Theory of Income II
Winter 2019

(Nancy Stokey)

Simon Sangmin Oh

University of Chicago

Note:

These lecture notes are based on Professor Stokey's lectures in Theory of Income II, Winter Quarter.

Contents

1 Deterministic dynamic programming

1.1 Economy with No Externality

We first re-visit the basic steps of dynamic programming via a Robinson Crusoe economy with human capital.

1.1.1 Formulation

Consider the following economy:

▷ Technology: $c + h' \leq Ah^\alpha$ with $A > 0, \alpha \in (0,1)$ where c is consumption; h, h' are human capital this and next period.

  * Note that human capital depreciates completely after one period, but this assumption is easily relaxable.
  * α is assumed to be strictly less than 1. (Why? Since we want the state space to be bounded.)

▷ Preferences:

$$U(\{c_t\}) = \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

where u is continuously differentiable, strictly increasing, strictly concave with $u'(0) = +\infty$.

  * For infinite horizons, you better have discounting; if things are growing, you better have enough discounting.

1. **Define a state space**: The state is human capital (h) which we constrain to lie in the interval $[0, h_{\max}]$ where $h_{\max}$ satisfies:

$$h_{\max} \leq Ah_{\max}^\alpha \Rightarrow h_{\max} = A^{1/(1-\alpha)}$$

2. **Formulate the sequence problem (SP):**

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{subject to } c_t + h_{t+1} \leq Ah_t^\alpha = f(h_t)$$
$$h_0 \text{ given}$$

3. **Formulate the Bellman equation (BE)**: it can be written as

$$v(h) = \max_{y \in [0, Ah^\alpha]} \left[u\left(Ah^\alpha - y\right) + \beta v(y)\right]$$

1.1.2 Properties

1. **Properties of the Value Function**

   (a) Solution to the above problem, denoted as v^*, is the same as $\max\{U\{c_t\}\}$.

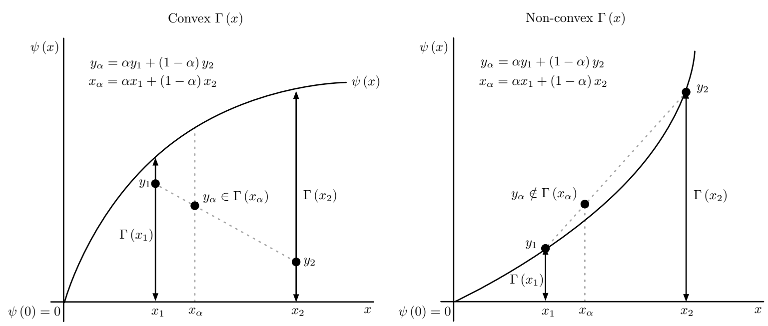
   (b) *Well-defined* (existence and uniqueness) and *continuous*

   ▷ *Boundedness*: Start by noting that since the state space is compact (closed and bounded) and $\beta < 1$, v must also be bounded.

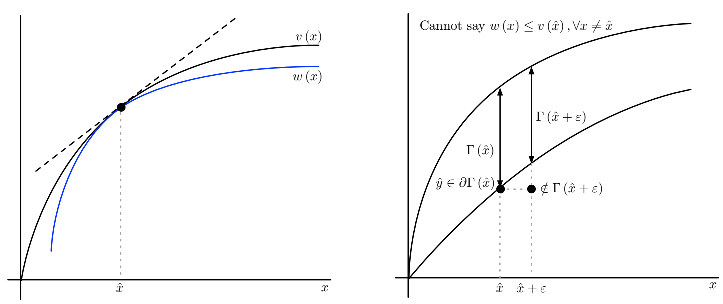
- ▷ *Existence and Uniqueness*: Suppose you plugged in a continuous and bounded $\hat{v}$ so $T\hat{v}$ (operator defined by the RHS of the BE) is also bounded. Furthermore, given the continuity of u, we have that $T\hat{v}$ is continuous. Sicne T maps the space of continuous and bounded functions to the space of continuous and bounded functions, we can use the Contraction Mapping Theorem to establish that v exists and is unique by using $\beta < 1$.
- ▷ *Continuity*: Using the fact that u is continuous and that the set $\Gamma(h) = [0, Ah^{\alpha}]$ varies continuously with h so establish that v *is* continuous.

(c) *Strictly increasing*: We rely on two facts: (1) u is strictly increasing and (2) the feasible set gets bigger as h gets bigger (i.e. Γ is monotone as a correspondence: $h' > h \implies \Gamma(h') \supseteq \Gamma(h)$). It wouldn't help that u is increasing if the feasible set was shrinking.

(d) *Strictly concave* (and the associated optimal policy is a single-valued, continuous function): We rely on two facts to show that the value function is strictly concave: (1) u is strictly concave, and (2) $\Gamma(x)$ is convex as a correspondence:



(e) *Once differentiable*: First, note that u is differentiable so v is differentiable at any point h where the solution y^* is interior, i.e. $y^* \in (0, Ah^{\alpha})$. The Theorem doesn't apply if it's not an interior solution – we can't say anything. To understand why this works, consider the following:

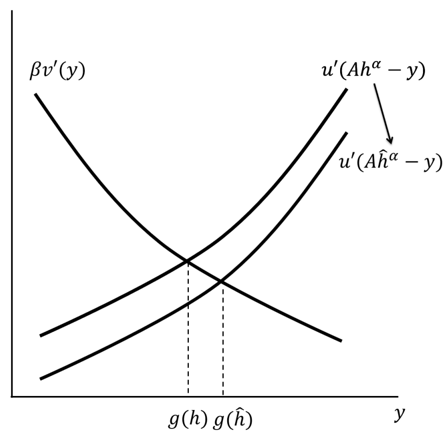


- ▷ We know v is continuous and concave, so what could go wrong? The worst-case scenario is that there could be a kink at some point.
- ▷ We address this by constructing a w that intersects at $\hat{x}$ and by differentiability, we know that there is a *unique* supporting hyperplane at $\hat{x}$. Since v lies above w, this hyperplane is also a supporting hyperplane for v as well. The important thing in constructing w is that you have to be able to perturb the optimal policy, i.e. it has to be interior.
- ▷ Can we have a corner solution? No !
  - * $y^* \neq Ah^{\alpha}$ thanks to the Inada condition.
  - * $y^* \neq 0$ since no future production is possible, thereby shutting down the problem. Note that if the economy starts above at 0, the agent will stay above 0.

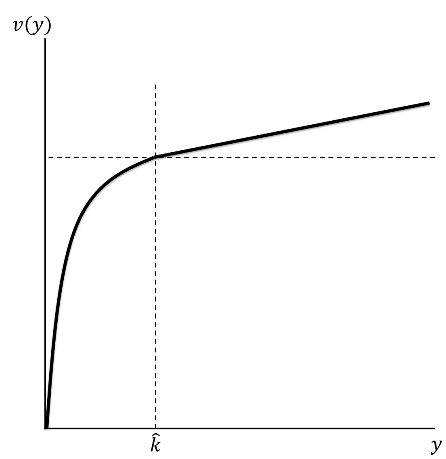
2. **Properties of the Optimal Policy Function** $g(h) = y^*$

   (a) *Single-valued and continuous*

   (b) *Strictly increasing with slope less than* $f'(h)$: If we increase h, the resulting increase $f(h)$ is divided among increase in consumption and increase in human capital. To see this, consider $\hat{h} > h$. Then by the shift in the graph, we see that $g(h) < g(\hat{h})$, which implies $\beta v'(\hat{y}^*) = u'\left(A\hat{h}^\alpha - g(\hat{h})\right) < u'(Ah^\alpha - g(h)) = \beta v'(y^*)$, leading to $\hat{c} > c$.



Exercise 1.1. When can a value function look like this i.e. have a kink:



Solution. This can occur when there is a binding non-negativity constraint, e.g. an inventory model. Another example is when the firm is subject to tax brackets. Focusing on the second example, think about the first-order conditions of the firm for an interior solution:

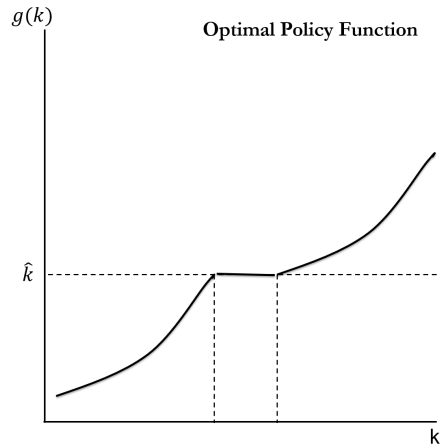
$$R_y(k, y) = \beta v'(y)$$

but for a solution $y = \hat{k}$, there are infinitely many derivatives at $\hat{k}$, i.e.

$$R_y(k, y) \in \beta D v'\left(\hat{k}\right)$$

which means that $R_y(k, y)$ can take multiple values.

What will happen to the policy function? It is still single-valued since the value function is still concave. The kink, however, leads to a flat region in the policy function:



1.1.3 Model Solution

1. **Write the first-order condition (FOC) and the envelope condition (EC)**:

$$\begin{aligned} \text{FOC } &: u'\left(f(h)-y^*\right)=\beta v'\left(y^*\right) \\ \text{EC } &: v'(h)=f'(h)\cdot u'\left(f(h)-y^*\right) \end{aligned}$$

First, let us focus on the FOC. Since v is strictly concave, v' is positive but decreasing – the RHS is thus decreasing in y. Since u is strictly concave, u' is positive but decreasing, so the LHS is increasing in y. The optimal choice $g(h)$ is at the intersection os these two graphs.

2. **Write the Euler Equation (EE)**: EE is simply a combination of FOC and EC to get rid of v. Plugging in $y^*=g(h)$:

$$u'\left(f(h)-g\left(h\right)\right)=\beta v'\left(g(h)\right)=\beta f'(g(h)\cdot u'\left(f(g(h))-g\left(g(h)\right)\right)$$

which is a second-order equation.

3. **Characterize the Steady States**: In the EE, replace the $g(h)$ with the steady-state value h_{ss} to yield:

$$\beta f'\left(h_{ss}\right)=1$$

   - h_{ss} is increasing in β.
   - h_{ss} is increasing in A and α ($f(h)$ shifts up $\Rightarrow$ $g(h)$ shifts up $\Rightarrow$ intersection of $g(h)$ and the 45-degree line increases, so h_{ss} increases)

4. **Linearize the EE**: Start with the following EE and using time subscripts:

$$u'\left(f(h)-g\left(h\right)\right)=\beta v'\left(g(h)\right)=\beta f\left(g(h)\right)\cdot u'\left(f(g(h))-g\left(g(h)\right)\right)$$

$$\Rightarrow u'\left(f(h_t)-h_{t+1}\right)=\beta f'(h_{t+1})\cdot u'\left(f(h_{t+1})-h_{t+2}\right)$$

and linearize each side.

(a) LHS: Denote $y = k(h_t, h_{t+1}) = u'\left(f(h_t) - h_{t+1}\right)$ and linearize it around the steady state (h_{ss}, h_{ss}):

$$\begin{aligned} L(h_t, h_{t+1}) &= k(h_{ss}, h_{ss}) + k_1\left(h_{ss}, h_{ss}\right)\left(h_t - h_{ss}\right) + k_2\left(h_{ss}, h_{ss}\right)\left(-h_{t+1} + h_{ss}\right) \\ &= u'\left(c_{ss}\right) + u''\left(c_{ss}\right) f'\left(h_{ss}\right)\left(h_t - h_{ss}\right) - u''\left(c_{ss}\right)\left(h_{t+1} - h_{ss}\right) \end{aligned}$$

(b) RHS: Denote $y = k\left(h_{t+1}, h_{t+2}\right) = \beta f'(h_{t+1}) \cdot u'\left(f(h_{t+1}) - h_{t+2}\right)$ and linearize it around the steady state (h_{ss}, h_{ss}):

$$\begin{aligned} L\left(h_{t+1}, h_{t+2}\right) &= k\left(h_{ss}, h_{ss}\right) + k_1\left(h_{ss}, h_{ss}\right)\left(h_{t+1} - h_{ss}\right) + k_2\left(h_{ss}, h_{ss}\right)\left(-h_{t+2} + h_{ss}\right) \\ &= \beta f'\left(h_{ss}\right) u'\left(c_{ss}\right) + \left[\beta f''\left(h_{ss}\right) u'\left(c_{ss}\right) + \beta f'\left(h_{ss}\right)^2 u''\left(c_{ss}\right)\right]\left(h_{t+1} - h_{ss}\right) \\ &\quad - \beta f'\left(h_{ss}\right) u''\left(c_{ss}\right)\left(h_{t+2} - h_{ss}\right) \end{aligned}$$

(c) LHS = RHS: Using the fact that $\beta f'\left(h_{ss}\right) = 1$ and denoting $z_t = h_t - h_{ss}$, we have

$$0 = -u''\left[f' z_t - z_{t+1}\right] + \beta\left\{f'' u' z_{t+1} + \left(f'\right)^2 u'' z_{t+1} - f' u'' z_{t+2}\right\}$$

1.2 Economy with Positive Externality

We introduce externalities and consider (1) the social planner's solution, and (2) a competitive solution in which individual decision makers all choose on their own. This would be an example of a recursive competitive equilibrium, where the equilibrium doesn't solve any optimization.

1.2.1 Social Planner's Problem

Now the technology is defined as $F(h, H) = Ah^\theta H^\gamma$ with $\theta, \gamma > 0, \theta + \gamma < 1$. Previously, it was just $f(h) = Ah^\alpha$. *Interpretation:* spillover in human capital. Denoting $\hat{\alpha} = \theta + \gamma$, the SP is as before with $f(h) = Ah^{\hat{\alpha}}$:

$$v(h) = \max_{y \in [0, Ah^\alpha]} \left[u\left(Ah^\alpha - y\right) + \beta v(y)\right]$$

1.2.2 Recursive CE with Individual Decision-making

An important point about the CE problem is that the individual agent, when he chooses the next period human capital, only looks at his private return in the future; the SP, on the other hand, took account of the external effect. Thus the individual agent's problem looks different:

1. Set up the Bellman Equation: Two state-variables: individual human capital h and average economy-wide human capital H, so

$$W(h, H) = \max_{y \in [0, F(h,H)]} \left[u\left(F(h, H) - y\right) + \beta W\left(y, \Psi\left(H\right)\right)\right]$$

where $\Psi\left(H\right)$ represents the agent's conjecture about the choices of other agents and the resulting average economy-wide human capital. Note that the agent's optimal (denoted by $*$) choice $\psi^*\left(h, H; \Psi\right))$ also depends on Ψ.

2. In equilibrium, we need a functional form of Ψ such that feeding it into ψ yields a similar functional form. More rigorously: a *recursive CE* is defined by a function $\Psi^e(h)$ with the property that

$$\psi^*\left(h, h, \Psi^e\right) = \Psi^e\left(h\right), \forall h$$

3. Take the FOC & EC and then evaluate at the equilibrium conjecture of $h = H$:

$$u'\left(F(h,h) - y\right) = \beta W_1\left(\Psi^e(h), \Psi^e(h)\right)$$
$$W_1(h,h) = F_1(h,h)u'\left(F(h,h) - \Psi^e(h)\right)$$

thereby yielding the following Euler Equation with time subscripts added:

$$u'\left(F\left(h_t, h_t\right) - h_{t+1}\right) = \beta F_1\left(h_{t+1}, h_{t+1}\right) u'\left(F\left(h_{t+1}, h_{t+1}\right) - h_{t+2}\right)$$

4. Contrast this with the SP's Euler Equation:

$$u'\left(f(h_t) - h_{t+1}\right) = \beta \underbrace{f'(h_{t+1})}_{=(SP):F_1+F_2} \cdot u'\left(f(h_{t+1}) - h_{t+2}\right)$$

Using our functional form, it's more obvious:

$$F_1(h,h) = \theta A h^{\theta+\gamma-1} \qquad vs. \qquad f'(h) = (\theta+\gamma)Ah^{\theta+\gamma-1}$$

5. Steady-state analysis for RCE: $1 = \beta F_1\left(h_{ss}, h_{ss}\right)$

6. Dynamics: Linearize the EE around the steady state.

7. Conclusion: In the CE problem, the steady state is *lower* than that of the SP problem.

Why do we need this? DP is only useful for single-agent problems; for multi-agent problems, we need machinery like this one.

1.3 Bellman Formulation Practice

Formulating Bellman Equations is non-trivial, so it's good to have multiple practice.

1.3.1 Example #1: Robinson Crusoe

Consider a Robinson Crusoe economy with capital stock that can be used fro either home production or market production. Labor is split between working and leisure. Home goods are only consumed, whereas market goods can be used for either consumption or investment. Two production functions – one for home and one for market – are CRS, Cobb-Douglas with share η for capital. Depreciation rate is δ.

1. Specify technology

   ▷ Notation: labor is ν (home work), n (market work), and $1-\nu-n$ (leisure). γ is the share of capital in home work, and $1-\gamma$ is the share of capital in market work.

   ▷ Denote c_h and c_m as the level of consumption and k as investment. k is the state variable. Then:

$$c_h \leq A_h\left[\gamma k\right]^\eta \nu^{1-\eta}$$
$$c_m + \left[k' - (1-\delta)k\right] \leq A_m\left[(1-\gamma)k\right]^\eta n^{1-\eta}$$

2. Preferences depend on consumption and leisure, i.e. $u\left(c_h, c_m, 1-\nu-n\right)$.

3. Bellman Formulation:

$$v(k) = \max_{\nu,n,\gamma,k'}\left\{u + \beta v\left(k'\right)\right\}$$

2 Stochastic Dynamic Programming

2.1 What changes as things become stochastic?

The dynamics for a stochastic DP model consist of a Markov process describing the (joint) behavior of the exogenous shock(s) and the endogenous state variable(s). Thus, a good understanding of Markov processes is important, especially the possible types of long run behavior.

2.1.1 Setup

We introduce an exogenous stochastic shock, z.

▷ *Which state space does z live in?*

  * Case (1): $z \in Z = \{z_1, ..., z_N\}$
  * Case (2): $z \in Z = [\underline{z}, \bar{z}]$.

▷ *How does z behave? $\{z_t\}$ is a first-order (stochastic) Markov process.*

  * Note that higher-orders can always be transformed into the first-order process by introducing more state variables.

Case (1): $z \in \{z_1, ..., z_N\}$

▷ Discrete shocks are summarized by the Markov (transition) matrix

$$Q = [q_{ij}]_{N\times N}, \qquad q_{ij} \geq 0, \qquad \sum_{j=1}^{N} q_{ij} = 1, \forall i$$

with q_{ij} representing the probability we have z_j in the next period given z_i today.

Case (2): $z \in \{\underline{z}, \bar{z}\}$

▷ Continuous shocks are summarized by the function $Q(z, z')$ with (continuous) density

$$q(z, z') = \frac{\partial Q}{\partial z'}(z, z'), \qquad \int_{z'}^{z} q(z, z')\, dz' = 1, \qquad z \in Z$$

Assumption 1. *For each $z \in Z$, $q(z, z')$ is continuous in its first argument.*

▷ What is the interpretation of the above assumption? If I change today's state a little bit, the density function moves a little bit. In economic problems, this is a very reasonable argument. This ensures that if $h(z)$ is continuous in z, then $(Mh)(z)$ defined by

$$(Mh)(z) = \int_{\underline{z}}^{\bar{z}} h(z')q(z, z')dz' = \text{expected value of } h(\cdot) \text{ next period}$$

is continuous in h. We say that q exhibits the *Feller Property* if the above holds for any continuous function h. (sufficient condition) And this is required for DP to work.

▷ Note that in Case (1), we don't have to worry about the Feller Property. The whole point about Feller is to preserve continuity; in discrete case, "continuity" is a vacuous property so the property still holds.

2.1.2 Stochastic Bellman Equation

Now we proceed to define the state variables to be $S = (x, z)$ with the typical Bellman Equation formulation:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left[F(x,y,z) + \beta \underbrace{\int_{\underline{z}}^{\bar{z}} v(y,z')q(z,z')dz'}_{\text{expected terminal value}} \right] \qquad \text{(BE)}$$

where $\Gamma(x, z)$ is the feasible set. Note that arguments similar to those for the deterministic case show that there exists a unique, continuous value function v satisfying (BE).

▷ Comment: Note that here is where DP proves to be immensely useful – stochastic programs are very difficult to formulate as sequence problems. BE allows us to take advantage of this Markov structure. Suppose F is increasing in x for each fixed z; strictly concave in x, and differentiable in x; $\Gamma(x, z)$ is (same assumptions as before for each fixed z) Suppose F and Γ are also strictly increasing in z for each fixed x.

An alternative BE formulation is the following:

$$v(x,y,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int v\left(\varphi\left(x,y,z'\right),z'\right) q\left(z,z'\right) dz' \right\} \qquad \text{(BE')}$$

▷ Comment: We can think of x as current assets, y as the amount of savings, and z' as the next-period interest rate, which together with x and y, determines the next period level of assets. The whole point of this formulation is that the agent cannot directly choose the next period state.

2.1.3 Properties of the Value & Policy Function

Denote $g(x, z)$ as the optimal policy.

1. The properties of v, g as functions of x given a fixed $z \in Z$ are:

   (a) v is increasing in x

   (b) v is concave (strictly) in x. If v is strictly concave, then $g(x, z)$ is single-valued.

   (c) v is differentiable in x .

2. Properties of v as functions of z : v is increasing (strictly) in z if

   (a) F is increasing (strictly) in z.

   (b) $z' \geq z \Rightarrow \Gamma(x, z') \geq \Gamma(x, z), \forall x$

   (c) transition function M is monotone, i.e. Mh is increasing in z if h is increasing in z. In the case where $z \in \mathbb{R}$, this condition is equivalent to first-order stochastic dominance:

   $$\hat{z} \geq z \Rightarrow F\left(z'|\hat{z}\right) \leq F\left(z'|z\right)$$

   This can be interpreted as a requirement that ensures "a higher shock today implies a better distribution tomorrow."

3. Properties of g as functions of z: examine the FOC

$$\underbrace{F_y\left(x,y^*,z\right)}_{\text{LHS}} = \underbrace{\beta E\left[v'(y,z')|z\right]}_{\text{RHS}} = \beta \int v_y\left(y^*,z'\right) q\left(z,z^*\right) dz$$

▷ Without iid assumptions about the shocks, we can't say much about the RHS because it's a conditional expectation.

Remark 2.1. **(Exam Tip)** When you get a question about the optimal policy function, use the FOC, not the EE. This is a very common mistake!

2.1.4 Steady State

We want to think about a stationary distribution for $s = (x, z)$. Assume z process has a unique ergodic set and stationary distribution and the system converges to it. Usually, the argument is specific to the model.

2.1.5 Euler Equations

The EC is $v_x(x, z) = \ldots$ and combined it with the FOC to get the stochastic EE, which yields a second-order differential equation. Once again, you can approximate the function around a point, but which point? Use the expected value under the stationary distribution, $\bar{z}$.

Remark 2.2. **(Perspective)** If you think this is messy, think about the worst-case scenario by formulating it as a sequence problem.

2.2 Application #1: The Commodity Model

Prices for many agricultural commodities (sugar, coffee, etc) fluctuate wildly from year to year. The following model has been used to describe their behavior. The key features of the model are the production is stochastic and goods can be stored.

Demand We model demand as stationary and fixed: $p = D(q)$ that is downward-sloping. This is designed to focus on the shocks to supply, not demand.

Supply Each period, sellers receive an exogenous endowment $\omega_t \in [m, M]$ which is i.i.d. Let $\mu(\omega)$ be the density and assume zero cost of production for simplicity. The commodity can be stored but at a cost. Assume that the cost of storing y units for one period is $\phi(y)$ where ϕ is continuously differentiable, strictly increasing, and strictly convex, with $\phi(0) = 0$.

CE via Social Planner To characterize the competitive equilibrium for this economy, we will appeal to the fact that competitive equilibria are Pareto efficient and look at the problem of maximizing the expected discounted value of total surplus: consumers' surplus minus storage costs. To this end, define:

$$U(q) = \int_0^q D(x)dx$$

with U strictly increasing, strictly concave, continuous, and differentiable. This follows from the fact that $D(q)$ is downward-sloping..

Problem 2.1. Write the Bellman equation for this surplus maximization problem.

Solution. Two formulations can be suggested:

1. BE with Two States: At the beginning of the period, the planner has inventory from the past period (x) and this period's supply shock (ω):
$$v(x,\omega) = \max_{y \in \Gamma(x,\omega)} \left\{ F(x,\omega,y) + \beta \int v(y,\omega')\mu(\omega')d\omega' \right\}$$
where
$$\Gamma(x,\omega) = [0, x+\omega]$$
$$F(x,\omega,y) = u(x+\omega-y) - \varphi(y)$$

2. BE with One State: Note that $x+\omega$ appear together, so we can reduce one state by defining an alternate state variable equal to $x+\omega$, so write:
$$\hat{v}(s) = \max_{y \in \Gamma(s)} \left\{ F(s,y) + \beta \int v(y+\omega')\mu(\omega')d\omega' \right\}$$
where
$$\Gamma(s) = [0, s]$$
$$F(s,y) = u(s-y) - \varphi(y)$$

Problem 2.2. Show that the inventory carried over to the next period $g(s)$ is non-decreasing in the beginning-of-period stock s and that consumption $c(s) = s - Y(s)$ is strictly increasing in s.

Solution. Examining the first-order condition:
$$-u'(s-y^*) - \varphi'(y^*) + \beta \int v'(y^* + \omega^*)\mu(\omega')d\omega' >=< 0$$
with equality holding at $s > y^* > 0$. As for corner solutions:

▷ $y = s$: consumption becomes zero, so the marginal utility of consumption shoots to the maximum. This corner will never bind.

▷ $y = 0$: this one could bind. In fact, if the marginal utility today is sufficiently high, you may even borrow against the future.

Note that given this fact, we expect a kink at the value function. To see

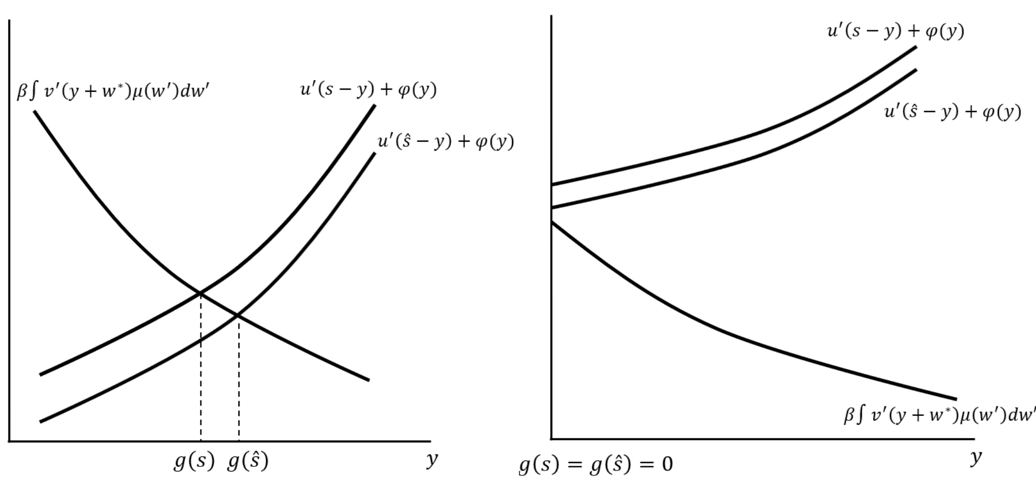


Figure 1: Comparative Statics

Referring to figure on comparative statics: if you're an interior solution and increase s to $\hat{s}$, the LHS shifts down (right) but more is saved and consumption also increases. Therefore, $g(s)$ is strictly increasing in s and strictly if the solution's away from the corner. If it's at the corner, $g(s)$ may stay the same. $c(s)$, on the other hand, must be strictly increasing.

Problem 2.3. Show that for any initial value of $s > 0$, there exists some $N \geq 1$ such that the stock hits zero with positive probability in N periods.

Solution. We proceed in the following steps:

1. If the initial value of s is the minimum m, then $g(m) = 0, c(m) = s = m$.

   ▷ To see this, consider a deviation from the policy $g(m) = 0, c(m) = s$. The marginal utility of deviating (consuming less today) is the increase in future utility; the marginal cost is the reduction in marginal utility today and an increase in the savings cost. Therefore, the agent will not deviate if and only if

   $$u'(m) + \varphi'(0) \geq \beta E\left[u'(w)\right]$$

   ▷ Since $u'(m)$ is the highest attainable marginal utility, $u'(m) > \beta E[u'(w)]$. Therefore, the above inequality holds strictly.

2. Since from the previous step

   $$u'(m) + \varphi'(0) > \beta E\left[u'(w)\right]$$

   and the continuity of all the functions, $\exists [m, m+\epsilon]$ with $\epsilon > 0$ such that $g(s) = 0, c(s) = s$. So if the stock s hits this interval with positive probability in the N periods, then our proof would be done.

3. Let θ be the probability of $\omega \in [m, m+\epsilon]$ and consider $x_0 > m + \epsilon$. Then $x_1 = g(x_0) + \omega \leq g(x_0) + m + \epsilon$ with probability θ. Note that $c(m+\epsilon) = m + \epsilon$ so $c(x_0) > m + \epsilon$ given that c is an increasing function. This implies that $g(x_0) < 0$ which implies $g(x_0) + m + \epsilon < m + \epsilon = x_0$. Thus we have established that $x_1 < x_0$ with probability θ. Through recursive operations, then we see that with positive probability θ, the stock hits zero with positive probability in finite periods.

Problem 2.4. Provide the sufficient conditions for inventories to be positive. Specifically, provide the conditions under which, given an initial inventory of zero, $x_0 = 0$, positive inventories are accumulated in some period with positive probability.

Solution. Consider the following:

$$\beta E[u'(w)] > u'(M) + \varphi'(0)$$

which means that from an initial inventory of zero, it's always desirable to store something (deviate from the policy of zero storage) when the endowment shock is high. This guarantees that positive inventories are accumulated in some period with some positive probability. (Note that other sufficient conditions can also be formulated).

Problem 2.5. Provide sufficient conditions for inventories to be always zero. Specifically, provide conditions under which any initial inventory $x_0 > 0$ is depleted in finite time, and no new inventories are ever accumulated.

Solution. We want to find a condition under which if the stock from the previous period is zero, no new stock will be accumulated. We want:

$$\beta E[u'(w)] \leq u'(M) + \varphi'(0)$$

which means that even if the agent receives the highest endowment shock possible, he will still consume everything.

2.3 Reflection on Assignment 01

These are Professor Stokey's reflections on Assignment 01.

2.3.1 Usage of Second Welfare Theorem

Macroeconomics relies heavily on the Second Welfare Theorem, but the conditions need to be surely met before we can use them. For the RBC model, this is okay. When there are externalities, taxes, learning by doing, money, or price-setting, then you cannot use it.

2.3.2 Finite Horizon Problems

How should we think about using DP given finite horizons?

▷ Fix the length of the horizon at N with $x \in X$ – a deterministic steup. The return function is given as $F(x, y)$ accompanied by the correspondence $\Gamma(x)$.

▷ To solve this, we have to use "backward induction." We also note that the value function depends on time.

▷ In period N, the value function is

$$v(x, N) = \max_{y \in \Gamma(x)} F(x, y)$$

In the context of the growth model, we have no investment. Useful for thinking about human capital allocation – in the last period, the agent will not be investing.

▷ For periods $n < N$, the value function is

$$v(x, n) = \max_{y \in \Gamma(x)} [F(x, y) + \beta v(x, n+1)]$$

We now consider the properties of the value function and the associated operator.

▷ Concavity, monotonicity, and differentiability of the value function are inherited from F and Γ.

▷ We can also show that the Bellman operator T preserves these properties, which was the same as in infinite horizon problems.

▷ Note that with finite horizons, T preserves differentiability because it is a finite sum of differentiable functions. We cannot make thes same argument for the infinite-horizon case, since it would require taking the limit of the finite case, but we don't know if the limiting function is differentiable. In particular, the space of differentiable functions is not closed, so the sequence of differentiable functions does not necessarily converge to a differentiable function.

2.4 Application #2: Real Business Cycles

The RBC model is a simply a growth model with stochastic shocks to the aggregate technology. Hence the competitive equilibrium is Pareto-efficient, and it can be studied as the solution to a standard (stochastic) DP problem, where the decision maker is the Social Planner. It took the profession by a surprise that a simple model was able to *quantitatively* match the stylized facts so nicely.

The main takeaway is that governments should not respond proactively to business cycles. The RBC model says that business cycles are an outcome of efficient response to productivity shocks.

2.4.1 Stylized Facts on Business Cycles

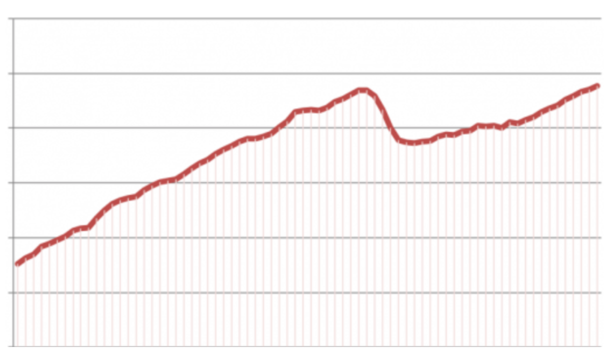
Lucas' main insight is that responses of the economy – the coherence and the amplitude of the series – are fairly common and standard. Others include:

- ▷ Length and height of business cycles differ.
- ▷ Output in all sectors move together (i.e. highly correlated).
- ▷ Prices are procyclical.
- ▷ Production of durables fluctuates more than production of non-durables; services even less.
- ▷ Short-term interest rates are procyclical, long-term interest rates are only slightly procyclical. Money velocity is procyclical (may not be true anymore).

We start with some stylized facts regarding business cycles:

1. Responses to business cycles have commonalities (based on data up to the 1990s):
   - ▷ Fluctuations in *output* and *hours worked* are nearly equal.
   - ▷ Employment fluctuates almost as much as output and total hours worked; average hours per work fluctuatues much less (which implies that labor adjusts at the extensive margin)
   - ▷ Consumption fluctuates less than output; investment fluctuates more than output; goverment expenditures is independent of output.
   - ▷ Consumption of non-durable goods is much smoother than output. Production of consumer durables fluctuates much more.
   - ▷ Capital stock fluctuates much less and is marginally uncorrelated with output.
   - ▷ Proudctivity is slightly pro-cyclical but considerably less than output.
   - ▷ * Wages vary less than productivity.
   - ▷ Imports are more procylical than exports.
2. Causes of business cycles are broadly in two buckets: real shocks and financial / monetary shocks. From an econometrician's perspective, distinguishing these two shocks is very difficult (for example, oil price shocks and the subsequent central bank policies)

Business cycles are fluctuations, but fluctuations around what? Up to 2008, the fluctuations were around a constant trend that can be seen from the graph of $\log y_t$ against time where y_t is the per capita output:



As we can we see in the graph, the per capita output has recovered its growth trend but not back on the level terms. Note that there are many ways to detrend – one way they do it is the Hodrick-Prescott (HP) filter.

2.4.2 Hodrick-Prescott (HP) Filter

Suppose $y_t = g_t + d_t$. Choose a decomposition $\{q_t, d_t\}$ to solve

$$\min \left[\sum_{t=1}^{T} d_t^2 + \lambda \sum_{t=1}^{T-1} \left[(g_{t+1} - g_t) - (g_t - g_{t-1}) \right]^2 \right]$$

where $\lambda > 0$. $\lambda = 0$ implies $q_t = y_t$; $\lambda = +\infty$ implies g_t is constant, i.e. a linear trend. The first term is the deviations from trend, and the second term is the changes in the trend line itself. λ is the penality we put on changing the slope of the trend line.

2.4.3 RBC Model Overview

For one doing an empirical analysis, these are the steps you would take:

1. Detrend the time series to get the "BC" part.
2. Choose preferences $u(c, l)$ and the technology $F(k, n)$ with household constraint that $n + l = 1$.
   - ▷ F: constant returns to scale
   - ▷ u: are these preferences are consistent with long-run growth?
     - * We saw that consumption has grown a lot, whereas leisure hasn't grown a lot.
     - * We want preferences that allow an increase in consumption and a relatively constant labor supply.
   - ▷ z: productivity shock
     - * We don't specify exactly what the source of this productivity shock is, but the response to such shock seems to be similar across countries. Even if the shock is a monetary shock, it could play out in a similar way.
3. Market Structure: Households own the capital stock directly, and they rent it out to firms. Assume households own the firms. Note that there are no profits since the market is perfectly competitive.

Household The representative household's problem is

$$\max_{\{c_t, k_{t+1}, n_t\}} E \left[\sum_{t=0}^{\infty} \beta u(c_t, 1 - n_t) \right]$$
$$\text{s.t. } c_t + k_{t+1} - (1 - \delta) k_t = w_t n_t + r_t k_t$$

Firm The representative firm's problem is

$$\max_{\{k_t, n_t\}} \left[e^{z_t} F(k_t, n_t) - w_t n_t - r_t k_t \right]$$

Note that this implies $e^{z_t} F_k = r_t$ and $e^{z_t} F_n = w_t$. All factors are employed.

Problem 2.6. Write the Social Planner's Bellman Equation.

Solution. Note that we can appeal to the Second Welfare Theorem. Stochastic shocks are okay; there are no taxes, externalities, or money.

- ▷ The state variables: (k, z)

▷ Then the BE is given as

$$v(k,z) = \max_{c,n,y} \left\{ u(c,1-n) + \beta \underbrace{E\left[v\left(y,z'\right)|z\right]}_{\int v(y,z')\varphi(z,z')dz'} \right\}$$
$$\text{s.t. } c + [y-(1-\delta)k] \le e^{z} F(k,n)$$

▷ One can reduce this BE into a two-step process:

* Choose $c^*(y), n^*(y)$ such that you maximize $u(c,1-n)$ subject to the above constraint.
* Then you plug c^*, n^* into the original BE such that you choose y.

Problem 2.7. Briefly explain what is needed to match the BC facts.

Solution. Empirically, $\{z_t\}$ is modeled as an AR(1) process with very high level of persistence. This fits well with the quarterly data. For the BE, we assume z to be a first-order Markov and is monotone (satisfies FOSD). Tax rebates are a poor candidate for z_t since they are not persistent. The persistence is key to the success of the RBC model, and this is something that's verified in the data.

What does this persistence introduce? If we have a high shock today, chances are things are good for the next few more periods. So households decide to build up capital and increase labor supply. The goal of RBC model was to create a model for which (c, h, k') move together with z, which can be interpreted as total factor productivity shock. Note that this is a *supply-side* shock. Why not a demand shock? The problem is that demand shocks will not produce a result in which c and k' move together—since income is not higher with higher demand, higher consumption implies lower saving, i.e. k'.

Problem 2.8. State the FOCs and the EC and obtain the Euler Equation.

Solution. Letting λ denote the Lagrange multiplier on the feasibility constraint, construct the Lagrangian:

$$u(c,1-n) + \beta E\left[v\left(y,z'\right)|z\right] + \lambda\left\{e^{z}F(k,n) - c - (y-(1-\delta)k)\right\}$$

Taking the derivative with respect to each choice variable, the first-order conditions are

$$\begin{aligned} &\{c\} \quad u_c(c,1-n) = \lambda \\ &\{n\} \quad u_\ell(c,1-n) = \lambda e^{z} F_n(k,n) \\ &\{k\} \quad \beta\mathbb{E}\left[v_k\left(k',z'\right)|z\right] = \lambda. \end{aligned}$$

and taking the derivative with respect to each endogenous state variable, which is just k, we get the envelope condition:

$$v_k(k,z) = \lambda\left(e^{z}F_k(k,n) + (1-\delta)\right)$$

Combining the FOC yields

$$MRS = \frac{u_\ell}{u_c} = e^{z}F_n(k,n) = MRT$$

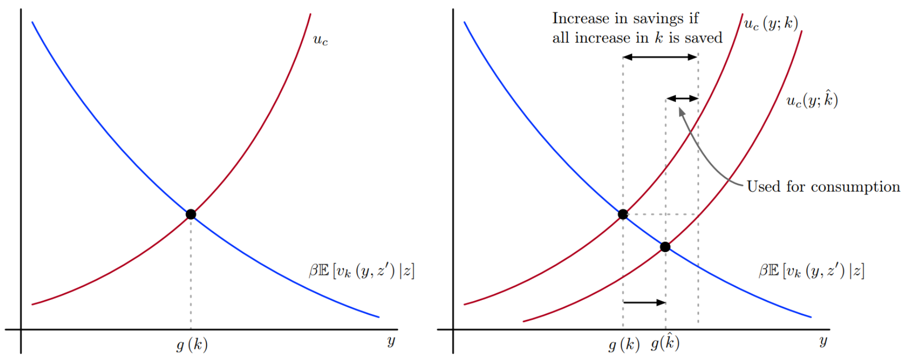
and the Euler Equation is

$$u_c(c,1-n) = \beta\mathbb{E}\left[u_c\left(c',1-n'\right)\left\{e^{z'}F_k\left(k',1-n'\right) + (1-\delta)\right\}|z\right]$$

Problem 2.9. Suppose labor supply is inelastic, $n = \bar{n}$. What happens to capital stock next period given an increase in current capital stock? What happens to capital stock next period if the current shock was higher?

Solution. Suppose current capital stock (k) increases. For interior solutions, the Euler Equation is

$$\beta\mathbb{E}\left[v_k\left(k', z'\right)|z\right] = u_c\left(c, 1-n\right) = u_c\left(e^z F\left(k, h\right) + \left(1-\delta\right)k - k', 1-h\right)$$



What about when current shock (z) was higher? Note that the u_c curve still shifts down. As for the LHS, we can estalish that v is increasing in z, but we don't know if the derivative is also increasing in z. Even the FOSD does not help us in showing this relationship since even though v may be increasing in z, we do not know if v_k is also increasing in z or whether the policy function is increasing in z.

Why is this the case? The FOSD – better distribution tomorrow – has two contrasting effects: (1) It induces the households to save less since the future is better, but a higher shock today implies higher savings today The net effect is therefore ambiguous.

Problem 2.10. Suppose labor supply is elastic. What happens to capital stock next period given an increase in current capital stock? What happens to capital stock next period if the current shock was higher?

Solution. Skipped. To approach this, we now fix (k, z) and write $c^*\left(k'; k, z\right)$ and $n^*\left(k'; k, z\right)$ as the optimal choice of the agent. Ultimately, we can show that the comparative statics are similar to the previous case assuming inelastic labor supply.

Now specialize to convenient functional forms:

$$F\left(k, n\right) = Ak^\alpha n^{1-\alpha}$$

and

$$\begin{aligned} u\left(c, 1-n\right) &= \ln c - \theta\frac{n^{1+\gamma}}{1+\gamma} \\ &= \ln c + \theta\ln\left(1-n\right) \end{aligned}$$

Note that the utility functions are popular since they are compatible with balanced growth.

Problem 2.11. Briefly explain how you would compute this model.

Solution. Start by assuming the shock process has a unique ergodic set. This is equivalent to saying that there is a unique stationary distribution.

1. Calibrate the LR expected value $\bar{z}$. For convenience, we can let $\bar{z} = 0$ so $e^{\bar{z}} = 1$.
2. Solve for the deterministic steady state with $\bar{z} = 0$, denoted as $\left(\bar{k}, \bar{c}, \bar{n}\right)$.
3. Take a Taylor series approximation around the Euler Equation, and you guess (and a verify a solution) that the solution is linear in z, i.e.
$$\hat{c} = a_1 \hat{k} + a_2 \hat{z}, \hat{n} = b_1 \hat{k} + b_2 \hat{z}, \hat{k} = c_1 \hat{k} + c_2 \hat{z}$$
where the hat denotes deviation from the steady state.
4. For quantitative calibration, obtain data for y_t, k_t, n_t, and α. Furthermore, since our model does not have growth, we would need to de-trend the data. Then we can back out z_t as the Solow residual.

Problem 2.12. Can you model this as a recursive EQ? When is this approach particularly useful?

Solution. Yes. The value function would look similar. The individual's budget is defined as a function of wages and returns on the capital (as we did in problem set 3.) In the end, you get the same set of equations. Note that using a recursive EQ approach would be necessary if an economy with price-setting firms, externalities, money, and taxes.

Problem 2.13. How would you extend this model?

Solution. You can differentiate between durable and non-durable consumption goods; different types of capital (residential vs. non-residential). This model is convenient for adding a lot of elaboration.

3 Asset Pricing

"Partial equilibrium doesn't make sense in asset pricing." – Nancy Stokey

3.1 Lucas Tree

The motivation was to provide macroeconomic setup into the efficients market ideas prevalent in finance. The idea was to build a GE that is convenient for addressing the topics in finance. The key questions are: how do asset prices depend on (1) preferences and (2) stochastic shocks that affect dividends?

You certainly need a GE model (as opposed to PE model) but a simplest possible model. Here, there is one consumption good and lots of assets. To make things even simpler, there is no production and only endowment. Therefore, the only trade here is trading consumption goods for shares in these assets.

3.1.1 Setup

In this model, there is a state variable, of which dividends are functions. There are no idiosyncratic shocks to individual people. The goal is to study asset prices as a function of the current shock. In general, we will adhere to the following steps:

1. Look at the maximizing problem of the representative household and obtain the FOCs, taking prices as given.
   - ▷ Here, the price refers to a function that maps current state into the prices.
2. Impose equilibrium conditions i.e. market clearing (such that no one wants to trade) and derive expressions for the prices.

Problem 3.1. What is the role of rational expectations in this setup?

Solution. *Rational Expectations* asserts that outcomes do not differ systematically from what people expected them to be. It does not deny that people often make forecasting errors, but it does suggest that errors will not persistently occur on one side or the other. In this problem, rational expectations implies that the agent knows what the price mapping is.

Notation There are J states and N assets. $\mathbf{1}$ is the endowment of assets – this is the only endowments agents receive, which can be used to purchase consumption goods and/or assets. $Q = [q_{kj}]_{J\times J}$ is the transition matrix that describes the probability of moving from state j to state k tomorrow. $Y = [y_{jn}]_{J\times N}$ is the dividends matrix, where the jth row $\mathbf{y}_j$ gives a vector of dividends for all the N assets in state j. $P = [p_{jn}]_{J\times N}$ is the price matrix, defined in terms of the contemporaneous consumption good. $\mathbf{z}$ is a portfolio.

One unique aspect of this setup is the following $Z = [1-\epsilon, 1+\epsilon]^N$ which is the (bounded) space of allowable portfolios. It takes this form because we know that in equilibrium consumers will hold $\mathbf{1}$ as their portfolio, so to obtain differentiability, we only need some values around $\mathbf{1}$.

Note that the states follow a first-order Markov process, but this does not imply that dividends follow a first-order Markov process. In this case, observing onyl the dividends does not allow us to identify the current state.

Preferences The preferences are given as

$$u(\mathbf{c}) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta u(c_t) \right], \beta \in (0,1)$$

where $u(\cdot)$ is strictly increasing, strictly concave, and once differentiable.

Assumptions

1. (Non-denegerate assets) Every asset pays some dividends in some state; $\forall n, y_{jn} > 0$ for some j.
2. (Non-degenerate states) Every state has at least one asset that pays dividends: $\forall j, y_{jn} > 0$ for some n.
3. If the agent holds the endowment $\mathbf{1}$, his income will be strictly positive; $\mathbf{y}_j \cdot \mathbf{1} > 0$ for all j;
4. Q – the transition matrix – has one ergodic set and no transient states. This is to guarantee that $\mathbf{p}_j \gg \mathbf{0}$ for any j.

Markets Each period, current consumption goods are traded for various assets. Here we will assume assets are traded *after* the current dividend is paid. This implies that the price of assets only includes the future expected dividends, not the current-period dividends.

1. At the beginning of period, the consumer holds $\mathbf{z}$ as his portfolio.
2. The state of the world j is realized and informed to the agents. This determines the probability distribution for the next period's state as governed by Q.
3. Agents receive their share of dividends: sum of dividends are given by $\mathbf{y}_j \cdot \mathbf{z}$.
4. Asset market opens and agents trade based on the price.
5. After trading, the agent hold a portfolio $\mathbf{z}'$ and he consumes c.

Since we already know the equilibrium outcome, our goal is to find prices P such that the agent chooses to walk out of the asset market with the portfolio that he walked in with, i.e. $\mathbf{z} = 1$ in all periods.

3.1.2 Bellman Equation

The state consists of a pair $(\mathbf{z}, j)$ where $\mathbf{z}$ is the current portfolio and j is the state. Denote future portfolio as $\mathbf{z}'$ and next state as k. We also use a p to denote that the individual maximization problem takes price p as given:

$$v^p(\mathbf{z}, j) = \max_{c \geq 0, \mathbf{z}' \in \Gamma(\mathbf{z}, j)} \left\{ u(c) + \beta \mathbb{E}_Q \left[v^p(\mathbf{z}', k) \,|j \right] \right\}$$

$$\text{s.t. } c + \underbrace{\sum_{n=1}^{N} p_n \left(z_n' - z_n \right)}_{= \mathbf{p}_j (\mathbf{z}' - \mathbf{z})} \leq \sum_{n=1}^{N} y_{jn} z_n$$

where $\Gamma(\mathbf{z}, j) = \{ x \in \mathcal{Z} : c_j = \mathbf{y}_j \mathbf{z} + \mathbf{p}_j (\mathbf{z} - x) \geq 0 \}$ adds the non-negativity constraint of consumption. Thus, the BE is

$$v^p(\mathbf{z}, j) = \max_{\mathbf{x} \in \Gamma(\mathbf{z}, j)} \left\{ u\left(\mathbf{y}_j . \mathbf{z} + \mathbf{p}_j . (\mathbf{z} - \mathbf{x}) \right) + \beta \sum_{k=1}^{J} q_{jk} v^P(\mathbf{x}, k) \right\}$$

where

$$\Gamma(\mathbf{z}, j) = \{ x \in \mathcal{Z} : c_j = \mathbf{y}_j \mathbf{z} + \mathbf{p}_j (\mathbf{z} - x) \geq 0 \}$$

Problem 3.2. State the FOC and the EE of the above BE. What is the interpretation?

Solution. The FOC is given as

$$u'\left[\mathbf{y}_{j\cdot}\mathbf{z}+\mathbf{p}_{j\cdot}\left(\mathbf{z}-\mathbf{x}^{*}\right)\right]p_{jn}=\beta\sum_{k=1}^{J}q_{jk}v_{k}^{p}\left(\mathbf{x}^{*},k\right),\forall j,n$$

and the EC is

$$v_{n}^{p}\left(\mathbf{z},j\right)=u'\left[\mathbf{y}_{j\cdot}\mathbf{z}+\mathbf{p}_{j\cdot}\left(\mathbf{z}-\mathbf{x}^{*}\right)\right]\left(y_{jn}+p_{jn}\right),\forall j,n$$

The interpretation of the EC is that the marginal value of having more of asset n is equal to the marginal increase in the form of dividends (y_{jn}) and the value of the endowment (p_{jn}), evaluated at the current marginal utility.

3.1.3 Setting up Recursive Equilibrium

A recursive equilibrium in this context is a price matrix P^e associated with the value function v^e and an optimal policy correspondence G^e such that the following market clearing conditions hold:

$$\mathbf{x}^{*}=\mathbf{1}\in G^{e}\left(\mathbf{1},j\right),\ \forall j\in\mathcal{J}.$$

That is, if the agent has $\mathbf{z}=\mathbf{1}$ as his endowment at the beginning of the period, it is optimal for him to choose the next period portfolio $\mathbf{x}^{*}=\mathbf{1}$ (note, it could also be optimal to choose other portfolios). Of course, the equilibrium conditions also include the optimality conditions—i.e. the first-order and the envelope conditions.

3.1.4 Solving for the Price Matrix

We want to do show that (1) there exists an RCE and (2) that the RCE is unique.

Problem 3.3. Characterize the equations for the price matrix P^e. Start by considering the equilibrium levels of consumption.

Solution. Suppose now that $P=P^e$. Define c_j^e as the level of consumption in state j given portfolio of $\mathbf{x}^{*}=\mathbf{z}=\mathbf{1}$:.

$$c_{j}^{e}=\mathbf{y}_{j\cdot}\mathbf{1}+\mathbf{p}_{j\cdot}^{e}\left(\mathbf{1}-\mathbf{1}\right)=\mathbf{y}_{j\cdot}\mathbf{1},\ \forall j\in\mathcal{J}$$

Then, the first-order condition becomes

$$u'\left(c_{j}^{e}\right)p_{jn}^{e}=\beta\sum_{k=1}^{J}q_{jk}v_{n}^{e}\left(\mathbf{1},k\right),\ \forall j,n.$$

The envelope condition evaluated at $\mathbf{z}=\mathbf{x}^{*}=\mathbf{1}$ and $j=k$ is

$$v_{n}^{e}\left(\mathbf{1},k\right)=u'\left(c_{k}^{e}\right)\left(y_{kn}+p_{kn}^{e}\right),\ \forall n.$$

Combining the two together yields the Euler equation:

$$u'\left(c_{j}^{e}\right)p_{jn}^{e}=\beta\sum_{k=1}^{J}q_{jk}u'\left(c_{k}^{e}\right)\left(y_{kn}+p_{kn}^{e}\right),\ \forall j,n.$$

This gives $J\times N$ equations in $J\times N$ unknowns (i.e. $[p_{ij}^{e}]$). We see that the price of asset n, $\mathbf{p}_{\cdot n}=(p_{1n},p_{2n},\ldots,p_{Jn})$ depends on

1. Its own prices and dividends in the next period

▷ Variability of assets n's dividends, $\mathbf{y}_{\cdot n} = (y_{1n}, y_{2n}, \ldots, y_{Jn})$;

2. $u'\left(c_j^e\right)$, $\forall j$; i.e. risk aversion and variability of consumption.

   ▷ Dividends of other assets $\mathbf{y}_{\cdot m}$ do not affect price of asset n directly, but only through the indirect effect via $u'(c_k^e)$.

3. Covariance terms between dividends and u'.

4. "persistence" in the state Q.

Problem 3.4. Prove that there exists an equilibrium price matrix P^e and that it is unique.

Solution. Lucas's idea was to compute the prices in terms of utilities, and then map them back into the consumption space. The intuition here is that the prices in the equation above always get multiplied by marginal utilities, so we first solve for "marginal utility prices" and show that these are unique. And then we come back to expressing prices as functions of consumption.

The sketch of the proof is shown below. Define:

$$\begin{aligned} a_j &:= u'\left(c_j^e\right), \ \forall j, \\ h_{jn} &:= \beta \sum_{k=1}^{J} q_{jk} a_k y_{kn}, \ \forall j, n, \\ \varphi_{jn}^e &:= a_j p_{jn}^e, \ \forall j, n, \end{aligned}$$

where h_{jn} is the discounted expected value of next-period dividends in utility terms and φ_{jn} is the price of asset n in state j in utility terms. Then rewrite the previous expression characterizing equilibrium prices:

$$\varphi_{jn}^e = h_{jn} + \beta \sum_{k=1}^{J} q_{jk} \varphi_{kn}^e, \ \forall j, n$$

which can be written in vector form (stacking across the j's)

$$\begin{aligned} \boldsymbol{\varphi}_{\cdot n}^e &= \mathbf{h}_{\cdot n} + \beta \sum_{k=1}^{J} \mathbf{q}_{\cdot k} \varphi_{kn}^e \\ &= \mathbf{h}_{\cdot n} + \beta Q \boldsymbol{\varphi}_{\cdot n}^e, \ \forall n \end{aligned}$$

This looks like a Bellman equation! For each asset n, the condition above defines an operator mapping $\mathbf{f}_{\cdot n}$ to $T\mathbf{f}_{\cdot n}$:

$$T\mathbf{f}_{\cdot n} = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n}.$$

Note, in particular, that this is a contraction.[1] Hence, we can interpret $\boldsymbol{\varphi}_{\cdot n}^e$ as a fixed point. This also allows us to use the Contraction Mapping Theorem. To emphasise, we can write above as

$$T\mathbf{f}_{[j]n} = \mathbf{h}_{\cdot n} + \beta \mathbb{E}\left[\mathbf{f}_{[j']n} | j\right].$$

[1] Check Blackwell's sufficient conditions.

(Monotone) If $\mathbf{f}_{\cdot n} \geq \mathbf{g}_{\cdot n}$, then

$$T\mathbf{f}_{\cdot n} = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n} \geq \mathbf{h}_{\cdot n} + \beta Q \mathbf{g}_{\cdot n} = T\mathbf{g}_{\cdot n}.$$

(Discounting) Let $a \geq 0$, then

$$\begin{aligned} T\left(\mathbf{f}_{\cdot n} + a\right) &= \mathbf{h}_{\cdot n} + \beta Q \left(\mathbf{f}_{\cdot n} + a\mathbf{I}\right) = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n} + a\beta Q \mathbf{I} \\ &= T\mathbf{f}_{\cdot n} + a\beta Q \mathbf{I} \leq T\mathbf{f}_{\cdot n} + a\beta \mathbf{I}, \end{aligned}$$

where the last (in)equality holds because the rows of Q sum to one by definition.

Since $\boldsymbol{\varphi}^e_{\cdot n}$ is in units of (contemporaneous) consumption good, we can write in matrix form:

$$\Phi^e = AP^e$$

$$\begin{bmatrix} \varphi^e_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{J1} & \cdots & \varphi^e_{JN} \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_J \end{bmatrix} \begin{bmatrix} p^e_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{J1} & \cdots & p^e_{JN} \end{bmatrix},$$

where A is a diagonal matrix of a_j's, Φ^e is the matrix of prices denominated in utilities. We can then obtain P^e as

$$P^e = A^{-1}\Phi^e$$

or

$$p^e_{jn} = \frac{1}{a_j}\varphi_{jn}, \forall j, n$$

Notice that since A is a diagonal matrix with strictly positive elements (utility functions are strictly increasing), A^{-1} is well-defined.

Problem 3.5. Prove that the prices $\varphi^e_{\cdot,n}$ (and therefore p^e_{jn}) are strictly positive.

Solution. We can first show that it is not equal to zero. Given our assumption about one ergodic set, we can assume that $\mathbf{h}_{\cdot n}$ will be positive in at least one of the elements, so this yields a contradiction – we started with a zero vector and ended with a non-zero vector.

Problem 3.6. Comment on how we might modify this model into a setup where two agents are allowed to trade.

Solution. We will have to specify the initial ownership, so the prices will inevitably depend on the initial ownership. It will be more complicated to work out, but you can do it. You could look for an initial endowment such that both agents are unwilling to trade. Suppose for example that the two agents have different utility functions – one more risk averse than the other. You probably want to give the same discount factor.

3.1.5 Complete Markets

Linear Dependence We define a portfolio to be a vector $\Lambda = (\lambda_1, ..., \lambda_N)$ where N is the number of assets. Suppose there exists a portfolio $\Lambda_0 \neq \mathbf{0}$ such that

$$Y\Lambda_0 = \mathbf{0}_{J\times 1}$$

which implies that you have an asset whose payoffs can be written as a linear combination of other assets. The price of this portfolio is then given by

$$P^e\Lambda_0 = A^{-1}\left(\beta Q + \beta^2 Q^2 + \cdots\right) AY\Lambda_0 = \mathbf{0}.$$

We call these assets to be lienarly dependent.

Spanning the Market Let $\mathbf{e}_j = (0, \ldots, 0, 1, 0 \ldots, 0)'$ be an $J \times 1$ vector with 1 in its jth element with zeros elsewhere. Suppose there exists a portfolio $\Lambda_{\cdot j}$ ($N \times 1$ vector) such that

$$Y\Lambda_{\cdot j} = \mathbf{e}_j,$$

then its price is given by

$$\left(\Pi^e_{\cdot j}\right)_{J\times 1} = \mathbf{P}^e \Lambda_{\cdot j},$$

where each row j gives the price of the portfolio in state j.

Problem 3.7. Show that if Y has the rank J, then the assets span the state space.

Solution. We need $\Lambda_{\cdot, j}$ such that $Y\Lambda_{\cdot j} = \mathbf{e}_j, \forall j$.

3.2 Reflections on the Midterm Exam

1. So many people like indicator functions. You lose a lot of style points. Indicator functions are ugly. Logical operations are slow in computers. If you go on the job market, I suggest you try and get rid of the indicator functions.
2. What's a state variable? It's not so precisely defined, but let me say that the point of dynamic programming is not to carry around the history of all the actions in the past. To say that all the actions in the past are state variables is to defeat the purpose of dynamic programming. What is a state variable? I would say it's something which is given at the beginning of the period, relevant for the decisions you make this period,... you know things which are constant forever are usually not the state variables.
3. There were some questions in the exam -- mostly professors are not looking to trip you up in the exams. We want you to learn. We are rooting for you to do well.

3.3 Building Intuition

Let's build some intuition about asset prices through simple examples.

3.3.1 Factors affecting Asset Prices

Recall that we had the equations:

$$AP^e = \beta Q A \left(Y + P^e\right)$$
$$P^e = A^{-1}\left(I - BQ\right)^{-1} BQAY$$

where A is a diagonal matrix of $a_j = u'(c_j)$; Q is the transition matrix; and Y are the dividends. Then the asset prices depend on:

1. A^{-1}, A : the marginal utility this period and next
2. $Y_{\cdot,n}$: the assets' own dividends
3. Variation in rows of Q
4. Covariance between margina utility and dividends

We will now illustrate through examples.

3.3.2 Example: $N = J = 1$

In this case, A is just a scalar and Q is equal to 1, so we have

$$P^e = \frac{\beta}{1-\beta} y$$

3.3.3 Example: $N = 1, J > 1$ with IID Dividends

If the shocks are IID, the rows of Q are all the same. Thus

$$\beta Q A \left(Y + P^e\right)$$

involves multiplying the rows of Q for each state, so it is identical across all states. The LHS has

$$AP^e$$

so the price is inversely related to the marginal utility. So high price implies low marginal utility, which indicates high endowment.

3.3.4 Example: $N = 1, J > 1$ with Positively Correlated Risky Dividend

Denote $y_1 < y_2 < \ldots < y_J$ with Q monotone. In this case, a higher dividend raises implies higher price in the future but lowers the marginal utility of consumption in the future as well. Since $N = 1$, we have $c_j = y_j$. Futher specifying our attention to CRRA preferences:

$$U(c) = \frac{1}{1-\theta} c^{1-\theta}, \theta > 0$$

this implies

$$a_j = e_j^{-\theta} = y_j^{-\theta}$$

Then

$$AY = \begin{bmatrix} y_1^{-\theta} & \\ & y_J^{-\theta} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_J \end{bmatrix} = \begin{bmatrix} y_1^{1-\theta} \\ \vdots \\ y_J^{1-\theta} \end{bmatrix}$$

If $\theta = 1$ (log utility) then AY is a constant. If $\theta < 1$ then AY is increasing in j. If $\theta > 1$ then AY is decreasing in j. Now from:

$$P^e = A^{-1} \left(I - \beta Q\right)^{-1} BQAY$$

One way to take a stab at this is to express the inverse as an infinite sum.

3.3.5 Example: N large, J large with IID Dividends and IID Assets

The aggregate consumption will be similar (same) across the state. Current dividends are not included in future prospects, so prices are the same across all states.

3.3.6 Example: N large, J large with Positively Correlated Dividends and IID Assets

The aggregate consumption will still be constant over time. But a high dividend today is a signal for higher dividends in the coming period, so the asset that currently has a higher dividend should have a higher price.

3.3.7 Example: $N = 1, J = 4$ and the Role of Expectations

Consider the stochastic payoff:

$$Y = \begin{bmatrix} L \\ L \\ H \\ H \end{bmatrix}, L < H$$

and

$$Q = \begin{bmatrix} 0.9 & 0.1 & & \\ 0.1 & 0.6 & 0.3 & \\ & 0.3 & 0.6 & 0.1 \\ & & 0.1 & 0.9 \end{bmatrix}$$

Between states 1 and 2, and states 3 and 4, the payoffs are the same but you have different expectations about the future. In state 1, it's likely that the bad times are likely to persist; in state 2, there's some chance that you get better. The high states are symmetric. In this case, the asset prices will depend not only on current dividend but also on the forecasts on what's going to happen in the economy. The pedagogical point here is that the current vector of dividends may not be only thing that's known.

3.4 *Importance of Aggregate Dividend

Consider two economies A and B with $J^A = J^B$ and same β, u, Q . But $Y^A \neq Y^B$. However $c = (c_j)$ and $A = [a_j]$ are the same. Now suppose for a certain asset, its payoff is the same in two economies:

$$y^A_{\cdot,n} = y^B_{\cdot,k}$$

Will they have the same prices? The marginal utilities and the dividends are the same in both economies; the transitions are the same in both economies; the covariance between the marginal utility of consumption and the dividend is the same in both economies. In this expression:

$$P^e = A^{-1} (I - \beta Q)^{-1} BQAY$$

the Ys are different but the specific column is the same for this asset.

The key point here is that the asset prices depend on the economy through the aggregate dividend. Therefore, the prices will be the same, even if one economy has a million assets and the other economy only has two assets.

Example 3.1. Suppose one asset in economy B has a payoff as a linear combination of the payoffs of assets in economy A :

$$y^B_{\cdot,h} = \sum_{n=1}^{N^A} \theta_n y^A_{\cdot,n}$$

Then the price of asset B must satisfy

$$p^B_{\cdot,h} = \sum_{n=1}^{N^A} \theta_n p^A_{\cdot,n}$$

The key thing to remember is that prices are linearin this case.

Example 3.2. Introduce a new asset with dividends $\epsilon y^B_{\cdot,k}$ where $\epsilon > 0$ is small into economy A . If $\epsilon > 0$, then the marginal utility of consumption will not change so we have

$$p^A_{\cdot,k} = \epsilon p^B_{\cdot,k}$$

Note that we haven't talked about spanning.

Example 3.3. Consider an economy with identical agents and a person with a different portfolio comes in. What would happen? (Left as food for thought).

4 Consumption and Savings

4.1 Aiyagari's Model

The key feature here is that markets are incomplete and that households face labour supply shocks that cannot be fully insured (i.e. no state-dependent bonds). Thus, unlike the RBC models which are driven by aggregate shocks (all individual shocks were insured by the complete market assumption), what drives the result in Aiyagari's model is the idiosyncratic shocks, as well as the borrowing constraints that the households face.

The factor ratios – rental rate $r = F_K$ and wage $w = F_L$ are constant since the aggregate levels of assets and labor are constant. The economy also features the same distribution of incomes across individuals, and people are bouncing around this invariant income distribution.

Broadly, the steps to solve this model are:

1. Characterise the optimal decisions of a household for some given (r, w), focusing on the decisions to accumulate assets.

2. Characterise the joint distribution of labour supply and assets across households.

3. Calculate average assets, k^{av}.

4. Check market clearing; i.e. ask if
$$r = F_K\left(k^{av}, 1\right), \; w = F_K\left(k^{av}, 1\right)?$$
If yes, we're done. If not, go back to step 1 with a different choice of (r, w).

Ultimately, we are interested in how much extra savings this plausibly leads to. This is a positive question, desgiend to deal with the aggregate wealth puzzle.

4.1.1 Setup

Labour

- ▷ Inelastically supplied by continuum of households of a unit measure.
- ▷ Labour supplied by each household, $\ell \in L = \left[\underline{\ell}, \overline{\ell}\right]$, follows a first-order Markov process.
- ▷ $G\left(\ell'; \ell\right)$ is the transition CDF with support $[\underline{\ell}, \overline{\ell}]$ with $\underline{\ell} > 0$ (density is denoted as $g\left(\ell'; \ell\right)$). Let $\varphi_t\left(\ell\right)$ be the marginal density, then:

$$\varphi_{t+1}\left(\ell'\right) = \int_{\underline{\ell}}^{\overline{\ell}} g\left(\ell'; \ell\right) \varphi_t\left(\ell\right) d\ell$$
$$\begin{aligned} \Rightarrow \Phi_{t+1}\left(\hat{\ell}\right) \equiv \int_{\underline{\ell}}^{\hat{\ell}} \varphi_{t+1}\left(\ell'\right) d\ell' &= \int_{\underline{\ell}}^{\hat{\ell}} \left(\int_{\underline{\ell}}^{\overline{\ell}} g\left(\ell'; \ell\right) \varphi_t\left(\ell\right) d\ell\right) d\ell' \\ &= \int_{\underline{\ell}}^{\overline{\ell}} \int_{\underline{\ell}}^{\hat{\ell}} g\left(\ell'; \ell\right) d\ell' \varphi_t\left(\ell\right) d\ell \\ &= \int_{\underline{\ell}}^{\overline{\ell}} G\left(\hat{\ell}; \ell\right) \varphi_t\left(\ell\right) d\ell \end{aligned}$$

* G has the Feller property and has a unique invariant distribution.

▷ $\Phi(\ell)$ denotes the invariant CDF with density φ (i.e. $\varphi_{t+1} = \varphi_t = \varphi$):

$$\Phi\left(\hat{\ell}\right) = \int_{\underline{\ell}}^{\hat{\ell}} \varphi\left(\ell'\right) d\ell' = \int_{\underline{\ell}}^{\overline{\ell}} G\left(\hat{\ell};\ell\right) \varphi\left(\ell\right) d\ell,$$

which can be interpreted as (i) an individual's long-run average labour supply; or (ii) cross-section distribution at a point in time.

▷ Normalise so that aggregate labour supply is one:

$$\int_{\underline{\ell}}^{\overline{\ell}} \ell\varphi\left(\ell\right) d\ell = 1.$$

Preferences

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right], \ \beta = \frac{1}{1+\rho} \in (0,1),$$

where u is strictly increasing, strictly concave, continuously differentiable and $u'(0) = \infty$.

Technology Define

$$f(k) = F(k,1)$$

then

$$\begin{aligned} r &= f'(k) \\ w &= f(k) - kf'(k) \end{aligned}$$

where we used the fact that F is CRS and we normalised aggregated labour supply to be one. K is the average level of assets held across households (given a unit measure of households).

4.1.2 Household Problem

Let $(a,\ell) \in A \times [\underline{\ell}, \overline{\ell}]$ denote a state. We will define A later. The budget constraint for the household is

$$a' + c = (1+r)\,a + w\ell.$$

Income consists of savings (which are allowed to be negative here) from the previous period, $(1+r)\,a$, and labour income (which is stochastic here), $w\ell$. Total income can be spent on consumption, c, and savings for the next period, a'. The Bellman equation is

$$\begin{aligned} v(a,\ell) &= \max_{a' \in \Gamma(a,\ell)} \left\{ u\left((1+r)\,a + w\ell - a'\right) + \beta \int_{\underline{\ell}}^{\overline{\ell}} v\left(a',\ell'\right) dG\left(\ell',\ell\right) \right\}, \\ \Gamma(a,\ell) &= \left[B, (1+r)\,a + w\ell\right], \end{aligned}$$

where B is the borrowing limit.

Assumption #1: Borrowing limit The upper bound for a' is clearly the income that the household has. What about the lower bound? We want to allow a' to be negative (so that households can borrow) but rule out bankruptcy. We set the borrowing limit B to be such that, even if the household received the worst labour shock, $\underline{\ell}$ (i.e. lowest level of labour income), it can pay the interest on its debt; i.e.

$$w\underline{\ell} \geq -rB \Leftrightarrow B \geq -\frac{w\underline{\ell}}{r}.$$

Notice that this is implied by requiring consumption to be nonnegative since

$$w\underline{\ell} \geq -rB \Leftrightarrow \underbrace{(1+r)\,B + w\underline{\ell} - B}_{=c} \geq 0$$

We therefore define the borrowing limit as

$$B := -\frac{w\underline{\ell}}{r},$$

which is often called the *natural borrowing limit*.

Assumption #2: Time preference vs interest rate Suppose that an agent has asset a today. If he saves everything, then the present value from doing so is

$$\frac{a\,(1+r)}{1+\rho}.$$

Suppose that $r \geq \rho$, then

$$r \geq \rho \Leftrightarrow a\frac{(1+r)}{1+\rho} \geq a;$$

i.e. the present value from saving a is greater. In this case, the household will always want to accumulate more assets. To make the problem economically interesting (so that assets do not diverge), we impose that

$$r < \rho.$$

Assumption #3: Labor Supply We need the labor supply to be variable enough to encourage precautionary savings. This is affected by the interest rate, but if the following is true for some $\hat{\ell}$:

$$u'\left(w, \hat{\ell}\right) < \frac{1+r}{1+\rho} \int u'\left(w, \tilde{\ell}\right) q\left(\tilde{\ell}, \hat{\ell}\right) d\tilde{\ell}$$

Note that the curvature of u also comes into play since it becomes problematic if u is linear and thus u' is a constant. We need enough risk aversion here.

Assumption #4: State Space and Risk Aversion We want the state space to be compact for the problem to be well-defined. To do so, we need to make it bounded, but the worry is that for a sufficiently risk-averse preference, such a^{max} might not exist. (If the agent is super risk-averse, the agent has all the incentive to continue accumulating the assets).

So the idea is that we do need some risk aversion, but not too much! A suffiicent condition is that there is some level of asset a^{max} that satisfies:

$$u'\left(w\bar{\ell} + ra^{max}\right) \geq \frac{1+r}{1+\rho} u'\left(w\underline{\ell} + ra^{max}\right)$$

Basically, the marginal utility from consuming, even at the highest wealth possible, is still bigger than the utility from saving at the lowest wealth possible (fixing a^{max}). This ensures that the agent will always consume at a^{max} which is exactly what we need to ensure the existence of a^{max}. This is placing an *upper bound* on risk aversion.

4.1.3 Properties of the Value Function

Recall our setup:

$$v\left(a,\ell\right)=\max_{a'\in\Gamma\left(a,\ell\right)}\left\{u\left(\left(1+r\right)a+w\ell-a'\right)+\beta\int_{\underline{\ell}}^{\overline{\ell}}v\left(a',\ell'\right)dG\left(\ell',\ell\right)\right\},$$
$$\Gamma\left(a,\ell\right)=\left[B,\left(1+r\right)a+w\ell\right],$$

Standard arguments show that

- ▷ v is strictly increasing in a (the period-return function is strictly increasing in a and the feasibility set is monotone in a);
- ▷ v is strictly increasing in ℓ if G is monotone (the period-return function is strictly increasing in ℓ, the feasibility set is monotone in ℓ);
- ▷ v is differentiable in a if $a'\in\operatorname{int}\Gamma\left(a,\ell\right)$ since we have that v is strictly concave and F is continuously differentiable in the interior.

For strict concavity of v, we cannot argue from the Bellman function directly. This is because the period return function

$$u\left((1+r)a+w\ell-a'\right)$$

is not (jointly) strictly concave in a and a'. One way to verify this:

- ▷ Compute the Hessian directly and check that $F_{aa}F_{a'a'}-F_{aa'}^{2}>0$. For a linear function in a such as the above, you only get a weak inequality which only gives us weak inequality.

Thus, to show concavity, directly argue from the sequence problem

$$v\left(a,\ell\right)=\max_{\left\{c_t\right\}_{t=0}^{\infty}}E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right]$$

where you fix ℓ and choose two values a_1,a_2. We can show by brute force that v is concave since carrying out all assets cannot be an optimal policy.

4.1.4 Solving the Problem

The optimal policy is characterised by the first-order condition:

$$u'\left(\left(1+r\right)a+w\ell-a'\right)\geq\beta\mathbb{E}_{\ell'}\left[v_a\left(a',\ell'\right)|\ell\right],$$

where the inequality holds with $\geq$ if $a'=B$ (recall we ruled out the other boundary case, $a'=\left(1+r\right)a+w\ell$, given Inada condition that $u'\left(0\right)=+\infty$).

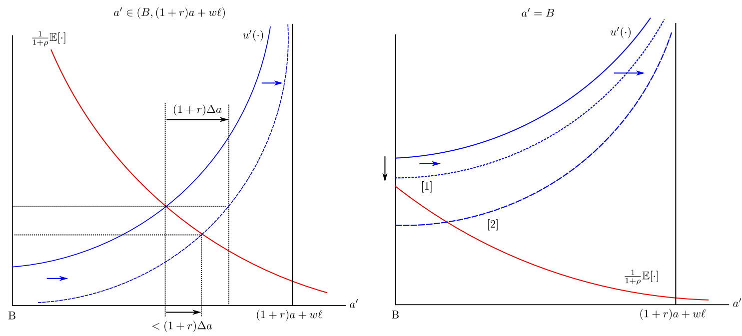
We also have the envelope condition:

$$v_a\left(a,\ell\right)=\left(1+r\right)u'\left(\left(1+r\right)a+w\ell-a'\right).$$

As usual, we can plot the two sides of the equation:

- ▷ LHS: Since u is strictly concave, u' is strictly decreasing. Thus, it is strictly increasing in a' and strictly decreasing in a. Also, as $a'\to\left(1+r\right)a+w\ell$, then $u'\left(\cdot\right)\to+\infty$ due to the Inada condition.
- ▷ RHS: Since v is strictly concave, v_a is strictly decreasing in a.

There are two cases to consider: $a' \in (B, \bar{a})$ and $a' = B$.



Effect of increase in a A higher a moves the LHS curve down and to the right. In the interior case, optimal policy a' is greater, although some of the increase in a translates to higher consumption. In the boundary case, optimal policy may not change (if the boundary condition is not broken) but consumption nevertheless increases. In case the LHS curve shifts to [1], then households consumes the entire increase in a. However, if the shift is to [2], then household splits the increase in a between consumption and savings.

Letting Δa denote the change in a, we can see from the figure that the increase in a', $\Delta a'$, is always less; i.e.

$$\Delta a' < (1+r)\,\Delta a.$$

This means that the slope of the optimal policy is less than $1+r$. So there are two cases:

▷ $\Delta a' < \Delta a < (1+r)\,\Delta a$ in which case the optimal policy has slope less than one;

▷ $\Delta a < \Delta a' < (1+r)\,\Delta a$ in which case the optimal policy ha slope between one and $(1+r)$.

The latter case holds if u' is sufficiently flat (i.e. utility is close to linear, which, in turn, implies that RHS is also flat). The significance of this case is explained next.

Effect of increase in ℓ A change in ℓ shifts both the LHS and RHS curves. Although we know that the LHS moves down and to the right, we do not know how the RHS moves. Therefore, the effect of the increase in ℓ on a' is ambiguous. If we assume i.i.d. shocks, then the RHS curve becomes independent of ℓ and so the comparative static becomes isomorphic to the case for a. Specifically, $s\,(a,\ell)$ is strictly increasing in ℓ with slope less than w.

Alternative Bellman equation The original BE was

$$v\,(a,\ell) = \max_{a' \in \Gamma(a,\ell)} \left\{ u\left((1+r)\,a + w\ell - a'\right) + \beta \int_{\underline{\ell}}^{\bar{\ell}} v\,(a',\ell')\,dG\,(\ell',\ell) \right\},$$
$$\Gamma\,(a,\ell) = [B, (1+r)\,a + w\ell]\,,$$

Alternately, we can set up the Bellman equation just after the point the household has observed its total cash-on-hand; i.e.

$$y = (1+r)\,a + w\ell.$$

In this case, the Bellman equation becomes

$$\hat{v}\,(y,\ell) = \max_{a' \in \hat{\Gamma}(y)} \left\{ u'\left(y - a'\right) + \beta \mathbb{E}_{\ell'}\left[\hat{v}\left((1+r)\,a' + w\ell', \ell'\right)|\ell\right] \right\},$$
$$\hat{\Gamma}\,(y) = [B, y]\,.$$

Notice that we can drop ℓ as a state variable if the shocks were i.i.d.

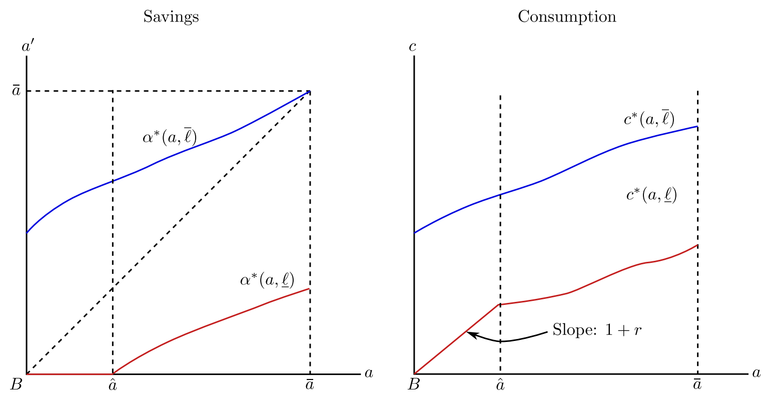
4.1.5 Optimal Policy

It's important to realize that the motive for saving in this economy is precautionary. Therefore, if we start out at ϵ assets and gets hit with $\underline{\ell}$, she will go to zero assets. It also helps that the states are ergodic, so the agent knows she is not going to be stuck at bad shocks. The ergodic distribution represents the cross-section in the entire economy. Let $\alpha^* (a, \ell)$ and $c^* (a, \ell)$ denote the optimal policies for asset and consumption respectively.

Slope of $\alpha^* \left(\cdot, \bar{\ell}\right)$ is less than 1 First, assume that the slope of $\alpha^* \left(\cdot, \bar{\ell}\right)$ is less than 1; i.e. the household saves less than one-to-one when a increases ($\Delta a' < \Delta a$). Recall $\bar{a}$ denotes the level of assets above which the agent has no more incentive to accumulate assets. Here, we defined this as

$$\bar{a} = \alpha^* \left(\bar{a}, \bar{\ell}\right);$$

i.e. we define $\bar{a}$ relative to $\bar{\ell}$ because we assume that $\alpha^* (\cdot, \ell)$ is ordered by ℓ.

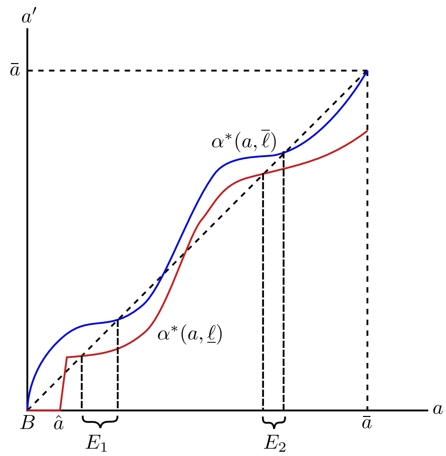


Observe that:

- ▷ Recall that we defined $\bar{a}$ as the point level of assets that no one would have incentive to save more—when ℓ is ordered, then this implies that $\alpha^* \left(a, \bar{\ell}\right)$ at $a = \overline{a}$ must intersect the 45 degree line so that $a' = a$; i.e. it is optimal not to save more.
- ▷ The flat portion of $\alpha^* (\alpha, \underline{\ell})$ represents boundary case in which the household would like to borrow but cannot, due to the borrowing constraint.
- ▷ The slope of $c^* (\alpha, \ell)$ is initially linear until $\tilde{a}$, the point at which the household begins accumulating assets. The slope before this point is $(1 + r)$ since a unit of a gives $(1 + r)$ units of consumption (see the budget constraint).

The ergodic set in this case is the entire state space $[B, \overline{a}]$. This means that, even in the long-run, there will be a positive "mass" of individuals who hold $a = 0$ assets.

Slope of $\alpha^* \left(\cdot, \bar{\ell}\right)$ is greater than 1 This means that $\Delta a' > \Delta a$. In this case, $\alpha^* \left(\cdot, \bar{\ell}\right)$ may cross the 45 degree line multiple times. The figure below shows the case in which there are 2 ergodic sets, which can be interpreted as representing a two-class society: E_2 is the rich, and E_1 is the poor. Observe that household in the transient states (T's) will diverge to the one of the surrounding ergodic set. More generally, there may be more ergodic and transient states (alternating) until $\bar{a}$.



Note that if the curve lies above the 45 degree line, then $a' > a$ so that households are accumulating assets (i.e. saving). On the other hand, if the curve lies below the 45 degree line, then $a' < a$ so that households are dissaving. The ergodic set is given by the interval defined by the point at which $\alpha^*\left(a,\bar{\ell}\right)$ crosses the 45 degree line from above to below, and the point at which $\alpha^*\left(a,\underline{\ell}\right)$ does the same. To the left (right) of these points, households are savings (dissaving) so that we are lead back to the same point.

Notice, in particular, that the interval $[B,\hat{a}]$ is not an ergodic set—hence, in this case, in the long run, everyone will hold some assets and we would not have any mass on $a=0$ as in the previous case.

In short, the ergodic set(s) is(are) the one where $\alpha^*\left(a,\bar{\ell}\right)$ stays above the 45-degree line and where $\alpha^*\left(a,\underline{\ell}\right)$ stays below the 45-degree line.

4.1.6 The Dynamics (with $B=0$ and IID shocks)

Now suppose that $B=0$ and that labour shocks are i.i.d. As we discussed above, in this case,

$$\begin{aligned}\hat{v}(y,\ell) &= \max_{a'\in\hat{\Gamma}(y)}\left\{u'\left(y-a'\right)+\beta\mathbb{E}_{\ell'}\left[\hat{v}\left(\left(1+r\right)a'+w\ell',\ell'\right)\right]\right\},\\ \hat{\Gamma}(y) &= [B,y].\end{aligned}$$

where $y=(1+r)a+w\ell$ and we may drop ℓ as the state variable.

1. We can write therefore write the optimal policy as

$$\alpha^*(a,\ell)=\sigma\left((1+r)a+w\ell\right),$$

   where σ is nondecreasing (strictly increasing if $\sigma>0$ since α^* is strictly increasing in both a and ℓ in the i.i.d. case with $B=0$).

2. Fix some asset level $a'=z$, then

$$\begin{aligned}\sigma\left((1+r)a+w\ell\right)\le z &\Leftrightarrow (1+r)a+w\ell\le\sigma^{-1}(z)\\ &\Leftrightarrow \ell\le\frac{1}{w}\left[\sigma^{-1}(z)-(1+r)a\right].\end{aligned}$$

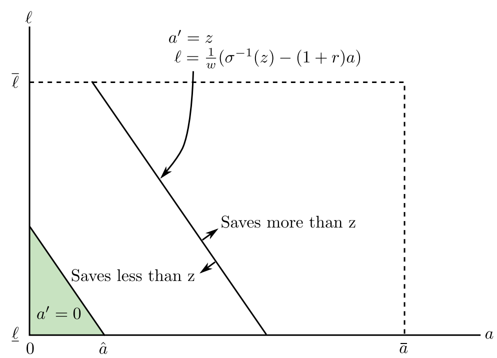
   ▷ For $z>0$, we may define σ^{-1} in the usual fashion since σ is strictly increasing.

- ▷ However, z, in general is zero, for a range of values of cash-in-hand. Thus, for $z = 0$, define σ^{-1} to be the largest value for $(1+r)\,a + w\ell$ that leads to $\sigma = 0$; i.e.

$$\sigma^{-1}(0) = \lim_{z \downarrow 0} \sigma^{-1}(z).$$

3. We can now write the "indifference curves" that depicts the combinations of a and ℓ that leads to the same value of savings $a' = z$. Note that the indifference curves are straight lines because the shocks are IID; if not, the picture will be qualitatively similar but definitely not a straight line.

   - ▷ Households with pairs of (a, ℓ) that lie above the indifference curve will save more than z.
   - ▷ Those that lie below the indifference curve will save less than z.
   - ▷ For $a' = z = 0$, observe that there will be mass of households (depicted by the green triangle) AND the entire left side of the rectangle (there is mass on this side as well)



4. Recall $\Phi(\ell)$ is the stationary distribution of labour supply and $\varphi(\ell)$ is its density. We are interested in $M_z(a)$, the stationary distribution of assets across agents (as-yet-unknown). For i.i.d. shocks, the product of the two gives the the joint distribution of (a, ℓ).

   - ▷ To solve for $M(a)$, we first compute the probability that a' is less than some $z \in [0, \overline{a}]$:

$$\begin{aligned}
\mathbb{P}\left(a' \le z\right) &= \mathbb{P}\left(\sigma\left((1+r)\,a + w\ell\right) \le z\right) \\
&= \mathbb{P}\left((1+r)\,a + w\ell \le \sigma^{-1}(z)\right) \\
&= \mathbb{P}\left(a \le \frac{\sigma^{-1}(z) - w\ell}{1+r}\right) \\
&= \int_{\underline{\ell}}^{\bar{\ell}} M_z\left(\frac{\sigma^{-1}(z) - w\ell}{1+r}\right) \varphi(\ell)\, d\ell,
\end{aligned}$$

   where the integral converts the distribution (a, ℓ) into a marginal distribution—the integral "sums" the probability that next period asset is less than z given a specific value of ℓ across all possible values of ℓ (since we may reach $a' \le z$ from many different levels of ℓ).

   - ▷ Now to characterize the stationary distribution, $M_z(a)$ can be characterised as a solution to fixed-point problems for each value of z:

$$M_z(a) = \int_{\underline{\ell}}^{\bar{\ell}} M_z\left(\frac{\sigma^{-1}(z) - w\ell}{1+r}\right) \varphi(\ell)\, d\ell, \ \forall z \in [0, \bar{a}].$$

5. With $M_z(a)$ in hand, we can calculate the average capital:

$$k^{av} = \int_0^{\bar{a}} a dM_z(a).$$

Then, we ask if the following conditions hold:

$$f'(k^{av}) = r,$$
$$f(k^{av}) - k^{av} f'(k^{av}) = w;$$

i.e. whether marginal product of capital is equated with the interest rate and whether the marginal product of labour is equated with the wage rate.

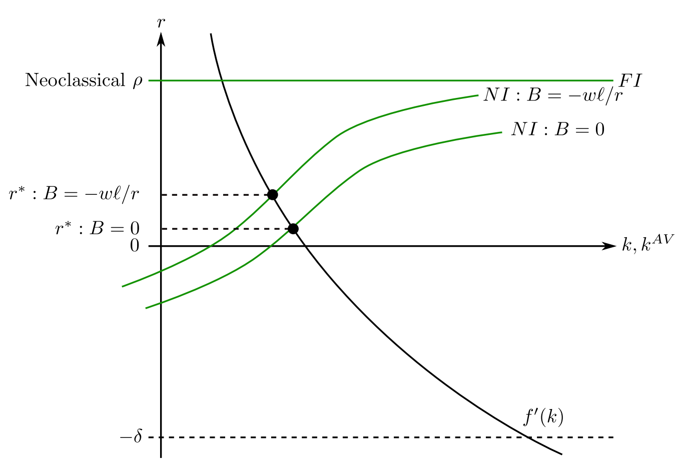
▷ In the full insurance/complete market case, the supply of capital is horizontal at ρ since they are willing to hold any amount of assets.

  * In this economy, you get the interest rate $r = \rho$ and a finite steady-state capital.

▷ In the incomplete market case, the interest rate will be lower but the magnitude depends on the borrowing limit.

  * The interest rate will be lower, and lower still for the case when $B = 0$ relative to $B = -w\ell/r$ (which is the borrowing limit at which even at the lowest income shock, the agent can still cover the interest rates). It will shift up as we make the borrowing constraints tighter.

▷ Note that supply of assets need not be increasing (i.e. the slope is sloping up; you just want them to cross). This is because an increase in k affects both the wage and the interest rate. An increase in k has both an income effect (from a higher w) and a substitution effect (from the lower interest). In the drawing, we will say the wealth effect dominates and draw it as an upward-sloping curve.



6. What if we want to compute r^* i.e. the crossing point? The $f'(k)$ curve is easy to compute, but how should we compute this?

▷ Newton's method; pick a point, compute the slope and take the next guess.

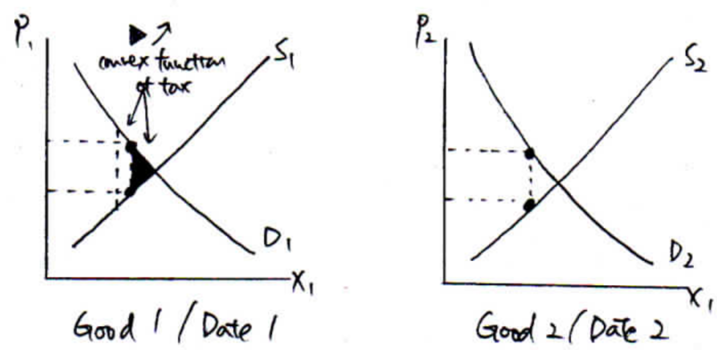
▷ Think linear; compute two points on each side by feeding in r, adjusting the w and finding the k such that they are sufficiently far away from the $f'(k)$ curve and draw a line. Find the point, compute the error and iterate.

5 Fiscal Policy

What's the best way to finance expenditure?

5.1 Motivation

We can easily think about taxing two different goods. We can easily see from the graph that tax creates dead-weight loss:



Now instead of thinking about them as separate goods, we can think of them as different dates.

Government Budget Constraint The budget constraint for the government is given as

$$\sum_{t=0}^{\infty} p_t \left(g_t + b_t^0 - T_t\right) \leq 0$$

b_t^0 is the amount of debt outstanding. The setup implies that the government has an incentive to manipulate the interest rates such that they can reduce the amount of debt outstanding.

Walras' Law If the households satisfy their budget constraints, *and* markets clear at every date, then the government BC holds.

Ricardian Equivalence If the taxes are lump-sum, only the total revenue matters since the taxes are non-distorting.

5.2 Recap of Ramsey Taxation

The setting is the taxation of capital and labor in a deterministic version of the NCG. We assume that the goverment has to fund a sequence of govenrment purchases using linear taxation. The Ramsey linear taxation refers to the representation of an equilibrium with taxes by the implementability constraint.

The eventual goal is to solve for equilibrium taxes $\{\tau_{\ell t}, \tau_{kt}\}$ that finances the exogenous stream of goverment purchases $\{g_t\}$. However, optimizing for $\{\tau_{\ell t}, \tau_{kt}\}$ is analytically difficult, we thus let goverment choose allocations instead of taxes. To do this, we need to eliminate taxes and prices from the household's budget constraint.

5.2.1 Solving a Ramsey Problem

A popular approach to solving a Ramsey problem is called the *primal approach*. The idea is to use first-order conditions for household optimization to eliminate taxes and prices in favor of quantities, then pose an optimization problem cast entirely in terms of quantities. After Ramsey quantities have been found, taxes and prices can then be unwound from the allocation. The primal approach uses four steps:

1. Obtain FOC of the household's problem and solve them for $\{p_s, \tau_s\}_{s=0}^{\infty}$ as functions of $\{c_s, n_s\}_{s=0}^{\infty}$.
2. Substitute these expressions for taxes and prices in terms of the allocation into the household's present-value budget constraint. This intertemporal constraint involves only the allocation and is regarded as an *implementability constraint.*
3. Find the allocation that maximizes the utility of the household subject to the feasibily constraint and the implementability condition. This is called the Ramsey allocation.
4. Use the Ramsey allocation with the formulas from step 1 to find taxes and prices.

5.3 Barro's Model

It's a nice warm-up exercise that gets to main features quite quickly through shortcuts. Barro will take the timing of expenditures as fixed, and in his world the government is going to choose just a sequence for revenues. He will have a function that describes the dead-weight loss. Given the contrived setup, the takeaways are not much but we have:

1. It took Ramsey's model and applied it to a macro model.
2. Constantly growing GDP is okay.
3. There is no need to keep a balanced budget every period.

5.3.1 Setup

We are given:

- ▷ $\{G_t\}_{t=0}^{\infty}$: exogenous government spending
- ▷ $\{Y_t\}_{t=0}^{\infty}$: exogenous income (ignore any effects from tax rates)
- ▷ b^0 : outstanding government debt at time $t = 0$
- ▷ r : constant (exogenous) interest rate (small open economy)

The government will choose:$\{R_t\}_{t=0}^{\infty}$: revenue sequence to minimize the loss function: $L_t = Y_t f(R_t/Y_t)$ where $f(0) = 0, f' > 0, f'' > 0$ and CRS.

5.3.2 Model

The government's problem is then to minimize the total value of distortions:

$$\min_{\{R_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t L_t$$
$$\text{s.t.} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t [G_t - R_t] + b^0 = 0$$

The first-order condition:

$$f'\left(\frac{R_t}{Y_t}\right) = \lambda, \forall t$$

which implies that $R_t/Y_t = \theta$ which is a constant, determined by the BC.

Exercise 5.1. $G_t = \bar{G}$ and $Y_t = \bar{Y}$ for all t, and $b_0 \geq 0$ is given. This implies $R_t = \bar{R} = \theta\bar{Y}$. Suppose $b_0 > 0$. What happens to the debt over time?

Solution. The optimal plan repays the interest on the debt but olls over the principle. Subtracting the two budget constraints for time 0 and 1, we have

$$b_1 - b_0 = (1+r)(G_0 - R_0) + rb_0$$

Exercise 5.2. Suppose $Y_t = \bar{Y}(1+r)^t$ and $G_t = \bar{G}(1+r)^t$. Then $R_t = (1+r)^t \bar{R}$. What happens to debt over time? (Assume that r is greater than the growth rate)

Solution. In this case, the government allows the debt to grow at the same rate that the real income is growing. In other words, b_t/Y_t ratio remains the constant.

There are more questions in PS6 (2018-19) that are useful for practice.

5.4 Lucas–Stokey's Model

This is a more elaborate (yet still a simple general equilibrium model) In constrast to Barro, we have complete markets. The key step in solving the model is instead of solving directly for tax rates, we solve for the allocation subject to the implementability constraint.

5.4.1 Setup

We adhere to the following notation:

- ▷ c_t is consumption; x_t is leisure; g_t is the government consumption; τ_t is the tax on labor income.
- ▷ Revenue is therefore given as $\tau_t(1 - \ell_t)$

The government takes as given $\{b_t^0\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$.

- ▷ Suppose at $t = 0$, the government can make binding commitments about future tax policy
- ▷ Assume that there are complete markets for state-contingent claims

5.4.2 Model

The resource constraint is given as

$$c_t + g_t = 1 - \ell_t$$

The representative agent's utility is given as

$$\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t)$$

In this case, the interest rates are endogenously determined.

Exercise 5.3. What if the government had access to a lump-sum tax?

Solution. Marginal utility of consumption and leisure will equal every period, and the government budget will be balanced every period. This is the "benchmark solution" that we will refer the subsequent solutions to.

5.4.3 Strategy for Solving the Model

1. Solve the Ramsey problem of the government.

    ▷ Government chooses the tax rates, and households optimize given the tax rates. So we first fix $\{\tau_t\}_{t=0}^{\infty}$ and $\{G_t\}_{t=0}^{\infty}$ and examine the HH's problem.

    ▷ Use the results to construct an "implementability constraint" (Key Step)

    ▷ Solve the government's problem and obtain $\{c_t, \ell_t\}_{t=0}^{\infty}$

    ▷ Back out the required tax rates to implement the solution

2. Show how to implement with segmented trading and limited markets

3. Ask about time consistency

1. Solving the Ramsey Problem By the *Ramsey* problem, we generally mean solving for an optimal policy in an environment where the initial government has a commitment technology that binds the actions of future governments. In this environment, therefore, the initial government chooses policy once and for all. Ramsey policies here consist of sequences of consumption taxes and money supplies.

1. Examine the household's problem: Given $\left\{b_s^0, p_s, \tau_s\right\}_{s=0}^{\infty}$:

$$\max_{\{c_s, x_s\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s U\left(c_s, \ell_s\right)$$
$$\text{s.t.} \sum_{s=0}^{\infty} p_s\left[c_s - b_s^0 - \left(1-\tau_s\right)\left(1-\ell_s\right)\right] \leq 0$$

    ▷ Interpretation: The demand for good is consumption; its resources are (1) labor income next of taxes and (2) payments on the bonds. (b_s^0 is some portfolio that it walks in with into period 0).

    ▷ First-order conditions: The Lagrangian is given as

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s U\left(c_s, \ell_s\right) - \lambda\left[\sum_{s=0}^{\infty} p_s\left[c_s - b_s^0 - \left(1-\tau_s\right)\left(1-\ell_s\right)\right]\right]$$

    * Taking the first-order condition with respect to c_t and ℓ_t:

$$\beta^t U_\ell\left(c_t, \ell_t\right) = p_t \lambda\left(1-\tau_t\right)$$
$$\beta^t U_c\left(c_t, \ell_t\right) = p_t \lambda$$

    * Taking the first-order condition with respect to c_t and c_0:

$$\beta^0 U_c\left(c_0, \ell_0\right) = \lambda p_0$$
$$\beta^t U_c\left(c_t, \ell_t\right) = \lambda p_t$$

    * Simplifying yields:

$$\frac{U_\ell\left(c_t, \ell_t\right)}{U_c\left(c_t, \ell_t\right)} = 1-\tau_t, \forall t$$
$$\beta^t \frac{U_c\left(c_t, \ell_t\right)}{U_c\left(c_0, \ell_0\right)} = \frac{p_t}{p_0}, \forall t$$

▷ Market must clear at each date (RC):

$$c_t + g_t + \ell_t = 1, \forall t$$

2. Cosntruct the implementability constraint (IC):

$$\sum_{s=0}^{\infty} \beta^s \left\{ U_c\left(c_s, \ell_s\right) \left[c_s - b_s^0\right] - U_\ell\left(c_s, \ell_s\right)\left[1 - \ell_s\right] \right\} \geq 0$$

▷ You can obtain this by setting the budget constraint to equality and substitute the ps and $(1 - \tau_s)$ from the FOC results.

Lemma 5.1. *Lemma 1: Given $\left\{g_t, b_t^0\right\}_{t=0}^{\infty}$, an allocation $\left\{c_t, \ell_t\right\}$ is implementable (with some tax policy $\left\{\tau_t\right\}_{t=0}^{\infty}$) if and only if it satisfies (IC) and RC)*

3. Solve the government's problem:

▷ The budget constraint is given as

$$\sum_{t=0}^{\infty} p_t \left[g_t + b_t^0 - \tau_t\left(1 - \ell_t\right)\right] \leq 0$$

* Use the previous results – $\tau_t = 1 - U_\ell / U_c$, $p_t = \beta^t U_c$, and $c_t + g_t + \ell_t = 1$ – to re-write:

$$\sum_{t=0}^{\infty} \beta^t \left[U_c\left(c_t, \ell_t\right)\left(c_t - b_t^0\right) - U_\ell\left(c_t, \ell_t\right)\left(1 - \ell_t\right)\right] \geq 0$$

▷ The Lagrangian for the government:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U\left(c_t, \ell_t\right) + \lambda_0 \sum_{t=0}^{\infty} \beta^t \left[U_c\left(c_t, \ell_t\right)\left(c_t - b_t^0\right) - U_\ell\left(c_t, \ell_t\right)\left(1 - \ell_t\right)\right] - \sum_{s=0}^{\infty} \underbrace{\beta^t \mu_t}_{=\hat{\mu}_t} \left[c_t + g_t + \ell_t - 1\right]$$

▷ The associated FOCs are:

$$\begin{aligned} \left(1 + \lambda_0\right) U_c + \lambda_0 \left[\left(c_t - b_t^0\right) U_{cc} - \left(1 - \ell_t\right) U_{c\ell}\right] &= \mu_t, \forall t \\ \left(1 + \lambda_0\right) U_\ell + \lambda_0 \left[\left(c_t - b_t^0\right) U_{c\ell} - \left(1 - \ell_t\right) U_{\ell\ell}\right] &= \mu_t, \forall t \end{aligned}$$

* Notice that the second derivatives appear in the equations.

▷ Combining the two:

$$\left(U_\ell - U_c\right) + \theta_0 \underbrace{\left[\left(c_t - b_t^0\right)\left(U_{cc} - U_{c\ell}\right) - \left(1 - \ell_t\right)\left(U_{\ell\ell} - U_{c\ell}\right)\right]}_{\text{distortion from the tax}} = 0, \forall t$$

where

$$\theta_0 = \frac{\lambda_0}{1 + \lambda_0}$$

* Observation 1: Symmetry (if all the periods are symmetric such that $g_t = \bar{g}$, $b_t^0 = \bar{b}$ for all t, the tax rates will be the same every period).
* Observation 2: Tax on labor income will distort away from c_t and labor supply and towards leisure ℓ_t. This would look identical if she used a consumption tax since there is no capital accumulation in this economy.

Exercise 5.4. What if $g_t = \bar{g}$, $b_t^0 = \bar{b}$ for all t?

Solution. The tax rates will be the same every period from symmetry. So $c_t = \bar{c}$ and $x_t = \bar{x}$ and $\tau_t = \bar{\tau}$ for all t.

Exercise 5.5. Suppose government spending is highly positive over n periods and 0 elsewhere. Suppose $b_t^0 = 0$ for all t:

$$g_t = \begin{cases} \bar{g} & t = T+1, ..., T+n \\ 0 & \text{otherwise} \end{cases}$$

Solution. There will be two pairs of (c, ℓ) and two values of the tax rate. The Ramsey planner will certainly tax during $g_t = 0$ period to minimize distortion. Govt will build a "war chest" during $g_t = 0$ period; it will draw down the war chest and once it runs out, it will issue debt. Then it will pay off the debt after $T + n$. So the govt is using debt policy to smooth tax distortions.

Exercise 5.6. (Cycles) Suppose there's a sequence $(\hat{g}_0, ..., \hat{g}_{k-1})$ – a cycle of length k – such that

$$\begin{aligned} g_{(nk)+j} &= \hat{g}_j, \quad j = 0, ..., k-1; n = 0, 1, 2... \\ b_t^0 &= 0, \forall t \end{aligned}$$

Solution. There will be k combinations of (c, ℓ) and repeat. Symmetry says, the government needs to come back to zero debt at the end of the cycle; budget needs to be balanced over the cycle.

2. Show how to implement with segmented trading and limited markets Previously, we assumed that the government implemented all these at time 0. But suppose the government wants to implement this over time. The main questions of concern are:

1. What are feasible strategies for rolling over debt?
2. Is there a strategy that answers the following question: suppose the government in period 1 has a chance to re-optimize (but this is the last time). Will it choose to carry out the plan that was formulated in period 0?

Suppose in period t, the government BC holds at date t:

$$\sum_{s=t}^{\infty} p_s \left[g_s + b_s^t - \tau_s (1 - \ell_s)\right] = 0$$

and the additional flow budget constraint in date t can be obtained by

$$\sum_{s=t}^{\infty} p_s \left[g_s + b_s^t - \tau_s (1 - \ell_s)\right] = \sum_{s=t+1}^{\infty} p_s \left[g_s + b_s^t - \tau_s (1 - \ell_s)\right] + p_t \left[g_t + b_t^t - \tau_t (1 - \ell_t)\right]$$

$$\begin{aligned} \Rightarrow p_t \left[g_t + b_t^t - \tau_t (1 - \ell_t)\right] &= \sum_{s=t+1}^{\infty} p_s \left[\tau_s (1 - \ell_s) - g_s - b_s^t\right] \\ &= \sum_{s=t+1}^{\infty} p_s \left[\tau_s (1 - \ell_s) - g_s\right] - \sum_{s=t+1}^{\infty} p_s \left[b_s^t\right] \\ &= \sum_{s=t+1}^{\infty} p_s b_s^{t+1} - \sum_{s=t+1}^{\infty} p_s b_s^t \\ &= \sum_{s=t}^{\infty} \beta p_s \left[b_s^{t+1} - b_s^t\right] \end{aligned}$$

Since $p_s = \beta^s U_c$, we have and thus

$$g_t + b_s^t - \tau_t (1 - \ell_t) = \sum_{s=t}^{\infty} \beta^t p_s \left[b_s^{t+1} - b_s^t \right]$$

▷ LHS: fiscal deficit / RHS: value of net new bond issues at date t

3. Issue of Time Consistency A strategy is said to be time-consistent if an agent finds it optimal from the point of view of some initial period 0 but finds it suboptimal in some subsequent period t. Specifically, this model shows that the incentive of the government to devalue its real debt typically also leads to a time consistency problem for optimal policies. As a solution, it shows that with a carefully chosen maturity structure for real government debt, optimal policies can be made time consistent in a real economy. If the government does not have this rich maturity structure, it is in trouble.

Consider reoptimizing at time t. Then the original conditions are modified accordingly:

$$(U_\ell - U_c) + \theta_1 \underbrace{\left[\left(c_t - b_t^1 \right) \left(U_{cc} - U_{c\ell} \right) - (1 - \ell_t) \left(U_{\ell\ell} - U_{c\ell} \right) \right]}_{\text{distortion from the tax}} = 0, \forall t$$

and

$$\sum_{s=1}^{\infty} \beta^s \left\{ U_c (c_s, \ell_s) \left[c_s - b_s^1 \right] - U_\ell (c_s, \ell_s) \left[1 - \ell_s \right] \right\} = 0$$

From the previous section, we also have

$$\left[g_0 + b_0^0 - \tau_0 (1 - \ell_0) \right] = \sum_{s=1}^{\infty} \beta \left[b_s^1 - b_s^0 \right]$$

Thus we need to find a debt sequence $\left\{ b_t^1 \right\}$ and θ_1 such that satisfies the above constraints. Define $\hat{a}_s = c_s + (1 - \ell_s) \frac{U_{\ell\ell} - U_{c\ell}}{U_{cc} - U_{c\ell}}, \forall s \geq 1$. Then we need

$$\theta_1 \left[\hat{a}_s - b_s^1 \right] = \theta_0 \left[\hat{a}_s - b_s^0 \right], \forall s$$

Thus the sequence $\left\{ b_t^1 \right\}$ depends on θ_0 / θ_1. The remaining part is to show that θ_1 is uniquely determined.

▷ If $\theta_0 = 0$, the government will set a zero tax rate every period. Then it will not doing any distorting at all, so its successors (also Ramsey governments) will not want to distort either. $\theta_1 = 0$ so the new debt sequence is indeterminate and any debt restructuring that satisfies the BC is fine.

▷ If $\theta_0 \neq 0$, then rearranging the above condition gives us

$$b_s^1 = b_s^0 + \left(1 - \frac{\theta_0}{\theta_1} \right) \left(\hat{a}_s - b_s^0 \right), \forall s \geq 1$$

for some θ_1. Substituting into

$$\left[g_0 + b_0^0 - \tau_0 (1 - \ell_0) \right] = \sum_{s=1}^{\infty} \beta \left[b_s^1 - b_s^0 \right]$$

to obtain

$$\begin{aligned} 0 &= \text{ constant } - \sum_{s=1}^{\infty} \beta^s b_s^1 U_c (c_s, \ell_s) \\ &= \text{ constant } + \frac{\theta_0}{\theta_1} \sum_{s=1}^{\infty} \beta^s \left(\hat{a}_s - b_s^0 \right) U_c (c_s, \ell_s) \end{aligned}$$

which tells us that θ_1 is indeed uniquely determined.

Exercise 5.7. Explain why $\hat{a}_s$ is like a consol.[2]

Solution. Suppose $b_t^0 = 0$. Then $b_t^1 = \Gamma_1 \hat{a}_s, \forall s \geq 1$ where $\Gamma_1 = 1 - \theta_0/\theta_1$. In any subsequent periods:

$$b_s^{t+1} = b_s^t + \left(1 - \frac{\theta_t}{\theta_{t+1}}\right)\left(\hat{a}_t - b_s^t\right) = \Gamma_{t+1}\hat{a}_{t+1}$$

They buy it up if they're running a surplus, and they issue more if they're running a deficit. The maturity structure never changes.

Notice that the government can affect the interest rate through its tax policy, thereby altering the marginal utility of consumption. This is why you have to e careful about the maturity structure to make the solution time-consistent.

Exercise 5.8. How would the model change if the shocks $\{g_t\}$ are stochastic?

Solution. In general, the same principles apply. At time t, we need to think about the history of shocks, denoted by $g^t = (g_0, ..., g_t)$. Fix the probability distribution $F(g)$ with the probability distribution over the tail of the sequence $(t+1, t+2, ...)$, given what we know i.e. $F_t\left(g^t\right), \forall t$.

- ▷ Then c_t, x_t, τ_t are functions of $f\left(g^t\right)$.
- ▷ Thus when the government issues debt, it's denominated not only by time but also the state.
- ▷ With complete markets, there will be smoothing not only over time but over the states of the world as well. It will self-insure by issuing contingent-claim debt.

5.5 Brief Musings on Literature on Tax

What's the best way to finance expenditure? Tax on intermediates, you move the production possibility frontier inwards. Taxing on capital income vs labor income. Tax on labor income only makes sense when labor is elastic – this affects the intratemporal choice between leisure (which is not taxed) and consumption (which is taxed). Tax on capital income, on the other hand, affects the intertemporal choice.

The two models on capital taxation – Chamley and another paper by Ken Judd. They both show that capital tax should go to zero. Chamley's a neoclassical growth model setup; Judd's is a two-class model (worker vs capitalists). Chamley also shows that in a society where workers are valued, zero tax is optimal. (r.f. Murphy's argument about MP_L going up in capital).

Lucas was interested in extending Chamley's model (where phyiscal capital is being accumulated) and adding human capital to ask about long-run growth and taxing labor income and capital in a setting where labor is not just a fixed factor but is used to accumulate human capital.

Another follow-up paper found that Chamley's result has to do with not destroying the incentive to accumulate capital, but to achieve this, you only need expected tax rate of zero, not necessarily zero in every period. Therefore the government can use a capital tax to raise revenue when spending shock is high, and use subsidy to capital when spending shock is low in order to maintain an incentive. This is useful for a governmen that may be averse to issuing debt for some reason. Of course, this necessitates a government that has the power to commit.

Another part where capital taxation comes in is about growth. Empirically, there is a sharp increase in the 40s where tax was levied to (partly) finance the war, and it didn't come back down after the war (presumably because there was still a lot of debt to pay off through tax revenues). If you look at the growth rate, the great depression shows up as a huge dip, then a huge spike during the war, and surprisingly it's back on track ona similar growth path.

[2] Essentially a perpetual bond without a maturity date. This is a debt instrument invented in Britain in the 1700s, infinitely-lived nominal-debt where it pays 1 pound stirling forever.

The so-called "developing" countries – or poor countries in general – don't grow becaues of the implicit taxes. If you assume that long-run growth is enabled by the development / importation of new technology, depressing this will affect the growth rates. In the model of Jaimovich and Rebelo (JPE, 2016), entrepreneurship drives growth so it's the taxation only at the very upper end of the distribution that affects the incentive to innovate.

There is a modest literature where you use capital tax to prevent over-accumulation. An OLG model will be an approriate choice if one was to write a theoretical model in this case. The idea is that society as a whole will accumulate too much capital, i.e. inefficient in that the rate of return on capital is lower than any individual's time preference. Is taxing capital the best way? The government can issue paper assets to soak up the demand for private wealth to insure against risk. This is probably a more efficient way to deal with over-accumulation than a simple tax.

5.6 Chamley's Model

What's the best way to finance expenditure? Chamley's model is basically a neoclassical growth model with a little more jazz. There is one good, produced with capital and labor. Chamley consider a larger set of preferences than is conventionally done for a neoclassical growth model.

5.6.1 Setup

The production function F is defined net of depreciation with

$$[RC]: \dot{k}\left(t\right) = F\left(k,n\right) - c\left(t\right) - g\left(t\right)$$

The government chooses $\{g\left(t\right) : t \geq 0\}$ and finances spending with flat rate taxes on capital and labor:

$$\{\tau_k\left(t\right) - \tau_n\left(t\right), t \geq 0\}$$

- ▷ Impose $\tau_k\left(t\right) \leq 1, \forall t$. The maximum is 100% of capital income; you can't tax more than 100% of income or tax any amount of stock.
- ▷ After-tax return on factors are:

$$\bar{r}\left(t\right) = \left[1 - \tau_k\left(t\right)\right] F_k\left(k(t), n(t)\right)$$
$$\bar{w}\left(t\right) = \left[1 - \tau_n\left(t\right)\right] F_n\left(k(t), n(t)\right)$$

- ▷ Define the integral of this instantaneous return:

$$R\left(t\right) = \int_0^t \bar{r}\left(s\right) ds$$

Then $e^{-R(t)}$ is the present discounted value of income at t. (You discount income at the interest rate). This is analogous to discounting at rate of time preference, $e^{-\rho t}$.

5.6.2 Household's Problem

Given $\{g_t, \bar{r}\left(t\right), \bar{w}\left(t\right), t \geq 0\}$ and wealth A_0,

$$\max_{\{c_t, n_t\}} \int_0^\infty e^{-\rho t} U\left(c\left(t\right), 1 - n(t), g(t)\right) dt$$
$$\text{subject to} \int_0^\infty e^{-R(t)} \left[c(t) - \bar{w}\left(t\right) n(t)\right] dt \leq A_0$$

Note that the constraint is in present value terms, but it can be transformed into a flow, period-by-period constraint. This formulation is useful because the transversality condition is baked into it. Now the FOCs for the HH are:

$$e^{-\rho t}U_c(t) = \lambda e^{-\bar{R}(t)}, \forall t$$
$$e^{-\rho t}U_\ell\left(t\right) = \lambda e^{-\bar{R}(t)}\bar{w}\left(t\right), \forall t$$

Note that in a steady state:

$$\frac{d\ln U_c}{dt} = \rho - \bar{r}\left(t\right) = 0 \Rightarrow \bar{r}\left(t\right) = \rho$$

To get the *implementability constraint*, rewrite the HH budget constraint as

$$[IC] : \int_0^\infty e^{-\rho t}\left[U_c\left(t\right)c(t) - U_\ell(t)n(t)\right]dt - U_c\left(0\right)A_0 \leq 0$$

Lemma 5.2. *An allocation $\{c(t), n(t), g(t)\}$ is implementable if and only if it satisfies $[IC]$ and $[RC]$.*

5.6.3 Government's Problem

The government chooses $\{c, k, n, g\}$ to maximize U such that $[IC]$ and $[RC]$ hold for all t. Define

$$W\left(c, n, g, \varphi\right) = U + \varphi\left[cU_c - nU_\ell\right]$$

where U and its derivatives are evaluated at $(c, 1-n, g)$. The Hamiltonian for the government's problem is then:

$$\mathcal{H} = W + \mu\left[F\left(k, n\right) - c - g\right]$$

where $\mu\left(t\right)$ is the co-state variable for capital. Note that this is exactly like the household's problem in the neoclassical growth model, so if we back out the tax rates from

$$\bar{r}\left(t\right) = \left[1 - \tau_k\left(t\right)\right]F_k\left(k(t), n(t)\right)$$
$$\bar{w}\left(t\right) = \left[1 - \tau_n\left(t\right)\right]F_n\left(k(t), n(t)\right)$$

it will be zero. φ measures the degree of distortion necessary to balance this budget. The conditions for a maximum include:

$$\frac{\partial\mathcal{H}}{\partial c} = W_c - \mu = 0$$
$$\frac{\dot{\mu}}{\mu} = \rho - F_k, \forall t$$

At a steady-state,

$$0 = \frac{d\ln W_c}{dt} = \frac{\dot{\mu}}{\mu} = \rho - F_k$$

so $\bar{r}\left(t\right) = \rho = F_k\left(t\right) \Rightarrow \tau_k = 0$.

5.6.4 Extensions

1. In Straub & Werning, they define Koupmans-Diamond-Williamson preferences (KDW):

$$W\left(c_0, c_1, ..\right) = V\left(c_0, W\left(c_1, c_2, ...\right)\right)$$

   for some function $V\left(c, W\right)$. This preference is convenient since if I think about the consumer's choice tomorrow, the agent will maximize $W\left(c_1, c_2, ...\right)$. This does not allow habit formation.

   ▷ *Contrast with E-Z utility:* If you try to put uncertainty in here, it doesn't have an expected utiltiy representation. E-Z have a way of keeping the convenience.

   ▷ Straub & Werning claim that if you take the Chamley result, for the KDW preferences, the conclusion that capital tax should go to zero in the long-run now becomes that capital stock goes to zero in the long-run.

2. In Lucas's model with human capital taxation, the representative individual allocates labor time between production and learning, and the steady-state equilibrium rate of growth of the economy depends on the allocation of time to acquiring education.

   ▷ Specifically, you have

$$\dot{H} = \varphi\left(z(t)\right) H(t)$$

   and taxes on earning won't affect this accumulation process.

3. In general, comparing steady states is a dangerous exercise to do. You have to know the transition dynamics.

6 Monetary Policy

"I thank your country [Argentina] for providing lots of data on inflation." – Nancy Stokey

6.1 Motivation

Some policy-oriented questions include:

▷ How should a monetary authority respond to shocks (Lehman, oil prices)

▷ What are the welfare costs of inflation?

To answer these questions, we need to understand how monetary decisions affect the real economy. Model seek to fit two kinds of data:

1. Time-series data on money demand

   ▷ In the U.S., up until 1990, if you looked at money demand relative to GDP, it looked like a demand curve for any other object; the price of holding cash is then nominal interest rate. Then in the 1990s, the stable demand curve just fell apart. People have been looking at institutional changes and technological development to explain these changes. Exampls include: ubiquitous use of credit cards, increased use of electronic transfers, and assets that pay higher interest rates than bank deposits.

   ▷ So the measure is not M1 anymore; it may include MMDAs, which may have stable properties like M1 in the modern era.

2. Short-run and long-run effects of money growth

   ▷ Inflation over the long run in many places and many time periods is linked very closely to the growth in M1. In the long-run, increase in money supply shows up as a higher price level. In the U.S., M1 does not work so well either in this arena. For recent examples, see Venezuela, Zimbabwe, and Argentina.

   ▷ In the 1980s, inflation has gone to double digits with around 20%, so Paul Volcker engineered a recession by tightening monetary policy very drastically. So you see a big drop in the inflation rate. For more, see Sargent's "The ends of four big inflations." He argues that bringing inflation rate down significantly without too much real effect requires a good fiscal policy. He also shows that hyper inflation at the core is a fiscal phenomenon since the government can't raise revenue to finance its expenditures.

We are interested in the real outcomes of monetary policy, so we need a model in which agents hold money.

▷ The first issue you run into is the question: why would anyone want to hold money? Everyone seems to do fine in an A-D economy, and you don't have a good reason why people hold money in these economies. Thus, we need to introduce a small friction.

  * One of the older models is Samuelson's OLG model where money is the only asset. People held money as a means of savings since there were no other assets, so it's not a terribly satisfactory model of money since there are other assets with positive rates of return in real life. The model is interesting for other reasons, but it's not a good starting place to talk about model of money.
  * There's another inventory theoretic model by Baumol and Tobin where the agent makes a trip to the bank, uses his money, and then makes a trip to the bank again.
  * Another one is by Strawsky who wrote a dissertation as Chicago PhD. He modified the utility function to include real cash balances.

* Another is a cash-in-advance model that says if a household wants to buy goods, it needs to pay with cash on spot. The simplest version is where all purchases are subject to this constraint. Of course, this is very constrained, so a modification can be made where some goods can be bought with credit or some goods can be paid with a mixture of cash and a little extra time.

Monetary policy does have a real effect in the short-run, but as of now we don't have good model of how.

6.2 Cash-in-Advance (CIA) Model

Pure endowment economy. We are interested in the nominal price level and the price of the nominal bond, or equivalently the interest rate.

6.2.1 Preliminaries

▷ Exogenous stock process given as $S_t = (y_t, w_t)$ where y_t is the endowment and w_t is the money growth

▷ Timing: the following events happen

1. Shocks S_t are realized at the beginning of the period.
2. Asset market meets and the households will come in with nominal assets and choose how much to hold as cash and how much to hold as nominal bonds. It will choose its portfolio.
3. Consumption occurs subject to a CIA constraint

Note that this is similar to the Lucas asset-pricing model. Any asset you can bring up, you can price it in this economy.

6.2.2 Plan of Action

1. Set up the model
2. Analyze HH problem
3. Define a recursive CE
4. Solve for equilibrium prices
5. Show how to add other assets
6. Examples

6.2.3 Setup

▷ Assets: The one asset of interest is a one-period nominal bond that is in zero net supply. So the question is what interest rate makes demand equal to zero.

▷ Money enters through lump-sum transfers in the asset-market. There are no Open-Market Operations (OMO); the government is not buying up bonds or selling bonds. Any changes in the money supply happen through lump-sum transfers.

▷ Money supply is characterized by $\bar{M}_t = w_t \bar{M}_{t-1}, \forall t$

▷ The model will be recursively defined, the shocks are Markov, and we will also hard-wire the quantity theory of money that money injection corresponds one-to-one with price increases in the long-run.

▷ Aggregate state: $\left(S_t, \bar{M}_t\right)$

6.2.4 Analyze the Household's Problem

- ▷ Preferences: $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t U\left(c_t\right)\right]$

- ▷ Asset market: To facilitate the model, it's convenient to think of the household comprising a worker and a shopper; they come to the asset market and jointly agree on a portfolio; they get the receipts from sales last period (they arrive with one period lag); the shopper shops using M_t and the worker sells Y_t. This bifurcation is the way to understand the friction in the household's problem in that it can't simply consume its own endowment (this is a constraint; endoed with a unit of "back-scratching" services):

$$M_t + Q_N\left(S_t, \bar{M}_t\right)N_t \le N_{t-1} + M_{t-1} + P\left(S_{t-1}, \bar{M}_{t-1}\right)\left(Y_{t-1} - C_{t-1}\right) + \left(w_t - 1\right)\bar{M}_{t-1}$$

  * HH will choose its cash holdings for this period (M_t) and how many nominal bonds to purchase (N_t) where each of these bonds is a claim to one dollar in the next period; Q is the current price of a nominal bond.
  * To conduct these operations, we have (1) the bonds from previous period are maturing (M_{t-1}), (2) yesterday they chose cash (M_{t-1}) and used it to buy goods (PC_{t-1}) with receipts from the sales (PY_{t-1}) and (3) receives the cash injection $(\omega_t - 1)\bar{M}_{t-1}$ which is the today's money supply minus yesterday's money supply.
  * M_t is the household's choice and $\bar{M}_t$ is the money supply.

6.2.5 Defining a Recursive CE

We characterize the household's BE. Start by defining

$$A_t \equiv N_{t-1} + M_{t-1} + P\left(S_{t-1}, \bar{M}_{t-1}\right)\left(Y_{t-1} - C_{t-1}\right) + \left(w_t - 1\right)\bar{M}_{t-1}$$

and write:

$$V\left(A_t; S_t, \bar{M}_t\right) = \max_{C_t, N_t, M_t}\left\{u\left(C_t\right) + \beta\mathbb{E}\left[V\left(A_{t+1}; S_{t+1}, \bar{M}_{t+1}\right)|S_t, \bar{M}_t\right]\right\}$$

subject to

$$\underbrace{M_t + Q_N\left(S_t, \bar{M}_t\right)N_t \le A_t,}_{=\text{Buget constraint}} \qquad \underbrace{P\left(S_t, \bar{M}_t\right)C_t \le M_t}_{=\text{CIA constraint}}$$

$$\underbrace{A_{t+1} = N_t + M_t + P\left(S_t, M_t\right)\left(Y_t - C_t\right) + \left(\omega_{t+1} - 1\right)\bar{M}_t}_{=\text{ Next-period asset}}$$

- ▷ Note that S_{t+1} is the only stochastic element here since $\bar{M}_{t+1}$ is determined by $\bar{M}_t$ given a realization of S_{t+1}.

Now to look for a recursive CE that satisfies QT, conjecture that prices must satisfy (both are prices):

$$P\left(S_t; \bar{M}_t\right) = p\left(S_t\right)\bar{M}_t, \forall t$$
$$Q_N\left(S_t; \bar{M}_t\right) = q_N\left(S_t\right), \forall t$$

where the RHS depends only on the "real" part of the state variable.

- ▷ Note that Q_N is the \$ today per \$ received next period. Q_N is pricing a nominal claim to next period in terms of money today; if I double the stock today, it will also double the stock of money next period, so it does not change this price.

- ▷ We don't expect the price of a nominal bond to fluctuate with the price *level* since the price level is scaled both in the present and the future, thereby cancelling the effect of money supply in determining the price. Thus, we can write q solely as a function of S_t.

Furthermore, the nominal interest rate $(i(S_t))$ satisfies:

$$Q_N\left(S_t;\bar{M}_t\right)=q_N\left(S_t\right)=\frac{1}{1+i\left(S_t\right)}$$

▷ Typically we argue that the nominal interest rate has to be non-negative, thus indicating $Q_N<1$. Note that Q_N is a way to transfer money today to money tomorrow.

Define variables normalized by today's money supply:

$$a_t=\frac{A_t}{\bar{M}_t},\quad m_t=\frac{M_t}{\bar{M}_t},\quad n_t=\frac{N_t}{\bar{M}_t}$$

Note that there are other normalizing choices as well. In equilibrium, $m_t=1$.

Re-write the BE, BC, CIA constraints and assets next period as :

$$v\left(a;s\right)=\max_{c,m,n}\left[u\left(c\right)+\beta\mathbb{E}\left[v\left(a';s'\right)|s\right]\right]$$

subject to

$$\underbrace{m+q_N\left(s\right)n\leq a}_{=\text{BC}},\qquad\underbrace{p\left(s\right)c\leq m}_{=\text{CIA}}$$

$$a'=\frac{1}{\omega'}\left[m+n+p\left(s_t\right)\left(y-c\right)+\left(\omega'-1\right)\right]$$

since from the original next-period asset equation, we have

$$\frac{A_{t+1}}{\bar{M}_{t+1}}=\frac{1}{\bar{M}_t}\left\{N_t+M_t+P\left(S_t,M_t\right)\left(Y_t-C_t\right)+\left(\omega_{t+1}-1\right)\bar{M}_t\right\}\cdot\left(\frac{\bar{M}_t}{\bar{M}_{t+1}}\right)$$

A recursive competitive equilibrium consists of functions $\{p\left(s\right),q\left(s\right)\},\{c\left(a,s\right),m\left(a,s\right),n\left(a,s\right)\}$ such that market clearing holds for goods, money, and bonds:

$$\begin{aligned}\text{[Goods]}&:c\left(1,s\right)=y\left(s\right)\\ \text{[Money]}&:m\left(1,s\right)=1\\ \text{[Bonds]}&:n\left(1,s\right)=0\end{aligned}$$

where $a=1$ can be derived by plugging in $m=1,n=0,y=c$ into the expression for the asset equation.

6.2.6 Solving for Equilibrium Prices

We want to characterize $p\left(s\right)$ and $q_N\left(s\right)$. We will focus on economies where the CIA constraint binds.

▷ Start by noting that from the CIA constraint, $p\left(s\right)y=1,\forall s$ which implies $y=1/p\left(s\right)$ so $p\left(\cdot\right)$ depends only on y.

▷ Velocity, total volume of transactions divided by the money supply, is unity since

$$\frac{p\left(s\right)y}{\bar{m}}\equiv v=1$$

i.e. how many times the money turns over in a period.

The HH problem gives 3 FOCs and 1EC. Use $\lambda(s)$ for BC and $\mu(s)$ for CIA. Then

$$\mathcal{L} = u(c) + \beta \mathbb{E}\left[v\left(a'; s'\right)|s\right] + \lambda(s)(a - m - q_N(s)n) + \mu(s)(m - p(s)c)$$

where

$$a' = \frac{1}{\omega'}\left[m + n + p(s)(y - c) + \left(\omega' - 1\right)\right]$$

▷ The FOCs are:

$$\begin{aligned}
[c] &: u'(y) = p(s)\mu(s) + \beta\frac{1}{\omega'}p(s)\mathbb{E}\left[v'\left(a'; s'\right)|s\right] = p(s)\overbrace{\left\{\mu(s) + \beta\frac{1}{\omega'}\mathbb{E}\left[v'\left(a'; s'\right)|s\right]\right\}}^{=\lambda(s)} \\
[m] &: \mu(s) = \lambda(s) - \beta\frac{1}{\omega'}\mathbb{E}\left[v'\left(a'; s'\right)|s\right] =^{[n]} \lambda(s)\left[1 - q_N(s)\right] \\
[n] &: q_N(s) = \frac{1}{\lambda(s)}\beta\frac{1}{\omega'}\mathbb{E}\left[v'\left(a'; s'\right)|s\right] =^{[\mathrm{EC}]} \beta\mathbb{E}\left[\frac{1}{\omega'}\frac{\lambda(s')}{\lambda(s)}|s\right]
\end{aligned}$$

▷ The EC is:

$$[EC] : v'(a) = \lambda(s)$$

Interpretation of each equation:

1. $\lambda(s)$: marginal value of having additional income:

$$u'(y) = p(s)\lambda(s)$$

2. $\mu(s)$: marginal value of having more cash

$$\mu(s) = \lambda(s)\left[1 - \beta\mathbb{E}\left[\frac{1}{\omega'}\frac{\lambda(s')}{\lambda(s)}|s\right]\right]$$

3. $q_N(s)$: bond price

$$q_N(s) = \beta\mathbb{E}\left[\frac{1}{\omega'}\lambda\left(s'\right)|s\right]/\lambda(s)$$

Recall that we assumed that

$$q_N(s) = \frac{1}{1 + i(s)}$$

where i is the nominal interest rate. Thus CIA constraint binding strictly ⇔ $\mu(s) > 0$ from complementary slackness. Since

$$\mu(s) = \lambda(s)\left[1 - q_N(s)\right]$$

this is ⇔ $q_N(s) < 1$ ⇔ $i(s) > 0$.

6.2.7 Examples

Example 6.1. Output is constant and money growth is constant, i.e. $y(s) = \bar{y}$ and $\omega = \bar{\omega}$. Then

- ▷ From $u'(y) = p(s)\lambda(s)$, $\lambda(s) = \bar{\lambda}$.
- ▷ From $q_N(s) = \beta\mathbb{E}\left[\frac{1}{\omega'}\lambda(s')\,|s\right]/\lambda(s)$, we have $q_N(s) = \bar{q} = \beta/\bar{\omega}$. Given our restriction that $q_N < 1, \bar{\omega} > \beta$ when $\beta < 1$. To interpret this result, note that

$$\beta = \frac{1}{1+\rho} < \bar{\omega} = 1+\pi$$
$$\Rightarrow (1+\pi)(1+\rho) > 1$$

  where π is the rate of money growth. This allows inflation to be negative, but not too much. If the rate of deflation is too large in absolute value, the consumer will just want to hold a lot of cash since the value of cash will be growing faster than his rate of time preference. So there is an incentive to hoard money if deflation is very rapid. This model is not suitable for thinking about this situation. This model is constructed to think about a world where holding cash is costly and people hold it for the transaction service that is conducted by cash.

Example 6.2. Constant $y = \bar{y}$ and money growth is stochastic.

- ▷ From $u'(y) = p(s)\lambda(s)$, $\lambda(s) = \bar{\lambda}$.
- ▷ From $q_N(s) = \beta\mathbb{E}\left[\frac{1}{\omega'}\lambda(s')\,|s\right]/\lambda(s)$, we have

$$q_N(\omega) = \beta\mathbb{E}\left[\frac{1}{\omega'}|\omega\right]$$

- ▷ If money growth was IID, then the RHS would be constant and the bond price would be constant.
- ▷ In general, money growth this period matters only through the effect of ω on expectations about ω'.

6.2.8 Stochastic (y, ω)

Assume CRRA preferences

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \theta > 0$$

Then

$$\lambda(y) = \frac{1}{p(s)}u'(s) = y^{1-\theta} \Rightarrow q_N(s) = \frac{1}{1+i} = \beta\mathbb{E}\left[\frac{1}{\omega'}\left(\frac{y'}{y}\right)^{1-\theta}|s\right]$$

so now the bond price depends on the expectations about (1) money growth and (2) real income growth.

- ▷ [1]: Higher real income today (y), ⇒ lower price (q_N) today. Fixing y', p' is also fixed while p is lowered, so ⇒ higher expected inflation ⇒ higher i
- ▷ $[-\theta]$: Higher real income ⇒ lower expected income growth ⇒ lower real interest rate ⇒ lower i

For log utility, these two effects cancel each other out.

6.2.9 Adding Other Assets

You can introduce two-period "seasoned bonds" or assets that are claims to goods. Then you have to add more components since the consumer may have more complicated portfolios. Particularly, the a part of $v(a, s)$.

6.3 McCallum & Goodfriend Model

Most of the model is like the CIA setup, but the transaction constraint (in place of CIA constraint) is:

$$c \leq \frac{m}{p}\psi\left(\frac{p\theta}{m}\right)$$

where m/p is the real cash balances and θ is the share of time endowment used. The idea is that you have to use a combination of cash m/p and time θ for transactions. Note that this implies $1-\theta$ time is spent for working at wage w.

▷ High inflation rates make holding cash more costly (since you forego interest), so the agent will switch away from cash towards using transaction time. But this time also has a real cost (since it's taken away from working) so there will be lower output in the economy with higher rate of inflation. This is how it gets the *real effect* of inflation on output.

6.3.1 Setup

▷ Agents: HH, firms, bank, and government

* HH: there is a fixed distribution of the size of purchases $G(z)$. Denote $g(z)$ is the density, and let

$$\hat{\Omega}(z) = \int_0^z \zeta g(\zeta)\, d\zeta$$

is the total value of transactions for goods with values smaller than z. Let $\nu \equiv \lim_{z\to\infty}\hat{\Omega}(z)$ is the total value. Define $\Omega(z) = \hat{\Omega}(z)/\nu$

▷ Assets: capital, government bonds, and money

▷ Goods: one good produced with capital & labor

▷ Choices are about means of payment

* 1) Cash: no time cost, but possibility of loss or theft
* 2) Cards: time cost k^d and liquidity requirement θ^d
* 3) Electronic: time cost k^e and liquidity requirement θ^e

▷ Assumptions:

* All adjustments in consumption as in the volume (the intensive margin) with the total number unchanged.
* Consumption level requires the number of purchases n^h (does ont vary with c) and volume of transactions, c.
* $0 < k^d < k^e$ and $\theta^e < \theta^d < \theta^c = 1$

6.3.2 Household's Problem

HH chooses which purchases to makew ith which means of payment, given c. Specifically, it will choose thresholds t_1, t_2 such that for $t < t_1$,she will use cash; for $t > t_2$ she will use electrocnic; for $t_1 < t < t_2$, she wil use cards. So it suffices for the household to choose these thresholds – denote them as γ and δ.

▷ Define

$$L^h(\gamma,\delta) = \Omega(\gamma)\theta^c + [\Omega(\delta) - \Omega(\gamma)]\theta^d + [1-\Omega(\delta)]\theta^e$$
$$T^h(\gamma,\delta) = G(\gamma)\cdot\theta + [G_h(\delta) - G_h(\delta)]k^d + [1-G_h(\delta)]k^e$$

for consumption level c, total required liquidity is $cL^n(\gamma,\delta)$ and total time needed is n_hT^h. Note that the total time does not depend on the total volume of the purchases (c).

▷ Therefore, total non-cash payment is $c\left[1-L^{n}\right]$.

The HH BE can be formulated as:

$$\begin{aligned} v\left(A_0;P_0\right) &= \max_{M_0,\gamma,\delta,\tau,c}\left[u\left(c\right)\frac{1}{\rho}\left(1-e^{-\rho\tau}\right)+e^{-\rho\tau}V^{h}\left(A';e^{\pi\tau}P_0\right)\right] \\ \text{subject to } & p_0 c L^{h}\left(\gamma,\delta\right)\frac{1}{\pi}\left(e^{\pi\tau}-1\right)\leq M_0 \\ A' &= \left[M_0-P_0 c L^{n}\left(\gamma',\delta\right)\frac{e^{\pi\tau}-1}{\pi}\right]+e^{i\tau}\left(A_0-M_0-P_0 w\varphi\right) \\ &\quad +\frac{e^{i\tau}-e^{\pi\tau}}{i-\pi}\left[w\left(1-n_h T^{h}\right)-\text{tax }-c\left(1-L^{h}\right)\right] \end{aligned}$$

Lemma 6.1. *$V\left(A,P\right)$ is homogeneous of degree zero in $\left(A,P\right)$ i.e. if you double the nominal price level and if you double the initial nominal wealth, her lifetime utility is unchanged.*

7 Bank Runs

"Last year, there was a student sitting at the back – a good student – who exclaimed, 'that's it?'." – Nancy Stokey

7.1 Diamond–Dybvig Model

7.1.1 Setup

There are three periods $t = 0, 1, 2$ and one technology by the bank that can take one unit of resource at date 0 (deposits received) and invest them. If it pulls out the investment at date 1, you get the principal back with no interest; if you pull it out at date 2, you get $R > 1$.

Preferences We have a continuum of households on $[0, 1]$ and they are subject to an i.i.d. preference shock at $\theta \in \{1, 2\}$. The utility function is

$$U(c_1, c_2, \theta) = \begin{cases} u(c_1) & \theta = 1 \\ \rho u(c_1 + c_2) & \theta = 2 \end{cases}$$

and assume $R\rho > 1$. Endowment is 1 in $T = 0$.

Autarky Solution Consider a no-trade solution assuming the individual has the same technology. Denote α the probability that $\theta = 1$. Then:

$$\mathbb{E}u = \alpha u(1) + (1 - \alpha) \rho u(R)$$

The problem with the autakry solution is that the agent has a consumption risk (utility risk) because of the shock. So the market is missing insurance against this shock. Since the shock is i.i.d. across the consumers, if the insurance company could come in, it can insure against it.

Perfect Insurance Market (θ is observable) Suppose the insurance company offers the contract $(-1, c_1, c_2)$ depending on θ. We want the contract to maximize the expected utility of the household:

$$\max_{c_1, c_2} [\alpha u(c_1) + (1 - \alpha) \rho u(c_2)]$$
$$\text{s.t. } \alpha c_1 + (1 - \alpha) \left(\frac{c_2}{R}\right) = 1$$

The FOCs are:

$$\alpha u'(c_1) = \alpha \lambda$$
$$(1 - \alpha) \rho u'(c_2) = (1 - \alpha) \frac{\lambda}{R}$$

Thus it follows that

$$\frac{u'(c_2)}{u'(c_1)} = \frac{1}{\rho R} < 1 \Leftrightarrow c_2^* > c_1^*$$

The next question is whether or not a bank can mimic this.

Introducing a Bank Assume RRA > 1 or $-cu''/u' > 1$. In fact, assume

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

which implies

$$\left(\frac{c_2^*}{c_1^*}\right)^{\sigma} = \rho R \Rightarrow c_2^* = (\rho R)^{1/\sigma} c_1^*$$

We also want $c_1^* > 1$. Thus from the budget constraint and specializing to the utility function, it is equivalent to

$$\alpha + (1-\alpha)\left(\frac{(\rho R)^{1/\sigma}}{R}\right) < 1$$

Rearranging, we have the following condition that ensures insurance company offering efficient $c_1^* > 1$.

$$(\rho R)^{1/\sigma} < R$$

Why do we need this condition? We are trying to say that a bank can mimic this. We want the bank to say to the depositors that you get some interest in date 1 but less than what you get in date 2. In other words, if the optimal insurance contract has

$$(c_1^*, c_2^*) \text{ with } c_1^* > 1, c_2^* < R$$

then a bank can mimic it. The bank's offer would be: any customer can withdraw at date 1 and get c_1^* or withdraw at date 2 and get c_2^*.

Possible Glitch What would a bank run look like in this setup? What if everyone tries to withdraw at $T = 1$? Then there is not enough to go around – some of them will get rationed. So we need a rationing rule. The D-D rationing rule is the *sequential service contraint:* if fraction f of deposits run, the first in line gets c_1^*; the rest gets 0. So only $1/c^* = f < 1$ can get paid. Is there a way to rule out this bad equilibrium?

1. Deposit insurance from outside

2. Suspension of convertibility: bank pays only the first α of those in line at $T = 1$. This prevents a bank run since the $\theta = 2$ have no incentive to run; if they are patient, they will get paid the full amount.