Price Theory II
Winter 2019

(Philip Reny)

Simon Sangmin Oh

University of Chicago

Note:

These proofs are based on JR 3rd edition and the lectures in-class.

Contents

1 Preference Relations and Axioms **4**
1.1 Theorem 1.1: Representation Theorem . . . 4
1.2 Theorem 1.4: Invariance . . . 5
1.3 Theorem 1.5: Link between u and $\succsim$ . . . 6

2 Basics of General Equilibrium – Consumption **7**
2.1 Theorem 5.1: Basic Properties of Demand . . . 7
2.2 Theorem 5.2: Properties of Aggregate Excess Demand . . . 8
2.3 Theorem 5.3: Aggregate Excess Demand and Walrasian Equilibrium . . . 9
2.4 Theorem 5.4: Utility and Aggregate Excess Demand . . . 10
2.5 Theorem 5.7: First Welfare Theorem . . . 11
2.6 Theorem $5.7^{1/2}$: No-Trade Theorem . . . 12
2.7 Theorem 5.8: Second Welfare Theorem . . . 13
2.8 Theorem: WEA in the Core . . . 14

3 Basics of General Equilibrium – Production **15**
3.1 Setup . . . 15
3.2 Theorem 5.9: Continuity of Production and Profits . . . 17
3.3 Theorem 5.12: Properties of Demand . . . 18
3.4 Theorem 5.13: Existence of Walrasian Equilibrium with Production . . . 19
3.5 Theorem 5.14: First Welfare Theorem with Production . . . 20
3.6 Theorem 5.15: Second Welfare Theorem with Production . . . 21

4 Matching **22**
4.1 Setup . . . 22
4.2 Theorem 5.1: Existence of a Stable Matching . . . 23
4.3 Theorem: Pareto Optimality of μ_M I . . . 24
4.4 Theorem: Pareto Optimality of μ_M II . . . 25

5 Social Choice **26**
5.1 Setup . . . 26
5.2 Theorem 5.1: Impossibility Theorem . . . 29
5.3 Lemma: PE and Monotonicty of Strategy-Proof Choice Function . . . 31
5.4 Theorem 5.3: Gibbard-Satterthwaite Theorem . . . 32

6 Decision-Making Under Uncertainty **36**
6.1 Setup . . . 36
6.2 Theorem 2.7: Expected Utility Property . . . 38
6.3 Theorem: Affine Trasformation . . . 39

7 Strategic Form Games **40**
7.1 Setup . . . 40
7.2 Theorem: Simplified Nash Equilibrium Tests . . . 42
7.3 Theorem: Existence of Nash Equilibriuma . . . 43

8 Games with Incomplete Information **45**
8.1 Setup 45
8.2 Theorem: Redefining Utility 50
8.3 Theorem: Equivalence among Strategies 51
8.4 Theorem: Equivalence among BNEs 52

9 Extensive Form Games **53**
9.1 Setup 53
9.2 Theorem: Equivalence among Strategies 61
9.3 Theorem: Sequential Equilibrium & SPNE 62
9.4 Theorem: Existence of Sequential Equilibrium 63

1 Preference Relations and Axioms

Axioms

- ▷ Axiom 1: Completeness – For all x^1 and x^2 in X, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$.
- ▷ Axiom 2: Transitivity – If $x^1 \succsim x^2 \succsim x^3$, then $x^1 \succsim x^3$.
- ▷ Axiom 3: Continuity – $\succsim (x)$ and $\precsim (x)$ are closed in $\mathbb{R}^n_+$.
- ▷ Axiom 4: Strict Monotonicity – If $x^0 \geq x^1$ then $x^0 \succsim x^1$ while if $x^0 \gg x^1$then $x^0 \succ x^1$.
- ▷ Axiom 5: Strict Convexity – If $x^1 \neq x^0$ and $x^1 \succsim x^0$, then $tx^1 + (1-t)x^0 \succ x^0$ for all $t \in (0,1)$.
- ▷ Axiom 4': Local Non-satiation – For all $x^0 \in \mathbb{R}^n_+$ and $\epsilon > 0$, there exists some $x \in B_\epsilon\left(x^0\right)$ such that $x \succ x^0$.
- ▷ Axiom 5': Convexity – If $x^1 \succsim x^0$ then $tx^1 + (1-t)x^0 \succsim x^0$ for all $t \in (0,1)$.

1.1 Theorem 1.1: Representation Theorem

Theorem. *(1.1. Representation Theorem) Let $X = \mathbb{R}^n_+$. Suppose that $\succsim$ satisfies Axioms 1-4. Then there exists a continuous real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that represents $\succsim$.*

Solution. It suffices to explicitly specify such utility function. Let $\mathbf{e}$ be a vector of ones and consider $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that $u(\mathbf{e})\,\mathbf{e} \sim \mathbf{x}$. We show that such u indeed exists and that it is uniquely determined so that u is a well-defined function.

- ▷ Existence: Define sets $A := \{t \geq 0 : t\mathbf{e} \succsim \mathbf{x}\}$ and $B := \{t \geq 0 : t\mathbf{e} \precsim \mathbf{x}\}$. It suffices to show that $A \cap B \neq \emptyset$.
  - * Continuity of $\succsim$ implies that both A and B are closed in $\mathbb{R}_+$. Strict monotonicity suggests that $t \in A$ implies $t' \in A$ for all $t' \geq t$. Similar argument for B yields that A must be of the form $[\underline{t}, \infty]$ and B of the form $[0, \bar{t}]$. Furthermore, completeness suggests that $t \in A \cup B$ which implies that $\mathbb{R}_+ = A \cup B$. Thus, $A \cup B \neq \emptyset$.
- ▷ Uniqueness: Since $t_1\mathbf{e} \sim \mathbf{x}$ and $t_2\mathbf{e} \sim \mathbf{x}$, then by the transitivity of $\sim$ we have $t_1\mathbf{e} \sim t_2\mathbf{e}$. So by strict monotonicity, it must be that $t_1 = t_2$.
- ▷ Representation of Preferences: Consider $\mathbf{x}^1, \mathbf{x}^2$ and the associated utilities $u\left(\mathbf{x}^1\right), u\left(\mathbf{x}^2\right)$. Then:

$$\mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right)\mathbf{e} \sim \mathbf{x}^1 \succsim \mathbf{x}^2 \sim u\left(\mathbf{x}^2\right)\mathbf{e} \Leftrightarrow u\left(\mathbf{x}^1\right) \geq u\left(\mathbf{x}^2\right)$$

- ▷ Continuity: Suffices to show that the inverse image under u of every open ball in $\mathbb{R}$ is open in $\mathbb{R}^n_+$.

1.2 Theorem 1.4: Invariance

Theorem. *(1.4 Invariance) Let $\succsim$ be a preference relation on $\mathbb{R}_+^n$ and $u(\mathbf{x})$ be a utility function that represents it. Then $v : \mathbb{R}_+^n \to \mathbb{R}$ also represents $\succsim$ if and only if*

$$v(\mathbf{x}) = f(u(\mathbf{x})), \forall \mathbf{x} \in \mathbb{R}_+^n$$

where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.

Solution. Proof Sketch:

- ($\Rightarrow$) Suppose such f exists. Then

$$x^1 \succsim x^2 \Leftrightarrow u\left(x^1\right) \geq u\left(x^2\right) \Leftrightarrow f\left(u\left(x^1\right)\right) \geq f\left(u\left(x^2\right)\right) \Leftrightarrow v\left(x^1\right) \geq v\left(x^2\right)$$

  and we are done.

- ($\Leftarrow$) Suppose u and v represent $\succsim$. Define f so that point-wise sets $f(u(x)) = v(x)$ in the image of u. So assume $v(x) = f(u(x))$. Assume f is not strictly increasing. This means there exists y and z such that

$$y = u\left(x^1\right) > z = u\left(x^1\right) \Rightarrow v\left(x^1\right) = f(y) \leq f(z) = v\left(x^2\right)$$

  which is also a contradiction since both represent the same preferences.

1.3 Theorem 1.5: Link between u and $\succsim$

Theorem. *Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:*

1. *$u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.*
2. *$u(\mathbf{x})$ is quasi-concave if and only if $\succsim$ is convex.*
3. *$u(\mathbf{x})$ is strictly quasi-concave if and only if $\succsim$ is strictly convex.*

Solution. This follows straight from the definitions of strictly increasing and quasi-concavity.

2 Basics of General Equilibrium – Consumption

2.1 Theorem 5.1: Basic Properties of Demand

Assumption. *(5.1) Each* $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ *is continuous, strongly increasing, and strictly quasi-concave.*

Theorem. *(5.1: Basic Properties of Demand) If* u^i *satisfies Assumption 5.1, then for each* $\mathbf{p} \gg 0$*, the consumer's problem has a unique solution* $\mathbf{x}^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right)$*. In addition,* $\mathbf{x}^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right)$ *is continuous in* $\mathbf{p}$ *on* $\mathbb{R}^n_{++}$*.*

Solution. The proof is as follows:

- ▷ Existence: follows because $\mathbf{p} \gg 0$ implies that the budget set is bounded.
- ▷ Uniqueness: follows from the strict quasi-concavity of u^i
- ▷ Continuity: follows from Theorem A2.21 (the theorem of the maximum).

Comments:

- ▷ *We can't have continuity in* $\mathbf{p}$ *on all of* $\mathbb{R}^n_+$ *because demand may well be infinite if one of the prices is zero. We need to do a little more work later to deal with this unpleasant, yet unavoidable, difficulty.*

2.2 Theorem 5.2: Properties of Aggregate Excess Demand

Theorem. *(5.2: Properties of Aggregate Excess Demand Functions) If for each consumer i, u^i satisfies Assumption 5.1, then for all* $\mathbf{p} \gg \mathbf{0}$*:*

1. $z(\cdot)$ *is continuous at* $\mathbf{p}$
2. $z(\lambda \mathbf{p}) = z(\mathbf{p})$ *for all* $\lambda > 0$.
3. $\mathbf{p} \cdot z(\mathbf{p}) = 0$

Solution. We proceed as following:

▷ Continuity: This follows from Theorem 5.1

▷ Homogeneity: Individual demands, and excess demands are homogeneous of degree zero in prices. Therefore, aggregate excess demand is also homogenous of degree zero in prices.

▷ Walras' Law: This is true becuase when u^i is strongly increasing, each consumer's budget constraint holds with equality. In other words, it must be the case that

$$\sum_{k=1}^{n} p_k \left(x_k^i \left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i \right) - e_k^i \right) = 0$$

Summing over individuals:

$$\sum_{i \in \mathcal{I}} \left\{ \sum_{k=1}^{n} p_k \left(x_k^i \left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i \right) - e_k^i \right) \right\} = 0$$

Exchanging the order of summation:

$$\sum_{k=1}^{n} \left\{ \sum_{i \in \mathcal{I}} p_k \left(x_k^i \left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i \right) - e_k^i \right) \right\} = 0$$

This is equivalent to the expression:

$$\sum_{k=1}^{n} p_k \left(\sum_{i \in \mathcal{I}} x_k^i \left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i \right) - \sum_{i \in \mathcal{I}} e_k^i \right) = \sum_{k=1}^{n} p_k z_k \left(\mathbf{p} \right) = 0$$

2.3 Theorem 5.3: Aggregate Excess Demand and Walrasian Equilibrium

Theorem. *(5.3: Aggregate Excess Demand and Walrasian Equilibrium) Suppose z satisfies the following three conditions:*

1. $z(\cdot)$ *is continuous on* $\mathbb{R}^n_{++}$
2. $p \cdot z(p) = 0$ *for all* $p \gg 0$.
3. *If* $\{p^m\}$ *is a sequence of price vectors in* $\mathbb{R}^n_{++}$ *converging to* $\bar{p} \neq 0$ *and* $\bar{p_k} = 0$ *for some good* k*, then for some good* k' *with* $\bar{p}_k = 0$*, the associated sequence of excess demands in the market for good* k'*,* $\{z_{k'}(p^m)\}$ *is unbounded above.*

Then there is a price vector $p^* \gg 0$ *such that* $z(p^*) = 0$.

Solution. We proceed as following:

▷ Fix $\epsilon \in (0,1)$ and let $S_\epsilon = \left\{ p \in [0,1]^n : \sum_{k=1}^n p_k = 1 \text{ and } p_k \geq \frac{\epsilon}{1+2n}, \forall k \right\}$. Roughly speaking, this is a set of prices that add up to 1 and strictly positive. Then it follows that S_ϵ is non-empty, compact, and convex.

Comments:

▷ *Under*

2.4 Theorem 5.4: Utility and Aggregate Excess Demand

Theorem. *(5.4: Utility and Aggregate Excess Demand) If each consumer's utility function satisfies Assumption 5.1, and if the aggregate endowment of each good is strictly positive, then aggregate excess demand satisfies conditions 1 through 3 of Theorem 5.3*

Solution. Condition 1 & 2 are satisfied, so we focus on condition 3.

▷ First, there is at least one consumer i for whom $\bar{p} \cdot e^i > 0$.

* Consider $\{p^m\}$ converging to some $\bar{p} \neq 0$ such that $\bar{p}_k = 0$ for some good k.
* Since $\sum_i e^i \gg 0$, we must have $\bar{p} \cdot \sum_i e^i > 0$ so there must be at least one consumer i for whom $\bar{p} \cdot e^i > 0$.

▷ Second, consider this consumer's demand $x^m \equiv x^i\left(p^m, p^m \cdot e^i\right)$ and we show that the sequence of demand vectors is unbounded. To show this, suppose by contradiction that the sequence of demand vectors is bounded.

* Consider the convergent subsequence of $\{x^m\}$ that converges to some value x^* (since it is bounded).
* At the limit, the following observations hold:
  - Since $p^m \cdot x^m = p^m \cdot e^i, \forall m$, it follows that $\bar{p} \cdot x^* = \bar{p} \cdot e^i > 0$.
  - Defining $\hat{x} = x^* + (0, ..., 0, 1, 0, ..., 0)$ with 1 at the kth index, we have $u^i\left(\hat{x}\right) > u^i\left(x^*\right)$ since utility is strongly increasing.
* Because $u\left(\cdot\right)$ is continuous, there is $t \in (0,1)$ close enough to 1 such that $u^i\left(t\hat{x}\right) > u^i(x^*)$ and at the same time $\bar{p} \cdot (t\hat{x}) < \bar{p} \cdot e^i$ meaning this new bundle is strictly affordable.
* Thus by continuity of $u^i(\cdot)$ and because $x^m \to x^*$ and $p^m \to \bar{p}$, there is a m large enough so that $u^i\left(t\hat{x}\right) > u^i\left(x^m\right)$ and $p^m \cdot (t\hat{x}) < p^m \cdot e^i$.
* This is a contradiction! Given a price vector p^m, the consumer chose x^m as her utility-maximizing bundle, but there exists another bundle that's better. Therefore, the sequence of demands should be unbounded.

▷ Since the supply of good k' is finite, $\{z_{k'}\left(p^m\right)\}$ is unbounded above.

▷ Because $p^m \cdot e^i \to \bar{p} \cdot e^i$, the sequence $\left\{p^m \cdot e^i\right\}$ is bounded, and so $p^m \cdot e^i \leq c, \forall m$ where c is a constant. Then

$$p^m_{k'} x^i_{k'}\left(p^m, p^m \cdot e^i\right) \leq p^m \cdot e^i \leq c$$

so $p^m_{k'} \to 0$ since this is the only way the demand for good k can be bounded above and affordable. Consequently, $\bar{p}_{k'} = 0$.

Key Concepts

1. Existence of a convergent subsequence
2. Continuity of the utility functions

2.5 Theorem 5.7: First Welfare Theorem

Theorem. *(5.7 First Welfare Theorem) For $\mathcal{E} = \left(u^i, e^i\right)_{i\in\mathcal{I}}$, if each u^i is strictly increasing, then every WEA is PE.*

Solution. Suppose by contradiction that $\hat{x} = \left(\hat{x}^1, ..., \hat{x}^{\mathcal{I}}\right)$ is a WEA at WE price p^*, but $\hat{x}$ is not PE.

1. Becaues $\hat{x}$ is feasible, $\exists y \in F(e)$ such that $u^i\left(y^i\right) \geq u^i\left(x^i\right), \forall i$ and at least one strict.

2. By Lemma 5.2 (which essentially states that better bundles are more expensive and weakly better bundles are weakly expensive), it follows that $p^* y^i \geq p^* x^i, \forall i \in \mathcal{I}$ and at least one strict. Thus, $p^* \cdot \sum_i \hat{y}^i > p^* \sum_i \hat{x}^i$ but this is a contradiction since $\sum_i \hat{x}^i = \sum_i \hat{y}^i = \sum e^i$.

Key Concepts:

- ▷ We don't even need utility to be continuous. The continuity was used to establish the existence of a WEA; now we're given WEA for free.
- ▷ The purpose of having "strictly increasing" – or even just "local non-satiation" – is to make sure that the person's tangent to the budget line.

2.6 Theorem $5.7^{1/2}$: No-Trade Theorem

Theorem. *($5.7^{1/2}$ No-Trade Theorem) Suppose A5.1 holds and $\sum_{i\in\mathcal{I}} e^i \gg 0$. If the endowment allocation is itself Pareto efficient, then the endowment allocation is the unique WEA i.e. there is no trade.*

Solution. We proceed as following:

1. By Theorem 5.5, $\exists$ WEA $\hat{x} \in F(e)$ at price p^*. Then it must be the case that $u^i(\hat{x}^i) \geq u^i\left(e^i\right), \forall i \in \mathcal{I}$.
2. Since e is PE, $u^i\left(\hat{x}^i\right) \leq u^i\left(e^i\right), \forall i$. Together, this implies $u^i\left(\hat{x}^i\right) = u^i(e^i), \forall i$.
3. Given the strict quasiconcavity of the utility function, $\hat{x}^i = e^i$.

Interpretation:

▷ You can open the markets and let things trade, but all agents will just sit home and consume their current endowments.

2.7 Theorem 5.8: Second Welfare Theorem

Theorem. *(5.8 Second Welfare Theorem) Suppose A5.1 holds and $\sum_{i \in \mathcal{I}} e^i \gg 0$. If $\bar{x} \in F(e)$ is PE and endowments are redistributed to $\bar{x}$, then the unique WEA of the new economy is $\bar{x}$.*

Solution. Proof follows immediately from Theorem $5.7^{1/2}$.

Implications:

- ▷ *If the WE price that supports $\bar{x}$ in $\bar{\mathcal{E}}$ is p^*, then redistributing endowments e to any $\bar{e}$ such that $p^* \cdot \bar{e}^i = p^* \cdot \bar{x}^i, \forall i$ will do. In the Edgeworth Box diagram, anywhere on the budget line (price vector line) would do.*
- ▷ *Another form of this would be saying: you can find a lump-sum tax that will make the allocations to a competitive equilibrium: $T^i + p^* \bar{e}^i = p^* \bar{x}^i$*

2.8 Theorem: WEA in the Core

Theorem. *If each u^i is strictly increasing, then every WEA is in the core.*

Solution. Assume by contradiction that WEA x is not in the core. Then there exists $\tilde{x}$ and a coalition S that blocks x. At prices p^*, however, it must be

$$p^* \cdot \sum_i \tilde{x}^i > p^* \cdot \sum_i x^i = p^* \cdot \sum_i e^i$$

which is a contradiction.

3 Basics of General Equilibrium – Production

3.1 Setup

We now introduce production. There are n products with J firms, and each firm has production set $Y^j \subseteq \mathbb{R}^n$.

Assumption. *(5.2) We make the following assumptions:*

1. $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$
2. Y^j is closed and bounded.
3. Y^j is strongly convex, i.e. $\forall y^1, y^2 \in Y^j$ and $\forall t \in (0,1)$, there exists $\bar{y} \in Y^j$ such that $\bar{y} \geq ty^1 + (1-t)\,y^2$ but never equal.
   - ▷ This assumption eliminates the possibility of CRS. It also eliminates the possibility of diminishing marginal rate of substitution.
   - ▷ Notice that $ty^1 + (1-t)y^2$ doesn't need to lie in the set i.e. it's perfectly fine to define the production set to be simply the boundary.

Note that these assumptions are the bare minimum – you can't rule out a production plan with all positive elements i.e. no inputs but positive output.

Now we add consumers. Let $\theta^{ij} \in [0,1]$ be the consumer i's ownership in firm j so we have

$$\sum_{i \in \mathcal{I}} \theta^{ij} = 1, \forall j \in \mathcal{J}$$

Thus the characterization of a production economy with private ownership is

$$\mathcal{E} = \left(u^i, e^i, \theta^{ij}, Y^j\right)_{i \in \mathcal{I}, j \in \mathcal{J}}$$

and the consumer budget constraint is now

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j\left(\mathbf{p}\right) \equiv m^i\left(\mathbf{p}\right)$$

Definition 3.1. (Profit Function) GIven any $\mathbf{p} \in \mathbb{R}^n_+$, each firm j solves

$$\max_{y^j \in Y^j} \mathbf{p} \cdot y^j \equiv \Pi^j\left(\mathbf{p}\right) where$$

prices are taken.

- ▷ Continuous: Theorem of the Maximum
- ▷ Existence of solution: Maximizing continuous function on a non-empty, compact set
- ▷ Uniqueness of solution: strong convexity of Y^j gives uniqueness immediately.

* Suppose by contradiction that there exists y^1, y^2 that maximize profits. By the strong convexity assumption, there exists $\bar{y}$ such that
$$\bar{y} \geq ty^1 + (1-t)\, y^2$$
Multiplying each side by $\mathbf{p}$, we have
$$\mathbf{p} \cdot \bar{y} > \mathbf{p} \cdot \left(ty^1 + (1-t)\, y^2\right) = \Pi^j\left(\mathbf{p}\right)$$
which is a contradiction.

Definition 3.2. $\mathbf{p}^* \in \mathbb{R}^n_+$ is a WE of $\mathcal{E}$ as defined above if and only if $\exists \hat{x}^1, ..., \hat{x}^I \in \mathbb{R}^n_+, \hat{y}^1 \in Y^1, ..., \hat{y}^J \in Y^J$ such that

1. $\hat{x}^i$ solves the consumer's maximization problem for all $i \in \mathcal{I}$
2. $\hat{y}^j$ solves the firm's problem for all $j \in \mathcal{J}$
3. market clears, i.e.
$$\sum_{i \in \mathcal{I}} \hat{x}^i = \sum_{i \in \mathcal{I}} e^i + \sum_{j \in \mathcal{J}} \hat{y}^j$$

And the corresponding $(\hat{x}, \hat{y})$ is a WEA.

Definition 3.3. (Excess Demand) For $\mathbf{p} \gg 0$, we may define
$$z\left(\mathbf{p}\right) = \sum_{i \in \mathcal{I}} \hat{x}^i\left(\mathbf{p}, m^i\left(\mathbf{p}\right)\right) - \sum_{i \in \mathcal{I}} e^i - \sum_{j \in \mathcal{J}} y^j\left(\mathbf{p}\right)$$
Furthermore if $\mathbf{p}^* \gg 0$ and $z\left(\mathbf{p}^*\right) = 0$, then $\mathbf{p}^*$ is a WE.

3.2 Theorem 5.9: Continuity of Production and Profits

Theorem. *(5.9 Continuity of Production and Profits) Under Assumption 5.2,* $\forall \mathbf{p} \gg 0$*, the solution to the profit maximization problem is unique and denoted by* $y^j(\mathbf{p})$*. Also,* $y^j(\cdot)$ *is continuous on* $\mathbb{R}^n_{++}$*. Finally,* $\Pi(\mathbf{p})$ *is well-defined and continuous on* $\mathbb{R}^n_+$*.*

Solution. Skipped.

3.3 Theorem 5.12: Properties of Demand

Theorem. *(5.12 Properties of Demand) Under Assumption 5.2 (about Y^j) and Assumption 5.1 (about u^i), a solution to each consumer i's maximization problem exists and is unique for all $\mathbf{p} \gg \mathbf{0}$. Denoting the solution as $x^i\left(\mathbf{p}, m^i\left(\mathbf{p}\right)\right)$, it is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$. In addition, $m^i\left(\mathbf{p}\right)$ is continuous on $\mathbb{R}^n_{+}$.*

Solution. Notice that each firm will earn non-negative profits because each can always choose the zero production vector. Consequently, $m^i\left(\mathbf{p}\right) \geq 0$. Therefore, the solution to the maximization problem always exists and will be unique whenever $\mathbf{p} \gg 0$.

From our result for endowment economy, we showed that the demand is continuous in $(\mathbf{p}, y)$ so here it suffices to show that $m^i\left(\mathbf{p}\right)$ is continuous in $\mathbf{p}$. Here we appeal to Theorem 5.9 to argue that $m^i\left(\mathbf{p}\right)$ is indeed continuous. (Theorem 5.9 states that given the conditions of Assumption 5.2, for every price $\mathbf{p} \gg 0$, the solution to the firm's problem (the production plan) and the profit function are continuous). Thus the proof is complete.

3.4 Theorem 5.13: Existence of Walrasian Equilibrium with Production

Theorem. *(5.13 Existence of Walrasian Production) Suppose A5.21 and A5.2 hold. If* $\exists \bar{y}^1 \in Y^1, ..., \bar{y}^J \in Y^J$ *such that* $\sum_{j \in \mathcal{J}} \bar{y}^j + \sum_{i \in \mathcal{I}} e^i \gg 0$ *, then* $\exists$*WE* $p^* \gg 0$.

Solution. Suffices to verify the conditions 1-3 of Theorem 5.3. The first two conditions are trivial, so we focus on the last one.

▷ We first show that there exists a consumer i whose income at the limit price is strictly positive: $m^i(\bar{p}) > 0$.

  * It suffices to show that $\sum_{i \in \mathcal{I}} m^i(\bar{p}) > 0$ since then there has to exist at least one consumer with $m^i(\bar{p}) > 0$.
  * To see this, consider the following sequence of equations:

$$\begin{aligned}\sum_{i \in \mathcal{I}} m^i(\bar{p}) &= \sum_{i \in \mathcal{I}} \left(\bar{p} \cdot e^i + \sum_{j \in \mathcal{J}} \theta^{ij} \Pi^j(\bar{p}) \right) \\ &= \sum_{i \in \mathcal{I}} \bar{p} \cdot e^i + \sum_{j \in \mathcal{J}} \Pi^j(\bar{p}) \\ &\geq \bar{p} \cdot \sum_{i \in \mathcal{I}} e^i + \sum_{j \in \mathcal{J}} \bar{p} \cdot \bar{y}^j \\ &> 0\end{aligned}$$

  where the first inequality follows from the fact that $\bar{y}$ is a feasible production choice and the second inequality follows from Theorem 5.3.

▷ We then show that this consumer i's demand for some good k' is unbounded above.

  * Suppose by contradiction that the demand $x^\ell := x^i(p^\ell, m^i(p^\ell))$ is bounded with $\{p^\ell\} \to \bar{p}$. Then there exists a converging subsequence of $\{x^\ell\}$ such that $x^\ell \to x^*$. So we assume without any loss (by reindexing the subsequence) that the original sequence of demands converges to x^* as well.
  * Since the budget constraint binds (given the property of utility function), it must be that $p^\ell \cdot x^\ell = m^i(p^\ell)$ for all values of ℓ.
  * Furthermore, $m^i(\cdot)$ is continuous, so $\bar{p} \cdot x^* = m^i(\bar{p})$ which is > 0 for this consumer i.
  * Now, consider a new bundle $\hat{x} = x^* + (0, ..., 0, 1, 0, ..., 0)$ where 1 is at the kth component.
    - Since u^i is strongly increasing, $u^i(\hat{x}) > u^i(x^*)$ and $\bar{p} \cdot \hat{x} = \bar{p} \cdot x^* = m^i(\bar{p}) > 0$.
    - Since u^i is continuous, there exists a sufficiently large $t \in (0, 1)$ so that $\bar{p} \cdot t\hat{x} < \bar{p} \cdot x^* = m^i(\bar{p})$ yet $u^i(t\hat{x}) > u^i(x^*)$.
    - Since $\{p^\ell\} \to \bar{p}$ and $\{x^\ell\} \to x^*$ and u^i is continuous, there exists a sufficiently large ℓ such that $p^\ell \cdot t\hat{x} < p^\ell \cdot x^\ell = m^i(p^\ell)$ and $u^i(t\hat{x}) > u^i(x^\ell)$. This is a *contradiction* to the fact that x^ℓ is the optimal bundle given price since $t\hat{x}$ achieves a higher level of utility while being feasible at the same time.

▷ Therefore, we conclude that there exists k' such that $\{x^\ell_{k'}\}$ is unbounded above, and the last condition in Theorem 5.3 is indeed satisfied.

3.5 Theorem 5.14: First Welfare Theorem with Production

Theorem. *(5.14 FWT with Production) If each u^i is strictly increasing, then every WEA is Pareto-efficient.*

Solution. We prove by contradiction.

- ▷ Suppose $\hat{x}, \hat{y}$ is a WEA at prices $p^* \in \mathbb{R}^n_+$ but is not Pareto-efficient. Since it is a WEA, it is feasible and markets clear. Furthermore, there exists $\tilde{x}, \tilde{y}$ such that they are feasible and

$$u^i\left(\tilde{x}^i\right) \geq u^i\left(\hat{x}^i\right) \forall i \text{ with at least one strict}$$

- ▷ Since $(\hat{x}, \hat{y})$ is a WEA, the consumer is already maximizing his utility, so it must be that by Lemma 5.2

$$p^* \cdot \tilde{x}^i \geq p^* \cdot \hat{x}^i, \forall i \text{ with at least one strict}$$

- ▷ Adding across all agents, it must be that

$$p^* \cdot \sum_{i \in \mathcal{I}} \tilde{x}^i > p^* \cdot \sum_{i \in \mathcal{I}} \hat{x}^i$$

- ▷ Since these are feasible allocations, above implies

$$p^* \cdot \left(\sum_i e^i + \sum_j \tilde{y}^j\right) > p^* \cdot \left(\sum_i e^i + \sum_j \hat{y}^j\right) \Leftrightarrow p^* \cdot \sum_j \tilde{y}^j > p^* \cdot \sum_j \hat{y}^j$$

  which implies that $\exists j$ such that $p^* \cdot \tilde{y}^j > p^* \cdot \hat{y}^j$. This ia contradiction to the profit-maximizing assumption!

Key Concepts:

- ▷ Contrast this with the proof for FWT without production. The last bullet point is modified such that there are only endowments without production. Once again, we obtain a contradiction.

3.6 Theorem 5.15: Second Welfare Theorem with Production

Theorem. *(5.15 SWT with Production) Suppose we have A5.1 & A 5.2 and the same assumptions for a WEA ($\exists \bar{y}^1 \in Y^1, ..., \bar{y}^J \in Y^J$ such that $\sum_j \bar{y}^j + \sum_i e^i \gg 0$). If $(\hat{x}, \hat{y})$ is PO, there exists income transfers $T_1, ..., T_I$ such that $\sum_i T_i = 0$ and $p^* \gg 0$ such that (1) $\hat{x}^i$ maximizes $u^i\left(x^i\right)$ subject to $p^* x^i \leq m^i\left(p^*\right) + T_i, \forall i$ and (2) $\hat{y}^i$ maximizes $p^* y^i$ subject to $y^j \in Y^j, \forall j$*

Solution. We prove by contradiction in three major steps. Suppose in addition that $\nabla u^i\left(x^i\right) \gg 0$ and $\hat{x}^i \gg 0, \forall i \in \mathcal{I}$.

▷ Step 1: Consumer Choice

* By Pareto efficiency, for each i, there exists $\lambda_i > 0$ so that

$$\frac{1}{\lambda_i} \nabla u^i\left(\hat{x}^i\right) = \frac{1}{\lambda_I} \nabla u^I\left(\hat{x}^I\right)$$

* So let $p^* = \frac{1}{\lambda_i} \nabla u^i\left(\hat{x}^i\right)$ which is independent of i. Note that $p^* \gg 0$. The quasi-concavity of u^i implies that the above FOC is sufficient to establish that $\hat{x}^i$ is the solution to the consumer i's problem.

▷ Step 2: Producer Choice

* By Pareto efficiency, $\forall i, \forall j, \nabla u^i\left(\hat{x}^i\right) \hat{y}^j \geq \nabla u^i\left(\hat{x}^i\right) y^j, \forall y^j \in Y^j$.
  - If there exists $\tilde{y}^j$ such that $\nabla u^i\left(\hat{x}^i\right)\left(\tilde{y}^j - \hat{y}^j\right) > 0$, then you can move in the direction of $\nabla u^i\left(\hat{x}^i\right)$, still be feasible, and attain higher utility.
* Above reasoning gives us $p^* \hat{y}^j \geq p^* y^j, \forall y^j \in Y^j$.

▷ Step 3: Defining T_is

* Define T_i such that

$$p^* \hat{x}^i = p^* e^i + \sum_j \theta_{ij}\left(p^* \cdot \hat{y}^j\right) + T_i$$

then (1) in the condition is met. Futhermore, summing the LHS and RHS over all i's ensures that $\sum_i T_i = 0$.

4 Matching

Does a stable matching function always exist? As we will explore, under some conditions, the answer is "yes."

4.1 Setup

Definition 4.1. A matching is an idempotent mapping $\mu : M \cup W \to M \cup W$ such that $\mu(m) \in W \cup \{m\}$ and $\mu(w) \in M \cup \{w\}, \forall m, w$.

Definition 4.2. A matching μ is *individually rational* (IR) or *not blocked by any individual* if and only if

$$\mu(x) >_x x \text{ for all } x : \mu(x) \neq x$$

In words, this means if the person is matched to you, that person is preferred to staying single (i.e. matched to yourself).

Definition 4.3. A matching μ is *blocked* by m and w if and only if $w >_m \mu(m)$ and $m >_w \mu(w)$ i.e. μ matches m to $\mu(m)$ but he prefers w and μ matches w to $\mu(w)$ but she prefers m.

Definition 4.4. A matching μ is *stable* if it is (1) individually rational and (2) not blocked by any pair.

Definition 4.5. m is *acceptable* to w if and only if $m >_w w$.

4.2 Theorem 5.1: Existence of a Stable Matching

Theorem 4.1. *A stable matching always exists in a one-to-one two-sided matching.*

Solution. We will define an algorithm whose output will be stable.

Algorithm 1. *(Men-Proposing DAA)*

- ▷ *All men & women start unengaged.*
- ▷ *Step k: Each unengaged one proposes to his most preferred acceptable woman among those who have not yet rejected him, if any.*
- ▷ *Step $k+1$: Each woman rejects all but her most preferred proposal and any unacceptable men.*
  - * *If a woman does not reject a man's proposal, each of them is engaged to the other. However, an engaged woman who receives proposals can reject the betrothed if she receives a proposal from a man she prefers more.*
- ▷ *Algorithm stops when no one proposes. When the algorithm stops, all engaged couples are matched and all singles are matched to themselves, so this produces a matching. This produces a matching that we denote as μ_M.*

It suffices to show that μ_M is well-defined and stable.

1. μ_M stops in finite steps: Each man can make at most $|W|$ proposals and no more than $|W||M|$ rounds.
2. The outcome is a matching: Each pair of engaged individual is only engaged to noe another, and if not engaged, you are matched to yourself.
3. μ_M is individually rational: No-one will be accepted to an unacceptable partner.
4. μ_M is not blocked by any pair.
   - ▷ Consider any pair (m, w) and such that $w >_m \mu_M(m)$ i.e. m prefers w to the matching that he has. We must show that $\mu_M(w) >_w m$ i.e. w would be worse off. If not, then the pair would block μ_M.
   - ▷ Given the nature of the algorithm, it must be that w rejected m at some point in the DAA. This happens if
     - * m is unacceptable to w: In this case, then w would not be a part of blocking pair.
     - * w was engaged or received a proposal from $m' \in M$ whom she preferred over m: In this case, note that the propsects of w improves (weakly) as the algorithm proceeds. Consequently,

$$\mu(w) >_w m' >_w m$$

       and hence w would not be a part of any blocking pair.
   - ▷ We thus conclude that no pair would block μ.

Reny's Comments:

- ▷ The existence result does not depend on strict preferences (the next Theorem does).
- ▷ The convergence of this algorithm depends on the fact that the prospects of w is weakly improving. This is analogous to Lyapunov functions, which are real-valued functions of a system's state which are monotonically non-increasing on every signal from the system's behavior set.
- ▷ A fixed point based approach to the theory of stable matchings is also possible; since things are discrete here, we would rely on Knaster-Tarski fixed point theorem rather than the one by Brauer.

4.3 Theorem: Pareto Optimality of μ_M I

Definition 4.6. For any pair $i, j \in M \cup W$ on opposite sides of the market, say i is achievable for j if and only if i and j are matched to one another in some stable matching.

Theorem 4.2. *Given strict preferences, the stable matching μ_M is Pareto preferred by the men to every other stable matching.*

Solution. It suffices to show that in men-proposing DAA, no man is ever rejected by an "achievable" woman. Otherwise, this would mean each man is matched to his most preferred achievable mate. To prove this, we use induction.

- ▷ Assume at step in men-proposing DAA, when it is the woman's turn and up to now, no man has been rejected by an achievable woman. Note that this is trivially true at step 0.
- ▷ Suppose w rejects m at this step. It suffices to show that w is not achievable for m.
  - * If m is not acceptable to w, then w is not achievable, so we are done.
  - * Otherwise, w must be rejecting m in favor of some m' to whom she is engaged. This implies $m' >_w m$
  - * By the induction hypothesis, m' was not rejected until now so it must be that m' was rejected by woman above w. So for m', w is strictly preferred to any woman except those who have so far rejected him, all of whom are unachievable for m'.
  - * Now by way of contradiction, suppose that w is achievable for m i.e. there exists a stable μ' such that $\mu'(m) = w$. Since μ' is stable, m' is also matched with an achievable mate, $\mu'(m')$ but we know that

    $$w >_{m'} \mu'(m')$$

    and that

    $$m' >_w m = \mu'(w)$$

    which implies that (m', w) would block μ', thereby contradicting the assertion that μ' is unstable. Hence, we conclude that w was not achievable for m.

4.4 Theorem: Pareto Optimality of μ_M II

Theorem 4.3. *Given strict preferences, if μ and μ' are stable, then μ is Pareto preferred by the men to μ' if and only if μ' is Pareto preferred by the women to μ.*

Solution. Suppose μ, μ' are stable and μ is Pareto preferred by the men to μ'. We prove by contradiction.

▷ Suppose by contradiction that μ' is not Pareto preferred by the women to μ. Because $\mu' \neq \mu$ (since μ is Pareto preferred by the men to μ'), there exists a woman $w \in W$ who strictly prefers μ to μ' i.e.

$$\mu\left(w\right) >_W \mu'\left(w\right)$$

▷ Since μ' is stable, it has to be individually rational so

$$\mu'\left(w\right) \geq_W w$$

▷ By the above inequality, it then must be that

$$\mu\left(w\right) >_W w$$

▷ Therefore, $\exists m \in M$ such that w is matched to m under μ but not in μ' i.e.

$$m = \mu(w) \neq \mu'\left(w\right)$$

This equivalently manes that m is matched to w under μ but not in μ' i.e.

$$\mu'\left(m\right) \neq w = \mu(m)$$

▷ Recall that the man Pareto prefers μ to μ' so he would rather be with w than the woman paired in w' i.e. $w = \mu(m) >_m \mu'(m)$. We also know that the woman prefers μ to μ' so she would prefer this man m.

▷ But this means that (m, w) blocks μ' and we have a contradiction.

Reny's Comments:

▷ Corollary of this theorem and the previous theroem is that

$$\mu_W \succsim_W \mu \succsim_W \mu_M$$

where μ is any stable matching.

▷ To see why strict preferences matter, consider:

$$\begin{aligned} >_{m_1} &: w_1, w_2 \\ >_{m_2} &: w_1, w_2 \\ >_{w_1} &: m_1 \sim m_2 \\ >_{w_2} &: m_1, m_2 \end{aligned}$$

and the following pairings:

$$\begin{aligned} \mu &: (m_1, w_1), (m_2, w_2) \\ \mu' &: \left(m_1, w_2\right), \left(m_2, w_1\right) \end{aligned}$$

5 Social Choice

5.1 Setup

The motivating question here is the following: is it possible to "coherently" aggregate individual preference relations into a preference relation for society? Here we introduce the setup to explore this question.

- ▷ X: finite set of social states / N: number of individuals where each individual i has a preference relation (ranking) over X denoted by R^i (instead of $\succsim^i$) and the strict relation is denoted P^i (instead of $\succ^i$).
- ▷ Let $\mathcal{R}$ be the set of all possible preference relations on X.

Our question: given any $R^1, ..., R^N$, can we "coherently" aggregate them to some $R \in \mathcal{R}$? Let's explore with a majority voting example.

Example 5.1. (Majority Voting) Suppose we have a set $X = \{x, y, z\}$ and the following breakdown of individuals:

$$20 : x, y, z \qquad 35 : y, z, x \qquad 45 : z, x, y$$

Then according to the majority vote, we have

$$xPy, \qquad zPx, \qquad yPz$$

so we see that the transitivity requirement does not hold.

Example 5.2. (Borda Rule) For any $x \in X$, the Borda score is given by

$$\mathbb{B}(x) = \sum_{i=1}^{N} \# \left\{ y \in X : xR^i y \right\}$$

which is the score for a state x given by the sum across individuals of the number of states ranked (weakly) below x. The Borda rule is defined by

$$\mathbb{B}(x) \geq \mathbb{B}(y) \Leftrightarrow xR^i y$$

which always satisfies transitivity, and therefore a valid social choice rule. In the example above, we have

$$\begin{aligned}
\mathbb{B}(x) &= 20 \times 3 + 35 \times 1 + 45 \times 2 = 185 \\
\mathbb{B}(y) &= 20 \times 2 + 35 \times 3 + 45 \times 1 = 190 \\
\mathbb{B}(z) &= 20 \times 1 + 35 \times 2 + 45 \times 3 = 225
\end{aligned}$$

thus implying $zPyPx$.

Definition 5.1. *(Social welfare function).* Given any $\mathcal{D} \subseteq \mathcal{R}$,we may say that $f : \mathcal{D}^N \to \mathcal{R}$ is a social welfare function (on $\mathcal{D}$). It is a mapping from individaul preference relations on X to a social preference relation on X.

Assumption 1. *(Arrow's requirements on the social welfare function).*

1. Unrestricted domain (U).

   The domain of f must include all possible combinations of individual preferences on X.

   $$f : \mathcal{R}^n \to \mathcal{R}.$$

   The induced social welfare function $R = f\left(R^1, R^2, \ldots, R^N\right)$ must be a preference relation (complete and transitive).

2. Weak Pareto Principle (WP).

   Whenever all individuals strictly prefer x over y, the social preference relation should also prefer x over y. That is, $\forall x, y \in X$, $\forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N$,
$$\left[xP^i y, \forall i\right] \Rightarrow [xPy].$$
   where xPy means xRy but not yRx.

3. Independent of Irrelevant Alternatives (IIA).

   Let $R = f(R^1, R^2, \ldots, R^N)$ and $\tilde{R} = f(\tilde{R}^1, \tilde{R}^2, \ldots, \tilde{R}^N)$. If every individual ranks $x, y \in X$ under R^i in the same way as under $\tilde{R}^i$, then R and $\tilde{R}$ must rank x and y in the same way. Society's rankings does not depend on z or w or anything else. [1] *That is, $\forall x, y \in X$, $\forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N$, $\forall \left(\tilde{R}^1, \ldots, \tilde{R}^N\right) \in \mathcal{R}^N$,*
$$\left[xR^i y \Leftrightarrow x\tilde{R}^i y, \forall i\right] \Rightarrow \left[xRy \Leftrightarrow x\tilde{R}y\right].$$

4. Non-dictatorship (D).

   There is no individual i such that if i prefers x over y, then the social preference relation prefer x over y regardless of the preferences R^j of all other individuals $j \neq i$. That is, $\forall x, y \in X$,
$$\nexists i : xP^i y \Rightarrow xPy, \ \forall \left(R^1, \ldots, R^N\right) \in \mathcal{R}^N.$$

Note that the Borda rule fails IIA – you can see this if you exchange the ranking of z and x for the second group of individuals.

Definition 5.2. *(Social choice function).* A social choice function is a functional mapping $c : \mathcal{R}^n \to X$ that is onto i.e. $\forall x \in X, \exists \left(\mathcal{R}^1, ..., \mathcal{R}^n\right) \in \mathcal{R}^n$ such that $c\left(\mathcal{R}^1, ..., \mathcal{R}^n\right) = x$. We say it is *dictatorial* if and only if there exists i such that
$$c\left(\mathcal{R}^1, ..., \mathcal{R}^n\right) \mathcal{R}^i y, \forall y \in X, \forall \left(\mathcal{R}^1, ..., \right)$$

Definition 5.3. *(Dictatorial).* A social choice function $c : \mathcal{R}^N \to X$ is dictatorial if and only if there exists i such that
$$c\left(\mathcal{R}^1, ..., \mathcal{R}^N\right) R^i y, \forall \mathcal{R}^1, ..., \mathcal{R}^N \in \mathcal{R}, \forall y \in X$$
i.e. there exists an individual who always ranks the social choice as its top for any set of preferences.

Definition 5.4. *(Strategy-Proof).* A social choice function $c : \mathcal{R}^N \to X$ is *strategy-proof* if and only if
$$\forall i, \forall \mathcal{R}^i, \tilde{\mathcal{R}}^i \in \mathcal{R}, \forall \mathcal{R}^{-i} \in \mathcal{R}^{N-1}, \forall x, y \in X$$
we have
$$\left[c\left(\mathcal{R}^i, \mathcal{R}^{-i}\right) = x \text{ and } c\left(\tilde{\mathcal{R}}^i, \mathcal{R}^{-i}\right) = y\right] \Rightarrow x\mathcal{R}^i y$$
In other words, no matter what other people report as their preferences, you never gain from lying about your preference.

[1] Put differently, IIA requires that if individuals' ranking of states $z \in X$ distinct from x and y changes relative to x and y, the social ranking between x and y does not change.

Definition 5.5. *(Pareto-Efficient).* A social choice function $c(\cdot)$ is *Pareto efficient* if and only if

$$\forall x \in X, \left[xP^i y, \forall y \in X \setminus \{x\}\right] \Rightarrow c\left(\mathcal{R}^1, ..., \mathcal{R}^N\right) = x$$

In other words, a social choice function $c(\cdot)$ is *Pareto efficient* if $c\left(R^1, R^2, \ldots, R^N\right) = x$ whenever $xP^i y$ for every individual i and every $y \in X$ distinct from x.

Definition 5.6. *(Monotonic social choice function).* A social choice function $c(\cdot)$ is *monotonic* if

$$\left[xR^i y \Rightarrow x\tilde{P}y,\ \forall i,\ \forall y \in X \setminus \{x\}\right] \Rightarrow \left[c\left(R^1, \ldots, R^N\right) = x \Rightarrow c\left(\tilde{R}^1, \ldots, \tilde{R}^N\right) = x\right].$$

Monotonicity says that the social choice does not change when individual preferences change so that every individual strictly prefers the social choice to any distinct social state that it was originally at least as good as. Loosely speaking, monotonicity says that the social choice does not change when the social choice rises in each individual's ranking. Notice that the individual rankings between pairs of social states other than the social choice are permitted to change arbitrarily.

5.2 Theorem 5.1: Impossibility Theorem

Theorem 5.1. *(Arrow's Impossibility Theorem) If there are at least three social states in X (and $N \geq 2$), then there is no social welfare function f that simultaneously satisfies conditions U, WP, IIA, and D.*

Solution. The general idea is to show that assumptions U, WP and IIA together imply the existence of dictator. We assume X is finite in this case. Suppose there exists a social welfare function $f : \mathcal{R}^n \to \mathcal{R}$ (which implies unrestricted domain) that satisfies WP and IIA. Then it suffices to show that f does not satisfy D.

▷ ***Step 1.*** First, note that assumptions U means that we are free to choose any individual preference relations. In particular, we are not restricted considering individual preference relations that we observe in the "data". With this in mind, let $c \in X$ be such that every individual ranks c at the bottom of his ranking; i.e. for all $i \in \mathcal{I}$, xP^ic for any $x \in X$ distinct from c. Then, by WP, xPc for any $x \in X$ distinct from c; i.e. c must be at the bottom of the social ranking.

R^1	R^2	$\cdots$	R^N	R
x^1	x^2	$\cdots$	x^N	$\vdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
c	c	$\cdots$	c	c

▷ **Step 2**. Consider the following procedure. First, we move c to the top of individual 1's ranking, leaving the ranking of all other states unchanged. Next, we move c to the top of individual 2's ranking, leaving the ranking of all others unchanged. And so on. Of course, if we move c to the top of every individual's ranking, WP implies that cPx for any $x \in X$ distinct from c. Thus, there must be a "first time" (i.e. an individual n) that c moves up from the bottom of the ranking to somewhere above. We now show that when c moves, it moves to the top of the social ranking—strictly preferred over all other social states.

* Suppose, by way of contradiction, that c does not move to the very top of the ranking. Then, there must exist $\beta \in X\setminus\{c\}$ such that $cR\beta$. Define β as the least preferred among all those; i.e.

$$\beta \in \{x \in X\setminus\{c\} : zRx,\ \forall z \in X\setminus\{c\}\ s.t.\ cRz\}.$$

  Similarly, if c does not move to the very top of the ranking, then there must exist $\alpha \in X\setminus\{c,\beta\}$ such that αRc. Define α as the most preferred among all those; i.e.

$$\alpha \in \{x \in X\setminus\{c,\beta\} : xRz,\ \forall z \in X\setminus\{c\}\ s.t.\ zRc\}.$$

  (We can find such an α since we assume that there are at least three social states in X.) Since R is a preference relation, it satisfies transitivity, which, in turn, implies that

$$\alpha Rc R\beta \Rightarrow \alpha R\beta, \tag{1}$$

  where each of α, c and β are distinct by construction.

* Now, consider switching the ranking of α and β of individuals (as necessary) so that $\beta P^i\alpha$ for all i. Notice that since c is either at the top or at the bottom of every individual's ranking, by IIA, such a switch does not affect the ranking of c relative to any $x \in X\setminus\{c\}$. Hence, IIA implies that (1) remains true after such a switch. However, since $\beta P^i\alpha$ for all i, WP implies that

$$\beta P\alpha,$$

  which contradicts (1). Thus, we conclude that when c moves away from the bottom of the ranking, it must move to the top of the ranking.

Figure A. Just before c moves

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	x^n	x^{n+1}	$\cdots$	x^N	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	c	c	$\cdots$	c	c

Figure B. Just after c moves

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	c	x^{n+1}	$\cdots$	x^N	c
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	c	$\cdots$	c	$\vdots$

Figure C. "New profile"

R^1	$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	R^N	R
c	$\cdots$	c	a	x^{n+1}	$\cdots$	x^N	a
$\vdots$	$\cdots$	$\vdots$	c	$\vdots$	$\cdots$	$\vdots$	c
$\vdots$	$\cdots$	$\vdots$	b	$\vdots$	$\cdots$	$\vdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	c	$\cdots$	c	$\vdots$

- ▷ **Step 3**. Consider now any two distinct social states a and b which are themselves distinct from c. Change individual n's ranking so that $aP^n cP^n b$. For the other individuals, ranking of a and b can be arbitrary so long as the position of c is unchanged for each individual (i.e. $cP^i a$ for all $i = 1, 2, \ldots, n-1$ and $aP^i c$ for all $i = n+1, n+2, \ldots, N$). We refer to this profile of preferences as the "new profile" (think of this as $\tilde{R}^i$'s).

  * Now compare Figure A and Figure C. Notice that the ranking of a and c under Figure C is the same for every individual as it was just before raising c to the top of individual n's ranking in step 2 (in Figure A). Thus, by IIA, the social ranking of a and c must be the same under the new profile as it was when c was at the bottom of the social ranking; i.e. aPc.
  * Comparing Figure B and C, we can see that the ranking of c to b in Figure C is the same for every individual as it was just after raising c to the top of every individual n's ranking in step 2 (in Figure B). Thus, by IIA, cPb.
  * By transitivity, we have aPb. Notice that, no matter how others rank a and b, the social ranking agrees with individual n's ranking. By IIA, since a and b were arbitrary, for any states a and b distinct from c, we have

  $$aP^n b \Rightarrow aPb, \ \forall a, b \in X \setminus \{c\}.$$

  That is, individual n is a dictator on all pairs of social states that does not involve c. It remains to show that n is also a dictator in pairs of social states that involve c. That is, we need to show that n is a dictator for the pair (c, x) for some $x \in X \setminus \{c\}$.

- ▷ **Step 4**. Let d be distinct from c and x (we are using the fact that $|X| \geq 3$ again). Following the same steps as above with d playing the role of c, we can conclude that some individual m (perhaps not the same n) is a dictator on all pairs not involving d. In particular, this means that m is a dictator for $\{x, c\}$. But in Figure A and B, we only changed n's ranking of c and all other $x \in X \backslash c$ and the result was a change in the society's ranking between the two. For the two to be both true, it must be that $m = n$. That is n is a dictator in all pairs of social states.

5.3 Lemma: PE and Monotonicty of Strategy-Proof Choice Function

Theorem 5.2. *A strategy-proof social choice function is Pareto efficient and monotonic.*

Solution. *(Monotonicity).* Let $(R^1, R^2, \ldots, R^N)$ be an arbitrary preference profile and suppose that $c(R^1, R^2, \ldots, R^N) = x$. Fix an individual, say i, and let $\tilde{R}^i$ be a preference for i such that, for every $y \in X \setminus \{x\}$, $xR^iy \Rightarrow x\tilde{P}^iy$.

▷ We first to show that $c(\tilde{R}^i, R^{-i}) = x$. Suppose, by way of contradiction, that $c(\tilde{R}^i, R^{-i}) = y \neq x$. Then, given that others report R^{-i}, individual i, when his preferences are R^i, can report truthfully and obtain the social state x or he can lie by reporting $\tilde{R}^i$ and obtain the social state y. Strategy-proofness requires that lying cannot be strictly better than telling the truth. Hence, we must have xR^iy. According to the definition of $\tilde{R}^i$, we then have $x\tilde{P}^iy$. Consequently, when individual i's preferences are $\tilde{R}^i$, he strictly prefers x to y and so, given that the others report R^{-i}, individual i strictly prefers lying (reporting R^i and obtaining x) to telling the truth (reporting $\tilde{R}^i$ and obtaining y), contradicting strategy-proofness. Thus, $c(\tilde{R}^i, R^{-i}) = x$.

▷ Let $(R^1, R^2, \ldots, R^N)$ and $(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N)$ be preference profiles such that $c(R^1, R^2, \ldots, R^N) = x$ and such that for every individual i and every $y \in X \setminus \{x\}$, $xR^iy \Rightarrow x\tilde{P}^iy$. To prove that $c(\cdot)$ is monotonic, we must show that $c(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N) = x$. But this follows immediately from the result just proven—simply change the preference profile from $(R^1, R^2, \ldots, R^N)$ to $(\tilde{R}^1, \tilde{R}^2 \ldots, \tilde{R}^N)$ by switching, one at a time, the preferences of each individual i from R^i to $\tilde{R}^i$. We conclude that $c(\cdot)$ is monotonic.

▷ (Pareto efficiency). Let x be an arbitrary social state and let $\hat{R}$ be a preference profile with x at the top of each individual's ranking. We must show that $c\left(\hat{R}\right) = x$.

▷ Because the range of $c(\cdot)$ is all of X, there is some preference profile R such that $c(R) = x$. Obtain the preference profile $\tilde{R}$ from R by moving x to the top of every individual's ranking. By monotonicity (proven above), $c(\tilde{R}) = x$. Because $\hat{R}$ places x at the top of every individual ranking and $c(\tilde{R}) = x$, we can again apply monotonicity (trivially, we have $x\tilde{R}^iy \Rightarrow x\hat{P}y,\ \forall i,\ \forall y \in X \setminus \{x\}$) and conclude that $c(\hat{R}) = x$, as desired.

5.4 Theorem 5.3: Gibbard-Satterthwaite Theorem

Theorem 5.3. *(Gibbard-Satterthwaite) If $|X| > 3$ then every strategy-proof social function is dictatorial.*

Solution. Similar to the way we proved the Arrow's Impossibility Theorem, we will show that $|X| \geq 3$, monotonicity and Pareto efficiency implies existence of a dictator. Although we use strict preferences in the proof, we are not ruling out indifference—it just so happens that we can prove the desired result by considering a subset of preferences (that do not exhibit indifference).We first establish the following two results which will use many times in the proof of the Theorem.

Claim 5.1. Suppose $c(R^1, \ldots, R^N) = x$ and that, for some individual i, R^i ranks y just below x. Let $\tilde{R}^i$ be identical to R^i except that the ranking of y and x are reversed. Then, $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$.

Solution 1. Suppose, by way of contradiction, that $c(\tilde{R}^i, R^{-i}) = z \notin \{x, y\}$. Note that $\tilde{R}^{-i} = R^{-i}$ as we only altered i's preference relation. Moreover, since we only changed the ordering between x and y for individual i, the ordering between z and any other element in X is unchanged. Then,

$$z\tilde{R}^j c \Rightarrow zR^j c,\ \forall j,\ \forall c \neq z.$$

Since we are dealing with strict preferences, in fact,

$$z\tilde{P}^j c \Rightarrow z\tilde{R}^j c \Rightarrow zP^j c,\ \forall j,\ \forall c \neq z.$$

Using monotonicity,

$$c(\tilde{R}^i, R^{-i}) = z \Rightarrow c(R^i, R^{-i}) = z.$$

But this contradicts the assumption that $c(R^1, \ldots, R^N) = x$. Hence, $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$.

Claim 5.2. Suppose $c(R^1, \ldots, R^N) = x$. Let $\tilde{R} = (\tilde{R}^1, \ldots, \tilde{R}^N)$ be strict rankings such that, for every individual i, the ranking of x versus any other social state is the same as under $\tilde{R}^i$ as it is under R^i. Then, $c(\tilde{R}) = x$.

Solution 2. Since the ranking of x has not changed, we have

$$xR^i y \Rightarrow x\tilde{R}^i y,\ \forall i,\ \forall y \in X \setminus \{x\}.$$

Since $\tilde{R}$ is a strict ranking, we may write

$$xR^i y \Rightarrow x\tilde{P}^i y,\ \forall i,\ \forall y \in X \setminus \{x\}.$$

Then, by monotonicity, since $c(R) = x$, we must have $c(\tilde{R}) = x$.

Now, consider any two distinct social states $x, y \in X$ and a profile of strict rankings in which x is ranked the highest for y lowest for every individual $i = 1, 2, \ldots, N$. Pareto efficiency implies that the social choice at this profile is x.

- ▷ Suppose we change individual 1's ranking by strictly raising y one position at a time. By monotonicity, the social choice remains equal to x so long as y is below x in 1's ranking. But when y rises above x, Claim 5.1 implies that the social choice either changes to y or remain as x. If the latter occurs, then we can begin the same process for individual 2, 3 etc. until for some individual n, the social choice does change from x to y when y rises above x in n's ranking. (Notice that such an individual exists since following this procedure, we will eventually have y at the top of everyone's ranking and so, by Pareto efficiency, the social choice would be y.)
- ▷ In Figure A, we show the ordering just before we change the ordering between x and y for individual n while figure B shows the ordering just after we change n's ranking. By construction, social changes is x in Figure A and y in Figure B.

▷ In Figure C, we reorder Figure A such that (i) we moved x to the bottom of ranking for all individuals $i < n$, and (ii) we moved x to be just above y for all $i > n$. Furthermore, consider Figure D in which: (i) we moved x to the bottom of ranking for all individuals $i < n$, and (ii) we moved x to be just above y for all $i > n$.

▷ Comparing Figure B and D: observe that the the individuals' rankings of y versus any other social state are the same between the two. Thus, by Claim 5.2 social choice in Figure D must be the same as in Figure B; i.e. y.

▷ Now compare Figures D and C: The two differ only in individual n's ranking of x and y. Thus, by Claim 5.1, because the social choice under $\tilde{R}$ is y, the social choice in Figure C must either be x or y. But if the social choice in Figure C is y, then, since the ordering of y versus any other social states are the same as in Figure A, by Claim 5.2, the social choice in Figure A must also be y—a contradiction. Hence, the social choice in Figure C must be x.

Figure A. Just *before* c changes

$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	$c(\cdot)$
$\cdots$	y	x	x	$\cdots$	
$\cdots$	x	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	y	$\cdots$	

$\Rightarrow$

Figure C. Reordered

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^n$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	y	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	$\vdots$	$\vdots$	x	$\cdots$	
$\cdots$	x	$\vdots$	y	$\cdots$	

Figure B. Just *after* c changes

$\cdots$	R^{n-1}	R^n	R^{n+1}	$\cdots$	$c(\cdot)$
$\cdots$	y	y	x	$\cdots$	
$\cdots$	x	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	y
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	y	$\cdots$	

$\Rightarrow$

Figure D. Reordered

$\cdots$	$\tilde{R}^{n-1}$	$\tilde{R}^n$	$\tilde{R}^{n+1}$	$\cdots$	$\tilde{R}$
$\cdots$	y	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	y
$\cdots$	$\vdots$	$\vdots$	x	$\cdots$	
$\cdots$	x	$\vdots$	y	$\cdots$	

▷ Let us choose another social state $z \in X$ that is distinct from x and y (recall that there are at least three social states). Take Figure C and add z as in Figure E below. Observe that the ranking of x versus any other social state in any individual rankings have not changed—hence, by Claim 5.2, the social choice must remain the same as x in Figure E.

▷ Now consider Figure F in which we reversed the ordering between x and y for individuals $i > n$. These are the only differences between Figure E and F so that, by Claim 5.1, social choice in Figure F must either be x or y. But the social choice cannot be y because z is ranked above y for all individuals and monotonicity would then imply that the social choice would remain y even if z were raised to the top of every individual's ranking—contradicting Pareto efficiency. Hence, the social choice in Figure F must be x.

Figure E. Modified Figure C

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^{n}$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	$\vdots$	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	z	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	z	$\vdots$	z	$\cdots$	
$\cdots$	y	$\vdots$	x	$\cdots$	
$\cdots$	x	$\vdots$	y	$\cdots$	

Figure F. Reordered

$\cdots$	$\hat{R}^{n-1}$	$\hat{R}^{n}$	$\hat{R}^{n+1}$	$\cdots$	$\hat{R}$
$\cdots$	$\vdots$	x	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	z	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	x
$\cdots$	z	$\vdots$	z	$\cdots$	
$\cdots$	y	$\vdots$	y	$\cdots$	
$\cdots$	x	$\vdots$	x	$\cdots$	

- ▷ An arbitrary profile of strict rankings with x at the top of individual n's ranking can be obtained from the profile in Figure F without reducing the ranking of x versus any other social state in any individual's ranking. Hence, Claim 5.2 implies that the social choice must be x whenever individual rankings are strict and x is at the top of individual n's ranking. We now show that this implies that, even when individual rankings are not strict and indifferences are present, the social choice must be at least as good as x for individual n whenever x is at least as good as every other social state for individual n.

- ▷ First, note that R denotes preferences that are all strict, and, as we showed above, $c\,(R) = x$. Let $\tilde{R}$ denote preferences that are not necessarily strict and denote $c^* := c(\tilde{R})$. We want to show that $c^* \tilde{R}^n x$ when $x \tilde{R}^n y$ for all $y \in X$. By way of contradiction, suppose that $x\tilde{R}^n y$ for all y but $xP^n c^*$ (which implies that $c^* \neq x$). We define a new set of strict preferences for all individuals, by changing $z\tilde{R}^i y$ to $z\tilde{P}^i y$ for all i, for all $z, y \in X$ and $z \neq y$. In particular, for all i, and for all $y \neq c^*$, we change $c^*\tilde{R}^i y$ to $c^* \tilde{P}^i y$. Then, by monotonicity, since $c(\tilde{R}) = c^*$, we must also have $c(\tilde{P}) = c^*$. But this would imply that $c^*\tilde{P}^n y$ for all $y \neq c^*$—in particular, $c^*\tilde{P}^n x$—a contradiction. Thus, we must have $c^*\tilde{R}^n x$ when $x\tilde{R}^n y$ for all $y \in X$.

- ▷ With this result, we may now say that individual n is a dictator for the social state x. Because x was chosen arbitrarily, we have shown that, for each social state $x \in X$, there is a dictator for x. The final step is to show that there cannot be distinct dictators for distinct states. We now show that if (i) x and y are distinct states, (ii) the social choice is at least as good as x for individual n whenever x is at least as good as every other social state for n, and (iii) the social choice is at least as good as y for individual m whenever y is at least as good as every other social state for m, then $n = m$. Combined with the previous result, this would tells us that there cannot be two distinct dictators for distinct social states.

- ▷ Let $c^* := c\,(R)$ denote the social choice and $x \neq y$. By the hypothesis of the claim, we have: (i) $c^* R^i x$ whenever $xR^i w$ for all $w \in X$; (ii) $c^* R^m y$ whenever $yR^i w$ for all $w \in X$. By way of contradiction, suppose $n \neq m$. Consider a profile P in which x is strictly preferred to all other social states by n and y is strictly preferred to all other social states by m. By (i), this implies that $c^* = x$ and by (ii), $c^* = y$. But this contradicts the fact that $x \neq y$. Hence, it must be that $n = m$.

- ▷ We may finally conclude that there is a single dictator for all social states and therefore the social choice function is dictatorial.

Reny's Comments:

- ▷ You can apply this theorem to the men-proposing DAA. The men will report truthfully, but the women may not report truthfully.

6 Decision-Making Under Uncertainty

6.1 Setup

Now we are interested in preferences over gambles. Let $A = \{a_1, ..., a_n\}$ be a finite set of outcomes. A simple gamble (over A) assigns a probability p_i to each outcome a_i such that $\sum_i p_i = 1$. We'll denote this simple gamble by

$$(p_1 \circ a_1, p_2 \cdot a_2, ..., p_n \circ a_n)$$

and if $p_3 = p_4 = \cdots = p_n = 0$, we will write

$$(p_1 \circ a_1, p_2 \circ a_2)$$

Define $\mathcal{G}_S = \{(p_1 \circ a_1, p_2 \cdot a_2, ..., p_n \circ a_n) \,|p_1, ..., p_n \geq 0, \sum_i p_i = 1\}$. If $g^1, g^2 \in \mathcal{G}_S$ then

$$\left(\lambda \circ g^1, (1-\lambda) \circ g^2\right), \lambda \in (0,1)$$

is an example of a compound gamble. Denote $\mathcal{G}$ as the set of all compound gambles and

$$\begin{aligned} \mathcal{G}^1 &\equiv \mathcal{G}_S \\ \mathcal{G}^{n+1} &\equiv \left\{ g \equiv \left(\alpha_1 \circ g^1, ..., \alpha_k \circ g^k\right) : k \in \mathbb{N}, g^1, ..., g^k \in \mathcal{G}^n, \alpha_1, ..., \alpha_k \geq 0, \sum_i \alpha_i = 1 \right\} \end{aligned}$$

Then the set of all compound gambles is:

$$\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}^n$$

We have the following assumptions about our decision maker's binary relation $\succsim$ on $\mathcal{G}$:

Assumption. *(G1) $\forall g, g' \in \mathcal{G}$, either $g \succsim g'$ or $g' \precsim g'$.*

Assumption. *(G2) $\forall g, g', g'' \in \mathcal{G}$, if $g \succsim g'$ and $g' \succsim g''$ then $g \succsim g''$.*

We say that $u : \mathcal{G} \to \mathbb{R}$ represensts $\succsim$ if and only if $g \succsim g' \Leftrightarrow u(g) \succsim u(g')$. The steps up to this part are the same as the axioms relevant to the consumer's utility problem. Now assume

$$a_1 \succsim ..., \succsim a_n$$

Assumption. *(G3, Continuity) $\forall g \in \mathcal{G}, \exists \alpha \in [0,1]$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$. a_1 is the "best" and a_n is the "worst" so any gamble should be in betweeen the two endpoints – hence the continuity.*

Assumption. *(G4, Monotonicity) $\forall \alpha, \beta \in [0,1]$, $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n) \Leftrightarrow \alpha \geq \beta$. This also includes saying indifference comes if and only if $\alpha = \beta$. Note that this is also saying that $a_1 \succ a_n$ (if not, the decision maker's indifferent across all possible gambles).*

Assumption. *(G5, Substitution) Suppose $\forall g \in \left(p_1 \circ g^1, \cdots, p_k \circ g^k\right), h \in \left(p_1 \circ h^1, \cdots, p_k \circ h^k\right)$ are in $\mathcal{G}$ and $g^i \sim h^i, \forall i = 1, ..., k$, then $g \sim h$.*

Note the following reduction of a complex gamble to a simple gamble: For any $g \in \mathcal{G}$, there is a unique $p = (p_1, \cdots, p_n) : \sum_i p_i = 1$ such that p_i is the probability of a_i under g. For example, if

$$g^1 = \left(\frac{1}{3} \circ a_1, \frac{2}{3} \circ a_2\right), \quad g^2 = \left(\frac{1}{4} \circ a_1, \frac{3}{4} \circ a_2\right), \quad g = \left(\frac{1}{2} \circ g^1, \frac{1}{2} \circ g^2\right)$$

Then

$$p_1 = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4}, \quad p_2 = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{3}{4}$$

Assumption. *(G6, Reduction) $\forall g \in \mathcal{G}$, the decision maker is indifferent between g and the simple gamble that it induces.*

We say that $u : \mathcal{G} \to \mathbb{R}$ has the *expected utility property* if $\forall \left(p_1 \circ a_1, ..., p_n \circ a_n\right) \in \mathcal{G}_S$,

$$u\left(\left(p_1 \circ a_1, ..., p_n \circ a_n\right)\right) = \sum_{i=1}^{n} p_i u\left(a_i\right)$$

The LHS is obviously the utility of the simple gamble.

6.2 Theorem 2.7: Expected Utility Property

Theorem 6.1. *If $\succsim$ satisfies G1 – G6, then $\exists u : \mathcal{G} \to \mathbb{R}$ that represents $\succsim$ and has the EU property.*

Solution. We will proceed similarly as before by constructing explicitly such utility function. For any $g \in \mathcal{G}$, define $u(g)$ so that

$$g \sim (u(g) \circ a, (1 - u(g)) \circ a_n)$$

By G3 (continuity), there exists at least one such number, and by G4 (monotonicity), there is at most one such number. Thus we have a well-defined function that represents $\succsim$ – this defines the $u(g), \forall g$. It now suffices to show that it represents the utility and satisfies the EU property.

To see that it represents the preference, consider $g, g' \in \mathcal{G}$. Then

$$g \succsim g' \Leftrightarrow (u(g) \circ a, (1 - u(g)) \circ a_n) \succsim \left(u\left(g'\right) \circ a, \left(1 - u\left(g'\right)\right) \circ a_n\right)$$

$$\Leftrightarrow u(g) \geq u\left(g'\right) \quad \text{by G4}$$

Now we wish to show that it satisfies the EU property. Consider any simple gamble $(p_1 \circ a_1, ..., p_n \circ a_n)$. Then from the definition of u we have,

$$\forall i, \quad a_i \sim \underbrace{(u(a_1) \circ a_1, (1 - u(a_n)) \circ a_n)}_{\equiv h_i}$$

Since the decision maker is indifferent between a_i and p_i, we have

$$(p_1 \circ a_1, ..., p_n \circ a_n) \sim (p_1 \circ h_i, ..., p_n \circ h_n)$$

by G5 (substitution). For h_i, there are only two choices, so the RHS essentially reduces to

$$\sim \left[\left(\sum_{i=1}^{n} p_i u(a_i)\right) \circ a_1, \left(1 - \left(\sum_{i=1}^{n} p_i u(a_i)\right)\right) \circ a_n\right]$$

by G6 (reduction). Therefore,

$$u(p_1 \circ a_1, ..., p_n \circ a_n) = \sum_{i=1}^{n} p_i u(a_i)$$

by the definition of u.

6.3 Theorem: Affine Trasformation

Theorem 6.2. *(Affine Transformation) If $u(\cdot)$ and $v(\cdot)$ represent $\succsim$ on $\mathcal{G}$ and both have the EU property, then $\exists \alpha \in \mathbb{R}, \beta > 0$ such that*

$$v(a_1) = \beta u(a_1) + \alpha, \forall i = 1, ..., n$$

Solution. *Recall that $a_1 \succsim a_2 \succsim ... \succsim a_n$.*

- If $a_1 \sim a_n$, then $u(a_1) = \cdots = u(a_n)$ and $v(a_1) = \cdots = v(a_n)$ and so $\beta = 1, \alpha = v(a_1) - u(a_1)$.
- If $a_1 \succ a_n$, then $u(a_1) > u(a_n)$ and $v(a_1) > v(a_n)$.
  * For all i, there exists $\alpha_i \in [0, 1]$ such that
  $$u(a_i) = (1 - \alpha_i) u(a_1) + \alpha_i u(a_n)$$
  * Since u has the EU property, it implies
  $$a_i \sim ((1 - \alpha_i) \circ a_1, \alpha_i \circ a_n)$$
  since u represents $\succsim$.
  * This implies
  $$v(a_i) = ((1 - \alpha_i) \circ a_1, \alpha_i \circ a_n)$$
  since v represents $\succsim$ and
  $$= (1 - \alpha_i) v(a_1) + \alpha_i v(a_n)$$
  because v has the EU proepty.
  * Thus it must be that
  $$\alpha_i = \frac{v(a_n) - v(a_1)}{v(a_i) - v(a_1)}$$
  Then we can write:
  $$v(a_i) - v(a_1) = -\alpha_i v(a_1) + \alpha_i v(a_n)$$
  Rearranging:
  $$v(a_i) = \left(\frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)}\right) v(a_i) + v(a_1) - \left(\frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)}\right) v(a_1)$$

Reny's Comments:

- Notice that α_i and its quantitative magnitude actually has an interpretation. It is a characteristic of your preferences.

7 Strategic Form Games

7.1 Setup

As always, we start with the basic definitions.

Definition 7.1. A strategic form game is a tuple $G = (S_i, u_i)_{i=1}^{N}$ where $\{1, ..., N\}$ is the set of players; S_i is the player i's non-empty set of strategies, and $u_i : X_{j=1}^{N} S_j \to \mathbb{R}$ is a payoff function. In this game, G, all players must simultaneously choose a strategy in their strategy set.

Example 7.1. $N = 2$ and $S_1 = \{U, M, D\}$ and $S_2 = \{L, R\}$. Then we have the following payoff matrix:

	L	R
U	$(3, 0)$	$(0, 4)$
M	$(1, -1)$	$(-2, 2)$
D	$(2, 4)$	$(-1, 8)$

In this case, we verify that player 1 indeed has a dominant strategy. On the other hand, consider the following payoff:

	L	M	R
U	$(3, 0)$	$(0, -5)$	$(0, -4)$
C	$(1, -1)$	$(3, 3)$	$(-2, 2)$
D	$(2, 4)$	$(4, 1)$	$(-1, 8)$

Then we can see that there is no dominant strategy for both players. However, we can go a little further. We see that D is strictly better than C no matter what.

Definition 7.2. A strategy profile (joint strategy) $\hat{s} = (\hat{s}_1, ..., \hat{s}_N)$ is a pure strategy Nash equilibrium if and only if

$$\forall i, u_i (\hat{s}) \geq u_i (s_i, \hat{s}_i), \forall s_i \in S_i$$

where $s_{-i} \equiv (s_1, ..., s_{i-1}, s_{i+1}, ..., s_N)$. In the example:

	L	H
L	$(0, 0)$	$(1, 2)$
H	$(2, 1)$	$(0, 0)$

then (L, L) and (H, H) are not pure strategy equilibrium. By similar reasoning, (L, H) and (H, L) are indeed pure strategy equilibrium.

Example 7.2. (Penalty Shootout) Consider the following payoff matrix with player as P1 and the goalkeeper as P2:

		$1-q$	q
		L	R
$1-p$	L	$(-1, 1)$	$(1, -1)$
p	R	$(1, -1)$	$(-\delta, \delta)$

where $\delta \in (0,1)$. Once again, we need $p, q \in (0,1)$ for a NE. By the same logic, if we are willing to mix, it must be the case that we are indifferent. The indifference condition for player 1 is given as

$$L : -(1-q) + q$$
$$R : (1-q) - \delta q$$

Solving for q yields $q = 2/(3+\delta)$ and $p = 2/(3+\delta)$.

Definition 7.3. (Mixed Strategies) A mixed strategy for the player i is a probability distribution over S_i. We assume now, that each S_i is finite. Let M_i denote the set of mixed strategies:

$$M_i := \left\{ m_i : S_i \to [0,1] \mid \sum_{s_i \in S_i} m_i(s_i) = 1 \right\}$$

In other words, the mixed strategy assigns a probability to each pure strategy. We will assume that $\forall i, \forall s = (s_1, ..., s_N) \in S := X_{i=1}^N s_i$, $u_i(s)$ is the vNM utility of the outcome that results under s. So, if the players choose joint mixed strategy

$$m = (m_1, ..., m_N) \in M = X_{i=1}^N m_i$$

Then

$$u_i(m) \equiv \sum_{s \in S} m_1(s_1) \cdots m_N(s_N) u_i(s_1, ..., s_N)$$

Definition 7.4. (Pure-strategy Nash Equilibrium) A joint strategy $\hat{s} \in S$ is a pure strategy Nash equilibrium of $G = (S_i, u_i)_{i=1}^{\infty}$ if and only if $\forall i, \forall s_i \in S$, we have

$$u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{-i})$$

Definition 7.5. (Mixed-strategy Nash Equilibrium) A mixed strategy $\hat{m} \in M$ is a Nash equilibrium of $G = (S_i, u_i)_{i=1}^N$ if and only if $\forall i, \forall m_i \in M_i$, we have

$$u_i(\hat{m}) \geq u_i(m_i, \hat{m}_{-i})$$

Thus in a Nash equilibrium, each player may be randomizing his choices, and no player can improve his expected payoff by unilaterally randomizing any differently.

7.2 Theorem: Simplified Nash Equilibrium Tests

Theorem 7.1. *The following statements are equivalent:*

1. $\hat{m} \in M$ is a Nash equilibrium.
2. For every player i, $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ such that $\hat{m}_i(s_i) > 0$ and $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$ given zero weight by $\hat{m}_i$.
3. For every player i, $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for every $s_i \in S_i$.

Solution. We will show that 1 implies 2 and then 3 implies 1. 2 implies 3 is obvious.

1. First, we show that 1 implies 2.
   - ▷ Suppose that $\hat{m}$ is a Nash equilibrium.

Reny's Comments:

- ▷ (2) says that a player must be indifferent between all pure strategies that are given positive weight by his mixed strategy (which are finite) and that each of these must be no worse than any of his pure strategies given zero weight.
- ▷ (3) says that it's enough to check, for each player, that no pure strategy yields a higher expected payoff than his mixed strategy in order that the vector of mixed strategies forms a Nash equilibrium.
- ▷ Usefulness:

Example 7.3. Consider the following payoff matrix:

		q	$1-q$
		H	L
p	H	$(0,0)$	$(2,1)$
$1-p$	L	$(1,2)$	$(0,0)$

For there to be a non-degenerate mixed strategies, we consider $p, q \in (0,1)$. For there to be a Nash equilibrium, both players would have to be indifferent between H and L.

- ▷ For Player 1, the expected utility from choosing H and L must be equal

$$EU_1^H = q \cdot 0 + (1-q) \cdot 2 = EU_1^L = q \cdot 1 + (1-q) \cdot 0 \Rightarrow q = \frac{2}{3}$$

  and for Player 2, the computation is symmetric and has $p = 2/3$.

- ▷ The expected utility for each player is $2/3$ which is worse than the coordinated outcome. So I'd rather be predictable here than unpredictable.
- ▷ The key is that you need to mix in order to make the other guy indifferent between the two. If you don't, then the other person will look at your probability and rather play a pure strategy.

7.3 Theorem: Existence of Nash Equilibriuma

Theorem 7.2. *Every finite strategic form game possesses at least one Nash equilibrium.*

Solution. WLOG, we are going to assume that every player has the same number of pure strategies, i.e.

$$|S_i| = |S_j|, \forall i, j$$

Let $G = (S_i, u_i)_{i=1}^N$ be a finite strategic form game and suppose there are n pure strategies. The set of possible mixed strategies M_i is defined as

$$M_i := \left\{ m_i \in [0,1]^n : \sum_{j=1}^{n} m_{ij} = 1 \right\}$$

where m_{ij} denotes the probability that player i puts on strategy j. Then M_i is non-empty, compact, and convex. We will show that a Nash equilibrium exists by demonstrating the existence of a fixed point of a function whose fixed points are necessarily equilibria of G.

1. Constructing the function

   ▷ Define $f : M \to M$ as follows: for each player i and for each $j \in S_i$ and for any $m \in M$, define

   $$f_{ij}(m) := \frac{m_{ij} + \max(0, u_i(j, m_{-i}) - u_i(m))}{1 + \sum_{f=1}^{n} \max(0, u_i(j', m_{-i}) - u_i(m))}$$

   * Basically, you're checking to see if a pure strategy j is better than keeping the mixed strategy, and if so, assign more probability weight to such pure strategy.

   ▷ Then define

   $$f_i(m) = (f_{i1}(m), f_{i2}(m), ..., f_{iN}(m)) \in M_i$$

   and

   $$f(m) = (f_1(m), ..., f_N(m)) \in M$$

2. Proving the existence of a fixed point

   ▷ f_{ij} is a continuous function of m for each i and j because:

   * utility functions are continuous; max of continuous functions are continuous; and denominator is bounded away from zero.
   * Consequently, f is a continuous function mapping the non-empty, compact, convex set M into itself.

   ▷ Thus by Brouwer's fixed-point theorem, f indeed has a fixed point, denoted by $\hat{m}$.

3. Demonstrating that this fixed point is indeed a Nash equilibrium of G.

   ▷ By the result above, we have

   $$f(\hat{m}) = \hat{m} \Rightarrow f_{ij}(\hat{m}) = \hat{m}_{ij}$$

   ▷ Plugging in, we have

   $$\hat{m}_{ij} = \frac{\hat{m}_{ij} + \max(0, u_i(j, \hat{m}_{-i}) - u_i(\hat{m}))}{1 + \sum_{f=1}^{n} \max(0, u_i(j', \hat{m}_{-i}) - u_i(\hat{m}))}$$

   and rearranging:

   $$\hat{m}_{ij} \sum_{j'=1}^{n} \max\left(0, u_i\left(j', \hat{m}_{-i}\right) - u_i(\hat{m})\right) = \max(0, u_i(j, \hat{m}_{-i}) - u_i(\hat{m}))$$

▷ Multiply both sides by the $u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right)$ and sum over $j = 1$ to n:

$$\sum_{j=1}^{n} \hat{m}_{ij} \left\{u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right)\right\} \underbrace{\sum_{j'=1}^{n} \max\left(0, u_i\left(j', \hat{m}_{-i}\right) - u_i\left(\hat{m}\right)\right)}_{=\text{constant}} = \sum_{j=1}^{n} \left\{u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right)\right\} \max\left(0, u_i\left(j, \hat{m}_{-i}\right.\right.$$

Notice that the term

$$\begin{aligned}\sum_{j=1}^{n} \hat{m}_{ij} \left\{u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right)\right\} &= \sum_{j=1}^{n} \hat{m}_{ij} u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right) \\ &= u_i\left(\hat{m}\right) - u_i\left(\hat{m}\right) = 0\end{aligned}$$

which implies that the RHS must also be equal to zero.

▷ The RHS can be zero only if

$$u_i\left(j, \hat{m}_{-i}\right) - u_i\left(\hat{m}\right) \leq 0, \forall j$$

If it is greater than zero for any j, then the jth term in the sum is strictly positive, and since no term in the sum is negative, this would render the entire sum strictly positive.

Hence by part 3 of the previous theorem, we conclude that $\hat{m}$ is indeed a Nash equilibrium.

8 Games with Incomplete Information

8.1 Setup

We can reduce incomplete information about available strategies to incomplete information about payoffs and so we can confie our attention to the latter.

Example 8.1. 2 players with $S_1 = \{L, R\}$ and $S_2 = \{L, R\}$. Consider the following payoff matrix for person 1 with two types:

Payoff ($t_1 = 1$)			
		L	R
	L	3	0
$\checkmark$	R	4	1

Payoff ($t_1 = 2$)			
		L	R
p	L	1	0
$1-p$	R	4	3

and for person 2:

Payoff ($t_2 = 1$)		
	$\checkmark$	
	L	R
L	1	0
R	4	3

Payoff ($t_2 = 2$)		
	L	R
	q	$1-q$
L	-1	1
R	1	-1

Thus, in addition to S_1 and S_2, we also have type sets $T_1 = \{1, 2\}$ and $T_2 = \{1, 2\}$ and the player's types provide information about their payoffs. Denote $p(\cdot)$ on the product of the sets of types, $T = T_1 \times T_2$ such that $p(t_1, t_2) \geq 0, \forall t_1, t_2$ and $\sum_{t_1, t_2} p(t_1, t_2) = 1$. For this example, let's suppose the types are independent and that P$(t_i = 1) = 1/3$. For type 1, each agent has a dominant strategy, but for type 2, there is no dominant strategy. Denote the probabilities accordingly as above.

- ▷ p cannot be zero: If $p = 0$, then 1 would choose R no matter what his type and so 2 knows that 1 chooses R. Then 2 will choose L no matter what his type is, so 1 should choose L.
- ▷ p cannot be one: If $p = 1$, then when 2's type is $t_2 = 2$, 2 knows that there is a probability $1/3$ that 1 chooses R and probability $2/3$ that 1 chooses L. So when her type is 2, then it is best for 2 to play R. But then it is best for 1 when his type is 2 to play R (i.e. $p = 0$) .

So there is no equilibrium with $p = 0$ or $p = 1$ and similarly, there is no "equilibrium" with $q = 0$ or $q = 1$. So if there is an equilibrium, $p, q \in (0, 1)$. So when $t_i = 2$, player 2 must be indifferent between L and R. Thus, when $t_1 = 2$, 1 must be indifferent between the two and when $t_2 = 2$, 2 must be indifferent between the two choices.

Using this fact, note that 1's expected utility when $t_1 = 2$ and chooses R is

$$\underbrace{\left(\frac{1}{3}\right)}_{t_2=1}(-1) + \underbrace{\left(\frac{2}{3}\right)}_{t_2=2}(q(-1) + (1-q)1) = \frac{1-4q}{3}$$

Similarly, the expected utility when $t_1 = 2$ and chooses L:

$$\left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)(q(1) + (1-q)(-1)) = \frac{4q-1}{3}$$

The player must be indifferent so we have

$$4q - 1 = 1 - 4q \Rightarrow q^* = \frac{1}{4}$$

Through a similar process, we can obtain p^*. Therefore:

- ▷ $t_1 = 1$ chooses R
- ▷ $t_1 = 2$ mixes $(p^*, 1 - p^*)$ on (L, R)
- ▷ $t_2 = 1$ chooses L
- ▷ $t_2 = 2$ mixes $\left(q^* = \frac{1}{4}, 1 - q^*\right)$ and (L, R)

Given other players' behavior for each of his type, no type of either can profitablty deviate.

Definition 8.1. A Bayesian game $BG = (A_i, T_i, u_i, p)_{i=1}^N$ consists of: for each player $i = 1, ..., N$, action set A_i, type set T_i, vNM utility function

$$u_i : A \times T \to \mathbb{R}, \quad A = X_{i=1}^N A_i, T = X_{i=1}^N T_i$$

and $p \in \Delta(T)$ is a probability distribution over T. When T is finite, we will assume $p(t) > 0, \forall t \in T$ and of course $\sum p(t) = 1$. For each i and each $t_i \in T_i$ (when T is finite), then by Bayes' rule:

$$p_i(t_{-i}|t_i) = \frac{p(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}}}$$

It's important to get information, conditional on your type, about the distribution of others' type.

Definition 8.2. A *pure* strategy for player i is $s_i : T_i \to A_i$ is a set of pure strategies $S_i = \{s_i : T_i \to A_i\}$. Thus, $|S_i| = |A_i|^{|T_i|}$.

Definition 8.3. A *mixed* strategy M_i can be defined as before:

$$M_i = \left\{ m_i : S_i \to [0,1] : \sum_{s_i \in S_i} m_i(s_i) = 1 \right\}$$

Definition 8.4. A *behavioral* strategy for player i is defined as $b_i : A_i \times T_i \to [0,1]$ such that

$$\sum_{a_i \in A_i} b_i(a_i|t_i) = 1$$

where $b_i(a_i|t_i)$ is the probability that type t_i chooses action a_i. Furthermore,

Definition 8.5. Let B_i denote the set of behavior strategies. Furthermore, for each player i, $\forall a_i \in A_i, t_i \in T_i, b_{-i} \in B_{-i}$, define expected payoff given the behaviors of others as:

$$V_i(a_i, b_{-i}|t_i) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}, t_i, t_{-i}) \, b_{-i}(a_i|t_i)$$

We say that $(b_1^*, ...b_N^*) \in \mathcal{B} = X_{i=1}^N B_i$ is a Bayes-Nash equilibrium if and only if $\forall i, \forall t_i \in T_i$ and $\forall a_i \in A_i$ such that $b_i^*(a_i|t_i) > 0$, then:

$$V_i\left(a_i, b_{-i}^*|t_i\right) \geq V_i\left(a_i', b_{-i}^*|t_i\right), \forall a_i' \in A_i$$

No matter what type you end up being, for a strategy that shows up in your mixed strategy, no other action will give you a higher expected payoff.

How do mixed $(m_i \in M_i)$ and behavior strategies (b_i) compare?

▷ Suppose players use the pure strategy profile $s = (s_1, s_2, ..., s_N)$. Then we can define player i's expected utility $U_i(s)$ as

$$U_i(s) = \sum_{(t_1,...,t_n)\in T_1\times\cdots\times T_N} p(t_1,...,t_n)\, u_i(s_1(t_1),...,s_N(t_N), t_1,...,t_N)$$

▷ So $G \equiv (S_i, U_i)_{i=1}^N$ is a strategic form game.

▷ Note that $S_i = \{s_i : T_i \to A_i\}$.

▷ Then for any $m \in M$, $u_i(m) = \sum_{s\in S} m(s)U_i(s)$ where $m(s) = \prod_{j=1}^N m_j(s_i)$

▷ How do the BNE (of BG) in behavior strategies compare with the NE (of G) in mixed strategies?

Remark 8.1. Here's an intuitive way to understand the difference:

▷ Mixed strategy: You have a lot of books (= pure strategies) to choose from. Each book outlines a mapping between your type and your action. You're randomizing over these books. You are randomizing over these pure strategies i.e. we are assigning a probability distribution over pure strategies.

▷ Behavioral strategy: You come with a probability distribution over your types. For each possible type, you contemplate which probabilities to assign to each action. This is explicitly assigning $b_i(a_i|t_i)$ i.e. independently for each information set, we are assigning a probability distribution over actions.

As a statistician, you're just interested in the ultimate $b_i(a_i|t_i)$. You don't really care how you got here. The subsequent results lend credibility to this claim.

Definition 8.6. For any player i, say that $m_i \in M_i$ and $b_i \in \mathcal{B}_i$ are equivalent if and only if $\forall \alpha \in A_{i,} \forall r \in T_i$:

$$b_i(\alpha|r) = \sum_{s_i\in S_i, s_i(\alpha)=r} m_i(s_i)$$

The idea is that no matter what player i's type turns out to be, they will give the same probability to action a_i.

Definition 8.7. We say that the profiles of strategies – $b \in \mathcal{B}$ and $m \in \mathcal{M}$ – are equivalent iff b_i is equivalent to m_i for all i.

Proposition 8.1. *If $b \in \mathcal{B}$ and $m \in \mathcal{M}$ are equivalent, then they have the same probability distribution over $A \times T$ because*

$$Pr(a,t|b) = p(t)\prod_{i=1}^N b_i(a_i|t_i)$$

$$Pr(a,t|b) = p(t)\prod_{i=1}^N \left(\sum_{s_i\in S_i, s_i(t_i)=a_i} m_i(s_i)\right)$$

In general, there can be many $m_i \in M_i$ that are requivalent to any one $b_i \in \mathcal{B}_i$. For example, $T_i = \left\{t^L, t^R\right\}, A_i = \{L, R\}$. Consider the behavioral strategy

$$p_i\left(k|t_i^k\right) = \frac{2}{3}, \quad k \in \{L, R\}$$

This b_i is equivalent to infinitely many $m_i \in M_i$:

$$\left(\frac{2}{9}, \frac{4}{9}, \frac{1}{9}, \frac{2}{9}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right)$$

where the vector denotes probability for $(L, L), (L, R), (R, L), (R, R)$ and each $(\cdot, \cdot)$ denotes the pure strategy. We can easily verify that this is true. Conditional on your type being right, the probability is $1/3$.

Example 8.2. Let $t_1 \sim U(0, 1)$ and $t_2 \sim U(1, 2)$ and $T_1 = [0, 1]$ and $T_2 = [1, 2]$

	L	R
U	$2 + t_1, 9 + t_2$	$4 + t_1, 3$
D	$5, t_2$	$1, 3$

- ▷ In this example, the types are independent, so t_1 provides no information about t_2. Thus, if 2's behavior strategy is b_2, the probability that 2 chooses L given that 1's type is t_1 is

$$q = \int_1^2 b_2(L|t_2)\, dt_2$$

and is independent of t_1. Similarly for t_2:

$$p = \int_0^1 b_1(U|t_1)\, dt_1$$

- ▷ Consider player 1 when type t_1 :

$$V_1(U, b_2|t_1) = t_1 + (2q + 4(1 - q)) = t_1 + 4 - 2q$$
$$V_1(D, b_2|t_1) = 5q + (1 - q) = 4q + 1$$

And thus only

$$t_1 = t_1^* = 6q - 3$$

is indifferent between U and D.

- ▷ Similarly, for player 2,

$$V_2(L, b_1|t_2) = t_2 + 9p$$
$$V_2(R, b_1|t_2) = 3$$

which yields

$$t_2 = t_2^* = 3 - 9p$$

- ▷ For player 1, for $t_1 > t_1^*$, you strictly prefer U and for $t_1 < t_1^*$, you strictly prefer D . This is b_1.
- ▷ For player 2, for $t_2 > t_2^*$, you strictly prefer L and for $t_2 < t_2^*$, you strictly prefer R. This is b_2.

▷ This leads

$$p = \int_0^1 b_1\left(U|t_1\right)dt_1 = \int_{t_1^*}^1 1dt_1 = 1 - t_1^*$$
$$q = \int_1^2 b_2\left(L|t_2\right)dt_2 = \int_{t_1^*}^2 1dt_2 = 2 - t_2^*$$

▷ For t_1^* to be indifferent between U and D, and for t_2^* to be indifferent between L and R:

$$t_1^* = 6q - 3 = 6\left(2 - t_2^*\right) - 3$$
$$t_2^* = 3 - 9p = 3 - 9\left(1 - t_1^*\right)$$

This yields

$$t_1^* = \frac{9}{11}, \quad t_2^* = \frac{15}{11}$$

8.2 Theorem: Redefining Utility

Theorem 8.1. *If $b \in \mathcal{B}$ and $m \in \mathcal{M}$ are equivalent, then for every i :*

$$U_i(m) = \sum_{r \in T_i, \alpha \in A_i} p_i(r)\, b_i(\alpha | r)\, V_i(\alpha, b_{-i} | r)$$

where $p_i(t_i) = \sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i})$.

Solution.

8.3 Theorem: Equivalence among Strategies

Theorem 8.2. *For any player i and for any $b_i \in \mathcal{B}_i$, the mixed strategy $m_i \in M_i$ defined by*

$$m_i\left(s_i\right) = \prod_{t_i \in T_i} b_i\left(s_i\left(t_i\right) | t_i\right), \forall s_i \in S_i$$

is equivalent to b_i.

Solution.

8.4 Theorem: Equivalence among BNEs

Theorem 8.3. *If $m^* \in M$ is a NE of G, then the (unique) $b^* \in \mathcal{B}$ that is equivalent to $m^* \in M$ is a BNE of $\mathcal{B}$. Conversely, if $b^* \in \mathcal{B}$ is a BNE of BG, then any $m^* \in M$ that is equvialent to b^* is a NE of G.*

Solution. Suppose $m^* \in M$ is a NE and b^* is equivalent to m^*. Let i be a player, and let b_i be any behavior strategy in B_i. Let m_i be any element of M_i that is equivalent to b_i (use Theorem 8.2). Then by Theorem 8.1,

$$\begin{aligned}\sum_{t_i, a_i} p_i\left(t_i\right) b_i^*\left(\alpha|r\right) V_i\left(\alpha, b_{-i}^*|t_i\right) &= U_i\left(m^*\right) \\ &\geq U_i\left(m_i, m_{-i}^*\right)\end{aligned}$$

since m^* is a NE (you're replacing just one guy's mixed strategy). Note that the last term is equal to

$$= \sum_{r \in T_i, \alpha \in A_i} p_i\left(r\right) b_i\left(\alpha|r\right) V_i\left(\alpha, b_{-i}^*|r\right)$$

It thus follows that

$$\sum_{r \in T_i, \alpha \in A_i} p_i\left(r\right)\left\{b_i^*\left(\alpha|r\right) - b_i\left(\alpha|r\right)\right\} V_i\left(\alpha, b_{-i}^*|t_i\right) \geq 0, \forall b_i \in \mathcal{B}_i \qquad (1)$$

since the choice of b_i was arbitrary. Consider any $\bar{a}_i, \bar{t}_i$ such that $b_i^*\left(\bar{a}_i|\bar{t}_i\right) > 0$ and any $a_i' \in A_i$. Then $\exists \epsilon > 0$ and $b_i \in \mathcal{B}_i$ such that

$$b_i\left(\alpha|r\right) = b^*\left(\alpha|r\right), \forall\left(\alpha, r\right) \in A_i \times T_i$$

and

$$\begin{aligned}b_i\left(\bar{a}_i|\bar{t}_i\right) &= b_i^*\left(\bar{a}_i|\bar{t}_i\right) - \epsilon \\ b_i\left(a_i'|\bar{t}_i\right) &= b_i^*\left(a_i'|\bar{t}_i\right) + \epsilon\end{aligned}$$

Then (1) becomes

$$p_i\left(r\right) \cdot \epsilon\left\{V_i\left(\bar{a}_i, b_{-i}^*|\bar{t}_i\right) - V_i\left(a_i', b_{-i}^*|\bar{t}_i\right)\right\} > 0$$

9 Extensive Form Games

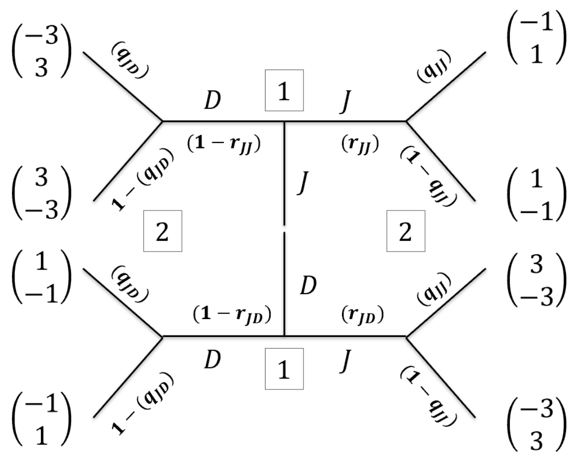
9.1 Setup

We start an example using a simple card game. There are 2 players, and the nature is a deck of cards consisting of 2 deuces and 1 Jack.

- ▷ Player 1 draws a card then claims either J or D. Then player 2 after hearing claim says J or D.
- ▷ The payoffs are as follows. If the card is $\theta \in \{J, D\}$, then

$$\begin{aligned}(u_1, u_2) &= (1, -1) \text{ if 1 tells truth and 2 is wrong}\\ &= (-1, 1) \text{ if 1 tells truth and 2 is right}\\ &= (3, -3) \text{ if 1 lies and 2 is wrong}\\ &= (-3, 3) \text{ if 1 lies and 2 is right}\end{aligned}$$

- ▷ To illustrate this game dynamics, we draw a game tree diagram:



- ▷ Denote r_{AB} as the probability that 1 chooses A after observing B.
- ▷ It can be shown that none of the qs and rs can be zeros or ones.

Now that we have established (claimed) that $r_{\cdot,\cdot} \in (0, 1)$, player 1 after observing J must be indifferent between J and D.

- ▷ Player 1's expected payoff from J given J is $(-1)\, q_{JJ} + (1 - q_{JJ})$.
- ▷ Player 1's expected payoff from D given J is $(-3)\, q_{JD} + (3)\, (-q_{JD})$

Similarly, player 1 after observing D must be indifferent between J and D .

- ▷ Then

$$q_{JJ}\,(3) + (1 - q_{JJ})\,(-3) = q_{JD}\,(1) + (1 - q_{JD})\,(-1)$$

The two systems of equations yields

$$q_{JJ} = q_{JD} = \frac{1}{2}$$

When player 1 hears 1's claim, she will update her beliefs about the true card using Bayes' rule:

$$Pr\left(\theta = J | a_1 = J\right) = \frac{Pr\left(\theta \in J, a_1 = J\right)}{Pr\left(a_1 = J\right)} = \frac{\frac{1}{3}r_{JJ}}{\frac{1}{3}r_{JJ} + \frac{2}{3}r_{JD}} = \frac{r_{JJ}}{r_{JJ} + 2r_{JD}}$$

Analogously, we have

$$Pr\left(\theta = D | a_1 = J\right) = \frac{2r_{JD}}{r_{JJ} + 2r_{JD}}$$

For 2 to be indifferent between $a_2 = J$ and D after $a_1 = J$, the expected utilities must be equated:

- ▷ 2's utility from $a_2 = J$ given $a_1 = J$ is

$$\left(\frac{r_{JJ}}{r_{JJ} + 2r_{JD}}\right)(1) + \left(\frac{2r_{JD}}{r_{JJ} + 2r_{JD}}\right)(-3)$$

- ▷ 2's utility from $a_2 = D$ given $a_1 = J$ is

$$\left(\frac{r_{JJ}}{r_{JJ} + 2r_{JD}}\right)(-1) + \left(\frac{2r_{JD}}{r_{JJ} + 2r_{JD}}\right)(3)$$

  This yields $r_{JJ} = 6r_{JD}$.

Thus we have

$$r_{JJ} = \frac{3}{8}, \quad r_{JD} = \frac{1}{16}$$

Definition 9.1. An extensive form game Γ consists of the following elements:

1. A finite number N of players
2. A set of actions A containing all actions that can ever occur in the game
3. A set of nodes (or histories) X such that
   - ▷ $x_0 \in X$ is the initial node (or empty history)
   - ▷ $X \backslash \{x_0\}$ is a collection of finite sequence of actions in A
   - ▷ If a sequence is in $X \backslash \{x_0\}$ then all its truncations are in $X \backslash \{x_0\}$.
4. $A(x_0) \subseteq A$ is nature's set of actions at the initial node x_0 , and $\pi \in \Delta(A(x_0))$ is nature's probability distribution over $A(x_0)$. Nature moves once and at the start of the game.
5. For any $x \in X \backslash \{x_0\}$, $A(x) = \{a \in A : (x, a) \in X\}$ is the set of available actions after the history x. Furthermore, define

$$E \equiv \{x \in X : A(x) = \emptyset\}$$

   is the set of end nodes, and

$$D \equiv X \backslash (E \cup \{x_0\})$$

   is the set of decision nodes.

6. $\iota : D \to \{1, ..., N\}$ indicates the player whose turn it is at any decision node. Subsequently:

$$X_i \equiv \{x \in D : \iota(x) = \iota\}$$

is player i's set of decision nodes.

7. For each player i, $\mathcal{I}_i$ is a partition of X_i such that $\forall I \in \mathcal{I}_i$, $\forall x, x' \in I$, $A(x) = A(x')$, and so we can define $A(I) = A(x), \forall x \in I$. Then

$$\mathcal{I} \equiv \bigcup_{i=1}^{N} \mathcal{I}_i$$

is a partition of D. For $x \in D$, the $\mathcal{I}(x)$ be the element of $\mathcal{I}$ that contains x i.e. the information set that contains a node. For example, $\mathcal{I}(c) = \{c, d\}$.

8. Payoffs are specified as $u_i : E \to \mathbb{R}$ is i's vNM utility function

Definition 9.2. A pure strategy for player i is a mapping from your information sets to actions, $s_i : \mathcal{I}_i \to A$, such that

$$s_i(I) \in A(I), \forall I \in \mathcal{I}_i$$

i.e. your action choice has to be feasible. Let S_i denote the set of pure strategies:

$$S = X_{i=1}^{N} S_i$$

Given $s = (s_1, ..., s_N) \in S$ and given any $a \in A(x_0)$, let $\eta(s, a)$ denote the end node that results. Then

$$U_i(s) = \sum_{a \in A(x_0)} \pi(a) u_i(\eta(s, a))$$

is the expected payoff under s. So $G = (S_i, U_i)_{i=1}^{N}$ is a strategic form game of Γ. Extensive form games actually generate strategic form games. In a general extensive form game, before the game starts you can go through every possible position in the nodes and write down all possible mappings.

Definition 9.3. A mixed strategy can be defined as

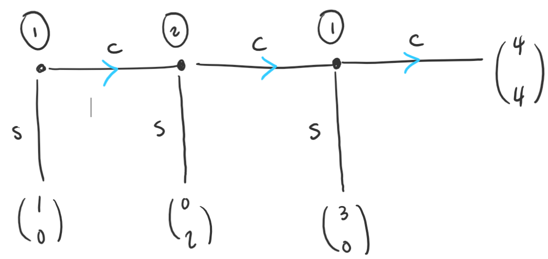
$$M_i = \left\{ m_i : S_i \to [0, 1] : \sum_{s_i \in S_i} m_i(s_i) = 1 \right\}$$

if S_i is finite (i.e. when X is finite).

Definition 9.4. A behavioral strategy for player i is a mapping $b_i(\cdot)$ as follows: For every $I \in \mathcal{I}_i$, $\forall a \in A(I)$, $b_i(a, I) \in [0, 1]$ where $\sum_{a \in A(I)} b_i(a', I) = 1$.

Definition 9.5. Γ has perfect information (or is a PI game) if and only if $\mathcal{I}(x) = \{x\}, \forall x \in D$.

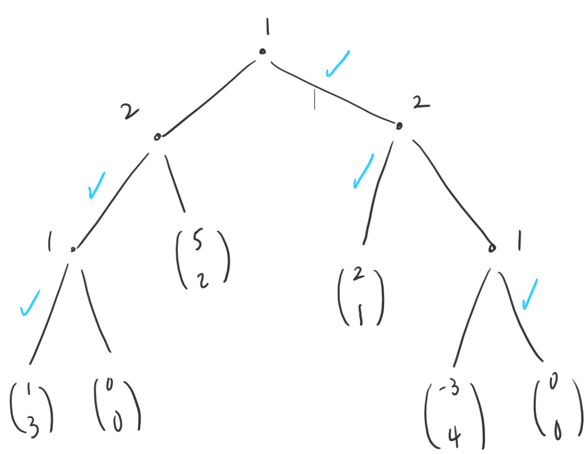
Example 9.1. (Example of Perfect Information) In the following example, each person can look ahead assuming the other is rational.



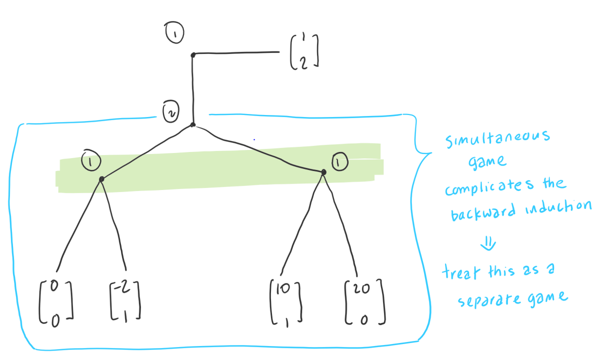
In PI games, we can capture the look-ahead capability of rational players by considering backward induction strategy profiles.

Definition 9.6. $s \in S$ is a backward induction strategy profile if and only if for every $x \in D$, player $i = \iota(x)$ can do no better than to choose the action $s_i(x)$ at x, given that all players, including i, make their choices according to s after x.

Example 9.2. (Example of Backward Induction)



Example 9.3. An illustrative example:



▷ There are no pure strategies, so consider mixed strategies.

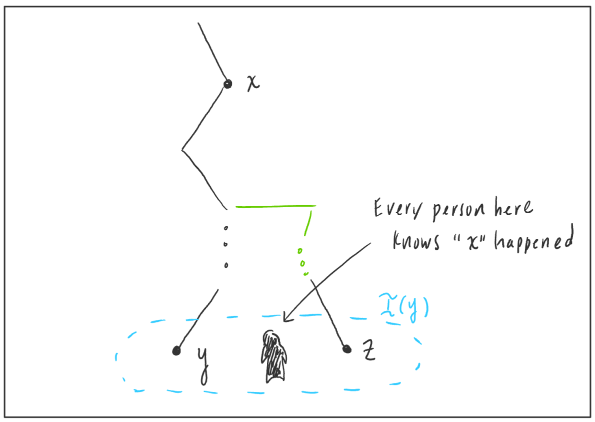
▷ For player 2 to be indifferent between the two choices, player 1 must mix with probability 1/2.

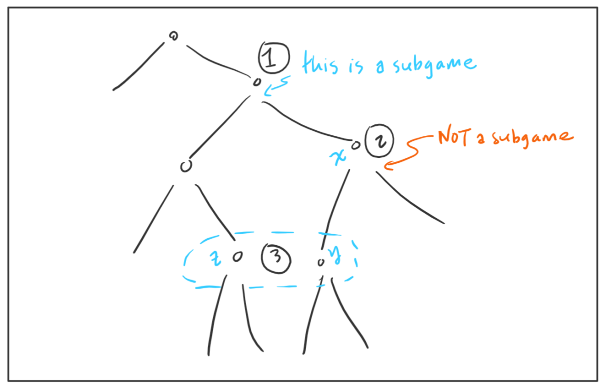
▷ For player 1 to be indifferent between the two choices, denote p is the probability with which agent 2 chooses the left. Then the expected payoffs are

$$V_1(L) = 10(1-p), \quad V_2(R) = -2p + 20(1-p)$$

Equating these values, we have $p = 5/6$, which implies 1's expected payoff at the highlighted part is $10/6$. So it will join the continuation game.

Definition 9.7. A node $x \in X \backslash E$ defines a subgame if and only if $x = x_0$ or $x \in \mathcal{D}$ and $\mathcal{I}(x) = \{x\}$ and $\forall y, z \in \mathcal{D}$, if $z \in \mathcal{I}(y)$ and $y \geq x$ (i.e. x is a truncation of y) then $z \geq x$. Figure-wise, the following is a sub-game:





Note that in a perfect information game, every node is a subgame.

Definition 9.8. (Behavioral Strategies) For $e \in E$ and any $b \in B$, if $e = (a_0, a_1, ..., a_k)$, then

$$Pr(e|b) = \pi(a_0) \prod_{k=1}^{K} b_{\iota(a_{<k})}(a_k, \mathcal{I}(a_{<k}))$$

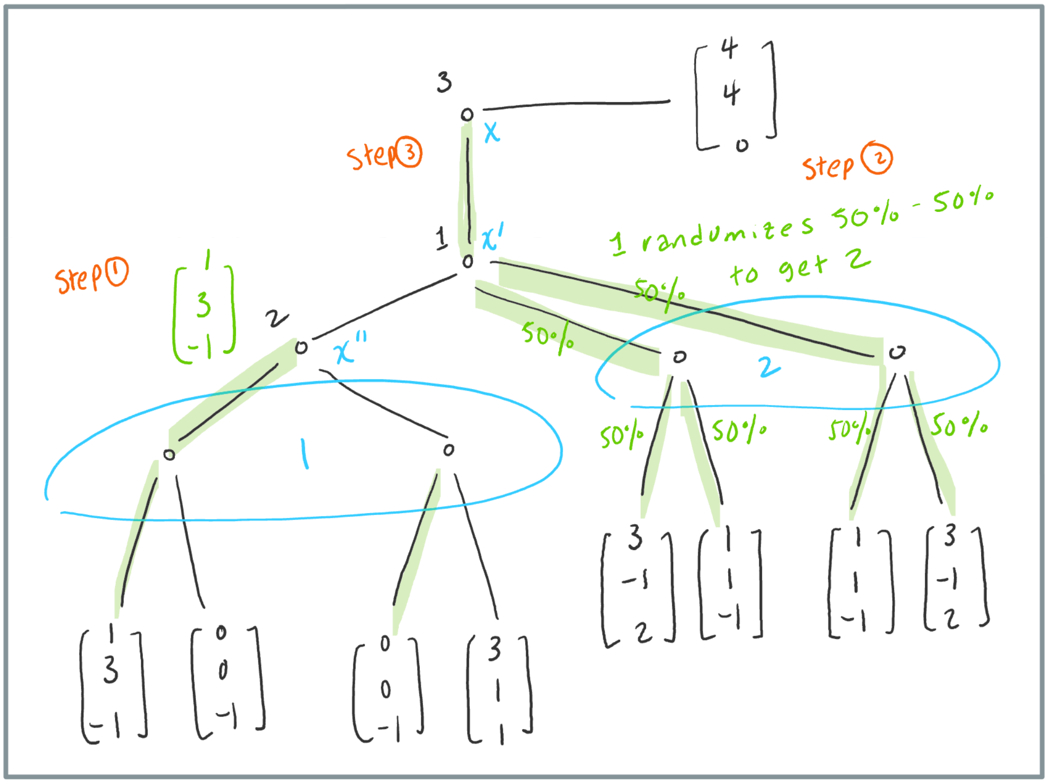
where a_0 can be interpreted as nature's moves and $a_{<k} \equiv (a_0, a_1, ..., a_{k-1}), k \geq 1$. Therefore, for any player i, we can define the expected utility

$$v_i(b) \equiv \sum_{e \in E} Pr(e|b) u_i(e)$$

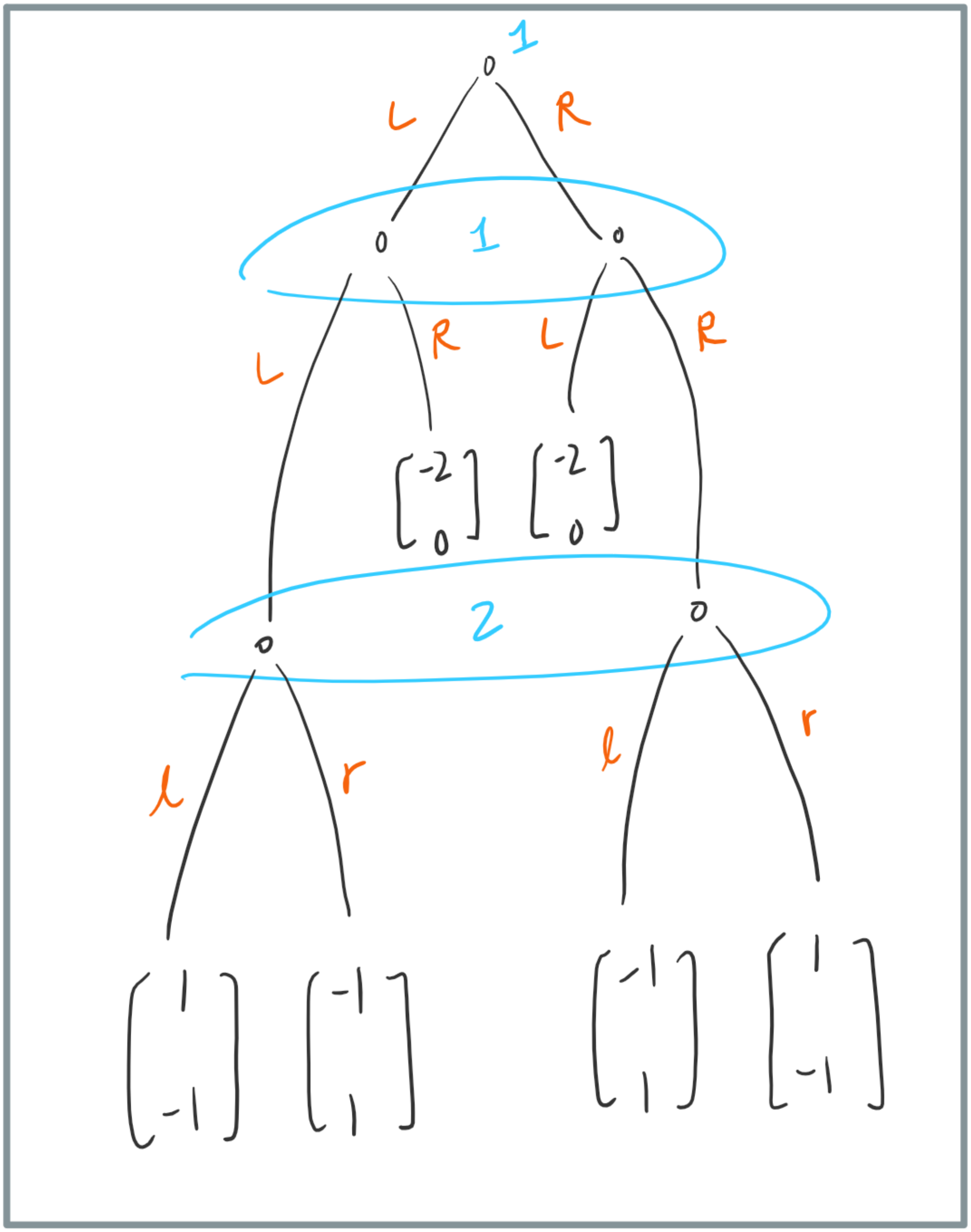
This allows us to define the nash equilibrium. We say $b^* \in B$ is a Nash Equilibrium of Γ (in behavior strategies) if and only if $\forall i, v_i(b^*) \geq v_i(b_i', b_{-i}^*)$ for all $b_i' \in B_i$.

Now define $v_i(b|x_0) \equiv v_i(b)$ – payoff when the game begins at the initial node and everybody used the behavioral strategies profile. Notice that for only $x \in X \backslash E$ that defines a subgame Γ_x, any $b \in B$ that defines behavioral strategies in Γ_x. So we may define $v_i(b|x)$ to be i's expected payoff in Γ_x given the b-induced behavior strategy in Γ_x.

Definition 9.9. (Subgame Perfect NE) We say that $b^* \in B$ is a subgame-perfect Nash equilibrium if and only if b^* induces a Nash equilibrium in every subgame of Γ. To find *a* subgame perfect NE:



Remark 9.1. (Game with Perfect Recall) A SPE may not exist, even in finite games! An example is shown below:



- ▷ In this game, we have a mixed strategy NE but no behavior strategy NE.
- ▷ If you are mixing: note that $s_1 = \{(L, L), (L, R), (R, L), (R, R)\}$ and $s_2 = \{\ell, r\}$. Player 1 will always put zeros on (L, R) and (R, L). So the only NE in such situation is where both players put $1/2$ on the two strategies that they ultimately have available. This is the unique NE with mixed strategies.
- ▷ But there is no behavioral strategy that can do this. It tells you how to mix between left and right at each point. At the second node, you end up getting positive probabilities on the (-2) outcomes. The issue here is that player 1 forgets. This is an example of a mixed strategy with no equivalent behavioral strategy.
- ▷ We are going to rule this out and regain equivalence between mixed and behavioral strategies.

Definition 9.10. Γ has perfect recall if and only if every player always recalls all of her past actions and anything she knew in the past.

Definition 9.11. (Folk Theorem) If the player is patient enough and far-sighted, then not only can repeated interaction allow many SPE outcomes, but actually SPE can allow virtually any outcome in the sense of average payoffs.

Definition 9.12. A system of beliefs is a mapping $p : D \to [0, 1]$ such that $\forall I \in \mathcal{I}$, $\sum_{x \in I} p(x) = 1$.

Definition 9.13. If p is a system of beliefs and $b \in \mathcal{B}$, then we call (p, b) an assessment for Γ.

Definition 9.14. We say (p, b) is sequentially rational if and only if $\forall i, \forall I \in \mathcal{I}_i$:

$$v_i(p, b|I) \geq v_i\left(p, b_i', b_{-i}|I\right), \forall b_i' \in \mathcal{B}_i$$

9.2 Theorem: Equivalence among Strategies

Theorem 9.1. *In any finite game with perfect recall, there is at least one subgame perfect equilibrium (SPE) in behavioral strategies.*

Solution.

9.3 Theorem: Sequential Equilibrium & SPNE

Theorem 9.2. *If (p, b) is a sequential equilibrium of Γ, then b is a SPNE of Γ and if Γ has perfect information, then b is a BI strategy if b is pure.*

Solution.

9.4 Theorem: Existence of Sequential Equilibrium

Theorem 9.3. *If Γ is a finite game with perfect recall, then Γ has at least one sequential equilibrium.*

Solution.