Theory of Income III

TA Session: Midterm review

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1 Midterm 2018: The Labor Wedge in a Two Sector Growth Model

Consider the following modification to the neoclassical growth model. There are two types of firms, those that produce consumption goods and those that produce capital goods. The representative consumption-goods-producing firm has a technology $C_t = K_{c,t}^{\alpha_c} \left(z_{c,t} H_{c,t}\right)^{1-\alpha_c}$ and the representative investment-goods-producing firm has a technology $X_t = K_{x,t}^{\alpha_x} \left(z_{x,t} H_{x,t}\right)^{1-\alpha_x}$, where $K_{c,t}$ and $K_{x,t}$ are the amount of capital used by the two types of firms, $H_{c,t}$ and $H_{x,t}$ are the amount of labor used, and $z_{c,t}$ and $z_{x,t}$ are the exogenous levels of technology. Assume $1 > \alpha_x > \alpha_c > 0$, so the capital-goods producing sector is more capital intensive.

Let p_t denote the price of investment goods in units of consumption goods in period t. This price is endogenous, but all firms take it as given when deciding how much labor to hire and how much to invest. Likewise, the wage w_t is the same in the two sectors and taken as given by all firms.

As in the standard neoclassical growth model, the objective of all firms is to maximize the present value of profits using the intertemporal price q_0^t, the price of a unit of consumption goods in period t in terms of the consumption good in period 0.

1.1 Question 1

Write the objective function for a consumption goods producing firm.

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We assume that each firms stocks capital but in order to do so by investment goods from the capital producing firm. Hence,

$$\max_{\{C_t, H_{c,t}, K_{c,t}\}} \sum_{t=0}^{\infty} \sum_{s^t} q_0^t \{C_t - w_t H_{c,t} - p_t X_{c,t}\} \tag{1}$$

$$s.t.\ ,\ C_t = K_{c,t}^{\alpha_c} (z_{c,t} H_{c,t})^{1-\alpha_c}$$

$$K_{c,t+1} = (1-\delta) K_{c,t} + X_{c,t}$$

1.2 Question 2

Write the objective function for a capital goods producing firm.

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Similarly,

$$\max_{\{K_{t+1}, K_{x,t}, H_{x,t}, X_t\}} \sum_{t=0}^{\infty} \sum_{s^t} q_0^t \{p_t (X_t - X_{x,t}) - w_t H_{x,t}\} \tag{2}$$

$$s.t.\ K_{x,t+1} = (1-\delta) K_{x,t} + X_{x,t}$$

$$X_t = K_{x,t}^{\alpha_x} (z_{x,t} H_{x,t})^{1-\alpha_x}\ .$$

1.3 Question 3

In a competitive equilibrium, both types of firms maximize profits, the representative household maximizes utility subject to a lifetime budget constraint, and all market clear. Define a competitive equilibrium.

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A competitive equilibrium is a sequence of quantities $\{C_t, H_{c,t}, H_{x,t}, K_{c,t}, K_{x,t}, X_{c,t}, X_{x,t}, X_t, K_t\}$ and prices $\{q_0^t, p_t, w_t\}_{t=0}^{\infty}$ such that:

- Given prices the sequences of quantities solves problem (1) and (2).
- The household maximizes utility subject to a lifetime budget constraint:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Pi_t (s^t)\, U(C_t, H_t)\ ,\ s.t.\ \sum_{t=0}^{\infty} \sum_{s^t} q_0^t (s^t) [C_t - w_t H_t] \leq a_0. \tag{3}$$

- Market clears:

$$C_t = K_{c,t}^{\alpha_c} \left(z_{c,t} H_{c,t}\right)^{1-\alpha_c}$$

$$K_t = K_{x,t} + K_{c,t}$$

$$H_t = H_{x,t} + H_{c,t}$$

$$X_t = X_{c,t} + X_{x,t}$$

1.4 Question 4

What are the first order conditions for labor demanded by the two types of firms and for labor supplied by the households?

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The household solves (3) which yields FOC:

$$\{C_t\}: \ \beta^t \Pi_t \left(s^t\right) U_c\left(t\right) = \lambda q_0^t \tag{4}$$

$$\{H_t\}: \ -\beta^t \Pi_t \left(s^t\right) U_H\left(t\right) = \lambda q_0^t w_t \tag{5}$$

Then labor supply by the firm solves

$$-\frac{U_H\left(t\right)}{U_C\left(t\right)} = w_t. \tag{6}$$

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The labor demanded by the firms solves instead:

$$\left(1-\alpha_c\right)\frac{C_t}{H_{c,t}} = \left(1-\alpha_c\right) K_{c,t}^{\alpha_c} \left(z_{c,t}\right)^{1-\alpha_c} H_{c,t}^{-\alpha_c} = w_t \tag{7}$$

$$\left(1-\alpha_c\right)\frac{X_t}{H_{x,t}} p_t = p_t \left(1-\alpha_x\right) K_{x,t}^{\alpha_x} \left(z_{x,t}\right)^{1-\alpha_x} H_{x,t}^{-\alpha_x} = w_t \tag{8}$$

1.5 Question 5

How would Chari, Kehoe, and McGrattan measure the labor wedge in this environment? Suppose that $H_{x,t+1}/H_{c,t+1} < H_{x,t}/H_{c,t}$. What happens to the measured labor wedge?

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CKM works with an aggregate production function which we assume Cobb-Doouglas. In that case the labor wedge is computed using the following equation:

$$-\frac{U_H(t)}{U_c(t)} = (1-\tau_h)(1-\alpha)\frac{Y_t}{H_t}.$$

So what we need to do know is to map this equation to our model. First from market clearance we can map total hours worked in the two model $H_t = H_{c,t} + H_{x,t}$. For output we use the value of production, thus $Y_t = C_t + p_t X_t$. So we have,

$$\begin{aligned}
\frac{Y_t}{H_t} &= \frac{C_t + p_t X_t}{H_t} \\
&= \left[\frac{H_{c,t}}{H_{ct}+H_{x,t}}\frac{C_t}{H_{c,t}} + \frac{H_{x,t}}{H_{ct}+H_{x,t}}\frac{p_t X_t}{H_{x,t}}\right] \\
&= \left[\frac{H_{c,t}}{H_{ct}+H_{x,t}}\frac{1}{1-\alpha_c}w_t + \frac{H_{x,t}}{H_{ct}+H_{x,t}}\frac{1}{1-\alpha_x}w_t\right] \\
&= \left[\frac{H_{c,t}}{H_{ct}+H_{x,t}}\frac{1}{1-\alpha_c} + \frac{H_{x,t}}{H_{ct}+H_{x,t}}\frac{1}{1-\alpha_x}\right]w_t \\
&= -\frac{U_H(t)}{U_c(t)}\left[\frac{1}{1+\gamma_t}\frac{1}{1-\alpha_c} + \frac{\gamma_t}{1+\gamma_t}\frac{1}{1-\alpha_x}\right],
\end{aligned}$$

where $\gamma_t \equiv \frac{H_{x,t}}{H_{c,t}}$. So we have,

$$(1-\tau_{ht}) \equiv \frac{1}{(1-\alpha)}\left[\frac{1}{1+\gamma_t}\frac{1}{1-\alpha_c} + \frac{\gamma_t}{1+\gamma_t}\frac{1}{1-\alpha_x}\right]^{-1}.$$

Note now that,

$$\begin{aligned}
\frac{\partial(1-\tau_{h,t})^{-1}}{\partial\gamma_t} &= (1-\alpha)\left[\frac{1}{1-\alpha_c}\frac{-1}{(1+\gamma_t)^2} + \frac{(1+\gamma_t)-\gamma_t}{(1+\gamma_t)^2}\frac{1}{1-\alpha_x}\right] \\
&= \frac{(1-\alpha)}{(1+\gamma_t)^2}\left[\frac{\alpha_x-\alpha_c}{(1-\alpha_x)(1-\alpha_c)}\right] > 0.
\end{aligned}$$

Hence, the the labor wedge is decreasing in time.

2 Midterm 2018: Interpretation of Log-linearized Solution

A social planner chooses $\{x_t, y_t, k_{t+1}, a_{t+!}\}_{t=0}^{\infty}$ to maximize:

$$\sum_{t=0}^{\infty}\beta^t f(x_t, y_t, k_t, a_t),$$

subject to the constraints,

$$k_{t+1} = g_1\left(x_t, y_t, k_t, a_t\right) \text{ and } a_{t+1} = g_2\left(x_t, y_t, k_t a_t\right),$$

taking as given k_0 and a_0. Assume that the solution to the planner's problem converges to a steady state (x^*, y^*, k^*, a^*) for any initial conditions. Suppose a log-linear approximation to the dynamics near tthe steady state can be represented as:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \\ \hat{k}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = B \cdot \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \\ \hat{k}_t \\ \hat{a}_t \end{bmatrix},$$

where the matrix B has eigenvalues 1.4, 1.1, 0.9, and 0.7. The associated eigenvectors are respectively:

$$\begin{bmatrix} -0.1 \\ 0.4 \\ 0.1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1.5 \\ 0.1 \\ 0.8 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.2 \\ 0.1 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

2.1 Question 1

Express $\hat{x}_0$ and $\hat{y}_0$ as functions of $\hat{k}_0$ and $\hat{a}_0$.

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Given the information above we can write:

$$\begin{bmatrix} \hat{x}_t \\ \hat{y}_t \\ \hat{k}_t \\ \hat{a}_t \end{bmatrix} = \mu_1 \lambda_1^t \begin{bmatrix} -0.1 \\ 0.4 \\ 0.1 \\ 1 \end{bmatrix} + \mu_2 \lambda_2^t \begin{bmatrix} -1.5 \\ 0.1 \\ 0.8 \\ 1 \end{bmatrix} + \mu_3 \lambda_3^t \begin{bmatrix} 0.8 \\ 0.2 \\ 0.1 \\ 1 \end{bmatrix} + \mu_4 \lambda_4^t \begin{bmatrix} -0.1 \\ 0.2 \\ 0 \\ 1 \end{bmatrix},$$

where $\{\mu_i\}$ are parameters to determine. Since λ_1 and λ_2 are both bigger than 1 it has to be that $\mu_1 = \mu_2 = 0$.

Now at time $t = 0$ we have:

$$\begin{bmatrix} \hat{x}_0 \\ \hat{y}_0 \\ \hat{k}_0 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} \mu_3 0.8 - \mu_4 0.1 \\ \mu_3 0.2 + \mu_4 0.2 \\ \mu_3 0.1 \\ \mu_3 + \mu_4 \end{bmatrix}.$$

Then,

$$\hat{x}_0 = \frac{0.8}{0.1}\hat{k}_0 - \left(\hat{a}_0 - \frac{1}{0.1}\hat{k}_0\right) 0.1$$
$$\hat{y}_0 = \hat{a}_0 0.2$$

2.2 Question 2

Suppose $\hat{k}_t = 0.9^t \hat{k}_0$ and $\hat{a}_t = 0.9^t \hat{a}_0$ for all t. What is $\hat{a}_0 / \hat{k}_0$?

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We know from the system above that:

$$\hat{k}_t = 0.1\mu_3 0.9^t$$
$$\hat{a}_t = \mu_3 0.9^t + \mu_4 0.7^t.$$

It follows then that, given the assumed solution,

$$\mu_4 = 0$$
$$\mu_3 = \hat{a}_0$$
$$0.1\mu_3 = \hat{k}_0.$$

Hence, $\hat{a}_0 / \hat{k}_0 = 10$.

2.3 Question 3

Suppose $\hat{k}_0 \neq 0$. What is $\lim_{t\to\infty} \hat{a}_t / \hat{k}_t$? $\lim_{t\to\infty} \hat{x}_t / \hat{k}_t$?

$\ldots\ldots$

$$\lim_{t\to\infty} \hat{a}_t/\hat{k}_t = \lim_{t\to\infty} \frac{\mu_3 0.9^t + \mu_4 0.7^t}{0.1\mu_3 0.9^t} = \lim_{t\to\infty} \frac{1}{0.1} + \frac{\mu_4}{0.1\mu_3}\left(\frac{0.7}{0.9}\right)^t = 10,$$

$$\lim_{t\to\infty} \hat{x}_t/\hat{k}_t = \lim_{t\to\infty} \frac{0.8\mu_3 (0.9)^t - 0.1\mu_4 (0.7)^t}{0.1\mu_3 0.9^t} = 8.$$

Note: the assumption $\hat{k}_0 \neq 0$ is to ensure that the ratios are well defined.

2.4 Question 4

Assume $\hat{a}_0$ and $\hat{k}_0$ are booth positive. For what initial values of $\hat{a}_0/\hat{k}_0$ are $\hat{x}_t$ and $\hat{y}_t$ always positive?

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Remember that $\mu_3 = \hat{k}_0/0.1 > 0$ and $\mu_4 = \hat{k}_0\left(\frac{\hat{a}_0}{\hat{k}_0} - 10\right)$. Also:

$$\hat{x}_t = 0.8\mu_3 (0.9)^t - 0.1\mu_4 (0.7)^t$$
$$\hat{y}_t = 0.2\hat{a}_t$$
$$\hat{a}_t = \mu_3 0.9^t + \mu_4 0.7^t$$

To make sure that $\hat{y}_t$ is always positive is enough to impose $\hat{a}_t > 0$

$$\mu_3 > -\mu_4\left(\frac{0.7}{0.9}\right)^t \iff 10 > \left(10 - \frac{\hat{a}_0}{\hat{k}_0}\right)\left(\frac{0.7}{0.9}\right)^t \iff 10 > \max_t \left(10 - \frac{\hat{a}_0}{\hat{k}_0}\right)\left(\frac{0.7}{0.9}\right)^t \iff 10 \geq \frac{\hat{a}_0}{\hat{k}_0}.$$

To ensure that $\hat{x}_t > 0$:

$$80 > \left(\frac{\hat{a}_0}{\hat{k}_0} - 10\right)\left(\frac{0.7}{0.9}\right)^t \iff 80 > \max_t \left(\frac{\hat{a}_0}{\hat{k}_0} - 10\right)\left(\frac{0.7}{0.9}\right)^t \iff 90 \geq \frac{\hat{a}_0}{\hat{k}_0}.$$

Hence, we require $\frac{\hat{a}_0}{\hat{k}_0} \leq 90$. Because of our assumption on the values at 0 we have in fact $\frac{\hat{a}_0}{\hat{k}_0} \in [0, 90]$.

3 Final 2017: Menu Costs and Productivity Growth

A representative household solves:

$$\max_{\{C_t, H_t\}} \sum_{t=0}^{\infty} \beta^t \left(\log C_t - V\left(H_t\right)\right), \text{ where } C_t \equiv \left(\int_0^1 \left(z_{j,t} c_{jt}\right)^{\frac{\eta-1}{\eta}} dj\right)^{\frac{\eta}{\eta-1}},$$

subject to

$$B_0 = \sum_{t=0}^{\infty} Q_0^t \left(\int_0^1 p_{jt} c_{jt} dj - W_t H_t \right)$$

where B_0 is initial nominal wealth, which is equal to the present value of firm's profits; $p_{j,t}$ is the price of good j ; $c_{j,t}$ is the consumption of good j; $z_{j,t}$ is a demand shifter; H_t is hours worked; and W_t is the nominal wage.

At the start of period t firm j has productivity a_{jt}, demand $z_{j,t}$ and a stale price $p_{j,t-1}$. It then decides whether to update its price, in which case it loses a fraction x of its profits in period t but can set any price $p_{j,t}$. Otherwise its price remains at $p_{j,t} = p_{jt-1}$.The firm then produces whatever amount of output is demanded by households at that price, hiring the necessary amount of labor: $c_{jt} = a_{jt} h_{jt}$.

Finally, at the start of period $t+1$ its productivity a and demand z remain constant with probability $q \in (0,1)$; while with probability $1-q$, productivity increases to $a_{j,t+1} = a_{jt} e^{\Delta}$ and demand falls to $z_{jt+1} = z_{jt} e^{-\Delta}$ for some fixed $\Delta > 0$. This shock is independent over time and across firms at a point in time. Assume $a_{jt} z_{jt} = a_{j't} z_{j't}$ for all firms j and j'.

The central bank chooses a path for nominal interptemporal prices Q_0^t.

3.1 Question 1

Using the household problem, define the ideal price index P_t in terms of individual prices, productivity, and demand, as well as model parameters.

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The household problem is given by:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\log \left(\int_0^1 (z_{j,t} c_{jt})^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} - V(H_t) \right) + \lambda \left[B_0 - \sum_{t=0}^{\infty} Q_0^t \left(\int_0^1 p_{jt} c_{jt} dj - W_t H_t \right) \right]$$

The FOC for this problem are:

$$\{c_{j,t}\} : \beta^t C_t^{\frac{1}{\eta}-1} z_{j,t}^{\frac{\eta-1}{\eta}} c_{j,t}^{\frac{\eta-1}{\eta}-1} = \lambda Q_0^t p_{j,t}$$

$$\{H_t\} : \beta^t V'(H_t) = \lambda Q_0^t W_t$$

Consider the first equation for two different varieties:

$$\left(\frac{z_{j,t}}{z_{0,t}}\right)^{\frac{\eta-1}{\eta}}\left(\frac{c_{jt}}{c_{0t}}\right)^{\frac{-1}{\eta}}=\frac{p_{j,t}}{p_{0,t}} \implies c_{jt}=c_{0t}\left(\frac{p_{j,t}}{p_{0,t}}\right)^{-\eta}\left(\frac{z_{j,t}}{z_{0,t}}\right)^{\eta-1}.$$

Substitute this into the definition of C_t :

$$C_t \equiv z_{0,t}c_{0t}\left(\int_0^1\left(\frac{p_{j,t}}{z_{j,t}}\right)^{1-\eta}dj\right)^{\frac{\eta}{\eta-1}}\left(\frac{z_{0,t}}{p_{0,t}}\right)^{-\eta}.$$

Define $P_t=\left(\int_0^1\left(\frac{p_{j,t}}{z_{j,t}}\right)^{1-\eta}dj\right)^{\frac{1}{1-\eta}}$, then-

$$c_{0t}=C_t\frac{P_t^{\eta}}{p_{0,t}^{\eta}}\frac{1}{z_{0,t}^{1-\eta}}.$$

3.2 Question 2

Assume $x=0$, so there is no cost of changing prices. How does the price of firm j relative to the nominal W_t depend on a_{jt} and z_{jt}? How does the quantity produced c_{jt}? How does expenditure $p_{jt}c_{jt}$?

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If there is no cost of adjustment each period the firm solves:

$$\begin{aligned}&\max p_{jt}c_{jt}-W_th_{tj}\\ &s.t.\ h_{jt}=\frac{c_{jt}}{a_{jt}}\\ &\qquad c_{jt}=C_t\frac{P_t^{\eta}}{p_{j,t}^{\eta}}\frac{1}{z_{j,t}^{1-\eta}}.\end{aligned}$$

Then, the optimal price solves:

$$\begin{aligned}&\{p_{jt}\}:(1-\eta)\,p_{jt}^{-\eta}+\eta p_{jt}^{-\eta-1}W_t\frac{1}{a_{jt}}=0\\ &\implies p_{jt}=\frac{\eta}{\eta-1}W_t\frac{1}{a_{jt}}.\end{aligned}$$

So the price does not depend on the preference shock and depends negatively on its productivity.

Plugging back into the demand function:

$$c_{jt} = \left[\frac{\eta}{\eta - 1}\right]^{-\eta} C_t \left(\frac{W_t}{P_t}\right)^{-\eta} a_{jt}^{\eta} z_{j,t}^{\eta-1},$$

the quantity depends positively on the productivity and demand shock. Finally expenditure in good j:

$$p_{jt}c_{jt} = \left[\frac{\eta}{\eta - 1}\right]^{1-\eta} C_t \left(W_t\right)^{1-\eta} P_t^{\eta} a_{jt}^{\eta-1} z_{j,t}^{\eta-1},$$

also depends positively on demand and productivity shocks.

3.3 Question 3

Still assuming $x = 0$, express the ideal price index in terms of individual productivity and demand (but no prices), as well as model parameters. What determines the growth rate of the ideal price index in equilibrium?

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Using the solution we found in the previous section and plugging it into the definition of the price index yields:

$$P_t = \frac{\eta}{\eta - 1} W_t \left(\int_0^1 \left(\frac{1}{a_{jt} z_{j,t}}\right)^{1-\eta} dj\right)^{\frac{1}{1-\eta}}.$$

Now from the problem we have that at the start of period $t+1$ its productivity a and demand z remain constant with probability $q \in (0, 1)$; while with probability $1 - q$, productivity increases to $a_{j,t+1} = a_{jt}e^{\Delta}$ and demand falls to $z_{jt+1} = z_{jt}e^{-\Delta}$ for some fixed $\Delta > 0$. This implies that the product $a_{j,t}z_{j,t}$ is constant in time. That means that the growth in price is determine by the growth in nominal wages.

Finally given the assumption $a_{jt}z_{jt} = a_{j't}z_{j't}$ for all firms j and j' in the question we have:

$$P_t = \frac{\eta}{\eta - 1} \frac{W_t}{a_t z_t}.$$

3.4 Question 4

For the remainder of the question, assume $x > 0$. Write the value function of a firm that starts period t with state price p, productivity a and demand z , conditional on the aggregate state of the system.

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If it decides to adjust prices:

$$V_A(p,a,z;W) = \max_{\tilde{p}} (1-x)\left[\tilde{p}^{1-\eta} - \tilde{p}^{-\eta}\frac{W}{a}\right] CP^{\eta}z^{\eta-1} + \beta\mathbb{E}\left[V(\tilde{p},a',z';,W')\right]$$
$$= \max_{\tilde{p}} (1-x)\left[\tilde{p}^{1-\eta} - \tilde{p}^{-\eta}\frac{W}{a}\right] CP^{\eta}z^{\eta-1} + \beta\left\{qV(\tilde{p},a,z;W') + (1-q)V\left(\tilde{p},ae^{\Delta},ze^{-\Delta};W\right)\right\}$$

If it decides not to adjust prices:

$$V_N(p,a,z;W) = \left[p^{1-\eta} - p^{-\eta}\frac{W}{a}\right] CP^{\eta}z^{\eta-1} + \beta\left\{qV(p,a,z;W') + (1-q)V\left(p,ae^{\Delta},ze^{-\Delta};W'\right)\right\}.$$

Then,

$$V(p,a,z;W) = \max_{\tilde{p}} \left(1-x\mathbb{I}_{\{\tilde{p}\neq p\}}\right)\left[\tilde{p}^{1-\eta} - \tilde{p}^{-\eta}\frac{W}{a}\right] CP^{\eta}z^{\eta-1} + \beta\left\{qV(\tilde{p},a,z;W') + (1-q)V\left(\tilde{p},ae^{\Delta},ze^{-\Delta};W\right)\right\}$$

Note that the price index is also an aggregate state but from what we have seen before the price level maps into the nominal wage.

3.5 Question 5

Conjecture that the nominal wage $W_t = W$ is constant. Also conjecture that whenever $a_{jt}p_{jt-1}/W \leq \underline{p}$ or $a_{jt,}p_{jt-1}/W \geqq \bar{p}$,the firm adjust its price to $p_{jt} = p^*W/a_{jt}$ for some threshold $\underline{p} < p^* < \bar{p}$. If all firms behave in this way, what is the growth in the identical price index? What fraction cut their price in each period? Raise their price?

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In each period the price each firms charge does not depend on z so any movement is due to movements in a_{jt}. According to the question the threshold depends on whether $\frac{a_{jt}p_{jt-1}}{W}$ moves in time. Since a_{jt} only moves up or stay the same it has to be that I change my price every time $\frac{a_{jt}p_{jt-1}}{W} > \bar{p}$ and I adjust to $\underline{p} = p^*$. So no-one raise their price. Define $\tilde{p}_j \equiv \frac{a_{j,t}p_{j,t}}{W}$, we know $\tilde{p}_j \in \left\{p^*, p^*e^{\Delta}, p^*e^{2\Delta}, \ldots, \bar{p}\right\}$ and takes each value with probability ϕ_j, $j = 0, 1, \ldots, n-1$. Then, $q\phi_{n-1}$ cut their price.

Finally, the price index. Using the definition found in question 1 we write:

$$\begin{aligned}
P_t^{1-\eta} &= \int_0^1 \left(\frac{p_{j,t}}{z_{j,t}}\right)^{1-\eta} dj \\
&= \left[\int_0^1 \left(\frac{a_{j,t}p_{j,t}}{W}\right)^{1-\eta} dj\right] \left[\frac{W}{a_{j,t}z_{j,t}}\right]^{1-\eta} \\
&= \left[\int_0^1 (\tilde{p})^{1-\eta} dj\right] \left[\frac{W}{a_{j,t}z_{j,t}}\right]^{1-\eta} \\
&= \left[\sum_{j=0}^{n-1} (\tilde{p}_j)^{1-\eta} \phi_j\right] \left[\frac{W}{a_{j,t}z_{j,t}}\right]^{1-\eta}.
\end{aligned}$$

Hence the price index is constant, since the first term is the expected value of $\tilde{p}^{1-\eta}$ under the stationary distribution, and the second term is constant given our conjecture of W and the process for $a_{jt}z_{jt}$.

3.6 Question 6

Maintain the same conjecture. What is the distribution of $a_{jt}p_{jt}/W$ in equilibrium?

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In the stationary distribution the ratio $a_j p_j/W$ can take the values $\left\{p^*, p^*e^{\Delta}, p^*e^{2\Delta}, \dots, \bar{p}\right\}$. Denote with $\phi(\cdot)$ the stationary distribution, then ϕ satisfies:

$$\begin{aligned}
\phi_0 &= (1-q)\,\phi_0 + q\phi_n, \\
\phi_j &= \phi_j\,(1-q) + \phi_{j-1}q,\ \forall j \in \{1, 2, \dots, n-1\} \\
\sum_{j=0}^{n-1} \phi\,(p_j) &= 1.
\end{aligned}$$

So from the first two equation we have:

$$\phi_j = \phi_{j-1} \equiv \bar{\phi},\ \forall j \in \{0, 1, 2, \dots, n-1\}.$$

Then from the third $\bar{\phi} = 1/n$