2015 Final Exam Q2

Two risk neutral partners seek to dissolve their partnership. Together they own and indivisible asset, and they wish to assign the asset to the one of them who values it most with the other being sufficiently compensated. Both partners own an equal share of the asset. The partners' private values for the asset are independent and drawn from a uniform distribution on $[0, 1]$. Each partner knows this and knows his own value, but does not know the values of the other partners. Assume throughout that if partner i's value is v_i, then, because he owns $1/2$ of the asset, he must obtain interim expected utility at least $v_i/2$ from the mechanism. Otherwise he will not participate and the deal will not go through. Show that there exists an expost efficient, incentive compatible, individually rational, expost budget balanced direct mechanism with which the two partners can dissolve their partnership. Does this contradict the Myerson-Satterthwaite theorem? Briefly explain.

Answer:

Let us consider a VCG mechanism. Whenever $v_i > v_j$, it is efficient for i to be assign the asset. So

$$\hat{x}(v_1, v_2) = \begin{cases} \{1\} & \text{if } v_1 > v_2 \\ \{2\} & \text{otherwise} \end{cases}.$$

Now let us think about what happens without one of the players. If $v_j > v_i$, and player i is not in the mechanism, then the outcome would not change—so player i imposes no externality on j. However, if $v_j < v_i$, then the presence of i changes the outcome and the externality imposed on j is v_j. Hence, the VCG (externality) cost function is

$$c_i^{VCG}(v_i, v_j) = \begin{cases} v_j & \text{if } v_j \leq v_i \\ 0 & \text{if } v_j > v_i \end{cases}.$$

The expected VCG cost function is

$$\begin{aligned} \bar{c}_i^{VCG}(v_i) &= \int_0^1 c_i^{VCG}(v_i, v_j) f(v_j) \, dv_j \\ &= \int_0^{v_i} v_j dv_j = \left[\frac{1}{2} v_j^2\right]_0^{v_i} = \frac{1}{2} v_i^2. \end{aligned}$$

The expected utility under the VCG mechanism is

$$\begin{aligned} u_i^{VCG}(v_i) &= \mathbb{E}_{v_j}\left[u_i\left(\hat{x}(v_i, v_j), v_i\right)\right] - \bar{c}_i^{VCG}(v_i) \\ &= \mathbb{E}_{v_j}\left[v_i \mathbf{1}\{v_i > v_j\}\right] - \bar{c}_i^{VCG}(v_i) \\ &= \int_0^{v_i} v_i f(v_j) \, dv_j - \frac{1}{2} v_i^2 \\ &= v_i^2 - \frac{1}{2} v_i^2 = \frac{1}{2} v_i^2. \end{aligned}$$

We have

$$IR_i(v_i) = \frac{v_i}{2}.$$

Let ψ_i denote the participation subsidy to i, then

$$\begin{aligned} u_i^{VCG}(v_i) + \psi_i &\geq IR_i(v_i), \ \forall v_i \in [0,1] \\ \Leftrightarrow \frac{1}{2}v_i^2 + \psi_i &\geq \frac{v_i}{2}, \ \forall v_i \in [0,1] \\ \Leftrightarrow \psi_i &\geq \frac{1}{2}v_i(1-v_i), \ \forall v_i \in [0,1]. \end{aligned}$$

This holds if

$$\psi_i^* = \frac{1}{8}.$$

since $v_i(1-v_i)$ is maximised when $v_i = 1/2$. We now wish to verify that the mechanism runs an expected surplus.

$$2\mathbb{E}_{v_i}\left[\bar{c}_i^{VCG}(v_i) - \psi_i^*\right] = 2\left(\int_0^1 \frac{1}{2}v_i^2 dv_i - \frac{1}{8}\right) = 2\left(\frac{1}{6} - \frac{1}{8}\right) > 0.$$

So the IR-VCG mechanism runs an expected surplus. Then, by the theorem from the class, we know that there exists cost functions such that the induced direct mechanism is: (i) incentive compatible, ex post efficient, budget balanced.

This does not contradict the Myerson-Satterthwaite theorem because in that theorem, we assumed that one party (the seller) has 100% of the property rights. In this question, we have that each individual has 50% of the property rights.

JR 9.11

Suppose all bidders' values are uniform on $[0,1]$. Construct a revenue maximising auction. What is the reserve price?

Answer:

Note that if values are distributed uniformly on $[0,1]$, then

$$F(v) = v, \ f(v) = 1.$$

So,

$$\mathcal{U}(v) \coloneqq v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v - 1.$$

This is clearly strictly increasing in $v \in [0,1]$. Define

$$v^* = \frac{1}{2}$$

so that virtual utility equals zero when $v = v^*$. Therefore we have the optimal assignment function

$$q_1(v_1 \dots v_N) = \begin{cases} 1 & \text{if } v_1 = \max_i\{2v_i - 1, 0\} \\ 0 & \text{ow} \end{cases},$$

and q_i, $i \neq 1$ constructed similarly (here I assumed tie is broken in favor of player 1). A revenue maximising auction is a second-price auction with reserve price $v^* = 1/2$. To see this, note that this produces the correct assignment function because it is still a dominant strategy to bid their own value, and the virtual value is strictly increasing in v. Now, to construct the cost function using the characterization theorem, we could always enforce that for all v_{-i}. (Recall that the characterization is in terms of $c_i(v_i) = E_{v_{-i}}[c_i(v_i, v_{-i})|v_i]$ but this is just a weighted sum of $c_i(v_i, v_{-i})$. By imposing the characterization for all v_{-i}, we can certainly satisfy the condition required.) Imposing the IR constraint ($c_i(0, v_{-i}) = 0 \ \ \forall v_{-i}$),

$$c_i(v_i, v_{-i}) = q_i(v_i, v_{-i})v_i - \int_0^{v_i} q_i(s, v_{-i})ds.$$

Now if $q_i(v_i, v_{-i}) = 0$, then $q_i(s, v_{-i}) = 0 \ \ \forall s \leq v_i$. So in that case $c_i(v_i, v_{-i}) = 0$. If $q_i(v_i, v_{-i}) = 1$,

$$q_i(s, v_{-i}) = \begin{cases} 1 & \text{if } s \geq \max\{v_{-i}, v^*\} \\ 0 & \text{ow} \end{cases}.$$

so

$$c_i(v_i, v_{-i}) = q_i(v_i, v_{-i})v_i - \int_0^{v_i} q_i(s, v_{-i})ds = v_i - (v_i - \max\{v_{-i}, v^*\}) = \max\{v_{-i}, v^*\},$$

which is exactly the cost function for second price auction with reserve price v^*.

9.12

Consider again the case of uniformly distributed values on $[0, 1]$. Is a first-price auction with the same reserve price as in the preceding question optimal for the seller? Prove your claim using the revenue equivalence theorem.

Answer:

Recall the revenue equivalence theorem.

Theorem. *(Revenue Equivalence Theorem). If 2 incentive compatible direct selling mechanism has the same probability assignment functions and that each bidder is indifferent between the two mechanisms when her value is zero, then the two mechanisms raise the same expected revenue.*

It is clear that when a bidder's value is zero, the expected utility for the bidder under the previous' question's second-price auction with reservation price and the first-price auction with the same reserve price are the same. That is, the second condition for the Revenue Equivalence Theorem is satisfied. It remains to verify whether the probability assignment functions are the same between the two mechanisms.

The probability assignment functions for the previous question is given by

$$p_i^{SPA}(v_1, v_2, \ldots, v_N) = \begin{cases} 1 & \text{if } v_i > v_j, v^* = \frac{1}{2},\ \forall j \neq i \\ 0 & \text{otherwise} \end{cases},$$

where v^* is the reservation price. Thus, the winner is the one with the highest valuation whose value is above the reservation price $1/2$ (to the extent that such a bidder exists).

Let us now consider the expected utility of a bidder that faces a first-price auction with reservation price/price $\underline{v}$ (i.e. the seller reveals his true private value as the reservation price price).

$$u(r_i, v_i) = \mathbf{1}\left\{\hat{b}(r_i) \geq \rho(\underline{v})\right\} F^{N-1}(r_i)\left(v_i - \hat{b}(r_i)\right),$$

where, as before, we assume that $\hat{b}$ is strictly increasing, and $\underline{v}$ denotes the seller's reservation value. We assume that $\hat{b}$ For $\hat{b}$ to be an equilibrium strategy, $u(r, v)$ must be maximised in r when evaluated at $r = v$. For $\hat{b}$ to be an equilibrium, $u(r, v)$ must be maximised in r when evaluated at $r = v$.

$$\begin{aligned} \left.\frac{\partial u(r, v)}{\partial r}\right|_{r=v} &= \left.(N-1) f(r)\left(v - \hat{b}(r)\right) F^{N-2}(r) - \hat{b}'(r) F^{N-1}(r)\right|_{r=v} \\ &= (N-1) f(v)\left(v - \hat{b}(v)\right) F^{N-2}(v) - \hat{b}'(v) F^{N-1}(v). \end{aligned}$$

Equating this to zero, we obtain that

$$\frac{d}{dv}\left[\hat{b}(v) F^{N-1}(v)\right] = (N-1) f(v) F^{N-2}(v) v.$$

Of course, this only holds when $v > \underline{v}$ (bidder do not bid if $v \leq \underline{v}$) so that

$$\begin{aligned} \hat{b}(v) F^{N-1}(v) - \hat{b}(\underline{v}) F^{N-1}(\underline{v}) &= \int_{\underline{v}}^{v} (N-1) f(x) F^{N-2}(x) x dx \\ \Leftrightarrow \hat{b}(v) &= \underline{v}\frac{F^{N-1}(\underline{v})}{F^{N-1}(v)} + \frac{1}{F^{N-1}(v)} \int_{\underline{v}}^{v} (N-1) f(x) F^{N-2}(x) x dx. \end{aligned}$$

Observe that bidding function in this case is the bidding function without reservation price plus another term. Since the bidding function is still increasing so that it remains the case that the bidder with the highest value bids the highest, and wins. If there are no bidders with values greater than the reservation price, then no sale occurs. Thus, the probability assignment function is unchanged from the case of the second price auction with the same reservation price as in the previous question. Then, by the Revenue Equivalence Theorem, we conclude that the first-price auction with the same reserve price as in the preceding question is optimal for the seller.

9.13

Suppose the bidders' values are iid, each according to a uniform distribution on $[1, 2]$. Construct a revenue-maximising auction for the seller.

Answer:

Now,

$$F(v) = v - 1, \ f(v) = 1.$$

The virtual utility is now

$$\mathcal{U}(v) := v - \frac{1 - F(v)}{f(v)} = v - \frac{1 - (v-1)}{1} = 2v - 2,$$

which is clearly increasing in v.

$$U(v) = 0 \Leftrightarrow v = 1.$$

Hence, the reservation price is $v^* = 1$, which equals the lower bound in the support of v. Hence, a standard second price auction (or with trivial reservation price of $v^* = 1$) would maximise the seller's expected revenue.

9.15

A drawback of the direct mechanism approach is that the seller must know the distribution of the bidders' values to compute the optimal auction. The following exercise provides an optimal auction that is distribution-free for the case of two asymmetric bidders, 1 and 2, with independent private values. Bidder i's strictly positive and continuous density of values on $[0,1]$ is f_i with distribution F_i. Assume throughout that

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

is strictly increasing for $i = 1, 2$.

The auction is as follows. In the first stage, the bidders each simultaneously submit a sealed bid. Before the second stage begins, the bids are publicly revealed. In the second stage, the bidders must simultaneously declare whether they are willing to purchase the object at the other bidder's revealed sealed bid. If one of them says "yes" and the other "no", then the the "yes" transaction is carried out. If they both say "yes" both say "no" then the seller keeps the object and no payments are made.

Note that the seller can run this auction without knowing the bidder's value distributions.

(a)

Consider the following strategies for the bidders. in the first stage, when her value is v_i, bidder $i \neq j$ submits the sealed bid $b_i^*(v_i) = b_i$, where b_i solves

$$r_j(b_i) = b_i - \frac{1 - F_j(b_i)}{f_j(b_i)} = \max\left(0, v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right).$$

(While such a b_i need not always exist, it will always exist if the functions $r_1(v_1) = v_1 - (1 - F_1(v_1))/f_1(v_1)$ and $r_2(v_2) = v_2 - (1 - F_2(v_2))/f_2(v_2)$ have the same range. So, assume this is the case.)

In the second stage, each bidder says "yes" if and only if her value is above the other bidder's first-stage bid.

Show that these strategies constitute an equilibrium of this auction.

(Also, note that while the seller need not know the distribution of values, each bidder needs to know the distribution of the other bidder's values in order to carry out her strategy. Hence, this auction shifts the informational burden from the seller to the bidders.)

(b)

(i) Show that in this equilibrium the seller's expected revenues are maximized. (ii) Is the outcome always efficient?

(c)

(i) Show that it is also an equilibrium for each bidder to bid his value and to then say "yes" if and only if his value is above the other's bid. (ii) Is the outcome always efficient in this equilibrium? (iii) Show that the seller's revenues are not maximal in this equilibrium.

Answer:

a)

Start from the second stage. We have the following structure:

	Y	N
Y	0,0	$v_1 - b_2$,0
N	$v_2 - b_1$,0	0,0

so what is described in the question is weakly dominant strategy.

Now consider the first stage, and player 1 deviating to $b_1' < b_1(v_1)$. If it were to change the outcome of the second stage, it must be that $b_1' < v_2 < b_1(v_1)$. When $v_1 < b_2(v_2)$ player 1 in the second stage always plays N, so it does not matter whether player 2 plays Y or N. On the other hand, if $v_1 > b_2(v_2)$, by inducing player 2 to play Y in the second stage, he is worse off. So the downward deviations are not profitable.

Let's think about the other case: $b_1' > b_1(v_1)$. The only time when the deviation seems profitable is when $v_1 > b_2(v_2)$ and $v_2 > b_1(v_1)$ (otherwise result will not change, or 1 plays N). We argue that this will not be the case, because $v_1 > b_2(v_2)$ implies

$$\begin{aligned} r_1(v_1) &> r_1(b_2(v_2)) \text{ because of monotonicity of } r_1 \\ &= \max\{0, r_2(v_2)\} \text{ by definion} \\ &\geq r_2(v_2). \end{aligned}$$

On the other hand, $v_2 > b_1(v_1)$ implies $r_2(v_2) > r_1(v_1)$. Therefore, upward deviations are not profitable for player 1 and by symmetry we conclude that this constitutes an equilibrium.

b)

- ▷ By the previous argument, we see that the mechanism allocates the good to the person with the highest virtual value. Also note that if both players have virtual value less than 0, the second stage play would be $\{Y, Y\}$ so the good will not be allocated to anyone. By the revenue equivalence theorem, and because cost to the lowest type player is 0, this mechanism is maximizing the seller's expected revenue.
- ▷ In general, revenue maximizing auctions are not efficient. There are cases where $v_1 > v_2 > 0$ but $r_1(v_1) < 0$ so that no transaction is carried over. In asymmetric cases, we could have cases where $v_1 > v_2$ but $r_1(v_1) < r_2(v_2)$.

c)

- ▷ The derivation would be exactly the same, and I am omitting it.
- ▷ It is efficient because the good is allocated to i whenever $v_i > 0$ and $v_i > v_j$.
- ▷ Efficiency usually means non-optimality in expected revenue, because pointwise maximization requires virtual values, not actual values.

9.31

Consider Example 9.3. Add the social state, "Don't Build" (D) to the set of social states so that $X = \{D, S, B\}$. Suppose that for each individual i,

$$v_i\left(D, t_i\right) = k_i$$

is independent of t_i.

(a)

Argue that one interpretation of k_i is the value of the leisure time individual i must give up towards the building of either the pool or the bridge. (For example, all the k_i might be zero except $k_1 > 0$, where individual 1 is the town's only engineer).

(b)

What are the interim individual rationality constraint if individuals have property rights over their leisure time?

(c)

When is it efficient to build the pool? The bridge?

(d)

Give sufficient conditions for the existence of ex post efficient mechanism both when individuals have property rights over their leisure time and when they do not. Describe the mechanism in both cases and show that the presence of property rights makes it more difficult to achieve ex post efficiency.

Answer:

a)

If the social state is either S or B, then i does not gain the utility of k_i. Hence, k_i represents the opportunity cost of building the pool or the bridge. If $k_1 > 0$ while $k_2 = k_3 = 0$, then we can interpret individual 1 as the engineer who would have to give up 10 utils worth of this time in building the pool or the bridge.

b)

We define

$$IR_i(t_i) \coloneqq k_i,\ \forall i.$$

Then, interim individual rationality constraint requires that

$$\sum_{x\in X} \bar{p}_i^x(t_i)\, v_i(x, t_i) - \bar{c}_i(t_i) \geq IR_i(t_i) = k_i,\ \forall t_i \in T_i,\ \forall i,$$

where

$$\bar{p}_i^x(t_i) \coloneqq \sum_{t_{-i}\in T_{-i}} q_{-i}(t_{-i})\, p^x(t_i, t_{-i}) \equiv \mathbb{E}_{t_{-i}}\left[p^x(t_i, t_{-i})\right],$$
$$\bar{c}_i(t_i) \coloneqq \sum_{t_{-i}\in T_{-i}} q_{-i}(t_{-i})\, c_i(t_i, t_{-i}) \equiv \mathbb{E}_{t_{-i}}\left[c_i(t_i, t_{-i})\right].$$

c)

Given quasilinear preferences, Pareto efficiency social outcome, given vector $t \in \times_{i=1,2,\ldots,N} T_i$ of individual types, $\hat{x}(t)$, solves

$$\hat{x}(t) \in \arg\max_{x\in X} \sum_{i=1}^{N} v_i(x, t_i),\ \forall t \in \times_{i=1,2,\ldots,N} T_i.$$

Since

$$v_i(x, t_i) = \begin{cases} t_i + 5 & \text{if } x = S \\ 2t_i & \text{if } x = B \\ k_i & \text{if } x = D \end{cases}.$$

It is efficient to build the pool if and only if

$$\begin{aligned}\sum_{i=1}^{N} v_i(S,t_i) &> \max\left\{\sum_{i=1}^{N} v_i(B,t_i), \sum_{i=1}^{N} v_i(D,t_i)\right\}\\ \Leftrightarrow \sum_{i=1}^{N} t_i + 5 &> \max\left\{\sum_{i=1}^{N} 2t_i, \sum_{i=1}^{N} k_i\right\}.\end{aligned}$$

It is efficient to build the bridge if and only if

$$\begin{aligned}\sum_{i=1}^{N} v_i(B,t_i) &> \max\left\{\sum_{i=1}^{N} v_i(S,t_i), \sum_{i=1}^{N} v_i(D,t_i)\right\}\\ \Leftrightarrow \sum_{i=1}^{N} 2t_i &> \max\left\{\sum_{i=1}^{N} t_i + 5, \sum_{i=1}^{N} k_i\right\}.\end{aligned}$$

d)

Although the question does not specify this, let us assume that the ex post efficient mechanism should satisfy budget balance. Recall from the class the following mechanism.

Theorem. *(IR-VCG expected surplus) Suppose that the IR-VCG mechanism runs an expected surplus so that*

$$\sum_{t\in T}\sum_{i=1}^{N} q(t)\left(c_i^{VCG}(t) - \psi_i^*\right) \geq 0.$$

Then, the following direct mechanism is incentive compatible, ex post efficient, budget balanced, and individually rational.

Each individual reports his type simultaneously. If the reported vector of types is $t \in T$, then the social state is $\hat{x}(t)$, and individual i pays the cost

$$\bar{c}_i^{VCG}(t_i) - \psi_i^* - \bar{c}_{i+1}^{VCG}(t_{i+1}) + \bar{c}_{i+1} - \frac{1}{N}\sum_{j=1}^{N}\left(\bar{c}_j^{VCG} - \psi_j^*\right),$$

where

$$\begin{aligned}\bar{c}_i^{VCG}(t_i) &:= \mathbb{E}_{t_{-i}}\left[c_i^{VCG}(t_i,t_{-i})\right],\\ \bar{c}_j^{VCG} &:= \mathbb{E}_{t_j}\left[\bar{c}_j^{VCG}(t_j)\right],\\ \psi_i^* &:= \max_{t_i\in T_i}\ \left(IR_i(t_i) - u_i^{VCG}(t_i)\right),\\ u_i^{VCG}(t_i) &= \mathbb{E}_{t_{-i}}\left[v_i(\hat{x}(t_i,t_{-i}),t_i) - \bar{c}_i^{VCG}(t_i)\right].\end{aligned}$$

and $\bar{c}_j^{VCG}$ is individual j's ex ante expected VCG cost (i.e. before j knows his own type).

Without property rights, we can set $\psi_i^* = 0$. Since $c_i^{VCG}(t) \geq 0$ by construction, it follows that the sufficient condition is automatically satisfied. This is not the case if $\psi_i^* > 0$; i.e. when individuals have property rights.

The intuition is that, without property rights, we were guaranteed that $c_i^{VCG}(t) \geq 0$ so that $\bar{c}_i^{VCG}(t_i) \geq 0$. This meant that, under the budget-balanced expected externality mechanism, individual i must pay individual $i+1$ an amount equal to $\bar{c}_i^{VCG}(t_i)$. However, since we are subtracting ψ_i^* from VCG cost functions, we are no longer guaranteed that $\bar{c}_i^{VCG}(t_i) \geq 0$ so that individual i might be paid by $i+1$ (i.e. $\bar{c}_i^{VCG}(t_i) < 0$). This might then lead to a violation of individual rationality for individual $i+1$, which, in turn, could lead to inefficient outcomes.

Core 2017 IV

Consider the standard independent private value auction model with n bidders. Assume that the distribution of valuation of each bidder is given by the CDF F supported on the interval $[1,2]$. (There is a pdf f on the interval $[1,2)$ corresponding to the CDF F.) In addition, the virtual valuation of each bidder is a weakly increasing function.

Consider the following allocation:

(i) if the valuation of each bidder is strictly smaller than two, the seller keeps the objective with probability one half and, with the remaining probability, the object is randomly assigned to a bidder with equal probabilities.

(ii) if there is at least one bidder whose valuation is two then the object is randomly assigned to one of those bidders whose valuation is two with equal probabilities.

(a)

Argue that this allocation can be implemented.

(b)

Suppose that a revenue-maximizing seller implements this allocation. What can one conclude about the CDF F?

Answer:

a)

Because the allocation function $q_i(v_i, v_{-i})$ is weakly increasing in v_i for all v_{-i}, there exists a cost function that implements it as an ICIR direct mechanism.

b)

Applying the pointwise maximizing principle (we can do this because of weak monotonicity of the virtual value function $r(v)$), we see from the allocation rule 1 that $r(v) = 0$ $\forall v \in [1,2)$ (otherwise the seller will not be indifferent between selling and not selling). Now, from the allocation rule 2, we have $r(2) \geq 0$.

MWG 23.E.7

Consider a bilateral trade setting in which the buyer's and seller's valuations are drawn independently from the uniform distribution on $[0,1]$.

(a)

Show that if f is a Bayesian IC and interim IR social choice function that is expost efficient then the sum of buyer's and seller's expected utility is no less than 5/6.

(b)

Show that there is no social choice function in which the sum of the utility is more than 2/3.

Answer:

a)

Let $q(v_b, v_s)$ be the buyer's allocation. Expost efficiency requires that $q(v_b, v_s) = I(v_b \geq v_s)$. Bayesian IC for the buyer requires

$$v_b = \arg\max_b E[q(b, v_s)v_b - c_b(b, v_s)|v_b] = P(v_s \leq b)v_b - c_b(b),$$

so the FOC is

$$v_b = c_b'(v_b).$$

From IR, we must have $c_b(0) \leq 0$, so interim expected utility is

$$U_b(v_b) \geq v_b^2 - \int_0^{v_b} s ds = \frac{1}{2}v_b^2.$$

On the other hand, seller's IC requires

$$v_s = \arg\max_b P(v_b \leq b)v_s - c_s(b),$$

so the FOC is

$$v_s = c_s'(v_s).$$

Here IR requires

$$v_s^2 - c_s(v_s) = \frac{1}{2}v_s^2 - c_s(0) \geq v_s \ \ \forall v_s.$$

Taking FOC, we see that IR needs to bind at $v_s = 1$, so $c_s(0) \leq -\frac{1}{2}$ and

$$U_s(v_s) \geq \frac{1}{2}v_s^2 + \frac{1}{2}.$$

Combining those results, we have

$$E[U_b + U_s] \geq \int_0^1 \int_0^1 U_b(v_b) + U_s(v_s) dv_s dv_b = \frac{1}{6} + \frac{1}{6} + \frac{1}{2} = \frac{5}{6}.$$

b)

Even if there is no incentive problem, and therefore the good is always allocated to the player with the highest value, we have

$$E[U_b + U_s] = \int_0^1 \int_0^1 \max\{v_b, v_s\} dv_s dv_b = 2 \int_0^1 \int_{v_b}^1 v_s dv_s dv_b = \frac{2}{3}.$$