Price Theory III
Spring 2018

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1 Risk sharing with syndicate

Let N denote the set of members of an investment partnership, or *syndicate*. The syndicate holds assets which yields random returns $\tilde{Y}$ that have some given probability distribution. Each individual $i \in N$ has utility function for money $u_i(\cdot)$. Let $x_i(y)$ denote the individual i's planned payoff when the assets return y. For the plan to be feasible,

$$\sum_{i \in N} x_i(y) = y \tag{1.1}$$

for all y in the support of $\tilde{Y}$, denoted $\mathcal{Y}$.

We consider an efficient allocation rule $\{x_i(y)\}_{i \in N, y \in \mathcal{Y}}$ that solves

$$\begin{aligned} \max_{\{x_i(y)\}_{i \in N, y \in \mathcal{Y}}} \quad & \sum_{i \in N} \lambda_i \mathbb{E}\left[u_i\left(x_i\left(\tilde{Y}\right)\right)\right] \\ s.t. \quad & \sum_{i \in N} x_i(y) = y, \forall y \in \mathcal{Y} \end{aligned}$$

and $\lambda_i \geq 0$ for all i and there exists at least one $\lambda_i > 0$. We can consider this problem for each possible realisation of $\tilde{Y}$. The Lagrangian is given by

$$\mathcal{L}_y = \sum_{i \in N} \lambda_i u_i(x_i(y)) + \mu_y \left[y - \sum_{i \in N} x_i(y)\right].$$

First-order condition with respect to x_i is

$$\lambda_i u_i'(x_i(y)) = \mu_y, \forall i \in N.$$

Hence, the weighted marginal utility must be equal across all $i \in N$. This implies that there exists some function $v(y)$ such that

$$\lambda_i u_i'(x_i(y)) = v(y), \forall i \in N, y \in \mathcal{Y}. \tag{1.2}$$

Differentiating with respect to y gives

$$\lambda_i u_i''(x_i(y)) \frac{\partial x_i(y)}{\partial y} = v'(y).$$

Eliminating λ_i using (1.2) yields

$$\begin{aligned} \frac{v(y)}{u'(x_i(y))} u_i''(x_i(y)) \frac{\partial x_i(y)}{\partial y} &= v'(y) \\ \Leftrightarrow \frac{u_i''(x_i(y))}{u'(x_i(y))} \frac{\partial x_i(y)}{\partial y} &= \frac{v'(y)}{v(y)} \\ \Leftrightarrow -r_i(x_i(y)) \frac{\partial x_i(y)}{\partial y} &= \frac{v'(y)}{v(y)}, \end{aligned}$$

where $r_i(x_i(y))$ is individual i's Arrow-Pratt index of risk aversion. Writing this in terms of risk tolerance, τ_i, we have

$$-\frac{1}{\tau_i\left(x_i\left(y\right)\right)}\frac{\partial x_i\left(y\right)}{\partial y}=\frac{v'\left(y\right)}{v\left(y\right)}\Leftrightarrow\frac{\partial x_i\left(y\right)}{\partial y}=-\tau_i\left(x_i\left(y\right)\right)\frac{v'\left(y\right)}{v\left(y\right)}. \tag{1.3}$$

The left-hand side is the *marginal share* of the variable risks held by the syndicate.

Differentiating the feasibility condition with respect to y gives

$$1=\sum_{i\in N}\frac{\partial x_i\left(y\right)}{\partial y}.$$

Substituting (1.3) yields

$$1=-\frac{v'\left(y\right)}{v\left(y\right)}\sum_{i\in N}\tau_i\left(x_i\left(y\right)\right).$$

Hence,

$$\frac{\partial x_i\left(y\right)}{\partial y}=\frac{-\tau_i\left(x_i\left(y\right)\right)\frac{v'(y)}{v(y)}}{-\frac{v'(y)}{v(y)}\sum_{i\in N}\tau_i\left(x_i\left(y\right)\right)}=\frac{\tau_i\left(x_i\left(y\right)\right)}{\sum_{i\in N}\tau_i\left(x_i\left(y\right)\right)}. \tag{1.4}$$

That is, individual i's marginal share of the syndicate's risk is proportional to i's risk tolerance. We can then back out $x_i\left(y\right)$ by integrating the expression above with respect to y, using (1.1) as the boundary condition.

1.1 Constant absolute risk aversion

Suppose that each individual i has utility function

$$u_i\left(w\right)=-\exp\left[-\frac{w}{T_i}\right]$$

for some given parameter T_i. Then,

$$u_i'\left(w\right)=\frac{1}{T_i}\exp\left[-\frac{w}{T_i}\right],$$
$$u''\left(w\right)=-\frac{1}{T_i^2}\exp\left[-\frac{w}{T_i}\right].$$

Define

$$T^*:=\sum_{j\in N}T_j,$$
$$u^*\left(y\right):=-\exp\left[\frac{-y}{T^*}\right].$$

As we saw above, in an efficient sharing rule,

$$\frac{\partial x_i\left(y\right)}{\partial y}=\frac{T_i}{\sum_{j=1}T_j}=\frac{T_i}{T^*},\forall y\in\mathcal{Y}.$$

Integrating both sides with respect to y gives

$$\begin{aligned} x_i(y) &= \int \frac{T_i}{T^*} dy = \frac{T_i}{T^*} y + C \\ &= x_i(0) + \frac{T_i}{T^*} y, \end{aligned}$$

where we assumed the boundary condition $x_i(0)$ is given to pin down C. Thus, individuals with constant risk tolerances share risks linearly in proportion to their risk tolerances.

To obtain an expression for $x_i(0)$, substitute the expression for $x_i(y)$ into (1.2),

$$v(y) = \lambda_i u_i' \left(x_i(0) + \frac{T_i}{T^*} y \right), \ \forall i \in N, y \in \mathcal{Y}.$$

Since the left-hand side does not depend on i, it follows that

$$\lambda_i u_i' \left(x_i(0) + \frac{T_i}{T^*} y \right) = \lambda_j u_j' \left(x_j(0) + \frac{T_j}{T^*} y \right), \ \forall i, j \in N, y \in \mathcal{Y}.$$

We have N unknowns $(x_1(0), \cdots, x_N(0))$ and this gives us $N-1$ equations. To solve for $x_i(0)$, we therefore also use the feasibility condition

$$\sum_{i \in N} x_i(y) = y, \ \forall y \in \mathcal{Y}.$$

We can also write expected utility for each agent in terms of certainty equivalents. Note first that

$$\begin{aligned} \mathbb{E}\left[u_i \left(x_i \left(\tilde{Y} \right) \right) \right] &= \mathbb{E}\left[-\exp\left[-\frac{x_i(0) + \frac{T_i}{T^*}\tilde{Y}}{T_i} \right] \right] \\ &= -\left(-\exp\left[-\frac{x_i(0)}{T_i} \right] \right) \mathbb{E}\left[-\exp\left[-\frac{\tilde{Y}}{T^*} \right] \right] \\ &= -u_i(x_i(0)) \, \mathbb{E}\left[u^*\left(\tilde{Y} \right) \right]. \end{aligned} \tag{1.5}$$

Let W_i and W^* be certainty equivalents; i.e.

$$\begin{aligned} u_i(W_i) &= \mathbb{E}\left[u_i \left(x_i \left(\tilde{Y} \right) \right) \right], \\ u^*(W^*) &= \mathbb{E}\left[u^* \left(\tilde{Y} \right) \right]. \end{aligned}$$

We can then write (1.5) as

$$
\begin{aligned}
u_i(W_i) &= -u_i(x_i(0))\,u^*(W^*) \\
\Leftrightarrow -\exp\left[-\frac{W_i}{T_i}\right] &= -\left(-\exp\left[-\frac{x_i(0)}{T_i}\right]\right)\left(-\exp\left[-\frac{W^*}{T^*}\right]\right) \\
\Leftrightarrow \exp\left[-\frac{W_i}{T_i}\right] &= \exp\left[-\frac{x_i(0)}{T_i}\right]\exp\left[-\frac{W^*}{T^*}\right] \\
\Leftrightarrow -\frac{W_i}{T_i} &= -\frac{x_i(0)}{T_i} - \frac{W^*}{T^*} \\
\Leftrightarrow W_i &= x_i(0) + \frac{T_i}{T^*}W^*.
\end{aligned}
$$

Therefore, we see that individuals want their syndicate to evaluate risks according to a collective utility function u^* that has a constant risk tolerance equal to the sum of their individual tolerances.

1.2 Constant relative risk aversion

Suppose that each individual i has utility function

$$
u_i(w) = \frac{w^{1-\alpha}}{1-\alpha},\ \alpha > 0.
$$

Then,

$$
\begin{aligned}
u_i'(w) &= w^{-\alpha}, \\
u_i''(w) &= -\alpha w^{-\alpha-1}.
\end{aligned}
$$

From (1.2),

$$
\begin{aligned}
\lambda_i (x_i(y))^{-\alpha} &= v(y) \\
\Rightarrow x_i(y) &= \lambda_i^{\frac{1}{\alpha}} v(y)^{-\frac{1}{\alpha}}
\end{aligned}
$$

By feasibility,

$$
\begin{aligned}
y = \sum_{j\in N} x_j(y) &= \sum_{j\in N} \lambda_j^{\frac{1}{\alpha}} v(y)^{-\frac{1}{\alpha}} \\
&= v(y)^{-\frac{1}{\alpha}} \sum_{j\in N} \lambda_j^{\frac{1}{\alpha}} \\
\Rightarrow v(y) &= y^{-\alpha}\left(\sum_{j\in N}\lambda_j^{\frac{1}{\alpha}}\right)^{\alpha}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
x_i(y) &= \lambda_i^{\frac{1}{\alpha}} \left(y^{-\alpha} \left(\sum_{j \in N} \lambda_j^{\frac{1}{\alpha}} \right)^{\alpha} \right)^{-\frac{1}{\alpha}} \\
&= \lambda_i^{\frac{1}{\alpha}} y \left(\sum_{j \in N} \lambda_j^{\frac{1}{\alpha}} \right)^{-1} \\
&= \frac{\lambda_i^{\frac{1}{\alpha}}}{\sum_{j \in N} \lambda_j^{\frac{1}{\alpha}}} y.
\end{aligned}$$

Hence, each individual i receives a constant fraction of the total y.

1.3 Implementation with a price system

Suppose that $\tilde{Y}$ has finitely many possible values y, each with some probability $p(y)$. We can use the marginal utility equation (1.2) to show that the efficient allocation can be implemented by a price system in which each player i has a budget I_i to spend on acquiring claims on income in each state of the total returns $\tilde{Y}$, and $\pi(y)$ is the price of one unit of income when the realisation of $\tilde{Y}$ is y.

In such a price system, i would choose the quantities $x_i(y)$ for all possible y's to maximise

$$\begin{aligned}
\max_{x_i(y)} \quad & \sum_{y \in \mathcal{Y}} p(y) u_i(x_i(y)) \\
s.t. \quad & \sum_{y \in \mathcal{Y}} \pi(y) x_i(y) \leq I_i.
\end{aligned}$$

We can write the Lagrangian as

$$\mathcal{L}_i = \sum_{y \in \mathcal{Y}} p(y) u_i(x_i(y)) + \mu_i \left[I_i - \sum_{y \in \mathcal{Y}} \pi(y) x_i(y) \right].$$

The first-order condition with respect to $x_i(y)$ is given by

$$p(y) u_i'(x_i(y)) = \pi(y) \mu_i, \ \forall i \in N, y \in \mathcal{Y}.$$

This coincides with the marginal utility equation (1.2) if and only if

$$u_i'(x_i(y)) = \frac{v(y)}{\lambda_i} = \frac{\pi(y) \mu_i}{p(y)}, \ \forall i \in N, y \in \mathcal{Y}.$$

Rearranging gives

$$\frac{v(y) p(y)}{\pi(y)} = \mu_i \lambda_i, \ \forall i \in N, y \in \mathcal{Y}.$$

Observe that the left-hand side does not depends on i, and the right-hand side does not depends on y. Therefore,

$$\frac{v(y) p(y)}{\pi(y)} = \mu_i \lambda_i = A, \ \forall i \in N, y \in \mathcal{Y},$$

where A does not depend either on i or y. The price is therefore given by

$$\pi(y) = \frac{v(y)\,p(y)}{A}.$$

A risk-free asset here pays one unit of income in each $y \in \mathcal{Y}$. Since $\pi(y)$ is the price of one unit of income when the realisation of $\tilde{Y}$ is y, a risk-free asset has price $\sum_{y\in\mathcal{Y}} \pi(y)$. So, we can define the risk-free rate of return ρ, on budgeted funds, as

$$1 + \rho \coloneqq \frac{1}{\sum_{y\in\mathcal{Y}} \pi(y)}.$$

Hence,

$$\begin{aligned}
\sum_{y\in\mathcal{Y}} \pi(y) = \sum_{y\in\mathcal{Y}} \frac{v(y)\,p(y)}{A} &= \frac{1}{A}\sum_{y\in\mathcal{Y}} v(y)\,p(y) \\
\Rightarrow A &= (1+\rho)\sum_{y\in\mathcal{Y}} v(y)\,p(y) \\
\Rightarrow \pi(y) &= \frac{v(y)\,p(y)}{(1+\rho)\sum_{y\in\mathcal{Y}} v(y)\,p(y)}.
\end{aligned}$$

Observe that the price of future income from returns y depends both on the probability $p(y)$ and on $v(y)$, the marginal social utility of future returns y. Since $\partial x_i(y)/\partial y \geq 0$ from (1.4), and u_i is strictly concave, from (1.2), we see that $v(y)$ is decreasing in y.

1.4 Optimal risk sharing among partners with constant risk tolerance

Consider a group of individuals who have formed a partnership to share the risky profits from some joint venture or gamble. Suppose each individual j in this group has a constant risk tolerance, denoted τ_j. Let T_o denote the sum of all the partners' risk tolerances; i.e.

$$T_o \coloneqq \sum_{j\in N} \tau_j.$$

These partners can maximise the sum of their certainty equivalents by sharing the risky profits among themselves in proportion to their risk tolerances, with each individual j taking the fractional share τ_j/T_o of the risky profits. The maximum sum of the partners' certainty equivalents that can be achieved by such efficient risk sharing is equal to the certainty equivalent of the whole gamble to an individual who has a constant risk tolerance equal to T_o. Thus, in making decisions about risky investments, the partnership should act as a corporate person with a risk tolerance equal to the sum of its members' risk tolerances.

Theorem 1.1. *(Coarse Theorem) If the partners were planning to share risks according to a sharing rule that does not maximise the sum of partners' certainty equivalents, then any partner j could propose another sharing rule that would increase j's own certainty equivalent and would not decrease the certainty equivalent of any other partners.*

Example 1.1. Consider a financial asset yielding a return $\tilde{Y}$ that is drawn from a normal distribution with mean $\mu = \$35,000$ and $\sigma = \$25,000$. Investor 1 has risk tolerance $T_1 = \$20,000$ and investor 2 has risk tolerance $T_2 = \$30,000$.

Since both individuals have constant risk tolerances, we can assume that their utility (over money) is given by

$$u_i\left(y\right)=-\exp\left[-\frac{y}{T_i}\right],\ \forall i\in\left\{1,2\right\},$$

where y is their (certain) income. The expected utility is thus

$$\mathbb{E}\left[u_i\left(\tilde{Y}\right)\right]=\mathbb{E}\left[-\exp\left[-\frac{\tilde{Y}}{T_i}\right]\right],\ \forall i\in\left\{1,2\right\},$$

where, here, we have $\tilde{Y}\sim N\left(\mu,\sigma^2\right)$. Since utility is log-normally distributed in this case, so

$$CE_i\left(\tilde{Y}\right)=\mu-\frac{1}{2}\frac{\sigma^2}{T_i},\ \forall i\in\left\{1,2\right\},$$

where CE_i denotes the certainty equivalent of the gamble $\tilde{Y}$ for each $i=1,2$.

If each $i\in\{1,2\}$ owns the whole asset, their certainty equivalent are:

$$\begin{aligned}CE_1\left(\tilde{Y}\right)&=\$35,000-\frac{1}{2}\frac{\left(\$25,000\right)^2}{\$20,000}=\$19,375,\\ CE_2\left(\tilde{Y}\right)&=\$35,000-\frac{1}{2}\frac{\left(\$25,000\right)^2}{\$30,000}=\$24,583.3.\end{aligned}$$

Thus, if 1 sold the asset to 2 for price x, then they would receive

$$\begin{aligned}CE_1\left(x\right)&=x,\\ CE_2\left(\tilde{Y}-x\right)&=\$24583.3-x.\end{aligned}$$

The transaction makes both better off if

$$\$19,175\leq x\leq\$24,583.3.$$

In this range $x=\$24,583.3$ would be best for investor 1 and $x=\$19,175$ would be best for investor 2.

Now consider the possibility that 1 could sell a 50% share to 2 for some price x (1 holds all the assets initially). Then, they get

$$\begin{aligned}CE_1\left(\frac{1}{2}\tilde{Y}+x\right)&=\frac{1}{2}\$35,000+x-\frac{1}{2}\frac{\left(\frac{1}{2}\$25,000\right)^2}{\$20,000}=\$13,593.8+x,\\ CE_2\left(\frac{1}{2}\tilde{Y}-x\right)&=\frac{1}{2}\$35,000-x-\frac{1}{2}\frac{\left(\frac{1}{2}\$25,000\right)^2}{\$20,000}=\$14,895.8-x.\end{aligned}$$

Thus, the transactions makes both better off if

$$\begin{aligned}\$13,593.8+x&=CE_1\left(\frac{1}{2}\tilde{Y}+x\right)\geq CE_1\left(\tilde{Y}\right)=\$19,375,\\ \$14,895.8-x&=CE_2\left(\frac{1}{2}\tilde{Y}-x\right)\geq CE_2\left(\tilde{Y}\right)=\$24,583.3;\end{aligned}$$

i.e.

$$\begin{aligned}
x &\geq \$19,375 - \$13,593.8 = \$5,781.2, \\
x &\leq \$14,895.8, \\
\Rightarrow \$5,781.2 &\leq x \leq \$14,895.8.
\end{aligned}$$

The sum of their certainty equivalent is

$$\begin{aligned}
CE_1\left(\frac{1}{2}\tilde{Y}+x\right)+CE_2\left(\frac{1}{2}\tilde{Y}-x\right) &= (\$13,593.8+x)+(\$14,895.8-x) \\
&= \$28,489.6,
\end{aligned}$$

which is higher than what either could get from the asset alone, which are $CE_1\left(\tilde{Y}\right)$ and $CE_2\left(\tilde{Y}\right)$.

More generally, if 1 sells a share θ to investor 2 for a price x, their certainty equivalents are

$$\begin{aligned}
CE_1\left((1-\theta)\tilde{Y}+x\right) &= (1-\theta)\mu + x - \frac{1}{2}\frac{((1-\theta)\sigma)^2}{T_1}, \\
CE_2\left(\theta\tilde{Y}-x\right) &= \theta\mu - x - \frac{1}{2}\frac{(\theta\sigma)^2}{T_2}.
\end{aligned}$$

The sum of the certainty equivalent is maximised if θ solves the following problem:

$$\begin{aligned}
&\max_{\theta\in[0,1]} \quad CE_1\left((1-\theta)\tilde{Y}+x\right)+CE_2\left(\theta\tilde{Y}-x\right) \\
\equiv &\max_{\theta\in[0,1]} \quad (1-\theta)\mu + x - \frac{1}{2}\frac{((1-\theta)\sigma)^2}{T_1} + \theta\mu - x - \frac{1}{2}\frac{(\theta\sigma)^2}{T_2} \\
= &\max_{\theta\in[0,1]} \quad \mu - \frac{1}{2}\sigma^2\left[\frac{(1-\theta)^2}{T_1} - \frac{\theta^2}{T_2}\right] \\
\equiv &\min_{\theta\in[0,1]} \quad \frac{(1-\theta)^2}{T_1} - \frac{\theta^2}{T_2}.
\end{aligned}$$

The first-order condition is

$$\begin{aligned}
-2\frac{1-\theta}{T_1} - 2\frac{\theta}{T_2} &= 0 \\
\Leftrightarrow (1-\theta)T_2 &= \theta T_1 \\
\Leftrightarrow \theta &= \frac{T_2}{T_1+t_2} \\
&= \frac{\$30,000}{\$20,000+\$30,000} = \frac{3}{5}.
\end{aligned}$$

Thus, if 1 sells the optimal 60% share to investor 2 for a price x,

$$\begin{aligned}
CE_1\left(\left(1-\frac{3}{5}\right)\tilde{Y}+x\right) &= \frac{2}{5}\$35,000 + x - \frac{1}{2}\frac{\left(\frac{2}{5}\$25,000\right)^2}{\$20,000} = \$11,500 + x \\
CE_2\left(\frac{3}{5}\tilde{Y}-x\right) &= \frac{3}{5}\$35,000 - x - \frac{1}{2}\frac{\left(\frac{3}{5}\$25,000\right)^2}{\$20,000} = \$17,250 - x.
\end{aligned}$$

The transaction makes both better off if

$$\begin{aligned} x &\geq \$19,375 - \$11,500 = \$7,785, \\ x &\leq \$17,250, \\ \Rightarrow \$7,785 \leq x &\leq \$17,250. \end{aligned}$$

The maximised sum of their certainty equivalent is

$$\begin{aligned} CE_1\left(\frac{1}{2}\tilde{Y} + x\right) + CE_2\left(\frac{1}{2}\tilde{Y} - x\right) &= (\$11,500 + x) + (\$17,250 - x) \\ &= \$28,750. \end{aligned}$$

For a hypothetical corporate person with risk tolerance

$$T_o = T_1 + T_2 = \$50,000,$$

the asset $\tilde{Y}$ is worth

$$CE_o\left(\tilde{Y}\right) = \$35,000 - \frac{1}{2}\frac{(\$25,000)^2}{\$50,000} = \$28,750.$$