Theory of Income, Fall 2009
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Problem Set 1

1 Endowment economy

Consider an OLG economy as described in the class notes. In this exercise we focus on an endowment economy and we study the effects of privatizing social security in such a way that everyone remains indifferent. In particular, we want to study the evolution of savings and government debt.

An agent born at period t cares only about consumption at period t and $t+1$ $\left(c_t^t, c_{t+1}^t\right)$, where c_j^i refers to consumption of an agent born at date i (also called of generation i) in period j. Agents born at date 0 are already old in period 1 and only care about consumption in period 1, c_1^0. As in the class notes, there is a mass 1 of agents in each generation.

The endowment stream of agent $t \geq 1$ is given by $\left(e_t^t, e_{t+1}^t\right)$ and the endowment of the initial old is e_1^0. The budget constraints of an agent of generation $t \geq 1$ are

$$\begin{aligned} s_t + c_t^t &= e_t^t - \tau_t^t \\ c_{t+1}^t &= s_t\left(1+r_t\right) + e_{t+1}^t - \tau_{t+1}^t, \end{aligned} \tag{1}$$

where s_t refers to savings, r_t is the interest rate between periods t and $t+1$, τ_t^t are the taxes levied on agents of generation t when young and τ_{t+1}^t are taxes levied on the same generation when they are old. The budget constraint of the initial old is

$$c_1^0 = s_0\left(1+r_0\right) + e_1^0 - \tau_1^0; \quad s_0 \text{ given.} \tag{2}$$

Let g_t denote government expenditures and B_t be the stock of government assets at period t. Then the government budget constraint is, for $t \geq 1$

$$B_t + g_t = B_{t-1}\left(1+r_{t-1}\right) + \tau_t^t + \tau_t^{t-1}. \tag{3}$$

The market clearing conditions in the goods market is

$$g_t + c_t^t + c_t^{t-1} = e_t^t + e_t^{t-1}, \tag{4}$$

and the market clearing condition in asset markets is

$$B_t + s_t = 0. \tag{5}$$

We also have the initial condition $B_0 + s_0 = 0$.

Question 1) *Walras' Law.* Use equations (1), (2), (3), (4) and the initial condition $B_0 + s_0 = 0$ to deduce that market clearing in assets market holds.

Ans: Take the budget constraints of a young person, of an old person and of the government at period t:

$$s_t + c_t^t = e_t^t - \tau_t^t,$$

$$c_t^{t-1} = s_{t-1}(1 + r_{t-1}) + e_t^{t-1} - \tau_t^{t-1},$$

$$B_t + g_t = B_{t-1}(1 + r_{t-1}) + \tau_t^t + \tau_t^{t-1}.$$

Add them to obtain

$$s_t + B_t + g_t + c_t^t + c_t^{t-1} = (s_{t-1} + B_{t-1})(1 + r_{t-1}) + e_t^t + e_t^{t-1},$$

using the market clearing condition (4) the last equation becomes

$$s_t + B_t = (s_{t-1} + B_{t-1})(1 + r_{t-1}). \tag{6}$$

We will prove the assertion by induction: first we show that (5) it is true for period $t = 1$. Second we assume that it holds at period t, if we can show that (4) also holds at $t + 1$, then it holds at all $t \geq 1$. So at period 1 we have

$$s_1 + B_1 = \underbrace{(s_0 + B_0)}_{=0}(1 + r_0) = 0,$$

where we used the initial condition $s_0 + B_0 = 0$. Now suppose $s_t + B_t = 0$, then using (6) at $t + 1$ we have

$$s_{t+1} + B_{t+1} = \underbrace{(s_t + B_t)}_{=0}(1 + r_t) = 0,$$

and the proof is finished.

Now we will analyze the social security *pay-as-you-go* system. The idea is that each period young agents are taxed and the earnings are redistributed to the current old. In terms of our notation, we assume

$$\begin{aligned} \tau_t^t &= \theta \\ \tau_{t+1}^t &= -\theta. \end{aligned}$$

Furthermore, consider equilibria with a constant interest rate $r_t = \bar{r}$ and assume

$$\begin{aligned} g_t &= 0 \\ e_t^t &= \alpha \\ e_t^{t-1} &= 1-\alpha \\ s_0 &= B_0 = 0, \end{aligned}$$

hence we immediately see from (3) that $B_t = 0$ for all $t \geq 1$. Since this is an endowment economy, from the class notes we also know that $s_t = 0$ for $t \geq 1$.

Ricardian Equivalence and Privatization of Social Security

Consider the following environment with different taxes

$$\begin{aligned} \tilde{\tau}_1^0 &= \tau_1^0 \\ \tilde{\tau}_t^t &= \tau_t^t + \frac{\tau_{t+1}^t}{(1+\tilde{r}_t)} \\ \tilde{\tau}_{t+1}^t &= 0, \end{aligned}$$

for all $t \geq 1$. A $\sim$ above any variable will refer to the equilibrium with the new tax system. Guess that in the new equilibrium the interest rate does not change, that is,

$$\tilde{r}_t = r_t = \bar{r} \text{ for all } t.$$

Question 2) Interpret $\left\{\tilde{\tau}_t^t, \tilde{\tau}_{t+1}^t\right\}$.

Ans: First notice that

$$\tilde{\tau}_t^t + \frac{\tilde{\tau}_{t+1}^t}{1+\tilde{r}_t} = \tau_t^t + \frac{\tau_{t+1}^t}{(1+\tilde{r}_t)} = \tau_t^t + \frac{\tau_{t+1}^t}{(1+\bar{r})},$$

so each agent of generation $t \geq 1$ pays the same present value of taxes in both systems. The difference is that in the second environment, everything is paid when young and nothing when old. In particular, since $\tau_{t+1}^t = -\theta$ is a subsidy, in the new environment each young of generation $t \geq 1$ is paying less than before. However, they don't receive the subsidy when old. Finally, $\tilde{\tau}_1^0 = \tau_1^0 = -\theta$ means that the initial old still receives the subsidy in the new environment.

Question 3) Show that $\tilde{c}_t^t = c_t^t$, $\tilde{c}_{t+1}^t = c_{t+1}^t$ for $t \geq 1$ and $\tilde{c}_1^0 = c_1^0$ is optimal.

Ans: The intertemporal budget constraint of each generation $t \geq 1$ is, from (1),

$$\begin{aligned} \tilde{c}_t^t + \frac{\tilde{c}_{t+1}^t}{1+\bar{r}} &= \alpha + \frac{(1-\alpha)}{1+\bar{r}} - \left(\tilde{\tau}_t^t + \frac{\tilde{\tau}_{t+1}^t}{1+\bar{r}}\right) = \\ & \alpha + \frac{(1-\alpha)}{1+\bar{r}} - \left(\tau_t^t + \frac{\tau_{t+1}^t}{1+\bar{r}}\right), \end{aligned}$$

thus the budget set is the same in both environment. Therefore the optimal choices must be the same: $\tilde{c}_t^t = c_t^t$ and $\tilde{c}_{t+1}^t = c_{t+1}^t$ for $t \geq 1$. Finally, since $c_1^0 = \tau_1^0$ and $\tilde{\tau}_1^0 = \tau_1^0$ then $\tilde{c}_1^0 = \tilde{\tau}_1^0 = c_1^0$.

Question 4) Show that $\tilde{B}_t = \tilde{B} = -\frac{\theta}{1+\bar{r}}$ for all $t \geq 1$. [Hint: Solve the government budget constraint].

Ans: The government budget at period 1 is

$$\tilde{B}_1 = \tilde{B}_0 (1 + r_{t-1}) + \tilde{\tau}_t^t + \tilde{\tau}_t^{t-1},$$

using

$$\tilde{B}_0 = 0, \quad \tilde{\tau}_1^1 = \theta - \frac{\theta}{1+\bar{r}} \quad \text{and } \tilde{\tau}_1^0 = -\theta,$$

we have

$$\tilde{B}_1 = \theta - \frac{\theta}{1+\bar{r}} - \theta = -\frac{\theta}{1+\bar{r}}.$$

For $t = 2$ we have

$$\begin{aligned} \tilde{B}_2 &= \tilde{B}_1 (1+\bar{r}) + \tilde{\tau}_2^2 + \tilde{\tau}_2^1 \\ &= -\frac{\theta}{1+\bar{r}} (1+\bar{r}) + \theta - \frac{\theta}{1+\bar{r}} \\ &= -\frac{\theta}{1+\bar{r}}. \end{aligned}$$

Continuing by induction we find $\tilde{B}_t = -\theta / (1+\bar{r})$ for all $t \geq 1$.

Question 5) Show that $\tilde{s}_t = \tilde{s} = \frac{\theta}{1+\bar{r}}$ solves the agent's problem.

Ans: The budget constraints of a young at period t after and before the change in taxes are

$$\begin{aligned} \tilde{s}_t + \tilde{c}_t^t &= e_t^t - \tilde{\tau}_t^t \\ s_t + c_t^t &= e_t^t - \tau_t^t. \end{aligned}$$

Subtracting both equations we have

$$\tilde{s}_t - s_t + \tilde{c}_t^t - c_t^t = e_t^t - \tilde{\tau}_t^t - \left(e_t^t - \tau_t^t\right).$$

Using $s_t = 0$, $\tilde{\tau}_t^t = \theta - \theta/(1+\bar{r})$, $\tau_t^t = \theta$ and that it is optimal to set $\tilde{c}_t^t = c_t^t$ we conclude that optimal savings are

$$\begin{aligned}\tilde{s}_t &= -\left(\theta - \frac{\theta}{1+\bar{r}}\right) + \theta \\ &= \frac{\theta}{1+\bar{r}}.\end{aligned}$$

Question 6) Show that $\tilde{B}_t$ and $\tilde{s}_t$ satisfy the assets market clearing condition.

Ans:

$$\tilde{B}_t + \tilde{s}_t = -\frac{\theta}{1+\bar{r}} + \frac{\theta}{1+\bar{r}} = 0 \text{ for all } t \geq 1.$$

Question 7) Interpret what happens in the assets market at time $t = 1$ and at $t \geq 2$ (who saves?, why?, etc.).

Ans: The government eliminates social security at period 1. It has to pay a subsidy of θ to the initial old, but only receives

$$\tilde{\tau}_1^1 = \theta - \frac{\theta}{1+\bar{r}} = \frac{\bar{r}}{1+\bar{r}}\theta < \theta,$$

from the young at period 1. So the government must issue debt in the amount

$$\tilde{\tau}_1^0 - \tilde{\tau}_1^1 = -\theta + \frac{\bar{r}}{1+\bar{r}}\theta = \frac{\theta}{1+\bar{r}},$$

so its holding of assets is $\tilde{B}_1 = -\theta/(1+\bar{r})$. That debt has to be hold by someone in the economy. The only agent willing to do so is the young, thus his/her savings are $s_1 = \theta/(1+\bar{r})$.

Since period 2 on, government debt stays constant forever at the level

$$\tilde{B}_t = -\frac{\theta}{1+\bar{r}}.$$

Notice that in order to have a constant debt, taxes have to pay the interest on the debt

$$\tilde{B}_{t+1} = \tilde{B}_t(1+\bar{r}) + \tilde{\tau}_t^t,$$

so $\tilde{B}_{t+1} = \tilde{B}_t = \tilde{B}$ if and only if

$$\tilde{\tau}_t^t = -\tilde{B}\bar{r} = \frac{\bar{r}}{1+\bar{r}}\theta.$$

This is exactly what the government does.

2 Production economy

In this problem we consider a production economy with capital and labor. Agents live for two periods. Young agents inelastically supply one unit of labor, consume and save, while old agents are retired and consume out of their savings. We will use the same notation as in the last problem, for example, c_j^i denotes consumption of a person born at date i in period j. Aggregate consumption at period t is given by

$$C_t = c_t^t + c_t^{t-1}. \tag{7}$$

We use K_t and I_t to denote the stock of capital installed at the beginning of period t and aggregate investment during period t respectively. So the stock of capital evolves as

$$K_{t+1} = I_t + (1-\delta) K_t, \tag{8}$$

where $0 < \delta < 1$ is the depreciation rate.

The utility function of an agent of generation $t \geq 1$ is $u\left(c_t^t, c_{t+1}^t\right)$ while the utility function of generation 0 (i.e. the initial old) is simply c_1^0. The budget constraint of an agent born at $t \geq 1$ is

$$\begin{aligned} c_t^t + s_t &= w_t - \tau_t^t \\ c_{t+1}^t &= s_t (1+r_t) - \tau_{t+1}^t, \end{aligned} \tag{9}$$

where w_t is the wage rate at t. The initial old's budget constraint is

$$c_1^0 = s_0 (1+r_0) - \tau_1^0.$$

The government's budget constraint is

$$B_t + g_t = B_{t-1} (1+r_{t-1}) + \tau_t^t + \tau_t^{t-1}. \tag{10}$$

Finally, output is produced with a constant return to scale technology $F(K_t, L_t)$. Technological feasibility is given by

$$g_t + I_t + C_t = F(K_t, L_t). \tag{11}$$

Firms rent capital and labor from the households. Let v_t denote the rental rate of capital

Then the firm's problem in any period t is

$$\max_{K_t, L_t} F(K_t, L_t) - w_t L_t - v_t K_t.$$

Question 1) Show that the firm's problem implies

$$\begin{aligned} F_L(K_t, L_t) &= w_t \\ F_K(K_t, L_t) &= v_{t.} \end{aligned}$$

Ans: The more direct (though not completely right) way of solving this is: take the two first order conditions and we obtain the above two equations. Given the linearity implied by the constant returns to scale assumption, a more formal argument would be like this: Given L, $F(K, L)$ is strictly concave in K, thus (for the given L), the first order condition w.r.t. K has to be satisfied:

$$F_K(K_t, L_t) = v_t.$$

Now, $F(K, L)$ being homogeneous of degree one implies that the partial derivatives (F_K and F_L) are homogeneous of degree zero, hence the last condition is equivalent to

$$F_K\left(\frac{K_t}{L_t}, 1\right) = v_t.$$

Thus we can see the last condition as determining the optimal capital to labor ratio $k(v_t) = K_t/L_t$. Now it rests to pick the optimal level of labor. Note that the objective function can be written as

$$\begin{aligned} & \max_{L_t} F(K_t, L_t) - w_t L_t - v_t K_t \\ = \; & \max_{L_t} L_t \left\{ F\left(\frac{K_t}{L_t}, 1\right) - w_t - v_t \frac{K_t}{L_t} \right\} \\ = \; & \max_{L_t} L_t \left\{ F(k(v_t), 1) - w_t - v_t k(v_t) \right\}. \end{aligned}$$

where we used the optimal capital to labor ratio $k(v_t)$. Since the term in braces does not depend on L_t, the objective function is linear in it. If the term in braces is not exactly zero, optimality requires either $L_t = \infty$ or $L_t = 0$. None of the cases is consistent with an equilibrium. Thus it rests to show that the term in braces being zero is equivalent to $F_L(K_t, L_t) - w_t = 0$. Note that

$$F(K, L) = LF\left(\frac{K}{L}, 1\right).$$

Thus

$$\begin{aligned} F_L(K_t, L_t) &= F\left(\frac{K_t}{L_t}, 1\right) - L_t F_K\left(\frac{K_t}{L_t}, 1\right)\frac{K_t}{L_t^2} \\ &= F(k(v_t), 1) - F_K(k(v_t), 1)\, k(v_t) \\ &= F(k(v_t), 1) - v_t k(v_t), \end{aligned}$$

where we used that at the optimum $K_t/L_t = k(v_t)$ and $F_K(k(v_t), 1) = v_t$. Thus the term in braces above is exactly

$$\begin{aligned} \{F(k(v_t), 1) - w_t - v_t k(v_t)\} &= F_L(K_t, L_t) - w_t \\ &= 0. \end{aligned}$$

Question 2) Show, by an arbitrage argument, that in any equilibrium we must have $v_{t+1} = r_t + \delta$.

Ans: Consider the following arbitrage: borrow one unit of good at period t and invest it to construct capital and rent it to the firm the following period. The next period the agent has to repay the amount $(1 + r_t)$ and receives the amount $(1 - \delta) + v_{t+1}$. That is, since a fraction δ of the capital depreciates, the agent receives $(1 - \delta)$ units of capital plus the rental rate v_{t+1}. Since net cash-flows at t are zero, they have to be zero at $t + 1$ as well, that is

$$1 - \delta + v_{t+1} - (1 + r_t) = 0,$$

or

$$v_{t+1} = r_t + \delta.$$

The agent's problem is

$$\max_{c_t^t, c_{t+1}^t} u\left(c_t^t, c_{t+1}^t\right),$$

subject to

$$c_t^t + \frac{c_{t+1}^t}{1 + r_t} = w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1 + r_t}.$$

Question 3) Argue that the solution to the agent's problem implies that savings can be written as a function of the interest rate r_t, current income $w_t - \tau_t$ and total wealth $w_t - \tau_t^t - \tau_{t+1}^t / (1 + r_t)$, as

$$s_t = s\left(r_t, w_t - \tau_t^t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1 + r_t}\right).$$

[Remark: If we formulate the problem in a different way, it is possible to write the savings policy function as $s\left(r_t, w_t - \tau_t^t, \tau_{t+1}^t\right)$, but for future purposes it is better to have it as above.]

Ans: From the agent's problem, we can obtain the demands for current and future consumption

$$c_t^t\left(r_t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right)$$
$$c_{t+1}^t\left(r_t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right).$$

Savings are defined as

$$s_t = w_t - \tau_t - c_t^t,$$

so that at the optimum

$$\begin{aligned} s_t &= (w_t - \tau_t) - c_t^t\left(r_t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right) \\ &= s\left(r_t, w_t - \tau_t^t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right). \end{aligned}$$

worth killing yourself
if you forget

Definition - *Equilibrium* : Given $B_0 + s_0 = K_1$, the quantities $\left\{K_t, c_t^t, c_t^{t-1}, I_t, B_t, g_t\right\}_{t=1}^{\infty}$ and the prices $\{w_t, r_t, v_t\}_{t=1}^{\infty}$ are an equilibrium if

i) Agents maximize utility.

ii) Firms maximize profits.

iii) Market clearing in goods is satisfied.

iv) The government budget constraint holds.

v) $L_t = 1$.

v) No arbitrage holds.

Question 4) - *Walras' law* - Show that in equilibrium

$$B_t + s_t = K_{t+1} \text{ for all } t.$$

[Hint: use the household's budget constraint, the government budget constraint, market clearing in goods market, constant returns to scale of $F(K, L)$, $L_t = 1$ for all t and $B_0 + s_0 = K_1$. Show that $B_1 + s_1 = K_2$ and then use induction in t].

Ans: Consider the budget constraints of a young person, of an old person and of the government at any period t:

$$\begin{aligned} s_t + c_t^t &= w_t - \tau_t^t \\ c_t^{t-1} &= s_{t-1}(1 + r_{t-1}) - \tau_t^{t-1} \\ B_t + g_t &= B_{t-1}(1 + r_{t-1}) + \tau_t^t + \tau_t^{t-1}. \end{aligned}$$

Add them to obtain

$$s_t + B_t + c_t^t + c_t^{t-1} + g_t = (s_{t-1} + B_{t-1})(1 + r_{t-1}) + w_t.$$

Notice that $c_t^t + c_t^{t-1} = C_t$. To save on notation, define $z_t = s_t + B_t$. So we have

$$z_t + C_t + g_t = z_{t-1}(1 + r_{t-1}) + w_t.$$

Now, from market clearing in final goods plus the capital evolution equation we have

$$C_t + g_t + K_{t+1} - (1 - \delta) K_t = F(K_t, 1).$$

Solving for $C_t + g_t$ and replacing the result into the previous equation we obtain

$$z_t + F(K_t, 1) - K_{t+1} + (1 - \delta) K_t = z_{t-1}(1 + r_{t-1}) + w_t.$$

Using that F is constant returns to scale we have $F(K_t, 1) = F_K(K_t, 1) K_t + F_L(K_t, 1)$ and from the firm's problem we have

$$\begin{aligned} F_K(K_t, 1) &= r_{t-1} + \delta \\ F_L(K_t, 1) &= w_t. \end{aligned}$$

Introducing this result into the last equation, we obtain

$$z_t + K_t(r_{t-1} + \delta) + w_t - K_{t+1} + (1 - \delta) K_t = z_{t-1}(1 + r_{t-1}) + w_t,$$

or

$$z_t + K_t r_{t-1} - K_{t+1} + K_t = z_{t-1}(1 + r_{t-1}). \tag{12}$$

Now, if we can show that $z_t = K_{t+1}$ for all $t \geq 1$ we are done. As in the last question, we will use induction in t. Evaluate (12) at $t = 1$ and use the initial condition to obtain

$$\begin{aligned} z_1 + K_1 r_0 - K_2 + K_1 &= z_0(1 + r_0) \\ z_1 + K_1 r_0 - K_2 + K_1 &= K_1(1 + r_0), \end{aligned}$$

thus $z_1 = K_2$. Now assume $z_{t-1} = K_t$. Introducing $z_{t-1} = K_t$ into (12) we immediately see that $z_t = K_{t+1}$, so that

$$B_t + s_t = K_{t+1} \text{ for all } t \geq 1.$$

Assume that $\tau_t^t = \tau_{t+1}^t = 0$, $g_t = 0$ for all t and $B_0 = 0$. Note that from the government budget constraint this implies $B_t = 0$ for all t.

Question 5) Show that

$$\begin{aligned} K_{t+1} &= s\left(F_K\left(K_{t+1}, 1\right) - \delta, F_L\left(K_t, 1\right), F_L\left(K_t, 1\right)\right) \\ K_1 &> 0, \end{aligned} \tag{13}$$

describes an equilibrium. [Hint: use the solution to question 3].

Ans: Using the result from questions 3 and 4, any equilibrium will satisfy

$$K_{t+1} = s_t = s\left(r_t, w_t, w_t\right).$$

From the firm's problem we have $r_t = F_K\left(K_{t+1}, 1\right) - \delta$ and $w_t = F_L\left(K_t, 1\right)$, thus

$$K_{t+1} = s\left(F_K\left(K_{t+1}, 1\right) - \delta, F_L\left(K_t, 1\right), F_L\left(K_t, 1\right)\right),$$

and K_1 is given by the initial condition $s_0 = K_1$. Given an initial K_1, (13) describes the equilibrium evolution of the stock of capital.

For the rest of the problem assume $g_t = 0$ for all t, $B_0 = 0$, $r_t > 0$ for all t (i.e. the equilibrium is such that interest rates are positive) and

$$\tau_t^t = \tau; \quad \tau_{t+1}^t = -\tau \quad \text{(i.e. social security)}.$$

Question 6) Show that if c_{t+1}^t is a normal good (actually, not an inferior good), then savings $s(\cdot)$ are strictly decreasing in τ. [Hint: It may be easier to solve this question without "taking derivatives". If $\tau' > \tau$ you can compute the change in wealth. Then use that in the "worst" case, the income effect is bounded by $-\left(\tau' - \tau\right)\frac{r_t}{1+r_t}$].

Ans: If $\tau_t^t = -\tau_{t+1}^t = \tau$, wealth of an agent born at t is

$$w_t - \tau + \frac{\tau}{1 + r_t} = w_t - \frac{r_t}{1 + r_t}\tau.$$

Savings as a function of τ are

$$s(\tau) = w_t - \tau - c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right).$$

We want to show that if $\tau' > \tau$, then $s(\tau') < s(\tau)$. This happens if and only if

$$w_t - \tau' - c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau'\right) < w_t - \tau - c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right),$$

or equivalently,

$$\tau' + c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau'\right) > \tau + c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right).$$

We will show that if c_{t+1}^t is not inferior, then the last condition must hold.

The next result is crucial: Take the budget constraints under taxes τ' and τ,

$$\begin{aligned} c_t^t(\tau') + \frac{c_{t+1}^t(\tau')}{1+r_t} &= w_t - \tau'\frac{r_t}{1+r_t} \\ c_t^t(\tau) + \frac{c_{t+1}^t(\tau)}{1+r_t} &= w_t - \tau\frac{r_t}{1+r_t} \end{aligned}$$

where we define $c_t^t(\tau) \equiv c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right)$. Subtracting the two equations we obtain

$$\left[c_t^t(\tau') - c_t^t(\tau)\right] + \frac{\left[c_{t+1}^t(\tau') - c_{t+1}^t(\tau)\right]}{1+r_t} = -\left(\tau' - \tau\right)\frac{r_t}{1+r_t}.$$

The term in the right hand side is the change in wealth (negative since $\tau' > \tau$). Now, since wealth with taxes τ' is lower than wealth with taxes τ, and c_{t+1}^t is not an inferior good, then $\left[c_{t+1}^t(\tau') - c_{t+1}^t(\tau)\right] \leq 0$, which implies

$$c_t^t(\tau') - c_t^t(\tau) \geq -\left(\tau' - \tau\right)\frac{r_t}{1+r_t}.$$

Intuitively, since consumption when old is not inferior, it remains constant or decreases, which implies that consumption when young cannot decrease more than the change in wealth.

The last inequality implies

$$\begin{aligned} c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau'\right) &\geq c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right) - \left(\tau' - \tau\right)\frac{r_t}{1+r_t} \\ &> c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right) - \left(\tau' - \tau\right), \end{aligned}$$

where the second (strict) inequality follows because $r_t/(1+r_t) < 1$ when $r_t > 0$. Rearranging we obtain

$$c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau'\right) + \tau' > c_t^t\left(r_t, w_t - \frac{r_t}{1+r_t}\tau\right) + \tau,$$

concluding that if $\tau' > \tau$, $s(\tau') < s(\tau)$.

Question 7) Show that if $\left.\frac{ds}{dr_t}\right|_{\tau=0} > 0$ (i.e. the substitution effect is strong enough), then there exist a function $s^*(\cdot)$ such that

$$K_{t+1} = s^*(K_t, \tau).$$

Ans: We know that in equilibrium

$$K_{t+1} = s\left(r_t, w_t - \tau_t^t, w_t - \tau_t^t + \frac{\tau_{t+1}^t}{1+r_t}\right).$$

Using $r_t = F_K(K_{t+1}, 1) - \delta$ and $w_t = F_L(K_t, 1)$ and $\tau_t^t = -\tau_{t+1}^t = \tau$ we have

$$K_{t+1} = s\left(F_K(K_{t+1}, 1) - \delta, F_L(K_t, 1) - \tau, F_L(K_t, 1) - \tau + \frac{\tau}{1 + F_K(K_{t+1}, 1) - \delta}\right).$$

Notice that this is an implicit difference equation of K_{t+1} as a function of K_t. We will use the implicit function theorem to argue that in any neighborhood of K_t and for small τ, there exists a function $K_{t+1} = s^*(K_t, \tau)$. This *implicit* function (if it exists), is defined as

$$s^*(K_t, \tau) = s\left(F_K(s^*(K_t, \tau), 1) - \delta, F_L(K_t, 1) - \tau, F_L(K_t, 1) - \tau + \frac{\tau}{1 + F_K(s^*(K_t, \tau), 1) - \delta}\right), \tag{14}$$

or equivalently

$$s^*(K_t, \tau) = s\left(r(s^*(K_t, \tau)), w(K_t) - \tau, w(K_t) - \tau + \frac{\tau}{1 + r(s^*(K_t, \tau))}\right), \tag{15}$$

where $r(s^*(K_t, \tau)) = F_K(s^*(K_t, \tau), 1) - \delta$ and $w(K_t) = F_L(K_t, 1)$.

If we are able to show that the derivative of s^* with respect to K_t is well defined (there is nothing divided by zero), by the implicit function theorem we know that such a function exists. Denote the derivatives of the function $s(\cdot)$ with respect to the i^{th} argument by s_i for

$i = 1, 2, 3$ (keeping the arguments implicit). Differentiating with respect to K_t we have

$$\frac{ds^*(K_t, 1)}{dK} = s_1 F_{KK}(s^*, 1) \frac{ds^*(K_t, 1)}{dK} + s_2 F_{LK}(s^*, 1) + s_3 \left[F_{LK}(s^*, 1) - \frac{F_{KK}(s^*, 1) \frac{ds^*(K_t,1)}{dK}}{(1 + F_K(s^*, 1) - \delta)^2} \right].$$

Notice that

$$\frac{ds}{dr_t} = s_1 - \frac{s_3 \tau}{(1+r)^2},$$

thus by assumption we have

$$0 < \frac{ds}{dr_t}|_{\tau=0} = s_1.$$

Then evaluating at $\tau = 0$ we obtain

$$\frac{ds^*}{dK_t}|_{\tau=0} = s_1 F_{KK}(s^*, 1) \frac{ds^*}{dK_t}|_{\tau=0} + s_2 F_{LK}(K_t, 1) + s_3 F_{LK}(K_t, 1),$$

or

$$\frac{ds^*}{dK_t}|_{\tau=0} [1 - s_1 F_{KK}(s^*, 1)] = (s_2 + s_3) F_{LK}(K_t, 1).$$

Since $s_1 > 0$ and $F_{KK}(s^*, 1) < 0$, the term in square brackets is strictly positive. Therefore by the implicitly function theorem there exist a function $s^*(K_t)$ for τ close to zero that satisfies

$$K_{t+1} = s^*(K_t, \tau).$$

Question 8) Show that

$$\left. \frac{ds^*}{d\tau} \right|_{\tau=0} < 0.$$

[Hint: Use the result from question 6: that if r_t is kept constant, then $\frac{ds}{d\tau} < 0$. Then you only have to show that the indirect effect of a change in τ through its effect on the equilibrium interest rate has the correct sign].

Ans: We will use (15) to show that the derivative of the implicit function $s^*(K_t, \tau)$ with respect to τ for small τ is negative. Notice that the effect of τ on s^* comes from two sources: the direct effect of the change in τ on $s(\cdot)$ and the indirect effect of the change in τ on the equilibrium interest rate. Using that insight we have

$$\frac{ds^*}{d\tau} = \frac{ds}{d\tau} + \frac{ds}{dr} \frac{dr}{ds^*} \frac{ds^*}{d\tau}.$$

But

$$\frac{ds}{dr}\frac{dr}{ds^*}\frac{ds^*}{d\tau} = \left[s_1 - \frac{s_3\tau}{(1+r)^2}\right] F_{KK}(s^*,1)\frac{ds^*}{d\tau}.$$

Then

$$\frac{ds^*}{d\tau} = \frac{ds}{d\tau} + \left[s_1 - \frac{s_3\tau}{(1+r)^2}\right] F_{KK}(s^*,1)\frac{ds^*}{d\tau},$$

or

$$\frac{ds^*}{d\tau}\left[1 - \left[s_1 - \frac{s_3\tau}{(1+r)^2}\right] F_{KK}(s^*,1)\right] = \frac{ds}{d\tau}.$$

Evaluating at $\tau = 0$ we obtain

$$\left.\frac{ds^*}{d\tau}\right|_{\tau=0}\left[1 - s_1 F_{KK}(s^*,1)\right] = \frac{ds}{d\tau}.$$

Finally, since by assumption $s_1 > 0$ (from the assumption $\left.\frac{ds}{dr}\right|_{\tau=0} > 0$) and $F_{KK} < 0$ we conclude that the term in square brackets is strictly positive. And as we showed in question 6, $\frac{ds}{d\tau} < 0$ concluding that

$$\left.\frac{ds^*}{d\tau}\right|_{\tau=0} < 0.$$

The last question showed that for each level of capital, next period stock of capital will be higher in an economy without social security. This, however, is not sufficient to argue that the steady state level of capital is lower in an economy with social security.

Consider an economy without social security in a steady state $K^* > 0$. We will analyze how does K^* changes if we introduce social security (for small τ).

Question 9) Assume that in a neighborhood K^* (i.e. close to K^*) we are in the "well behaved case":

$$0 < \frac{ds^*}{dK_t} < 1.$$

Show that an increase in τ reduces the steady state level of capital. [Hint: a graphic is sufficient].

Ans: As shown above, an increase in τ around $\tau = 0$ reduces $s^*(K,\tau)$ for each K. So in a graphic with K in the $x-$axis and s^* in the $y-$axis an increase in τ shifts the curve s^* down. If $0 < \frac{ds^*}{dK_t} < 1$ we immediately see that the stock of capital is lower in the economy with social security (see Figure 1-A).

Question 10) Argue that, if close to K^* the dynamic system has $\frac{ds^*}{dK_t} > 1$, then the steady state capital level is higher in the economy with social security. Is that steady state stable? [Hint: suppose you start at that steady state, and imagine that the stock of capital is changed a little bit. Does the dynamic system drives you back to the original steady state?]

Ans: An increase in τ shifts the curve s^* down for each level of capital. But as $\frac{ds^*}{dK_t} > 1$, we can see in Figure 1-B that the steady state capital stock is higher if τ is higher. The initial steady state K^* is unstable since if we increase K a little bit, the stock of capital continues increasing, while if we decrease K a little bit, the stock of capital continues decreasing. In both cases, small movements away from K drives the system to some other place (i.e. it doesn't come back to K^*), so the steady state is unstable (Figure 2).

Figure 1

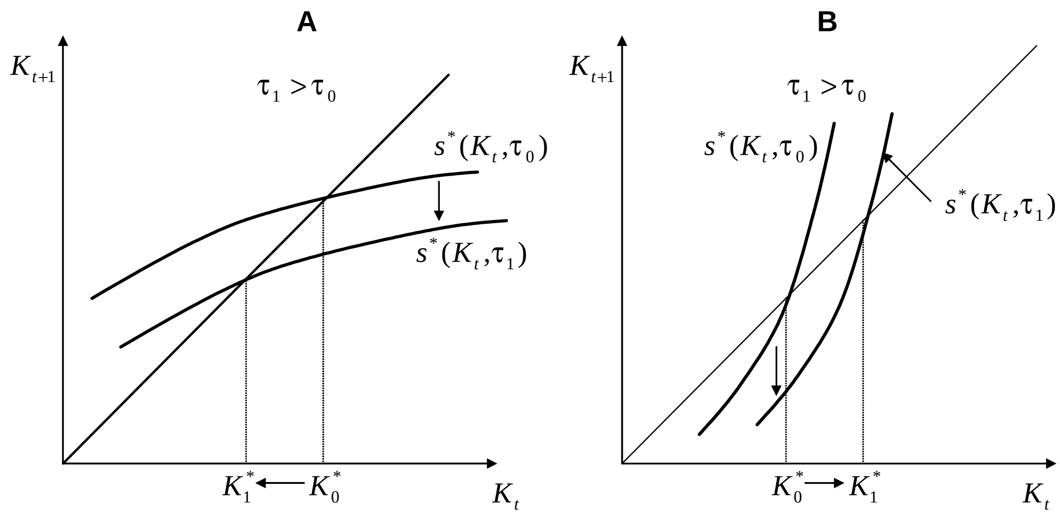
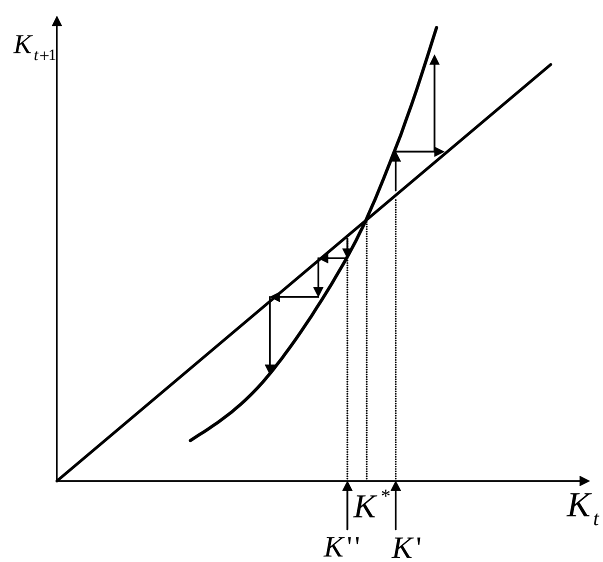


Figure 2



3 Production Economy: An Example

In the class notes we saw that in a pure endowment economy without population growth, if the equilibrium interest rate is positive, the introduction of social security makes the initial old better off and all the subsequent generations worse off. In this problem we will show that this need not be the case in a production economy with capital. In particular, we will construct an example where the generation born at period 1 is strictly better off with the introduction of social security.

Consider the following modification to the last problem: The production function is Cobb-Douglas $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$ and consumers only care about consumption when old, so that $u\left(c_t^t, c_{t+1}^t\right) = c_{t+1}^t$. Also assume $g_t = 0$ for all t and $B_0 = 0$.

Question 1) Argue that $s_t = w_t - \tau_t^t$ for all $t \geq 1$.

Ans: Young agents do not value current consumption. They will save all their income, so $s_t = w_t - \tau_t^t$.

Since $B_t = 0$ for all t, then the stock of capital is completely owned by the old agents, so $K_{t+1} = s_t$, or

$$K_{t+1} = w_t - \tau_t^t. \tag{16}$$

Question 2). Show that in equilibrium the stock of capital evolves as.

$$K_{t+1} = (1-\alpha) K_t^{\alpha} - \tau_t^t. \tag{17}$$

Ans: The solution to the firm's problem we have $w_t = F_L(K_t, 1) = (1-\alpha) K_t^{\alpha}$. Introducing this into (16) we have

$$K_{t+1} = (1-\alpha) K_t^{\alpha} - \tau_t^t.$$

Question 3) Assume $\tau_t^t = \tau_{t+1}^t = 0$ for all t. Solve for the (unique) steady state with positive capital (denote it by K^*).

Ans: From (17) we have that

$$K^* = (1-\alpha)\left(K_t^*\right)^{\alpha}.$$

Thus

$$K^* = (1-\alpha)^{1/(1-\alpha)}.$$

Suppose that at the beginning of period 1 the economy is at the steady state without social security and the government announces that it will introduce a pay-as-you-go system with $\tau_t^t = \tau$ and $\tau_{t+1}^t = -\tau$ for all t. Denote consumption of generation i in period j as a function of τ by $c_j^i(\tau)$. We will construct an example where agents born at period 1 are strictly better off in the new environment.

Question 4) Show that consumption of generation 1 at period 2 is given by

$$\begin{aligned} c_2^1(\tau) &= (w_1 - \tau)(1 + r_1(\tau)) + \tau \\ &= (K^* - \tau)\left(1 - \delta + \alpha (K^* - \tau)^{\alpha - 1}\right) + \tau. \end{aligned} \tag{18}$$

Ans: Since $s_1 = w_1 - \tau$ (i.e. young agents save their whole income) then

$$c_2^1(\tau) = (w_1 - \tau)(1 + r_1(\tau)) + \tau.$$

Notice that the wage rate w_1 is independent of τ since it only depends on the stock of capital at period 1, which is given by $K^* = (1 - \alpha)^{1/(1-\alpha)}$, thus

$$\begin{aligned} w_1 &= F_L(K^*, 1) \\ &= (1 - \alpha)(1 - \alpha)^{\frac{\alpha}{1-\alpha}} \\ &= (1 - \alpha)^{1/(1-\alpha)} \\ &= K^*. \end{aligned}$$

On the other hand the interest rate between period 1 and 2 depends on τ since tomorrow's capital depends on τ, thus

$$\begin{aligned} r_1(\tau) &= F_K(K_2, 1) - \delta \\ &= \alpha (K_2)^{\alpha - 1} - \delta \\ &= \alpha \left[(1 - \alpha)(K^*)^{\alpha} - \tau\right]^{\alpha - 1} - \delta \\ &= \alpha \left[K^* - \tau\right]^{\alpha - 1} - \delta. \end{aligned}$$

Replacing those equations above we obtain

$$c_2^1(\tau) = (K^* - \tau)\left(1 - \delta + \alpha (K^* - \tau)^{\alpha - 1}\right) + \tau.$$

Question 5) Show that given $\delta > 0$, $\left.\frac{dc_2^1(\tau)}{d\tau}\right|_{\tau=0}$ can be positive for sufficiently small α.

Ans: Rewrite equation (18) as

$$c_2^1(\tau) = (K^* - \tau)(1-\delta) + \alpha (K^* - \tau)^{\alpha} + \tau$$

Differentiating with respect to τ we have

$$\begin{aligned}
\frac{dc_2^1(\tau)}{d\tau} &= \delta - 1 - \frac{\alpha^2}{(K^* - \tau)^{1-\alpha}} + 1 \\
&= \delta - \frac{\alpha^2}{(K^* - \tau)^{1-\alpha}} \\
&= \delta - \frac{\alpha^2}{\left((1-\alpha)^{1/(1-\alpha)} - \tau\right)^{1-\alpha}}.
\end{aligned}$$

At $\tau = 0$ the derivative becomes

$$\frac{dc_2^1(\tau)}{d\tau}|_{\tau=0} = \delta - \frac{\alpha^2}{(1-\alpha)}.$$

Now consider the term $\alpha^2/(1-\alpha)$. It is zero for $\alpha = 0$, it tends to $+\infty$ when α tends to 1, it is continuous and strictly increasing. Therefore, since $0 < \delta < 1$, there must exist a $\bar{\alpha}$ such that

$$\delta - \frac{\bar{\alpha}^2}{(1-\bar{\alpha})} = 0,$$

such that if $\alpha < \bar{\alpha}$, then $\delta - \alpha^2/(1-\alpha) > 0$ and if $\alpha > \bar{\alpha}$, $\delta - \alpha^2/(1-\alpha) < 0$.

In other words, if $\alpha < \bar{\alpha}$,then $\frac{dc_1^2(\tau)}{d\tau}|_{\tau=0} > 0$. Furthermore, we can give an explicit formula for $\bar{\alpha}$. Direct computation shows that

$$\bar{\alpha} = \frac{1}{2}\sqrt{4\delta + \delta^2} - \frac{1}{2}\delta,$$

satisfies the condition.

Question 6) Is it possible that *every* generation is made better off by introducing social security? [Hint: Remember that the First Welfare Theorem holds in all equilibria with positive interest rate].

Ans: If interest rates are positive, the First Welfare Theorem holds without policy interventions. This means that after introducing social security at least one agents has to be worse-off.

Question 7) In the class notes we saw that the introduction of social security in an endowment economy makes everyone worse-off except the initial old, who receive the transfer without paying anything. Above we constructed an example where generation 1 is strictly better off with the introduction of social security. Can you give an intuitive explanation for this result?

Ans: In the endowment economy the interest rate always adjust in order to have zero savings in equilibrium. The main difference in our case is that with production, agents are able to save. For example, take two tax levels $\tau' > \tau$, then

$$\begin{aligned} c_2^1(\tau') - c_2^1(\tau) &= \left(w_1 - \tau'\right)\left(1 + r_1(\tau')\right) - (w_1 - \tau)(1 + r_1(\tau)) + \tau' - \tau \\ &= \left(w_1 - \tau'\right) r_1\left(\tau'\right) - (w_1 - \tau) r_1(\tau) + \left(w_1 - \tau'\right) - (w_1 - \tau) + \tau' - \tau \\ &= \left(w_1 - \tau'\right) r_1\left(\tau'\right) - (w_1 - \tau) r_1(\tau). \end{aligned}$$

Then $c_2^1(\tau') - c_2^1(\tau) > 0$ if and only if

$$\left(w_1 - \tau'\right) r_1\left(\tau'\right) > (w_1 - \tau) r_1(\tau),$$

that is, if and only if the *net return* on savings under τ' are higher than that under τ. In other words, given the interest rate, initial savings decline by $(w_1 - \tau') - (w_1 - \tau) = \tau' - \tau$. However benefits when old increase by the same amount. Thus if there is a difference it must be in terms of net returns. Even though savings decline, the interest rate increases. There is a trade-off between the two effects. In our example the second effect outweighs the first.

4 Heterogeneity*

Consider the OG model seen in class. Agents are now indexed by their date of birth and their endowment:

$$\begin{aligned} e_t^{t,n} &= 1 - \alpha_n, \qquad e_{t+1}^{t,n} = \alpha_n, \\ e_s^{t,n} &= 0, \qquad \text{for all } s \neq t, t+1, \end{aligned}$$

where $e_s^{t,n}$ denotes the endowment at time s of an individual of type n born at time t. Preferences are described by

$$u^{t,n}\left(c_1^{t,n}, c_2^{t,n}, ...\right) = (1 - \beta)\log c_t^{t,n} + \beta \log c_{t+1}^{t,n}, \qquad 0 < \beta < 1.$$

The initial old generation receives an endowment at time $t = 1$ only,

$$\begin{aligned} e_1^{0,n} &= \alpha_n, \\ e_s^{0,n} &= 0, \qquad \text{for all } s \neq 1. \end{aligned}$$

As usual, the initial old only care about their consumption when old, with preferences described by

$$u^{0,n}\left(c_1^{0,n}, c_2^{0,n}, ...\right) = c_1^{0,n}.$$

We will assume that each generation has N different types of endowments α_n, so that we can write $n = 1, 2, ..., N$. The set of agents I for this economy consists of the pairs

$$I = \{i = (t, n) : t = 0, 1, 2, ... \text{ and } n = 1, 2, ..., N\}.$$

(a) Write down the market clearing constraint in this economy, as it applies for any given period t.

Ans: Market clearing in the goods market at time t requires the sum of the consumption of those agents alive at time t to equal the sum of the endowments available at time t. Thus,

$$\sum_{n=1}^{N} c_t^{t,n} + \sum_{n=1}^{N} c_t^{t-1,n} = \sum_{n=1}^{N} e_t^{t,n} + \sum_{n=1}^{N} e_t^{t-1,n} = N. \tag{19}$$

(b) Show that the competitive equilibrium is such that there is no trade *across* generations. (Hint: Start by arguing that the initial old generation will consume its endowment. Then explore the implication of this result for the initial young generation. Finally, use an inductive argument).

Ans: Since the initial old of type n only values consumption in period 1, they will simply eat their endowment:

$$c_1^{0,n} = e_1^{0,n}.$$

Aggregating this equation across all types we obtain

$$\sum_{n=1}^{N} c_1^{0,n} = \sum_{n=1}^{N} e_1^{0,n}. \tag{20}$$

But from (19) we know that market clearing at time $t = 1$ requires that

$$\sum_{n=1}^{N} c_1^{1,n} + \sum_{n=1}^{N} c_1^{0,n} = \sum_{n=1}^{N} e_1^{1,n} + \sum_{n=1}^{N} e_1^{0,n}.$$

Using (20) in the above equation yields

$$\sum_{n=1}^{N} c_1^{1,n} = \sum_{n=1}^{N} e_1^{1,n},$$

which implies that aggregate savings of generation 1 are equal to zero.

Now, assume that aggregate savings of generation t are equal to zero:

$$\sum_{n=1}^{N} c_t^{t,n} = \sum_{n=1}^{N} e_t^{t,n}. \tag{21}$$

The budget constraint of this generation can be written as

$$p_t c_t^{t,n} + p_{t+1} c_{t+1}^{t,n} = p_t e_t^{t,n} + p_{t+1} e_{t+1}^{t,n}.$$

Aggregating this constraint across all types we obtain

$$p_t \sum_{n=1}^{N} c_t^{t,n} + p_{t+1} \sum_{n=1}^{N} c_{t+1}^{t,n} = p_t \sum_{n=1}^{N} e_t^{t,n} + p_{t+1} \sum_{n=1}^{N} e_{t+1}^{t,n}.$$

Using (21) in the above equation and dividing by p_t (recall that prices are strictly positive since the utility function is strictly increasing) yields

$$\sum_{n=1}^{N} c_{t+1}^{t,n} = \sum_{n=1}^{N} e_{t+1}^{t,n}. \tag{22}$$

But, again, from (19) we know that market clearing at time $t+1$ requires that

$$\sum_{n=1}^{N} c_{t+1}^{t+1,n} + \sum_{n=1}^{N} c_{t+1}^{t,n} = \sum_{n=1}^{N} e_{t+1}^{t+1,n} + \sum_{n=1}^{N} e_{t+1}^{t,n}$$

Plugging (22) in the above expression we obtain

$$\sum_{n=1}^{N} c_{t+1}^{t+1,n} = \sum_{n=1}^{N} e_{t+1}^{t+1,n},$$

which implies that aggregate savings of generation $t+1$ also equal zero. We thus conclude that there is no trade across generations.

(c) Find an expression for *aggregate savings*, $s\left(r; \{\alpha_n\}_{n=1}^{N}, \beta\right)$, the savings of all the young of a generation in terms of the parameters of the model (α's and β's) and the net interest rate, r. (Hint: Use the FOCs to find an expression for $c_t^{t,n}$ and then proceed to aggregate). Find an expression for the optimal consumption choice of an agent when young and when old if her endowment is characterized by $\alpha_n = \alpha$ and if she faces an interest rate r. Denote these optimal choices by $c_y(\alpha, r)$ and $c_o(\alpha, r)$.

Ans: The problem of the agent of type (t, n) consists of

$$\max_{c_t^{t,n}, c_{t+1}^{t,n}} (1-\beta)\log c_t^{t,n} + \beta \log c_{t+1}^{t,n},$$

$$\text{s.t.}: c_t^{t,n} + \frac{c_{t+1}^{t,n}}{1+r_t} = (1-\alpha_n) + \frac{\alpha_n}{1+r_t},$$

where $1 + r_t \equiv p_t/p_{t+1}$. The FOCs associated with this problem are

$$\begin{aligned} \left[c_t^{t,n}\right] &: \quad \frac{1-\beta}{c_t^{t,n}} = \mu \\ \left[c_{t+1}^{t,n}\right] &: \quad \frac{\beta}{c_{t+1}^{t,n}} = \frac{\mu}{1+r_t}, \end{aligned}$$

where μ is the multiplier of the budget constraint. Combining these two equations we arrive at

$$c_{t+1}^{t,n} = \frac{\beta}{1-\beta}(1+r_t)\, c_t^{t,n}.$$

Using this result in the budget constraint we obtain

$$c_t^{t,n} = (1-\beta)\left[(1-\alpha_n) + \frac{\alpha_n}{1+r_t}\right], \tag{23}$$

and

$$c_{t+1}^{t,n} = \beta(1+r_t)\left[(1-\alpha_n) + \frac{\alpha_n}{1+r_t}\right]. \tag{24}$$

Now, aggregate savings at time t are given by

$$s_t = \sum_{n=1}^{N}(1-\alpha_n) - \sum_{n=1}^{N} c_t^{t,n}.$$

Thus, using (23) we can write

$$s_t\left(r_t; \{\alpha_n\}_{n=1}^{N}, \beta\right) = \sum_{n=1}^{N}(1-\alpha_n) - (1-\beta)\left[\sum_{n=1}^{N}(1-\alpha_n) + \frac{\sum_{n=1}^{N}\alpha_n}{1+r_t}\right]. \tag{25}$$

When an agent faces an endowment $\alpha_n = \alpha$ and a constant interest rate r, from (23) and (24) we know that her optimal choices will be

$$c_y(\alpha, r) = (1-\beta)\left[(1-\alpha) + \frac{\alpha}{1+r}\right], \tag{26}$$

and

$$c_o(\alpha, r) = \beta(1+r)\left[(1-\alpha) + \frac{\alpha}{1+r}\right]. \tag{27}$$

(d) Let $\bar{\alpha} \equiv (1/N)\sum_{n=1}^{N}\alpha_n$. Show that

$$\begin{aligned} c_y(\bar{\alpha}, r) &= (1-\beta)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right], \\ c_o(\bar{\alpha}, r) &= \beta\ (1+r)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right], \end{aligned}$$

$$\begin{aligned} \frac{c_y(\alpha, r)}{c_y(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \\ \frac{c_o(\alpha, r)}{c_o(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \end{aligned}$$

$$\sum_{n=1}^{N} c_y(\alpha_n, r) = N\ c_y(\bar{\alpha}, r)\ .$$

Ans: The first four results follow from straightforward algebra using equations (26) and (27). The last expression can also be easily derived from (26) as follows:

$$\begin{aligned} \sum_{n=1}^{N} c_y(\alpha_n, r) &= (1-\beta)\left[\sum_{n=1}^{N}(1-\alpha_n) + \frac{\sum_{n=1}^{N}\alpha_n}{1+r_t}\right] \\ &= N(1-\beta)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r_t}\right] \\ &= N\ c_y(\bar{\alpha}, r). \end{aligned} \tag{28}$$

Interestingly, notice that if $(1-\alpha) > (1-\bar{\alpha})$ and $r > 0$, then $c_y(\alpha, r) > c_y(\bar{\alpha}, r)$ and $c_o(\alpha, r) > c_o(\bar{\alpha}, r)$. That is, if interest rates are positive, an agent that is relatively rich when young will enjoy higher levels of lifetime consumption and lifetime utility. This result is due to the fact that the individual is relatively rich in a period where goods are relatively more expensive (recall that if $r > 0$ then prices are falling over time).

(e) Using the above results, characterize the equilibrium interest rate $\bar{r}$, i.e. write an equation for $\bar{r}$, which is the solution to

$$s\left(\bar{r}; \{\alpha^n\}_{n=1}^{N}, \beta\right) = 0.$$

Show that if two economies have the same value of $\bar{\alpha} \equiv (1/N)\sum_{n=1}^{N}\alpha_n$, but possible different distribution of the α's, then they still have the same equilibrium interest rate $\bar{r}$, so that we can write $\bar{r}(\bar{\alpha}, \beta)$. Show also that if $\bar{\alpha}$ is the same for two economies, then the equilibrium average aggregate consumption for the young is the same, i.e. $(1/N)\sum_{n=1}^{N} c_t^{t,n} \equiv (1/N)\sum_{n=1}^{N} c_y(\alpha_n, \bar{r})$ is the same regardless of the distribution of the α's.

Ans: From (25) aggregate savings can be written as

$$s\left(r; \{\alpha^n\}_{n=1}^{N}, \beta\right) = N\left\{(1-\bar{\alpha}) - (1-\beta)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right]\right\}.$$

Thus,

$$s\left(\bar{r}; \{\alpha^n\}_{n=1}^{N}, \beta\right) = 0,$$

iff the interest rate $\bar{r}$ satisfies

$$\begin{aligned}
(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+\bar{r}} &= \frac{(1-\bar{\alpha})}{1-\beta} \\
\frac{\bar{\alpha}}{(1-\bar{\alpha})(1+\bar{r})} &= \frac{1}{1-\beta} - 1 \\
\bar{r} &= \frac{\bar{\alpha}(1-\beta)}{\beta(1-\bar{\alpha})} - 1,
\end{aligned}$$

or

$$\bar{r}(\bar{\alpha}, \beta) = \frac{\bar{\alpha} - \beta}{\beta(1-\bar{\alpha})}. \tag{29}$$

Hence, the equilibrium interest rate $\bar{r}$ is a function of $\bar{\alpha}$ and β only. Finally, from (28) we see that

$$\frac{1}{N}\sum_{n=1}^{N} c_y(\alpha_n, \bar{r}) = c_y(\bar{\alpha}, \bar{r}).$$

Thus, since $\bar{r}$ is a function of $\bar{\alpha}$ and β only, the equilibrium average aggregate consumption for the young is a function of $\bar{\alpha}$ and β only, as was to be shown.

(*f*) Compute the "best aggregate symmetric allocation" for this economy; that is, the welfare-maximizing allocation that only depends on whether a particular agent is young or old, so that $c_t^{t,n} = c_y^*$ and $c_{t+1}^{t,n} = c_o^*$ for all t and n.

Ans: The best symmetric allocation is the solution to the following program:

$$\max_{c_y, c_o} (1-\beta)\log c_y + \beta \log c_o,$$

$$\text{s.t.}: c_y + c_o = 1.$$

The FOCs associated with this problem are

$$\begin{aligned} [c_y] &: \frac{1-\beta}{c_y} = \lambda \\ [c_o] &: \frac{\beta}{c_o} = \lambda. \end{aligned}$$

Thus,

$$c_o = \frac{\beta}{1-\beta} c_y.$$

Introducing this result in the feasibility constraint we obtain

$$c_y^* = 1-\beta,$$

and

$$c_o^* = \beta.$$

(g) Show that if the following condition holds:

$$\bar{\alpha} \equiv \frac{1}{N} \sum_{n=1}^{N} \alpha_n < \beta,$$

the equilibrium interest rate is negative and the allocation is not Pareto Optimal. To do so consider the following allocation:

$$\begin{aligned} c_t^{*t,n} &= c_y^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}}, \\ c_{t+1}^{*t,n} &= c_o^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}}, \end{aligned}$$

for $t \geq 1$ and $n = 1, .., N$ (all the current and future young) and for the current old:

$$c_1^{*0,n} = \alpha_n \ + (\beta - \bar{\alpha}),$$

for $n = 1, .., N$, where $\bar{r}$ is the interest rate that corresponds to the equilibrium of the economy with $\bar{\alpha}$ and β.

i) Show that the "$*$" allocation is feasible. (Hint: Show that $(1/N) \sum_{n=1}^{N} c_t^{*t,n} = 1-\beta$ for all $t \geq 1$ and $(1/N) \sum_{n=1}^{N} c_{t+1}^{*t,n} = \beta$ for all $t \geq 0$).

ii) Show that this allocation Pareto dominates the equilibrium allocation for the economy with $\bar{\alpha}$ and β. Make sure that you show that this allocation is preferred for each type n of the initial old and for each type n of young agents. (Hint: For the young you will have to compare the utility of the equilibrium vector

$$\left(c_y\left(\alpha_n,\bar{r}\right),\ c_o\left(\alpha_n,\bar{r}\right)\right)=\left(c_y\left(\bar{\alpha},\bar{r}\right)\ \frac{c_y\left(\alpha_n,\bar{r}\right)}{c_y\left(\bar{\alpha},\bar{r}\right)}\ ,\ c_o\left(\bar{\alpha},\bar{r}\right)\ \frac{c_o\left(\alpha_n,\bar{r}\right)}{c_o\left(\bar{\alpha},\bar{r}\right)}\right),$$

with the one of the proposed "$*$" allocation

$$\left(c_y^*\ \frac{1+\left(1-\alpha_n\right)\bar{r}}{1+\left(1-\bar{\alpha}\right)\bar{r}}\ ,\ c_o^*\ \frac{1+\left(1-\alpha_n\right)\bar{r}}{1+\left(1-\bar{\alpha}\right)\bar{r}}\right),$$

and, using your answer to (d), (e) and (f), conclude that the second is preferred to the first if and only if the young prefers $\left(c_y^*,c_o^*\right)$ to the bundle $\left(c_y\left(\bar{\alpha},\bar{r}\right),c_o\left(\bar{\alpha},\bar{r}\right)\right)$. Finally, to show the last statement argue that $\left(c_y\left(\bar{\alpha},\bar{r}\right),c_o\left(\bar{\alpha},\bar{r}\right)\right)$ is a feasible symmetric allocation, so that $\left(c_y^*,c_o^*\right)$ is indeed preferred by the young).

Ans: First, from (29) we see that

$$\bar{r}\left(\bar{\alpha},\beta\right)<0\iff\bar{\alpha}<\beta.$$

Second, notice that

$$\frac{1}{N}\sum_{n=1}^{N}c_t^{*t,n}=\left(1-\beta\right)\frac{\frac{1}{N}\sum_{n=1}^{N}\left[1+\left(1-\alpha_n\right)\bar{r}\right]}{1+\left(1-\bar{\alpha}\right)\bar{r}}=1-\beta,$$

$$\frac{1}{N}\sum_{n=1}^{N}c_{t+1}^{*t,n}=\beta\frac{\frac{1}{N}\sum_{n=1}^{N}\left[1+\left(1-\alpha_n\right)\bar{r}\right]}{1+\left(1-\bar{\alpha}\right)\bar{r}}=\beta,$$

and

$$\frac{1}{N}\sum_{n=1}^{N}c_1^{*0,n}=\frac{1}{N}\sum_{n=1}^{N}\alpha_n\ +\left(\beta-\frac{1}{N}\sum_{n=1}^{N}\bar{\alpha}\right)=\beta,$$

so we see that the "$*$" allocation is indeed feasible. Now we need to show that this allocation Pareto dominates the equilibrium allocation for the economy with $\bar{\alpha}$ and β. To do so, consider first the change in consumption for each initial old of type n:

$$c_1^{*0,n}-c_1^{0,n}=\beta-\bar{\alpha}.$$

Since $\bar{\alpha}<\beta$ by assumption, then each initial old is better off in the "$*$" allocation. Finally, we need to compare the competitive equilibrium allocation

$$\left(c_y\left(\alpha_n,\bar{r}\right),\ c_o\left(\alpha_n,\bar{r}\right)\right)=\left(c_y\left(\bar{\alpha},\bar{r}\right)\frac{c_y\left(\alpha_n,\bar{r}\right)}{c_y\left(\bar{\alpha},\bar{r}\right)}\ ,\ c_o\left(\bar{\alpha},\bar{r}\right)\frac{c_o\left(\alpha_n,\bar{r}\right)}{c_o\left(\bar{\alpha},\bar{r}\right)}\right),$$

with the proposed "$*$" allocation

$$\left(c_y^* \frac{1 + (1 - \alpha_n)\, \bar{r}}{1 + (1 - \bar{\alpha})\, \bar{r}} \ , c_o^* \frac{1 + (1 - \alpha_n)\, \bar{r}}{1 + (1 - \bar{\alpha})\, \bar{r}} \ \right).$$

Using our previous results, the second allocation is preferred to the first if and only if each agent prefers $\left(c_y^*, c_o^*\right)$ to the bundle $\left(c_y\left(\bar{\alpha}, \bar{r}\right), c_o\left(\bar{\alpha}, \bar{r}\right)\right)$. But $\left(c_y\left(\bar{\alpha}, \bar{r}\right), c_o\left(\bar{\alpha}, \bar{r}\right)\right)$ is a feasible symmetric allocation and $\left(c_y\left(\bar{\alpha}, \bar{r}\right), c_o\left(\bar{\alpha}, \bar{r}\right)\right) \neq \left(c_y^*, c_o^*\right)$, so the bundle $\left(c_y^*, c_o^*\right)$ is preferred by all young agents. Thus, we conclude that the "$*$" allocation Pareto dominates the competitive equilibrium allocation.

5 OG Model with Multiple Periods*

We will now introduce agents living multiple periods in the basic model with homogenous generations. We will allow for general preferences and endowments. In this problem we will show that if equilibrium interest rate are negative, then the equilibrium is not Pareto Optimal.

The first time period is $t = 1$. Preferences are given by

$$u^t \left(c_1^t, c_2^t, ...\right) = v^t \left(c_t^t, c_{t+1}^t, ..., c_{t+N-1}^t\right) \ ,$$

for the generations born after $t \geq 1$, and

$$u^t \left(c_1^t, c_2^t, ...\right) = v^t \left(c_1^t, c_2^t, ..., c_{t+N-1}^t\right),$$

for the old generations born at $t = -N + 2, -N + 3, ..., -1, 0$.

We let a denote the age of the agents, $a = 1, 2, ..., N$. Endowments are positive only while agents are alive, so that for $t \geq 1$

$$\begin{aligned} e_{t+a-1}^t &> 0, \qquad \text{all } a: \ 1 \leq a \leq N, \\ e_{t+a-1}^t &= 0, \qquad \text{otherwise,} \end{aligned}$$

and for generations born at $t = -N + 2, \ -N + 3, ..., 0$,

$$e_1^t, \ e_2^t, ..., e_{t+N-1}^t > 0,$$

and zero otherwise.

(a) Write down the market clearing constraint in this economy, as it applies for any given period t.

Ans: Market clearing in the goods market at time t requires the sum of the consumption of those agents alive at time t to equal the sum of the endowments available at time t. Thus,

$$\sum_{a=1}^{N} c_t^{t-a+1} = \sum_{a=1}^{Y} e_t^{t-a+1} + \sum_{a=Y+1}^{N} e_t^{t-a+1} = Y(1-\alpha) + (N-Y)\alpha.$$

(b) Let $\{\bar{c}^t\}$ denote the equilibrium consumption allocation of the generation born at time t. Let λ^t be the Lagrange multiplier of the budget constraint:

$$\sum_{a=1}^{N} p_{t+a-1}\ c_{t+a-1}^t = \sum_{a=1}^{N} p_{t+a-1}\ e_{t+a-1}^t,$$

for $t \geq 1$ and

$$p_1 c_1^t + p_2 c_2^t + ... + p_{t+N-1} c_{t+N-1}^t = p_1 e_1^t + p_2 e_2^t + ... + p_{t+N-1} e_{t+N-1}^t,$$

for $t = -N+2,\ -N+3, ..., -1, 0$.

Write down the first order conditions that the optimal choice $\bar{c}^t$ for generation t must satisfy.

Ans: The FOC of the problem of an agent born at time $t \geq 1$ is

$$\left[c_{t+a-1}^t\right] : v_a^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right) = \lambda^t p_{t+a-1}, \tag{30}$$

for all $a = 1, ..., N$, and where $v_a^t(\cdot)$ denotes the derivative of $v^t(\cdot)$ with respect to its $a-$th argument. Similarly, the FOC of the problem of an agent born at time $t = -N+2,\ -N+3, ..., 0$ is

$$\left[c_{t+a-1}^t\right] : v_{t+a-1}^t\left(\bar{c}_1^t, \bar{c}_2^t, ..., \bar{c}_{t+N-1}^t\right) = \lambda^t p_{t+a-1}, \tag{31}$$

for all $a = 2-t, ..., N$.

(c) We will define a new allocation using the equilibrium allocation $\{\bar{c}^t\}$ and two parameters τ and Y. The "$*$" allocation is defined as

$$c_{t+a-1}^{*t}(\tau) = \begin{cases} \bar{c}_{t+a-1}^t - \tau, & \text{all } a:\ 1 \leq a \leq Y, \\ \bar{c}_{t+a-1}^t + \frac{Y}{N-Y}\tau, & \text{all } a:\ Y+1 \leq a \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

so that when agents are young (age $a \leq Y$), we are reducing their consumption by an amount τ each period relative to the equilibrium consumption. When agents are old (age $a > Y$), we are adding an amount $\frac{Y}{N-Y}\tau$ to their consumption.

Let $P(\tau, Y)$ be the present value (at the time of birth) of the transfer described by parameters (τ, Y) for a generation born at time t using prices $\{p_t\}$. Show that for generations born after time $t \geq 1$

$$P^t(\tau, Y) = \frac{1}{p_t}\left[\sum_{a=1}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right].$$

Show that for generations born at $t = -N+2,\ -N+3, ..., -1, 0$ we have two cases: one for those that at time $t = 1$ are $a = 2, ..., Y$ years old (they were born at time t satisfying $-Y+2 \leq t \leq 0$, so they are young):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right]$$

and one for those that at time $t = 1$ are $a = Y+1, ..., N$ years old (they were born at time t satisfying $-N+2 \leq t \leq -Y+1$, so they are old):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right].$$

Argue that if the prices p_t are increasing in t for all $t \geq 1$ (so that the implied interest rates are negative) then $P^t(\tau, Y) > 0$.

Ans: Consider those generations born at time $t \geq 1$. They pay an amount τ during their first Y periods of life, and receive $\frac{Y}{N-Y}\tau$ during their last $N-Y$ periods of life. Thus, the time t value of the net transfers they receive is

$$\begin{aligned} P^t(\tau, Y) &= \frac{1}{p_t}\left[p_t(-\tau) + \cdots + p_{t+Y-1}(-\tau) + p_{t+Y}\left(\frac{Y}{N-Y}\tau\right) + \cdots + p_{t+N-1}\left(\frac{Y}{N-Y}\tau\right)\right] \\ &= \frac{\tau}{p_t}\left[-\sum_{a=1}^{Y} p_{t+a-1} + \left(\frac{Y}{N-Y}\right)\sum_{a=Y+1}^{N} p_{t+a-1}\right], \end{aligned}$$

as was to be shown. Now assume that prices are constant over time, $p_t = p$ for all $t \geq 1$. Then, from the above equation we obtain

$$P^t(\tau, Y) = \frac{\tau}{p}\left[-Yp + \left(\frac{Y}{N-Y}\right)(N-Y)p\right] = 0.$$

This result implies that if prices p_t are increasing in t for all $t \geq 1$ then $P^t(\tau, Y) > 0$, as desired. We can think of $P^t(\tau, Y)$ as the present (as of time t) discounted value of social security for an individual born at time t.

A similar analysis applies for those generations born at time t satisfying $-N+2 \leq t \leq 0$.

(d) Consider an equilibrium $\left\{\bar{c}^t, p_t\right\}$. Define the interest rates:

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t},$$

for $t \geq 1$. Show that if the equilibrium interest rates are negative for all $t \geq 1$ then the CE allocation $\left\{\bar{c}^t\right\}$ is not Pareto Optimal. To establish this show that the allocation $\left\{c^{*t}(\tau)\right\}$ is feasible and, at least for small τ, $\left\{c^{*t}(\tau)\right\}$ Pareto dominates $\left\{\bar{c}^t\right\}$. In particular,

$i)$ Show that this allocation is feasible for any τ.

$ii)$ Compute the marginal welfare gain (or loss) of changing the allocation by introducing τ small. Define the function $U^t(\tau)$

$$U^t(\tau) = v^t\left(c_t^{*t}(\tau), c_{t+1}^{*t}(\tau), ..., c_{t+N-1}^{*t}(\tau)\right), \tag{32}$$

for $t \geq 1$ and

$$U^t(\tau) = v^t\left(c_1^{*t}(\tau),\ c_2^{*t}(\tau),\ ...,\ c_{t+N-1}^{*t}(\tau)\right),$$

for $t = -N+2,\ -N+3, ..., -1, 0$. Differentiate $U^t(\tau)$ with respect to τ and evaluate this derivative at $\tau = 0$. Use your answer to (c) and the fact that $r_t < 0$ to argue that $dU^t/d\tau > 0$.

Ans: We will first show that the "$*$" allocation is feasible for any τ. Consider the aggregate consumption of all agents alive at time t:

$$\begin{aligned}
\sum_{a=1}^{N} c_t^{*t-a+1} &= \sum_{a=1}^{Y}\left(\bar{c}_t^{t-a+1} - \tau\right) + \sum_{a=Y+1}^{N}\left(\bar{c}_t^{t-a+1} + \frac{Y}{N-Y}\tau\right) \\
&= \sum_{a=1}^{Y}\bar{c}_t^{t-a+1} + \sum_{a=Y+1}^{N}\bar{c}_t^{t-a+1} - \tau Y + \frac{Y}{N-Y}\tau(N-Y) \\
&= \sum_{a=1}^{N}\bar{c}_t^{t-a+1}.
\end{aligned}$$

Thus, since the competitive equilibrium allocation $\left\{\bar{c}^t\right\}$ is feasible, it follows that the allocation $\left\{c^{*t}(\tau)\right\}$ is feasible for any τ. Finally, we will compute the marginal gain of changing the competitive equilibrium allocation by introducing τ small. From (32) we can write

$$\begin{aligned}
\frac{dU^t(\tau)}{d\tau} &= v_1^t(\cdot)(-1) + \cdots + v_Y^t(\cdot)(-1) + v_{Y+1}^t(\cdot)\left(\frac{Y}{N-Y}\right) + \cdots + v_N^t(\cdot)\left(\frac{Y}{N-Y}\right) \\
&= \sum_{a=1}^{Y} v_a^t\left(\bar{c}_t^t(\tau), ..., c_{t+N-1}^{*t}(\tau)\right)(-1) + \sum_{a=Y+1}^{N} v_a^t\left(c_t^{*t}(\tau), ..., c_{t+N-1}^{*t}(\tau)\right)\left(\frac{Y}{N-Y}\right).
\end{aligned}$$

Thus,

$$\left.\frac{dU^t(\tau)}{d\tau}\right|_{\tau=0} = -\sum_{a=1}^{Y} v_a^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right) + \left(\frac{Y}{N-Y}\right)\sum_{a=Y+1}^{N} v_a^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right). \quad (33)$$

Now, using (30), the marginal utilities of consumption at ages 1 and a of an individual born at time t are related as follows:

$$v_a^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right) = \frac{p_{t+a-1}}{p_t} v_1^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right).$$

Plugging this result in (33) yields

$$\left.\frac{dU^t(\tau)}{d\tau}\right|_{\tau=0} = \frac{v_1^t\left(\bar{c}_t^t, \bar{c}_{t+1}^t, ..., \bar{c}_{t+N-1}^t\right)}{p_t}\left[-\sum_{a=1}^{Y} p_{t+a-1} + \left(\frac{Y}{N-Y}\right)\sum_{a=Y+1}^{N} p_{t+a-1}\right].$$

If the equilibrium interest rates are negative for all $t \geq 1$, then we know from our previous results that $P^t(\tau, Y) > 0$. But $P^t(\tau, Y) > 0$ iff the term in brackets above is strictly positive. Hence, we conclude that $dU^t/d\tau > 0$ for small τ. This is just another way to derive the result that when interest rates are negative, then the competitive equilibrium allocation can be improved upon by the introduction of social security.

A similar analysis applies for those generations born at time t satisfying $-N+2 \leq t \leq 0$.

6 Population Growth and Social Security*

Return again to the basic model where each generation lives for two periods, preferences are logarithmic, and endowments are $\left(e_t^t, e_{t+1}^t\right) = (1-\alpha, \alpha)$ for all $t \geq 1$ and $e_1^0 = \alpha$. We will examine the welfare effects of changing demographic patterns in the presence of Social Security. To this end, let N_t denote the number of young agents at time t. Population grows at the rate n, so that

$$N_{t+1} = (1+n) N_t, \qquad N_0 = 1,$$

where we have normalized the initial population to one.

(a) Write down the market clearing constraint in this economy, as it applies for any given period t.

Ans: Market clearing in the goods market at time t requires the sum of the consumption of those agents alive at time t to equal the sum of the endowments available at time t. Thus,

$$N_t c_t^t + N_{t-1} c_t^{t-1} = N_t e_t^t + N_{t-1} e_t^{t-1},$$

or

$$c_t^t + \frac{c_t^{t-1}}{1+n} = (1-\alpha) + \frac{\alpha}{1+n}.$$

(*b*) Briefly argue that the competitive equilibrium is such that there is no trade across and within generations.

Ans: Since agents are identical there will not be trade within generations. Since the initial old only care about their consumption when old, they will eat their endowment. By market clearing, this implies that the initial young cannot engage in trade either (i.e., their aggregate savings are equal to zero). Then, using an inductive argument it is straightforward to show that this will hold for all future generations as well.

(*c*) Find an expression for *aggregate savings,* $s(r; \alpha, \beta, N)$, the savings of all the young of a generation in terms of the parameters of the model (α, β and N) and the net interest rate, r. (Hint: Use the FOCs to find an expression for c_t^t and then proceed to aggregate). Characterize the equilibrium interest rate $\bar{r}$, i.e. write an equation for $\bar{r}$, which is the solution to

$$s(\bar{r}; \alpha, \beta, N) = 0.$$

Ans: The problem of the agent of type t consists of

$$\max_{c_t^t, c_{t+1}^t} (1-\beta)\log c_t^t + \beta \log c_{t+1}^t,$$

$$\text{s.t.} : c_t^t + \frac{c_{t+1}^t}{1+r_t} = (1-\alpha) + \frac{\alpha}{1+r_t},$$

where $1 + r_t \equiv p_t / p_{t+1}$. The FOCs associated with this problem are

$$\begin{aligned} \left[c_t^t\right] &: \frac{1-\beta}{c_t^{t,n}} = \mu \\ \left[c_{t+1}^t\right] &: \frac{\beta}{c_{t+1}^{t,n}} = \frac{\mu}{1+r_t}, \end{aligned}$$

where μ is the multiplier of the budget constraint. Combining these two equations we arrive at

$$c_{t+1}^t = \frac{\beta}{1-\beta}(1+r_t)\, c_t^t.$$

Using this result in the budget constraint we obtain

$$c_t^t = (1-\beta)\left[(1-\alpha) + \frac{\alpha}{1+r_t}\right], \tag{34}$$

and

$$c_{t+1}^{t}=\beta\left(1+r_{t}\right)\left[(1-\alpha)+\frac{\alpha}{1+r_{t}}\right]. \tag{35}$$

Now, aggregate savings at time t are given by

$$s_{t}=N_{t}\left[(1-\alpha)-c_{t}^{t}\right].$$

Hence, using (34) we can write

$$s_{t}\left(r_{t};\alpha,\beta,N_{t}\right)=N_{t}\left\{(1-\alpha)-(1-\beta)\left[(1-\alpha)+\frac{\alpha}{1+r_{t}}\right]\right\}. \tag{36}$$

Thus,

$$s\left(\bar{r};\alpha,\beta,N\right)=0,$$

iff the interest rate $\bar{r}$ satisfies

$$\begin{aligned}
(1-\alpha)+\frac{\alpha}{1+\bar{r}} &= \frac{(1-\alpha)}{1-\beta} \\
\frac{\alpha}{(1-\alpha)\left(1+\bar{r}\right)} &= \frac{1}{1-\beta}-1 \\
\bar{r} &= \frac{\alpha\left(1-\beta\right)}{\beta\left(1-\alpha\right)}-1,
\end{aligned}$$

or

$$\bar{r}\left(\alpha,\beta\right)=\frac{\alpha-\beta}{\beta\left(1-\alpha\right)}. \tag{37}$$

(d) Compute the best symmetric allocation for this economy; that is, the welfare-maximizing allocation that only depends on whether a particular agent is young or old, so that $c_t^t = c_y$ and $c_{t+1}^t = c_o$ for all t.

Ans: The best symmetric allocation is the solution to the following program:

$$\max_{c_y,c_o}\,(1-\beta)\log c_y+\beta\log c_o,$$

$$\text{s.t.}: c_y+\frac{c_o}{1+n}=(1-\alpha)+\frac{\alpha}{1+n}.$$

The FOCs associated with this problem are

$$\begin{aligned}
[c_y] &: \quad \frac{1-\beta}{c_y}=\lambda \\
[c_o] &: \quad \frac{\beta}{c_o}=\frac{\lambda}{1+n}.
\end{aligned}$$

Thus,

$$c_o = \frac{\beta}{1-\beta}\left(1+n\right)c_y.$$

Introducing this result in the feasibility constraint we obtain

$$c_y^* = \left(1-\beta\right)\left[\left(1-\alpha\right)+\frac{\alpha}{1+n}\right],$$

and

$$c_o^* = \beta\left(1+n\right)\left[\left(1-\alpha\right)+\frac{\alpha}{1+n}\right].$$

(e) Compute the level of per-capita tax collection in a pay-as-you-go system that would implement the best symmetric allocation. How does this level depend on n? Briefly explain.

Ans: In a pay-as-you-go system, the after-tax consumption of each agent will be given by:

$$\begin{aligned} c_y^{ss} &= \left(1-\alpha\right)-\tau \\ c_o^{ss} &= \alpha+\left(1+n\right)\tau, \end{aligned} \tag{38}$$

where τ denotes the level of per-capita tax collection. If we want to implement the best symmetric allocation $\left(c_y^{ss}, c_o^{ss}\right) = \left(c_y^*, c_o^*\right)$, then τ must satisfy

$$\begin{aligned} \tau &= \left(1-\alpha\right)-c_y^* \\ &= \left(1-\alpha\right)-\left(1-\beta\right)\left[\left(1-\alpha\right)+\frac{\alpha}{1+n}\right] \\ &= \beta\left(1-\alpha\right)-\frac{\alpha\left(1-\beta\right)}{1+n} \\ &= \beta\left(1-\alpha\right)-\frac{\beta\left(1-\alpha\right)\left(1+\bar{r}\right)}{1+n} \qquad \text{[using (37)]}, \end{aligned}$$

or,

$$\tau^* = \beta\left(1-\alpha\right)\left(\frac{n-\bar{r}}{1+n}\right).$$

Of course, since we are in a social security system we must have that $n > \bar{r}$ so that $\tau^* > 0$. From the above equation we obtain

$$\frac{\partial\tau^*}{\partial n} = \beta\left(1-\alpha\right)\frac{1+\bar{r}}{\left(1+n\right)^2} > 0.$$

We then see that the optimal level of per-capita tax collection rises with population growth. From (38) it then follows that the consumption of young agents falls and that of old agents rises.

To understand this result, notice that changes in n are isomorphic to changes in the price of future consumption: $1/(1+n)$ is the price of future consumption relative to current consumption from the point of view of the best symmetric allocation. Hence, we can use the standard Hicksian decomposition of the total effect of a price change to explore the effects of a rise in n (that is, a fall in the relative price of future consumption). By the separability and concavity of the agent's utility function, combined with the fact that we only have two goods, we know that consumption in both periods are normal goods and, moreover, they are net substitutes (i.e., in the compensated sense). Thus, a fall in the relative price of future consumption generates two effects:

- A *positive* income effect, given that agents are net savers in the best symmetric allocation (and thereby they benefit from a rise in the rate of return of social security, n). By normality, this positive income effect induces a rise in both current and future consumption.

- A substitution effect due to the fact that future consumption is relatively cheaper. This effect induces a rise in future consumption (since the compensated demand curve of a good is negatively sloped with respect to its own price) and a fall in current consumption (since the goods are net substitutes).

We can summarize these results in the following table:

Effect of a rise in n

Effect:	Income	+	Substitution	=	**Total**
c_y^{ss}	$\uparrow$		$\downarrow$		Uncertain
c_o^{ss}	$\uparrow$		$\uparrow$		$\uparrow$

In our particular example the substitution effect on current consumption outweighs the income effect, so that consumption of young agents fall with n. Since $c_y^{ss} = (1-\alpha) - \tau$, this immediately implies that per-capita taxes (that is, per-capita savings) should rise with n, which is exactly what we found above.

(g) Assume that population growth falls permanently to $n' < n$ but the tax collected from each young agent is held fixed. Calculate the effect of this change in the welfare of current and future young and old generations. Is it possible to change the tax system so as to keep the level of utility for the current old and the current and future young in the economy with n' at the same level than the economy with n?

Ans: Let $c^{ss}(n', \tau(n))$ denote the consumption of an agent when population growth is n' but taxes are collected as if population growth were n. Then, the change in consumption of the current and future old is:

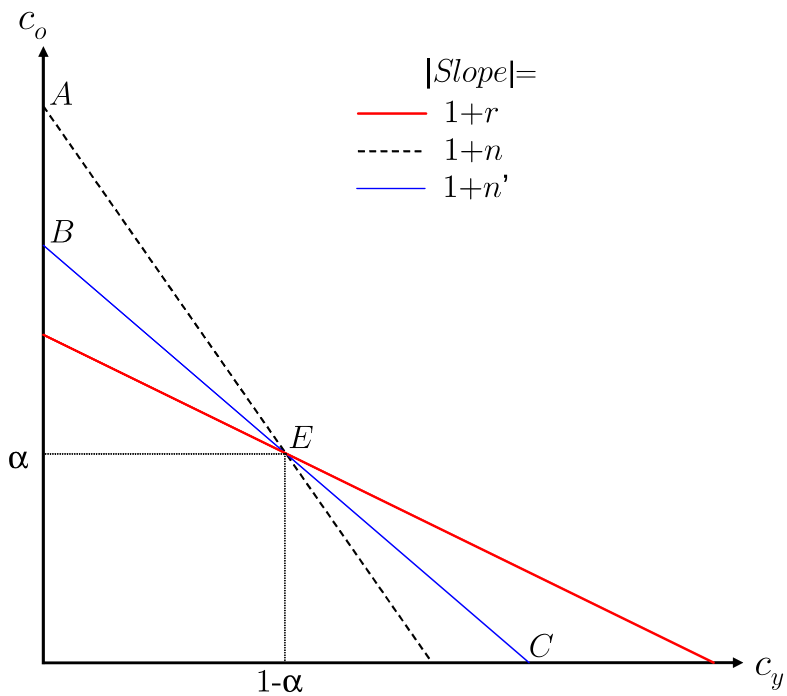
$$c_o^{ss}\left(n', \tau(n)\right) - c_o^{ss}(n, \tau(n)) = \left(1+n'\right)\tau(n) - (1+n)\tau(n) < 0.$$

In turn, the change in consumption of the current and future young is:

$$c_y^{ss}\left(n', \tau(n)\right) - c_y^{ss}(n, \tau(n)) = -\tau(n) - (-\tau(n)) = 0.$$

Thus, if population growth falls permanently to $n' < n$ but the tax collected from each young agent is held fixed, then every current and future generation is made worse off.

Figure 3



A simple revealed-preference argument can be used to show that it is impossible to keep all generations at the same level of utility they enjoyed before population growth fell from n to n'. To do so, consider Figure 3 above which depicts three budget constraints with slopes $1+r < 1+n' < 1+n$ in the (c_y, c_o) space. All these constraints must go through the initial endowment $(1-\alpha, \alpha)$, point E, since consuming the initial endowment is affordable at any interest rate. Since $n > r$, there must exist an allocation $\left(c_y^*, c_o^*\right)$ lying in the interval $[A, E)$ that is strictly preferred to the initial allocation $(1-\alpha, \alpha)$. This is what we call the best symmetric allocation. Let $U\left(c_y^*, c_o^*\right)$ denote the lifetime utility that an agent derives from this allocation.

Now, assume that population growth falls to n' and that it is possible to find an allocation $(\tilde{c}_y, \tilde{c}_o)$ such that

$$U(\tilde{c}_y, \tilde{c}_o) = U\left(c_y^*, c_o^*\right) > U(1-\alpha, \alpha). \tag{39}$$

Notice that such allocation cannot lie in the interval $[B, E)$, since all these allocations were affordable when population growth was n and, since they were not chosen, they must yield a lifetime utility strictly lower than $U\left(c_y^*, c_o^*\right)$. Moreover, $(\tilde{c}_y, \tilde{c}_o)$ cannot lie in point E since $U\left(c_y^*, c_o^*\right) > U(1-\alpha, \alpha)$. Thus, such allocation could only lie in the interval $(E, C]$. But notice that all the allocations lying in this interval were affordable in the competitive equilibrium, but they were not chosen, so we must have that

$$U(1-\alpha, \alpha) > U(\tilde{c}_y, \tilde{c}_o).$$

This clearly contradicts (39), so we conclude that a fall in population growth will always make at least one generation worse off.