Economics Camp 2018

Contents

Part I. Show me the meaning of having a utility function

1 Lecture 2018.09.14 (Friday)

We will be focusing on individuals. Suppose Anne **has** a utility function over apples and bananas: $u(a, b) = a + \ln(b)$. What does it exactly mean for a person to "have" a utility function? What does this mean? We start by considering the following setup:

- A set of alternatives X
  - There's no way around imposing what this set of alternatives looks like. If you think color is important, add it to the set
- $P(X)$: set of all non-empty subsets of X

This leads to our first definition:

Definition 1. A choice structure (β, C) is a pair consisting of:
(1) A family of budget sets $\beta \subseteq P(X)$
(2) A choice rule $C : \beta \rightrightarrows X$ such that $\forall B \in \beta, \emptyset \neq C(B) \subset B$

Here's an example. $X = \{a, b, c\}$ and $\beta = \{\{a, b\}, \{c\}, \{a, c\}\}$. Then $C(\{a, b\}) = \{a\}$ means that when I see a choice of a and b, a was chosen. We need a correspondence structure because it is possible to have $C(\{a, c\}) = \{a, c\}$.

Definition 2. A choice structure (β, C) satisfies the Weak Axiom of Revealed Preference (WARP) if for some $B \in \beta$ with $x, y \in B$, we have $x \in C(B)$, then $\forall B' \in \beta$ with $x, y \in B'$, $y \in C(B') \Rightarrow x \in C(B')$.

In other words, if x was in the choice set when x, y were available, then if we find a case where y is in the choice set, then x must also be in the choice set under WARP.

Remark 1. It turns out that the axiom is violated in many instances. For example, consider: $\beta = \{\{a, b\}, \{a, b, c\}\}$, $C(\{a, b\}) = \{a\}, C(\{a, b, c\}) = \{b\}$. Compromise effect is one way to violate; asymmetric dominance is another. Choice overload may also be possible.

What is surprising about WARP is that it is the only thing you need before we can start talking about utility functions. We now define a preferenc relation:

Definition 3. A preference relation $\succsim$ is a binary relation on X.

Remark 2. Recall that a binary relation $\geq$ on X is a subset of $X \times X$. For example, $a \geq b \Leftrightarrow \{a, b\} \in \geq$. This can be thought of as a list of answers to a yes/no question

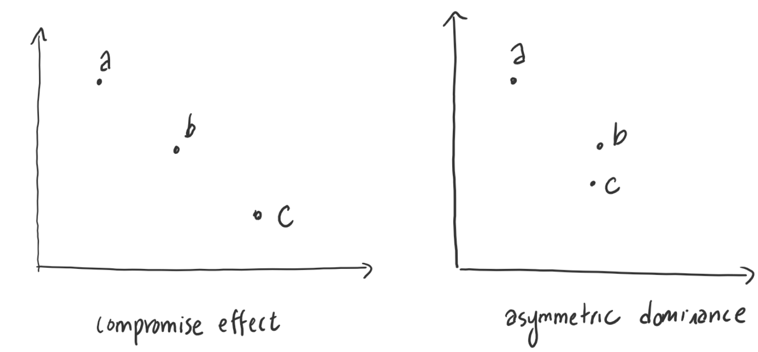

Fig. 1.1: Compromise Effect and Asymmetric Dominance

Definition 4. A preference relation is rational if it is
(1) complete; $\forall x, y \in X, x \succsim y$ or $y \succsim x$
(2) transitive; if $x \succsim y, y \succsim z$, then $x \succsim z$

Definition 5. Given $\succsim$, define:
(1) strict preference: $x \succ y$ if $x \succeq y$ and $y \not\succeq x$
(2) indifference: $x \succeq y$ and $y \succeq x$

Definition 6. A function $u : X \to \mathbb{R}$ is a utility function representing $\succsim$ if $\forall x, y \in X, x \succeq y \Leftrightarrow u(x) \geq u(y)$

Remark 3. Remark: Suppose u represents $\succsim$. Suppose $v(x) = f(u(x)$ for some strictly increasing f. Then v represents $\succsim$.

Proposition 1. *If $\succsim$ is represented by some $u : X \to \mathbb{R}$, then $\succsim$ is rational.*

Definition 7. A complete $\succsim$ generates the choice structure $(P(X), C)$ if $C(B) = \{x \in B | x \succsim y, \forall y \in B\}, \forall B \in P(X)$

Definition 8. A $\succsim$ rationalizes (β, C) if $\succsim$ is rational, generates $(P(X), C^*)$ such that $C^*(B) = C(B), \forall B \in \beta$

Proposition 2. *A choice structure generated by a rational $\succsim$ satsifies WARP.*

Proof. Suppose $(P(X), C)$ is generated by a rational $\succsim$ and suppose $x, y \in B$ and $X \in C(B)$. Then $x \succsim y$. Suppose $x, y \in B'$ and $y \in C(B')$. Then:
$y \succsim z, \forall z \in B' \Rightarrow x \succsim z, \forall z \in B' \Rightarrow x \in C(B')$ □

Conjecture 1. *If (β,C) satisfies WARP, then there is a $\succsim$ that rationalies it. (FALSE, but almost true).*

Conjecture 2. *If $\succsim$ is rational, there is a utility function that represents it. (FALSE, but almost true)*

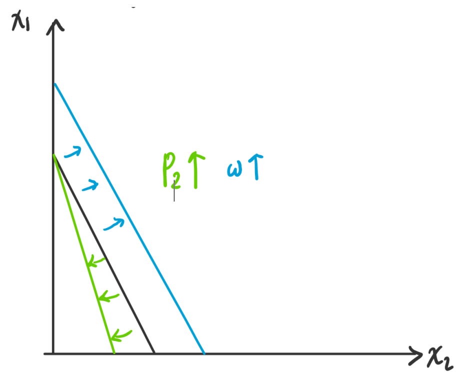

Fig. 1.2: Example of a Budget Set

Let us first consider Conjecture 1. The counter example is the following: $X = \{x,y,z\}, \beta = \{\{x,y\},\{y,z\},\{z,x\}\}$ and $C(\{x,y\}) = \{x\}, C(\{x,z\}) = \{y\}, C(\{x,z\}) = \{z\}$. Now we have a way to get around this issue:

Proposition 3. *If (β,C) satisfies WARP and $\{B \in P(X)||B| \leq 3\} \subseteq \beta$, then there is a $\succsim$ that rationalizes (β, C).*

This means that if you have a "data point" on choice for all pairs and triplets, then we're good. An implication of this result is the following:

Definition 9. A Walrusian budget set given $X = \mathbb{R}^n_+$ is $B_{p,w} = \{x \in X | p \cdot x \leq w\}$.

Notice that if the consumer choice is given in terms of price and wealth, then we won't have the pair and triplet comparisons necessary to satisfy Proposition 3. This is logical impossibility because using prices and wealth will generate much larger number of choices by construction.

So what should we do about Walrasian budget sets? Consider the graph of a budget set in Figure 1.2.

Definition 10. A Walrasian demand correspondence $X(p,w) : \mathbb{R}^n_+ \times \mathbb{R} \rightrightarrows X$ assigns a set of consumption bundles to each price-wealth pair.

Note that $X(p,w)$ is a homogeneous of degree zero. The intution here is that the choice set, which is expressed in terms of the units of goods, remains the same when you scale the price of all goods and the wealth by the same amount. Ideally, we would like to see this in the data as well. But this homogeneity of degree zero is often violated and is referred to as "money illusion." This has been proposed as one reason why nominal prices are slow to change even where inflation has caused real prices/costs to rise.

Definition 11. $X(p,w)$ satisfies Walras' Law if for every $p > 0$ and $w > 0$, we have $p \cdot x = w, \forall x \in X(p,w)$

This is rather an assumption - you spend all your money. Weird name.

2 Lecture 2018.09.17 (Monday)

We're going to switch notations such that $X(p,w)$ is essentially the same as $C(B)$. Then notice the following properties:

- Homogeneous of degree zero: $X(p,w) = X(p',w')$ if $B_{p,w} = B_{p',w'}$ saying that there is no framing, i.e. the denominations of price quotations do not matter.
- Walras' Law: $X(p,w) \cdot p = w$ saying that the total amount of money you spend is the amount of money you have. In other words, you will never choose an interior point of a given budget set.

There's no reason to expect that these two conditions will give us WARP, which is something that $C(B)$ can (or cannot) satisfy. So the ultimate question is what WARP gives us for $X(p,w)$. To get there, we start with some definitions:

Definition 12. Suppose $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is single-valued and smooth. Let η_i be the income elasticity for good i:

$$\eta_i \equiv \frac{\partial X_i(p,w)/X_i(p,w)}{\partial w/w} = \frac{\partial X_i(p,w)}{\partial w} \cdot \frac{w}{X_i(p,w)}$$

and let ϵ_{ij} be the cross-price elasticity for good i and price of j:

$$\epsilon_{ij} \equiv \frac{\partial X_i(p,w)/X_i(p,w)}{\partial p_j/p_j}$$

and let $b_i \equiv p_i X_i(p,w)/w$, the share of income for good i.

Proposition 4. *Suppose $X(p,w)$ satisfies homogeneity of degree zero. Then if $X_i(p,w) > 0$,*

$$(\sum_{j=1}^{n} \epsilon_{ij}) + \eta_i = 0$$

Proof. We know that $X_i(\alpha p, \alpha w) - X_i(p,w) = 0$ for some α. Differentiating each side with respect to α:

$$\frac{\partial X_i(\alpha p, \alpha w)}{\partial \alpha} = \sum_{j=1}^{n} \frac{\partial X_i(\alpha p, \alpha w)}{\partial p_j} p_j + \frac{\partial X_i(\alpha p, \alpha w)}{\partial w} w = 0$$

Letting $\alpha = 1$ and dividing each side by $X_i(p,w)$, the proof is complete. □

Proposition 5. *If $X(p,w)$ satisfies Walras' Law:*
(1) $\sum_i b_i \eta_i = 1$
(2) $b_i + \sum_j b_j \epsilon_{ij} = 0$

Proof. Since $X(p,w) \cdot p = w$, differentiate each side with respect to w, we immediately get (1):

$$\sum_i \left(\frac{\partial X_i(p,w)}{\partial w} \right) p_i = \sum_i b_i \eta_i = 1$$

and differentiating each side with respect to p_i, we immediately get (2):

$$\frac{\partial X_i(p,w)}{\partial p_i} + \sum_j \left(\frac{\partial X_j(p,w)}{\partial p_i} p_j \right) = 0$$

□

Theorem 1. *$X(p,w)$ satisfies WARP if and only if the following holds for any two price-wealth pairs (p,w), (p',w') : if $p \cdot X(p',w') \leq w$ and $X(p,w) \neq X(p',w')$, then $p' \cdot X(p,w) > w'$.*

To understand this, suppose C is single-valued. Then (β,C) satisfies WARP if and only if: if $x, y \in B, C(B) = x$, then $\forall B'$ such that $x, y \in B', C(B') \neq y$. Above theorem is essentially the same as this result. If your new choice is in the old budget, and hence both old choice and the new choice are affordable in the old budget, you chose the old choice (meaning the old choice was better). Now if you consider the new budget, it must be the case that the old choice is not affordable in the new budget. Now to our main result:

Theorem 2. *(Law of Compensated Demand) $X(p,w)$ satisfies WARP if and only if the following holds: Given (p,w) and p', let $w' = p' \cdot X(p,w)$. Then*

$$(p' - p) \cdot \left[X(p',w') - X(p,w) \right] \leq 0$$

with strict inequality if $X(p,w) \neq X(p',w')$.

This is essentially the law of demand: you will not be able to derive $\partial X_i(p,w)/\partial p_i$ from rationality, since there's a wealth effect. This is a shocking result - you shouldn't able to expect WARP to give us a compensated downward-sloping demand curve. The key component here is $w' = p' \cdot X(p,w)$ - this is essentially eliminating the wealth effect. Note that we can also eliminate the wealth effect by imposing some additional properties on the preferences.

Definition 13. (Slutsky Matrix) Given X that is differentiable in prices and wealth, define the Slutsky matrix:

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1n}(p,w) \\ \vdots & & \vdots \\ s_{n1}(p,w) & \cdots & s_{nn}(p,w) \end{bmatrix}$$

where

$$s_{ij}(p,w) = \frac{\partial X_i(p,w)}{\partial p_j} + \frac{\partial X_i(p,w)}{\partial w} X_j(p,w)$$

Interpretation: suppose price goes up a little bit, and we also give you some wealth to compensate for the fact that you have a little less now, such that you can afford the old budget. The Slutsky matrix is an encoding of all compensated price elasticities.

Theorem 3. *Suppose X is differentiable. $X(p,w)$ satisfies WARP if and only if the followin holds: at any (p,w), S is negative semi-definite. In particular, $s_{ii} \leq 0, \forall i$.*

Now we return to using utility functions to achieve the same results. Start with the following conjecture:

Conjecture 3. *If $\succsim$ is rational, there is a $u : X \to \mathbb{R}$ that represents it. (FALSE)*

A counterexample: $X = \mathbb{R}^2_+$. Consider a preference relation $x \succsim_L y$ if $x_1 > y_1$ or $x_1 = y_{1,} x_2 > y_2$, commonly known as a lexicographical preference. You can never get enough of x_2 to make up for a deficiency in x_1. Note that $\succsim$ is complete and transitive but $\nexists u : X \to \mathbb{R}$ that represents $\succsim_L$. To see this, assume the contrary and suppose u represents $\succsim_L$. Given any $x_1 \in \mathbb{R}$, $\exists r(x_1) \in Q_1$ such that $u(x_1, 2) > r(x_1) > u(x_{1,}1)$. Moreover, if x_1 is bigger than x_1', we have $r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1)'$. This implies that $r : \mathbb{R} \to \mathbb{Q}$ such that $x \neq x'$ implies $r(x) \neq r(x')$, hence a contradiction.Note that if we restrict X to be finite, then the conjecture would be TRUE.

Another intuition here: with lexicographical preference, you can't be indifferent between two choices.

Definition 14. Given X and a $\succsim$ on X , let $\succsim (x) = \{y \in X | y \succsim x\}$ and $\precsim (x) = \{y \in X | y \precsim x\}$. A $\succsim$ on X is continuous if $\forall x \in X, \succsim (x)$ and $\precsim (x)$ are closed.

Definition 15. Suppose $X \subset \mathbb{R}^n$. $\succsim$ is monotone if $x \geq y$ (meaning $x_i \geq y_i, \forall i$) implies $x \succsim y$.

Theorem 4. *If $\succsim$ on X is rational and continuous, there is a (continuous) $u : X \to \mathbb{R}$ that represents it.*

Proof. For the proof, we will assume monotonocity. Let $e = (1, ..., 1)^T$. The overall steps are as follows: □

1. $\forall x \in X, \exists \lambda \in \mathbb{R}^+$ such that $x \sim \lambda e$
   - Given x_i, let $\Lambda_B \equiv \{\lambda \geq 0 | \lambda e \succsim x\}$ and $\Lambda_W \equiv \{\lambda \geq 0 | x \succsim \lambda e\}$. By completeness of $\succsim, \forall \lambda \in \mathbb{R}^+$, either $\lambda e \succsim x$ or $x \succsim \lambda e$. Therefore, $\Lambda_B \cup \Lambda_W = \mathbb{R}_+$.
   - Notice that this step is impossible for lexicographical preferences; hence, we need continuity.
   - By continuity and monotonocity, each is a closed interval.
   - Hence, $\Lambda_B = [\underline{\lambda}, \infty)$and $\Lambda_W = [0, \bar{\lambda}]$. Since the union of the two sets is $\mathbb{R}_+$, we have $\underline{\lambda} \leq \bar{\lambda}$. So for any $\lambda \in [\underline{\lambda}, \bar{\lambda}]$,we have $\lambda e \sim x$.

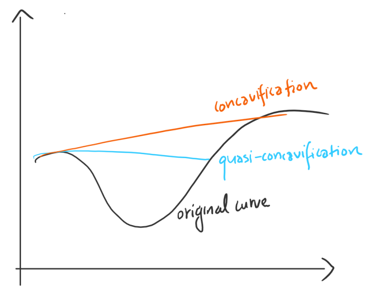

Fig. 3.1: Quasi-concavification and Concavification

2. Define $\lambda : X \to \mathbb{R}$ by $x \sim \lambda(x)e$ (essentially showing that λ is unique for a given x)
   - By transitivity, $\underline{\lambda}$e~x~$\bar{\lambda}e$,we have $\underline{\lambda}$=$\bar{\lambda}$. Hence there is a unique λ which allows us to define $\lambda : X \to \mathbb{R}$
3. Show $\lambda(\cdot)$ represents $\succsim$.
   - $x \succsim y \Leftrightarrow \lambda(x)e \sim x \succsim y \sim \lambda(y)e \Leftrightarrow \lambda(x)e \succsim \lambda(y)e \Leftrightarrow \lambda(x) \succsim \lambda(y)$
4. Show $\lambda(\cdot)$ is continuous.

3 Lecture 2018.09.18 (Tuesday)

Suppose we have a rational, continuous $\succsim$. We know $\exists u : X \to \mathbb{R}$ such that $u(x) \geq u(y) \Leftrightarrow x \succsim y$. Is u necessarily continuous? Not quite - consider the following example. Let $X = \mathbb{R}$ and define a preference relation to be $x \succsim y \Leftrightarrow u(x) = x \geq u(y) = y$. But consider another utility function: $\tilde{\tilde{u}}(x) = x$ if $x \leq 5$ and $x + 7$ if $x > 5$. This is a valid utility function but not continuous.

Another question is whether u is necessarily quasi-concave.

Definition 16. $f : X \to \mathbb{R}$ is quasi-concave if $\forall x, y \in X$ and $\forall \alpha \in (0, 1)$, $f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}$.

Definition 17. $f : X \to \mathbb{R}$ is concave if $\forall x, y \in X$ and $\forall \alpha \in (0, 1)$, $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$

No - because you can always apply a monotone transformation and kill off the quasi-concavity. (**Is this actually true?**) Then we ask: can we always find a quasi-concave utility function? The answer is also no. To arrive here, we need an additional assumption that the preference relation is convex:

Definition 18. $\succsim$ is convex if $\forall x, \{y \in X | y \succsim x\}$ is convex.

Why do we care so much about concave utility functions? Because local optimum is the global optimum. It turns out that it's difficult to get strict concavity, but we can easily get to quasi-concavity through some additional assumptions on the utility function and preferences. The existence of the optimum is simply given by the fact that the utility function is continuous and defined on a compact set.

Definition 19. Walrasian demand is defined as $X(p, w) \equiv \arg\max u(x)$ such that $p \cdot x \leq w$.

Definition 20. $\succsim$ satisfies local non-satiation if $\forall x$ and $\forall \epsilon > 0$, there exists a $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succsim x$.

Note that local non-satiation is implied by monotonicity of preferences, but the alternate direction is not true.

Proposition 6. *$X(p, w)$ is homogeneous of degree zero.*

1. If u is monotone, $X(p, w)$ satisfies Walras' Law.
2. If u is quasi-concave, $X(p, w)$ is convex-valued.
3. If u is strictly quasi-concave, $X(p, w)$ is single-valued.

Definition 21. Indirect utility function $V(p, w) \equiv \max u(x)$ such that $p \cdot x \leq w$. Consequently, $V(p, w) = u(X(p, w))$.

This will be like a utility function expressing preferences over budget sets, parameterized by prices and wealth. For instance, suppose $B = \{x, y, z\}, B' = \{y, z, w\}, B'' = \{x, w\}$ and you are given $C(B) = \{x\}, C(B') = \{y\}, C(B'') = \{x\}$. Suppose you ask a person to choose between B and B', and the person will choose B over B'.

Proposition 7. *Note:*

1. V is continuous
2. V is homogeneous of degree zero.
3. V is strictly increasing in w (which wouldn't be true if Walras' Law was not satisfied) and weakly decreasing in p (weakly becaues I may not buy some goods).
4. V is quasi-convex in p.

Note property #4: suppose with probability $1/2$, you face price vector p and with probability $1/2$, you face price vector p'. Alternatively, you face $0.5p + 0.5p'$ with certainty. If V is convex in p, you would prefer the lottery (which is clearly not true). Because V is quasi-convex, it says that a mixture of prices can't be worse than its average.

Recall that we needed a convex preference relation to obtain the existence of a quasi-concave utility function. Keeping that in mind, consider the following theorem:

Theorem 5. *Suppose $\succsim$ is represented by u with $V(p,x) = \max u(x)$ such that $p \cdot x \leq w$ and $X(p,w) = \arg\max_x u(x)$ subject to the same constraint. Then there exists a continuous, weakly increasing, quasi-concave $\tilde{u}$ such that*

$$V(p,w) = \max_x \tilde{u}(x) \text{ subject to } p \cdot x \leq w$$

$$\tilde{u}(X(p,w)) = u(X(p,w))$$

Moreover, $\tilde{u}(x) = \min_p V(p,w)$ such that $p \cdot x \leq w$ and if u is quasi-concave, then $u(x) = \tilde{u}(x), \forall x$.

This says that you can find an equivalent utility function that only disagree on goods that you would never buy.

Now we define a few concepts critical for arriving at our goal of Slutsky's equation.

1. $h(p,u_0) = \arg\min p \cdot x$ such that $u(x) \geq u_0$ (Hicksian Demand)
2. $e(p_0,u_0) = \min p \cdot x$ such that $u(x) \geq u_0$ (Expenditure function)

It turns out that Hicksian Demand will serve as a concise analog of the whole business of compensating the person when prices change. Seeing how $X(p,w)$ changes with respect to p with a subsdiy in w is exactly the same as seeing how $h(p,u_0)$ changes with respect to p. There's no wealth in Hicksian Demand. Also:

Proposition 8. *Results on the expenditure function:*

1. e is continuous.
2. e is homongeoues of degree 1 in p.
3. e is strictly increasing in u and weakly increasing in p.
4. e is concave in p.

A brief note on quasi-concavity: if u is monotone, then it is quasi-concave. It is also quasi-convex. But monotonocity is way too strong - a weaker condition suffices. For whatever you are consuming, there is some bundle out there that makes you better off and requires buying more things.

Remark 4. Note:

1. $e(p,u) = p \cdot h(p,u)$, $V(p,w) = u(X(p,w))$
2. $e(p,V(p,w)) = w$, $V(p,e(p,u)) = u$
3. $X(p,w) = h(p,V(p,w))$, $h(p,u) = X(p,e(p,u))$
   - Notice that when you change p in $h(p,u)$, you are also changing p in $e(p,u)$ such that the original utility is preserved.

Fig. 3.2: Example of Quasi-concave Function

Theorem 6. *(Properties of Hicksian Demand)*

1. h is weakly decreasing in p. (Demand curve slopes downward)
2. Let

$$S(p,u) = \begin{bmatrix} s_{11}(p,u) & \cdots & s_{1n}(p,u) \\ \vdots & & \vdots \\ s_{n1}(p,u) & \cdots & s_{nn}(p,u) \end{bmatrix}, s_{ij}(p,u) = \frac{\partial h_i(p,u)}{\partial p_j}$$

then S is negative semi-definite and **symmetric**: $s_{ij} = s_{ji}$.

Previously, we saw that WARP is equivalent to having a negative semi-definite Slutsky matrix and the compensated law of demand ($\Delta X \cdot \Delta p \leq 0$). We took X as given. But now we took a new approach – we started with a utility function with completeness, transitivity, and continuity of preferences. This implies compensated law of demand (h decreasing in p) and negative semi-definite and symmetric S.

Note that symmetry is a very strong property. This holds for any two goods. This is the most substantive empirical concent of saying people have utility functions. What is so special about the properties of utility functions that give rise to symmetry of the Slutsky matrix? Xavier Gabaix has a paper on sparse maximization that breaks the symmetry of the Slutsky matrix by driving small terms to zero.

We also have a result in the other direction. If you are given an S, you can get back the u. There are many papers estimating demand and make comments on welfare implications. For instance, suppose you are interested in estimating a from the following demand equation:

$$X_1(p_1, p_{2,} w) = p_2 + a \cdot \frac{w}{p_1}$$

and ponder the implications of putting tax on p_1 and looking at its implications on the utility functions. This inherently assumes that there is some utility function that led to the demand that looked like it. Is there a u that leads to this demand function? All you need to check is whether the function has a negative semidefinite Slutsky matrix and that is all you have to check.

Back to Hicksian: why did we need h? Because the proof of symmetry is very instructive and the proof is trivial once you have expressed S in terms of h and not in terms of X.

Lemma 1. *$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$ (The Hicksian demand is simply how the expenditure function varies with p_i).*

Proof. Since $e(p,u) = p \cdot h(p,u)$, by the Envelope Theorem:

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u)$$

□

Proof. (Theorem 6) From the previous result, the symmetry is given to us:

$$\frac{\partial h_i(p,w)}{p_j} = \frac{\partial}{\partial p_j}\left[\frac{\partial e(p,u)}{\partial p_i}\right] = \frac{\partial^2 e(p,u)}{\partial p_j p_j} = \frac{\partial}{\partial p_i}\left[\frac{\partial e(p,u)}{\partial p_j}\right] = \frac{\partial h_j(p,u)}{\partial p_i}$$

□

A brief review of **Envelope Theorems**: suppose you have an optimization problem of the form:

$$W(\theta) = \max_x f(x,\theta)$$

which of course implies $x^*(\theta) = \arg\max_x f(x,\theta)$. Then write $W(\theta) = f(x^*(\theta),\theta)$. The Envelope Theorem says that

$$\frac{\partial W(\theta)}{\partial \theta} = f_2(x^*(\theta),\theta)$$

and don't bother moving around $x^*(\theta)$. Or more simply:

$$W(\theta+\epsilon) - W(\theta) \approx f(x^*(\theta),\theta+\epsilon) - f(x^*(\theta),\theta)$$

since f has zero slope at the optimum. So arbitrarily small changes at the optimum do not impact f.

Part II. Preference for Preference

4 Lecture 2018.09.18 (Tuesday)

One interesting question is the following: how do indifference curves differ for different levels of wealth? This is essentially the wealth effect. We've looked away by compensating people with wealth. Now we look at two special classes that will allow you to erase away the effect of wealth. The first class is called homothetic preferences; the second class is called quasi-linear preferences.

Definition 22. Preferences are homothetic if they can be represented by a utility function that is homogeneous of degree 1. Essentially, this means that the trade-off between two goods does not differ depending on your income. (Think of rice and yachts as a function of your wealth)

Proposition 9. *If preferences are homothetic, $\exists \tilde{X}$ such that $X(p,w) = \tilde{X}(p) \cdot w$. This means that wealth only affects how much you buy of each but not what you buy.*

Remark 5. Cobb-Douglas Preferences: $u(x) = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ with $\sum_{\beta_i} = 1$. This is homogeneous of degree 1 since $u(\lambda x) = \lambda u(x)$. Consider $\tilde{u}(x) = \ln u(x) = \sum \beta_i \ln x_i$ – this is a very popular in the literature.

Definition 23. Preferences are quasi-linear if they can be represented by a utility function of a form

$$u(x) = x_0 + \tilde{u}(x_{1,...,} x_n)$$

Remark 6. Essentially, there is one good where the utility is linear. WLOG, set $p_0 = 1$. You see this all the time as a workhorse model in IO:

$$u_i(x) = \theta_i q_x - p_x$$

Note that the price is in the utility function. What they mean here is that in the background, you have quasi-linear preference, where x_0 is interpreted as everything else in the world.

Part III. Dive into Uncertainty

5 Lecture 2018.09.20 (Thursday)

Key Idea: Looking at $\succsim$ over $\Delta(X)$ tells us a lot more about the utility functions. We start off with a expected utility theory. Let us go back to basics – consider $\geq$ on X . Suppose $X = X_1 \times X_2 \times \cdots \times X_n$. Suppose separability:

$$u(\boldsymbol{x}) = \sum_{i=1}^{n} u(x_i)$$

for which it does not satisfy the following:

$$\left\{ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\} \succ \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right\} \prec \left\{ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\}$$

Note that WARP is about the availability of the third good not affecting your choices; here, both of them are available. So when can we assume separability?

Suppose $X = (\boldsymbol{x}_0, ..., \boldsymbol{x}_T)$ where $\boldsymbol{x}_t$ represents what your consume in time t. Then

$$u(\boldsymbol{x}) = \sum_{t=0}^{T} \delta^t u(\boldsymbol{x}_t)$$

which does imply some independence across time. This implies that you can assume two periods ignoring the history or further future. This is not a trivial assumption - for example, whether or not you'd like to delay food consumption an hour later, that totally depends on whether you've already eaten or not.

This is everywhere. One example is housing price - if a house was sold in 2013, the same house would not be sold in 2014 unless something happened to it. You also risk getting inaccurate measures of housing prices if you take a cross-sectional average and compare across different years. So people instead formulate an equation

$$P(H) = \alpha_0 + \alpha_1(\text{ \# of bedrooms }) + \alpha_2(\text{ \# of bathrooms }) + \cdots$$

which is crazy! The # of bedrooms you want is not independent of # of bathrooms. But we have no choice here. This is assuming separability across things, not time, but they are essentially equivalent.

Now we introduce some changes in notation. Previously, we talked about $\succsim$ on X; now it's $\succsim$ on $\mathcal{L}$ where $\mathcal{L}$ is a set of lotteries on X. Mathematically:

$$\mathcal{L} = \Delta(X)$$

where $\Delta(X)$ denotes the set of probability distributions over X. For example, if $X = \{a, b\}$ then $\mathcal{L} = \Delta(X) = [0, 1]$. If $X = \{a, b, c\}$ then $\mathcal{L} = \{(p_a, p_b, p_c | p_a, p_b, p_c \in [0, 1], p_a + p_b + p_c = 1\}$. If this is the case, what assumptions on $\succsim$ do we need to obtain a $u : \mathcal{L} \to \mathbb{R}$ that represents $\succsim$? The same assumptions that we had for X! The utility function was defined for all sets in X.

Definition 24. $U : \mathcal{L} \to \mathbb{R}$ has the expected utility form if

$$U(\mathcal{L}) = \sum_{x \in X} p_x^L u(x)$$

for some u, where p_x^L is the probability of getting $x \in X$ under lottery L. U is the vN-M utility function and u is Bernoulli utility.

So the important task is figuring out the conditions under which the utility function has an expected utility form. Note that once these conditions are established, it also allows us to express utility as a discounted sum of future utilities, since the two problems are essentially equivalent.

Remark 7. Given $L, L' \in \mathcal{L}$ for any $\alpha \in [0, 1]$, $L'' = \alpha L + (1 - \alpha) L' \in \mathcal{L}$ with $p_x^{L''} = \alpha p_x^L + (1 - \alpha) p_x^{L'}$.

Definition 25. $\succsim$ on $\mathcal{L}$ satisfies the independence axiom if $\forall L, L', L'' \in \mathcal{L}, \forall \alpha = (0, 1)$,

$$L \succsim L' \Leftrightarrow \alpha L + (1 - \alpha) L'' \succsim \alpha L' + (1 - \alpha) L''$$

In essence, unrealized outcomes cannot impact your trade-off of realized outcomes. Here's an example that does not satisfy the indepencen axiom. Suppose L is watching documentaries about Hawaii; L' is watching Bojack Horseman; L'' is a free trip to Hawaii. The counterfactual world really matters.

Here's another example which violates the independence axiom. Suppose you have twins – Bob and Anne – and have $L =$ (1 lollipop for Anne, 0 lollipop for Bob) and L' = (0 to Anne, 1 to Bob). There's also $L^* = 0.5L + 0.5L'$. Suppose you had $L^* \succ L$ and $L^* \succ L'$. This violates the independence axiom. Why? By the independence axiom, $L \succ L' \Rightarrow L > L^*$, $L \prec L' \Rightarrow L' > L^*$, and $L \sim L' \Rightarrow L^* \sim L \sim L'$.

Proposition 10. *U has an expected utility form if and only if it is linear, i.e.,*

$$U(\alpha L + (1-\alpha)L') = \alpha U(L) + (1-\alpha)U(L')$$

Proof. Skipped. $\square$

Proposition 11. *Suppose $U : \mathcal{L} \to \mathbb{R}$ is a vN-M utility function representing $\succsim$. Then $\tilde{U} : \mathcal{L} \to \mathbb{R}$ is also a vN-M utility function representing $\succsim$ if and only if*

$$\tilde{U} = \beta U + \gamma$$

for some $\beta > 0$ and $\gamma \in \mathbb{R}$.

Remark 8. Suppose $U(L) = \sum_x p_x^L \ln x$ represents $\succsim$. Does $\tilde{U}(L) = \ln(U(L)) = \ln(\sum_x p_x^L \ln x)$ represent the preference relation? Yes! But $\nexists \tilde{u} : X \to \mathbb{R}$ such that $\ln(\sum_x p_x^L \ln x) = \sum_x p_x^L \tilde{u}(x)$? No, and this follows from the proposition. The proposition says that if $f\left(\sum_x p_x^L u(x)\right) = \sum p_x^L \tilde{u}(x)$, then $\tilde{u} = \beta u + \gamma$ with $f =$ identity.

Remark 9. We started from preference over lotteries and arrived at a utility function for goods. Then we say that the utility function over lotteries is induced by the utility function over goods.

Theorem 7. *Suppose rational, continuous $\succsim$ on $\mathcal{L}$ satisfies the independence axiom. Then $\exists$ a vN-M $U : \mathcal{L} \to \mathbb{R}$ that represents $\succsim$, i.e. $\exists u : X \to \mathbb{R}$ such that*

$$L \succsim L' \Leftrightarrow \sum_x p_x^L u(x) \geq \sum_i p_x^{L'} u(x)$$

Proof. Note that we are going to have a finite X in the background. Note $\exists \overline{L}, \underline{L}$ such that $\overline{L} \succsim L \succsim \underline{L}, \forall L \in \mathcal{L}$. $\square$

- Step 1: If $L > L'$ and $\alpha \in (0,1)$, $L \succ \alpha L + (1-\alpha)L' > L'$. To see this:
  - $L = \alpha L + (1-\alpha)L \succ \alpha L + (1-\alpha)L' \succ \alpha L' + (1-\alpha)L' = L'$
- Step 2: Let $\alpha, \beta = [0,1]$. Then $\beta\overline{L} + (1-\beta)\underline{L} \succsim \alpha\overline{L} + (1-\alpha)\underline{L} \Leftrightarrow \beta \geq \alpha$
- Step 3: $\forall L \in \mathcal{L}$, $\exists \lambda \in (0,1)$ such that $L \sim \lambda\overline{L} + (1-\lambda)\underline{L}$
  - This is analogous to what we did for $\succsim$ rational and continuous implies the existence of a utility function.
  - This implies that $\exists \lambda : \mathcal{L} \to \mathbb{R}$ such that $\lambda(L)\overline{L} + (1-\lambda(L))\underline{L} = L$

- Step 4: $U(L) = \lambda(L)$ represents $\succsim$.
  - $L \succsim L' \Leftrightarrow \lambda(L)\overline{L} + \lambda(L)\underline{L} \succsim \lambda(L')\overline{L} + \lambda(L')\underline{L} \Leftrightarrow \lambda(L) \geq \lambda(L')$
- Step 5: $U(L)$ is a vN-M utility function.
  - We previously showed that if $U(\cdot)$ is linear, it has an expected utility form.
  - Showing U is linear is a simply algebra exercise.

So what are we really assuming when we assume expected utility form? We are assuming that unrealized options do not matter. One alternative to expected utility is prospect theory. Another alternative is decision theory – regret models for example.

If you're just starting from scratch, you usually start from the standard model of expected utility. If you're trying to explain a puzzle, you start with modificadtions to the standard theory. You could also start with these non-standard topics when you are interested in deriving implications of such setups.

Now it's time to talk about **risk-aversion**. And for no good reason, we are going to change notation to follow the textbook setup.

Let $X \subseteq \mathbb{R}$. It turns out that for multi-dimensional settings, it's quite tricky to think about risk-aversion. Our usual interpretation would be that X is money. How is this so? Recall that $V(p, w) = \max u(x), p \cdot x \leq w$. Given p, then $V(\omega) \equiv V(p, \omega)$ can be considered as a utility function of money. We are also going to denote $\mathcal{L}$ as a set of random variables and $p \in \mathcal{L}$ under which $E_p[x]$ can finally be defined, since x is part of a cardinal set. Note that if $\succsim$ on $\mathcal{L}$ satisfies rationality, continuity, and the independence axiom, $\exists u : \mathbb{R} \to \mathbb{R}$ such that $p \succsim p' \Leftrightarrow E_p[u(x)] \geq E_p[u(x)]$. Note that in our old notation, $U[p] = E_p[u(x)]$.

Definition 26. An individual is risk-averse (risk-loving) if $\forall$non-degenerate p, $E_p[u(x)] < (>) u(E_p[x])$. If they are equal, the individual is risk-neutral.

Proposition 12. *(Jensen's Inequality) Given $f : \mathbb{R} \to \mathbb{R}$, f is strictly concave if and only if $E_p[f(x)] < f(E_p[x]), \forall$ non-degenerate p. f is linear if and only if $E_p[f(x)] = f(E_p[x])$. f is strictly convex if $E_p[f(x)] > f(E_p[x])$.*

Remark 10. We know have a proposition that links concavitiy/convexity/linearity of the Bernoulli utility function to an attitude towards risk.

Definition 27. The Arrow-Pratt measure of absolute risk aversion for an individual with utility u_i is defined by

$$R_i^A(x) = -\frac{u_i''(x)}{u_i'(x)}$$

Why do we divide by $u_i'(x)$? Because it allows us to be immune to the affine distribution of the utility functions. We also have a result $R_i^A(x) > R_j^A(x) \Leftrightarrow R_i^R(x) > R_j^R(x)$.

Definition 28. The relative risk aversion for an individual with utility u_i is defined by

$$R_i^R(x) = -x\frac{u_i''(x)}{u_i'(x)}$$

Note that we only talk about $R_i^R(x)$ only for $x > 0$.

Note that we have two important classes of preferences:

- CARA: $u_i : R_i^A(x) = \rho$
- CRRA: $u_i : R_i^R(x) = \gamma$

6 Lecture 2018.09.21 (Friday)

Start the day strong with another definition:

Definition 29. $f : \mathbb{R} \to \mathbb{R}$ is more concave than $g : \mathbb{R} \to \mathbb{R}$ if and only if

$$f(x) = h(g(x))$$

for some strictly increasing, concave h.

This is another way to think about the levels of concavity between two different functions.

Definition 30. The certainty equivalent of a lottery P, denoted as $CE_i(p)$, for an individual with utility u_i is defined by

$$u_i(CE_i(p)) = E_p[u(x)]$$

The risk premium $\rho_i(p)$ is defined by

$$\rho_i(p) = E_p[x] - CE_i(p)$$

Theorem 8. *The following are equivalent:*

1. $R_i^A(x) < R_j^A(x), \forall x$
2. $R_i^R(x) < R_j^R(x), \forall x$
3. u_j is more concave than u_i. In other words, $\exists h$ strictly increasing and concave such that $u_j(x) = h(u_i(x))$.
4. $\rho_j(p) \geq \rho_i(p), \forall p$

Note that insurance is not designed to protect you against simply bad states; it is designed to protect you against states when the marginal utility of money is higher. For example, it is definitely a bad state if your child dies but we don't have an insurance against this event. Why? Because the marginal utility of money is not higher once your child dies. Similarly, you can have insurance "against" good states – for example, marginal utility of money will be higher if you plan on living a long life. That is why annuities are considered a form of insurance.

Another note on risk aversion: suppose Anne loves activities that have a high probability of dying whereas Bob is scared of such activities. In normal English parlance, we would say that Bob is "risk-averse." But this has nothing to do with risk-aversion in an economic sense. Bob is simply expressing a preference between two types of activities. Would people like Anne have more stocks than bonds on average than people like Bob?

Now we will establish a framework that allows to talk about payoffs. We know that a random variable p is identified with some cumulative distribution function (CDF):

$$F_p[x] \equiv Pr(p \leq x)$$

Note that the following are equivalent:

$$\begin{aligned} U(L) &= \sum_x p_x^L u(x) \\ U(P) &= E_p[u(x)] \\ U(F) &= \int u(x) dF(x) \end{aligned}$$

So here's a question. I want to give a lottery F or G for Anne's birthday present. All I know about her utility function u_i is that it's increasing. There are cases in which we would know which one Anne would prefer without much information about her utility function. This motivates the following definition

Definition 31. F first-order stochastically dominates G if

$$\int u(x) dF(x) \geq \int u(x) dG(x)$$

for every increasing $u : \mathbb{R} \to \mathbb{R}$.

Here's another question - suppose F is $N(100, 2)$ and G is $N(10, 1)$. Does $F \succ_{FSD} G$? We need some way to check the first-order stochastic dominance in a convenient way. And there's a surprisingly easy way to check.

Theorem 9. *$F \succ_{FSD} G$ if and only if $F(x) \leq G(x), \forall x$.*

This is why we are working in the space of CDFs – they are much easier to work with!

Proof. We'll do part of the proof: $(\Rightarrow)$. Suppose $\exists x_0$ such that $F(x_0) > G(x_0)$. Consider

$$u(x) = \begin{cases} 0 & (x \leq x_0) \\ 1 & (x > x_0) \end{cases}$$

Then:

$$\int u(x) dF(x) = Pr_F(x > x_0) = 1 - F(x_0) < 1 - G(x_0) = Pr_G(x > x_0) = \int u(x) dG(x)$$

which is a contradiction. The proof for ($\Leftarrow$) is in the textbook and is quite mechanical through integration by parts. $\square$

Definition 32. F second-order stochastically dominates G if

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

for every increasing concave $u : \mathbb{R} \to \mathbb{R}$.

Theorem 10. *Suppose F and G have the same mean. Then $F \succ_{SSD} G$ if and only if*

$$\int_{-\infty}^{t} F(x)dx \leq \int_{-\infty}^{t} G(x)dx, \forall t$$

Now we can take more about information and uncertainty. A very important idea in economics is the notation of a **state space**, often denoted as $\omega \in \Omega$. There's also the **action space**, denoted as $a \in A$, which represents the actions that an agent takes. This allows you to express your utility as $u(a, \omega)$. Notice that we have transitioned from the utility from lotteries, but they are essentially the same thing.

To see this, let $\omega \in \Omega =$ {rain, shine} and $a \in A =$ {umbrella, no umbrella}. Then X can be defined as

$$X = \left\{ \begin{array}{c} \text{wet, free hands} \\ \text{dry, free hands} \\ \text{dry, umbrella} \end{array} \right\}$$

and the payoff as the following

$$L = \begin{pmatrix} 0.2 \\ 0.8 \\ 0 \end{pmatrix}, L' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where L represents the lottery when you don't bring an umbrella and L' the one when you bring an umbrella.

Now let us formalize this notion. First we consider the concept of exogenous beliefs. Given some belief $\mu \in \Delta(\Omega)$, the agent solves

$$\max_{a \in A} \mathbb{E}_{\mu}[u(a, \omega)]$$

Note that this is no different from

$$\max_{L \in \mathcal{L}} U(L) = \max_{L \in \mathcal{L}} \sum_{x} p_x^L u(x)$$

or

$$\max_{F \in \mathcal{F}} \int u(x)dF(x)$$

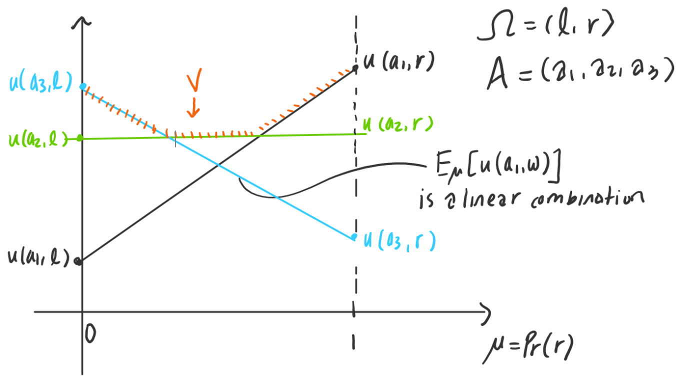

Fig. 6.1: Convexity of $V(\mu)$

Theorem 11. *Let $V(\mu) = \max_{a \in A} \mathbb{E}_\mu[u(a, w)]$. Then V is convex.*
This is a deep result that contains a substantial amount of economics.

Consider an example: $\Omega = \{l, r\}$ and $A = \{a_1, a_2, a_3\}$. Let's see this graphically in Figure 6:

Definition 33. Let X be a random variable associated with a CDF F and Y be a random variable associated with a CDF G . Then F is a mean-preserving spread of G if we have that

$$X = Y + Z_Y$$

with $\mathbb{E}[Z_Y] = 0$.

Theorem 12. *Suppose F and G have the same mean. Then the following are equivalent:*

1. $F \succ_{SSD} G \Leftrightarrow \mathbb{E}_F[u(x)] \geq \mathbb{E}_G[u(x)], \forall$ concave u
2. G is a mean-preserving spread of F
3. Moreover, if $X \subseteq \mathbb{R}$, 1. and 2. are equivalent to

$$\int_0^t F(x)dx \leq \int_0^t G(x)dx$$

Theorem 13. *Fix Ω. For every convex $V : \Delta(\Omega) \to \mathbb{R}$, $\exists A : \mu : A \times \Omega \to \mathbb{R}$ such that*

$$V(\mu) = \max_{a \in A} \mathbb{E}_\mu[u(a, \omega)]$$

Let's try to understand this. Let $\tau_1 \in \Delta(\Delta(\Omega))$ with $\mathbb{E}_{\tau_1}[\mu] = \mu_1$ and $\tau_2 \in \Delta(\Delta(\Omega))$ with $\mathbb{E}_{\tau_2}[\mu] = \mu_2$. Suppose $\mu_1 = \mu_2$. Then we have the following result:

Corollary 1. *Every agent (for any $A, \mu : A \times \Omega \to \mathbb{R}$) is better off under τ_1 than under τ_2 if and only if τ_1 is a mean-preserving spread of τ_2*

Note this is because the value function over your beliefs is convex. So you want to have your beliefs as dispersed as possible.

Definition 34. A signal (a.k.a. signal structure, information structure, experiment, Blackwell experiment, data generating process (DGP)) is a map $\Pi : \Omega \to \Delta(S)$ for some S.

This is the way we think about information – we think of it as a map from the truth to the distribution of what the agent sees. $\omega \in \Omega$ is the "truth" and $s \in S$ is the "data." $\pi \in \Pi$ represents the relationship between the truth and the data (=information).

Let $\Omega = \{L, R\}$ and suppose we have a prior belief $\mu_0 = Pr(R) = 0.5$. The signal $\pi(L)$ and $\pi(R)$ is given as

$$\pi(L) = \begin{cases} l & \text{w.p. } 3/4 \\ r & \text{w.p. } 1/4 \end{cases}$$

$$\pi(R) = \begin{cases} l & \text{w.p. } 1/4 \\ r & \text{w.p. } 3/4 \end{cases}$$

Are these signals valuable? Yes! They are telling you something about the state. Given a prior μ_0, and a signal π, a signal realization $s \in S$ induces a posterior belief:

$$\mu(R|l) = 1/4, \mu(L/r) = 3/4$$

Bayes' rule:

$$\mu(\omega|s) = \frac{\mu(s|\omega)\mu_0(\omega)}{\sum_{w'} \mu(s|\omega')\mu_0(\omega')}$$

It's important to note that the constraint that DP has knowledge of the signal is without loss of generality. Suppose the agent is uncertain about which data-generating process is and attaches a probability to each different DGP. Well, that itself is a belief about the DGP!

Now fix Ω and $\mu_0 \in \Delta(\Omega)$. Consider $\pi : \Omega \to \Delta(S)$. Then each $s \in S$ induces $\mu_s \in \Delta(\Omega)$. So a belief induces a lottery over beliefs. Δ^2! π induces $\tau \in \Delta(\Delta(\Omega))$. Formally,

$$\tau(\mu_s) = \sum_{\omega} \pi(s|\omega)\mu_0(\omega)$$

where $\tau(\mu_s)$ represents the probability of μ_s under τ . **Every signal** (π) **induces some lottery over beliefs** $(\tau \in \Delta(\Delta(\Omega))$**.** Denote τ induced by π by $\tau =< \pi >$. Note: for any $\pi : \Omega \to \Delta(S)$,

$$\mathbb{E}_{<\pi>}[\mu_s] = \mu_0$$

This is saying that on average, you can't information to push you in a certain direction. Note that τ is the mean-preserving spread of the prior – it noises up the prior. Now we arrive at the most important result in information and uncertainty literature:

Theorem 14. *(Blackwell's Theorem) Fix $\Omega, \mu_0 \in \Delta(\Omega)$. Given two signals π and π', the following are equivalent:*

1. Every agent (regardless of A and $u : A \times \Omega \to \mathbb{R}$) is better off observing π than π'.

2. $< \pi >$, the distribution of posteriors induced by π, is a mean-preserving spread of $< \pi' >$.

Intuition: you always have a convex V. Then you like noisier lotteries over the beliefs you are going to have. So the signal is better if it's noisier. This is transferring an object into a space that's easily comparable. This happens a lot in economics.

Part IV. TA Sessions

7 TA Session 2018.09.17 (Monday)

Problem 1. You have a household with three normal goods: x, y, z with prices p_x, p_y, p_z and 2 members in the household M, W. The demand functions are given as $x = x_M, z = z_W, y = y_M + y_W$. They also have income (I_M, I_W) and utilities $u_M(y_M, x_M)$ and $u_W(y_W, z_W)$.

1. Does real household consumption satisfy the law of demand for each good?

2. Is there a well-defined houeshold demand function for each of these goods?

3. Assume that we have a household data observing x, y, z, I and data for singles, $x_M^S, \cdots z_W^S$. How can you use individual data to understand household consumption?

Solution 1. See below:

1. Yes, assuming normal good, slutsky's equation always gives you a downward-sloping demand curve.

2. No, for there to be a well-defined household demand function, it must be the case that $x = x(p_x, p_y, p_{z,} I)$ but it is in fact only a function of I_m. A good counterexample is one using Cobb-Douglas.

3. Step-by-step approach:

(a) Start by constructing a demand function for $x_M^S = g(p_x, p_Y, I_M^S) = x_M(p_x, p_Y, I_M^S)$ and invert it to estimate $I_M^S = f(p_X, p_Y, x_M^S)$.

(b) Because single men and married men are no different, we have $I_M = f(p_X, p_Y, x_M)$ and use this to estimate I_W.

(c) Now we turn attention to disentangling y. We estimate $y_M^S = y(p_Y, p_X, I_M^S)$ and once again married men and single men are identical, we have $y_M = y(p_X, p_Y, I_M)$, which would give us $y_W = y - y_M$

8 TA Session 2018.09.18 (Tuesday)

Problem 2. Now men and women care about each other. That is, they maximize the sum of utilities: $u_M + u_W$. In other words, the problem is

$$\max_{y_M, y_W, x_M, z_W} u_M(x_M, y_M) + u_W(y_W, z_W)$$

subject to $p_y(y_M + y_W) + p_x x_M + p_z z_W \leq I_M + I_W$.

1. Will household consumption satisfy law of demand?
2. Does there exist a proper household demand function for the total household? (i.e., in terms of total variables I, x, y, z; not $I_M, I_W, \ldots$)
3. (Marshallian Demand I) When p_y goes down, how do x_M, z_W, y_W, u_M, u_W change?
4. (Marshallian Demand II) When p_x goes down, how do x_M, z_W, y_W, u_M, u_W change?
5. (Hicksian Demand I) When compensated p_y goes down (i.e. you increase other prices so that given a fixed I, you can exactly afford the same bundle), how do all the variables change?
6. (Hicksian Demand II) When compensated p_x goes down, how do all the variables change?

Solution 2. We can rewrite the problem by defining a family utility:

$$u(y, x, z) = \max_{y_M, y_W} u_M(y_M, x) + u_W(y_W, z)$$

subject to $y_M + y_W \leq y$. Now the problem changes to

$$\max_{x,y,z} u(x, y, z) \text{ such that } p_x x + p_y y + p_z z \leq I$$

and therefore there should be a well-specified demand function forms $y(p_x, p_y, p_z, I)$.

1. Yes, since the Slutsky's equation stays the same. You set up the Slutsky's equation and state why the derivative of the Hicksian demand is always negative, they are normal goods, so the demand is downward-sloping.

2. For there to be a well-defined household demand function, it must be the case that $x = x(p_x, p_y, p_z, I)$.

3. Suppose $p_y \downarrow$. Then $y \uparrow$ from the law of demand since the good is normal. For x and z, it depends on the cross price elasticity:

   (a) For x and z, the income effect is positive and the substitution effect is negative, so the direction is unclear.

   (b) Note that for normal goods, the income elasticity is greater than zero; the opposite concept is that of inferior goods. Within normal goods, there are luxury goods for which the income elasticity is greater than one. This gives rise to another form of Slutsky's equation:
   $$\epsilon^M_{yy} = \epsilon^H_{yy} - s_Y \mu_x$$
   where the first term is the income effet and the second term is the substitution effect.

   (c) For y_M and y_W, consider the following example. Let $u_M(x_M, y_M) = \min\{a_M y_M, x_M\}$ and $u_W(y_W, z_W) = \min\{a_W y_W, z_W\}$. If you draw this graphically, you immediately see that $x^*_M = a_M y^*_M$ and $z^*_W = a_W y^*_W$ in which case the optimization problem changes to
   $$\max_{y_M, y_W} a_M y_M + a_W y_W \text{ s.t. } y_M(p_y + a_M p_x) + y_W(p_y + a_W p_z) \le I_m + I_w$$
   Now this utility function is a line (perfect substitute) and the solutions are the corners, depending on the slope of the budget constraint. So the answer to this maximization problem is
   $$y_W = \begin{cases} 0 & p_y > p^*_y \\ \frac{I}{p_y + a_W p_z} & p_y < p^*_y \end{cases}, y_M = \begin{cases} \frac{I}{p_y + a_M p_x} & p_y > p^*_y \\ 0 & p_y < p^*_y \end{cases}$$
   where $p^*_y = \frac{a_M a_W}{a_M - a_W}(p_x - p_z)$. You can plug this into u_M and $\underline{u_W}$ and show that a similar result holds.

4. Suppose $p_x \downarrow$. Then $x \uparrow$ from the law of demand. Not sure about y and z but we know that $u_M \uparrow$. Not sure about u_W but total utility $u \uparrow$

5. Suppose $p_y \downarrow$ compensated, so there is no income effect. In the matrix of elasticities, the diagonals are all negative and the off-diagonals are all positive. Note that for the marshallian elasticity matrix, Therefore, $y \uparrow$, $x \downarrow, z \downarrow$. To get an idea for y_M, y_W, u_M, u_W, consider the family's problem as the following:
   $$\min_{u_M, u_W} E_M(p_x, p_y, u_M) + E_W(p_y, p_z, u_W) \text{ s.t. } u_M + u_W = u$$
   where the equality constraint follows from the compensation. Also, note that we can use the Hicksian demand to provide the following definition

of normal goods:

$$\frac{\partial y_M^H}{\partial u_M} > 0$$

and furthermore by the Envelope Theorem, we have

$$\frac{\partial E}{\partial p_y} = y^H$$

and we combine the two to get

$$\frac{\partial^2 E_i}{\partial u_i \partial p_y} > 0$$

6. Suppose $p_x \downarrow$ compensated. Then $x \uparrow, y \downarrow, z \downarrow$. $u_M \uparrow$ and $u_W \downarrow$ because u is fixed. For y_M and y_W, we are not sure.

9 TA session 2018.09.20 (Thursday)

Problem 3. Assume $\exists$consumption efficiency for households s.t. $t > 1$, i.e. $y_M + y_W = ty$ ($y = 10$; when you were together, you only used 10 lbs of rice; $y_M + y_W = 15$; when you were separate households, you'd buy 15 lbs. In this case, $t = 1.5 > 1$ so there exists a consumption efficiency).

1. If t increases, what happens to y, y_M, y_W, x_M, z_W?
2. Compare effective income of people in households with individuals. (Hint: use Hicksian expenditure function to see if you can use less income for the same level of utility in a household vs. an individual)
3. If the # of families increase (i.e. reduce individual HH, increase joint HH), would the effective income of society differ? (Discussion question; no math)

Solution 3. Note that we are still maximizing $u(ty, x, z)$ subject to $p_y y + p_x x + p_z z \leq I$

1. Define consumption of y as $c \equiv ty$ and then $p_c \equiv p_y/t$. Then taking the Lagrangian:

$$\frac{\partial u(ty, x, z)}{\partial y} = \frac{\lambda p_y}{t} \Rightarrow \frac{\partial u(c, x, z)}{\partial y} = \lambda p_c \Rightarrow ty = c(p_y/t, p_x, p_z, t)$$

and take the logs and differentiating:

$$d \log y = -d \log t + \epsilon_c(-d \log t) = -(1 + \epsilon_c) d \log t$$

so if $|\epsilon_c| > 1$ then $t \uparrow$ and $y \uparrow$. Note that elasticity is defined as

$$\epsilon_c = \frac{\partial c}{\partial p_c} \times \frac{p_c}{c} = \frac{d \log c}{d \log p_c}$$

2. Solve the expenditure minimization problem

$$E(p_y, p_x, p_z, u) = \min_{x,y,z} p_x x + p_y y + p_z z$$

such that $u(ty, x, z) \geq u$. The Lagrangian:

$$\mathcal{L} = p_y y + p_x x + p_z z - \mu[u(ty, x, z) - u]$$

and the first-order constraints:

$$[y] : p_y = \mu t \frac{\partial u}{\partial y}$$

and the Envelope theorem:

$$\frac{\partial E}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}_* = \frac{\partial(-\mu u(ty, x, z))}{\partial t}|_* = -\mu y \frac{\partial u}{\partial y} = -\mu y \frac{p_y}{\mu t} = -\frac{y p_y}{t}$$

which gives us the elasticity as the share of expenditure of good y:

$$\epsilon_{E,t} = \frac{d \log E}{d \log t} = \frac{t}{E} \cdot \frac{\partial E}{\partial t} = \frac{t}{E} \cdot \frac{-y p_y}{t} = -s_y$$

$$\Rightarrow d \log E = -s_y d \log t$$

$$\Rightarrow \frac{\Delta E}{E} \approx -s_y \cdot \frac{\Delta t}{t}$$

Plug in $\Delta E = E_2 - E_1$ and $E = (E_1 + E_2)/2$, and let $\Delta t = 1 - t$ you get the following result:

$$\frac{E_1}{E_2} = s_y t + (1 - s_y)$$

3. Make some assumptions about (1) frictions in the market, (2) whether the existing distribution was voluntary. Also consider the people on the margin.

10 TA Session 2018.09.20 (Thursday)

Problem 4. Now we want think about composite goods, rather than two simple goods. Specifically, we have childcare and everything else. Consider CES (constant elasticity of substitution), CRS (constant returns to scale) production function with goods input and time input.

1. If $p_c \downarrow$, how does this affect the quantity of childcare and quantity of other goods?

2. Would there be a difference in the effect of decline in price of childcare on hours worked and child quantity depend on college education where the only source of heterogeneity is in income, i.e. $w_e > w_u$?

3. Now assume the only difference between educated and uneducated is $A_e > A_u$ similarly for both goods. Do the same exercise.
4. Combine 2. and 3. and analyze.
5. What if educated people have some outside income or endowment? How does that affect?

Solution 4. Setup: define childcare as C and other goods as Z. Use x for goods and h for time. Then the utility function is $U(C(x_c, h_c), Z(x_z, h_z))$ where prices are given as p_c, p_z for each good and w for hour. Define t_c as the time needed to work to produce the goods $t_c w = x_c p_c$. This gives rise to the budget constraint that $(t_c + h_c) + (t_z + h_z) = T_c + T_z \leq 1$ where the equality would hold at the optimum. Also note that for composite prices, we use shadow prices denoted by π.

We are interested in how C, Z, and h are affected. We can set up a maximization problem as

$$V(\pi_C, \pi_Z) = \max U(C, Z)$$

$$\text{subject to } \pi_C C + \pi_Z Z \leq 1$$

but we have to understand how C is constructed and compute the shadow prices to solve this maximization problem. This is the need to set up the production function:

$$Q = A(\alpha x^{\frac{\sigma-1}{\sigma}} + (1-\alpha)h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

or

$$Q = A(\alpha x^{\rho} + (1-\alpha)h^{\rho})^{\frac{1}{\rho}}$$

where σ is the elasticity of substitution. It is simpler to use the second formulation and then change to a function of elasticity at the very end. We then minimize the cost

$$\min px + wh$$

$$\text{subject to } A(x^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \geq Q$$

The first-order constraint:

$$[x] : \mu \left(\frac{\sigma}{\sigma-1}\right) x^{\frac{\sigma-1}{\sigma}-1} A \left(\frac{\sigma-1}{\sigma}\right) (x^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma-1}{\sigma}-1} = p$$

$$[h] : \mu \left(\frac{\sigma}{\sigma-1}\right) h^{\frac{\sigma-1}{\sigma}-1} A \left(\frac{\sigma-1}{\sigma}\right) (x^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma-1}{\sigma}-1} = w$$

Dividing the two constraints:

$$\frac{h}{x} = \left(\frac{p}{w}\right)^{\sigma} \Rightarrow h = \left(\frac{p}{w}\right)^{\sigma} x$$

Plugging this back into Q:

$$Q = A\left[x^{\frac{\sigma-1}{\sigma}}\left(1 + \left(\frac{p}{w}\right)^{\sigma-1}\right)\right]^{\frac{\sigma}{\sigma-1}}$$

Re-write the equation as

$$x = \left(1 + \left(\frac{w}{p}\right)^{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \frac{Q}{A}$$

$$h = \left(1 + \left(\frac{p}{w}\right)^{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \frac{Q}{A}$$

This is how much it takes to produce one more unit:

$$\frac{x_c p_c + h_c w}{Q} = \pi_C w$$

since π is in units of time. Therefore:

$$\pi_C = \frac{1}{Aw}\left(p_c^{1-\sigma} + w^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

1. Now we are ready to answer the question. Note that if $\sigma = 0$, then they are complements; if $\sigma = \infty$ they are perfect substitutes. Therefore,

$$p_c \downarrow \Rightarrow \pi_C \downarrow \Rightarrow C \uparrow$$

and for z, recall the Slutsky's equation:

$$\epsilon_{z,c} = \epsilon^H_{z,c} - s_c \eta_z$$

so the consumption of z would depend on the income elasticity of z (η_z). If the substitution effect dominates income effect, $Z \downarrow$. Note that $T_z = Z\pi_Z$ and $T_C = 1 - T_Z$ and linearizing at T_Z:

$$t_c + t_z = (1 - T_Z)\frac{t_c}{T_C} + T_Z \frac{t_z}{T_Z}$$

So $Z \downarrow \Rightarrow T_Z \downarrow \Rightarrow T_C \uparrow$ and assuming $t_c/T_C < t_z/T_Z$, then $t \downarrow$. Note that the ratio is the market-time intensity. Note that since

$$\frac{h}{x} = \left(\frac{p}{w}\right)^{\sigma}$$

then using the relationship $x = wt/p$:

$$\frac{h+t}{t} = 1 + \left(\frac{p}{w}\right)^{\sigma-1}$$

and inverting it formulate the original condition as:

$$\frac{t}{T} = \frac{t}{t+h} = \frac{1}{1 + \left(\frac{p}{w}\right)^{\sigma-1}}$$

2. Let $w_e > w_u$. Recall that

$$\pi_C = \frac{1}{A}\left(1 + \left(\frac{p_C}{w}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

We are interested in the change of p_C but childcare is influenced by π_C so we first look at:

$$\epsilon_C = \frac{\partial \log \pi_C}{\partial \log p_C} = \frac{1}{1 + \left(\frac{w}{p_C}\right)^{1-\sigma}}$$

which depends on the value of σ. If $\sigma > 1$, then $\epsilon_C(w_e) > \epsilon_C(w_u)$, and vice versa. So for $\sigma > 1$, the educated group will purchase more than does the uneducated group.

3. Let $A_e > A_u$. There is no substitution effect since the relative prices of the shadow prices move together. The person is better off becaues the prices are cheaper, so there is only the income effect and the consumption of both goods increase.

4. The conditions in 2. are essentially the substitution effect and the conditions in 3. are essentially the income effect.

5. This is identical to 3. 3. was an income effect through productivity, although we have more time used in production compared to 3. In 5., we spend more time because we are not as productive. With endowment, $x \uparrow$ and $h \uparrow$; with the productivity increase, $C(x, h) \uparrow$.

11 TA Session 2018.09.21 (Friday)

Problem 5. There is a club good where you can reduce the price by paying a fixed cost.

1. Characterize the demand for this good.

2. How will the consumption of the other good vary with income? Could they sharply increase or decrease at some income levels even if the underlying demand function is continuous?

3. Suppose p_0 increases by ϵ and p_1 increases by ϵ. How will this affect the decision?

4. Suppoes F goes up. What happens to each group of customers?

Solution 5. Start with notations.

- Write the problem as $v(p, q, I) = \max u(x, y)$ such that $px + qy \leq I$. Let F denote the fixed price and p_1 the discount price. Let p_0 be the original price.

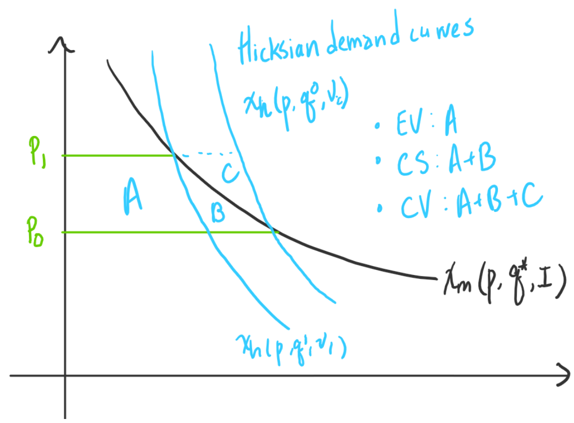

Fig. 11.1: Compensating Variation

- To choose the option of paying, the individual must satisfy

$$v(p_1, q, I - F) \geq v(p_0, q, I)$$

- The concept of compensating variation (CV) is the willingness to pay for a price change:

$$v(p_1, q, I - CV) = v(p_0, q, I)$$

  which can be expressed as:

$$CV = -\int_{p_0}^{p_1} x_h(p, q_0, v_0)dp$$

  whereas equivalent variation (EV) is the willignes to pay to avoid a price change: (compute after the price change)

$$EV = -\int_{p_0}^{p_1} x_h(p, q_1, v_1)dp$$

  Note that for normal goods, $EV < CS < CV$; for inferior goods, $EV > CS > CV$.

- Since $CV = -(e(p_1, q_0, v_0) - e(p_0, q_0, v_0))$ and since $\partial e / \partial p = x^h$, we can combine these two to get the expression for EV.

1. To estimate the demand function, write

$$x(p_1, p_0, q, I, F) = \begin{cases} x_m(p, q, I - F) & F < CV \\ x_m(p_0, q_0, I) & F > CV \end{cases}$$

   where CV is expressed in terms of the Hicksian demand x^h.

2. If you think about the demand for x and the person at the margin, he is buying
$$x_0 \leq \frac{F}{p_o - p_1} \leq x_F$$
where x_0 is the quantity purchased for those who did not pay the fixed cost and x_F is the quantity purchased for those who did pay the fixed cost. Now for the other good, we have
$$y_m(p_1, p_0, q, I, F) = \begin{cases} y_m(p, q, I - F) & F < CV \\ y_m(p_0, q_0, I) & F > CV \end{cases}$$
Let I^* be the income level at which $v(p_1, q, I^* - F) = v(p_0, q, I)$. With this income level, we can look at the Marshallian demand for ys:
$$y(p_1 q, I^* - F) \neq y(p_0, q, I^*)$$
In this case, the consumption of y will have a jump.

3. As the price changes, the compensating variation gets higher. Only people whose CV is less than F will be willing to pay. The proportion of people whose CV will be less than F gets smaller since CV is increasing.

4. Suppose F goes up.
   - For those $I < I_0^*$where I_0^* is when $F = F_0$, they do not change their behavior.
   - For those $I > I_1^*$ where I_1^* is when $F = F_1$, the substitution effect is 0 and there is only income effect. So both goods are normal, purchase less of both x and y.
   - For those $I_0^* < I < I_1^*$ (people who used to pay but are not paying anymore)
     - You're going from $x(p_1 q, I - F_0)$ to $x(p_0, q, I)$. So the income has increased but the relative price has increased, so the substitution effect and the income effect are in opposite directions. For y, the relative price has decreased so the income effect and the substitution effect are in the same direction.

Problem 6. Suppose the marginal cost of a monopolist increases.

1. Assume constant marginal cost c. If c rises, does p^* rise more than c or less than c?

2. Now there's a unit tax and we see that the change in price increased more than the tax. Is this possible that this market is competitive?

Solution 6. Setup:

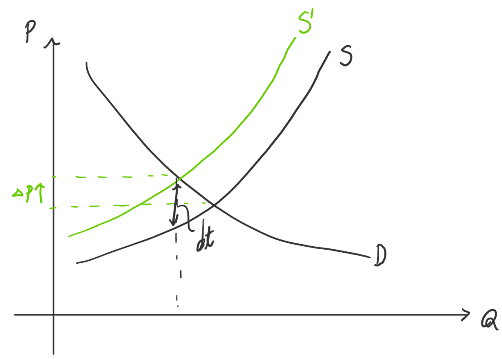

Fig. 11.2: Taxation in a competitive market

1. We want to know if
$$\frac{\partial p^*}{\partial c} > 1$$
The monopolist solves
$$\max_p D(p)(p-c)$$
for which the FOC is
$$[p] : (p^* - c)D'(p^*) + D(p^*) = 0$$
We would rather have elasticities rather than demands, so multiply each side by $p^*/D(p^*)$:
$$p^* = \frac{\epsilon^D(p^*)}{\epsilon^D(p^*) - 1} c$$
since
$$\epsilon^D(p) = -\frac{D'(p^*)p^*}{D(p^*)}$$
We now take the partial derivative of the optimal price p^*:
$$\frac{\partial p^*}{\partial c} = \frac{\epsilon}{\epsilon - 1} - c\frac{\frac{\partial \epsilon}{\partial p^*}}{(\epsilon - 1)^2}\frac{\partial p}{\partial c} = \frac{\epsilon}{\epsilon - 1} - \frac{p^* \frac{\partial \epsilon}{\partial p^*}}{(\epsilon - 1)\epsilon}\frac{\partial p}{\partial c} = \frac{\epsilon}{\epsilon - 1} - \epsilon_\epsilon \frac{1}{\epsilon - 1}\frac{\partial p}{\partial c} \Rightarrow \frac{\partial p^*}{\partial c} = \frac{\epsilon}{\epsilon - 1 + \epsilon_\epsilon}$$
If $\epsilon_\epsilon(p^*) < 1$ then it's possible than the optimal price can rise more than 1. Note that the above quantity is called "pass through."

2. Note that in an competitive industry, we have a supply curve. (For a monopolist, we were merely choosing points). Note that dt represents the tax. See Figure 11.2 for a graphical illustration.
$$\Delta\%Q^D = -\epsilon^D \Delta$$

$$\Delta\%Q^S = \epsilon^S \left(\frac{\Delta P}{P} - \frac{\Delta t}{P} \right)$$

At very small changes,

$$dP = \frac{1/\epsilon_D}{1/\epsilon_D + 1/\epsilon_S} dt \Rightarrow \frac{dP}{dt} < 1$$