid V_\mu, \Sigma_\mu) \\ &= \Phi(0 \mid V_Z, \Sigma_Z)\end{aligned}$$

- Where

- 1 V_Z is the mean of $\Delta_j \tilde{u} = \Delta_j \tilde{Z} \bar{\beta}$
- 2 Σ_Z is the variance of $\Delta_j \tilde{Z} \Sigma_\beta \tilde{Z}' \Delta_j' + \Delta_j \Sigma_\eta \Delta_j'$
- 3 V_Z is $(J - 1) \times 1$
- 4 Σ_Z : $(J - 1) \times (J - 1)$

- We reduce dimensions of the integral by one.

- This says that all of the information exists in the contrasts.
- Can't identify all the components because we only observe the contrasts.
- Now define $\tilde{\Delta}_j$ as Δ_j with J th column removed and choose J as the reference alternative with corresponding Δ_J .
- Then can verify that

$$\Delta_j = \tilde{\Delta}_j \cdot \Delta_J$$

- For example, with three goods:

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

- $\tilde{\Delta}_j$, ($j = 2$, included) Δ_J , ($J = 3$, included) Δ_j , ($j = 2$, 3rd column removed)

3rd column removed) reference alt.)

- Therefore, we can write

$$\begin{aligned} V_Z &= \Delta_j \tilde{Z} \bar{\beta} \\ \Sigma_Z &= \Delta_j \tilde{Z} \Sigma_{\beta} \tilde{Z}' \Delta_j' + \tilde{\Delta}_j \Delta_J \Sigma_{\eta} \Delta_J' \tilde{\Delta}_j' \end{aligned}$$

- where $C_J = \Delta_J \Sigma_{\eta} \Delta_J'$ and $(J-1) \times (J-1)$ has $\frac{(J-1)^2 - (J-1)}{2} + (J+1)$ parameters = $\frac{J(J-1)}{2}$ total.
- Since original model can always be expressed in terms of a model with $(\beta, \Sigma_{\beta}, C_J)$, it follows that some of the parameters in the original model are not identified.

How many parameters not identified?

- Original model:

$$K + \frac{K(K+1)}{2} + \frac{J(J+1)}{2}$$

- Now:

$$K + \frac{K(K+1)}{2} + \frac{J(J-1)}{2}, \quad \frac{J^2 + J - (J^2 - J)}{2} \\ = J \text{ not identified}$$

- Turns out that one additional parameter not identified.
- Total: $J + 1$
- Note:** Evaluation of $\Phi(0 \mid kv_Z, k^2\Sigma_Z)$ $k > 0$ gives same result as evaluating $\Phi(0 \mid v_Z, \Sigma_Z)$ can eliminate one more parameter by suitable choice of k .



Illustration

$$J = 3 \quad \Sigma_{\eta} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$$\begin{aligned} C_2 &= \Delta_2 \Sigma_{\eta} \Delta_2' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \cdot \Sigma_{\eta} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}' \\ &= \begin{pmatrix} \sigma_{11} & -2\sigma_{21} & +\sigma_{22}, & \sigma_{21} & -\sigma_{31} & -\sigma_{32} & +\sigma_{22} \\ \sigma_{21} & -\sigma_{31} & -\sigma_{32} & +\sigma_{22}, & \sigma_{33} & -2\sigma_{31} & +\sigma_{22} \end{pmatrix} \end{aligned}$$

Illustration

$$C_2 = \tilde{\Delta}_2 \Delta_3 \Sigma_\eta \Delta_3' \Delta_2' = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & -2\sigma_{21} & +\sigma_{33}, & \sigma_{21} & -\sigma_{31} & -\sigma_{32} & +\sigma_{33} \\ \sigma_{21} & -\sigma_{31} & -\sigma_{32} & +\sigma_{33} & \sigma_{22} & -2\sigma_{32} & \sigma_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Normalization Approach of Albreit, Lerman, and Manski (1978)

- **Note:** Need $J + 1$ restrictions on VCV matrix.
- Fix J parameters by setting last row and last column of Σ_η to 0
- Fix scale by constraining diagonal elements of Σ_η so that trace $\frac{\Sigma_\epsilon}{J}$ equals variance of a standard Weibull. (To compare estimates with MNL and independent probit)

How do we solve the forecasting problem?

- Suppose that we have 2 goods and add a 3rd

$$\begin{aligned}\Pr(1 \text{ chosen}) &= \Pr(u^1 - u^2 \geq 0) \\ &= \Pr(1((Z^1 - Z^2)\bar{\beta} \geq \omega^2 - \omega^1))\end{aligned}$$

- Define η_1, η_2, η_3 as random choice specific shocks independent of Z^1, Z^2 and Z^3 .
- $(\beta - \bar{\beta})$ arises from variability in slope coefficients.

- Define:

$$\omega^1 = Z^1 (\beta - \bar{\beta}) + \eta^1, \quad \omega^2 = Z^2 (\beta - \bar{\beta}) + \eta^2$$

$$= \int_{-\infty}^{\infty} \frac{(z^1 - z^2)^{\bar{\beta}}}{[\sigma_{11} + \sigma_{22} - 2\sigma_{12} + (z^2 - z^1)\Sigma_{\beta}(z^2 - z^1)']^{1/2}} \frac{1}{\sqrt{2\pi}} e^{-t/2} dt$$

- $Pr(1 \text{ chosen})$.
- Now add a 3rd good

$$u^3 = Z^3 \bar{\beta} + Z^3 (\beta - \bar{\beta}) + \eta^3.$$

- **Problem:** η^3 comes out of the blue: We don't know correlation of η^3 with other errors.
- Suppose that $\eta^3 = 0$ (i.e. only preference slope heterogeneity). Then

$$\Pr(1 \text{ chosen}) = \int_{-\infty}^a \int_{-\infty}^b B.V.N. \, dt_1 dt_2$$

$$\text{when } a = \frac{(Z^1 - Z^2) \bar{\beta}}{[\sigma_{11} + \sigma_{22} - 2\sigma_{12} + (Z^2 - Z^1) \Sigma_{\beta} (Z^2 - Z^1)']^{1/2}}$$

$$\text{and } b = \frac{(Z^1 - Z^3) \bar{\beta}}{[\sigma_{11} + (Z^3 - Z^1) \Sigma_{\beta} (Z^3 - Z^1)']^{1/2}}$$

- We could also solve the forecasting problem if we make an assumption like $\eta^2 = \eta^3$.
- We solve red-bus/blue-bus problem if $\eta^2 = \eta^1 = 0$ and $z^3 = z^2$.

$$\Pr(1 \text{ chosen}) = \Pr(u^1 - u^2 \geq 0, u^1 - u^3 \geq 0)$$

- but $u^1 - u^2 \geq 0 \wedge u^1 - u^3 \geq 0$ are the same event.
- \therefore adding a third choice does not change the choice of 1.

- Models tend to be difficult to estimate because of high dimensional integrals.
- Integrals need to be evaluated at each stage of estimating the likelihood.
- Simulation provides a means of estimating $P_{ij} = \Pr(i \text{ chooses } j)$

Computation and Estimation

[Link to Appendix](#)

References:

- Amemiya, Ch. 10
- Different types of sampling (previously discussed)
 - a random sampling
 - b censored sampling
 - c truncated sampling
 - d other non-random (exogenous stratified, choice-based)

Standard Tobit Model (Tobin, 1958) “Type I Tobit”

$$y_i^* = x_i\beta + u_i$$

- Observe, i.e.,

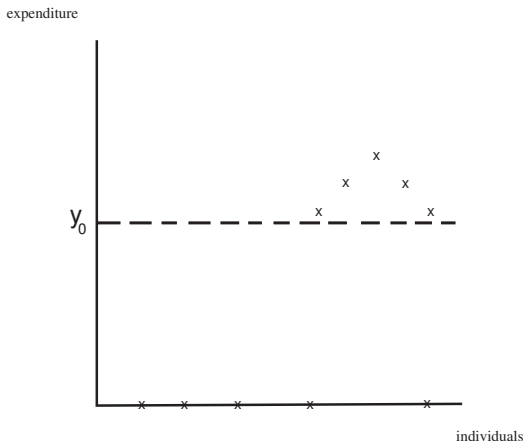
$$y_i = y_i^* \quad \text{if } y_i^* \geq y_0 \text{ or } y_i = 1 \text{ (} y_i^* \geq y_0 \text{) } y_i^*$$

$$y_i = 0 \quad \text{if } y_i^* < y_0$$

$$y_i = 1 \quad (y_i^* < y_0) y_i^*$$

- Tobin's example-expenditure on a durable good only observed if good is purchased

Figure 1



Note: Censored observations might have bought the good if price had been lower.

- Estimator. Assume $y_i^*/x_i \sim N(0, \sigma_u^2)$ $y_i^*/x_i \sim N(x_i\beta, \sigma_u^2)$

Density of Latent Variables

$$g(y^*) = f(y_i^* | y_i^* < y_0) \Pr(y_i^* < y_0) + f(y_i^* | y_i^* \geq y_0) \cdot \Pr(y_i^* \geq y_0)$$

$$\Pr(y_i^* < y_0) = \Pr(x_i\beta + u_i < y_0) = \Pr\left(\frac{u_i}{\sigma_u} < \frac{y_0 - x_i\beta}{\sigma_u}\right) = \Phi\left(\frac{y_0 - x_i\beta}{\sigma_u}\right)$$

$$f(y_i^* | y_i^* \geq y_0) = \frac{\frac{1}{\sigma_u} \phi\left(\frac{y_i^* - x_i\beta}{\sigma_u}\right)}{1 - \Phi\left(\frac{y_0 - x_i\beta}{\sigma_u}\right)}$$

- **Question: Why?**

$$\begin{aligned} & \Pr(y^* = y_i^* | y_0 \leq y^*) \\ &= \Pr(x\beta + u = y_i^* | y_0 \leq x\beta + u) \\ & \Pr\left(\frac{u}{\sigma_u} = \frac{y_i^* - x\beta}{\sigma_u} \mid \frac{u}{\sigma_u} \geq \frac{y_0 - x\beta}{\sigma_u}\right) \end{aligned}$$



- Note that likelihood can be written as:

$$\mathcal{L} = \underbrace{\Pi_0 \Phi \left(\frac{y_0 - x_i \beta}{\sigma_u} \right) \Pi_1 \left(1 - \Phi \left(\frac{y_0 - x_i \beta}{\sigma_u} \right) \right)}_{\text{This part you would set with just a simple probit}} \underbrace{\Pi_1 \frac{\frac{1}{\sigma_u} \phi \left(\frac{y_i^* - x_i \beta}{\sigma_u} \right)}{\left\{ 1 - \Phi \left(\frac{y_0 - x_i \beta}{\sigma_u} \right) \right\}}}_{\text{Additional information}}$$

- You could estimate β up to scale using only the information on whether $y_i \gtrless y_0$, but will get more efficient estimate using additional information.
 * if you know y_0 , you can estimate σ_u .

Truncated Version of Type I Tobit

Observe $y_i = y_i^*$ if $y_i^* > 0$
(observe nothing for truncated observations)
example: only observe wages for workers

$$\begin{aligned}\text{Likelihood: } \mathcal{L} &= \prod_1 \frac{\frac{1}{\sigma_u} \phi\left(\frac{y_i^* - x_i \beta}{\sigma_u}\right)}{\Phi\left(\frac{x_i \beta}{\sigma_u}\right)} \\ \Pr(y_i^* > 0) &= \Pr(x\beta + u > 0) \\ &= \Pr\left(\frac{u}{\sigma_u} > \frac{-x\beta}{\sigma_u}\right) \\ &= \Pr\left(u < \frac{x\beta}{\sigma_u}\right)\end{aligned}$$

Different Ways of Estimating Tobit

- a if censored, could obtain estimates of $\frac{\beta}{\sigma_u}$ by simple probit
- b run OLS on observations for which y_i^* is observed

$$E(y_i | x_i\beta + u_i \geq 0) = x_i\beta + \sigma_u E\left(\frac{u_i}{\sigma_u} \mid \frac{u_i}{\sigma_u} > \frac{-x_i\beta}{\sigma_u}\right) \quad (y_0 = 0)$$

- where $E(y_i | x_i\beta + u_i \geq 0)$ is the conditional mean for truncated normal r.v and

$$\sigma_u E\left(\frac{u_i}{\sigma_u} \mid \frac{u_i}{\sigma_u} > \frac{-x_i\beta}{\sigma_u}\right) \longrightarrow \lambda\left(\frac{x_i\beta}{\sigma_u}\right) = \frac{\phi\left(\frac{-x_i\beta}{\sigma_u}\right)}{\Phi\left(\frac{\pi_i\beta}{\sigma_u}\right)}$$

- $\lambda\left(\frac{x_i\beta}{\sigma_u}\right)$ known as “Mill’s ratio” ; bias due to censoring, can be viewed as an omitted variables problem



Heckman Two-Step procedure

- Step 1: estimate $\frac{\beta}{\sigma_u}$ by probit
- Step 2:

$$\text{form } \hat{\lambda} \left(\frac{x_i \hat{\beta}}{\sigma} \right)$$

regress

$$y_i = x_i \beta + \sigma \hat{\lambda} \left(\frac{x_i \beta}{\sigma} \right) + v + \varepsilon$$

$$v = \sigma \left\{ \lambda \left(\frac{x_i \beta}{\sigma} \right) - \hat{\lambda} \left(\frac{x_i \beta}{\sigma} \right) \right\}$$

$$\varepsilon = u_i - E(u_i | u_i > x_i \beta)$$

- Note: errors ($v + \varepsilon$) will be heteroskedastic;
- need to account for fact that λ is estimated (Durbin problem)
- Two ways of doing this:
 - a Delta method
 - b GMM (Newey, Economic Letters, 1984)
 - c Suppose you run OLS using all the data

$$\begin{aligned}
 E(y_i) &= \Pr(y_i^* \leq 0) \cdot 0 + \Pr(y_i^* > 0) \left[x_i\beta + \sigma_u E\left(\frac{u_i}{\sigma_u} \mid \frac{u_i}{\sigma_u} > \frac{-x_i\beta}{\sigma}\right) \right] \\
 &= \Phi\left(\frac{x_i\beta}{\sigma}\right) \left[x_i\beta + \sigma_u \lambda\left(\frac{x_i\beta}{\sigma}\right) \right]
 \end{aligned}$$

- Could estimate model by replacing Φ with $\hat{\phi}$ and λ with $\hat{\lambda}$.
- For both (b) and (c), errors are heteroskedastic, meaning that you could use weights to improve efficiency.
- Also need to adjust for estimated regressor.
 - (d) Estimate model by Tobit maximum likelihood directly.

Variations on Standard Tobit Model

$$\begin{aligned}y_{1i}^* &= x_{1i}\beta + u_{1i} \\y_{2i}^* &= x_{2i}\beta + u_{2i} \\y_{2i} &= y_{2i}^* \quad \text{if} \quad y_{1i}^* \geq 0 \\&= 0 \quad \text{else}\end{aligned}$$

- Example
 - y_{2i} student test scores
 - y_{1i}^* index representing parents propensity to enroll students in school
 - Test scores only observed for proportion enrolled

$$\begin{aligned}
\mathcal{L} &= \Pi_1 [\Pr(y_{1i}^* > 0) f(y_{2i}|y_{1i}^* > 0)] \Pi_0 [\Pr(y_{1i}^* \leq 0)] \\
f(y_{2i}^*|y_{1i}^* \geq 0) &= \frac{\int_0^\infty f(y_{1i}^*, y_{2i}^*) dy_{1i}^*}{\int_0^\infty f(y_{1i}^*) dy_{1i}^*} \\
&= \frac{f(y_{2i}) \int_0^\infty f(y_{1i}^*|y_{2i}^*) dy_{1i}^*}{\int_0^\infty f(y_{1i}^*) dy_{1i}^*} \\
&= \frac{1}{\sigma^2} \phi\left(\frac{y_{2i}^* - x_{2i}\beta_2}{\sigma^2}\right) \cdot \frac{\int_0^\infty f(y_{1i}^*|y_{2i}^*) dy_{1i}^*}{\Pr(y_{1i}^* > 0)}
\end{aligned}$$

$$\begin{aligned}
y_{1i} &\sim N(x_{1i}\beta_1, \sigma^2) \\
y_{2i} &\sim N(x_{2i}\beta_2,)
\end{aligned}$$

$$y_{1i}^* \mid y_{2i}^* \sim N \left(x_{1i}\beta_1 + \frac{\sigma_{12}}{\sigma_2^2} (y_{2i} - x_{2i}\beta_2), \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} \right)$$

$$E(y_{1i}^* \mid u_{2i} = y_{2i}^* - x_{2i}\beta) = x_{1i}\beta_1 + E(u_{1i} \mid u_{2i} = y_{2i}^* - x_{2i}\beta)$$

$$L = \Pi_0 \left[1 - \Phi \left(\frac{x_{1i}\beta}{\sigma_1} \right) \right] \Pi_1 \frac{1}{\sigma_2} \cdot \phi \left(\frac{y_{2i}^* - x_{2i}\beta_2}{\sigma_2} \right) \\ \cdot \left\{ 1 - \Phi \left(\frac{- \left\{ x_{1i}\beta_1 + \frac{\sigma_{12}}{\sigma_2^2} (y_{2i} - x_{2i}\beta_2) \right\}}{\sigma^x} \right) \right\}$$

Estimation by Two-Step Approach

- Using data on y_{2i} for which $y_{1i} > 0$

$$\begin{aligned} E(y_{2i} | y_{1i} > 0) &= x_{2i}\beta + E(u_{2i} | x_i\beta + u_{1i} > 0) \\ &= x_{2i}\beta + \sigma_2 E\left(\frac{u_{2i}}{\sigma_2} \mid \frac{u_{1i}}{\sigma_1} > \frac{-x_{1i}\beta_1}{\sigma_1}\right) \\ &= x_{2i}\beta + \sigma_2 \frac{\sigma_{12}}{\sigma_1 \sigma_2} E\left(\frac{u_{1i}}{\sigma_1} \mid \frac{u_{1i}}{\sigma_1} > \frac{-x_{1i}\beta_1}{\sigma_1}\right) \\ &= x_{2i}\beta_2 + \frac{\sigma_{12}}{\sigma_1} \lambda\left(\frac{-x_{1i}\beta_1}{\sigma_1}\right) \end{aligned}$$

Example: Female labor supply model

$$\begin{aligned} & \max \quad u(L, x) \\ \text{s.t.} \quad & x = wH + v \quad H = 1 - L \\ \text{where } & H : \text{ hours worked} \\ & v : \text{ asset income} \\ & w : \text{ given} \\ & P_x = 1 \\ & L : \text{ time spent at home for child care} \end{aligned}$$

$$\frac{\frac{\partial u}{\partial L}}{\frac{\partial u}{\partial x}} = w \quad \text{when } L < 1$$

$$\text{reservation wage} = MRS \big|_{H=0} = w_R$$



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Example: Female labor supply model

- We don't observe w_R directly.

Model

$$\begin{aligned}w^0 &= x\beta + u \quad (\text{wage person would earn if they worked}) \\w^R &= z\gamma + v \\w_i &= w_i^0 \quad \text{if } w_i^R < w_i^0 \\&= 0 \quad \text{else}\end{aligned}$$

- Fits within previous Tobit framework if we set

$$\begin{aligned}y_{1i}^* &= x\beta - z\gamma + u - v = w^0 - w^R \\y_{2i} &= w_i\end{aligned}$$

Incorporate choice of H

$$\begin{aligned}w^0 &= x_{2i}\beta_2 + u_{2i} \quad \text{given} \\MRS &= \frac{\frac{\partial u}{\partial L}}{\frac{\partial u}{\partial x}} = \gamma H_i + z_i'\alpha + v_i\end{aligned}$$

(Assume functional form for utility function that yields this)

$$\begin{aligned}
w^r(H_i = 0) &= z_i' \alpha + v_i \\
\text{work if } w^0 &= x_{2i} \beta_2 + u_{2i} > z_i \alpha + v_i \\
\text{if work, then } w_i^0 &= MRS \implies x_{2i} \beta_2 + u_{2i} = \alpha H_i + z_i \alpha + v_i \\
&\implies H_i = \frac{x_{2i} \beta_2 - z_i' \alpha + u_{2i} - v_i}{\gamma} \\
&= x_{1i} \beta_1 + u_{1i} \\
\text{where } x_{1i} \beta_1 &= (x_{2i} \beta_2 - z_i \alpha) \gamma^{-1} \\
u_{1i} &= u_{2i} - v_i
\end{aligned}$$

Type 3 Tobit Model

$$y_{1i}^* = x_{1i}\beta_1 + u_{1i} \longleftarrow \text{hours}$$

$$y_{2i}^* = x_{2i}\beta_1 + u_{2i} \longleftarrow \text{wage}$$

$$\begin{aligned} y_{1i} &= y_{1i}^* && \text{if } y_{1i}^* > 0 \\ &= 0 && \text{if } y_{1i}^* \leq 0 \end{aligned}$$

$$\begin{aligned} y_{2i} &= y_{2i}^* && \text{if } y_{1i}^* > 0 \\ &= 0 && \text{if } y_{1i}^* \leq 0 \end{aligned}$$

$$\begin{aligned} \text{Here } H_i &= H_i^* && \text{if } H_i^* > 0 \\ &= 0 && \text{if } H_i^* \leq 0 \end{aligned}$$

$$\begin{aligned} w_i &= w_i^0 && \text{if } H_i^* > 0 \\ &= 0 && \text{if } H_i^* \leq 0 \end{aligned}$$

- Note: Type IV Tobit simply adds

$$\begin{aligned} y_{3i} &= y_{3i}^* && \text{if } y_{1i}^* > 0 \\ &= 0 && \text{if } y_{1i}^* \leq 0 \end{aligned}$$

- Can estimate by
 - ① maximum likelihood
 - ② Two-step method

$$E(w_i^0 \mid H_i > 0) = \gamma H_i + z_i \alpha + E(v_i \mid H_i > 0)$$

Type V Tobit Model of Heckman (1978)

$$\begin{aligned}y_{1i}^* &= \gamma y_{2i} + x_{1i}\beta + \delta_2 w_i + u_{1i} \\ y_{2i} &= \gamma_2 y_{1i}^* + x_{2i}\beta_2 + \delta_2 w_i + u_{2i}\end{aligned}$$

- Analysis of an antidiscrimination law on average income of African Americans in i th state.
- Observe x_{1i} , x_{2i} , y_{2i} and w_i

$$\begin{aligned}w_i &= 1 \quad \text{if } y_{1i}^* > 0 \\ w_i &= 0 \quad \text{if } y_{1i}^* \leq 0\end{aligned}$$

- y_{2i} = average income of African Americans in the state
- y_{1i}^* = unobservable sentiment towards African Americans
- w_i = if law is in effect

- Adoption of Law is endogenous
- Require restriction $\gamma\delta_2 + \delta_1 = 0$ so that we can solve for y_{1j}^* as a function that does not depend on w_j .
- This class of models known as “dummy endogenous variable” models.

Coherency Problem (Suppose Restriction Does Not Bind?)

- See notes on “Dummy Endogenous Variables in simultaneous equations.”

References:

- Heckman (AER, 1990) “Varieties of Selection Bias”
- Heckman (1980), “Addendum to Sample Selection Bias as Specification Error”
- Heckman and Robb (1985, 1986)

$$y_1^* = x\beta + u$$

$$y_2^* = z\gamma + v$$

$$y_1 = y_1^* \text{ if } y_2^* > 0$$

$$\begin{aligned} E(y_1^* \mid \text{observed}) = & x\beta + E(u \mid x, z\gamma + u > 0) \\ & + [u - E(u \mid x, z\gamma + u > 0)] \\ & \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{-z\gamma} uf(u, v \mid x, z) dv du}{\int_{-\infty}^{\infty} \int_{-\infty}^{-z\gamma} f(uv \mid x, z) dv du} \end{aligned}$$

- Note:

$$\Pr(y_2^* > 0 \mid z) = \Pr(z\gamma + u > 0 \mid z) = P(Z) = 1 - F_v(-z\gamma)$$

$$\Rightarrow F_v(-z\gamma) = 1 - P(Z)$$

$$\Rightarrow -z\gamma = F_v^{-1}(1 - P(Z)) \quad \text{if } F_v$$

- Can replace $-z\gamma$ in integrals by $F_v^{-1}(1 - P(Z))$ if in addition $f(u, v | x, z) = f(u, v | z\gamma)$ (index sufficiency)
- Then

$$E(y_1^* | y_2 > 0) = x\beta + g(P(z)) + \varepsilon \text{ where } g(P(Z))$$

is bias or “control function.”

- Semiparametric selection model-Approximate bias function by Taylor series in $P(z\gamma)$, truncated power series.

Airum Models

Notes on McFadden Chapter/Integrating Discrete Continuous (see Heckman, 1974b, 1978, change notation)

- Notation:
 - I : enumeration of discrete alternatives
 - x : divisible goods
 - w : attributes of discrete choices
 - r : price of x
 - q_i : price of good i
 - y : income
 - $y: rx + q_i$
 - $\tilde{u}: \tilde{x} \times \omega \times I \rightarrow [0, 1]$ utility

- Define indirect utility function

$$v(y - q, r, w_i, i, \tilde{u}) = \max_x \tilde{u}(x, w_i, i \mid rx \leq y - q_i)$$

- Maximize out over continuous goods so we are left with discrete goods

Assumptions

- We assume v has usual properties of an indirect utility function
- Continuous, twice differentiable, homogeneous degree 0 in $(y, q - r)$, quasiconcave in r , $\frac{dv}{d(y-q)} > 0$
- Then get

$$x(y - q, r, w_i, i; \tilde{u}) = \frac{\frac{-\partial v}{\partial r}}{\frac{\partial v}{\partial y}}. \text{ (Roy's Identity)}$$

Assumptions

- For discrete alternative, we also get something like Roy's Identity

$$\delta_j = D(j \mid B, s; \tilde{u}) = \frac{\frac{-\partial v^*}{\partial q_j}}{\frac{\partial v^*}{\partial y}}$$

where

$$\begin{aligned} v^*(y - q_B, r, w_R, B; \tilde{u}) &= \max_{i \in B} v(y - q_i, r, w_i, i; \tilde{u}) \\ \delta_j &= \begin{cases} 1 & \text{if } j \in B \text{ } v_j \geq v_k \text{ } \forall k \\ 0 & \end{cases} \end{aligned}$$

- If IU assumptions satisfied, can write relationship between the probability of choosing j and the utility function as

$$P(j \mid B, s) = E_{u|s} D(j \mid B, s; \tilde{u})$$

- We seek sufficient conditions on preferences u such that we can integrate out over characteristics and come up with probabilities
- McFadden Shows that v takes AIRUM form

$$v(y - q, r, w_i, i; \tilde{u}) = \frac{y - q - \alpha(r, w_i, i, \tilde{u})}{\beta(r)}$$

where $y > q + \alpha$ α, β homogeneous of degree one wrt r

- Then

$$\begin{aligned}\bar{v} &= E_{u|s} \max_{i \in B} v(y - q_i, r, w_i, i; \tilde{u}) \\ &= \frac{1}{\beta(r)} \left[y + \max_{i \in B} (E_{u|s} (-q_i - \alpha(r, w, i; \tilde{u}))) \right]\end{aligned}$$

and

$$P(j) = E_{u|s} D(j \mid B, s) = \frac{-\frac{\partial \bar{v}}{\partial q_j}}{\frac{\partial \bar{v}}{\partial y}}$$

- \bar{v} is a utility function yielding the PCS
- Demand distribution can be analyzed as if it were generated by a population with common tastes, with each representative consumer having fractional consumption rates for the discrete alternative.

- Let

$$\tilde{G}(q_B, r, w_B, B, s) = E_{u|s} \max_{i \in B} [-q_i - \alpha(r, w_B, i; \tilde{u})] \quad (*)$$

“Social surplus function”

- Then

$$P(j | B, s) = \frac{-\partial \tilde{G}(q_B, r, w_B, B, s)}{\partial q_j} \quad (**)$$

under SS conditions given in Mcfadden's chapter

- I.e., choice probabilities given by the gradient of the SS function.

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Appendix

Variety of Simulation Methods

- Simulated method of moments
- Method of simulated scores
- Simulated maximum likelihood

References:

- Lerman and Manski (1981), *Structural Analysis* (online at McFadden's website)
- McFadden (1989), *Econometrica*
- Ruud (1982), *Journal of Econometrics*
- Hajivassilon and McFadden (1990)
- Hajivassilon and Ruud (Ch. 20), *Handbook of Econometrics*
- Stern (1992), *Econometrica*
- Stern (1997), Survey in JEL
- Bayesian MCMC (Chib *et al.* on reading list)

Early Simulation Method: “Crude Frequency Method”

Model

$$u_j = Z_j \beta + \eta_j \quad \text{with } \beta \text{ fixed, } \eta_j \sim N(0, \Omega), \quad J \text{ choices}$$

$$P_{ij} = \text{prob } i \text{ chooses } j$$

$$Y_{ij} = 1 \text{ if } i \text{ chooses } j, 0 \text{ else}$$

$$\$ = \prod_{i=1}^N \prod_{j=1}^J (P_{ij})^{Y_{ij}}$$

$$\log \$ = \sum_{i=1}^N \sum_{j=1}^J Y_{ij} \log P_{ij}$$

Simulation Algorithm

- i For given β, Ω generate Monte Carlo draws (R of them)

$$u_j^r, j = 1 \dots J, r = 1 \dots R$$

- ii Let $\tilde{P}_k = \frac{1}{R} \sum_{r=1}^R 1(u_k^r = \max\{u_1^r, \dots, u_J^r\})$ where \tilde{P}_k is a “frequency simulator” of $\Pr(k \text{ chosen}; \beta, \Omega)$

- iii Maximize $\sum_{i=1}^N \log \tilde{P}_{ik}$ over alternative values for β, Ω

- **Note:** Lerman and Manski found that this procedure performs poorly and requires a large number of draws, particularly when P is close to 0 or 1.

$$\text{var} \left(\frac{1}{R} \sum 1(\cdot) \right) = \frac{1}{R^2} \sum_{i=1}^R \text{var} 1(\cdot) \text{ with } \text{var} 1(\cdot) \text{ at true values}$$

- McFadden (1989) provided some key insights into how to improve the simulation method. He showed that simulation is viable even for a small number of draws provided that:
 - an unbiased simulator is used
 - functions to be simulated appear linearly in the conditions defining the estimator
 - same set of random draws is used to simulate the model at different parameter values
- **Note:** Condition (b) is violated for crude frequency method which had $\log \tilde{P}_{ik}$

Simulated Method of Moments (McFadden, Econometrica, 1989)

$$u_{ij} = Z_{ij}\beta = Z_{ij}\bar{\beta} + Z_{ij}\varepsilon_i \quad \beta = \bar{\beta} + \varepsilon_i$$

(earlier model with only preference heterogeneity)

$$P_{ij}(\gamma) = P(i \text{ chooses } j \mid w_i, \gamma) \quad (w_i \text{ are regressors})$$

- Define $Y_{ij} = 1$ if i chooses j , $Y_{ij} = 0$ otherwise.

$$\begin{aligned} \log \mathcal{L} &= \frac{1}{N_0} \sum_{i=1}^N \sum_{j=1}^J Y_{ij} \ln P_{ij}(\gamma) \quad N_0 = NJ \\ \frac{\partial \log \mathcal{L}}{\partial \gamma} &= \frac{1}{N_0} \sum_{i=1}^N \sum_{j=1}^J Y_{ij} \left[\frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}(\gamma)} \right] = 0 \end{aligned} \tag{3}$$



Simulated Method of Moments (McFadden, Econometrica, 1989)

$$\sqrt{N_0}(\hat{\gamma}_{MLE} - \gamma_0) \sim N(0, I_f^{-1})$$

$$\hat{I}_f = \frac{1}{N_0} \sum_{i=1}^N \left[\sum_{j=1}^J Y_{ij} \left[\frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}(\gamma)} \right] \right] \left[\sum_{j=1}^J Y_{ij} \left[\frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}(\gamma)} \right] \right]'$$

(outer product of score vector)

- Now use the fact that $\sum_{j=1}^J P_{ij}(\gamma) = 1$

$$\Rightarrow \sum_{j=1}^J \frac{\partial P_{ij}}{\partial \gamma} = 0 \quad \Rightarrow \sum_{j=1}^J \frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}} P_{ij} = 0$$

- Rewrite 3 as

$$\frac{1}{N_0} \sum_{i=1}^N \sum_{j=1}^J (Y_{ij} - P_{ij}) \frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}} = 0$$

- Note:** $E(Y_{ij}) = P_{ij}$.

- Letting $\varepsilon_{ij} = Y_{ij} - P_{ij}$, and $Z_{ij} = \frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}}$, we have

$$\frac{1}{N_0} \sum_{i=1}^N \sum_{j=1}^J \varepsilon_{ij} Z_{ij} = \frac{\frac{\partial P_{ij}}{\partial \gamma}}{P_{ij}}$$

- like a moment condition using Z_{ij} as the instrument but so far P_{ij} still a $J - 1$ dimensional integral.

- **Model**

$$u_{ij} = Z_{ij}\bar{\beta} + Z_{ij} \cdot \varepsilon_i \quad J \text{ choices, } K \text{ characteristics}$$

$$u_{ij} : 1 \times 1 \quad Z_{ij} : 1 \times K \quad \bar{\beta} : K \times 1$$

$$Z_{ij} : 1 \times K \quad \varepsilon_i : K \times 1$$

- Rewrite as

$$\tilde{u}_i = \tilde{Z}_i \bar{\beta} + \tilde{Z}_i \Gamma \tilde{\varepsilon}_i \quad \text{where } \Gamma \Gamma' = \Sigma_\varepsilon \text{ (Cholesky decomposition),}$$

$$\tilde{\varepsilon}_i \sim N(0, I_K), \varepsilon_i = \Gamma \tilde{\varepsilon}_i$$

$$\tilde{u}_i : J \times 1 \quad \tilde{Z}_i : J \times K \quad \bar{\beta} : K \times 1$$

$$\tilde{Z}_i : J \times K \quad \Gamma : K \times K \quad \tilde{\varepsilon}_i : K \times 1$$

- **Step (i).** Generate \tilde{e}_i for each i such that \tilde{e}_i are iid across persons and distributed $N(0, I_k)$. In total, generate

$$N(\text{sample size}) \cdot K(\text{vector length}) \cdot R$$

(number of Monte Carlo draws)

- **Step (ii).** Fix matrix Γ and obtain $\eta_{ij} = Z_{ij}\Gamma\tilde{e}_i$, where $Z_{ij} : 1 \times K$; $\Gamma : K \times K$; \tilde{e}_i is $K \times 1$.

- Form vector $\begin{pmatrix} Z_{i1}\Gamma\tilde{e}_i \\ Z_{i2}\Gamma\tilde{e}_i \\ \vdots \\ Z_{iJ}\Gamma\tilde{e}_i \end{pmatrix}$ for each person.

- **Step (iii).** Fix $\bar{\beta}$ and generate

$$\tilde{u}_{ij} = Z_{ij}\bar{\beta} + \eta_{ij} \quad \forall i.$$

- **Step (iv).** Find relative frequency that i th person chooses alternative j across Monte Carlo draws

$$\tilde{P}_{ij}(\gamma) = \frac{1}{R} \sum_{r=1}^R 1(\tilde{u}_{ij} > \tilde{u}_{im}; \quad \forall m \neq j)$$

- where $\tilde{P}_{ij}(\gamma)$ is the “simulator” for $P_{ij}(\gamma)$. Stack to get $\tilde{P}_i(\gamma)$.

- **Step (v).** To get $\tilde{P}_i(\gamma)$ for different values of γ , repeat steps (ii) through (iv) using the same r.v.'s \tilde{e}_i generated in step (i).
- **Step (vi).** Define

$$w_{ij} = \frac{\frac{\partial P_{ij}(\gamma)}{\partial \gamma}}{P_{ij}}$$

- and w_{ij} as corresponding stacked vector simulator for w_{ij} can be obtained by a numerical derivative

$$\frac{\partial P_{ij}(\gamma)}{\partial \gamma} = \frac{P_{ij}(\gamma + h_{lm}) - P_{ij}(\gamma - h_{lm})}{2h}$$

where $m = 1 \dots J$, $lm =$ vector with 1 in m th place.

- Apply Gauss-Newton Method, iterate to convergence

$$\begin{aligned}\gamma_1 &= \gamma_0 + \left\{ \frac{1}{N} \sum w_i(\gamma) \{y_i - \tilde{P}_i(\gamma_0)\} \{y_i - \tilde{P}_i(\gamma_0)\}^{-1} \right. \\ &\quad \cdot \left. \frac{1}{N} \sum_{i=1}^N w_i(\gamma_0) \{y_i - \tilde{P}_i(\gamma_0)\} \right\}\end{aligned}$$

Digression on Gauss-Newton

- Suppose problem is

$$S = \min_{\beta} \frac{1}{N} \sum_{i=1}^N [y_i - f_i(\beta)]^2 \quad (\text{nonlinear least squares}).$$

$$f_i(\beta) = f_i(\hat{\beta}_1) + \left. \frac{\partial f_i}{\partial \beta} \right|_{\beta_1} (\beta - \hat{\beta}_1) + \dots$$

by Taylor expansion around initial guess $\hat{\beta}_1$

$$= f_i(\hat{\beta}_1) + \left. \frac{\partial f}{\partial \beta} \right|_{\hat{\beta}_1} (\beta - \hat{\beta}_1) + \dots$$

(terms ignored)

- Substitution gives

$$\min_{\beta} \frac{1}{N} \sum_{i=1}^N [y_i - f_i(\hat{\beta}_1) - \left. \frac{\partial f}{\partial \beta} \right|_{\hat{\beta}_1} (\beta - \hat{\beta}_1)]^2$$

- Solve for $\hat{\beta}_2$ to get

$$\hat{\beta}_2 = \hat{\beta}_1 + \left[\sum_{i=1}^N \left. \frac{\partial f_i}{\partial \beta} \right|_{\hat{\beta}_1} \left. \frac{\partial f_i}{\partial \beta} \right|'_{\hat{\beta}_1} \right]^{-1} \left. \frac{\partial S}{\partial \beta} \right|_{\hat{\beta}_1}$$

- Repeat until convergence (problem if matrix is singular).

Disadvantages of Simulation Methods

- 1 \tilde{P}_{ij} is not smooth due to indicator function (causes difficulties in deriving asymptotic distribution; need to use methods developed by Pakes and Pollard (1989) for nondifferentiable functions). Smoothed SMOM methods developed by Stern, Hajivassiliou, and Ruud.
- 2 \tilde{P}_{ij} cannot be 0 (causes problems in denominator when close to 0)
- 3 Simulating small P_{ij} may require large number of draws

- Refinement: “Smoothed Simulated Method of Moments” replaces indicator with a smooth function (Stern (1992), *Econometrica*).

$$\int \text{ instead of}$$
$$\tilde{P}_{ij}(\gamma) = \frac{1}{R} \sum_{r=1}^R R(\Phi(\tilde{u}_{ij} - \tilde{u}_{im}))$$

How does simulation affect the asymptotic distribution?

- Without simulation get

$$\sqrt{\eta}(\hat{\gamma}_{mme} - \gamma_0) \sim N\left(0, \left[\text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum w_i (y_i - P_i)' w_i'\right]^{-1}\right)$$

- with simulation, the variance is slightly higher due to simulation error

$$\sqrt{n}(\hat{\gamma}_{msm} - \gamma_0) \sim N\left(0, \text{plim}_{N \rightarrow \infty} C^{-1} \left\{1 + \frac{1}{\eta}\right\}\right)$$

How does simulation affect the asymptotic distribution?

- where

$$C = -\frac{1}{N} \sum_{i=1}^N w_i (y_i - P_i) (y_i - P_i)' w_i'$$

- as $N \rightarrow \infty$, $\sqrt{\eta} (\gamma_{msm} - \gamma) \sim N(0, C^{-1})$.
- **Note:** Method does not require that number of draws go to infinity.

Choice-Based Sampling (See Heckman in New Palgrave)

- **Reference:**

- Chs. 1-2 of Manski and McFadden volume
- Manski and Lerman (1978 *Econometrica*)
- Amemiya

- **Examples:**

- ① Suppose we gather data on transportation mode choice at the
 - train station
 - subway station
 - car checkpoints (toll booths etc.)

- We observe characteristics on populations conditioned on the choice that they made (this type of sampling commonly arises)
 - ② Evaluating effects of a social program; have data on participants and non-participants; usually participants oversampled relative to frequency in the population.
- Distinguish between exogenous stratification and endogenous stratification, the latter of which is choice-based. (But a special type of endogenous stratification)
- Oversampling in high population areas (as is commonly done to reduce sampling costs or to increase representation of some groups) could be exogenous stratification (depending on phenomenon being studied).

Notation:

- Let $P_i = P(i | Z)$ in a random sample P_i^* in a choice-based sample (CBS)
- Under CBS, sampling is assumed to be random within i partitions of the data

$$P(Z | i) = P^*(Z | i) \text{ but } P(Z) \neq P^*(Z)$$

- Suppose that we want to recover $P(i | Z)$ from choice-based data
- We observe

$P^*(i | Z)$ (assume Z are discrete conditioning cells)

$P^*(Z)$

$P^*(i)$

By Bayes' Rule

$$P(A, B) = \frac{P(A, B)}{P(B)} = \frac{P(B | A) \cdot P(A)}{P(B)}$$

$$P^*(i | Z) = \frac{P^*(Z | i) \cdot P^*(i)}{P^*(Z)}$$

$$P(i | Z) = \frac{P(Z | i) \cdot P(i)}{P(Z)}$$

$$P(i | Z) = \frac{\left[\frac{P^*(i|Z) \cdot P^*(Z)}{P^*(i)} \right] P(i)}{P(Z)}$$

$$P(Z) = \sum_j P(Z | j) P(j)$$

$$P(Z | j) = P^*(Z | j)$$

$$P^*(Z | j) = \frac{P^*(j | Z) \cdot P^*(Z)}{P^*(j)}$$

$$P(i | Z) = \frac{\frac{P^*(i|Z)P^*(Z)}{P^*(i)} P(i)}{\sum_j \frac{P^*(j|Z)P^*(Z)}{P^*(j)} P(j)} = \frac{P^*(i | Z) \frac{P(i)}{P^*(i)}}{\sum_j P^*(j | Z) \frac{P(j)}{P^*(j)}}$$



- To recover $P(i | Z)$ from choice-based sampled data, you need to know $P(j)$, $P^*(j) \forall j$. $P^*(j)$ can be estimated from sample, but $P(j)$ requires outside information. Need weights $\frac{P(i)}{P^*(i)}$.
- **Note:** Problem set asks you to consider how CBS biases the coefficients and intercept in a logit model. (Can show that bias only in the constant)

Application and Extension: Berry, Levinsohn, and Pakes (1995)

- Develop equilibrium model and estimation techniques for analyzing demand and supply in differentiated product markets
- Use to study automobile industry
- Goal is to estimate parameters of both the demand and cost functions incorporating own and cross price elasticities and elasticities with respect to product attributes (car horse power, MPG, air conditioning, size,...) using only aggregate product level data supplemented with data on consumer characteristic distributions (income distribution from CBS)
- Want to allow for flexible substitution patterns

Key assumptions

- i joint distribution of observed and unobserved product and consumer characteristics
- ii price taking for consumers, Nash eq assumptions on producers in oligopolistic, differentiated products market.

Notation

- ζ : individual characteristics

x (observed)

ξ (unobserved) : product characteristics

p (price)

$u_{ij} = u(\zeta_i, p_j, x_j, \theta)$: utility if person i chooses j
(Cobb-Douglas assumption here)

$j = 0, 1, \dots, J$

0 = not purchasing any

- Define

$$A_j = \{ \zeta : u_{ij}(\zeta, p_j, x_j, \xi_j ; \theta) \geq u(\zeta, p_r, x_r, \xi_r ; 0) \quad r = 0, \dots, J \},$$

- the set of ζ that induces choice of good j . This is defined over individual characteristics which may be observed or unobserved.

$$s_j(p, x, \xi; \theta) = \int_{\zeta \in A_j} f(\zeta) d\zeta; \text{ (s is vector of market shares)}$$

- Special functional form:

$$u_{ij} = u(\zeta_i, p_j, x_j, \xi_j; \theta) = x_j\beta - \alpha p_j + \xi_j + \epsilon_{ij} = \delta_j + \epsilon_{ij}$$

Market Share

- $\delta_j = x_j\beta - \alpha p_j + \xi_j =$ mean utility from good j
- $\bar{\xi}_j$ is mean across consumers of unobserved component of utility
- ϵ_{ij} are the only elements representing consumer characteristics
- **Special Case:**

$$\begin{aligned}\xi_j &= 0 \quad (\text{no unobserved characteristic}) \\ \epsilon_{ij} &\text{ iid over } i, j, \text{ independent of } x_j\end{aligned}$$

- Then share

$$s_j = \int_{-\infty}^{\infty} \Pi_{q \neq j} F(\delta_j - \delta_q + \epsilon) f(\epsilon) d\epsilon$$

- Unidimensional integral; has closed form solution under extreme value.

Why is assumption that utility is additively separable and iid in consumer and product characteristics highly restrictive?

- a Implies that all substitution effects depend only on the δ s (since there is a unique vector of market shares associated with each δ vector). Therefore, conditional on market shares, substitution patterns don't depend on characteristics of the product.
Example: if Mercedes and Yugo have some market share then they must have the same δ s and some cross derivative with respect to any 3rd car (BMW).

$$\frac{\partial s_i}{\partial p_k} = \int \Pi_{q \neq k} F(\delta_k - \delta_q + \epsilon) F'(\delta_k - \delta_q + \epsilon) \frac{\partial \delta_k}{\partial p_k} f(\epsilon) d\epsilon$$

(same if δ s same)

Why is assumption that utility is additively separable and iid in consumer and product characteristics highly restrictive?

- Two products with same market share have same own price derivatives (not good because you expect product markup to depend on more than market share)
- Also assumes that individuals value product characteristics in same way (no preference heterogeneity)

Alternative Model (Random Coefficients Versions)

$$u_{ij} = x_j \bar{\beta} - \alpha p_j + \xi_j + \sum_k \sigma_k x_{jk} \nu_{ik} + \epsilon_{ij}$$

$$\beta_k = \bar{\beta}_k + \sigma_k \nu_k$$

$$E(\nu_{ik}) = 1$$

- Could still assume ϵ_{ij} has iid extreme value.

Model Actually Used

- Impose alternative functional form assumption because they want to incorporate prior info on distribution of relevant consumer characteristics and on interactions between consumer and product characteristics.

$$u_{ij} = (y - p_j)^\alpha G(x_i, \xi_j, \nu_i) e^{\epsilon(i,j)}$$

- Assume G is log linear

$$\tilde{u}_{ij} = \log u_{ij} = \alpha \log (y_i - p_j) + x_j \bar{\beta} + \xi_j + \sum_k \sigma_k x_{jk} \nu_{ik} + \epsilon_{ij}$$

- No good utility:

$$= \alpha \log y_i + \xi_0 + \sigma_0 \nu_{io} + \epsilon_{io}$$

- **Note:** Prices are likely to be correlated with unobserved product attributes, ξ , which leads to an endogeneity problem. (ξ may represent things like style, prestige, reputation, etc.)

quantity demanded, $q_j = M_{s_j}(x, \xi, p; \theta)$ (share)

- ξ enters nonlinearly, so we need to use some transformation to be able to apply instrumental variables (Principle of Replacement Function).

- Multiproduct firms $1, \dots, F$. Each produces subset τ_F of J possible products. Cost of producing good assumed to be independent of output levels and log linear in vector of cost characteristics (W_j, ω_j) .

$$\ln mc_j = W_j \gamma + \omega_j \Rightarrow \Pi_f = \sum_{j \in \tau_F} (p_j - mc_j) ms_j(p, x, \xi; \theta)$$

Nash Assumption

- Firm chooses prices that maximize profit taking as given attributes of its products and the prices and attributes of its competitor's products. P_j satisfies

$$s_j(p, x, \xi; \theta) + \sum_{r \in \tau_F} (p_r - mc_r) \frac{\partial s_r(p, x, \xi; \theta)}{\partial p_j} = 0$$



- Define

$$\Delta_{jr} = \left\{ \begin{array}{c} \frac{-\partial s_r}{\partial p_j} \\ 0 \end{array} \right\} \text{ if } r \text{ and } j \text{ produced by same firm}$$

$$\Rightarrow s(p, x, \xi; \theta) - \Delta(p, x, \xi; \theta) [p - mc] = 0$$

$$\Rightarrow p = mc + \Delta(p, x, \xi; \theta)^{-1} s(p, x, \xi; \theta) \text{ (market)}$$

- Market term depends on parameters of demand system and on equilibrium price vector

$$p = mc + \Delta(p, x, \xi; \theta)^{-1} s(p, x, \xi; \theta)$$

- Mark-up depends only on the parameters of the demand system and equilibrium price vector.
- Since p is a function of w , $b(p, x, \xi; \theta)$ also a function of w (unobs cost determinants)
- Let

$$\begin{aligned} \ln mc_j &= W_j \gamma + \omega_j \\ &\Rightarrow p \exp \{W \gamma + \omega\} + b(p, x, \xi; \theta) \\ \ln (p - b(p, x, \xi, \theta)) &= W \gamma + \omega \quad (\text{pricing equation}) \end{aligned}$$

- Need instruments for both demand and pricing equations. *i.e.* need variables correlated with (x, ω) uncorrelated with ξ and ω .
Let

$$Z = (X, W) \quad (p, q \text{ not included in } Z)$$

- Assume

$$\begin{aligned} E(\xi_j | Z_j) &= E(\omega_j | Z) = 0 \\ E((\xi_j, \omega_j) | Z) &= \Omega(Z_j) \quad (\text{finite almost every } Z_j) \end{aligned}$$

- Note that demand for any product is a function of characteristics of all products, so don't have any exclusion restrictions.

- J vectors $(x_j, \omega_j, p_j, q_j)$
- n : number of households sampled
- s^n : vector of sampled market shares
- Assume that a true θ_0 population abides by models.
- Decision Rules
- s^n converges to s^0 (multinomial)
- $\sqrt{n}(s^n - s^0) = O_p(1)$

- Assume we could calculate

$$\{\varepsilon_j(\theta, s, p_0), \omega_j(\theta, s, p_0)\}_{j=1}^J$$

for alternative values of θ

- They show that any choice of
 - a observed vector of market shares, s
 - b distribution of consumer characteristics, P
 - c parameter of model
- implies a unique sequence of estimates for the two unobserved characteristics

$$\xi_j(\theta, s, P), \omega_j(\theta, s, P)$$

- Then any function of Z must be uncorrelated with the vector

$$\{\xi(\theta, s^0, P_0), \omega(\theta, s^0, P_0)\}_{j=1}^J \text{ when } \theta = \theta_0$$

- Can use GMM
- Note: Conditional moment restriction implies infinite number of unconditional restrictions

$$\min \frac{1}{J} \sum H_j(Z) \begin{pmatrix} \xi_j(\theta, s^0, P_0) \\ w_j(\theta, s^0, P_0) \end{pmatrix}$$

- Logit ϵ_{ij} is Weibull.

$$\begin{aligned}\delta_j &= x_j\beta - \alpha p_j + \varepsilon_j \\ u_{ij} &= x_j\beta - \alpha p_j + \varepsilon_j + \epsilon_{ij} \\ s_j(p, x, \varepsilon) &= \frac{e^{\delta_j}}{1 + \sum_{j=1}^J e^{\delta_j}} \\ \delta_j &= \ln s_j - \ln s_0, \quad j = 1, \dots, J \\ \varepsilon_j &= \ln s_j - \ln s_0 - x_j\beta + \alpha p_j\end{aligned}$$

- See paper for more details.

Generalized Method of Moments (GMM)

References:

- Hansen (1982), *Econometrica*
- Hansen and Singleton (1982,1988)
- Also known as “minimum distance estimators”
- Suppose that we have some data $\{x_t\}$ $t = 1 \dots T$ and we want to test hypotheses about $E(x_t = \mu)$.
- How do we proceed? By a CLT

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T (x_t - \mu) \sim N(O, V_0)$$

$$\begin{aligned} V_0 &= E((x_t - \mu)(x_t - \mu)') \quad \text{if } x_t \text{ iid} \\ &= \lim_{t \rightarrow \infty} E\left(\frac{1}{\sqrt{T}} \sum_t (x_t - \mu) \frac{1}{\sqrt{T}} (x_t - \mu)'\right) \quad (\text{general case}) \end{aligned}$$

- We can decompose $V_0 = QDQ'$ where
 $QQ' = I$, $Q^{-1} = Q'$, $D = \text{matrix of eigenvalues}$

$$\begin{aligned} V_0 &= QD^{1/2}D^{1/2}Q' \\ Q'V_0Q &= D^{1/2}D^{1/2} \\ D^{-1/2}Q'V_0QD^{-1/2} &= I \end{aligned}$$

- under rule H_0 ,

$$\begin{aligned} &\left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right]' V_0^{-1} \left[\frac{1}{\sqrt{T}} \sum (x_t - u_0) \right] \\ &= \left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right]' QD^{-1/2}D^{-1/2}Q' \\ &\left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right] \sim X^2(n) \end{aligned}$$

- where n is the number of moment conditions.
- How does test statistic behave under alternative? ($\mu \neq \mu_0$)
- should get large

- Write as

$$\left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right]' V_0^{-1} \left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right] \quad (4)$$

$$= \left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right]' V_0^{-1} \left[\frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right]$$

$$+ \frac{2}{\sqrt{T}} \sum_t (\mu - \mu_0)' V_0^{-1} \frac{1}{\sqrt{T}} \sum_t (x_t - \mu) + \dots$$

$$+ \frac{1}{\sqrt{T}} \sum_t (\mu - \mu_0)' V_0^{-1} \frac{1}{\sqrt{T}} \sum_t (\mu - \mu_0) \quad (5)$$

- last term * is $0(T)$.

$\lambda = T (\mu - \mu_0)' V_0^{-1} (\mu - \mu_0)$ is the noncentrality parameter

- Problems:
 - ① V_o is not known *a priori*.
- Estimate $V_T \longrightarrow V_0$
- In the setting, use sample covariance matrix.
- In general setting, approximate limit by finite T
 - ① μ not known
- Suppose we want to test $\mu = \varphi(\beta)$
 - φ specified
 - β unknown

- Can estimate by *min- x^2 estimation*.

$$\min_{\beta \in B} \left[\frac{1}{\sqrt{T}} \sum (x_t - \varphi(\beta)) \right]' V_T^{-1} \left[\frac{1}{\sqrt{T}} \sum (x_t - \varphi(\beta)) \right]$$

$$\sim x^2(n - k)$$

k = dimension of β

n = number of moments

- Note: Searching over k dimensions, you lose one degree of freedom (will show next).

- Find distribution theory for $\hat{\beta}$:
Q: This is a M -estimator, so how do you proceed?
- FOC

$$\sqrt{T} \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}_T}' V_T^{-1} \frac{1}{\sqrt{T}} \left[x_t - \varphi(\hat{\beta}_T) \right] = 0$$

- Taylor expand $\varphi(\hat{\beta}_T)$ around $\varphi(\hat{\beta}_0)$

$$\varphi(\hat{\beta}_T) = \varphi(\hat{\beta}_0) + \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}^*} (\hat{\beta}_T - \beta_0) \quad \beta^* \text{ between } \beta_0, \hat{\beta}_T$$

- get

$$\sqrt{T} \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}_T}' V_T^{-1} \frac{1}{\sqrt{T}} \sum_t \left[x_t - \varphi(\beta_0) - \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}^*} (\hat{\beta}_T - \beta_0) \right] = 0$$

- Rearrange to solve for $(\hat{\beta}_T - \beta_0)$

$$\begin{aligned}
 & + \left[\sqrt{T} \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T}' V_T^{-1} \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}^*} \right] (\hat{\beta}_T - \beta_0) \frac{T}{\sqrt{T}} \\
 = & \sqrt{T} \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T}' V_T^{-1} \frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T)
 \end{aligned}$$

$$\begin{aligned}
 \text{if } \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T}' & \rightarrow D_0 \text{ (Convergence of random function)} \\
 V_T & \rightarrow V_0 \\
 \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}^*} & \rightarrow D_0
 \end{aligned}$$

- Apply CLT to $\frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) \sim N(0, V_0)$
- Then $\sqrt{T}(\hat{\beta}_T - \beta_0) \sim N\left(0, (D_0' V_0^{-1} D_0)^{-1} (D_0' V_0^{-1} D_0)^{-1'}\right)$

- Why is the limiting distribution $\chi^2(n - k)$?
- Write

$$\frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) = \frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) + \frac{1}{\sqrt{T}} \sum_t (\mu_0 - \varphi(\hat{\beta}_T))$$

- Recall, we had

$$\varphi(\hat{\beta}_T) = \varphi(\beta_0) + \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}^*} \quad (6)$$

$$\left\{ \left(\left. \frac{\partial \varphi}{\partial \beta} \right|'_{\hat{\beta}_T} V_T^{-1} \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}^*} \right)^{-1} \cdot \left. \frac{\partial \varphi}{\partial \beta} \right|_{\hat{\beta}_T} \cdot V_T^{-1} \frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0) \right\}$$

$$\text{definition of } \sqrt{T}(\hat{\beta}_T - \beta_0) \text{ derived easier.} \quad (7)$$

- Note that the second term (6) is a linear combination of the first.

$$\Rightarrow \frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) = B \frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0)$$

- where $B = I - \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0} \left[\frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0}' V_0^{-1} \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0} \right]^{-1} \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0}' V_0^{-1} = B_0$

$$\text{Note that } \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0}' V_0^{-1} B_0 = 0$$

- This tells us that certain linear combinations of B will give a degenerate distribution (along k dimensions)
- This needs to be taken into account when testing.
- Recall that we had $V_0 = QDQ'$
 $QQ' = I \quad V_0^{-1} = QD^{-1/2}D^{-1/2}Q'$

$$D^{-1/2}Q' \frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) = D^{-1/2}Q'B \frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0)$$

- where

$$\begin{aligned} D^{-1/2}Q'B &= D^{-1/2}Q' - D^{-1/2}Q' \cdot \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T} \left[\frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T}' QD^{-1/2}D^{-1/2}Q' \frac{\partial \varphi}{\partial \beta} \Big|_{\hat{\beta}_T} \right]^{-1} \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0}' \\ &\quad QD^{-1/2}D^{-1/2}Q' \\ &= (I - A(A'A)^{-1}) D^{-1/2}Q' \text{ (idempotent matrix } M_A) \end{aligned}$$

- Thus

$$D^{-1/2} Q' \frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) = M_A \cdot D^{-1/2} \cdot Q' \cdot \frac{1}{\sqrt{T}} \sum_t (x_t - \mu_0)$$

- This matrix M_A accounts for the fact that we performed the minimization over β

How is distribution theory affected?

- Have a quadratic form in normal r.v's with idempotent matrix

$$\text{ex. } \hat{\varepsilon} \cdot \hat{\varepsilon} = \varepsilon' M_x \varepsilon \quad M_x = I - x(x'x)^{-1}x'$$

- Me facts

(i) Theorem

- Let $Y \sim N(\theta, \sigma^2 I_n)$ and let P be a symmetric matrix of rank γ
- Then $Q = \frac{(Y-B)'P(Y-B)}{\sigma^2} \sim \chi_r^2$ iff $p^2 = p$ (i.e. P idempotent)
- See Seber, p.37

- (ii) if $Q_i \sim \chi_{r_i}^2$ $i = 1, 2$ $r_1 > r_2$ and $Q = Q_1 - Q_2$ is independent of Q_2 , then $Q \sim \chi_{r_1-r_2}^2$

- Apply these results to

$$\frac{\hat{\varepsilon}'\hat{\varepsilon}}{\sigma^2} = \frac{\varepsilon' M_x \varepsilon}{\sigma^2} \sim \chi^2 (\text{rank } M_x)$$

- Rank of an idempotent matrix is equal to its trace and

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i \quad \lambda_i \text{ eigenvalues} \quad (8)$$

for idempotent matrix, eigenvalues are all 0 or 1.

- (note rank = #non-zero eigenvalues for idempotent eigenvalues all 0 or 1)

$$\text{rank} \left(I - x (x'x)^{-1} x' \right) = \text{rank} (I) - \text{rank} \left(x (x'x)^{-1} x' \right)$$

where $\text{rank} (I) = n$

$$\begin{aligned} \text{rank} \left(x (x'x)^{-1} x' \right) &= \text{trace} \left(x'x (x'x)^{-1} \right) \text{ since } \text{tr} (AB) = \text{tr} (BA) \\ &= \text{trace} (I_k) \end{aligned}$$

$$\text{rank} \left(I - x (x'x)^{-1} x' \right) = nk$$

- Thus, by same reasoning limiting distribution of

$$\left[\frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) \right] V_T^{-1} \left[\frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\hat{\beta}_T) \right] \\ \sim \chi^2(n - k) = \text{rank}(A)$$

- We preserve χ^2 but lose degrees of freedom in estimating β .
- In case where $n = k$ (just identified case)
- we can estimate β but have no degrees of freedom left to perform the test.
- *Would GMM provide a method for estimating β if we used a weighting matrix other than V_T^{-1} ?*
- Why not replace V_0^{-1} by w_0 ?

$$\min_{\beta \in B} \frac{1}{\sqrt{T}} \sum_t \left(x_t - \varphi \left(\hat{\beta}_T \right) \right)' w_0 \frac{1}{\sqrt{T}} \sum_t x_t - \varphi(\beta)$$

- Could choose $w_0 = I$ (avoid need to estimate weighting matrix)
- Result: Asymptotic covariance is altered and asymptotic distribution of criterion is different, but $\frac{1}{\sqrt{T}} \sum_t \left(x_t - \varphi \left(\hat{\beta}_T \right) \right)$ will still be normal.

- *What is the advantage of focusing on minimum x^2 estimation?*
- Choosing $w_0 = V_0^{-1}$ gives smallest covariance matrix. Get most efficient estimator for β and most powerful test of restrictions.
- *Show this:* Show $(D_0' w_0 D_0)^{-1} (D_0' w_0 V_0^{-1} w_0 D_0) (D_0' w_0 D_0^{-1})^{-1} - (D_0' V_0 D_0)^{-1}$ is P.S.D
- where $(D_0' w_0 D_0)^{-1} (D_0' w_0 V_0^{-1} w_0 D_0) (D_0' w_0 D_0^{-1})^{-1}$ is the covariance matrix for $\sqrt{T} (\hat{\beta}_T - \beta_0)$ when general weighting matrix is used.
- Equivalent to showing

$$(D_0 V_0^{-1} D_0) - (D_0' w_0 D_0) (D_0' w_0 V_0^{-1} w_0 D_0)^{-1} (D_0' w_0 D_0)$$

is P.S.D

Show that it can be written as a quadratic form

- Take any vector α

$$\begin{aligned}
& \alpha' \left[D' V_0^{-1} D_0 - (D'_0 W_0 D_0) (D'_0 W_0 V_0 W_0 D_0)^{-1} (D'_0 W_0 D_0) \right] \alpha \\
&= \alpha' D'_0 V_0^{-1/2} \left[I - V_0'^{1/2} W_0 D_0 \left(D'_0 W_0 V_0^{1/2} \cdot Y_0^{1/2} W_0 D_0 \right)^{-1} D'_0 W_0 V_0^{1/2} \right] Y_0^{-1/2} D_0 \alpha \\
&= \alpha' D'_0 V_0^{-1/2} \left[I - \tilde{V} \left(\tilde{V}' \tilde{V} \right)^{-1} \tilde{V}' \right] V^{-1/2} D_0 \alpha \geq 0 \quad (= 0 \text{ if } W_0 = V_0^{-1})
\end{aligned}$$

- Therefore $W_0 = V_0^{-1}$ is the optimal choice for the weighting matrix.

Many standard estimators can be interpreted as GMM estimators

- Some examples:

① OLS

$$y_t = x_t\beta + u_t \quad E(u_t x_t) = 0 \quad \Rightarrow \quad E((y_t - x_t\beta) x_t) = 0$$

$$\min_{\beta \in B} \left(\frac{1}{\sqrt{T}} \sum_t (y_t - x_t\beta) x_t \right)' V_T^{-1} \left(\frac{1}{\sqrt{T}} \sum_t (y_t - x_t\beta) x_t \right)$$

- where y_t is the estimator for $E(x_t u_t u_t' x_t')$ for iid case = $\sigma^2 E(x_t x_t')$ if homoskedastic.

(2) Instrumental Variables

$$y_t = x_t\beta + u_t$$

$$E(u_t x_t) \neq 0$$

$$E(u_t z_t) = 0$$

$$E(x_t z_t) \neq 0$$

$$\hat{\beta}_T = \arg \min_{\beta \in B} \left(\frac{1}{\sqrt{T}} \sum (y_t - x_t\beta) z_t \right)' V_T^{-1} \frac{1}{\sqrt{T}} \sum (y_t - x_t\beta)$$

where $V_T = \hat{E}(z_t u_t u_t' z_t')$ in iid case

- Estimator for

$$\lim_{T \rightarrow \infty} E \left(\frac{1}{\sqrt{T}} \sum_t z_t u_t \left(\frac{1}{\sqrt{T}} \sum_s z_s u_s \right)' \right) \quad (\text{time series case})$$

- Suppose

$$E(u_t u_t' | z_t) = \sigma^2 I$$

- then

$$W_0 = \sigma^2 E(z_t z_t')$$

- Can verify that 2SLS and GMM give same estimator

$$\hat{\beta}_{2SLS} = \left(x'z (z'z)^{-1} z'x \right)^{-1} \left(x'z (z'z)^{-1} z'y \right)$$

- Note: In first stage regress x on Z

$$\hat{x} = z (z'z)^{-1} z'x$$

$$\hat{y} = z (z'z)^{-1} z'y$$

$$\begin{aligned} \text{var} \left(\hat{\beta}_{2SLS} \right) &= \left[E \left(x_i z_i \right) E \left(z_i z_i' \right)^{-1} E \left(z_i x_i \right) \right]^{-1} E \left(x_i z_i \right) E \left(z_i z_i' \right)^{-1} E \left(z_i u_i u_i' z_i' \right) \\ &\quad \cdot E \left(z_i z_i' \right)^{-1} E \left(x_i z_i \right)' \left[E \left(x_i z_i \right) E \left(z_i z_i' \right)^{-1} E \left(z_i x_i \right) \right]^{-1} \end{aligned}$$

- Under GMM

$$\text{var} \left(\hat{\beta}_{\text{GMM}} \right) = \left(D_0 V_0^{-1} D_0 \right)^{-1} \quad (\text{when } W_0 = V_0^{-1})$$

$$D_0 = \frac{\partial \varphi}{\partial \beta} \Big|_{\beta_0} = \text{plim}_{\frac{1}{n}} \sum x_i z_i' = E(x_i z_i')$$

$$W_0 = (\sigma^2)^{-1} E(z_i z_i')^{-1}$$

In the presense of heterskedastocity, weighting matrix would be different (and 2SLS and GMM not the same)

$$W_0 = E(z'uu'z)^{-1} = E(z'E(uu' | z)z)^{-1} = E(z'vz)^{-1}$$

- with panel data could have

$$E(uu' | z) = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ & v_2 & \cdots & \\ & & \ddots & 0 \\ 0 & & & v_T \end{pmatrix} = y$$

- allow for correlation over time for given individual, but iid across individuals.

$$y_t = \varphi(x_t, \beta) + u_t \quad E(u_t \varphi(x_t; \beta)) = 0$$

$$\min_{\beta \in B} \left[\frac{1}{\sqrt{T}} \sum (y_t - \varphi(x_t; \beta)) \varphi(x_t; \beta) \right]$$
$$V_T^{-1} \left[\frac{1}{\sqrt{T}} \sum (y_t - \varphi(x_t; \beta)) \varphi(x_t; \beta) \right]$$

- General Method of Moments

$$\min_{\beta \in B} \sum_t f_t(\beta)' V_0^{-1} \left[\sum_t f_t(\beta) \right]$$

- where

$$\frac{1}{\sqrt{T}} \sum f_t(\beta_0) \rightarrow N(O, V_0)$$
$$\frac{1}{\sqrt{T}} \sum f_t \rightarrow Ef_t$$

- In general, f_t is a random function.

- Suppose we want to estimate β 5×1 and we have 6 potential instruments. Can we test validity of the instruments? What if we have 5 instruments? If we assume $E(\varepsilon_i | x_i) = 0$ instead of $E(\varepsilon_i x_i) = 0$ (i. e. conditional instead of unconditional) then have infinite number of moment conditions.

$$E(\varepsilon_i f(x_i)) = E(E(\varepsilon_i | x_i) f(x_i)) = 0 \quad \text{any } f(x_i)$$

- How to optimally choose which moment conditions to use is current area of research. How might you use GMM to check if a variable is normally distributed?

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