1 PS5 Q1

Consider the managerial contracting problem in MWG 14.C, but with continuously distributed efforts. Specifically, suppose that a firm hires a manager of type θ, distributed with CDF $F(\theta)$ on $[\underline{\theta}, \bar{\theta}]$ where $[1 - F(\theta)]/f(\theta)$ is nondecreasing. The manager chooses effort at personal cost $g(e, \theta)$, which is increasing and strictly convex in $e \in [0, \infty)$ and decreasing in θ. Effort increases profit in a deterministic way, $\pi(e)$, where $\pi(\cdot)$ is strictly increasing and concave. Assume g and π three times continuously differentiable, $g_{e\theta}(e, \theta) < 0$ for all $e > 0$, $\pi(0) = g(0, \theta) = 0$ for all θ, $\pi'(0) - g_e(0, \theta) = \infty$ and $\lim_{e\to\infty} \pi(e) - g(e, \theta) = -\infty$ for all θ.
The type-θ manager's payoff from a wage w and a choice of effort, e, is

$$w - g(e, \theta).$$

We assume that the manager is risk neutral for positive payoffs (i.e., for all $w - g(e, \theta) \geq 0$), but is infinitely risk averse of negative payoffs: $u = -\infty$ if $w - g(e, \theta) < 0$. As such, any contract will need to promise the manager a nonnegative return in exchange for participation. The firmâs profits are simply $\pi(e) - w$, and the firm maximizes expected profits. Profits are observable and contractible.
The timing is standard: nature chooses the managerâs type; the firm offers the manager a contract; the manager accepts (or rejects) and chooses effort to maximize his payoff.

Problem 1.1. Characterize the first-best level of effort when θ is observable and contractible using the first-order condition.

Solution. Note that when θ is observable, the firm knows the exact cost of effort of the manager, $g(e, \theta)$, per effort level e. Hence, the firm can maximize profit by directly setting wage as low as possible to $g(e, \theta)$ just enough to induce manager's participation (IR). Hence, the firm's problem is given as:

$$\max_{w(\cdot), e} \pi(e) - w(e)$$
$$s.t. \quad w(e) \geq g(e, \theta) \quad \text{(IR)}$$

Hence, by the FOC condition of the objective function with respect to e where (IR) binds, the first-best effort level $e^*(\theta)$ satisfies:

$$\pi'(e^*(\theta)) = g_e(e^*(\theta), \theta).$$

Problem 1.2. From now on, assume θ is private information to the manager. NOte that because $\pi(e)$ is strictly increasing, it is invertible and the firm can contract on e as well as π.

State the revelation principle for this problem using deterministic direct mechanisms. Specifically, consider direct-revelation mechanisms of the form $\{e(\cdot), w(\cdot)\}$ where $e(\hat{\theta})$ is the required effort (i.e. $\pi(\hat{\theta}) = \pi(e(\hat{\theta}))$ is the required profit level) for a worker who reports $\hat{\theta}$ in exchange for wage $w(\hat{\theta})$.

Solution. The revelation principle for this problem using deterministic direct mechanisms would say that for every mechanism Γ and every equilibrium manager's strategy $\sigma^*(\theta)$ with deterministic equilibrium outcome $\{e^*(\theta), w^*(\theta)\}$, there exists a direct mechanism $\tilde{\Gamma}$ with equilibrium manager's strategy $\tilde{\sigma}^*$ with deterministic equilibrium outcome $\{\tilde{e}^*(\cdot), \tilde{w}^*(\cdot)\}$ such that

- ▷ Every manager finds it optimal to report one's true type: $\tilde{\sigma}^*(\theta) = \theta$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, i.e.

$$\tilde{w}^*(\theta) - g(\tilde{e}^*(\theta), \theta) \geq \tilde{w}^*(\theta') - g(\tilde{e}^*(\theta'), \theta) \quad \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}]$$

- ▷ $\tilde{\sigma}^*$ induces an identical equilibrium outcome: $\tilde{e}^*(\cdot) = e^*(\theta)$ and $\tilde{w}^*(\theta) = w^*(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

Problem 1.3. Define the effort-utility allocation as $\{e(\cdot), U(\cdot)\}$. State and prove the two conditions for such a profile to be implementable with wage payment $w(\theta) = U(\theta) + g(e(\theta), \theta)$.

Solution. $\{e(\cdot), U(\cdot)\}$ would be implementable with wage payment $w(\theta) = U(\theta) + g(e(\theta), \theta)$ if $\forall \theta \in [\underline{\theta}, \bar{\theta}]$,

$$U(\theta) \geq w(\theta') - g(e(\theta'), \theta), \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}].$$

The two sufficient conditions for this are:

1. $e(\cdot)$ is nondecreasing
2. $U(\theta) = U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} g_\theta(e(s), s) ds,$

and we prove why these are sufficient.

Proof. Note that, because $w(\theta) = U(\theta) + g(e(\theta), \theta)$, the following are equivalent conditions for implentability: for $\forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}]$,

$$\begin{aligned} U(\theta) \geq w(\theta') - g(e(\theta'), \theta) &\iff U(\theta) \geq U(\theta') + g(e(\theta'), \theta') - g(e(\theta'), \theta) \\ &\iff U(\theta) - U(\theta') \geq g(e(\theta'), \theta') - g(e(\theta'), \theta) \\ &\iff \int_{\theta}^{\theta'} g_\theta(e(s), s) ds \geq \int_{\theta}^{\theta'} g_\theta(e(\theta'), s) ds, \end{aligned}$$

where the last line follows from condition 2.

Consider $\theta' > \theta$. Since $e(\cdot)$ is nondecreasing, $e(s) \leq e(\theta')$ for all $s \in [\theta, \theta']$. And since $g_{e\theta} < 0$, we must have $g_\theta(e(s), s) \geq g_\theta(e(\theta'), s)$ for all $s \in [\theta, \theta']$, and thus the last equation above holds. The symmetric argument holds for $\theta' < \theta$, so the two conditions stated above are sufficient for implementability. ■

Problem 1.4. Solve for the optimal effort allocation, $e(\cdot)$. You may make any additional assumptions on π and g, but be explicit about them. How does this compare to the first-best allocation in (a).

Solution. A revised firm's problem with incentive compatible mechanisms (the two sufficient conditions for the implementability condition) and implementable allocation $\{e(\cdot), U(\cdot)\}$ with wage payment $w(\theta) = U(\theta) + g(e(\theta), \theta)$ becomes

$$\begin{aligned}
\max_{e(\cdot), w(\cdot)} \quad & E_\theta[\pi(e(\theta)) - U(\theta) - g(e(\theta), \theta)], \\
s.t. \quad & U(\theta) = U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} g_\theta(e(s), s) ds \\
& e(\cdot) \text{ nondecreasing in } \theta \\
& U(\theta) \geq 0 \quad \text{(IR)}
\end{aligned}$$

Notice that because $g_\theta < 0$, IR holds if $U(\underline{\theta}) \geq 0$. And since lower $U(\underline{\theta})$ deterministically yields higher profit for the firm, the firm will set $U(\underline{\theta}) = 0$. Hence, setting $U(\underline{\theta}) = 0$ and substituting in the first constraint into the objective function, the firm's problem now becomes

$$\begin{aligned}
\max_{w(\cdot)} \quad & \int_{\underline{\theta}}^{\bar{\theta}} \left(\pi(e(\theta)) + \int_{\underline{\theta}}^{\theta} g_\theta(e(s), s) ds - g(e(\theta), \theta) \right) f(\theta) d\theta, \\
s.t. \quad & e(\cdot) \text{ nondecreasing in } \theta.
\end{aligned}$$

Note that $\int_{\underline{\theta}}^{\bar{\theta}} (\int_{\underline{\theta}}^{\theta} g_\theta(e(s), s) ds) f(\theta) d\theta$ can be simplified using integration by parts:

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} g_\theta(e(s), s) ds \right) f(\theta) d\theta &= \underbrace{\left(\int_{\underline{\theta}}^{\bar{\theta}} g_\theta(e(s), s) ds \right)}_{=0 \text{ if } \theta = \underline{\theta}} \underbrace{(F(\theta) - 1)}_{=0 \text{ if } \theta = \bar{\theta}} \Bigg|_{\theta = \underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} g_\theta(e(s), s)(F(\theta) - 1) d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} g_\theta(e(\theta), \theta))(1 - F(\theta)) d\theta.
\end{aligned}$$

Hence, the firm's problem finally reduces to:

$$\begin{aligned}
\max_{w(\cdot)} \quad & \int_{\underline{\theta}}^{\bar{\theta}} \left(\pi(e(\theta)) + \frac{1 - F(\theta)}{f(\theta)} g_\theta(e(\theta)), \theta)) - g(e(\theta), \theta) \right) f(\theta) d\theta, \\
s.t. \quad & e(\cdot) \text{ nondecreasing in } \theta.
\end{aligned}$$

Let $\Lambda(e, \theta) := \pi(e(\theta)) + \frac{1-F(\theta)}{f(\theta)} g_\theta(e(s), s) - g(e(\theta), \theta)$. We impose the following regularity assumptions on Λ:

1. Λ is twice continuously differentiable (which we already have since g and π are three times continuously differentiable.
2. Λ is strictly quasiconcave over $e \in [0, \infty)$
3. $\Lambda_{e\theta}(e, \theta) \geq 0$,

Because $\frac{1-F(\theta)}{f(\theta)}$ is nondecreasing in θ, the program is regular so we can use the FOC condition with respect to e to solve for the optimal effort $e^*(\cdot)$. Hence, for $\forall \in [\underline{\theta}, \bar{\theta}]$, $e^*(\cdot)$ satisfies:

$$\pi'(e^*(\theta)) = g_e(e^*(\theta), \theta) - \frac{1 - F(\theta)}{f(\theta)} g_{e\theta}(e^*(\theta)), \theta))$$

We compare the above with the first-best outcome:

$$\begin{aligned}
\pi'(e^{fb}(\theta)) &= g_e(e^{fb}(\theta), \theta) \\
\pi'(e^*(\theta)) &= g_e(e^*(\theta), \theta) - \frac{1 - F(\theta)}{f(\theta)} g_{e\theta}(e^*(\theta)), \theta))
\end{aligned}$$

Note that g is strictly increasing and convex and π is strictly increasing and concave and because $g_{e\theta} < 0$, so only $e^*(\theta) < e^{fb}(\theta)$ is consistent with above. (If $e^*(\theta) > e^{fb}(\theta)$, then the LHS of the first equation decreases while the RHS increases and also the positive second term adds onto it, so the equality in the second equation cannot hold.) Hence, the second-best effort level is less than the first-best.

2 PS5 Q3

(MWG, Exercise 14.C.8 - variation) Air Shangri-la is the only airline allowed to fly between the islands of Shangri-la and Nirvana. There are two types of passengers, tourist and business. Business travelers are willing to pay more than tourists. The airline, however, cannot tell directly whether a ticket purchaser is a tourist or a business traveler. The two types do differ, though, in how much they are willing to pay to avoid having to purchase their tickets in advance. (Passengers do not like to commit themselves in advance to traveling at a particular time.) More specifically, the utility levels of each of the two types net of the price of the ticket, p, for any given amount of time w prior to the flight that the ticket is purchased are given by

▷ Business: $v - \theta_b p - w$

▷ Tourist: $v - \theta_t p - w$

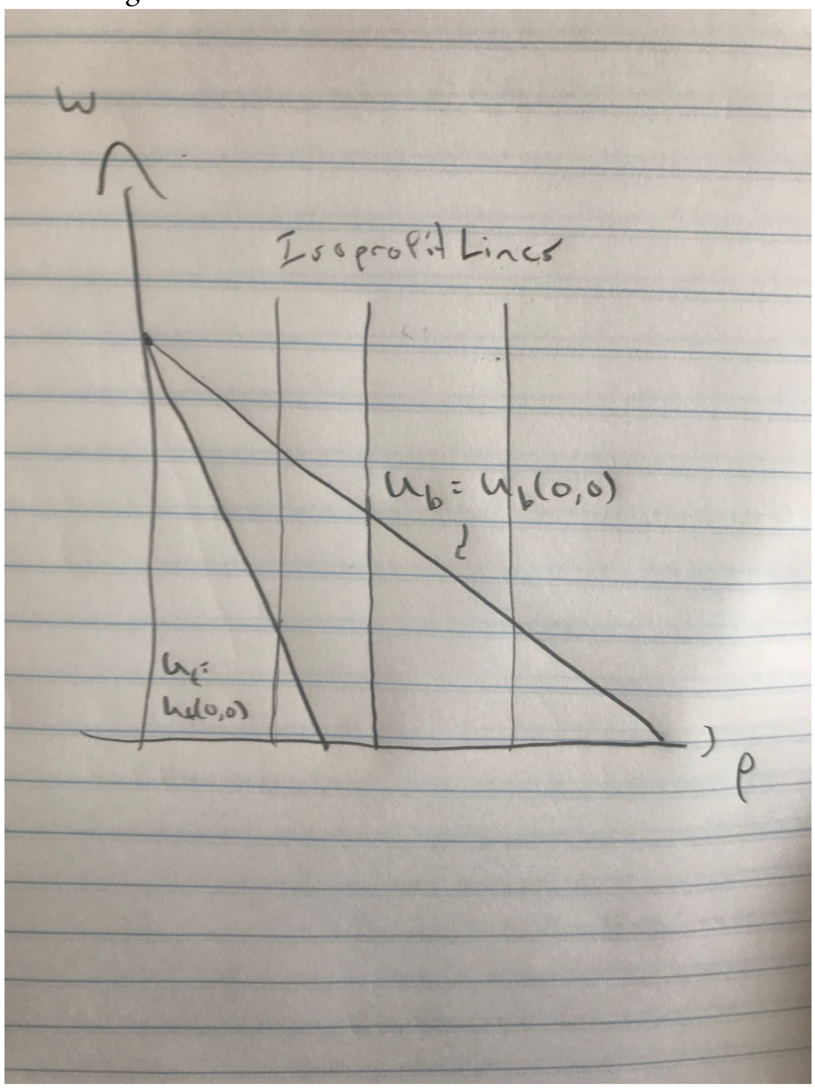
where $\theta_t > \theta_b > 0$. (Note that for any given level of w, the business traveler is willing to pay more for his ticket. Also, the business traveler is willing to pay more for any given reduction in w.) The proportion of travelers who are business customers is $\phi \in (0, 1)$ and the proportion who are tourists is $1 - \phi$. Assume that the cost of transporting a passenger is c where $c \leq \frac{v}{\theta_t}$ so that it is optimal to serve both types in a full-information world.
Assume in (a)-(c) below that Air Shangri-la wants to carry both types of passengers so the per-unit cost of transporting a passenger is immaterial.

Problem 2.1. Draw the indifference curves of the two types in (p, w)-space. Draw the airline's isoprofit curves. Now formulate the optimal (profit-maximizing) price discrimination problem mathematically that Air Shangri-la would want to solve. [Hint: Impose nonnegativity of prices as a constraint since, if it charged a negative price, it would sell an infinite number of tickets at this price.]

Solution. Below is the figure with the indifference curves and iso-profit lines. Iso-profit lines are increasing profits as you move to the right.

The airline will solve

Figure 1: Indifference Curves and Iso-Profit Lines



$$
\begin{aligned}
\max_{p_b,p_t,w_b,w_t} \; & \phi p_b + (1-\phi)p_t && (1)\\
\text{s.t. } & \theta_b p_b + w_b \leq v, && (2)\\
& \theta_t p_t + w_t \leq v, && (3)\\
& \theta_b p_b + w_b \leq \theta_b p_t + w_t, && (4)\\
& \theta_t p_t + w_t \leq \theta_t p_b + w_b, && (5)\\
& p_b, p_t, w_b, w_t \geq 0. && (6)
\end{aligned}
$$

Problem 2.2. Show that in the optimal solution, tourists are indifferent between buying a ticket and not going at all.

Solution. Note that (3) and (4) combined with $\theta_t > \theta_b$ give us

$$\theta_b p_b + w_b \leq \theta_b p_t + w_t < \theta_t p_t + w_t \leq v.$$

and so (2) is redundant and is slack. This tells us that (3) must bind. If not, then we would have

$$
\begin{aligned}
& \theta_t p_t + w_t < v,\\
& \theta_b p_b + w_b \leq \theta_b p_t + w_t,\\
& \theta_t p_t + w_t \leq \theta_t p_b + w_b,\\
& p_b, p_t, w_b, w_t \geq 0
\end{aligned}
$$

as the constraints. Since these functions are continuous and the first inequality is strict, there exists some $\delta, \epsilon > 0$ such that $p_t' = p_t + \epsilon$ and $p_b' = p_b + \delta$ still satisfy these constraints and profits will be higher, contradicting p_t, p_b were part of the optimal solution. Thus,

$$\theta_t p_t + w_t = v = u_t(0,0)$$

and tourists are indifferent between buying a ticker and not going at all.

Problem 2.3. Show that in the optimal solution, business travelers never buy their ticket prior to the flight and are just indifferent between doing this and buying when tourists buy.

Solution. Assume otherwise: the optimal solution involves $w_b > 0$. We showed before that

$$\theta_b p_b + w_b < v.$$

Let $\tilde{w}_b = w_b - \epsilon$ and $\tilde{p}_b = p_b + \frac{\epsilon}{\theta_b}$ so this constraint still holds and the airline can charge business travelers more money and earn higher profits. Note that

$$\theta_t \tilde{p}_b + \tilde{w}_b > \theta_b \tilde{p}_b + \tilde{w}_b = \theta_b p_b + w_b \geq \theta_t p_t + w_t$$

and hence this new price and time for business travelers is incentive compatible and results in higher profits for the airline, a contradiction to our optimal solution. Since ϵ was general, we must have that in the optimal solution $w_b = 0$.

We also have that business travelers are indifferent between doing this and buying when tourists buy. If they were not indifferent, we would have

$$\theta_b p_b < \theta_b p_t + w_t$$

and the airline could simply raise p_b and make more profit without affecting any other constraint, a contradiction. Thus, it must be

$$\theta_b p_b = \theta_b p_t + w_t$$

and hence business travelers are indifferent between their their package and the tourist package.

Problem 2.4. Describe fully the optimal price discrimination scheme under the assumption that the airline sells to both types.

Solution. The problem can now be simplified to

$$\begin{aligned} \max_{p_b, p_t, w_t} \ & \phi p_b + (1-\phi) p_t && (7) \\ \text{s.t. } & \theta_t p_t + w_t = v, && (8) \\ & \theta_b p_b = \theta_b p_t + w_t, && (9) \\ & \theta_t p_t + w_t \leq \theta_t p_b, && (10) \\ & p_b, p_t, w_t \geq 0. && (11) \end{aligned}$$

Note that we have for any (p_t, w_t), $(\tilde{p}_t, \tilde{w}_t) = (p_t - \frac{\epsilon}{\theta_t}, w_t + \epsilon)$ where $\epsilon > 0$ will also satisfy (8) and give us $\tilde{p}_b$ where

$$\begin{aligned}\tilde{p}_b &= \tilde{p}_t + \frac{1}{\theta_b}\tilde{w}_t \\ &= p_t + \frac{1}{\theta_b}w_t + \epsilon(\frac{1}{\theta_b} - \frac{1}{\theta_t}) > p_b.\end{aligned}$$

Since we have increased p_b in this new allocation, all other constraints still hold. Since our constraints are linear this trade-off exists everywhere thus is a profitable change in prices if

$$\begin{aligned}&\phi\tilde{p}_b + (1-\phi)\tilde{p}_t = \phi p_b + (1-\phi)p_t + \epsilon(\phi\frac{\theta_t - \theta_b}{\theta_t\theta_b} - (1-\phi)\frac{1}{\theta_t}) > \phi p_b + (1-\phi)p_t \\ \implies &\phi\frac{\theta_t - \theta_b}{\theta_t\theta_b} - (1-\phi)\frac{1}{\theta_t} > 0 \\ \implies &\frac{\phi}{1-\phi} > \frac{\theta_b}{\theta_t - \theta_b}.\end{aligned}$$

If it is profitable, then it is always profitable to lower prices of tourists all the way to zero and only service business travelers. Here we want both to be served and thus we must have

$$\frac{\phi}{1-\phi} \leq \frac{\theta_b}{\theta_t - \theta_b}.$$

Note by constraint (9), $p_b \geq p_t$. If $p_b > p_t$, it is profitable to implement the same scheme as above but now $\epsilon < 0$ (i.e., we now reduce the price for business travelers and increase it for tourists). Given the constraint (10), this results in the prices being equal and our problem is now

$$\max_{p,w_t} \phi p + (1-\phi)p \tag{12}$$

$$\text{s.t. } \theta_t p = v, \tag{13}$$

$$0 = w_t, \tag{14}$$

$$w_t \leq 0, \tag{15}$$

$$p, w_t \geq 0. \tag{16}$$

and thus we have $(p_t, w_t) = (p_b, w_b) = (\frac{v}{\theta_t}, 0)$.

Problem 2.5. Describe fully the optimal price discrimination scheme under the assumption that the airline sells to only the high type. Give a condition as a function of the underlying parameters for when selling to both is more profitable than selling only to the high types.

Solution. As we showed above, the airline will only serve the business travelers when

$$\frac{\phi}{1-\phi} > \frac{\theta_b}{\theta_t - \theta_b}$$

and thus we get

$$\begin{aligned}
&\max_{p_b, w_t} \phi p_b && (17)\\
&\text{s.t. } w_t = v, && (18)\\
&\quad \theta_b p_b = w_t, && (19)\\
&\quad w_t \leq \theta_t p_b, && (20)\\
&\quad p_b, w_t \geq 0. && (21)
\end{aligned}$$

which gives us the optimal scheme $(p_b, w_b) = (\frac{v}{\theta_b}, 0)$ and $(p_t, w_t) = (0, v)$.

3 PS5 Q5

Consider the price discrimination model considered in the lecture, except that there are now three types, where $\theta_3 > \theta_2 > \theta_1 > 0$. The utility of agent θ_i is

$$u(q, \theta_i) = \theta_i q - t$$

Denote the allocations to agent θ_i by (q_i, t_i). Now there are three (IR) constraints, one for each type. There are also six (IC) constraints, since we must ensure type θ_1 does not want to copy type θ_2 or θ_3 and similary for other agents. For example, (IC12) says that θ_1 must not want to copy θ_2, i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2$$

The firm's profit is

$$\sum_{i=1}^{3} \phi(t_i - C(q_i))$$

where ϕ_i is the proportion of type θ_i agents and $C(q)$ is increasing and convex.

Problem 3.1. Show that (IR2) and (IR3) can be ignored.

Solution. To see that (IR2) and (IR3) can be ignored, notice that these two constraints are guaranteed to hold from (IR1), (IC21), (IC31), (IR12), and (IR13). In particular, we have that

$$\begin{aligned}
\theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 && \text{(IC21)} \\
&\geq \theta_1 q_1 - t_1 && \text{since } \theta_2 > \theta_1 > 0 \\
&\geq 0 && \text{from (IR1)}
\end{aligned}$$

and likewise we have that

$$\begin{aligned}
\theta_3 q_3 - t_3 &\geq \theta_3 q_1 - t_1 && \text{(IR31)} \\
&\geq \theta_1 q_1 - t_1 && \text{since } \theta_3 > \theta_1 > 0 \\
&\geq 0 && \text{from (IR1)}
\end{aligned}$$

Problem 3.2. Show that $q_3 \geq q_2 \geq q_1$

Solution. By way of of contradiction, suppose that this. From (IC21) we know that

$$\begin{aligned} \theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 \\ \theta_2 (q_2 - q_1) &\geq t_2 - t_1 \end{aligned}$$

Now, from the (IC12) constraint, we know that

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq \theta_1 q_2 - t_2 \\ t_2 - t_1 &\geq \theta_1 (q_2 - q_1) \end{aligned}$$

then we have that

$$\theta_2 \underbrace{(q_2 - q_1)}_{<0} \geq t_2 - t_1 \geq \theta_1 \underbrace{(q_2 - q_1)}_{<0} > \theta_2 (q_2 - q_1)$$

where the final inequality holds since $\theta_1 > \theta_2 > 0$ which is a contradiction.

Problem 3.3. Assume that $q_3 \geq q_2 \geq q_1$ in (b) holds. Using (IC12) and (IC23) show that we can ignore (IC13). Using (IC32) and (IC21) show that we can ignore (IC31).

Solution. Using (IC12) we have that

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \tag{IC12}$$

now notice that (IC23) implies

$$\begin{aligned} \theta_2 q_2 - t_2 &\geq \theta_2 q_3 - t_3 \\ \theta_2 q_2 - q_2 (\theta_2 - \theta_1) - t_2 &\geq \theta_2 q_3 - q_2 (\theta_2 - \theta_1) - t_3 \\ \theta_2 q_2 - q_2 (\theta_2 - \theta_1) - t_2 &\geq \theta_2 q_3 - q_3 (\theta_2 - \theta_1) - t_3 && \text{since } q_3 > q_2 \\ \theta_1 q_2 - t_2 &\geq \theta_1 q_3 - t_3 \end{aligned}$$

so combining expressions, we have that

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq \theta_1 q_2 - t_2 \geq \theta_1 q_3 - t_3 \\ \theta_1 q_1 - t_1 &\geq \theta_1 q_3 - t_3 \end{aligned}$$

which is precisely (IR13).

From (IR32) we have that

$$\theta_3 q_3 - t_3 \geq \theta_3 q_2 - t_2 \tag{IR32}$$

Now, from (IC21) we have that

$$\begin{aligned}
\theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 \qquad \text{(IR21)} \\
\theta_2 q_2 + q_2 (\theta_3 - \theta_2) - t_2 &\geq \theta_2 q_1 + q_2 (\theta_3 - \theta_2) - t_1 \\
\theta_2 q_2 + q_2 (\theta_3 - \theta_2) - t_2 &\geq \theta_2 q_1 + q_1 (\theta_3 - \theta_2) - t_1 \\
\theta_3 q_2 - t_2 &\geq \theta_3 q_1 - t_1
\end{aligned}$$

Combining these expressions, we have that

$$\theta_3 q_3 - t_3 \geq \theta_2 q_2 - t_2 \geq \theta_3 q_1 - t_1$$

which is precisely (IR31)

Problem 3.4. Show that (IR1) will bind.

Solution. This follows directly from the firm's problem. In particular, the firm's problem is given by

$$\begin{aligned}
&\max \sum_{i=1}^{3} \phi (t_i - C(q_i)) \\
&\Leftrightarrow \max \phi_1 (t_1 - C(q_1)) + \phi_2 (t_2 - C(q_2)) + \phi_3 (t_3 - C(q_3))
\end{aligned}$$

subject to the IC and IR constraints. Now suppose that (IR1) did not bind and that $\theta_1 q_1 > t_1$. The the firm could adjust the three contracts offered by first offering a new contract $(q_1 - \epsilon, t_1)$ and earn strictly higher profits on the lowest type. Further, the firm can also offer the new contracts $(q_2 - \delta, t_2)$ and $(q_3 - \gamma, t_3)$ as well such that the IC constraints still hold for each type and also earn strictly higher profits from the contracts offered to every type.

Problem 3.5. Show that (IC21) will bind.

Solution. Returning to the firm's problem, suppose that (IC21) did not bind. I.e., that

$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1$$

then the firm could offer the contract $(q_2 - \epsilon, t_2)$ where $\epsilon = \frac{1}{2}((\theta_2 q_2 - t_2) - (\theta_2 q_1 - t_1))$ and earn strictly higher profits from type two. Analogous to the previous part, the firm can now also offer a worse contract $(q_3 - \delta, t_3)$ to the highest type such that the IC constraints for the second and third types still hold. Therefore the firm will make higher profits from the contracts offered to the second and third types.

Problem 3.6. Consider

Solution. Suppose that (IC32) did not bind. I.e., that

$$\theta_3 q_3 - t_3 > \theta_3 q_2 - t_2$$

then the firm could offer the contract $(q_3 - \epsilon, t_3)$ where $\epsilon = \frac{1}{2}\left((\theta_3 q_3 - t_2) - (\theta_3 q_2 - t_2)\right)$ and earn strictly higher profits from type two. Again, notice that this will not break the IC constraints for the other types as type 3's contract is now strictly worse.

Problem 3.7. Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC12) and (IC23) can be ignored.

Solution. Suppose that (IC12) was violated. Then we have that

$$\theta_1 q_1 - t_1 < \theta_1 q_2 - t_2$$

but we know that since (IC21) binds

$$\theta_2 q_1 - t_1 = \theta_2 q_2 - t_2$$

and so, combining these, we have that

$$\begin{aligned}
\theta_1 q_1 - t_1 - (\theta_2 q_1 - t_1) &< \theta_1 q_2 - t_2 - (\theta_2 q_2 - t_2) \\
q_1 (\theta_1 - \theta_2) &< q_2 (\theta_1 - \theta_2) \\
q_1 &> q_2
\end{aligned}$$

which is a contradiction. Therefore (IC12) must hold given $q_3 \geq q_2 \geq q_1$ and (IC21) binds.

The same logic gives that (IC23) can be ignored. In particular, suppose that (IC23) was violated. Then we have that

$$\theta_2 q_2 - t_2 < \theta_2 q_3 - t_3$$

Since we know that (IC32) binds we have that

$$\theta_3 q_2 - t_2 = \theta_3 q_3 - t_3$$

and so, combining these, we have that

$$\begin{aligned}
\theta_2 q_2 - t_2 - (\theta_3 q_2 - t_2) &< \theta_2 q_3 - t_3 - (\theta_3 q_2 - t_2) \\
q_2 (\theta_2 - \theta_3) &< q_3 (\theta_2 - \theta_3) \\
q_2 &> q_3
\end{aligned}$$

which is again a contradiction and so (IC23) will hold.

Problem 3.8. Suppose that $C(q) = \frac{1}{2}q^2$, $\theta_1 = 4$, $\theta_2 = 5$, $\theta_3 = 6$, $\phi_1 = \phi_2 = \phi_2 = \frac{1}{3}$. State the firm's optimization program given your conclusions in (a)-(g) and find the optimal qualities to sell, (q_3, q_2, q_1).

Solution. The firm's problem is now

$$\max \sum_{i=1}^{3} \phi_i \left(t_i - \frac{1}{2} q_i^2 \right)$$

subject to

$$\theta_1 q_1 - t_1 = 0 \tag{IR1}$$
$$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1 \tag{IC21}$$
$$\theta_3 q_3 - t_3 = \theta_3 q_2 - t_2 \tag{IR32}$$

Now we can plug in the actual values into these constraints and solve the maximization problem. In particular

$$\begin{aligned} \theta_1 q_1 - t_1 &= 0 \\ \theta_1 q_1 &= t_1 \end{aligned}$$

then we have that

$$\begin{aligned} \theta_2 q_2 - t_2 &= \theta_2 q_1 - t_1 \\ \theta_2 q_2 - t_2 &= \theta_2 q_1 - \theta_1 q_1 \\ \theta_2 (q_2 - q_1) + \theta_1 q_1 &= t_2 \end{aligned}$$

and then

$$\begin{aligned} \theta_3 q_3 - t_3 &= \theta_3 q_2 - t_2 \\ \theta_3 q_3 - t_3 &= \theta_3 q_2 - \theta_2 (q_2 - q_1) - \theta_1 q_1 \\ \theta_3 (q_3 - q_2) + \theta_2 (q_2 - q_1) + \theta_1 q_1 &= t_3 \end{aligned}$$

Now we have that the firm's maximization problem is given by

$$\begin{aligned} \max_{q_1, q_2, q_3} \quad & \frac{1}{3} \left(\theta_1 q_1 - \frac{1}{2} q_1^2 \right) \\ & + \frac{1}{3} \left(\theta_2 (q_2 - q_1) + \theta_1 q_1 - \frac{1}{2} q_2^2 \right) \\ & + \frac{1}{3} \left(\theta_3 (q_3 - q_2) + \theta_2 (q_2 - q_1) + \theta_1 q_1 - \frac{1}{2} q_3^2 \right) \end{aligned}$$

Now, the first-order condition with respect to q_1 is given by

$$\begin{aligned}
\frac{1}{3}(\theta_1 - q_1) + \frac{1}{3}(-\theta_2 + \theta_1) + \frac{1}{3}(-\theta_2 + \theta_1) &= 0 \\
3\theta_1 - 2\theta_2 &= q_1 \\
3 \times 4 - 2 \times 5 &= q_1 \\
2 &= q_1
\end{aligned}$$

Next, the first-order condition with respect to q_2 is given by

$$\begin{aligned}
\frac{1}{3}(\theta_2 - q_2) + \frac{1}{3}(-\theta_3 + \theta_2) &= 0 \\
2\theta_2 - \theta_3 &= q_2 \\
2 \times 5 - 6 &= q_2 \\
4 &= q_2
\end{aligned}$$

Finally, the first-order condition with respect to q_3 is given by

$$\begin{aligned}
\frac{1}{3}(\theta_3 - q_3) &= 0 \\
\theta_3 &= q_3 \\
6 &= q_3
\end{aligned}$$

so the optimal quantities to sell are $(q_3, q_2, q_1) = (6, 4, 2)$.