

proposed policy is

$$F_y(y, x) + \beta F_x(x, y) = 0,$$

which is also satisfied by assumption. Thus, the two-period cycle is indeed optimal.

8. Show that  $(0.29, 0.18)$  is a two-period optimal cycle for the previous parameters.

**Ans:** We know that the proposed two-period cycle must satisfy

$$F_y(x, y) + \beta F_x(y, x) = (1 - \theta)(1 - y)^{\alpha(1-\theta)-1} x^{\alpha\theta} - \beta\theta(1 - x)^{\alpha(1-\theta)} y^{\alpha\theta-1} = 0,$$

and

$$F_y(y, x) + \beta F_x(x, y) = (1 - \theta)(1 - x)^{\alpha(1-\theta)-1} y^{\alpha\theta} - \beta\theta(1 - y)^{\alpha(1-\theta)} x^{\alpha\theta-1} = 0.$$

Plugging  $(x, y) = (.29, .18)$  into these equations for the parameter values described above, we find that both equations are equal to zero. Thus, the proposed two-period cycle is optimal.

## 2 A model of durable and non-durable goods

Consider an economy where in each period every one of the consumers has an endowment  $y$ . This endowment can be used for investment in durable goods or for consumption of non-durables. Then the technology for this economy is:

$$x(t) + c(t) = y$$

for all  $t \geq 0$ , where  $x(t)$  denote the investment in durables and  $c(t)$  the consumption of non-durables. The stock of durable goods have a law of motion:

$$\dot{d}(t) = x - \delta d(t)$$

where  $\delta$  is the depreciation rate of durables per unit of time.

The period utility function depends on the flow of nondurable purchases and on the stock of durables, and is given by  $U(c, d)$ . We assume that  $U$  is strictly quasi-concave in  $(c, d)$ . In some cases we will specialize to

$$U(c, d) = \frac{[h(c, d)]^{1-\gamma} - 1}{1 - \gamma} \tag{1}$$

for  $\gamma \geq 0$ , and where

$$h(c, d) = \left[ c^{-\theta} + \frac{1}{A} d^{-\theta} \right]^{-1/\theta}$$

for  $\theta \geq -1$ . The agent's utility is the discounted value of  $U(c, d)$ , using discount rate  $\rho$ . With this parameterization the elasticity of substitution between  $c$  and  $d$  is  $1/(1 + \theta)$ , and the inter-temporal elasticity of substitution between the bundle  $h$  of  $(c, d)$  is  $1/\gamma$ .

Thus problem of the planner for this economy is

$$\max_{c, d} \int_0^\infty e^{-\rho t} U(c(t), d(t)) dt$$

subject to

$$\dot{d}(t) + c(t) = y - \delta d(t),$$

and  $d(0) > 0$  given.

Q0. To better understand the utility function in 1. show that if

$$\begin{aligned} U_{cd} &> 0 \text{ if } \frac{1}{\gamma} > \frac{1}{1 + \theta}, \\ U_{cd} &= 0 \text{ if } \frac{1}{\gamma} = \frac{1}{1 + \theta}, \\ U_{cd} &< 0 \text{ if } \frac{1}{\gamma} < \frac{1}{1 + \theta}. \end{aligned}$$

And hence that if  $\sigma \equiv 1/\gamma = 1/(1 + \theta)$  the utility function is additively separable in  $c, d$ :

$$U(c, d) = \frac{c^{1-\frac{1}{\sigma}} + \frac{1}{A} d^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}}.$$

A. The first derivative is

$$U_c = \left[ c^{-\theta} + \frac{1}{A} d^{-\theta} \right]^{-\frac{1-\gamma}{\theta}-1} c^{-\theta-1}$$

so the cross derivative is

$$U_{cd} = (1 - \gamma + \theta) \left\{ \left[ c^{-\theta} + \frac{1}{A} d^{-\theta} \right]^{-\frac{1-\gamma}{\theta}-2} c^{-\theta-1} \frac{1}{A} d^{-\theta-1} \right\}$$

so

$$1 - \gamma + \theta \geq 0 \iff 1 + \theta \geq \gamma \iff \frac{1}{\gamma} \geq \frac{1}{1 + \theta}.$$

Finally, if  $1/\gamma = \sigma = \frac{1}{1+\theta}$  then

$$\begin{aligned} -\theta &= 1 - \frac{1}{\sigma} \\ 1 - \gamma &= 1 - \frac{1}{\sigma} \end{aligned}$$

so

$$U(c, d) = \frac{[c^{-\theta} + \frac{1}{A}d^{-\theta}]^{-\frac{1-\gamma}{\theta}}}{1-\gamma} = \frac{c^{1-\frac{1}{\sigma}} + \frac{1}{A}d^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}$$

Q1. Write the Hamiltonian of the problem, using  $\lambda$  for the co-state,  $d$  for the state, and  $c$  for the control.

A:

$$H(c, d) = U(c, d) + \lambda[-d\delta + y - c]$$

Q2. Write the f.o.c. w.r.t  $c$  and  $d$ .

A:

$$\dot{\lambda} = \rho\lambda - H_d : \dot{\lambda} = \lambda(\rho + \delta) - U_d(c, d),$$

and

$$H_c = 0 : U_c(c, d) = \lambda.$$

Q3. Let  $\bar{v} \equiv \delta + \rho$  be the steady state user cost of the durable good. Write two equations in two unknowns for the steady state values of  $(\bar{c}, \bar{d})$  in terms of  $U_d$ ,  $U_c$ ,  $\rho$  and  $\bar{v}$ .

A:

The steady state is then:

$$\begin{aligned} \frac{U_d}{U_c}(\bar{c}, \bar{d}) &= \bar{v}, \\ \delta\bar{d} + \bar{c} &= y. \end{aligned}$$

Q4. Use the equation  $H_c = 0$  to obtain a differential equation linking  $\dot{\lambda}$ ,  $\dot{c}$  and  $\dot{d}$ .

A: From  $U_c(c(t), d(t)) = \lambda(t)$  we obtain:

$$\dot{c}U_{cc} + \dot{d}U_{cd} = \dot{\lambda},$$

Q5. Using this last expression, replace the law of motion for the co-state variable and the law of motion of the state variable to find the law of motion of the control  $\dot{c}$  as a function of parameters  $c$  and  $d$ .

A:

$$\dot{c}U_{cc}(c, d) + \dot{d}U_{cd}(c, d) = U_c(c, d)(\delta + \rho) - U_d$$

and using the budget constraint for  $\dot{d}$ :

$$\dot{c}U_{cc}(c, d) = U_c(c, d)(\delta + \rho) - [y - d\delta - c]U_{cd}(c, d) - U_d(c, d)$$

Q6. Linearize this last ODE around the steady state, i.e.:  $(\dot{c}, \dot{d}, c, d) = (0, 0, \bar{c}, \bar{d})$  and replacing  $\dot{d}$  by using the resource constraint of the economy. Your answer should be of the type  $\dot{c} = a_{11}(c - \bar{c}) + a_{12}(d - \bar{d})$  for two constants  $a_{11}$  and  $a_{12}$ . This constant are functions of the second derivatives of  $U$  evaluated at the steady state, and of  $\delta$  and  $\rho$ .

Note on linearization: Suppose we want to linearize the function  $g(x, y)$  around  $\bar{x}$  and  $\bar{y}$ . We get:

$$g(x, y) = g_x(\bar{x}, \bar{y})(x - \bar{x}) + g_y(\bar{x}, \bar{y})(y - \bar{y})$$

A: The ODE that we want to linearize is:

$$\dot{c} = \frac{U_c(c, d)(\delta + \rho) - [y - d\delta - c]U_{cd}(c, d) - U_d(c, d)}{U_{cc}(c, d)}$$

Linearizing around the steady state

$$\begin{aligned} \frac{U_d(\bar{c}, \bar{d})}{U_c(\bar{c}, \bar{d})} &= \bar{v}, \\ \delta\bar{d} + \bar{c} &= y. \end{aligned}$$

$$\begin{aligned} \dot{c} &= \left( \frac{(U_{cc}(\delta + \rho) + U_{cd} - [y - \delta\bar{d} - \bar{c}]U_{ccd} - U_{cd})U_{cc} - (U_c(\delta + \rho) - [y - \delta\bar{d} - \bar{c}]U_{cd} - U_d)U_{ccc}}{(U_{cc})^2} \right) (c - \bar{c}) \\ &+ \left( \frac{(U_{cd}(\delta + \rho) + \delta U_{cd} - [y - \delta\bar{d} - \bar{c}]U_{cdd} - U_{dd})U_{cc} - (U_c(\delta + \rho) - [y - \delta\bar{d} - \bar{c}]U_{cd} - U_d)U_{ccd}}{(U_{cc})^2} \right) (d - \bar{d}) \end{aligned}$$

Evaluated at the steady state we get:

$$\begin{aligned} \dot{c} &= \left( \frac{U_{cc}(\delta + \rho) + U_{cd} - U_{dc}}{U_{cc}} \right) (c - \bar{c}) \\ &+ \left( \frac{U_{cd}(\delta + \rho) + \delta U_{cd} - U_{dd}}{U_{cc}} \right) (d - \bar{d}) \end{aligned}$$

or:

$$\begin{aligned} \dot{c} &= (\delta + \rho) \frac{U_{cc}}{U_{cc}} (c - \bar{c}) + (\delta + \rho) \frac{U_{cd}}{U_{cc}} (d - \bar{d}) + \frac{U_{cd}}{U_{cc}} (c - \bar{c}) \\ &+ \frac{\delta U_{cd}}{U_{cc}} (d - \bar{d}) - \frac{U_{dd}}{U_{cc}} (d - \bar{d}) - \frac{U_{dc}}{U_{cc}} (c - \bar{c}) \end{aligned}$$

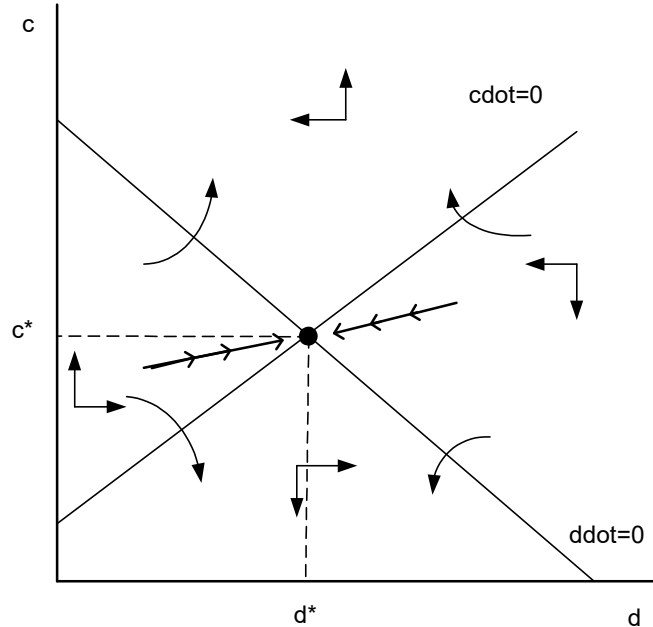
where  $U_{cc}, U_{cd}, U_{dd}$  are evaluated at the steady state.

Q7. Using your previous answer to write the two **linear** differential equations that characterize the dynamics of this economy, one for  $\dot{c} = a(c, d)$  and one for  $\dot{d} = b(c, d)$ .

A: Summarizing we have the linear system:

$$\begin{aligned}\dot{c} &= a(c, d) \equiv [\delta + \rho](c - \bar{c}) + \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{U_{cc}} \right] (d - \bar{d}) \\ \dot{d} &= b(c, d) \equiv y - d\delta - c\end{aligned}$$

Q8. Assume that  $U_{cd}(\delta + \bar{v}) - U_{dd} > 0$ . Draw the phase diagram with  $c$  in the y-axis and  $d$  in the x-axis. Label the axis, label the steady states, draw all the arrows for the field, and indicate clearly where the stable arm (saddle-path) is.



Q9. We are looking for a solution of the form

$$c = \psi(d) = \bar{c} + \psi'(\bar{d})(d - \bar{d})$$

thus, we are looking for the value of the constant  $\psi'(\bar{m})$ . Use the method of undetermined coefficients to find a quadratic equation for  $\psi'$  as a function of:  $[\bar{v} + \delta]$  and

$$\Delta \equiv \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{-U_{cc}} \right]$$

where the second derivatives are evaluated at the steady state. Hint: You need to use L'Hopital's rule. Recall what we did in pset 5 problem 1.

A:

$$\begin{aligned}\frac{\partial \psi(\bar{d})}{\partial d} &\equiv \psi'(\bar{d}) = \frac{\dot{c}}{\dot{d}} = \frac{a(\psi(\bar{d}), \bar{d})}{b(\psi(\bar{d}), \bar{d})} = \frac{0}{0} \\ &= \frac{(\delta + \rho)\psi'(\bar{d}) + \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{U_{cc}} \right]}{-\delta - \psi'(\bar{d})},\end{aligned}$$

where the second line uses L'Hopital. The quadratic equation for  $\psi'$  is then

$$(\psi'(\bar{d}))^2 + (2\delta + \rho)\psi'(\bar{d}) + \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{U_{cc}} \right] = 0$$

Q10. Show that the stable solution is given by

$$\psi' = \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 4 \frac{U_{cd}(\bar{v} + \delta) - U_{dd}}{(-U_{cc})}}}{2},$$

(hint: this is trivial once you have the solution of Q9, and the figure for the saddle path, that help you to find the "right" solution of the quadratic equation).

A: Solve the quadratic equation.

$$(\psi'(\bar{d}))^2 + (2\delta + \rho)\psi'(\bar{d}) + \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{U_{cc}} \right] = 0$$

the quadratic equation is:

$$\psi'(\bar{d}) = \frac{-[\bar{v} + \delta] \pm \sqrt{[\bar{v} + \delta]^2 - 4 \left[ \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{U_{cc}} \right]}}{2}$$

where the positive root is given by:

$$\psi' = \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 4 \frac{U_{cd}(\bar{v} + \delta) - U_{dd}}{(-U_{cc})}}}{2}.$$

Elasticity of the optimal consumption function.

For this we specialize the utility function  $U$  to (1).

We will show how the elasticity of the policy function is related to the inter-temporal and intra-temporal elasticities of substitution. Recall that the intra-temporal elasticity of substitution between  $c$  and  $d$  is  $1/(1 + \theta)$ , and the inter-temporal elasticity of substitution between bundles of  $(c, d)$  given by  $h$  is  $1/\gamma$ .

We will parameterized the problem as a function of  $(\theta, \gamma, \delta, \bar{v}, \bar{d}/\bar{c})$ . The interpretation of  $\bar{d}/\bar{c}$  as a parameter, is that we solve for the constant  $A$  using the steady-state equation derived above as a function of the paramters  $\theta, \bar{v}$ , so that  $\bar{d}/\bar{c}$ . We obtain the following result:

Keeping the steady state value  $\bar{d}/\bar{c}$  fixed, the elasticity of the optimal consumption function evaluated at steady state is a function of  $\gamma/(1+\theta)$  and satisfies

$$\frac{d}{c} \frac{\partial c(d)}{\partial d} \Big|_{d=\bar{d}} \equiv \frac{\bar{d}}{\bar{c}} \psi'(\bar{d}) = \begin{cases} 1 & \text{for } \frac{\gamma}{1+\theta} = 0 \\ < 1 & \text{for } \frac{\gamma}{1+\theta} > 0 \end{cases}$$

and  $\frac{\bar{d}}{\bar{c}} \psi'(\bar{d})$  is decreasing in  $\frac{\gamma}{1+\theta}$ .

As an intermediate step to see why  $(\bar{d}/\bar{c}) (\partial c(\bar{d})/\partial d)$  depends on the ratio of  $\gamma$  to  $1+\theta$  only, and to develop a formulat for  $\Delta(\gamma/(1+\theta))$  do the following:

Q11. To show this, first show that when  $h$  is a CES we have that:

$$\begin{aligned} \frac{h_{dd}}{h_{cc}} &= \frac{1}{(d/c)^2}, \\ \frac{h_{cd}}{h_{cc}} &= -\frac{1}{d/c}, \end{aligned}$$

$$\frac{h_c h_c}{-h h_{cc}} = \frac{1}{(1+\theta) \bar{v} (d/c)},$$

and that for  $U(c, d) = h(c, d)^{1-\gamma} / (1-\gamma)$

$$\begin{aligned} \frac{U_{dd}}{U_{cc}} &= \frac{h_{dd}/h_{cc} + \gamma \bar{v}^2 (h_c h_c) / (-h h_{cc})}{1 + \gamma (h_c h_c) / (-h h_{cc})}, \\ \frac{U_{cd}}{U_{cc}} &= \frac{h_{cd}/h_{cc} + \gamma \bar{v} (h_c h_c) / (-h h_{cc})}{1 + \gamma (h_c h_c) / (-h h_{cc})} \end{aligned}$$

and

$$\frac{h_c h_c}{-h h_{cc}} = \frac{1}{(1+\theta) \bar{v} (d/c)}.$$

Q12. First assume that  $\gamma = 0$ . Using part of the results of Q11 show that

$$\psi'(\bar{d}) = \frac{\bar{c}}{\bar{d}}$$

A. we have

$$\psi' = \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 4 \frac{h_{cd}(\bar{v} + \delta) - h_{dd}}{-h_{cc}}}}{2}$$

so

$$\frac{h_{cd}(\bar{v} + \delta) - h_{dd}}{-h_{cc}} = \frac{(\delta + \bar{v})}{d/c} + \left(\frac{1}{d/c}\right)^2$$

Thus

$$\begin{aligned} & [\bar{v} + \delta]^2 + 4 \frac{h_{cd}(\bar{v} + \delta) - h_{dd}}{-h_{cc}} \\ = & [\bar{v} + \delta]^2 + 2(\delta + \bar{v}) \left(\frac{2}{d/c}\right) + \left(\frac{2}{d/c}\right)^2 \\ = & \left(\bar{v} + \delta + \left(\frac{2}{d/c}\right)\right)^2 \end{aligned}$$

and hence

$$\begin{aligned} \psi' &= \frac{-[\bar{v} + \delta] + \sqrt{\left([\bar{v} + \delta] + \left(\frac{2}{d/c}\right)\right)^2}}{2} \\ &= \frac{-[\bar{v} + \delta] + [\bar{v} + \delta] + \left(\frac{2}{d/c}\right)}{2} \\ &= \frac{1}{d/c}. \end{aligned}$$

Q13. Assume that  $\gamma > 0$  and that  $1/\gamma = \sigma$  and  $-\theta = 1 - \frac{1}{\sigma}$ , or  $\frac{\gamma}{1+\theta} = 1$  so that  $U$  is additively separable. What is the value of  $\Delta\left(\frac{\gamma}{1+\theta}\right) = \Delta(1)$  for this case? (hint: compute  $U_{cd}$ ,  $U_{dd}$  and  $U_{cc}$  at the steady state values of  $c, d$ ). Verify that  $\Delta$ , and hence  $-(d/c) \partial c / \partial d$  does depend on the particular value of  $\sigma$ , given  $\bar{c}/\bar{d}$  and  $\bar{v}$ . Show that the value of  $(d/c) \psi'$  is smaller than the one for  $\gamma = 0$  and  $\theta > -1$ .

A. Since  $U_{cd} = 0$ , and

$$\Delta\left(\frac{\gamma}{1+\theta}\right) = \Delta(1) = \frac{U_{dd}}{U_{cc}} = \frac{-\sigma \frac{1}{A} d^{-\frac{1}{\sigma}-1}}{-\sigma c^{-\frac{1}{\sigma}-1}} = \frac{c}{d} \frac{\frac{1}{A} d^{-\frac{1}{\sigma}}}{c^{-\frac{1}{\sigma}}} = \left(\frac{c}{d}\right) \bar{v}$$



Comparing with the case of  $\gamma/(1+\theta) = 0$  we have:

$$\begin{aligned}
 \psi' &= \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 4\left(\frac{\bar{c}}{\bar{d}}\right)\bar{v}}}{2} \\
 &< \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 2(\bar{v} + \delta)2\left(\frac{\bar{c}}{\bar{d}}\right) + 4\left(\frac{\bar{c}}{\bar{d}}\right)^2}}{2} \\
 &= \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta + 2\left(\frac{\bar{c}}{\bar{d}}\right)]^2}}{2} \\
 &= \frac{-[\bar{v} + \delta] + (\bar{v} + \delta + 2(\bar{c}/\bar{d}))}{2} \\
 &= \frac{\bar{c}}{\bar{d}}.
 \end{aligned}$$

Q14. Assume that  $\gamma > 0$ , what assumptions are required for  $1/(1+\theta)$  such that you also find

$$\psi'(\bar{d}) = \frac{\bar{c}}{\bar{d}}$$

Hint: look at the formula for  $\psi'$ .

A: We need  $\frac{\gamma}{1+\theta} = 0$ , so if  $\gamma > 0$  then  $\theta \rightarrow \infty$ .

Q15. Give an intuitive interpretation for this last two results. (2 lines max).

A: The elasticity of the optimal decision rule for consumption depends on the ratio between the elasticity of substitution between durable and non durable goods,  $1/(1+\theta)$ , and the intertemporal elasticity of substitution  $1/\gamma$ . If durables and non durables are very poor substitutes, the elasticity  $(d/c)\psi'$  is one. To understand this effect, consider an agent that starts with a stock of durables 1 percent below its steady state, so it must decrease consumption of non durables to reach the higher level of durables in steady state. If  $1/(1+\theta)$  is close to zero, so that durables and non durables are Leontief, then on impact it will decrease consumption by 1 percent. In this case, durables and non-durables are, essentially, the same. If instead, they are good substitutes, so that  $1/(1+\theta)$  is high, the effect of durables in non-durables is smaller, and hence non-durables consumption will not decrease that much.

Q16. Assume that  $\gamma > 0$ . We will like to show that  $\frac{\bar{d}}{\bar{c}}\psi'(\bar{d})$  is decreasing in  $\frac{\gamma}{1+\theta}$ .

For this, show that  $\Delta\left(\frac{\gamma}{1+\theta}\right)$  is decreasing in  $\gamma$  provided that  $\delta > 0$ , where  $\Delta(\gamma/(1+\theta))$  is given by

$$\Delta\left(\frac{\gamma}{1+\theta}\right) \equiv \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{(-U_{cc})}.$$

A:

$$\begin{aligned}
\Delta\left(\frac{\gamma}{1+\theta}\right) &\equiv \frac{U_{cd}(\delta + \bar{v}) - U_{dd}}{(-U_{cc})} \\
&= \frac{\left(\frac{1}{d/c} - \frac{\gamma}{(1+\theta)(d/c)}\right)[\delta + \bar{v}] + \left(\frac{1}{d/c}\right)^2 + \frac{\gamma\bar{v}}{(1+\theta)(d/c)}}{1 + \frac{\gamma}{(1+\theta)\bar{v}(d/c)}} \\
&= \frac{\frac{1}{d/c}[\delta + \bar{v}] - \frac{\gamma}{(1+\theta)(d/c)}\delta + \left(\frac{1}{d/c}\right)^2}{1 + \frac{\gamma}{(1+\theta)\bar{v}(d/c)}} \\
&= \frac{\frac{1}{d/c}[\delta + \bar{v}] + \left(\frac{1}{d/c}\right)^2}{1 + \frac{\gamma}{1+\theta} \frac{1}{\bar{v}(d/c)}} - \frac{\frac{\delta}{(d/c)}}{\frac{(1+\theta)}{\gamma} + \frac{1}{\bar{v}(d/c)}} \\
&\quad \frac{\partial \Delta\left(\frac{\gamma}{1+\theta}\right)}{\partial \left(\frac{\gamma}{1+\theta}\right)} < 0
\end{aligned}$$

Q17. Argue that if  $\Delta(\gamma/(1+\theta))$  is decreasing in  $\gamma$  then  $\frac{\bar{d}}{\bar{c}}\psi'(\bar{d})$  is decreasing in  $\frac{\gamma}{1+\theta}$ .

A: Since

$$\psi'\left(\bar{d}/\bar{c}, \frac{\gamma}{1+\theta}\right) = \frac{-[\bar{v} + \delta] + \sqrt{[\bar{v} + \delta]^2 + 4\Delta\left(\frac{\gamma}{1+\theta}\right)}}{2}$$

then  $\psi'\left(\frac{\gamma}{1+\theta}\right)$  is decreasing in  $\frac{\gamma}{1+\theta}$ .

Q18. Give an intuitive interpretation of this result. (2 lines max).

A: This answer follows the answer to Q14. If  $\gamma$  is very large, the agent does not want to substitute the bundle  $h$  intertemporally, so that consumption reacts very little. Or, equivalently, if  $1/(1+\theta)$  is very large, so the agent substitutes durable and non durable easily, non durable consumption will also react by a small amount.