Theory of Income I

Fall 2018

Solution to the Midterm

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1 OLG with finite horizon

Time is discrete and runs $t = 1, 2, \ldots, T$, where $T < \infty$. Consider an OLG model where agents live for two periods. There is a measure one of agents born in each period. Agents born in period t have an endowment $e_t^t > 0$ and $e_{t+1}^t > 0$ in period t and $t+1$, and zero in all other periods (the superscript indexes the period that the agent was born and the subscript indexes the time period) for agents born at $t = 1, \ldots, T-1$. For those born at $t = 0$ they only have non-zero endowment at e_1^0. For those born at $t = T$ they only only have non-zero endowment at e_T^T . The utility function of agents born in period t is $u^t\left(c_t^t, c_{t+1}^t\right)$, which is strictly increasing and strictly concave for $t = 1, \ldots, T-1$. The utility function of initial old and those born in period T are given respectively by $u^0\left(c_0^0\right)$ and $u^T\left(c_T^T\right)$; they are both strictly increasing. Feasibility is given by $c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t$ for all $t = 1, 2, \ldots, T$

1.1 Section 1

Is autarky (i.e. period consumption equals period endowment for every agent) the only equilibrium? If you say yes, sketch an argument - maximum three lines. If you say no, give an argument - also maximum of three lines.

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Yes, autarky is the only equilibrium.

For the initial old, since they only value consumption at $t = 1$, we have that, in any equilibrium, they consume their entire endowment; i.e.

$$c_1^0 = e_1^0 \tag{1}$$

Similarly, for those born in $t = T$,

$$c_T^T = e_T^T \tag{2}$$

Feasibility at $t = 1$ give us the consumption of the first young,

$$c_1^1 = e_1^1 + e_1^0 - c_1^0 = e_1^1 \tag{3}$$

But then, from the budget constraint of the young we get that $c_2^1 = e_2^1$. Then, for any t, if the old generation consumes their entire endowment then the young consume their endowment as well. This follows from the same argument—at time t, if $c_t^{t-1} = e_t^{t-1}$ then feasibility implies $c_t^t = e_t^t$ and generation t budget constraint $c_{t+1}^t = e_{t+1}^t$ (proof by induction as we saw in class, though since it's finite, we don't really need induction).

Hence, in any equilibrium must be autarky; i.e.

$$\left\{\bar{c}_1^0, \left\{\bar{c}_t^t, \bar{c}_{t+1}^t\right\}_{t=1}^{T-1}, \bar{c}_T^T\right\} = \left\{e_1^0, \left\{e_t^t, e_{t+1}^t\right\}_{t=1}^{T-1}, e_T^T\right\}. \tag{4}$$

Comment: When I went over the proof in one of the TA class, I forgot to mention that you need to use the budget constraint. I have since updated the TA class notes to reflect this so please do take a look. Given this, I did not take any marks off for not mentioning the budget constraint.

1.2 Section 2

Assume that $u^t\left(c_t^t, c_{t+1}^t\right) = (1-\beta)\log c_t^t + \beta \log c_{t+1}^t$ for $t = 1, 2, \ldots, T-1$, $u^0\left(c_1^0\right) = \log c_1^0$ and $u^T\left(c_T^T\right) = \log c_T^T$. Also, assume that $e_t^t = 1-\alpha$ and $e_{t+1}^t = \alpha$ for $t = 1, 2, \ldots, T-1$, $e_1^0 = \alpha$ and $e_T^T = 1-\alpha$. Compute the equilibrium interest rate for this case.

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Generation $t \in \{1, 2, \ldots, T-1\}$ solves

$$\begin{aligned} \max_{c_t^t, c_{t+1}^t} \quad & (1-\beta)\log c_t^t + \beta \log c_{t+1}^t \\ s.t. \quad & c_t^t + s_t = e_t^t, \\ & c_{t+1}^t = e_{t+1}^t + (1+r_t)\, s_t. \end{aligned}$$

Expressing consumption in terms of savings using the constraints, we can write the Lagrangian for this problem as

$$\mathcal{L} = (1-\beta)\log\left(e_t^t - s_t\right) + \beta \log\left(e_{t+1}^t + (1+r_t)\, s_t\right).$$

The first-order condition given by

$$\{s_t\}: \quad \frac{1-\beta}{\bar{c}_t^t} = \frac{\beta}{\bar{c}_{t+1}^t}(1+r_t), \tag{5}$$

where we've used the constraints again.

In question 1, we already solved for the equilibrium allocations. Substituting for the equilibrium allocations gives us the equilibrium interest rate

$$1 + r_t = \frac{1-\beta}{\beta} \frac{e_{t+1}^t}{e_t^t} = \frac{1-\beta}{\beta} \frac{\alpha}{1-\alpha}. \tag{6}$$

We can also write

$$\begin{aligned} r_t &= \frac{\alpha(1-\beta) - \beta(1-\alpha)}{\beta(1-\alpha)} \\ &= \frac{\alpha - \beta}{\beta(1-\alpha)} \end{aligned}$$

Comment: I think the way we derive the expression here is the quickest way. Time can be a constraint so it's good to know how you can solve for the interest quickly.

1.3 Section 3

Using the specification in the previous question, is the equilibrium allocation necessarily Pareto optimal, regardless of the value of α and β?

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Yes, the competitive equilibrium is Pareto optimal no matter what values the parameters α and β take. This is because the time horizon is finite, so even if the interest rate were negative the present value of endowments would still be finite (because we have finite individuals in finite number of periods) so that the First Welfare Theorem holds.

Comment: Full marks were awarded for the correct answer. I did see a number of people referring to the infinite horizon case—i.e. whether competitive equilibrium is Pareto optimal depends on the sign of the interest rate. Here, this is not the case since the value of endowments are necessarily finite.

1.4 Section 4

Assume that $\beta > \alpha$. Are interest rates positive or negative? What are the welfare consequences of introducing a pay-as-you-go social security system in which young agents are taxed $\beta - \alpha$ and the old agents receive a transfer of $\beta - \alpha$. Make sure you consider all the generations.

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Given the assumption on the parameters α and β, combined with equation (6), we have

$$r_t = \frac{\alpha - \beta}{\beta (1 - \alpha)} < 0. \tag{7}$$

That is, when $\beta > \alpha$, the interest rate is negative.

Suppose now we introduce a pay-as-you-go system, where young agents are taxed $\beta - \alpha$ and old agents receive a transfer of $\beta - \alpha$. Under this policy, consumption in equilibrium is given by

$$\left\{ \tilde{c}_1^0, \left\{ \tilde{c}_t^t, \tilde{c}_{t+1}^t \right\}_{t=1}^{T-1}, \tilde{c}_T^T \right\} = \left\{ \beta, \left\{ 1 - \beta, \beta \right\}_{t=1}^{T-1}, 1 - \beta \right\}$$

Remember that, in the competitive equilibrium, we had

$$\left\{ \bar{c}_1^0, \left\{ \bar{c}_t^t, \bar{c}_{t+1}^t \right\}_{t=1}^{T-1}, \bar{c}_T^T \right\} = \left\{ \alpha, \left\{ 1 - \alpha, \alpha \right\}_{t=1}^{T-1}, 1 - \alpha \right\}$$

Since $\beta > \alpha$, the initial old is better off, while the end-of-time young is strictly worst off. For generations in $t \in \{1, 2, \dots, T-1\}$, note that the solution of

$$\max_{c_t m c_{t+1}} \quad (1 - \beta) \log c_t + \beta \log c_{t+1} \tag{8}$$
$$s.t. \quad c_t + c_{t+1} = 1$$

is given by

$$c_t = 1 - \beta, \; c_{t+1} = \beta. \tag{9}$$

Hence, the new equilibrium gives generation $t \in \{1, 2, \dots, T-1\}$ their most preferred feasible allocation, so they are also better off.

We can also answer this question intuitively. Since the pay-as-you-go system is a transfer from the young to the old, the initial old benefits (since he's never young and so never has to pay into the system) while the final young loses (since he's only ever young and so never receives from the system). For the intermediate generations, note that when the interest rate is negative, income in the future is worth more—recall that the present value of endowment when old is calculated as

$e^t_{t+1}/\left(1+r\right)$. So a one-to-one transfer from young to old results in an increase in the present value of income for each of the intermediate generation—so they end up being better off.

Comment: There were three "sets" of generations to consider. The initial old, the intermediate generations and the final young. I saw many of you saying that the intermediate generations are equally as well off (since it's a transfer). But, observe that their present value of total income is greater if they receive income in the later year when the interest rate is negative—so they must be better off under the proposed pay-as-you-go system (when interest rate is negative, we're dividing the endowment when old by a positive number smaller than 1).

2 Aggregation

Consider a pure endowment economy with $m > 1$ goods so that $L = \mathbb{R}^m$. Assume that the utility function for every agent is given by:

$$u^i(x) = \sum_{l=1}^{m} U_i(x_l)\,\pi_l \tag{10}$$

for every agent i and where

$$U_i(c) = \begin{cases} \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{c}{\gamma} + \tau_i\right)^{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c + \tau_i) & \text{otherwise} \end{cases} \tag{11}$$

and $\pi_l \geq 0$ are common across agents and satisfy $\sum_{l=1}^{m} \pi_l = 1$. We assume that

$$X^i = \left\{ x \in \mathbb{R}^m : \tau_i + \frac{x_l}{\gamma} > 0,\ \forall l = 1, 2, \ldots, m \right\} \tag{12}$$

2.1 Section 1

Show that this utility function is increasing and concave for any values of τ and γ as long as consumption belongs to the sets described above for each of the cases.

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For $\gamma = 1$, we have that $U_i(c)$ is the logarithm of $c + \tau_i$, since composition of strictly increasing and strictly concave function is increasing and concave, we have that the utility function is increasing and concave. In this case, the requirement of $c \in X^i$ ensures that the logarithm is well defined.

For any $\gamma \neq 1$, we have

$$\frac{\partial U_i(c)}{\partial c} = \left(\frac{c}{\gamma} + \tau_i\right)^{-\gamma} > 0,\ \forall c \in X^i \tag{13}$$

and

$$\frac{\partial^2 U_i(c)}{\partial c^2} = -\left(\frac{c}{\gamma} + \tau_i\right)^{-\gamma-1} < 0,\ \forall c \in X^i. \tag{14}$$

Finally, since $u(x)$ is a sum of increasing and concave functions it is also increasing and concave.

2.2 Section 2

Compute the reciprocal of the curvature of $U_i(c)$, given by minus the ratio of the first and the second derivative evaluated at c. Recall that, we argued that this describe the slope of the Pareto optimal allocations with respect to different values of the aggregate endowment. Based on this computation, what type of utility function $U_i(c)$ do we get as we let $\gamma \to \infty$?

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The risk tolerance for consumer i is given by

$$T_i(c) = -\frac{U_{i,c}(c)}{U_{i,cc}(c)}.$$

(Incidentally, recall that risk tolerance is the reciprocal of the Arrow-Pratt coefficient of absolute risk aversion.) Using equations (13) and (14), we get

$$T_i(c) = \frac{c}{\gamma} + \tau_i \tag{15}$$

Letting $\gamma \to \infty$ yields

$$\lim_{\gamma \to \infty} T_i(c) = \tau_i;$$

i.e. the utility function becomes CARA/exponential utility.

2.3 Section 3

What utility function do we get if we set $\gamma = -1$ and $\tau_i > 0$?

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From equation (11), at $\gamma = -1$, we have

$$U_i(c) = -\frac{1}{2}(\tau_i - c)^2, \tag{16}$$

which is a quadratic utility function.

2.4 Section 4

What utility function do we get if we set $0 < \gamma < \infty$?

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With γ in $[0, \infty]$, we have decreasing absolute risk aversion (DARA) utility function, which is called power utility. Note that (only) if $\tau_i = 0$, the utility function has CRRA.

2.5 Section 5

Write down the social planner's problem using weights λ_i for each agent. Use p_l for the Lagrange multiplier of the feasibility constraint of good l, for $l = 1, 2 \ldots, m$. Denote by $\bar{e}_l$ the aggregate endowment of commodity l.

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The social planner's problem is given by

$$\begin{aligned} \max_{\{x_l^i\}} \quad & \sum_{i=1}^{I} \lambda_i \left(\sum_{l=1}^{m} U_i\left(x_l^i\right) \pi_l \right) \qquad (17) \\ s.t. \quad & \sum_{i=1}^{I} x_l^i = \bar{e}_l, \ \forall l = 1, 2, \ldots m. \end{aligned}$$

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{i=1}^{I} \left(\sum_{l=1}^{m} \lambda_i U_i\left(x_l^i\right) \pi_l \right) + \sum_{l=1}^{m} p_l \left[\bar{e}_l - \sum_{i=1}^{I} x_l^i \right]. \qquad (18)$$

Comment: Many people were unclear about the fact that there is a feasibility constraint for each good.

2.6 Section 6

Write down the first-order condition for the planner's problem with respect to the consumption of good l assigned to agent i.

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The first-order conditions of problem (18) are

$$\left\{x_l^i\right\}: \quad \lambda_i U_{i,c}\left(x_l^i\right)\pi_l = p_l,\ \forall i, l. \tag{19}$$

Substituting equation (13) into (19), we obtain

$$p_l = \lambda_i \pi_l \left(\frac{x_l^i}{\gamma} + \tau_i\right)^{-\gamma}. \tag{20}$$

2.7 Section 7

Use the previous answer and the functional form for $U_i'(\cdot)$ to solve for x_l^i as a function of λ_i, the Lagrange multiplier p_l, and the preference parameters of agent i.

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From equation (20), we solve for x_l^i,

$$x_l^i = \gamma\left[\left(\frac{p_l}{\lambda_i \pi_l}\right)^{-\frac{1}{\gamma}} - \tau_i\right]. \tag{21}$$

2.8 Section 8

Use the previous answer as well as the feasibility condition for good l to solve for p_l. Express your answer in term of $\bar{e}$, the $\sum_{i=1}^I \tau_i$, the parameters γ and π_l, and the sum $\sum_{i=1}^I \lambda_i^{\frac{1}{\gamma}}$.

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From feasibility and equation (21):

$$\begin{aligned}\sum_{i=1}^I x_l^i &= \sum_{i=1}^I \gamma\left[\left(\frac{p_l}{\lambda_i \pi_l}\right)^{-\frac{1}{\gamma}} - \tau_i\right] \\ \Leftrightarrow \bar{e}_l &= \gamma\left[\left(\frac{p_l}{\pi_l}\right)^{-\frac{1}{\gamma}} \sum_{i=1}^I \lambda_i^{\frac{1}{\gamma}} - \sum_{i=1}^I \tau_i\right].\end{aligned}$$

Solving for p_l gives

$$p_l = \frac{\pi_l\left(\sum_{i=1}^I \lambda_i^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(\frac{\bar{e}_l}{\gamma} + \sum_{i=1}^I \tau_i\right)^{\gamma}}. \tag{22}$$

2.9 Section 9

Do the values of the ratio of any two Lagrange multipliers p_l/p_k depend on the vector of $\{\lambda_i\}$? In other words, do we have Gorman aggregation?

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From equation (22), for goods l and k we have

$$\frac{p_l}{p_k} = \frac{\pi_l}{\pi_k} \frac{\left(\frac{\bar{e}_k}{\gamma} + \sum_{i=1}^{I} \tau_i\right)^{\gamma}}{\left(\frac{\bar{e}_l}{\gamma} + \sum_{i=1}^{I} \tau_i\right)^{\gamma}}, \tag{23}$$

which does not depend on $\{\lambda_i\}$. Hence, we have Gorman aggregation.

2.10 Section 10

Use the first-order condition with respect to x_l^i for agent i good l and the answer for the previous question to solve for x_l^i as a function of λ_i, $\bar{e}_l$, γ, and $\sum_{i=1}^{I} \tau_i$. Hint: give an expression for it in the following form:

$$\frac{x_l^i}{\gamma} + \tau_i = \kappa^i \left(\frac{\bar{e}_l}{\gamma} - \bar{\tau}\right)$$

and write down a formula for $\bar{\tau}$ and κ^i.

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From equations (21) and (22),

$$\begin{aligned} x_l^i &= \gamma \left(\left[\frac{\pi_l \left(\sum_{i=1}^{I} \lambda_i^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(\frac{\bar{e}_l}{\gamma} + \sum_{i=1}^{I} \tau_i\right)^{\gamma}} \frac{1}{\lambda_i \pi_l} \right]^{-\frac{1}{\gamma}} - \tau_i \right) \\ \Leftrightarrow \frac{x_l^i}{\gamma} + \tau_i &= \left(\frac{\bar{e}_l}{\gamma} + \sum_{i=1}^{I} \tau_i\right) \frac{\lambda_i^{\frac{1}{\gamma}}}{\sum_{i=1}^{I} \lambda_i^{\frac{1}{\gamma}}}. \end{aligned}$$

Thus,

$$\frac{x_l^i}{\gamma} + \tau_i = \kappa^i \left(\frac{\bar{e}_l}{\gamma} - \bar{\tau}\right) \tag{24}$$

where

$$\kappa^i := \frac{\lambda_i^{\frac{1}{\gamma}}}{\sum_{i=1}^{I} \lambda_i^{\frac{1}{\gamma}}}, \ \bar{\tau} := -\sum_{i=1}^{I} \tau_i.$$

2.11 Section 11

Using your answer to the previous question, what is the utility function of the representative agent $u(x) : \mathbb{R}^m \to \mathbb{R}$? (i.e. the agent whose marginal rate of substitution, evaluated at $\bar{e}$, gives the equilibrium relative prices).

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Rewrite equation (23) as

$$\frac{p_l}{p_k} = \frac{\pi_l \left(\frac{\bar{e}_l}{\gamma} - \bar{\tau}\right)^{-\gamma}}{\pi_k \left(\frac{\bar{e}_k}{\gamma} - \bar{\tau}\right)^{-\gamma}}$$

This equation tells us that relative prices equal the marginal rate of substitution for and individual with preferences given by

$$U(c) = \frac{\gamma}{1-\gamma}\left(\frac{c}{\gamma} - \bar{\tau}\right)^{1-\gamma} \tag{25}$$

$$\Rightarrow u(x) = \sum_{l=1}^{m} \pi_l U(x_l)$$

This is the preferences of our representative consumer.

Comment: Many of you stated the planner's problem as the answer here. However, the representative agent's utility function must be a function of aggregate consumption (and aggregate endowment). In particular, it should not be a function of $\left\{x_l^i\right\}$ so the planner's problem cannot be the utility function of the representative agent.