TA session makeup

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Consider functions which maps a function into a function.

$$\mathbb{T}f(x) = \mathbf{E}\left[f(X_{t+1})|X_t = x\right]$$
$$\mathbb{M}f(x) = (1-\delta)\sum_{j=0}^{\infty}\delta^j\mathbb{T}^j f(x)$$

Note that the choice of t does not matter for $\mathbb{T}$ since we are only talking about Markov processes here. Also, note that $\mathbb{T}^j$ is well-defined. For example,

$$\begin{aligned}
\mathbb{T}^2 f(x) &= \mathbf{E}\left[\mathbb{T}f(X_{t+1})|X_t = x\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[f(X_{t+2})|X_{t+1}\right]|X_t = x\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[f(X_{t+2})|X_{t+1}, X_t = x\right]|X_t = x\right] \quad \because \text{Markov property} \\
&= \mathbf{E}\left[f(X_{t+2})|X_t = x\right] \quad \because \text{law of iterated expectation}
\end{aligned}$$

and so on. From now, we only consider the stationary distribution of $\{X_t\}$, Q.

Proposition 3.4.1.

Suppose

$$\mathbb{T}\tilde{f} = \tilde{f} \quad \Rightarrow \quad \tilde{f} \text{ is a constant function with } Q \text{ measure one.}$$

Then, it is possible to construct the (Markov) process $\{X_t\}$ by using a transformation $\mathbb{S}$ that is measure-preserving and ergodic.

Proposition 3.4.2.

Suppose

$$f \geq 0 \text{ and } \int f(x)Q(dx) > 0 \quad \Rightarrow \quad \mathbb{M}f > 0 \text{ with } Q \text{ measure one.}$$

Then,

$$\mathbb{T}\tilde{f} = \tilde{f} \quad \Rightarrow \quad \tilde{f} \text{ is a constant function with } Q \text{ measure one.}$$

Thus, by combining those two, we can see that

$$f \geq 0 \text{ and } \int f(x)Q(dx) > 0 \quad \Rightarrow \quad \mathbb{M}f > 0 \text{ with } Q \text{ measure one.}$$

is a sufficient condition for ergodicity. Now, let's move on to finite state case (i.e. X_t can take only finite values) with transition matrix $\mathbb{P}$. WLOG Denote the set of states with $\mathcal{N} = \{1, 2, \cdots, n\}$. Suppose we have

$$\text{any state can go to any state in finite times} \quad \Leftrightarrow \quad \forall x, y \in \mathcal{N},\ \exists\, t \text{ s.t. } \mathbb{P}^t(x,y) > 0.$$

Then choose an arbitrary f such that

$$f \geq 0 \text{ and } \int f(x) Q(dx) > 0.$$

Since its expectation is strictly positive, there should at least one $x \in \mathcal{N}$ such that $f(x) > 0$. Otherwise, $\int f(x)Q(dx) = 0$. Denote such x with x_0. Since we have

$$\forall x, y \in \mathcal{N},\ \exists\, t \text{ s.t. } \mathbb{P}^t(x,y) > 0,$$

for any $x \in \mathcal{N}$, there is some t_x such that $\mathbb{P}^{t_x}(x, x_0) > 0$. Then,

$$\begin{aligned}
\mathbb{T}^{t_x} f(x) &= \mathbf{E}[f(X_{t+t_x}) | X_t = x] \\
&\geq f(x_0) \Pr\{X_{t+t_x} = x_0 | X_t = x\} \\
&= f(x_0) \mathbb{P}^{t_x}(x, x_0) > 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{M} f(x) &= (1-\delta) \sum_{j=0}^{\infty} \delta^j \mathbb{T}^j f(x) \\
&\geq (1-\delta) \delta^{t_x} \mathbb{T}^{t_x} f(x) > 0
\end{aligned}$$

Thus, we have $\mathbb{M} f > 0$ with Q measure one. This is why

$$\text{any state can go to any state in finite times} \quad \Leftrightarrow \quad \forall x, y \in \mathcal{N},\ \exists\, t \text{ s.t. } \mathbb{P}^t(x,y) > 0.$$

is a sufficient condition for ergodicity in finite Markov process.

Now let me fill in the gap from today. I needed one more step to show this and I was totally forgetting about that. First, let me show $\mathbb{T}\tilde{f} = \tilde{f}$ implies that $\Pr\left\{\tilde{f}(X_{t+1}) = \tilde{f}(X_t)\right\} = 1$. (This was the step I was missing!) Note that

$$\begin{aligned}
\mathbf{E}\left[\tilde{f}(X_{t+1})\tilde{f}(X_t)\right] &= \mathbf{E}\left[\mathbf{E}\left[\tilde{f}(X_{t+1})|X_t\right]\tilde{f}(X_t)\right] \quad \because \text{law of iterated expectation} \\
&= \mathbf{E}\left[\mathbb{T}\tilde{f}(X_t)\tilde{f}(X_t)\right] \\
&= \mathbf{E}\left[\tilde{f}(X_t)\tilde{f}(X_t)\right] \quad \because \mathbb{T}\tilde{f} = \tilde{f}
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{E}\left[\left(\tilde{f}(X_{t+1}) - \tilde{f}(X_t)\right)^2\right] &= \mathbf{E}\left[\tilde{f}(X_{t+1})^2\right] + \mathbf{E}\left[\tilde{f}(X_t)^2\right] - 2\mathbf{E}\left[\tilde{f}(X_{t+1})\tilde{f}(X_t)\right] \\
&= \mathbf{E}\left[\tilde{f}(X_t)^2\right] + \mathbf{E}\left[\tilde{f}(X_t)^2\right] - 2\mathbf{E}\left[\tilde{f}(X_{t+1})\tilde{f}(X_t)\right] \quad \because Q \text{ is stationary.} \\
&= \mathbf{E}\left[\tilde{f}(X_t)^2\right] + \mathbf{E}\left[\tilde{f}(X_t)^2\right] - 2\mathbf{E}\left[\tilde{f}(X_t)^2\right] \\
&= 0.
\end{aligned}$$

Thus, $\tilde{f}(X_{t+1}) = \tilde{f}(X_t)$ with probability one. Consider some $\tilde{f}$ such that $\mathbb{T}\tilde{f} = \tilde{f}$. Take some Borel set $\tilde{B} \subset \mathbb{R}$. Let

$$f(x) = \begin{cases} 1, & \text{if } \tilde{f}(x) \in \tilde{B} \\ 0, & \text{if } \tilde{f}(x) \notin \tilde{B} \end{cases}$$

We want to show $\mathbb{T}f = f$.

$$\begin{aligned}
\mathbb{T}f(x) &= \mathbf{E}\left[f(X_{t+1}) | X_t = x\right] \\
&= \mathbf{E}\left[\mathbb{I}\{\tilde{f}(X_{t+1}) \in \tilde{B}\} | X_t = x\right] \\
&= \mathbf{E}\left[\mathbb{I}\{\tilde{f}(X_t) \in \tilde{B}\} | X_t = x\right] \quad \because \tilde{f}(X_{t+1}) = \tilde{f}(X_t) \\
&= \mathbf{E}\left[\mathbb{I}\{\tilde{f}(x) \in \tilde{B}\} | X_t = x\right] \\
&= \mathbb{I}\{\tilde{f}(x) \in \tilde{B}\} = f(x).
\end{aligned}$$

Ta-da! The gap is filled!