1 Q1

Consider a principal-agent setting with moral hazard and a finite number of outputs and efforts. There are n possible outputs, $x_i \in \mathcal{X} = \{x_1, \ldots, x_n\}, x_n > x_{n-1} > \cdots > x_1$. There are m possible effort levels, $e_j \in \mathcal{E} = \{e_1, \ldots, e_m\}, e_m > e_{m-1} > \cdots > e_1$. The probability of output x_i given effort e_j is $\phi_i(e) = \phi(x_i|e_j) > k > 0$ (i.e., bounded away from zero). Assume that $\phi_i(\cdot)$ satisfies MLRP: i.e., for all $e > \tilde{e}$,

$$\frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}, \text{ is increasing in } i$$

Assume that the principal is risk neutral and the agent is risk averse with preferences $U = u(w) - \psi(e)$ where $u(\cdot)$ is increasing and concave, and $\psi(\cdot)$ is increasing and convex. Assume the agent's outside option is $U = 0$ and $\psi(e_1) = 0$. Finally, the principal's wage schedule must satisfy $w_i = w(x_i) \in [\underline{w}, \overline{w}]$, but assume that these constraints are slack at the optimum and ignore them.

Procblem 1.1.

State the principal's program for the optimal e and $(w_1, \ldots, w_n)$, and write the Lagrangian for the program using $\{\mu_j\}_j$ as multipliers for the IC constraints and λ as the multiplier for the IR constraint.

Solution. The principal's problem is

$$\max_{e, w_1, \ldots, w_n} \sum_{i=1}^{n} \phi_i(e)(x_i - w_i)$$

$$\text{s.t. } \sum_{i=1}^{n} \phi_i(e)u(w_i) - \psi(e) \geq 0 \text{ (IR)}$$

$$\text{and } \sum_{i=1}^{n} \phi_i(e)u(w_i) - \psi(e) \geq \sum_{i=1}^{n} \phi_i(e_j)u(w_i) - \psi(e_j), \forall j \text{ (IC)}$$

where e is the optimal level of effort. So the Lagrangean is

$$L = \sum_{i=1}^{n} \phi_i(e)(x_i - w_i) + \lambda\left[\sum_{i=1}^{n} \phi_i(e)u(w_i) - \psi(e)\right] + \sum_{j=1}^{n} \mu_j\left[\sum_{i=1}^{n} \phi_i(e)u(w_i) - \psi(e) - \left(\sum_{i=1}^{n} \phi_i(e_j)u(w_i) - \psi(e_j)\right)\right]$$

Procblem 1.2.

Prove that the agent's IR constraint is binding (i.e., $\lambda > 0$) in the solution to the program.

Solution. BWOC, assume $\lambda = 0$, so the IR constraint is slack at the optimal wage schedule $w(\cdot)$. Let $\epsilon > 0$, and define $w_\epsilon(x_i) = w(x_i) - \delta_i, \forall i$, where $\delta_i > 0$ is such that $u(w_\epsilon(x_i)) = u(w(x_i)) - \epsilon$ (note that we can do this by continuity of u). Thus, we've reduced the expected wage payment by $\sum_{i=1}^n \delta_i > 0$ while we haven't affected any of the IC constraints at all (i.e. we've reduced the LHS and RHS by ϵ). So for sufficiently small $\epsilon > 0$, the IR constraint will still be slack for $w_\epsilon(\cdot)$. Note, however, that wage schedule $w_\epsilon(\cdot)$ is strictly preferable to the principal, which contradicts the optimally of the original optimal wage schedule $w(\cdot)$.

$\therefore$ the IR constraint must bind at the optimal wage schedule, so $\lambda = 0$.

Procblem 1.3.

Suppose that the optimal choice of effort is $e^* = e_m,$ the maximum effort level. Prove that the optimal wage schedule which induces e^m is monotonic, $w_1^* \leq w_2^* \cdots \leq w_n^*$ with strict inequality for some i (i.e., $w_i^* < w_{i+1}^*$)

Solution. BWOC, assume $\exists i, j$ such that $i > j$ and $w_i^* < w_j^*$. Note that the FOC in w_i of the principal's problem form part a is

$$0 = -\phi_i(e) + \lambda\phi_i(e)u'(w_i) + \sum_{i=1}^n \mu_j(\phi_i(e) - \phi_i(e_j))u'(w_i)$$
$$\leftrightarrow \frac{1}{u'(w_i)} = \lambda + \sum_{i=1}^n \mu_j \frac{(\phi_i(e) - \phi_i(e_j))}{\phi_i(e)}$$

By concavity $u'(w_i^*) > u'(w_j^*)$, so $1/u'(w_i^*) < 1/u'(w_j^*)$. Plugging this relation into the FOC we see:

$$\lambda + \sum_{k=1}^n \mu_k \frac{(\phi_i(e_m) - \phi_i(e_k))}{\phi_i(e_m)} = \frac{1}{u'(w_i)} < \frac{1}{u'(w_j)} = \lambda + \sum_{k=1}^n \mu_k \frac{(\phi_j(e_m) - \phi_j(e_k))}{\phi_j(e_m)},$$

which contradicts that

$$\frac{(\phi_i(e_m) - \phi_i(e_k))}{\phi_i(e_m)} > \frac{(\phi_j(e_m) - \phi_j(e_k))}{\phi_j(e_m)},$$

by MLRP since $e_m \geq e_k$. Thus, $w_j^* \leq w_i^*, \forall i \geq j$.

Now BWO assume $w_{1=\ldots=w_n^*}^*$. Then since ψ is increasing (assume strictly increasing for some e) let $e_i < e_m$ such that $\psi(e_i) < \psi(e_m)$. Then

$$\sum_{j=1}^{n} \phi_j(e_i)u(w_j^*) - \psi(e_i) = w_i^* - \psi(e_i) > w_m^* - \psi(e_m) = \sum_{j=1}^{n} \phi_j(e_)u(w_j^*) - \psi(e_m),$$

which violates the IC constraint, contradicting the optimality of w^*.

Procblem 1.4.

> Prove that the MLRP implies first-order stochastic dominance for the case of a discrete distribution, $(\phi_1(e), \ldots, \phi_n(e))$.

Solution. Here FOSD means $\forall e_i > e_j, \forall x$

$$P(X > x|e_i) \geq P(X > x|e_j)$$
$$\leftrightarrow P(X \leq x|e_i) \leq P(X \leq x|e_j)$$

and strict for some x. Equivalently, $\forall k = 1, \ldots, n$

$$\sum_{l=1}^{k} \phi_l(e_i) \leq \sum_{l=1}^{k} \phi_l(e_j)$$
$$\leftrightarrow \sum_{l=1}^{k} \phi_l(e_i) - \phi_l(e_j) \leq 0.$$

Note that

$$\sum_{l=1}^{n} \frac{\phi_l(e_i) - \phi_l(e_j)}{\phi_l(e_i)} \phi_l(e_i) = \sum_{l=1}^{n} \phi_l(e_i) - \phi_l(e_j) = 1 - 1 = 0. \tag{1}$$

Since $e_i > e_j$,

$$\frac{\phi_l(e_i) - \phi_l(e_j)}{\phi_l(e_i)} \tag{2}$$

is increasing in i by MLRP. Thus, (2) must start negative and end positive so that (1) holds. Hence, $\forall k < n$:

$$\sum_{l=1}^{k} \frac{\phi_l(e_i) - \phi_l(e_j)}{\phi_l(e_i)} \phi_l(e_i) = \sum_{l=1}^{k} \phi_l(e_i) - \phi_l(e_j) < 0.$$

This last statement is exaclty the statement of FOSD in this discrete setting. Thus, MLRP implies FOSD here.

Procblem 1.5.

Suppose that the optimal choice of effort $e^* > e_1$. Prove that there must exist some i such that $w_i^* < w_{i+1}^*$ (i.e., prove that $w_1^* \geq w_2^* \geq \cdots \geq w_n^*$ cannot be optimal). [Hint: use the fact in (d) above.]

Solution. BWOC, assume $w_1^* \geq \ldots w_n^*$ is optimal. Note that $\psi(e)$ is increasing (assume strictly increasing for some e). Then note that by part d, MLRP implies FOSD, so we have

$$\sum_{l=1}^{n} \phi_l(e_i) w_l \geq \sum_{l=1}^{n} \phi_l(e_j) w_l, \forall i \leq j,$$

since $\phi(e_i)$ puts more weight on higher outcomes of x_l, which, by assumption, receive weakly lower wages. So $\forall i$ such that $e_i < e^*$ (such e_i exist because $e^* > e_1$), we have

$$\sum_{l=1}^{n} \phi_l(e_i) w_l - \psi(e_i) \geq \sum_{l=1}^{n} \phi_l(e^*) w_l - \psi(e^*). \tag{3}$$

If $\psi(e_i) < \psi(e^*)$ for any $e_i < e^*$, then (3) violates IC, which contradicts the optimality of e^*.

2 Q3

Consider the setting like in Holmstrom and Milgrom, in which a risk neutral principal contracts with a risk-averse (CARA utility with r) agent. Specifically, suppose that a school wishes to design an optimal compensation contract for its teachers. Teachers engage in two tasks:

$$x_1 = e_1 + \epsilon_1,$$
$$x_2 = e_2 + \epsilon_2,$$

where ϵ_i is indeoendently distributed according to $N(0, \sigma_i^2)$. The cost of the teacher's effort is $\gamma(e_1 + e_2)$ where

$$\gamma(e) = \begin{cases} 0 & \text{if } e \leq \hat{e} \\ \frac{1}{2}(e - \hat{e})^2 & \text{if } e \geq \hat{e} \end{cases}$$

Thus, $\hat{e} > 0$ represents an amount of free labor that a teacher will supply.
Suppose that task 1 (teaching math and reading skills) is easy to measure via test scores in math and reading, but task 2 (teaching creativity) is almost impossible to measure. Hence, σ_1^2 is small and σ_2^2 is large. Additionally, suppose that the school cares about the benefits of e_1 and e_2 for its students and these benefits are characterized by the function, $B(e_1, e_2) \geq 0$. $B(\cdot, \cdot)$ is strictly concave, and strictly increasing in both arguments for $e_1 > 0$ and $e_2 > 0$; but assume that $B(e_1, 0) = B(0, e_2) = 0$ for all e_1 and e_2. That is, teaching math and reading is pointless without teaching a little bit of creativity, and vice versa. For reasons given in Holmstrom and Milgrom (1987), the school chooses a linear compensation function,

$$w(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 + \beta.$$

The school maximizes the expectation of $B(e_1, e_2) - w(x_1, x_2)$ subject to incentive compatibility and the teacher's outside option, $\underline{\text{U}}$. The teachers are solely motivated by the wage contract and their cost of aggregate effort. They do not care about B or how effort is allocated across tasks, except insofar as it impacts the wage. If they are indifferent between two allocations of effort, however, we assume that they will choose the one preferable to the school.

Procblem 2.1. Suppose in the extreme that $\sigma_2^2 = \infty$; i.e. e_2 cannot be measured. What is the optimal incentive scheme for the school to offer the teachers?

Solution. First note that up to an effort level $e = (e_1, e_2)'$ such that $e_1 + e_2 \leq \hat{e}$, the cost of effort is zero. And because $w = \alpha_1 x_1 + \alpha_2 x_2 + \beta$ and $E[x_i|e_i] = E[e_i + \epsilon_i|e_i] = e_i$, the expected wage, $E[w(x)|e]$, is increasing in e, so the minimum effort level the teacher will provide would be $e_1 + e_2 = \hat{e}$. Hence, we can safely assume that the optimal effort provided by the teacher would be $e^* = (e_1^*, e_2^*)'$ such that $e_1^* + e_2^* \geq \hat{e} \Rightarrow \gamma(e^*) = \frac{1}{2}(e_1^* + e_2^* - \hat{e})^2$.

The general set-up of the problem should be that the school chooses $\alpha = (\alpha_1, \alpha_2)'$ of the wage schedule, $w(x) = \alpha_1 x_1 + \alpha_2 x_2 + \beta$, that induces $e = (e_1, e_2)'$ such that it satisfies teacher's IR and IC constraint and maximizes the school's expected profit:

$$\begin{aligned} \max_{\alpha, e} \quad & E[B(e) - w(x)|e] \\ s.t. \quad & E[u(w(x) - \gamma(e))|e] \geq \underline{U} && \text{(IR)} \\ & e \in \arg\max_{\tilde{e}} E[u(w(x) - \gamma(\tilde{e}))|\tilde{e}] && \text{(IC)} \end{aligned}$$

Notice that, in this problem, we have:

- ▷ The principal (school) is risk-neutral,
- ▷ The agent (teacher) is risk-neutral with CARA utility with risk aversion, r,
- ▷ The performance, x_i, follows a Brownian motion with the drift being the effort provided, e_i, and the normal error ϵ_i,
- ▷ The wage schedule is linear in observed performances (x_1, x_2), i.e. $w = \alpha_1 x_1 + \alpha_2 x_2 + \beta$,

so we can borrow the results of Holmstrom and Milgrom (1991) on multi-task incentive contracts elaborated in the lecture notes (page 31) where the optimal contract (α, e) solves

$$\begin{aligned} \max_{\alpha, e} \quad & B(e) - \gamma(e) - \frac{r}{2}\alpha' \Sigma \alpha, \\ s.t. \quad & e \in \arg\max_{\tilde{e}} \quad \alpha' \tilde{e} - \gamma(\tilde{e}), \end{aligned}$$

where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

Note that the linearity of the wage schedule and CARA utility of the teacher ensures β be set to meet the teacher's IR constraint:

$$\beta = \underline{w} - \alpha' e + \gamma(e) + \frac{r}{2}\alpha' \Sigma \alpha,$$

where $u(\underline{w})$ = $\underline{U}$ (i.e. the certainty equivalent wage under the teacher's reservation utility, $\underline{U}$) and $\frac{r}{2}\alpha' \Sigma \alpha$ describes the teacher's risk premium. Using the functional form of $\gamma(\cdot)$ and expanding out into a scalar form:

$$\begin{aligned} \max_{\alpha_1, \alpha_2, e_1, e_2} \quad & B(e_1, e_2) - \frac{1}{2}(e_1 + e_2 - \hat{e})^2 - \frac{r}{2}(\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2), \\ s.t. \quad (e_1, e_2)' \in \arg\max_{\tilde{e}_1, \tilde{e}_2} \quad & \alpha_1 \tilde{e}_1 + \alpha_2 \tilde{e}_2 - \frac{1}{2}(\tilde{e}_1 + \tilde{e}_2 - \hat{e})^2 && \text{(IC)} \end{aligned}$$

Solving the IC constraint, we get:

$$\begin{aligned}
[e_1]: \quad & \alpha_1 = \frac{\partial}{\partial e_1}\frac{1}{2}(e_1 + \tilde{e}_2 - \hat{e})^2 = e_1 + \tilde{e}_2 - \hat{e} \\
[e_2]: \quad & \alpha_2 = \frac{\partial}{\partial e_2}\frac{1}{2}(\tilde{e}_1 + e_2 - \hat{e})^2 = \tilde{e}_1 + e_2 - \hat{e} \\
[\text{at optimal } e = (e_1, e_2)] \Rightarrow \quad & \alpha_1 = \alpha_2 = e_1 + e_2 - \hat{e}.
\end{aligned}$$

Now, consider $\sigma_2^2 = \infty$. Recall that the IR constraint in this problem satisfies:

$$\begin{aligned}
\beta &= \underline{w} - \alpha' e + \gamma(e) + \frac{r}{2}\alpha'\Sigma\alpha \\
&= \underline{w} - \alpha_1 e_1 - \alpha_2 e_2 + \frac{1}{2}(e_1 + e_2 - \hat{e})^2 + \frac{r}{2}(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2).
\end{aligned}$$

Since β cannot be finite, α_2 must be set to 0 since $\sigma_2^2 = \infty$. But recall that $\alpha_1 = \alpha_2 = e_1 + e_2 - \hat{e}$. Therefore, in the optimal contract, we must have $\alpha_1 = 0$ as well and thus have the optimal efforts be such that $e_1 + e_2 = \hat{e}$.

Now, how should e_1 and e_2 be allocated? Note that $B(e_1, 0) = B(0, e_2) = 0$, so a contract is only meaningful if it enforces the teacher to provide a nonzero effort on both tasks, i.e. $e_1, e_2 > 0$. We also know that the teacher will provide $e_1 + e_2 = \hat{e}$, where the cost of such effort is 0 since $\gamma(e) = \frac{1}{2}(e_1 + e_2 - \hat{e})^2 = 0$. Hence, there is no incentive for the teacher to deviate from $e_1, e_2 > 0$ to either $e_1 = 0$ or $e_2 = 0$ since her utility is the same under $\alpha_1 = \alpha_2 = 0$ as long as $e_1 + e_2 = \hat{e}$ (in fact, we are told that if the teachers are indifferent between two allocation of effort, we assume that they will choose the one preferable to the school, which is $e_1, e_2 > 0$). Hence, under the optimal contract, we must have $e > 0$, so we can use the results in the lecture notes (page 32), where the optimal contract satisfies $\alpha_2 = 0$ and

$$\alpha_1 = \left(B_1(e_1, e_2) - B_2(e_1, e_2)\frac{\gamma_{12}(e)}{\gamma_{22}(e)}\right)\left(1 + r\sigma_1^2\left(\gamma_{11}(e) - \frac{\gamma_{12}(e)^2}{\gamma_{22}(e)}\right)\right)^{-1}.$$

Note that

$$\begin{aligned}
\gamma(e) &= \frac{1}{2}(e_1 + e_2 - \hat{e})^2 \\
\Rightarrow \gamma_i(e) &= e_1 + e_2 - \hat{e} \quad \forall i = 1, 2 \\
\Rightarrow \gamma_{ij}(e) &= 1 \quad \forall i, j \\
\Rightarrow \alpha_1 &= B_1(e_1, e_2) - B_2(e_1, e_2).
\end{aligned}$$

But we also know that $\alpha_1 = 0$, so we must have $B_1(e_1, e_2) = B_2(e_1, e_2)$. Hence, the optimal effort would be given as $e_1 + e_2 = \hat{e}$ such that $B_1(e_1, e_2) = B_2(e_1, e_2)$, i.e. the marginal returns are equated.

Procblem 2.2. Suppose that σ_2^2 is finite. State the principal's optimal program. Prove that in any solution, $\alpha_1 = \alpha_2 = \alpha$, and prove that $\alpha \notin (0, \hat{e}]$.

Solution. Now, both σ_1^2 and σ_2^2 are finite. Recall from part (a) that the principal's program is given as:

$$\begin{aligned} \max_{\alpha_1,\alpha_2,e_1,e_2} \quad & B(e_1,e_2) - \frac{1}{2}(e_1+e_2-\hat{e})^2 - \frac{r}{2}(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2), \\ s.t. \quad (e_1,e_2)' \in & \arg\max_{\tilde{e}_1,\tilde{e}_2} \quad \alpha_1\tilde{e}_1 + \alpha_2\tilde{e}_2 - \frac{1}{2}(\tilde{e}_1+\tilde{e}_2-\hat{e})^2. \end{aligned}$$

Recall from part a that the FOCs of the IC constraint result in $\alpha_1 = \alpha_2 = e_1 + e_2 - \hat{e}$. Having shown that $\alpha_1 = \alpha_2$, the optimal contract α satisfies (using the results straight from the lecture notes, page 31):

$$\begin{aligned} \alpha &= (I + r[\gamma_{ij}(e)\Sigma])^{-1}B'(e) \\ \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &= \begin{pmatrix} 1+r\sigma_1^2 & r\sigma_2^2 \\ r\sigma_1^2 & 1+r\sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} B_1(e_1,e_2) \\ B_2(e_1,e_2) \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 1+r\sigma_1^2 & r\sigma_2^2 \\ r\sigma_1^2 & 1+r\sigma_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &= \begin{pmatrix} B_1(e_1,e_2) \\ B_2(e_1,e_2) \end{pmatrix} \\ [\alpha_1 = \alpha_2 = e_1+e_2-\hat{e}] \Rightarrow (1+r\sigma_1^2+r\sigma_2^2)(e_1+e_2-\hat{e}) &= B_1(e_1,e_2) = B_2(e_1,e_2) \\ \Rightarrow \alpha_i &= \frac{B_i(e_1,e_2)}{1+r\sigma_1^2+r\sigma_2^2} \quad \forall i = 1,2. \end{aligned}$$

So, the optimal contract is such that $\alpha_1 = \alpha_2 = e_1 + e_2 - \hat{e}$ where the effort levels satisfy $e_1 + e_2 - \hat{e} = \frac{B_1(e_1,e_2)}{1+r\sigma_1^2+r\sigma_2^2} = \frac{B_2(e_1,e_2)}{1+r\sigma_1^2+r\sigma_2^2}$. Based on our discussion with the TAs, we have agreed that with the original cost function we cannot prove $\alpha_1 = \alpha_2 \notin (0, \hat{e}]$.

With the "corrected" cost function:

$$\begin{aligned} \gamma(e_1+e_2) &= \max\{\frac{1}{2}(e_1+e_2)^2 - \frac{1}{2}\hat{e}^2, 0\} \\ &= \begin{cases} \frac{1}{2}(e_1+e_2)^2 - \frac{1}{2}\hat{e}^2 & \text{if } e_1+e_2 > \hat{e} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence, as before:

$$[e_1]: \quad \alpha_1 = \frac{\partial}{\partial e_1}\gamma(e_1+\tilde{e}_2) = \begin{cases} e_1+\tilde{e}_2 & \text{if } e_1+\tilde{e}_2 > \hat{e} \\ 0 & \text{otherwise} \end{cases}$$

$$[e_2]: \quad \alpha_2 = \frac{\partial}{\partial e_1}\gamma(\tilde{e}_1+e_2) = \begin{cases} \tilde{e}_1+e_2 & \text{if } \tilde{e}_1+e_2 > \hat{e} \\ 0 & \text{otherwise} \end{cases}$$

$$[\text{at optimal } e = (e_1,e_2)] \Rightarrow \quad \alpha_1 = \alpha_2 = \begin{cases} e_1+e_2 & \text{if } e_1+e_2 > \hat{e} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \alpha_1 = \alpha_2 \notin (0, \hat{e}].$$

3 Q5

Consider the case of a single firm contracting with two agents, each with a CARA parameter of r in the standard linear-contracts framework of Holmstrom and Milgrom. The outputs of the two agents are $x_1 = e_1 + \varepsilon_1$ and $x_2 = e_2 + \varepsilon_2$, where e_i is agent i's effort and ε_i is a normally distributed noise term, $\varepsilon_i \sim \mathcal{N}\left(0, \sigma_i^2\right)$. The cost of effort to each agent i is $\frac{1}{2}e_i^2$. Finally, the measurement errors across the two agents may be positively correlated, with $\operatorname{Cov}\left(\varepsilon_1, \varepsilon_2\right) = \sigma_{12} = \rho\sigma_1\sigma_2$, $\rho \geq 0$. Assume for reasons given in Holmstrom and Milgrom that optimal contracts are linear.

Procblem 3.1. Consider the benchmark where a firm is restricted to allowing only an individual worker's own output to affect compensation:

$$w_i\left(x_1, x_2\right) = \alpha_i^i x_i + \beta^i$$

Superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i).

Solution. The firm's problem is

$$\max_{\alpha_1^1, \alpha_2^2, e_1, e_2} \mathbb{E}\left[x_1 + x_2 - \alpha_1^1 x_1 - \beta^1 - \alpha_2^2 x_2 - \beta^2\right]$$

$$\begin{aligned}
\text{s.t. } e_1 \in \arg\max_{e_1} \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\beta^1-\frac{1}{2}e_1^2\right)}\right] & \qquad \text{IC for Agent 1} \\
\underline{U}_1 \leq \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\beta^1-\frac{1}{2}e_1^2\right)}\right] & \qquad \text{IR for Agent 1} \\
e_2 \in \arg\max_{e_2} \mathbb{E}\left[-e^{-r\left(\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2\right)}\right] & \qquad \text{IC for Agent 2} \\
\underline{U}_2 \leq \mathbb{E}\left[-e^{-r\left(\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2\right)}\right] & \qquad \text{IR for Agent 2}
\end{aligned}$$

The certainty equivalence of agent 1 c_1 satisfies

$$\begin{aligned}
-e^{-rc_1} &= \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\beta^1-\frac{1}{2}e_1^2\right)}\right] \\
&= -e^{-r\left(\alpha_1^1 e_1+\beta^1-\frac{1}{2}e_1^2\right)}\mathbb{E}\left[e^{-r\alpha_1^1\varepsilon_1}\right] \\
&= -e^{-r\left(\alpha_1^1 e_1+\beta^1-\frac{1}{2}e_1^2\right)}e^{\frac{r^2}{2}\left(\alpha_1^1\right)^2\sigma_1^2} \\
&= -e^{-r\left(\alpha_1^1 e_1+\beta^1-\frac{1}{2}e_1^2-\frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2\right)} \\
\Rightarrow c_1 &= \alpha_1^1 e_1 + \beta^1 - \frac{1}{2}e_1^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2
\end{aligned}$$

By symmetry, agent 2's certainty equivalence c_2 is

$$c_2 = \alpha_2^2 e_2 + \beta^2 - \frac{1}{2}e_2^2 - \frac{r}{2}\left(\alpha_2^2\right)^2 \sigma_2^2$$

Both agents would choose an effort level to maximize their certainty equivalence. Since the certainty equivalence are strictly concave in e_i for both agents, for each fixed wage offer, there is only one level of effort that satisfies IC for both agents, which is solved by the f.o.c.

$$\alpha_1^1 - e_1 = 0 \quad \Rightarrow \quad e_1 = \alpha_1^1$$

$$\alpha_2^2 - e_2 = 0 \quad \Rightarrow \quad e_2 = \alpha_2^2$$

Substitute in $e_1 = \alpha_1^1$ and $e_2 = \alpha_2^2$, then the firm's objective function becomes

$$\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^2} \mathbb{E}\left[\left(\alpha_1^1+\varepsilon_1\right)+\left(\alpha_2^2+\varepsilon_2\right)-\alpha_1^1\left(\alpha_1^1+\varepsilon_1\right)-\beta^1-\alpha_2^2\left(\alpha_2^2+\varepsilon_2\right)-\beta^2\right] \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_2^2-\left(\alpha_1^1\right)^2-\left(\alpha_2^2\right)^2-\beta^1-\beta^2\right] + \underbrace{\mathbb{E}\left[\varepsilon_1+\varepsilon_2-\alpha_1^1\varepsilon_1-\alpha_2^2\varepsilon_2\right]}_{=0} \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_2^2-\left(\alpha_1^1\right)^2-\left(\alpha_2^2\right)^2-\beta^1-\beta^2\right] \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^2} \alpha_1^1+\alpha_2^2-\left(\alpha_1^1\right)^2-\left(\alpha_2^2\right)^2-\beta^1-\beta^2 \qquad (4)
\end{aligned}$$

The binding IR for agent 1 gives us

$$\begin{aligned}
-e^{-rc_1} &= \underline{U}_1 \\
\Rightarrow \alpha_1^1 e_1 + \beta^1 - \frac{1}{2}e_1^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 &= \frac{1}{-r}\ln\left(-\underline{U}_1\right) \\
\Rightarrow \alpha_1^1 e_1 - \frac{1}{2}e_1^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 &= \underbrace{\frac{1}{-r}\ln\left(-\underline{U}_1\right)-\beta^1}_{\equiv m_1} \\
\Rightarrow \alpha_1^1 e_1 - \frac{1}{2}e_1^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 &= m_1 \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 &= m_1 \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 + \frac{1}{r}\ln\left(-\underline{U}_1\right) &= -\beta^1 \qquad (5)
\end{aligned}$$

Similarly, the binding IR for agent 2 gives us

$$\frac{1}{2}\left(\alpha_2^2\right)^2 - \frac{r}{2}\left(\alpha_2^2\right)^2\sigma_2^2 + \frac{1}{r}\ln\left(-\underline{U}_2\right) = -\beta^2 \qquad (6)$$

Substitute (5) and (6) into (4), we can get the objective function as

$$
\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \left(\alpha_1^1\right)^2 - \left(\alpha_2^2\right)^2 - \beta^1 - \beta^2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \left(\alpha_1^1\right)^2 - \left(\alpha_2^2\right)^2 + \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 + \frac{1}{r}\ln\left(-\underline{U}_1\right) \\
&\quad + \frac{1}{2}\left(\alpha_2^2\right)^2 - \frac{r}{2}\left(\alpha_2^2\right)^2\sigma_2^2 + \frac{1}{r}\ln\left(-\underline{U}_2\right) \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{1}{2}\left(\alpha_2^2\right)^2 - \frac{r}{2}\left(\alpha_1^1\right)^2\sigma_1^2 - \frac{r}{2}\left(\alpha_2^2\right)^2\sigma_2^2
\end{aligned}
$$

The first-order condition for α_1^1 gives us

$$1 - \alpha_1^1 - r\sigma_1^2\alpha_1^1 = 0 \quad \Leftrightarrow \quad \alpha_1^1 = \frac{1}{1 + r\sigma_1^2}$$

Similary, the first-order condition for α_2^2 gives us

$$1 - \alpha_2^2 - r\sigma_2^2\alpha_2^2 = 0 \quad \Leftrightarrow \quad \alpha_2^2 = \frac{1}{1 + r\sigma_2^2}$$

So, the optimal contract parameters are

$$\alpha_1^1 = \frac{1}{1 + r\sigma_1^2}, \quad \alpha_2^2 = \frac{1}{1 + r\sigma_2^2}$$

Procblem 3.2. Now suppose that the firm can use relative performance evaluation. That is

$$w_1\left(x_1, x_2\right) = \alpha_1^1 x_1 + \alpha_2^1 x_2 + \beta^1$$

$$w_2\left(x_1, x_2\right) = \alpha_1^2 x_1 + \alpha_2^2 x_2 + \beta^2$$

Again, superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i). Explain how and why α_i^i varies with respect to ρ. Does it matter if the correlation is positive or negative?

Hint: Hint: if ε_1 and ε_2 are normal random variables with covariance σ_{12}, then

$$\mathbb{E}\left[-e^{r(a\varepsilon_1 + b\varepsilon_2)}\right] = -e^{\frac{r^2}{2}\left(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_{12}\right)}$$

Solution. The firm's problem is

$$\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2,e_1,e_2} \mathbb{E}\left[x_1 + x_2 - \alpha_1^1 x_1 - \alpha_2^1 x_2 - \beta^1 - \alpha_1^2 x_1 - \alpha_2^2 x_2 - \beta^2\right]$$

$$\begin{aligned}
\text{s.t. } e_1 \in \arg\max_{e_1} \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}e_1^2\right)}\right] && \text{IC for Agent 1}\\
\mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}e_1^2\right)}\right] \geq \underline{U}_1 && \text{IR for Agent 1}\\
e_2 \in \arg\max_{e_2} \mathbb{E}\left[-e^{-r\left(\alpha_1^2(e_1+\varepsilon_1)+\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2\right)}\right] && \text{IC for Agent 2}\\
\mathbb{E}\left[-e^{-r\left(\alpha_1^2(e_1+\varepsilon_1)+\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2\right)}\right] \geq \underline{U}_2 && \text{IR for Agent 1}
\end{aligned}$$

The certainty equivalence of agent 1 c_1 satisfies

$$\begin{aligned}
-e^{-rc_1} &= \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}e_1^2\right)}\right]\\
&= -e^{-r\left(\alpha_1^1e_1+\alpha_2^1e_2+\beta^1-\frac{1}{2}e_1^2\right)}\mathbb{E}\left[-e^{-r\left(\alpha_1^1\varepsilon_1+\alpha_2^1\varepsilon_2\right)}\right]\\
&= -e^{-r\left(\alpha_1^1e_1+\alpha_2^1e_2+\beta^1-\frac{1}{2}e_1^2\right)}e^{\frac{r^2}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)}\\
&= -e^{-r\left(\alpha_1^1e_1+\alpha_2^1e_2+\beta^1-\frac{1}{2}e_1^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)\right)}\\
\Rightarrow c_1 &= \alpha_1^1e_1+\alpha_2^1e_2+\beta^1-\frac{1}{2}e_1^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)
\end{aligned}$$

By symmetry, agent 2's certainty equivalence c_2 is

$$c_2 = \alpha_1^2e_1+\alpha_2^2e_2+\beta^2-\frac{1}{2}e_2^2-\frac{r}{2}\left(\left(\alpha_2^2\right)^2\sigma_2^2+\left(\alpha_1^2\right)^2\sigma_1^2+2\alpha_2^2\alpha_1^2\sigma_{12}\right)$$

Same as in part (a), the f.o.c. give us

$$\alpha_1^1 - e_1 = 0 \quad \Rightarrow \quad e_1 = \alpha_1^1$$

$$\alpha_2^2 - e_2 = 0 \quad \Rightarrow \quad e_2 = \alpha_2^2$$

Again, since the question says we can ignore β^i, we are only solving for the α_1^1, α_2^1, α_2^2 and α_1^2 for the optimal contract.

Substitute in $e_1 = \alpha_1^1$ and $e_2 = \alpha_2^2$, then the firm's objective function becomes

$$\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\left(\alpha_1^1+\varepsilon_1\right)+\left(\alpha_2^2+\varepsilon_2\right)-\left(\alpha_1^1+\alpha_1^2\right)\left(\alpha_1^1+\varepsilon_1\right)-\beta^1-\left(\alpha_2^1+\alpha_2^2\right)\left(\alpha_2^2+\varepsilon_2\right)-\beta^2\right]\\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_2^2-\left(\alpha_1^1+\alpha_1^2\right)\alpha_1^1-\left(\alpha_2^1+\alpha_2^2\right)\alpha_2^2-\beta^1-\beta^2\right]\\
&\qquad + \underbrace{\mathbb{E}\left[\varepsilon_1+\varepsilon_2-\left(\alpha_1^1+\alpha_1^2\right)\varepsilon_1-\left(\alpha_2^1+\alpha_2^2\right)\varepsilon_2\right]}_{=0}\\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_2^2-\left(\alpha_1^1+\alpha_1^2\right)\alpha_1^1-\left(\alpha_2^1+\alpha_2^2\right)\alpha_2^2-\beta^1-\beta^2\right]\\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1+\alpha_2^2-\left(\alpha_1^1\right)^2-\alpha_1^2\alpha_1^1-\left(\alpha_2^2\right)^2-\alpha_2^1\alpha_2^2-\beta^1-\beta^2
\end{aligned} \tag{7}$$

The binding IR for agent 1 gives us

$$
\begin{aligned}
-e^{-rc_1} &= \underline{U}_1 \\
\Rightarrow \alpha_1^1 e_1 + \alpha_2^1 e_2 + \beta^1 - \frac{1}{2}e_1^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right) &= \frac{1}{-r}\ln\left(-\underline{U}_1\right) \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2 + \alpha_2^1\alpha_2^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right) &= \underbrace{\frac{1}{-r}\ln\left(-\underline{U}_1\right) - \beta^1}_{\equiv m_1} \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2 + \alpha_2^1\alpha_2^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right) &= m_1 \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right) - m_1 &= -\alpha_2^1\alpha_2^2 \qquad (8)
\end{aligned}
$$

Similarly, the binding IR for agent 2 gives us

$$
\frac{1}{2}\left(\alpha_2^2\right)^2 - \frac{r}{2}\left(\left(\alpha_2^2\right)^2\sigma_2^2 + \left(\alpha_1^2\right)^2\sigma_1^2 + 2\alpha_2^2\alpha_1^2\sigma_{12}\right) - m_2 = -\alpha_1^2\alpha_1^1 \qquad (9)
$$

Substitute (8) and (9) into (7), we can get the objective function as

$$
\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \left(\alpha_1^1\right)^2 - \alpha_1^2\alpha_1^1 - \left(\alpha_2^2\right)^2 - \alpha_2^1\alpha_2^2 - \beta^1 - \beta^2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \left(\alpha_1^1\right)^2 - \left(\alpha_2^2\right)^2 + \frac{1}{2}\left(\alpha_2^2\right)^2 - \frac{r}{2}\left(\left(\alpha_2^2\right)^2\sigma_2^2 + \left(\alpha_1^2\right)^2\sigma_1^2 + 2\alpha_2^2\alpha_1^2\sigma_{12}\right) \\
&\qquad + \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right) - m_1 - m_2 - \beta^1 - \beta^2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1 + \alpha_2^2 - \frac{1}{2}\left(\alpha_1^1\right)^2 - \frac{1}{2}\left(\alpha_2^2\right)^2 \\
&\qquad - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_1^2\right)^2\sigma_1^2 + \left(\alpha_2^2\right)^2\sigma_2^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_2^2\alpha_1^2\sigma_{12} + 2\alpha_1^1\alpha_2^1\sigma_{12}\right)
\end{aligned}
$$

The f.o.c. for α_1^1 is

$$
1 - 2\alpha_1^1 - r\left(\alpha_1^1\sigma_1^2 + \alpha_2^1\sigma_{12}\right) = 0
$$

The f.o.c. for α_2^1 is

$$
\begin{aligned}
2\left(\alpha_2^1\right)\sigma_2^2 + 2\alpha_1^1\sigma_{12} &= 0 \\
\alpha_2^1 &= \frac{-\alpha_1^1\sigma_{12}}{\sigma_2^2} \\
\alpha_1^1 &= \frac{-\sigma_2^2}{\sigma_{12}}\alpha_2^1
\end{aligned}
$$

Plug in $\alpha_1^1 = \frac{-\sigma_2^2}{\sigma_{12}}\alpha_2^1$ we can get

$$\begin{aligned}
1 - 2\alpha_1^1 - r\left(\alpha_1^1\sigma_1^2 + \alpha_2^1\sigma_{12}\right) &= 0 \\
\Rightarrow 1 + 2\frac{\sigma_2^2}{\sigma_{12}}\alpha_2^1 + r\frac{\sigma_2^2}{\sigma_{12}}\alpha_2^1\sigma_1^2 - r\alpha_2^1\sigma_{12} &= 0 \\
\Rightarrow \alpha_2^1 &= \frac{-1}{\frac{\sigma_2^2}{\sigma_{12}} + r\frac{\sigma_2^2}{\sigma_{12}}\sigma_1^2 - r\sigma_{12}} \\
&= \frac{-1}{\frac{\sigma_2^2}{\rho\sigma_1\sigma_2} + r\frac{\sigma_2^2}{\rho\sigma_1\sigma_2}\sigma_1^2 - r\rho\sigma_1\sigma_2} \\
&= \frac{-1}{\frac{\sigma_2}{\rho\sigma_1} + r\frac{\sigma_2}{\rho\sigma_1}\sigma_1^2 - r\rho\sigma_1\sigma_2} \\
&= \frac{-\rho\sigma_1}{\sigma_2 + r\sigma_2\sigma_1^2 - r\rho^2\sigma_1^2\sigma_2} \\
\Rightarrow \alpha_2^1 &= \frac{-\rho\sigma_1}{\sigma_2 + r\sigma_2\sigma_1^2\left(1-\rho^2\right)}
\end{aligned}$$

Then,

$$\begin{aligned}
\alpha_1^1 &= \frac{-\sigma_2^2}{\sigma_{12}}\alpha_2^1 \\
&= \frac{-\sigma_2^2}{\rho\sigma_1\sigma_2}\frac{-\rho\sigma_1}{\sigma_2 + r\sigma_2\sigma_1^2\left(1-\rho^2\right)} \\
\Rightarrow \alpha_1^1 &= \frac{\sigma_2}{\sigma_2 + r\sigma_2\sigma_1^2\left(1-\rho^2\right)}
\end{aligned}$$

By symmetry, we have

$$\begin{aligned}
\alpha_2^2 &= \frac{\sigma_1}{\sigma_1 + r\sigma_1\sigma_2^2\left(1-\rho^2\right)} \\
\alpha_1^2 &= \frac{-\rho\sigma_2}{\sigma_1 + r\sigma_1\sigma_2^2\left(1-\rho^2\right)}
\end{aligned}$$

So, in conclusion, we have the optimal contract designed as

$$\begin{cases}
\alpha_1^1 = \frac{\sigma_2}{\sigma_2 + r\sigma_2\sigma_1^2(1-\rho^2)} \\
\alpha_2^1 = \frac{-\rho\sigma_1}{\sigma_2 + r\sigma_2\sigma_1^2(1-\rho^2)}
\end{cases}
\qquad
\begin{cases}
\alpha_1^2 = \frac{-\rho\sigma_2}{\sigma_1 + r\sigma_1\sigma_2^2(1-\rho^2)} \\
\alpha_2^2 = \frac{\sigma_1}{\sigma_1 + r\sigma_1\sigma_2^2(1-\rho^2)}
\end{cases}$$

Note that α_i^i increases in ρ because we assumed $r > 0$ and we have $\sigma_1 > 0$ and $\sigma_2 > 0$. The sign of ρ doesn't matter.

Procblem 3.3. Continue to assume that the firm can use relative performance evaluation, as in (b), but that now $\rho = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$. In addition to choosing e_2 to increase output, agent two can also engage in a second "helpful" activity, h_2. This activity does not affect the output level directly, but rather reduces the effort cost of the other agent. The interpretation is that agent 2 can help agent 1 (but not the other way round). The cost functions of the agents are given by:

$$\psi_1(e_1, h_2) = \frac{1}{2}(e_1 - h_2)^2$$

$$\psi_2(e_2, h_2) = \frac{1}{2}e_2^2 + \frac{1}{2}h_2^2$$

Agent 1 chooses his effort level e_1 only after he has observed the level of help h_2. Compute the optimal incentive scheme and effort levels. Explain your result.

Solution. The firm's problem is

$$\max_{\alpha_1^1, \alpha_2^1, \alpha_1^2, \alpha_2^2, e_1, e_2} \mathbb{E}\left[x_1 + x_2 - \alpha_1^1 x_1 - \alpha_2^1 x_2 - \beta^1 - \alpha_1^2 x_1 - \alpha_2^2 x_2 - \beta^2\right]$$

$$\text{s.t. } e_1 \in \arg\max_{e_1} \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}(e_1-h_2)^2\right)}\right] \qquad \text{IC for Agent 1}$$

$$\mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}(e_1-h_2)^2\right)}\right] \geq \underline{U}_1 \qquad \text{IR for Agent 1}$$

$$e_2 \in \arg\max_{e_2} \mathbb{E}\left[-e^{-r\left(\alpha_1^2(e_1+\varepsilon_1)+\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2\right)}\right] \qquad \text{IC for Agent 2}$$

$$\mathbb{E}\left[-e^{-r\left(\alpha_1^2(e_1+\varepsilon_1)+\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2\right)}\right] \geq \underline{U}_2 \qquad \text{IR for Agent 1}$$

The certainty equivalence of agent 1 c_1 satisfies

$$\begin{aligned}
-e^{-rc_1} &= \mathbb{E}\left[-e^{-r\left(\alpha_1^1(e_1+\varepsilon_1)+\alpha_2^1(e_2+\varepsilon_2)+\beta^1-\frac{1}{2}(e_1-h_2)^2\right)}\right] \\
&= -e^{-r\left(\alpha_1^1 e_1+\alpha_2^1 e_2+\beta^1-\frac{1}{2}(e_1-h_2)^2\right)} \mathbb{E}\left[-e^{-r\left(\alpha_1^1\varepsilon_1+\alpha_2^1\varepsilon_2\right)}\right] \\
&= -e^{-r\left(\alpha_1^1 e_1+\alpha_2^1 e_2+\beta^1-\frac{1}{2}(e_1-h_2)^2\right)} e^{\frac{r^2}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)} \\
&= -e^{-r\left(\alpha_1^1 e_1+\alpha_2^1 e_2+\beta^1-\frac{1}{2}(e_1-h_2)^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)\right)} \\
\Rightarrow c_1 &= \alpha_1^1 e_1 + \alpha_2^1 e_2 + \beta^1 - \frac{1}{2}(e_1-h_2)^2 - \frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2 + \left(\alpha_2^1\right)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12}\right)
\end{aligned}$$

Agent 2's certainty equivalence c_2 is

$$\begin{aligned}
-e^{-rc_2} &= \mathbb{E}\left[-e^{-r\left(\alpha_1^2(e_1+\varepsilon_1)+\alpha_2^2(e_2+\varepsilon_2)+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2\right)}\right] \\
&= -e^{-r\left(\alpha_1^2 e_1+\alpha_2^2 e_2+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2\right)} \mathbb{E}\left[e^{-r\left(\alpha_1^2\varepsilon_1+\alpha_2^2\varepsilon_2\right)}\right] \\
&= -e^{-r\left(\alpha_1^2 e_1+\alpha_2^2 e_2+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2\right)} e^{\frac{r^2}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right)} \\
&= -e^{-r\left(\alpha_1^2 e_1+\alpha_2^2 e_2+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right)\right)} \\
\Rightarrow c_2 &= \alpha_1^2 e_1 + \alpha_2^2 e_2 + \beta^2 - \frac{1}{2}e_2^2 - \frac{1}{2}h_2^2 - \frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2 + \left(\alpha_2^2\right)^2\sigma_2^2 + 2\alpha_1^2\alpha_2^2\sigma_{12}\right)
\end{aligned}$$

The f.o.c. for agent 2 w.r.t. e_2 gives us

$$\alpha_2^2 - e_2 = 0 \quad \Rightarrow \quad e_2 = \alpha_2^2$$

The f.o.c. for agent 1 w.r.t. e_1 gives us

$$\alpha_1^1 - (e_1 - h_2) = 0 \quad \Rightarrow \quad e_1 = \alpha_1^1 + h_2$$

So, agent 2's certainty equivalence becomes

$$\Rightarrow c_2 = \alpha_1^2\left(\alpha_1^1 + h_2\right) + \alpha_2^2 e_2 + \beta^2 - \frac{1}{2}e_2^2 - \frac{1}{2}h_2^2 - \frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2 + \left(\alpha_2^2\right)^2\sigma_2^2 + 2\alpha_1^2\alpha_2^2\sigma_{12}\right)$$

and the f.o.c. for agent 2 w.r.t. h_2 gives us

$$\alpha_1^2 - h_2 = 0 \quad \Rightarrow \quad h_2 = \alpha_1^2$$

So, we have

$$e_1 = \alpha_1^1 + \alpha_1^2 \text{ and } e_2 = \alpha_2^2$$

Again, since the question says we can ignore β^i, we are only solving for the α_1^1, α_2^1, α_2^2 and α_1^2 for the optimal contract.

Substitute in $e_1 = \alpha_1^1 + \alpha_1^2$ and $e_2 = \alpha_2^2$, then the firm's objective function becomes

$$\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2,e_1,e_2} \mathbb{E}\left[(e_1+\varepsilon_1) + (e_2+\varepsilon_2) - \left(\alpha_1^1+\alpha_1^2\right)(e_1+\varepsilon_1) - \beta^1 - \left(\alpha_2^1+\alpha_2^2\right)(e_2+\varepsilon_2) - \beta^2\right] \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\left(\alpha_1^1+\alpha_1^2+\varepsilon_1\right) + \left(\alpha_2^2+\varepsilon_2\right) - \left(\alpha_1^1+\alpha_1^2\right)\left(\alpha_1^1+\alpha_1^2+\varepsilon_1\right) - \beta^1 - \left(\alpha_2^1+\alpha_2^2\right)\left(\alpha_2^2+\varepsilon_2\right) - \beta^2\right] \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_1^2+\alpha_2^2 - \left(\alpha_1^1+\alpha_1^2\right)\left(\alpha_1^1+\alpha_1^2\right) - \beta^1 - \left(\alpha_2^1+\alpha_2^2\right)\alpha_2^2 - \beta^2\right] \\
&\qquad + \underbrace{\mathbb{E}\left[\varepsilon_1+\varepsilon_2 - \left(\alpha_1^1+\alpha_1^2\right)\varepsilon_1 - \left(\alpha_2^1+\alpha_2^2\right)\varepsilon_2\right]}_{=0} \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \mathbb{E}\left[\alpha_1^1+\alpha_1^2+\alpha_2^2 - \left(\alpha_1^1+\alpha_1^2\right)^2 - \beta^1 - \left(\alpha_2^1+\alpha_2^2\right)\alpha_2^2 - \beta^2\right] \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1+\alpha_1^2+\alpha_2^2 - \left(\alpha_1^1+\alpha_1^2\right)^2 - \beta^1 - \left(\alpha_2^1+\alpha_2^2\right)\alpha_2^2 - \beta^2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2} \alpha_1^1+\alpha_1^2+\alpha_2^2 - \left(\alpha_1^1\right)^2 - \left(\alpha_1^2\right)^2 - \left(\alpha_2^2\right)^2 - 2\alpha_1^1\alpha_1^2 - \alpha_2^1\alpha_2^2 \qquad (10)
\end{aligned}$$

The binding IR for agent 2 gives us

$$\begin{aligned}
-e^{-rc_2} &= \underline{U}_2 \\
\Rightarrow \alpha_1^2\left(\alpha_1^1+h_2\right)+\alpha_2^2 e_2+\beta^2-\frac{1}{2}e_2^2-\frac{1}{2}h_2^2-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right) &= \frac{1}{-r}\ln\left(-\underline{U}_2\right) \\
\Rightarrow \alpha_1^2\left(\alpha_1^1+\alpha_1^2\right)+\frac{1}{2}\left(\alpha_2^2\right)^2-\frac{1}{2}\left(\alpha_1^2\right)^2-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right) &= \underbrace{\frac{1}{-r}\ln\left(-\underline{U}_2\right)-\beta^2}_{\equiv m_2} \\
\Rightarrow \alpha_1^2\alpha_1^1+\frac{1}{2}\left(\alpha_2^2\right)^2+\frac{1}{2}\left(\alpha_1^2\right)^2-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right) &= m_2 \\
\Rightarrow \frac{1}{2}\left(\alpha_2^2\right)^2+\frac{1}{2}\left(\alpha_1^2\right)^2-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right)-m_2 &= -\alpha_1^2\alpha_1^1 \qquad (11)
\end{aligned}$$

The binding IR for agent 1 gives us

$$\begin{aligned}
-e^{-rc_1} &= \underline{U}_1 \\
\Rightarrow \alpha_1^1 e_1+\alpha_2^1 e_2+\beta^1-\frac{1}{2}\left(e_1-h_2\right)^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right) &= \frac{1}{-r}\ln\left(-\underline{U}_1\right) \\
\Rightarrow \alpha_1^1\left(\alpha_1^1+\alpha_1^2\right)+\alpha_2^1\alpha_2^2-\frac{1}{2}\left(\alpha_1^1\right)^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right) &= \underbrace{\frac{1}{-r}\ln\left(-\underline{U}_1\right)-\beta^1}_{\equiv m_1} \\
\Rightarrow \alpha_1^1\left(\alpha_1^1+\alpha_1^2\right)-\frac{1}{2}\left(\alpha_1^1\right)^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)-m_1 &= -\alpha_2^1\alpha_2^2 \\
\Rightarrow \frac{1}{2}\left(\alpha_1^1\right)^2-\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)-m_1 &= -\alpha_1^1\alpha_1^2-\alpha_2^1\alpha_2^2 \qquad (12)
\end{aligned}$$

Substitute (12) and (11) into (10), we can get the objective function as

$$\begin{aligned}
&\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2}\alpha_1^1+\alpha_1^2+\alpha_2^2-\left(\alpha_1^1\right)^2-\left(\alpha_1^2\right)^2-\left(\alpha_2^2\right)^2-2\alpha_1^1\alpha_1^2-\alpha_2^1\alpha_2^2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2}\alpha_1^1+\alpha_1^2+\alpha_2^2-\left(\alpha_1^1\right)^2-\left(\alpha_1^2\right)^2-\left(\alpha_2^2\right)^2+\frac{1}{2}\left(\alpha_1^1\right)^2+\frac{1}{2}\left(\alpha_2^2\right)^2+\frac{1}{2}\left(\alpha_1^2\right)^2 \\
&\qquad -\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}\right)-\frac{r}{2}\left(\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^2\alpha_2^2\sigma_{12}\right)-m_1-m_2 \\
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2}\alpha_1^1+\alpha_1^2+\alpha_2^2-\frac{1}{2}\left(\alpha_1^1\right)^2-\frac{1}{2}\left(\alpha_1^2\right)^2-\frac{1}{2}\left(\alpha_2^2\right)^2 \\
&\qquad -\frac{r}{2}\left(\left(\alpha_1^1\right)^2\sigma_1^2+\left(\alpha_1^2\right)^2\sigma_1^2+\left(\alpha_2^1\right)^2\sigma_2^2+\left(\alpha_2^2\right)^2\sigma_2^2+2\alpha_1^1\alpha_2^1\sigma_{12}+2\alpha_1^2\alpha_2^2\sigma_{12}\right)
\end{aligned}$$

Plug in $\sigma_1=\sigma_2=\sigma$ and $\rho=0$, we get

$$\begin{aligned}
\Leftrightarrow &\max_{\alpha_1^1,\alpha_2^1,\alpha_1^2,\alpha_2^2}\alpha_1^1+\alpha_1^2+\alpha_2^2-\frac{1}{2}\left(\alpha_1^1\right)^2-\frac{1}{2}\left(\alpha_1^2\right)^2-\frac{1}{2}\left(\alpha_2^2\right)^2 \\
&\qquad -\frac{r}{2}\left(\left(\alpha_1^1\right)^2+\left(\alpha_1^2\right)^2+\left(\alpha_2^1\right)^2+\left(\alpha_2^2\right)^2\right)\sigma^2
\end{aligned}$$

The f.o.c. for α_1^1 is

$$\begin{aligned} 1 - \alpha_1^1 - r\sigma^2 \left(\alpha_1^1\right) = 0 \\ \Rightarrow \alpha_1^1 = \frac{1}{1 + r\sigma^2} \end{aligned}$$

The f.o.c. for α_1^2 is

$$\begin{aligned} 1 - \alpha_1^2 - r\sigma^2 \left(\alpha_1^2\right) = 0 \\ \Rightarrow \alpha_1^2 = \frac{1}{1 + r\sigma^2} \end{aligned}$$

The f.o.c. for α_2^2 is

$$\begin{aligned} 1 - \alpha_2^2 - r\sigma^2 \left(\alpha_2^2\right) = 0 \\ \Rightarrow \alpha_2^2 = \frac{1}{1 + r\sigma^2} \end{aligned}$$

The f.o.c. for α_2^1 is

$$\begin{aligned} -\frac{r}{2} \left(2\alpha_2^1\right) \sigma^2 = 0 \\ \Rightarrow \alpha_2^1 = 0 \end{aligned}$$

So, in conclusion, we have the optimal contract designed as

$$\begin{cases} \alpha_1^1 = \frac{1}{1+r\sigma^2} \\ \alpha_2^1 = 0 \end{cases} \qquad \begin{cases} \alpha_1^2 = \frac{1}{1+r\sigma^2} \\ \alpha_2^2 = \frac{1}{1+r\sigma^2} \end{cases}$$

The optimal effort level is

$$\begin{aligned} e_1^\star = \alpha_1^1 + \alpha_1^2 = \frac{2}{1 + r\sigma^2} \\ e_2^\star = \alpha_2^2 = \frac{1}{1 + r\sigma^2} \end{aligned}$$

This result is very intuitive:

- ▷ Agent 2 can help agent 1, so we should give agent 2 some payoff for helping agent 1.
- ▷ But since agent 1 cannot help agent 2, optimally agent 1 receives nothing for agent 2's output.
- ▷ Since agent 2 pays some cost to help agent 1, agent 1 exerts less effort for his own output than agent 1.
- ▷ Since the firm only cares about the total output and the cost function for e_1 and h_2 is quadratic and additively separable, it is optimal for the firm to give the same incentive to e_1 and h_2.