Search Model

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Road Map

- Linear search model - Planner's Problem
  - Continuous time Bellman Equation — (HJB)
- Pissarides model with aggregate productivity shocks.

Linear search model

- The model is the one described in class:
- Planner's chooses the share of unemployed workers and vacancies to maximize discounted total welfare subject to feasibility.
- Here feasibility $\implies$ law of motion for unemployment (remember the exogenous nature of the separation rate).
- Welfare here $\implies$ Total net output.

Planner's Problem

Feasibility:

$$u_{t+\Delta} = u_t - m(u_t, v_t)\Delta + (1-u_t)\chi\Delta.$$

Rearrange terms to write:

$$\frac{u_{t+\Delta} - u_t}{\Delta} = -m(u_t, v_t) + (1-u_t)\chi,$$

and let $\Delta \to 0$, thus:

$$\dot{u}(t) = -m(u(t), v(t)) + (1-u(t))\chi.$$

Total output per dt **time:**

$$(1-u(t))(z-\gamma) - v(t)c$$

Planner's Problem (cont.)

The planner's solve:

$$W(u_0) = \max_{u,v} \int_0^{\infty} e^{-rt} \{(1-u(t))(z-\gamma) - v(t)c\} dt$$
$$s.t.\ \dot{u}(t) = -m(u(t), v(t)) + (1-u(t))\chi$$
$$u(0) = u_0.$$

Planner's Problem (cont.)

It is easiest to work with the sequence problem. As a reminder we derive the continuous time version of the Bellman equation.
If time where discrete:

$$W(u_t) = \max_{u_{t+\Delta}, v_t} \left\{ \Delta \left[(1-u_t)(z-\gamma) - v_t c \right] + \frac{1}{1+r\Delta} W(u_{t+\Delta}) \right\}$$
$$s.t.\ u_{t+\Delta} = u_t + \Delta \left[(1-u_t)\chi - m(u_t, v_t) \right].$$

We need to solve:

$$W(u_t) = \max_{v_t} \left\{ (1-u_t)(z-\gamma) - v_t c + \dots \right.$$
$$\left. \dots + \frac{1}{1+r\Delta} W(u_t + \Delta \left[(1-u_t)\chi - m(u_t, v_t) \right]) \right\}$$

Planner's Problem (cont.)

Use Taylor's Theorem to write $W\left(u_t + \Delta\left[\left(1-u_t\right)\chi - m\left(u_t, v_t\right)\right]\right)$ as:

$$W\left(u_t\right) + W'\left(u_t\right)\left[u_t + \Delta\left[\left(1-u_t\right)\chi - m\left(u_t, v_t\right)\right] - u_t\right] + o\left(\Delta\right)$$

The BE is then:

$$\begin{aligned} W\left(u_t\right)\left(1+r\Delta\right) = \max_{v_t}\{&\left(1+r\Delta\right)\Delta\left[\left(1-u_t\right)\left(z-\gamma\right) - v_t c\right] + \dots \\ &\dots + W\left(u_t\right) + \Delta W'\left(u_t\right)\left[\left(1-u_t\right)\chi - m\left(u_t, v_t\right)\right] + \dots \\ &\dots + o\left(\Delta\right)\}. \end{aligned}$$

Since $W\left(u_t\right)$ is given in the max operator we can re write this as:

$$\begin{aligned} rW\left(u_t\right) = \max_{v_t}\{&\left(1+r\Delta\right)\left[\left(1-u_t\right)\left(z-\gamma\right) - v_t c\right] \\ &+ W'\left(u_t\right)\left[\left(1-u_t\right)\chi - m\left(u_t, v_t\right)\right] + \frac{o\left(\Delta\right)}{\Delta}\right\}. \end{aligned}$$

Finally, letting $\Delta \to 0$:

$$rW\left(u\right) = \max_{v}\left(1-u\right)\left(z-\gamma\right) - vc + W'\left(u\right)\left[\left(1-u\right)\chi - m\left(u, v\right)\right].$$

Planner's Problem (cont.)

The first order condition and Envelope are:

$$\begin{aligned}
\{FOC\} &: -c - W'(u)\left[m_v(u,v)\right] = 0 \\
\{EC\} &: rW'(u) = -(z-\gamma) + W''(u)\left[(1-u)\chi - m(u,v)\right] + \dots \\
&\quad \dots - W'(u)\left[\chi + m_u(u,v)\right]
\end{aligned}$$

Conjecture linear solution $\implies$ $W'(u) = -k$ for some k to be determined. Then:

$$\begin{aligned}
\{FOC\} &: c = k\left[m_v(u,v)\right] \\
\{EC\} &: -rk = -(z-\gamma) + k\left[\chi + m_u(u,v)\right].
\end{aligned}$$

Hence the solution satisfies:

$$\frac{(z-\gamma)}{c} - \frac{m_u(u,v)}{m_v(u,v)} = \frac{\chi + r}{m_v(u,v)}.$$

Planner's Problem (cont.)

Assume $m(u,v) = u^{\eta} v^{1-\eta}$. Then:

$$m_v = (1-\eta) u^{\eta} v^{-\eta} = (1-\eta) \frac{u^{\eta} v^{1-\eta}}{v} = (1-\eta) \mu(\theta)$$
$$m_u = \eta u^{\eta-1} v^{1-\eta} = \eta f(\theta) = \eta \theta \mu(\theta).$$

The planner's solution satisfies:

$$(1-\eta) \frac{(z-\gamma)}{c} - \theta \eta = \frac{\chi + r}{\mu(\theta)}.$$

- Note then that if $\phi = \eta$ (bargaining power of worker equal to the elasticity of matching function with respect to unemployment), the equilibrium in the search model coincides with the planner's solution. This condition is feasible since both $\phi, \eta \in (0,1)$, but nothing is ensuring it in the decentralized equilibrium (*Hosios Condition*).
- In general decentralized equilibrium is inefficient: inefficiently high or inefficiently low?
  - If $\phi < \eta$ too few vacancies.
  - If $\phi > \eta$ too many vacancies.

Pissarides with Aggregate Productivity Shocks

- We add aggregate productivity shocks to the standard search model.
- We follow a simplified version of Shimer (AER, 2005) with a fixed in notation.
  - No shocks to separation rate.
- Motivation: the typical search model does poorly generating the business cycle frequency fluctuations in unemployment and job vacancies in response to aggregate shocks.

Shimer (2005)

- Labor productivity z follows a first order Markov process in continuous time.
  - Poisson process with arrival rate λ, $z \to z'$ from a state dependent distribution.
- Agents are risk-neutral and discount utility at rate r.
- Workers can either be:
  - Unemployed $\implies$ flow utility γ and searches for a job.
  - Employed $\implies$ earns wage w faces a separation rate χ.
- Firms uses labor to produce with stochastic productivity $z > \gamma$. To hire needs to open a vacancy at flow cost c.
- Free entry for firms. $\implies$ value of open a vacancy is zero.
- Let $u(t)$ and $v(t)$ denote unemployment and vacancy at t.
- Let $m(u(t), v(t))$ be the matching function, same assumptions we made in class.
- We define θ, $\mu(\theta)$ and $f(\theta)$ in the same way.
- Nash bargaining to determine wages $\implies$ ϕ bargaining power of worker.

Characterization of Equilibrium

We look for an equilibrium in which θ depends only on z.
Define $V^{u}(z), V^{e}(z), V^{J}(z)$ and $V^{v}(z)$ the state-contingent present value of an unemployed worker, employed worker, filled job and vacancy. This values satisfy:

$$rV^{u}(z) = \gamma + f(\theta)\left[V^{e}(z) - V^{u}(z)\right] + \lambda\left\{\mathbb{E}_{z}\left[V^{u}(z')\right] - V^{u}(z)\right\} \tag{1}$$
$$rV^{e}(z) = w(z) + \chi\left[V^{u}(z) - V^{e}(z)\right] + \lambda\left\{\mathbb{E}_{z}\left[V^{e}(z')\right] - V^{e}(z)\right\} \tag{2}$$
$$rV^{J}(z) = z - w(z) - \chi V^{J}(z) + \lambda\left\{\mathbb{E}_{z}\left[V^{J}(z')\right] - V^{J}(z)\right\} \tag{3}$$
$$rV^{v}(z) = -c + \mu(\theta)\left[V^{J}(z) - V^{v}(z)\right] + \lambda\left\{\mathbb{E}_{z}\left[V^{v}(z')\right] - V^{v}(z)\right\}. \tag{4}$$

Remember $V^{v}(z) = 0\ \forall z$.

Characterization of Equilibrium (cont.)

- Sum equation (2) and (3) and subtract equation (1).
- Defining $V(z) = V^J(z) + V^e(z) - V^u(z)$:

$$
\begin{aligned}
rV(z) = z - \gamma - f(\theta)\left[V^e(z) - V^u(z)\right] + \dots \\
\dots - \chi V(z) + \lambda \left\{\mathbb{E}_z\left[V\left(z'\right)\right] - V(z)\right\}.
\end{aligned} \tag{5}
$$

Nash Bargaining solution

The equilibrium wage stipulated between the worker and the firm solves:

$$\max_{w}\left(V^{e}(z)-V^{u}(z)\right)^{\eta}V^{J}(z)^{1-\eta}.$$

From equation (3) :

$$V^{J}(z)=\frac{z-w}{r+\chi+\lambda}+\frac{\lambda}{r+\chi+\lambda}\mathbb{E}_{z}\left[V^{J}(z')\right].$$

From equation (2):

$$V^{e}(z)-V^{u}(z)=\frac{w}{(r+\chi+\lambda)}+\frac{\lambda\mathbb{E}_{z}\left[V^{e}(z')\right]-(r+\lambda)V^{u}(z)}{(r+\chi+\lambda)}$$

FOC implies:

$$\frac{V^{J}(z)}{(1-\eta)}=\frac{V^{e}(z)-V^{u}(z)}{\eta}. \tag{6}$$

Finally, note that using the definition of $V(z)$ yields:

$$\frac{V(z)-V^{J}(z)}{\eta}=\frac{V^{J}(z)}{(1-\eta)}\implies V(z)=\frac{1}{(1-\eta)}V^{J}(z). \tag{7}$$

Characterization of Equilibrium (cont.)

Combining equations (5), (6) and (7) to substitute out $V^e(z) - V^u(z)$:

$$rV(z) = z - \gamma - f(\theta)\eta V(z) - \chi V(z) + \lambda \left\{ \mathbb{E}_z \left[V(z') \right] - V(z) \right\}. \tag{8}$$

Characterization of Equilibrium (cont.)

From the free entry condition and equation (4):

$$c = \mu(\theta) V^J(z).$$

Or using equation (7):

$$c = \mu(\theta)(1-\eta) V(z). \tag{9}$$

Characterization of Equilibrium (cont.)

Combining equations (8) and (9) to eliminate $V(z)$, yields the characterization of the v-u ratio:

$$(r+\chi+\lambda)\frac{c}{\mu(\theta)(1-\eta)} = z-\gamma-\frac{f(\theta)\eta c}{\mu(\theta)(1-\eta)}+\frac{\lambda c}{(1-\eta)}\mathbb{E}_z\left[\frac{1}{\mu(\theta')}\right].$$

Using that $f(\theta)=\mu(\theta)\theta$:

$$\frac{(r+\chi+\lambda)}{\mu(\theta)} = (1-\eta)\frac{z-\gamma}{c}+\lambda\mathbb{E}_z\left[\frac{1}{\mu(\theta')}\right]-\theta\eta.$$