10.7 A Model of Search Unemployment

Consider a worker who begins each period with a current wage offer and has two alternative actions. He can work at that wage or he can search for a new wage offer. If he chooses to search, he earns nothing during the current period, and his new wage is drawn according to some fixed probability measure. He cannot divide his time within a period between searching and working. Moreover, if a worker chooses to work during the current period, then with probability $1 - \theta$ the same wage is available to him next period. But with probability θ he will lose his job at the beginning of next period and begin next period with a "wage" of zero.

The worker does not value leisure, and his preferences over random consumption sequences $\{c_t\}$ are given by

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right]$$

The worker cannot borrow or lend, so his consumption is equal to his earnings during each period.

The decision problem for this worker is defined by β, U, θ, and the probability distribution over new wage offers. Assume that $0 < \beta < 1$, and let $U: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be continuously differentiable, strictly increasing, and strictly concave, with $U(0) = 0$ and $U'(0) < \infty$. Assume that all wage offers lie in the interval $W = [0, \bar{w}]$, and let f be a density on that interval.

It is possible, but awkward, to set up this problem in sequence form. To do so we need two exogenous state variables. The first is $d \in D = \{0, 1\}$, where $d = 0$ or 1 is interpreted respectively as meaning that the worker does or does not lose his job at the beginning of the current period, given that he chose to work last period. The second is $z \in Z = [0, \bar{w}]$, where z is interpreted as the worker's current wage offer, given that he chose to search last period. In addition there is one endogenous state variable, the current wage $w \in W$. In each period, given his current wage offer w, the worker chooses an action $y \in Y = \{0, 1\}$, where $y = 0$ or 1 is interpreted respectively as meaning that the worker chooses to search or to work at his current job.

Exercise 10.7 a. What is the law of motion $\phi: W \times Y \times D \times Z \rightarrow W$ for this model? That is, describe w_{t+1} in terms of $(w_t, y_t, d_{t+1}, z_{t+1})$.

b. Formulate the worker's decision problem as a choice of functions mapping partial histories $(d^t, z^t) = [(d_1, z_1), \ldots, (d_t, z_t)]$ of exogenous shocks into actions and current wage offers that satisfy the law of motion above, given the initial state (w_0, d_0, z_0). Show that the supremum function v^* for this problem is well defined and depends only on w_0 (not on d_0 or z_0).

The recursive formulation of this problem is much simpler and more natural. For notational convenience, drop the asterisk and let v be the supremum function for the problem in (b). Suppose that a worker's current wage offer is w, that he chooses to work at this wage for one period, and that he will follow an optimal policy (if one exists) forever after. Then his expected present discounted value of utility is

$$U(w) + \beta[(1 - \theta)v(w) + \theta v(0)].$$

If he chooses to search instead, his expected utility is

$$0 + \beta \int_0^{\bar{w}} v(w')f(w')dw'$$

Combining these possibilities, we find that v must satisfy

$$(1) \qquad v(w) = \max\left\{U(w) + \beta[(1-\theta)v(w) + \theta v(0)],\ \beta \int_0^{\bar{w}} v(w')f(w')dw'\right.$$

Exercise 10.7 c. Show that there exists a unique bounded continuous function v satisfying (1) and that v is the supremum function for the problem in part (b). Show that v is weakly increasing.

The value function v can be characterized more sharply by exploiting special features of (1). First, define

$$(2) \qquad A = \beta \int_0^{\bar{w}} v(w')f(w')dw',$$

and note that $v(0) = A$.

Exercise 10.7 d. Show that there is a unique $w^* \in W$ such that

$$v(w^*) = U(w^*) + \beta[(1-\theta)v(w^*) + \theta A] = A$$

It follows from part (d) that w^* is the unique value satisfying

$$(3) \qquad U(w^*) = (1-\beta)A$$

Exercise 10.7 e. Show that v has the form

$$(4) \qquad v(w) = \begin{cases} A & \text{if } w < w^* \\ \dfrac{U(w) + \beta\theta A}{1 - \beta(1-\theta)} & \text{if } w \geq w^* \end{cases}$$

as shown in Figure 10.4.

Equation (4) gives the solution v to (1) in terms of A and w^*. The value of A is in turn given in terms of v by (2), and the value of w^* by (3).

Exercise 10.7 f. Use (2) and (4) to show that

$$(5) \qquad (1-\beta)A = \frac{\beta}{1 + \beta\theta - \beta F(w^*)} \int_{w^*}^{\bar{w}} U(w)f(w)dw;$$

where $F(w) = \int_0^w f(w')dw'$ is the cumulative distribution function corresponding to f.

Equations (3) and (5) are now two equations in the unknown parameters w^* and A. Combining them to eliminate A gives

$$(6) \qquad [1 + \beta\theta - \beta F(w^*)]U(w^*) = \beta \int_{w^*}^{\bar{w}} U(w)f(w)\,dw:$$

Exercise 10.7 g. Show that there is a unique value w^* satisfying (6).

Parts (d)–(g) completely characterize the value function v: it is given by (4) with w^* as determined in part (g) and A then given by (3) or (5). From (1) we see that the optimal decision rule for the worker is simply: if the current wage is at least w^*, work; if not, search. Call w^* the *reservation wage.*

Exercise 10.7 h. How does the reservation wage depend on the parameters β and θ?

i. What can be said about the effect of changes in the variance of the wage distribution on the expected utility of the worker?

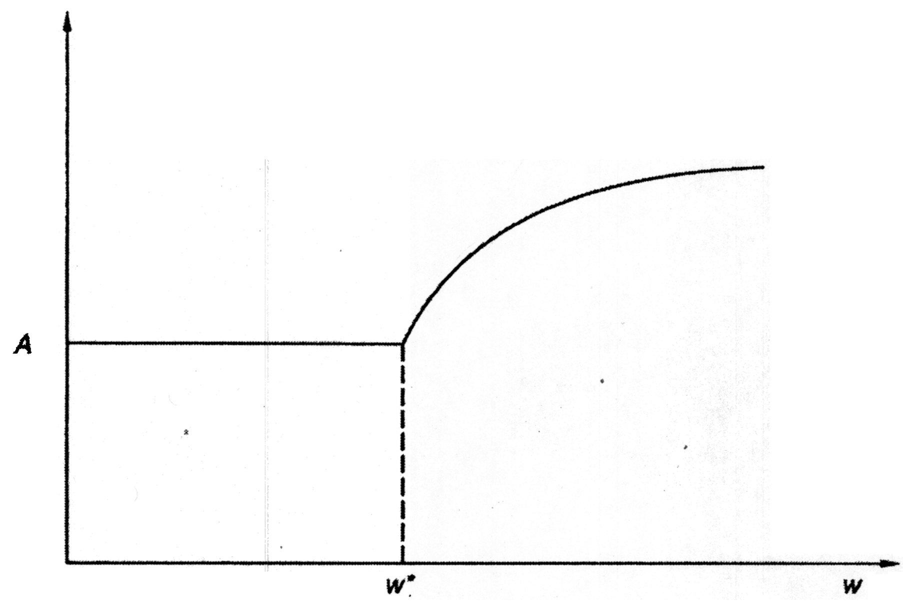

Figure 10.4

The Dynamics of the Search Model

Let $W = [0, \bar{w}]$ with its Borel subsets $\mathcal{W}$, and let μ be a probability measure on $(W, \mathcal{W})$. Let $w^* \in (0, \bar{w}]$, and let $A = [w^*, \bar{w}]$. In the search model of the last problem, we showed that if a worker follows an optimal strategy, then his wage offers $\{w_t\}$ are a Markov process on $(W, \mathcal{W})$ with transition function

$$P(w, B) = \mu(B), \quad \text{all } B \in \mathcal{W}, \text{ if } w \in A^c;$$

$$P(w, B) = \begin{cases} 0 & \text{if } 0 \notin B \text{ and } w \notin B \\ \theta & \text{if } 0 \in B \text{ and } w \notin B \\ 1 - \theta & \text{if } 0 \notin B \text{ and } w \in B \\ 1 & \text{if } 0 \in B \text{ and } w \in B, \end{cases} \quad \text{if } w \in A$$

Here $W = [0, \bar{w}]$ is the set of possible wage offers; μ is the probability measure over wage offers if the worker is searching; $A = [w^*, \bar{w}]$ is the set of acceptable wage offers; and $A^c = [0, w^*)$ is the set of unacceptable offers. (We adopt the convention that the worker accepts the wage w^*.) Here we study the long-run behavior of this Markov process. That is, we ask: Given an initial probability measure λ_0 on $(W, \mathcal{W})$, what can we say about the sequence of probability measures $\lambda_n = T^{*n}\lambda_0$, $n = 1, 2, \ldots$, where T^* is the adjoint operator associated with P (cf. Section 8.1)? Because the transition function P is so simple, it is possible to answer this question very explicitly using ad hoc arguments.

Let λ_0 be an initial probability measure on $(W, \mathcal{W})$, and define the sequence $\{\lambda_t\}$ as above. Then the probability that the worker is unemployed (searching) in any period t is $\lambda_t(A^c)$. We begin by determining the sequence $\{\lambda_t(A^c)\}_{t=0}^{\infty}$ of unemployment probabilities. To do this, we note that the probability that the worker is unemployed in period $t + 1$ is equal to the probability that he is unemployed in period t and draws an unacceptable wage, plus the probability that he is employed in period t and loses his job. Hence, as (1a) and (1b) imply,

$$\begin{aligned} \lambda_{t+1}(A^c) &= \int P(w, A^c)\lambda_t(dw) \\ &= \lambda_t(A^c)\mu(A^c) + \lambda_t(A)\theta \\ &= \lambda_t(A^c)\mu(A^c) + [1 - \lambda_t(A^c)]\theta \\ &= \theta + \lambda_t(A^c)[\mu(A^c) - \theta], \quad t = 0, 1, \end{aligned} \tag{2}$$

That is, the sequence $\{\lambda_t(A^c)\}$ is described by the first-order difference equation (2).

Exercise 10.8 a. Show that the difference equation (2) is stable and that

$$\lim_{t\to\infty} \lambda_t(A^c) = \frac{\theta}{\theta + \mu(A)} \tag{3}$$

b. Let $C \subseteq A^c$ be any measurable set of unacceptable wage offers the same reasoning as above to show that if $0 \in C$, then

$$\lambda_{t+1}(C) = \theta + \lambda_t(A^c)[\mu(C) - \theta], \quad t = 0, 1, . \qquad \text{and}$$

$$\lim_{t\to\infty} \lambda_t(C) = \frac{\theta}{\theta + \mu(A)}\mu(C) + \frac{\mu(A)}{\theta + \mu(A)}\theta. \tag{4}$$

Show that if $0 \notin C$, then

$$\lambda_{t+1}(C) = \lambda_t(A^c)\mu(C), \quad t = 0, 1, \qquad \text{and}$$

$$\lim_{t\to\infty} \lambda_t(C) = \frac{\theta}{\theta + \mu(A)}\mu(C). \tag{5}$$

For any measurable set $C \subseteq A$, the probabilities are also easily determined. The probability that in period $t + 1$ the worker has a wage in the set $C \subseteq A$ is simply the probability that he is searching in period t and draws a wage in the set C, plus the probability that he had a wage in the set C last period and retained his job. Thus, as (1a) and (1b) imply,

$$\begin{aligned}\lambda_{t+1}(C) &= \int P(w, C)\lambda_t(dw) \\ &= \lambda_t(A^c)\mu(C) + \lambda_t(C)[1 - \theta], \quad t = 0, 1,\end{aligned}$$

Exercise 10.8 c. Show that for any measurable set $C \subseteq$ acceptable wage offers,

$$\lim_{t\to\infty} \lambda_t(C) = \frac{\theta}{\theta + \mu(A)} \frac{\mu(C)}{\theta}. \tag{6}$$

d. Interpret the results in (3)–(6). What is the average wage for employed workers in this economy? What is the distribution of the length of unemployment spells?

Variations on the Search Model

Once the basic structure of the search model in Sections 10.7 and 10.8 is understood, it is easy to think of variations that capture realistic features that are abstracted from in the original version. It would be tedious to work through all such variations in detail, but it is instructive to think some of them through at least to the point of formulating the appropriate analogue to the functional equation.

Exercise 10.9 a. Suppose wage offers follow a Markov process with transition function Q on $(W, \mathcal{W})$. Assume that Q is monotone and has the Feller property. How does this change the functional equation [Equation (1) in Section 10.7] and Figure 10.4?

b. Suppose the worker is endowed with one unit of time each period, which he divides between l_t units of leisure and $1 - l_t$ units of work or search. His utility function is $U(c_t, l_t)$. He can choose his hours, if he works at all. Thus, if his current wage is w_t and he chooses to work $1 - l_t$ units of time, his current consumption (equal to his current earnings) is $c_t = (1 - l_t)w_t$. If he searches, the probability of obtaining *any* offer is $1 - l_t$, so with probability l_t he draws a wage offer of zero. Reformulate the functional equation for this case.

c. Suppose that the job-loss probability θ is zero, but that each worker spends exactly $T + 1$ periods in the work force. Specifically, a worker enters the labor force at age $t = 0$ with no job (an initial wage offer of zero) and hence will spend at least one period searching. The objective function for a worker just entering the labor force is thus $\mathrm{E}[\Sigma_{t=1}^{T} \beta^t U(c_t)]$. What is the functional equation for the value $v_t(w)$ for a worker of age t who begins with a wage offer w? What can be said about the sequence $w_1^*, \ldots, w_T^*$ of age-specific reservation wages?

d. Retain the assumptions of part (c), and assume the following demographics. In each period, all age $T + 1$ workers retire, and an equal number of age zero workers enter. Thus, in each period there are an equal number of workers at each of the ages $t = 0, 1, \ldots, T$. What do the age-specific unemployment rates look like for such an economy? What is the shape of the age-earnings profile, the sequence of age-specific average wages? What shapes of age-specific unemployment rate functions could one obtain by combining this model with the original model of Section 10.7?

e. Suppose that $x \in X = [0, 1]$ is an index of labor market conditions and that $f(\cdot, x)$, $x \in X$, is a family of density functions on the interval $W = [0, \bar{w}]$. Thus $f(\cdot, x)$ describes the distribution of offers the worker faces if market conditions are x. Assume that this family of density functions has the monotone likelihood ratio property. That is, for any $x' > x$, the ratio $f(w, x')/f(w, x)$ is increasing in w. Let $g: X \rightarrow \mathbf{R}_+$ also be a density function.

Modify the search model in Section 10.7 as follows. Assume that whenever a worker becomes unemployed, there is a random draw of x from the distribution given by g. The value of x is fixed as long as the worker continues searching, but the worker does not observe x directly. (He does know that it is a random draw from the distribution g.) However, the worker can make inferences about x based on the wage offers he observes. Thus the worker can use Bayes's rule to update his beliefs about x while he is searching.

How must the functional equation (1) in Section 10.7 be modified to incorporate x? How does the reservation wage of a worker who has been searching for n periods depend on the offers he has received?

10.10 A Model of Job Matching

Rather than thinking of a job as being characterized by its wage rate, one may think of it as being described by a productivity variable that is specific to a particular worker-task "match." Here is a simple, discrete version of this idea.

A worker must choose among a continuum of possible tasks. At any given task, in any period he produces a return of 1 with probability θ or 0 with probability $1 - \theta$. Returns on a given task are serially independent. There is no way to tell one's proficiency θ at a particular task short of trying it out, but one's θ on any task is drawn from a known distribution with the density function μ on $[0, 1]$. Once a worker chooses a task, he can keep it as long as he wants or he can leave it and draw a new task from μ.

Suppose a worker has engaged in a specific task for n periods and has achieved $k \in \{0, 1, \ldots, n\}$ successes. His probability of a success at this