Price Theory, III | **Professor Lars Stole**
Economics 33003 | **Spring 6019**

Assignment 8
(Due Friday, June 7, prior to the start of the Review session)

Problem 1 (Full surplus extraction with correlated signals.) A seller has a single unit to sell at zero opportunity cost, $\theta_0 = 0$, to one of two bidders. Each bidder has two possible types, $\{30, 60\}$, but the valuations are correlated according to the joint probability function, $f(\theta_1, \theta_2)$. Specifically, the probability that both are high or low is $f(30, 30) = f(30, 30) = \frac{1}{3}$, and the probability that the valuations are different is $f(30, 60) = f(60, 30) = \frac{1}{6}$.

Without loss of generality, suppose that the seller offers a direct mechanism $\{\phi_i, t_i\}_{i=1,2}$. Further, suppose that the seller wants to implement the efficient allocation of the good:

$$\phi_i(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_i > \theta_j, \\ 0 & \text{if } \theta_i < \theta_j, \\ \frac{1}{2} & \text{if } \theta_i = \theta_j. \end{cases}$$

Restrict attention to symmetric payments which are paid as a function of the bidder's reported type and are independent from whether or not the bidder gets the good if there is a tie. Thus, the seller needs to determine four numbers: $t_1(30, 30)$, $t_1(30, 60)$, $t_1(60, 30)$ and $t_1(60, 60)$.

(a). Show that the seller can implement the efficient allocation while simultaneously extracting all of the bidders' surplus.

[Hint: write down the two IC constraints for bidder i as a function of the four transfer payments using the conditional probabilities $\text{Prob}[\theta_j = 60|\theta_i = 60] = \frac{2}{3}$, etc. Set the transfers so that the high type, $\theta_1 = 60$, is indifferent to telling the truth or pretending to be $\theta_1 = 30$ and show in your solution that the other IC constraint is slack. Write down the two IR constraints for bidder i and set these equal to zero (i.e., no surplus is left). Note that you will have one degree of freedom in your linear qualities. Set $t_1(60, 60) = 30$ to find a unique solution for the remaining three transfers.]

(b). Reinterpret the answer in (a) as the following game. Each bidder is required to bid either 30 or 60. The winning bidder must pay her bid. In addition, any bidder who bids 30 is also required to take a side bet with the seller which has zero expected payoff if the bidder's true type is $\theta_i = 30$ and a large negative payoff if the bidder's type is actually $\theta_i = 60$. Describe the side bet for this reinterpretation of the mechanism in (a). Not sure if our answer is correct

Problem 2 (Bilateral trade with discrete types.) Consider a bilateral trading situation similar to Myerson and Satterthwaite (1983), except that the distributions of types are discrete. Specifically, suppose $\theta_s \in \{1, 4\}$ with equal probability, and $\theta_b \in \{0, 3\}$ with equal probability. Hence, trade is efficient for some pairs of types, but not all.

(a). Do the conditions of the Myerson-Satterthwaite impossibility theorem apply to this situation? Explain.

(b). If your answer to (a) is "No," then attempt to provide a simple counter example in which efficient trade can be implemented. If your answer to (a) is "Yes," then for which type pairs is trade necessarily inefficient?

check answers

Problem 3 (Impossibility of efficient public good provision.) Consider a public goods setting in which two citizens must collectively decide whether or not to produce a public good which costs c (known to both parties). The value of the public good to each citizen is θ_i, distributed according to $F(\cdot)$ on $[0,1]$. An ex post efficient mechanism produces the public good if $\theta_1 + \theta_2 \geq c$ and does not otherwise. Suppose that $c \in (0,2)$ which implies that it is efficient to build the public good for some type profiles, but not for all.

Suppose that the government designs a direct revelation mechanism to implement the public good with probability $\phi(\theta_1, \theta_2)$ and taxes each citizen the amount $t_i(\theta_1, \theta_2)$, where we require that $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = c$ if the public good is built and $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0$ otherwise. (This is a form of ex post budget balance.) Assume that each agent has the right to refuse to participate in the mechanism after learning type θ_i, in which case the public good is not built and no taxes are charged to either agent.

Prove that there does not exist an IC, IR mechanism which is ex post efficient.

[Hint: Convert the public-goods problem into the Myerson-Satterthwaite bilateral trading problem. Imagine agent 1 in the role of buyer, $t_1 = t$ is a transfer from agent 1 to agent 2, and assume that agent 2 must build the public good using personal funds, c, if $\phi = 1$. Check all of the MS conditions. Then apply the MS impossibility theorem.]

Problem 4 (Required subsidy for efficient bilateral trade.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0,2]$.

how does probability go away?

(a). A necessary condition for IC (as shown in class) requires

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right] \geq 0.$$

Using the fact that types are uniformly distributed on $[0,2]$, show that, in any mechanism, conditional on trade, the expected difference between θ_b and θ_s must be at least 1.

(b). Compute the expected difference between θ_b and θ_s, conditional on efficient trade occurring (i.e., conditional on $\theta_b \geq \theta_s$). Using (a), conclude directly that an ex post efficient, budget-balanced, IC mechanism cannot satisfy IR.

(c). Compute the value of

$$E\left[\phi(\theta_b, \theta_s)\left(\left(\theta_b - \frac{1 - F_b(\theta_b)}{f_b(\theta_b)}\right) - \left(\theta_s + \frac{F_s(\theta_s)}{f_s(\theta_s)}\right)\right)\right]$$

when ϕ is ex post efficient. Using your answers to (a) and (b), explain why this number is a measure of the minimum amount of external subsidy which must be introduced to obtain ex post efficient trade in an IC and IR mechanism.

Problem 5 (Bilateral trading game, Chatterjee-Samuelson (1983).) Consider a setting of bilateral trade in which the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$. Suppose the buyer and seller play the following game. Both buyer and seller simultaneously bid a price for trade. If the buyer's bid, p_b, weakly exceeds the seller's bid, p_s, then trade takes place at the average price bid, $p = \frac{p_b+p_s}{2}$. Otherwise, trade does not take place.

A pure-strategy Bayesian-Nash equilibrium is a pair of functions, $\{\overline{p}_b(\cdot), \overline{p}_s(\cdot)\}$, which is a map from a player's type to a bid.

(a). Given $\overline{p}_s(\cdot)$ is played by the seller, the Buyer's optimization program is

$$\max_{p_b} \left(\theta_b - \frac{p_b + E_{\theta_s}[\overline{p}_s(\theta_s)|p_b \geq \overline{p}_s(\theta_s)]}{2} \right) \text{Prob}_{\theta_s}[p_b \geq \overline{p}_s(\theta_s)].$$

Write down the Seller's optimization program, given $\overline{p}_b(\cdot)$.

(b). Show that the following is a discontinuous equilibrium to the game for any $x \in (0, 2)$:

$$\overline{p}_b(\theta_b) = \begin{cases} x & \text{if } \theta_b \geq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{p}_s(\theta_s) = \begin{cases} x & \text{if } \theta_s \leq x \\ 2 & \text{otherwise.} \end{cases}$$

(c). Now consider only linear equilibria: $\overline{p}_b(\theta_n) = \alpha_b + \beta_b\theta_b$ and $\overline{p}_s(\theta_s) = \alpha_s + \beta_s\theta_s$. Show that a linear equilibrium exists and compute the equilibrium bidding functions.

(d). In the equilibrium in (c), trade takes place if and only if $\theta_b - \theta_s \geq \alpha$. What is the value of the gap α?

Problem 6 (Maximizing welfare in bilateral trade.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

(a). Solve for the welfare-maximizing trading allocation direct mechanism, $\{\phi(\cdot), t(\cdot)\}$, that is incentive compatible, individually rational and ex post budget-balanced (i.e., the seller receives exactly the payment that the buyer pays – there are no additional subsidies). For this question, it is sufficient to characterize $\phi(\cdot)$; you do not need to characterize $t(\cdot)$.

(b). How does the efficiency gap in (a) compare to the one you found in Exercise 5 in the linear equilibrium to the Chatterjee-Samuelson bilateral trading game?

Problem 7 (Maximizing profit for a bilateral trading platform.) Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

Suppose that a trading platform designs a profit-maximizing mechanism which allocates the seller's good to the buyer with probability $\phi(\theta_b, \theta_s)$, the buyer pays the platform $t_b(\theta_b, \theta_s)$, the seller receives $t_s(\theta_b, \theta_s)$ and the platform keeps the difference in payments as profit:

$$t_b(\theta_b, \theta_s) - t_s(\theta_b, \theta_s).$$

(a). Write down the monotonicity and integral conditions which are necessary and sufficient for the platform's mechanism to be incentive compatible.

(b). Using your answer in (a), compute $E[U_b(\theta_b)]$ and $E[U_s(\theta_s)]$ as functions of $\phi(\theta_b, \theta_s)$, $U(0)$ and $U_s(2)$.

(c). Using your result in (b), write the platform's objective entirely in terms of $\phi(\theta_b, \theta_s)$, $U(\underline{\theta}_b)$ and $U_s(\overline{\theta}_s)$.

(d). Solve for the profit-maximizing trading rule, ϕ.

(e). How does your answer in (d) compare to the welfare-maximizing rule found in Exercise 6?