*Theory of Income, Fall 2017*
*Fernando Alvarez, U of C*

# Problem Set 1

## 1 Endowment economy

Consider an OLG economy as described in the class notes. In this exercise we focus on an endowment economy and we study the effects of privatizing social security in such a way that everyone remains indifferent. In particular, we want to study the evolution of savings and government debt.

An agent born at period $t$ cares only about consumption at period $t$ and $t+1$ $\left(c_t^t, c_{t+1}^t\right)$, where $c_j^i$ refers to consumption of an agent born at date $i$ (also called of generation $i$) in period $j$. Agents born at date 0 are already old in period 1 and only care about consumption in period 1, $c_1^0$. As in the class notes, there is a mass 1 of agents in each generation.

The endowment stream of agent $t \geq 1$ is given by $\left(e_t^t, e_{t+1}^t\right)$ and the endowment of the initial old is $e_1^0$. The budget constraints of an agent of generation $t \geq 1$ are

$$\begin{aligned} s_t + c_t^t &= e_t^t - \tau_t^t \\ c_{t+1}^t &= s_t\left(1+r_t\right) + e_{t+1}^t - \tau_{t+1}^t, \end{aligned} \tag{1}$$

where $s_t$ refers to savings, $r_t$ is the interest rate between periods $t$ and $t+1$, $\tau_t^t$ are the taxes levied on agents of generation $t$ when young and $\tau_{t+1}^t$ are taxes levied on the same generation when they are old. The budget constraint of the initial old is

$$c_1^0 = s_0\left(1+r_0\right) + e_1^0 - \tau_1^0; \quad s_0 \text{ given.} \tag{2}$$

Let $g_t$ denote government expenditures and $B_t$ be the stock of government assets at period $t$. Then the government budget constraint is, for $t \geq 1$

$$B_t + g_t = B_{t-1}\left(1+r_{t-1}\right) + \tau_t^t + \tau_t^{t-1}. \tag{3}$$

The market clearing conditions in the goods market is

$$g_t + c_t^t + c_t^{t-1} = e_t^t + e_t^{t-1}, \tag{4}$$

and the market clearing condition in asset markets is

$$B_t + s_t = 0. \tag{5}$$

We also have the initial condition $B_0 + s_0 = 0$.

**Question 1)** *Walras' Law.* Use equations (1), (2), (3), (4) and the initial condition $B_0 + s_0 = 0$ to deduce that market clearing in assets market holds.

Now we will analyze the social security *pay-as-you-go* system. The idea is that each period young agents are taxed and the earnings are redistributed to the current old. In terms of our notation, we assume

$$\begin{aligned} \tau_t^t &= \theta \\ \tau_{t+1}^t &= -\theta. \end{aligned}$$

Furthermore, consider equilibria with a constant interest rate $r_t = \bar{r}$ and assume

$$\begin{aligned} g_t &= 0 \\ e_t^t &= \alpha \\ e_t^{t-1} &= 1-\alpha \\ s_0 &= B_0 = 0, \end{aligned}$$

hence we immediately see from (3) that $B_t = 0$ for all $t \geq 1$. Since this is an endowment economy, from the class notes we also know that $s_t = 0$ for $t \geq 1$.

*Ricardian Equivalence and Privatization of Social Security*

Consider the following environment with different taxes

$$\begin{aligned} \tilde{\tau}_1^0 &= \tau_1^0 \\ \tilde{\tau}_t^t &= \tau_t^t + \frac{\tau_{t+1}^t}{(1+\tilde{r}_t)} \\ \tilde{\tau}_{t+1}^t &= 0, \end{aligned}$$

for all $t \geq 1$. A $\sim$ above any variable will refer to the equilibrium with the new tax system. Guess that in the new equilibrium the interest rate does not change, that is,

$$\tilde{r}_t = r_t = \bar{r} \text{ for all } t.$$

**Question 2)** Interpret $\left\{\tilde{\tau}_t^t, \tilde{\tau}_{t+1}^t\right\}$.

**Question 3)** Show that $\tilde{c}_t^t = c_t^t$, $\tilde{c}_{t+1}^t = c_{t+1}^t$ for $t \geq 1$ and $\tilde{c}_1^0 = c_1^0$ is optimal.

**Question 4)** Show that $\tilde{B}_t = \tilde{B} = -\frac{\theta}{1+\bar{r}}$ for all $t \geq 1$. [Hint: Solve the government budget constraint].

**Question 5)** Show that $\tilde{s}_t = \tilde{s} = \frac{\theta}{1+\bar{r}}$ solves the agent's problem.

**Question 6)** Show that $\tilde{B}_t$ and $\tilde{s}_t$ satisfy the assets market clearing condition.

**Question 7)** Interpret what happens in the assets market at time $t = 1$ and at $t \geq 2$ (who saves?, why?, etc.).

## 2 Production economy

In this problem we consider a production economy with capital and labor. Agents live for two periods. Young agents inelastically supply one unit of labor, consume and save, while old agents are retired and consume out of their savings. We will use the same notation as in the last problem, for example, $c_j^i$ denotes consumption of a person born at date $i$ in period $j$. Aggregate consumption at period $t$ is given by

$$C_t = c_t^t + c_t^{t-1}. \tag{6}$$

We use $K_t$ and $I_t$ to denote the stock of capital installed at the beginning of period $t$ and aggregate investment during period $t$ respectively. So the stock of capital evolves as

$$K_{t+1} = I_t + (1-\delta) K_t, \tag{7}$$

where $0 < \delta < 1$ is the depreciation rate.

The utility function of an agent of generation $t \geq 1$ is $u\left(c_t^t, c_{t+1}^t\right)$ while the utility function of generation 0 (i.e. the initial old) is simply $c_1^0$. The budget constraint of an agent born at $t \geq 1$ is

$$\begin{aligned} c_t^t + s_t &= w_t - \tau_t^t \\ c_{t+1}^t &= s_t (1+r_t) - \tau_{t+1}^t, \end{aligned} \tag{8}$$

where $w_t$ is the wage rate at $t$. The initial old's budget constraint is

$$c_1^0 = s_0 (1+r_0) - \tau_1^0.$$

The government's budget constraint is

$$B_t + g_t = B_{t-1}(1+r_{t-1}) + \tau_t^t + \tau_t^{t-1}. \tag{9}$$

Finally, output is produced with a constant return to scale technology $F(K_t, L_t)$. Technological feasibility is given by

$$g_t + I_t + C_t = F(K_t, L_t). \tag{10}$$

Firms rent capital and labor from the households. Let $v_t$ denote the rental rate of capital Then the firm's problem in any period $t$ is

$$\max_{K_t, L_t} F(K_t, L_t) - w_t L_t - v_t K_t.$$

**Question 1)** Show that the firm's problem implies

$$\begin{aligned} F_L(K_t, L_t) &= w_t \\ F_K(K_t, L_t) &= v_{t.} \end{aligned}$$

**Question 2)** Show, by an arbitrage argument, that in any equilibrium we must have $v_{t+1} = r_t + \delta$.

The agent's problem is

$$\max_{c_t^t, c_{t+1}^t} u\left(c_t^t, c_{t+1}^t\right),$$

subject to

$$c_t^t + \frac{c_{t+1}^t}{1+r_t} = w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}.$$

**Question 3)** Argue that the solution to the agent's problem implies that savings can be written as a function of the interest rate $r_t$, current income $w_t - \tau_t$ and total wealth $w_t - \tau_t^t - \tau_{t+1}^t / (1+r_t)$, as

$$s_t = s\left(r_t, w_t - \tau_t^t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right).$$

[Remark: If we formulate the problem in a different way, it is possible to write the savings policy function as $s\left(r_t, w_t - \tau_t^t, \tau_{t+1}^t\right)$, but for future purposes it is better to have it as above.]

Definition - *Equilibrium* : Given $B_0 + s_0 = K_1$, the quantities $\left\{K_t, c_t^t, c_t^{t-1}, I_t, B_t, g_t\right\}_{t=1}^{\infty}$ and the prices $\{w_t, r_t, v_t\}_{t=1}^{\infty}$ are an equilibrium if

*i*) Agents maximize utility.

*ii*) Firms maximize profits.

*iii*) Market clearing in goods is satisfied.

*iv*) The government budget constraint holds.

*v*) $L_t = 1$.

*v*) No arbitrage holds.

**Question 4)** - *Walras' law* - Show that in equilibrium

$$B_t + s_t = K_{t+1} \text{ for all } t.$$

[Hint: use the household's budget constraint, the government budget constraint, market clearing in goods market, constant returns to scale of $F(K, L)$, $L_t = 1$ for all $t$ and $B_0 + s_0 = K_1$. Show that $B_1 + s_1 = K_2$ and then use induction in $t$].

Assume that $\tau_t^t = \tau_{t+1}^t = 0$, $g_t = 0$ for all $t$ and $B_0 = 0$. Note that from the government budget constraint this implies $B_t = 0$ for all $t$.

**Question 5)** Show that

$$\begin{aligned} K_{t+1} &= s\left(F_K\left(K_{t+1}, 1\right) - \delta, F_L\left(K_t, 1\right), F_L\left(K_t, 1\right)\right) \\ K_1 &> 0, \end{aligned} \tag{11}$$

describes an equilibrium. [Hint: use the solution to question 3].

For the rest of the problem assume $g_t = 0$ for all $t$, $B_0 = 0$, $r_t > 0$ for all $t$ (i.e. the equilibrium is such that interest rates are positive) and

$$\tau_t^t = \tau; \quad \tau_{t+1}^t = -\tau \quad \text{(i.e. social security)}.$$

**Question 6)** Show that if $c_{t+1}^t$ is a normal good (actually, not an inferior good), then savings $s(\cdot)$ are strictly decreasing in $\tau$. [Hint: It may be easier to solve this question without "taking derivatives". If $\tau' > \tau$ you can compute the change in wealth. Then use that in the "worst" case, the income effect is bounded by $-\left(\tau' - \tau\right)\frac{r_t}{1+r_t}$].

**Question 7)** Show that if $\left.\frac{ds}{dr_t}\right|_{\tau=0} > 0$ (i.e. the substitution effect is strong enough), then there exist a function $s^*(\cdot)$ such that

$$K_{t+1} = s^*\left(K_t, \tau\right).$$

**Question 8)** Show that

$$\left.\frac{ds^*}{d\tau}\right|_{\tau=0} < 0.$$

[Hint: Use the result from question 6: that if $r_t$ is kept constant, then $\frac{ds}{d\tau} < 0$. Then you only have to show that the indirect effect of a change in $\tau$ through its effect on the equilibrium interest rate has the correct sign].

The last question showed that for each level of capital, next period stock of capital will be higher in an economy without social security. This, however, is not sufficient to argue that the steady state level of capital is lower in an economy with social security.

Consider an economy without social security in a steady state $K^* > 0$. We will analyze how does $K^*$ changes if we introduce social security (for small $\tau$).

**Question 9)** Assume that in a neighborhood $K^*$ (i.e. close to $K^*$) we are in the "well behaved case":

$$0 < \frac{ds^*}{dK_t} < 1.$$

Show that an increase in $\tau$ reduces the steady state level of capital. [Hint: a graphic is sufficient].

**Question 10)** Argue that, if close to $K^*$ the dynamic system has $\frac{ds^*}{dK_t} > 1$, then the steady state capital level is higher in the economy with social security. Is that steady state stable? [Hint: suppose you start at that steady state, and imagine that the stock of capital is changed a little bit. Does the dynamic system drives you back to the original steady state?]

## 3 Production Economy: An Example

In the class notes we saw that in a pure endowment economy without population growth, if the equilibrium interest rate is positive, the introduction of social security makes the initial old better off and all the subsequent generations worse off. In this problem we will show that this need not be the case in a production economy with capital. In particular, we will construct an example where the generation born at period 1 is strictly better off with the introduction of social security.

Consider the following modification to the last problem: The production function is Cobb-Douglas $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$ and consumers only care about consumption when old, so that $u\left(c_t^t, c_{t+1}^t\right) = c_{t+1}^t$. Also assume $g_t = 0$ for all $t$ and $B_0 = 0$.

**Question 1)** Argue that $s_t = w_t - \tau_t^t$ for all $t \geq 1$.

Since $B_t = 0$ for all $t$, then the stock of capital is completely owned by the old agents, so $K_{t+1} = s_t$, or

$$K_{t+1} = w_t - \tau_t^t. \tag{12}$$

**Question 2)**. Show that in equilibrium the stock of capital evolves as.

$$K_{t+1} = (1-\alpha) K_t^{\alpha} - \tau_t^t. \tag{13}$$

**Question 3)** Assume $\tau_t^t = \tau_{t+1}^t = 0$ for all $t$. Solve for the (unique) steady state with positive capital (denote it by $K^*$).

Suppose that at the beginning of period 1 the economy is at the steady state without social security and the government announces that it will introduce a pay-as-you-go system with $\tau_t^t = \tau$ and $\tau_{t+1}^t = -\tau$ for all $t$. Denote consumption of generation $i$ in period $j$ as a function of $\tau$ by $c_j^i(\tau)$. We will construct an example where agents born at period 1 are strictly better off in the new environment.

**Question 4)** Show that consumption of generation 1 at period 2 is given by

$$\begin{aligned} c_2^1(\tau) &= (w_1 - \tau)(1 + r_1(\tau)) + \tau \\ &= (K^* - \tau)\left(1 - \delta + \alpha (K^* - \tau)^{\alpha-1}\right) + \tau. \end{aligned} \tag{14}$$

**Question 5)** Show that given $\delta > 0$, $\left.\frac{dc_2^1(\tau)}{d\tau}\right|_{\tau=0}$ can be positive for sufficiently small $\alpha$.

**Question 6)** Is it possible that *every* generation is made better off by introducing social security? [Hint: Remember that the First Welfare Theorem holds in all equilibria with positive interest rate].

**Question 7)** In the class notes we saw that the introduction of social security in an endowment economy makes everyone worse-off except the initial old, who receive the transfer without paying anything. Above we constructed an example where generation 1 is strictly better off with the introduction of social security. Can you give an intuitive explanation for this result?

## 4 Heterogeneity

Consider the OG model seen in class. Agents are now indexed by their date of birth and their endowment:

$$\begin{aligned} e_t^{t,n} &= 1-\alpha_n, \qquad e_{t+1}^{t,n} = \alpha_n, \\ e_s^{t,n} &= 0, \qquad \text{for all } s \neq t, t+1, \end{aligned}$$

where $e_s^{t,n}$ denotes the endowment at time $s$ of an individual of type $n$ born at time $t$. Preferences are described by

$$u^{t,n}\left(c_1^{t,n}, c_2^{t,n}, \ldots\right) = (1-\beta)\log c_t^{t,n} + \beta \log c_{t+1}^{t,n}, \qquad 0 < \beta < 1.$$

The initial old generation receives an endowment at time $t = 1$ only,

$$\begin{aligned} e_1^{0,n} &= \alpha_n, \\ e_s^{0,n} &= 0, \qquad \text{for all } s \neq 1. \end{aligned}$$

As usual, the initial old only care about their consumption when old, with preferences described by

$$u^{0,n}\left(c_1^{0,n}, c_2^{0,n}, \ldots\right) = c_1^{0,n}.$$

We will assume that each generation has $N$ different types of endowments $\alpha_n$, so that we can write $n = 1, 2, \ldots, N$. The set of agents $I$ for this economy consists of the pairs

$$I = \{i = (t,n) : t = 0, 1, 2, \ldots \text{ and } n = 1, 2, \ldots, N\}.$$

$(a)$ Write down the market clearing constraint in this economy, as it applies for any given period $t$.

$(b)$ Show that the competitive equilibrium is such that there is no trade *across* generations. (Hint: Start by arguing that the initial old generation will consume its endowment. Then explore the implication of this result for the initial young generation. Finally, use an inductive argument).

$(c)$ Find an expression for *aggregate savings*, $s\left(r; \{\alpha_n\}_{n=1}^N, \beta\right)$, the savings of all the young of a generation in terms of the parameters of the model ($\alpha$'s and $\beta$'s) and the net interest rate, $r$. (Hint: Use the FOCs to find an expression for $c_t^{t,n}$ and then proceed to aggregate). Find an expression for the optimal consumption choice of an agent when young and when old if her endowment is characterized by $\alpha_n = \alpha$ and if she faces an interest rate $r$. Denote these optimal choices by $c_y(\alpha, r)$ and $c_o(\alpha, r)$.

($d$) Let $\bar{\alpha} \equiv (1/N)\sum_{n=1}^{N} \alpha_n$. Show that

$$\begin{aligned} c_y(\bar{\alpha}, r) &= (1-\beta)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right], \\ c_o(\bar{\alpha}, r) &= \beta\ (1+r)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right], \end{aligned}$$

$$\begin{aligned} \frac{c_y(\alpha, r)}{c_y(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \\ \frac{c_o(\alpha, r)}{c_o(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \end{aligned}$$

$$\sum_{n=1}^{N} c_y(\alpha_n, r) = N\ c_y(\bar{\alpha}, r)\ .$$

($e$) Using the above results, characterize the equilibrium interest rate $\bar{r}$, i.e. write an equation for $\bar{r}$, which is the solution to

$$s\left(\bar{r}; \{\alpha^n\}_{n=1}^{N}, \beta\right) = 0.$$

Show that if two economies have the same value of $\bar{\alpha} \equiv (1/N)\sum_{n=1}^{N} \alpha_n$, but possible different distribution of the $\alpha$'s, then they still have the same equilibrium interest rate $\bar{r}$, so that we can write $\bar{r}(\bar{\alpha}, \beta)$. Show also that if $\bar{\alpha}$ is the same for two economies, then the equilibrium average aggregate consumption for the young is the same, i.e. $(1/N)\sum_{n=1}^{N} c_t^{t,n} \equiv (1/N)\sum_{n=1}^{N} c_y(\alpha_n, \bar{r})$ is the same regardless of the distribution of the $\alpha$'s.

($f$) Compute the "best aggregate symmetric allocation" for this economy; that is, the welfare-maximizing allocation that only depends on whether a particular agent is young or old, so that $c_t^{t,n} = c_y^*$ and $c_{t+1}^{t,n} = c_o^*$ for all $t$ and $n$.

($g$) Show that if the following condition holds:

$$\bar{\alpha} \equiv \frac{1}{N}\sum_{n=1}^{N} \alpha_n < \beta,$$

the equilibrium interest rate is negative and the allocation is not Pareto Optimal. To do so consider the following allocation:

$$
\begin{aligned}
c_t^{*t,n} &= c_y^* \frac{1+(1-\alpha_n)\bar{r}}{1+(1-\bar{\alpha})\bar{r}}, \\
c_{t+1}^{*t,n} &= c_o^* \frac{1+(1-\alpha_n)\bar{r}}{1+(1-\bar{\alpha})\bar{r}},
\end{aligned}
$$

for $t \geq 1$ and $n = 1,..,N$ (all the current and future young) and for the current old:

$$
c_1^{*0,n} = \alpha_n \ + (\beta - \bar{\alpha}),
$$

for $n = 1,..,N$, where $\bar{r}$ is the interest rate that corresponds to the equilibrium of the economy with $\bar{\alpha}$ and $\beta$.

*i*) Show that the "$*$" allocation is feasible. (Hint: Show that $(1/N)\sum_{n=1}^{N} c_t^{*t,n} = 1-\beta$ for all $t \geq 1$ and $(1/N)\sum_{n=1}^{N} c_{t+1}^{*t,n} = \beta$ for all $t \geq 0$).

*ii*) Show that this allocation Pareto dominates the equilibrium allocation for the economy with $\bar{\alpha}$ and $\beta$. Make sure that you show that this allocation is preferred for each type $n$ of the initial old and for each type $n$ of young agents. (Hint: For the young you will have to compare the utility of the equilibrium vector

$$
(c_y(\alpha_n,\bar{r}),\ c_o(\alpha_n,\bar{r})) = \left( c_y(\bar{\alpha},\bar{r}) \ \frac{c_y(\alpha_n,\bar{r})}{c_y(\bar{\alpha},\bar{r})} \ , c_o(\bar{\alpha},\bar{r}) \ \frac{c_o(\alpha_n,\bar{r})}{c_o(\bar{\alpha},\bar{r})} \right),
$$

with the one of the proposed "$*$" allocation

$$
\left( c_y^* \ \frac{1+(1-\alpha_n)\bar{r}}{1+(1-\bar{\alpha})\bar{r}} \ , c_o^* \ \frac{1+(1-\alpha_n)\bar{r}}{1+(1-\bar{\alpha})\bar{r}} \right),
$$

and, using your answer to ($d$), ($e$) and ($f$), conclude that the second is preferred to the first if and only if the young prefers $\left(c_y^*, c_o^*\right)$ to the bundle $\left(c_y(\bar{\alpha},\bar{r}), c_o(\bar{\alpha},\bar{r})\right)$. Finally, to show the last statement argue that $\left(c_y(\bar{\alpha},\bar{r}), c_o(\bar{\alpha},\bar{r})\right)$ is a feasible symmetric allocation, so that $\left(c_y^*, c_o^*\right)$ is indeed preferred by the young).

($h$) Consider introducing a pay-as-you-go social security system where young agents are taxed by $\tau$ when young, and receive $\tau$ when old, regardless of their type (i.e. regardless of n). Assume that the initial allocation is such that $\bar{r} < 0$. This will amount to change the endowments to $e_{t+1}^{t,n} = \alpha_n + \tau$ and $e_t^{t,n} = 1 - \alpha_n - \tau$.

*i*) How would the equilibrium interest rate will change? (Hint: compute the new value of $\bar{\alpha}$ )

*ii*) Is the new equilibrium allocation necessarily Pareto superior to the old equilibrium allocation? In particular, does it necessarily improve the welfare of all the young, independently of their type ($n$).

*iii*) What is the implication of the finding in *ii*) for the design of a pay-as-you-go social security system?

## 5 OG Model with Multiple Periods

We will now introduce agents living multiple periods in the basic model with homogenous generations. We will allow for general preferences and endowments. To simplify we will assume, without loss of generality for our purposes, that it is a pure-endowment economy.In this problem we will show that if equilibrium interest rate are negative, then the equilibrium is not Pareto Optimal.

We assume that agents live for $N$ periods, so that $N \geq 2$, and hence the case of $N = 2$ corresponds to the Samuelson OLG case analyzed before. The first time period is $t = 1$. Preferences are given by

$$u^t \left( c_1^t, c_2^t, ... \right) = v^t \left( c_t^t, c_{t+1}^t, ..., c_{t+N-1}^t \right) \ ,$$

for the generations born after $t \geq 1$, and

$$u^t \left( c_1^t, c_2^t, ... \right) = v^t \left( c_1^t, c_2^t, ..., c_{t+N-1}^t \right),$$

for the old generations born at $t = -N + 2, -N + 3, ..., -1, 0$.

We let $a$ denote the age of the agents, $a = 1, 2, ..., N$. Endowments are positive only while agents are alive, so that for $t \geq 1$

$$\begin{aligned} e_{t+a-1}^t &> 0, \qquad \text{all } a: \ 1 \leq a \leq N, \\ e_{t+a-1}^t &= 0, \qquad \text{otherwise}, \end{aligned}$$

and for generations born at $t = -N + 2, \ -N + 3, ..., 0$,

$$e_1^t, \ e_2^t, ..., e_{t+N-1}^t > 0,$$

and zero otherwise.

$(a)$ Write down the market clearing constraint in this economy, as it applies for any given period $t$.

$(b)$ Let $\left\{ \bar{c}^t \right\}$ denote the equilibrium consumption allocation of the generation born at time $t$. Let $\lambda^t$ be the Lagrange multiplier of the budget constraint:

$$\sum_{a=1}^{N} p_{t+a-1} \ c_{t+a-1}^t = \sum_{a=1}^{N} p_{t+a-1} \ e_{t+a-1}^t,$$

for $t \geq 1$ and

$$p_1 c_1^t + p_2 c_2^t + ... + p_{t+N-1} c_{t+N-1}^t = p_1 e_1^t + p_2 e_2^t + ... + p_{t+N-1} e_{t+N-1}^t,$$

for $t = -N+2,\ -N+3, ..., -1, 0$.

Write down the first order conditions that the optimal choice $\bar{c}^t$ for generation $t$ must satisfy.

($c$) We will define a new allocation using the equilibrium allocation $\left\{\bar{c}^t\right\}$ and two parameters $\tau$ and $Y$. The "$*$" allocation is defined as

$$c^{*t}_{t+a-1}(\tau) = \begin{cases} \bar{c}^t_{t+a-1} - \tau, & \text{all } a:\ 1 \leq a \leq Y, \\ \bar{c}^t_{t+a-1} + \frac{Y}{N-Y}\tau, & \text{all } a:\ Y+1 \leq a \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

so that when agents are young (age $a \leq Y$), we are reducing their consumption by an amount $\tau$ each period relative to the equilibrium consumption. When agents are old (age $a > Y$), we are adding an amount $\frac{Y}{N-Y}\tau$ to their consumption.

Let $P(\tau, Y)$ be the present value (at the time of birth) of the transfer described by parameters $(\tau, Y)$ for a generation born at time $t$ using prices $\{p_t\}$. Show that for generations born after time $t \geq 1$

$$P^t(\tau, Y) = \frac{1}{p_t}\left[\sum_{a=1}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\ \tau\right)\right].$$

Show that for generations born at $t = -N+2,\ -N+3, ..., -1, 0$ we have two cases: one for those that at time $t=1$ are $a = 2, ..., Y$ years old (they were born at time $t$ satisfying $-Y+2 \leq t \leq 0$, so they are young):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\ \tau\right)\right]$$

and one for those that at time $t=1$ are $a = Y+1, ..., N$ years old (they were born at time $t$ satisfying $-N+2 \leq t \leq -Y+1$, so they are old):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\ \tau\right)\right].$$

Argue that if the prices $p_t$ are increasing in $t$ for all $t \geq 1$ (so that the implied interest rates are negative) then $P^t(\tau, Y) > 0$.

($d$) Consider an equilibrium $\left\{\bar{c}^t, p_t\right\}$. Define the interest rates:

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t},$$

for $t \geq 1$. Show that if the equilibrium interest rates are negative for all $t \geq 1$ then the CE allocation $\left\{\bar{c}^t\right\}$ is not Pareto Optimal. To establish this show that the allocation $\left\{c^{*t}(\tau)\right\}$ is feasible and, at least for small $\tau$, $\left\{c^{*t}(\tau)\right\}$ Pareto dominates $\left\{\bar{c}^t\right\}$. In particular,

*i*) Show that this allocation is feasible for any $\tau$.

*ii*) Compute the marginal welfare gain (or loss) of changing the allocation by introducing $\tau$ small. Define the function $U^t(\tau)$

$$U^t(\tau) = v^t\left(c_t^{*t}(\tau), c_{t+1}^{*t}(\tau), ..., c_{t+N-1}^{*t}(\tau)\right), \tag{15}$$

for $t \geq 1$ and

$$U^t(\tau) = v^t\left(c_1^{*t}(\tau),\ c_2^{*t}(\tau),\ ...,\ c_{t+N-1}^{*t}(\tau)\right),$$

for $t = -N+2,\ -N+3, ..., -1, 0$. Differentiate $U^t(\tau)$ with respect to $\tau$ and evaluate this derivative at $\tau = 0$. Use your answer to $(c)$ and the fact that $r_t < 0$ to argue that $dU^t/d\tau > 0$.

$(e)$ Suppose that the economy has intertemporal possibilities of production, so that aggregate consumption in period $t$ does not need to be equal to the aggregate endowment in period $t$ or formally $Y \neq \varnothing$. Does your conclusion changes?, i.e. if interest rates are negative does $\{c^{*t}\}$ Pareto dominates $\{\bar{c}^t\}$? [Hint: is $\{c^{*t}\}$ feasible?]

## 6 Population Growth and Social Security

Return again to the basic model where each generation lives for two periods, preferences are logarithmic, and endowments are $\left(e_t^t, e_{t+1}^t\right) = (1-\alpha, \alpha)$ for all $t \geq 1$ and $e_1^0 = \alpha$. We will examine the welfare effects of changing demographic patterns in the presence of Social Security. To this end, let $N_t$ denote the number of young agents at time $t$. Population grows at the rate $n$, so that

$$N_{t+1} = (1+n)\,N_t, \qquad N_0 = 1,$$

where we have normalized the initial population to one.

$(a)$ Write down the market clearing constraint in this economy, as it applies for any given period $t$.

$(b)$ Briefly argue that the competitive equilibrium is such that there is no trade across and within generations.

$(c)$ Find an expression for *aggregate savings*, $s\,(r; \alpha, \beta, N)$, the savings of all the young of a generation in terms of the parameters of the model ($\alpha$, $\beta$ and $N$) and the net interest rate, $r$. (Hint: Use the FOCs to find an expression for $c_t^t$ and then proceed to aggregate). Characterize the equilibrium interest rate $\bar{r}$, i.e. write an equation for $\bar{r}$, which is the solution to

$$s\,(\bar{r}; \alpha, \beta, N) = 0.$$

$(d)$ Compute the best symmetric allocation for this economy; that is, the welfare-maximizing allocation that only depends on whether a particular agent is young or old, so that $c_t^t = c_y$ and $c_{t+1}^t = c_o$ for all $t$.

$(e)$ Compute the level of per-capita tax collection in a pay-as-you-go system that would implement the best symmetric allocation. How does this level depend on $n$? Briefly explain.

$(g)$ Assume that the competitive equilibrium with no transfers has negative real interest rates. Assume that population growth falls permanently to $n' = 0 < n$ but the transfers given to each old are held fixed, and hence the tax to each young will have to change. (Note that we are comparing two economies with different growth rates, $n$ and $n'$). Draw a graph with per-capita consumption of young and old in the $x - y$ axis. Draw two straight lines with the corresponding sets of feasible symmetric allocations for each of the two growth rates. Locate the equilibrium with no social security in your graph, the equilibrium with the best symmetric allocation for $n > 0$ , and the equilibrium with the social security system described above for the $n' = 0$ case. How is the welfare of the initial old affected when population growth decreases under the social security system with constant transfer to the old? How is the welfare of the young of the current and each future generation affected?