Price Theory II
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1 Preferences and utility

1.1 Preferences

We represent a consumer's *preference* by a binary relation $\succsim$, which we interpret as the "at-least-as-good as" relation. Suppose there are n (i.e. finite) goods. A *bundle* of goods is given by

$$\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in \mathbf{X} \coloneqq \mathbb{R}^n_+,$$

where $\mathbf{X}$ can be interpreted as the *consumption set*. Here, we restrict $\mathbf{X}$ to be the nonnegative real numbers. For any $\mathbf{x}^1, \mathbf{x}^2 \in \mathbf{X}$,

$$\mathbf{x}^1 \succsim \mathbf{x}^2$$

means that $\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$ for the consumer. Using $\succsim$, we can define the "strictly preferred", $\succ$, and "as-good-as", $\sim$, relations, respectively, as

$$\mathbf{x}^1 \succ \mathbf{x}^2 \Leftrightarrow \left(\mathbf{x}^1 \succsim \mathbf{x}^2\right) \wedge \neg \left(\mathbf{x}^2 \succsim \mathbf{x}^1\right),$$
$$\mathbf{x}^1 \sim \mathbf{x}^2 \Leftrightarrow \left(\mathbf{x}^1 \succsim \mathbf{x}^2\right) \wedge \left(\mathbf{x}^2 \succsim \mathbf{x}^1\right).$$

1.1.1 Restrictions on $\succsim$: Axioms

We would like to impose some basic properties on $\succsim$. First, we would like to ensure that the consumer is able to make a decision given any bundles in the consumption set. We also want to ensure that preferences is consistent across pairs of goods (in particular, we rule out preference *cycles*).

Axiom 1. *(Completeness) For any* $\mathbf{x}^1, \mathbf{x}^2 \in \mathbf{X}$, *either* $\mathbf{x}^1 \succsim \mathbf{x}^2$ *or* $\mathbf{x}^2 \succsim \mathbf{x}^1$.

Axiom 2. *(Transitivity) For any* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbf{X}$,

$$\mathbf{x}^1 \succsim \mathbf{x}^2, \mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^3.$$

Definition 1.1. *(Preference relation).* The binary relation on $\succsim$ on the consumption set $\mathbf{X}$ is called a *preference relation* if it satisfies Axioms 1 and 2.

With Axioms 1 and 2, for any $\mathbf{x^0} \in \mathbf{X}$,we can define the "at-least-as-good-as" set as

$$\succsim\left(\mathbf{x}^0\right) \coloneqq \left\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim \mathbf{x}^0\right\}.$$

The "no-better-than", "worse-than", "preferred to" and the "indifference" sets can be defined by replacing $\succsim$ with $\precsim$, $\succ$, $\prec$ and $\sim$, respectively.

Proposition 1.1. *(Properties of $\succ$ and $\sim$) Suppose $\succsim$ satisfies Axioms 1 and 2. Then, for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbf{X}$,*

(i) $\succ$ *is both* irreflexive *($\mathbf{x}^1 \succ \mathbf{x}^1$ never holds) and* transitive ($\mathbf{x}^1 \succ \mathbf{x}^2, \mathbf{x}^2 \succ \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succ \mathbf{x}^3$);

(ii) $\sim$ *is* reflexive *($\mathbf{x}^1 \sim \mathbf{x}^1$),* transitive *($\mathbf{x}^1 \sim \mathbf{x}^2, \mathbf{x}^2 \sim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \sim \mathbf{x}^3$) and* symmetric *$\left(\mathbf{x}^1 \sim \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \sim \mathbf{x}^1\right)$;*

(iii) If $\mathbf{x}^1 \succ \mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succ \mathbf{x}^3$.

Proposition 1.2. *For any $\mathbf{x}^1, \mathbf{x}^2 \in \mathbf{X}$, exactly one of $\mathbf{x}^1 \succ \mathbf{x}^2$, $\mathbf{x}^1 \sim \mathbf{x}^2$, $\mathbf{x}^1 \prec \mathbf{x}^2$ holds.*

Proof. (Proposition 1.1). Axiom 1 ensures that we can compare pairs of bundles $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$.

(i) By definition $\mathbf{x}^1 \succ \mathbf{x}^1 \Leftrightarrow \left(\mathbf{x}^1 \succsim \mathbf{x}^1\right) \wedge \neg\left(\mathbf{x}^1 \succsim \mathbf{x}^1\right)$—a contradiction. Suppose $\mathbf{x}^1 \succ \mathbf{x}^2, \mathbf{x}^2 \succ \mathbf{x}^3$ but $\mathbf{x}^3 \succsim \mathbf{x}^1$. Since $\mathbf{x}^1 \succ \mathbf{x}^2 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^2$, by transitivity of $\succsim$, $\mathbf{x}^3 \succsim \mathbf{x}^1 \succsim \mathbf{x}^2 \Rightarrow \mathbf{x}^3 \succsim \mathbf{x}^2$, which contradicts $\mathbf{x}^2 \succ \mathbf{x}^3$.

(ii) By definition, $\mathbf{x}^1 \sim \mathbf{x}^1 \Leftrightarrow \left(\mathbf{x}^1 \succsim \mathbf{x}^1\right) \wedge \left(\mathbf{x}^1 \succsim \mathbf{x}^1\right)$. Suppose $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \sim \mathbf{x}^3$; i.e. $\left(\mathbf{x}^1 \succsim \mathbf{x}^2\right) \wedge \left(\mathbf{x}^2 \succsim \mathbf{x}^1\right)$ and $\left(\mathbf{x}^2 \succsim \mathbf{x}^3\right) \wedge \left(\mathbf{x}^3 \succsim \mathbf{x}^2\right)$. By transitivity of $\succsim$, $\mathbf{x}^1 \succsim \mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^3$ and $\mathbf{x}^3 \succsim \mathbf{x}^2 \succsim \mathbf{x}^1 \Rightarrow \mathbf{x}^3 \succsim \mathbf{x}^1$. Thus, $\mathbf{x}^1 \sim \mathbf{x}^3$ by definition. Finally, $\mathbf{x}^1 \sim \mathbf{x}^2 \Leftrightarrow \left(\mathbf{x}^1 \succsim \mathbf{x}^2\right) \wedge \left(\mathbf{x}^2 \succsim \mathbf{x}^1\right) \Leftrightarrow \left(\mathbf{x}^2 \succsim \mathbf{x}^1\right) \wedge \left(\mathbf{x}^1 \succsim \mathbf{x}^2\right) \Leftrightarrow \mathbf{x}^2 \sim \mathbf{x}^1$.

(iii) Split into two: (a) $\mathbf{x}^1 \succ \mathbf{x}^2 \succ \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succ \mathbf{x}^3$ and (b) $\mathbf{x}^1 \succ \mathbf{x}^2 \sim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succ \mathbf{x}^3$. (a) Suppose not, so that $\mathbf{x}^3 \succsim \mathbf{x}^1$. Since $\mathbf{x}^1 \succ \mathbf{x}^2 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^2$, by transitivity of $\succsim$, $\mathbf{x}^3 \succsim \mathbf{x}^2$, which contradicts $\mathbf{x}^2 \succ \mathbf{x}^3$. (b) Since $\mathbf{x}^1 \succ \mathbf{x}^2 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \sim \mathbf{x}^3 \Rightarrow \mathbf{x}^2 \succsim \mathbf{x}^3$, then by transitivity of $\succsim$, $\mathbf{x}^1 \succsim \mathbf{x}^3$. Suppose that $\mathbf{x}^1 \sim \mathbf{x}^3$, then by transitivity of $\sim$, $\mathbf{x}^1 \sim \mathbf{x}^3 \sim \mathbf{x}^2 \Rightarrow \mathbf{x}^1 \sim \mathbf{x}^2$, which contradicts $\mathbf{x}^1 \succ \mathbf{x}^2$. Thus, $\mathbf{x}^1 \succ \mathbf{x}^3$. ■

Proof. (Proposition 1.2). By completeness, $(x \succsim y) \vee (y \succsim x), \forall x, y \in \mathbb{R}^l$.There are three cases to consider: (i) $x \succsim y$ and $y \succsim x$, then, by definition, $x \sim y$; (ii) $x \succsim y$ and $\neg(y \succsim x)$, then, by definition, $x \succ y$; (iii) $\neg(x \succsim y)$ and $y \succsim x$, then, by definition, $y \succ x$. The case $\neg(x \succsim y)$ and $\neg(y \succsim x)$ is ruled out by completeness. ■

The following axiom imposes a kind of topological regularity on preferences.

Axiom 3. *(Continuity) For any $\mathbf{x} \in \mathbf{X}$, the sets $\succsim(\mathbf{x})$ and $\precsim(\mathbf{x})$ are closed.*

Proposition 1.3. *Assume Axiom 3, the sets $\succ(\mathbf{x})$ and $\prec(\mathbf{x})$ are open and $\sim(\mathbf{x})$ is closed.*

Proof. Since the complement of a closed set is open, it follows that the sets $\succ(\mathbf{x})$ and $\prec(\mathbf{x})$ are open. Since intersection of finite closed sets are closed, the set $\sim(\mathbf{x})$ is closed. ■

Remark. Recall that a set is closed if any converging sequence in the set has its limit inside the set. Consider a sequence $\{y_n\} \in \succsim(\mathbf{x})$ with $\mathbf{y}_n \to \mathbf{y}$. Continuity axiom ensures that $\mathbf{y} \in \succsim(\mathbf{x})$. Thus, the axiom ensures that preference relations remain consistent in the limit; i.e. $\mathbf{y}^n \succsim \mathbf{x}, \forall n \in \mathbb{N} \Rightarrow \mathbf{y} \succsim \mathbf{x}$.

The following axiom captures that idea that more is preferred over less of goods. Specifically, it says that: (i) if one bundle contains at least as much of every goods as another bundle, then one is at least as good as the other; and (ii) if one bundle contains strictly more of every good as another bundle, then one strictly preferred over the other.

Axiom 4. *(Strict monotonicity) For all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbf{X}$*:*

(i) $\mathbf{x}^0 \geq \mathbf{x}^1 \Rightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$*;*

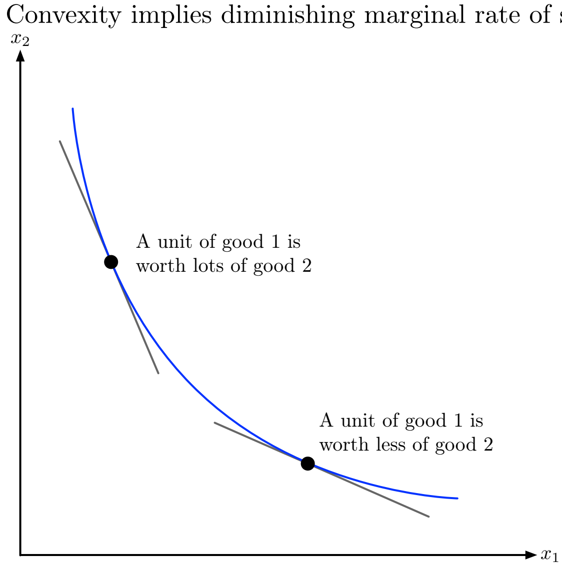
(ii) $\mathbf{x}^0 \gg \mathbf{x}^1 \Rightarrow \mathbf{x}^0 \succ \mathbf{x}^1$.

The next restriction requires that a convex combination of any two bundles are preferred than at one of the two goods. One interpret is that consumers to prefer a more "balanced" combination of bundles than the "extremes". Another is that the marginal rate of substitution is diminishing (see figure below)

Axiom. *(Convexity) For any* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbf{X}$*, if* $\mathbf{x}^1 \succsim \mathbf{x}^0$ *then* $t\mathbf{x}^1 + (1-t)\,\mathbf{x}^0 \succsim \mathbf{x}^0, \forall t \in [0,1]$.

Axiom 5. *(Strict convexity) For any* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbf{X}$*, if* $\mathbf{x}^1 \neq \mathbf{x}^0$ *and* $\mathbf{x}^1 \succsim \mathbf{x}^0$*, then* $t\mathbf{x}^1 + (1-t)\,\mathbf{x}^0 \succ \mathbf{x}^0, \forall t \in (0,1)$.

Figure 1.1: Convexity implies diminishing marginal rate of substitution



1.2 Utility representation

Binary relations are hard to work with—we would rather work with real-valued functions, or, utility functions.

Definition 1.2. *(Utility function)*. A real-valued function $u : \mathbf{X} \to \mathbb{R}$ is called a utility function representing the preference relation $\succsim$ if

$$u\left(\mathbf{x}^0\right) \geq u\left(\mathbf{x}^1\right) \Leftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1, \forall \mathbf{x}^0 \succsim \mathbf{x}^1 \in \mathbf{X}.$$

1.2.1 Finite $\mathbf{X}$

In case $\mathbf{X}$ is finite, then the following theorem guarantees that Axioms 1 and 2 are sufficient for a utility representation.

Theorem 1.1. *Suppose $\succsim$ satisfies Axioms 1 and 2 and that $\mathbf{X}$ is finite. Then there exists a utility function $u : \mathbf{X} \to \mathbb{R}$ representing $\succsim$.*

Proof. It suffices to find an example of a utility function representing $\succsim$. With Axioms 1 and 2, the set $\precsim(\mathbf{x})$ is well defined for any $\mathbf{x} \in \mathbf{X}$. Since $\mathbf{X}$ is finite, the cardinality of the set $\precsim(\mathbf{x})$ is always finite. Then, define

$$u(\mathbf{x}) := |\precsim(\mathbf{x})| = |\{\mathbf{y} \in \mathbf{X} | \mathbf{x} \succsim \mathbf{y}\}|$$

To verify that this represents $\succsim$,

$$\begin{aligned} u\left(\mathbf{x}^0\right) \geq u\left(\mathbf{x}^1\right) &\Leftrightarrow \left|\precsim\left(\mathbf{x}^0\right)\right| \geq \left|\precsim\left(\mathbf{x}^1\right)\right| \\ &\Leftrightarrow \left|\left\{\mathbf{y} \in \mathbf{X} | \mathbf{x}^0 \succsim \mathbf{y}\right\}\right| \geq \left|\left\{\mathbf{y} \in \mathbf{X} | \mathbf{x}^1 \succsim \mathbf{y}\right\}\right| \\ &\Leftrightarrow \left\{\mathbf{y} \in \mathbf{X} | \mathbf{x}^1 \succsim \mathbf{y}\right\} \subseteq \left\{\mathbf{y} \in \mathbf{X} | \mathbf{x}^0 \succsim \mathbf{y}\right\} \\ &\Leftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1. \end{aligned}$$

∎

1.2.2 $\mathbf{X} = \mathbb{R}^n_+$

The following characterises the necessary and sufficient conditions for the existence of a continuous utility function.

Theorem 1.2. *A binary relation $\succsim$ can be represented by a continuous utility function if and only if $\succsim$ satisfies Axioms 1–3.*

We will prove a slightly less general result, which, in addition to Axioms 1–3, assumes Axiom 4.

Theorem 1.3. *(Representation theorem). Let $\mathbf{X} = \mathbb{R}^n_+$. Suppose $\succsim$ satisfies Axioms 1–4. Then, there exists a continuous real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that represents $\succsim$.*

Proof. As before, it suffices to find an example of a utility function representing $\succsim$. Let $\mathbf{e} := (1, 1, \ldots, 1) \in \mathbb{R}^l_+$ be a vector of ones and consider $u : \mathbb{R}^n_+ \to \mathbb{R}$:

$$u(\mathbf{x})\,\mathbf{e} \sim \mathbf{x}.$$

This is equivalent to finding out where the indifference curve to $\mathbf{x}$ crosses the "45 degree" line and defining the util value as the coordinate of such a bundle. We first prove existence of such an u, and proceed to prove that it is uniquely determined so that u is a well-defined function.

Existence: Fix $\mathbf{x} \in \mathbb{R}^n_+$ and define

$$\begin{aligned} \mathbf{A} &:= \{t \geq 0 : t\mathbf{e} \succsim \mathbf{x}\}, \\ \mathbf{B} &:= \{t \geq 0 : t\mathbf{e} \precsim \mathbf{x}\}. \end{aligned}$$

We wish to show that $\mathbf{A} \cap \mathbf{B} \neq \emptyset$ since then, for any $t \in \mathbf{A} \cap \mathbf{B}$, $t\mathbf{e} \succsim \mathbf{x}$ and $t\mathbf{e} \precsim \mathbf{x}$ so that $t\mathbf{e} \sim \mathbf{x}$ and we can define $u(\mathbf{x}) = t$.

- ▷ **Step 1: A and B must be of the form $[\underline{t}, \infty)$ and $[0, \bar{t}]$.** We first show that $\mathbf{A}$ is closed. Consider any converging sequence $\{t^n\}$ in $\mathbf{A}$ such that $t^n \to t^*$. By definition, $\{t^n \mathbf{e}\} \in \succsim (\mathbf{x})$ for all n; i.e. $\{t^n \mathbf{e}\}$ is a sequence in the set $\succsim (\mathbf{x})$. Since $\succsim (\mathbf{x})$ is closed by Axiom 3, the limit of any converging sequence in $\succsim (\mathbf{x})$ is contained in the set. Thus, $\{t^* \mathbf{e}\} \in \succsim (\mathbf{x})$ so that $t^* \in \mathbf{A}$.

  By Axiom 4, $t \in \mathbf{A}$ implies that $t' \in \mathbf{A}$ for all $t' \geq t$. Thus, $\mathbf{A}$ must be a closed interval of the form $[\underline{t}, \infty)$. Similarly, $\mathbf{B}$ must be a closed interval of the form $[0, \bar{t}]$ (the lower bound is zero since Axiom 4 implies that $\mathbf{0} \precsim \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n_+$).

- ▷ **Step 2: $\underline{t} \leq \bar{t}$.** Axiom 1 ensures that, for any $t \geq 0$, either $t\mathbf{e} \succsim \mathbf{x}$ or $t\mathbf{e} \precsim \mathbf{x}$; i.e. $t \in \mathbf{A} \cup \mathbf{B}$. That is, $\mathbf{A} \cup \mathbf{B} = [0, \bar{t}] \cup [\underline{t}, \infty) = \mathbb{R}_+$. Thus, $\underline{t} \leq \bar{t}$, which, in turn, implies that $\mathbf{A} \cap \mathbf{B} \neq \emptyset$.

Uniqueness: Suppose $t_1 \mathbf{e} \sim \mathbf{x}$ and $t_2 \mathbf{e} \sim \mathbf{x}$, then by Axiom 2 (in particular, the transitivity of $\sim$), $t_1 \mathbf{e} \sim t_2 \mathbf{e}$. We want to show that $t_1 = t_2$. By way of contradiction (and without loss of generality), suppose that $t_1 > t_2$. By Axiom 3, $t_1 \mathbf{e} \gg t_2 \mathbf{e} \Rightarrow t_1 \mathbf{e} \succ t_2 \mathbf{e}$. Then, by Axiom 4,

$$\theta t_1 \mathbf{e} + (1 - \theta) t_2 \mathbf{e} \equiv t_\theta \mathbf{e} \succ t_2 \mathbf{e}, \ \forall \theta \in (0, 1).$$

But since $t_1 > t_\theta > t_2$, by Axiom 3, we must have

$$t_1 \mathbf{e} \succ t_\theta \mathbf{e}, \ t_\theta \mathbf{e} \succ t_2 \mathbf{e}.$$

By transitivity, above implies $t_1 \mathbf{e} \succ t_2 \mathbf{e}$; i.e. $t_1 \mathbf{e} \sim \mathbf{x} \succ \mathbf{x} \sim t_2 \mathbf{e}$, which is a contradiction.

We have therefore shown that, for every $\mathbf{x} \in \mathbb{R}^n_+$, there exists exactly one number, $u(\mathbf{x})$, such that $u(\mathbf{x}) \mathbf{e} \sim \mathbf{x}$. We now verify that this utility function represents the preferences $\succsim$. Consider two bundles $\mathbf{x}^1$ and $\mathbf{x}^2$, and their associated utility numbers $u(\mathbf{x}^1)$ and $u(\mathbf{x}^2)$. By construction, $u(\mathbf{x}^1) \mathbf{e} \sim \mathbf{x}^1$ and $u(\mathbf{x}^2) \mathbf{e} \sim \mathbf{x}^2$. Then,

$$\begin{aligned}
\mathbf{x}^1 \succsim \mathbf{x}^2 &\Leftrightarrow u(\mathbf{x}^1) \mathbf{e} \sim \mathbf{x}^1 \succsim \mathbf{x}^2 \sim u(\mathbf{x}^2) \mathbf{e} \\
&\Leftrightarrow u(\mathbf{x}^1) \mathbf{e} \succsim u(\mathbf{x}^2) \mathbf{e} \\
&\Leftrightarrow u(\mathbf{x}^1) \geq u(\mathbf{x}^2),
\end{aligned}$$

where the first line follows from the definition, the second follows from the fact Axiom 2 (since $\sim$ is reflexive). To see the line, suppose, by way of contradiction that $u(\mathbf{x}^1) \mathbf{e} \succsim u(\mathbf{x}^2) \mathbf{e}$ but $u(\mathbf{x}^1) < u(\mathbf{x}^2)$. By Axiom 3, we must have $u(\mathbf{x}^1) \mathbf{e} \ll u(\mathbf{x}^2) \mathbf{e} \Rightarrow u(\mathbf{x}^1) \mathbf{e} \prec u(\mathbf{x}^2) \mathbf{e}$, which is a contradiction. Now suppose (again, by way of contradiction) that $u(\mathbf{x}^1) \geq u(\mathbf{x}^2)$ but $u(\mathbf{x}^1) \mathbf{e} \prec u(\mathbf{x}^2) \mathbf{e}$. Then, by Axiom 4,

$$\theta u(\mathbf{x}^1) \mathbf{e} + (1 - \theta) u(\mathbf{x}^2) \mathbf{e} \equiv u_\theta \mathbf{e} \succ u(\mathbf{x}^1) \mathbf{e}, \ \forall \theta \in (0, 1).$$

But since $u(\mathbf{x}^1) \geq u_\theta \geq u(\mathbf{x}^2)$, Axiom 3 implies that

$$u(\mathbf{x}^1) \mathbf{e} \succsim u_\theta \mathbf{e}, \ u_\theta \mathbf{e} \succsim u(\mathbf{x}^2) \mathbf{e} \Rightarrow u(\mathbf{x}^1) \mathbf{e} \succsim u(\mathbf{x}^2) \mathbf{e},$$

where the implication follows by transitivity. But this contradicts our assumption that $u(\mathbf{x}^2) \mathbf{e} \succ u(\mathbf{x}^1) \mathbf{e}$.

It remains to show that $u(\mathbf{x})$ is continuous. It suffices to show that the inverse image under u, denoted $u^{-1}(\cdot)$, of every open ball in $\mathbb{R}$ is open in $\mathbb{R}^n_+$.[1] Since open balls in $\mathbb{R}$ are open intervals, this is equivalent to showing that $u^{-1}((a,b))$ is open in $\mathbb{R}^n_+$ for every $a<b$. Note

$$\begin{aligned} u^{-1}((a,b)) &= \left\{\mathbf{x}\in\mathbb{R}^n_+ : a < u(\mathbf{x}) < b\right\} \\ &= \left\{\mathbf{x}\in\mathbb{R}^n_+ : a\mathbf{e} \prec u(\mathbf{x})\,\mathbf{e} \prec b\mathbf{e}\right\} \\ &= \left\{\mathbf{x}\in\mathbb{R}^n_+ : a\mathbf{e} \prec \mathbf{x} \prec b\mathbf{e}\right\}, \end{aligned}$$

where the first equality follows from the definition of the inverse image, the second from the fact that u represents $\succsim$, and the last line follows from the fact that $\mathbf{x}\sim u(\mathbf{x})\,\mathbf{e}$ by construction and by Axiom 2. We can rewrite the last line as

$$u^{-1}((a,b)) = \succ(a\mathbf{e})\bigcap \prec(b\mathbf{e}).$$

By Axiom 2, $\succ(a\mathbf{e})$ and $\prec(b\mathbf{e})$ are open in $\mathbb{R}^n_+$ and we note that intersection of two open sets in $\mathbb{R}^n_+$ is open in $\mathbb{R}^n_+$. ■

We did not show that there is a unique utility function that represents $\succsim$. In fact, there are arbitrary many continuous utility functions that can represent $\succsim$ satisfying Axioms 1–4 (e.g. take $u(\mathbf{x})$ as we found above and let $v(\mathbf{x}) = u(\mathbf{x}) + c$, where $c\in\mathbb{R}$). This is driven by the fact that what matters utility representation is its ordinal properties, and not its cardinal properties, as shown below.

Definition 1.3. *(Increasingness).* $u(\mathbf{x})$ is *strictly increasing* if $u\left(\mathbf{x}^0\right)\ge u\left(\mathbf{x}^1\right)$ whenever $\mathbf{x}^0\ge\mathbf{x}^1$, and the inequality is strict whenever $\mathbf{x}^0\gg\mathbf{x}^1$.

Theorem 1.4. *(Invariance). Let $\succsim$ be a preference relation on $\mathbb{R}^n_+$ and $u(\mathbf{x})$ be a utility function that represents it. Then $v:\mathbb{R}^n_+\to\mathbb{R}$ also represents $\succsim$ if and only if*

$$v(\mathbf{x}) = f(u(\mathbf{x})), \forall\mathbf{x}\in\mathbb{R}^n_+,$$

where $f:\mathbb{R}\to\mathbb{R}$ is strictly increasing on the set of values taken on by u.

Proof. ($\Rightarrow$) Suppose such f exists. Then, for any $\mathbf{x}^0,\mathbf{x}^1\in\mathbb{R}^l_+$,

$$\begin{aligned} \mathbf{x}^0\succsim\mathbf{x}^1 &\Leftrightarrow u\left(\mathbf{x}^0\right)\ge u\left(\mathbf{x}^1\right) \\ &\Leftrightarrow f\left(u\left(\mathbf{x}^0\right)\right)\ge f\left(u\left(\mathbf{x}^1\right)\right) \\ &\Leftrightarrow v\left(\mathbf{x}^0\right)\ge v\left(\mathbf{x}^1\right), \end{aligned}$$

where the first line follows from the fact that u represents $\succsim$, the second follows from the fact that f is strictly increasing, and the last follows by definition.

[1] Let D be a subset of $\mathbb{R}^m$. The following conditions are equivalent:

(i) $f:D\to\mathbb{R}^n$ is continuous.

(ii) For every open ball B in $\mathbb{R}^n$, $f^{-1}(B)$ is open in D.

(iii) For every open set S in $\mathbb{R}^n$, $f^{-1}(S)$ is open in D.

($\Leftarrow$) Suppose u and v represents $\succsim$. Then,

$$\mathbf{x}^0 \succsim \mathbf{x}^1 \Leftrightarrow u\left(\mathbf{x}^0\right) \geq u\left(\mathbf{x}^1\right) \Leftrightarrow v\left(\mathbf{x}^0\right) \geq v\left(\mathbf{x}^1\right).$$

We want to show that this implies that there exists a strictly increasing function f that satisfies $v(\mathbf{x}) = f(u(\mathbf{x}))$; i.e. $y \geq z \Rightarrow f(y) \geq f(z)$. This is equivalent to requiring that

$$\begin{aligned} y = z &\Rightarrow f(y) = f(z), \\ y > z &\Rightarrow f(y) > f(z). \end{aligned}$$

Note that we can always create a function that satisfies the equality so the question is whether such a function is strictly increasing or not (i.e. define f so that point-wise sets $f(u(x)) = v(x)$ in the image of u). So, assume $v(\mathbf{x}) = f(u(\mathbf{x}))$. Suppose f is not strictly increasing, then letting $y = u\left(\mathbf{x}^0\right)$ and $z = u\left(\mathbf{x}^1\right)$, this means that there exists y and z such that

$$y = u\left(\mathbf{x}^0\right) > z = u\left(\mathbf{x}^1\right) \Rightarrow v\left(\mathbf{x}^0\right) = f(y) \leq f(z) = v\left(\mathbf{x}^1\right)$$

but this is a contradiction—$u\left(\mathbf{x}^0\right) > u\left(\mathbf{x}^1\right) \Leftrightarrow \mathbf{x}^0 \succ \mathbf{x}^1$ but $v\left(\mathbf{x}^0\right) \leq v\left(\mathbf{x}^1\right) \Leftrightarrow \mathbf{x}^1 \succsim \mathbf{x}^0$ (since both represent the same preferences $\succsim$). We lead to a similar contradiction if $y = z$ so we may conclude f must be strictly increasing. ■

Remark 1.1. The condition that f be strictly increasing *"on the set of values taken on by u"* is important. To see why, consider

$$u(x, y) = \begin{cases} xy & \text{if } xy \leq 5 \\ xy + 4 & \text{if } xy > 5 \end{cases}.$$

Let

$$f(z) := \begin{cases} z & \text{if } z \leq 5 \\ z - 4 & \text{if } z > 5 \end{cases},$$

then

$$v(x, y) := f(u(x, y)) = xy.$$

Notice that $f(z)$ is *not* strictly increasing everywhere (in particular, when $z = 5$). However, $f(z)$ is strictly increasing if we consider only the values taken on by $u(x, y)$: $[0, 5] \cup (9, \infty)$.

1.2.3 Links between properties of u and $\succsim$

Theorem 1.5. *Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:*

(i) $u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.

(ii) $u(\mathbf{x})$ is quasiconcave if and only if $\succsim$ is convex.

(iii) $u(\mathbf{x})$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.

Proof. (i) Suppose $u(\mathbf{x})$ is strictly increasing, then

$$\begin{aligned}\mathbf{x}^1 \geq \mathbf{x}^0 &\Rightarrow u\left(\mathbf{x}^1\right) \geq u\left(\mathbf{x}^0\right) \Leftrightarrow \mathbf{x}^1 \succsim \mathbf{x}^0, \\ \mathbf{x}^1 \gg \mathbf{x}^0 &\Rightarrow u\left(\mathbf{x}^1\right) > u\left(\mathbf{x}^0\right) \Leftrightarrow \mathbf{x}^1 \succ \mathbf{x}^0.\end{aligned}$$

Hence, $\succsim$ is strictly monotonic. Since the relationship between u and $\succsim$ is if and only if, we may reverse the last two implications.

(ii) Suppose $u(\mathbf{x})$ is quasiconcave, then

$$u\left(t\mathbf{x}^0 + (1-t)\mathbf{x}^1\right) \geq \min\left\{u\left(\mathbf{x}^0\right), u\left(\mathbf{x}^1\right)\right\}, \ \forall t \in [0,1].$$

Without loss of generality, suppose $u\left(\mathbf{x}^0\right) \geq u\left(\mathbf{x}^1\right)$,

$$\begin{aligned}& u\left(t\mathbf{x}^0 + (1-t)\mathbf{x}^1\right) \geq u\left(\mathbf{x}^1\right), \ \forall t \in [0,1] \\ \Leftrightarrow\ & t\mathbf{x}^0 + (1-t)\mathbf{x}^1 \succsim \mathbf{x}^1, \ \forall t \in [0,1].\end{aligned}$$

Thus, $\succsim$ is convex. Reverse the steps in the proof.

(iii) Suppose $u(\mathbf{x})$ is strictly quasiconcave and $\mathbf{x}^0 \neq \mathbf{x}^1$, then

$$u\left(t\mathbf{x}^0 + (1-t)\mathbf{x}^1\right) > \min\left\{u\left(\mathbf{x}^0\right), u\left(\mathbf{x}^1\right)\right\}, \ \forall t \in (0,1).$$

Without loss of generality, suppose $u\left(\mathbf{x}^0\right) \geq u\left(\mathbf{x}^1\right) \Leftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$, then,

$$\begin{aligned}& u\left(t\mathbf{x}^0 + (1-t)\mathbf{x}^1\right) > u\left(\mathbf{x}^1\right), \ \forall t \in (0,1) \\ \Leftrightarrow\ & t\mathbf{x}^0 + (1-t)\mathbf{x}^1 \succ \mathbf{x}^1, \ \forall t \in (0,1).\end{aligned}$$

Thus, $\succsim$ is strictly convex. Reverse the steps in the proof. ■

1.3 TA session: Debreu's representation theorem*

We provide a proof of (slightly less general version of) Debreu's representation theorem. We begin with some definitions.

1.3.1 Preliminaries

Definition 1.4. A subset $D \subseteq X$ is *dense* in the metric space (X, d) if, for any $x \in X$, and for any $\delta > 0$, there exists $x_0 \in D$ such that $x \in N_\delta(x_0) \coloneqq \{y \in X : d(y, x) < \delta\}$.

That is, if D is dense in the metric space (X, d), then any point in X can be approximated by a point from D—in particular, any point in X can be contained in any open ball around a point in D.

Definition 1.5. *(Separable space).* A metric space (X, d) is *separable* if there exists a countable subset $D \subset X$ that is dense in X.

For example, $\mathbb{R}$ (with the usual Euclidean metric) is a separable space since it contains the set of rational numbers $\mathbb{Q}$, which is both countable and dense in $\mathbb{R}$. That $\mathbb{Q}$ is dense in $\mathbb{R}$ means that, for every nonempty open interval $(a, b) \subseteq \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $q \in (a, b)$.

Definition 1.6. (Connected metric space). A metric space (X, d) is *connected* if there does not exist two nonempty open sets $U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \emptyset$.

An interval $X = [a, b]$ in $\mathbb{R}$ is an example of a connected space.[2] We use the following alternative definition of connected spaces.

Proposition 1.4. *A metric space (X, d)* is connected*, if for any $A \subseteq X$, A is both open and closed if and only if $A = \emptyset$ or $A = X$.*

Proof. Let $A \subseteq X$ and suppose (X, d) is a metric space.

First, suppose A is both open and closed. That A is closed means that its complement, $X \backslash A$, is open. Therefore, $A \cup X \backslash A = X$ and $A \cap X \backslash A = \emptyset$ (by definition of complement). Since (X, d) is connected, it must be the case that $A = \emptyset$ or $A = X$.

To show the converse, recall that the empty set and the whole set (X) are both open and closed. ■

1.3.2 Debreu's representation theorem

Lemma 1.1. *(Denseness). Let (X, d) be a connected metric space and $\succsim$ be a binary relation that is complete, transitive and continuous. Suppose that there exists $x, y \in X$ such that $x \succ y$, then there exists $z \in X$ such that $x \succ z \succ y$.*

Proof. Consider $\succsim(x)$ and $\precsim(y)$. Suppose that $\succsim(x) \cup \precsim(y) = X$, which would imply that $z \in X$ such that $x \succ z \succ y$ does not exist. Note that $\succsim(x) \neq \emptyset$ and $\precsim(y) \neq \emptyset$ since x and y are respective members of the sets. Moreover, since $x \succ y$, $\succsim(x) \cap \precsim(y) = \emptyset$. This, together with the assumption that $\succsim(x) \cup \precsim(y) = X$ imply that the two sets form a partition on X; i.e. $\succsim(x) = X \backslash \precsim(y)$.

By continuity, $\succsim(x)$ is closed. Continuity also implies that $\precsim(y)$ is closed and its complement, $\succsim(x)$, must be open by definition. Therefore, $\succsim(x)$ is both closed and open. From Definition 1.4, this implies that $\succsim(x) = \emptyset$ or $\succsim(x) = X$. The former contradicts the fact that x is nonempty. The latter would imply that $\precsim(y) = \emptyset$, which is also a contradiction. ■

We first establish the following corollary of Lemma 1.1. This is is stronger than Lemma 1.1 since it requires z to be in a countable set.

Corollary 1.1. *Let (X, d) be a connected and separable metric space and $\succsim$ be a binary relation that is complete, transitive and continuous. Then, there exists a countable set $Z \subseteq X$ such that whenever $x, y \in X$ and $x \succ y$, there exists $z \in Z$ such that $x \succ z \succ y$.*

Proof. Since (X, d) is separable, by definition, there exists $Z \subseteq X$ which is countable and dense in X. Take any $x, y \in X$ such that $x \succ y$. Since (X, d) is connected, from Lemma 1.1, there exists $\tilde{z} \in X$ such that $x \succ \tilde{z} \succ y$. Since Z is dense in X, there exists a sequence in Z, $\{z_n\}$, such that $z_n \to \tilde{z}$ as $n \to \infty$. By continuity, $\succ(y)$ and $\prec(x)$ are open so that there exists $N \in \mathbb{N}$ such that $x \succ z_n \succ y$ for all $n \geq N$. ■

[2] To prove this, suppose not; i.e. there exists U and V that are nonempty and open subsets of X that are disjoint $(U \cap V = \emptyset)$ and form a partition on X $(U \cup V = X)$. Pick any $a \in U$ and $b \in V$, and (wlog) let $a < b$. Define $c := \sup\{t \in U : t < b\}$. Notice that $a \leq c \leq b$ so $c \in X$ and c is either in U or in V.

- ▷ If $c \in U$, then $c \neq b$ since $U \cap V = \emptyset$ so $c < b$. Since U is open, there exists $\delta > 0$ such that $b > c + \varepsilon \in U$. But this contradicts the choice of c as the supremum.
- ▷ If $c \notin U$, then $a < c$ since $U \cup V = X$. Given that V is open, $(c - \delta, c) \subseteq V$ and hence $(c - \delta, c) \cap U = \emptyset$. However, by the definition of c, $c - \delta \in U$—otherwise, we must have $c = \sup\{t \in U : t < b\} \leq c - \delta$—a contradiction.

Theorem 1.6. *(Debreu). Let (X, d) be a connected and separable metric space and $\succsim$ be a binary relation that is complete, transitive and continuous. Then, there exists $u : X \to \mathbb{R}$ that represents $\succsim$.*

Proof. Using the corollary, take a countable set $Z \subseteq X$, and for any $x \in X$, define

$$\begin{aligned}\succsim^z(x) &\coloneqq \{z \in Z : z \succsim x\},\\ \precsim^z(x) &\coloneqq \{z \in Z : z \precsim x\}.\end{aligned}$$

Because Z is countable, we may write all its elements as a sequence $\{z_n\}$. Define

$$u(x) \coloneqq \sum_{\{j : z_j \leq x\}} 2^{-j} - \sum_{\{j : z_j \geq x\}} 2^{-j}.$$

To see that $u(x)$ is well define, by the Triangle Inequality,

$$\begin{aligned}|u(x)| &= \left| \sum_{\{j : z_j \leq x\}} 2^{-j} - \sum_{\{j : z_j \geq x\}} 2^{-j} \right|\\ &\leq \left| \sum_{\{j : z_j \leq x\}} 2^{-j} \right| + \left| \sum_{\{j : z_j \geq x\}} 2^{-j} \right|\\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.\end{aligned}$$

We now show that u represents ; i.e. $x \succsim y \Leftrightarrow u(x) - u(y) \geq 0$. Suppose that $x \succsim y$.

- ▷ If $x \sim y$, then
$$\succsim^z(x) = \succsim^z(y), \precsim^z(x) = \precsim^z(y)$$
so that $u(x) - u(y) = 0$. Moreover, by transitivity, $\succsim^z(x) \subseteq \succsim^z(y)$ and $\precsim^z(y) \subseteq \precsim^z(x)$.
- ▷ If $x \succ y$, then by the corollary, there exists $z \in Z$ such that $x \succ z \succ y$, which implies that
$$\succsim^z(x) \subsetneq \succsim^z(y), \precsim^z(y) \subsetneq \precsim^z(x)$$
so that $u(x) - u(y) > 0$. ∎

2 Exchange economy

2.1 Walrasian equilibrium

There are I consumers. Let $\mathcal{I} = \{1, 2, \ldots, I\}$ denote the set of consumers. Each consumer $i \in \mathcal{I}$ has a utility function $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ that represents his preferences that satisfies Axioms 1–4, and an endowment vector $\mathbf{e}^i = \left(\mathbf{e}^i_1, \mathbf{e}^i_2, \ldots, \mathbf{e}^i_n\right) \in \mathbb{R}^n_+$. An exchange economy is defined as

$$\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}.$$

Definition 2.1. *(Walrasian equilibrium)* $\mathbf{p}^* \in \mathbb{R}^n_+$ is a *Walrasian equilibrium* of $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ if there exists $\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^I \in \mathbb{R}^n_+$ such that:[a]

(i) for all $i \in \mathcal{I}$, $\hat{\mathbf{x}}^i$ maximises $u^i\left(\mathbf{x}^i\right)$ subject to $\mathbf{p}^* \cdot \mathbf{x}^i \leq \mathbf{p}^* \cdot \mathbf{e}^i$;

(ii) market clears; i.e. $\sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i$.

We will say that $\hat{\mathbf{x}} = \left\{\hat{\mathbf{x}}^i\right\}_{i\in\mathcal{I}}$ is a *Walrasian equilibrium allocation* (WEA).

[a] The book assumes instead that $\mathbf{p}^* \in \mathbb{R}^n_{++}$.

Thus, a Walrasian equilibrium (also called a competitive equilibrium) is a price vector. In particular, we need not specify bundles $\left\{\hat{\mathbf{x}}^i\right\}_{i\in\mathcal{I}}$ (unlike in macro). Notice also that Walrasian equilibrium price need not be strictly positive as a matter of definition. We now analyse the conditions that guarantee the existence of a Walrasian equilibrium $\mathbf{p}^*$.

2.2 Excess demand function

Definition 2.2. *(Strong increasingness).* We say that u^i is *strongly increasing* if

$$\mathbf{x} \geq \mathbf{y},\ \mathbf{x} \neq \mathbf{y} \Rightarrow u^i(\mathbf{x}) > u^i(\mathbf{y}).$$

Definition 2.3. *(Strict quaisconcavity).* For any $\mathbf{x} \neq \mathbf{y}$,

$$u^i\left(t\mathbf{x} + (1-t)\mathbf{y}\right) > \min\left\{u^i(\mathbf{x}), u^i(\mathbf{y})\right\},\ \forall t \in (0,1).$$

Assumption 1. *Each u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.*

Remark 2.1. In fact, if u^i is strictly increasing and strict quasiconcave, then u^i is strongly increasing. To see this, let $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$. Since u^i is strictly increasing, this implies that $u^i(\mathbf{x}) \geq u^i(\mathbf{x})$. Then strict quasiconcavity gives us that

$$u^i\left(t\mathbf{x} + (1-t)\mathbf{y}\right) > \min\left\{u^i(\mathbf{x}), u^i(\mathbf{y})\right\} = u^i(\mathbf{y}),\ \forall t \in (0,1).$$

Since $\mathbf{x} \geq t\mathbf{x} + (1-t)(\mathbf{y})$ and u^i is strictly increasing,

$$u^i(\mathbf{x}) \geq u^i\left(t\mathbf{x} + (1-t)(\mathbf{y})\right).$$

Together, we must have $u^i(\mathbf{x}) > u^i(\mathbf{y})$.

That u^i is continuous is implied by Axioms 1–3. Strict quasiconcavity is implied by the strict convexity axiom and means that the indifference curves are strictly convex. Strong increasingness is a "strong" assumption and rules out, for example, Cobb-Douglas utility functions.

Theorem 2.1. *If u^i satisfies Assumption 1, then, for all $\mathbf{p} \gg \mathbf{0}$, the problem*

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} \quad u^i\left(\mathbf{x}^i\right) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i$$

has a unique solution $x^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right)$, which is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.

Proof. That a solution exists follows from the Weierstrass theorem.[3] Since $\mathbf{p} \gg \mathbf{0}$, the budget set

$$\Gamma(\mathbf{x}) = \left\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i\right\}$$

is closed and bounded, which in $\mathbb{R}^n_+$ implies that $\Gamma(\mathbf{x})$ is compact. Since u^i is continuous, the Weierstrass theorem guarantees existence of a maximum.

That the solution is unique follows from the strict quasiconcavity of u^i. To see this, suppose there are two optima—$\tilde{\mathbf{x}}^i$ and $\hat{\mathbf{x}}^i$. Consider taking a convex combination of these bundles. By strict quasiconcavity of u^i,

$$u^i\left(t\tilde{\mathbf{x}}^i + (1-t)\hat{\mathbf{x}}^i\right) > \min\left\{u^i\left(\tilde{\mathbf{x}}^i\right), u^i\left(\hat{\mathbf{x}}^i\right)\right\},$$

which contradicts the assumption that $\tilde{\mathbf{x}}^i$ and $\hat{\mathbf{x}}^i$ are maximum.

Finally, continuity of $\mathbf{x}^i$ follows from the Theorem of the Maximum. Note that continuity at $p_k = 0$ is not guaranteed since demand may well be infinite if one of the prices are zero. ■

For convenience, we describe each separate market by a single *excess demand function.* We define this as a vector-valued function

$$\mathbf{z}(\mathbf{p}) \coloneqq \left(z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p})\right),$$

where $z_k(\mathbf{p})$ denotes the aggregate excess demand function for good k:

$$z_k(\mathbf{p}) = \sum_{i \in \mathcal{I}} x^i_k\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right) - \sum_{i \in \mathcal{I}} e^i_k.$$

Theorem 2.2. *(Properties of aggregate excess demand function). Suppose $\left\{u^i\right\}_{i \in \mathcal{I}}$ satisfies Assumption 1, then*

- ▷ *(continuity) $\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}_{++}$;*
- ▷ *(homogeneity of degree zero) $\mathbf{z}(\lambda\mathbf{p}) = z(\mathbf{p})$ for all $\lambda > 0$ and for all $\mathbf{p} \gg \mathbf{0}$;*
- ▷ *(Walras' law) $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$.*

Proof. That $\mathbf{z}(\cdot)$ is continuous at $\mathbf{p}$ follows from the fact that x^i is continuous at $\mathbf{p}$ and since continuity is preserved by adding/subtracting constants.

That $\mathbf{z}(\cdot)$ is homogeneous of degree zero follows from the fact that multiplying prices by $\lambda > 0$ leaves the budget set (and everything else) unchanged.

[3] The Weierstrass theorem states that a continuous function $f : S \to \mathbb{R}$ where S is compact subset of $\mathbb{R}^n$ attains a maximum and minimum value on S.

Since u^i is strongly increasing, we know that, at the optimum, the budget constraint must bind for each $i \in \mathcal{I}$ (if not, we can increase demand for any good by some amount and obtain a strictly higher utility). That is,

$$\mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i, \ \forall i \in \mathcal{I}.$$

Summing across all i:

$$\begin{aligned}
\sum_{i \in \mathcal{I}} \mathbf{p} \cdot \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{p} \cdot \mathbf{e}^i &\Leftrightarrow \mathbf{p} \cdot \sum_{i \in \mathcal{I}} \mathbf{x}^i = \mathbf{p} \cdot \sum_{i \in \mathcal{I}} \mathbf{e}^i \\
&\Leftrightarrow \mathbf{p} \cdot \left(\sum_{i \in \mathcal{I}} \mathbf{x}^i - \sum_{i \in \mathcal{I}} \mathbf{e}^i \right) = 0 \\
&\Leftrightarrow \mathbf{p} \cdot \mathbf{z}\left(\mathbf{p}\right) = 0.
\end{aligned}$$

■

Remark 2.2. Some remarks.

- ▷ That $\mathbf{z}\left(\mathbf{p}\right)$ is homogeneous of degree zero means that only relative prices matter in consumer's choices.
- ▷ *(Walras' law).* Consider a two-good economy and suppose that prices are strictly positive. By Walras' law,

  $$p_1 z_1 \left(\mathbf{p}\right) = -p_2 z_2 \left(\mathbf{p}\right),$$

  so that, if there is excess demand in, say, market 1 (i.e. $z_1\left(p\right) > 0$), then we know immediately that $z_2\left(p\right) < 0$; i.e. there is excess supply in market 2. By the same logic, if we know market 1 is in equilibrium (i.e. $z_1\left(\mathbf{p}\right) = 0$), then it follows that market 2 is also in equilibrium with $z_2\left(\mathbf{p}\right) = 0$. More generally, if at some prices $n-1$ markets are in equilibrium, Walras' law ensures that the nth market is in equilibrium.
- ▷ *(Partial vs general equilibrium).* Note that excess demand in any particular market, $z_k\left(\mathbf{p}\right)$, may depend on the prevailing prices in *every* market. Thus, the system of market we are considering here are completely interdependent. We say that there is a *partial equilibrium* in the single market k when $z_k\left(\mathbf{p}\right) = 0$ (but not necessarily $z_{k'}\left(\mathbf{p}\right) = 0$ for some other good k'). Only if $z\left(\mathbf{p}\right) = \mathbf{0}$ we can say that the system of markets is in a general equilibrium.

Prices that equate demand and supply in every market is called *Walrasian.* We can define a Walrasian equilibrium using the aggregate excess demand function.

Definition 2.4. *(Walrasian equilibrium).* A vector $\mathbf{p}^* \in \mathbb{R}^n_+$ is a Walrasian equilibrium if $\mathbf{z}\left(\mathbf{p}^*\right) = \mathbf{0}$.

2.3 Existence of a Walrasian equilibrium

We now turn to the question of existence of a Walrasian equilibrium.

Theorem 2.3. *(Aggregate excess demand and Walrasian equilibrium). Suppose* $\mathbf{z} : \mathbb{R}^n_{++} \to \mathbb{R}$ *satisfies the following three conditions:*

(i) (continuity) $\mathbf{z}(\cdot)$ *is continuous on* $\mathbb{R}^n_{++}$*;*

(ii) (Walras' law) $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ *for all* $\mathbf{p} \gg \mathbf{0}$*;*

(iii) if $\{\mathbf{p}^m\}$ *is a sequence of prices in* $\mathbb{R}^n_{++}$ *and* $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ *and* $\bar{p}_k = 0$ *for some good* k*, then for some good* k' *with* $\bar{p}_{k'} = 0$*, the sequence* $\{z_{k'}(\mathbf{p}^m)\}$*is unbounded above.*[a]

Then, there exists $\mathbf{p}^* \gg \mathbf{0}$ *such that* $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

[a] The last condition says that if prices of some, but not all goods, is arbitrarily close to zero, then the (excess) demand for at least one of those goods is arbitrarily high.

Before proceeding with the proof, let us consider how we might go about proving this theorem. Recall that $z_k(\mathbf{p}) > 0$ means that there is excess demand for good k. Intuition tells us then that p_k should be higher to clear the market. Thus, one idea is to consider the price of good k to be $f_k(\mathbf{p})$

$$f_k(\mathbf{p}) = p_k + z_k(\mathbf{p}).$$

This would increase the price of good k when there is excess demand and reduce it when there is excess supply.[4] Let $f(\mathbf{p}) = (f_1(\mathbf{p}), f_2(\mathbf{p}), \ldots, f_n(\mathbf{p}))$ be a vector-valued function. Then, finding $\mathbf{p}^* \gg 0$ such that $z_k(\mathbf{p}^*) = 0$ for all k is equivalent to

$$\mathbf{f}(\mathbf{p}^*) = \mathbf{p}^* \Leftrightarrow f_k(p^*) = p^*_k, \ \forall k.$$

Thus, we are looking for a fixed point $\mathbf{p}^*$. We will use the Brouwer's Fixed Point Theorem.

Theorem 2.4. *(Brouwer's Fixed Point Theorem). Let* $f : C \to C$ *be continuous, self-mapping function, where* C *is a nonempty compact subset of* $\mathbb{R}^n$*. Then, there exists* $x^* \in C$ *such that* $f(x^*) = x^*$.

However, we face two problems.

- ▷ Since $z_k(\mathbf{p})$ can be negative, $f_k(\mathbf{p})$ could also be negative, and thus we may not be able to satisfy the requirement for f to be a self map (since we want $C = \mathbb{R}^n_{++}$ here).
- ▷ C is not compact here since is $\mathbb{R}^n_{++}$ is unbounded.

Thus, we need to close the set and be careful around zero.

Proof. (Proof of Theorem 2.3). We first restrict our search for p^* to the following set. Fixing $\varepsilon \in (0, 1)$,

$$S_\varepsilon = \left\{ \mathbf{p} \in \mathbb{R}^n_{++} : \sum_{k=1}^{n} p_k = 1 \text{ and } p_k \geq \frac{\varepsilon}{1 + 2n}, \ \forall k \right\}.$$

Note, in particular, that S_ε is compact—it is closed (the inequalities are all weak) and bounded (since all prices has to sum to one with all prices being weakly larger than a strictly positive

[4] This is a purely mathematical construct—the units of p_k and z_k are not consistent.

number)—nonempty and convex.[5]

We now "patch up" $f_k(\mathbf{p}) = p_k + z_k(\mathbf{p})$. We would like to ensure that $f_k(\mathbf{p})$ is nonnegative—the possibility arose because $z_k(\mathbf{p})$ could be negative. We can fix this by instead using $\max\{z_k(\mathbf{p}), 0\}$. At the same time, we would like to be able to take limits—for reasons that will become clear later, we also bound $z(\mathbf{p})$ by considering the function

$$\bar{z}_k(\mathbf{p}) = \min\{z_k(\mathbf{p}), 1\};$$

(it is not important the upper bound is set to 1 here). Thus, let us redefine $f_k(\mathbf{p})$ as

$$f_k(\mathbf{p}) \coloneqq \frac{\varepsilon + p_k + \max\{\bar{z}_k(\mathbf{p}), 0\}}{n\varepsilon + 1 + \sum_{m=1}^{n} \max\{\bar{z}_m(\mathbf{p}), 0\}}.$$

We wish to show that f_k is a self-mapping continuous function. Since

$$\begin{aligned}\sum_{k=1}^{n} f_k(\mathbf{p}) &= \sum_{k=1}^{n} \frac{\varepsilon + p_k + \max\{\bar{z}_k(\mathbf{p}), 0\}}{n\varepsilon + 1 + \sum_{m=1}^{n} \max\{\bar{z}_m(\mathbf{p}), 0\}} \\ &= \frac{n\varepsilon + \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} \max\{\bar{z}_k(\mathbf{p}), 0\}}{n\varepsilon + 1 + \sum_{m=1}^{n} \max\{\bar{z}_m(\mathbf{p}), 0\}} \\ &= 1,\end{aligned}$$

we know that the first condition for S_ε is satisfied. To see that the second condition is satisfied, we wish to make $f_k(\mathbf{p})$ as small as possible: notice that $\max\{\cdot\}$ and p_k are both bounded below by zero, and the summation in the denominator is bounded above by n (upper bound of each $\bar{z}_m(\mathbf{p}) = 1$)), so

$$\begin{aligned}f_k(p) = \frac{\varepsilon + p_k + \max\{\bar{z}_k(\mathbf{p}), 0\}}{n\varepsilon + 1 + \sum_{m=1}^{n} \max\{\bar{z}_m(\mathbf{p}), 0\}} &\geq \frac{\varepsilon + 0 + 0}{n\varepsilon + 1 + n} \\ &= \frac{\varepsilon}{1 + (\varepsilon + 1) n} \\ &\geq \frac{\varepsilon}{1 + 2n},\end{aligned}$$

where the last inequality follows because $\varepsilon \in (0, 1)$. Thus, we conclude that $f(\mathbf{p})$ is a self map on S_ε; i.e. $f : S_\varepsilon \to S_\varepsilon$. It remains to show that f_k is continuous to be able to use the Brouwer's Fixed Point Theorem. Since $z(\cdot)$ is continuous on $\mathbb{R}^n_{++}$, $z_k(\cdot)$ is also continuous. Moreover, 1 is a constant which is continuous, and taking the minimum is a continuous operation so that $\bar{z}_k(\cdot)$ is

[5]That S_ε is nonempty follows from the fact that $p_k = (2 + 1/n) / (1 + 2n)$ for all k is always a member of the set S_ε:

$$\sum_{k=1}^{n} \frac{2n + 1/n}{1 + 2n} = \frac{2n + 1}{1 + 2n} = 1,$$
$$\frac{2 + 1/n}{1 + 2n} > \frac{\varepsilon}{1 + 2n}, \ \forall n \ \because \varepsilon \in (0, 1).$$

To see that S_ε is convex, consider $p, p' \in S_\varepsilon$ and define $\tilde{p} = tp + (1 - t) p'$ for any $t \in (0, 1)$. Then,

$$\tilde{p}_k = tp_k + (1 - t) p'_k, \forall k$$
$$\Rightarrow \sum_{k=1}^{n} \tilde{p}_k = t\left(\sum_{k=1}^{n} p_k\right) + (1 - t)\left(\sum_{k=1}^{n} p'_k\right) = 1$$

and since p'_k and p_k are all weakly larger than $\varepsilon / (1 + 2n)$, any convex combination of the two must also be larger than $\varepsilon / (1 + 2n)$.

also continuous. Thus, both the numerator and the denominator of f_k are continuous. Moreover, since the denominator is bounded away from zero (because it always takes on the value of at least one), we conclude that f_k is continuous.

Together, we showed that $f_k : S_\varepsilon \to S_\varepsilon$ is a self-mapping continuous function where S_ε is a nonempty compact subset of $\mathbb{R}^n$. Hence, by Brouwer's Fixed Point Theorem, we conclude that there exists $\mathbf{p}^\varepsilon \in S_\varepsilon$ such that

$$f(\mathbf{p}^\varepsilon) = \mathbf{p}^\varepsilon;$$

i.e. $f_k(\mathbf{p}^\varepsilon) = \mathbf{p}^\varepsilon$ for all k. Using the definition of $f_k(\cdot)$, for all k, we obtain

$$p_k^\varepsilon \left[n\varepsilon + \sum_{m=1}^{n} \max\left\{ \bar{z}_m(\mathbf{p}^\varepsilon), 0 \right\} \right] = \varepsilon + \max\left\{ \bar{z}_k(\mathbf{p}^\varepsilon), 0 \right\}. \tag{2.1}$$

We have so far showed that, for every $\varepsilon \in (0,1)$, there exists a price vector $\mathbf{p}^\varepsilon \in S_\varepsilon$ that satisfies (2.1).

Notice that it is possible that $\mathbf{p}^* \notin S_\varepsilon$. To check this, we take the limit as $\varepsilon \to 0$ and consider the associated sequence of price vectors $\{\mathbf{p}^\varepsilon\}$ satisfying (2.1). This price sequence is bounded since $\mathbf{p}^\varepsilon \in S_\varepsilon$ implies that the price in every market always lies between zero and one. Since every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence, it follows that some subsequence of $\{\mathbf{p}^\varepsilon\}$ must converge. Let $\{\mathbf{p}^\varepsilon\}$ denote such a subsequence so that it converges to $\mathbf{p}^*$. Then, it must be that $\mathbf{p}^* \in S_0 \subseteq \mathbb{R}^n_+$; i.e. $\mathbf{p}^* \geq \mathbf{0}$ and $\mathbf{p}^* \neq \mathbf{0}$ (since prices has to sum to one, they cannot all be zero). We would like to show that, in fact, $\mathbf{p}^* \gg 0$.

Suppose not, then, for some $\bar{k}$, $p_{\bar{k}}^* = 0$. Then, by assumption (iii), there exists k' with $p_{k'}^* = 0$ such that $\{z_k(\mathbf{p}^*)\}$ is unbounded above as $\varepsilon \to 0$. Now consider the left-hand side of (2.1) for good k'. Since $\mathbf{p}^\varepsilon \to \mathbf{p}^*$, $p_{k'}^* = 0$ implies that $p_{k'}^\varepsilon \to p_{k'}^* = 0$. Since the terms inside the square brackets is bounded (recall that $\bar{z}_k$ is bounded above by 1 by construction), the left-hand side must tend to zero as $\varepsilon \to 0$. In contrast, the right-hand does not, since the unboundedness above of $z_{k'}(\mathbf{p}^\varepsilon)$ implies that $\bar{z}_k(\mathbf{p}^\varepsilon)$ attains its maximum value (of 1) infinitely often, which is a contradiction since (2.1) must hold for all values of ε. We therefore conclude that $\mathbf{p}^* \gg \mathbf{0}$.

We now have that $\mathbf{p}^\varepsilon \to \mathbf{p}^* \gg \mathbf{0}$ as $\varepsilon \to 0$. Since $\bar{z}(\cdot)$ inherits continuity on $\mathbb{R}^n_{++}$ from $z(\cdot)$, we may take the limit as $\varepsilon \to 0$ in (2.1) to obtain that

$$p_k^* \sum_{m=1}^{n} \max\left\{ \bar{z}_m(\mathbf{p}^*), 0 \right\} = \max\left\{ \bar{z}_k(\mathbf{p}^*), 0 \right\}, \ \forall k.$$

We want to use Walras' law so we multiply both sides by $z_k(p^*)$ and sum across k, which yields

$$\begin{aligned} \sum_{k=1}^{n} z_k(p^*) \max\left\{ \bar{z}_k(\mathbf{p}^*), 0 \right\} &= \sum_{k=1}^{n} p_k^* z_k(\mathbf{p}^*) \sum_{m=1}^{n} \max\left\{ \bar{z}_m(\mathbf{p}^*), 0 \right\} \qquad (2.2) \\ &= \mathbf{p}^* \cdot z(\mathbf{p}^*) \left(\sum_{m=1}^{n} \max\left\{ \bar{z}_m(\mathbf{p}^*), 0 \right\} \right) \\ &= 0, \end{aligned}$$

where we used Walras' law which says that $\mathbf{p}^* \cdot z(\mathbf{p}^*) = 0$. We now claim that the expression above holds only if $z_k(\mathbf{p}^*) = 0$ for all k. To do so, we first show that $z_k(\mathbf{p}^*) \leq 0$ for all k and use the Walras' law and the fact that $\mathbf{p}^* \gg \mathbf{0}$.

Suppose $z_k(\mathbf{p}^*) > 0$, then $\bar{z}_k(\mathbf{p}^*) > 0$ so that $z_k(\mathbf{p}^*) \max\{\bar{z}_k(\mathbf{p}^*), 0\} > 0$. Suppose instead that $z_k(\mathbf{p}^*) < 0$, then $z_k(\mathbf{p}^*) \max\{\bar{z}_k(\mathbf{p}^*), 0\} = 0$. Thus, for the left-hand side of (2.2) to equal zero, it must be that $z_k(\mathbf{p}^*) \leq 0$ for all k. Recall Walras' law, which states that

$$\sum_{k=1}^{n} p_k^* z_k(\mathbf{p}^*) = 0.$$

Since $\mathbf{p}^* \gg \mathbf{0}$ (i.e. $p_k^* > 0$ for all k), for Walras' law to hold, $z_k(\mathbf{p}^*)$ cannot be negative; i.e. $z_k(\mathbf{p}^*) = 0$ for all k so that

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{0}.$$ ■

Using the theorem above, we can prove the (sufficient) conditions under which the existence of a Walrasian equilibrium is guaranteed.

Theorem 2.5. *(Existence of Walrasian equilibrium). Suppose each u^i satisfies Assumption 1 and $\sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$. Then, there exists at least one Walrasian equilibrium; i.e. at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.*

Proof. We will show that the excess demand function $\mathbf{z}(\cdot)$ satisfies conditions of the Theorem 2.3. By Theorem 2.2, we know that $\mathbf{z}$ is continuous at each $\mathbf{p} \gg \mathbf{0}$ and that it satisfies Walras' law; i.e. $\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$ for any $\mathbf{p} \gg \mathbf{0}$. Thus, it remains to show that condition (iii) of Theorem 2.3 is satisfied.

Let $\{\mathbf{p}^m\}$ be a sequence of prices in $\mathbb{R}^n_{++}$, $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k. We wish to show that this implies that there exists some good k' with $\bar{p}_{k'} = 0$ and the sequence $\{z_{k'}(\mathbf{p}^m)\}$ is unbounded above. We will proceed by way of contradiction.

Since $\sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ and $\bar{\mathbf{p}} \neq \mathbf{0}$,

$$\bar{\mathbf{p}} \cdot \sum_{i\in\mathcal{I}} \mathbf{e}^i = \sum_{i\in\mathcal{I}} \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0.$$

Thus, there exists $i \in \mathcal{I}$ such that

$$\bar{\mathbf{p}} \cdot \mathbf{e}^i > 0.$$

Consider such a consumer.

Let $\mathbf{x}^m := x^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i)$. We would like to show that i's demand for some good k' is unbounded above. Suppose, by way of contradiction, that $\mathbf{x}^m$ is bounded. This means that there exists a converging subsequence of $\mathbf{x}^m$. Let $\mathbf{x}^m$ denote such a subsequence; i.e. $\mathbf{x}^m \to \mathbf{x}^*$. Since u^i is strongly increasing, recall that the budget constraint must bind at the optimal $\mathbf{x}^m$; i.e.

$$\mathbf{p}^m \cdot \mathbf{x}^m = \mathbf{p}^m \cdot \mathbf{e}^i, \ \forall m.$$

Taking the limit as $m \to \infty$,

$$\bar{\mathbf{p}} \cdot \mathbf{x}^* = \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0,$$

where the inequality follows from our choice of consumer i.

Let $\hat{\mathbf{x}} = \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$ denote a bundle composed of adding 1 to the kth component of $\mathbf{x}^*$. Since $\bar{p}_k = 0$ by assumption,

$$\bar{\mathbf{p}} \cdot \hat{\mathbf{x}} = \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0. \tag{2.3}$$

Moreover, since u^i is strongly increasing, we also have that

$$u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x}^*). \tag{2.4}$$

By continuity of u^i, for sufficient large $t \in (0,1)$,

$$\begin{aligned} \bar{\mathbf{p}} \cdot t\hat{\mathbf{x}} < \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0 \\ u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^*). \end{aligned}$$

But because $\mathbf{p}^m \to \bar{\mathbf{p}}$, $\mathbf{x}^m \to \mathbf{x}^*$ and u^i is continuous, this means that for sufficiently large m,

$$\begin{aligned} \mathbf{p}^m \cdot t\hat{\mathbf{x}} < \mathbf{p}^m \cdot \mathbf{x}^m > 0 \\ u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m), \end{aligned}$$

which contradicts the fact that $\mathbf{x}^m$ is optimal demand at prices $\mathbf{p}^m$. Thus, $\mathbf{x}^m$ must be unbounded.

Since $\mathbf{x}^m$ is nonnegative, that $\{\mathbf{x}^m\}$ is unbounded means that there exists k' such that $\{\mathbf{x}^m_{k'}\}$ is unbounded above. However, since i's income converges to $\bar{\mathbf{p}} \cdot \mathbf{e}^i$, which is finite, the sequence of i's income $\{\mathbf{p}^m \cdot \mathbf{e}^i\}$ must be bounded. The only way to reconcile these two observation is that there exists k' whose demand, $\{\mathbf{x}^m_{k'}\}$ is unbounded above and $p^m_{k'} \to 0$.

Finally, since the aggregate endowment is finite and all consumers demand nonnegative amount of good k', that $\{\mathbf{x}^m_{k'}\}$ is unbounded above means that $\{\mathbf{z}_{k'}(\mathbf{p}^m)\}$ must be unbounded above. ■

Remark. Equations (2.3) and (2.4) do not lead to a contradiction that $\mathbf{x}^*$ is optimal since we only know from Theorem (2.1) that the demand function is continuous at $\mathbf{p} \gg \mathbf{0}$ but $\mathbf{x}^*$ is the demand for price $\mathbf{p} \neq \mathbf{0}$ with $p_k = 0$.

Remark. Cobb-Douglas utility functions is not strongly increasing on $\mathbb{R}^n_+$. However, a Walrasian equilibrium exists in this case since it can be shown that such utility functions satisfy the conditions of Theorem 2.3. In particular, we show that condition (iii) of the theorem is satisfied. Consider $u^i(\mathbf{x}) = x_1^{\alpha_1^i} x_2^{\alpha_2^i} \cdots x_n^{\alpha_n^i}$, where $\alpha_k^i > 0$ for all $k = 1, 2, \ldots, n$, and $\sum_{k=1}^n \alpha_k^i = 1$. Since the solution must be interior if income is strictly positive (if any $x_k^i = 0$ at the optimum, $u^i(\mathbf{x})$ is zero so that the agent can increase his utility by spending less on other goods and ensuring that $x_k^i > 0$). Thus, it satisfies the first-order condition of the consumer maximisation problem. This gives the demand function to be

$$x_k^i(\mathbf{p}) = \frac{\alpha_k}{p_k} \sum_{j=1}^n p_j e_j^i, \ \forall k.$$

Hence, it's clear that if $p_k \to 0$, then $x_k^i(\mathbf{p}) \to \infty$ so that condition (iii) is satisfied (with $k' = k$).

Remark 2.3. *(Exercise 5.22/Problem Set 2).* We can relax the assumptions on the utility functions if we were willing to strengthen the assumption regarding endowments. Specifically, if each consumer in an exchange economy is endowed with a positive amount of each good ($\mathbf{e}^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$) and has a continuous, quasiconcave and strictly increasing utility function, a Walrasian equilibrium exists.

2.4 Efficiency

We introduce a notation for the set of feasible allocation—set of allocations for which demand equals supply—for convenience.[6]

Definition 2.5. *(Feasible allocation set).* Let $\mathbf{e} = \left(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^I\right)$ be the vector of endowments for each agent $i \in \mathcal{I}$. The set of feasible allocation is given by

$$F(\mathbf{e}) := \left\{ \mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^i\right) \in \mathbb{R}_+^{nI} : \sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i \right\}.$$

We now define Pareto optimality, which is a requirement that it is not possible to make someone strictly better off without making someone else strictly worse off. Note that Pareto optimality places no restriction on the distributive properties of an allocation—for example, an allocation that gives all endowments in the economy to a single consumer can be Pareto optimal.

Definition 2.6. *(Pareto efficient allocation).* We say that an allocation $\mathbf{x} = \left\{\mathbf{x}^i\right\}_{i \in \mathcal{I}} \in \mathbb{R}_+^{nI}$ is *Pareto efficient* if it is feasible (i.e. $\left\{\mathbf{x}^i\right\}_{i \in \mathcal{I}} \in F(\mathbf{e})$) and there is no feasible allocation $\mathbf{y} \in F(\mathbf{e})$ such that $u^i\left(\mathbf{y}^i\right) \geq u^i\left(\mathbf{x}^i\right)$ for all $i \in \mathcal{I}$ with at least one strict inequality.

In an exchange economy, each agent can always consume his endowment. In other words, it defines the minimum level of utility that the consumer can attain. It would be reasonable to expect that, in equilibrium, no consumers can be made worse off than consuming their own endowment. When there are multiple consumers in the economy, the idea generalises—although each consumer alone may be unable to block trades, consumers may be able to block together with others.

Definition 2.7. *(Blocking coalition).* Let $S \subseteq \mathcal{I}$ denote a *coalition of consumers.* We say that S blocks allocation $\mathbf{x} \in F(\mathbf{e})$ if there is an allocation $\mathbf{y} \in \mathbb{R}_+^{nI}$ such that:

(i) $\mathbf{y}$ is feasible among those in the coalition—i.e. $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$;

(ii) at least one member of the coalition is strictly better off—i.e. $u^i\left(\mathbf{y}^i\right) \geq u^i\left(\mathbf{x}^i\right)$ for all $i \in S$ with at least one strict inequality.

We can now define a subset of feasible allocations that are robust to blocks by coalition(s) of consumers.

Definition 2.8. *(Core).* The core of an exchange economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i \in \mathcal{I}}$ is the set of all unblocked feasible allocations, and is denoted $C(\mathbf{e}) \subseteq F(\mathbf{e})$.

Notice that Pareto optimality is equivalent to the requirement that an allocation is not blocked by the *grand coalition*—i.e. the set of all consumers, $\mathcal{I}$. Thus, it follows that the core of an exchange economy is a subset of Pareto optimal set (denoted as $P(\mathbf{e})$), which is itself a subset of $F(\mathbf{e})$.

We will now show that the set of WEAs, $W(\mathbf{e})$, is in the core. Together, therefore, we would have

$$W(\mathbf{e}) \subseteq C(\mathbf{e}) \subseteq P(\mathbf{e}) \subseteq F(\mathbf{e}).$$

[6] We do not define feasibility with a weak inequality (i.e. $\sum_{i \in \mathcal{I}} \mathbf{x}^i \leq \sum_{i \in \mathcal{I}} \mathbf{e}^i$). There are at least two ways to motivate this. First, given the assumption that u^i is strongly increasing (i.e. all goods are, in fact, goods), in equilibrium, the inequality will, in any case, bind. A second interpretation is that using equality means that we avoid the difficult question of what consumers do with goods that are "left over". If we consider such left-overs to be, say, rotten food, we may define it to be another good in the economy (e.g. it could be used for compost) so that in equilibrium, feasibility holds with equality.

Thus, it is immediate that every WEA is a Pareto optimal allocation—this result is called the First Welfare Theorem.

Lemma 2.1. *Suppose that u^i is strictly increasing on $\mathbb{R}^n_+$, that consumer i's demand is well-defined at $\mathbf{p} \geq 0$ and equal to $\hat{\mathbf{x}}^i$, and that $\mathbf{x}^i \in \mathbb{R}^n_+$. Then,*

$$u^i\left(\mathbf{x}^i\right) > u^i\left(\hat{\mathbf{x}}^i\right) \Rightarrow \mathbf{p} \cdot \mathbf{x}^i > \mathbf{p} \cdot \hat{\mathbf{x}}^i,$$
$$u^i\left(\mathbf{x}^i\right) \geq u^i\left(\hat{\mathbf{x}}^i\right) \Rightarrow \mathbf{p} \cdot \mathbf{x}^i \geq \mathbf{p} \cdot \hat{\mathbf{x}}^i.$$

Proof. Consider the first implication. Suppose, by way of contradiction, that $u^i\left(\mathbf{x}^i\right) > u^i\left(\hat{\mathbf{x}}^i\right)$ implies that $\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \hat{\mathbf{x}}^i$. This means that $\mathbf{x}^i$ is affordable and gives strictly higher utility than $\hat{\mathbf{x}}^i$, which contradicts the fact that $\hat{\mathbf{x}}^i$ is the solution to the consumer's maximisation problem.

We now consider the second implication. Suppose again, by way of contradiction, that $u^i\left(\mathbf{x}^i\right) \geq u^i\left(\hat{\mathbf{x}}^i\right)$ implies $\mathbf{p} \cdot \mathbf{x}^i < \mathbf{p} \cdot \hat{\mathbf{x}}^i$. Thus, we can increase the amount of every good consumed by a small enough amount so that the resulting bundle, $\bar{\mathbf{x}}^i$, remains strictly less expensive that $\hat{\mathbf{x}}^i$; i.e. $\mathbf{p}\cdot\bar{\mathbf{x}}^i < \mathbf{p}\cdot\hat{\mathbf{x}}^i$. However, since u^i is strictly increasing,[7] $u^i\left(\bar{\mathbf{x}}^i\right) > u^i\left(\mathbf{x}^i\right) \geq u^i\left(\hat{\mathbf{x}}^i\right)$, which contradicts the first implication with $\mathbf{x}^i$ replaced by $\bar{\mathbf{x}}^i$. ∎

Theorem 2.6. *(Core and Walrasian equilibria). Consider an exchange economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$. If each u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is in the core; i.e.*

$$W(\mathbf{e}) \subseteq C(\mathbf{e}).$$

Corollary 2.1. *(First Welfare Theorem). If each u^i is strictly increasing on $\mathbb{R}^n_+$, every WEA of $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ is a Pareto efficient allocation.*

Proof. The theorem states that if $\mathbf{x}(\mathbf{p}^*) = \left(x^1(\mathbf{p}^*), \ldots, x^I(\mathbf{p}^*)\right)$ is a WEA for equilibrium prices $\mathbf{p}^*$, then $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. To prove this, suppose that $\mathbf{x}(\mathbf{p}^*)$ is WEA, and assume that $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.

By definition, WEA satisfies feasibility so that $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$. Since $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, there must exists a coalition $S \subseteq \mathcal{I}$ and another allocation $\mathbf{y}$ such that

$$\sum_{i\in S} \mathbf{y}^i = \sum_{i\in S} \mathbf{e}^i, \tag{2.5}$$
$$u^i\left(\mathbf{y}^i\right) \geq u^i\left(\mathbf{x}^i\right),\ \forall i \in S \tag{2.6}$$

with at least one strict inequality.

From (2.5) (since u^i is strictly increasing, the budget constraint must bind),

$$\mathbf{p}^* \cdot \sum_{i\in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i\in S} \mathbf{e}^i. \tag{2.7}$$

From (2.6), Lemma 2.1 implies that

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \cdot \mathbf{e}^i,\ \forall i \in S$$

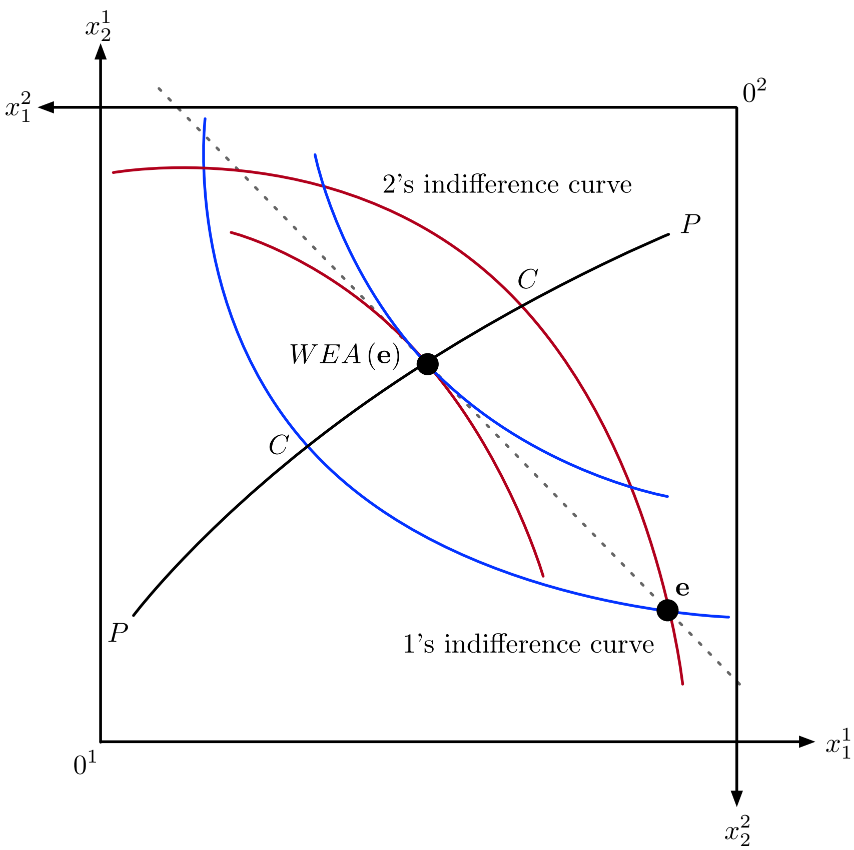
[7] Recall that u^i is strictly increasing on $\mathbb{R}^n_+$ if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ with $\mathbf{x} \geq \mathbf{y}$ implies $u^i(\mathbf{y}) \geq u^i(\mathbf{y})$ and $\mathbf{x} \gg \mathbf{y}$ implies $u^i(\mathbf{x}) > u^i(\mathbf{y})$.

with at least one inequality strict. But then summing across all $i \in S$,

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i,$$

which contradicts (2.7). Thus, $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. ■

Example 2.1. *(2 by 2 case).* Suppose there are two consumers and two goods so that we can represent the economy using an Edgeworth box. In this case, the only possible coalition is the grand coalition. Hence, given the contract curve (i.e. Pareto optima allocations) P-P, the core of the economy is given by the segment C-C, which falls inside the "lens" created by the two indifference curves that goes through the endowment point $\mathbf{e}$. Observe that the Walrasian equilibrium lies inside the core of the economy.



We now ask if the "reverse" holds; i.e. whether Pareto optimal allocations are WEA. Of course, not all Pareto optimal allocations are WEA, however, we can show that any Pareto optimal allocations can be a WEA for some initial endowments. This is called the Second Welfare Theorem. We can view this as an answer to the following question: Is a system that depends on decentralised, self-interested decision making by large number of consumers capable of sustaining the socially "best" allocation of resources (if we could agree on the definition of "best"). The Second Welfare Theorem says yes, as long as socially "best" requires, at least, Pareto efficiency.

Theorem 2.7. *(Second Welfare Theorem). Consider an exchange economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ with aggregate endowment $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, and with each utility function u^i satisfying Assumption 1 (continuity, strictly quasiconcave and strongly increasing). Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation for $\mathcal{E}$, and that endowments are redistributed so that the new endowment vector is $\bar{\mathbf{x}}$. Then, $\bar{\mathbf{x}}$ is a Walrasian equilibrium allocation of the resulting exchange economy $\bar{\mathcal{E}} = \left(u^i, \bar{\mathbf{x}}^i\right)_{i\in\mathcal{I}}$.*

Corollary 2.2. *Under the assumption of the previous theorem, if $\bar{\mathbf{x}}$ is Pareto efficient, then $\bar{\mathbf{x}}$ is a WEA for some Walrasian equilibrium $\bar{\mathbf{p}}$ after redistribution of initial endowments to any allocation $\mathbf{e}^* \in F(\mathbf{e})$ such that $\bar{\mathbf{p}} \cdot \mathbf{e}^{*i} = \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i$ for all $i \in \mathcal{I}$.*

Proof. (Theorem 2.7). Suppose that $\bar{\mathbf{x}}$ is a Pareto optimal allocation. By definition, $\bar{\mathbf{x}} = \left(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^I\right)$ must be feasible; i.e.

$$\sum_{i\in\mathcal{I}} \bar{\mathbf{x}}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg 0,$$

where strict inequality follows from the assumption that $\sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$. By Theorem 2.5, we know that $\bar{\mathcal{E}}$ has WEA, say, $\hat{\mathbf{x}}$. It remains to show that $\hat{\mathbf{x}} = \bar{\mathbf{x}}$. Since each consumer $i \in \mathcal{I}$ can consume his endowments in the economy $\bar{\mathcal{E}}$, it has to be that

$$u^i\left(\hat{\mathbf{x}}^i\right) \geq u^i\left(\bar{\mathbf{x}}^i\right), \forall i \in \mathcal{I}.$$

But, since $\hat{\mathbf{x}}$, by definition, must be feasible for the economy $\bar{\mathcal{E}}$,

$$\sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i = \sum_{i\in\mathcal{I}} \bar{\mathbf{x}}^i. \tag{2.8}$$

Thus, $\hat{\mathbf{x}}^i$ is also feasible in the economy $\mathcal{E}$. Since $\bar{\mathbf{x}}$ is Pareto optimal in the economy $\mathcal{E}$ and $\hat{\mathbf{x}}$ is feasible in the economy $\mathcal{E}$, it follows that

$$u^i\left(\hat{\mathbf{x}}^i\right) = u^i\left(\bar{\mathbf{x}}^i\right), \ \forall i \in \mathcal{I}.$$

That is, $\hat{\mathbf{x}}$ is also Pareto optimal in the economy $\mathcal{E}$. $\hat{\mathbf{x}}$ must also be affordable given (2.8). But since u^i's are strictly quasiconcave, we know that the solution to the consumers' maximisation problems must be unique (Theorem 2.1). Therefore, we conclude that

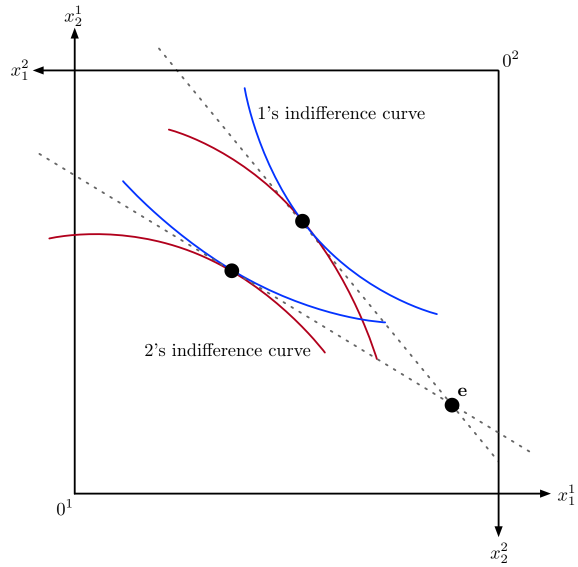
$$\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i, \ \forall i \in \mathcal{I}.$$

■

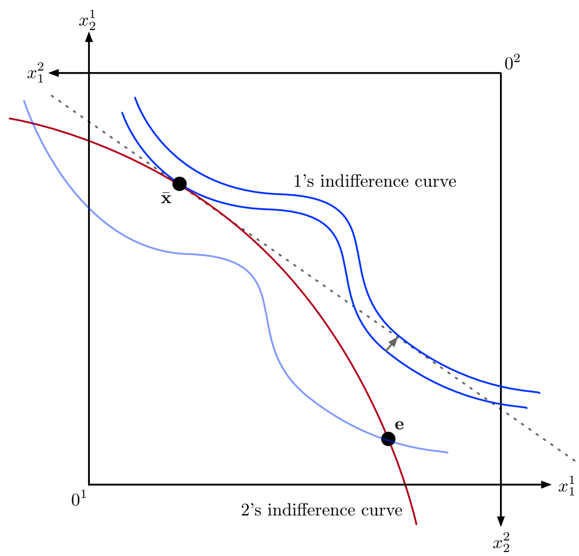
Proof. (Corollary 2.2). Let $\bar{\mathbf{p}}$ denote the Walrasian equilibrium price for the economy $\bar{\mathcal{E}}$. Suppose $\bar{\mathbf{p}} \cdot \mathbf{e}^{*i} = \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i = \bar{\mathbf{p}} \cdot \mathbf{e}^i$. Then, the solution to the consumer's maximisation problem given budget constraint $\bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \mathbf{e}^{*i}$ is exactly $\bar{\mathbf{x}}^i$. We are also given that $\mathbf{e}^*$ is feasible. Thus, the WAE for economies $\bar{\mathcal{E}}$ and $\mathcal{E}^* = \left(u^i, \mathbf{e}^{*i}\right)_{i\in\mathcal{I}}$ coincide. ■

2.5 Counterexamples

Exercise 2.1. Provide an Edgeworth box showing the existence of two Walrasian equilibria.

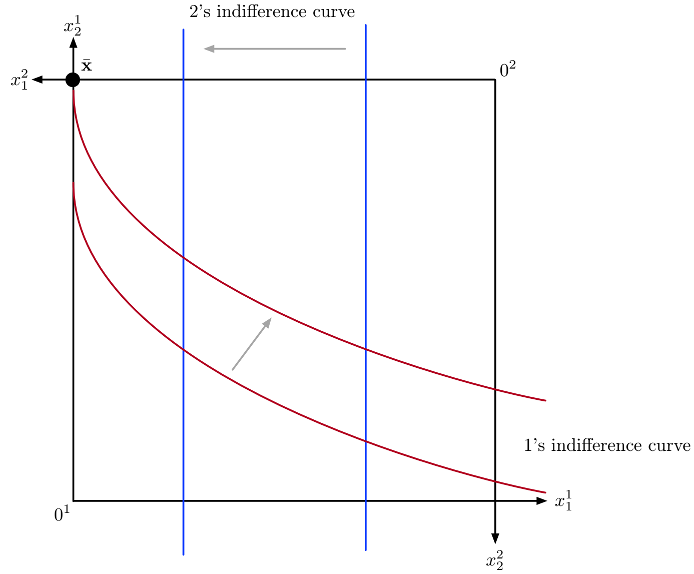


Exercise 2.2. Provide an Edgeworth box showing a Pareto efficient allocation that is in the interior of the box but that cannot be a Walrasian equilibrium allocation because one of the consumer's utility function is not quasiconcave.



Exercise 2.3. Provide an Edgeworth box showing an allocation that cannot be part of Walrasian equilibrium allocation because one of the consumer's utility function is not quasiconcave.

Solution. Consider the case in which the endowment, $\bar{\mathbf{x}}$, lies on the boundary of the Edgeworth box so that consumer 1 has all the endowment of good 2 while consumer 2 has all the endowment of good 1. Preferences are such that consumer 1 only values good 1 while consumer 2's indifferent curve has an infinite slope whenever he consumes zero amounts of good 2. A Walrasian equilibrium does not exist in this case because, although consumer 2 is willing to pay an "infinite" price for good 2 from consumer 1 by selling good 1, consumer 1 will not accept such a trade because he does not value good 1.



As a particular example, consider utility functions that are given by

$$u^1\left(x_1^1, x_2^1\right) = \sqrt{x_1^1} + x_2^1,$$
$$u^2\left(x_1^2, x_2^2\right) = x_1^2,$$
$$\bar{\mathbf{x}} = \left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right) = \left(\left(0, 1\right), \left(1, 0\right)\right).$$

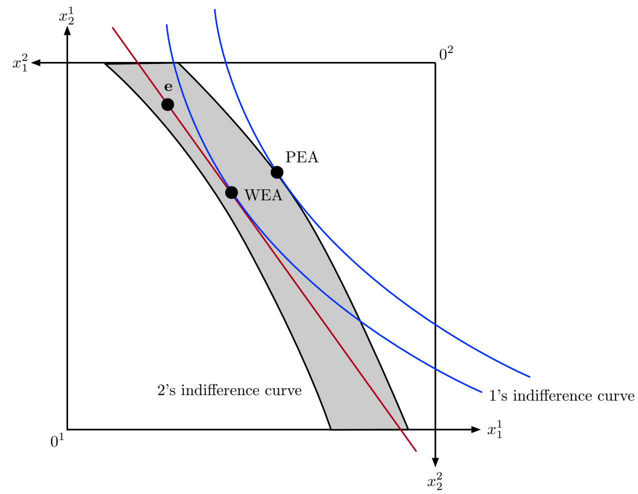
Observe that $\bar{\mathbf{x}}$ is Pareto efficient—to increase u^2, we must either increase x_1^2 or x_2^2 but this would either violate feasibility condition or lower u^2. Similarly, to increase u^1, we must increase x_2^2 but this would violate feasibility.

Observe that 1's marginal utility for good 1 is given by

$$\frac{\partial u^1\left(x_1^1, x_2^1\right)/\partial x_1}{\partial u^1\left(x_1^1, x_2^1\right)/\partial x_2} = -\frac{1}{2\sqrt{x_1^1}}.$$

Hence, the indifference curve for 1 has infinite slope when $x_2^1 = 0$. This is an example of a violation of the Second Welfare Theorem (Theorem 2.7). Notice that u^1 does not satisfy Assumption 1 (it is neither strongly increasing nor strictly quasiconcave).

Exercise 2.4. Provide an Edgeworth box example in which the First Welfare Theorem does not hold because, even though both consumers' utility functions are increasing, one consumer's utility function is not strictly increasing.



2.6 A systematic way of finding WEA*

Consider the Edgeworth box economy described by

$$\begin{aligned} &u^1(x_1, x_2), \quad \mathbf{e}^1 = \left(e_1^1, e_2^1\right), \\ &u^2(x_1, x_2), \quad \mathbf{e}^2 = \left(e_1^2, e_2^2\right). \end{aligned}$$

Proposition 2.1. *An allocation is Pareto efficient if and only if it solves the following problem for some $k \in \mathbb{R}$.*

$$\max_{\mathbf{x}^1, \mathbf{x}^2} \quad u^1\left(x_1^1, x_2^1\right) \tag{2.9}$$

$$s.t. \quad x_1^1 + x_1^2 = e_1^1 + e_1^2, \tag{2.10}$$

$$x_2^1 + x_2^2 = e_2^1 + e_2^2, \tag{2.11}$$

$$u^2\left(x_1^2, x_2^2\right) \geq k, \tag{2.12}$$

where $\mathbf{x}^1, \mathbf{x}^2 \geq 0$.

Proof. ($\Rightarrow$) Let $(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ be a Pareto efficient allocation. Then, by definition, the allocation is feasible so that (2.10) and (2.11) are satisfied. Let $\tilde{k} = u^2(\tilde{x}_1^2, \tilde{x}_2^2)$. Then, (2.12) is satisfied trivially. Finally, by the definition of Pareto efficiency, there does not exist another feasible allocation $(\mathbf{x}^1, \mathbf{x}^2)$ such that

$$u^1\left(x_1^1, x_2^1\right) > u^1\left(\tilde{x}_1^1, \tilde{x}_2^1\right), \quad u^2\left(x_1^2, x_2^2\right) \geq u^2\left(\tilde{x}_1^2, \tilde{x}_2^2\right) = \tilde{k}.$$

Thus, $(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ solves the problem.

($\Leftarrow$) Let $\hat{\mathbf{x}}$ be the solution to (2.9) with some $\hat{k} \in \mathbb{R}$. We first show that (2.12) is binding. Suppose not, so that $u^2(\hat{x}_1^2, \hat{x}_2^2) > \hat{k}$, then by continuity, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &u^2\left(\hat{x}_1^2 - \varepsilon, \hat{x}_2^2 - \varepsilon\right) > \hat{k}, \\ &u^1\left(\hat{x}_1^1 + \varepsilon, \hat{x}_2^1 + \varepsilon\right) > u^1\left(\hat{x}_1^1, \hat{x}_2^1\right), \end{aligned}$$

which contradicts the fact that $\hat{\mathbf{x}}$ solves (2.9). Hence, $u^2(\hat{x}_1^2, \hat{x}_2^2) = \hat{k}$. We now show that $\hat{\mathbf{x}}$ is Pareto efficient.

Since $\hat{\mathbf{x}}$ solves the problem

$$u^1\left(\hat{x}_1^1, \hat{x}_2^1\right) \geq u^1\left(x_1^1, x_2^1\right)$$

for all $\mathbf{x}^1 = \left(\mathbf{x}_1^1, x_2^1\right)$ and $\mathbf{x}^2 = \left(x_1^2, x_2^2\right)$ that are feasible (i.e. satisfies (2.10) and (2.11)).

Now suppose instead that, for some feasible $\left(\mathbf{x}^1, \mathbf{x}^2\right)$,

$$u\left(x_1^1, x_2^1\right) \geq u\left(\hat{x}_1^1, \hat{x}_2^1\right), \; u^2\left(x_1^2, x_2^2\right) > u^2\left(\hat{x}_1^2, \hat{x}_2^2\right) = \bar{k}.$$

But for sufficiently small $\varepsilon > 0$, we can have

$$u\left(x_1^1 + \varepsilon, x_2^1 + \varepsilon\right) > u\left(\hat{x}_1^1, \hat{x}_2^1\right), \; u^2\left(x_1^2 - \varepsilon, x_2^2 - \varepsilon\right) > u^2\left(\hat{x}_1^2, \hat{x}_2^2\right) = \bar{k},$$

where the first inequality follows from the fact that u is strongly increasing. But this contradicts the optimality of $\hat{\mathbf{x}}$. Thus, we conclude that $\hat{\mathbf{x}}$ is Pareto efficient. ■

3 Production economy

We consider a private ownership economy in which consumers own shares in firms, and firm profits are distributed to shareholders.

3.1 The set up

3.1.1 Firms

There are J firms. Let $\mathcal{J} = \{1, 2, \ldots, J\}$ denote the set of firms. Let $\mathbf{y}^j \in \mathbb{R}^n$ be a production plan for some firm, where $y_k^j < 0$ if commodity k is an input to production, and $y_k^j > 0$ if it is an output of production. We suppose that each firm possesses a production possibly set Y^j.

Assumption 2. *(The individual firm).*

(i) $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$.

(ii) Y^j *is closed and bounded.*

(iii) Y^j *is strongly convex; i.e. for any distinct* $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$*, for all* $t \in (0,1)$*, there exists* $\bar{\mathbf{y}} \in Y^j$ *such that* $\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\,\mathbf{y}^2$ *and the equality does not hold.*

Note:

- ▷ $\mathbf{0} \in Y^j$: means that firms profits are bounded from below by zero (since they can always produce nothing using no inputs).
- ▷ Allowing $Y^j \subseteq \mathbb{R}^n$ to contain negative elements captures the idea that production of output (positive elements) requires inputs.
- ▷ Y^j is closed: imposes continuity; i.e. limits of possible production plans are themselves production plans.
- ▷ Y^j is bounded: a simplifying assumption that is very restrictive (though dispensable)—it is not to say that resources are limited.
- ▷ Strong convexity (with $\mathbf{p} \gg \mathbf{0}$) ensures that the firm's profit maximising production plan is unique. However, it is restrictive as it rules out constant and increasing returns to scale in production.

Each firm faces fixed commodity prices $\mathbf{p} \geq \mathbf{0}$ and chooses a production plan to maximise profit.

Theorem 3.1. *Let*

$$\Pi^j(\mathbf{p}) \coloneqq \max_{\mathbf{y}^j \in Y^j} \quad \mathbf{p} \cdot \mathbf{y}^j.$$

If Y^j *satisfies Assumption 2, then,*

- ▷ *for all* $\mathbf{p} \geq \mathbf{0}$, $\Pi^j(\mathbf{p})$ *exists and is continuous on* $\mathbb{R}^n_+$*;*
- ▷ *the maximisation problem has a unique solution* $\mathbf{y}^j(\mathbf{p})$ *whenever* $\mathbf{p} \gg \mathbf{0}$*, which is continuous in* $\mathbf{p}$ *on* $\mathbb{R}^n_{++}$.

Proof. That the solution to the problem $\mathbf{y}^j(\mathbf{p})$ exists follows from the Weierstrass theorem since Y^j is compact (closed and bounded) and the objective function is continuous. This also implies that $\Pi^j(\mathbf{p})$ exists. That $\Pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$ and follows from the Theorem of the Maximum. To show that $\mathbf{y}^j(\mathbf{p})$ is unique, suppose there are two distinct solutions to the problem—$\tilde{\mathbf{y}}^j$ and $\hat{\mathbf{y}}^j$. Denote the maximising profit as $\Pi^j(\mathbf{p})$. Then by the fact that Y^j is strongly convex means that there exists $\bar{\mathbf{y}}^j \in Y^j$ such that $\bar{\mathbf{y}}^j \geq t\tilde{\mathbf{y}}^j + (1-t)\hat{\mathbf{y}}^j$ and the equality does not hold. If $\mathbf{p} \gg \mathbf{0}$, then this implies that

$$\begin{aligned}\mathbf{p}\cdot\bar{\mathbf{y}}^j &> \mathbf{p}\cdot\left(t\tilde{\mathbf{y}}^j + (1-t)\hat{\mathbf{y}}^j\right)\\ &= t\left(\mathbf{p}\cdot\tilde{\mathbf{y}}^j\right) + (1-t)\left(\mathbf{p}\cdot\hat{\mathbf{y}}^j\right) = \Pi^j(\mathbf{p})\end{aligned}$$

contradicting the assumption that $\tilde{\mathbf{y}}^j$ and $\hat{\mathbf{y}}^j$ maximises firms profits. Note that if $\mathbf{p} \geq \mathbf{0}$, then the strict inequality above may not hold (e.g. the price of the commodity for which $\bar{y}^j_k > t\tilde{y}^j_k + (1-t)\hat{y}^j_k$ may be zero). That $\mathbf{y}^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_{++}$ follows from the Theorem of the Maximum. Note that if some prices were zero, then there are infinitely many possible $y_k(\mathbf{p})$ and so continuity is not guaranteed. ■

Remark 3.1. Notice that $\Pi^j(\mathbf{p})$ is homogenous of degree one in prices, and $\mathbf{y}^j(\mathbf{p})$ is homogenous of degree zero in prices (since multiplying the objective function by a constant does not affect the optimal $\mathbf{y}^j$).

Remark 3.2. We can also show that $\Pi^j(\mathbf{p})$ is strictly convex in $\mathbf{p}$. To see this, consider two distinct prices $\mathbf{p}', \mathbf{p}'' \in \mathbb{R}^n_{++}$ and $\mathbf{p}^t = t\mathbf{p}' + (1-t)\mathbf{p}''$ for any $t \in (0,1)$. Then

$$\begin{aligned}\Pi^j\left(\mathbf{p}^t\right) &= \mathbf{p}^t\cdot\mathbf{y}^{jt} = \left(t\mathbf{p}' + (1-t)\mathbf{p}''\right)\cdot\mathbf{y}^{jt}\\ &= t\left(\mathbf{p}'\cdot\mathbf{y}^{jt}\right) + (1-t)\left(\mathbf{p}''\cdot\mathbf{y}^{jt}\right).\end{aligned}$$

Since profit maximising plan $\mathbf{y}^j$ is unique (under Assumption 2) for any given price:

$$\begin{aligned}\Pi^j\left(\mathbf{p}'\right) &= \mathbf{p}'\cdot\mathbf{y}^{j\prime} > \mathbf{p}'\cdot\mathbf{y}^j, \forall\mathbf{y}^j \in Y^j,\\ \Pi^j\left(\mathbf{p}''\right) &= \mathbf{p}''\cdot\mathbf{y}^{j\prime\prime} > \mathbf{p}''\cdot\mathbf{y}^j, \forall\mathbf{y}^j \in Y^j\end{aligned}$$

which imply that

$$\begin{aligned}\Pi^j\left(\mathbf{p}^t\right) &= t\left(\mathbf{p}'\cdot\mathbf{y}^{jt}\right) + (1-t)\left(\mathbf{p}''\cdot\mathbf{y}^{jt}\right)\\ &< t\left(\mathbf{p}'\cdot\mathbf{y}^{j\prime}\right) + (1-t)\left(\mathbf{p}''\cdot\mathbf{y}^{j\prime\prime}\right)\\ &= t\Pi^j\left(\mathbf{p}'\right) + (1-t)\Pi^j\left(\mathbf{p}''\right).\end{aligned}$$

Suppose there are no externalities in production between firms (Y^j does not depend on $\mathbf{y}^k$ $k \neq j$), and define the aggregate production possibilities set as

$$Y := \left\{\mathbf{y} : \mathbf{y} = \sum_{j\in\mathcal{J}}\mathbf{y}^j, \mathbf{y}^j \in Y^j,\ \forall j \in \mathcal{J}\right\}.$$

The set Y inherits all the properties of the individual production sets.

Theorem 3.2. *If each Y^j satisfies Assumption 2, then the aggregate production possibility set Y also satisfies Assumption 2.*

Proof. (i) Since $\mathbf{0} \in Y^j$ for all $j \in \mathcal{J}$, then $\mathbf{0} = \sum_{j\in\mathcal{J}} \mathbf{0} \in Y$. Take any element $\mathbf{y}^1 \in Y^1$ and $\mathbf{y}^2 \in Y^2$, then $\mathbf{y}_1 + \mathbf{y}_2 \in \mathbb{R}^n$. By induction, it follows that any element in Y is an element of $\mathbb{R}^n$; i.e. $Y \subseteq \mathbb{R}^n$.

(ii) Since Y^j is bounded, let $\underline{\mathbf{y}}^j$ and $\bar{\mathbf{y}}^j$ be the bounds. Then, since finite sum of finite numbers are finite,

$$\begin{aligned}\underline{\mathbf{y}} &\coloneqq \sum_{j\in\mathcal{J}} \underline{\mathbf{y}}^j < \infty,\\ \bar{\mathbf{y}} &\coloneqq \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j < \infty.\end{aligned}$$

So that Y is bounded by $\underline{\mathbf{y}}$ and $\bar{\mathbf{y}}$. Note that sum of closed sets need not be closed.[8] However, here, we are considering sum of compact sets. Since summation is a continuous operation and compactness is preserved under continuous operations, it follows that $\mathbf{y}$ is compact—in particular, $\mathbf{y}$ is closed.

To show that Y^j is strongly convex, for each j, take any distinct $\mathbf{y}^{j1}, \mathbf{y}^{j2} \in Y^j$, then since Y^j is strictly convex, for any $t \in (0,1)$, there exists $\bar{\mathbf{y}}^j \in Y^j$ such that $\bar{\mathbf{y}}^j \geq t\mathbf{y}^{1j} + (1-t)\,\mathbf{y}^{2j}$ and the equality does not hold. Define

$$\bar{\mathbf{y}} \coloneqq \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j \in Y^j,$$

then

$$\begin{aligned}\bar{\mathbf{y}} &\geq t\sum_{j\in\mathcal{J}} \mathbf{y}^{1j} + (1-t)\sum_{j\in\mathcal{J}} \mathbf{y}^{2j}\\ &= t\mathbf{y}^1 + (1-t)\,\mathbf{y}^2, \forall t \in (0,1)\end{aligned}$$

and the equality does not hold, and $\mathbf{y}^1$ and $\mathbf{y}^2$ can be any distinct elements in Y. ■

We now show that aggregate profit is maximised if and only if individual firms profit maximises.

Theorem 3.3. *(Aggregate profit maximisation). For any prices* $\mathbf{p} \geq \mathbf{0}$*,*

$$\mathbf{p}\cdot\bar{\mathbf{y}} \geq \mathbf{p}\cdot\mathbf{y},\ \forall \mathbf{y} \in Y$$

if and only if for some $\bar{\mathbf{y}}^j \in Y^j$*,* $j \in \mathcal{J}$*, we may write* $\bar{\mathbf{y}} = \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j$ *and*

$$\mathbf{p}\cdot\bar{\mathbf{y}}^j \geq \mathbf{p}\cdot\mathbf{y}^j,\ \forall \mathbf{y}^j \in Y^j,\ \forall j \in \mathcal{J}.$$

Proof. Suppose $\bar{\mathbf{y}} \in Y$ maximises aggregate profits at price $\mathbf{p}$, where $\bar{\mathbf{y}} = \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j$. If $\bar{\mathbf{y}}^k$ does not maximise profit, then there exists $\tilde{\mathbf{y}}^k \in Y^k$ that gives strictly higher profits. But then aggregate production vector composed of $\tilde{\mathbf{y}}^k$ and $\bar{\mathbf{y}}^j$ for all $j \in \mathcal{J}\backslash k$ would imply higher profits, which contradicts the assumption that $\bar{\mathbf{y}}$ maximised profits.

[8] For example, consider the closed sets $A = \{n|n = 1,2,\ldots\}$ and $B = \left\{-n + \frac{1}{n}|n = 1,2,\ldots\right\}$. Then

$$A + B = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = \left\{\frac{1}{n}|n = 1,2,\ldots\right\}.$$

Then a limit point of $A+B$ is 0 but this is not contained in the sumset $A+B$.

Now suppose that $\bar{\mathbf{y}}^j$ maximise profits at prices $\mathbf{p}$ for each firm $j \in \mathcal{J}$. Then,

$$\begin{gathered}\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}^j,\ \forall \mathbf{y}^j \in Y^j,\ \forall j \in \mathcal{J} \\ \Rightarrow \sum_{j \in \mathcal{J}} \mathbf{p} \cdot \bar{\mathbf{y}}^j = \mathbf{p} \cdot \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j \geq \sum_{j \in \mathcal{J}} \mathbf{p} \cdot \mathbf{y}^j = \mathbf{p} \cdot \sum_{j \in \mathcal{J}} \mathbf{y}^j,\ \forall \mathbf{y}^j \in Y^j,\ \forall j \in \mathcal{J} \\ \Rightarrow \mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y},\ \forall \mathbf{y} \in Y.\end{gathered}$$

■

3.1.2 Consumers

Consumers are as before, except that they now receive profits from firms that they own shares in. That we let $\mathbf{X} = \mathbb{R}^n_+$ does not preclude the possibility that consumers supply goods and services to the market—we simply need to give endowments of goods and services that consumers provide to the market.

Let consumer i's share in firm j be $\theta^{ij} \in [0,1]$ so that he is entitled to a share θ^{ij} of the profits of firm j. The shares, of course, has to sum to one:

$$\sum_{i \in \mathcal{I}} \theta^{ij} = 1,\ \forall j \in \mathcal{J}.$$

The consumer's budget constraint is now

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \Pi^j(\mathbf{p}) \coloneqq m^i(\mathbf{p}),$$

where $m^i(\mathbf{p})$ denotes the consumer's total income.

Theorem 3.4. *Suppose each Y^j satisfies Assumption 2 and each u^i satisfies Assumption 1. Then, for all $\mathbf{p} \gg \mathbf{0}$, the problem*

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} \quad u^i\left(\mathbf{x}^i\right) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$$

has a unique solution, $\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right)$, which is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$. Moreover, $m_i(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$.

Proof. Under Assumption 2, each firm earns non-negative profits. Since $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{e} \geq \mathbf{0}$, it follows that $m^i(\mathbf{p}) \geq 0$. By Weierstrass theorem, the solution to the maximisation problem exists whenever $\mathbf{p} \gg \mathbf{0}$. That $m_i(\mathbf{p})$ is continuous follows from Theorem 3.1, we know that $\Pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$. Then, by the Theorem of the Maximum, $\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right)$ is continuous on $\mathbb{R}^n_{++}$. ■

3.1.3 Production economy

The production economy is summarised by the collection

$$\mathcal{E} = \left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i \in \mathcal{I}, j \in \mathcal{J}}.$$

3.2 Walrasian equilibrium

In a production economy, the aggregate excess demand for commodity k is given by

$$z_k(\mathbf{p}) \coloneqq \sum_{i\in\mathcal{I}} x_k^i\left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{j\in\mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i\in\mathcal{I}} e_k^i.$$

The aggregate excess demand vector is then defined as

$$\mathbf{z}(\mathbf{p}) \coloneqq \left(z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p})\right).$$

Definition 3.1. *(Walrasian equilibrium).* $\mathbf{p}^* \in \mathbb{R}_+^n$ is a *Walrasian equilibrium* of $\mathcal{E} = \left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i\in\mathcal{I}, j\in\mathcal{J}}$ if there exists $\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^I \in \mathbb{R}_+^n$ and $\hat{\mathbf{y}}^1 \in Y^1, \hat{\mathbf{y}}^2 \in Y^2, \ldots, \hat{\mathbf{y}}^J \in Y^J$ such that:

(i) for all $i \in \mathcal{I}$, $\hat{\mathbf{x}}^i$ maximises $u^i\left(\mathbf{x}^i\right)$ subject to $\mathbf{p}^* \cdot \mathbf{x}^i \leq m^i(\mathbf{p}^*) = \mathbf{p}^* \cdot \mathbf{e}^i + \sum_{j\in\mathcal{J}} \theta^{ij}\Pi^j(\mathbf{p}^*)$;

(ii) for all $j \in \mathcal{J}$, $\hat{\mathbf{y}}^j$ maximises $\mathbf{p}^* \cdot \mathbf{y}^j$ subject to $\mathbf{y}^j \in Y^j$;

(iii) markets clear; i.e. $\sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j$.

We will say that $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \left(\left\{\hat{\mathbf{x}}^i\right\}_{i\in\mathcal{I}}, \left\{\hat{\mathbf{y}}^i\right\}_{j\in\mathcal{J}}\right)$ is a *Walrasian equilibrium allocation* (WEA).

Remark 3.3. As in the case for exchange economies, we can define the vector $\mathbf{p}^* \in \mathbb{R}_+^n$ as a Walrasian equilibrium if $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

The definition of Walrasian equilibrium remains unchanged: it is a price vector $\mathbf{p}^* \geq \mathbf{0}$ that clears all markets, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$. Since, as before, the excess demand vector is homogenous of degree zero in prices, it follows that Walrasian equilibrium prices are not unique.

Theorem 3.5. *(Existence of Walrasian equilibrium with Production). Consider the economy $\mathcal{E} = \left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i\in\mathcal{I}, j\in\mathcal{J}}$. If each u^i satisfies Assumption 1 and each Y^j satisfies Assumption 2, and $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j\in\mathcal{J}} Y^j$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.*

Proof. We show that $\mathbf{z}(\mathbf{p})$ satisfies the conditions of Theorem 2.3.

(i) Continuity. By Theorems 3.1 and 3.4, $\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right)$ and $\mathbf{y}^j(\mathbf{p})$ are continuous on $\mathbb{R}_{++}^n$. Since adding and subtracting are continuous operations, it follows that $\mathbf{z}(\mathbf{p})$ is continuous on $\mathbb{R}_{++}^n$.

(ii) Since u^i is strongly increasing, at the optimum, budget constraint for each $i \in \mathcal{I}$ binds. That is,

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{x}^i &= \mathbf{p}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\mathbf{p}),\ \forall i\in\mathcal{I}\\
\Rightarrow \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{x}^i &= \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{e}^i + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\mathbf{p})\\
&= \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\underbrace{\Pi^j(\mathbf{p})}_{=\mathbf{p}\cdot\mathbf{y}^j(\mathbf{p})}\underbrace{\sum_{i\in\mathcal{I}}\theta^{ij}}_{=1}\\
\Rightarrow \mathbf{p}\cdot\sum_{i\in\mathcal{I}}\mathbf{x}^i &= \mathbf{p}\cdot\sum_{i\in\mathcal{I}}\mathbf{e}^i + \mathbf{p}\cdot\sum_{j\in\mathcal{J}}\mathbf{y}^j(\mathbf{p})\\
\mathbf{0} &= \mathbf{p}\cdot\left(\sum_{i\in\mathcal{I}}\mathbf{x}^i - \sum_{j\in\mathcal{J}}\mathbf{y}^j(\mathbf{p}) - \sum_{i\in\mathcal{I}}\mathbf{e}^i\right)\\
&= \mathbf{p}\cdot\mathbf{z}(\mathbf{p}).
\end{aligned}$$

(iii) It remains to show that if $\{\mathbf{p}^r\}$ is a sequence of prices in $\mathbb{R}^n_{++}$ and $\mathbf{p}^r\to\bar{\mathbf{p}}\neq\mathbf{0}$ and $\bar{p}_k=0$ for some good k, then for some good k' with $\bar{p}_{k'}=0$, the sequence $\{z_{k'}(\mathbf{p}^r)\}$is unbounded above. By Assumption 2, Y^j is bounded and the endowments are bounded so that we only need to show that some consumer's demand for some good is unbounded as $\mathbf{p}^r\to\bar{\mathbf{p}}$.

Since $\mathbf{y}+\sum_{i\in\mathcal{I}}\mathbf{e}^i\gg\mathbf{0}$ for some aggregate production vector $\mathbf{y}$ and since $\bar{\mathbf{p}}$ has no negative components and $\mathbf{p}\neq\mathbf{0}$, we must have

$$\bar{\mathbf{p}}\cdot\left(\mathbf{y}+\sum_{i\in\mathcal{I}}\mathbf{e}^i\right)>0.$$

Hence,

$$\begin{aligned}
\sum_{i\in\mathcal{I}}m^i(\bar{\mathbf{p}}) &= \sum_{i\in\mathcal{I}}\left(\bar{\mathbf{p}}\cdot\mathbf{e}^i+\sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\bar{\mathbf{p}})\right)\\
&= \sum_{i\in\mathcal{I}}\bar{\mathbf{p}}\cdot\mathbf{e}^i+\sum_{i\in\mathcal{I}}\Pi^j(\bar{\mathbf{p}})\\
&\geq \bar{\mathbf{p}}\cdot\left(\mathbf{y}+\sum_{i\in\mathcal{I}}\mathbf{e}^i\right)>0, \qquad (3.1)
\end{aligned}$$

where the weak inequality follows from the fact that sum of individual firm maximised profits must be at least as large as the maximised aggregate profits (Theorem 3.3), and, hence, at least as large as the aggregate profits from $\mathbf{y}$. Thus, there must exist at least one consumer whose income at prices $\bar{\mathbf{p}}$, $m^i(\bar{\mathbf{p}})$, is strictly positive. We consider such a consumer i. The rest of the proof mimics that of Theorem 2.5.

Let $\mathbf{x}^r \coloneqq x^i\big(\mathbf{p}^r, m^i(\mathbf{p}^r)\big)$. We want to show that i's demand for some good k' is unbounded above. Suppose, by way of contradiction, that $\mathbf{x}^r$ is bounded. Then, there exists a converging subsequence of $\mathbf{x}^r$, which we denote as $\mathbf{x}^r$ so that $\mathbf{x}^r\to\mathbf{x}^*$. Since the budget constraint must bind at the optimal $\mathbf{x}^r$;

$$\mathbf{p}^r\cdot\mathbf{x}^r = m^i(\mathbf{p}^r),\ \forall r.$$

Since $m^i(\cdot)$ is continuous on $\mathbb{R}^n_+$ by Theorem 3.4, as $r \to \infty$,

$$\bar{\mathbf{p}} \cdot \mathbf{x}^* = m^i(\bar{\mathbf{p}}) > 0,$$

where the strict inequality follows from (3.1).

Let $\hat{\mathbf{x}} = \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$ denote a bundle composed of adding 1 to the kth component of $\mathbf{x}^*$. Since $\bar{p}_k$ by assumption, and u^i is strongly increasing,

$$\begin{aligned}\bar{\mathbf{p}} \cdot \hat{\mathbf{x}} &= \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0,\\ u^i(\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*).\end{aligned}$$

By continuity of u^i, for sufficiently large $t \in (0, 1)$,

$$\begin{aligned}\bar{\mathbf{p}} \cdot t\hat{\mathbf{x}} &< \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0,\\ u^i(t\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*).\end{aligned}$$

Since $\mathbf{p}^r \to \bar{\mathbf{p}}$, $\mathbf{x}^r \to \mathbf{x}^*$ and u^i is continuous, for sufficiently large r,

$$\begin{aligned}\mathbf{p}^r \cdot t\hat{\mathbf{x}} &< \mathbf{p}^r \cdot \mathbf{x}^r > 0\\ u^i(t\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*),\end{aligned}$$

which contradicts that $\mathbf{x}^r$ is optimal demand at prices $\mathbf{p}^r$. Thus, $\mathbf{x}r$ must be bounded.

Finally, since $\mathbf{x}^r$ is nonnegative, that $\{\mathbf{x}^r\}$ is unbounded means that there exists k' such that $\{\mathbf{x}^r_{k'}\}$ is unbounded above. To reconcile this with the fact that $\left\{\mathbf{p}^r \cdot m^i(\mathbf{p}^r)\right\}$ is bounded, it must be that $p^r_{k'} \to 0$. ■

3.3 Efficiency

In the production economy, a Walrasian equilibrium allocation given equilibrium price vector $\mathbf{p}^*$ is a pair $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ such that: (i) each consumer's commodity bundle is the most preferred in his budget set at prices $\mathbf{p}^*$; (ii) each firm's production plan is profit maximising in its production possibility set at prices $\mathbf{p}^*$; and (iii) demand equals supply in every market.

An allocation $(\mathbf{x}, \mathbf{y})$ is feasible if

$$\sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i + \sum_{j \in \mathcal{J}} \mathbf{y}^j.$$

where $\mathbf{x}^i \in \mathbb{R}^n_+$ for all $i \in \mathcal{I}$ and $\mathbf{y}^j \in Y^j$ for all $j \in \mathcal{J}$.

Definition 3.2. *(Pareto efficiency).* A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that $u^i\left(\bar{\mathbf{x}}^i\right) \geq u^i\left(\mathbf{x}^i\right)$ for all $i \in \mathcal{I}$ with at least one strict inequality.

We are now ready to state the welfare theorems in the context of a production economy.

Theorem 3.6. *(First Welfare Theorem with production). Consider a production economy* $\mathcal{E} = \left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i\in\mathcal{I}, j\in\mathcal{J}}$. *If each* u^i *is strictly increasing on* $\mathbb{R}^n_+$*, then every Walrasian equilibrium allocation is Pareto efficient.*

Theorem 3.7. *(Second Welfare Theorem with production). Suppose that (i) each* u^i *satisfies Assumption 1; (ii) each* Y^j *satisfies Assumption 2; (iii)* $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ *for some aggregate production vector* $\mathbf{y}$*; and (iv) the allocation* $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is Pareto efficient. Then, there are income transfers,* $T_1, \ldots, T_I$ *satisfying* $\sum_{i\in\mathcal{I}} T_i = 0$ *and a price vector* $\bar{\mathbf{p}}$ *such that*

(i) $\hat{\mathbf{x}}^i$ *maximises* $u^i\left(\mathbf{x}^i\right)$ *subject to the budget constraint* $\bar{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\bar{\mathbf{p}}) + T_i$ *for all* $i \in \mathcal{I}$*;*

(ii) $\hat{\mathbf{y}}^j$ *maximises* $\bar{\mathbf{p}} \cdot \mathbf{y}^j$ *subject to* $\mathbf{y}^j \in Y^j$ *for all* $j \in \mathcal{J}$*.*

Proof. (Proof of Theorem 3.6). The idea is to use Lemma 2.1, which says that if something is more preferred, then it must cost more. But with production, we need to do a little more work than before become consumer's income includes profits from the firm—so when we pick a different feasible allocation, consumer's income can change (whereas in the economy, income was always given by the value of the endowments).

Let $(\mathbf{x}, \mathbf{y})$ be a WEA at prices $\mathbf{p}^*$. Suppose, by way of contradiction, that the allocation is not Pareto efficient. That is, there must exist feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i\left(\hat{\mathbf{x}}^i\right) \geq u^i\left(\mathbf{x}^i\right), \ \forall i \in \mathcal{I}$$

with at least one strict inequality. By Lemma 2.1, this implies that

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i, \ \forall i \in \mathcal{I}$$

with at least one strict inequality. Summing across all agents,

$$\begin{aligned} \sum_{i\in\mathcal{I}} \mathbf{p}^* \cdot \hat{\mathbf{x}}^i &> \sum_{i\in\mathcal{I}} \mathbf{p}^* \cdot \mathbf{x}^i \\ \Leftrightarrow \mathbf{p}^* \cdot \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i &> \mathbf{p}^* \cdot \sum_{i\in\mathcal{I}} \mathbf{x}^i. \end{aligned}$$

Since $(\mathbf{x}, \mathbf{y})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ are both feasible, above implies that

$$\begin{aligned} \mathbf{p}^* \cdot \left(\sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j + \sum_{j\in\mathcal{J}} \mathbf{e}^i\right) &> \mathbf{p}^* \cdot \left(\sum_{j\in\mathcal{J}} \mathbf{y}^j + \sum_{j\in\mathcal{J}} \mathbf{e}^i\right) \\ \Rightarrow \mathbf{p}^* \cdot \sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j &> \mathbf{p}^* \cdot \sum_{j\in\mathcal{J}} \mathbf{y}^j. \end{aligned}$$

Above means that there exists some firm j such that $\mathbf{y}^j$ was not profit maximising, which contradicts the assumption that $(\mathbf{x}, \mathbf{y})$ was a Walrasian equilibrium. ■

Proof. (Proof of Theorem 3.7). For each $j \in \mathcal{J}$, let $\bar{Y}^j := Y^j - \left\{\hat{\mathbf{y}}^j\right\}$ (i.e. subtract $\hat{\mathbf{y}}^j$ from each element of Y^j). Note that each $\bar{Y}^j$ satisfies Assumption 2 (Theorem 3.2). Consider now the economy $\bar{\mathcal{E}} = (u^i, \hat{\mathbf{x}}^i, \theta^{ij}, \bar{Y}^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$ obtained from the original economy by: (i) replacing consumer i's endowment, $\mathbf{e}^i$, with the endowment $\hat{\mathbf{x}}^i$; (ii) replacing each production set, Y^j, with the production set $\bar{Y}^j$. Then, by Theorem 3.5, $\bar{\mathcal{E}}$ possesses a Walrasian equilibrium, $\bar{\mathbf{p}} \gg \mathbf{0}$, and an associated WEA, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Since $\mathbf{0} \in Y^j$ for each firm, profits are nonnegative in equilibrium. Thus, each consumer can afford his endowment vector so that

$$u^i\left(\bar{\mathbf{x}}^i\right) \geq u^i\left(\hat{\mathbf{x}}^i\right), \ \forall i \in \mathcal{I}. \tag{3.2}$$

Next, we argue that, for some aggregate production vector $\tilde{\mathbf{y}}$, $(\bar{\mathbf{x}}, \tilde{\mathbf{y}})$ is feasible for the original economy. To see this, note that each $\bar{\mathbf{y}}^j \in \bar{Y}^j$ is of the form $\bar{\mathbf{y}}^j = \tilde{\mathbf{y}}^j - \hat{\mathbf{y}}^j$ for some $\tilde{\mathbf{y}}^j \in Y^j$. Since $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a WEA for $\bar{\mathcal{E}}$, it must be feasible in the economy. Therefore,

$$\begin{aligned}
\sum_{i\in\mathcal{I}} \bar{\mathbf{x}}^i &= \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j \\
&= \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} \left(\tilde{\mathbf{y}}^j - \hat{\mathbf{y}}^j\right) \\
&= \left(\sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i - \sum_{j\in J} \hat{\mathbf{y}}^j\right) + \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}} \\
&= \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}},
\end{aligned}$$

where the last inequality follows from the fact that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible in the original economy. Above shows that $(\bar{\mathbf{x}}, \tilde{\mathbf{y}})$ is feasible in the original economy, where $\tilde{\mathbf{y}} = \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}}^j$.

It follows that every inequality in (3.2) must be an equality as otherwise $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ would not have been Pareto efficient. Moreover, since (strict) quasiconcavity of u^i imply unique demand for each consumer, we therefore obtain that

$$\bar{\mathbf{x}}^i = \hat{\mathbf{x}}^i, \ \forall i \in \mathcal{I}.$$

Since $\bar{\mathbf{x}}^i$ maximises utility given prices $\bar{\mathbf{p}}$, it follows that

$$\hat{\mathbf{x}}^i = \arg\max_{\mathbf{x}^i} \ u^i\left(\mathbf{x}^i\right) \quad \text{s.t.} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} \theta^{ij}\left(\bar{\mathbf{p}} \cdot \bar{\mathbf{y}}^j\right), \ \forall i \in \mathcal{I}.$$

But because utility is strongly increasing, the budget constraint binds at $\mathbf{x}^i = \hat{\mathbf{x}}^i$, which implies that

$$\begin{aligned}
\sum_{j\in\mathcal{J}} \theta^{ij}\left(\bar{\mathbf{p}} \cdot \bar{\mathbf{y}}^j\right) = 0 &\Rightarrow \bar{\mathbf{p}} \cdot \bar{\mathbf{y}}^j = 0, \ \forall j \in \mathcal{J} \\
&\Rightarrow \bar{\mathbf{y}}^j = \mathbf{0}, \ \forall j \in \mathcal{J}.
\end{aligned}$$

where the first implication follows from the fact that profits are nonnegative in equilibrium, and the second follows from $\bar{\mathbf{p}} \gg \mathbf{0}$. Then,

$$\begin{aligned} \bar{\mathbf{p}} \cdot \bar{\mathbf{y}}^j &\geq \bar{\mathbf{p}} \cdot \mathbf{y}^j, \ \forall \mathbf{y}^j \in \bar{Y}^j \\ \Rightarrow \bar{\mathbf{p}} \cdot \left(\bar{\mathbf{y}}^j + \hat{\mathbf{y}}^j\right) &\geq \bar{\mathbf{p}} \cdot \left(\mathbf{y}^j + \hat{\mathbf{y}}^j\right) \\ \Rightarrow \bar{\mathbf{p}} \cdot \hat{\mathbf{y}}^j &= \bar{\mathbf{p}} \cdot \tilde{\mathbf{y}}^j, \ \forall \tilde{\mathbf{y}}^j \in Y^j, \end{aligned}$$

where the last implication follows from the fact that $\bar{\mathbf{y}}^j = \mathbf{0}$ and the definition of $\bar{Y}^j$. In other words, we showed that, since $\bar{\mathbf{y}}^j = \mathbf{0}$ maximises firm j's profits at prices $\bar{\mathbf{p}}$ when its production set is $\bar{Y}^j$, then (by the definition of $\bar{Y}^j$), $\hat{\mathbf{y}}^j$ maximises firm j's profits at prices $\bar{\mathbf{p}}$ when its production set is Y^j (i.e. in the original economy).

So altogether, we have shown the following

$$\begin{aligned} \hat{\mathbf{x}}^i &= \arg\max_{\mathbf{x}^i} \ u^i\left(\mathbf{x}^i\right) \quad \text{s.t.} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \ \forall i \in \mathcal{I}, \\ \hat{\mathbf{y}}^j &= \arg\max_{\mathbf{y}^j \in Y^j} \ \bar{\mathbf{p}} \cdot \mathbf{y}^j, \ \forall j \in \mathcal{J}. \end{aligned} \tag{3.3}$$

We can set

$$T_i := \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i - m^i\left(\bar{\mathbf{p}}\right) = \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i - \bar{\mathbf{p}} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij}\left(\bar{\mathbf{p}} \cdot \hat{\mathbf{y}}^j\right)$$

and notice that

$$\sum_{i \in \mathcal{I}} T_i = 0$$

since $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible. Finally, notice that

$$\begin{aligned} \bar{\mathbf{p}} \cdot \mathbf{x}^i &\leq m^i\left(\bar{\mathbf{p}}\right) + T_i \\ &= \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \end{aligned}$$

so that the budget constraint coincides with that in (3.3). ■

4 Replica economies*

Consider an economy with I agents, and think of each agent $i \in \mathcal{I}$ as a type, distinguished by the preferences and endowments. A twofold replica of this economy is one in which there are two consumers of each type.

Definition 4.1. *(r-fold replica economy).* Let there be I types of consumers in the basic exchange economy $\mathcal{E}$ indexed by the set $\mathcal{I} = \{1, , \dots, I\}$. An r-fold replica economy, denoted $\mathcal{E}_r$, is an economy with r consumers of each type, consisting of rI consumers. For any type $i \in \mathcal{I}$, all r consumers of that type share the common preferences $\succsim^i$ on $\mathbb{R}^n_+$ and have identical endowments $\mathbf{e}^i \gg \mathbf{0}$. We further assume that for $i \in \mathcal{I}$ that preferences $\succsim^i$ can be represented by a utility function u^i satisfying Assumption 1.

Proposition 4.1. *The core of $\mathcal{E}_r$ is nonempty for all r.*

Proof. Given the definition of an r-fold economy, all of the hypotheses of Theorem 2.5 are satisfied so that a WEA exists and it will be in the core. ■

Let $\mathbf{x}^{iq}$ denote the bundle of the qth ($q = 1, 2, \dots, r$) consumer of type $i \in \mathcal{I}$. An allocation in a replica economy is denoted

$$\mathbf{x} = \left(\mathbf{x}^{11}, \mathbf{x}^{12}, \dots, \mathbf{x}^{1r}, \dots\dots, \mathbf{x}^{I1}, \mathbf{x}^{I2}, \dots, \mathbf{x}^{Ir}\right).$$

Since each r consumers of type i has the same endowment vector $\mathbf{e}^i$, the allocation is feasible if

$$\sum_{i \in \mathcal{I}} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i \in \mathcal{I}} \mathbf{e}^i.$$

Theorem 4.1. *(Equal treatment in the Core). If $\mathbf{x}$ is an allocation in the core of $\mathcal{E}_r$, then every consumer of type i must have the same bundle according to $\mathbf{x}$. That is, for every $i = 1, 2, \dots, I$, $\mathbf{x}^{iq} = \mathbf{x}^{iq'}$ for every $q, q' = 1, 2, \dots, r$.*

Proof. We give a proof for the case when $I = 2$ and $r = 2$. Suppose that

$$\mathbf{x} := \left(\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22}\right)$$

is an allocation in the core of $\mathcal{E}_2$ so that $\mathbf{x}$ is feasible; i.e.

$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2. \tag{4.1}$$

Suppose, by way of contradiction, that $\mathbf{x}$ does not assign identical bundles to some pair of identical types, say, consumers 11 and 12 so that $\mathbf{x}^{11} \neq \mathbf{x}^{12}$. Since each consumer of type 1 has identical preferences, $\succsim^1$, which is complete, it must rank one of the two bundles as being at least as good as the other. Without loss of generality, suppose

$$\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12},$$

where we note that the preference relation may be strict or indifferent. We want to show that, because $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ are distinct, $\mathbf{x}$ cannot be in the core of $\mathcal{E}_2$. To do so, we will show that $\mathbf{x}$ can be blocked by a coalition.

Consider the two consumers of type 2. Since they have identical preferences $\succsim^2$ that is complete, at least one of the bundle must be at least as good as the other. Without loss of generality, suppose

$$\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}.$$

Therefore, we assume that consumer 2 of type 1 is the worst-off type-1 consumer, and consumer 2 of type 2 is the worst-off type 2 consumer. We form a coalition with these worst-off consumers and show that they can block allocation $\mathbf{x}$.

Define two "average" bundles $\bar{\mathbf{x}}^{12}$ and $\bar{\mathbf{x}}^{22}$ as

$$\bar{\mathbf{x}}^{12} := \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2}, \ \bar{\mathbf{x}}^{22} := \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}.$$

By Assumption 1, utility functions are strictly quasiconcave, which implies that $\succsim^i$ are strictly convex (see Theorem 1.5). Thus, since $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ are distinct, it follows that

$$\bar{\mathbf{x}}^{12} \succ \mathbf{x}^{12}$$

so that consumer 2 of type 1 strictly prefers the average bundle. For consumer 2 of type 2, since we have not assumed that $\mathbf{x}^{21}$ and $\mathbf{x}^{22}$ are distinct, we have

$$\bar{\mathbf{x}}^{22} \succsim \mathbf{x}^{22}.$$

Thus, $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ makes consumer 12 strictly better off and consumer 22 no worse off than the allocation $\mathbf{x}$. Therefore, if this pair of bundles can be achieved by the two consumers, then they can block allocation $\mathbf{x}$. Note, by feasibility (4.1),

$$\begin{aligned}\bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{22} &= \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2} \\ &= \frac{1}{2}\left(2\mathbf{e}^1 + 2\mathbf{e}^2\right) = \mathbf{e}^1 + \mathbf{e}^2.\end{aligned}$$

Thus, $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ is feasible among the coalition $S = \{12, 22\}$ so that they can block $\mathbf{x}$. This contradicts the fact that $\mathbf{x}$ is in the core. ■

The equal treatment property allows us to describe any allocation in the core of $\mathcal{E}_r$ by reference to a similar allocation in the basic economy $\mathcal{E}_1$. By the property, an allocation $\mathbf{x}$ that is in the core of $\mathcal{E}_r$ must have the form:

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}}\right)$$

so that core allocations in $\mathcal{E}_r$ are just *r-fold copies* of allocation

$$\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I\right)$$

in $\mathcal{E}_1$. Note that because $\mathbf{x}$ is in the core allocation in $\mathcal{E}_r$

$$\sum_{i\in\mathcal{I}}\sum_{q=1}^{r}\mathbf{x}^{iq} = r\sum_{i\in\mathcal{I}}\mathbf{x}^{i} = r\sum_{i\in\mathcal{I}}\mathbf{e}^{i}$$
$$\Rightarrow \sum_{i\in\mathcal{I}}\mathbf{x}^{i} = \sum_{i\in\mathcal{I}}\mathbf{e}^{i}.$$

That is, $\left(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^I\right)$ is feasible in the economy $\mathcal{E}_1$. Define

$$C_r \coloneqq \left\{\mathbf{x} = \left(\mathbf{x}^1,\ldots\mathbf{x}^I\right)\in F\left(\mathbf{e}\right) : \left(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^1}_{r\text{ times}},\ldots,\underbrace{\mathbf{x}^I,\ldots,\mathbf{x}^I}_{r\text{ times}}\right) \text{ is in the core of } \mathcal{E}_r\right\}.$$

The following lemma states that the core of the r-fold replica gets no larger as r increases.

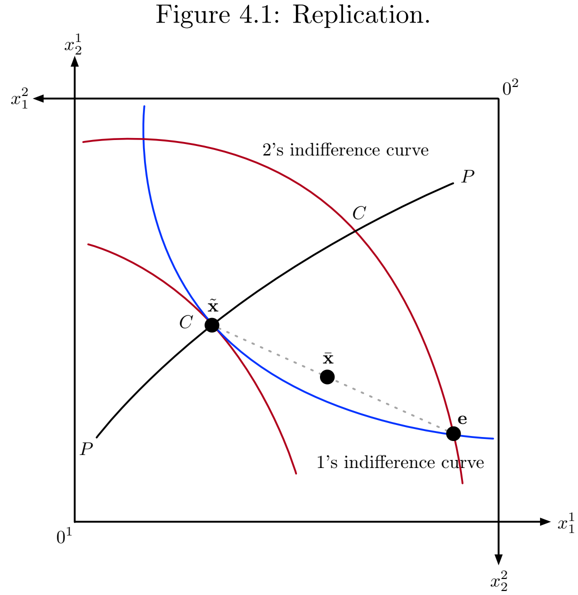
Lemma 4.1. *The sequences of sets $C_1, C_2, \ldots$ is decreasing; i.e. $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_r \supseteq \ldots$.*

Proof. By induction, it suffices to show that for $r > 1$, $C_r \subseteq C_{r-1}$. Let $\mathbf{x}\in C_r$ so that its r-fold copy cannot be blocked in the r-fold replica economy. But, since r-fold replica economy contains all possible coalitions in the $(r-1)$-fold economy, it follows that $\mathbf{x}\in C_{r-1}$. Hence, $C_r \subseteq C_{r-1}$. ■

The following lemma says that, as we replicate the economy, the set of Walrasian equilibrium allocations remains "constant" in the sense that it consists purely of copies of Walrasian equilibria of the basic economy.

Example 4.1. $(r = 2)$. Consider In the basic economy with one consumer of each type, the core of $\mathcal{E}_1$ is the segment

Figure 4.1: Replication.



C-C.The core contains some allocations that are WEA and others that are not. For example, the allocation $\tilde{\mathbf{x}}$ is not a WEA because the price line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumer's indifference curves at $\tilde{\mathbf{x}}$. If we replicate this economy once, can the replication of this allocation be in the core of the larger four-consumer economy?

First, notice that, since preferences are strict convex, any point along the line joining $\mathbf{e}$ and $\tilde{\mathbf{x}}$ are preferred over $\mathbf{e}$ and $\tilde{\mathbf{x}}$ by type-1 consumers. Take, for example, $\bar{\mathbf{x}}$ that lies in the middle of this line. Consider now the three-consumer coalition, $S = \{11, 12, 21\}$ consisting of the two type-1 consumers and one of the type-2 consumers. We let each type-1 consumer have a bundle corresponding to type-1 bundle at $\bar{\mathbf{x}}$ and let the lone type-2 consumer have a type-2 bundle at $\tilde{\mathbf{x}}$. We already know that each type-1 consumer prefers the bundle $\bar{\mathbf{x}}$ while the type-2 consumer is equally well off; i.e.

$$\begin{aligned}
\bar{\mathbf{x}}^{11} &:= \frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}\right) \succ^1 \tilde{\mathbf{x}}^{11}, \\
\bar{\mathbf{x}}^{12} &:= \frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}\right) \succ^1 \tilde{\mathbf{x}}^{12}, \\
&\qquad\qquad\quad \tilde{\mathbf{x}}^{21} \sim^2 \tilde{\mathbf{x}}^{21}.
\end{aligned}$$

We now check that this bundle is feasible among S:

$$\begin{aligned}
\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} &= 2\left[\frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}\right)\right] + \tilde{\mathbf{x}}^{21} \\
&= \mathbf{e}^1 + \tilde{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21}.
\end{aligned}$$

Since $\tilde{\mathbf{x}}$ is in the core of $\mathcal{E}_1$, it must be feasible in the two-consumer economy so that $\tilde{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2$. Thus,

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^1 + \mathbf{e}^2$$

so that the proposed allocation is feasible for the coalition S . Since at least one member of the coalition is strictly better off, the coalition S will block $\tilde{\mathbf{x}}$ in the four-consumer economy—i.e. $\tilde{\mathbf{x}}$ is not in the core of $\mathcal{E}_2$.

As we show below, if we continue to replicate the economy, so that more consumers can form more coalitions, we can shrink the core further. Moreover, we will find that WEA are never ruled out.

Our goal is to show that the set of core allocations for $\mathcal{E}_r$ converges to its set of Walrasian equilibrium allocations as r increases. Through the equal treatment property, we have been able to describe the core allocations for $\mathcal{E}_r$ as r-fold copies of allocations in the basic economy. We now do the same for WEA of the economy $\mathcal{E}_r$.

The lemma below says that, as we replicate the economy, the set of Walrasian equilibria remains "constant"—they always consists purely of copies of Walrasian equilibria of the basic economy.

Lemma 4.2. *An allocation* $\mathbf{x}$ *is a WEA for* $\mathcal{E}_r$ *if and only if it is of the form*

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r\ times}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r\ times}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r\ times}\right)$$

and the allocation $\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I\right)$ *is a WEA for* $\mathcal{E}_1$.

Proof. If $\mathbf{x}$ is a WEA for $\mathcal{E}_r$, then by Theorem 2.5, it is in the core of $\mathcal{E}_r$. Then, by Theorem 4.1, it must satisfy the equal treatment property. Hence, it must be an r-fold copy of some allocation in $\mathcal{E}_1$.

That $\mathbf{x}$ is a WEA in $\mathcal{E}_r$ means that there exists $\mathbf{p}^*$ such that $\mathbf{z}^r(\mathbf{p}^*) = \mathbf{0}$, where $\mathbf{z}^r$ is the excess demand function for the economy $\mathcal{E}_r$. Since each consumer of the same type have the same bundle $\mathbf{x}^i$ (by the equal treatment property) and endowment $\mathbf{e}^i$ (by construction), for any good $k = 1, 2, \ldots, n$,

$$0 = \mathbf{z}_k^r(\mathbf{p}^*) = \sum_{i=1}^{I}\left(\sum_{q=1}^{r}\mathbf{x}_k^{r,iq}\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right)\right) - \sum_{i=1}^{I}\left(\sum_{q=1}^{r} e_k^{iq}\right)$$
$$= r\sum_{i=1}^{I}\left(\mathbf{x}_k^{r,i}\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) - e_k^i\right).$$

We wish to show that that

$$\mathbf{x}_k^{r,i}\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{x}_k^{1,i}\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right).$$

But since prices are the same, and the endowments are the same between $\mathcal{E}_1$ and $\mathcal{E}_r$ economies, each consumer solves the same problem so the equation above must hold with equality. Hence,

$$\mathbf{z}_k^1(\mathbf{p}^*) = 0 \Leftrightarrow \mathbf{z}_k^r(\mathbf{p}^*) = 0, \ \forall k, r > 0. \qquad \blacksquare$$

Given Lemma 4.1, as we replicate the economy and consider C_r, in the limit, only those allocations satisfying$\mathbf{x} \in C_r$ for all r will remains. Thus, to say that the core shrinks to the set of competitive equilibria is to say that, if $\mathbf{x} \in C_r$ for every r, then $\mathbf{x}$ is a competitive equilibrium allocation for $\mathcal{E}_1$.

Theorem 4.2. *(Edgeworth-Debreu-Scarf: A limit theorem on the Core). If $\mathbf{x} \in C_r$ for every $r = 1, 2, \ldots$, then $\mathbf{x}$ is a Walrasian equilibrium allocation for $\mathcal{E}_1$.*

Before proving the theorem, let us consider the 2-by-2 case and suppose that, by way of contradiction, some non-Walrasian equilibrium allocation, $\tilde{\mathbf{x}}$, is in C_r for every r. In particular, then, $\tilde{\mathbf{x}}$ is in the core of the basic economy so that it must be on the line segment C-C in Figure 4.1. Since $\tilde{\mathbf{x}}$ is not a WEA, the line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ cannot be tangent to the indifference curves crossing at $\tilde{\mathbf{x}}$. Then, as we did before, we can select a point in the middle of the line that connect $\tilde{\mathbf{x}}$ and $\mathbf{e}$ such that, by strict convexity of preferences, individuals of type 1 would strictly prefer this bundle. We then construct a coalition that include all individuals of type 1 and all but one of type-2 individuals, while not changing the bundle for the type-2 individuals in the coalition the same. Then we only need to show that such an allocation is feasible among those in the coalition.

We now give the general argument under two additional hypotheses:

(i) if $\mathbf{x} \in C_1$, then $\mathbf{x} \gg \mathbf{0}$;

(ii) for each $i \in \mathcal{I}$, the utility function u^i representing $\succsim^i$ is differentiable on $\mathbb{R}^n_{++}$ with a strictly positive gradient vector there.

Proof. (Theorem 4.2). Suppose that $\tilde{\mathbf{x}} \in C_r$ for every r. We must show that $\tilde{\mathbf{x}}$ is a WEA for $\mathcal{E}_1$.

We first establish that

$$u^i\left((1-t)\,\tilde{\mathbf{x}}^i + t\mathbf{e}^i\right) \le u^i\left(\tilde{\mathbf{x}}^i\right), \ \forall t \in [0,1], \ \forall i \in \mathcal{I}.$$

Suppose, by way of contradiction, that the equality above does not hold. That is, for some $\bar{t} \in [0,1]$ and some $i \in \mathcal{I}$,

$$u^i\left((1-t)\,\tilde{\mathbf{x}}^i + t\mathbf{e}^i\right) > u^i\left(\tilde{\mathbf{x}}^i\right).$$

By strict quasiconcavity of u^i, this implies that (think strict quasiconcavity means single peaked):

$$u^i\left((1-t)\,\tilde{\mathbf{x}}^i + t\mathbf{e}^i\right) > u^i\left(\tilde{\mathbf{x}}^i\right),\ \forall t \in (0,\bar{t}].$$

Consequently, by continuity of u^i, there exists some integer r sufficiently large such that

$$u^i\left(\left(1-\frac{1}{r}\right)\tilde{\mathbf{x}}^i + \frac{1}{r}\mathbf{e}^i\right) > u^i\left(\tilde{\mathbf{x}}^i\right).$$

[See the book.] ■

5 Social choice and welfare

5.1 Arrow's Theorem

Let X denote the set of mutually exclusive "social states". For example, in the context of an exchange economy, a social state is an allocation of endowments to individuals in the economy. Each individual $i \in \mathcal{I}$ has a preference relation (i.e. complete and transitive), R^i, defined over X. Suppose there are $N \geq 2$ individuals. What "should" society's preference relation over X be?

Definition 5.1. *(Social preference relation).* A social preference relation, R, is a complete and transitive binary relation on X. For any $x, y \in X$, xRy means that x is socially at least as good as y. P and I denotes the associated relations of strict social preference and social indifference, respectively.

The next example shows that a social relation based on majority voting can fail to satisfy transitivity when there are more than two alternatives.

Example 5.1. *(Condorcet's paradox).* Let $N = 100$ and $X = \{x, y, z\}$. Let P be a social preference relation based on pair-wise majority voting. Individuals have the following preference relations:

Ranking\No. of individuals	20	35	45
1	x	y	z
2	y	z	x
3	z	x	y

- ▷ Choice between x and y. $20 + 45 = 65$ individuals prefer x over y, against 35 individuals who prefer y over x. xPy.
- ▷ Choice between y and z. $20 + 35 = 55$ individuals prefer y over z, against 45 individuals who prefer z over y. yPz.
- ▷ Choice between x and z. $35 + 45 = 80$ individuals prefer z over x, against 20 individuals who prefer $x$over z. zPx.

The first two implies that xPz by transitivity but this contradicts the outcome when society is presented with a choice between $\{x, z\}$. That is, transitivity fails, and so no social preference relation exists.

Example 5.2. *(Borda rule).* For any $x \in X$, the Borda score is given by

$$\mathbb{B}(x) = \sum_{i=1}^{n} \# \left\{ y : xR^i y \right\}.$$

Thus, Border score for a social state x is give by the sum across individuals of the number of states ranked (weakly) below x. The Borda rule is given by

$$xRy \Leftrightarrow \mathbb{B}(x) \geq \mathbb{B}(y).$$

Since $\geq$ satisfies transitivity, the preference relation always satisfies transitivity; i.e. the Borda rule is a social preference relation. For example, the Borda scores for each social state in Example 5.1

are:

$$\begin{aligned}
\mathbb{B}(x) &= 20 \times 3 + 35 \times 1 + 45 \times 2 = 185 \\
\mathbb{B}(y) &= 20 \times 2 + 35 \times 3 + 45 \times 1 = 190 \quad \Rightarrow \quad zPyPx. \\
\mathbb{B}(z) &= 20 \times 1 + 35 \times 2 + 45 \times 3 = 225
\end{aligned}$$

Our task is to find a mapping that takes as input a profile of individuals' preferences $(R^i)_{i=1}^N$ and gives as output a social preference relation R for the society with "good" properties.

Definition 5.2. *(Social welfare function).* A social welfare function f

$$R = f\left(R^1, R^2, \ldots, R^N\right)$$

is a mapping from individual preference relations on X to a social preference relation on X.

What might constitute as "good" properties? Arrow proposed the following. First, let

$$\mathcal{R} = \{\text{set of all possible preference relations on } X\}.$$

Assumption 3. *(Arrow's requirements on the social welfare function).*

(i) Unrestricted domain (U).

The domain of f must include all possible combinations of individual preferences on X.

$$f : \mathcal{R}^n \to \mathcal{R}.$$

The induced social welfare function $R = f\left(R^1, R^2, \ldots, R^N\right)$ must be a preference relation (complete and transitive).

(ii) Weak Pareto Principle (WP).

Whenever all individuals strictly prefer x over y, the social preference relation should also prefer x over y. That is, $\forall x, y \in X$, $\forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N$,

$$\left[xP^i y, \forall i\right] \Rightarrow [xPy].$$

(iii) Independent of Irrelevant Alternatives (IIA).

Let $R = f(R^1, R^2, \ldots, R^N)$ and $\tilde{R} = f(\tilde{R}^1, \tilde{R}^2, \ldots, \tilde{R}^N)$. If every individual ranks $x, y \in X$ under R^i in the same way as under $\tilde{R}^i$, then R and $\tilde{R}$ must rank x and y in the same way.[a] *That is, $\forall x, y \in X$, $\forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N$, $\forall \left(\tilde{R}^1, \ldots, \tilde{R}^N\right) \in \mathcal{R}^N$,*

$$\left[xR^i y \Leftrightarrow x\tilde{R}^i y, \forall i\right] \Rightarrow \left[xRy \Leftrightarrow x\tilde{R}y\right].$$

(iv) Non-dictatorship (D).

There is no individual i such that if i prefers x over y, then the social preference relation prefer x over y regardless of the preferences R^j of all other individuals $j \neq i$. That is, $\forall x, y \in X$,

$$\nexists i : xP^i y \Rightarrow xPy, \ \forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N.$$

[a] Put differently, IIA requires that if individuals' ranking of states $z \in X$ distinct from x and y changes relative to x and y, the social ranking between x and y does not change.

Example 5.3. *(Borda rule fails IIA).* Suppose

Ranking\No. of individuals	3	2
1	x	y
2	y	x
3	z	z

The Borda scores are

$$\begin{aligned}
\mathbb{B}(x) &= 3 \times 3 + 2 \times 2 = 13 \\
\mathbb{B}(y) &= 3 \times 2 + 2 \times 3 = 12 \quad \Rightarrow \quad xPyPz. \\
\mathbb{B}(z) &= 3 \times 1 + 2 \times 1 = 5
\end{aligned}$$

Consider changes to the ranking of each social states that do not change the ordering of y and x. IIA implies that such changes should not result in changes in the ranking of x and y for the society. Let us exchange the ranking of x and z for the second group of individuals.

Ranking\No. of individuals	3	2
1	x	y
2	y	z
3	z	x

The Borda scores are now

$$\begin{aligned} \mathbb{B}(x) &= 3 \times 3 + 2 \times 1 = 11 \\ \mathbb{B}(y) &= 3 \times 2 + 2 \times 3 = 12 \quad \Rightarrow \quad yPxPz. \\ \mathbb{B}(z) &= 3 \times 1 + 2 \times 2 = 7 \end{aligned}$$

Notice that the ranking of x and y are now reversed. Hence, we conclude that Borda rule does not satisfy IIA.

Theorem 5.1. *(Arrow's Impossibility Theorem). If there are at least three social states in X (and $N \geq 2$), then there is no social welfare function f that simultaneously satisfies conditions* U*,* WP*,* IIA*, and* D*.*

Proof. The idea is to show that assumptions U, WP and IIA together imply the existence of a dictator. We assume X is finite in this case.

Step 1. First, note that assumptions U means that we are free to choose any individual preference relations. In particular, we are not restricted considering individual preference relations that we observe in the "data". With this in mind, let $c \in X$ be such that every individual ranks c at the bottom of his ranking; i.e. for all $i \in \mathcal{I}$, $xP^i c$ for any $x \in X$ distinct from c. Then, by WP, xPc for any $x \in X$ distinct from c; i.e. c must be at the bottom of the social ranking.

R^1	R^2	$\cdots$	R^N	R
x^1	x^2	$\cdots$	x^N	$\vdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
c	c	$\cdots$	c	c

Step 2. Consider the following procedure. First, we move c to the top of individual 1's ranking, leaving the ranking of all other states unchanged. Next, we move c to the top of individual 2's ranking, leaving the ranking of all others unchanged. And so on. Of course, if we move c to the top of every individual's ranking, WP implies that cPx for any $x \in X$ distinct from c. Thus, there must be a "first time" (i.e. an individual n) that c moves up from the bottom of the ranking to somewhere above. We now show that when c moves, it moves to the top of the social ranking—strictly preferred over all other social states.

Suppose, by way of contradiction, that c does not move to the very top of the ranking. Then, there must exist $\beta \in X \backslash \{c\}$ such that $cR\beta$. Define β as the least preferred among all those; i.e.

$$\beta \in \{x \in X \backslash \{c\} : zRx, \ \forall z \in X \backslash \{c\} \ \ s.t. \ cRz\}.$$

Similarly, if c does not move to the very top of the ranking, then there must exist $\alpha \in X \backslash \{c, \beta\}$ such that $\alpha R c$. Define α as the most preferred among all those; i.e.

$$\alpha \in \{x \in X \backslash \{c, \beta\} : xRz,\ \forall z \in X \backslash \{c\}\ \ s.t.\ zRc\}.$$

(We can find such an α since we assume that there are at least three social states in X.) Since R is a preference relation, it satisfies transitivity, which, in turn, implies that

$$\alpha R c R \beta \Rightarrow \alpha R \beta, \tag{5.1}$$

where each of α, c and β are distinct by construction.

Now, consider switching the ranking of α and β of individuals (as necessary) so that $\beta P^i \alpha$ for all i. Notice that since c is either at the top or at the bottom of every individual's ranking, by IIA, such a switch does not affect the ranking of c relative to any $x \in X \backslash \{c\}$. Hence, IIA implies that (5.1) remains true after such a switch. However, since $\beta P^i \alpha$ for all i, WP implies that

$$\beta P \alpha,$$

which contradicts (5.1). Thus, we conclude that when c moves away from the bottom of the ranking, it must move to the top of the ranking.

Figure A. Just before c moves

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	x^n	x^{n+1}	$\cdots$	x^N	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	c	c	$\cdots$	c	c

Figure B. Just after c moves

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	c	x^{n+1}	$\cdots$	x^N	c
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	c	$\cdots$	c	$\vdots$

Figure C. "New profile"

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	a	x^{n+1}	$\cdots$	x^N	a
$\vdots$	$\cdots$	$\vdots$	c	$\vdots$	$\cdots$	$\vdots$	c
$\vdots$	$\cdots$	$\vdots$	b	$\vdots$	$\cdots$	$\vdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	c	$\cdots$	c	$\vdots$

Step 3. Consider now any two distinct social states a and b which are themselves distinct from c. Change individual n's ranking so that $aP^n cP^n b$. For the other individuals, ranking of a and b can be arbitrary so long as the position of c is unchanged for each individual (i.e. $cP^i a$ for all $i = 1, 2, \ldots, n-1$ and $aP^i c$ for all $i = n+1, n+2, \ldots, N$). We refer to this profile of preferences as the "new profile" (think of this as $\tilde{R}^i$'s).

Now compare Figure A and Figure C. Notice that the ranking of a and c under Figure C is the same for every individual as it was just before raising c to the top of individual n's ranking in step 2 (in Figure A). Thus, by IIA, the social ranking of a and c must be the same under the new profile as it was when c was at the bottom of the social ranking; i.e. aPc.

Comparing Figure B and C, we can see that the ranking of c to b in Figure C is the same for every individual as it was just after raising c to the top of every individual n's ranking in step 2 (in Figure B). Thus, by IIA, cPb.

By transitivity, we have aPb. Notice that, no matter how others rank a and b, the social ranking agrees with individual n's ranking. By IIA, since a and b were arbitrary, for any states a and b distinct from c, we have

$$aP^n b \Rightarrow aPb,\ \forall a, b \in X \setminus \{c\}\,.$$

That is, individual n is a dictator on all pairs of social states that does not involve c. It remains to show that n is also a dictator in pairs of social states that involve c. That is, we need to show that n is a dictator for the pair (c, x) for some $x \in X \setminus \{c\}$.

Step 4: Let d be distinct from c and x (we are using the fact that $|X| \geq 3$ again). Following the same steps as above with d playing the role of c, we can conclude that some individual m (perhaps not the same n) is a dictator on all pairs not involving d. In particular, this means that m is a dictator for $\{x, c\}$. But in Figure A and B, we only changed n's ranking of c and all other $x \in X \backslash c$ and the result was a change in the society's ranking between the two. For the two to be both true, it must be that $m = n$. That is n is a dictator in all pairs of social states. ■

Example 5.4. Let us show that when $X = 2$, the Arrow's Impossibility Theorem does not hold. It suffices to show that there exists f that satisfies all four properties when $X = 2$. Let us consider a majority voting rule in the case where $X = 2$—the social state with the highest number of votes wins.

- ▷ Unrestricted domain: the induced social preference is complete (since we can always add) and trivially transitive (since there are only two social states, transitivity has no bite). Hence, the majority voting rule satisfies U.

- ▷ WPP: If everyone prefers one outcome over the other, then majority rule implies that the most preferred state is chosen for the society. Hence, WPP is satisfied.

- ▷ IIA: When $X = 2$, the IIA assumption has no bite. This is because the IIA ensures that ordering between a pair of states does not change when we change the ordering between other pairs—but when there are only two states, there are no such thing as "other pairs". So IIA is trivially satisfied.

- ▷ ND: Suppose, by way of contradiction, that individual n is a dictator. Then, n is the pivotal voter under all possible preference orderings. However, this cannot be true since we can always reverse the ordering of preference of another individual to make n non-pivotal. Thus, ND is also satisfied by the majority voting rule.

5.2 Social choice and the Gibbard-Satterthwaite Theorem

We have so far implicitly assumed that the true preference of each individual can be obtained. But how do we obtain the individuals' true preferences? If we were to simply ask individuals, we have to consider the possibility that people may lie. Thus, in addition to coherently aggregating individual rankings into a social ranking, there is also the problem of finding out individual preferences in the first place.

Assumption 4. *Let X be a finite set of social states. Assume unrestricted domain; i.e. the N individuals in the society are permitted to have any preference relation on X.*

Definition 5.3. (*Social choice function*) A s*ocial choice function* is a mapping $c : \mathcal{R}^n \to X$ such that $c(\mathcal{R}^n) = X$ (i.e. $c(\cdot)$ is onto X).

In contrast to a social welfare function which gives a social preference relation (i.e. mapping from $\mathcal{R}^N \to \mathcal{R}$), the social choice function selects a particular state $x \in X$. That $c(\cdot)$ is onto X means that, for each element in X, there exists some individual preference relations $\left(R^1, \ldots, R^N\right) \in \mathcal{R}^N$ such that $c\left(R^1, \ldots, R^N\right) = x$ (if not, we may as well remove such an element of X from the set).

Definition 5.4. *(Dictatorial social choice function).* A social choice function $c(\cdot)$ is *dictatorial* if there exists i such that

$$c\left(R^1, \ldots, R^N\right) R^i y, \ \forall y \in X, \ \forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N.$$

That is, c is dictatorial if there exists an individual for whom the social choice is always at the top (weakly) of i's ranking.

We want to make sure that the preference relation input into the social choice function is truthful. To see why this might be important, suppose we fix R^{-i}. Consider i whose true preference relation is given by R^i and he is thinking about some other preference relation $\tilde{R}^i$. Suppose that the choice if i reports truthfully is

$$c\left(R^i, R^{-i}\right) = x \in X$$

but the choice if he "lied" was

$$c\left(\tilde{R}^i, R^{-i}\right) = y \neq x.$$

Then, i would have an incentive to "lie" if preferred y over x; i.e. yP^ix. We want to rule this out.

Definition 5.5. *(Strategy-proofness).* A social choice function $c : \mathcal{R}^N \to X$ is *strategy proof* if $\forall i$, $\forall R^i, \tilde{R}^i \in \mathcal{R}$, $\forall R^{-i} \in \mathcal{R}^{N-1}$,

$$\begin{bmatrix} c\left(R^i, R^{-i}\right) = x \\ c\left(\tilde{R}^i, R^{-i}\right) = y \end{bmatrix} \Rightarrow xR^iy.$$

If a social choice function is strategy proof, no individual can ever strictly gain by misreporting his preferences no matter what others report (even if others lie). Conversely, if a social choice function is not strategy proof, then there is at least one circumstance under which some individual can strictly gain by misreporting his preferences. Thus, strategy proof ensures that it is optimal for individuals to report their preferences honestly and so society's choice will be based upon the true preferences of its individual members.

Theorem 5.2. *(Gibbard-Satterthwaite Theorem). If $|X| \geq 3$, then every strategy-proof social choice function is dictatorial.*

To prove the result, we first show that a strategy-proof social choice function must exhibit two properties: Pareto efficiency and monotonicity. We then show that any monotonic and Pareto efficient social choice function is dictatorial.

Definition 5.6. *(Pareto-efficient social choice function).* A social choice function $c(\cdot)$ is *Pareto efficient* if $c\left(R^1, R^2, \ldots, R^N\right) = x$ whenever xP^iy for every individual i and every $y \in X$ distinct from x.

Thus, a social choice function is Pareto efficient if, whenever x is at the top of every individual's ranking, the social choice is x.

Definition 5.7. *(Monotonic social choice function).* A social choice function $c(\cdot)$ is *monotonic* if

$$\left[xR^iy \Rightarrow x\tilde{P}y,\ \forall i,\ \forall y \in X\setminus\{x\}\right] \Rightarrow \left[c\left(R^1, \ldots, R^N\right) = x \Rightarrow c\left(\tilde{R}^1, \ldots, \tilde{R}^N\right) = x\right].$$

Monotonicity says that the social choice does not change when individual preferences change so that every individual strictly prefers the social choice to any distinct social state that it was originally at least as good as. Loosely speaking, monotonicity says that the social choice does not change when the social choice rises in each individual's ranking. Notice that the individual rankings between pairs of social states other than the social choice are permitted to change arbitrarily.

Lemma 5.1. *A strategy-proof social choice function is Pareto efficient and monotonic.*

Proof. (Monotonicity). Let $(R^1, R^2, \ldots, R^N)$ be an arbitrary preference profile and suppose that $c(R^1, R^2, \ldots, R^N) = x$. Fix an individual, say i, and let $\tilde{R}^i$ be a preference for i such that, for every $y \in X\setminus\{x\}$, $xR^iy \Rightarrow x\tilde{P}^iy$.

We first to show that $c(\tilde{R}^i, R^{-i}) = x$. Suppose, by way of contradiction, that $c(\tilde{R}^i, R^{-i}) = y \neq x$. Then, given that others report R^{-i}, individual i, when his preferences are R^i, can report truthfully and obtain the social state x or he can lie by reporting $\tilde{R}^i$ and obtain the social state y. Strategy-proofness requires that lying cannot be strictly better than telling the truth. Hence, we must have xR^iy. According to the definition of $\tilde{R}^i$, we then have $x\tilde{P}^iy$. Consequently, when individual i's preferences are $\tilde{R}^i$, he strictly prefers x to y and so, given that the others report R^{-i}, individual i strictly prefers lying (reporting R^i and obtaining x) to telling the truth (reporting $\tilde{R}^i$ and obtaining y), contradicting strategy-proofness. Thus, $c(\tilde{R}^i, R^{-i}) = x$.

Let $(R^1, R^2, \ldots, R^N)$ and $(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N)$ be preference profiles such that $c(R^1, R^2, \ldots, R^N) = x$ and such that for every individual i and every $y \in X\setminus\{x\}$, $xR^iy \Rightarrow x\tilde{P}^iy$. To prove that $c(\cdot)$ is monotonic, we must show that $c(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N) = x$. But this follows immediately from the result just proven—simply change the preference profile from $(R^1, R^2, \ldots, R^N)$ to $(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N)$ by switching, one at a time, the preferences of each individual i from R^i to $\tilde{R}^i$. We conclude that $c(\cdot)$ is monotonic.

(Pareto efficiency). Let x be an arbitrary social state and let $\hat{R}$ be a preference profile with x at the top of each individual's ranking. We must show that $c\left(\hat{R}\right) = x$.

Because the range of $c(\cdot)$ is all of X, there is some preference profile R such that $c(R) = x$. Obtain the preference profile $\tilde{R}$ from R by moving x to the top of every individual's ranking. By monotonicity (proven above), $c(\tilde{R}) = x$. Because $\hat{R}$ places x at the top of every individual ranking and $c(\tilde{R}) = x$, we can again apply monotonicity (trivially, we have $x\tilde{R}^iy \Rightarrow x\hat{P}y,\ \forall i,\ \forall y \in X\setminus\{x\}$) and conclude that $c(\hat{R}) = x$, as desired. ■

We now proceed to prove Theorem 5.2. Similar to the way we proved the Arrow's Impossibility Theorem, we will show that $|X| \geq 3$, monotonicity and Pareto efficiency implies existence of a dictator. Although we use strict preferences in the proof, we are not ruling out indifference—it just so happens that we can prove the desired result by considering a subset of preferences (that do not exhibit indifference).

We first establish the following two results which will use many times in the proof of Theorem 5.2.

Claim 5.1. Suppose $c(R^1, \ldots, R^N) = x$ and that, for some individual i, R^i ranks y just below x. Let $\tilde{R}^i$ be identical to R^i except that the ranking of y and x are reversed. Then, $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$.

Proof. Suppose, by way of contradiction, that $c(\tilde{R}^i, R^{-i}) = z \notin \{x, y\}$. Note that $\tilde{R}^{-i} = R^{-i}$ as we only altered i's preference relation. Moreover, since we only changed the ordering between x and y for individual i, the ordering between z and any other element in X is unchanged. Then,

$$z\tilde{R}^j c \Rightarrow zR^j c, \ \forall j, \ \forall c \neq z.$$

Since we are dealing with strict preferences, in fact,

$$z\tilde{P}^j c \Rightarrow z\tilde{R}^j c \Rightarrow zP^j c, \ \forall j, \ \forall c \neq z.$$

Using monotonicity,

$$c(\tilde{R}^i, R^{-i}) = z \Rightarrow c(R^i, R^{-i}) = z.$$

But this contradicts the assumption that $c(R^1, \ldots, R^N) = x$. Hence, $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$. ■

Claim 5.2. Suppose $c(R^1, \ldots, R^N) = x$. Let $\tilde{R} = (\tilde{R}^1, \ldots, \tilde{R}^N)$ be strict rankings such that, for every individual i, the ranking of x versus any other social state is the same as under $\tilde{R}^i$ as it is under R^i. Then, $c(\tilde{R}) = x$.

Proof. Since the ranking of x has not changed, we have

$$xR^i y \Rightarrow x\tilde{R}^i y, \ \forall i, \ \forall y \in X \backslash \{x\}.$$

Since $\tilde{R}$ is a strict ranking, we may write

$$xR^i y \Rightarrow x\tilde{P}^i y, \ \forall i, \ \forall y \in X \backslash \{x\}.$$

Then, by monotonicity, since $c(R) = x$, we must have $c(\tilde{R}) = x$. ■

Proof. (Proof of Theorem 5.2). Consider any two distinct social states $x, y \in X$ and a profile of strict rankings in which x is ranked the highest for y lowest for every individual $i = 1, 2, \ldots, N$. Pareto efficiency implies that the social choice at this profile is x.

Suppose we change individual 1's ranking by strictly raising y one position at a time. By monotonicity, the social choice remains equal to x so long as y is below x in 1's ranking. But when y rises above x, Claim 5.1 implies that the social choice either changes to y or remain as x. If the latter occurs, then we can begin the same process for individual 2, 3 etc. until for some individual n, the social choice does change from x to y when y rises above x in n's ranking. (Notice that such an individual exists since following this procedure, we will eventually have y at the top of everyone's ranking and so, by Pareto efficiency, the social choice would be y.)

In Figure A, we show the ordering just before we change the ordering between x and y for individual n while figure B shows the ordering just after we change n's ranking. By construction, social changes is x in Figure A and y in Figure B. Now, consider Figure D in which: (i) we moved x to the bottom of ranking for all individuals $i < n$, and (ii) we moved x to be just above y for all $i > n$. Comparing Figure B and D, observe that the the individuals' rankings of y versus any other social state are the same between the two. Thus, by Claim 5.2 social choice in Figure D must be the same as in Figure B; i.e. y.

Now compare Figures D and C. The two differ only in individual n's ranking of x and y. Thus, by Claim 5.1, because the social choice under $\tilde{R}$ is y, the social choice in Figure C must either be x or y. But if the social choice in Figure C is y, then, since the ordering of y versus any other social states are the same as in Figure A, by Claim 5.2, the social choice in Figure A must also e y—a contradiction. Hence, the social choice in Figure C must be x.

Figure A. Just *before* c changes

$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	$c(\cdot)$
$\cdots$	y	x	x	$\cdots$	
$\cdots$	x	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	y	$\cdots$	

Figure C. Reordered

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^n$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	y	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	$\vdots$	$\vdots$	x	$\cdots$	
$\cdots$	x	$\vdots$	y	$\cdots$	

Figure B. Just *after* c changes

$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	$c(\cdot)$
$\cdots$	y	y	x	$\cdots$	
$\cdots$	x	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	y
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	y	$\cdots$	

Figure D. Reordered

$\cdots$	$\tilde{R}^{n-1}$	$\tilde{R}^n$	$\tilde{R}^{n+1}$	$\cdots$	$\tilde{R}$
$\cdots$	y	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	y
$\cdots$	$\vdots$	$\vdots$	x	$\cdots$	
$\cdots$	x	x	y	$\cdots$	

Let us choose another social state $z \in X$ that is distinct from x and y (recall that there are at least three social states). Take Figure C and add z as in Figure E below. Observe that the ranking of x versus any other social state in any individual rankings have not changed—hence, by Claim 5.2, the social choice must remain the same as x in Figure E.

Now consider Figure F in which we reversed the ordering between x and y for individuals $i > n$. These are the only differences between Figure E and F so that, by Claim 5.1, social choice in Figure F must either be x or y. But the social choice cannot be y because z is ranked above y for all individuals and monotonicity would then imply that the social choice would remain y even if z were raised to the top of every individual's ranking—contradicting Pareto efficiency. Hence, the social choice in Figure F must be x.

Figure E. Modified Figure C

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^{n}$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	$\vdots$	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	z	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	z	$\vdots$	z	$\cdots$	
$\cdots$	y	$\vdots$	x	$\cdots$	
$\cdots$	x	$\vdots$	y	$\cdots$	

Figure F. Reordered

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^{n}$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	$\vdots$	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	z	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	z	$\vdots$	z	$\cdots$	
$\cdots$	y	$\vdots$	y	$\cdots$	
$\cdots$	x	$\vdots$	x	$\cdots$	

An arbitrary profile of strict rankings with x at the top of individual n's ranking can be obtained from the profile in Figure F without reducing the ranking of x versus any other social state in any individual's ranking. Hence, Claim 5.2 implies that the social choice must be x whenever individual rankings are strict and x is at the top of individual n's ranking. We now show that this implies that, even when individual rankings are not strict and indifferences are present, the social choice must be at least as good as x for individual n whenever x is at least as good as every other social state for individual n.

First, note that R denotes preferences that are all strict, and, as we showed above, $c(R) = x$. Let $\tilde{R}$ denote preferences that are not necessarily strict and denote $c^* := c(\tilde{R})$. We want to show that $c^* \tilde{R}^n x$ when $x \tilde{R}^n y$ for all $y \in X$. By way of contradiction, suppose that $x \tilde{R}^n y$ for all y but $x P^n c^*$ (which implies that $c^* \neq x$). We define a new set of strict preferences for all individuals, by changing $z \tilde{R}^i y$ to $z \tilde{P}^i y$ for all i, for all $z, y \in X$ and $z \neq y$. In particular, for all i, and for all $y \neq c^*$, we change $c^* \tilde{R}^i y$ to $c^* \tilde{P}^i y$. Then, by monotonicity, since $c(\tilde{R}) = c^*$, we must also have $c(\tilde{P}) = c^*$. But this would imply that $c^* \tilde{P}^n y$ for all $y \neq c^*$—in particular, $c^* \tilde{P}^n x$—a contradiction. Thus, we must have $c^* \tilde{R}^n x$ when $x \tilde{R}^n y$ for all $y \in X$.

With this result, we may now say that individual n is a dictator for the social state x. Because x was chosen arbitrarily, we have shown that, for each social state $x \in X$, there is a dictator for x. The final step is to show that there cannot be distinct dictators for distinct states. We now show that if (i) x and y are distinct states, (ii) the social choice is at least as good as x for individual n whenever x is at least as good as every other social state for n, and (iii) the social choice is at least as good as y for individual m whenever y is at least as good as every other social state for m, then $n = m$. Combined with the previous result, this would tells us that there cannot be two distinct dictators for distinct social states.

Let $c^* := c(R)$ denote the social choice and $x \neq y$. By the hypothesis of the claim, we have: (i) $c^* R^i x$ whenever $x R^i w$ for all $w \in X$; (ii) $c^* R^m y$ whenever $y R^i w$ for all $w \in X$. By way of contradiction, suppose $n \neq m$. Consider a profile P in which x is strictly preferred to all other social states by n and y is strictly preferred to all other social states by m. By (i), this implies that $c^* = x$ and by (ii), $c^* = y$. But this contradicts the fact that $x \neq y$. Hence, it must be that $n = m$.

We may finally conclude that there is a single dictator for all social states and therefore the social choice function is dictatorial. ■

The Gibbard-Satterthwaite theorem tell us that it is impossible to design a non-dictatorial system in which social choices are made based upon self-reported preferences without introducing the possibility that individuals can gain by lying. However, in Price Theory III, we will see that we can restrict the domain (in particular, to quasilinear preferences) which would allow us to escape the conclusion of the Gibbard-Satterthwaite theorem.

Example 5.5. As we did for the case for the Arrow's Impossibility Theorem, let us show that when there are just two alternatives, $X = \{x, y\}$, the majority voting rule social choice function is: (i) Pareto efficient, strategy proof and non-dictatorial. We assume that the there are odd number of individuals.

- ▷ Pareto efficiency. If xP^iy for every i , then $c(R) = x$ given the majority voting rule. So $c(\cdot)$ is Pareto efficient.
- ▷ Strategy proof. Call individual i as being *pivotal* whenever the preferences of the other individuals, R^{-i}, leads to a tie. Consider first the case in which i is not pivotal, and the social choice is $c(R^i, R^{-i}) = y$. Since i is not pivotal, lying about this preferences would not affect the outcome. Hence, he has no incentive to lie. Suppose now that i is pivotal, and without loss of generality, suppose that xR^iy. Then, $c\left(R^i, R^{-i}\right) = x$ while lying and reporting xP^iy yields (call it $\tilde{R}^i$) $c(\tilde{R}^i) = y$. But since xR^iy, i has no incentive to lie and report $\tilde{R}^i$. Thus, strategy proof is satisfied.
- ▷ Non-dictatorial. Suppose $c^*(R) = x$. This must mean that for at least $(n+1)/2$ individuals, xR^iy. But none of these individuals are dictators since for any such individual i, we may always change the preferences of others such that for $(n+1)/2$ individuals, yR^ix so that $c^*(R) = y$.