Price Theory II
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(Phil Reny)

Finding Mixed Strategy Equilibria

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1 Computing randomised Nash equilibria for games larger than 2×2

Notation differ slightly from class.

1.1 General strategy

Notation: $N = \{1, 2, \ldots, n\}$, $u_i : \times_{j \in N} C_j \to \mathbb{R}$, $\sigma \in \times_{i \in N} \Delta(C_i)$.

Step 1: Guess the supports for each player $i \in N$ denoted $S_i(\sigma) = \{c_i \in C_i : \sigma_i(c_i) > 0\} \neq \emptyset$.

Step 2: Solve for the equilibrium in the reduced game where the action set for each player i is reduced to S_i

The unknowns For each player $i \in N$, and each action $s_i \in S_i$, let $\sigma_i(s_i)$ denote is probability of choosing s_i, and let w_i denote player i's expected payoff in equilibrium; i.e.

$$w_i := u_i(\sigma_{-i}; [s_i]), \ \forall s_i \in S_i, \ \forall i \in N.$$

The number of unknowns equal the sum of the number of pure strategies in the support for each player (i.e. $\sigma_i(s_i)$'s) plus the number of players (i.e. w_i's).

The equations For each player i, $\sigma_i(s_i)$'s must be strictly positive, and the sum of $\sigma_i(s_i)$ must equal one (which, in turn, implies, $\sigma_i(a_i) = 0$ for all $a_i \in C_i \backslash S_i$):

$$\begin{aligned} \sigma_i(s_i) &> 0, \ \forall s_i \in S_i, \ \forall i \in N, \\ \sum_{s_i \in S_i} \sigma_i(s_i) &= 1, \ \forall i \in N. \end{aligned} \tag{1.1}$$

For each player i, and for each action $s_i \in S_i$, player i's expected payoff when he chooses s_i but all other players randomise independently according to their σ_j probabilities must equal w_i; i.e.

$$w_i = u_i(\sigma_{-i}; [s_i]), \ \forall s_i \in S_i, \ \forall i \in N, \tag{1.2}$$

where $u_i(\sigma_{-i}; [s_i]) = \mathbb{E}[u_i(s_i|\sigma_{-i})]$. We have N equations from (1.1) and sum of the number of pure strategies in the support for each player equation from (1.2); i.e. we have as many equations as the unknowns $(w_i, \sigma_i(s_i))$.

The check If the equations have no solution, the guess for the support was wrong. Return to step 1 with a different guess for the support. If we find a solution, then it is a randomised equilibrium of the reduced game where each player chooses actions in S_i.

[1] I have borrowed materials from Roger Myerson's lecture notes heavily. All mistakes are my own.

Step 3: Check that equilibrium for the reduced game is an equilibrium of the original game We must now check that players would not prefer to play pure strategies outside the guessed support. That is, we check, for each player i, and for each action $a_i \in C_i \backslash S_i$, whether player i could do better than w_i by choosing a_i when all other players randomise independently according to their σ_j probabilities; i.e. we must check

$$u_i \left(\sigma_{-i}; [a_i]\right) \leq w_i, \ \forall a_i \in C_i \backslash S_i, \ \forall i \in N. \tag{1.3}$$

If the solution does not satisfy this, then go back to step 1 with a different guess for the support.

Proposition 1.1. *A Nash equilibrium*

$$\sigma = \left(\left(\sigma_i\left(a_i\right)\right)_{a_i \in C_i}\right)_{i \in N}$$

with payoffs $w = (w_i)_{i \in N}$ *must satisfy*

$$\begin{aligned} \sum_{s_i \in S_i} \sigma_i\left(s_i\right) = 1, \ \forall i \in N \\ \sigma_i\left(a_i\right) \geq 0, \ \forall a_i \in C_i, \ \forall i \in N, \\ u_i \left(\sigma_{-i}; [a_i]\right) \leq w_i, \ \forall a_i \in C_i, \ \forall i \in N \text{ with at least one equality.} \end{aligned}$$

The support for each player i is the set of actions $s_i \in C_i$ for which $\sigma_i(c_i) > 0$ so that $u_i(\sigma_{-i}; [s_i]) = w_i$.

Remark 1.1. The last condition is equivalent to saying that $u_i(\sigma_{-i}; [a_i]) \leq w_i$ holds with equality if $\sigma_i(a_i) > 0$; i.e. it can be interpreted as a complementary slackness condition as in the Lagrangian method.

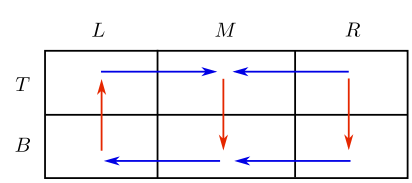
1.2 Example

Consider the following game.

Player 1\Player 2	L (q)	M $(1-q-r)$	R (r)
T (p)	$7, 2$	$2, 7$	$3, 6$
B $(1-p)$	$2, 7$	$7, 2$	$4, 5$

It is helpful to draw a figure below which shows the direction that each player prefers given other player's play. For example, player 1 prefers to play T given player 2 plays L.

Bottom-right blue arrow is wrong way around...



Let k_i be the number of pure strategies for player i, the number of possible support is given by $2^{k_i} - 1$. Therefore, for player 1, there are 3 possible supports ($\{T\}$, $\{B\}$ and $\{T, B\}$), and for player 2, there are 7 possible supports ($\{L\}$, $\{M\}$, $\{R\}$, $\{L, M\}$, $\{L, R\}$, $\{M, R\}$, $\{L, M, R\}$). So, there are $3 \times 7 = 21$ possible combinations of support for equilibria. But, since none of the pure actions are strongly dominant, we know that each player will be randomising over at least two supports. Then, the possible combinations are:

$$\begin{aligned}
&\{T, B\}, \{L, M, R\}, \\
&\{T, B\}, \{M, R\}, \\
&\{T, B\}, \{L, M\}, \\
&\{T, B\}, \{L, R\}.
\end{aligned}$$

Guess $\{T, B\}, \{L, M, R\}$.

$$\begin{aligned}
w_1 &= 7q + 2(1-q-r) + 3r = 2q + 7(1-q-r) + 4r, \\
w_2 &= 2p + 7(1-p) = 7p + 2(1-p) = 6p + 5(1-p).
\end{aligned}$$

Observe that

$$\begin{aligned}
2p + 7(1-p) = 7p + 2(1-p) &\Leftrightarrow p = \frac{1}{2} \\
&\Rightarrow w_2 = 4.5.
\end{aligned}$$

but this contradicts

$$6p + 5(1-p)\,|_{p=\frac{1}{2}} = 5.5 \neq 4.5.$$

Hence, there is no p such that player 2 will be willing to randomise across all three of her strategies. In general, it is rare for the number of pure strategies in the support to not be equal between the two players. Here, player 2 would be willing to randomise across all three of her strategies if the payoff from (B, R) was 3 instead of 5.

Guess $\{T, B\}, \{M, R\}$, which implies $q = 0$. Then,

$$\begin{aligned}
w_1 &= 2(1-r) + 3r = 7(1-r) + 4r \\
w_2 &= 7p + 2(1-p) = 6p + 5(1-p).
\end{aligned}$$

This implies that $r = 5/4 > 1$, which violates (1.1). Thus, this cannot be an equilibrium. This is because, for player 1, strategy T is strongly dominated by strategy B in the reduced game. This is clear from the fact that, for either players, the direction of the arrows are the same between the two strategies in the support: for player 1, the arrows both point down, and for player 2, the arrows both point to the left.

Guess $\{T, B\}, \{L, M\}$, which implies $r = 0$. Then,

$$\begin{aligned}
w_1 &= 7q + 2(1-q) = 2q + 7(1-q), \\
w_2 &= 2p + 7(1-p) = 7p + 2(1-p),
\end{aligned}$$

which implies

$$p = q = \frac{1}{2}.$$

So (1.1) is not violated. We now move on to step 3. Notice that

$$u_2(\sigma_1; [R]) = 6p + 5(1-p) = 5.5 > 4.5 = w_1.$$

That is, if player 2 is willing to randomise over $\{L, M\}$, she would prefer to play R (no need to check for player 1 since he has only two strategies). Hence, (1.3) is violated. So this cannot be an equilibrium.

Finally, guess $\{T, B\}, \{L, R\}$, which implies $1 - q - r = 0 \Leftrightarrow r = 1 - q$. Then,

$$w_1 = 7q + 3(1-q) = 2q + 4(1-q),$$
$$w_2 = 2p + 7(1-p) = 6p + 5(1-p),$$

which implies

$$q = \frac{1}{6}, \quad w_1 = \frac{11}{3},$$
$$p = \frac{1}{3}, \quad w_2 = \frac{16}{3}.$$

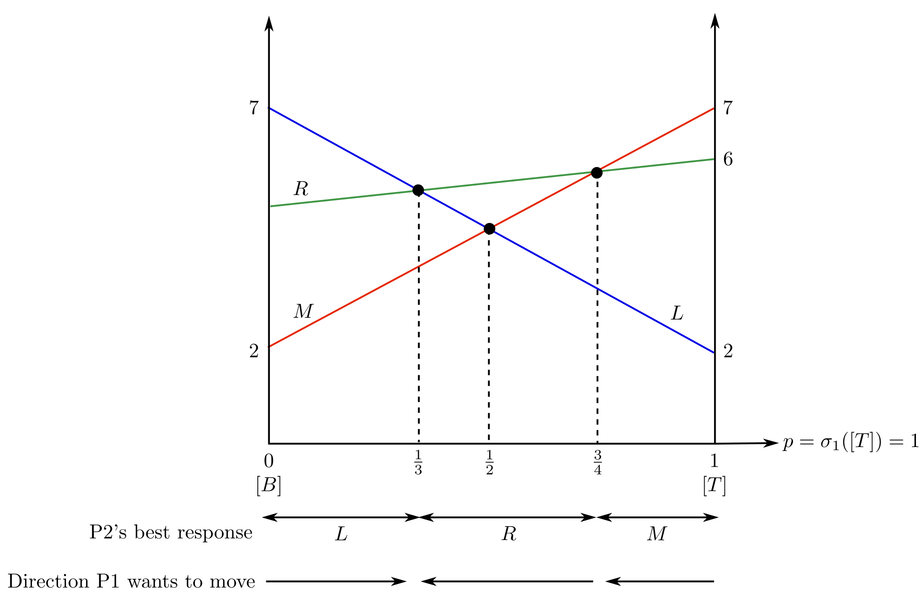
So (1.1) is not violated. We now move on to step 3. Notice that

$$u_2(\sigma_1; [M]) = 7p + 2(1-p) = \frac{11}{3} < \frac{16}{3} = w_2.$$

Hence, (1.3) is not violated, and we have found an equilibrium:

$$\left(\frac{1}{3}[T] + \frac{2}{3}[B], \frac{1}{6}[L] + \frac{5}{6}[R]\right).$$

Figure below plots the expected payoffs for player 2 from his pure strategies as a function of $p = \sigma_1(T)$. Since expected utilities are linear, the expected payoffs as a function of p from each of player 2's strategies are linear.



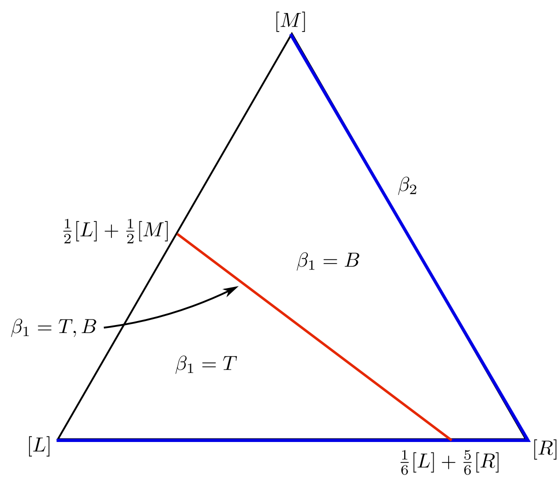
The player 2's best response is given by the upper envelope of the three payoffs:

$$\beta_2(\sigma_1) = \begin{cases} \{L\} & \text{if } p \in [0, 1/3] \\ \{R\} & \text{if } p \in [1/3, 3/4] \\ \{M\} & \text{if } p \in [3/4, 1] \end{cases}.$$

Since player 1 prefers: (i) T if player 2 plays L; (ii) B if player 2 plays M; (iii) B if player 2 plays R (recall the table with arrows), we can see that randomised equilibria will be at $p = 1/3$ in which player 2 randomises over L and R.

Remark 1.2. When we are checking for possible support for player 2, the only possibility are where the lines cross

Now, to see how player 1 should randomise, consider the following simplex.



First, observe that player 2 would not randomise between L and M—the expected payoffs from these two strategies are equal (to 3.5) only if $p = 1/2$, in which case expected payoff from R is strictly higher (5.5). Hence, player 2's best response in the simplex is the line that connects $[M]$ to $[R]$ to $[L]$. Note that, for example, if player 2 is indifferent between playing strategies $[M]$ and $[R]$, she is indifferent between any point on the line segment between $[M]$ and $[R]$.

To represent player 1's best response, notice that player 1 is indifferent between the playing T and B if player 2 plays L and M with equal probabilities, and he prefers to play B if player 2 plays L with greater probability. Similarly, player 1 is indifferent between his strategies if player 2 plays L and R with probabilities 1/6 and 5/6 (an easy way to see is to observe that player 1: (i) gains one util from playing B (over playing T) if player 2 plays R; (ii) player 1 gains 5 from playing T (over playing B) if player 2 plays L. Thus, for player 1 to be indifferent, the ratio of probabilities between L and R must be $1 : 5$, which implies probabilities 1/6 and 5/6 for L and R respectively). [Remember this trick!]

Notice that we need not consider the case when player 2 randomises between M and R since in both cases, player 1 wishes to play B (look at the direction of the arrows in the table above). Thus, above the red line drawn in the figure, player 1 prefers to play B and below, he prefers to play T.

Finally, observe that for both players to randomise in equilibrium, we must be on the edge of the simplex on the best response for player 2, and the indifference line for player 1; i.e. the point given by $(1/6)[L] + (5/6)[R]$.