Price Theory
Proposed Solution to Problem Set 4, Question 1

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Consider the market for housing. Assume that housing is heterogeneous with a range of quality levels. New housing (which you can assume is high quality) is produced by a competitive industry with constant return to scale and there is an existing stock of old housing. Individuals differ in their level of income but otherwise have the same preferences for housing.

A housing model related to the one presented in this question is the compensating differentials model of housing prices explained by Murphy in this lecture. It corresponds to chapter 8 of the new textbook.

This question examines equilibrium in a market with both product heterogeneity and consumer heterogeneity.

1 *Setting*

We will model a discreet, finite amount of housing quality levels.[2]

[2] It is also feasible to model a continuous range of quality levels.

We think of housing quality levels as providing various rates of housing services: higher quality houses provide more housing services (e.g. jacuzzi baths) in any given month. I will use this interpretation, so that the good that we have in mind is monthly housing services.

Because of the nature of the housing market, we will assume that individuals can rent at most one house, so that really they choose among a finite set of housing service levels, each of which has its own price.

Finally, we are not going to consider the case where there is more old housing than families/population: the new housing industry exists in equilibrium.

Some Notation

We will take a housing quality level to be $q \in \{q_0, \ldots, q_{n-1}, q_n\}$, so that there are n quality levels for old houses and new houses have quality q_n. We assume that $q_0 < \cdots < q_{n-1} < q_n$. The amount of old housing available of any given quality level q is $h(q)$.

On the consumers' side, there will be a continuum of income levels and will denote the income cumulative distribution by $F()$. We could carry around a population size P, but will normalize it to one to keep things simple. Since there is less old housing than population, we have that

$$\sum_{i=0}^{n-1} h(q_i) < 1.$$

Next, we will let $D(q; I)$ denote the willingness to pay of an individual with income I for housing quality q.

We think of housing services as a normal good, so that (i) $D(q;I)$ is increasing in I, and (ii) the willingness to pay for additional services – $D(q_i, I) - D(q_{i-1}, I)$ – is increasing and continuous in I for every $i \in \{0, \ldots, n\}$.

Finally, there will be two kinds of prices we will deal with: rental rates $r_0, \ldots, r_n$ and house prices $p_0, \ldots, p_n$. For most of the question we will deal with rental rates.

a.

Describe the equilibrium to this model. Who would tend to live in the new houses? What will determine the price of the old houses (which range in quality)?

The equilibrium of this model is based on a simple idea: poorer people have worse housing. We now construct the equilibrium by first positing it and then verifying that (i) people are behaving optimally and (ii) supply equals demand across quality levels.

Consider quality q_0. There are $h(q_0)$ homes with this quality level. This same amount of people will purchase this quality, and they will come from the bottom of the income distribution. Formally, define I_0 as:

$$F(I_0) = h(q_0) \iff I_0 = F^{-1} \circ h(q_0).$$

The equilibrium rental rate difference between qualities q_0 and q_1 will be such that the individual with income I_0 is indifferent between them:[3]

$$D(q_1, I_0) - D(q_0, I_0) = r_1 - r_0.$$

We call the individual with income I_0 a marginal individual.

Going further, consider quality level q_1. Define I_1 as:

$$F(I_1) - F(I_0) = h(q_1).$$

Again, rental rate differences will be pinned down by the marginal individual with income I_1:[4]

$$D(q_2, I_1) - D(q_1, I_1) = r_2 - r_1.$$

Proceeding this way, we can construct income levels $I_0, I_1, \ldots, I_{n-1}$. Marginal individuals then pin down rental rate differences $r_1 - r_0, r_2 - r_1, \ldots, r_n - r_{n-1}$.

To get rental rate levels, recall that new housing is produced by a competitive industry with constant returns to scale. Because of constant returns to scale, the marginal cost of producing new housing is constant. Because the industry is competitive, the supply of new housing is perfectly elastic. Thus, in equilibrium there can be only one price

[3] Otherwise there is either excess supply or excess demand of the lowest quality housing. A more extensive reasoning on this is provided below.

[4] If the individual with income I_1 were not indifferent between q_1 and q_2, there would be either excess demand or excess supply of housing with quality q_1.

that supports a finite amount of new housing produced. Call this price p_n^*, and let r_n^* be the associated rental rate of new housing. Rental rates $r_{n-1}, \ldots, r_0$ are then pinned down.

We now have to verify that the rental rates so constructed are indeed equilibrium rental rates. Consider an individual with income $I \in [0, I_0)$. We will argue she is best off choosing housing quality q_0. To see this, let q_k be any other quality level. Now note that purchasing quality q_k over q_0 costs more than she's willing to pay:

$$
\begin{aligned}
r_k - r_0 &= \sum_{l=1}^{k} r_l - r_{l-1} \\
&= \sum_{l=1}^{k} D(q_l, I_{l-1}) - D(q_{l-1}, I_{l-1}) \\
&> \sum_{l=1}^{k} D(q_l, I) - D(q_{l-1}, I) \\
&= D(q_k, I) - D(q_0, I),
\end{aligned}
$$

where the inequality comes from the fact that housing services are a normal good.

This argument is easily extended, so that any individual with income $I \in [I_{k-1}, I_k)$ is best off when choosing housing quality q_k, $k \in \{1, \ldots, n-1\}$ and people with income at or above I_{n-1} are best off purchasing new housing.[5]

[5] A minor but important detail remains: are the resulting housing rental rates below each stratum's willingness to pay for them? This will be the case if the marginal cost of production of new housing is low enough, and we assume this is the case. Otherwise, some people would not rent housing, a possibility we discard in this exercise by extending the definition of what a house can be: even a blanket and a piece of cardboard provide housing services and could constitute quality q_0.

Because everyone is behaving optimally and supply matches the demand for housing at every quality level, we have found our equilibrium. As stated in the beginning of this question: poor people rent low quality housing, and rich people rent high quality housing.

b.

How would an increase in the income of a group of consumers (say those in a given range of the income distribution) affect housing prices and the welfare of consumers at the different levels of income?

The structure of the equilibrium we have found will remain unchanged with changes in the income distribution: the outcome is always that poor people have worse housing and equilibrium prices must clear the market.

Suppose that the income of a segment of the population were to increase, so that the income range in this group goes from $[\underline{I}, \overline{I}]$ to $[\underline{I}', \overline{I}']$, where $\underline{I}' > \underline{I}$ and $\overline{I}' > \overline{I}$. Let F' denote the new income distribution. A first aspect to notice is that if $I_k > \overline{I}'$ – where I_k was defined in the previous question – then the groups of people purchasing qualities

$q_{k+1}, \ldots, q_n$ do not change, and therefore equilibrium rental rate differences $r_{k+1} - r_k, \ldots, r_n - r_{n-1}$ remain the same. Similarly, if $I_k < \underline{I}$ then the groups of people purchasing qualities $q_0, \ldots, q_k$ do not change, and therefore equilibrium rental rate differences $r_1 - r_0, \ldots, r_{k+1} - r_k$ also remain the same.

Next, fix the quality level $q_{\underline{k}}$, such that $\underline{I} < I_{\underline{k}}$ but $I_{\underline{k}-1} < \underline{I}$. Because of the income increase, there will be a new marginal individual with income level $I'_{\underline{k}}$ given by

$$F'(I'_{\underline{k}}) - F'(I_{\underline{k}-1}) = h(q_{\underline{k}}).$$

We can see that $I'_{\underline{k}}$ must exceed $I_{\underline{k}}$: otherwise supply of housing of quality $q_{\underline{k}}$ would exceed demand. Equilibrium rental rate differences will then be pinned down by the indifference condition:

$$D(q_{\underline{k}+1}, I'_{\underline{k}}) - D(q_{\underline{k}}, I'_{\underline{k}}) = r'_{\underline{k}+1} - r'_{\underline{k}}.$$

Because $I'_{\underline{k}} > I_{\underline{k}}$, it follows that $r'_{\underline{k}+1} - r'_{\underline{k}} > r_{\underline{k}+1} - r_{\underline{k}}$.

We can extend this procedure for every quality level q_k such that $\underline{I} < I_k < \overline{I}'$ so that

$$r'_{k+1} - r'_k = D(q_{k+1}, I'_k) - D(q_k, I'_k),$$

where I'_k is such that $F'(I'_k) - F'(I'_{k-1}) = h(q_k)$, so that $I'_k > I_k$ and thus $r'_{k+1} - r'_k > r_{k+1} - r_k$.

Since new house prices cannot change, rental rate levels will remain the same for the rich people: those who purchase quality levels q_k, where $I_k > \overline{I}'$. Because the new marginal consumers now have higher income, rental rates will fall for everybody else, and hence they will be better off.

c.

What would happen if the government replaced a number of the oldest houses with new ones through a process of urban renewal? Who would gain from that? Why? Could some people lose? How would the answers differ if individuals owned versus rented their housing?

In equilibrium, taking x of the oldest houses and replacing them by new ones shifts exactly x of the wealthiest people within each housing quality group of buyers to the next housing quality group: the x richest people previously renting housing quality q_0 now go for q_1, the x richest people previously renting quality q_1 now go to q_2, and so on. Thus, the income of the new marginal individuals is now lower: $I'_0 < I_0, I'_1 < I_1, \ldots, I'_{n-1} < I_{n-1}$. Thus, equilibrium rental rate differences go down and equilibrium rental rates increase, since the price of new housing remains the same.

Thus, people who used to rent old housing are definitely worse off. However, people who own old housing are better off: the value of their assets increases, and thus they become richer.

d.

Now assume that other aspects of life differ across houses in that the level of city services (e.g. police protection) differs across houses with the lower quality homes also receiving less police protection. How would a policy that mandated equal levels of services across areas affect outcomes? Who would gain from such a policy? Assume the amount of service is set equal to that of the highest quality homes, and that any additional spending is financed by lump sum taxes on consumers.

This last exercise posits a change in house qualities, or equivalently of the amount of housing services each type of house provides. We are concerned with the equilibrium effects of increasing qualities $q_0, q_1, \ldots, q_{n-1}$ while keeping q_n and the quality ordering fixed.

It is easiest to start the discussion with an extreme case: suppose this policy brings all housing qualities to the new housing level. In this case, the rental rates of all houses are determined by the marginal cost of production of new houses. Is any renter worse off? It is easy to see that, in fact, all of them are.[6] The reason is that everyone could already rent the highest quality housing *at that rental rate* before the policy took place. The reason people did not do this is that they had a better choice available: lower quality housing. In other words, because the supply of new housing is perfectly elastic, this policy is removing alternatives from the consumers' choice set. In this story, the winners are clearly the owners of old housing.

[6] Except those who were already renting the highest quality housing.

The intermediate case is similar. For concreteness, assume that the policy effectively shifts the quality of each housing level one tier up, so that the new housing quality distribution is $\hat{h}$, where

$$\hat{h}(q_0) = 0 \quad \text{and} \quad \hat{h}(q_i) = h(q_{i-1})$$

for all $i \in \{1, \ldots, n-1\}$. Now construct the equilibrium as in question *a*. In this new equilibrium, the marginal individuals have a lower income than under the housing distribution h, so that rental rate differences are smaller and thus rental rates are higher at all remaining quality levels except q_n.

Welfare implications are then easily spelled out: rental rates are higher at quality levels $q_1, \ldots, q_{n-1}$, and housing of quality q_0 has ceased to exist. Renters must therefore necessarily be worse off. As in our initial discussion, it's the homeowners who benefit from the higher rental rates.

This analysis has not paid attention to the lump sum taxes required to increase the level of city services. Recall that a lump sum tax does not affect peoples' decision-making but does reduce peoples' income levels. Thus, the structure of the equilibrium under these taxes does not change, but the equilibrium rental rates end up even higher, since the income of the marginal individuals – and thus the rental rate differences – is even lower. Therefore, under lump sum taxes, renters are clearly worse off, and homeowners could be better off or worse off, depending on the magnitude of such taxes.

Political economy twist: renters are a less cohesive group than real estate owners. In this world, it should therefore not be surprising if real estate lobbies were behind such government programs which sound good when spelled out in public.