

OLG Perpetual Youth

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Set up

Demographics

- ▶ Continuous time. Agent die period of length dt with prob. $p dt$.
- ▶ Note that $p \in (0, \infty)$. Expected lifetime $1/p$.
- ▶ If measure p is born at date s , fraction $p e^{-p(t-s)}$ survive at t .
 - ▶ Let $N(s, t)$ be the size of the cohort, so $N(s, t + \Delta) = N(s, t)(1 - p\Delta)$ so
 - ▶ $dN(s, t)/dt = -N(s, t)p$ and boundary $N(s, s) = p$: initial cohort size.
 - ▶ Then solution is $N(s, t) = p e^{-p(t-s)}$.
- ▶ Agents that die replaced by newborns.
- ▶ Adding all cohort alive at time t : $\int_{-\infty}^t N(s, t) ds = \int_{-\infty}^t p e^{-p(t-s)} ds = 1$.

Insurance, Annuities

- ▶ All agents have same time expectancy, no young and old.
- ▶ Unrealistic, but tractable. Effects of currently alive vs. those yet not born.
- ▶ Agents can insure against debt, at fair prices.
- ▶ Let r be the net risk-less interest rate.
- ▶ Invest v at t , gets $v \frac{1+\Delta r}{1-\rho\Delta}$ if alive at $t + \Delta$, and zero if dead.
- ▶ Insurance company selling annuities breaks even.
- ▶ Continuous time (as $\Delta \downarrow 0$) : $v \frac{1+\Delta r}{1-\rho\Delta} = v + v(r + p)\Delta + o(\Delta)$

Preferences

- ▶ Expected discounted utility. Utility only while alive.
- ▶ Agent born at t maximizes: $\mathbb{E} \left[\int_t^\infty u(c(z)) e^{-\theta(z-t)} dz \right]$
- ▶ Discount rate $\theta \in (0, \infty)$, so 1 util at time $t + \Delta$ is worth $\frac{1}{1+\Delta\theta}$ at t .
- ▶ Expectation only with respect to realization of death.
- ▶ Time remaining alive exponentially distributed:

$$\mathbb{E} \left[\int_t^\infty u(c(z)) e^{-\theta(z-t)} dz \right] = \int_t^\infty u(c(z)) e^{-(p+\theta)(z-t)} dz$$

Household Problem

- ▶ Specialize $u(\cdot)$ to log utility.
- ▶ $\max \mathbb{E} \left[\int_t^\infty u(c(z)) e^{-\theta(z-t)} dz \right] = \int_t^\infty \log(c(z)) e^{-(p+\theta)(z-t)} dz$
- ▶ subject to $\int_t^\infty [c(z) - y(z)] R(t, z) dz = v(t)$ where
 - ▶ $R(t, z)$: price of a good in z in terms of goods in t ,
 - ▶ $v(t)$: non-human (financial) wealth at time t ,
 - ▶ $y(z)$: labor income at time z .

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 - ▶ $v(t)$: non-human (financial) wealth at time t ,
 - ▶ $y(z)$: labor income at time z .
- ▶ define human wealth as $h(t) = \int_t^\infty y(z) R(t, z) dz$
- ▶ Solution $c(t) = (\theta + p)(v(t) + h(t))$.

Preferences, log case

- Recall that $\max_{c_1, c_2, \dots, c_n} a_1 \log(c_1) + a_1 \log(c_2) + \dots + a_n \log(c_n)$
subject to $R_1 c_1 + R_1 c_2 + \dots + R_n c_n = h + v$.

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Optimal demand gives: $c_1 = a_1 [h + v]$.

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Optimal demand gives: $c_1 = a_1 [h + v]$.
- ▶ Since weights add up to one $\int_t^\infty \log(c(z)) e^{-(p+\theta)(z-t)} dz$ and
 $R(t, t) = 1$ by definition, so $c(t) = (\theta + p)(v(t) + h(t))$,
notice that “weights add up to one”: $\int_t^\infty e^{-(\theta+p)(z-t)} dz = 1/(\theta + p)$.

Deriving ODEs for HH problem

- ▶ Given Intertemporal prices $R(t, z)$ define interest rates $r(\mu) + p$ as :

$$R(t, z) = \exp \left(- \int_t^z [r(\mu) + p] d\mu \right) .$$

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- ▶ Budget Constraint $\int_t^\infty [c(z) - y(z)] R(t, z) dz = v(t)$ equivalent to:

$$\frac{dv(z)}{dz} = [r(z) + p] v(z) + y(z) - c(z) \text{ and } \lim_{z \rightarrow \infty} R(t, z) v(z) = 0 .$$

- ▶ Definition of human wealth $h(t) = \int_t^\infty y(z) R(t, z) dz$ equivalent to:

$$\frac{dh(z)}{dz} = [r(z) + p] h(z) - y(z) \text{ and } \lim_{z \rightarrow \infty} R(t, z) h(z) = 0 .$$

- ▶ Differentiate f.o.c. for $c(z)$: $\frac{\exp(-(\theta+p)(z-t))}{c(z)} = \lambda \exp \left(- \int_t^z [r(\mu) + p] d\mu \right)$

Obtain: $\frac{dc(z)}{dz} = [r(z) - \theta] c(z)$ (where λ was multiplier in BC).

Deriving ODEs for HH problem, algebra I

- ▶ Given Intertemporal prices $R(t, z)$ define interest rates $r(\mu) + p$ as :

$$R(t, z) = \exp \left(- \int_t^z [r(\mu) + p] d\mu \right)$$

- ▶ Consider discrete time case with Δ the length of time period.
- ▶ There are $(z - t)/\Delta$ discrete time periods between t and z , and let $p = 0$:

$$R(t, z)^{-1} = (1 + r_{t+1}\Delta)(1 + r_{t+2}\Delta) \cdots (1 + r_z\Delta) = \prod_{i=1}^{(t-z)/\Delta} (1 + r_{t+i}\Delta)$$

$$-\log(R(t, z)) = \sum_{i=1}^{(t-z)/\Delta} \log(1 + r_{t+i}\Delta) = \sum_{i=1}^{(t-z)/\Delta} \left[r_{t+i} + \frac{o(\Delta)}{\Delta} \right] \Delta$$

- ▶ $\lim_{\Delta \downarrow 0} \log(R(t, z)) = - \int_t^z r(\mu) d\mu$.
- ▶ In general with $p > 0$: $\lim_{\Delta \downarrow 0} \log(R(t, z)) = - \int_t^z [r(\mu) + p] d\mu$.



Deriving ODEs for HH problem, algebra II

- ▶ Let $v(t) = \int_t^T [c(z) - y(z)] R(t, z) dz + R(t, T)v(T)$
- ▶ Use expression for $R(t, z)$ in terms of $r + p$:
- ▶ $v(t) = \int_t^T [c(z) - y(z)] e^{-\int_t^z (r(\mu)+p)d\mu} dz + e^{-\int_t^T (r(\mu)+p)d\mu} v(T)$
- ▶ Differentiate previous expression with respect to t :

$$\frac{dv(t)}{dt} = -[c(t) - y(t)] + [r(t) + p] \int_t^T [c(z) - y(z)] e^{-\int_t^z (r(\mu)+p)d\mu} dz + [r(t) + p] e^{-\int_t^T (r(\mu)+p)d\mu} v(T) .$$

- ▶ Replace the expression for $v(t)$: $\frac{dv(t)}{dt} = y(t) - c(t) + [r(t) + p] v(t)$.
capital gain
- dividend
current income



Boundary of ODEs for HH problem, interpretation

- For ODE (sequential BC) to be equivalent to Present Value BC:

$$\lim_{T \rightarrow \infty} v(T) e^{-\int_t^T (r(\mu) + \rho) d\mu} = 0 .$$

- If this is strictly positive, the agent did not maximize its utility. (Why?)
- If this is strictly negative, the agent the plan was not budget feasible.

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- If this is strictly negative, the agent the plan was not budget feasible.

- For $h(t) = \lim_{T \rightarrow \infty} \int_t^T y(z) R(t, z) dz$ to converge requires:

$$\lim_{T \rightarrow \infty} y(T) e^{-\int_t^T (r(\mu) + \rho) d\mu} = 0 .$$

- Interpretation: if this limit does not converge, the agent human wealth is unbounded.
- If human wealth is unbounded, the problem has no solution.

Consumption/Savings Behaviour of Households given Human and Non-Human Wealth

- Individuals: $c(s, t) = (p + \theta) (h(s, t) + v(s, t))$ or optimal decision rule

$$\frac{dc(s, t)}{dt} = [r(t) - \theta] c(s, t) \text{ BC + and boundary condition.}$$

- Define aggregate: $C(t) \equiv \int_{-\infty}^t N(s, t) c(s, t) ds$.
- Use $c(s, t)$ above and definitions of $H(t)$ and $V(t)$ as aggregates:

$$H(t) \equiv \int_{-\infty}^t N(s, t) h(s, t) ds, \quad \text{human wealth}$$

$$V(t) \equiv \int_{-\infty}^t N(s, t) v(s, t) ds. \quad \text{non-human wealth}$$

- Aggregate: $C(t) = (p + \theta) (H(t) + V(t))$



Exponential model of labor services

- Model endowment of labor services as decreasing with age at rate α .
- Labor income = labor services \times wage.
- $Y(t)$: aggregate labor income of agents alive at time t .
- $y(s, t)$: labor income of agent born at s alive at t .
- $y(s, t) = a Y(t) e^{-\alpha(t-s)}$ for some constant $a > 0$.

$$Y(t) = \int_{-\infty}^t N(s, t) y(s, t) ds = \int_{-\infty}^t p e^{-\rho(t-s)} y(s, t) ds$$

$$= \int_{-\infty}^t p e^{-\rho(t-s)} a Y(t) e^{-\alpha(t-s)} ds \implies a = \frac{\rho + \alpha}{\rho}.$$



Individual and Aggregate Human Wealth

- ▶ Using definitions $h(t, s) = \int_t^\infty y(s, z) R(t, z) dz$,

intertemporal price: $R(t, z) = \exp \left(- \int_t^z [r(\mu) + p] d\mu \right)$, and

exponentially declining labor services: $y(s, t) = a Y(t) e^{-\alpha(t-s)}$:

- ▶ define Aggregate Human Wealth $H(t) = \int_{-\infty}^t h(s, t) N(s, t) ds$.

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- ▶ define Aggregate Human Wealth $H(t) = \int_{-\infty}^t h(s, t) N(s, t) ds$.

- ▶ Aggregate Human Wealth satisfies: $\frac{dH(t)}{dt} = [r(t) + p + \alpha] H(t) - Y(t)$,

and $\lim_{z \rightarrow \infty} H(z) \exp \left(- \int_t^z [r(\mu) + \alpha + p] d\mu \right) = 0$.

- ▶ $H(t)$: discounted present value of those currently alive:

discounted for death (p) and for decline on labor services (α).

Individual and Aggregate Human Wealth, algebra

► $h(t, s) = \int_t^\infty a Y(z) e^{-\alpha(z-s)} R(t, z) dz$

$$= a \left[\int_t^\infty Y(z) e^{-\alpha(z-t)} R(t, z) dz \right] e^{-\alpha(t-s)} .$$

► $H(t) = \int_{-\infty}^t h(s, t) p e^{-\rho(t-s)} ds$

$$= \int_{-\infty}^t a p \left[\int_t^\infty Y(z) e^{-\alpha(z-t)} R(t, z) dz \right] e^{-\alpha(t-s)} e^{-\rho(t-s)} ds .$$

- Using the characterization of $R(t, z)$ and a :

$$H(t) = \int_t^\infty Y(z) \exp \left\{ - \int_t^z (\alpha + \rho + r(\mu)) d\mu \right\} dz .$$

- This present value relationship implies the ODE (diff. w.r.t. t):

$$dH(t) = [\alpha + r(t) + \rho] H(t) - Y(t) \text{ plus boundary.}$$



Individual and Aggregate Non-Human Wealth

► Define $V(t) = \int_{-\infty}^t v(s, t) N(s, t) ds = \int_{-\infty}^t v(s, t) p e^{-\rho(t-s)} ds .$

- Differentiate $V(t)$ w.r.t. to t to obtain:

$$\frac{dV(t)}{dt} = \rho v(t, t) - \rho V(t) + \int_{-\infty}^t \left[\frac{dv(s, t)}{dt} \right] p e^{-\rho(t-s)} ds,$$

- $v(t, t) = 0$ since it is the wealth at birth (OLG with no altruism!),

► $\frac{dv(s, t)}{dt} = [r(t) + \rho] v(s, t) + y(t) - c(t)$, i.e. budget constraint,

► Thus $\frac{dV(t)}{dt} = r(t) V(t) + Y(t) - C(t) .$

- Aggregate wealth accumulates at r , individual at $r + \rho$.

- Difference, $\rho V(t)$ is the transfer from annuities.

insurance does not add to aggregate wealth.



Summary Aggregate Behavior

- ▶ $C(t) = (p + \theta)(H(t) + V(t))$
- ▶ $\frac{dV(t)}{dt} = r(t) V(t) + Y(t) - C(t)$
- ▶ $\frac{dH(t)}{dt} = [r(t) + p + \alpha] H(t) - Y(t)$
- ▶ and boundary condition as $t \rightarrow \infty$.

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- ▶ and boundary condition as $t \rightarrow \infty$.
- ▶ Differentiating $C(t)$ and replacing $\frac{dH(t)}{dt}$, $\frac{dV(t)}{dt}$:

$$\begin{aligned} \frac{dC(t)}{dt} &= [r(t) + \alpha - \theta] C(t) - (p + \alpha)(p + \theta) V(t), \\ \frac{dV(t)}{dt} &= r(t) V(t) + Y(t) - C(t), \end{aligned}$$

and boundary condition as $t \rightarrow \infty$.

Summary Aggregate Behavior, algebra

- ▶ $C(t) = (p + \theta)(H(t) + V(t))$
- ▶ $\dot{V}(t) = r(t) V(t) + Y(t) - C(t)$
- ▶ $\dot{H}(t) = [r(t) + p + \alpha] H(t) - Y(t)$
- ▶ Differentiating C , replacing \dot{H} , \dot{V} , and $C = (\theta + p)(H + V)$:

$$\begin{aligned}
 \dot{C} &= (p + \theta)r[H + V] - (p + \theta)C + (p + \alpha)(p + \theta)H, \\
 &= [r - p - \theta] C + (p + \alpha)(p + \theta)H, \\
 &\quad \text{adding and subtracting } (p + \alpha)(p + \theta)V \\
 &= [r - p - \theta] C + (p + \alpha)(p + \theta)(H + V) - (p + \alpha)(p + \theta)V, \\
 &= [r - p - \theta] C + (p + \alpha)C - (p + \alpha)(p + \theta)V, \\
 &= [r + \alpha - \theta] C - (p + \alpha)(p + \theta)V.
 \end{aligned}$$

Summary Aggregate Behavior, details on boundaries

- ▶ For the agent problem to have finite wealth we require:
- ▶ $Y(t) \lim_{T \rightarrow \infty} e^{-\int_t^T [r(z) + \alpha + p] dz} = 0$.
- ▶ If this is violated agents have unbounded human wealth.
- ▶ If this is violated agents can borrow an unbounded amount.
- ▶ Note that for $\alpha + p > 0$, we can have $r < 0$.

General Equilibrium: 3 cases

- Interest rates on pure endowment:

$$Y(t) = C(t) \text{ and } V(t) = 0.$$

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- Add capital accumulation, study capital accumulation:

$$F(K) = \mathbb{F}(K, 1) - \delta K, \quad \mathbb{F}(\cdot, \cdot) \text{ CRTS neoclassical,}$$

$$Y(t) = F(K(t)) - F'(K(t))K(t), \quad r(t) = F'(K(t)),$$

$$V(t) = K(t).$$

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- ▶ Add government debt:

$$V(t) = B(t) + K(t).$$

Pure Endowment

- ▶ $Y(t) = C(t) = Y$.

- ▶ $V(t) = 0$, since some agents borrow and others lend:

$$V(t) = \int_{-\infty}^t N(s, t) v(s, t) ds = 0.$$

- ▶ $v(s, t)$ value of cumulated savings (net assets) of cohort born at s at t .

- ▶ $0 = \frac{dC(t)}{dt} = [r(t) + \alpha - \theta] C(t) - (p + \alpha)(p + \theta) V(t)$ implies

- ▶ Equilibrium interest rate $r(t) = \theta - \alpha$.

- ▶ Individuals: $\frac{dc(s, t)}{dt} = [r(t) - \theta] c(s, t) = -\alpha c(s, t)$.

- ▶ Thus equilibrium is autarky! $c(s, t) = y(s, t)$ and $v(s, t) = 0$.

Pure Endowment, cont.

- ▶ What did we assume for the distribution of wealth among those agents alive at time $t = 0$?
- ▶ What is the equilibrium interest rate for $\alpha = 0$? How are the individual consumptions?
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Pure Endowment, cont.

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- ▶ What is the equilibrium interest rate for $\alpha = 0$? How are the individual consumptions?
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- ▶ What is the equilibrium interest rate for $p = 0$? How are the individual consumptions?
- ▶ Consider the class of allocations with $c(s, t) = Y$. Are these Pareto superior to the equilibrium?
- ▶ Question: can $r(t) = \theta$ be an equilibrium?

Capital Accumulation, Technology

- ▶ Neoclassical Production Function $\mathbb{F}(K, L)$.
- ▶ Time t Feasibility: $I(t) + C(t) = \mathbb{F}(K(t), L(t))$.
- ▶ Discrete time with length Δ gives: $K_{t+\Delta} = \Delta I_t + K_t(1 - \Delta\delta)$.
- ▶ Limit of discrete time as $\Delta \downarrow 0$: $\frac{dK(t)}{dt} = I(t) - \delta K(t)$.
- ▶ Replacing Law of Motion of capital into time t Feasibility:
- ▶ $\frac{dK(t)}{dt} + C(t) = \mathbb{F}(K(t), L(t)) - \delta K(t)$.
- ▶ We also replace that labor supply is inelastic at $L = 1$:
- ▶ Define $F(K) \equiv \mathbb{F}(K, 1) - \delta K$ and assume no depreciation:

$$\frac{dK(t)}{dt} = F(K(t)) - C(t).$$



Capital Accumulation, Equilibrium payment to factors

- ▶ Aggregate Labor Income $Y(t) = w(t) L = F(K(t)) - F'(K(t)) K(t)$ for $L = 1$.
- ▶ Interest Rate = Marginal Productivity capital: $r(t) = F'(K(t))$.
- ▶ Discrete time : $\nu_{t+\Delta} = r_t + \delta$ and $\mathbb{F}_k(K_{t+\Delta}, 1) = \nu_{t+\Delta}$.
where ν_t is the rental rate of capital at t .
- ▶ Limit of discrete time as $\Delta \downarrow 0$:

$$\mathbb{F}_k(K(t), 1) = \nu(t) = r(t) + \delta \text{ or } F'(K(t)) = r(t).$$



Capital Accumulation, Equilibrium

- ▶ $v(s, t)$ value of cumulated savings (net assets) of cohort born at s at t :

$$V(t) = \int_{-\infty}^t N(s, t) v(s, t) ds = K(t) .$$

- ▶ Interest payments plus labor income = Output:

$$r(t) K(t) + Y(t) = F'(K(t)) K(t) + F(K(t)) - F'(K(t)) K(t) = F(K(t))$$

$$\frac{dC(t)}{dt} = [r(t) + \alpha - \theta] C(t) - (p + \alpha)(p + \theta) V(t) ,$$

$$\frac{dV(t)}{dt} = r(t) V(t) + Y(t) - C(t) ,$$

- ▶ Becomes

$$\frac{dC(t)}{dt} = [F'(K(t)) + \alpha - \theta] C(t) - (p + \alpha)(p + \theta) K(t) ,$$

$$\frac{dK(t)}{dt} = F(K(t)) - C(t) .$$



Capital Accumulation, Dynamics

- ▶ Phase diagram on (K, C) space.
- ▶ Locate (K, C) such that $\dot{C} \equiv \frac{dC(t)}{dt} = 0$. Line given by: $C = \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$
- ▶ Locate (K, C) such that $\dot{K} \equiv \frac{dK(t)}{dt} = 0$. Line given by: $C = F(K)$
- ▶ Locate steady state(s): (\bar{K}, \bar{C}) .
- ▶ Find field (direction of movements everywhere)
- ▶ Discard paths that takes system to points that eventually are not feasible.



Dynamics, more details

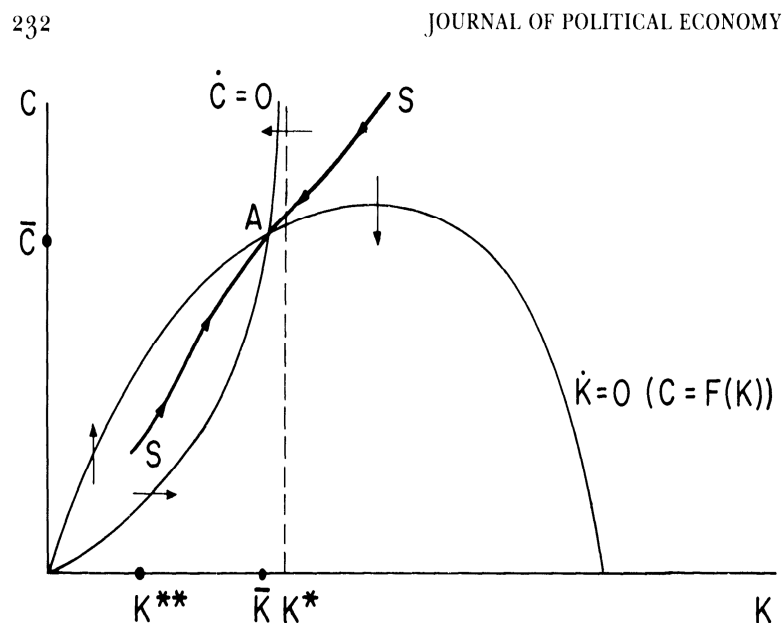
- ▶ $\dot{K} = 0$: $C = F(K)$.
 - ▶ Concave, initially increasing. Maximum at $F'(\bar{K}) = 0$, for all $K \geq 0$.
 - ▶ Starts very steep, and converges to zero with Inada conditions.
 - ▶ If $C > F(K)$, then capital decreases.
- ▶ $\dot{C} = 0$: $C = \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$, for all $K \geq \hat{K}$
 - ▶ Convex increasing function of K , since $F'(\cdot)$ is decreasing.
 - ▶ Asymptotes to $+\infty$ as K converges to \hat{K} defined as: $F'(\hat{K}) = \theta - \alpha$.
 - ▶ If $C > \frac{(p+\alpha)(p+\theta)}{F'(K)+\alpha-\theta} K$, consumption decreases.
- ▶ Unique Interior Steady state solves: $F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta} \bar{K}$.



Dynamics, saddle path and field

- ▶ Locate the direction of change at the lines $\dot{C} = 0$ and $\dot{K} = 0$.
- ▶ Locate the direction of change close to the axes: $C = 0$ and $K = 0$.
- ▶ Follow the motion of the system in four quadrants defined by the intersections of the $\dot{C} = 0$ and $\dot{K} = 0$ lines.
- ▶ Argue that there is an upward sloping line $C = \mathbf{S}(K)$ such that :
 - ▶ If $C(0) = \mathbf{S}(K(0))$, and the system follows the ODE, converges to SS.
 - ▶ Any other path converges in finite time to negative C or K .
 - ▶ Thus, in a perfect foresight equilibrium $C(t) = \mathbf{S}(K(t))$ for all $t \geq 0$.
 - ▶ What is the relationship between $C = \mathbf{S}(K)$ and $C = (\theta + p)(K + H)$?
 - ▶ Is $\mathbf{S}(\cdot)$ necessary linear?



Figure: Case of $\alpha = 0$ 

$$F'(K^*) = \theta \text{ and } F'(K^{**}) = p + \theta$$

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Adjustment to Steady State

- ▶ Assume that $0 < K(0) < \bar{K}$.
- ▶ What are the characteristics of the time path for $C(t)$ and $K(t)$?
Are $C(t)$ and $K(t)$ monotone?
- ▶ What are the characteristics of the time path for labor income $Y(t)$?
Are $Y(t)$ monotone?
- ▶ What are the characteristics of the time path for interest rates $r(t)$?
Are $r(t)$ monotone?
- ▶ What are the characteristics of the time path for human wealth $H(t)$?
Are $H(t)$ monotone?

Comparative Statics of Steady States

- ▶ Unique Interior Steady state $\bar{K} : F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta} \bar{K}$ is
 - ▶ Decreasing in p .
 - ▶ Increasing in α .
 - ▶ Decreasing in θ .
- ▶ Interest rate $r = F'(\bar{K})$. If $r > (<) \theta$, individual consumption \uparrow (\downarrow).
- ▶ Set $\alpha = 0$. If individual consumption is constant ($r = \theta$), zero savings.
- ▶ If individual consumption increases, then agents save when young.
- ▶ Slope of consumption profile depends only on $r - \theta$: thus as $p \downarrow$, longer lifetime, smaller turn-around of capital across generations.
- ▶ If $\alpha > 0$, agents save, even if their consumption is constant.

Comparative Statics of Steady States, cont.

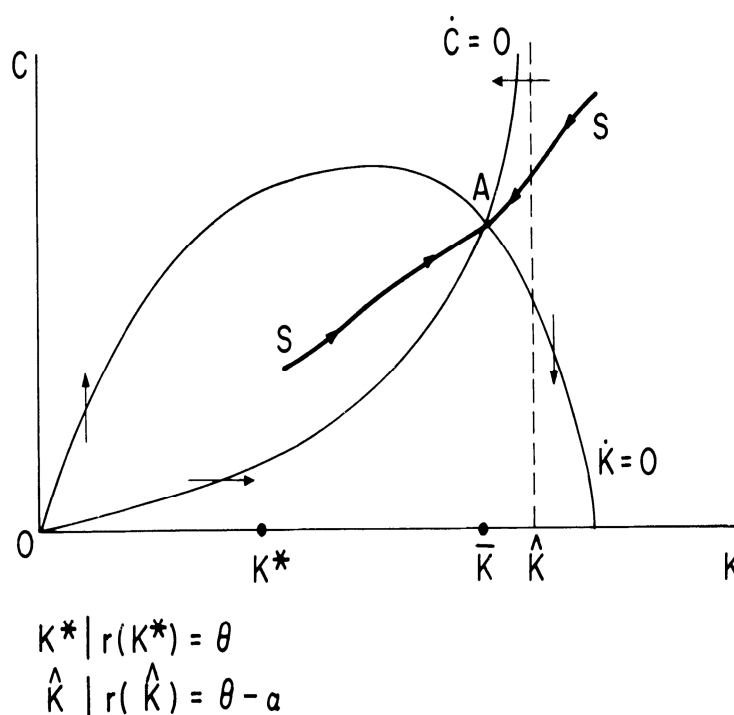
- ▶ Unique Interior Steady state $\bar{K} : F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta} \bar{K}$
 - ▶ If $\alpha = 0$, then \bar{K} is smaller than golden rule, and hence $r > 0$.
 - ▶ If $\alpha = 0$, then $\theta < F'(\bar{K}) < \theta + p$.
 - ▶ Define $F'(K^{**}) = \theta + p$. Replace on expression for \bar{K} .
 - ▶ Use that, by concavity of $F(\cdot)$: $F(K^{**}) < (p + \theta)K^{**} = F'(K^{**})K^{**}$.
 - ▶ If $\alpha > \theta$, then \bar{K} can be larger than golden rule, and hence $r < 0$.
 - ▶ If $\alpha > \theta$, for p large enough, then \bar{K} larger than golden rule.
- ▶ Recall that if $p = 0$, we must set $\alpha = 0$.
 - ▶ If $p = 0$ then $F'(\bar{K}) = \theta$, smaller than golden rule, and hence $r > 0$.
 - ▶ If $p = 0$ then $\dot{C} = 0$ is a vertical line.

Comparative Statics of Steady States, details

- ▶ Mathematical argument to show that \bar{K} is increasing in α .
- ▶ Steady state solves: $F(\bar{K}) = \frac{(p+\alpha)(p+\theta)}{F'(\bar{K})+\alpha-\theta} \bar{K}$
- ▶ Suffices to show that $\frac{p+\alpha}{F'(\bar{K})+\alpha-\theta}$ is decreasing in α .
- ▶ We have $\frac{\partial}{\partial \alpha} \frac{p+\alpha}{F'(\bar{K})+\alpha-\theta} = \frac{F'(\bar{K})-\theta-p}{(F'(\bar{K})+\alpha-\theta)^2}$.
- ▶ Recall that if $\alpha = 0$ then $0 < F'(\bar{K}) < \theta + p$.
- ▶ Thus for $\alpha = 0$, steady state \bar{K} is increasing in α .
- ▶ For higher values of α , capital \bar{K} is higher, and thus $F'(\bar{K})$ even smaller.
- ▶ Hence, \bar{K} is increasing for all non-negative values of α .

Navigation icons: back, forward, search, etc.

Figure: Case of $\alpha > 0$ large enough



Navigation icons: back, forward, search, etc.

Infinite Lifetime, limit as $p \downarrow 0$

- ▶ If $p = 0$, stationarity requires $\alpha = 0$.
- ▶ System becomes:

$$\begin{aligned}\frac{dC(t)}{dt} &= [F'(K(t)) - \theta] C(t), \\ \frac{dK(t)}{dt} &= F(K(t)) - C(t).\end{aligned}$$

= neoclassical

- ▶ $\dot{C} = 0$ locus vertical.
- ▶ Steady state $F'(\bar{K}) = \theta$.
- ▶ Slightly different argument to rule out paths outside saddle path.
- ▶ Similar adjustment to steady state.

infinite elastic supply of capital

Government Budget Constraint

- ▶ Let $T(t)$ denote taxes levied to all agents alive at time t .
- ▶ Let $G(t)$ denote government purchases at time t .
- ▶ Let $B(t)$ denote government debt at time t .
- ▶ Sequential B.C.: $\frac{dB(z)}{dz} = G(z) - T(z) + r(z)B(z)$.
- ▶ Present value B.C: $B(t) = \int_t^\infty [T(z) - G(z)] e^{-\int_t^z r(\mu) d\mu} dz$ or
- ▶ Sequential B.C. and boundary $\lim_{T \rightarrow \infty} B(T) e^{-\int_T^z r(\mu) d\mu} = 0$ are equivalent to present value.

Ricarding Equivalence

- ▶ Effect on taxes and debt on determination of consumption $C(t)$ as a function of human wealth $H(t)$ and non-human wealth $V(t)$.
- ▶ To simplify notation and discussion, for now set $\alpha = 0$.
- ▶ Labor income $Y(t)$ becomes labor income after taxes $Y(t) - T(t)$:

$$\begin{aligned} C(t) &= (p + \theta) (H(t) + V(t)) \\ H(t) &= \int_t^\infty [Y(z) - T(t)] \exp \left\{ - \int_t^z (p + r(\mu)) d\mu \right\} dz \\ \frac{dV(t)}{dt} &= r(t) V(t) + Y(t) - T(t) - C(t). \end{aligned}$$

- ▶ or ODE for agg. human wealth: $\frac{dH(t)}{dt} = [r(t) + p] H(t) - [Y(t) - T(t)]$.

Ricarding Equivalence, Partial Equilibrium

- ▶ Change in taxes at time t offset with change at time $t + s$
Use Gov. B.C.: $dT(t + s) = - \exp \left[\int_t^{t+s} r(\mu) d\mu \right] dT(t)$
- ▶ $dT(t)$ and $dT(t + s)$ used heuristically, each has negligible impact.
- ▶ Effect on human wealth $H(t)$, and hence, $C(t)$: Use ODE for H :
$$dH(t) = -dT(t) - dT(t + s) \left\{ \exp \left[- \int_t^{t+s} [r(\mu) + p] d\mu \right] \right\}$$
- ▶ Replace into Gov. BC: $dH(t) = -dT(t) [1 - \exp(-ps)]$.
- ▶ If $dT(t) < 0$, then $dH(t) > 0$ and $dC(t) > 0$ if $p > 0$:
or, cut on taxes increases consumption if $p > 0$. Why?

- ▶ Do you get same equilibrium C 's and r 's?

budget constraint

- ▶ Is this a failure of Ricarding Equivalence? No.

- ▶ Compare with experiment in problem set.

Government Debt on Pure Endowment case

- ▶ Still $Y(t) - G = C(t)$ for $G = 0$, but $V(t) = B(t)$:
- ▶ Private sector previous cohorts cumulated savings equal gov't debt

$$V(t) = \int_{-\infty}^t N(s, t) v(s, t) ds = B(t) .$$

- ▶ $0 = \frac{dC(t)}{dt} = [r(t) + \alpha - \theta] (Y - G) - (p + \alpha)(p + \theta)B(t)$ implies
- ▶ $r(t) = (p + \alpha)(p + \theta) \frac{B(t)}{Y - G} + [\theta - \alpha] .$
- ▶ Evolution of debt $\frac{dB(t)}{dt} = r(t)B(t) + G - T .$
- ▶ Replacing interest rate into dynamics for debt:

$$\dot{B} = (p + \alpha)(p + \theta) \frac{B^2}{Y - G} + B[\theta - \alpha] + G - T .$$

Simple debt dynamics

- ▶ Set $T = G = 0$

$$\dot{B} = (p + \alpha)(p + \theta) \frac{B^2}{Y} + B[\theta - \alpha] .$$

- ▶ Two steady states: $B^* = 0$ and $\bar{B} = \frac{\alpha - \theta}{(p + \alpha)(p + \theta)} Y$
- ▶ Autarky steady state interest rate $\bar{r} = \theta - \alpha$.
- ▶ If $\bar{r} < 0$, interior steady state \bar{B} is positive.
- ▶ Interior positive steady state \bar{B} is not stable:
 - ▶ If $B(0) < \bar{B}$, then $\lim_{t \rightarrow \infty} B(t) = 0$.
 - ▶ If $B(0) > \bar{B}$, there is no equilibrium.
- ▶ Identical to the case of 2 period OLG model, log utility. Try it.
- ▶ How will the analysis change if $T - G \neq 0$? Try it.
- ▶ Which s.s. is stable when $T - G \neq 0$? How does it depend on $\theta - \alpha$?

Debt dynamics and capital

- ▶ Set $\alpha = 0$ for simplicity (but it rules out some important cases)
- ▶ Dynamical System with $B + K = V$, keeping T, G constant:

$$\begin{aligned}\frac{dC}{dt} &= (r - \theta)C - p(p + \theta)(B + K) \\ \frac{dK}{dt} &= F(K) - C - G, \\ \frac{dB}{dt} &= rB + G - T, \\ r &= F'(K).\end{aligned}$$

- ▶ Steady States for given G, T :

$$\begin{aligned}[F'(K^*) - \theta] C^* &= p(p + \theta)(B^* + K^*), \\ F(K^*) &= C^* + G, \quad \text{before } C = F(K) \\ F'(K^*)B^* &= T - G.\end{aligned}$$

Debt and capital: steady states (fixed G, T)

- ▶ Steady States capital K^* solves:

$$F(K^*) - G = \frac{p(p + \theta)}{[F'(K^*) - \theta]} \left(\frac{T - G}{F'(K^*)} + K^* \right).$$

- ▶ Interpretation of comparative static w.r.t. G and T .
Does increases in taxes T increase the debt?
- ▶ Is there a unique steady state? Signs of B and r at s.s.
Compare with pure endowment.
- ▶ How do K^*, B^* and r^* vary with G and or T ?
What is the difference between increases in G and decreases in T .
- ▶ What is the effect on steady state quantities if p increases?
What is the limit case as $p \downarrow 0$?

Debt dynamics and capital: balanced budget

- ▶ Consider Fiscal Policy of keeping debt constant, adjusting taxes.
- ▶ Dynamical System with $B + K = V$, keeping B, G constant:

$$\begin{aligned}\frac{dC}{dt} &= (F'(K) - \theta) C - p(p + \theta)(B + K) \\ \frac{dK}{dt} &= F(K) - C - G, \\ T &= F'(K)B + G.\end{aligned}$$

- ▶ Steady States for given G, B :

$$\begin{aligned}[F'(K^*) - \theta] C^* &= p(p + \theta)(B + K^*), \\ F(K^*) &= C^* + G\end{aligned}$$

- ▶ Is there a unique steady state K^* ?
- ▶ What is the effect on K^*, r^* of an increase in the initial level B ?
- ▶ What is the effect on K^*, r^* of an increase on G ?
- ▶ How do the answers depend on p ?