Price Theory III
Spring 2019

(Lars Stole)

Solutions to
Assignment 6
by Takuma Habu

University of Chicago

Contents

1 Problem 1 **3**

1.1 Part (a) 3

1.2 Part (b) 4

1.3 Part (c) 5

2 Problem 2 **5**

2.1 Part (a) 5

2.2 Part (b) 6

2.3 Part (c) 7

2.4 Part (d) 8

2.5 Part (e) 10

3 Problem 3 (from Balazs Szentes) **12**

3.1 Part (a) 12

3.2 Part (b) 12

3.3 Part (c) 13

3.4 Part (d) 13

3.5 Part (e) 14

3.6 Part (f) 14

4 Problem 4 **15**

4.1 Part (a) 15

4.2 Part (b) 15

4.3 Part (c) 16

5 Problem 5: JR, Exercise 9.8 **17**

5.1 Part (a) 17

5.2 Part (b) 18

5.3 Part (c) 18

5.4 Part (d) 19

6 Problem 6: JR, Exercise 9.9 **20**

6.1 Part (a) 20

6.2 Part (b) 21

6.3 Part (c) 22

6.4 Part (d) 24

For typos/comments, email me at takumahabu@uchicago.edu.

v1.0 Initial version

1 Problem 1

Consider a natural (but regulated) monopolist with a constant marginal cost of production equal to c, which is private information at the time of contracting and uniformly distributed on $[1,2]$. The monopolist's profit for producing output q at marginal cost c with transfer t from the regulator is

$$u(q,c,t) \coloneqq t - cq.$$

Note that the firm does not get any revenue from selling q; all payments are through t. The firm's participation constraint is $\underline{U} = 0$.

The regulator's objective function is to maximise

$$\int_{\underline{c}}^{\overline{c}} \left[vq(c) - \frac{1}{2} q(c)^2 - t(c) \right] f(c)\,\mathrm{d}c$$

over the class of DRMs which satisfy IC and IR constraints. Assume $v \geq 3$. Among other things, this implies that the first-best, full informatino output would be $q^{fb}(c) = v - c > 0$.

1.1 Part (a)

What are the IC and IR constraints for this problem? State and prove a characterisation theorem for the set of all IC mechanism $\{q(\cdot), t(\cdot)\}$ in terms of a monotonicity condition and an integral condition.

. . . . . .

Let $\{q(\cdot), t(\cdot)\}$ be some direct mechanism. We define IC and IR constraints at the interim stage—i.e. when the monopolist knows its type c, but when the regulator does not. The interim IR constraint must ensure that the monopolist wishes to participate in the direct mechanism:

$$u(c) \coloneqq t(c) - cq(c) \geq 0, \ \forall c \in [1,2].$$

The interim IC constraint ensure that a type-c monopolist has the incentive to report truthfully; i.e.

$$u(c) \equiv t(c) - cq(c) \geq t(\hat{c}) - cq(\hat{c}) \eqqcolon u(\hat{c}|c), \ \forall \hat{c} \in [1,2].$$

Note that the single crossing property here is "flipped around". So we would expect the properties on q and $t(c)$ to be "flipped" around also.

Claim 1.1. A direct mechanism $\{q(\cdot), t(\cdot)\}$ is incentive compatible if and only if $q(\cdot)$ is nonincreasing and

$$t(c) = \overline{c}q(\overline{c}) + \int_{\tilde{c}=c}^{\overline{c}} q'(c)\,\mathrm{d}\tilde{c}.$$

Proof. ($\Rightarrow$) Adding together the type-c and type-$\hat{c}$ monopolists' IC constraints

$$t(c) - cq(c) \geq t(\hat{c}) - cq(\hat{c}),$$
$$t(\hat{c}) - \hat{c}q(\hat{c}) \geq t(c) - \hat{c}q(c),$$

gives

$$t(c) - cq(c) + t(\hat{c}) - \hat{c}q(\hat{c}) \geq t(\hat{c}) - cq(\hat{c}) + t(c) - \hat{c}q(c)$$
$$\Leftrightarrow (\hat{c} - c)\, q(c) \geq (\hat{c} - c)\, q(\hat{c}).$$

Three cases:

- ▷ $c = \hat{c}$. The inequality holds vacuously.
- ▷ $\hat{c} > c$. Dividing by $\hat{c} - c > 0$ both sides yields $q(c) \geq \hat{q}(c)$.
- ▷ $\hat{c} < c$. Dividing by $\hat{c} - c < 0$ both sides yields $q(c) \leq \hat{q}(c)$.

In all cases, we see that $q(\cdot)$ is nonincreasing. We can write the IC constraint as

$$u(c) = \max_{\hat{c} \in [\underline{c}, \overline{c}]} t(\hat{c}) - cq(\hat{c}).$$

The envelope theorem gives us that

$$u'(c) = q(c).$$

Integrating both sides from c to $\overline{c}$ yields

$$u(c) = \int_{\tilde{c}=c}^{\overline{c}} q(c)\, \mathrm{d}\tilde{c} + C.$$

To find out C, evaluate above at $c = \overline{c}$:

$$u(\overline{c}) = \int_{\tilde{c}=\overline{c}}^{\overline{c}} q(c)\, \mathrm{d}\tilde{c} + C = C.$$

Therefore,

$$u(c) = u(\overline{c}) + \int_{\tilde{c}=c}^{\overline{c}} q(c)\, \mathrm{d}\tilde{c}.$$ ■

($\Leftarrow$) Suppose $q(\cdot)$ is nonincreasing and $t(c)$ is as given in the statement of the theorem.

1.2 Part (b)

Solve for the regulator's optimal revelation mechanism, $\{q(\cdot), t(\cdot)\}$, and compare $q(c)$ to $q^{fb}(c)$. Explain the inequality that you find.

. . . . . .

[See solution.]

1.3 Part (c)

Set $v = 3$. Compute an indirect tariff, $T(q)$, with the property that the regulator can offer the firm this schedule, $T(q)$, letting the firm freely choose its output q in exchange for $T(q)$, and the resulting choice and payments are equivalent to those from (b).

……

[See solution.]

2 Problem 2

Suppose that an airline is selling tickets to business customers (b) and tourists (t). Each category of customer draws its value from a different distribution: $F_b(\theta)$ and $F_t(\theta)$. For now, we don't make any assumptions about how these relate. At the time of purchase, the customer knows whether they are a business or tourist flyer, but the airline does not; the customer does not know his ex post type, θ. Business customers represent the proportion $\phi \in (0,1)$ of potential consumers.

The airline has decided to restrict attention to a simple pricing scheme where a ticket costs p at the time of purchase, but also has a refund provision that allows the customer to return the ticket for r after the consumer learns θ. Hence, a ticket is defined by its price and the amount that is refundable, (p, r). The airline wants to design a menu of tickets, $\{(p_b, r_b), (p_t, r_t)\}$, to maximise its profits. It has constant unit cost of serving either customer of c.

2.1 Part (a)

Write down the airline's program, including the IC and IR constraints for both types.

……

Let's think about the customers first and think backwards. Having purchased (p_j, r_j), when the customer finds out θ, he prefers to have a refund only if $r_j \geq \theta$ for $j \in \{b, t\}$. So ex post payoff is $\max\{r_j, \theta\}$. Ex ante, the type-i consumer only knows that $\theta \sim F_i(\theta)$ and has to pay p_j to purchase the contract. Hence, expected utility is

$$\mathbb{E}_{F_i(\theta)}\left[\max\{r_j, \theta\} - p_j\right] = \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_j, \theta\} F_i(\mathrm{d}\theta) - p_j$$

where I assumed $\operatorname{Supp}(F) = [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}$ (note that $F_i(\mathrm{d}\theta) \equiv f_i(\theta)\,\mathrm{d}\theta$ if F has density f_i). Normalising the outside option to be zero, the IR constraints are, $\forall i \in \{b, t\}$,

$$\int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_j, \theta\} F_i(\mathrm{d}\theta) - p_j \geq 0.$$

In order to ensure that type-i would purchase (p_i, r_i), we need that, $\forall i, j \in \{b, t\}$ with $i \neq j$,

$$\int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_i, \theta\} F_i(\mathrm{d}\theta) - p_i \geq \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_j, \theta\} F_i(\mathrm{d}\theta) - p_j,$$

which gives us the IC constraints.

With probability ϕ $(1-\phi)$, the airline faces a business customer (tourist). In each case, the airline earns the price of the ticket p_i and incurs a cost r_i in case the customer chooses the refund (which happens with probability $F_i(r_i)$) and incurs a cost mc in case the customer chooses to fly. Hence, the airline's problem is

$$\begin{aligned}
\max_{\{(p_b,r_b),(p_t,r_t)\}} \quad & \phi\left(p_b - r_b F_b\left(r_b\right) - c\left(1 - F_b\left(r_b\right)\right)\right) + \phi\left(p_t - r_t F_t\left(r_t\right) - c\left(1 - F_t\left(r_t\right)\right)\right) \\
\text{s.t.} \quad & \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\left\{r_i, \theta\right\} F_i\left(\mathrm{d}\theta\right) - p_i \geq 0,\ \forall i \in \{b,t\}, \\
& \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\left\{r_i, \theta\right\} F_i\left(\mathrm{d}\theta\right) - p_i \\
& \qquad \geq \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\left\{r_j, \theta\right\} F_i\left(\mathrm{d}\theta\right) - p_j,\ \forall i,j \in \{b,t\}, i \neq j.
\end{aligned}$$

2.2 Part (b)

Argue that the IR constraint of the business customer can be ignored if either F_b first-order stochastically dominates F_t, or F_b is a mean preserving spread of F_t.

......

Let IR_i and IC_i denote the IR and IC constraints for type-i customer, respectively. Here are two facts of life that is worth remembering.

Definition 2.1 (Stochastic dominance)**.** Let X and Y be random vectors in $\mathbb{R}^n$ with cumulative distribution functions F_X and F_Y, respectively. Then,

▷ X *first-order stochastically dominates* Y, written $X \geq_{FOSD} Y$ if and only if

$$\int_{\mathbb{R}^n} u\left(x\right) dF_X\left(x\right) \geq \int_{\mathbb{R}^n} u\left(x\right) dF_Y\left(x\right)$$

for all increasing function $u : \mathbb{R}^n \to \mathbb{R}$;

▷ X *second-order stochastically dominates* Y, written $X \geq_{SOSD} Y$ if and only if

$$\int_{\mathbb{R}^n} u\left(x\right) dF_X\left(x\right) \geq \int_{\mathbb{R}^n} u\left(x\right) dF_Y\left(x\right)$$

for every concave function $u : \mathbb{R}^n \to \mathbb{R}$.

Remark. Some more facts of life.

▷ If $n = 1$, then we get the familiar condition: $X \geq_{FOSD} Y \Leftrightarrow F_X\left(x\right) \leq F_Y\left(x\right),\ \forall x \in \mathbb{R}$.

▷ If $\mathbb{E}\left[X\right] = \mathbb{E}\left[Y\right]$, then $X \geq_{SOSD} Y \Leftrightarrow Y \geq_{MPS} X$, where the latter means that Y is a mean preserving spread of X.

▷ If u is concave, then $-u$ is convex. Thus, we can state that $X \geq_{SOSD} Y$ if and only if

$$\int_{\mathbb{R}^n} u(x)\, dF_X(x) \leq \int_{\mathbb{R}^n} u(x)\, dF_Y(x)$$

for every convex function $u : \mathbb{R}^n \to \mathbb{R}$.

In particular, the last two remarks together mean that if $X \geq_{MPS} Y$, then $\int_{\mathbb{R}^n} u(x)\, dF_X(x) \geq \int_{\mathbb{R}^n} u(x)\, dF_Y(x)$ for every convex function $u : \mathbb{R}^n \to \mathbb{R}$.

Claim 2.1. Suppose $F_b \geq_{FOSD} F_t$ or $F_b \geq_{MPS} F_t$.[1] Then, IC_b and IR_t imply IR_b.

Proof. IC_b gives us that

$$\int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_b, \theta\} F_b(\mathrm{d}\theta) - p_b \geq \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_t, \theta\} F_b(\mathrm{d}\theta) - p_t.$$

Observe that $\max\{r_t, \theta\}$ is (i) an increasing function of θ; and (ii) a convex function of θ. Thus, the definitions that I gave above imply that

$$\begin{aligned}\int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_b, \theta\} F_b(\mathrm{d}\theta) - p_b &\geq \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_t, \theta\} F_b(\mathrm{d}\theta) - p_t \\ &\geq \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\{r_t, \theta\} F_t(\mathrm{d}\theta) - p_t.\end{aligned}$$

Finally, we know that the expression on the second line is nonnegative by IR_t. Hence, we've shown that IC_b and IR_t imply IR_b. ■

2.3 Part (c)

Consider the relaxed program in which the IC constraint for the type-t consumer is ignored. Show in the relaxed program that IR for t must bind and IC for b must bind.

......

Consider the relaxed program in which we ignore IC_t. We maintain the assumptions from part (b) that either $F_b \geq_{FOSD} F_t$ or $F_b \geq_{MPS} F_t$. In particular, this means that we can ignore IR_b.

Suppose, by way of contradiction, that IR_t does not bind. Then, the airline can increase p_t. which only makes IC_b more likely to hold (it also affects IC_t but we're ignoring that), but increases its expected profit. Hence, at the optimal, IR_t must bind at any optima.

Now suppose that IC_b does not bind at some optimum. If not, we can increase p_b which leads to a higher expected profit for the airline without affecting any other constraints (we're using the act that IC_b and IR_t imply IR_b so that we can ignore the effect of increasing $p_b$here on IR_b).

[1] $\geq_{FOSD}$ and F_{MPS} are (partial) orders over distributions with respect to first-order stochastic dominance (FOSD) and mean preserving spread (MPS).

2.4 Part (d)

State and solve the relaxed program in which IR constraint for type t and IC constraint for type b bind (and IC constraint for type t is ignored). Describe the optimal refund policy for the two classes of tickets for the case when $F_b \geq_{FOSD} F_t$. Given your solution, why might this not be a such a great model of airline pricing we observe in the real world?

......

The relaxed program is

$$\begin{aligned}
\max_{\{(p_b,r_b),(p_t,r_t)\}} \quad & \phi\left(p_b - r_b F_b\left(r_b\right) - c\left(1 - F_b\left(r_b\right)\right)\right) + \phi\left(p_t - r_t F_t\left(r_t\right) - c\left(1 - F_t\left(r_t\right)\right)\right) \\
\text{s.t.} \quad & \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\left\{r_t, \theta\right\} F_t\left(\mathrm{d}\theta\right) = p_t \\
& \int_{\theta=\underline{\theta}}^{\overline{\theta}} \max\left\{r_b, \theta\right\} F_b\left(\mathrm{d}\theta\right) - p_b = \int_{\theta=\underline{\theta}}^{\theta} \max\left\{r_t, \theta\right\} F_b\left(\mathrm{d}\theta\right) - p_t.
\end{aligned}$$

Observe that:[2] for any $i, j \in \{t, b\}$,

$$\begin{aligned}
\mathbb{E}_{F_i(\theta)}\left[\max\left\{r_j, \theta\right\} - p_j\right] &= \int_{\theta=\underline{\theta}}^{\overline{\theta}} \left(\max\left\{r_j, \theta\right\} - p_j\right) F_i\left(\mathrm{d}\theta\right) \\
&= \int_{\theta\in[\underline{\theta}, r_j]} \max\left\{r_j, \theta\right\} F_i\left(\mathrm{d}\theta\right) + \int_{\theta\in(r_j, \overline{\theta}]} \max\left\{r_j, \theta\right\} F_i\left(\mathrm{d}\theta\right) - p_j \\
&= r_j F_i\left(r_j\right) + \int_{\theta\in(r_j, \overline{\theta}]} \theta F_i\left(\mathrm{d}\theta\right) - p_j.
\end{aligned}$$

Thus, the constraints give us that

$$\begin{aligned}
p_t &= r_t F_t\left(r_t\right) + \int_{\theta\in(r_t, \overline{\theta}]} \theta F_t\left(\mathrm{d}\theta\right), \\
p_b &= \left[r_b F_b\left(r_b\right) + \int_{\theta\in(r_b, \overline{\theta}]} \theta F_b\left(\mathrm{d}\theta\right)\right] - \left[r_t F_b\left(r_t\right) + \int_{\theta\in(r_t, \overline{\theta}]} \theta F_b\left(\mathrm{d}\theta\right)\right] + p_t \\
&= r_b F_b\left(r_b\right) + \int_{\theta\in(r_b, \overline{\theta}]} \theta F_b\left(\mathrm{d}\theta\right) + r_t\left(F_t\left(r_t\right) - F_b\left(r_t\right)\right) + \int_{\theta\in(r_t, \overline{\theta}]} \theta\left(F_t\left(\mathrm{d}\theta\right) - F_b\left(\mathrm{d}\theta\right)\right).
\end{aligned}$$

Integration by parts gives us that

$$\begin{aligned}
\int_{\theta\in(r_t, \overline{\theta}]} \theta\left(F_t\left(\mathrm{d}\theta\right) - F_b\left(\mathrm{d}\theta\right)\right) &= \left[\theta\left(F_t\left(\theta\right) - F_b\left(\theta\right)\right)\right]_{r_t^+}^{\overline{\theta}} - \int_{\theta\in(r_t, \overline{\theta}]} \left(F_t\left(\theta\right) - F_b\left(\theta\right)\right) \mathrm{d}\theta \\
&= -r_t\left(F_t\left(r_t^+\right) - F_b\left(r_t^+\right)\right) - \int_{\theta\in(r_t, \overline{\theta}]} \left(F_t\left(\theta\right) - F_b\left(\theta\right)\right) \mathrm{d}\theta,
\end{aligned}$$

[2] If you're wondering about the half open intervals in the limits, recall that CDFs are right-continuous and we haven't technically assumed that F is continous.

where we $F_i\left(r_i^+\right) = \lim_{x\uparrow r_i} F_i(x)$. To simplify, suppose that F is (absolutely) continuous, then $F_i\left(r_i^+\right) = F_i(r_i)$ and so we can write

$$p_b = r_b F_b(r_b) + \int_{\theta=r_b}^{\overline{\theta}} \theta f_b(\theta)\,\mathrm{d}\theta - \int_{\theta=r_t}^{\overline{\theta}} \left(F_t(\theta) - F_b(\theta)\right)\mathrm{d}\theta.$$

Substituting these expressions into the objective function yields

$$\phi \int_{\theta=r_b}^{\overline{\theta}} (\theta - c) f_b(\theta)\,\mathrm{d}\theta - \phi \int_{\theta=r_t}^{\overline{\theta}} \left(F_t(\theta) - F_b(\theta)\right)\mathrm{d}\theta + (1-\phi)\left(\int_{\theta=r_t}^{\overline{\theta}} (\theta - c) f_t(\theta)\,\mathrm{d}\theta\right).$$

The problem is now to maximise above with respect to r_b and r_t. Using Leibniz's rule, the first-order condition with respwct to r_b and r_t are

$$\begin{aligned}
0 &= -\phi\left(r_b^* - c\right) f_b\left(r_b^*\right), \\
0 &= \phi\left(F_t\left(r_t^*\right) - F_b\left(r_t^*\right)\right) - (1-\phi)\left(r_t^* - c\right) f_t\left(r_t^*\right).
\end{aligned}$$

Assuming $f_b, f_t > 0$ (i.e. full support), then

$$\begin{aligned}
r_b^* &= c, \\
r_t^* &= c + \frac{\phi}{1-\phi}\frac{F_t\left(r_t^*\right) - F_b\left(r_t^*\right)}{f_t\left(r_t^*\right)} \geq r_b^*,
\end{aligned}$$

where the inequality follows from the fact that $F_b \geq_{FOSD} F_t$. From IC_b, using integration by parts,

$$\begin{aligned}
p_b^* - p_t^* &= \left[r_b^* F_b\left(r_b^*\right) + \int_{\theta=r_b^*}^{\overline{\theta}} \theta f_b(\theta)\,\mathrm{d}\theta\right] - \left[r_t^* F_b\left(r_t^*\right) + \int_{\theta=r_t^*}^{\overline{\theta}} \theta f_b(\theta)\,\mathrm{d}\theta\right] \\
&= r_b^* F_b\left(r_b^*\right) - r_t^* F_b\left(r_t^*\right) + \int_{\theta=r_b^*}^{r_t^*} \theta f_b(\theta)\,\mathrm{d}\theta \\
&= r_b^* F_b\left(r_b^*\right) - r_t^* F_b\left(r_t^*\right) + \left(\left[\theta F_b(\theta)\right]_{r_b^*}^{r_t^*} - \int_{\theta=r_b^*}^{r_t^*} F_b(\theta)\,\mathrm{d}\theta\right) \\
&= -\int_{\theta=r_b^*}^{r_t^*} F_b(\theta)\,\mathrm{d}\theta \leq 0.
\end{aligned}$$

Observe the following.

- ▷ since $r_b^* \geq r_t^*$, tourist customer is less likely to take a flight once they learn their ex post type. This outcome, that tourists fly less than business customers is not surprising given they have lower valuations on average ($\Leftarrow F_b \geq_{FOSD} F_t$).
- ▷ but $p_t^* \geq p_b^*$; i.e. tourists pay more than business customers. Business customers do not value refund terms as much since they are more certain they will fly, and hence they have a lower purchase price but a lower refund as well. In practice, we observe business customers buying more expensive tickets without refund penalties and tourists buying cheaper non-refundable tickets.

Remark 2.1. We can show that IC_t is slack at the optimal. Since IC_b binds, as shown above, we have that

$$p_t^* - p_b^* = \int_{\theta=r_b^*}^{r_t^*} F_b(\theta)\,\mathrm{d}\theta.$$

IC_t is given by

$$r_t^* F_t(r_t^*) + \int_{\theta=r_t^*}^{\overline{\theta}} \theta f_t(\theta)\,\mathrm{d}\theta - p_t^* \geq r_b^* F_t(r_b^*) + \int_{\theta=r_b^*}^{\overline{\theta}} \theta f_t(\theta)\,\mathrm{d}\theta - p_b^*,$$

which we can rewrite as

$$\begin{aligned} p_t^* - p_b^* &\leq \left[r_t^* F_t(r_t^*) + \int_{\theta=r_t^*}^{\overline{\theta}} \theta f_t(\theta)\,\mathrm{d}\theta\right] - \left[r_b^* F_t(r_b^*) + \int_{\theta=r_b^*}^{\overline{\theta}} \theta f_t(\theta)\,\mathrm{d}\theta\right] \\ &= \int_{\theta=r_b^*}^{r_t^*} F_t(\theta)\,\mathrm{d}\theta. \end{aligned}$$

Given that IC_b binds, above holds if and only if

$$\int_{\theta=r_b^*}^{r_t^*} \left(F_b(\theta) - F_t(\theta)\right)\mathrm{d}\theta \leq 0.$$

Toward a contradiction, suppose this inequality does not hold, and consider the alternative offer of $\tilde{r}_t = \tilde{r}_b = c$.

Remark 2.2. [???]

The surplus of the t type is higher in the alternative menu and the rent to type b is smaller because

$$\int_{r_t}^{\overline{\theta}} \left(F_t(\theta) - F_b(\theta)\right)\mathrm{d}\theta > \int_{r_t}^{\overline{\theta}} \left(F_t(\theta) - F_b(\theta)\right)\mathrm{d}\theta + \int_{r_t}^{\overline{\theta}} \left(F_t(\theta) - F_b(\theta)\right)\mathrm{d}\theta$$

Hence, in the relaxed program, suppose that the solution violates IC_t leads to a contradiction.

2.5 Part (e)

Suppose instead that F_b is a mean-preserving spread of F_t. In particular, suppose that $F_b(b) = \theta$ on $[0, 1]$, but the tourists have a triangular distribution:

$$F_t(\theta) = \begin{cases} 2\theta^2 & \text{if } \theta < 1/2 \\ 4\theta - 2\theta^2 - 1 & \text{if } \theta > 1/2 \end{cases}.$$

Let $c = 3/8$ and $\phi = 1/2$. Describe the optimal refund policy for the two classes of tickets. [Hint: Look for the solution toe the first-order condition for r_t where $r_t \leq 1/2$.] Describe your solution. Does this do a better job at fitting actual airline pricing?

……

Given the functional form

$$f_t(\theta) = \begin{cases} 4\theta & \text{if } \theta < 1/2 \\ 4(1-\theta) & \text{if } \theta > 1/2 \end{cases}.$$

Recall from part (d) that

$$r_b^* = c = \frac{3}{8}$$

and

$$\begin{aligned} r_t^* &= c + \frac{\phi}{1-\phi} \frac{F_t(r_t^*) - F_b(r_t^*)}{f_t(r_t^*)} \\ &= \frac{3}{8} + \frac{1/2}{1/2} \frac{2(r_t^*)^2 - r_t^*}{4r_t^*} \\ \Leftrightarrow 0 &= 4(r_t^*)^2 - r_t^* \\ &= r_t^*(4r_t^* - 1) \\ \Leftrightarrow r_t^* &= 0 \text{ of } \frac{1}{4} \leq \frac{1}{2} \end{aligned}$$

so we've verified the hint. Notice that

$$r_b^* = \frac{3}{8} > \frac{1}{4} = r_t^*.$$

The refund policy is efficient for the business traveler, but encourages too much travel for the tourist (because the refund is less generous). The business traveler pays more for the better refund policy up front because she values the option to cancel her flight given the greater uncertainty in θ_b. This seems to be closer to the real world practice of selling expensive refundable tickets to business customers and cheap, nonrefundable tickets to tourists. Still, we have assumed that the means are the same, which is probably not realistic, so we would want to mix the two models (FOSD and something like SOSD into the same model.

3 Problem 3 (from Balazs Szentes)

Consider a trade of a divisible good between a seller (principal) and a buyer (agent). The buyer's payoff is $vq - t$, where v is her valuation, q is the quantity traded, and t is the payment to the seller. The seller's payoff is $t - q^2/2$, where $q^2/2$ is the cost of producing q units of the good.

Suppose that the buyer's valuation is

$$v = \lambda\theta + (1 - \lambda)\,\varepsilon,$$

where θ is the buyer's private information and ε is a publicly observable and contractible shock. Assume that both θ and ε are independently and uniformly distributed on $[0, 1]$. The seller can offer a contract to the buyer prior to the realisation of ε, but θ is privately known by the buyer at the time of the contract. The buyer's outside option is zero.

3.1 Part (a)

What is the set of contracts to which it is without loss of generality to restrict attention?

......

By the revelation principle, it is without loss of generality to restrict attention to direct mechanisms and truthful equilibria. Specifically, a direct mechanism is $\{q(\cdot), t(\cdot)\}$. where $q : [0, 1]^2 \to \mathbb{R}_+$ and $t : [0, 1]^2 \to \mathbb{R}$, where $q\left(\hat{\theta}, \varepsilon\right)$ is the quantity traded if $\hat{\theta}$ is reported and ε is later observed, and $t\left(\hat{\theta}, \varepsilon\right)$ is the corresponding payment.

3.2 Part (b)

Consider an incentive compatible contract and let $U(\theta)$ denote the equilibrium payoff of the buyer with type θ. What is $U(\theta) - U(0)$?

......

Let $U(\theta)$ be the interim expected utility from truthful reporting for a type-θ buyer:Recall from class that

$$U(\theta) = U(\underline{\theta}) + \int_{s=\underline{\theta}}^{\theta} u_\theta\left(q(s), s\right) \mathrm{d}s.$$

We need to modify this slightly here since we have ε. Note

$$\begin{aligned} U(\theta) &\coloneqq \mathbb{E}_\varepsilon\left[\theta q(\theta, \varepsilon) - t(\theta, \varepsilon)\right] = \int_0^1 \left[\theta q(\theta, \varepsilon) - t(\theta, \varepsilon)\right] \mathrm{d}\varepsilon, \\ u\left(q(\theta, \varepsilon), \theta\right) &\coloneqq \mathbb{E}_\varepsilon\left[\left(\lambda\theta + (1 - \lambda)\,\varepsilon\right) q(\theta, \varepsilon)\right] \\ &\Rightarrow u_\theta = \mathbb{E}_\varepsilon\left[\lambda q(\theta, \varepsilon)\right]. \end{aligned}$$

Then,

$$U(\theta) = U(0) + \lambda \int_{s=0}^{\theta} \mathbb{E}_\varepsilon\left[q(s, \varepsilon)\right] \mathrm{d}s.$$

3.3 Part (c)

Use your result in part (b) to express the buyer's ex ante expected payoff, $\mathbb{E}_\theta [U(\theta)]$.

……

Given what we found in part (b),

$$\begin{aligned}
\mathbb{E}_\theta [U(\theta)] &= U(0) + \mathbb{E}_\theta \left[\int_{s=0}^{\theta} \mathbb{E}_\varepsilon [\lambda q(s,\varepsilon)] \, \mathrm{d}s \right] \\
&= U(0) + \underbrace{\int_{\theta=0}^{1} \int_{s=0}^{\theta}}_{0 \le s \le \theta \le 1} \mathbb{E}_\varepsilon [\lambda q(s,\varepsilon)] \, \mathrm{d}s \mathrm{d}\theta \\
&= U(0) + \underbrace{\int_{s=0}^{1} \int_{\theta=s}^{1}}_{0 \le s \le \theta \le 1} \mathrm{d}\theta \mathbb{E}_\varepsilon [\lambda q(s,\varepsilon)] \, \mathrm{d}s \\
&= U(0) + \lambda \int_{\theta=0}^{1} (1-\theta) \int_{0}^{1} q(\theta,\varepsilon) \, \mathrm{d}\varepsilon \mathrm{d}\theta \\
&= U(0) + \lambda \int_{\theta=0}^{1} \int_{0}^{1} (1-\theta) q(\theta,\varepsilon) \, \mathrm{d}\varepsilon \mathrm{d}\theta \\
&\equiv U(0) + \lambda \mathbb{E}_{\theta,\varepsilon} [(1-\theta) q(\theta,\varepsilon)] .
\end{aligned}$$

where we exchanged integrals in the penultimate line and replaced the dummy variable s with θ in the last line.

3.4 Part (d)

Express the seller's expected payoff as the difference between social surplus and the buyer's payoff.

……

Seller's expected payoff is

$$\mathbb{E}_{\theta,\varepsilon} \left[t(\theta,\varepsilon) - \frac{1}{2} (q(\theta,\varepsilon))^2 \right],$$

where Recall from class that (combining envelope characterisation and implementability) $t(\theta) = u(q(\theta), \theta) - U(\theta)$. In this context, this becomes

$$\begin{aligned}
\mathbb{E}_\varepsilon [t(\theta,\varepsilon)] &= u(q(\theta,\varepsilon), \theta) - U(\theta), \\
\mathbb{E}_{\theta,\varepsilon} [t(\theta,\varepsilon)] &= \mathbb{E}_{\theta,\varepsilon} [(\lambda\theta + (1-\lambda)\varepsilon) q(\theta,\varepsilon)] - \mathbb{E}_\theta [U(\theta)] \\
&= \mathbb{E}_{\theta,\varepsilon} [q(\theta,\varepsilon)(\lambda\theta + (1-\lambda)\varepsilon - \lambda(1-\theta))] - U(0).
\end{aligned}$$

Hence, we can rewrite the seller's expected payoff as

$$\mathbb{E}_{\theta,\varepsilon} \left[q(\theta,\varepsilon)(\lambda\theta + (1-\lambda)\varepsilon - \lambda(1-\theta)) - \frac{1}{2} (q(\theta,\varepsilon))^2 \right] - U(0).$$

3.5 Part (e)

Use your result is part (d) to derive the optimal quantity produced as a function of θ and ε.

......

We maximise the expression we derived in (d) pointwise over (θ, ε). So our problem is

$$\max_{\tilde{q}(\theta,\varepsilon)\geq 0} \quad \tilde{q}(\theta,\varepsilon)\left(\lambda\theta + (1-\lambda)\varepsilon - \lambda(1-\theta)\right) - \frac{1}{2}\left(\tilde{q}(\theta,\varepsilon)\right)^2.$$

The objective is strictly concave so that the first-order condition is necessary and sufficient for an interior maximum.

$$\lambda\theta + (1-\lambda)\varepsilon - \lambda(1-\theta) - q(\theta,\varepsilon) = 0.$$

Since $\tilde{q}(\theta,\varepsilon) \geq 0$, it follows that

$$q(\theta,\varepsilon) = \max\left\{\lambda\theta + (1-\lambda)\varepsilon - \lambda(1-\theta), 0\right\}.$$

3.6 Part (f)

Suppose that ε is not observable by the seller, but instead remains private information to the buyer. Otherwise, the timing is as before: the contract is offered after the buyer has observed θ but before the buyer observes ε. Using a result from class, argue that the same allocation in (e) will be implemented by the seller.

......

Recall from the lecture on dynamic screening that, in a two-period consumption model, we know that, if $G_{\theta_1}(\theta_2|\theta_1) = 0$, then q_2 is set at first best. Here, because θ and ε are independently distributed (i.e. θ_1 and θ_2 independently distributed), we obtain the first-best allocation. Effectively, the seller "pre-sells" the efficient quantity related to ε, and the buyer obtains no information rents from private information arising from ε.

4 Problem 4

Consider an IPV auction environment with two bidders, one "strong" and one "weak". The strong bidder's type is $\theta_s \sim \text{Uniform}\,[2,3]$ and weak bidder's type is $\theta_w \sim \text{Uniform}\,[0,1]$.

4.1 Part (a)

Compute the equilibrium bidding functions in the second-price auction. Compute the expected revenue to the seller.

. . . . . .

Even if the support of value for each bidder differs, it remains the case that, in a second-price auction, it is a weakly dominant strategy for each bidder to bid his or her own true value. Thus,

$$\bar{b}\,(\theta_i) = \theta_i,\ \forall i \in \{s, w\}\,.$$

The strong type always win the auction and the expected price is the expected value of the weak type; i.e. 1/2. The expected revenue is also 1/2.

Remark 4.1. For the sake of completeness, here's the proof that bidding true value is a weakly dominant strategy. Observe that the proof does not refer to the support of the other's bids!

Recall that the rule is that when you win, you pay the highest bid among others, denoted $v_{-i} = \max\{v_j\}_{j \neq i}$. Suppose the bidder i bids $r_i < v_i$.

If $r_i < v_i < v_{-i}$, then the bidder does not win and obtains zero utility;

If $r_i < v_{-i} < v_i$, then reporting v_i gives the bidder utility of

$$v_i - v_{-i} > 0$$

and reporting r_i gives zero utility.

If $v_{-i} < r_i < v_i$, then the bidder wins whether he reports r_i or v_i and the utility is given by

$$v_i - v_{-i} > 0$$

in both cases.

Hence, bidding v_i is a weakly dominant strategy.

Put differently, a bidder wants to win if $v_i > v_{-i}$ and wants to lose if $v_i < v_{-i}$. By bidding v_i, this is exactly what would happen—you will win when you would want to and lose when you would want to. Hence, every bidder bidding their own values is an equilibrium in the second-price auction.

4.2 Part (b)

Compute the equilibrium bidding functions in the first-price auction for the equilibrium in which the weak player bids $\bar{b}_w\,(\theta_w) = \theta_w$. Compute the expected revenue to the seller.

. . . . . .

Given that the weak player bids according to $\bar{b}_w(\theta_w) = \theta_w$, the type-$\theta_s$ strong player's problem is

$$\max_{b_s \in [0,1]} F_w(b_s)(\theta_s - b_s) = \max_{b_s \in [0,1]} b_s(\theta_s - b_s).$$

Note that since $\theta_w \in [0,1]$, it is without loss to bound $b_s \leq 1$. The derivative with respect to b_s is

$$\theta_s - 2b_s$$

which is positive for all $\theta_2 \in [2,3]$ and $b_s \in [0,1]$. Hence, bidding $b_s = 1$ is optimal. Given this, type-θ_w weak player loses with probability one so that he is indifferent across any bidding strategy; in particular, this means that bidding true value is optimal.

The expected revenue in this case is 1.

Remark 4.2. There are other equilibria that generate even more revenue. e.g. suppose the weak player always bids $2 - \epsilon$.

4.3 Part (c)

Compare the expected revenues. Explain why they are the same (i.e., explain how the revenue equivalence theorem applies to this setting), or explain why they are different (i.e., why the revenue equivalence theorem does not apply to this situation).

. . . . . .

Clearly, the revenues are different! In other words, revenue equivalence does not hold here. This is because the expected utility of the worst type of strong buyer $\theta_s = 2$ is not the same in the two auction formats. In the second-price auction, expected utility is $2 - 1 = 3$; in the first-price auction, expected utility is $2 - 1 = 1$.

Remark 4.3. In this example, the high type bidder always wins. If the supports overlap, however, this would not generally hold. The weak bidder might sometimes win with a lower type than the strong bidder. Consequently, the revenue equivalence theorem cannot be applied in that circumstance either. Interesting results comparing the two auction formats exist, however. See Maskin and Riley (ReStud, 2000).

5 Problem 5: JR, Exercise 9.8

In a first-price, all-pay auction, the bidders simultaneously submit sealed bids. The highest bid wins the object and every bidder pays the seller the amount of his bid. Consider the independent private values model with symmetric bidders whose values θ_i are each distributed according to the distribution function F, with density f.

5.1 Part (a)

Find the unique symmetric equilibrium bidding function.

......

The derivation is isomorphic to the first-price auction.

Denote $\hat{b} : [0,1] \to [0,1]$ as an increasing function that represents the symmetric equilibrium strategy. The expected utility from bidding $r \in [0,1]$ given value v is now given by

$$u(r,v) = F^{N-1}(r)\, v - \hat{b}(r),$$

where $\hat{b}$ is no longer multiplied by the probability of winning to reflect the fact that bids must be paid no matter the outcome.

$$\begin{aligned}
\left.\frac{\partial u(r,v)}{\partial r}\right|_{r=v} &= (N-1) f(r) F^{N-2}(r)\, v - \hat{b}'(r)\Big|_{r=v} \\
&= (N-1) f(v) F^{N-2}(r)\, v - \hat{b}'(v) \\
&= \frac{dF^{N-1}(v)}{dv} v - \hat{b}'(v).
\end{aligned}$$

Equating this derivative with zero, we obtain

$$\begin{aligned}
\hat{b}'(v) &= \frac{dF^{N-1}(x)}{dv} v \\
\Rightarrow \int_0^v \hat{b}'(x)\, dx &= \int_0^v x \frac{dF^{N-1}(x)}{dx} dx = \int_0^v x dF^{N-1}(x) \\
\Rightarrow \hat{b}(v) &= \int_0^v x dF^{N-1}(x),
\end{aligned}$$

where we used $\hat{b}(0) = 0$.

Clearly, $\hat{b}$ is strictly increasing given that $f(v) > 0$. Moreover,

$$\begin{aligned}
\frac{\partial u(r,v)}{\partial r} &= (N-1) f(r) F^{N-2}(r)\, v - \hat{b}'(r) \\
&= (N-1) f(r) F^{N-2}(r)\, v - (N-1) f(r) F^{N-2}(r)\, r \\
&= (N-1) f(r) F^{N-2}(r)(v-r).
\end{aligned}$$

The derivative is positive if $v > r$ and negative if $v < r$; i.e. $u(r,v)$ is strictly quasiconcave. Hence, $u(r,v)$ achieves a unique maximum at $r = v$.

5.2 Part (b)

Do bidders bid higher or lower than in a first-price, all pay auction?

......

In the standard first-price auction, the biding function was

$$\begin{aligned}\hat{b}_{FPA}(v) &= \frac{1}{F^{N-1}(v)}\int_0^v \frac{dF^{N-1}(x)}{dx}xdx \\ &= \int_0^v xd\frac{F^{N-1}(x)}{F^{N-1}(v)}.\end{aligned}$$

Hence,

$$\hat{b}_{FPA}(v) \equiv \frac{1}{F^{N-1}(v)}\hat{b}(v).$$

Since $(F(v))^{N-1} \in (0,1)$, we realise that the bidders bid higher in the standard first-price auction; i.e.

$$\hat{b}_{FPA} \geq \hat{b}(v).$$

In a first-price auction, a bidder has an incentive to raise his bid to increase his chances of winning the auction, yet he has an incentive to reduce his bid to lower the price he pays when he does win. In an all-pay first-price auction, the incentive to reduce the bid is even greater since the bidder must pay it no matter the outcome of the auction. Hence, bidders bid lower in the all-pay case, relative to the standard case.

5.3 Part (c)

Find an expression for the seller's expected revenue.

......

Each bidder bids according to $\hat{b}(v_i)$ and the seller receives money from every bidder. Thus, the seller's expected revenue from the first-price all-pay auction (FPAPA) is

$$R_{FPAPA} = N\mathbb{E}\left[\hat{b}(v_i)\right].$$

Given symmetry,

$$R_{FPAPA} = N\int_0^1 \left(\int_0^v xdF^{N-1}(x)\right) f(v)\,dv.$$

Since

$$\begin{aligned}dF^{N-1}(x) &= \frac{dF^{N-1}(x)}{dx}dx \\ &= (N-1)\,f(x)\,F^{N-2}(x)\,dx,\end{aligned}$$

we can write

$$\begin{aligned} R_{FPAPA} &= N \int_0^1 \left(\int_0^v x \left(N-1\right) f\left(x\right) F^{N-2}\left(x\right) dx \right) f\left(v\right) dv \\ &= N\left(N-1\right) \int_0^1 \int_0^v x F^{N-2}\left(x\right) f\left(x\right) f\left(v\right) dx dv. \end{aligned}$$

5.4 Part (d)

Both with and without using the revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

. . . . . .

In a first-price all-pay auction, the probability assignment function is the same as in the standard first-price auction; i.e. whoever bids highest wins. Moreover, bidders whose value is zero would bid zero so $\bar{c}_i\left(0\right) = 0$—the same as in the standard first-price auction. Thus, by the Revenue Equivalence Theorem, the seller's expected revenue should be the same between the two auctions.

Let us verify this explicitly. We exchange the integral, taking care about the bounds:

$$\begin{aligned} R_{FPAPA} &= N\left(N-1\right) \int_0^1 x F^{N-2}\left(x\right) f\left(x\right) \left(\int_x^1 f\left(v\right) dv \right) dx \\ &= N\left(N-1\right) \int_0^1 x F^{N-2}\left(x\right) f\left(x\right) \left(1 - F\left(x\right)\right) dx \\ &= R_{FPA}. \end{aligned}$$

6 Problem 6: JR, Exercise 9.9

Suppose there are just two bidders. In a second-price, all-pay auction, the two bidders simultaneously submit sealed bids. The highest bid wins the objective and both bidders pay the second-highest bid.

6.1 Part (a)

Find the unique symmetric equilibrium bidding function. Interpret.

......

Let F_j denote player j's CDF. If player i bids r_i given v_i, then there is probability $F_j(r_i)$ that i wins. In this case, he obtains the value v_i less the second highest bid; i.e. bid by player j, who we assume bids according to his true value, v_j. With probability $1 - F_j(r_i)$, player i loses, in which case, he has to pay $\hat{b}(r_i)$ (his own bid that is lower than j's). Hence, we can write player i's expected utility as

$$\begin{aligned}
u_i(r_i, v_i) &= F(r_i)\, v_i - \int_0^1 \min\left\{\hat{b}(r_i), \hat{b}(v_j)\right\} f(v_j)\, dv_j \\
&= F(r_i)\, v_i - \left(\int_0^{r_i} \hat{b}(v_j) f(v_j)\, dv_j + \int_{r_i}^1 \hat{b}(r_i) f(v_j)\, dv_j\right) \\
&= F(r_i)\, v_i - \int_0^{r_i} \hat{b}(v_j) f(v_j)\, dv_j - (1 - F(r_i))\, \hat{b}(r_i)
\end{aligned}$$

where we assumed symmetry; i.e. $F_i(r) = F(r)$ for all $i \in \{1, 2\}$, and assumed that $\hat{b}(\cdot)$ is strictly increasing. We want this function to be maximised when $r_i = v_i$:

$$\begin{aligned}
\left.\frac{\partial u_i(r_i, v_i)}{\partial r_i}\right|_{r_i = v_i} &= \left. f(r_i)\, v_i - \hat{b}(r_i) f(r_i) + f(r_i)\, \hat{b}(r_i) - (1 - F(r_i))\, \hat{b}'(r_i)\right|_{r_i = v_i} \\
&= f(v_i)\, v_i - f(v_i)\, \hat{b}(v_i) - F(v_i)\, \hat{b}'(v_i) + \hat{b}(v_i) f(v_i) \\
&= f(v_i)\, v_i - (1 - F(v_i))\, \hat{b}'(v_i)
\end{aligned}$$

Equating the derivative to be zero

$$\begin{aligned}
\hat{b}'(v_i) &= \frac{f(v_i)}{1 - F(v_i)} v_i \\
\Rightarrow \int_0^{v_i} \hat{b}'(x)\, dx &= \int_0^{v_i} \frac{f(x)}{1 - F(x)} x dx \\
\Rightarrow \hat{b}(v_i) &= \int_0^{v_i} \frac{x f(x)}{1 - F(x)} dx \\
&= \int_0^{v_i} \frac{1}{1 - F(x)} x \frac{dF(x)}{dx} dx \\
&= \int_0^{v_i} \frac{x}{1 - F(x)} dF(x).
\end{aligned}$$

Since $f(v_i) > 0$, it is immediate that $\hat{b}$ is strictly increasing. Moreover, since

$$\begin{aligned}\frac{\partial u_i(r_i, v_i)}{\partial r_i} &= f(r_i)\,v_i - (1 - F(r_i))\,\hat{b}'(r_i)\\ &= f(r_i)\,v_i - (1 - F(r_i))\,\frac{f(r_i)}{(1 - F(r_i))} r_i\\ &= f(r_i)(v_i - r_i).\end{aligned}$$

and we have $f(r_i) > 0$, we conclude that $u_i(r_i, v_i)$ is maximised at $r_i = v_i$.

6.2 Part (b)

Do bidders bid higher or lower than in a first-price, all pay auction?

......

The bidding function in this case is

$$\hat{b}_{SPAPA}(v) = \int_0^v \frac{xf(x)}{1 - F(x)} dx.$$

The bidding function from standard first-price and second-price auctions with $N = 2$ are

$$\begin{aligned}\hat{b}_{FPA}(v) &= \int_0^v \frac{xf(x)}{F(v)} dx,\\ \hat{b}_{SPA}(v) &= v,\\ \hat{b}_{FPAPA}(v) &= \int_0^v xf(x)\,dx.\end{aligned}$$

So, we can see immediately that

$$\hat{b}_{SPAPA}(v) \geq \hat{b}_{FPAPA}(v).$$

That is, it remains the case that bidders bid more aggressively in the second-price auction as, conditional on winning the auction, the bids do not affect the amount the winning bidder pays.

Consider

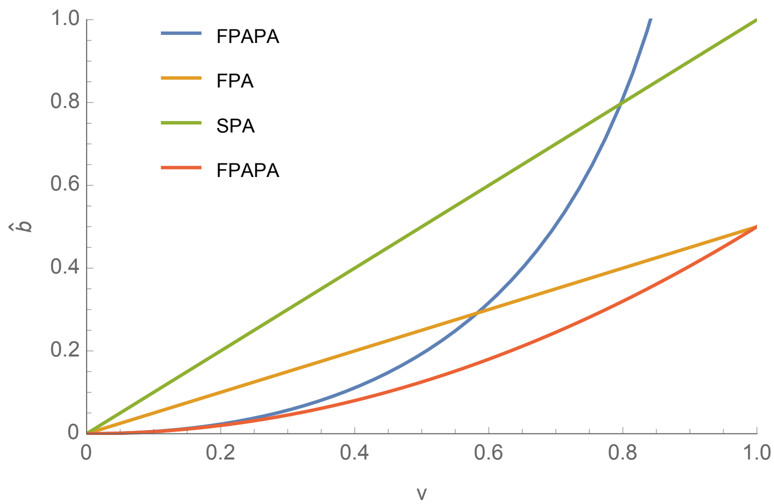
$$\begin{aligned}\hat{b}_{SPAPA}(v) - \hat{b}_{FPA}(v) &= \int_0^v \frac{xf(x)}{1 - F(x)} dx - \int_0^v \frac{xf(x)}{F(v)} dx\\ &= \int_0^v xf(x)\left(\frac{1}{1 - F(x)} - \frac{1}{F(v)}\right) dx\\ &= \int_0^v xf(x)\left(\frac{F(v) - (1 - F(x))}{(1 - F(x))\,F(v)}\right) dx\\ &= \int_0^v \frac{xf(x)}{(1 - F(x))\,F(v)} (F(v) - (1 - F(x)))\,dx.\end{aligned}$$

The sign of the difference depends upon F.

Consider the case in which values are distributed uniformly in the interval $[0, 1]$; so $F(v) = v$ and $f(v) = 1$. Then,

$$\begin{aligned}
\hat{b}_{SPAPA}(v) &= \int_0^v \frac{x}{1-x} dx = \int_0^v \left(-1 + \frac{1}{1-x}\right) dx \\
&= [-x - \ln(1-x)]_0^v = -v - \ln(1-v), \\
\hat{b}_{FPA}(v) &= \int_0^v \frac{x}{v} dx = \frac{v}{2}, \\
\hat{b}_{SPA}(v) &= v, \\
\hat{b}_{FPAPA}(v) &= \int_0^v x dx = \frac{v^2}{2}.
\end{aligned}$$

See the figure below.



6.3 Part (c)

Find an expression for the seller's expected revenue.

......

In this case, both bidders pay the second highest bid. But, ex ante, neither v_1 nor v_2 is known, so the seller's expected revenue is given by

$$R_{SPAPA} = 2 \int_0^1 \int_0^1 \min\left\{\hat{b}(v_1), \hat{b}(v_2)\right\} f(v_1) f(v_2) dv_1 dv_2.$$

Then,

$$
\begin{aligned}
R_{SPAPA} &= 2\int_0^1 \int_0^1 \min\left\{\hat{b}\left(v_1\right), \hat{b}\left(v_2\right)\right\} f\left(v_1\right) dv_1 f\left(v_2\right) dv_2 \\
\Leftrightarrow \frac{1}{2} R_{SPAPA} &= \int_0^1 \left[\int_0^{v_2} \hat{b}\left(v_1\right) f\left(v_1\right) dv_1 + \int_{v_2}^1 \hat{b}\left(v_2\right) f\left(v_1\right) dv_1\right] f\left(v_2\right) dv_2 \\
&= \int_0^1 \left[\int_0^{v_2} \hat{b}\left(v_1\right) f\left(v_1\right) dv_1 + \hat{b}\left(v_2\right) \int_{v_2}^1 f\left(v_1\right) dv_1\right] f\left(v_2\right) dv_2 \\
&= \int_0^1 \int_0^{v_2} \hat{b}\left(v_1\right) f\left(v_1\right) dv_1 f\left(v_2\right) dv_2 + \int_0^1 \hat{b}\left(v_2\right)\left(1 - F\left(v_2\right)\right) f\left(v_2\right) dv_2 \\
&= \int_0^1 \left(\int_{v_1}^1 f\left(v_2\right) dv_2\right) \hat{b}\left(v_1\right) f\left(v_1\right) dv_1 + \int_0^1 \hat{b}\left(v_2\right)\left(1 - F\left(v_2\right)\right) f\left(v_2\right) dv_2 \\
&= \int_0^1 \left(1 - F\left(v_1\right)\right) \hat{b}\left(v_1\right) f\left(v_1\right) dv_1 + \int_0^1 \hat{b}\left(v_2\right)\left(1 - F\left(v_2\right)\right) f\left(v_2\right) dv_2 \\
&= \int_0^1 \left(1 - F\left(v\right)\right) \hat{b}\left(v\right) f\left(v\right) dv + \int_0^1 \hat{b}\left(v\right)\left(1 - F\left(v\right)\right) f\left(v\right) dv \\
\Leftrightarrow R_{SPAPA} &= 4\int_0^1 \hat{b}\left(v\right)\left(1 - F\left(v\right)\right) f\left(v\right) dv.
\end{aligned}
$$

Substituting the functional form for $\hat{b}\left(v\right)$ yields

$$
\begin{aligned}
R_{SPAPA} &= 4\int_0^1 \int_0^v \frac{x f\left(x\right)}{1 - F\left(x\right)} dx \left(1 - F\left(v\right)\right) f\left(v\right) dv \\
&= 4\int_0^1 \left(\int_x^1 \left(1 - F\left(v\right)\right) f\left(v\right) dv\right) \frac{x f\left(x\right)}{1 - F\left(x\right)} dx.
\end{aligned}
$$

Consider the inner integration first:

$$
\begin{aligned}
\int_x^1 \left(1 - F\left(v\right)\right) f\left(v\right) dv &= \int_x^1 f\left(v\right) dv - \int_x^1 F\left(v\right) f\left(v\right) dv \\
&= \left(1 - F\left(x\right)\right) - \int_x^1 F\left(v\right) f\left(v\right) dv.
\end{aligned}
$$

Using integration by parts, we can write the the second term on the right-hand side as

$$
\begin{aligned}
\int_x^1 F\left(v\right) f\left(v\right) dv &= \left[F\left(v\right) F\left(v\right)\right]_x^1 - \int_x^1 F\left(v\right) f\left(v\right) dv \\
&= \left(1 - F^2\left(x\right)\right) - \int_x^1 F\left(v\right) f\left(v\right) dv \\
\Leftrightarrow \int_x^1 F\left(v\right) f\left(v\right) dv &= \frac{1}{2}\left(1 - F^2\left(x\right)\right) = \frac{1}{2}\left(1 + F\left(x\right)\right)\left(1 - F\left(x\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_x^1 \left(1 - F\left(v\right)\right) f\left(v\right) dv &= \left(1 - F\left(x\right)\right) - \frac{1}{2}\left(1 + F\left(x\right)\right)\left(1 - F\left(x\right)\right) \\
&= \frac{1}{2}\left(1 - F\left(x\right)\right)^2.
\end{aligned}
$$

Substituting this expression back into R_{SPAPA} gives

$$\begin{aligned}
R_{SPAPA} &= 4\int_0^1 \left(\frac{1}{2}\left(1-F\left(x\right)\right)^2\right)\frac{xf\left(x\right)}{1-F\left(x\right)}dx \\
&= 2\int_0^1 xf\left(x\right)\left(1-F\left(x\right)\right)dx
\end{aligned}$$

6.4 Part (d)

Both with and without using the revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

. . . . . .

In a second-price all-pay auction, the probability assignment function remains the same as in the standard first-price auction; i.e. whoever bids highest wins. Moreover, bidders whose value is zero would bid zero so $\bar{c}_i\left(0\right) = 0$—the same as in the standard first-price auction. Thus, by the Revenue Equivalence Theorem, the seller's expected revenue should be the same between the two auctions.

When $N = 2$,

$$\begin{aligned}
R_{FPA} &= 2\int_0^1 \int_0^v xf\left(x\right)f\left(v\right)dxdv \\
&= 2\int_0^1 xf\left(x\right)\left(\int_x^1 f\left(v\right)dv\right)dx \\
&= 2\int_0^1 xf\left(x\right)\left(1-F\left(x\right)\right)dx = R_{SPAPA}.
\end{aligned}$$