Theory of Income
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(Fernando Alvarez)

Notes on OLG models

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1 Maths preliminaries

1.1 CRS production functions

Definition 1.1. (*Homogeneity*). The function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous of degree* $k \in \mathbb{N}$ *if*

$$F(r\mathbf{x}) \equiv r^k F(\mathbf{x}), \ \forall r > 0,$$

where $\mathbf{x} \in \mathbb{R}^n$.

When F is homogeneous of degree one, we say that F exhibits c*onstant returns to scale.* A classic example is the Cobb-Douglas production function,

$$F(K, L) \coloneqq K^\alpha L^{1-\alpha}, \ \alpha \in [0, 1].$$

In Theory of Income, you will almost always work with production function of the form $F(K, L)$ so let us work with this.

Fact 1.1. *Suppose $F(K, L)$ exhibits constant returns to scale. Then, its partial derivatives are homogenous of degree zero.*

Proof. By definition of constant returns to scale,

$$rF(K, L) \equiv F(rK, rL), \ \forall r > 0. \tag{1.1}$$

Differentiating with respect to $i \in \{L, K\}$ yields

$$\begin{aligned} rF_i(K, L) &= rF_i(rK, rL) \\ \Leftrightarrow F_i(K, L) &= F_i(rK, rL), \end{aligned}$$

where $F_i \coloneqq \partial F/\partial i$, $i \in \{L, K\}$. (In fact, if F is homgenous of degree k, then its partial derivatives will always be homogenous of degree $k - 1$.) ∎

You will use this *all the time* when you're dealing with CRS production functions in the context of RBC/NCG models. No need to remember the proof but remember this as a fact!

To be specific, from the firm's problem, you will usually get something like

$$F_K(K_t, L_t) = \nu_t.$$

To pin down the equilibrium, you will then use the fact that F_K is homogenous of degree zero so that we can write

$$F_K\left(\frac{K_t}{L_t}, 1\right) \equiv F_K(K_t, L_t) = \nu_t,$$

and deal mainly with the capital-to-labour ratio, $\kappa_t \coloneqq K_t/L_t$. Very often, you will assume inelastic supply of labour in which case we generally assume $L_t \equiv 1$.

Fact 1.2. *(Euler's theorem) Suppose $F(K, L)$ exhibits constant returns to scale. Then,*

$$F(K, L) = KF_K + LF_L$$

Proof. Differentiate both sides of (1.1) with respect to r. ∎

Used less often than the previous claim but still useful. Why? Because in 'standard' setting F_K equals the rental rate (r) and F_L equals the wage rate (w). So this allows us to write the production function as

$$F(K, L) = rK + wL.$$

1.2 Strict concavity

When we check for strict concavity, we generally just check the sign of the second derivative (strictly negative). But let's just make sure we know what it means for multivariate functions to be strictly concave. Now, the official definition is the following.

Definition 1.2. (*Strict concavity*) Suppose X is a convex subset of $\mathbb{R}^n$. Then, the function $F : X \to \mathbb{R}$ is said to be strictly concave on the set X if

$$F((1-\lambda)x + \lambda x') > (1-\lambda)F(x) + \lambda F(x'), \ \forall x, x' \in X, \ \forall \lambda \in (0, 1).$$

The important takeaway from above is that the definition does not rely on derivatives! Let us now suppose that F is twice-differentiable. Then, the following is a *sufficient* (but not necessary!) condition for strict concavity.[1]

Proposition 1.1. *Suppose X is a convex subset of $\mathbb{R}^n$. Then, the twice-differentiable function $F : X \to \mathbb{R}$, is said to be strictly concave on the set X if the Hessian of F, denoted $H(x)$, is negative definite for all $x \in X$.*

If we are only concerned about weak concavity, then that $H(x)$ is negative semidefinite is both necessary and sufficient (we're still assuming F to be twice differentiable). So what goes wrong?

Example 1.1. Suppose $F(x) = -x^4$. This is a strictly concave function (plot) BUT,

$$F_1 = -4x^3, \ F_{11} = -12x^2$$

and so at $F_{11}(0) = 0$. Hence, its Hessian is not negative definite for all $x \in X$.

So what does this mean?

▷ If the Hessian is negative definite for all x, then the function is strictly concave.

[1] The Hessian of F at point x is given by

$$H(x) := \begin{bmatrix} F_{11}(x) & F_{12}(x) & \cdots & F_{1n}(x) \\ F_{21}(x) & F_{22}(x) & \cdots & F_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1}(x) & F_{n2}(x) & \cdots & F_{nn}(x) \end{bmatrix}.$$

- ▷ If the Hessian is not negative semidefinite for all x, then the function is not concave and so it is of course not strictly concave.
- ▷ If the Hessian is negative semidefinite for all x but the Hessian is not negative definite for all values of x, then the function may or may not be strictly concave—you need to go back to the initial definition of strict concavity to determine whether the function is strictly concave or not.

Exercise 1.1. Is the Cobb-Douglas production function, $F(K,L) = K^{\alpha}L^{1-\alpha}$, $\alpha \in [0,1]$, strictly concave? What if we restricted the domain to (i) either $K > 0$ or $L > 0$; (ii) $K, L > 0$?

Hint: The Hessian is given by

$$H(K,L) = \begin{bmatrix} F_{KK}(x) & F_{KL}(x) \\ F_{LK}(x) & F_{LL}(x) \end{bmatrix}.$$

For this to be negative definite, we need

$$\begin{aligned} F_{KK} &< 0, \\ F_{KK}F_{LL} - F_{KL}F_{LK} &> 0. \end{aligned}$$

1.3 Inada conditions

This is one of the innate knowledge in economics that Fernando assumes you have—something that I hadn't heard of at all before coming to UChicago!

Inada conditions are regularity conditions on the production function that ensures that macro models behave nicely (stability/uniqueness).

Fact 1.3. *(Inada conditions) We say that the production function $F(K,L)$ satisfies the Inada conditions if:*

- ▷ $F(0,0) = 0$;
- ▷ F *is concave in* (K,L); *i.e.* $F_{xx} \leq 0$ *and the Hessian is negative semidefinite.*
- ▷ $\lim_{x\to 0} F_x = +\infty$ *for all* $x \in \{K,L\}$;
- ▷ $\lim_{x\to \infty} F_x = 0$ *for all* $x \in \{K,L\}$.

Concavity assumption can be thought of as ensuring uniqueness and the last two conditions on the partial derivatives ensure existence. [Why?]

Remark 1.1. Depending on the professors/questions, Inada conditions can encompass different assumptions. But the point is, if you see Inada conditions, then 99% of the time you can assume uniqueness and existence!

1.4 Proof by induction

You will come across proof by induction in a number of topics (in both Theory of Income and Price Theory).

Suppose we want to prove a statement about an arbitrary number $n \in \mathbb{N}$.

(i) Show that the statement holds when $n = 1$.

(ii) Now, make an *induction hypothesis* that the statement holds for when $n = k$. Then, show that the statement holds for $n = k + 1$.

(iii) Having followed these two steps, write 'Therefore, by induction, the statement holds'.

So let's do a purely mathematical proof followed by the proof we did in the OLG model in class.

Claim 1.1. $\sum_{i=1}^{n} i = n(n+1)/2$.

Proof. We prove by induction.

▷ Step 1: Let $n = 1$, then

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = 1.$$

Thus, the statement holds.

▷ Step 2: Suppose $n = k$ hold; i.e.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

We want to show that the statement holds when $n = k + 1$. Observe that

$$\begin{aligned}\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ = \frac{k(k+1) + 2(k+1)}{2} &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

Hence, we see that the statement does indeed hold for when $n = k + 1$.

▷ Step 3: By induction, the claim is true. ■

Proposition 1.2. *In the OLG model from class, any competitive equilibrium is autarky (i.e. no trade occurs).*

Proof. We will show that

$$x_j^i = e_j^i, \forall i, j.$$

We prove by induction.

▷ Step 1: Let us first consider the initial old. Since the initial old only cares about consumption in period 1, he consumes all his endowment in period 1; i.e.

$$x_1^0 = e_1^0.$$

The market-clearing condition in period 1 tell us that

$$x_1^0 + x_1^1 = e_1^0 + e_1^1.$$

Combining the two, we get that

$$x_1^1 = e_1^1.$$

So what we have shown is that

$$x_1^0 = e_1^0 \Rightarrow x_1^1 = e_1^1$$

and that the left-hand side is true.

- ▷ Step 2: Now, consider period $t = k$ and make the induction hypothesis that $x_k^{k-1} = e_k^{k-1}$. Substituting this into the market-clearing condition gives

$$x_k^{k-1} + x_k^k = e_k^{k-1} + e_k^k,$$

which implies that

$$x_k^k = e_k^k.$$

That is, we've now shown that

$$x_k^{k-1} = e_k^{k-1} \Rightarrow x_k^k = e_k^k.$$

- ▷ Step 3: By induction, it follows that

$$\begin{aligned}\left(x_t^t, x_{t+1}^t\right) &= \left(e_t^t, e_{t+1}^t\right), \ \forall t \geq 1 \\ x_1^0 &= e_1^0.\end{aligned}$$

That is, every agent consumes their own endowment in every period; i.e. no trade occurs in the competitive equilibrium. ■

1.5 Implicit function theorem

We usually use the implicit function theorem to see how the optimal choice changes with parameters. Informally, we just assume that the optimal choice variable is a function of the parameter of interest, and then take the (partial) derivative.

In PS1 Q2, you'll use the implicit function theorem slightly differently. You'll use it to argue that the optimal savings can be expressed a function of current capital stocks exists (locally) by showing that the derivatives of the optimum savings with respect to K_t is well defined (i.e. no division by zero).

1.6 Elasticity of substitution

Another innate economics knowledge that Fernando assumes. Let c_1 and c_2 denote consumption in period 1 and 2. Elasticity of substitution is defined as

$$\begin{aligned}\sigma &:= \frac{d \ln \left(c_2/c_1\right)}{d \ln \left(MU_1/MU_2\right)} \\ &\equiv \frac{d\left(c_2/c_1\right)}{d\left(MU_1/MU_2\right)} \frac{MU_1/MU_2}{c_2/c_1}.\end{aligned}$$

The elasticity of substitution of tells you how the MRS between two goods change with the ratio of the quantities. You'll see this also in Price Theory I.

2 OLG

2.1 Takeaways

- ▷ In the OLG model, the First Welfare Theorem (i.e. any CE is Pareto optimal) fails whenever the interest rate is 'sufficiently low'.
  - ▹ Without population/productivity growth, FWT fails whenever the interest rate is negative.
  - ▹ With population/productivity growth, FWT fails whenever the interest rate is below population plus productivity growth (PS1: Q6)
  - ▹ Note that the fact that the 'proof' of FWT fails is not a sufficient condition that competitive equilibrium (CE) is inefficient (e.g. CE is still optimum when interest rate is exactly zero in the model from class).
- ▷ Since the competitive equilibrium is not Pareto optimal, there is scope for intervention.
  - ▹ With Fernando, we'll see that introducing pay-as-you-go social security (i.e. payments from the young to the old) can lead to a Pareto improvement.
  - ▹ With Shimer, you'll see that introducing fiat money can also lead to a Pareto improvement.

2.2 (Hopefully) Useful concepts from the OLG model

2.2.1 Interpreting Arrow-Debreu prices as intertemporal prices/interest rates

In the OLG model, we wrote down generation $t \geq 1$'s problem as follows

$$\max_{x_t^t, x_{t+1}^t \in \mathbb{R}} \quad u\left(x_t^t, x_{t+1}^t\right)$$
$$s.t. \quad p_t x_t^t + p_{t+1} x_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t. \tag{2.1}$$

In this problem, you should interpret p_t as the price of period-t good relative to some base period (usually, we set $p_1 := 1$). So the ratio of p_t's represents the relative price of goods across periods—i.e. the interest rate.

We define the time-t interest rate as:[2]

$$\frac{1}{1+r_t} \equiv \frac{p_{t+1}}{p_t}. \tag{2.2}$$

Does this definition make sense? The left-hand side is how much one unit of period-$t+1$ goods is worth in terms of period-t goods. The numerator on the right-hand side says how much a unit of period-$t+1$ good costs. Dividing this by how much a unit of good is worth in period-t then gives you the interest rate.

Here's another way to see (2.2). Let us rewrite the agent's per-period budget constraints. When he is young, he can either consume his endowment or save and he has endowment of e_t^t. When he

[2]Some people will define it as $(1+r_{t+1})^{-1} \equiv p_{t+1}/p_t$ so just be careful which convention each professor uses.

is old, he can either consume out of his endowment e_{t+1}^t or from his saving (of course, he has no incentive to save when he is old). So the *dynamic budget* constraints are

$$\begin{aligned} x_t^t + s_t &= e_t^t, \\ x_{t+1}^t &= (1 + r_t)\, s_t + e_{t+1}^t, \end{aligned}$$

where I've assumed that the budget constraints bind. Notice that prices do not appear in the budget constraints because we are defining a budget constraint for each period. We can eliminate s_t from the first budget constraint using the second. Rearranging the second budget constraint gives you

$$s_t = \frac{x_{t+1}^t}{1 + r_t} - \frac{e_{t+1}^t}{1 + r_t}.$$

Substituting this into the first budget constraint gives you

$$\begin{aligned} x_t^t + \left(\frac{x_{t+1}^t}{1 + r_t} - \frac{e_{t+1}^t}{1 + r_t} \right) &= e_t^t \\ \Leftrightarrow x_t^t + \frac{x_{t+1}^t}{1 + r_t} &= e_t^t + \frac{e_{t+1}^t}{1 + r_t}. \end{aligned}$$

Observe that above is equivalent to (2.1) when we divide both sides by p_t and use (2.2).

Remark 2.1. (Answer to the question in the TA class). In the first Arrow-Debreu set up, we are assuming that all trading takes place at the beginning of time (through purchase/sale of contingent plans). No need for ex post trading in this setting.

In the second, we are instead assuming that trade takes sequentially in spot markets for consumption goods that open in each period. We also are assuming that there is an asset market through which individuals save.

The two approaches are equivalent!

2.2.2 Finding the competitive equilibrium

Let's go through the maths at least once in full. As in class, we assume log time-separable utility

$$u^t \left(x_t^t, x_{t+1}^t \right) \equiv v\left(c_y, c_o\right) := (1 - \beta) \ln c_y + \beta \ln c_o, \ \beta \in (0, 1)\,,$$

and endowment as

$$e_t^t = 1 - \alpha, \ e_{t+1}^t = \alpha, \ \alpha \in (0, 1)\,.$$

We already know that in any CE, consumption equals endowment in each period; i.e.

$$c_y^* = 1 - \alpha, \ c_o^* = \alpha. \tag{2.3}$$

But remember that, to define a competitive equilibrium, we also need to define a vector of prices. Given that we know the competitive equilibrium allocations, we therefore need to find prices such that the agents would find it optimal to choose not to trade.

The agent's problem is

$$\begin{aligned}\max_{c_y, c_o} \quad & (1-\beta)\ln c_y + \beta \ln c_o \\ s.t. \quad & c_y + \frac{c_o}{1+r_t} = 1 - \alpha + \frac{\alpha}{1+r_t}.\end{aligned}$$

Write the Lagrangian as

$$\mathcal{L} = (1-\beta)\ln c_y + \beta \ln c_o + \lambda \left[1 - \alpha + \frac{\alpha}{1+r_t} - c_y - \frac{c_o}{1+r_t}\right].$$

The first-order conditions are

$$\begin{aligned}\frac{1-\beta}{c_y^*} &= \lambda, \\ \frac{\beta}{c_o^*} &= \frac{\lambda}{1+r_t}.\end{aligned}$$

Eliminating λ gives

$$\frac{c_y^*}{c_o^*} = \frac{1-\beta}{\beta}\frac{1}{1+r_t}. \tag{2.4}$$

To obtain the equilibrium interest rate, we use (2.3), which allows us to write the above condition as

$$\begin{aligned}\frac{1-\alpha}{\alpha} &= \frac{1-\beta}{\beta}\frac{1}{1+r_t} \\ \Leftrightarrow r_t &= \frac{\alpha-\beta}{\beta(1-\alpha)}, \ \forall t. \end{aligned} \tag{2.5}$$

So the equilibrium interest rate is a constant and can be positive or negative.

Remark 2.2. You can also get to the same conclusion by solving explicitly for the optimal c_y^* and c_o^* (substitute (2.4) into the budget constraint). Then use the market clearing condition (i.e. $c_y^* + c_o^* = 1$) to obtain the equilibrium interest rate.

Rewrite (2.4) as

$$c_y^* = \frac{1-\beta}{\beta}\frac{1}{1+r_t}c_o^*$$

and substituting into the budget constraint yields

$$\begin{aligned}\frac{1-\beta}{\beta}\frac{1}{1+r_t}c_o^* + \frac{c_o^*}{1+r_t} &= 1 - \alpha + \frac{\alpha}{1+r_t} \\ \Leftrightarrow \left(\frac{1-\beta}{\beta} + 1\right)c_o^* &= ((1-\alpha)(1+r_t) + \alpha) \\ \Leftrightarrow c_o^* &= \beta(1+r_t)\left(1 - \alpha + \frac{\alpha}{1+r_t}\right).\end{aligned}$$

This, in turn, implies that

$$c_y^* = (1-\beta)\left(1 - \alpha + \frac{\alpha}{1+r_t}\right).$$

The market clearing condition tells us that

$$\begin{aligned}
1 &= \alpha + 1 - \alpha \\
&= c_o^* + c_y^* \\
&= \beta\left(1 + r_t\right)\left(1 - \alpha + \frac{\alpha}{1 + r_t}\right) + \left(1 - \beta\right)\left(1 - \alpha + \frac{\alpha}{1 + r_t}\right) \\
&= \left[\beta\left(1 + r_t\right) + \left(1 - \beta\right)\right]\left(1 - \alpha + \frac{\alpha}{1 + r_t}\right). \qquad (2.6)
\end{aligned}$$

We can rearrange this to obtain r_t but let's take a slightly easier route—let's just check that when we subsitute in (2.5), which we can rearrange as,

$$1 + r_t = \frac{1 - \beta}{\beta}\frac{\alpha}{1 - \alpha},$$

to the right-hand side of (2.6) equals one:

$$\begin{aligned}
&\left[\beta\frac{1 - \beta}{\beta}\frac{\alpha}{1 - \alpha} + \left(1 - \beta\right)\right]\left(1 - \alpha + \alpha\frac{\beta}{1 - \beta}\frac{1 - \alpha}{\alpha}\right) \\
&= \left(1 - \beta\right)\left(1 + \frac{\alpha}{1 - \alpha}\right)\left(\left(1 + \frac{\beta}{1 - \beta}\right)\left(1 - \alpha\right)\right) \\
&= \frac{1 - \beta}{1 - \alpha}\frac{1 - \alpha}{1 - \beta} = 1
\end{aligned}$$

Exercise 2.1. Is the competitive equilibrium unique?

2.2.3 Solving for the optimal savings rate

Above, we found the interest rate such that agents would maximise utility by consuming exactly their endowment. This is equivalent to solving for the interest that ensures that the optimal savings rate is zero. To see this, define saving (by the young) as $s_t = e_t^t - c_y$. We can then write the agent's problem as

$$\max_{s_t} \quad \left(1 - \beta\right)\ln\left(\left(1 - \alpha\right) - s_t\right) + \beta\ln\left(\left(1 + r_t\right)s_t + \alpha\right).$$

The first-order condition then gives us that

$$\begin{aligned}
\beta\left(1+r_t\right)\frac{1}{\left(1+r_t\right)s_t^*+\alpha} &= \left(1-\beta\right)\frac{1}{1-\alpha-s_t^*} \\
\Leftrightarrow \beta\left(1+r_t\right)\left(1-\alpha-s_t^*\right) &= \left(1-\beta\right)\left(\left(1+r_t\right)s_t^*+\alpha\right) \\
\Leftrightarrow \beta\left(1+r_t\right)\left(1-\alpha\right)-\alpha\left(1-\beta\right) &= \beta\left(1+r_t\right)s_t^*+\left(1-\beta\right)\left(1+r_t\right)s_t^* \\
\Leftrightarrow s_t^*\left(1+r_t\right) &= \beta\left(1+r_t\right)\left(1-\alpha\right)-\alpha\left(1-\beta\right) \\
\Leftrightarrow s_t^* &= \left(1-\alpha\right)\beta-\frac{\alpha\left(1-\beta\right)}{\left(1+r_t\right)} \qquad (2.7) \\
&= \left(1-\alpha\right)\beta+\left[\left(1-\alpha\right)-\left(1-\alpha\right)\right]-\frac{\alpha\left(1-\beta\right)}{\left(1+r_t\right)} \\
&= \left(1-\alpha\right)-\left(1-\alpha\right)\left(1-\beta\right)-\frac{\alpha\left(1-\beta\right)}{\left(1+r_t\right)} \\
&= \left(1-\alpha\right)-\left(1-\beta\right)\left[\left(1-\alpha\right)+\frac{\alpha}{\left(1+r_t\right)}\right]. \qquad (2.8)
\end{aligned}$$

Now, let's figure out the interest rate that makes $s_t^*=0$.

$$\begin{aligned}
0 &= \left(1-\alpha\right)-\left(1-\beta\right)\left[\left(1-\alpha\right)+\frac{\alpha}{\left(1+r_t\right)}\right] \\
\Leftrightarrow \frac{1-\alpha}{1-\beta} &= \left(1-\alpha\right)+\frac{\alpha}{\left(1+r_t\right)} \\
\Leftrightarrow 1+r_t &= \frac{\alpha\left(1-\beta\right)}{\left(1-\alpha\right)-\left(1-\alpha\right)\left(1-\beta\right)} \\
&= \frac{\alpha\left(1-\beta\right)}{\left(1-\alpha\right)\beta} \\
\Leftrightarrow r_t &= \frac{\alpha-\beta}{\beta\left(1-\alpha\right)},\ \forall t.
\end{aligned}$$

Voila!

2.2.4 Income and substitution effects

Changes in interest rates have both income and substitution effects. Consider an increase in the interest rate in a two-period setting (we also assume consumption in both periods are normal goods).

- ▷ Income effect: If the agent is a saver, a higher interest rate means that saving is worth more, so increases consumption in both periods. If the agent is a borrower, a higher interest rate reduces consumption in both periods.
- ▷ Substitution effect: a higher interest rate means that next period consumption is relatively cheaper. So this incentives agents to substitute today's consumption with tomorrow's.

With log utility, the two effects 'cancel out'. What does this mean? It means that agent's always consume a constant share of the present value of their wealth.

Specifically, recall (2.8). Notice that the term inside the square brackets, $(1-\alpha)+\alpha/(1+r_t)$, is the present value of the total endowment that each agent receives. The expression above means that the optimal saving for the young is his endowment, $1-\alpha$, less a share of the present value of total endowments, where the share depends on his preference for consumption when young, $(1-\beta)$.

This is a feature of using the log utility function, which means that the substitution effect and the income effect from changes in the interest rate exactly offset each other.

2.2.5 Best symmetric allocation

When we solve for the symmetric equilibrium, we are essentially solving the planner's problem—maximising each generation's utility subject to feasibility—i.e. no prices!

To show that, when interest rates were negative, introduction of a pay-as-you-go social security system that implements the best symmetric allocation can improve welfare, we only checked that the initial old was made better off by such a system. Why? This uses the fact that the competitive equilibrium allocation is a symmetric allocation. So when we solve for the best symmetric allocation and find that it is different from the CE allocation, it must be the case that the best symmetric allocation is preferred over the CE allocation. This is a *revealed preference* argument. Of course, you can do the maths but that takes a lot of effort!

The agent's problem is

$$\max_{c_y, c_o} \quad (1-\beta)\ln c_y + \beta \ln c_o$$
$$s.t. \quad c_y + c_o = 1$$

and the solution is

$$c_y^* = 1 - \beta.$$
$$c_o^* = \beta.$$

2.3 Social security

Consider a tax policy indexed by a single parameter τ whereby the young are taxed τ and this amount is given to the old. For positive τ, this tax policy resembles a pay-as-you-go social security system. By construction, this policy is feasible since tax equals subsidies in each period.

With this policy, the (post-tax) endowments are now:

$$\begin{aligned} e_i^i &= (1-\alpha) - \tau, \quad && \forall i \geq 1, \\ e_{i+1}^i &= \alpha + \tau, && \forall i \geq 0. \end{aligned}$$

Since the consumers consume all their endowments in each period, consumption equal to the endowments above represents the competitive equilibrium with the social security system.

Exercise 2.2. Suppose you propose a pay-as-you-go social security system for the OLG economy as described above.

- ▷ Under what parameter values will this policy produce a competitive equilibrium that Pareto dominates the competitive equilibrium without the policy?
- ▷ What interest rate does the competitive equilibrium without the policy need to be so that the introduction of social security produces a competitive equilibrium that Pareto dominates the one without policy?

Solution. Recall that when $\beta > \alpha$, the best symmetric allocation Pareto dominates the competitive equilibrium allocation. Thus, in such a situation, we can set $\tau = \beta - \alpha > 0$ to achieve the best symmetric allocation under competitive equilibrium. Note that interest rate is negative when $\beta > \alpha$; i.e. the policy intervention can lead to a Pareto improvement when interest rates are negative.

Suppose we introduce a social security system when interest rate is positive. Who gains/loses? Clearly, since $\tau > 0$ under a social security system, the initial old is better off with than without the system since he receives something (τ) for nothing. Note that it is still the case that the equilibrium is autarky; i.e. equilibrium interest rate is such that optimal savings is zero.

The quantity $s_t \coloneqq e_t^t - \tau - x_t^t$ can be interpreted as the saving by generation i while they are young. Note that

$$e_t^t - \tau - s_t = (1-\alpha) - \tau - s_t.$$

In this case, the budget constraint is given by

$$\begin{aligned}
p_t x_t^t + p_{t+1} x_{t+1}^t &= p_t \left(e_t^t - \tau\right) + p_{t+1}\left(e_{t+1}^t + \tau\right) \\
\Rightarrow \frac{p_{t+1}}{p_t} x_{t+1}^t &= \underbrace{\left(e_t^t - \tau - x_t^t\right)}_{\coloneqq s_t} + \frac{p_{t+1}}{p_t}\left(e_{t+1}^t + \tau\right) \\
\Rightarrow x_{t+1}^t &= \left(\frac{p_{t+1}}{p_t^t}\right)^{-1} \left(e_t^t - \tau - x_t^t\right) + \left(e_{t+1}^t + \tau\right) \\
&= \left(1 + \bar{r}\right) s_t + \alpha + \tau.
\end{aligned}$$

Hence, we can rewrite the agent's utility function in terms of s_t:

$$\max_{s_t}\ (1-\beta)\ln\left((1-\alpha) - \tau - s_t\right) + \beta \ln\left[\left(1+\bar{r}\right) s_t + \alpha + \tau\right].$$

First-order condition is then

$$\begin{aligned}
\frac{\beta\left(1+r_t\right)}{\left(1+r_t\right) s_t^* + \alpha + \tau} &= \frac{1-\beta}{(1-\alpha) - \tau - s_t^*} \\
\Rightarrow \beta\left(1+r_t\right)\left((1-\alpha) - \tau - s_t^*\right) &= (1-\beta)\left(\left(1+r_t\right) s_t^* + \alpha + \tau\right) \\
\Leftrightarrow s_t^{t*}\left(1+r_t\right)\left(\beta + (1-\beta)\right) &= \beta\left(1+r_t\right)\left(1 - (\alpha+\tau)\right) - (1-\beta)(\alpha+\tau) \\
&= s_t^*\left(1+r_t\right) \\
\Rightarrow s_t^* &= \beta\left(1 - (\alpha+\tau)\right) - \frac{(1-\beta)(\alpha+\tau)}{1+r_t} \\
&= \beta\left(1 - (\alpha+\tau)\right) + \left[\left(1 - (\alpha+\tau)\right) - \left(1 - (\alpha+\tau)\right)\right] \\
&\quad - \frac{(1-\beta)(\alpha+\tau)}{1+r_t} \\
&= (1-\alpha-\tau) - (1-\beta)(1-\alpha-\tau) \\
&\quad - \frac{(1-\beta)(\alpha+\tau)}{1+r_t} \\
&= (1-\alpha-\tau) - (1-\beta)\left[(1-\alpha-\tau) + \frac{\alpha+\tau}{1+r_t}\right].
\end{aligned}$$

This is the same as (2.8) when $\tau = 0$. The equilibrium interest rate that ensures $s_t^* = 0$ is given by

$$
\begin{aligned}
\frac{\beta}{1-\beta}\frac{1-\alpha-\tau}{\alpha+\tau} &= \frac{1}{1+r_t} \\
\Rightarrow r_t &= \frac{1-\beta}{\beta}\frac{\alpha+\tau}{1-\alpha-\tau} - 1 \\
&= \frac{(1-\beta)(\alpha+\tau) - \beta(1-\alpha-\tau)}{\beta(1-\alpha-\tau)} \\
&= \frac{\alpha-\beta+\tau}{\beta(1-\alpha-\tau)},
\end{aligned}
$$

which is the same as the case without social security when $\tau = 0$.

3 Practice questions

3.1 Problem set 1

* indicates question that, in my opinon, are more relevant for exams.

Problem 1: Endowment economy*

- ▷ Q1.1 Walras' Law: if there are N budget constraints in the economy, and if $N-1$ constraints are satisfied, then the last constraint is automatically satisfied.
- ▷ Q1.2 Ricardian Equivalence: changes in tax system that leaves the total wealth of agents unchanged has no effect on consumption.
- ▷ Q1.2.6 Describes how the government finances the social security system.
- ▷ Technique
  - ▷ Q1.2.3/4 Use induction

Problem 2: Production economy*

- ▷ Introduces production (with capital and labour) to the OLG model from class. Labour supply is exogenous.
- ▷ The question examines whether the steady-state stock of capital is lower in an economy with or without social security.
- ▷ Q2.4.6 The result depends on how the optimal savings change with capital, ds^*/dK_t:
  - ▷ if $ds^*/dK_t \in (0, 1)$, then a higher social security transfers, τ, reduces the steady-state level of capital.
  - ▷ if $ds^*/dK_t > 1$, then a higher τ increases the steady-state level of capital. However, steady-state here is not stable.
- ▷ Techniques
  - ▷ Q2.1 How to optimise element by element when we only know that the production function is CRS (and not concave).
  - ▷ Q2.2 No arbitrage argument to show that $v_{t+1} = r_t + \delta$. Suppose that an agent borrows money to fund a purchase of (a unit of) capital in period t. The agent rents the capital out. In period $t + 1$, the agent: (i) receives the rental rate v_{t+1}; (ii) pays r_t in interest; and (iii) loses δ in depreciation. For there to be no arbitrage, then it must be that $v_{t+1} = r_t + \delta$. If, in fact, $v_{t+1} > r_t + \delta$, the agent would borrow more to purchase more capital. If, on the other hand, $v_{t+1} < r_t + \delta$, then those who would owns capital would wish to sell.
  - ▷ Q2.2.4 Use of the Implicit Function Theorem to check for the existence of an implicit function.

Problem 3: Production economy: An example*

- ▷ In the pure-endowment economy, interest rate always adjust to ensure zero savings in equilibrium. But here, agents save in the form of capital.

Problem 4: Heterogeneity

- ▷ Heterogeneity (in endowments) among agents . Each generation has N different types of endowments α_n so each cohort is a pair $i = (t, n)$.
- ▷ Q4.2 In equilibrium, there is no trade across generations.
- ▷ Q4.4 If interest rates are positive, an agent that is relatively rich when young enjoy higher levels of lifetime consumption and lifetime utility. If $r > 0$ then since $1 + r_t = p_t/p_{t+1} > 1$, it means $p_t > p_{t+1}$; i.e. prices are falling over time. That is, individual is relatively rich in a period where goods are relatively more expensive (i.e. when they are young).
- ▷ Q4.5 Interest rate is determined by the average α (does not depend on distribution of α).
- ▷ Q4.7 $\bar{\alpha} < \beta$ means interest rate is negative and CE is not PO.

Problem 5: OLG model with multiple periods

- ▷ Agents living multiple periods in the basic model with homogeneous generations. We will allow for general preferences and endowments.
- ▷ Confusing time subscripts. Each individual lives for N periods and are born in period t. So they die after period $t + N - 1$. The initial old were born in$t = -N + 2, -N + 3, \dots, -1, 0$.
- ▷ Q5.4 If equilibrium interest rates are negative, then the equilibrium is not Pareto optimal.

Problem 6: Population growth and social security*

- ▷ Introduce population growth: $N_{t+1} = (1 + n) N_t$, $N_0 = 1$.
- ▷ Q6.5 Changes in n are isomorphic to changes in the price of future consumption: $1/(1 + n)$ is the price of future consumption relative to current consumption from the point of view of the best symmetric allocation. By separability and concavity of the agent's utility function, combined with the fact that we only have two goods, we know that consumption in both periods are normal goods and, moreover, they are net substitutes (in the compensated sense). Thus, a fall in relative price of future consumption generates two effects:
  - ▷ a positive income effect—given that agents are net savers in the best symmetric allocation (and thereby they benefit from a rise in the rate of return of social security, n), By normality, this positive income effect induces a rise in both current and future consumption.
  - ▷ a substitution effect—due to the fact that future consumption is relatively cheaper. This effect induces a rise in future consumption (since the compensated demand curve of a good is negatively sloped with respect to its own price, i.e. law of demand holds) and a fall in current consumption (since the goods are net substitutes).

In this case, the substitution effect on current consumption outweighs the income effect so that consumption of the young falls with n. Since $c_y = c_t^t - \tau$, this implies that per-capital taxes (i.e. per capita savings) should rise with n as we found above.

▷ Q6.6 A fall in population growth will always make at least one generation worse off

3.2 2016 Mid-Term Q1: OLG with finite horizon economy

Next TA class.

3.3 2016 Mid-Term Q2: 'OLG-like' on a circle

Next TA class.