Adjustment Costs and Q-Theory
Lecture Note 9

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- This note considers the investment decision of a firm subject to adjustment costs in the stock of capital.
- In particular, we study the dynamics, stability, long run behavior and speed of convergence of the dynamical system derived from the optimal policies under different assumptions about the adjustment costs function and of the production function.
- The models used are all deterministic.
- This notes are based on Blanchard and Fischer's, Lectures on Macroeconomics, Chapter 2.4.
- The notes are developed as a simple set of exercises.

Consider the following planning problem: maximize the discounted (at rate ρ) value of output net of investment expenditures:

$$V(k_0) = \max_{i(t),t\geq 0} \int_0^\infty e^{-\rho t} \left[f(k(t)) - i(t) \right] dt$$

subject to

$$\begin{aligned} \dot{k}(t) &= i(t) - \delta k(t), \\ 0 &\leq i(t) \leq f(k(t)) \end{aligned}$$

for all $t \geq 0$, and k_0 given.

The second inequality restricts investment to be non-negative and not to exceed output. In this sense there are adjustment cost of investing, i.e. investment can not be too small or too large.

Assume that f is strictly increasing and strictly concave in k, and that it satisfies Inada conditions.

Exercise. Consider the Hamiltonian

$$H(k, i, q) = f(k) - i + q[i - \delta k]$$

Maximize it taking into account the restriction on the control i that $0 \leq i \leq f(k)$. You will have to consider 3 cases when you differentiate H with respect to i, depending on whether i is in a corner or not.

Exercise. Draw the phase diagram for the following system in the (q, k) space. Let investment i be given by

$$\begin{aligned} i &= 0 \text{ if } q < 1 \\ i &= \delta k \text{ if } q = 1 \\ i &= f(k) \text{ if } q > 1 \end{aligned}$$

Draw the phase diagram for:

$$\begin{aligned} \dot{q} &= q(\rho + \delta) - f'(k) \\ \dot{k} &= i - \delta k \end{aligned}$$

Exercise. Consider a policy of the form: $i = u(k)$

$$\begin{aligned} u(k) &= 0 \text{ if } k > k^* \\ u(k) &= \delta k \text{ if } k = k^* \\ u(k) &= f(k) \text{ if } k < k^* \end{aligned}$$

where

$$f'(k^*) = \rho + \delta$$

Show that trajectories generated by this system satisfy

$$\begin{aligned} H_i(k,i,q) &= 0 \text{ if } i > 0 \\ H_i(k,i,q) &\leq 0 \text{ if } i = 0 \\ H_i(k,i,q) &> 0 \text{ if } i = f(k) \end{aligned}$$

Exercise. Using the previous exercises argue that the optimal policy is of the form

$$\begin{aligned} i(t) &= 0,\ q(t) < 1,\ \text{if } k(t) > k^* \\ i(t) &= \delta k,\ q(t) = 1 \text{ if } k(t) = k^* \\ i(t) &= f(k(t)) \text{ and } q(t) > 1,\ \text{if } k(t) < k^* \end{aligned}$$

and the system for (q,k) can be solved recursively by solving

$$\dot{k} = u(k) - \delta k$$

with initial condition $k(0)$ and then letting

$$q(t) = \int_t^\infty e^{-(\rho+\delta)(s-t)} f'(k(s))\, ds$$

Smooth adjustment cost

Consider the problem of maximizing the present value of output net of investment cost:

$$V(k_0) = \max_{i(t),\, t \geq 0} \int_0^\infty e^{-\rho t} \left[f(k(t)) - i(t) \left[1 + \phi\left(\frac{i(t)}{k(t)} \right) \right] \right] dt$$

subject to

$$\dot{k}(t) = i(t) - \delta k(t)$$

for $t \geq 0$ with $k(0) = k_0$ given. There are no other restrictions on i.

The quantity $\phi(i/k)$ represent the adjustment cost per unit of investment of investing at rate i/k, above and beyond the purchase of the investment expenditures i, which cost i.

Assumptions

We assume that

$$\phi\left(i/k\right) \geq 0, \;\; \phi\left(\delta\right) = \phi'\left(\delta\right)\delta = 0, \;\; \phi''\left(i/k\right) \geq 0$$

We also assume that the second derivative of $i/k\left[1+\phi\left(i/k\right)\right]$ is positive, i.e.

$$2\phi'\left(i/k\right) + \frac{i}{k}\phi''\left(i/k\right) > 0$$

This condition guarantees that the adjustment function is convex.

Exercise. Set up the Hamiltonian letting q be the multiplier of the $\dot{k}$ equation. Obtain all the relevant first order conditions.

Exercise. Let $x = i/k$. Show that $H_i = 0$ implies that

$$q = 1 + \phi(x) + x\phi'(x) \equiv \theta(x),$$

which is equivalent to

$$x = \gamma(q)$$

with

$$\gamma'(q) = \frac{1}{2\phi'(\gamma(q)) + \gamma(q)\phi''(\gamma(q))} > 0 .$$

Exercise. Show that the system of differential equations for k and q can be written as:

$$\begin{aligned} \dot{q} &= q(\rho+\delta) - [\gamma(q)]^2 \phi'(\gamma(q)) - f'(k) \\ \dot{k} &= k\ [\gamma(q) - \delta] \end{aligned}$$

Exercise. Use $H_i = 0$ into $\dot{q} = q\rho - H_k$ to obtain the following differential equation

$$\dot{q}(t) = q(t)(\rho + \delta - x(t)) - [f'(k(t)) - x(t)(1 + \phi'(x(t)))] .$$

What is the interpretation of the term $f'(k(t)) - x(t)(1 + \phi'(x(t)))$? Show that this differential equation has a solution given by

$$q(t) = \int_t^\infty e^{-\int_t^r (\rho+\delta-x(s))ds} [f'(k(r)) - x(r)(1 + \phi(x(r)))]\, dr$$

To verify this differentiate this expression for $q(t)$ with respect to time. Give an economic interpretation for this expression for $q(t)$. In your interpretation recall that $x(t) - \delta$ is the proportional growth rate of $k(t)$.

Exercise. Show that the values of (q, k) such that $\dot{q} = 0$, i.e.

$$q(\rho + \delta) - [\gamma(q)]^2 \phi'(\gamma(q)) = f'(k)$$

defines a function η such that $\dot{q} = 0$ if and only if

$$q = \eta(k).$$

We now consider, until further notice, the case where f is strictly concave and satisfy Inada conditions.

Exercise. Show that η' is given by

$$\frac{dq}{dk} = \eta'(k) = \frac{f''(k)}{\rho + \delta - \gamma(\eta(k))}$$

[Hint. Totally differentiate the expression in the previous exercise and use that $\gamma' = 1/\theta'$.]

Exercise. Use $\dot{k} = 0$ to show that $\gamma(q^*) = \delta$ gives $q^* = 1$ and $i^* = \delta k^*$. Show that the only steady state of the (q, k) dynamical system is given by

$$\begin{aligned} f'(k^*) &= \rho + \delta, \\ q^* &= 1 \\ i^* &= \delta k^* \end{aligned}$$

Exercise. Evaluate $\eta'(k^*)$, the slope of the $\dot{q} = 0$ locus, at steady state to show that

$$\frac{dq}{dk} = \eta'(k^*) = \frac{f''(k^*)}{\rho}$$

Exercise. Use the previous exercises to draw a phase diagram for the differential equations for q, k with k in the horizontal axes. Make sure to label the steady state values, the $\dot{k} = 0$ and $\dot{q} = 0$ locus, the saddle-path, the direction of the flows in all the relevant quadrants defined by the $\dot{k} = 0$ and $\dot{q} = 0$ locus, as well as the direction of typical trajectories that do not converge to the steady state.

Exercise. Linearize the dynamical system around $(q,k)=(1,k^*)$. Use that $\gamma(1)=\delta,\ \gamma'(1)=1/\theta'(\delta)$, and $f'(k^*)=\delta+\rho$ to obtain

$$\begin{aligned}\dot{q} &= \rho\,[q-1]-f''(k^*)\,[k-k^*] \\ \dot{k} &= \frac{k^*}{2\phi'(\delta)+\delta\phi''(\delta)}\,[q-1]\end{aligned}$$

Show that the negative eigenvalue of this system is given by the solution to

$$(\lambda-\rho)\,\lambda-(-f''(k^*))\,\frac{k^*}{2\phi'(\delta)+\delta\phi''(\delta)}=0$$

or

$$\begin{aligned}\lambda &= \left[\rho-\left(\rho^2-\frac{4k^*f''(k^*)}{2\phi'(\delta)+\delta\phi''(\delta)}\right)^{1/2}\right]/2 \\ &= \left[\rho-\left(\rho^2+4\frac{(\rho+\delta)\,\varepsilon}{2\phi'(\delta)+\delta\phi''(\delta)}\right)^{1/2}\right]/2\end{aligned}$$

where

$$\varepsilon=-\frac{k^*f''(k^*)}{f'(k^*)}>0.$$

Exercise (cont).

How does the speed of convergence of capital depend on the curvature of the production function measured by ε? That is, if the production function is less concave is the speed of convergence higher or smaller? How does the speed of convergence depend on the second derivative of $[1 + \phi(x)]\, x$ evaluated at the steady state $x = \delta$, i.e. $2\phi'(\delta) + \delta\phi''(\delta)$?

Exercise. Slope of the saddle path. We are looking for an expression for $q = s(k)$ giving q as a function of k in the saddle path. In particular we want $s' = ds(k^*)/dk$. To obtain s' notice that

$$s' = \frac{dq}{dk} = \frac{dq/dt}{dk/dt} \equiv \frac{\dot{q}}{\dot{k}}$$

We do have explicit expressions for $\dot{q}$ and $\dot{k}$, but, of course, at $k = k^*$, both $\dot{q} = \dot{k} = 0$, so [as explained in the class notes, and done in other problem sets] we must use L'Hopital rule to evaluate this ratio at k^*:

$$s' = \frac{d\dot{q}/dk}{d\dot{k}/dk} = \frac{(d\dot{q}/dq)\, s' + d\dot{q}/dk}{\left(d\dot{k}/dq\right) s' + \left(d\dot{k}/dk\right)}$$

Since $d\dot{k}/dk = 0,\ s'$ solves a quadratic expression:

$$(s')^2 \left(d\dot{k}/dq\right) = (d\dot{q}/dq)\, s' + d\dot{q}/dk$$

or

$$(s')^2 \frac{k^*}{2\phi'(\delta) + \delta\phi''(\delta)} = \rho s' - f''$$

where $ds(k^*)/dk$ is the negative root of this quadratic equation. How does s' changes if $-f''$ increases? What is the implication for the speed to which capital adjust to steady state if $-f''$ is higher? What is the economic intuition for this result? What is the limit of s' as f'' goes to 0, what is the economic intuition for this result? How does s' changes if $2\phi'(\delta) + \delta\phi''(\delta)$ increases, what is the economic intuition for this result? What is the limit of s' as $2\phi'(\delta) + \delta\phi''(\delta) \to \infty$, what is the economic intuition for this result (relate to phase diagram)?

Exercise. Let $q(k)$ be the value of the costate q for the given state and $V(k)$ the value function. Is V increasing? Is V concave? What is the economic interpretation of $V(k)/k$. What is the relationship between $V(k)/k$ and $q(k)$?

Consider now the case where

$$f(k) = \pi k$$

for some $\rho < \pi < \rho + \delta$.

Exercise. Assume that $F(k, l)$ is a neoclassical production function, strictly quasi-concave satisfying Inada conditions. Consider the indirect profit function, i.e. the maximum value of sales net of wage payments. Show that

$$\pi k = \max_{l} F(k, l) - wl$$

for a positive constant π satisfying

$$\begin{aligned} \pi &= F(1, \lambda(w)) - \lambda(w) \\ w &= F_l(1, \lambda(w)) \end{aligned}$$

Exercise. Show that

$$\begin{aligned}
\dot{q}(t) &= q(t)(\rho+\delta) - \pi - [\gamma(q(t))]^2 \phi'(\gamma(q(t))) \\
q(t) &= \theta(x(t)) \\
\dot{k}(t)/k(t) &= x(t) - \delta
\end{aligned}$$

are all the relevant first order conditions. Argue that they can be solved recursively. That is, the first equation is time homogenous and does not involve x or k.

Consider now a balanced growth path, i.e. a path where

$$\begin{aligned}
q(t) &= q^* \\
x(t) &= x^* \\
\dot{k}/k &= x^* - \delta
\end{aligned}$$

Exercise. Show that in a balanced growth path the solution is

$$\begin{aligned}\theta(x^*)(\rho+\delta)-[x^*]^2\phi'(x^*) &= \pi \\ q^* &= \theta(x^*) \\ \dot{k}/k &= x^*-\delta\end{aligned}$$

i.e. show that the following path satisfies the optimality conditions $H_i = 0$ and $\dot{q} = q\rho - H_k$

$$\begin{aligned}i(t) &= x^* k(t) \\ k(t) &= k_0 e^{(x^*-\delta)t} \\ q(t) &= q^*\end{aligned}$$

for $t \geq 0$. Moreover, if $x^* < \rho + \delta$ it also satisfies the transversality conditions. How does x^* depend on π?

Exercise. Use the results of the previous exercise to show that

$$V(k_0) = q^* k_0$$

Hint: recall that

$$\begin{aligned}
V(k_0) &= \max_{i(t),\, t \geq 0} \int_0^\infty e^{-\rho t} \left[f(k(t)) - i(t) \left[1 + \phi \left(\frac{i(t)}{k(t)} \right) \right] \right] dt \\
&= \int_0^\infty e^{-\rho t} \left(\pi - x^* \left[1 + \phi(x^*) \right] \right) k_0 e^{(x^* - \delta) t} dt \\
&= k_0 \frac{\left(\pi - x^* \left[1 + \phi(x^*) \right] \right)}{(\rho + \delta) - x^*}
\end{aligned}$$

and that

$$\begin{aligned}
q^* (\rho + \delta) - [x^*]^2 \phi'(x^*) &= \pi, \\
1 + \phi(x^*) + \phi'(x^*) x^* &= q^*.
\end{aligned}$$

Exercise. What is the market interpretation of $V(k_0)$? What is the market interpretation of $V(k_0)/k_0$? What is the interpretation of q^*?

Consider the case where $w(t)$ changes through time so that the profits depend on time. In this case we write $\pi(t)$. We will assume that $0 < \pi(t) < \rho + \delta$ for all t.

Exercise. Use the equation obtained previously for $q(t)$

$$q(t) = \int_t^\infty e^{-\int_t^r (\rho+\delta - x(s))ds} [\pi(t) - x(r)(1 + \phi(x(r)))] \, dr$$

to argue that $q(t)$ and $x(t)$ does not depend on k_0. Show that

$$V(k(t), t) = q(t) k(t)$$

Is q necessarily constant?