Class Note I: Theory of Income

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Fall 2018

Basic Math Review

- The Hyperplane Separation Theorem
- Maximization with inequality constraints
- The Envelope Theorem

Hyperplane Separation Theorem

Given two convex disjoint sets A, B contained in R^m we can always find a hyperplane (i.e. a line in R^2, a plane in R^3, etc) such that A is fully contained in one side of the hyperplane while B is fully contained in the other side of the hyperplane.

Theorem: Let A, B be subsets of R^m. Assume that A and B are **convex** and have **no element in common**, i.e. $A \cap B = \emptyset$. Then, there exist a vector $p \in R^m, \ p \neq 0$, such that

$$p \cdot x \geq p \cdot y \text{ for all } x \in A, \text{ and all } y \in B.$$

Maximization Problem

- Objective function: $f : R^m \times R^l \rightarrow R$
- n constraints, $g : R^m \times R^l \rightarrow R^n$
- Problem P:

$$P : L(\theta) = \max_x f(x, \theta) \text{ subject to } g(x, \theta) \geq 0$$

- m choice variables x. The l variables θ are parameters. $L(\theta)$ gives the maximized value of the objective function.

If g is concave, then the set $\{x : g(x, \theta) \geq 0\}$ is convex.

Necessary and sufficient conditions

If f and g are **concave** (plus technical conditions), then x^* solves problem P if and only if there is a vector $\lambda \in R^n$ such that:

$$\begin{aligned} \frac{\partial f(x^*,\theta)}{\partial x} + \lambda \frac{\partial g(x^*,\theta)}{\partial x} &= 0, \\ \lambda\, g(x^*,\theta) &= 0. \end{aligned} \tag{1}$$

Remark. Concavity of f and g can be relaxed for the necessary conditions. In particular, concavity of g is not needed. Consider the case of one constraint, $m = 2$. By the implicity function theorem, providing that $g(x_1^*, x_2^*) = 0$ and that $g_1(x_1^*, x_2^*) \neq 0$ at that point, there is a differentiable function γ satisfying

$$g(\gamma(x_2), x_2) = 0$$

for all x_2 in a neigborhoood of x_2^*. Then differentiating

$$g_1(\gamma(x_2), x_2)\gamma'(x_2) + g_2(\gamma(x_2), x_2) = 0$$

or

$$\gamma'(x_2) = -\frac{g_2(\gamma(x_2), x_2)}{g_1(\gamma(x_2), x_2)}$$

and evaluating at x_2^*

$$\gamma'(x_2^*) = -\frac{g_2(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)}$$

Using this function to solve for the constraint in the objective function we have that if (x_1^*, x_2^*) is the solution of problem P, then x_2^* must solve

$$\max_{x_2} f(\gamma(x_2), x_2)$$

that has foc

$$f_1(x_1^*, x_2^*)\gamma'(x_2^*) + f_2(x_1^*, x_2^*) = 0 \tag{2}$$

or, using the expression for γ' :

$$\frac{f_2(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)} = \frac{f_1(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)}.$$

Divide the foc (2) by $\gamma'(x_2^*)$, and divide and multiply by g_1

$$f_1(x_1^*, x_2^*) + \frac{f_2(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)\gamma'(x_2^*)} g_1(x_1^*, x_2^*) = 0,$$

use the expression for γ' to obtain

$$f_1(x_1^*, x_2^*) + \left[-\frac{f_2(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}\right] g_1(x_1^*, x_2^*) = 0 \tag{3}$$

Take the foc (2) and divide it and multiply by g_2,

$$f_2(x_1^*, x_2^*) + \frac{f_1(x_1^*, x_2^*)\gamma'(x_2^*)}{g_2(x_1^*, x_2^*)} g_2(x_1^*, x_2^*) = 0,$$

and using the expression for γ' obtain

$$f_2(x_1^*, x_2^*) + \left[-\frac{f_1(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)}\right] g_2(x_1^*, x_2^*) = 0 \tag{4}$$

Setting

$$\lambda = \left[-\frac{f_1(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)}\right] = -\left[\frac{f_2(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}\right]$$

then (3) and (4) give

$$\frac{\partial f(x^*)}{\partial x} + \lambda \frac{\partial g(x^*)}{\partial x} = 0 .$$

Remark. Concavity of f and g can not be ignored for sufficiency of the foc.

Assume

$$\hat{g}(x_1, x_2, \theta) = g(x_1, x_2) - \theta$$

Write $x_1 = \gamma(x_2, \theta)$, then

$$\frac{\partial x_1(x_2^*, \theta)}{\partial \theta} = \frac{\partial \gamma(x_2^*, \theta)}{\partial \theta} = -\frac{1}{g_1(x^*)}$$

Interpretation of λ :

$$\lambda = f_1(x_1^*, x_2^*) \left[-\frac{1}{g_1(x^*)} \right] = \frac{\partial f(x^*)}{\partial x_1} \frac{\partial x_1(x_2^*, \theta)}{\partial \theta}$$

and using (1):

$$\lambda = f_2(x_1^*, x_2^*) \left[-\frac{1}{g_2(x^*)} \right] = \frac{\partial f(x^*)}{\partial x_2} \frac{\partial x_2(x_1^*, \theta)}{\partial \theta}$$

Remark. Both x^* and λ are functions of θ, so we write $x^*(\theta), \lambda(\theta)$. Notice, since the foc (1) must hold for any θ, we have

$$L(\theta) = f(x^*(\theta), \theta) + \lambda(\theta) g(x^*(\theta), \theta)$$

Envelope Theorem:

Under further technical conditions:

$$\frac{dL(\theta)}{d\theta} = \frac{\partial f(x^*(\theta), \theta)}{\partial \theta} + \lambda(\theta) \frac{\partial g(x^*(\theta), \theta)}{\partial \theta}$$

Proof (sketch): Assuming that x^* and λ are differentiable, totally differentiate the Lagrangean with respect to θ:

$$\begin{aligned}\frac{dL(\theta)}{d\theta} &= \left[\frac{\partial f(x^*(\theta),\theta)}{\partial x} + \lambda(\theta)\frac{\partial g(x^*(\theta),\theta)}{\partial x}\right]\frac{\partial x^*(\theta)}{\partial\theta} \\ &+ [g^*(x^*(\theta),\theta)]\frac{\partial\lambda(\theta)}{\partial\theta} \\ &+ \frac{\partial f(x^*(\theta),\theta)}{\partial\theta} + \lambda(\theta)\frac{\partial g(x^*(\theta),\theta)}{\partial\theta}\end{aligned}$$

Using the first order conditions

$$\frac{\partial f(x^*,\theta)}{\partial x} + \lambda\frac{\partial g(x^*,\theta)}{\partial x} = 0$$

and that either

$$g(x^*(\theta),\theta) = 0$$

or

$$\lambda(\theta) \equiv 0$$

we obtain the desired result

$$\partial\lambda(\theta)/\partial\theta = 0$$

Of course, this sketch presumes that x^* and λ are differentiable, a fact that needs to be established for this proof to be rigorous.

Main concepts of General Equilibrium Theory

- Definition of an economy: Commodity space, households, firms, ownership structure
- Feasible allocation
- Competitive Equilibrium (CE)
- Pareto Optimal allocations (PO)
- Welfare Theorems:
  1. First Welfare theorem: CE are PO
  2. Second Welfare theorem: PO allocations are CE
- Classical (as in old but good) reference: Debreu's theory of value `http://cowles.econ.yale.edu/P/cm/m17/`

Definition of an Economy

- *Commodity space L*. Typically we use $L = R^m$ so that there are m goods. Sometimes we will let $m = \infty$, e.g. in the overlapping generations model.

  Interpretation of Commodity space. Differentiate commodities in terms of characteristic of the good (classical), location (geography and trade), time (dynamics), state of nature (uncertainty).

- *Households*. We index households by $i = 1, 2, ..., I$, so that there are I agents. Abusing notation, denote also by I the set of agents. Sometimes we will deal with an infinite number of agents, either with a countably infinite such as $i = 1, 2, 3, ...$, or with a continuum.

- Each agent has utility function u^i defined on her consumption possibility set $X^i \subset L$.

- The interpretation is that household i is able to consume any vector $x \in X^i$. This set may include constraints such as non-negativity of consumptions, a maximum on the number of hours that a household can supply per period, etc.

- The standard convention in general equilibrium analysis is that positive entries of x denotes quantities demanded (purchases) for the households, and negative entries denotes quantities offered (sales) by the households.

- *Firms.* We index firms by $j = 1, 2, ..., J$, so that there are J firms. We abuse notation and we sometimes denote the set of firms by J.

- The technology of each firms is represented by a set $Y^j \subset L$. The interpretation of this set is that firm j can produce any vector $y \in Y^j$. These sets include, for instance, the description of the production functions of each firm.

- The standard convention in general equilibrium analysis is that positive entries of y denote quantities offered by the firms and negative entries of y denotes quantities demanded (purchased) by the firms.

- *Ownership*. We let consumer i own a fraction $\theta_j^i \geq 0$ of firm j.
- Thus, for each firm j we require that it will be owned entirely by households,

$$\sum_{i \in I} \theta_j^i = 1 \text{ for all } j \in J.$$

- We also let agent i own a quantity $e^i \in L$, which we refer as to the endowment of agent i.

Feasible allocations

Definition. A feasible allocation $\{x^i, y^j\}$ satisfies three conditions: each consumer can consume x^i,

$$x^i \in X^i \text{ for all } i \in I$$

each firm can produce y^j,

$$y^j \in Y^j \text{ for all } j \in J$$

and that demand equals supply,

$$\sum_{i \in I} x^i = \sum_{j \in J} y^j + \sum_{i \in I} e^i.$$

Key concept for Competitive Equilibrium: price vector p

- Let $L = \mathbb{R}^m$ so that there are m commodities.
- We use $p = (p_1, \ldots, p_m)$ for the vector of m prices.
- $p\,x$ for $x \in L$ means $p\,x = p_1 x_1 + p_2 x_2 + \cdots + p_m x_m \equiv \sum_{s=1}^m p_s x_s$
- $p\,x$ is the value of the commodity vector x in terms of the numeraire.
- $p\,x$ is the inner product of p and x.
- $p\,x^a + p\,x^b = p\,(x^a + x^b)$ for any two $x^a, x^b \in L$.

Competitive equilibrium (CE)

Definition. We denote prices by a vector $p \in R^m$. A competitive equilibrium is a price vector p, and a feasible allocation $\{x^i, y^j\}$ such that

i) each firm $j \in J$ maximize its profits. We denote the profit of firm j by π^j.

$$y^j \text{ solves } \quad \pi^j \equiv \max_y p\,y \quad \text{s.t. } \; y \in Y^j$$

and ii) consumers maximize utility subject to her budget constraint:

$$x^i \text{ solves } \max_{x \in X^i} u^i\left(x^i\right) \text{ s.t. } p\,x \leq p\,e^i + \sum_{j \in J} \theta^i_j \pi^j.$$

The LHS of the budget constraint has the value of the net purchases of the households. The RHS of the budget constraint contains the source of funds for the net purchases.

Pareto Optimal Allocations (PO)

Definition. A feasible allocation $\{\bar{x}^i, \bar{y}^j\}$ is P.O. if there is no other feasible allocation $\{x^i, y^i\}$ preferred by everybody, i.e. one such that

$$\begin{aligned} u^i\left(x^i\right) &\geq u^i\left(\bar{x}^i\right) \text{ for all } i \in I \\ u^i\left(x^i\right) &> u^i\left(\bar{x}^i\right) \text{ for some } i \in I \end{aligned}$$

Welfare Theorems: CE vs PO

1st Welfare Thm: CE allocations are PO allocations

2nd Welfare Thm: (under convexity assumptions) PO allocations are CE

First Welfare Theorem: CE are PO

Assumption. Local Non-satiation. u^i, X^i satisfies local non satiation if for any $x \in X^i$ and any neighborhood of x, $B_\varepsilon(x)$, there is an $\hat{x} \in B_\varepsilon(x) \cap X^i$ such that $u^i(\hat{x}) > u^i(x)$, where given $\varepsilon > 0$,

$$B_\varepsilon(x) = \left\{x' \in X^i : \|x - x'\| < \varepsilon\right\}$$

and $\|x\| = \sum_{k=1}^m x_k^2$ is the Euclidean norm.

1st Welfare Theorem:

Let $\{p, \bar{x}^i, \bar{y}^j\}$ be a CE equilibrium.

Assume that the preferences for all agents satisfies Local Non-satiation.

Then $\{\bar{x}^i, \bar{y}^j\}$ is a PO allocation.

Proof. By contradiction, assume that there is a feasible allocation $\{x^i, y^j\}$ that Pareto dominates $\{\bar{x}^i, \bar{y}^j\}$, i.e.

$$\begin{aligned} u^i\left(x^i\right) &\geq u^i\left(\bar{x}^i\right) \text{ for all } i \in I \text{ and} \\ u^{i'}\left(x^{i'}\right) &> u^{i'}\left(\bar{x}^{i'}\right) \text{ for some } i' \in I \end{aligned}$$

Then, it must be that

$$\begin{aligned} p\, x^i &\geq p\, \bar{x}^i \text{ for all } i \in I && (5) \\ p\, x^{i'} &> p\, \bar{x}^{i'} \text{ for } i' && (6) \end{aligned}$$

To see why, notice that if, by contradiction of (5), $p\, x^i < p\, \bar{x}^i$, then, by the Local-non satiation assumption, there must be a $\hat{x} \in X^i$ in a neighborhood of x^i such that $u^i(\hat{x}) > u\left(x^i\right)$, and by choosing the neighborhood small enough, $p\, \hat{x} \leq p\, \bar{x}^i$. This will contradict that $\bar{x}^i$ maximizes utility, and hence (5) must hold. To see that (6) must hold, notice that if, by contradiction, $p\, x^{i'} \leq p\, \bar{x}^{i'}$, then $x^{i'}$ is budget-feasible, and hence it contradicts that $\bar{x}^{i'}$ solves the consumer problem.

Then, using (5), and adding across consumers

$$\sum_{i \in I} p\, x^i \geq \sum_{i \in I} p\, \bar{x}^i$$

and using that (6) holds with strict inequality for consumer i',

$$\sum_{i \in I} p\, x^i > \sum_{i \in I} p\, \bar{x}^i$$

or

$$p \sum_{i \in I} x^i > p \sum_{i \in I} \bar{x}^i \qquad (7)$$

Also, since $\bar{y}^j$ maximizes profits for all j, and y^j is feasible,

$$p\, \bar{y}^j \geq p\, y^j$$

and thus, adding across j,

$$p \sum_{j \in J} \bar{y}^j \geq p \sum_{j \in J} y^j\,. \qquad (8)$$

Since both $\{x^i, y^j\}$ and $\{\bar{x}^i, \bar{y}^j\}$ are feasible,

$$\begin{aligned} p \sum_{i \in I} x^i &= p \sum_{j \in J} y^j + p \sum_{i \in I} e^i, \\ p \sum_{i \in I} \bar{x}^i &= p \sum_{j \in J} \bar{y}^j + p \sum_{i \in I} e^i \end{aligned}$$

a contradiction with (7) and (8). QED.

- *Question.* If u is continuous and strictly increasing, does it satisfy local-non satiation?
- *Question.* If u is continuous and increasing, does it satisfy local-non-satiation?
- *Question.* If u is strictly increasing, but not necessarily continuous, does it satisfy local non-satiation?
- *Question.* Suppose that all u are convex (instead of concave). Does the first welfare theorem hold?

- *Question.* Suppose that Y is the production possibility set corresponding to a firm with increasing returns to scale. Does the first welfare theorem hold?
- *Question.* Suppose that there is only one firm providing one of the goods (literally one, not just one type). Does the welfare theorem hold?

- *Question.* Suppose that there are external effects in consumption. For instance, the consumption of agent i enter, directly, into the utility function of the consumption of agent i'. In a CE, agent i' takes that equilibrium consumption of agent i as given. Does the first welfare theorem hold?
- *Question.* Suppose that there are external effects in production. For instance, the production of firm j affects directly, the production possibility set of firm j'. In a CE assume that firm j' takes firm j production plan as given. Does the first welfare theorem hold?

- *Question.* Suppose that there are three goods but there are market for only two (markets are incomplete). Does the first welfare theorem hold? What is the notion of Efficiency to which we are comparing the equilibrium with?
- *Question.* Does the first welfare theorem requires that agents agree on their view of the word? Does it requires that their view of the world is the reasonable?
- *Question.* What do agents have to agree about the world in the classical notion of competitive equilibrium?
- *Question.* What is the interpretation of inter-temporal budget constraint? (To be discussed later).

- *Question.* Can we incorporate “frictions", such as difficulties to find a job, a house, etc on this notion of equilibrium? What are the implications for efficiency? (To be discussed later and more in detailed in a different class)
- *Question.* Can we incorporate a role for “money" on this notion of equilibrium? What are the implications for efficiency? (To be discussed later)
- *Question.* Can we incorporate a nominal contracting on this notion of equilibrium? What are the implications for efficiency? (To be discussed later, or in a different class)

Second Welfare Theorem

Assumption HH. X^i are convex for all i, and u^i are continuous, strictly quasi-concave, i.e. the upper contour sets of u

$$\left\{x \in X^i : u^i(x) \geq u^i(\bar{x})\right\}$$

are **strictly convex** for all i and all $\bar{x} \in X^i$.

Assumption FF. The aggregate production set of the economy is convex, i.e.

$$Y = \left\{ y \in L : y = \sum_{j \in J} y^j, \text{ for some } y^j \in Y^j, \text{ all } j \in J \right\}$$

Definition: "**Strictly convex**" means:

If

$$\begin{aligned} x, x' &\in X^i, \text{ and} \\ u^i(x') &> u^i(x) \end{aligned}$$

then

$$u^i\left(\ \theta x + (1-\theta)\, x'\ \right) > u^i(x)$$

$$\text{for } \theta \in (0,1)$$

Striclty convex implies convex.

2nd Welfare Theorem (main part). Let $\{\bar{x}^i, \bar{y}^j\}$ be a PO allocation. Then, there exists a price vector p such that,

i) all firms maximize profits

$$p\,\bar{y}^j \geq p\,y \text{ for all } y \in Y^j \text{ all } j$$

and

ii) given the allocation $\bar{x}^i$, consumers minimize expenditure subject to attain at least the same utility obtained with $\bar{x}^i$, i.e.

$$\bar{x}^i \text{ solves } \min_{x \in X^i} p\,x \text{ subject to } u^i(x) \geq u^i(\bar{x}^i)$$

Proof. Let A be the set-sum of all the upper contour set of the households, i.e.

$$A = \left\{ x \in L : x = \sum_{i \in I} x^i, x^i \in X^i, u^i(x^i) \geq u^i(\bar{x}^i), \text{ all } i \in I \right\}$$

Notice that, since by hypothesis, $\{\bar{x}^i, \bar{y}^j\}$ is a PO allocation,

$$Int(A) \cap B = \emptyset, \tag{9}$$

$$B \equiv Y + \left\{ \sum_{i \in I} e^i \right\}$$

Where $Int(\cdot)$ denotes the interior of the set. Recall that Y is the aggregate feasible production set.

If the intersection between *Int* (A) and B were not empty, then one could find a feasible allocation (i.e., an allocation in $Y + \left\{\sum_{i \in I} e^i\right\}$) that it is also in *Int* (A). And since this allocation is *Int* (A), then it Pareto dominates $\left\{\bar{x}^i\right\}$.

Notice that by Assumptions HH and FF, the sets A and B are both convex. Using the hyperplane separation theorem, and the conditions (9), we can find a vector $p \neq 0$, such that

$$p\,x \;\geq\; p\,z \tag{10}$$

for all $x \in A$ and $z \in B$. Notice that since $\left\{\bar{x}^i, \bar{y}^j\right\}$ is feasible,

$$p \sum_{i \in I} \bar{x}^i \;= p \sum_{j \in J} \bar{y}^j \;+\; p \sum_{i \in I} e^i \tag{11}$$

Thus,

$$p\,\bar{y}^j \geq p\,y$$

for all $y \in Y^j$, i.e. firms maximize profits. If this were not the case, say if there were a $\hat{y}$ such that

$$p\,\hat{y} > p\,\bar{y}^j \text{ for } \hat{y} \in Y^j$$

then one can construct a z that will violate (11) and (10).

Likewise,

$$\bar{x}^i \text{ solves } \min_{x \in X^i} p\, x \text{ subject to } u^i(x) \geq u^i\left(\bar{x}^i\right).$$

If this were not the case, say if there were a $\hat{x} \in X^i$ such that

$$p\,\hat{x} \;<\; p\,\bar{x}^i \text{ and } u^i(\hat{x}) \geq u^i\left(\bar{x}^i\right)$$

then one can construct a x that will violate (11) and (10). QED.

For the next result we strengthen the notion of quasi-concavity.

Arrow's remark.

Assume that u^i is strictly quasi-concave and continuous. Assume that, $\bar{x}^i$ is not the "cheapest" point in the budget set of household i, i.e. assume that for all i, there is a $\tilde{x}^i \in X^i$ such that

$$p\,\tilde{x}^i < p\,\bar{x}^i.$$

Then, the PO allocation $\left\{\bar{x}^i, \bar{y}^j\right\}$ and price vector p is a competitive

equilibrium (i.e. households not only minimize expenditure, but they also maximize utility subject to their budgets constraints).

Proof. We want to show that $\bar{x}^i$, which minimizes expenditure subject to attaining utility at least $u^i\left(\bar{x}^i\right)$, also maximizes utility subject to a budget constraint with expenditure not higher than $p\,\bar{x}^i$.

By way of contradiction, suppose that there is a $\hat{x} \in X^i$, $p\,\hat{x} \leq p\,\bar{x}^i$ and $u^i\left(\hat{x}\right) > u\left(\bar{x}^i\right)$. Then, let $x^\theta = \theta\,\tilde{x} + (1-\theta)\;\hat{x}$ for $\theta \in (0,1)$.

Since $u^i\left(\hat{x}\right) > u^i\left(\bar{x}^i\right)$, then, using the continuity and strict quasiconcavity of u^i, we have that for all θ close enough to 0,

$$u^i\left(x^\theta\right) \geq u^i\left(\bar{x}\right).$$

Moreover,

$$\begin{aligned} p\,x^\theta &= \theta\,p\,\tilde{x} + (1-\theta)\;p\,\hat{x} \\ &< \theta\,p\,\bar{x}^i + (1-\theta)\;p\,\bar{x}^i \end{aligned}$$

where the inequality is strict, since $\tilde{x}$ is the cheapest point. Thus, x^θ has an expenditure strictly smaller than $\bar{x}^i$, a contradiction.

QED.

- *Question.* Suppose that $\{\bar{x}^i, \ \bar{y}^j\}$ is a P.O. allocation, and p is the price that (by virtue of the 2nd welfare theorem) decentralizes this allocation. Describe an ownership structure, i.e. $e's$ and $\theta's$, for which $\{\bar{x}^i, \ \bar{y}^j\}$ and p is a competitive equilibrium. Is there a unique one?

- *Question.* Construct an example with $m = 2$ where $\bar{x}$ minimizes expenditure subject to a given level of utility. Make sure that the utility function is quasi-concave. Display your example graphically, plotting the corresponding indifference curves and budget constraint. Hint: Obviously, in your example the cost minimizing choice must be the cheapest point. Try with one good with zero price and u increasing, but without Inada conditions.

- *Question.* What assumptions on u^i and X^i will be sufficient to ensure that in any PO allocation $\{\bar{x}^i, \bar{y}^j\}$, each agent will have a cheapest point.

- *Question.* Sketch an Edgeworth box (two consumers, two goods) where a PO allocation can not be decentralized as a CE. Draw the indifference curves for one agent whose utility function is NOT quasi-concave. (i.e. the 2nd welfare theorem does not hold)

- *Question.* Draw a graph of a production possibility set with one firm and two goods, consumption and labor. Assume that there is only one consumer with utility function between consumption and leisure that is quasi-concave. Draw the production possibility set that will correspond to a non-concave production function; in particular that it will correspond to a production function that has a fixed cost and an increasing marginal cost. Make sure that your diagram is such that the Pareto optimal allocation can not be decentralized (i.e. the 2nd welfare theorem does not hold)