Recursive Methods in Economic Dynamics

NANCY L. STOKEY
AND
ROBERT E. LUCAS, JR.

with Edward C. Prescott

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and hence that $N(x_2) > N(x_1)$. Use (4) to show that this, in turn, impli that

$$\int \omega v'[Y(x_1) + N(x_1)\omega]\, d\mu < \int \omega v'[Y(x_2) + N(x_2)\omega]\, d\mu. \tag{6}$$

Next we will use (5) and (6) to obtain a contradiction to the hypothes that $Y(x_2) \leq Y(x_1)$ and $Y(x_1) > 0$. Define

$$\gamma(\omega) = v'[Y(x_2) + N(x_2)\omega] - v'[Y(x_1) + N(x_1)\omega].$$

Since $Y(x_2) \leq Y(x_1)$ and $N(x_2) > N(x_1)$, it follows immediately that

$$\gamma(\omega) \gtreqless 0 \text{ as } \omega \lesseqgtr A, \quad \text{where } A = [Y(x_1) - Y(x_2)]/[N(x_2) - N(x_1)] >$$

Then (5) and (6) respectively state that $\mathrm{E}[\gamma(\omega)] < 0$ and $\mathrm{E}[\gamma(\omega)\omega] > ($

Exercise 10.5 f. Show that this is a contradiction.

Hence if $x_2 > x_1$, then either $Y(x_2) > Y(x_1)$ or $Y(x_2) = Y(x_1) = 0$.

Exercise 10.5 g. Show that $N(x) > 0$, all x, so that (4) alwa holds with equality. Use this fact and the result established in part (e) show that $x_2 > x_1$ implies that $N(x_2) < N(x_1)$. How would this result l changed if $c'(0) > 0$?

10.6 Asset Prices in an Exchange Economy

In this problem we study the determination of equilibrium asset prices a pure exchange economy. There are a finite number of producti assets, each in fixed supply, that produce random quantities of the sing consumption good each period; we call these dividends. Thus, an asset a claim to a stochastic dividend stream. We normalize units of assets : that there is one unit of each asset per consumer. There are a lar number of consumers, all with identical tastes and with equal endo ments of all the assets. The consumption good is not storable.

In each period there are spot markets for the single consumptic good and for shares in the assets. Since all agents are identical, tl competitive equilibrium quantities are trivial: in each period each co sumer holds one unit of each asset and consumes all of the dividends (consumption goods) produced by those assets. That is, although competitive markets open each period, no consumer actually chooses a nonzero trade. Our goal is to characterize the prices that support this allocation as a competitive equilibrium.

The preferences of the representative consumer over random consumption sequences are

$$\text{(1)} \qquad \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right],$$

where U: $\mathbf{R}_+ \to \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $U(0) = 0$, and where $\beta \in (0, 1)$.

There are $i = 1, \ldots, k$ productive assets. Let Z be a compact subset of $\mathbf{R}^k_+$, with its Borel subsets $\mathscr{Z}$. Dividends in any period are described by a vector $z = (z_1, \ldots, z_k) \in Z$, where z_i denotes the dividend (the quantity of current consumption good) paid by one unit of asset i. The dividends follow a Markov process, with stationary transition function Q on $(Z, \mathscr{Z})$, and Q has the Feller property.

Ownership of assets is determined by trading on a competitive stock market, where in each period the current consumption good and shares in all of the assets are traded. Prices in each period are normalized so that the price of the current consumption good is unity. Our goal is to characterize asset prices in a stationary competitive equilibrium. In any such equilibrium, asset prices in each period are described by a (stationary) function p: $Z \to \mathbf{R}^k_+$, where $p(z) = [p_1(z), \ldots, p_k(z)]$ is the vector of asset prices that prevail if the current state of the economy is $z \in Z$. Each consumer takes the function p as given. (Note that we are making use of the fact that in equilibrium the distribution of wealth across consumers does not change. If consumers were heterogeneous in either preferences or endowments of assets or both, then in general the distribution of wealth would change over time, and the joint distribution of tastes and wealth would be another state variable.)

The consumer's asset holdings in any period are described by a vector $x = (x_1, \ldots, x_k) \in \mathbf{R}^k_+$. His problem, viewed in period 0, is as follows. Given the price function p, the initial state of the economy z_0, and his initial asset holdings x_0, he wants to choose a sequence of plans for consumption and end-of-period asset holdings that maximizes his present discounted expected utility (1) subject to the constraints

$$c_t + p(z_t) \cdot x_{t+1} \leq [z_t + p(z_t)] \cdot x_t, \quad \text{all } z^t, \text{ all } t,$$

$$c_t, x_{t+1} \geq 0, \quad \text{all } z^t, \text{ all } t.$$

Exercise 10.6 a. Formulate the consumer's problem in sequen form. Show that given any continuous price function p, the supremu function v^{p*} for the consumer's problem is well defined.

In order to study the functional equation corresponding to the pro lem in (a), it is convenient to impose an upper bound on the consumei holdings of any asset. Since we know that in equilibrium the consum chooses to hold exactly one unit of each asset, any bound $\bar{x} > 1$ nonbinding in equilibrium. Then let $X = [0, \bar{x}]^k$ with its Borel subsets and define the correspondence $\Gamma: X \times Z \to X$ by

$$\Gamma(x, z) = \{y \in X: p(z) \cdot y \leq [z + p(z)] \cdot x\}.$$

Thus $\Gamma(x, z)$ is the set of feasible end-of-period portfolios if the co sumer's beginning-of-period portfolio is x and the current state is z. F simplicity, we consider the case where all assets have strictly positi dividends in all states, $Z \subset \mathbf{R}^k_{++}$, and hence the price function is strict positive in all states.

Exercise 10.6 b. Show that for any continuous, strictly positi function $p: Z \to \mathbf{R}_{++}$, there exists a unique bounded continuous functic $v^p : X \times Z \to \mathbf{R}_+$ such that

$$v^p(x, z) = \max_{y \in \Gamma(x,y)} \left\{U[z \cdot x + p(z) \cdot (x - y)] + \beta \int v^p(y, z')Q(z, dz')\right.$$

Show that for each $z \in Z$, v^p is strictly increasing and weakly concave in Show that the associated policy correspondence G^p is nonempty, cor pact-valued, convex-valued, and u.h.c. Explain briefly why v^p is on weakly concave in x, and why there may be multiple optimal portfolic Show that for any $(x, z) \in X \times Z$,

$$p(z) \cdot y = p(z) \cdot y', \quad \text{all } y, y' \in G^p(x, z).$$

c. Let p be any continuous, strictly positive price function, and let (v G^p) be as defined in part (b). Fix $(x, z) \in X \times Z$. Show that if $x \in \text{int}\, X$ ar if $G^p(x, z) \cap \text{int}\,\Gamma(x, z)$ is nonempty, then $v^p(\cdot, z)$ is differentiable with respect to x at (x, z), with derivatives given by

$$(1) \qquad v_i^p(x, z) = \partial v^p(x, z)/\partial x_i = U'[z \cdot x + p(z) \cdot (x - y)][z_i + p_i(z)],$$

$$i = 1, \ldots, k, \text{ all } y \in G^p(x, z).$$

Show that if $G^p(x, z) \cap \text{int}\,\Gamma(x, z)$, is nonempty, then

$$(2) \qquad U'[z \cdot x + p(z) \cdot (x - y)]p_i(z) = \beta \int v_i^p(y, z')Q(z, dz'),$$

$$i = 1, \ldots, k, \text{ all } y \in G^p(x, z).$$

To compute the equilibrium prices for this economy, we will use the fact that (1) and (2) hold at equilibrium and that we know the equilibrium quantities in advance. Specifically, we know that if each consumer begins with one unit of each asset, then in equilibrium he must choose to hold that same portfolio in every successive period, under any realization for the endowment shocks, and hence consume the dividends from that portfolio. Note that since $Z \subset \mathbf{R}^k_{++}$ it follows that $z \cdot \underline{1} > 0$, all $z \in Z$, so $\underline{1} \in \text{int}\,\Gamma(\underline{1}, z)$, all $z \in Z$.

Suppose that p^* is an equilibrium price function and that p^* is continuous. Let v^* be the value function described in (b), and let v_i^*, $i = 1, \ldots, k$, denote the partial derivatives of v^*.

Exercise 10.6 d. Show that

$$(3) \qquad v_i^*(\underline{1}, z) = U'(z \cdot \underline{1})[z_i + p_i^*(z)], \quad i = 1, \ldots, k; \quad \text{and}$$

$$(4) \qquad U'(z \cdot \underline{1})p_i^*(z) = \beta \int v_i^*(\underline{1}, z')Q(z, dz'), \quad i = 1, \ldots, k.$$

Substituting from (3) into (4) to eliminate the v_i^*'s, we find that

$$(5) \qquad U'(z \cdot \underline{1})p_i^*(z) = \beta \int U'(z' \cdot \underline{1})[z_i' + p_i^*(z')]Q(z, dz'), \quad i = 1, \ldots, k.$$

Any continuous function p^* satisfying (5) is an equilibrium price function. Our final task is to show that there exists exactly one such function.

To this end, define the functions

$$h_i(z) = \beta \int U'(z' \cdot \underline{1}) z_i' Q(z, dz'), \quad i = 1, \ldots, k.$$

Then finding an equilibrium price function $p^*(z) = [p_1^*(z), \ldots, p_k^*(z)]$ equivalent to finding functions $\phi_1(z), \ldots, \phi_k(z)$ satisfying the k (indepe dent) functional equations

$$\phi_i(z) = h_i(z) + \beta \int \phi_i(z')Q(z, dz'), \quad i = 1, \ldots, k. \tag{6}$$

If a solution to (6) can be found, then $p_i^*(z) = \phi_i(z)/U'(z \cdot \underline{1})$, $i = 1, \ldots$ is a solution to (5).

Exercise 10.6 e. Show that for each $i = 1, \ldots, k$, there exist unique bounded continuous function ϕ_i satisfying (6). Show that each is strictly positive. Express the solution $\phi_i(z)$ in terms of the function and the iterates Q^n of the transition function.

f. What is the interpretation of (5)? What can be said about as prices in the case where utility is linear, $U(c) = c$? What can be said in t case where there are a very large number of assets, with i.i.d. return

g. How can dividends and asset prices of zero be incorporated?

10.7 A Model of Search Unemployment

Consider a worker who begins each period with a current wage of and has two alternative actions. He can work at that wage or he c search for a new wage offer. If he chooses to search, he earns nothi during the current period, and his new wage is drawn according to so fixed probability measure. He cannot divide his time within a per between searching and working. Moreover, if a worker chooses to wo during the current period, then with probability $1 - \theta$ the same wag available to him next period. But with probability θ he will lose his jo the beginning of next period and begin next period with a "wage" zero.

The worker does not value leisure, and his preferences over rand consumption sequences $\{c_t\}$ are given by

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right].$$