10.4 Industry Investment under Uncertainty

Consider an industry in which costs of production and of investment are known with certainty and are constant over time, but in which exogenous shocks to demand follow a stationary, first-order Markov process. In addition, the technologies for both production and investment display constant returns to scale. Here we begin studying the problem of total (consumers' plus producers') surplus maximization for this industry; we complete the analysis in Section 13.4, where the issue of convergence to a stationary distribution is addressed. Then in Section 16.4 we show that the results of the surplus-maximization model can be interpreted as the competitive equilibrium of an industry with many small producers, each with the same (constant-returns-to-scale) technology.

Let $Z = [\underline{z}, \bar{z}]$ be an interval in $\mathbf{R}_+$, with its Borel subsets $\mathcal{Z}$. The exogenous state variable $z \in Z$ is an index of the strength of demand. Specifically, demand is described by the inverse demand curve $D: \mathbf{R}_+ \times Z \to \mathbf{R}_+$. That is, $p = D(q, z)$ is the market-clearing price when q is the aggregate quantity supplied and z is the state of demand. Assume that D is continuous, strictly decreasing in q, and strictly increasing in z, with

$$D(0, z) \quad 0 \quad \text{and} \quad \lim_{q \to \infty} D(q, z) = 0, \quad \text{all } z \in Z.$$

Define $L \quad \mathbf{R}_+ \times Z \to \mathbf{R}_+$ as the integral

$$U(q, z) = \int_0^q D(\nu, z) d\nu, \quad \text{all } q \in \mathbf{R}_+, z \in Z.$$

$U(q, z)$ is total consumers' surplus (the area under the demand curve) when q is the quantity consumed and z is the state of demand. Assume that U is uniformly bounded; that is, for some $A < \infty$,

$$\lim_{q \to \infty} U(q, z) \leq A, \quad \text{all } z \in$$

The endogenous state variable for the system is x, the total industry capital stock. Output is produced using capital as the only input, so that there are no direct costs of production. Without loss of generality we can choose units so that aggregate industry output is equal to the aggregate industry capital stock, $q = x$.

Investment costs are the only costs, and the unit cost of investment is assumed to depend on the (percentage) rate of increase in the capital stock. Specifically, if the current capital stock is $x > 0$ and next period's stock is $y > 0$, then the cost of investment is $xc(y/x)$. Assume that $c: \mathbf{R}_+ \to \mathbf{R}_+$ is continuously differentiable and that for some $\delta \in (0, 1)$, $c(a) = 0$ on $[0, 1 - \delta]$ and c is strictly increasing and strictly convex on $(1 - \delta, +\infty)$. (See Figure 10.1.) The parameter δ is interpreted as the rate of depreciation of capital. Note that since δ is strictly less than one, for $x > 0$ the strictly positive capital stock $y = (1 - \delta)x$ can be carried over at no cost. Since costs of investment are not defined for $x = 0$, we will exclude that point from the state space and let $X = \mathbf{R}_{++}$.

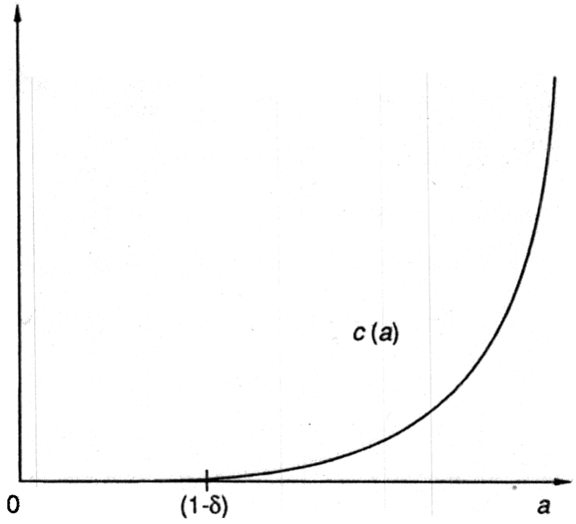

Figure 10.1

Also note that since there are constant returns to scale in both production and investment, the distribution of the capital stock over firms does not matter. For our purposes here, it may be easier to think of this problem as one involving just one firm. We return to this issue in Section 16.4, where we show that the solution developed here can be interpreted as a competitive equilibrium outcome.

Let Q be a transition function on $(Z, \mathscr{Z})$; assume that Q is monotone and has the Feller property. Given an initial value $z_0 \in Z$, the evolution of demand conditions over time is described by the transition function Q. Assume that the interest rate $r > 0$ is constant over time. Consider the problem of maximizing total expected discounted consumers' plus producers' surplus, given the initial state (x_0, z_0):

$$\sup \mathrm{E}\left\{\sum_{t=0}^{\infty} (1 + r)^{-t}[U(x_t, z_t) - x_t c(x_{t+1}/x_t)]\right\}$$

Exercise 10.4 a. Give a precise statement of the problem in (1) in sequence form, and show that the supremum function for that problem is well defined. What can be said about the relationship between the supremum function and solutions to the functional equation

$$(2) \qquad v(x, z) = \sup_{y \in X} \; U(x, z) - xc(y/x) + (1 + r)^{-1} \int v(y, z')Q(z, dz')\Big]?$$

What can be said about the relationship between optimal plans for the sequence problem and the policy correspondence for (2)?

Our next task is to establish the existence and uniqueness of a function $v: X \times Z \to \mathbf{R}$ satisfying (2) and to characterize that function as precisely as possible. Note that since $X = \mathbf{R}_{++}$, the maximization in (2) is over a set that is neither closed nor bounded. To sidestep this problem, it is useful to define $\Gamma: \mathbf{R}_{++} \to \mathbf{R}_{++}$ by

$$\Gamma(x) = [(1 - \delta')x, M], \quad x \in (0, M],$$

$$\Gamma(x) = [(1 - \delta')x, x], \quad x \in (M, +\infty).$$

where $\delta < \delta' <$, and M is a suitably chosen (very large) number.

Exercise 10.4 b. Show that there exists a unique bounded continuous function $v: X \times Z \to \mathbf{R}_+$ satisfying

$$(2') \qquad v(x, z) = \max_{y \in \Gamma(x)} \Big[U(x, z) - xc(y/x) + (1 + r)^{-1} \quad v(y, z')Q(z, dz') \Big]$$

Show that the function v satisfying (2′) also satisfies (2).

c. Show that v is strictly increasing in both arguments and strictly concave in its first argument. Show that the optimal policy correspondence associated with v is a (single-valued) continuous function; call this function g. Show that for each $(x, z) \in X \times Z$, $g(x, z)$ lies in the interior of the set $\Gamma(x)$. Show that for each $z \in Z$, $v(\cdot, z)$ is continuously differentiable on X.

d. Show that for each $z \in Z$, $g(x, z)$ is strictly increasing in x, but with a slope strictly less than one. (Refer to Figure 10.2.) Notice that these facts imply that the growth rate in aggregate capacity, $g(x, z)/x$, is strictly decreasing in current capacity x.

e. Show that for each $x \in X$, $g(x, z)$ is nondecreasing in z, and is strictly increasing at points where gross investment is strictly positive: where

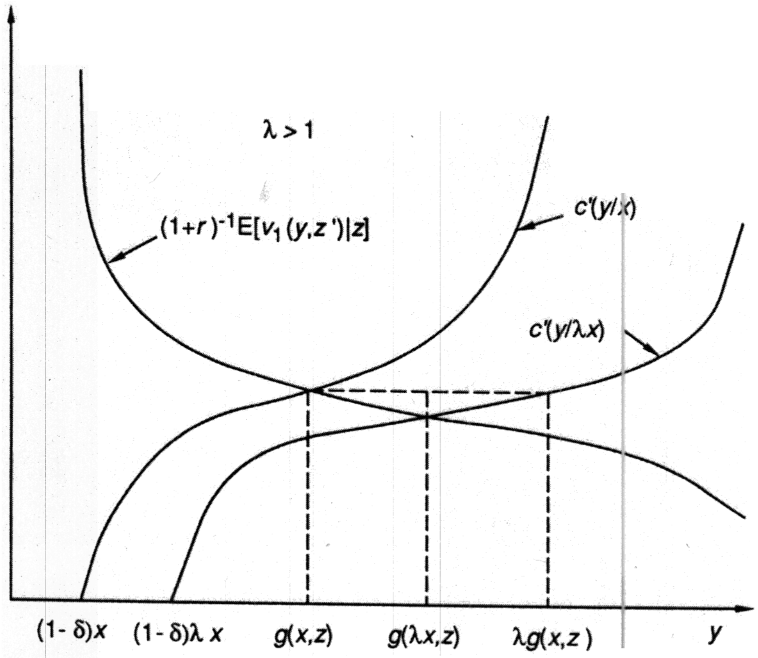

Figure 10.2

$g(x, z) \gg (1 - \delta)x$. (Refer to Figure 10.3.) Notice that these facts imply that the growth rate in aggregate capacity, $g(x, z)/x$, is strictly increasing in the state of current demand, z.

Next, consider the long-run behavior of the aggregate capital stock under the assumption that the demand shocks are i.i.d. That is, suppose that there is a probability measure μ on $(Z, \mathscr{Z})$ such that $Q(z, \cdot) = \mu(\cdot)$, all $z \in Z$.

Exercise 10.4 f. Which, if any, of the conclusions in parts (a)–(e) are changed under the assumption of i.i.d. shocks to demand? Show that under this assumption, the optimal policy function does not depend on z; that is, it can be written as simply $g(x)$. Hence for any $x_0 > 0$ the unique optimal plan is given by the deterministic difference equation $x_{t+1} = g(x_t)$, $t = 0, 1, \ldots$. Show that for any $x_0 > 0$ the optimal sequence $\{x_t\}$ converges to a stationary point $\hat{x}$ that is independent of x_0. Under what assumptions on demand and costs is $\hat{x}$ strictly positive?

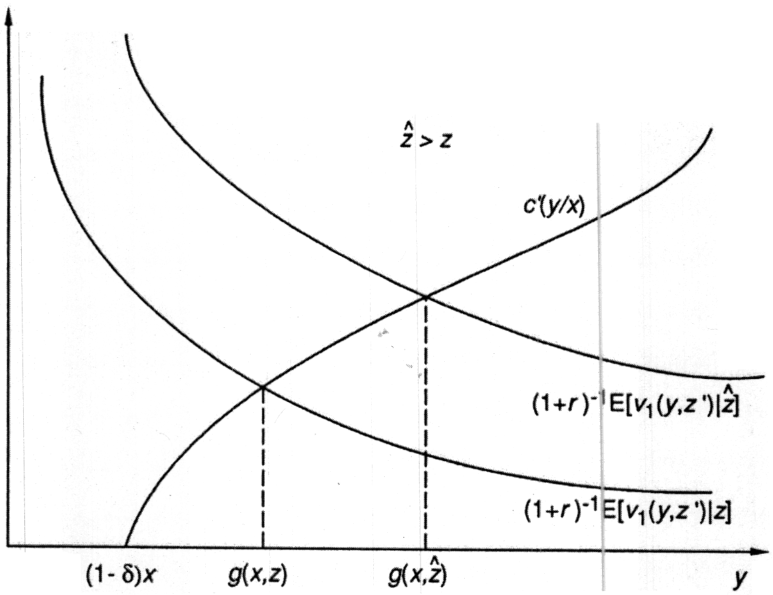

Figure 10.3

10.5 Production and Inventory Accumulation

In markets for many agricultural commodities, inventories play an important role in smoothing the stochastic shocks to supply that result from fluctuations in the weather. In this section we study the determination of consumption, production, and inventories in such a setting. Here, as in Section 10.4, we study the problem of maximizing total (consumers' plus producers') surplus. As noted there, the arguments to be discussed in Chapter 15 can be used to show that the solution to this problem can be interpreted as a competitive equilibrium allocation.

Assume that demand is constant over time and is described by the inverse demand curve $D: \mathbf{R}_+ \to \mathbf{R}_+$. That is, $D(q)$ is the market-clearing price when $q > 0$ is the quantity supplied. Assume that D is continuous and strictly decreasing, with $0 < D(0) < \infty$, and $\lim_{q\to\infty} D(q) = 0$. Define the integral $U: \mathbf{R}_+ \to \mathbf{R}_+$ by

$$U(q) = \int_0^q D(\nu)d\nu, \quad \text{all } q \in \mathbf{R}_+,$$