Price Theory, III **Professor Lars Stole**
Economics 33003 **Spring 2019**

Assignment 4
(Due Friday, May 3, prior to the start of the Review session)

Problem 1 Consider a principal-agent setting with moral hazard and a finite number of outputs and efforts. There are n possible outputs, $x_i \in \mathcal{X} = \{x_1, \ldots, x_n\}$, $x_n > x_{n-1} > \cdots > x_1$. There are m possible effort levels, $e_j \in \mathcal{E} = \{e_1, \ldots, e_m\}$, $e_m > e_{m-1} > \cdots > e_1$. The probability of output x_i given effort e_j is $\phi_i(e) = \phi(x_i|e_j) > k > 0$ (i.e., bounded away from zero). Assume that $\phi_i(\cdot)$ satisfies MLRP: i.e., for all $e > \tilde{e}$,

$$\frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}, \text{ is increasing in } i.$$

Assume that the principal is risk neutral and the agent is risk averse with preferences $U = u(w) - \psi(e)$, where $u(\cdot)$ is increasing and concave, and $\psi(\cdot)$ is increasing and convex. Assume the agent's outside option is $\underline{U} = 0$ and $\psi(e_1) = 0$. Finally, the principal's wage schedule must satisfy $w_i = w(x_i) \in [\underline{w}, \overline{w}]$, but assume that these constraints are slack at the optimum and ignore them.

(a). State the principal's program for the optimal e and $(w_1, \ldots, w_n)$, and write the Lagrangian for the program using $\{\mu_j\}_j$ as multipliers for the IC constraints and λ as the multiplier for the IR constraint.

(b). Prove that the agent's IR constraint is binding (i.e., $\lambda > 0$) in the solution to the program.

(c). Suppose that the optimal choice of effort is $e^* = e_m$, the maximum effort level. Prove that the optimal wage schedule which induces e^m is monotonic, $w_1^* \leq w_2^* \cdots \leq w_n^*$ with strict inequality for some i (i.e., $w_i^* < w_{i+1}^*$).

(d). Prove that the MLRP implies first-order stochastic dominance for the case of a discrete distribution, $(\phi_1(e), \ldots, \phi_n(e))$.

(e). Suppose that the optimal choice of effort $e^* > e_1$. Prove that there must exist some i such that $w_i^* < w_{i+1}^*$ (i.e., prove that $w_1^* \geq w_2^* \geq \cdots \geq w_n^*$ cannot be optimal). [Hint: use the fact in (d) above.]

Problem 2 Consider the following moral hazard problem. The worker can choose one of two (unobservable) effort levels, $e \in \{1, 2\}$. If the worker exerts effort e and receives wage w, her payoff is $\sqrt{w} - e$. The outside option of the worker is zero. The owner of the firm is risk neutral. There are two possible realizations of the output, $x_1 = 1$ and $x_2 = 16$.

(a). Suppose that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	α	0
$x_2 = 16$	$1 - \alpha$	1

where $\alpha \in (0, 1)$. Characterize the set of those α's for which the owner of the firm can achieve the same payoff as if the effort of the worker was observable and contractible.

(b). Suppose now that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	1	β
$x_2 = 16$	0	$1 - \beta$

where $\beta \in (0, 1)$. Again, characterize the set of those β's for which the owner of the firm can achieve the same payoff as if the effort of the worker was contractible.

(c). Suppose that the stochastic relationship between effort and output is described by the probability matrix given in part (b) and $\beta = \frac{1}{4}$. Characterize the optimal contract for the owner of the firm.

(d). Suppose that the production technology is the same as in part (c). Assume that the owner of the firm can invest in a monitoring technology which perfectly reveals the effort of the worker. The outcome of this monitoring is contractible. The owner has to make this (observable) investment decision prior to offering a contract and it costs p. For what values of p should the owner invest in monitoring?

Problem 3 Consider the a setting like in Holmstrom and Milgrom, in which a risk neutral principal contracts with a risk-averse (CARA utility with r) agent. Specifically, suppose that a school wishes to design an optimal compensation contract for its teachers. Teachers engage in two tasks:

$$x_1 = e_1 + \varepsilon_1,$$

$$x_2 = e_2 + \varepsilon_2,$$

where ε_i is independently distributed according to $N(0, \sigma_i^2)$. The cost of the teacher's effort is $\psi(e_1 + e_2)$ where

$$\psi(e) = \begin{cases} 0 & \text{if } e \leq \hat{e} \\ \frac{1}{2}(e - \hat{e})^2 & \text{if } e \geq \hat{e}. \end{cases}$$

Thus, $\hat{e} > 0$ represents an amount of free labor that a teacher will supply.

Suppose that task 1 (teaching math and reading skills) is easy to measure via test scores in math and reading, but task 2 (teaching creativity) is almost impossible to measure. Hence, σ_1^2 is small and σ_2^2 is large. Additionally, suppose that the school cares about the benefits of e_1 and e_2 for its students and these benefits are characterized by the function, $B(e_1, e_2) \geq 0$. $B(\cdot, \cdot)$ is strictly concave, and strictly increasing in both arguments for $e_1 > 0$ and $e_2 > 0$; but assume that $B(e_1, 0) = B(0, e_2) = 0$ for all e_1 and e_2. That is, teaching math and reading is pointless without teaching a little bit of creativity, and vice versa. For reasons given in Holmstrom and Milgrom (1987), the school chooses a linear compensation function,

$$w(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 + \beta.$$

The school maximizes the expectation of $B(e_1, e_2) - w(x_1, x_2)$ subject to incentive compatibility and the teacher's outside option, $\underline{U}$. The teachers are solely motivated by the wage contract and their cost of aggregate effort. They do not care about B or how effort is allocated across tasks, except insofar as it impacts the wage. If they are indifferent between two allocations of effort, however, we assume that they will choose the one preferable to the school.

(a). Suppose in the extreme that $\sigma_2^2 = \infty$; i.e., e_2 cannot be measured. What is the optimal incentive scheme for the school to offer the teachers?

(b). Suppose that σ_2^2 is finite. State the principal's optimal program. Prove that in any solution, $\alpha_1 = \alpha_2 = \alpha$, and prove that $\alpha \notin (0, \hat{e}]$.

Problem 4 Consider the modified linear managerial-incentive-scheme problem, where the manager's effort, e, affects current profits, $x_1 = e + \varepsilon_1$, and future profits, $x_2 = e + \varepsilon_2$, where ε_i are i.i.d. with normal distribution $N(0, \sigma_\varepsilon^2)$. Note that the single effort equally impacts the short run (period 1) and the long run (period 2). The manager retires at the end of the first period, and the manager's compensation cannot be based on x_2. However, the company can issue stock that she must hold for one year after retirement. The price of the stock one year after retirement is $p = x_1 + x_2 + \eta$, where η is distributed normally, $N(0, \sigma_\eta^2)$, and η is independently distributed from ε_t. The firm maximizes the expectation of $x_1 + x_2 - w - sp$, where s are the shares of stock given to the manager. There is no time discounting and the manager only cares about the total value of compensation once the stock is sold one year after retirement. The manager's utility is CARA with risk parameter, r, and her monetary cost of effort is $\frac{1}{2}e^2$. Her outside option is $\underline{U}$. Assume for reasons given in Holmstrom and Milgrom (1987) that the optimal contract is linear. Derive the optimal compensation contract for period 1 output

$$w(x_1) = \alpha x_1 + \beta,$$

and derive the optimal amount of stock, s, to give to the manager. Explain the differences in the optimal α and s. In particular, what happens if the stock market price is a perfect aggregator of $x_1 + x_2$ with $\sigma_\eta^2 = 0$?

Problem 5 Consider the case of a single firm contracting with two agents, each with a CARA parameter of r in the standard linear-contracts framework of Holmstrom and Milgrom. The outputs of the two agents are $x_1 = e_1 + \varepsilon_1$ and $x_2 = e_2 + \varepsilon_2$, where e_i is agent i's effort and ε_i is a normally distributed noise term, $\varepsilon_i \sim \mathcal{N}(0, \sigma_i^2)$. The cost of effort to each agent i is $\frac{1}{2}e_i^2$. Finally, the measurement errors across the two agents may be positively correlated, with $\mathrm{Cov}(\varepsilon_1, \varepsilon_2) = \sigma_{12} = \rho\sigma_1\sigma_2$, $\rho \geq 0$. Assume for reasons given in Holmström and Milgrom that optimal contracts are linear.

(a). Consider the benchmark where a firm is restricted to allowing only an individual worker's own output to effect compensation:

$$w_i(x_1, x_2) = \alpha_i^i x_i + \beta^i.$$

Superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i).

(b). Now suppose that the firm can use relative performance evaluation. That is

$$w_1(x_1, x_2) \equiv \alpha_1^1 x_1 + \alpha_2^1 x_2 + \beta^1,$$

$$w_2(x_1, x_2) \equiv \alpha_1^2 x_1 + \alpha_2^2 x_2 + \beta^2.$$

Again, superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i). Explain how and why α_i^i varies with respect to ρ. Does it matter if the correlation is positive or negative?

Hint: if ε_1 and ε_2 are normal random variables with covariance σ_{12}, then

$$E[-e^{-r(a\varepsilon_1+b\varepsilon_2)}] = -e^{-r(a^2\sigma_1^2+b^2\sigma_2^2+2ab\sigma_{12})}.$$

(c). Continue to assume that the firm can use relative performance evaluation, as in (b), but that now $\rho = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$. In addition to choosing e_2 to increase output, agent two can also engage in a second "helpful" activity, h_2. This activity does not affect the output level directly, but rather reduces the effort cost of the other agent. The interpretation is that agent 2 can help agent 1 (but not the other way round). The cost functions of the agents are given by:

$$\psi_1(e_1, h_2) = \frac{1}{2}(e_1 - h_2)^2,$$

$$\psi_2(e_2, h_2) = \frac{1}{2}e_2^2 + \frac{1}{2}h_2^2.$$

Agent 1 chooses his effort level e_1 only after he has observed the level of help h_2.

Compute the optimal incentive scheme and effort levels. Explain your result.

Problem 6 (MWG, 14.B.6) Amend the two-effort-level model with a risk-neutral principal as follows: Suppose now that effort has distinct effects on revenues, R, and costs, C, where $x = R - C$. Let $f_R(R, a)$ and $f_C(C, a)$ denote the density functions of R and C conditional on a, and assume that, conditional on a, R and C are independently distributed. Assume $R \in [R_0, R_1]$, $C \in [C_0, C_1]$, and that for all a, $f_R(R, a) > 0$ for all $R \in [R_0, R_1]$ and $f_C(C, a) > 0$ for all $C \in [C_0, C_1]$.

The two effort choices are now $\{a_R, a_C\}$, where a_R is an effort choice that devotes more time to revenue enhancement and less to cost reduction, and the opposite is true for a_C. In particular, assume that $F_R(R, a_R) < F_R(R, a_C)$ for all $R \in (R_0, R_1)$ and that $F_C(C, a_C) > F_C(C, a_R)$ for all $C \in (C_0, C_1)$. Moreover, assume that the monotone likelihood ratio property holds for each of these variables in the following form: $f_R(R, a_R)/f_R(R, a_C)$ is increasing in R, and $f_C(C, a_R)/f_C(C, a_C)$ is increasing in C. Finally, the agent prefers revenue enhancement over cost reduction: that is, $\psi(a_C) > \psi(a_R)$.

(a). Suppose that the owner wants to implement effort choice a_C and that both R and C are observable. Derive the first-order condition for the optimal compensation scheme $w(R, C)$. How does it depend on R and C?

(b). How would your answer to (a) change if the agent could always unobservably reduce the revenues of the firm (in a way that is of no direct benefit to him)?

(c). What if, in addition, costs are now unobservable by a court (so that compensation can be made contingent only on revenues)?