1 PS6 Q2

Suppose that an airline it selling tickets to business customers (b) and tourists (t). Each category of customer draws its value from a different distribution: $F_b(\theta)$ and $F_t(\theta)$. For now, we don't make any assumptions about how these relate. At the time of purchase, the customer knows whether they are a business or tourist flyer, but the airline does not; the customer does not know his ex post type, θ. Business customers represent the proportion $\phi \in (0, 1)$ of potential consumers.
The airline has decided to restrict attention to a simple pricing scheme where. A ticket costs p at the time of purchase, but also has a refund provision that allows the consumer to return the ticket for r after the consumer learns θ. Hence, a ticket is defined by its price and the amount that is refundable, (p, r). The airline wants to design a menu of tickets, $\{(p_b, r_b), (p_t, r_t)\}$, to maximize its profits. It has constant unit cost of serving either customer of c.

Problem 1.1. Write down the airline's program, including the IC and IR constraints for both types.

Solution. A traveler can either purchase the business plan, (p_b, r_b), or the tourist plan, (p_t, r_t). Consider a traveler buying (p_i, r_i) where $i \in \{b, t\}$. Before realizing his value, the traveler pays p_i to purchase the flight. Ex post, the traveler realizes her value θ randomly drawn from the distribution of her own type $j \in \{b, t\}$, i.e. $\theta \sim F_j(\theta)$. She would get a refund, r_i if her value is less than the refundable amount, i.e. $\theta < r_i$, but if her value is higher, then she would not get a refund and get on the flight. Hence, ex-post payoff would be given as:

$$u(\theta, p_i, r_i) = \max\{\theta, r_i\} - p_i.$$

However, note that she decides on her purchase before realizing her value. Hence, the ex-ante expected payoff of a traveler of type j purchasing the plan (p_i, r_i) is given as:

$$\begin{aligned} E_j[u(\theta, p_i, r_i)] &= E_j[\max\{\theta, r_i\}] - p_i \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \max\{\theta, r_i\} f_j(\theta) d\theta - p_i. \end{aligned}$$

Note that the airline company gets p_i from traveler type i and incurs cost c if the traveler decides to travel and pays out r_i if the traveler decides to get a refund upon realizing her value θ. Hence, the airline's company's ex-post payoff from plan i would be given as:

$$\pi_i = \begin{cases} p_i - c & \text{if } \theta \geq r_i \\ p_i - r_i & \text{if } \theta < r_i. \end{cases}$$

Note that $Pr(\theta < r_i) = F_i(r_i)$ and $Pr(\theta \geq r_i) = 1 - F_i(r_i)$, and let π_i be the profit of the airline company from type i travelers. Assuming that both types of travelers truthfully buy their respective plan, the expected profit of the airline company from the business travelers would be given as:

$$\begin{aligned} E[\pi_b] &= (p_b - r_b)Pr(\theta < r_b) + (p_b - c)Pr(\theta \geq r_b) \\ &= p_b - F_b(r_b)r_b - (1 - F_b(r_b))c, \end{aligned}$$

and that from the tourists would be given as:

$$\begin{aligned} E[\pi_t] &= (p_t - r_t)Pr(\theta < r_t) + (p_t - c)Pr(\theta \geq r_t) \\ &= p_t - F_b(r_t)r_t - (1 - F_t(r_t))c. \end{aligned}$$

Since the fraction of business travelers among potential travelers is $\phi \in (0,1)$, the airline's program with IC and IR constraints for both types of travelers is as follows:

$$\begin{aligned} &\max_{p_b, p_t, r_b, r_t} && \phi[p_b - F_b(r_b)r_b - (1 - F_b(r_b))c] + (1-\phi)[p_t - F_t(r_t)r_t - (1 - F_t(r_t))c] \\ s.t. \quad &(IR_b) && E_b[u(\theta, p_b, r_b)] \geq 0 \\ &(IR_t) && E_t[u(\theta, p_t, r_t)] \geq 0 \\ &(IC_b) && E_b[u(\theta, p_b, r_b)] \geq E_b[u(\theta, p_t, r_t)] \\ &(IC_t) && E_t[u(\theta, p_t, r_t)] \geq E_t[u(\theta, p_b, r_b)]. \end{aligned}$$

Problem 1.2. Argue that the IR constraint of the business customer can be ignored if either F_b first-order stochastically dominates F_t, or F_b is a mean-preserving spread of F_t.

Solution. Note that either of the following

1. $F_b \geq_{\text{FOSD}} F_t$
2. F_b is a mean-preserving spread of F_t

implies $E_b[u(\cdot)] > E_t[u(\cdot)]$. The first sufficiency condition trivially follows from the properties of FOSD with increasing utility ($u(\cdot)$ is weakly increasing in θ). The second sufficiency condition is also true because $u(\cdot)$ is a nondecreasing convex function of θ; if F_b is a mean-preserving spread of F_t, then F_b puts more weight on the right tail of the distribution where $u(\cdot)$ is weakly increasing in θ while $u(\cdot)$ are equal on the left tail of the distributions. Hence, we have $E_b[u(\cdot)] \geq E_t[u(\cdot)]$.

Then, we have:

$$\begin{aligned} E_b[u(\theta, p_b, r_b)] &\geq E_b[u(\theta, p_t, r_t)] && \because (IC_b) \\ &> E_t[u(\theta, p_t, r_t)] && \because E_b[u(\cdot)] \geq E_t[u(\cdot)] \text{ from above} \\ &\geq 0 && \because (IR_t) \\ \Rightarrow E_b[u(\theta, p_b, r_b)] &> 0, \end{aligned}$$

which implies IR_b is slack. Hence, with F_d dominating F_t in FOSD sense or MPS sense, IR_b is slack and can be ignored.

Problem 1.3. Consider the relaxed program in which the IC constraint for the t consumer is ignored. Show in the relaxed program that IR for t must bind and IC for b must bind.

Solution. Recall that for $i, j \in \{b, t\}$:

$$E_j[u(\theta, p_i, r_i)] = \int_{\underline{\theta}}^{\bar{\theta}} \max\{\theta, r_i\} f_j(\theta) d\theta - p_i \quad \forall j \in \{b, t\},$$

and

$$E[\pi_i] = p_i - F_b(r_b)r_i - (1 - F_b(r_b))c.$$

We see that $E_j[u(\theta, p_i, r_i)]$ is weakly decreasing in p_i and weakly increasing in r_i while $E[\pi_i]$ is weakly increasing in p_i and weakly decreasing in r_i.

Note that in this problem we ignore IC_t. If IR_t is slack, i.e. $E_t[u(\theta, p_t, r_t)] > 0$, then the airline company can lower r_t by a small amount $\epsilon > 0$, which would increase $E[\pi_t]$ while further relaxing IC_b : $E_b[u(\theta, p_b, r_b)] \geq E_b[u(\theta, p_t, r_t)]$ since $E_b[u(\theta, p_t, r_t)]$ is increasing in r_i. Hence, this violates optimality of (p_t, r_t), so IR_t must bind: $\mathrm{E}_t[u(\theta, p_t, r_t)] = 0$.

Now, suppose IC_b is slack, i.e. $E_b[u(\theta, p_b, r_b)] > E_b[u(\theta, p_t, r_t)]$. Then, the airline company can raise p_b by a small amount $\epsilon > 0$ to raise $E[\pi_b]$. Note that this is feasible because we've seen in the previous part that IR_b is always slack: $E_b[u(\theta, p_b, r_b)] > 0$. Hence, p_b can be raised until IC_b binds: $E_b[u(\theta, p_b, r_b)] = E_b[u(\theta, p_t, r_t)]$, leaving the firm with higher profit while the business traveller unaffected.

Therefore, if IC_t is ignored, then both IR_t and IC_b bind.

Problem 1.4. State and solve the relaxed program in which IR_t and IC_b bind (and IC_t is ignored). Describe the optimal refund policy for the two classes of tickets for the case when $F_b \geq_{\text{FOSD}} F_t$. [You may assume that IC_t is slack, though this can be proved.] Given your solution, why this may not be such a great model of the airline pricing we observe in the real world.

Solution. The relaxed program with IR_t and IC_b binding and IC_t ignored is as follows:

$$\begin{aligned}
&\max_{p_b, p_t, r_b, r_t} && \phi[p_b - F_b(r_b)r_b - (1 - F_b(r_b))c] + (1 - \phi)[p_t - F_t(r_t)r_t - (1 - F_t(r_t))c] \\
s.t. \quad &(IR_t) && E_t[u(\theta, p_t, r_t)] = 0 \\
&(IC_b) && E_b[u(\theta, p_b, r_b)] = E_b[u(\theta, p_t, r_t)]
\end{aligned}$$

Because IR_t binds, we have:

$$\begin{aligned} 0 &= E_t[u(\theta, p_t, r_t)] \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \max\{\theta, r_t\} f_t(\theta) d\theta - p_t \\ &= \int_{\underline{\theta}}^{r_t} r_t f(\theta) d\theta + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta) d\theta - p_t \\ &= F_t(r_t) r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta) d\theta - p_t \\ \Rightarrow p_t &= F_t(r_t) r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta) d\theta \end{aligned}$$

Because IC_b binds, we have:

$$\begin{aligned} E_b[u(\theta, p_b, r_b)] &= E_b[u(\theta, p_t, r_t)] \\ \Rightarrow F_b(r_b) r_b + \int_{r_b}^{\bar{\theta}} \theta f_b(\theta) d\theta - p_b &= F_b(r_t) r_t - \int_{r_t}^{\bar{\theta}} \theta f_b(\theta) d\theta - p_t \\ \Rightarrow p_b - p_t &= F_b(r_b) r_b + \int_{r_b}^{\bar{\theta}} \theta f_b(\theta) d\theta - F_b(r_t) r_t - \int_{r_t}^{\bar{\theta}} \theta f_b(\theta) d\theta \\ &= F_b(r_b) r_b - F_b(r_t) r_t + \int_{r_b}^{r_t} \theta f_b(\theta) d\theta \\ \text{[integration by parts]} \Rightarrow &= F_b(r_b) r_b - F_b(r_t) r_t + \theta F_b(\theta) \Big|_{r_b}^{r_t} - \int_{r_b}^{r_t} F_b(\theta) d\theta \\ &= - \int_{r_b}^{r_t} F_b(\theta) d\theta \\ \Rightarrow p_b &= p_t - \int_{r_b}^{r_t} F_b(\theta) d\theta \\ &= F_t(r_t) r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta) d\theta - \int_{r_b}^{r_t} F_b(\theta) d\theta \end{aligned}$$

Plugging the above two expressions for p_t and p_b into the objective function:

$$
\begin{aligned}
&\max_{r_b, r_t} \quad \phi[p_b - F_b(r_b)r_b - (1 - F_b(r_b))c] + (1 - \phi)[p_t - F_t(r_t)r_t - (1 - F_t(r_t))c] \\
\Rightarrow &\max_{r_b, r_t} \quad \phi[F_t(r_t)r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta)d\theta - \int_{r_b}^{r_t} F_b(\theta)d\theta - F_b(r_b)r_b - (1 - F_b(r_b))c] \\
&\qquad + (1 - \phi)[F_t(r_t)r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta)d\theta - F_t(r_t)r_t - (1 - F_t(r_t))c] \\
\Rightarrow &\max_{r_b, r_t} \quad \phi[F_t(r_t)r_t + \int_{r_t}^{\bar{\theta}} \theta f_t(\theta)d\theta - \int_{r_b}^{r_t} F_b(\theta)d\theta - F_b(r_b)r_b - (1 - F_b(r_b))c] \\
&\qquad + (1 - \phi)[\int_{r_t}^{\bar{\theta}} \theta f_t(\theta)d\theta - (1 - F_t(r_t))c].
\end{aligned}
$$

Note that Leibniz's rule states that for any functions $a(x)$ and $b(x)$ and $g(x, \theta)$:

$$
\frac{d}{dx} \int_{a(x)}^{b(x)} g(x, \theta)d\theta = g(x, b(x))b'(x) - g(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} g(x, \theta)d\theta.
$$

Hence, using Leibniz's rule and product rule for differentiation, the FOCs of the optimization program yield:

$$
\begin{aligned}
[r_b]: \quad 0 &= \phi(F_b(r_b) - F_b(r_b) - f_b(r_b)r_b + f_b(r_b)c) \\
&= f_b(r_b)(c - r_b) \\
\Rightarrow r_b &= c, \\
[r_t]: \quad 0 &= \phi(f_t(r_t)r_t + F_t(r_t) - r_t f_t(r_t) - F_b(r_t)) + (1 - \phi)(-r_t f_t(r_t) + f_t(r_t)c) \\
&= \phi(F_t(r_t) - F_b(r_t)) + (1 - \phi)f_t(r_t)(c - r_t) \\
\Rightarrow r_t &= c - \frac{\phi(F_b(r_t) - F_t(r_t))}{(1 - \phi)f_t(r_t)} \\
\Rightarrow r_t &> c,
\end{aligned}
$$

where we establish the last inequality using the FOSD assumption, $F_b(r_t) - F_t(r_t) < 0$. Hence, we have $r_t > r_b = c$. Also, notice from before that:

$$
\begin{aligned}
p_b &= p_t - \int_{r_b}^{r_t} F_b(\theta)d\theta \\
[r_b < r_t] &\Rightarrow p_b < p_t.
\end{aligned}
$$

Hence, it follows that the tourists pay more for airfare and get more refund than do business travelers, which doesn't quite fit what we observe in the real world where business class flights are normally more expensive and subject to higher refund.

Problem 1.5. Suppose instead that F_b is a mean-preserving spread of F_t. In particular, let's suppose that $F_b(\theta) = \theta$ on $[0, 1]$, but the tourists have a triangular distribution,

$$F_t(\theta) = \begin{cases} 2\theta^2 & \text{if } \theta \leq \frac{1}{2} \\ 4\theta - 2\theta^2 - 1 & \text{if } \theta > \frac{1}{2}. \end{cases}$$

Let $c = \frac{3}{8}$ and $\phi = \frac{1}{2}$. Describe the optimal refund policy for the two classes of tickets. [Hint: Look for the solution to the FOC for r_t where $r_t \leq \frac{1}{2}$.] Describe your solution. Does this do a better job at fitting actual airline pricing?

Solution. Recall the results from the previous part:

$$\begin{aligned} r_b &= c \\ r_t &= c - \frac{\phi(F_b(r_t) - F_t(r_t))}{(1-\phi)f_t(r_t)}. \end{aligned}$$

As before, we have $r_b = c$. Using the values we've been given, and assuming $r_t \leq \frac{1}{2}$, we get:

$$\begin{aligned} r_b &= c = \frac{3}{8} \\ r_t &= c - \frac{\phi(F_b(r_t) - F_t(r_t))}{(1-\phi)f_t(r_t)} \\ &= \frac{3}{8} - \frac{\frac{1}{2}(r_t - 2r_t^2)}{\frac{1}{2}4r_t} \\ &= \frac{3}{8} - \frac{1}{4} + \frac{1}{2}r_t \\ \Rightarrow \frac{1}{2}r_t = \frac{1}{8} &\Rightarrow r_t = \frac{1}{4}. \end{aligned}$$

Notice that $r_t < r_b$, and because $p_t - p_b = \int_{r_b}^{r_t} F_b(\theta)d\theta < 0$, we have $p_t < p_b$. Here, tourists pay less for airfare and get less refund than do business travellers, which is more consistent with actual airline pricing than what we had in part d.

2 PS6 Q4

Consider an IPV auction environment with two bidders, one "strong" and one "weak". The strong bidder's type θ_s, is uniformly distributed on [2, 3] and the weak bidder's type θ_w is uniformly distributed on [0, 1].

Problem 2.1. Compute the equilibrium bidding functions in the second-price auction. Compute the expected revenue to the seller.

Solution. In fact, strong type can bid anything $b_s \in (1, \infty)$, and weak type can bid anything $b_w \in (0, 1)$, and it would be an equilibrium. The reason is that in a second-price auction, given that weak type bids $b_w \in (0, 1)$, strong type always win the good and pay the weak type's bid, so it doesn't make any difference for the strong type to bid anything $b_s \in (1, \infty)$. Now, let's consider the weak type. Given that strong type bids $b_s \in (1, \infty)$, the payoff is always zero if weak type bids within $(0, 1)$, so he won't have incentive to deviate.

Therefore, one equilibrium is that both agents bid their true valuation θ_i, $i \in \{s, w\}$. And thus both of them won't have incentive to deviate from the equilibrium bidding equation. So, in that case, the bidding function is $\bar{b}_w(\theta) = \bar{b}_s(\theta) = \bar{b}(\theta) = \theta$. The expected revenue to the seller is

$$\mathbb{E}[\theta_w] = \frac{1}{2}$$

Problem 2.2. Compute the equilibrium bidding functions in the first-price auction for the equilibrium in which the weak player bids $\bar{b}_w(\theta_w) = \theta_w$. Compute the expected revenue to the seller.

Solution. ▷ First, denote agent s's utility of bidding b_s with type θ_s by $U_s(b_s, \theta_s)$ where

$$U_s(b_s, \theta_s) = \mathbb{P}(b_s > \theta_w)(\theta_s - b_s)$$

▷ We focus on the case where agent s will only bid $b_s \in [0, 1]$ because for any bid lower than 0, his payoff is definitely 0, which is not better than bidding 0 and getting 0, and for any bid over 1, agent s is losing money by overbidding.

▷ Note that given $b_s \in [0, 1]$

$$\mathbb{P}(b_s > \theta_w) = \mathbb{P}(\theta_w \leq b_s) = b_s$$

▷ So, agent s's utility is

$$U_s(b_s, \theta_s) = b_s(\theta_s - b_s)$$

▷ The f.o.c. is

$$\theta_s - 2b_s = 0$$
$$\Rightarrow b_s = \frac{1}{2}\theta_s$$

Since $b_s \in [0,1]$ and $\theta_s \in [2,3]$, we have that $b_s = 1$ for $\forall \theta_s$

▷ So, the equilibrium bidding function is

$$\bar{b}_w(\theta_w) = \theta_w \text{ and } \bar{b}_s(\theta_s) = 1$$

▷ The expected payoff to the seller is

$$\begin{aligned} &\mathbb{E}[\theta_w \mid \theta_w > 1]\,\mathbb{P}(\theta_w > 1) + \mathbb{E}[1 \mid \theta_w < 1]\,\mathbb{P}(\theta_w < 1) \\ =&0 + 1 \\ =&1 \end{aligned}$$

Problem 2.3. Compare the expected revenues. Explain why they are the same (i.e., explain how the revenue equivalence theorem applies to this setting), or explain why they are different (i.e., why the revenue equivalence theorem does not apply to this situation).

Solution. In (a), the second-price auction generates expected revenue $\frac{1}{2}$ while in (b) the first-price auction generates 1, which are different. Revenue equivalence theorem does not apply here because the condition $U_i^{\mathrm{I}}(\underline{\theta}_i) = U_i^{\mathrm{II}}(\underline{\theta}_i)$ doesn't hold. In particular, consider the strong type:

▷ Under first price auction,

$$\begin{aligned} U_s^{\mathrm{FPA}}(\underline{\theta}_s) &= \bar{\phi}_s^{\mathrm{FPA}}(\underline{\theta}_s)\,\underline{\theta}_s - \bar{t}_s^{\mathrm{FPA}}(\underline{\theta}_s) \\ &= 1 \times 2 - 1 \\ &= 1 \end{aligned}$$

▷ Under second price auction,

$$\begin{aligned} U_s^{\mathrm{SPA}}(\underline{\theta}_s) &= \bar{\phi}_s^{\mathrm{SPA}}(\underline{\theta}_s)\,\underline{\theta}_s - \bar{t}_s^{\mathrm{SPA}}(\underline{\theta}_s) \\ &= 1 \times 2 - \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

▷ Clearly $1 \neq \frac{3}{2}$, so $U_s^{\mathrm{FPA}}(\underline{\theta}_s) \neq U_s^{\mathrm{SPA}}(\underline{\theta}_s)$, and therefore the revenue equivalence theorem fails.

3 PS6 Q6

Suppose there are just two bidders. In a second-price, all-pay auction, the two bidders simultaneously submit sealed bids. The highest bid wins the object and both bidders pay the second-highest bid.

Problem 3.1. Find the unique symmetric equilibrium bidding function. Interpret.

Solution. We then have that the utility of the bidder is given by

$$\begin{aligned}
u_i(r_i, v_i) &= F(r_i)\, v_i - \mathbb{E}\left[\min\left\{b(r_i), b(r_j)\right\}\right] \\
&= F(r_i)\, v_i - \left[(1 - F(r_i))\, b(r_i) + F(r_i)\, \mathbb{E}\left[b(r_j) \mid r_j < r_i\right]\right] \\
&= F(r_i)\, v_i - \left[(1 - F(r_i))\, b(r_i) + F(r_i) \int_0^{r_i} b(x)\, f(x) \frac{1}{F(r_i) - 0} dx\right]
\end{aligned}$$

we can then take the derivative of this expression with respect to r_i to get

$$\begin{aligned}
\frac{\partial u_i(r_i, v_i)}{\partial r_i} &= f(r_i)\, v_i - \left[\begin{array}{c} b'(r_i)(1 - F(r_i)) + b(r_i)(-f(r_i)) + f(r_i) \int_0^{r_i} b(x)\, f(x) \frac{1}{F(r_i)} dx \\ +F(r_i)\left[\int_0^{r_i} b(x)\, f(x) \left(-\frac{f(r_i)}{F(r_i)^2}\right) dx + b(r_i)\, f(r_i) \frac{1}{F(r_i)}\right] \end{array}\right] \\
&= f(r_i)\, v_i - \left[\begin{array}{c} b'(r_i)(1 - F(r_i)) + b(r_i)(-f(r_i)) + \int_0^{r_i} b(x)\, f(x) \frac{f(r_i)}{F(r_i)} dx \\ - \int_0^{r_i} b(x)\, f(x) \frac{f(r_i)}{F(r_i)} dx + b(r_i)\, f(r_i) \end{array}\right]
\end{aligned}$$

we can then plug in $r_i = v_i = v$ and set the derivative equal to zero to get that

$$\begin{aligned}
f(v)\, v &= b'(v)(1 - F(v)) - b(v)\, f(v) + b(v)\, f(v) \\
f(v)\, v &= b'(v)(1 - F(v)) \\
b'(v) &= \frac{f(v)\, v}{1 - F(v)}
\end{aligned}$$

Then we can solve for $b(v)$ by taking the integral and exploiting that the equilibrium bid of the lowest type is zero, in particular

$$b(v) = \int_0^v \frac{f(x)}{1 - F(x)} x dx$$

notice that $\frac{f(x)}{1-F(x)}$ is the survival function, so the bidding function is essentially the survival function weighted by the values of x integrated from the lower limit of the support of v to the bidder's private valuation.

Problem 3.2. Do bidders bid higher or lower than in a first-price, all-pay auction?

Solution. Instead of arguing directly, we can invoke the Revenue Equivalence Theorem to show that bidders will bid higher than in the first-price, all-pay auction. In particular, we know that from the revenue equivalence theorem that if two incentive-compatible direct selling mechanisms have the same probability assignment functions and every bidder with value zero is indifferent between the two mechanisms, then the two mechanisms generate the same expected revenue for the seller.

From the derivation of the optimal bidding function we know that both auctions are incentive-compatible direct selling mechanisms. Further, because in both cases the highest type will win the auction with certainty the two auctions have the same probability assignment functions. Finally, we can see from the optimal bidding functions for parts (5) and parts (6) the optimal bid for the lowest type will be zero, therefore a bidder with value zero will pay the same amount in both mechanisms (zero), and is indifferent between the two mechanisms.

Now, we know that the seller's expected revenue must be the same across auctions. Therefore, in expectation, the bid of the second highest bidder in the second auction must equal the highest bid in the first auction $\Rightarrow$ bidders bid higher than in the first-price, all-pay auction.

Problem 3.3. Find an expression for the seller's expected revenue.

Solution. To solve for this we can use the density of the second order statistic (specialized to the two person case). This is given by

$$\begin{aligned} g(n,\theta) &= n(n-1) f(\theta) F(\theta)^{n-2} (1-F(\theta)) \\ &= 2f(\theta) F(\theta) (1-F(\theta)) \end{aligned}$$

then we have that the seller's expected revenue is given by the following expression:

$$\begin{aligned} ER &= 2\int_{\underline{\theta}}^{\bar{\theta}} \underbrace{\int_{\underline{\theta}}^{v} \frac{f(x)}{1-F(x)} x dx}_{b(x)} \times 2f(v)(1-F(v))\, dv \\ &= 2\int_{\underline{\theta}}^{\bar{\theta}} \int_{x}^{\bar{\theta}} 2f(v)(1-F(v))\, dv \, \frac{f(x)}{1-F(x)} x dx \\ &= 2\int_{\underline{\theta}}^{\bar{\theta}} \left[-(1-F(v))^2\right]_x^{\bar{\theta}} \frac{f(x)}{1-F(x)} x dx \\ &= 2\int_{\underline{\theta}}^{\bar{\theta}} (1-F(x))^2 \, \frac{f(x)}{1-F(x)} x dx \\ &= 2\int_{\underline{\theta}}^{\bar{\theta}} x f(x)(1-F(x))\, dx \end{aligned}$$

Problem 3.4. Both with and without the revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

Solution. We've already argued (in part B) that the conditions for the revenue equivalence theorem holds and that the expected revenue must be the same as in the all-pay first-price auction. It remains to show the result directly. To show this notice the following:

$$\begin{aligned}
ER^{\text{First Price All Pay}} &= 2\int_{\underline{\theta}}^{\bar{\theta}}\left(\int_{\underline{\theta}}^{\theta} x dF\left(x\right)\right)dF\left(\theta\right) \\
&= 2\int_{\underline{\theta}}^{\bar{\theta}}\left(\int_{x}^{\bar{\theta}} dF\left(\theta\right)\right)x dF\left(x\right) \\
&= 2\int_{\underline{\theta}}^{\bar{\theta}} x f\left(x\right)\left(1-F\left(x\right)\right)dx \\
&= ER^{\text{Second Price All Pay}}
\end{aligned}$$