Continuous-Time Dynamic Optimization

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Last updated: September 17, 2018

1 Dynamic Optimization Problems

1.1 Finite Horizon

1.1.1 Dynamic Problems with Two Boundary Conditions

$$\begin{aligned} &\max_{x(\cdot)\in C^\infty} \int_{t_0}^{t_1} F\big(t, x(t), x'(t)\big)dt \\ &\text{s.t.} \quad x(t_0) = x_0 \\ &\qquad\; x(t_1) = x_1, \end{aligned} \tag{P1}$$

where x_0, $x_1 \in R$, and $F \in C^\infty$ are given. The objective is to find a smooth function $x(\cdot)$ defined on $[t_0, t_1]$ satisfying the two boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$ to maximize the objective function, $\int_{t_0}^{t_1} F\big(t, x(t), x'(t)\big)dt$.

Necessary Conditions If $x^*(\cdot)$ is the solution to the problem above, then the following conditions must be true

(1) $F_x\big(t, x^*(t), x^{*'}(t)\big) = \frac{d}{dt}F_{x'}\big(t, x^*(t), x^{*'}(t)\big)$ for all $t \in [t_0, t_1]$

(2) $x^*(t_0) = x_0$

(3) $x^*(t_1) = x_1$

Condition (1) is called the **Euler equation**. If we further develop the RHS of condition (1), we will get a second-order differential equation (DE) for $x^*(t)$

$$F_x = F_{x't} + F_{x'x}x' + F_{x'x'}x'' \quad \text{for all} \quad t \in [t_0, t_1].$$

Note that conditions (1)-(3) are necessary conditions (may not be sufficient) to reach extremal values, just like the first order conditions in the static problems, so the solution to the DE above might be a minimizer instead of a maximizer.

1.1.2 Dynamic Problems with One Boundary Condition

Now, consider a problem without any restriction on the value of x at the endpoint when $t = t_1$.

$$\begin{aligned} \max_{x(\cdot)\in C^\infty} \quad & \int_{t_0}^{t_1} F\big(t, x(t), x'(t)\big)dt \\ \text{s.t.} \quad & x(t_0) = x_0 \end{aligned} \tag{P2}$$

Necessary Conditions If $x^*(\cdot)$ is the solution to the problem above, then the following conditions must be true

(1) $F_x\big(t, x^*(t), x^{*'}(t)\big) = \frac{d}{dt}F_{x'}\big(t, x^*(t), x^{*'}(t)\big)$ for all $t \in [t_0, t_1]$

(2) $x^*(t_0) = x_0$

(3) $F_{x'}\big(t_1, x^*(t_1), x^{*'}(t_1)\big) = 0$

Condition (3) is called the **transversality condition** for this dynamic problem. It says that the marginal value of x' at the endpoint $t = t_1$ has to be zero in the optimal. Intuitively speaking, an infinitesimal changes in the last moment does not raise the value. Otherwise, the optimal is not reached.

In some cases, we might require x to be non-negative at the endpoint, that is, $x(t_1) \geq 0$. For example, if households are not allowed to die with debts, then they will have to hold a positive value for assets at the end of lifetime when $t = t_1$. Consider the following problem.

$$\begin{aligned} \max_{x(\cdot)\in C^\infty} \quad & \int_{t_0}^{t_1} F\big(t, x(t), x'(t)\big)dt \\ \text{s.t.} \quad & x(t_0) = x_0 \\ & x(t_1) \geq 0 \end{aligned} \tag{P3}$$

Necessary Conditions If $x^*(\cdot)$ is the solution to the problem above, then the following conditions must be true

(1) $F_x\big(t, x^*(t), x^{*'}(t)\big) = \frac{d}{dt}F_{x'}\big(t, x^*(t), x^{*'}(t)\big)$ for all $t \in [t_0, t_1]$

(2) $x^*(t_0) = x_0$

(3) $F_{x'}\big(t_1, x^*(t_1), x^{*'}(t_1)\big) \cdot x^*(t_1) = 0$

The transversality condition requires either $F_{x'}|_{t=t_1} = 0$ or $x(t_1) = 0$. This is similar to the complementary slackness in the static optimization problems. If the constraint is not binding in the optimal,

$x^*(t_1) > 0$, we go back to (P2) and $F_{x'}|_{t=t_1} = 0$ must hold. If $F_{x'}|_{t=t_1} \neq 0$, it must be the case that the constraint is strictly binding, $x^*(t_1) = 0$.

Sufficient Conditions The necessary conditions for the optimal solutions to each of the three problems discussed above are also sufficient if $F(t, x, x')$ is concave in (x, x') for all t.

1.2 Infinite Horizon

1.2.1 Infinite-Horizon Dynamic Problems

$$\begin{aligned} &\max_{x \in C^\infty} \int_{t_0}^{\infty} F\big(t, x(t), x'(t)\big) dt \\ &\text{s.t.} \quad x(t_0) = x_0 \end{aligned} \tag{P4}$$

Necessary Conditions If $x^*(\cdot)$ is the solution to the problem above, then the following conditions must be true

(1) $F_x\big(t, x^*(t), x^{*'}(t)\big) = \frac{d}{dt} F_{x'}\big(t, x^*(t), x^{*'}(t)\big)$ for all $t \in [t_0, \infty)$

(2) $x^*(t_0) = x_0$

(3) $\lim_{t_1 \to \infty} F_{x'}\big(t_1, x^*(t_1), x^{*'}(t_1)\big) \cdot x^*(t_1) = 0$

Condition (3) is the transversality condition for the infinite-horizon dynamic problem. Note that $F_{x'}$ is multiplied by x. When the time horizon is infinite, $[t_0, \infty)$, we generally need additional inequality constraints at the endpoint to guarantee that the integral is finite and the optimal solution exists. One example of these constraints is no-Ponzi-scheme constraint.

1.2.2 Stationary Problems

Denote the solution to the following dynamic problem by $x^*\big(t \big| x_0, t_0\big)$,

$$\begin{aligned} V(t_0, x_0) := &\max_{x \in C^\infty} \int_{t_0}^{\infty} F\big(t, x(t), x'(t)\big) dt \\ &\text{s.t.} \quad x(t_0) = x_0 \end{aligned}$$

Definition The dynamic problem is **stationary** if for all x_0 and for all t_0 and t_0' such that $t_0 > t_0'$,

$$V(t_0, x_0) = V(t_0', x_0)$$

and

$$x^*\big(t+t_0-t_0'\big|x_0,t_0'\big) = x^*\big(t\big|x_0,t_0\big) \quad \text{for all } \ t \in [t_0,\infty).$$

If an infinite-horizon dynamic problem is stationary, the path of the optimal solution and the value function do not depend on the initial calendar time t_0, and they only depends on the initial value x_0.

Example Consider the following problem

$$\begin{aligned} V(t_0,x_0) := \max_{x\in C^\infty} \quad & \int_{t_0}^{\infty} e^{-\rho(t-t_0)} f\big(x(t),x'(t)\big)dt \\ \text{s.t.} \quad & x(t_0)=x_0 \end{aligned}$$

Define $s=t-t_0$. Substituting t with s, we obtain

$$\begin{aligned} V(t_0,x_0) = \max_{x\in C^\infty} \quad & \int_{0}^{\infty} e^{-\rho s} f\big(x(s),x'(s)\big)ds \\ \text{s.t.} \quad & x(0)=x_0 \end{aligned}$$

This problem is independent of the initial calendar time t_0, so it is a stationary problem.

2 Optimal Control

2.1 Optimal Control Problems

In the optimal control problems there are two categories of variables: state variables and control variables. The **state variables** are governed by the law of motion, while the **control variables** are chosen explicitly to maximize the objective function. In the following example, variable u is the control variable. The state variable x is determined by a law of motion $\dot{x}(t)=g(t,x(t),u(t))$.

$$\begin{aligned} V(t_0,x_0) = \max_{u(\cdot)} \quad & \int_{t_0}^{\infty} F\big(t,x(t),u(t)\big)dt \\ \text{s.t.} \quad & x(t_0)=x_0 \\ & \dot{x}(t)=g(t,x(t),u(t)) \ \ t\geq t_0 \end{aligned} \tag{OC}$$

Example Consider a household's consumption/saving problem. Given the initial stock of asset $a(0)=a_0\in R$. The household decides how much to consume and how much to save or borrow. The instantaneous utility function is $ln\big(c(t)\big)$, where $c(t)$ is the instantaneous consumption at time t, and the time preference rate is $\rho > 0$. The instantaneous income flow is $y(t)$ with the entire path $\{y(t)\}_{t\in[0,\infty)}$ given. The instantaneous interest rate is $r(t)$ with the entire path $\{r(t)\}_{t\in[0,\infty)}$ given. The dynamics of asset follows $\dot{a}(t) = y(t) - c(t) + r(t)a(t)$. This problem is characterized by the following model:

$$\begin{aligned}
&\max_{c(\cdot)} \quad \int_0^\infty e^{-\rho t} ln\big(c(t)\big)dt \\
&\text{s.t.} \quad a(0) = a_0 \\
&\qquad\quad \dot{a}(t) = y(t) - c(t) + r(t)a(t)
\end{aligned}$$

To insure that the solution exists, we need to impose the **no-Ponzi-scheme constraint**:

$$\lim_{t\to\infty} \exp\Big(-\int_0^t r(s)ds\Big)\, a(t) \geq 0.$$

There are two possibility to satisfy this constraint. The first one is $\lim_{t\to\infty} a(t) \geq 0$, which means that the household holds a positive amount of assets eventually. The second possibility is that $\lim_{t\to\infty} a(t) < 0$, but $\lim_{t\to\infty} \exp\Big(-\int_0^t r(s)ds\Big)\, a(t) = 0$. Even though the household holds a positive amount of debts eventually, the growth rate of debts is smaller than the interest rate $r(t)$. This implies that the present value of debts is close to zero. If no-Ponzi-scheme constrain is not imposed, the household could reach any large amount of consumption by choosing a path of negative value of asset such that $r(t)a(t) - \dot{a}(t) > 0$, where the household keeps increasing consumption and repay debts $r(t)a(t)$ by borrowing even more ($\dot{a}$ become even more negative).

Necessary Conditions If $u^*(\cdot)$ is the solution to the problem (OC), then there exists some function $\mu(\cdot) \in C^\infty$ such that

(1) $F_u\big(t, x(t), u^*(t)\big) + \mu(t)g_u\big(t, x(t), u^*(t)\big) = 0$ for all $t \in [t_0, \infty)$

(2) $\dot{x}(t) = g(t, x(t), u^*(t))$ for all $t \in [t_0, \infty)$

(3) $-\dot{\mu}(t) = F_x\big(t, x(t), u^*(t)\big) + \mu(t)g_x\big(t, x(t), u^*(t)\big) = 0$ for all $t \in [t_0, \infty)$

(4) $x(t_0) = x_0$

(5) $\lim_{t\to\infty} \mu(t)x(t) = 0$

Interpretations The function μ is called the **co-state variable,** which measures the marginal value of x. Specifically,

(i) $\mu(t) = V_x\big(t, x(t)\big)$ for all $t \in [t_0, \infty)$, where $x(\cdot)$ is generated by the optimal control variable $u^*(\cdot)$.

This implies that $\mu(t)$ is the marginal value of $x(t)$, in the same unit as the unit of $V(\cdot)$.

(ii) The tranversality condition (5) $\lim_{t\to\infty}\mu(t)x(t) = 0$ implies the value of x should be zero eventually (as $t \to \infty$) in the optimal.

NC with Hamiltonian If we define a function,

$$H\big(t, x(t), u(t), \mu(t)\big) := F\big(t, x(t), u(t)\big) + \mu(t)g(t, x(t), u(t)),$$

then the necessary conditions can be rewritten as

(1) $H_u = 0$ for all $t \in [t_0, \infty)$

(2) $H_\mu = \dot{x}(t)$ for all $t \in [t_0, \infty)$

(3) $H_x = -\dot{\mu}(t)$ for all $t \in [t_0, \infty)$

(4) $x(t_0) = x_0$

(5) $\lim_{t\to\infty} \mu(t)x(t) = 0$

This function H is called **Hamiltonian**, which is analogous to the Lagrangian in the static optimization problems. Condition (1) is similar to the first-order-condition. Condition (2) is the law of motion for the state variable x, given exogenously. Condition (3) is the law of motion for μ, the co-state variable. Conditions (4) and (5) are the initial condition and the transversality condition, respectively.

Sufficient Conditions The necessary conditions with Hamiltonian are also sufficient for the optimal solutions if F and g are concave in (x, u) for all t.

Assuming F and g are concave, the optimal solution can be characterized by a system of 2-by-2 ordinary differential equations (ODE). To do this, we use condition (1) to rewrite $u(\cdot)$ as a function of $\mu(\cdot)$ and $x(\cdot)$, and replace $u(\cdot)$ in conditions (2) and (3) by this function. A 2-by-2 ODE system with two variables (μ, x) is obtained. Along with two boundary conditions (4) and (5), we are able to solve for $(x(t), \mu(t))$ based on this 2-by-2 ODE system. Finally, we obtain $u(\cdot)$ by (1). Therefore, these five conditions give us the optimal path $\{x(t), u(t), \mu(t)\}_{t\in[t_0,\infty)}$ for this problem.

From now on, we will focus on the following problem with a discount factor $e^{-\rho t}$, and $f(t, x, u)$ is interpreted as the instantaneous value flow:

$$\begin{aligned} V(t_0, x_0) = \max_{u \in C^\infty} \quad & \int_{t_0}^{\infty} e^{-\rho t} f\big(t, x(t), u(t)\big) dt \\ \text{s.t.} \quad & x(t_0) = x_0 \\ & \dot{x}(t) = g(t, x(t), u(t)) \ \ t \geq t_0 \end{aligned} \tag{D1}$$

Note that the way we define problem (D1) implies that V is expressed in the unit of value at time 0 (a present value). This is because all the values $f\big(t, x(t), u(t)\big)$ in time $t \geq t_0$ are discounted back to the value in time 0 by $e^{-\rho t}$.

2.2 Present- and Current-Value Hamiltonians

The Hamiltonian associated with problem (D1) is

$$H(t, x, u, \mu) = e^{-\rho t} f(t, x, u) + \mu g(t, x, u)$$

Recall that $\mu(t) = V_x(t, x(t))$, so $\mu(t)$ is the marginal value, of $x(t)$, in the unit of value at time 0 (a marginal present value). Therefore, we call $\mu(\cdot)$ the **present-value multiplier**. Note that the first term on RHS $e^{-\rho t} f(t, x, u)$ in the Hamiltonian $H(\cdot)$ is the instantaneous value in the unit of value in time 0, instead of in time t. Therefore, $H(\cdot)$ is expressed in the unit of value in time 0, and we call H the **present-value Hamiltonian**.

Now rewrite the present-value Hamiltonian as

$$H(t, x, u, \mu) = e^{-\rho t} \big[f(t, x, u) + e^{\rho t} \mu g(t, x, u) \big].$$

Define $\lambda(t) := e^{\rho t} \mu(t)$. Since $\mu(t)$ is the marginal value of $x(t)$ in the unit of value at time 0, $\lambda(t)$ will be the marginal value of $x(t)$ in the unit of value at time t. Thus, we call $\lambda(\cdot)$ the **current-value multiplier**.

Furthermore, define

$$\tilde{H}(t, x, u, \lambda) := f(x, u) + \lambda g(x, u).$$

Then, $\tilde{H}(t, x, u, \lambda) = e^{\rho t} H(t, x, u, \mu)$. Since $H(\cdot)$ is in the unit of value in time 0, $\tilde{H}(\cdot)$ will be in the unit of value in time t. We call $\tilde{H}(\cdot)$ the **current-value Hamiltonian**.

NC with Current-Value Hamiltonian Assume $u^*(\cdot)$ is the solution to problem (D1) and define a function as

$$\tilde{H}\big(t, x(t), u(t), \lambda(t)\big) := f\big(t, x(t), u(t)\big) + \lambda(t) g(t, x(t), u(t)).$$

Then there exists some function $\lambda(\cdot) \in C^\infty$ such that

(1) $\tilde{H}_u = 0$ for all $t \in [t_0, \infty)$

(2) $\tilde{H}_\lambda = \dot{x}(t)$ for all $t \in [t_0, \infty)$

(3) $\tilde{H}_x = -\dot{\lambda}(t) + \rho\lambda(t)$ for all $t \in [t_0, \infty)$

(4) $x(t_0) = x_0$

(5) $\lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0$,

where the argument in all $\tilde{H}$ is $\big(t, x(t), u^*(t), \lambda(t)\big)$.

Time-Variant Discount In some cases, the time discount rate changes through time. Consider the following problem:

$$\begin{aligned} \max_{u\in C^\infty} \quad & \int_0^\infty e^{-\int_0^t r(s)ds} f\big(x(t), u(t)\big) dt \\ \text{s.t.} \quad & x(0) = x_0 \\ & \dot{x}(t) = g(x(t), u(t)) \;\; t \geq 0 \end{aligned}$$

where $r(t)$ is the instantaneous interest rate so that the interest rate from time 0 to t is $e^{\int_0^t r(s)ds}$, and hence $e^{-\int_0^t r(s)ds}$ is the discount factor from time t to time 0.[1] In this case, the necessary condition (3) with current-value Hamiltonian will be

$$\tilde{H}_x = -\dot{\lambda}(t) + r(t)\lambda(t) \quad \text{for all } \; t \in [t_0, \infty).$$

3 Bellman Equations

The Bellman equation is another tool to solve the dynamic optimization problems. The Hamiltonian characterizes the optimal solutions by a system of ordinary differential equations, while the Bellman equation does the same by a partial differential equation.

[1] Note that if $r(t) = r$ then the time discount becomes e^{-rt}.

Define the value function $V(\cdot)$ as the value of the following problem:

$$
\begin{aligned}
V(t_0, x_0) := \max_{u(\cdot)} \quad & \int_{t_0}^{\infty} e^{-\int_{t_0}^{s} r(v)dv} f\big(s, x(s), u(s)\big) ds \\
\text{s.t.} \quad & x(t_0) = x_0 \\
& \dot{x}(s) = g(s, x(s), u(s)) \ \ s \geq t_0
\end{aligned}
$$

The corresponding Bellman equation is defined as follows:

$$
r(t)V(t,x) = \max_u \Big\{ \ f(t,x,u) + V_x(t,x)g(t,x,u) + V_t(t,x) \Big\}
$$

Example 5.1 Consider the following problem

$$
\begin{aligned}
V(t_0, x_0) = \max_{u \in C^\infty} \quad & \int_{t_0}^{\infty} e^{-\rho(t-t_0)} f\big(x(t), u(t)\big) dt \\
\text{s.t.} \quad & x(t_0) = x_0 \\
& \dot{x}(t) = g(x(t), u(t)) \ \ t \geq t_0
\end{aligned}
$$

The corresponding Bellman equation is

$$
\rho V(t,x) = \max_u \Big\{ \ f(x,u) + V_x(t,x)g(x,u) + V_t(t,x) \Big\}
$$

Note that the problem is stationary, so V does not depend on time t and hence $V_t(t,x) = 0$. The value function becomes:

$$
\rho V(x) = \max_u \ \Big\{ f(x,u) + V'(x)g(x,u) \Big\}
$$

The first order condition (FOC) is

$$
f_u + V' g_u = 0.
$$

The envelop condition (EC) is

$$
\rho V'(x) = f_x + V'' g + V' g_x.
$$

Recall that the co-state variable in Hamiltonian equals the marginal value of $x(t)$, $\lambda(t) = V'(x(t))$ and $\dot{x}(t) = g\big(x(t), u(t)\big)$. It follows that FOC is equivalent to $H_u = 0$, while EC is equivalent to $H_x = -\dot{\lambda} + \rho\lambda$.

Example 5.2 Using the Bellman equation, solve the following problem

$$V(x) := \int_0^\infty e^{-\rho t} \frac{c(s)^{1-\gamma}}{1-\gamma} ds$$
$$\text{s.t.} \quad n(0) = n$$
$$\dot{n}(t) = rn(t) - c(t) \;\; t \geq 0$$

where $n(t)$ represents the net worth, n is a given initial net worth and r is a given constant interest rate.

Example 5.3 Consider a model where no decision is made:

$$V(t_0, x_0) := \int_{t_0}^\infty e^{-\rho t} f\big(s, x(s)\big) ds$$
$$\text{s.t.} \quad x(t_0) = x_0$$
$$\dot{x}(s) = g(s, x(s)) \;\; s \geq t$$

the corresponding Bellman equation is

$$\rho V(t,x) = f(t,x) + g(x,u)\frac{\partial}{\partial x}V(t,x) + \frac{\partial}{\partial t}V(t,x)$$

which is a partial differential equation.

4 Stochastic Models: Poisson Arrivals

If an event has a Poisson arrival rate $\lambda > 0$, then the probability that the event occurs in the interval $[t, t+\Delta]$ is $\Delta\lambda$, for all $t \geq$ and all $\Delta > 0$ small enough. This implies that the probability that the event does not occur in the interval $[0, t]$ is $e^{\lambda t}$. Formally, let T be a random variable describing the length of time before the event occurs, then

$$\Pr(T > t) = e^{\lambda t},$$

which is just an exponential distribution with parameter λ. It has the memoryless property that for all $s, t, \geq 0$,

$$\Pr(T > t + s | T > s) = \Pr(T > t).$$

Example 6.1 An agent with time preference rate $\rho > 0$ obtain utility $u > 0$ when being alive and utility 0 when being dead. Assume the event of ding has a Poisson arrival rate $\lambda > 0$. Then, the utility path $\{\tilde{u}(s)\}_{s\geq 0}$ will be a stochastic process and has the stationary property in the sense that given an initial value $\tilde{u}(s) = u$, the process of $\{u(t+s\})_{s\geq 0}$ is the same for all $t \geq 0$. The value of being dead at time t will be

$$V_{dead}(t) := \int_t^\infty e^{-\rho(s-t)}\, 0\, ds = 0$$

The value of being alive at time t will be

$$V(t) := E\Big[\int_t^\infty e^{-\rho(s-t)}\, \tilde{u}(s)\, ds \big| \tilde{u}(t) = u\Big]$$

which is equivalent to

$$V(t) := E\Big[\int_0^\infty e^{-\rho v}\, \tilde{u}(v+t)\, dv \big| \tilde{u}(t) = u\Big]$$

Since $\tilde{u}()$ is a stationary process, RHS does not depend on t at all and hence $V(t)$ is independent of t:

$$V(t) \equiv V := E\Big[\int_0^\infty e^{-\rho v}\, \tilde{u}(v)\, dv \big| \tilde{u}(0) = u\Big]$$

To simplify V:

$$\begin{aligned} V &= \int_0^\infty e^{-\rho t}\Big[\underbrace{e^{-\lambda t} \cdot u}_{\text{still alive at } t} + \underbrace{\lambda e^{-\lambda t} \cdot V_{dead}(t)}_{\text{dead at } t}\Big] dt \\ &= \frac{u}{\rho + \lambda} \end{aligned}$$

It is also equivalent to

$$\rho V = u + \lambda(V_{dead} - V) \tag{1}$$

Example 6.2 An agent with time preference rate $\rho > 0$ obtain utility u_i when the state for the economy is i, where $i \in \{0, 1\}$. If the current state for the economy is i, the economy switches to state $1-i$ at the Poisson arrival rate $\lambda_i > 0$. It follows that the utility path $\tilde{u}$ is a stationary process. Define the value at state i as

$$V_i := E\Big[\int_0^\infty e^{-\rho t}\, \tilde{u}(t)\, dt \big| \tilde{u}(0) = u_i\Big]$$

Then, for $i \in \{0, 1\}$,

$$\rho V_i = u_i + \lambda_i (V_{1-i} - V_i) \tag{2}$$

and

$$\rho V_i = \frac{\rho + \lambda_{1-i}}{\rho + \lambda_{1-i} + \lambda_i} u_i + \frac{\lambda_i}{\rho + \lambda_{1-i} + \lambda_i} u_{1-i}$$

Example 6.3 An unemployed worker with time preference rate $\rho > 0$ is searching for a job. When being unemployed, the utility she gets is $b > 0$. With a Poisson arrival rate $\lambda > 0$, she finds a job and she draws a wage w from a distribution F. Once the wage is realized, she decides whether to accept or not. If she accepts an offer with wage w, she keeps the job forever and obtains utility w. If sh rejects the offer, then she becomes unemployed and search again. Let $V(w)$ be the values when working for a job with wage w, and U the value of being unemployed. Then,

$$\begin{aligned}
U &= \int_0^\infty e^{-\rho t}\Big[\underbrace{e^{-\lambda t}\cdot b}_{\text{no job found in }[0,t)} + \underbrace{\lambda e^{-\lambda t}\cdot \int \max\{V(w),U\}dF(w)}_{\text{find a job at }t}\Big]dt \\
V(w) &= \int_0^\infty e^{-\rho t} w dt
\end{aligned}$$

With some simplification,

$$\begin{aligned}
\rho U &= b + \lambda\Big[\int \max\{V(w),U\}\, dF(w) - U\Big] \\
\rho V(w) &= w
\end{aligned} \tag{3}$$

Example 6.4 A seller who can produces a good at zero cost is waiting for buyers to purchase the good. At any point of time, she can only hold exactly one unit of good. Higher-type buyer arrives at rate λ_h while low type buyers arrive at rate λ_l. Assume that the arrivals of two types of buyers are independent and no more than one buyer will arrive in any interval small enough. When high type buyer arrives, the seller derives utility u_h and produces one unit of good immediately, waiting for the next buyer. Similarly, u_l for the low type. Finally, there is an opportunity cost of waiting for customers at a constant value c, so the instantaneous utility is $-c$. Then, the value for the agent will be

$$V = \int_0^\infty e^{-\rho t}\Bigg[\underbrace{e^{-\alpha t}e^{-\beta t}\cdot(-c)}_{\text{no one arrives in }[0,t)} + \underbrace{\lambda_h e^{-\lambda_h t}e^{-\lambda_l t}\cdot(u_h+V)}_{\text{high type arrives at }t} + \underbrace{\lambda_l e^{-\lambda_l t}e^{-\lambda_h t}\cdot(u_h+V)}_{\text{low type arrives at }t}\Bigg]dt$$

The associated Bellman equation is

$$\rho V = -c + \lambda_h\big(u_h + V - V\big) + \lambda_l\big(u_l + V - V\big) \tag{4}$$

From the examples above, especially from equations (1) to (4), we learn some pattern with stochastic Poisson arrivals. We conclude this section by the following general formula.

Stochastic Dynamic Optimization with Poisson Arrivasl Consider a general stochastic optimal control problem as follows. Note that the instantaneous value f_j and the law of motion g_j depends on the current state j.

$$
\begin{aligned}
V_i(t,x_0) := \max_{u(\cdot)} \quad & E_{i,t}\Big[\int_t^{\infty} e^{-\rho(s-t)} f_{\tilde{j}(s)}\big(s,x(s),u(s)\big)ds\Big] \\
\text{s.t.} \quad & x(t) = x_0 \\
& \dot{x}(s) = g_{\tilde{j}(s)}(s,x(s),u(s)) \quad s \geq t
\end{aligned}
$$

where the expectation is taken over the stochastic future state of the economy $\tilde{j}(s) \in \{1,2,...,N\}$, conditional on the current state i and time t, i.e. $\tilde{j}(t) = i$. The arrival rate of switching from state i to j is λ_{ij}. The associated Bellman equation with this problem is

$$
\rho V_i(t,x) = \max_u \quad \left\{ f_i\big(t,x,u\big) + \frac{\partial V_i(t,x)}{\partial x} g_i(t,x,u) + \frac{\partial V_i(t,x)}{\partial t} + \sum_{j=1}^{N} \lambda_{ij}\Big(V_i(t,x) - V_j(t,x)\Big) \right\}
$$

If we further assume that there is additional value d_{ij} when switching from state i to j, the associated Bellman equation with this problem becomes

$$
\rho V_i(t,x) = \max_u \quad \left\{ f_i\big(t,x,u\big) + \frac{\partial V_i(t,x)}{\partial x} g_i(t,x,u) + \frac{\partial V_i(t,x)}{\partial t} + \sum_{j=1}^{N} \lambda_{ij}\Big(d_{ij} + V_i(t,x) - V_j(t,x)\Big) \right\}
$$