Steady States
Neoclassical Growth Model, deterministic case
Lecture Notes 10

Fernando Alvarez

University of Chicago

Fall 2018

(colors denote importance: blue > black > gray)

- In this note we study the neoclassical growth model with variable labor supply and some extensions of it.
- Define the competitive equilibrium we endow firms and households with an objective function and a feasible set of actions.
- We discuss two extreme versions: in the first, firms owns and accumulates the whole stock of capital of the economy.
- In the second, the stock of capital is owned by the households, which is rented by the firms. The definition of the consumption possibility set, budget constraint and production possibility sets differs in the two interpretations of the competitive equilibrium.
- Notwithstanding the latter fact, in both versions (as in any intermediate case), the equilibrium prices and aggregate quantities are identical.

- Next the steady state is analyzed, paying particular attention to the fact that labor supply is endogenous.
- We show that if leisure is normal, there is a unique steady state.
- We introduce government purchases, financed with lump-sum taxes. Using normality we can analyze the effect of them on output.
- We solve a parametric example calibrated to roughly match the aggregates in the US economy.

- The core of the note is the analysis of the steady state effects of linear taxes to labor and (net) capital income.
- We adapt the arguments used before for existence and uniqueness of steady state with distortive taxation.
- The main result is a comparative static results of the effect of labor and (net) capital income tax rates and of government purchases on the steady state level of consumption, labor supply, and investment.
- Finally the note discusses different interpretations and alternative formulations of the utility function (e.g. Uzawa's endogenous discount rate).

Households

There is a representative household who derives utility from sequences of a unique consumption good and leisure $\{c_t, 1-n_t\}_{t=0}^{\infty}$ whose preferences are represented by the utility function

$$u\left(\{c_t, 1-n_t\}_{t=0}^{\infty}\right) = u\left(c_0, 1-n_0, c_1, 1-n_1, c_2, 1-n_2 ...\right)$$

where c_t is consumption, n_t is labor and total time has been normalized to 1 per period, so that the quantity $1-n_t$ is leisure at time t.

Planning problem

We start with the following planning problem

$$\max_{\{c_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty}} u\left(c_0, 1-n_0, c_1, 1-n_1, c_2, 1-n_2 ...\right)$$

subject to

$$\begin{aligned} c_t + x_t &= F\left(k_t, n_t\right) \\ k_{t+1} &= x_t + k_t\left(1-\delta\right) \end{aligned}$$

for all $t \geq 0$, k_0 given.

As evident from the description, x_t is investment and k_t is capital. F is a neoclassical, constant returns to scale production function and $0 \leq \delta \leq 1$ is the depreciation rate.

We typically use

$$u(c_0, 1-n_0, c_1, 1-n_1, ...) = \sum_{t=0}^{\infty} \beta^t v(c_t, 1-n_t)$$

Now we consider two versions of competitive equilibrium corresponding to this economy.

In both we assume that all trade takes place at time $t = 0$. We will later return to this issue.

The technology used to produce the consumption good is represented by a neoclassical constant returns to scale production function.

This implies that the number of firms is irrelevant.

Version I: Firms owns and accumulate capital

- *Commodity Space*:
  Let $L = R^{2T}$ for $T = \infty$, or more formally

$$L = \left\{ \{c_t, n_t\}_{t=0}^{\infty} \left| (c_t, n_t) \in R^2 \right. \right\}$$

  that is, the commodity space is the set of all (pairs) of real sequences.

- *Production possibility set of the firm*:

$$\begin{aligned} Y &= \{\{c_t, n_t\} : x_t + c_t \leq F(n_t, k_t), \\ &\quad k_{t+1} = x_t + k_t(1-\delta), \ x_t \in R \text{ for all } t \geq 0, \ k_0 \text{ given}\} \end{aligned}$$

- *Consumption possibility set*:

$$X = \{\{c_t, n_t\} : c_t \geq 0, 0 \leq n_t \leq 1\}$$

- *Household budget constraint*:

$$\sum_{t=0}^{\infty} p_t \left[c_t + w_t l_t\right] = \pi + \sum_{t=0}^{\infty} p_t w_t$$

where p_t is the price of a consumption good at time t, and $p_t w_t$ is the price of a unit of labor at time t. Both prices are in units of the numeraire. Thus, w_t is the time t real wage in terms of time t consumption units.

- *Firm's problem*:

$$\pi = \max_{(c,n) \in Y} \sum_{t=0} p_t \left[c_t - w_t n_t\right]$$

Exercise. Show that from the firm's problem can be written as

$$\pi = \max_{\{k_{t+1}, n_t\}} \sum_{t=0} p_t \left[F\left(k_t, n_t\right) - w_t n_t - \left(k_{t+1} - k_t\left(1 - \delta\right)\right)\right]$$

and hence it must satisfy

$$\begin{aligned} w_t &= F_n\left(k_t, n_t\right) \\ \delta + r_t &= F_k\left(k_{t+1}, n_{t+1}\right) \\ \pi &= F_k\left(\kappa_0, 1\right) k_0 + \left(1 - \delta\right) k_0 \end{aligned}$$

for all $t \geq 0$ where κ_0 solves $w_0 = F_n\left(\kappa_0, 1\right)$ and where $1 + r_t = p_t / p_{t+1}$. Thus, r_t is the time t interest rate.

(Hint: first use that F has CRTS and use the foc w.r.t. n_t to show that

$$\pi = \max_{\{k_{t+1}\}} \sum_{t=0} p_t \left[F_k\left(\kappa_t, 1\right) k_t - \left(k_{t+1} - k_t\left(1-\delta\right)\right)\right]$$

where κ_t solves $F_n\left(\kappa_t, 1\right) = w_t$ and then write

$$\begin{aligned} \pi &= F_k\left(\kappa_0, 1\right) k_0 + \left(1-\delta\right) k_0 \\ &+ \max_{k_{t+1}} \sum_{t=0} p_t k_{t+1} \left\{ -1 + \frac{1}{1+r_t}\left(F_k\left(\kappa_{t+1}, 1\right) + \left(1-\delta\right)\right)\right\} \end{aligned}$$

Exercise. Show that the foc for the consumer's problem give

$$\begin{aligned} \frac{u_{c_{t+1}}\left(c_0, 1-n_0, c_1, 1-n_1, \ldots\right)}{u_{c_t}\left(c_0, 1-n_0, c_1, 1-n_1, \ldots\right)}\left(1+r_t\right) &= 1 \\ \frac{u_{l_t}\left(c_0, 1-n_0, c_1, 1-n_1, \ldots\right)}{u_{c_t}\left(c_0, 1-n_0, c_1, 1-n_1, \ldots\right)} &= w_t \end{aligned}$$

And for the additively separable case:

$$\begin{aligned} \beta \frac{v_c\left(c_{t+1}, l_{t+1}\right)}{v_c\left(c_t, l_t\right)}\left(1+r_t\right) &= 1 \\ \frac{v_l\left(c_t, l_t\right)}{v_c\left(c_t, l_t\right)} &= w_t \end{aligned}$$

Version II.
Households own capital, which is rented to the firm

- *Commodity Space:*
  Let $L = R^{3T}$ for $T = \infty$, or more formally
  $$L = \left\{ \{c_t, n_t, k_t\}_{t=0}^{\infty} \left| (c_t, n_t, k_t) \in R^3 \right. \right\}$$
- *Production possibility set of the firm:*
  $$Y = \{\{c_t, n_t, k_t\} : c_t + x_t \leq F(n_t, k_t)\}$$

- *Consumption possibility set:*
  $$\begin{aligned} X &= \{\{c_t, n_t, k_t\} : 0 \leq n_t \leq 1, \\ &\quad k_{t+1} = x_t + k_t(1-\delta) \text{ for all } t \geq 0, \ k_0 \text{ given}\} \end{aligned}$$
- *Household budget constraint:*
  $$\sum_{t=0}^{\infty} p_t [x_t + c_t + w_t l_t] = \sum_{t=0}^{\infty} p_t [w_t + \upsilon_t k_t]$$
  where p_t and w_t are as before, and where $p_t \upsilon_t$ is the rental price of capital at time t. Thus, υ_t is the time t real rental price of capital in terms of time t consumption units.
- *Firm's problem:*
  $$\pi = \max_{(c,n,k) \in Y} \sum_{t=0} p_t [c_t - w_t n_t - \upsilon_t k_t]$$

Exercise. Show that in an equilibrium $\pi = 0$ and

$$\begin{aligned} w_t &= F_n(k_t, n_t) \\ \upsilon_t &= F_k(k_t, n_t) \end{aligned}$$

for all $t \geq 0$.
Exercise: Show that in equilibrium

$$\upsilon_{t+1} = r_t + \delta \text{ where } 1 + r_t = \frac{p_t}{p_{t+1}}$$

for $t \geq 0$. (Hint: collect the terms in the budget constraint with k_{t+1} after using that $x_t = k_{t+1} - k_t(1-\delta)$, to obtain

$$k_{t+1}\{p_t - p_{t+1}[\upsilon_{t+1} + (1-\delta)]\}$$

and use that $1 + r_t = p_t/p_{t+1}$)

Steady states

Definition. We call an steady state an equilibrium with

$$\begin{aligned} r_t &= r,\; w_t = w,\; \upsilon_t = \upsilon \\ l_t &= l,\; k_t = k,\; x_t = x,\; c_t = c \end{aligned}$$

for all $t \geq 0$ with $k_0 = k$.

Alternatively, an steady state is an initial condition k_0 such that if the economy starts at that value it continues with it.

Exercise. Assume that u is additively separable, with discount factor β and utility v.

Show that $\{w, v, r, l, c, x, k\}$ is an steady state if and only if it satisfies the following equations:

$$
\begin{aligned}
k &= x + k(1-\delta) \\
c + x &= F(k, n) \\
w &= F_n(k, n) \\
\upsilon &= F_k(k, n) \\
r + \delta &= \upsilon \\
(1+r)\beta \frac{v_c(c, l)}{v_c(c, l)} &= 1 \\
\frac{v_l(c, l)}{v_c(c, l)} &= w
\end{aligned}
$$

Assumption: additively separable From now on, unless otherwise stated, assume u is given by the present value of $v(c_t, 1-n_t)$, with discount rate β.

Assumption: c and l normal goods. From now on, unless otherwise stated, assume that both consumption and leisure are normal goods in $v(c, 1-n)$.

Proposition. If $v(c, 1-n)$ is such that leisure is a normal good, then there is a **unique** steady state.

Proof. Define ρ as $1/(1+\rho) = \beta$ then using that

$$(1+r)\,\beta\frac{v_c(c,l)}{v_c(c,l)} = 1$$

we have that

$$r = \rho$$

Using $\upsilon = \rho + \delta$ and $\upsilon = F_k(k,n)$ we have that κ is the solution of

$$\rho + \delta = F_k(\kappa, 1)$$

We also have from the foc for n

$$w = F_n(\kappa, 1)$$

Using $x = \delta k$ into $c + x = F(k,n)$, and by virtue of Euler's Theorem we have

$$c + \delta k = F_k(\kappa,1)\,k + F_n(\kappa,1)\,n$$

or using $F_k(\kappa,1) = \rho + \delta$, and $w = F_n(\kappa,1)$, we have

$$c + \delta k = (\rho + \delta)\,k + wn$$

or

$$c + lw = \rho k + w$$

Finally, consider the problem

$$\max_{c,l} v(c,l) \text{ subject to } c + lw = \rho k + w$$

Notice that its solution satisfies,

$$\frac{v_l(c,l)}{v_c(c,l)} = w$$

and the resource constraint $c + \delta k = F(k,n)$ provided that $k/n = \kappa$.

We now find the value of k for which this is true. Denote the solution of the maximization problem stated above by $c = C(k)$ and $l = L(k)$.

Notice that since leisure is normal, L is strictly increasing, and hence $N(k) \equiv 1 - L(k)$ is strictly decreasing. Now, k is the solution of

$$N(k) = (1/\kappa)\, k$$

Since $N(0) > 0$ and N is decreasing, there is a unique solution k. QED

Add government purchases to the model

.
The feasibility constraint is now

$$c_t + x_t + g_t = F(k_t, l_t)$$

and households are charged lump sum taxes τ_t per period to finance these purchases.

The household budget constraint is now

$$\sum_{t=0}^{\infty} p_t \left[x_t + c_t + w_t l_t + \tau_t\right] = \sum_{t=0}^{\infty} p_t \left[w_t + \upsilon_t k_t\right]$$

and the government budget constraint is

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \tau_t$$

Assume that $g_t = g$ are constant.

Exercise. Show that in this case the steady state has the same capital output ratio, higher n, and lower consumption c.

[Hint: modify the previous proof, considering the problem:

$$\max_{c,l} v(c,l) \text{ subject to } c + lw = \rho k + w - g$$

with solution $L(k,g)$ and $N(k,g) = 1 - L(k,g)$.]

Exercise. **Neutral Productivity shock.**

Replace the production function $F(k,n)$ by $AF(k,n)$ with $A > 1$. What is the effect on the steady state values of r, w, k/n, and k. What can you say of the effect on n?

[Hint: Repeat the analysis of the problem

$$\max_{c,l} v(c,l) \text{ subject to } c + lw(A) = \rho k + w(A)$$

with solution $L(k,A)$ and $N(k,A) = 1 - L(k,A)$ and with $\kappa(A)$ solving

$$r + \delta = AF_k(\kappa, 1).]$$

Exercise. **Productivity shock.**

Replace the production function $F(k, n)$ by $F(k, An)$ with $A > 1$.

- What is the effect on the steady state values of r, w, $\kappa = k/A$, and k.
- What can you say of the effect on n?
- What is the answer if

$$\begin{aligned} v(c, l) &= \frac{c^{1-\sigma}}{1-\sigma} h(n) \text{ for } \sigma > 0 \text{ and } \sigma \neq 1 \\ v(c, l) &= \log c + h(n) \quad \text{for } \sigma = 1. \end{aligned}$$

[Hint: Repeat the analysis of the problem

$$\max_{c,l} v(c, l) \text{ subject to } c + lw(A) = \rho k + w(A)$$

with solution $L(k, A)$ and $N(k, A) = 1 - L(k, A)$ and with $\kappa(A)$ solving

$$r + \delta = F_k\left(\frac{\kappa}{A}, 1\right).]$$

Exercise. **Infinite elastic savings in the long run.**

Suppose you compute the steady state for economies with different:

1. utility function u,
2. production function F,
3. depreciation rate δ,
4. discount factor β (or discount rate ρ).

When is that the steady state interest rate change?

What does this said about the long run (i.e. steady state) elasticity of the supply of capital (i.e. savings) in this model?

Parametric Example.

Let the utility function and production functions be

$$\begin{aligned} v(c, l) &= \alpha \log c + (1-\alpha) \log l \\ F(k, n) &= k^{\theta} n^{1-\theta} \end{aligned}$$

We will solve the steady state of this model assuming that government purchases, as a fraction of gnp are equal to $g/y = \eta$.

The values of $r,\ v,\ k/y,\ x/y,\ c/y$ and n as a function of the parameters are:

$$\begin{aligned} r &= \rho \\ \upsilon &= \rho + \delta \end{aligned}$$

$$\begin{aligned} \frac{k}{y} &= \frac{k\,\upsilon}{y}/\upsilon = \frac{\theta}{\rho+\delta} \\ \frac{x}{k} &= \delta \\ \frac{x}{y} &= \frac{x}{k}\frac{k}{y} = \delta\,\frac{\theta}{\rho+\delta} \\ \frac{c}{y} &= 1 - \frac{x}{y} - \frac{g}{y} = 1 - \delta - \eta \end{aligned}$$

$$\begin{aligned} \frac{n}{1-n} &= \frac{(1-\theta)}{(1-\delta-\eta)}\frac{\alpha}{1-\alpha} \text{ or} \\ n &= \frac{(1-\theta)\,\alpha}{(1-\delta-\eta)(1-\alpha) + (1-\theta)\,\alpha} \end{aligned}$$

Except the last, the first six equations follow quite directly from the answer of a previous exercise.

To see where we get the expression for n, notice that

$$w = \frac{v_l}{v_c} = \frac{1-\alpha}{\alpha} \frac{c}{1-n}$$

so

$$\frac{(1-\theta)\, y}{(1 - x/y - g/y)\, y} \frac{\alpha}{1-\alpha} = \frac{wn}{c} \frac{\alpha}{1-\alpha} = \frac{n}{1-n}$$

Numerical Example

Consider the following numerical values. Set the share parameters $\alpha = 1/3, \ \theta = 0.3$.

Assume that the depreciation rate and discount rate are 7.5%, i.e. $\delta = 0.075$ and $\rho = 0.075$.

Finally, set the steady state where government purchases as 15% of gnp, i.e. $\eta = 0.15$.

In this case we have:

$$\begin{aligned} r &= 0.075 \\ \upsilon &= 0.15 \end{aligned}$$

$$\begin{aligned} \frac{k}{y} &= 2 \\ \frac{x}{k} &= 0.075 \\ \frac{x}{y} &= 0.15 \\ \frac{c}{y} &= 0.7 \\ n &= 1/3 \end{aligned}$$

Linear Taxes in Steady State

- Assume utility is additively separable: $\sum_{t=0}^{\infty} \beta^t v(c_t, l_t)$
- Linear taxes on labor and (net) capital income.
- Obtain budget constraints (own and operate capital)

  (will return to this when discussing Ramsey problem)
- Specialize to Steady State

Consumer's budget constraint

$$\sum_{t=0}^{\infty} p_t \left[c_t + x_t + \tau_t\right] = \sum_{t=0}^{\infty} p_t \left[\left(1 - \tau_{lt}\right) w_t n_t + k_t \upsilon_t - k_t \left(\upsilon_t - \delta\right) \tau_{kt}\right],$$

and the law of motion of capital

$$k_{t+1} = x_t + \left(1 - \delta\right) k_t$$

where p_t is the Arrow-Debreu price of consumption goods at time t in terms of time zero consumption good, and w_t and υ_t are the before tax real wage and rental rate of capital in terms of consumption t goods.

There are two tax rates, τ_{lt} is the tax rate on labor income and τ_{kt} is that tax rate on capital and a lump sum tax τ_t, denominated in time t units of the consumption good.

The time t income from renting capital equals

$$\upsilon_t k_t - \tau_{kt} \left(\upsilon_t - \delta\right) k_t = \upsilon_t k_t \left(1 - \tau_{kt}\right) + \tau_{kt} \delta k_t$$

The firm's problem is

$$\max_{\{k_t, n_t\}} \sum_{t=0}^{\infty} p_t \left[F\left(k_t, n_t\right) - w_t n_t - \upsilon_t k_t\right]$$

Government purchases are denoted by g_t.

The government budget constraint is

$$\sum_{t=0}^{\infty} p_t \left[\tau_t + \tau_{lt} w_t n_t + k_t \left(\upsilon_t - \delta\right) \tau_{kt}\right] = \sum_{t=0}^{\infty} p_t g_t$$

Long run effect of taxes

- We will analyze the steady state as a function of the assumed constant g, τ_k and τ_l.
- We assume that the lump sum taxes τ adjust to satisfy the government budget constraint.
- We will assume that v is C^2, strictly concave and strictly increasing in (c, l), and that satisfies the standard Inada conditions for so that so that we don't have to worry about corner solutions for (c, l).

Question

Using λ for the multiplier of the budget constraint of the agent, write down the foc with respect to c_t and n_t.

Use these first order conditions to obtain an expression for the marginal rate of substitution of c_{t+1} with c_t and for the marginal rate of substitution of c_t with n_t.

Answer:

$$\begin{aligned} 0 &= \beta^t v_c(c_t, l_t) - \lambda p_t \\ 0 &= \beta^t v_l(c_t, l_t) - \lambda p_t w_t (1 - \tau_{lt}) \end{aligned}$$

thus

$$\begin{aligned} \beta \frac{v_c(c_{t+1}, l_{t+1})}{v_c(c_t, l_t)} &= \frac{p_{t+1}}{p_t} \\ \frac{v_l(c_t, l_t)}{v_c(c_t, l_t)} &= w_t (1 - \tau_{lt}) \end{aligned}$$

Question.

Write down the first order conditions for the firm's maximization problem.

Answer:

$$\begin{aligned} F_k(k_t, n_t) &= \upsilon_t \\ F_n(k_t, n_t) &= w_t \end{aligned}$$

Question.

Using the households budget constraint, find an expression for υ_{t+1} in terms of δ, r_t and τ_{kt+1} that must hold if agents will find optimal to choose $0 < k_{t+1} < \infty$. From here on we let r_t be the time t real interest rate, i.e. $p_{t+1}/p_t = 1/(1 + r_t)$.

Answer:

$$\upsilon_{t+1} = \delta + \frac{r_t}{1 - \tau_{kt+1}}$$

since:

$$\begin{aligned} & ... - p_t k_{t+1} + ... + p_{t+1} k_{t+1} \left[(1-\delta) + \upsilon_{t+1}(1 - \tau_{kt+1}) + \tau_{kt+1}\delta\right] + ... \\ = \quad & ... - k_{t+1} + ... + \frac{p_{t+1}}{p_t} k_{t+1} \left[(1-\delta) + \upsilon_{t+1}(1 - \tau_{kt+1}) + \tau_{kt+1}\delta\right] + \end{aligned}$$

thus

$$1 + r_t = (1-\delta) + \upsilon_{t+1}(1 - \tau_{kt+1}) + \tau_{kt+1}\delta$$
$$r_t = (\upsilon_{t+1} - \delta)(1 - \tau_{kt+1})$$

Question.

Assume that the tax rate are constant, i.e. $\tau_{kt} = \tau_k$ and $\tau_{lt} = \tau_l$ for all t. Let κ be the capital labor ratio, $\kappa \equiv k/n$, and ρ be the time discount factor $\beta = 1/(1+\rho)$.

Use the answer for the previous question, as well as feasibility and the law of motion of capital to find system of 7 equations in the following 7 variables: $r,\ \kappa,\ w,\ x,\ c,\ l$ and k that characterize a steady state.

In particular write them so that equation i) relates r and ρ, eqn. ii) relates $\kappa,\ \rho,\ \tau_k\ ,\ \delta$ and $F_k(\kappa, 1)$, eqn iii) relates $w,\ \kappa$ and $F_n(\kappa, 1)$, eqn. iv) relates $v_l/v_c(\cdot)$, w and τ_l, eqn. v) relates $c,\ \kappa,\ \delta,\ l,\ g$ and $F(\kappa, 1)$, eqn. vi) relates x, δ, κ and l, and eqn vi) relates k, κ, and l.

Answer:

$$
\begin{aligned}
i) &\ :\ r = \rho \\
ii) &\ :\ F_k(\kappa, 1) = \frac{\rho}{1-\tau_k} + \delta \\
iii) &\ :\ w = F_n(\kappa, 1) \\
iv) &\ :\ \frac{v_l(c,l)}{v_c(c,l)} = w(1-\tau_l) \\
v) &\ :\ c = (F(\kappa,1) - \delta\kappa)(1-l) - g \\
vi) &\ :\ x = \delta\kappa\ (1-l) \\
vii) &\ :\ k = \kappa\ (1-l)
\end{aligned}
$$

Check that these equations coincides with lump-sum taxes and without linear taxes.

Question. [Normal good characterization]

Consider the optimal choice for a static problem with utility v for two level of income and the same relative prices. If c and l are normal goods,then

$$\frac{\partial}{\partial c}\frac{v_l(c,l)}{v_c(c,l)} > 0 \text{ and } \frac{\partial}{\partial l}\frac{v_l(c,l)}{v_c(c,l)} < 0 \text{ . Why?}$$

a. Show that if c and l are normal goods, one can define a function $\phi(l,\omega)$ such that

$$\frac{v_l(\phi(l,\omega),l)}{v_c(\phi(l,\omega),l)} = \omega.$$

b. Show that ϕ is increasing in ω and l.

c. Suppose that $v(c,l) = \left(\left[c^{(1-\alpha)}l^{\alpha}\right]^{1-\gamma} - 1\right)/(1-\gamma)$. Find and expression for $\phi(l,\omega)$ in this case.

Linear Taxes steady state characterization

Answer a. That ϕ is well defined follows since

$$\sigma(x,l) \equiv \frac{v_l(x,l)}{v_c(x,l)}$$

is increasing by assumption that l is a normal good, and hence there is a unique solution to:

$$\sigma(\phi,l) = \omega$$

That there is a solution it follows from the Inada conditions.

Answer b.

$$\frac{\partial}{\partial c}\left(\frac{v_l(\phi(l,\omega),l)}{v_c(\phi(l,\omega),l)}\right)\frac{\partial\phi(l,\omega)}{\partial l}+\frac{\partial}{\partial l}\left(\frac{v_l(\phi(l,\omega),l)}{v_c(\phi(l,\omega),l)}\right)=0$$

or

$$\frac{\partial\phi(l,\omega)}{\partial l}=\frac{-\frac{\partial}{\partial l}\left(\frac{v_l}{v_c}\right)}{\frac{\partial}{\partial c}\left(\frac{v_l}{v_c}\right)}>0$$

and

$$\frac{\partial}{\partial c}\left(\frac{v_l(\phi(l,\omega),l)}{v_c(\phi(l,\omega),l)}\right)\frac{\partial\phi(l,\omega)}{\partial\omega}=1$$

or

$$\frac{\partial\phi(l,\omega)}{\partial\omega}=\frac{1}{\frac{\partial}{\partial c}\left(\frac{v_l}{v_c}\right)}>0$$

Answer c.

$$\phi(l,\omega)=\omega\frac{1-\alpha}{\alpha}l$$

Question. [Steady state characterization, existence & uniqueness]

Consider the system of two equations

$$\begin{aligned}\frac{v_l(c,l)}{v_c(c,l)} &= F_n(\kappa,1)(1-\tau_l)\\ c &= (F(\kappa,1)-\delta\kappa)(1-l)-g\end{aligned}$$

in two unknowns, c, l for given values of $\kappa,\ g$, and τ_l. Assume that $v(c,l)$ is such that c and l are both normal goods.

Using the previous results write this system as

$$\begin{aligned}c &= \phi(l\ ,\ F_n(\kappa,1)(1-\tau_l)\ )\\ c &= (F(\kappa,1)-\delta\kappa)\ (1-l)-g\end{aligned}$$

Question.

Show that, given κ, g, and τ_l, there is a unique c and l that solve this equation. Define $C(\kappa, g, \tau_l)$ and $L(\kappa, g, \tau_l)$ as the solution.

It may help plotting the RHS of the two expressions with c in the vertical axis and l in the horizontal axis. Mark the values of $C(\kappa, g, \tau_l)$ and $L(\kappa, g, \tau_l)$ in your graph.

Answer.

The function $\phi(l\ ,\ F_n(\kappa, 1)(1-\tau_l)\ )$ is increasing in l. The function $(F(\kappa, 1) - \delta\kappa)\ (1-l) - g$ is linear in l, and decreasing.

For the next three parts of the question, draw three different set of plots, in the first draw the functions for two values of τ_l, in the the second for two values of κ, and in the third for two values of g.

Mark the intersection of the two functions (i.e. the solution of the system) for the baseline parameters and for the new ones.

Question

Using your first plot, answer whether $C(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of τ_l and whether $L(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of τ_l

Answer

The function $\phi(l\ ,\ F_n(\kappa, 1)(1-\tau_l)\ )$ is increasing in its second argument, so it is decreasing in τ_l, i.e. the function shifts down for higher $\tau_{l.}$. So C is decreasing in τ_l and L increasing in τ_l.

Question

Using the second plot, answer whether $C(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of κ and whether $L(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of κ

Answer

The function $\phi(l\ ,\ F_n(\kappa, 1)(1 - \tau_l)\ )$ is increasing in its second argument, and F_n is increasing in κ, so the function shifts up higher κ.

The function $(F(\kappa, 1) - \delta\kappa)\ (1 - l) - g$ is increasing in κ.

Thus, C is increasing in κ and L can be increasing or decreasing in κ.

Linear Taxes steady state characterization

Question

Using the third plot, answer whether $C(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of g and whether $L(\kappa, g, \tau_l)$ is a decreasing, increasing or possible a non-monotone function of g.

Answer

The function $(F(\kappa, 1) - \delta\kappa)\ (1 - l) - g$ is decreasing in g.

Thus, C is decreasing in κ, and L is decreasing in g.

For the next question, define the functions $C^*(\tau_k, \tau_l, g)$ and $L^*(\tau_k, \tau_l, g)$ as

$$\begin{aligned} C^*(\tau_k, \tau_l, g) &= C(\kappa^*(\tau_k), g, \tau_l) \\ L^*(\tau_k, \tau_l, g) &= L(\kappa^*(\tau_k), g, \tau_l) \end{aligned}$$

where $\kappa^*(\tau_k)$ solves

$$F_k(\kappa^*(\tau_k), 1) = \delta + \frac{\rho}{1-\tau_k}$$

The functions C^* and L^* give the steady state values of consumption and leisure in terms of the fiscal policy parameters (τ_k, τ_l, g).

Question. [Summary of long run effects of taxes]

Based on the answer to the previous question(s), complete the following table. In each entry write $+$, $-$, $=$ or ? if the quantity increases, decreases, stay constant, or its effect can not be determined.

That is, examine what happens with the steady states values C^*, L^*, κ^* as well as other steady state objects (r, w, $w(1-\tau_l)$, v, $(v-\delta)(1-\tau_k)$, n, and x) as a function of the fiscal policy parameters τ_l, τ_k and g.

Assume that c and l are normal goods.

[Hint: First find the effect of the fiscal policy on κ^*, and then use this info to the RHS of the equations defined above]

Answers

steady-state quantitiy fiscal policy parameter	τ_l	τ_k	g
interest rate, r			
before tax wages, w			
after-tax wages $w(1-\tau_l)$			
capital-labor ratio, κ			
before tax rental rate of capital v			
after tax net rental rate of capital $(v-\delta)(1-\tau_k)$			
consumption, c			
labor supply, n			
capital stock, k			
investment, x			

Sequence budget constraint (skip)

Consider the following sequence of budget constraints

$$q_t a_{t+1} + c_t + l_t w_t = w_t + a_t$$

for all $t \geq 0$ and a_0 given. In these constraints, a_t are beginning of period assets, q_t is the time t price of a good at time $t+1$. Purchasing a_{t+1} assets at time t gives the agent a_{t+1} consumption goods at time $t+1$.

Proposition. Let $q_t = 1/(1+r_t)$ and $\pi = a_0$. 1) If $\{c_t, l_t\}$ satisfy the sequence budget constraints described above and if

$$\lim_{t\to\infty} [q_0 q_1 \ldots q_t]\, a_{t+1} = 0$$

then $\{c_t, l_t\}$ satisfies the present value budget constraint

$$\sum_{t=0}^{\infty} p_t [c_t + w_t l_t] = \pi + \sum_{t=0}^{\infty} p_t w_t$$

2) If $\{c_t, l_t\}$ satisfies the present value budget constraint, then it satisfies the sequence budget constraints.

Proof. To show that the present value budget constraint implies the sequence budget constraint, define a_t as

$$a_t = \frac{1}{p_t} \sum_{s=t}^{\infty} p_s \left[c_s + w_s l_s - w_s \right]$$

for all $t \geq 0$. Compute

$$q_t a_{t+1} = \frac{q_t}{p_{t+1}} \sum_{s=t+1}^{\infty} p_s \left[c_s + w_s l_s - w_s \right]$$

using that $q_t = 1/\left(1 + r_t\right) = p_{t+1}/p_t$ so that

$$\begin{aligned} q_t a_{t+1} - a_t &= \frac{1}{p_t} \sum_{s=t+1}^{\infty} p_s \left[c_s + w_s l_s - w_s \right] - \frac{1}{p_t} \sum_{s=t}^{\infty} p_s \left[c_s + w_s l_s - w_s \right] \\ &= -\frac{p_t}{p_t} \left[c_t + w_t l_t - w_t \right] = - \left[c_t + w_t l_t - w_t \right] \end{aligned}$$

or

$$q_t a_{t+1} + c_t + w_t l_t - w_t = a_t.$$

To show that the sequence budget constraint implies the present value budget constraint, take the time t sequence budget constraint and multiply it by $q_{t-1} q_{t-2} \ldots q_0$, add them up from time $t = 0$ to time $t = T$. For instance, let $T = 2$ and let $z_t = c_t + w_t l_t - w_t$, so we have

$$\begin{aligned} q_0 a_1 + z_0 &= a_0 \\ q_0 q_1 a_2 + q_0 z_1 &= q_0 a_1 \\ q_1 q_0 q_2 a_3 + q_1 q_0 q_2 z_2 &= q_1 q_0 a_2 \end{aligned}$$

and adding them up

$$q_1 q_0 q_2 a_3 + q_1 q_0 q_2 z_2 + q_0 z_1 + z_0 = a_0$$

In general, we have

$$\frac{p_T}{p_0} a_{T+1} + \sum_{t=0}^{T} p_t \left[c_t + l_t w_t - w_t \right] = a_0.$$

QED.

Discounted utility I. Additive Separable

Interpretation of

$$u(c_0, l_0, c_1, l_1, c_2, l_2, ...) = \sum_{t=0}^{\infty} \beta^t v(c_t, l_t)$$

Suppose that a period is a generation and that parents are altruistic toward their children. We can capture this by assuming that parent care about their consumption and their children well-being. In this case we can write the utility of generation t as

$$U_t = v(c_t, l_t) + \beta U_{t+1}$$

Solving this forward by repeated substitutions we arrive to the previous expression.

Discounted utility II. Recursive formulation

A feature of the discounted utility defined above, is that it makes the marginal rate of substitution between consumption at dates t' and t, for $t' > t$ independent of the consumptions at dates before t.

This feature simplify the analysis a lot. This feature will be shared for any utility function that satisfies

$$U_t = A(c_t, l_t, U_{t+1})$$

where again we obtain U_t as a function of $(c_0, l_0, c_1, l_1, c_2, l_2, ...)$ by repeated substitution.

Notice that the previous case A was given by

$$A(c_t, l_t, U_{t+1}) = v(c_t, l_t) + \beta U_{t+1}.$$

Uzawa's Model

The following is an example of a function A

$$A(c_t, l_t, U_{t+1}) = v(c_t, l_t) + e^{-[\rho + v(c_t, l_t)]} U_{t+1}.$$

Exercise. Show that solving this forward gives

$$U_t = \sum_{t=0}^{\infty} v(c_t, l_t) \, e^{-\sum_{s=0}^{t-1}[\rho + v(c_s, l_s)]}$$

Intuitively, and as a first approximation, the main difference of this preferences with the standard ones, is that in the Uzawa's preferences the rate of discount $[\rho + v(c, l)]$ is variable.

So that if agent's consume a lot, and hence, have a high $v(c, l)$, then their discount rate is higher, and thus they are more impatient.

Discounted Utility: interpretations

Exercise. Consider the problem of maximizing the utility

$$U_t = \sum_{t=0}^{\infty} v(c_t) \, e^{-\sum_{s=0}^{t-1}[\rho + \theta v(c_s)]}$$

subject to

$$\frac{1}{1+r} a_{t+1} + c_t = w + a_t$$

(Notice that if $\theta = 0$ we have the standard case). We suppress labor to simplify the problem.

Exercise (cont).

1) Show that $\frac{\partial U_0}{\partial c_t}(c_0, c_1, c_2, ...)$ can be written as

$$\frac{\partial U_0}{\partial c_t}(c_0, c_1, c_2, ...) = e^{-\sum_{s=0}^{t-1}[\rho+\theta v(c_s)]} v'(c_t) \left\{ 1 - \theta \sum_{r=t+1}^{\infty} v(c_r) e^{-\sum_{s=t}^{r-1}[\rho+\theta v(c_s)]} \right\}$$

2) Write down the first order conditions for c_t this problem. Use λ for the Lagrange multiplier on the present value budget constraint

$$\sum_{t=0} \left(\frac{1}{1+r}\right)^t (c_t - w) = a_0$$

Show that this foc can be written as

$$\begin{aligned}\frac{1}{1+r} &= \frac{\frac{\partial U_0}{\partial c_{t+1}}(c_0, c_1, c_2, ...)}{\frac{\partial U_0}{\partial c_t}(c_0, c_1, c_2, ...)} \\ &= \frac{e^{-[\rho+\theta v(c_t)]}\left\{1 - \theta \sum_{r=t+2}^{\infty} v(c_r) e^{-\sum_{s=t+1}^{r-1}[\rho+\theta v(c_s)]}\right\}}{\left\{1 - \theta \sum_{r=t+1}^{\infty} v(c_r) e^{-\sum_{s=t}^{r-1}[\rho+\theta v(c_s)]}\right\}} \frac{v'(c_{t+1})}{v'(c_t)}\end{aligned}$$

Discounted Utility: interpretations

Exercise (cont).

3) Solve for the value of c that is consistent with the optimal solution satisfying $c_t = c$ for all c, so that

$$\frac{\frac{\partial U_0}{\partial c_{t+1}}(c_0, c_1, c_2, ...)}{\frac{\partial U_0}{\partial c_t}(c_0, c_1, c_2, ...)} = \frac{\frac{\partial U_0}{\partial c_{t+1}}(c, c, c, ...)}{\frac{\partial U_0}{\partial c_t}(c, c, c, ...)} = \frac{1}{1+r}$$

Use that if $c_t = c$ for all t,

$$\begin{aligned}\frac{v'(c_{t+1})}{v'(c_t)} &= 1, \\ \beta(c) &= e^{-[\rho+\theta v(c)]},\end{aligned}$$

and

$$\sum_{r=t+1}^{\infty} e^{-\sum_{s=t}^{r-1}[\rho+\theta v(c)]} = \sum_{r=t+1}^{\infty} \beta(c)^{r-1-t} = \sum_{s=0}^{\infty} \beta(c)^s = \frac{1}{1-\beta(c)},$$
$$\sum_{r=t+2}^{\infty} e^{-\sum_{s=t+1}^{r-1}[\rho+\theta v(c)]} = \sum_{r=t+2}^{\infty} \beta(c)^{r-t-2} = \sum_{s=0}^{\infty} \beta(c)^s = \frac{1}{1-\beta(c)}$$

so that

$$\frac{\frac{\partial U_0}{\partial c_{t+1}}(c,c,c,...)}{\frac{\partial U_0}{\partial c_t}(c,c,c,...)} = \frac{e^{-[\rho+\theta v(c)]}\left\{1-\theta v(c)\sum_{r=t+2}^{\infty} e^{-\sum_{s=t+1}^{r-1}[\rho+\theta v(c)]}\right\}}{\left\{1-\theta v(c)\sum_{r=t+1}^{\infty} e^{-\sum_{s=t}^{r-1}[\rho+\theta v(c)]}\right\}}\frac{v'(c)}{v'(c)}$$
$$= \beta(c)$$

Exercise (cont).

4) Let $\hat{r}$ be defined as

$$\frac{1}{1+r} = e^{-\hat{r}}$$

show that c solves

$$\hat{r} = \rho + \theta v(c).$$

5) Consider first the case where $\theta = 0$.How is that c is determined for different w? [Hint: use the budget constraint]. Notice that this expression does not depend on w,and if $\theta > 0$ determines uniquely c. What is the interpretation of this equation in terms of the long run distribution of consumption if there is heterogeneity of w?

Steady state in Uzawa's model.

Continue with the Uzawa set up of the previous exercise. Add a neoclassical production function $A\,F(k,n)$. Continue to assume that labor is supplied inelastically, say at $n=1$.

Notice that from our previous analysis we still have that in steady state

$$\begin{aligned} v &= r+\delta \\ F_k(k,1) &= v \\ c+\delta k &= F(k,1) \end{aligned}$$

with the extra addition of the condition equating the marginal rate of substitution to the interest rate in steady state,

$$\beta(c)(1+r)=1$$

where $\beta(\cdot)$ is given by

$$\beta(c)=e^{-[\rho+\theta v(c)]}$$

Discounted Utility: interpretations

In this case we can combine these equations to obtain on equation on k :

$$\beta(F(k,1)-\delta k)(1-\delta+F_k(k,1))=1$$

or

$$\rho+\theta v(F(k,1)-\delta k)=\log(1-\delta+F_k(k,1))$$

Since the LHS is increasing in k and the RHS is decreasing in k there is at most one solution.

Notice also that the LHS of this equation equals $\hat{r}$, where $e^{\hat{r}}=1+r$.

One can regard the LHS as the steady state supply of capital, and the RHS as the steady state demand of capital.

Exercise. Productivity shock in the Uzawa model.

Suppose the production function $F(k, 1)$ changes to $A\, F(k, 1)$ for $A > 1$ in the previous set up.

What is the effect on the steady state value of k and in the steady state interest rate r?

Compare this case with the standard one where $\theta = 0$.

What is the economic intuition behind this difference?