1 Short Answers

Skipped.

2 Saving for Retirement

Skipped.

3 Growth with a Stochastic Price for Investment Goods

Skipped.

4 Financing Infrastructure

Consider an economy where taxes are used to finance purchase of government capital – infrastructure – that is used in production. Specifically, the government levies a tax at flat rate $\tau \in (0,1)$ on output and uses all of the revenue to finance investment in government capital. Let G be the stock of government capital. The representative household uses its own private capital and government capital to produce goods, and the household's capital stock evolves in the usual way. Therefore, the household's resource constraint is

$$c + \left[k' - (1-\delta)\,k\right] - (1-\tau)\,F\,(k,G) \leq 0, \quad 0 < \delta < 1$$

where F is strictly increasing, strictly quasi-concave, homogeneous of degree one, and continuously differentiable with $\lim_{k\to 0} F\,(k,G) = +\infty$.

The household is infinitely lived, with discount factor $\beta \in (0,1)$ and maximizes the present discounted value of its lifetime utility. Its period utility function $u\,(c)$ is strictly increasing, strictly concave, and continuously differentiable with Inada conditions holding. There is a continuum of identical households of measure one. In addition to its own capital stock k, there are two aggregate stocks of interest to an individual household: the government capital stock G and the aggregate capital stock K. The latter is important because it affects the rate at which government capital is accumulated.

Since he government uses all of the revenue to invest in additional government capital, the stock of infrastructure evolves as:

$$G' = (1-\delta)\,G + \tau F\,(K,G)$$

Assume investment is reversible. Then the household's Bellman equation is

$$v\,(k;G,K) = \max_{k' \in [0,(1-\delta)k+(1-\tau)F(k,G)]} u\left\{(1-\tau)\,F\,(k,G) - \left[k' - (1-\delta)\,k\right]\right\} + \beta v\left(k';G',\gamma\,(G,K)\right)$$

where $\gamma\,(G,K)$ describes the households' conjecture about the evoluation of K. Assume that v is strictly increasing and strictly concave in k for each fixed (G,K) and that the optimal policy function $h\,(k;G,K)$ is single-valued and continuous.

Problem 4.1. Does h depend on γ? Explain your answer briefly.

Solution. Recall that h is the policy function and γ is the conjecture.

▷ h depends on γ in the sense that the household optimizes assuming that K evolves with respect to $\gamma\,(G,K)$.

▷ In the end, for the RCE to exist, h must equal γ since there is a unit continuum of households.

■

Problem 4.2. Write the FOC for the choice of $k' = h\,(k;G,K)$.

Solution. Recall that the BE is given as:

$$v\,(k;G,K) = \max_{k' \in [0,(1-\delta)k+(1-\tau)F(k,G)]} u\left\{(1-\tau)\,F\,(k,G) - \left[k' - (1-\delta)\,k\right]\right\} + \beta v'\left(k';G',\gamma\,(G,K)\right)$$

The FOC is then:

$$u'\left((1-\tau)\,F\,(k,G) - \left[k' - (1-\delta)\,k\right]\right) >=< \beta v'\left(k';G',\gamma\,(G,K)\right)$$

▷ $=$ holds for interior values of k'.

▷ Both boundary values are ruled out by the Inada conditions.

Therefore, the FOC holds with equality:

$$u'\left((1-\tau)F(k,G)-\left[k'-(1-\delta)k\right]\right)=\beta v\left(k';G',\gamma(G,K)\right)$$

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Problem 4.3. Is h increasing (strictly increasing) in G for fixed (k,K)? Explain your answer.

Solution. We can't say how $\beta v_1(\cdot)$ reacts to G, so we cannot say h is increasing in G. ■

Problem 4.4. What are the requirements on (γ,h) for a RCE for this economy?

Solution. The conjecture must be correct, i.e.

$$\gamma(G,K)=h(k;G,K)$$

The rates of return on capital must be equal:

$$F_1(K,G)=F_1(k,G)$$

■

Problem 4.5. Write the Envelop condition and the Euler equation for the household.

Solution. The EC is given as:

$$v'(k;G,K)=[(1-\delta)+(1-\tau)F_1(k,G)]\,u'\left((1-\tau)F(k,G)-\left[k'-(1-\delta)k\right]\right)$$

Recall the FOC we had:

$$u'\left((1-\tau)F(k,G)-\left[k'-(1-\delta)k\right]\right)>=<\beta v'\left(k';G',\gamma(G,K)\right)$$

Therefore, the EE is:

$$u'\left((1-\tau)F(k,G)-\left[k'-(1-\delta)k\right]\right)=\beta\left[(1-\delta)+(1-\tau)F_1\left(k',G'\right)\right]u'\left((1-\tau)F\left(k',G'\right)-\left[k''-(1-\delta)k'\right]\right)$$

where

$$\begin{aligned}k'&=h(k;G,K)\\k''&=h\left(h(k;G,K),G',K'\right)\end{aligned}$$

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Problem 4.6. What condition describes the steady state of the economy?

Solution. Denote k^* as the steady state and plug it into the EE to obtain:

$$(1-\delta)+(1-\tau)\, F_1\left(k^*, G^*\right)=\frac{1}{\beta}$$

Furthermore, for the government: we have

$$G^*=\frac{\tau}{\delta} F\left(k^*, G^*\right)$$

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Problem 4.7. What is the rate of return on private capital in the steady state?

Solution. Since F is CRS, the rate of return should be equal to $F_1\left(k^*, G^*\right)$, which yields:

$$\begin{aligned}
&(1-\delta)+(1-\tau)\, F_1\left(k^*, G^*\right)=\frac{1}{\beta} \\
&\Rightarrow F_1\left(k^*, G^*\right)=\frac{1}{1-\tau}\left[\frac{1}{\beta}-(1-\delta)\right]
\end{aligned}$$

■

For the next questions, for convenience assume:

$$F(K, G)=A K^{1-\theta} G^\theta, \quad A>0, \quad \theta<1/2$$

where the restriction of θ insures private capital has a larger share than public capital.

Problem 4.8. Write the steady-state conditions using these functional forms. Does a steady state exist? If so, is it unique?

Solution. Recall our equations for the steady state:

$$\begin{aligned}
&[1]:(1-\delta)+(1-\tau)\, F_1\left(k^*, G^*\right)=\frac{1}{\beta} \\
&[2]: G^*=\frac{\tau}{\delta} F\left(k^*, G^*\right)
\end{aligned}$$

▷ Specializing to the functional form:

$$\begin{aligned}
F(K, G) &= A K^{1-\theta} G^\theta \\
F_1(K, G) &= (1-\theta)\, A\left(\frac{G}{K}\right)^\theta
\end{aligned}$$

▷ Plugging in yields:

$$[1] : (1-\delta) + (1-\tau)(1-\theta) A \left(\frac{G^*}{K^*}\right)^{\theta} = \frac{1}{\beta}$$
$$[2] : G^* = \frac{\tau}{\delta} A K^{1-\theta} G^{\theta}$$

▷ Rearranging [2]:

$$\left(\frac{G^*}{K^*}\right)\frac{\tau}{\delta} = A\left(\frac{G^*}{K^*}\right)^{\theta}$$

and plugging into [1] yields:

$$(1-\delta) + (1-\tau)(1-\theta)\left(\frac{G^*}{K^*}\right)\frac{\tau}{\delta} = \frac{1}{\beta}$$

which implies that G^* and K^* are only identified up to a ratio.

▷ Rearranging yields:

$$\frac{G^*}{K^*} = \frac{\delta}{\tau(1-\tau)(1-\theta)}\left[\frac{1}{\beta} - (1-\delta)\right]$$

■