1 PS7 Q2

(Auctioning procurement contracts.) (This is closely related to Problem 1 from last week's Problem Set 6. You may want to consult that question and solution before tackling this one.)
A monopsony buyer is interested in purchasing a large quantity of output from one of n possible suppliers. Each supplier i has a constant marginal cost of production equal to c_i which is private information to the supplier and is uniformly distributed on $[1, 2]$. Supplier i's payoff from producing $q \in [0, Q]$ units of output for a transfer of t dollars is

$$t - c_i q.$$

Each supplier's outside option is 0. The buyer's payoff from purchasing q units of output at a total price of t dollars is

$$vq - \frac{1}{2}q^2 - t,$$

where we assume $v \geq 3$.
The buyer's objective is to design an optimal direct-revelation mechanism, $\{\phi_i, q_i, t_i\}_{i=1}^n$, where each component is a mapping from cost reports, $c = (c_1, ..., c_n)$ to probabilities of selecting firm i, output for the selected firm, and transfers to each supplier, respectively, in order to maximize

$$E_c\left[\sum_{i=1}^n \phi_i(c)\left(vq_i(c) - \frac{1}{2}q_i(c)^2 - c_i q_i(c)\right) - U_i(c_i)\right].$$

Problem 1.1. State the two conditions that any incentive compatible mechanism must satisfy. [Hint: the monotonicity condition will now involve both ϕ_i and q_i.]

Solution. First, note that all cost functions are uniformly distributed on $[1, 2]$, so the pdfs for each firm are given by:

$$f_i(c_i) = \begin{cases} 1 & \text{if } c_i \in [1, 2] \\ 0 & \text{otherwise .} \end{cases}$$

Since the cost of each firm is i.i.d, joint densities are given by:

$$f(c) = \prod_{i=1}^n f(c_i),$$
$$f_{-i}(c_{-i}) = \prod_{j \neq i} f(c_j).$$

We define the following variables as the expected value counterpart of the DRM variables:

$$\begin{aligned}
\bar{\phi}_i(c_i) &:= E_{c_{-i}}[\phi_i(c_i, c_{-i})] \\
\bar{q}_i(c_i) &:= E_{c_{-i}}[q_i(c_i, c_{-i})] \\
\bar{t}_i(c_i) &:= E_{c_{-i}}[t_i(c_i, c_{-i})],
\end{aligned}$$

where $E_{c_{-i}}[g(\cdot)] = \int_{C_{-i}} g(\cdot) f_{-i}(c_{-i}) dc_{-i}$ with $C_{-i} = [1,2]^{n-1}$ for any function $g(\cdot)$.

The IR and IC constraint for each firm i are given by:

$$\begin{aligned}
&\text{(IR):} \quad \bar{\phi}_i(c_i)(\bar{t}_i(c_i) - c_i \bar{q}_i(c_i)) \geq 0, && \forall c_i \in [1,2], \forall i \\
&\text{(IC):} \quad \bar{\phi}_i(c_i)(\bar{t}_i(c_i) - c_i \bar{q}_i(c_i)) \geq \underbrace{\bar{\phi}_i(\hat{c})(\bar{t}_i(\hat{c}) - c_i \bar{q}_i(\hat{c}))}_{:=U_i(\hat{c}|c_i)}, && \forall c_i, \hat{c} \in [1,2], \forall i
\end{aligned}$$

where we defined

$$U_i(\hat{c}|c_i) := \bar{\phi}_i(\hat{c})(\bar{t}_i(\hat{c}) - c_i \bar{q}_i(\hat{c}))$$

as the expected payoff of firm i with realized cost c_i reporting their cost as $\hat{c}$. We will provide and prove two necessary conditions for the IC constraint. The two necessary conditions are, for all $\forall i$:

1. $\bar{\phi}_i(\cdot)\bar{q}_i(\cdot)$ is nonincreasing,
2. $U_i(c_i) = U_i(2) + \int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s) ds$,

where $U_i(c_i) = U_i(c_i|c_i)$. (So the implementable transfer function, $t_i(\cdot)$, satisfies $U_i(c_i) = \bar{\phi}_i(c_i)(\bar{t}_i(c_i) - c_i \bar{q}_i(c_i))$.)

Now, we prove these necessary conditions for IC.

Proof. To avoid confusion when taking partial derivatives, we will switch notation: $U_i \Rightarrow U^i$. For truth-telling by each firm to be optimal, the first-order condition yields, $\forall c_i \in [1,2]$:

$$\begin{aligned}
0 &= U_1^i(c_i|c_i) \\
&= [\bar{\phi}_i(c_i)\bar{t}_i'(c_i) + \bar{\phi}_i'(c_i)\bar{t}_i(c_i)] - c_i[\bar{\phi}_i(c_i)\bar{q}_i'(c_i)) + \bar{\phi}_i'(c_i)\bar{q}_i(c_i))].
\end{aligned}$$

Totally differentiating the above yields:

$$0 = U_{11}^i(c_i|c_i) + U_{12}^i(c_i|c_i).$$

Note that because $U_i(\cdot|\cdot)$ is a concave in the first argument, we must have $U_{11}^i(c_i|c_i) \leq 0$. Hence, we must have

$$\begin{aligned}
&0 \leq U_{11}^i(c_i|c_i) \\
\Rightarrow\ &0 \leq -[\bar{\phi}_i(c_i)\bar{q}_i'(c_i)) + \bar{\phi}_i'(c_i)\bar{q}_i(c_i))] \\
\Rightarrow\ &0 \geq \bar{\phi}_i(c_i)\bar{q}_i'(c_i)) + \bar{\phi}_i'(c_i)\bar{q}_i(c_i)) \\
\Rightarrow\ &0 \geq \frac{\partial}{\partial c_i}(\bar{\phi}_i(c_i)\bar{q}_i(c_i)),
\end{aligned}$$

so $\bar{\phi}_i(\cdot)\bar{q}_i(\cdot)$ is non-increasing, proving the first necessary condition.

Recall $U^i(c_i) = U^i(c_i|c_i)$. By envelope theorem,

$$\begin{aligned} \frac{d}{dc_i}U^i(c_i) &= \underbrace{U_1^i(c_i|c_i)}_{=0} + U_2^i(c_i|c_i) \\ &= U_2^i(c_i|c_i) \\ &= -\bar{\phi}_i(c_i)\bar{q}_i(c_i) \leq 0. \end{aligned}$$

Because $\frac{d}{dc_i}U^i(c_i) \leq 0$ implies that the highest cost type is worst off, the optimal mechanism should only bind its IR to ensure that the IR constraints for all other types are satisfied. Switching the notation back to the original notation ($U^i \Rightarrow U_i$) and integrating both sides yields of the above equation (second equality) yields

$$U_i(c_i) = U_i(2) + \int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds,$$

proving the second necessity condition, so we are done.

■

Problem 1.2. Using you conditions in (a), find an expression $E_c[U_i(c_i)]$ that is entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

Solution. Using results from part (a):

$$\begin{aligned} U_i(c_i) &= U_i(2) + \int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds \\ \Rightarrow E_c[U_i(c_i)] &= E_{c_i}[U_i(c_i)] \quad \because \forall i \sim \text{i.i.d} \\ &= E_{c_i}[U_i(2) + \int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds] \\ &= U_i(2) + E_{c_i}[\int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds] \\ &= U_i(2) + \int_{1}^{2} \left(\int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds \right) f_i(c_i)dc_i \\ &= U_i(2) + \int_{1}^{2} \left(\int_{c_i}^{2} \bar{\phi}_i(s)\bar{q}_i(s)ds \right) f_i(c_i)dc_i. \end{aligned}$$

Now, consider the second expression on RHS:

$$
\begin{aligned}
&\int_1^2 \left(\int_{c_i}^2 \bar{\phi}_i(s)\bar{q}_i(s)ds \right) f_i(c_i)dc_i \\
&= \int_1^2 \left(\int_1^s f_i(c_i)dc_i \right) \bar{\phi}_i(s)\bar{q}_i(s)ds \\
&= \int_1^2 \bar{\phi}_i(s)\bar{q}_i(s)F_i(s)ds \\
&= \int_1^2 \bar{\phi}_i(s)\bar{q}_i(s)\frac{F_i(s)}{f_i(s)}f_i(s)ds \\
&= \int_1^2 \bar{\phi}_i(c_i)\bar{q}_i(c_i)\frac{F_i(c_i)}{f_i(c_i)}f_i(c_i)dc_i \\
&= \int_1^2 \left(\bar{\phi}_i(c_i)\bar{q}_i(c_i)(c_i - 1) \right) f_i(c_i)dc_i \\
&= E_{c_i}[E_{c_{-i}}[\phi_i(c_i, c_{-i})]E_{c_{-i}}[q_i(c_i, c_{-i})](c_i - 1)] \\
&= E_c[\phi_i(c_i, c_{-i})q_i(c_i, c_{-i})(c_i - 1)]
\end{aligned}
$$

Hence, we can express $E_c[U_i(c_i)]$ entirely in terms of $\phi_i(\cdot)$, $q_i(\cdot)$, and $U_i(2)$:

$$
\begin{aligned}
E_c[U_i(c_i)] &= U_i(2) + E_c[\phi_i(c_i, c_{-i})q_i(c_i, c_{-i})(c_i - 1)] \\
&= U_i(2) + E_c[\phi_i(c)q_i(c)(c_i - 1)],
\end{aligned}
$$

where $c = (c_i, c_{-i})$.

Problem 1.3. Using your results in (b), substitute into the buyer's objective function to obtain a maximization program that is expressed entirely in terms of $\phi(\cdot)$, $q_i(\cdot)$, and $U_i(2)$.

Solution. Recall the buyer's objective function:

$$
E_c\left[\sum_{i=1}^n \phi_i(c) \left(vq_i(c) - \frac{1}{2}q_i(c)^2 - c_iq_i(c) \right) - U_i(c_i) \right].
$$

Substituting in the expression from part (b) into the objective function,

$$
\begin{aligned}
&= E_c\left[\sum_{i=1}^{n}\phi_i(c)\left(vq_i(c)-\frac{1}{2}q_i(c)^2-c_iq_i(c)\right)\right]-E_c\left[\sum_{i=1}^{n}U_i(c_i)\right]\\
&= E_c\left[\sum_{i=1}^{n}\phi_i(c)\left(vq_i(c)-\frac{1}{2}q_i(c)^2-c_iq_i(c)\right)\right]-E_c\left[\sum_{i=1}^{n}U_i(2)+E_c[\phi_i(c)q_i(c)(c_i-1)]\right]\\
&= E_c\left[\sum_{i=1}^{n}\phi_i(c)\left(vq_i(c)-\frac{1}{2}q_i(c)^2-c_iq_i(c)\right)\right]-\sum_{i=1}^{n}U_i(2)-E_c\left[\sum_{i=1}^{n}\phi_i(c)q_i(c)(c_i-1)\right]\\
&= -\sum_{i=1}^{n}U_i(2)+E_c\left[\sum_{i=1}^{n}\phi_i(c)\left(vq_i(c)-\frac{1}{2}q_i(c)^2-c_iq_i(c)-q_i(c)(c_i-1)\right)\right]\\
&= -\sum_{i=1}^{n}U_i(2)+E_c\left[\sum_{i=1}^{n}\phi_i(c)\left((v-2c_i+1)q_i(c)-\frac{1}{2}q_i(c)^2\right)\right]
\end{aligned}
$$

Since IR binds for the highest type of all firms, we have

$$U_i(2)=0 \quad \forall i,$$

so the maximization program now becomes

$$\max_{\{\phi_i(\cdot),q_i(\cdot)\}_{i=1}^n} E_c\left[\sum_{i=1}^{n}\phi_i(c)\left((v-2c_i+1)q_i(c)-\frac{1}{2}q_i(c)^2\right)\right]$$

subject to $\sum_{i=1}^n \phi_i(c)=1$, and $\bar{\phi}_i(\cdot)\bar{q}_i(\cdot)$ nonincreasing.

Problem 1.4. Find the optimal $\phi_i(c)$ and $q_i(c)$ components of the optimal mechanism. Make whatever regularity assumptions you use to this end explicit.

Solution. Recall the maximization problem from part c:

$$\max_{\{\phi_i(\cdot),q_i(\cdot)\}_{i=1}^n} E_c\left[\sum_{i=1}^{n}\phi_i(c)\left((v-2c_i+1)q_i(c)-\frac{1}{2}q_i(c)^2\right)\right]$$

subject to $\sum_{i=1}^n \phi_i(c)=1$, and $\bar{\phi}_i(\cdot)\bar{q}_i(\cdot)$ nonincreasing.

Let

$$\Lambda(q_i(c)) := (v-2c_i+1)q_i(c)-\frac{1}{2}q_i(c)^2,$$

which is the virtual type analogue of $q_i(c)$ from Myerson's optimal auction design. Note that $\Lambda(q_i(c))$ is strictly concave in $q_i(c)$, so the FOC approach to $q_i(c)$ is valid in finding globally optimal $q_i(c)$. Let's solve the FOCs with respect to $q_i(c)$:

$$\begin{aligned}
[q_i(c)]: \quad 0 &= E_c[(v - 2c_i + 1)\phi_i(c_i) - \phi_i(c_i)q_i^*(c)] \\
&= E_c[\phi_i(c_i)(v - 2c_i + 1 - q_i^*(c))] \\
\Rightarrow q_i^*(c) &= v - 2c_i + 1.
\end{aligned}$$

Also, note that at optimal $q_i^*(c) = v - 2c_i + 1$:

$$\begin{aligned}
\Lambda(q_i^*(c)) &= (v - 2c_i + 1)q_i^*(c) - \frac{1}{2}q_i^*(c)^2 \\
[q_i^*(c_i) = v - 2c_i + 1] \Rightarrow &= \frac{1}{2}(v - 2c_i + 1) = \frac{1}{2}q_i^*(c) \\
\Rightarrow \frac{d}{dc_i}\Lambda(q_i^*(c)) &= -1 < 0, \quad \frac{d}{dq_i^*(c)}\Lambda(q_i^*(c)) = \frac{1}{2} > 0,
\end{aligned}$$

so $\Lambda(q_i^*(c))$ increasing in $q_i^*(c)$, so we have satisfied the regularity assumption for the optimal $\phi_i^*(c)$ satisfying below:

$$\phi_i^*(c) = \begin{cases} 1 & \text{if } \Lambda(q_i^*(c)) > \max_{j \neq i} \Lambda(q_j^*(c)) \\ 0 & \text{if } \Lambda(q_i^*(c)) < \max_{j \neq i} \Lambda(q_j^*(c)) \\ \frac{1}{k} & \text{if } k \text{ ties in } \Lambda(q_i^*(c)) = \max_{j \neq i} \Lambda(q_j^*(c)) \end{cases}$$

Also note that $\Lambda(q_i^*(c))$ was strictly decreasing in c_i, so we further have

$$\phi_i^*(c) = \begin{cases} 1 & \text{if } c_i < \min_{j \neq i} c_j \\ 0 & \text{if } c_i > \min_{j \neq i} c_j \\ \frac{1}{k} & \text{if } k \text{ ties in } c_i = \min_{j \neq i} c_j. \end{cases}$$

Intuitively, the buyer would put as much weight as possible ($\phi_i^*(c) = 1$) on the product from a seller yielding highest utility (one that can produce with the least cost) and 0 on products of all other less efficient firms, or randomize over the set of products whose firms share the least marginal cost ($c_i = \min_{j \neq i} c_j$).

2 PS7 Q4

The economics department is trying to procure teaching services from one of n potential lecturers. Candidate i has an outside opportunity of $\theta_i \in [0,1]$ with distribution $F(\cdot)$. This opportunity is private information and can be thought of as the candidate's type. The department gets teaching value $v(\theta_i)$ from a lecturer with type θ_i; the function $v(\cdot)$ is increasing and differentible.

Consider a direct revelation mechanism consisting of an allocation function $\phi_i(\theta_1, \ldots, \theta_n) \in [0,1]$ for each lecturer i, and a transfer function $t_i(\theta_1, \ldots, \theta_n)$ which is the payment to each lecturer i. i's utility from reporting $\hat{\theta}_i$ when her true type is θ_i is

$$\begin{aligned} U_i\left(\hat{\theta} \mid \theta\right) &= \mathbb{E}_{\theta_{-i}}\left[t_i\left(\hat{\theta}_i, \theta_{-i}\right) - \phi_i\left(\hat{\theta}_i, \theta_{-i}\right)\theta_i\right] \\ &= \bar{t}_i(\theta_i) - \bar{\phi}_i(\theta_i)\theta_i \end{aligned}$$

[Note that the lecturer utility is decreasing in θ_i and the single-crossing term is negative.] The department's objective is to maximize

$$\Pi = \mathbb{E}\left[\sum_{i=1}^{n} \phi_i(\theta) v(\theta_i) - t_i(\theta)\right]$$

Problem 2.1. Characterize incentive compatability in terms of an integral equation for the agent's utility, $U_i(\theta_i)$ and a monotonicity constraint.

Solution. Incentive compatibility requires that

$$\begin{aligned} U_i(\theta \mid \theta) &\geq U\left(\hat{\theta} \mid \theta\right) \\ U_i(\theta \mid \theta) &\geq \bar{t}_i\left(\hat{\theta}_i\right) - \bar{\phi}_i\left(\hat{\theta}_i\right)\theta_i \\ U_i(\theta \mid \theta) &\geq \bar{t}_i\left(\hat{\theta}_i\right) - \bar{\phi}_i\left(\hat{\theta}_i\right)\hat{\theta}_i + \bar{\phi}_i\left(\hat{\theta}_i\right)\hat{\theta}_i - \bar{\phi}_i\left(\hat{\theta}_i\right)\theta_i \\ U_i(\theta \mid \theta) &\geq U_i\left(\hat{\theta}_i \mid \hat{\theta}_i\right) + \bar{\phi}_i\left(\hat{\theta}_i\right)\left(\hat{\theta}_i - \theta_i\right) \\ U_i(\theta \mid \theta) - U_i\left(\hat{\theta}_i \mid \hat{\theta}_i\right) &\geq \bar{\phi}_i\left(\hat{\theta}_i\right)\left(\hat{\theta}_i - \theta_i\right) \end{aligned}$$

through a similar exercise we find that

$$
\begin{aligned}
U_i\left(\hat{\theta} \mid \hat{\theta}\right) &\geq U_i\left(\theta \mid \hat{\theta}\right) \\
U_i\left(\hat{\theta} \mid \hat{\theta}\right) &\geq \bar{t}_i\left(\theta_i\right) - \bar{\phi}_i\left(\theta_i\right)\hat{\theta}_i \\
U_i\left(\hat{\theta} \mid \hat{\theta}\right) &\geq \bar{t}_i\left(\theta_i\right) - \bar{\phi}_i\left(\theta_i\right)\theta_i + \bar{\phi}_i\left(\theta_i\right)\theta_i - \bar{\phi}_i\left(\theta_i\right)\hat{\theta}_i \\
U_i\left(\hat{\theta} \mid \hat{\theta}\right) &\geq U_i\left(\theta \mid \theta\right) + \bar{\phi}_i\left(\theta_i\right)\theta_i - \bar{\phi}_i\left(\theta_i\right)\hat{\theta}_i \\
U_i\left(\hat{\theta}_i \mid \hat{\theta}_i\right) - U_i\left(\theta \mid \theta\right) &\geq \bar{\phi}\left(\theta_i\right)\left(\theta_i - \hat{\theta}_i\right) \\
U_i\left(\theta \mid \theta\right) - U_i\left(\hat{\theta}_i \mid \hat{\theta}_i\right) &\leq \bar{\phi}\left(\theta_i\right)\left(\hat{\theta}_i - \theta_i\right)
\end{aligned}
$$

and so combining these two conditions we find that

$$
\begin{aligned}
\bar{\phi}_i\left(\hat{\theta}_i\right)\left(\hat{\theta}_i - \theta_i\right) &\leq \bar{\phi}\left(\theta_i\right)\left(\hat{\theta}_i - \theta_i\right) \\
\bar{\phi}_i\left(\hat{\theta}_i\right) &\leq \bar{\phi}\left(\theta_i\right)
\end{aligned}
$$

and so our monotonicity condition is that $\bar{\phi}\left(\theta_i\right) \leq \bar{\phi}\left(\hat{\theta}_i\right)$ for $\theta_i \geq \hat{\theta}_i$ which is our monotonicity condition.

Next, using the inequalities we derived above again, we find that

$$
\frac{\bar{\phi}_i\left(\theta_i\right)\left(\hat{\theta}_i - \theta_i\right)}{\hat{\theta}_i - \theta_i} \geq \frac{U_i\left(\theta \mid \theta\right) - U_i\left(\hat{\theta}_i \mid \hat{\theta}_i\right)}{\hat{\theta}_i - \theta_i} \geq \frac{\bar{\phi}_i\left(\hat{\theta}_i\right)\left(\hat{\theta}_i - \theta_i\right)}{\hat{\theta}_i - \theta_i}
$$

Which implies, by absolute continuity, that our integral condition is given by

$$
\begin{aligned}
U\left(\theta_i\right) &= U\left(0\right) - \int_0^{\theta_i} c_\theta\left(q\left(s\right), s\right) ds \\
&= U\left(1\right) + \int_{\theta_i}^{1} c_\theta\left(q\left(s\right), s\right) ds
\end{aligned}
$$

where

$$
\begin{aligned}
c\left(\bar{\phi}_i\left(\hat{\theta}_i\right), \theta_i\right) &= \bar{\phi}_i\left(\hat{\theta}_i\right)\theta_i \\
c_\theta\left(\bar{\phi}_i\left(\hat{\theta}_i\right), \theta_i\right) &= \bar{\phi}_i\left(\hat{\theta}_i\right)
\end{aligned}
$$

denotes the expected cost to the lecturer and the derivative of the expected cost respectively.

Problem 2.2. Using (a), what is the department's profit expressed in terms of $\phi_i\left(\cdot\right)$ and $U_i\left(1\right)$?

Solution. Now, we first need to further characterize the following expression

$$\begin{aligned}\mathbb{E}_\theta\left[U(\theta)\right] &= \mathbb{E}_\theta\left[U(1) + \int_\theta^1 \bar{\phi}_i(s)\,ds\right] \\ &= \int_0^1 \left(U(1) + \int_\theta^1 \bar{\phi}_i(s)\,ds\right) f(\theta)\,d\theta \\ &= U(1) + \underbrace{\int_0^1 \int_\theta^1 \bar{\phi}_i(s)\,ds f(\theta)\,d\theta}_{\text{Term A}}\end{aligned}$$

Using integration by parts we have the following explicit expression for Term A:

$$\begin{aligned}\int_0^1 \int_\theta^1 \bar{\phi}_i(s)\,ds f(\theta)\,d\theta &= \left[(F(\theta)+K)\int_\theta^1 \bar{\phi}_i(s)\,ds\right]_0^1 + \int_0^1 \bar{\phi}_i(\theta)(F(\theta)+K)\,d\theta \\ &= -(F(0)+K)\int_0^1 \bar{\phi}_i(s)\,ds + \int_0^1 \bar{\phi}_i(\theta) F(\theta)\,d\theta + \int_0^1 \bar{\phi}_i(\theta) K d\theta \\ &= -\int_0^1 \bar{\phi}_i(s) K ds + \int_0^1 \bar{\phi}_i(\theta) F(\theta)\,d\theta + \int_0^1 \bar{\phi}_i(\theta) K d\theta \\ &= \int_0^1 \bar{\phi}_i(\theta) F(\theta)\,d\theta \\ &= \int_0^1 \bar{\phi}_i(\theta_i) \frac{F(\theta_i)}{f(\theta_i)} f(\theta_i)\,d\theta \\ &= \mathbb{E}_{\theta_i}\left[\bar{\phi}_i(\theta_i)\frac{F(\theta_i)}{f(\theta_i)}\right]\end{aligned}$$

Next, we want to use our two equivalent expressions for $\mathbb{E}_\theta\left[U(\theta)\right]$ to solve for the transfer. In particular,

$$\begin{aligned}\bar{t}_i(\theta_i) - \bar{\phi}_i(\theta_i)\theta_i &= U(1) + \mathbb{E}_{\theta_i}\left[\bar{\phi}_i(\theta_i)\frac{F(\theta_i)}{f(\theta_i)}\right] \\ \bar{t}_i(\theta_i) &= U(1) + \mathbb{E}_{\theta_i}\left[\bar{\phi}_i(\theta_i)\frac{F(\theta_i)}{f(\theta_i)}\right] + \bar{\phi}_i(\theta_i)\theta_i\end{aligned}$$

Then we can write the seller's expected profit as

$$\begin{aligned}\Pi &= \mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta) v(\theta_i) - \left(\mathbb{E}_{\theta_i}\left[\bar{\phi}_i(\theta_i)\frac{F(\theta_i)}{f(\theta_i)}\right] + \bar{\phi}_i(\theta_i)\theta_i + U(1)\right)\right] \\ &= \mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta) v(\theta_i) - \phi_i(\theta_i)\frac{F(\theta_i)}{f(\theta_i)} - \phi_i(\theta_i)\theta_i - U(1)\right] \\ &= \mathbb{E}_\theta\left[\sum_{i=1}^n \phi_i(\theta)\left(v(\theta_i) - \frac{F(\theta_i)}{f(\theta_i)} - \theta_i\right) - U(1)\right]\end{aligned}$$

Problem 2.3. If $v'(\theta_i) \le 1$, what is the department's optimal hiring policy, $\{\phi_i(\cdot)\}_i$?

Solution. We can define the virtual type as $J_i(\theta_i) = v(\theta_i) - \frac{F(\theta_i)}{f(\theta_i)} - \theta_i$. The optimal hiring policy will then satisfy

$$\phi_i(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max_{j \neq i} J_j(\theta_j) \text{ and } J_i(\theta_i) \geq 0 \\ 0 & \text{if } J_i(\theta_i) < \max_{j \neq i} J_j(\theta_j) \text{ or } J_i(\theta_i) < 0 \\ \frac{1}{k} & \text{if } J_i(\theta_i) = \max_{j \neq i} J_j(\theta_j) \geq 0 \text{ if } k \text{ ties in virtual type} \end{cases}$$

Notice, however, that now the virtual type is decreasing in type. So the department will hire the lecturer with precisely the lowest type. This is the sense in which there is adverse selection, the lecturer who is hired is precisely the worst from the perspective of the department.

Problem 2.4. Suppose instead that $v'(\theta_i) > 2$ and $\mathbb{E}[v(\theta_i)] \geq 1$. What is the department's optimal hiring policy? [Hint: if an unrelaxed program violates the required monotonicity at every point of θ_i, then the constrained-optimal solution must be constant.]

Solution. Now the problem collapses as the condition that $v'(\theta_i) > 2$ guarantees that the highest type lecturer will also have the highest virtual type since we have $\frac{\partial}{\partial \theta} v(\theta_i) > \frac{\partial}{\partial \theta}\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right)$ and we have that the virual type is increasing in the actual type and so the highest type is hired.

3 PS7 Q6

Consider an IPV setting with $n = 2$ and θ_i uniformly distributed on $[0, 1]$ for both bidders. Suppose that $\theta_0 = 0$ (the seller's opportunity cost of the object is zero).

Problem 3.1. In the Myerson optimal auction, what is the allocation function $\phi_i(\theta, \theta_{-i})$ and what is the reserve type, $\theta^\star$ (i.e., what is the highest type bidder θ_i such that $\bar{\phi}_i(\theta_i) = 0$). In describing the optimal allocation function, you may ignore situations with zero probability (i.e., ignore ties).

Solution. The standard optimal auction result tells us that that the mechanism must satisfy

$$(1) \qquad \bar{\phi}_i(\theta_i) \text{is non-decreasing in } \theta_i \ \forall\, i$$

$$(2) \qquad U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{\phi}_i(s)\, ds$$

and then that

$$(1) \qquad U_i(\underline{\theta}_i) = 0 \quad \forall\, i$$

$$(2) \qquad \phi_i(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max_{j \neq i} J_j(\theta_j) \text{ and } J_i(\theta_i) \geq \theta_0 \\ 0 & \text{if } J_i(\theta_i) < \max_{j \neq i} J_j(\theta_j) \text{ or } J_i(\theta_i) < \theta_0 \\ \frac{1}{k} & \text{if } J_i(\theta_i) = \max_{j \neq i} J_j(\theta_j) \geq \theta_0 \text{ if } k \text{ ties in virtual type} \end{cases}$$

where the virtual type is defined as $J_i(\theta_i) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$. We can then solve for the reserve type, we just need to solve for $r^\star$ such that $J(r^\star) = 0$. Imposing our functional form assumptions, this is given by

$$\begin{aligned} J(r^\star) &= r^\star - \frac{1 - F(r^\star)}{f(r^\star)} \\ 0 &= r^\star - \frac{1 - r^\star}{1} \\ 0 &= 2r^\star - 1 \\ r^\star &= \frac{1}{2} \end{aligned}$$

Problem 3.2. Show that a first-price auction with an appropriately chosen reserve price is also optimal. Argue that the optimal reserve price is set at $r^\star = \theta^\star$.

Solution. To show that a first-price auction with an appropriately chosen reserve price is optimal, we just need show that it satisfies the conditions in Myerson's theorem. In particular, we know that in a first-price auction the lowest type will never win, and so we have that $U_i\left(\underline{\theta}_i\right) = 0$.

To see that the second requirement is satisfied, notice that the term the virtual type is increasing in the type, i.e.

$$\begin{aligned}\frac{\partial}{\partial \theta_i} J\left(\theta_i\right) &= \frac{\partial}{\partial \theta_i}\left(2\theta_i - 1\right) \\ &= 2\end{aligned}$$

So we have that the highest type will win both the first-price auction and the optimal auction described in Myerson's theorem. Finally, we just need to require that in the first price auction, the bidder only wins if his type is above the seller's type. This follows if we set the reserve price to be the seller's type, i.e. $r^\star = \theta^\star$.

Problem 3.3. Compute the equilibrium bidding function in (b) given the optimal reserve. [Hint: it is not linear. Try using the envelope theorem to find $\bar{b}\left(\cdot\right)$].

Solution. To solve for the equilibrium bidding function, first notice that

$$\begin{aligned}\bar{\phi}_i\left(s\right) &= \mathbb{E}_{\theta_{-i}}\left[\phi_i\left(\theta_i, \theta_{-i}\right)\right] \\ &= F\left(\theta_i\right) \\ &= \theta_i\end{aligned}$$

And then also notice that (with the reserve price) we get that

$$\begin{aligned}U_i\left(\theta_i\right) &= U_i\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{\theta_i} \bar{\phi}_i\left(s\right) ds \\ &= 0 + \int_{\frac{1}{2}}^{\theta_i} s ds \\ &= \frac{1}{2} s^2 \mid_{\frac{1}{2}}^{\theta_i} \\ &= \frac{1}{2}\theta_i^2 - \frac{1}{8}\end{aligned}$$

Finally, equating our two expressions for expected utility, we get that=

$$
\begin{aligned}
\frac{1}{2}\theta_i^2 - \frac{1}{8} &= F(\theta_i)\left(\theta_i - \bar{b}(\theta_i)\right) \\
\frac{1}{2}\theta_i^2 - \frac{1}{8} &= \theta_i\left(\theta_i - \bar{b}(\theta_i)\right) \\
-\frac{1}{2}\theta_i^2 - \frac{1}{8} &= -\theta_i \bar{b}(\theta_i) \\
\bar{b}(\theta_i) &= \frac{1}{2}\theta_i + \frac{1}{8\theta_i} \\
\bar{b}(\theta_i) &= \frac{\left(\frac{1}{2}\right)^2 + \theta_i^2}{2\theta_i}
\end{aligned}
$$

Problem 3.4. In order to implement the optimal auction allocation in a standard all-pay auction (i.e. highest bidder wins but everyone pays their bids), what is the optimal reserve price, $r^\star$, that must be set. [Hint: for the all-pay auction, the answer is not the same as in (b). The equilibrium bid function will be of the form $\bar{b}(\theta) = 0$ for all $\theta < \theta^\star$, jump at $b(\theta^\star) = r^\star$, and $\bar{b}(\theta) > r^\star$ for all $\theta > \theta^\star$. Try using the envelope theorem.]

Solution. We can repeat a similar procedure as in part (c). In particular, following the steps above, we know that

$$
\begin{aligned}
U_i(\theta_i) &= 0 + \int_r^{\theta_i} \bar{\phi}_i(s)\, ds \quad \theta_i > r \\
&= \left[\frac{1}{2}s^2\right]_r^{\theta_i} \\
&= \frac{1}{2}\left(\theta_i^2 - r^2\right)
\end{aligned}
$$

We also have that the expression for surplus is given as

$$
\max F(\theta_i)\,\theta_i - b(\theta_i) \qquad \theta_i > r
$$

and so simplifying this expression and equating it to the expression we found before, we have that

$$
\begin{aligned}
\frac{1}{2}\left(\theta_i^2 - r^2\right) &= \theta_i^2 - \bar{b}_i(\theta_i) \\
-\frac{1}{2}\theta_i^2 - \frac{1}{2}r^2 &= -\bar{b}_i(\theta_i) \\
\bar{b}_i(\theta_i) \frac{1}{2}&\left(\theta_i^2 + r^2\right)
\end{aligned}
$$

Now, we want that $\bar{b}\left(\theta_i\right) > r \Leftrightarrow \theta_i > \theta^2 = \frac{1}{2}$. To find this, we just need that

$$\begin{aligned}\frac{1}{2}\theta_i^2 + \frac{1}{2}r^2 &> r \\ \Leftrightarrow \theta_i^2 &> r\left(1 - \frac{1}{2}r\right)\end{aligned}$$

setting $\theta_i^2 = \left(\frac{1}{2}\right)^2$, i.e. the square of the reservation type and solving for where the inequality holds with equality, we have that

$$\begin{aligned}r - \frac{1}{2}r^2 &= \frac{1}{4} \\ r^2 - 2r + \frac{1}{2} &= 0 \\ r &= \frac{2 \pm \sqrt{4 - 4\frac{1}{2}}}{2} \\ &= \frac{2 \pm \sqrt{2}}{} \\ &= 1 \pm \frac{\sqrt{2}}{2}\end{aligned}$$

and so we get that

$$r^{\star} = 1 - \frac{\sqrt{2}}{2}$$