Price Theory, III **Professor Lars Stole**
Economics 33003 **Spring 6019**

Assignment 8
(Due Friday, June 7, prior to the start of the Review session)

Problem 1 (Full surplus extraction with correlated signals.) A seller has a single unit to sell at zero opportunity cost, $\theta_0 = 0$, to one of two bidders. Each bidder has two possible types, $\{30, 60\}$, but the valuations are correlated according to the joint probability function, $f(\theta_1, \theta_2)$. Specifically, the probability that both are high or low is $f(30,30) = f(30,30) = \frac{1}{3}$, and the probability that the valuations are different is $f(30,60) = f(60,30) = \frac{1}{6}$.

Without loss of generality, suppose that the seller offers a direct mechanism $\{\phi_i, t_i\}_{i=1,2}$. Further, suppose that the seller wants to implement the efficient allocation of the good:

$$\phi_i(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_i > \theta_j, \\ 0 & \text{if } \theta_i < \theta_j, \\ \frac{1}{2} & \text{if } \theta_i = \theta_j. \end{cases}$$

Restrict attention to symmetric payments which are paid as a function of the bidder's reported type and are independent from whether or not the bidder gets the good if there is a tie. Thus, the seller needs to determine four numbers: $t_1(30,30)$, $t_1(30,60)$, $t_1(60,30)$ and $t_1(60,60)$.

(a). Show that the seller can implement the efficient allocation while simultaneously extracting all of the bidders' surplus.

[Hint: write down the two IC constraints for bidder i as a function of the four transfer payments using the conditional probabilities $\text{Prob}[\theta_j = 60 | \theta_i = 60] = \frac{2}{3}$, etc. Set the transfers so that the high type, $\theta_1 = 60$, is indifferent to telling the truth or pretending to be $\theta_1 = 30$ and show in your solution that the other IC constraint is slack. Write down the two IR constraints for bidder i and set these equal to zero (i.e., no surplus is left). Note that you will have one degree of freedom in your linear qualities. Set $t_1(60,60) = 30$ to find a unique solution for the remaining three transfers.]

(b). Reinterpret the answer in (a) as the following game. Each bidder is required to bid either 30 or 60. The winning bidder must pay her bid. In addition, any bidder who bids 30 is also required to take a side bet with the seller which has zero expected payoff if the bidder's true type is $\theta_i = 30$ and a large negative payoff if the bidder's type is actually $\theta_i = 60$. Describe the side bet for this reinterpretation of the mechanism in (a).

Problem 2 (Bilateral trade with discrete types.) Consider a bilateral trading situation similar to Myerson and Satterthwaite (1983), except that the distributions of types are discrete. Specifically, suppose $\theta_s \in \{1, 4\}$ with equal probability, and $\theta_b \in \{0, 3\}$ with equal probability. Hence, trade is efficient for some pairs of types, but not all.

(a). Do the conditions of the Myerson-Satterthwaite impossibility theorem apply to this situation? Explain.

(b). If your answer to (a) is "No," then attempt to provide a simple counter example in which efficient trade can be implemented. If your answer to (a) is "Yes," then for which type pairs is trade necessarily inefficient?

Problem 3 (Impossibility of efficient public good provision.) Consider a public goods setting in which two citizens must collectively decide whether or not to produce a public good which costs c (known to both parties). The value of the public good to each citizen is θ_i, distributed according to $F(\cdot)$ on $[0, 1]$. An ex post efficient mechanism produces the public good if $\theta_1 + \theta_2 \geq c$ and does not otherwise. Suppose that $c \in (0, 2)$ which implies that it is efficient to build the public good for some type profiles, but not for all.

Suppose that the government designs a direct revelation mechanism to implement the public good with probability $\phi(\theta_1, \theta_2)$ and taxes each citizen the amount $t_i(\theta_1, \theta_2)$, where we require that $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = c$ if the public good is built and $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0$ otherwise. (This is a form of ex post budget balance.) Assume that each agent has the right to refuse to participate in the mechanism after learning type θ_i, in which case the public good is not built and no taxes are charged to either agent.

Prove that there does not exist an IC, IR mechanism which is ex post efficient.

[Hint: Convert the public-goods problem into the Myerson-Satterthwaite bilateral trading problem. Imagine agent 1 in the role of buyer, $t_1 = t$ is a transfer from agent 1 to agent 2, and assume that agent 2 must build the public good using personal funds, c, if $\phi = 1$. Check all of the MS conditions. Then apply the MS impossibility theorem.]

Problem 4 (Required subsidy for efficient bilateral trade.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

(a). A necessary condition for IC (as shown in class) requires

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right] \geq 0.$$

Using the fact that types are uniformly distributed on $[0, 2]$, show that, in any mechanism, conditional on trade, the expected difference between θ_b and θ_s must be at least 1.

(b). Compute the expected difference between θ_b and θ_s, conditional on efficient trade occurring (i.e., conditional on $\theta_b \geq \theta_s$). Using (a), conclude directly that an ex post efficient, budget-balanced, IC mechanism cannot satisfy IR.

(c). Compute the value of

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right]$$

when ϕ is ex post efficient. Using your answers to (a) and (b), explain why this number is a measure of the minimum amount of external subsidy which must be introduced to obtain ex post efficient trade in an IC and IR mechanism.

Problem 5 (Bilateral trading game, Chatterjee-Samuelson (1983).) Consider a setting of bilateral trade in which the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$. Suppose the buyer and seller play the following game. Both buyer and seller simultaneously bid a price for trade. If the buyer's bid, p_b, weakly exceeds the seller's bid, p_s, then trade takes place at the average price bid, $p = \frac{p_b+p_s}{2}$. Otherwise, trade does not take place.

A pure-strategy Bayesian-Nash equilibrium is a pair of functions, $\{\overline{p}_b(\cdot), \overline{p}_s(\cdot)\}$, which is a map from a player's type to a bid.

(a). Given $\overline{p}_s(\cdot)$ is played by the seller, the Buyer's optimization program is

$$\max_{p_b} \left(\theta_b - \frac{p_b + E_{\theta_s}[\overline{p}_s(\theta_s) | p_b \geq \overline{p}_s(\theta_s)]}{2} \right) \text{Prob}_{\theta_s}[p_b \geq \overline{p}_s(\theta_s)].$$

Write down the Seller's optimization program, given $\overline{p}_b(\cdot)$.

(b). Show that the following is a discontinuous equilibrium to the game for any $x \in (0, 2)$:

$$\overline{p}_b(\theta_b) = \begin{cases} x & \text{if } \theta_b \geq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{p}_s(\theta_s) = \begin{cases} x & \text{if } \theta_s \leq x \\ 2 & \text{otherwise.} \end{cases}$$

(c). Now consider only linear equilibria: $\overline{p}_b(\theta_n) = \alpha_b + \beta_b \theta_b$ and $\overline{p}_s(\theta_s) = \alpha_s + \beta_s \theta_s$. Show that a linear equilibrium exists and compute the equilibrium bidding functions.

(d). In the equilibrium in (c), trade takes place if and only if $\theta_b - \theta_s \geq \alpha$. What is the value of the gap α?

Problem 6 (Maximizing welfare in bilateral trade.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

(a). Solve for the welfare-maximizing trading allocation direct mechanism, $\{\phi(\cdot), t(\cdot)\}$, that is incentive compatible, individually rational and ex post budget-balanced (i.e., the seller receives exactly the payment that the buyer pays – there are no additional subsidies). For this question, it is sufficient to characterize $\phi(\cdot)$; you do not need to characterize $t(\cdot)$.

(b). How does the efficiency gap in (a) compare to the one you found in Exercise 5 in the linear equilibrium to the Chatterjee-Samuelson bilateral trading game?

Problem 7 (Maximizing profit for a bilateral trading platform.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

Suppose that a trading platform designs a profit-maximizing mechanism which allocates the seller's good to the buyer with probability $\phi(\theta_b, \theta_s)$, the buyer pays the platform $t_b(\theta_b, \theta_s)$, the seller receives $t_s(\theta_b, \theta_s)$ and the platform keeps the difference in payments as profit:

$$t_b(\theta_b, \theta_s) - t_s(\theta_b, \theta_s).$$

(a). Write down the monotonicity and integral conditions which are necessary and sufficient for the platform's mechanism to be incentive compatible.

(b). Using your answer in (a), compute $E[U_b(\theta_b)]$ and $E[U_s(\theta_s)]$ as functions of $\phi(\theta_b, \theta_s)$, $U(0)$ and $U_s(2)$.

(c). Using your result in (b), write the platform's objective entirely in terms of $\phi(\theta_b, \theta_s)$, $U(\underline{\theta}_b)$ and $U_s(\overline{\theta}_s)$.

(d). Solve for the profit-maximizing trading rule, ϕ.

(e). How does your answer in (d) compare to the welfare-maximizing rule found in Exercise 6?

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Answers to Assignment 8

1 (a). The two IC constraints are

$$\frac{2}{3}\left(\phi_1(30,30)30 - t_1(30,30)\right) + \frac{1}{3}\left(\phi_1(30,60)30 - t_1(30,60)\right)$$
$$\geq \frac{2}{3}\left(\phi_1(60,30)30 - t_1(60,30)\right) + \frac{1}{3}\left(\phi_1(60,60)30 - t_1(60,60)\right),$$

$$\frac{2}{3}\left(\phi_1(60,60)60 - t_1(60,60)\right) + \frac{1}{3}\left(\phi_1(60,30)60 - t_1(60,30)\right)$$
$$\geq \frac{2}{3}\left(\phi_1(30,60)60 - t_1(30,60)\right) + \frac{1}{3}\left(\phi_1(30,30)60 - t_1(30,30)\right).$$

The two IR constraints (satisfied with equality) are

$$\frac{2}{3}\left(\phi_1(30,30)30 - t_1(30,30)\right) + \frac{1}{3}\left(\phi_1(30,60)30 - t_1(30,60)\right) = 0,$$
$$\frac{2}{3}\left(\phi_1(60,60)60 - t_1(60,60)\right) + \frac{1}{3}\left(\phi_1(60,30)60 - t_1(60,30)\right) = 0.$$

The two IR constraints (satisfied with equality) are

Using the fact that ϕ is efficient and using the hint that the IC constraint will bind for the $\theta_i = 60$ type, these four conditions are simplified to

$$\frac{2}{3}\left(15 - t_1(30,30)\right) + \frac{1}{3}\left(-t_1(30,60)\right) \geq \frac{2}{3}\left(30 - t_1(60,30)\right) + \frac{1}{3}\left(15 - t_1(60,60)\right),$$
$$\frac{2}{3}\left(30 - t_1(60,60)\right) + \frac{1}{3}\left(60 - t_1(60,30)\right) = \frac{2}{3}(-t_1(30,60) + \frac{1}{3}(30 - t_1(30,30)).$$
$$\frac{2}{3}\left(15 - t_1(30,30)\right) + \frac{1}{3}\left(-t_1(30,60)\right) = 0,$$
$$\frac{2}{3}\left(30 - t_1(60,60)\right) + \frac{1}{3}\left(60 - t_1(60,30)\right) = 0.$$

Substituting in $t_1(60,60) = 30$, we have three equations and three unknowns:

$$\frac{1}{3}\left(60 - t_1(60,30)\right) = \frac{2}{3}(-t_1(30,60) + \frac{1}{3}(30 - t_1(30,30)).$$
$$\frac{2}{3}\left(15 - t_1(30,30)\right) + \frac{1}{3}\left(-t_1(30,60)\right) = 0,$$
$$\frac{1}{3}\left(60 - t_1(60,30)\right) = 0.$$

The third equation implies $t_1(60,30) = 60$. Substituting this into the first two equations we have

$$0 = \frac{2}{3}(-t_1(30,60) + \frac{1}{3}(30 - t_1(30,30)).$$

$$\frac{2}{3}\left(15 - t_1(30,30)\right) + \frac{1}{3}\left(-t_1(30,60)\right) = 0.$$

Solving these two equations in two unknowns yields $t_1(30,30) = 10$ and $t_1(30,60) = 10$.

Lastly, we need to check that the low types IC constraint is in fact satisfied:

$$\frac{2}{3}\left(15 - t_1(30,30)\right) + \frac{1}{3}\left(-t_1(30,60)\right) \geq \frac{2}{3}\left(30 - t_1(60,30)\right) + \frac{1}{3}\left(15 - t_1(60,60)\right),$$

and thus we require that

$$\frac{2}{3}(15 - 10) + \frac{1}{3}(-10) \geq \frac{2}{3}(30 - 60) + \frac{1}{3}(15 - 30),$$

or

$$0 \geq \frac{2}{3}(-30) + \frac{1}{3}(15) = -15.$$

(b). We reinterpret the solution as a requirement that bidders bid either 30 or 60 in which winning bidders pay their bid, and bids of $b = 30$ require that the bidder take on a side bet. We want to use our transfers from (a) to figure out this side bet.

We have $t_1(60,60) = 30$ (regardless of who wins the coin toss), we can simply reinterpret this as a requirement that the winning bidder in a tie with $(60,60)$ pays 60 conditional on winning. Consider $t_1(60,30) = 60$. Here, the winning bidder pays her bid which is also consistent with the mechanism in (a).

Now consider bids of 30. We have $t_1(30,30) = 10$ which we can reinterpret as the winning bidder in a tie with $(30,30)$ paying 30, but both bidders get a prize of 5 from the seller when bidding $(30,30)$. In contrast, consider $t_1(30,60)$. The losing bidder in the mechanism in (a) pays 10. Thus, we can think of the following side bet for $b = 30$ bids: If the other bidder bids 30 (which happens 2/3 of the time under honest reporting), the bidder gets a prize of 5; if the other bidder bids 60 (which happens 1/3 of the time), the bidder pays 10. This has an expected value of 0, and so a risk neutral bidder with type $\theta_i = 30$ is willing to take on the bet. On the other hand, if $\theta_i = 60$, then this bet loses money: $(2/3)(-10) + (1/3)(5) = -15/3 = -5$. You'll notice that this expected loss, -5, is exactly equal to the expected gain from bidding $b = 30$ when $\theta = 60$ in the underlying auction without the side bet:

$$\frac{2}{3}(0) + \frac{1}{3}\left(\frac{1}{2}\right)(60 - 30) = 5.$$

2 (a). The conditions to Myerson-Satterthwaite are not satisfied because continuous distributions over intervals were used in the proof. It is not enough that the discrete distributions in this question overlap. Requiring continuous distributions effectively implies that a monopoly screening problem has continuous margins. With a smooth objective function, the buyer and seller will always find it optimal to distort trade (as any monopolist would) in order to increase surplus, and hence inefficiency will emerge. With discrete types, this is no longer the case.

(b). Here is a simple trading mechanism described in Myerson-Satterthwaite (1983) for precisely this setting: "sell at price 2 if both are willing (otherwise no trade)." This is IC and IR.

3 Here is a conversion that works. Let agent 1 be the "buyer" and $t = t_1$ is the transfer 1 pays directly to agent 2. Agent 1's utility is therefore

$$\phi\theta_1 - t.$$

For agent 2 (who has the role of the seller and must expend c to produce the public good), define $\tilde{\theta}_2 = c - \theta_2$, and assume that t_2 is paid by agent 2 to herself, so it is immaterial. Because agent 2 must produce the public good at personal cost c but receives agent 1's transfer, we can write the payoff to agent 2 as

$$t - \phi\tilde{\theta}_2.$$

Note that $\tilde{\theta}_2$ is distributed on the support $[c-1, c]$ and this support intersects with the buyer's support, $[0, 1]$. Efficient trade (i.e., $\theta_1 > \tilde{\theta}_2$) corresponds to to efficient production of the public good. All of the conditions of the MS impossibility theorem are satisfied. We conclude that it is not possible to implement efficient provision of the public good.

4 (a). Rewriting the necessary condition using the uniform distributions, we have

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right] \geq 0.$$

$$\frac{1}{4}\int_0^2\int_0^2 \phi(\theta_b, \theta_s)\left(\theta_b - (2 - \theta_b) - 2\theta_s\right) d\theta_s d\theta_b = \frac{1}{2}\int_0^2\int_0^2 \phi(\theta_b, \theta_s)\left((\theta_b - \theta_s) - 1\right) d\theta_s d\theta_b.$$

Thus, conditional on trade, the expected value of $\theta_b - \theta_s$ must be at least 1.

(b).

$$E[\theta_b - \theta_s | \theta_b \geq \theta_s] = \frac{1}{4}\int_0^2\int_0^{\theta_b} (\theta_b - \theta_s) d\theta_s d\theta_s = \frac{1}{4}\int_0^2 (\theta_b\theta_s - \frac{1}{2}\theta_s^2)|_0^{\theta_b} d\theta_b = \frac{1}{4}\int_0^2 \frac{1}{2}\theta_b^2 d\theta_b = \frac{1}{3}.$$

(c). The necessary condition, using the ex post efficient mechanism, requires

$$\frac{1}{2}\int_0^2\int_0^{\theta_b} \left((\theta_b - \theta_s) - 1\right) d\theta_s d\theta_b \geq 0.$$

Computing the left-hand side, we have $-\frac{1}{3}$. Thus the condition is violated.

Notice that it is efficient that trade takes place half of the time. From (a), we know that a difference of 1 is needed (on average) for the efficient trades which occur with probability (1/2). The expected surplus in (b) shows that only (1/3) is generated on average with the efficient trades that occur with probability (1/2). Thus, in expectation, $(1/2)(1) - (1/2)(1/3) = (1/3)$ is the shortfall to implement efficient trade. But this is exactly the number computed above.

5 (a). The seller's program is

$$\max_{p_s} \left(\frac{E_{\theta_b}[\overline{p}_b(\theta_b) | \overline{p}_b(\theta_b) \geq p_s]}{2} - \theta_s\right) \text{Prob}_{\theta_b}[\overline{p}_b(\theta_b) \geq p_s].$$

(b). Given the seller follows the given strategy, the seller will choose $p_s = x$ with probability x and 2 with probability $1 - x$. The buyer would never bid $p_b \in (x, 2)$ because the probability of winning would be the same, s, but the price paid would increase. Furthermore, it is never optimal for the buyer to bid 2; the probability of trade would be 1, but the buyer's surplus would be nonpositive (zero only if $\theta_b = 2$). Hence, a buyer will either bid x, or will bid below x and certainly not trade.

Hence, the buyer may as well bid either 0 or x. Given these two possibilities, it is optimal for all buyers with types $\theta_b \geq x$ to bid x and all types $\theta_b < x$ to bid $p_b = 0$. A similar line of reasoning applies to the seller's equilibrium strategies, given $\overline{p}_b(\cdot)$. I have omitted that argument here, but for full marks you needed to give the argument on both sides.

(c). With linear strategies and uniform distributions, the optimization programs from (a) can be greatly simplified. If the seller follows the given linear strategy, then p_s is uniformly distributed on $[\alpha_s, \alpha_s + 2\beta_s]$. The buyer chooses p_b to solve pointwise

$$\max_{p_b} \left(\theta_b - \frac{1}{2}\left(p_b + \frac{\alpha_s + p_b}{2}\right)\right)\left(\frac{p_b - \alpha_s}{2\beta_s}\right).$$

Similarly, if the buyer follows the given linear strategy, then p_b is uniformly distributed on $[\alpha_b, \alpha_b{+}2\beta_b]$ and the seller chooses p_s to solve pointwise

$$\max_{p_s} \left(\frac{1}{2}\left(p_s + \frac{p_s + \alpha_b + 2\beta_b}{2}\right) - \theta_s\right)\left(\frac{\alpha_b + 2\beta_b - p_s}{2\beta_b}\right).$$

The first-order condition for the buyer's program implies

$$p_b = \frac{2}{3}\theta_b + \frac{1}{3}\alpha_s.$$

The first-order condition for the seller's program implies

$$p_s = \frac{2}{3}\theta_s + \frac{1}{3}(\alpha_b + 2\beta_b).$$

Thus, $\alpha_b = \alpha_s = \frac{2}{3}$. Additionally, we have

$$\alpha_b = \frac{1}{3}\alpha_s, \quad \text{and} \quad \alpha_s = \frac{1}{3}(\alpha_b + \frac{4}{3}).$$

Thus, we conclude the equilibrium strategies are

$$\overline{p}_b(\theta_b) = \frac{1}{6} + \frac{2}{3}\theta_b,$$

$$\overline{p}_s(\theta_s) = \frac{1}{2} + \frac{2}{3}\theta_s.$$

(d). Using the strategies in (c), trade takes place if and only if

$$\overline{p}_b(\theta_b) \geq \overline{p}_s(\theta_s) \Longleftrightarrow \theta_b - \theta_s \geq \alpha \equiv \frac{1}{2}.$$

6 (a). The program we want to solve is

$$\max_{\{\phi(\cdot)\}} E[\phi(\cdot)(\theta_b - \theta_s)]$$

subject to

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right] \geq 0,$$

$\overline{\phi}_b$ nondecreasing and $\overline{\phi}_s(\cdot)$ nonincreasing. We know from class that if the inequality above is satisfied and $\overline{\phi}_i$ is appropriately monotonic, that there is a transfer $t(\cdot)$ such that the mechanism satisfies IR constraints (i.e., $U_b(\underline{\theta}_b) \geq 0$ and $U_s(\overline{\theta}_s) \geq 0$). We proceed by ignoring the monotonicity conditions (they will not be a problem given the assumption of uniform distributions ensures regularity).

Form the Lagrangian using $\lambda \geq 0$ on the inequality constraint. After some simplification, we have

$$\mathcal{L} = E\left[\phi(\theta_b, \theta_s)\left(\left((1+\lambda)\theta_b - \lambda\frac{1-F_b(\theta_b)}{f_b(\theta_b)}\right) - \left((1+\lambda)\theta_s + \lambda\frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right].$$

Divide through by $(1+\lambda)$ to simplify the program:

$$\mathcal{L} = E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \left(\frac{\lambda}{1+\lambda}\right)\frac{1-F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \left(\frac{\lambda}{1+\lambda}\right)\frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right].$$

The optimal mechanism solves the program pointwise over (θ_b, θ_s):

$$\phi(\theta_b, \theta_s) = \begin{cases} 1 & \theta_b - \theta_s \geq \left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1-F_b(\theta_b)}{f_b(\theta_b)} + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right) \\ 0 & \text{otherwise.} \end{cases}$$

This allocation is increasing in θ_b and decreasing in θ_s, and so $\overline{\phi}_i$ satisfy the monotonicity requirements.

Applying our uniform distributions to this problem characterization, we have

$$\left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1-F_b(\theta_b)}{f_b(\theta_b)} + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right) = \left(\frac{\lambda}{1+\lambda}\right)(2 - \theta_b + \theta_s).$$

The condition for $\phi = 1$ is therefore $\theta_b - \theta_s \geq \alpha \equiv \frac{2}{1+2\lambda} \in (0, 2)$. What remains to be determined is the value of α. This number must be set so that the IC inequality is exactly satisfied:

$$\frac{1}{4}\int_\alpha^2 \int_0^{\theta_b - \alpha} ((2\theta_b - 2) - (2\theta_s))\, d\theta_s d\theta_b = 0,$$

which simplifies to

$$\int_\alpha^2 \int_0^{\theta_b - \alpha} (\theta_b - \theta_s - 1)\, d\theta_s d\theta_b = 0.$$

Thus,

$$(2\alpha - 1)(\alpha - 2)^2 = 0.$$

The unique root satisfying $\alpha \in (0, 2)$ is $\alpha = \frac{1}{2}$. Thus, the welfare-maximizing trading allocation is

$$\phi(\theta_b, \theta_s) = \begin{cases} 1 & \theta_b - \theta_s \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(b). Efficient trade is distorted by a gap of 1/2, exactly the same as the linear equilibrium outcome in the game from Exercise 5.

7 (a). A direct mechanism $\{\phi, t\}$ is IC iff $\overline{\phi}_b(\cdot)$ is nondecreasing, $\overline{\phi}_s(\cdot)$ is nonincreasing,

$$U_b(\theta_b) = U_b(0) + \int_0^{\theta_b} \overline{\phi}_b(x)dx,$$

$$U_s(\theta_s) = U_s(2) + \int_{\theta_s}^{2} \overline{\phi}_s(x)dx.$$

(b). Integrating by parts ,we obtain

$$E[U_b(\theta_b)] = U_b(0) + E\left[\overline{\phi}_b(\theta_b)\frac{1-F_b(\theta_b)}{f_b(\theta_b)}\right] = U_b(0) + E\left[\phi_b(\theta_b,\theta_s)\frac{1-F_b(\theta_b)}{f_b(\theta_b)}\right],$$

$$E[U_s(\theta_s)] = U_s(2) + E\left[\overline{\phi}_s(\theta_s)\frac{F_s(\theta_s)}{f_s(\theta_s)}\right] = U_s(2) + E\left[\phi_s(\theta_b,\theta_s)\frac{F_s(\theta_s)}{f_s(\theta_s)}\right].$$

(c). The broker maximizes

$$\begin{aligned} &E\left[\phi(\theta_b,\theta_s)\left(\theta_b-\theta_s\right) - U_b(\theta_b) - U_s(\theta_s)\right] \\ &\quad = E\left[\phi(\theta_b,\theta_s)\left(\left(\theta_b - \frac{1-F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right) - U_b(0) - U_s(2)\right]. \end{aligned}$$

(d). Ignoring monotonicity conditions for now, the pointwise optimal ϕ is

$$\phi(\theta_b,\theta_s) = \begin{cases} 1 & \text{if } \theta_b - \theta_s \geq \left(\frac{1-F_b(\theta_b)}{f_b(\theta_b)} + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that the type distributions are uniform on $[0, 2]$, the condition for $\phi = 1$ reduces to

$$\theta_b - \theta_s \geq (2-\theta_b) + \theta_s,$$

or

$$\theta_b - \theta_s \geq 1.$$

The trade gap that a profit-maximizing broker chooses is 1 while the welfare-optimal gap is 1/2 in Exercise 6.