## 9.2 Bounded Returns

In this section we study functional equations of the two forms discussed in the introduction to this chapter:

$$(1) \qquad v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\}, \quad \text{and}$$

$$(2) \qquad v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int_Z v[\phi(x, y, z'), z'] Q(z, dz') \right\}$$

under the assumption that the return function $F$ is bounded and continuous, the discount factor $\beta$ is strictly less than one, and the transition function $Q$ has the Feller property. Mathematically, the difference between the functional equation studied in Chapter 4 and (1) and (2) above is the presence in the latter of the exogenous stochastic shocks $z$, $z'$, and the resulting integral with respect to $Q(z, \cdot)$. But it is shown below that, under the same assumptions about $X$ used in Chapter 4 and suitable restrictions on the shocks, the required mathematical properties of the function $v$ are preserved under integration. Thus, the results for the deterministic model carry over virtually without change. Our approach in this section is first to study (1) and then to indicate how the arguments can be modified to fit (2).

As in the last section, let $(X, \mathscr{X})$ and $(Z, \mathscr{Z})$ be measurable spaces of possible values for the endogenous and exogenous state variables, respectively; let $(S, \mathscr{S}) = (X \times Z, \mathscr{X} \times \mathscr{Z})$ be the product space; let $Q$ be a transition function on $(Z, \mathscr{Z})$; let $\Gamma: S \to X$ be a correspondence describing the feasibility constraints; let $A$ be the graph of $\Gamma$; let $F: A \to \mathbf{R}$ be the one-period return function; and let $\beta \geq 0$ be the discount factor. Thus, the givens for the problem we will study are $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$. We will use $A_z$, $A_{yz}$, and so on to denote the sections of $A$.

Our first assumption, which restricts $X$, is precisely the one used throughout Chapter 4. The second puts restrictions on $Z$ and $Q$.

ASSUMPTION **9.4** *$X$ is a convex Borel set in $\mathbf{R}^l$, with its Borel subsets $\mathscr{X}$.*

ASSUMPTION **9.5** *One of the following conditions holds:*

*a. $Z$ is a countable set and $\mathscr{Z}$ is the $\sigma$-algebra containing all subsets of $Z$; or*

*b. $Z$ is a compact (Borel) set in $\mathbf{R}^k$, with its Borel subsets $\mathscr{Z}$, and the transition function $Q$ on $(Z, \mathscr{Z})$ has the Feller property.*

If $Z \subset \mathbf{R}^k$, we require that the Markov operator associated with $Q$ map the space of bounded continuous functions on $Z$ into itself (cf. Section 8.1). If $Z$ is a countable set, we use the discrete metric and all functions on $Z$ are continuous, so this requirement would be vacuous. Notice that a sequence $\{s_n = (x_n, z_n)\}$ in $S$ converges to $s = (x, z) \in S$ if and only if $x_n \to x$ and $z_n \to z$. Therefore, if $Z$ is a countable set, a function on $S$ is continuous if and only if each $z$-section $f(\cdot, z): X \to \mathbf{R}$ is continuous. As before we take $C(S)$ to be the space of bounded continuous functions $f: S \to \mathbf{R}$ with the sup norm, $\|f\| = \sup_{s \in S} |f(s)|$. We stress, as we did in Chapter 4, that many of the results below apply much more broadly, and the arguments here can easily be adapted to other situations.

The following lemma shows that, under these two assumptions, integration preserves the required properties of the integrand in (1)—boundedness, continuity, monotonicity, and concavity. This lemma is the basis for showing that the arguments presented in Chapter 4 can be applied here as well.

**LEMMA 9.5** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, and $Q$ satisfy Assumptions 9.4 and 9.5. If $f: X \times Z \to \mathbf{R}$ is bounded and continuous, then $Mf$ defined by*

$$(Mf)(y, z) = \int f(y, z')Q(z, dz' \qquad \text{all } (y, z) \in X \times Z,$$

*is also; that is, $M: C(S) \to C(S)$. If $f$ is increasing (strictly increasing) in each of its first $l$ arguments, then so is $Mf$; and if $f$ is concave (strictly concave) jointly in its first $l$ arguments, then so is $Mf$.*

*Proof.* Suppose that $f$ is bounded and continuous. Since for each $z \in Z$, $Q(z, \cdot)$ is a probability measure, it is clear that $\|Mf\| \le \|f\|$. Hence $Mf$ is bounded. To see that $Mf$ is continuous, choose a sequence $(y_n, z_n) \to (y, z)$. Then

$$|(Mf)(y, z) - (Mf)(y_n, z_n)|$$

$$Mf)(y, z) - (Mf)(y, z_n)| + |(Mf)(y, z_n) - (Mf)(y_n, z_n)|$$

$$\le |(Mf)(y, z) - (Mf)(y, z_n)| + \quad |f(y, z' \quad f(y_n, z')|Q(z_n, dz' \tag{3}$$

There are two possibilities, corresponding to the two possibilities for the space $Z$ admitted by Assumption 9.5.

If $Z$ is a countable set, then $z_n \to z$ implies that $z_n = z$ for all $n$ sufficiently large. Hence as $n \to \infty$ the first term in (3) vanishes and $Q(z_n, \cdot) = Q(z, \cdot)$ is a fixed probability measure. Moreover, the functions $h_n(z') =$

$|f(y, z') - f(y_n, z')|$, $n = 1, 2, \ldots$, are all measurable; the sequence of functions $\{h_n\}$ converges pointwise to the zero function; and each term in the sequence is bounded above by the constant function $2\|f\|$. Hence by the Lebesgue Dominated Convergence Theorem (Theorem 7.10),

$$\lim_{n\to\infty} \int h_n(z')Q(z, dz') = \int \lim_{n\to\infty} h_n(z')Q(z, dz') = 0,$$

and the second term in (3) also vanishes.

Alternatively, suppose that $Z$ is a compact set in $\mathbf{R}^k$. The fact that $Q$ has the Feller property implies that the first term in (3) vanishes as $n \to \infty$. Moreover, since $y_n \to y$, it follows that there exists a compact set $D \subseteq X$ such that $y_n \in D$, all $n$, and $y \in D$. Since $f$ is continuous, it is uniformly continuous on the compact set $D \times Z$. That is, for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$|f(y, z') - f(y_n, z')| < \varepsilon, \quad \text{all } n > N, \text{ all } z' \in Z.$$

Hence the second term in (3) vanishes as $n \to \infty$.

That weak monotonicity in $y$ is preserved is obvious. To see that strict monotonicity is preserved, choose $y, \hat{y} \in X$ such that $y \leq \hat{y}$ and $y \neq \hat{y}$. Then $f(y, z') < f(\hat{y}, z')$, all $z' \in Z$. The desired conclusion then follows from Exercises 7.18 and 7.24.

To see that concavity is preserved, choose $y, \hat{y} \in X$, with $y \neq \hat{y}$, and for any $\theta \in (0, 1)$, let $y_\theta = \theta y + (1 - \theta)\hat{y}$. If $f$ is concave in $y$, then

$$\begin{aligned}
(Mf)(y_\theta, z) &= \int f(y_\theta, z')Q(z, dz') \\
&\geq \int [\theta f(y, z') + (1 - \theta)f(\hat{y}, z')]Q(z, dz') \qquad (4) \\
&= \theta(Mf)(y, z) + (1 - \theta)(Mf)(\hat{y}, z), \\
&\qquad \text{all } z \in Z, \text{ all } \theta \in (0, 1).
\end{aligned}$$

If $f$ is strictly concave in $y$, then

$$f(y_\theta, z') > \theta f(y, z') + (1 - \theta)f(\hat{y}, z'), \quad \text{all } z' \in Z, \text{ all } \theta \in (0, 1),$$

and it follows from Exercises 7.18 and 7.24 that the inequality in (4) is also strict. ∎

In some situations the requirement that the set $Z \subset \mathbf{R}^k$ be compact is very unattractive. In fact, it can be dispensed with; but the proof of Lemma 9.5 becomes more complicated. We defer this proof until Section 12.6, when the required mathematical tools will have been developed.

With Lemma 9.5 in hand, it is straightforward to show that all of the results proved for deterministic dynamic programs have analogues when stochastic shocks are added. The next two assumptions are analogues to those used throughout Section 4.2.

**ASSUMPTION 9.6** *The correspondence* $X \times Z \to X$ *is nonempty, compact-valued, and continuous.*

**ASSUMPTION 9.7** *The function* $F: A \to \mathbf{R}$ *is bounded and continuous, and* $\beta \in (0, 1)$.

If $Z$ is a countable set, we interpret Assumption 9.6 to mean that for each fixed $z \in Z$, the correspondence $\Gamma(\cdot, z)$: $X \to X$ is nonempty, compact-valued, and continuous. Similarly, in this case Assumption 9.7 means that for each fixed $z \in Z$, the function $F(\cdot, \cdot, z)$: $A_z \to \mathbf{R}$ (the $z$-section of $F$) is continuous.

The following exercise shows that under these assumptions, Theorems 9.2 and 9.4 hold.

*Exercise 9.6* Show that under Assumptions 9.4–9.7, Assumptions 9.1–9.3 are satisfied. [*Hint.* Use the Measurable Selection Theorem (Theorem 7.6).]

Under these same assumptions, we have the following basic result.

**THEOREM 9.6** *Let* $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, *and* $\beta$ *satisfy Assumptions 9.4–9.7, and define the operator* $T$ *on* $C(S)$ *by*

$$(5) \qquad (Tf)(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int f(y, z')Q(z, dz') \right.$$

*Then* $T$: $C(S) \to C(S)$; $T$ *has a unique fixed point* $v$ *in* $C(S)$ *and for any* $v_0 \in C(S)$,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = \quad 2$$

*Moreover, the correspondence $G: S \to X$ defined by*

$$G(x, z) = \left\{ y \in \Gamma(x, z): v(x, z) = F(x, y, z) + \beta \int v(y, z')Q(z, dz') \right\}, \tag{6}$$

*is nonempty, compact-valued, and u.h.c.*

*Proof.* Fix $f \in C(S)$. Then it follows from Lemma 9.5 that

$$(Mf)(y, z) = \int f(y, z')Q(z, dz')$$

is a bounded continuous function of $(y, z)$. Moreover, since $Q(z, \cdot)$ is a probability measure, $M(f + c) = Mf + c$, for any constant function $c$. Hence the proof of Theorem 4.6 applies without change. ∎

To obtain sharper characterizations of the unique fixed point of $T$, more structure is needed. We examine in turn the consequences of monotonicity, concavity, and differentiability.

**ASSUMPTION 9.8** *For each $(y, z) \in X \times Z$, $F(\cdot, y, z): A_{yz} \to \mathbf{R}$ is strictly increasing.*

**ASSUMPTION 9.9** *For each $z \in Z$, $\Gamma(\cdot, z): X \to X$ is increasing in the sense that $x \leq x'$ implies $\Gamma(x, z) \subseteq \Gamma(x', z)$.*

**THEOREM 9.7** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.4–9.9, and let $v$ be the unique fixed point of the operator $T$ in (5). Then for each $z \in Z$, $v(\cdot, z): X \to \mathbf{R}$ is strictly increasing.*

*Proof.* Let $C'(S) \subset C(S)$ be the set of bounded continuous functions $f$ on $S$ that are nondecreasing in their first $l$ arguments, and let $C''(S) \subset C'(S)$ be the set of functions that are strictly increasing in those arguments. Since $C'(S)$ is a closed subspace of the complete metric space $C(S)$, by Corollary 1 to the Contraction Mapping Theorem (Theorem 3.2), it is sufficient to show that $T[C'(S)] \subseteq C''(S)$. Under Assumptions 9.8 and 9.9, Lemma 9.5 ensures that this is so. ∎

Next we consider concavity. Assumption 9.10 is a concavity restriction on $F$, and Assumption 9.11 is a convexity restriction on $\Gamma$.

ASSUMPTION **9.10** *For each $z \in Z$, $F(\cdot, \cdot, z)$: $A_z \to \mathbf{R}$ satisfies*

$$F[\theta(x, y) + (1 - \theta)(x', y'), z] \geq \theta F(x, y, z) + (1 - \theta)F(x', y', z),$$

$$\text{all } \theta \in (0, 1), \quad \text{and all } (x, y), (x', y') \in A_z$$

*and the inequality is strict if $x \neq x'$.*

ASSUMPTION **9.11** *For all $z \in Z$ and all $x, x' \in X$,*

$$y \in \Gamma(x, z) \quad \text{and} \quad y' \in \Gamma(x', z) \text{ implies}$$

$$\theta y + (1 - \theta)y' \in \Gamma[\theta x + (1 - \theta)x', z], \quad \text{all } \theta \in [0, 1].$$

Since the set $X$ is convex, Assumption 9.11 is equivalent to assuming that for each $z \in Z$, the set $A_z$ is convex. In particular, Assumption 9.11 implies that $\Gamma(s)$ is a convex set for each $s \in S$, and that there are no increasing returns.

THEOREM **9.8** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.4–9.7 and 9.10–9.11; let $v$ be the unique fixed point of the operator $T$ in (5); and let $G$ be the correspondence defined by (6). Then for each $z \in Z$, $v(\cdot, z)$: $X \to \mathbf{R}$ is strictly concave and $G(\cdot, z)$: $X \to X$ is a continuous (single-valued) function.*

*Proof.* Let $C'(S) \subset C(S)$ be the set of bounded continuous functions on $S$ that are weakly concave jointly in their first $l$ arguments, and let $C''(S) \subset C'(S)$ be the subset consisting of functions that are strictly concave jointly in those arguments. Since $C'(S)$ is a closed subspace of the complete metric space $C(S)$, by Corollary 1 to the Contraction Mapping Theorem (Theorem 3.2), it is sufficient to show that $T[C'(S)] \subseteq C''(S)$. Under Assumptions 9.10 and 9.11, Lemma 9.5 ensures that this is so. ∎

As it does in the deterministic case, concavity ensures that the sequence of approximate policy functions $\{g_n\}$ converges to the optimal policy function $g$.

THEOREM **9.9** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.4–9.7 and 9.10–9.11; let $C'(S) \subset C(S)$ be the set of bounded continuous functions on $S$ that are weakly concave jointly in their first $l$ arguments; let $v \in C'(S)$ be the* *unique fixed point of the operator $T$ in (5); and let $g = G$ be the (single-valued) function defined by (6). Let $v_0 \in C'(S)$, and define $\{(v_n, g_n)\}$ by*

$$v_n = Tv_{n-1}, \quad \text{and}$$

$$g_n(x, z) = \underset{y \in \Gamma(x,z)}{\operatorname{argmax}} \left\{F(x, y, z) + \beta \int v_n(y, z')Q(z, dz')\right\}$$

$$n = 1, 2, \ldots .$$

*Then $g_n \to g$ pointwise. If $X$ and $Z$ are both compact, then the convergence is uniform.*

*Proof.* Let $C''(S) \subset C'(S)$ be as defined in the proof of Theorem 9.8; as shown there $T[C'(S)] \subseteq C''(S)$ and $v \in C''(S)$. Let $v_0 \in C'(S)$, and define the functions $\{f_n\}$ and $f$ by

$$f_n(x, z, y) = F(x, y, z) + \beta \int v_n(y, z')Q(z, dz'), \quad n = \quad 2,$$

and

$$f(x, z, y) = F(x, y, z) + \beta \int v(y, z')Q(z, dz').$$

Since $v_0 \in C'(S)$, each function $v_n$, $n = 1, 2, \ldots$, is in $C''(S)$, as is $v$. Hence for any $s \in S = X \times Z$, the functions $\{f_n(s, \cdot)\}$ and $f(s, \cdot)$ are all strictly concave in $y$. Therefore Theorem 3.8 applies. ∎

For concave problems with interior solutions, the differentiability of the value function can also be established.

**ASSUMPTION 9.12** *For each fixed $z \in Z$, $F(\cdot, \cdot, z)$ is continuously differentiable in $(x, y)$ on the interior of $A_z$.*

**THEOREM 9.10** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.4–9.7 and 9.10–9.12; let $v \in C'(S)$ be the unique fixed point of the operator $T$ in (5), and let $g = G$ be the function defined by (6). If $x_0 \in \operatorname{int} X$ and $g(x_0, z_0) \in \operatorname{int} \Gamma(x_0, z_0)$, then $v(\cdot, z_0)$ is continuously differentiable in $x$ at $x_0$, with derivatives given by*

$$v_i(x_0, z_0) = F_i[x_0, g(x_0, z_0), z_0], \quad i = 1, \quad , l.$$

*Proof.* Let $x_0 \in \text{int } X$ and $g(x_0, z_0) \in \text{int } \Gamma(x_0, z_0)$. Then there is some open neighborhood $D$ of $x_0$ such that $g(x_0, z_0) \in \text{int } \Gamma(x, z_0)$, all $x \in D$. Hence we can define $W: D \to \mathbf{R}$ by

$$W(x) = F[x, g(x_0, z_0), z_0] + \beta \quad v[g(x_0, z_0), z']Q(z_0 \quad dz')$$

Clearly $W$ is concave and continuously differentiable on $D$ and

$$W(x) \leq v(x, z_0), \quad \text{all } x \in D,$$

with equality at $x_0$. Hence Theorem 4.10 applies, establishing the desired result. ■

In some applications it is reasonable to expect that the value function is monotone in $z$ as well as in $x$. Clearly this requires that $Z$ be a set for which monotonicity is well defined; thus, if $Z$ is a countable set, we will assume that $Z = \{1, 2, \ldots\}$. We will also need restrictions on $F$ and $\Gamma$ analogous to Assumptions 9.8 and 9.9, and an additional restriction on the transition function $Q$.

ASSUMPTION **9.13** *For each $(x, y) \in X \times X$, $F(x, y, \quad A_{xy} \to \mathbf{R}$ is strictly increasing.*

ASSUMPTION **9.14** *For each $x \in X$, $\Gamma(x, \cdot): Z \to X$ is increasing in the sense that $z \leq z'$ implies $\Gamma(x, z) \subseteq \Gamma(x, z')$.*

ASSUMPTION **9.15** *$Q$ is monotone; that is, if $f: Z \to \mathbf{R}$ is nondecreasing, then the function $(Mf)(z) = \int f(z')Q(z, dz')$ is also nondecreasing.*

THEOREM **9.11** *Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.4–9.7 and 9.13–9.15; and let $v \in C(S)$ be the unique fixed point of the operator $T$ in (5). Then for each $x \in X$, $v(x, \cdot): Z \to \mathbf{R}$ is strictly increasing.*

*Proof.* Let $C'(S) \subset C(S)$ be the set of bounded continuous functions on $X \times Z$ that are nondecreasing in $z$, and let $C''(S) \subset C'(S)$ be the subset consisting of functions that are strictly increasing in $z$. Since $C'(S)$ is a closed subspace of the complete metric space $C(S)$, by Corollary 1 to the Contraction Mapping Theorem, it is sufficient to show that $T[C'(S)] \subseteq C''(S)$.

Fix $x \in X$; suppose that $f(x, \cdot)$: $Z \to \mathbf{R}$ is nondecreasing; and choose $z_1 < z_2$. Let $y_1 \in \Gamma(x, z_1)$ attain the maximum in (5) for $z = z_1$. Then

$$\begin{aligned}(Tf)(x, z_1) &= F(x, y_1, z_1) + \beta \int f(y_1, z')Q(z_1, dz') \\ &< F(x, y_1, z_2) + \beta \int f(y_1, z')Q(z_2, dz') \\ &\leq \max_{y \in \Gamma(x, z_2)} \left[F(x, y, z_2) + \beta \int f(y, z')Q(z_2, dz')\right] \\ &= (Tf)(x, z_2),\end{aligned}$$

where the second line uses Assumptions 9.13 and 9.15 and the third uses Assumption 9.14. Hence $(Tf)(x, \cdot)$ is strictly increasing, as was to be shown. ■

In the remainder of this section we show that the results above all have close parallels for the case where the functional equation has the form in (2). Let $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $(S, \mathscr{S})$, $Q$, and $\beta$ be as specified above. In addition let $(Y, \mathscr{Y})$ be a measurable space of actions available to the decision-maker; let $\Gamma$: $X \times Z \to Y$ be a correspondence describing the feasibility constraints; let $A$ be the graph of $\Gamma$; let $F$: $A \to \mathbf{R}$ be the one-period return function; let

$$D = \{(x, y) \in X \times Y: y \in \Gamma(x, z), \text{ for some } z \in Z\};$$

and let $\phi$: $D \times Z \to X$ be the law of motion for $x$.

To characterize solutions to the functional equation (2), we are interested in the operator $T$ defined by

$$(Tf)(x, z) = \sup_{y \in \Gamma(x, z)} \left[F(x, y, z) + \beta \int f[\phi(x, y, z'), z']Q(z, dz')\right]. \tag{7}$$

Clearly we must retain Assumptions 9.4 and 9.5. We also need to restrict the set of feasible actions $Y$ and to place a continuity assumption on the law of motion $\phi$.

ASSUMPTION **9.16** *$Y$ is a convex Borel set in $\mathbf{R}^m$, with its Borel subsets $\mathscr{Y}$.*

ASSUMPTION **9.17** *$\phi$: $D \times Z \to X$ is continuous.*

If $Z$ is a countable set, then we interpret Assumption 9.17 to mean that for each $z \in Z$, the $z$-section of $\phi$, the function $\phi(\cdot, \cdot, z)$: $D \to X$ is continuous. With these additional assumptions, we have the following parallel to Lemma 9.5.

LEMMA **9.5′** *Let $(X, \mathscr{X})$, $(Y, \mathscr{Y})$, $(Z, \mathscr{Z})$, $Q$, and $\phi$ satisfy Assumptions 9.4, 9.5, 9.16, and 9.17. Then for any continuous function $f$: $X \times Z \to \mathbf{R}$, the function $h$: $D \times Z \to \mathbf{R}$ defined by*

$$h(x, y, z) = \int f[\phi(x, y, z'), z']Q(z, dz')$$

*is also continuous.*

*Proof.* Let $u = (x, y)$ and define $\psi(u, z') = f[\phi(u, z'), z']$; since $f$ and $\phi$ are continuous, so is $\psi$. It then follows from Lemma 9.5 that

$$h(u, z) = \int \psi(u, z')Q(z, dz')$$

is continuous.

With this result in hand, it is straightforward to mimic the results in Exercise 9.6 and in Theorems 9.6–9.11; the required steps are presented in the following exercise. Note that the range of $\Gamma$ is now $Y$, so $A$ is now a subset of $X \times Y \times Z$. Rather than restate Assumptions 9.6–9.14, however, we merely note that the appropriate modifications must be made.

***Exercise 9.7*** a. Let $(X, \mathscr{X})$, $(Y, \mathscr{Y})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, $\beta$, and $\phi$ satisfy Assumptions 9.4–9.7 and 9.16–9.17. Show that Assumptions 9.1′–9.3′ are satisfied.

b. Let the assumptions in part (a) hold, and let $T$ be the operator defined in (7). Show that $T$: $C(S) \to C(S)$; that $T$ has a unique fixed point $v \in C(S)$; and that for any $v_0 \in C(S)$,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 1, 2, \ldots$$

Also show that the correspondence $G: X \times Z \to Y$ defined by

$$G(x, z) = \left\{y \in \Gamma(x, z): v(x, z) = F(x, y, z) + \beta \int v[\phi(x, y, z'), z']Q(z, dz')\right\},$$

is nonempty, compact-valued, and u.h.c.

c. Show that if, in addition, Assumptions 9.8 and 9.9 hold and $\phi$ is nondecreasing in each of its first $l$ arguments, then $v$ is strictly increasing in each of its first $l$ arguments.

d. Suppose that, in addition to the assumptions in part (a), Assumptions 9.10 and 9.11 hold and that, for each $z' \in Z$, the function $\phi(\cdot, \cdot, z')$ is concave. Show that $v$ is strictly concave jointly in its first $l$ arguments and that $G$ is a continuous (single-valued) function.

e. Show that under the assumptions in part (d) the sequence of policy functions $\{g_n\}$ defined as in Theorem 9.9 converges pointwise to the optimal policy function $g$; show that if $X \times Z$ is compact the convergence is uniform.

f. Let the assumptions in part (d) hold, let Assumption 9.12 hold, and assume that the law of motion $\phi(y, z')$ does not depend on $x$. Suppose that $(x_0, z_0) \in \text{int}(X \times Z)$ and $g(x_0, z_0) \in \text{int } \Gamma(x_0, z_0)$. Show that $v(\cdot, z_0)$ is differentiable in $x$ at $x_0$ and that $v_i(x_0, z_0) = F_i[x_0, g(x_0, z_0), z_0]$, $i = 1, \ldots, l$.

g. Suppose that, in addition to the assumptions in part (a), Assumptions 9.13–9.15 hold. Show that $v(x, z)$ is strictly increasing in $z$.