Theory of Income, Fall 2008
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Problem Set 7

1 Varieties of Growth Models

In this problem set we consider three different models. One with no steady states, the Lucas-Uzawa endogenous growth model,a growth model with steady states that are not necessarily stable and a model of durable and non durable goods.

1.1 Lucas-Uzawa endogenous growth model

Let $N(t)$ be the size of population at time t and let

$$\dot{N}(t) = nN(t),$$

so the population growth rate is n. Let the "dynasty" preferences be given by

$$\int_0^\infty N(t) e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt,$$

where $c(t)$ is per-capita consumption. Production is done using a Cobb-Douglas function of capital and time, in efficiency units, i.e.,

$$\dot{K}(t) + N(t) c(t) = A [K(t)]^\alpha [h(t) u(t) N(t)]^{1-\alpha},$$

where $K(t)$ is the stock of physical capital, $h(t)$ is the per-person stock of human capital, $u(t)$ is the fraction of time devoted to production. Thus, $\dot{K}$ is investment (we assume that the depreciation rate is zero) and hN is the total units of time available for production, in efficiency units. The accumulation of human capital is as follows

$$\dot{h}(t) = \delta h(t) [1 - u(t)],$$

where $1 - u(t)$ is the fraction of time devoted to learning.

1. Write down the Hamiltonian and all the relevant first order conditions for this problem. [Hint: These are two differential equations for the co-states of K and h, and two FOCs w.r.t. to the controls c, and u].

Ans:

$$\mathcal{H}(c,u,K,h) = N\frac{c^{1-\sigma}}{1-\sigma} + \lambda\left[AK^{\alpha}(Nhu)^{1-\alpha} - Nc\right] + \mu\left[\delta h(1-u)\right].$$

FOCs:

$$\begin{aligned}
c &: \; Nc^{-\sigma} - N\lambda, \\
u &: \; \lambda(1-\alpha)AK^{\alpha}(Nhu)^{-\alpha}Nh - \mu\delta h = 0, \\
k &: \; \dot{\lambda} = \rho\lambda - \lambda\alpha AK^{\alpha-1}(Nhu)^{1-\alpha}, \\
h &: \; \dot{\mu} = \rho\mu - \lambda(1-\alpha)AK^{\alpha}(Nhu)^{-\alpha}Nu - \mu\delta(1-u),
\end{aligned}$$

or

$$\begin{aligned}
c &: \; c^{-\sigma} = \lambda, \\
u &: \; \lambda(1-\alpha)AK^{\alpha}(Nhu)^{-\alpha}Nu = \mu\delta u, \\
k &: \; \dot{\lambda} = \rho\lambda - \lambda\alpha AK^{\alpha-1}(Nhu)^{1-\alpha}, \\
h &: \; \dot{\mu} = \rho\mu - \mu\delta u - \mu\delta(1-u),
\end{aligned}$$

where the last line uses the FOC for u.

Definition. We say that the numbers N, K, h, c, u and κ are a balanced growth path if

$$\begin{aligned}
K(t) &= e^{(n+\kappa)t}K, \\
h(t) &= e^{\kappa t}h, \\
c(t) &= e^{\kappa t}c, \\
u(t) &= u,
\end{aligned}$$

is optimal for initial conditions $K(0) = K$, $h(0) = h$, $c(0) = 0$ and $N(0) = N$.

2. Write down a formula for the growth rate κ, the time devoted to production u, and the savings rate $\dot{K}/\left(AK^{\alpha}[huN]^{1-\alpha}\right)$ in a balanced growth path. Your expressions should be a function of the parameters ρ, n, δ, σ, α. How does κ change as a function of ρ, δ, σ and n? How does u change as function of α? Give an intuitive explanation for each of these cases.

Ans: The general definition of a balanced growth path (BGP) only requires c, h and K to grow at constant rates and u to be constant. We will prove that, in fact, the growth rates are related as above, i.e., that they satisfy

$$\frac{\dot{K}}{K} - n = \frac{\dot{c}}{c} = \frac{\dot{h}}{h} = \kappa,$$

for some parameter κ.

Start by taking logs and derivatives w.r.t time of the FOCs for c and u, and rewrite the FOCs for K and H to arrive at the following system of differential equations:

$$-\sigma\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}, \tag{1}$$

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{\lambda}}{\lambda} + \alpha\frac{\dot{K}}{K} - \alpha\left(n + \frac{\dot{h}}{h}\right) + n, \tag{2}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha A K^{\alpha-1}\left(Nhu\right)^{1-\alpha}, \tag{3}$$

$$\frac{\dot{\mu}}{\mu} = \rho - \delta. \tag{4}$$

Note that in deriving equation (2) we have used that in a BGP the time devoted to production must be constant, $u\left(t\right) = u$. Now, let κ denote the rate of growth of consumption in a BGP, $\dot{c}/c = \kappa$. From equation (1) we then see that the co-state λ falls at the rate $-\sigma\kappa$, which is constant over time. Using this result in (3) yields

$$\alpha A K^{\alpha-1}\left(Nhu\right)^{1-\alpha} = \rho + \sigma\kappa, \tag{5}$$

or, taking logs and derivatives w.r.t. time,

$$\frac{\dot{K}}{K} = n + \frac{\dot{h}}{h} = n + \eta,$$

where η denotes the rate of growth of human capital. Using this result together with equation (4) in equation (2) it follows that

$$\kappa = \frac{1}{\sigma}\left(n + \delta - \rho\right). \tag{6}$$

It only rests to show that $\eta = \kappa$. To do this, divide the feasibility condition by K to obtain

$$\left(n + \eta\right) + \frac{Nc}{K} = AK^{\alpha-1}\left(Nhu\right)^{1-\alpha},$$

or, taking logs and derivatives w.r.t. time,

$$n - \kappa - \left(n + \eta\right) = \left(\alpha - 1\right)\left(n + \eta\right) + \left(1 - \alpha\right)\left(n + \eta\right),$$

which implies that $\eta = \kappa$.

To derive an expression for u, notice that from the law of motion for capital we obtain

$$\frac{\dot{h}}{h} = \delta (1 - u),$$

or

$$1 - u = \frac{\kappa}{\delta} = \frac{1}{\sigma}\left(1 + \frac{n - \rho}{\delta}\right). \tag{7}$$

Finally, the savings rate can be written as

$$\frac{\dot{K}}{AK^{\alpha}(huN)^{1-\alpha}} = \frac{\dot{K}/K}{AK^{\alpha-1}(huN)^{1-\alpha}} = \frac{n + \kappa}{(\rho + \sigma\kappa)/\alpha},$$

where the last equality uses expression (5).

i) κ increasing in δ: A higher δ implies that each additional unit of time invested in education is more productive.

ii) κ decreasing in ρ: A higher ρ is equivalent to more impatience: agents discount future flows of consumption more heavily, so the incentives to increase future consumption (which is done by accumulating human capital) decreases. Thus, as ρ increases agents devote less time to the accumulation of human capital and more time to production activities. This can be seen from equation (7): $1 - u$, the fraction of time devoted to education, is decreasing in ρ.

iii) κ decreasing in σ: $(1/\sigma)$ is the intertemporal elasticity of substitution. When σ increases agents are less willing to substitute intertemporally. The allocation of time to the accumulation of human capital can be thought of as substituting current consumption (since working time decreases) for future consumption. As σ increases, agents are less willing to do so.

iv) κ is increasing in n: A higher n means that more people will belong to the dynasty in the future relative to today. Thus, devoting more time to human capital accumulation increases the utility of all future generations, which increases the discounted value of utilities today. In this sense a positive n acts as if the effective discount rate were lower.

v) u does not depend on α: The increases in labor productivity driven by increases in human capital are labor augmenting. This makes α to exactly cancel out in the equation that determines the amount of time devoted to accumulating human capital.

3. Write down the relevant transversality conditions. What conditions are needed on the parameters ρ, n, δ, σ, α, so that the balanced growth path characterized in 2 satisfies the transversality conditions?

Ans:

$$\begin{aligned}\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t) &= 0,\\ \lim_{t\to\infty} e^{-\rho t}\mu(t)h(t) &= 0,\end{aligned}$$

or

$$\begin{aligned}\lim_{t\to\infty} e^{-\rho t}e^{-\sigma\kappa t}\lambda(0)e^{(n+\kappa)t}K(0) &= 0,\\ \lim_{t\to\infty} e^{-\rho t}e^{(\rho-\delta)t}\mu(0)e^{\kappa t}h(0) &= 0.\end{aligned}$$

The first condition requires that

$$-\rho-\sigma\kappa+n+\kappa<0,$$

or, since $-\sigma\kappa=\rho-n-\delta$,

$$\kappa<\delta,$$

which is also the requirement for the second condition to be satisfied. Thus, the BGP characterized in exercise 2 satisfies the transversality conditions iff

$$\frac{1}{\sigma}(n+\delta-\rho)<\delta.$$

1.2 A one sector growth model where capital is produced with labor only

Consider the following discrete time growth model, where labor, inelastically supplied, is used to produced capital. Feasibility is given by

$$\begin{aligned}c_t &= n_t f(k_t/n_t),\\ k_{t+1} &= 1-n_t,\\ 0 &\le n_t\le 1,\end{aligned}$$

where f is a neoclassical production function (strictly increasing and strictly concave, C^2, with $f(0)=0$, and satisfying Inada conditions) and n_t is labor used in production of consumption goods, so that $1-n_t$ is the time used in production of capital goods, which we assume equal to next period capital stock (i.e. 100% depreciation). Preferences are standard discounted utility, with period utility function U (strictly increasing, strictly concave, satisfying Inada conditions, C^2, and $U(0)=0$), and the discount rate is $\beta\in(0,1)$.

The corresponding Bellman Equation is

$$v(k) = \max_{y \in [0,1]} \{U[(1-y)\ f(k/(1-y))] + \beta v(y)\},$$

for any $k \in [0,1]$, where we use that $y = 1 - n$.

Preliminaries. Notice that the period return function is

$$F(k,y) = U[(1-y)\ f(k/(1-y))].$$

Moreover, letting $\kappa \equiv k/n$ denote the capital labor ratio, we know that the production function satisfies the following conditions:

$$\begin{aligned} f'(\kappa) &> 0, \qquad f''(\kappa) < 0, \\ \lim_{\kappa \to 0} f'(\kappa) &= \lim_{\kappa \to \infty} \left[f(\kappa) - \kappa f'(\kappa)\right] = \infty, \\ \lim_{\kappa \to \infty} f'(\kappa) &= \lim_{\kappa \to 0} \left[f(\kappa) - \kappa f'(\kappa)\right] = 0. \end{aligned}$$

The last four conditions are the Inada conditions written in intensive form, and say that the marginal product of capital and labor go to infinity (zero) as capital and labor tend to zero (infinity), respectively. To check that $f(\kappa) - \kappa f'(\kappa)$ is indeed the marginal product of labor, simply differentiate $G(k,n) \equiv nf(k/n)$ w.r.t. n.

We will now show that the value function is strictly increasing, strictly concave and differentiable.

$v(k)$ is strictly increasing. The feasibility correspondence is $\Gamma(k) = [0,1]$, whose graph is the square $[0,1] \times [0,1]$. Evidently, Γ is monotone, since for any $k' > k$ we have that $\Gamma(k) \subseteq \Gamma(k')$ (in fact, in our case $\Gamma(k) = \Gamma(k') = [0,1]$ since Γ is independent of k). Moreover, for each y, $F(k,y)$ is strictly increasing in k, since

$$F_x(k,y) = U'(c)\ f'(\kappa) > 0,$$

given that U and f are strictly increasing. Thus, by Theorem 4.7 of RMED we conclude that $v(k)$ is strictly increasing.

$v(k)$ is strictly concave. Γ is clearly convex, since for any $\theta \in [0,1]$, $y \in \Gamma(k) = [0,1]$ and $y' \in \Gamma(k') = [0,1]$ we have that

$$\theta y + (1-\theta) y' \in \Gamma\left(\theta k + (1-\theta) k'\right) = [0,1].$$

It only rests to show that $F(k,y)$ is strictly concave. To do this, it suffices to show that

$G(k,n) = nf(k/n)$ is concave, since U is strictly increasing and strictly concave. But, since the production function is neoclassical, we know that $G(k,n)$ is indeed concave. Thus, by Theorem 4.8 of RMED we conclude that $v(k)$ is strictly concave. This also implies that the optimal policy $g(k)$ is a continuous, single-valued function.

$v(k)$ *is differentiable.* F is differentiable since U and f are twice differentiable. Thus, by Theorem 4.9 of RMED we conclude that $v(k)$ is differentiable as well.

1. Write down the Euler Equation and the Envelope condition for this problem. Denote the optimal decision rule by g, i.e., $k' = g(k)$.

Ans: To economize on notation, we will let

$$C(k) = (1 - g(k)) f(\kappa(k)),$$

denote the optimal consumption function, where $\kappa(k) = k/(1 - g(k))$ is the optimal capital to labor ratio.

The FOC is

$$y : U'(c)\left[-f(k/(1-y)) + \frac{k}{1-y} f'(k/(1-y))\right] + \beta v'(y) = 0,$$

or

$$U'(C(k))\left[\,f(\kappa(k)) - \kappa(k) f'(\kappa(k))\right] = \beta v'(g(k)). \tag{8}$$

The Envelope condition is

$$\text{EC} : v'(k) = U'(C(k)) f'(\kappa(k)). \tag{9}$$

Thus, the Euler equation of this problem can be written as

$$U'(C(k))\left[\,f(\kappa(k)) - \kappa(k) f'(\kappa(k))\right] = \beta U'(C(g(k))) f'(\kappa(g(k))). \tag{10}$$

2. Write down the equation that an interior steady state k^* must satisfy. Show that a solution k^* exists and it is unique.

Ans: In a steady state we must have that $k^* = g(k^*)$. Thus,

$$U'(C(k^*))\left[f(\kappa(k^*)) - \kappa(k^*) f'(\kappa(k^*))\right] = \beta U'(C(g(k^*))) f'(\kappa(g(k^*))),$$

or

$$f(\kappa(k^*)) - \kappa(k^*) f'(\kappa(k^*)) = \beta f'(\kappa(k^*)), \tag{11}$$

with

$$\kappa(k^*) = \frac{k^*}{1-k^*}.$$

The LHS of equation (11) is the marginal product of labor, which is strictly increasing in κ, and by the Inada conditions, tends to zero as $\kappa \to 0$ and to infinity as $\kappa \to \infty$. In turn, the RHS of (11) is (β times) the marginal product of capital, which is strictly decreasing in κ, and by the Inada conditions, tends to infinity as $\kappa \to 0$ and to zero as $\kappa \to \infty$. Thus, there exists a unique positive and finite $\bar{\kappa}$ that solves equation (11). Since $\kappa(k^*) = k^*/(1-k^*)$, this implies that there exists a unique k^*, that satisfies

$$k^* = \frac{\bar{\kappa}}{1+\bar{\kappa}} \in (0,1).$$

3. Show that $g(0) = 1$.

Ans: The value function with $k = 0$ is

$$v(0) = \max_{y\in[0,1]} \{U[(1-y)\ f(0/(1-y))] + \beta v(y)\} = \max_{y\in[0,1]} \{\beta v(y)\}.$$

Since the value function is strictly increasing, the above problem is solved by letting $y = 1$, that is, $g(0) = 1$.

4. Let $U(c) = c^\alpha$ for $0 < \alpha < 1$, $f(z) = z^\theta$, $0 < \theta < 1$. Show that g is strictly decreasing in $k \in [0,1]$. [Hint: In a previous problem set or exercise you showed that this depends on the sign of F_{xy}]. What is the intuition for this result? Why is this so different from the neoclassical growth model?

Ans: Using the given functional forms, the FOC (8) can be written as

$$\alpha(C(k))^{\alpha-1}\left[(\kappa(k))^\theta - \kappa(k)\,\theta(\kappa(k))^{\theta-1}\right] = \beta v'(g(k)),$$

or, since $C(k) = (1-g(k))(\kappa(k))^\theta$,

$$\alpha(1-\theta)(1-g(k))^{\alpha-1}(\kappa(k))^{\theta\alpha} = \beta v'(g(k)). \tag{12}$$

Now, assume by way of contradiction that $k' > k$ implies that $g(k') \geq g(k)$ (and thus $\kappa(k') > \kappa(k)$). Then,

$$
\begin{aligned}
\beta v'\left(g\left(k'\right)\right) &= \alpha\left(1-\theta\right)\left(1-g\left(k'\right)\right)^{\alpha-1}\left(\kappa\left(k'\right)\right)^{\theta\alpha}, \qquad \text{(from [12])}, \\
&> \alpha\left(1-\theta\right)\left(1-g\left(k\right)\right)^{\alpha-1}\left(\kappa\left(k\right)\right)^{\theta\alpha}, \qquad \left(\text{since } \kappa\left(k'\right)>\kappa\left(k\right)\right), \\
&= \beta v'\left(g\left(k\right)\right), \qquad \text{(from [12])},
\end{aligned}
$$

which implies that $g\left(k'\right)<g\left(k\right)$ since $v\left(y\right)$ is strictly concave. Thus, we have reached a contradiction, so we conclude that $g\left(k\right)$ is strictly decreasing.

5. Show that the linearized Euler equation is

$$
\left(k_{t+2}-k^*\right)+B\left(k_{t+1}-k^*\right)+\frac{1}{\beta}\left(k_t-k^*\right)=0,
$$

where

$$
B=\frac{1-\alpha\left(1-\theta\right)}{\alpha\left(1-\theta\right)}+\frac{1-\alpha\theta}{\alpha\theta\beta}.
$$

Ans: Using the given functional forms, the Envelope condition (9) can be written as

$$
v'\left(k\right)=\alpha\theta\left(C\left(k\right)\right)^{\alpha-1}\left(\kappa\left(k\right)\right)^{\theta-1},
$$

or, since $C\left(k\right)=\left(1-g\left(k\right)\right)\left(\kappa\left(k\right)\right)^{\theta}$,

$$
v'\left(k\right)=\alpha\theta\left(1-g\left(k\right)\right)^{\alpha-1}\left(\kappa\left(k\right)\right)^{\alpha\theta-1}.
$$

Thus, the Euler equation of this problem can be written as

$$
\left(1-\theta\right)\left(1-g\left(k\right)\right)^{\alpha-1}\left(\kappa\left(k\right)\right)^{\alpha\theta}=\beta\theta\left(1-g\left(g\left(k\right)\right)\right)^{\alpha-1}\left(\kappa\left(g\left(k\right)\right)\right)^{\alpha\theta-1}. \tag{13}
$$

In a steady state we must have that $k^*=g\left(k^*\right)$. Thus,

$$
\left(1-\theta\right)\left(1-k^*\right)^{\alpha-1}\left(\kappa\left(k^*\right)\right)^{\alpha\theta}=\beta\theta\left(1-k^*\right)^{\alpha-1}\left(\kappa\left(k^*\right)\right)^{\alpha\theta-1},
$$

or

$$
\kappa\left(k^*\right)=\frac{\beta\theta}{1-\theta}.
$$

Since $\kappa\left(k^*\right)=k^*/\left(1-k^*\right)$, we obtain

$$
k^*=\frac{\beta\theta}{1-\theta+\beta\theta}.
$$

Now, from (13), the Euler equation can be expressed as

$$H(k_t, k_{t+1}, k_{t+2}) = (1-\theta)(1-k_{t+1})^{\alpha(1-\theta)-1} k_t^{\alpha\theta} - \beta\theta(1-k_{t+2})^{\alpha(1-\theta)} k_{t+1}^{\alpha\theta-1} = 0,$$

where $\tilde{\kappa}(k_t, k_{t+1}) = k_t/(1-k_{t+1})$. We want to linearize this equation around k^*:

$$H(k_t, k_{t+1}, k_{t+2}) \cong H_1(k^*, k^*, k^*)(k_t - k^*) + H_2(k^*, k^*, k^*)(k_{t+1} - k^*) + H_3(k^*, k^*, k^*)(k_{t+2} - k^*) = 0,$$

where we use that $H(k^*, k^*, k^*)$ by definition of the steady state. The other terms are as follows:

$$\begin{aligned} H_1(k^*, k^*, k^*) &= \alpha\theta(1-\theta)(1-k^*)^{\alpha(1-\theta)-1}(k^*)^{\alpha\theta-1}, \\ H_2(k^*, k^*, k^*) &= (1-\alpha(1-\theta))(1-\theta)(1-k^*)^{\alpha(1-\theta)-2}(k^*)^{\alpha\theta} + (1-\alpha\theta)\beta\theta(1-k^*)^{\alpha(1-\theta)}(k^*)^{\alpha\theta-2}, \\ H_3(k^*, k^*, k^*) &= \alpha(1-\theta)\beta\theta(1-k^*)^{\alpha(1-\theta)-1}(k^*)^{\alpha\theta-1}, \end{aligned}$$

Hence,

$$\frac{H_1(k^*, k^*, k^*)}{H_3(k^*, k^*, k^*)} = \frac{1}{\beta},$$

and

$$\frac{H_2(k^*, k^*, k^*)}{H_3(k^*, k^*, k^*)} = \frac{(1-\alpha(1-\theta))(1-\theta)k^*}{\alpha(1-\theta)\beta\theta(1-k^*)} + \frac{(1-\alpha\theta)(1-k^*)}{\alpha(1-\theta)k^*} = \frac{1-\alpha(1-\theta)}{\alpha(1-\theta)} + \frac{1-\alpha\theta}{\alpha\beta\theta},$$

since $(1-\theta)k^*/(1-k^*) = \beta\theta$. We then conclude that the desired linearization is

$$(k_{t+2} - k^*) + B(k_{t+1} - k^*) + \frac{1}{\beta}(k_t - k^*) = 0, \tag{14}$$

where

$$B \equiv \frac{1-\alpha(1-\theta)}{\alpha(1-\theta)} + \frac{1-\alpha\theta}{\alpha\beta\theta}.$$

6. Let $\beta = 0.1$, $\alpha = 0.8$ and $\theta = 0.75$. Show that $k^* = 0.23$, approximately. Show that this steady state is unstable. [Hint: Show that both roots of the previous equation are larger than one in absolute value].

Ans: Note that with these parameter values we have that

$$B = \frac{1-(0.8)(0.25)}{(0.8)(0.25)} + \frac{1-(0.8)(0.75)}{(0.8)(0.1)(0.75)} = 10.667,$$

and

$$k^* = \frac{(0.1)(0.75)}{1-(0.75)+(0.1)(0.75)} = 0.23077.$$

Moreover, the roots of equation (14) are the solutions of the characteristic equation

$$\lambda^2 + B\lambda + \frac{1}{\beta} = 0,$$

which are determined by

$$\begin{aligned}
\lambda &= \frac{-10.67 \pm \sqrt{(10.67)^2 - 4(1)(10)}}{2(1)} = \frac{-10.67 \pm 8.59}{2(1)} \\
(\lambda_1, \lambda_2) &= (-1.03859, -9.62841).
\end{aligned}$$

Given that both roots are larger than one in absolute value, we conclude that the steady state is unstable.

7. Define a two-period cycle as a sequence $\{k_t\}$ such that

$$\begin{aligned}
k_t &= x \text{ if } t \text{ is odd}, \\
k_t &= y \text{ if } t \text{ is even},
\end{aligned}$$

for $x, y \in [0,1]$, $x \neq y$. Show that a two-period cycle is optimal if $k_0 = x$ and

$$\begin{aligned}
0 &= F_y(x,y) + \beta F_x(y,x), \\
0 &= F_y(y,x) + \beta F_x(x,y),
\end{aligned}$$

for F the period return function, i.e. $F(x,y) = U[(1-y)\ f(k/(1-y))]$.

Ans: We know that any optimal policy has to satisfy the Euler equation

$$F_y(k_t, k_{t+1}) + \beta F_x(k_{t+1}, k_{t+2}) = 0, \qquad \text{all } t,$$

and the transversality condition

$$\lim_{t\to\infty} \beta^t F_x(k_t, k_{t+1})\, k_t = 0.$$

First notice that the transversality condition is always satisfied for any feasible sequence, since $F(k,y)$ is bounded for any $0 \leq k \leq 1$ and $0 \leq k \leq 1$. Now, start with t equal to an odd number. Then, the Euler equation under the proposed policy is

$$F_y(x,y) + \beta F_x(y,x) = 0,$$

which is satisfied by assumption. In turn, when t is even, the Euler equation under the proposed policy is

$$F_y(y,x) + \beta F_x(x,y) = 0,$$

which is also satisfied by assumption. Thus, the two-period cycle is indeed optimal.

8. Show that $(0.29, 0.18)$ is a two-period optimal cycle for the previous parameters.

Ans: We know that the proposed two-period cycle must satisfy

$$F_y(x,y) + \beta F_x(y,x) = (1-\theta)(1-y)^{\alpha(1-\theta)-1} x^{\alpha\theta} - \beta\theta(1-x)^{\alpha(1-\theta)} y^{\alpha\theta-1} = 0,$$

and

$$F_y(y,x) + \beta F_x(x,y) = (1-\theta)(1-x)^{\alpha(1-\theta)-1} y^{\alpha\theta} - \beta\theta(1-y)^{\alpha(1-\theta)} x^{\alpha\theta-1} = 0.$$

Plugging $(x,y) = (.29, .18)$ into these equations for the parameter values described above, we find that both equations are equal to zero. Thus, the proposed two-period cycle is optimal.

2 A model of durabe and non-durable goods

Consider an economy where in each period every one of the consumers has an endowment y. This endowment can be used for investment in durable goods or for consumption of non-durables. Then the technology for this economy is:

$$x(t) + c(t) = y$$

for all $t \geq 0$, where $x(t)$ denote the investment in durables and $c(t)$ the consumption of non-durables. The stock of durable goods have a law of motion:

$$\dot{d}(t) = x - \delta d(t)$$

where δ is the depreciation rate of durables per unit of time.

The period utility function depends on the flow of nondurable purchases and on the stock of durables, and is given by $U(c,d)$. We assume that U is strictly quasi-concave in (c,d). In some cases we will specialize to

$$U(c,d) = \frac{[h(c,d)]^{1-\gamma} - 1}{1-\gamma} \tag{1}$$

for $\gamma \geq 0$, and where

$$h(c,d) = \left[c^{-\theta} + \frac{1}{A} d^{-\theta}\right]^{-1/\theta}$$

for $\theta \geq -1$.The agent's utility is the discounted value of $U(c,d)$, using discount rate ρ. With this parameterization the elasticity of substitution between c and d is $1/(1+\theta)$, and the inter-temporal elasticity of substituion between the bundle h of (c,d) is $1/\gamma$.

Thus problem of the planner for this economy is

$$\max_{c,d} \int_0^\infty e^{-\rho t} U(c(t), d(t))\, dt$$

subject to

$$\dot{d}(t) + c(t) = y - \delta d(t),$$

and $d(0) > 0$ given.

Q0. To better understand the utility function in 1. show that if

$$\begin{aligned} U_{cd} &> 0 \text{ if } \frac{1}{\gamma} > \frac{1}{1+\theta}, \\ U_{cd} &= 0 \text{ if } \frac{1}{\gamma} = \frac{1}{1+\theta}, \\ U_{cd} &< 0 \text{ if } \frac{1}{\gamma} < \frac{1}{1+\theta}. \end{aligned}$$

And hence that if $\sigma \equiv 1/\gamma = 1/(1+\theta)$ the utility function is additively separable in c, d :

$$U(c,d) = \frac{c^{1-\frac{1}{\sigma}} + \frac{1}{A} d^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}.$$

A. The first derivative is

$$U_c = \left[c^{-\theta} + \frac{1}{A} d^{-\theta} \right]^{-\frac{1-\gamma}{\theta}-1} c^{-\theta-1}$$

so the cross derivative is

$$U_{cd} = (1-\gamma+\theta) \left\{ \left[c^{-\theta} + \frac{1}{A} d^{-\theta} \right]^{-\frac{1-\gamma}{\theta}-2} c^{-\theta-1} \frac{1}{A} d^{-\theta-1} \right\}$$

so

$$1-\gamma+\theta \geq 0 \iff 1+\theta \geq \gamma \iff \frac{1}{\gamma} \geq \frac{1}{1+\theta}.$$

Finally, if $1/\gamma = \sigma = \frac{1}{1+\theta}$ then

$$\begin{aligned} -\theta &= 1 - \frac{1}{\sigma} \\ 1-\gamma &= 1 - \frac{1}{\sigma} \end{aligned}$$

so

$$U(c,d)=\frac{\left[c^{-\theta}+\frac{1}{A}d^{-\theta}\right]^{-\frac{1-\gamma}{\theta}}}{1-\gamma}=\frac{c^{1-\frac{1}{\sigma}}+\frac{1}{A}d^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}$$

Q1. Write the Hamiltonian of the problem, using λ for the co-state, d for the state, and c for the control.

A:

$$H(c,d)=U(c,d)+\lambda\left[-d\delta+y-c\right]$$

Q2. Write the f.o.c. w.r.t c and d.

A:

$$\dot{\lambda}=\rho\lambda-H_d:\dot{\lambda}=\lambda(\rho+\delta)-U_d(c,d),$$

and

$$H_c=0:U_c(c,d)=\lambda.$$

Q3. Let $\bar{v}\equiv\delta+\rho$ be the steady state user cost of the durable good. Write two equations in two unknowns for the steady state values of $\left(\bar{c},\bar{d}\right)$ in terms of $U_d,\ U_c,\ \rho$ and $\bar{v}$..

A:

The steady state is then:

$$\begin{aligned}\frac{U_d}{U_c}\left(\bar{c},\bar{d}\right) &= \bar{v},\\ \delta\bar{d}+\bar{c} &= y\ .\end{aligned}$$

Q4. Use the equation $H_c=0$ to obtain a differential equation linking $\dot{\lambda},\dot{c}$ and $\dot{d}$.

A:From $U_c(c(t),d(t))=\lambda(t)$ we obtain:

$$\dot{c}U_{cc}+\dot{d}U_{cd}=\dot{\lambda},$$

Q5. Using this last expression, replace the law of motion for the co-state variable and the law of motion of the state variable to find the law of motion of the control $\dot{c}$ as a function of parameters c and d.

A:

$$\dot{c}U_{cc}(c,d)+\dot{d}U_{cd}(c,d)=U_c(c,d)(\delta+\rho)-U_d$$

and using the budget constraint for $\dot{d}$:

$$\dot{c}U_{cc}(c,d)=U_c(c,d)(\delta+\rho)-\left[y-d\delta-c\right]U_{cd}(c,d)-U_d(c,d)$$

Q6. Linearize this last ODE around the steady state, i.e.: $\left(\dot{c}, \dot{d}, c, d\right) = \left(0, 0, \bar{c}, \bar{d}\right)$ and replacing $\dot{d}$ by using the resource constraint of the economy.Your answer should be of the type $\dot{c} = a_{11}\left(c - \bar{c}\right) + a_{12}\left(d - \bar{d}\right)$ for two constants a_{11} and a_{12}. This constant are functions of the second derivatives of U evaluated at the steady state, and of δ and ρ.

Note on linearization: Suppose we want to linearize the function $g\left(x, y\right)$ around $\bar{x}$ and $\bar{y}$. We get:

$$g\left(x, y\right) = g_x\left(\bar{x}, \bar{y}\right)\left(x - \bar{x}\right) + g_x\left(\bar{x}, \bar{y}\right)\left(y - \bar{y}\right)$$

A:The ODE that we want to linearize is:

$$\dot{c} = \frac{U_c\left(c, d\right)\left(\delta + \rho\right) - \left[y - d\delta - c\right]U_{cd}\left(c, d\right) - U_d\left(c, d\right)}{U_{cc}\left(c, d\right)}$$

Linearizing around the steady state

$$\begin{aligned} \frac{U_d\left(\bar{c}, \bar{d}\right)}{U_c\left(\bar{c}, \bar{d}\right)} &= \bar{v}, \\ \delta\bar{d} + \bar{c} &= y\ . \end{aligned}$$

$$\dot{c} = \left(\frac{\left(U_{cc}\left(\delta + \rho\right) + U_{cd} - \left[y - \delta\bar{d} - \bar{c}\right]U_{ccd} - U_{cd}\right)U_{cc} - \left(U_c\left(\delta + \rho\right) - \left[y - \delta\bar{d} - \bar{c}\right]U_{cd} - U_d\right)U_{ccc}}{\left(U_{cc}\right)^2}\right)\left(c - \bar{c}\right)$$

$$+\left(\frac{\left(U_{cd}\left(\delta + \rho\right) + \delta U_{cd} - \left[y - \delta\bar{d} - \bar{c}\right]U_{cdd} - U_{dd}\right)U_{cc} - \left(U_c\left(\delta + \rho\right) - \left[y - \delta\bar{d} - \bar{c}\right]U_{cd} - U_d\right)U_{ccd}}{\left(U_{cc}\right)^2}\right)\left(d - \bar{d}\right)$$

Evaluated at the steady state we get:

$$\begin{aligned} \dot{c} &= \left(\frac{U_{cc}\left(\delta + \rho\right) + U_{cd} - U_{dc}}{U_{cc}}\right)\left(c - \bar{c}\right) \\ &\quad + \left(\frac{U_{cd}\left(\delta + \rho\right) + \delta U_{cd} - U_{dd}}{U_{cc}}\right)\left(d - \bar{d}\right) \end{aligned}$$

or:

$$\begin{aligned} \dot{c} &= \left(\delta + \rho\right)\frac{U_{cc}}{U_{cc}}\left(c - \bar{c}\right) + \left(\delta + \rho\right)\frac{U_{cd}}{U_{cc}}\left(d - \bar{d}\right) + \frac{U_{cd}}{U_{cc}}\left(c - \bar{c}\right) \\ &\quad + \frac{\delta U_{cd}}{U_{cc}}\left(d - \bar{d}\right) - \frac{U_{dd}}{U_{cc}}\left(d - \bar{d}\right) - \frac{U_{dc}}{U_{cc}}\left(c - \bar{c}\right) \end{aligned}$$

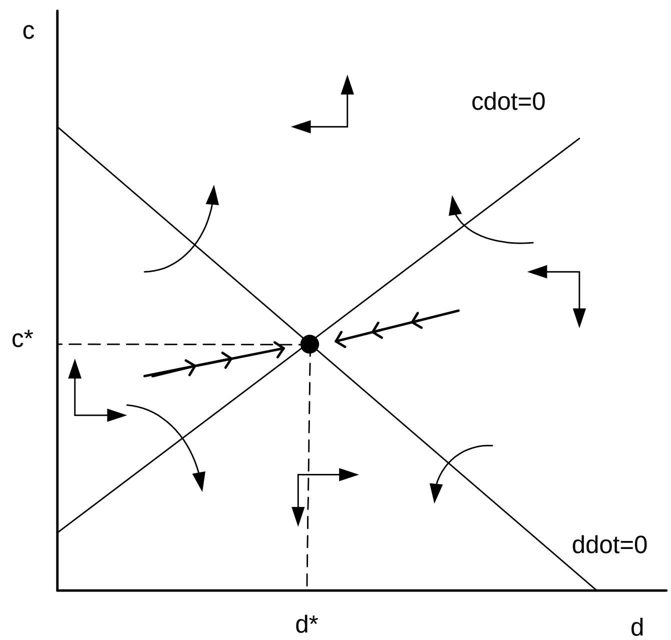
where U_{cc}, U_{cd}, U_{dd} are evaluated at the steady state.

Q7. Using your previous answer to write the two **linear** differential equations that characterize the dynamics of this economy, one for $\dot{c} = a\ (c, d)$ and one for $\dot{d} = b\,(c, d)$.

A:Summarizing we have the linear system:

$$\begin{aligned}\dot{c} &= a\,(c,d) \equiv [\delta + \rho]\,(c - \bar{c}) + \left[\frac{U_{cd}\,(\delta + \bar{v}) - U_{dd}}{U_{cc}}\right]\left(d - \bar{d}\right) \\ \dot{d} &= b\,(c,d) \equiv y - d\delta - c\end{aligned}$$

Q8. Assume that $U_{cd}\,(\delta + \bar{v}) - U_{dd} > 0$. Draw the phase diagram with c in the y-axis and d in the x-axis. Label the axis, label the steady states, draw all the arrows for the field, and indicate clearly where the the stable arm (saddle-path) is.



Q9. We are looking for a solution of the form

$$c = \psi\,(d) = \bar{c} + \psi'\left(\bar{d}\right)\left(d - \bar{d}\right)$$

thus, we are looking for the value of the constant $\psi'\,(\bar{m})$. Use the method of undetermined coefficients to find a quadratic equation for ψ' as a function of: $[\bar{v} + \delta]$ and

$$\Delta \equiv \left[\frac{U_{cd}\,(\delta + \bar{v}) - U_{dd}}{-U_{cc}}\right]$$

where the second derivatives are evaluated at the steady state. Hint: You need to use L'Hopital's rule. Recall what we did in pset 5 problem 1.

A:

$$\begin{aligned}\frac{\partial \psi\left(\bar{d}\right)}{\partial d} &\equiv \psi'\left(\bar{d}\right)=\frac{\dot{c}}{\dot{d}}=\frac{a\left(\psi\left(\bar{d}\right),\bar{d}\right)}{b\left(\psi\left(\bar{d}\right),\bar{d}\right)}=\frac{0}{0}\\ &= \frac{\left(\delta+\rho\right)\psi'\left(\bar{d}\right)+\left[\frac{U_{cd}\left(\delta+\bar{v}\right)-U_{dd}}{U_{cc}}\right]}{-\delta-\psi'\left(\bar{d}\right)},\end{aligned}$$

where the second line uses L'Hopital. The quadratic equation for ψ' is then

$$\left(\psi'\left(\bar{d}\right)\right)^2+\left(2\delta+\rho\right)\psi'\left(\bar{d}\right)+\left[\frac{U_{cd}\left(\delta+\bar{v}\right)-U_{dd}}{U_{cc}}\right]=0$$

Q10. Show that the stable solution is given by

$$\psi'=\frac{-\left[\bar{v}+\delta\right]+\sqrt{\left[\bar{v}+\delta\right]^2+4\frac{U_{cd}\left(\bar{v}+\delta\right)-U_{dd}}{\left(-U_{cc}\right)}}}{2},$$

(hint: this is trivial once you have the solution of Q9, and the figure for the saddle path, that help you to find the "right" solution of the quadratic equation).

A: Solve the quadratic equation.

$$\left(\psi'\left(\bar{d}\right)\right)^2+\left(2\delta+\rho\right)\psi'\left(\bar{d}\right)+\left[\frac{U_{cd}\left(\delta+\bar{v}\right)-U_{dd}}{U_{cc}}\right]=0$$

the quadratic equation is:

$$\psi'\left(\bar{d}\right)=\frac{-\left[\bar{v}+\delta\right]\pm\sqrt{\left[\bar{v}+\delta\right]^2-4\left[\frac{U_{cd}\left(\delta+\bar{v}\right)-U_{dd}}{U_{cc}}\right]}}{2}$$

where the positive root is given by:

$$\psi'=\frac{-\left[\bar{v}+\delta\right]+\sqrt{\left[\bar{v}+\delta\right]^2+4\frac{U_{cd}\left(\bar{v}+\delta\right)-U_{dd}}{\left(-U_{cc}\right)}}}{2}.$$

Elasticity of the optimal consumption function.

For this we specialize the utility function U to (1).

We will show how the elasticity of the policy function is related to the inter-temporal and intra-temporal elasticities of substitution. Recall that the intra-tempral elasticity of substitution between c and d is $1/\left(1+\theta\right)$, and the inter-temporal elasticity of substituion between bundles of (c,d) given by h is $1/\gamma$.

We will parameterized the problem as a function of $\left(\theta,\gamma,\delta,\bar{v},\bar{d}/\bar{c}\right)$. The interpretation of $\bar{d}/\bar{c}$ as a parameter, is that we solve for the constant A using the steady-state equation derived above as a function of the paramters $\theta,\bar{v}$, so that $\bar{d}/\bar{c}$. We obtain the following result:

Keeping the steady state value $\bar{d}/\bar{c}$ fixed, the elasticity of the optimal consumption function evaluated at steady state is a function of $\gamma/\left(1+\theta\right)$ and satisfies

$$\frac{d}{c}\frac{\partial c\left(d\right)}{\partial d}|_{d=\bar{d}}\equiv\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)=\begin{cases}1 & \text{for } \frac{\gamma}{1+\theta}=0\\ <1 & \text{for } \frac{\gamma}{1+\theta}>0\end{cases}$$

and $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

As an intermediate step to see why $\left(\bar{d}/\bar{c}\right)\left(\partial c\left(\bar{d}\right)/\partial d\right)$ depends on the ratio of γ to $1+\theta$ only, and to develop a formulat for $\Delta\left(\gamma/\left(1+\theta\right)\right)$ do the following:

Q11. To show this, first show that when h is a CES we have that:

$$\begin{aligned}\frac{h_{dd}}{h_{cc}} &= \frac{1}{\left(d/c\right)^{2}},\\ \frac{h_{cd}}{h_{cc}} &= -\frac{1}{d/c},\end{aligned}$$

$$\frac{h_{c}h_{c}}{-hh_{cc}}=\frac{1}{\left(1+\theta\right)\bar{v}\left(d/c\right)},$$

and that for $U\left(c,d\right)=h\left(c,d\right)^{1-\gamma}/\left(1-\gamma\right)$

$$\begin{aligned}\frac{U_{dd}}{U_{cc}} &= \frac{h_{dd}/h_{cc}+\gamma\bar{v}^{2}\left(h_{c}h_{c}\right)/\left(-hh_{cc}\right)}{1+\gamma\left(h_{c}h_{c}\right)/\left(-hh_{cc}\right)},\\ \frac{U_{cd}}{U_{cc}} &= \frac{h_{cd}/h_{cc}+\gamma\bar{v}\left(h_{c}h_{c}\right)/\left(-hh_{cc}\right)}{1+\gamma\left(h_{c}h_{c}\right)/\left(-hh_{cc}\right)}\end{aligned}$$

and

$$\frac{h_{c}h_{c}}{-hh_{cc}}=\frac{1}{\left(1+\theta\right)\bar{v}\left(d/c\right)}\ .$$

Q12. First assume that $\gamma=0$. Using part of the results of Q11 show that

$$\psi'\left(\bar{d}\right)=\frac{\bar{c}}{\bar{d}}$$

A. we have

$$\psi'=\frac{-\left[\bar{v}+\delta\right]+\sqrt{\left[\bar{v}+\delta\right]^{2}+4\frac{h_{cd}\left(\bar{v}+\delta\right)-h_{dd}}{-h_{cc}}}}{2}$$

so

$$\frac{h_{cd}(\bar{v}+\delta)-h_{dd}}{-h_{cc}}=\frac{(\delta+\bar{v})}{d/c}+\left(\frac{1}{d/c}\right)^2$$

Thus

$$\begin{aligned}
&[\bar{v}+\delta]^2+4\frac{h_{cd}(\bar{v}+\delta)-h_{dd}}{-h_{cc}}\\
=\;&[\bar{v}+\delta]^2+2(\delta+\bar{v})\left(\frac{2}{d/c}\right)+\left(\frac{2}{d/c}\right)^2\\
=\;&\left(\bar{v}+\delta+\left(\frac{2}{d/c}\right)\right)^2
\end{aligned}$$

and hence

$$\begin{aligned}
\psi' &= \frac{-[\bar{v}+\delta]+\sqrt{\left([\bar{v}+\delta]+\left(\frac{2}{d/c}\right)\right)^2}}{2}\\
&= \frac{-[\bar{v}+\delta]+[\bar{v}+\delta]+\left(\frac{2}{d/c}\right)}{2}\\
&= \frac{1}{d/c}.
\end{aligned}$$

Q13. Assume that $\gamma>0$ and that $1/\gamma=\sigma$ and $-\theta=1-\frac{1}{\sigma}$, or $\frac{\gamma}{1+\theta}=1$ so that U is additively separable. What is the value of $\Delta\left(\frac{\gamma}{1+\theta}\right)=\Delta(1)$ for this case? (hint: compute U_{cd}, U_{dd} and U_{cc} at the steady state values of c,d). Verify that Δ, and hence $-(d/c)\,\partial c/\partial d$ does depend on the particular value of σ, given $\bar{c}/\bar{d}$ and $\bar{v}$. Show that the value of $(d/c)\,\psi'$ is smaller than the one for $\gamma=0$ and $\theta>-1$.

A. Since $U_{cd}=0$, and

$$\Delta\left(\frac{\gamma}{1+\theta}\right)=\Delta(1)=\frac{U_{dd}}{U_{cc}}=\frac{-\sigma\frac{1}{A}d^{-\frac{1}{\sigma}-1}}{-\sigma c^{-\frac{1}{\sigma}-1}}=\frac{c}{d}\frac{\frac{1}{A}d^{-\frac{1}{\sigma}}}{c^{-\frac{1}{\sigma}}}=\left(\frac{c}{d}\right)\bar{v}$$

Comparing with the case of $\gamma/(1+\theta)=0$ we have:

$$
\begin{aligned}
\psi' &= \frac{-[\bar{v}+\delta]+\sqrt{[\bar{v}+\delta]^2+4\left(\frac{\bar{c}}{\bar{d}}\right)\bar{v}}}{2} \\
&< \frac{-[\bar{v}+\delta]+\sqrt{[\bar{v}+\delta]^2+2(\bar{v}+\delta)\,2\left(\frac{\bar{c}}{\bar{d}}\right)+4\left(\frac{\bar{c}}{\bar{d}}\right)^2}}{2} \\
&= \frac{-[\bar{v}+\delta]+\sqrt{\left[\bar{v}+\delta+2\left(\frac{\bar{c}}{\bar{d}}\right)\right]^2}}{2} \\
&= \frac{-[\bar{v}+\delta]+\left(\bar{v}+\delta+2(\bar{c}/\bar{d})\right)}{2} \\
&= \frac{\bar{c}}{\bar{d}}.
\end{aligned}
$$

Q14. Assume that $\gamma>0$, what assumptions are required for $1/(1+\theta)$ such that you also find

$$\psi'\left(\bar{d}\right)=\frac{\bar{c}}{\bar{d}}$$

Hint: look at the formula for ψ'.

A: We need $\frac{\gamma}{1+\theta}=0$, so if $\gamma>0$ then $\theta\to\infty$.

Q15. Give an intuitive interpretation for this last two results. (2 lines max).

A: The elasticity of the optimal decision rule for consumption depends on the ratio between the elasticity of substitution between durable and non durable goods, $1/(1+\theta)$, and the intertemporal elasticity of substitution $1/\gamma$. If durables and non durables are very poor substitutes, the elasticity $(d/c)\psi'$is one. To understand this effect, consider an agent that starts with a stock of durables 1 percent below its steady state, so it must decrease consumption of non durables to reach the higher level of durables in steady state. If $1/(1+\theta)$ is close to zero, so that durables and non durables are Leontief, then on impact it will decrease consumption by 1 percent. In this case, durables and non-durables are, essentially, the same. If instead, they are good substitutes, so that $1/(1+\theta)$ is high, the effect of durables in non-durables is smaller, and hence non-durables consumption will not decreae that much.

Q16. Assume that $\gamma>0$. We will like to show that $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

For this, show that $\Delta\left(\frac{\gamma}{1+\theta}\right)$ is decreasing in γ provided that $\delta>0$, where $\Delta(\gamma/(1+\theta))$ is given by

$$\Delta\left(\frac{\gamma}{1+\theta}\right)\equiv\frac{U_{cd}(\delta+\bar{v})-U_{dd}}{(-U_{cc})}.$$

A:

$$
\begin{aligned}
\Delta\left(\frac{\gamma}{1+\theta}\right) &\equiv \frac{U_{cd}\left(\delta+\bar{v}\right)-U_{dd}}{\left(-U_{cc}\right)} \\
&= \frac{\left(\frac{1}{d/c}-\frac{\gamma}{(1+\theta)(d/c)}\right)\left[\delta+\bar{v}\right]+\left(\frac{1}{d/c}\right)^{2}+\frac{\gamma\bar{v}}{(1+\theta)(d/c)}}{1+\frac{\gamma}{(1+\theta)\bar{v}(d/c)}} \\
&= \frac{\frac{1}{d/c}\left[\delta+\bar{v}\right]-\frac{\gamma}{(1+\theta)(d/c)}\delta+\left(\frac{1}{d/c}\right)^{2}}{1+\frac{\gamma}{(1+\theta)\bar{v}(d/c)}} \\
&= \frac{\frac{1}{d/c}\left[\delta+\bar{v}\right]+\left(\frac{1}{d/c}\right)^{2}}{1+\frac{\gamma}{1+\theta}\frac{1}{\bar{v}(d/c)}}-\frac{\frac{\delta}{(d/c)}}{\frac{(1+\theta)}{\gamma}+\frac{1}{\bar{v}(d/c)}}
\end{aligned}
$$

$$
\frac{\partial\Delta\left(\frac{\gamma}{1+\theta}\right)}{\partial\left(\frac{\gamma}{1+\theta}\right)}<0
$$

Q17. Argue that if $\Delta\left(\gamma/\left(1+\theta\right)\right)$ is decreasing in γ then $\frac{\bar{d}}{\bar{c}}\psi'\left(\bar{d}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

A: Since

$$
\psi'\left(\bar{d}/\bar{c},\frac{\gamma}{1+\theta}\right)=\frac{-\left[\bar{v}+\delta\right]+\sqrt{\left[\bar{v}+\delta\right]^{2}+4\Delta\left(\frac{\gamma}{1+\theta}\right)}}{2}
$$

then $\psi'\left(\frac{\gamma}{1+\theta}\right)$ is decreasing in $\frac{\gamma}{1+\theta}$.

Q18. Give an intuitive interpretation of this result. (2 lines max).

A: This answer follows the answer to Q14. If γ is very large, the agent does not want to substitute the bundle h intertemporally, so that consumption reacts very little. Or, equivalently, if $1/(1+\theta)$ is very large, so the agent substitutes durable and non durable easily, non durable consumption will also react by a small amount.