Theory of Income II
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(Nancy Stokey)

Notes by Takuma Habu

University of Chicago

Contents

1 Deterministic dynamic programming

1.1 9-step method

We will focus on stationary problems; i.e. return function and law of motions are time invariant.

9-step method for dynamic programming

(i) Write the sequence problem (SP)

- ▷ Choose state variables x and state space X
- ▷ Describe the feasibility set $\Gamma(x)$, return function F, and discounting β

(ii) Check basic conditions: feasible set always non-empty + discounting

(iii) Formulate the Bellman equation (BE)

(iv) Check that the Contraction Mapping Theorem (CMT) applies—check conditions for X, Γ, F

(v) Check properties of v (value function) and G (optimal policy correspondence).

- ▷ Monotonicity of v
- ▷ Concavity of v (if strict, G is single valued, in which case write g)
- ▷ Differentiability of v and g

(vi) Euler equation.

(vii) Characterise steady states. Linearise Euler equation to study local stability

(viii) Global stability

(ix) Comparative statics

1.2 Example: Renewable resource problem

1.2.1 The "givens"

- ▷ Discount factor $\beta \in (0,1)$, where $\beta = 1/(1+\rho)$ and ρ reflects time preference (not interest rate).
- ▷ Demand curve given by $p(q)$ that is stationary and satisfies law of demand (i.e. downward sloping). Utility $S(q)$ is given by the area under the demand curve:

$$S(q) = \int_0^q p(z)\, dz.$$

- ▷ State variable is x—the stock of "fish" at the beginning of the period.
- ▷ Timing: (i) Beginning-of-period stock is x; (ii) q is harvested.
- ▷ Law of motion $x_{t+1} = \psi(x_t - q_t)$ gives the next period's stock, where ψ is continuous, strictly increasing, weakly concave and differentiable with $\psi(0) = 0$.

1.2.2 Step 1: Write SP

The sequence problem is given by

$$\begin{aligned} \bar{v}(x) \coloneqq \max_{\{q_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t S(q_t) \\ s.t. \quad & x_{t+1} = \psi(x_t - q_t), \forall t \\ & q_t \in [0, x_t], \end{aligned}$$

where $[0, x_t]$ is the feasibility set, and we have discounting according to β.

In each period, the agent can consume all stock of fish or consume none. So, the feasibility set that describes the next-period stock is given by $\Gamma(x) = [\psi(0), \psi(x)] = [0, \psi(x)]$. Since ψ is continuous and strictly increasing $\varphi \coloneqq \psi^{-1}$ is well defined. We can therefore write

$$\varphi(x_{t+1}) = x_t - q_t \Leftrightarrow q_t = x_t - \varphi(x_{t+1}).$$

We can interpret φ as the amount we leave for reproduction tomorrow. Define the period-return function as

$$F(x, y) = S(x_t - \varphi(x_{t+1})),$$

where x denotes current state and y denotes next-period state.

1.2.3 Step 2: Check basic conditions

For each $x \in X$, the feasible set is $\Gamma(x)$ is nonempty. This is sufficient if F is bounded. However, if F is unbounded, some additional conditions are needed to ensure that growth is not "too fast" (relative to β).

1.2.4 Step 3: Formulate BE

$$v(x) \coloneqq \max_{y \in \Gamma(x)} \{S(x - \varphi(y)) + \beta v(y)\},$$

where $\beta v(y)$ is the discounted continuation value.

1.2.5 Step 4: Check that CMT applies

Let $C(X)$ be the space of bounded, continuous function, $\tilde{v}(x) \in C(X)$ for all $x \in X$. Define the Bellman operator as

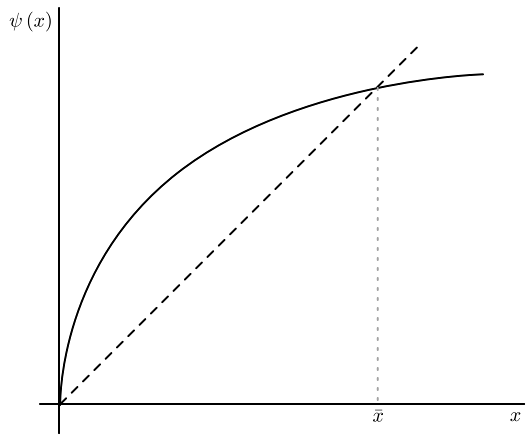
$$(T\tilde{v})(x) \coloneqq \max_{y \in \Gamma(x)} \{S(x - \varphi(y)) + \beta \tilde{v}(y)\}.$$

Suppose $\tilde{v}$ is bounded. For CMT to apply, we need that (i) $T\tilde{v}$ is bounded and continuous; (ii) T is a contraction.

- ▷ $T\tilde{v}$ is bounded:
    - ▷ Either, period return S is bounded—$\lim_{q \to \infty} S(q) \leq \bar{S}$ (S is bounded below by zero by definition);
    - ▷ Or, state space X is bounded—e.g. there exists some $\bar{x} > 0$ such that

$$\psi(x) < x, \forall x > \bar{x}.$$

We can think of this condition as saying that $\psi(x)$ crosses the 45 degree line at some finite point.[1]



Define $X = [0, \bar{x}]$ as the state space.

- $T\tilde{v}$ is continuous
  - F varies continuously with x and y—follows from the definition of S as an integral (which guarantees that S is continuous), and φ being continuous.
  - $\Gamma(x)$ varies continuously with state (i.e. Γ is continuous as a correspondence)—follows from the fact that ψ is continuous.
- T is a contraction: Check Blackwell's sufficient condition (T monotone and discounts).

We therefore established that CMT applies, which means that

- Principle of optimality holds; i.e. $v = \bar{v}$.
- T has a unique fixed point v such that $Tv = v$.
- $G(x) \subseteq \Gamma(x)$ is compact-valued and upper hemicontinuous.

1.2.6 Step 5: Check properties of v and G

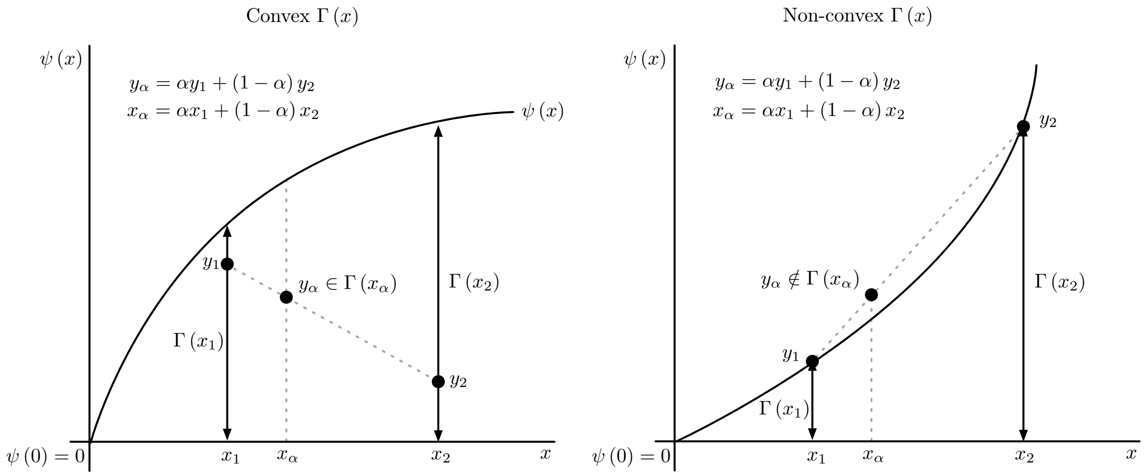
Monotonicity:

- Requirement 1. $\Gamma(x)$ monotone (i.e. $x' \geq x \Rightarrow \Gamma(x) \subseteq \Gamma(x')$)—holds, since $\Gamma(x) = [0, \psi(x)]$ and ψ is strictly increasing,
- Requirement 2. $F(x, y)$ weakly increasing in x—let $\hat{C}(X) \subset C(X)$ be the set of weakly increasing functions (so the set is closed). Want to show that $\tilde{v} \in \hat{C}(X) \Rightarrow T\tilde{v} \in \hat{C}(X)$—i.e. $T : \hat{C}(X) \to \hat{C}(X)$. Holds since the law of demand holds for $p(q)$ so that S is weakly increasing in x.[2]
- If $F(x, y)$ is strictly increasing in x, then $T\tilde{v}$ is strictly increasing (i.e. the fixed point v lies in the interior of $\hat{C}(x)$).

[1] Think of x as δk and $\psi(x)$ as $f(k)$ in the neoclassical growth model.
[2] S is strictly increasing until the demand curve "hits" the x-axis.

Concavity:

- Let $\tilde{C}(x) \subset C(x)$ be the set of functions that are weakly concave (so the set is closed). We want to show that $\tilde{v} \in \tilde{C} \Rightarrow T\tilde{v} \in \tilde{C}$—i.e. $T : \tilde{C}(X) \to \tilde{C}(X)$.
- Requirement 1. $F(x,y)$ weakly concave in (x,y)—holds since ψ is weakly concave (which implies that φ is weakly convex and $-\varphi$ is weakly concave).
- Requirement 2. $\Gamma(x)$ is convex.



- If F is strictly concave in x, then $T\tilde{v}$ is strictly concave (i.e. the fixed point v lies in the interior of $\tilde{C}(x)$). Then, there exists a unique maximiser which is continuous in x (i.e. the optimal policy correspondence G is, in fact, a single-valued function).

Differentiability at $\hat{x}$

- Requirement 1. v is strictly concave.
- Requirement 2. $\hat{x} \in \text{int}(X)$.
- Requirement 3. $g(\hat{x}) \in \text{int}(\Gamma(\hat{x}))$.
- Requirement 4. F is differentiable in x.
- We cannot use the method we used before since the set of differentiable function is not closed (with respect to the sup norm). The idea is to construct a strictly concave, differentiable W that lies everywhere below v and $w(\hat{x}) = v(\hat{x})$. Then W has the same supporting hyperplane as v so that v is differentiable at $\hat{x}$.[3] Fixing $\hat{y} = g(\hat{x})$ and if $\hat{y}$ is feasible in the neighbourhood

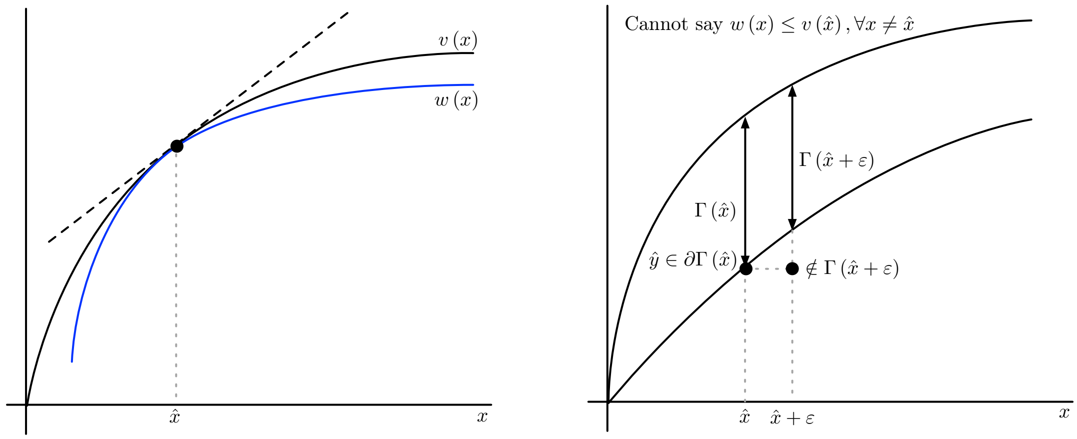
[3] At $x = \hat{x}$, the slope of w and v coincide:

$$\left.\frac{dw(x)}{dx}\right|_{x=\hat{x}} = F_x(\hat{x},\hat{y}) + \beta v'(\hat{y}) = v'(x).$$

of $\hat{x}$ (i.e. $\hat{y} \in \operatorname{int}(\Gamma(\hat{x}))$), we can use $w(x) = F(x, \hat{y}) + \beta v(\hat{y})$. Since $\hat{y}$ maximises at $\hat{x}$,[4]

$$w(x) \leq w(\hat{x}) = v(\hat{x}), \ \forall x \neq \hat{x}.$$

Notice that $\hat{y}$ may not be feasible in the neighbourhood of $\hat{x}$ if v has a kink at $\hat{x}$. In such a case, the inequality above may not hold.



▷ Differentiability of v does not give differentiability of g. That would require twice differentiability of v. The proof is a “jungle” but it does require at least twice differentiability of F in x.

1.2.7 Step 6: Euler equation

Write the first-order conditions and the envelope condition.

First-order condition The first-order condition with respect to y is given by

$$-S'(x - \varphi(y^*))\,\varphi'(y^*) + \beta v'(y^*) \lesseqqgtr 0,$$

where: (i) $\leq$ if $y^* = 0 = \psi(0)$ (i.e. $q = x$); (ii) $=$ if $y^* \in (0, \psi(x))$; and (iii) $\geq$ if $y^* = \psi(x)$ (i.e. $q = 0$). Notice that the first-order condition equates the marginal value of consumption today $-S'(\cdot)\,\varphi'(\cdot)$ against the marginal value of stock in the future $\beta v'(y^*)$.

Envelope condition Substituting in $y^* = g(x)$ into the BE:

$$v(x) = S(x - \varphi(g(x))) + \beta v(g(x)).$$

[4] By construction, $w(\hat{x}) = v(\hat{x})$. Finally,

$$\begin{aligned} v(\hat{x}) &= \max_{y \in \Gamma(\hat{x})} F(\hat{x}, y) + \beta v(y) = F(\hat{x}, \hat{y}) + \beta v(\hat{y}) \\ &\geq F(x, \hat{y}) + \beta v(\hat{y}) = w(x) \end{aligned}$$

for all x in the neighbourhood of $\hat{x}$ such that $\hat{y}$ is feasible.

Differentiating above with respect to x, while noting that the Envelope Theorem implies that we need not consider derivatives with respect to $g(x)$:

$$v'(x) = S'(x - \varphi(g(x))).$$

Euler equation Euler equation is given by combining the first-order and the envelope conditions. From the envelope condition:

$$v'(g(x)) = S'(g(x) - \varphi(g(g(x)))).$$

Substituting this into the first-order condition gives the Euler equation:

$$S'(x - \varphi(g(x)))\varphi'(g(x)) = \beta S'(g(x) - \varphi(g(g(x)))).$$

Corner solutions We can think about what conditions would rule out corner solutions.

Consider first the case $y^* = g(x) = 0$. This is optimal if

$$S'(x)\varphi'(0) \geq \beta S'(g(x) - \varphi(g(g(x)))) = \beta S'(-\varphi(g(0))).$$

where we also used the fact that $\varphi(0) = 0$ and $g(x) = 0$. If the inequality does not hold, then we can rule out this corner solution.

Now consider the case $y^* = g(x) = \psi(x)$. This is optimal if

$$\begin{aligned} S'(x - \varphi(\psi(x)))\varphi'(\psi(x)) &\leq \beta S'(\psi(x) - \varphi(g(\psi(x)))) \\ \Leftrightarrow S'(0)\varphi'(\psi(x)) &\leq \beta S'(\psi(x) - \varphi(g(\psi(x)))) \end{aligned}$$

where we used the fact that $\varphi = \psi^{-1}$. Again, if the inequality does not hold, then we can rule out this corner solution. For example, if we have an Inada condition so that $S'(0) = +\infty$, then we can rule out this corner solution.

Remark 1.1. Look up 2015 Mid-term review question 7 ("Commodity prices") which describes a similar set up with stochastic endowment.

1.2.8 Step 7: Characterise steady states

Steady state: Interior In the steady state, $x = g(x) = \bar{x}$, then the Euler equation becomes

$$S'(\bar{x} - \varphi(\bar{x}))\varphi'(\bar{x}) = \beta S'(\bar{x} - \varphi(\bar{x})),$$

which simplifies to

$$\varphi'(\bar{x}) = \beta.$$

Above pins down the steady-state value of stock. This can be seen as a rate of return condition—the steady stock is such that the reproduction rate offsets discounting.

Steady state: Corner solutions Steady states at corner solution are possible. If

$$S'(x)\varphi'(0) > \beta s'(0),$$

then

$$\bar{x} = 0.$$

Similarly, if

$$S'(0)\,\varphi'(\psi(x)) < \beta S'(\psi(x) - \varphi(\psi(x)))$$

then $\bar{x} = \psi(\bar{x})$.

1.2.9 Step 8: Global stability

N/A.

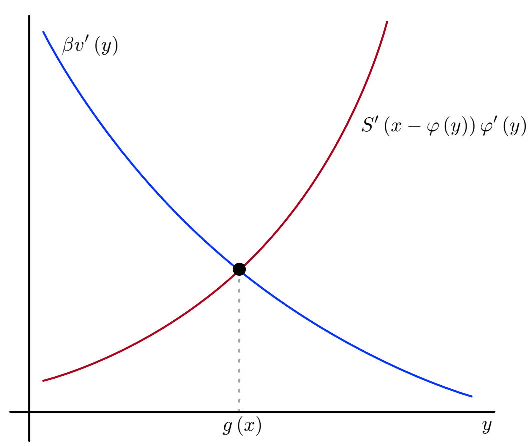
1.2.10 Step 9: Comparative statics

Recall that we are not guaranteed that v is twice differentiable or that $g(x)$ is differentiable. So we need to conduct comparative statics without taking derivatives.

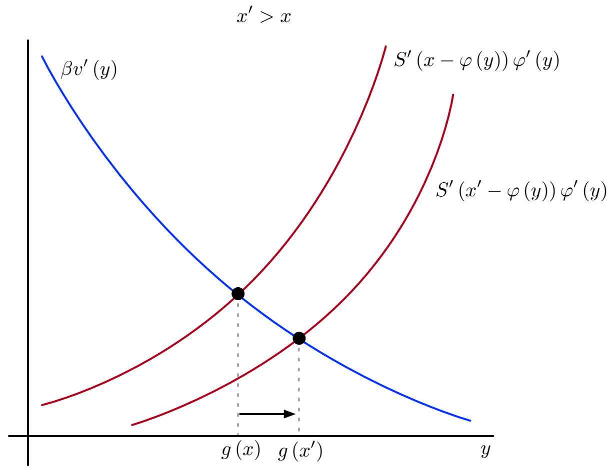
Notice that, from the first-order condition, $y^* = g(x)$ is given by the intersection of $\beta v'(y)$ and $S'(x - \varphi(y))\,\varphi'(y)$, and

- ▷ $\beta v'(\cdot)$ is strictly decreasing in y since v is strictly concave;
- ▷ $S'(\cdot)\,\varphi'(\cdot)$ is increasing in y since: (i) $-\varphi$ is strictly decreasing in y, and we assume that S is strictly concave; (ii) $S' > 0$ since S is (assumed to be) strictly increasing; and (iii) φ is strictly increasing and strictly convex so that $\varphi' > 0$ and φ' is increasing in y.

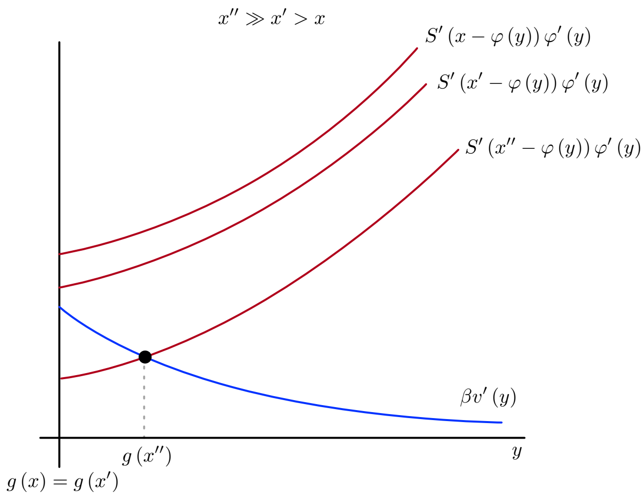
We can plot the two curves.

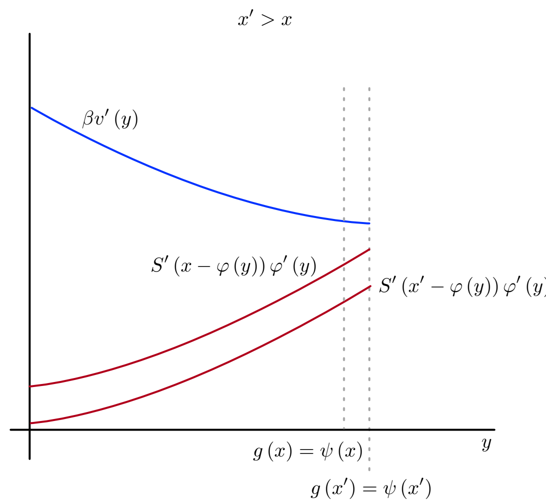


How would g change if stock today was higher; i.e. $x' > x$. Since S is strictly concave, a higher x implies S' is lower; i.e. the locus $S'(\cdot)\,\varphi'(\cdot)$ moves down/right. $\beta v'(\cdot)$ is unaffected by x. As can be seen from the figure, this implies that $g(x)$ is increasing in x.



Note that if $g(x)$ is a corner solution, $g(x)$ may not be strictly increasing with x.





- ▷ $g(x)$ is at the lower bound zero (figure on the left). The initial value of state is x.
  - ▷ Suppose x increases to x'. The figure shows the case in which the two curves do not cross at x' so that $g(x) = g(x')$ and g is not strictly increasing. However, observe that $S'(x'-\varphi(y))\varphi(y) < S'(x-\varphi(y))\varphi(y)$ since y remains unchanged. Since S is strictly concave, S' is decreasing so that

$$q' = x' - \varphi(y) > x - \varphi(y) = q,$$

    where we used the fact that $x - \varphi(y)$ equals consumption, q. Hence, we realise that consumption is increasing in x.
  - ▷ Suppose x increases to x'' such that the two curves now intersect. Then, as we can see from the figure, $g(x'') > g(x)$.
- ▷ $g(x)$ is at the upper bound $\psi(x)$ (figure on the right). The initial value of state is x. In this case, an increase in state from x to x' increases the upper bound from $\psi(x)$ to $\psi(x')$ so that $g(x') > g(x)$.

Remark. In general, the lower bound could also move but we may not have monotonicity of $\Gamma(x)$ in such a case.

1.3 Prescott, AER (2002): Prosperity and Depression

1.3.1 Motivation

Facts to explain:

- ▷ *growth rates are similar* across developed countries
- ▷ however, absolute levels (e.g. output per person) differ

Thesis: Can differences in absolute levels be explained by higher tax rates and/or differences in productivity?

1.3.2 Model

We want to use a growth model to develop a system of accounting for differences in output per working-age person.

Preferences A unit measure of households have preferences over consumption and leisure with utility function that permits balanced growth path:

$$u\left(\{c_t, h_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \left(\frac{1+n_i}{1+\rho}\right)^t \left[\ln(c_t) + \alpha \ln(1-h_t)\right].$$

Size of household grows at a constant rate $n_i \in [0, \rho)$, which may differ across countries. Assume that ρ, α and utility function is the same across countries. In particular, this means that discount factor across countries can only differ due to population growth, not due to time preferences (ρ).

Technology Representative firm with constant returns to scale technology:

$$F(K, H) = (A_i H)^{1-\theta} K^{\theta}, \tag{1.1}$$

where H and K denote aggregate labour supply and capital respectively, and labour-augmenting productivity A_i may differ across countries. Production function and θ are assumed to be the same across countries.

Government Taxes levied on consumption, labour and capital income at rates τ_{ic}, τ_{ih} and τ_{ik} respectively, which is assumed to be constant over time (but differ across countries). All tax revenue is assumed to be rebated to the households as lump-sum transfer T. Government purchases are implicitly assumed to be a perfect substitute for consumption good (more likely to hold for government purchases that are at the margin).

Factor markets Markets are assumed to be competitive. Let w and r be the wage and rental rates respectively.

1.3.3 Setting up the recursive problem

The household solves the following sequence problem

$$\begin{aligned}
\max_{\{c_t,,k_{t+1}\geq 0,h_t\in[0,1]\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty}\left(\frac{1+n_i}{1+\rho}\right)^t\left[\ln\left(c_t\right)+\alpha\ln\left(1-h_t\right)\right] \\
s.t. \quad & \left(1+\tau_{ic}\right)c_t+\left(\left(1+n_i\right)k_{t+1}-k_t\right) \\
& =\left(1-\tau_{ih}\right)w_t h_t+\left(1-\tau_{ik}\right)\left(r_t-\delta\right)k_t+T_t, \forall t.
\end{aligned}$$

We have (distortionary) taxes in the model, which means that the welfare theorems do not apply; i.e. competitive equilibrium will not be Pareto optimal and neither can we solve the planner's problem to obtain a competitive equilibrium allocation.

We solve the problem by way of a *recursive competitive equilibrium*. This involves setting up a recursive problem supposing that individual households have conjectures about aggregate variables, which they take as given. We then impose an equilibrium condition (also called "representativeness" condition) to ensure that households' belief/conjectures are correct. Implicitly, we are assuming that the households are individually too small to affect the aggregate (e.g. assume a measure one of households).

Here, the state variable is capital—individual capital k which the household choose, as well as the capital held by all others in the economy, K. The pair (k, K) is the state variable for the household's recursive problem. Define the following conjectures:

- ▷ next-period per-capita stock of capital held by others: $K' = G_K(K)$;
- ▷ per-capital labour supplied by others: $G_H(K)$;
- ▷ per-capita consumption by others: $G_C(K)$;
- ▷ transfers $T(K)$;
- ▷ wage rate $w(K)$ and rental rate $r(K)$.

Let $\mathbf{G}$ denote the collection of all conjectures (which include the functions themselves).

Solving the firm's problem gives

$$\begin{aligned}
w\left(K\right) &= F_H\left(K, G_H\left(K\right)\right)=\left(1-\theta\right)A_i^{1-\theta}\left(\frac{K}{G_H\left(K\right)}\right)^{\theta}, \\
r\left(K\right) &= F_K\left(K, G_H\left(K\right)\right)=\theta A_i^{1-\theta}\left(\frac{K}{G_H\left(K\right)}\right)^{-(1-\theta)}.
\end{aligned} \tag{1.2}$$

We can now define the recursive problem for an individual household as

$$\begin{aligned}
v\left(k, K;\mathbf{G}\right) &= \max_{\{c,h\in[0,1].k'\}}\ln c+\alpha\ln\left(1-h\right)+\frac{1+n_i}{1+\rho}v\left(k', G_K\left(K\right);\mathbf{G}\right) \\
s.t. \quad & c=\frac{1}{1+\tau_{ic}}\left(k+\left(1-\tau_{ik}\right)\left(r\left(K\right)-\delta\right)k-\left(1+n_i\right)k'+\left(1-\tau_{ih}\right)w\left(K\right)h+T\left(K\right)\right)
\end{aligned} \tag{1.3}$$

We can verify that above is a well-defined Bellman equation. Suppose, for now, that taxes and transfers are zero, then define the one-period return function as

$$F\left(k, k'\right)=\max_{h\in[0,1]}\left\{\ln\left(rk+wh+\left(1-\delta\right)k-\left(1+n_i\right)k'\right)+\alpha\ln\left(1-h\right)\right\} \tag{1.4}$$

for $k > 0$ and

$$k' \in \Gamma(k) := \left[0, \frac{rk + wh + (1-\delta)k}{1+n_i}\right],$$

where the lower bound is obtained by assuming that all capital stock is consumed, while the upper bound is obtained by assuming that consumption is zero. Notice that $\Gamma(k)$ is nonempty, monotone and convex.

Since (1.4) is continuous in h, and the feasible set for h is compact, by the Weirstrass Theorem, we know that a maximum exists—i.e. F is bounded. Moreover, since the right-hand side is strictly concave in h, we know that the maximiser is unique.

$$\begin{aligned}
\frac{\partial RHS}{\partial h} &= \frac{w}{rk + wh + (1-\delta)k - (1+n_i)k'} - \frac{\alpha}{1-h,} \\
\frac{\partial^2 RHS}{\partial h^2} &= -\frac{w^2}{(rk + wh + (1-\delta)k - (1+n_i)k')^2} - \frac{\alpha}{(1-h)^2} < 0.
\end{aligned}$$

By the Theorem of the Maximum, we know that F is continuous. We can also see that $F(k, k')$ is strictly increasing in k and strictly decreasing in k'. To verify that F is concave, choose k_i and $k'_i \in \Gamma(k_i)$ for $i = 1, 2$, and let h_i be the maximiser for (k_i, k'_i). Fix $\lambda \in (0, 1)$ and define the weighted averages

$$x_\lambda = \lambda x_1 + (1-\lambda) x_2,$$

for $x = k, k', h$. Since $\Gamma(k)$ is convex, $k'_\lambda \in \Gamma(k)$. Now, define the function

$$\phi(k, k', h) := \ln(rk + wh + (1-\delta)k - (1+n_i)k') + \alpha \ln(1-h).$$

We already verified that this is strictly concave in h. It is also strictly concave in k and k':

$$\begin{aligned}
\frac{\partial \phi(k, k', h)}{\partial k} &= \frac{r + (1-\delta)}{rk + wh + (1-\delta)k - (1+n_i)k'}, \\
\frac{\partial^2 \phi(k, k', h)}{\partial k^2} &= -\frac{(r + (1-\delta))^2}{(rk + wh + (1-\delta)k - (1+n_i)k')^2} < 0, \\
\frac{\partial \phi(k, k', h)}{\partial k'} &= -\frac{1+n_i}{rk + wh + (1-\delta)k - (1+n_i)k'}, \\
\frac{\partial^2 \phi(k, k', h)}{\partial (k')^2} &= -\frac{(1+n_i)^2}{(rk + wh + (1-\delta)k - (1+n_i)k')^2} < 0.
\end{aligned}$$

Since $F(k_\lambda, k'_\lambda)$ is the maximised value,

$$\begin{aligned}
F(k_\lambda, k'_\lambda) &\geq \lambda F(k_1, k'_1) + (1-\lambda) F(k_2, k'_2) \\
&= \lambda \phi(k_1, k'_1, h_1) + (1-\lambda) \phi(k_2, k'_2, h_2)
\end{aligned}$$

which holds with equality if and only if

$$(k_1, k'_1, h_1) = (k_2, k'_2, h_2) \Leftrightarrow (k_1, k'_1) = (k_2, k'_2),$$

because h is uniquely maximised. Therefore, we may conclude that F is strictly concave.

1.3.4 Solving the problem

Substituting the expression for consumption, the first-order and the envelope conditions are given by

$$\begin{aligned}
\{h\} \quad & \frac{1-\tau_{ih}}{1+\tau_{ic}} w(K) \frac{1}{c} = \frac{\alpha}{1-h}, && (1.5)\\
\{k'\} \quad & \frac{1+n_i}{1+\tau_{ic}} \frac{1}{c} = \frac{1+n_i}{1+\rho} v_k\left(k', G_K(K); \mathbf{G}\right), && (1.6)\\
\{EC\} \quad & v_k(k, K; \mathbf{G}) = \frac{1}{c} \frac{1}{1+\tau_{ic}} \left[1 + (1-\tau_{ik})(r(K) - \delta)\right]. && (1.7)
\end{aligned}$$

Combining (1.6) and (1.7) gives the familiar Euler equation:

$$\begin{aligned}
\frac{1+n_i}{1+\tau_{ic}} \frac{1}{c} &= \frac{1+n_i}{1+\rho} \frac{1}{c'} \frac{1}{1+\tau_{ic}} \left[1 + (1-\tau_{ik})(r(G_K(K)) - \delta)\right] \\
\Rightarrow \frac{c'}{c} &= \frac{1}{1+\rho} \left[1 + (1-\tau_{ik})(r(G_K(K)) - \delta)\right], \qquad (1.8)
\end{aligned}$$

which states that consumption growth equals the net-of-tax return on capital.

Denote the household's optimal policy functions for next-period capital, labour and consumption respectively as

$$g(k, K; \mathbf{G}), \ \eta(k, K; \mathbf{G}), \ \gamma(k, sK; \mathbf{G}).$$

We now impose the recursive equilibrium condition (the representativeness condition) that household's conjectures are correct. That is, for all k (remember, there is measure one of household),

$$\begin{aligned}
K &= k, \\
G_K(k) &\equiv g(k, k; \mathbf{G}), \\
G_H(k) &\equiv \eta(k, k; \mathbf{G}), \\
G_C(k) &\equiv \gamma(k, k; \mathbf{G}).
\end{aligned}$$

The equilibrium condition differs from the steady-state condition in that they must hold at all times—even in transition periods. Equilibrium wages, rental rates and transfers are given by

$$\begin{aligned}
w(k) &= (1-\theta) A_i^{1-\theta} \left(\frac{k}{G_H(k)}\right)^{\theta}, \\
r(k) &= \theta A_i^{1-\theta} \left(\frac{k}{G_H(k)}\right)^{-(1-\theta)}, \\
T(k) &= \tau_{ic} G_C(k) + \tau_{ih} w(k) G_H(k) + \tau_{ik} k (r(k) - \delta).
\end{aligned}$$

1.3.5 Steady state

Steady state condition:

$$k^{ss} = k' = k,$$

which implies that

$$\begin{aligned} k^{ss} &= k' = g\left(k^{ss}, k^{ss}; \mathbf{G}\right) = k \\ h^{ss} &= \eta\left(k^{ss}, k^{ss}; \mathbf{G}\right), \\ c^{ss} &= \gamma\left(k^{ss}, k^{ss}; \mathbf{G}\right), \\ w^{ss} &= w\left(k^{ss}\right), \\ r^{ss} &= r\left(k^{ss}\right). \end{aligned}$$

Substituting the condition into the Euler equation gives

$$\begin{aligned} 1 &= \frac{1}{1+\rho}\left[1 + \left(1-\tau_{ik}\right)\left(r^{ss}-\delta\right)\right] \\ \Rightarrow r^{ss} &= \frac{\rho}{1-\tau_{ik}} + \delta. \end{aligned}$$

We refer to $1-\tau_{ik}$ as the *intertemporal tax wedge.*

1.3.6 Calibration

We can use data on tax rates to ask whether the model fits other countries.

We want to use observables (for the US) to calibrate α, θ, δ and ρ. Information on θ (capital share of output), δ (depreciation) and ρ are generally available. How do we obtain α (relative preference towards leisure)?

From the intratemporal condition, (1.5), we have

$$\begin{aligned} \frac{1-\tau_{ih}}{1+\tau_{ic}}\frac{w^{ss}}{c^{ss}} &= \frac{\alpha}{1-h^{ss}} \\ \Leftrightarrow 1-h^{ss} &= \frac{1+\tau_{ic}}{1-\tau_{ih}}\frac{\alpha c^{ss}}{w^{ss}} \end{aligned} \tag{1.9}$$

We refer to $\left(1-\tau_{ih}\right)/\left(1+\tau_{ic}\right)$ as the *intratemporal tax wedge.* Note that per-capita output is given by

$$y^{ss} = \left(Ah^{ss}\right)^{1-\theta}\left(k^{ss}\right)^{\theta}.$$

From the firm's first-order condition, (1.2), and the steady-state condition, we can express the steady-state wage rate as

$$w^{ss} = \left(1-\theta\right)A^{1-\theta}\left(\frac{k^{ss}}{h^{ss}}\right)^{\theta} = \left(1-\theta\right)\frac{\left(Ah^{ss}\right)^{1-\theta}\left(k^{ss}\right)^{\theta}}{h^{ss}} = \left(1-\theta\right)\frac{y^{ss}}{h^{ss}}.$$

Substituting into (1.9) gives

$$\begin{aligned} 1-h^{ss} &= \frac{1+\tau_{ic}}{1-\tau_{ih}}\frac{\alpha c^{ss}}{\left(1-\theta\right)\frac{y^{ss}}{h^{ss}}} \\ \Leftrightarrow \alpha &= \frac{1-h^{ss}}{h^{ss}}\frac{1-\tau_{ih}}{1+\tau_{ic}} - \theta\left(\frac{c^{ss}}{y^{ss}}\right)^{-1}. \end{aligned} \tag{1.10}$$

We can use the expression above to recover α.

1.3.7 Conclusion

Rearranging (1.10) gives

$$\frac{1-h^{ss}}{h^{ss}}=\frac{1+\tau_{ic}}{1-\tau_{ih}}\frac{\alpha}{1-\theta}\left(\frac{c^{ss}}{y^{ss}}\right),$$

which says that the ratio of leisure to labour supply depends on the intratemporal tax wedge (since c^{ss}/y^{ss} does not depend on the intratemporal tax wedge).

We can express the last ratio in terms of capital to output ratio. Since all tax revenues are refunded to the household, household's income net of transfer and depreciation (i.e. replacement investment in the steady state) gives consumption. To see this, we can use the expression for c in (1.3),

$$\begin{aligned}c^{ss}&=\frac{1}{1+\tau_{ic}}\left(k^{ss}+\left(1-\tau_{ik}\right)\left(r^{ss}-\delta\right)k^{ss}-\left(1+n_i\right)k^{ss}+\left(1-\tau_{ih}\right)w^{ss}h^{ss}\right.\\&\qquad\left.\tau_{ic}c^{ss}+\tau_{ih}w^{ss}h^{ss}+\tau_{ik}k^{ss}\left(r^{ss}-\delta\right)\right)\\&=\frac{1}{1+\tau_{ic}}\left(\tau_{ic}c^{ss}+r^{ss}k^{ss}+w^{ss}h^{ss}-\left(n_i+\delta\right)k^{ss}\right)\\&=r^{ss}k^{ss}+w^{ss}h^{ss}-\left(n_i+\delta\right)k^{ss}.\end{aligned}$$

Since technology is CRS, we have

$$r^{ss}k^{ss}+w^{ss}h^{ss}=F_Kk^{ss}+F_Nh^{ss}=F\left(k^{ss},h^{ss}\right)=y^{ss}.$$

Hence,

$$\begin{aligned}c^{ss}&=y^{ss}-\left(n_i+\delta\right)k^{ss}\\\Rightarrow\frac{c^{ss}}{y^{ss}}&=1-\left(n_i+\delta\right)\frac{k^{ss}}{y^{ss}}.\end{aligned}$$

So,

$$\frac{1-h^{ss}}{h^{ss}}=\frac{1+\tau_{ic}}{1-\tau_{ih}}\frac{\alpha}{1-\theta}\left(1-\left(n_i+\delta\right)\frac{k^{ss}}{y^{ss}}\right)$$

Thus, we see that variation in the labour supply (i.e. the left-hand side) across countries depends on: (i) intratemporal tax wedge; (iii) population growth rate; and (iv) capital stock to output ratio.

Table summarises the result from the calibrated model.

	% relative to the US			
Country	GDP	Productivity (A)	Capital factor (k/y)	Labour factor (h)
France	−31	6	1	−37
Japan	−31	−33	3	−1
United Kingdom	−41	−29	2	−13

	Average tax rates and tax wedge		
Country	τ_c	τ_h	Intratemporal tax wedge
France	33	49	2.60
United Kingdom	26	31	1.82
United States	13	32	1.66

▷ Capital factor is not important in explaining cross-country differences in GDP per capita.

- ▷ France vs US: Labour factor is the major reason why France is depressed relative to the US—France's intertemporal tax wedge is significantly higher relative to the US (2.60 vs 1.66).
- ▷ UK vs US: Both labour and productivity factor explains why UK is depressed relative to the US—UK's tax wedge is slightly higher than US (1.82 vs 1.66).
- ▷ Japan vs US: Productivity is the major reason why Japan is depressed relative to the US.
- ▷ More generally, "productivity accounts for much of the current differences in income across the OECD countries today and changes in relative incomes of these countries over time."

2 Stochastic dynamic programming

We wish to study the Bellman equation of the form

$$v(x,z) = \max_{y\in\Gamma(x,z)} \{F(x,y,z) + \beta\mathbb{E}[v(y,z')|z]\}, \tag{2.1}$$

where x are the endogenous state variables (as in the discrete case) while z are the exogenous (random) state variables.

Definition 2.1. (Time-homogenous/Stationary Markov chains). A stochastic process is a stationary Markov chain if

$$\mathbb{P}(x_{t+1} = s_j|x_t = s_i) = \mathbb{P}(x_t = s_j|x_{t-1} = s_i),\ \forall t.$$

Markov process fits the recursive nature of dynamic programming. Note that we can always redefine the state space and express n-order Markov process as first-order.

There are two basic cases to consider.

Discrete shocks State space $Z = \{z_1, z_2, \ldots\}$ is countable and transition probabilities are described by a matrix

$$Q = [q_{ij}],\ q_{ij} \geq 0,\ \sum_{j=1}^{\infty} q_{ij} = 1,\ \forall i.$$

The transition probability q_{ij} describes the probability of state z_j occurring in the next period, given current state of z_i.

$Z \subseteq \mathbb{R}^m$ is a closed rectangle In other words, Z is a product of closed intervals. For each $z \in Z$, let $q(z'|z)$ be the density for the shocks next period. In this case, we need an additional assumption to ensure that the value function is continuous.

Definition. (Feller property) The conditional density $q(z'|z)$ is said to have the Feller property if, for any fixed $z' \in Z$, $q(z'|z)$ is continuous in z.

Theorem 2.1. *Let $h(z)$ be a continuous function defined on Z and that $q(z'|z)$ satisfies the Feller property. Then,*

$$(\mathbb{M}h)(z) = \int h(z')\, q(z'|z)\, dz' = \mathbb{E}[h(z')|z]$$

is continuous on Z.

Thus, the Feller property, in addition to the conditions required in the discrete dynamic programming case (F varies continuously with x and y, and $\Gamma(x,z)$ varies continuously with x for any fixed z), guarantees that the Bellman operator is continuous. Note that, if Z is discrete, then the Feller property is trivially satisfied. For this continuous Bellman operator to be well-defined, we also need it to be bounded and for it to be a contraction (as in the deterministic case).

The arguments for monotonicity, continuity, and differentiability of v and G in x are the same as in the discrete case (the must hold for any fixed $z \in Z$).

To show that $v(x,z)$ is increasing in z (not x!), assume, in addition:

- ▷ F is increasing in z;
- ▷ $\Gamma(x,z)$ is monotone in z (i.e. $\hat{z} \geq z \Rightarrow \Gamma(x,z) \subseteq \Gamma(x,\hat{z})$)

▷ transition function is monotone; i.e. conditional expectation of h, $\mathbb{M}h$, is increasing in z if $h(z)$ is increasing in z. In the case $z \in \mathbb{R}$, this condition is equivalent to first-order stochastic dominance; i.e. $\hat{z} \geq z \Rightarrow F(z'|\hat{z}) \leq F(z'|z)$.

We can interpret the last condition as a requirement that ensures that a higher shock today implies a "better" distribution tomorrow.

We often set up the problem of the following form:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_Z v[\phi(x,y,z'),z']\, Q(z,dz') \right\},$$

where ϕ is the law of motion of the endogenous state variable. For example, we can think of x as the current level of assets, y as the amount of saving and z' as the next-period interest rate which, together with the current level of assets and savings, determine the next-period level of assets.

2.1 Real Business Cycle "facts"

2.1.1 Lucas

(i) Length and height of business cycles differ.

(ii) Output in all sectors move together (i.e. highly correlated).

(iii) Production of durables fluctuates more than production of non-durables, services even less.

(iv) Prices are procyclical.

(v) Short-term interest rates are procyclical, long-term interest rates are only slightly procyclical. Money velocity is procyclical (may not be true anymore).

2.1.2 Cooley and Prescott (FCBR)

(i) Fluctuations in *output* and *hours worked* are *similar.*

(ii) Fluctuations in *employment* and *output* are *similar*, but fluctuations in *weekly hours* worked are *less* (suggests labour adjusts at the extensive margin).

(iii) *Consumption* fluctuates *less* than output.

(iv) *Investment* fluctuates *more* than output.

(v) Fluctuations in *capital stock* is *less* and *independent* of output.

(vi) Productivity is slightly procyclical.

(vii) Wages fluctuations are small.

(viii) Government expenditure is independent of output.

(ix) Imports are more procyclical than exports.

2.2 RBC model

Should government respond proactively to business cycles? The canonical RBC model says no—i.e. business cycles are an outcome of efficient response to productivity shocks.

2.2.1 The model

- ▷ Representative household with preferences given by $u(c,h)$.
- ▷ Representative firm with CRS technology $y = zf(k,h)$.
- ▷ Stochastic process $z \in Z$ with transition density $q(z'|z)$. Assume that the transition function has the Feller property and that it is monotone.
- ▷ Since the shocks are aggregate shocks, households cannot insure against shocks even if the markets were complete.
- ▷ General observation: Shocks tend to be very persistent—AR(1) coefficient of around 0.9 to 0.95.
- ▷ No taxes—we can solve the planner's problem.
- ▷ No money.

The Bellman equation is given by

$$\begin{aligned} v(k,z) = \max_{c,h,k'} \quad & u(c,1-h) + \beta \int v(k',z')\, q(z'|z)\, dz' \\ s.t. \qquad & c + k' - e^z F(k,h) - (1-\delta)k \leq 0. \end{aligned}$$

The goal of RBC model was to create a model for which (c,h,k') move together with z, which can be interpreted as total factor productivity shock. Note that this is a *supply-side* shock. Why not a demand shock? The problem is that demand shocks will not produce a result in which c and k' move together—since income is not higher with higher demand, higher consumption implies lower saving, i.e. k'.

2.2.2 FOC and EC

Letting λ denote the Lagrange multiplier on the feasibility constraint, the first-order conditions are

$$\{c\} \qquad u_c(c,1-h) = \lambda, \tag{2.2}$$

$$\{h\} \qquad u_\ell(c,1-h) = \lambda e^z F_h(k,h), \tag{2.3}$$

$$\{k'\} \quad \beta \mathbb{E}\left[v_k(k',z')\,|z\right] = \lambda. \tag{2.4}$$

The envelope condition is

$$v_k(k,z) = \lambda\left(e^z F_k(k,h) + (1-\delta)\right). \tag{2.5}$$

2.2.3 Inelastic labour supply

We focus on the interior solution case. To start with, suppose that labour supply is inelastic; i.e. $h = \bar{h}$. Then, combining (2.2) and (2.4) gives the Euler equation

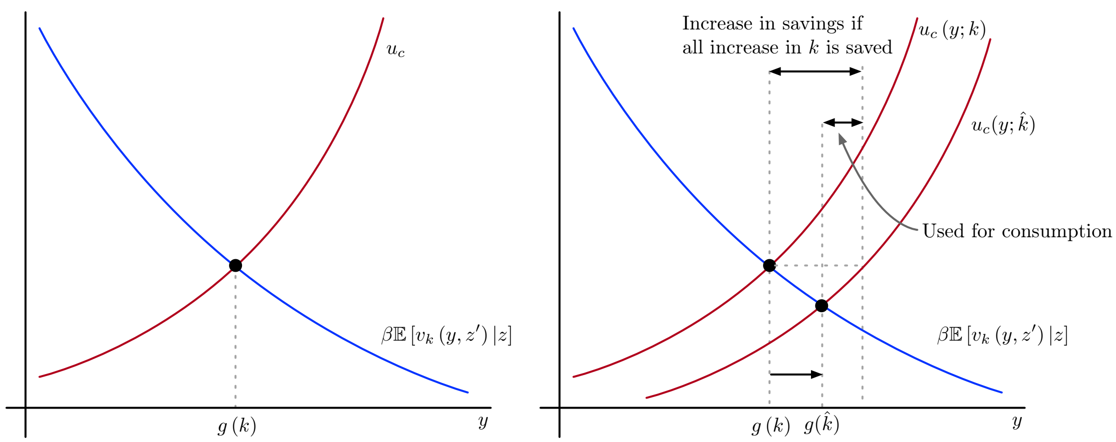
$$\begin{aligned} \beta \mathbb{E}\left[v_k(k',z')\,|z\right] &= u_c(c,1-h) \\ &= u_c\left(e^z F(k,h) + (1-\delta)k - k', 1-h\right). \end{aligned} \tag{2.6}$$

As usual the Euler equation equates the marginal (discounted) expected utility from investment with marginal utility from consumption.

From (2.6), we can see that k' is given by the intersection of two functions, expressed as functions of y;

$$\beta \mathbb{E}\left[v_k\left(y, z'\right)|z\right],$$
$$u_c\left(e^z F\left(k, h\right) + \left(1-\delta\right)k - y\right).$$

Given the assumptions, we know that v is a strictly increasing, concave function. Thus, v_k is positive and strictly decreasing in y. On the other hand, u_c is strictly positive and increasing in y. Thus, there can be at most one solution. The solution exists since we are maximising a continuous function in a compact set. Full Inada conditions would guarantee that the solution is interior.



What happens to $g\left(k\right)$ (i.e. capital next period) if current capital stock, k, increases from k to $\hat{k}$? Then only the u_c curve moves—since F is increasing in k and utility function is strictly concave, a higher k shifts the u_c curve to down/right. As the figure shows, $g\left(k\right)$ increases to $g(\hat{k})$. However, notice that the change in $g\left(k\right)$ is less than the horizontal movement of the u_c curve. If household were to save all of the increase in k (in the form of higher $g\left(k\right)$), then the movement in the curve would coincide with the movement of $g\left(k\right)$. The fact that it moves by less means that households are, in fact, consuming some of the higher stock of capital today. Letting $\Delta k = \hat{k} - k$, we have that

$$\left(e^z F_k + 1 - \delta\right)\Delta k < \Delta g\left(k\right).$$

What happens to $g\left(k\right)$ if current shock ($\hat{z}$) were higher? Since u_c is increasing in z, as in the case with an increase in k, u_c curves moves right/down. However, with a higher z, $\beta\mathbb{E}\left[v_k|z\right]$ could also shift (assuming shocks are not iid).

Empirical data suggests that positive technology shocks such as those represented by y has a positive effect (i.e. a higher shock implies a better distribution tomorrow), implying that CDFs (i.e. transition functions) are ordered by FOSD. This, in turn, implies that v is increasing in z; however, we do not know how v_k changes with z (nor do we know how the policy function changes with z). This is because a "better" distribution has two effects.

On the one hand, a better distribution in the future incentivises households to save *less*, but, on the other hand, a higher shock today means more income today, which incentivises households to save *more*. The net effect is therefore ambiguous.

Data suggests that a positive shock increases both savings and capital, meaning that the movement of the $\beta\mathbb{E}[\cdot]$ curve should be small (either to the left, or to the right).

2.2.4 Elastic labour supply

Now we suppose that labour supply is elastic.

Fix (k, z). For any fixed (potential) choice of k', write $c^*(k'; k, z)$ and $h^*(k'; k, z)$ as the choices that maximises current utility then we consider $u_c(c^*(k'; k, z), h^*(k'; k, z))$; i.e.

$$\max_{c,h} \quad u(c, 1-h) \quad s.t. \quad c + k' - e^z F(k, h) - (1 - \delta k) = 0,$$

which is equivalent to

$$\max_{h} \quad u(e^z F(k, h) + (1-\delta) k - k', 1 - h).$$

The first-order conditions imply that

$$\begin{aligned} u_c(c, 1-h) e^z F_h(k, h) &= u_\ell(c, 1-h) \\ \Rightarrow \frac{u_\ell(c, 1-h)}{u_c(c, 1-h)} &= e^z F_h(k, h), \end{aligned} \tag{2.7}$$

where we can interpret the right-hand side as the marginal product of labour. To make headway, specialise the functions to

$$\begin{aligned} u(c, 1-h) &= \theta \ln c - \frac{1}{1+\gamma} h^{1+\gamma}, \\ F(k, h) &= A k^\alpha h^{1-\alpha}. \end{aligned}$$

Then, (2.7) becomes

$$\begin{aligned} \frac{h^\gamma}{\theta / c} &= e^z (1-\alpha) A \left(\frac{k}{h}\right)^\alpha \\ \Rightarrow c &= \theta e^z (1-\alpha) A k^\alpha h^{-(\alpha+\gamma)}. \end{aligned} \tag{2.8}$$

We can solve for $c^*(k')$ and $h^*(k')$ using the expression above and the feasibility condition:

$$\begin{aligned} c &= e^z F(k, h) + (1-\delta) k - k' \\ &= e^z A k^\alpha h^{1-\alpha} + (1-\delta) k - k'. \end{aligned} \tag{2.9}$$

Equating the two

$$\begin{aligned} (1-\delta) k - k' &= \theta e^z (1-\alpha) A k^\alpha h^{-(\alpha+\gamma)} - e^z A k^\alpha h^{1-\alpha} \\ &= e^z A k^\alpha \left[\theta (1-\alpha) h^{-(\alpha+\gamma)} - h^{1-\alpha}\right] \end{aligned}$$

Differentiating h with respect to k', then

$$\begin{aligned}
-1 &= e^z A k^\alpha \left[-\theta(1-\alpha)(\alpha+\gamma) h^{-(1+\alpha+\gamma)} \frac{\partial h(k')}{\partial k'} - (1-\alpha) h^{-\alpha} \frac{\partial h(k')}{\partial k'}\right] \\
&= -(1-\alpha)\left[\theta(\alpha+\gamma) h^{-(1+\alpha+\gamma)} + h^{-\alpha}\right] \frac{\partial h(k')}{\partial k'} \\
\Rightarrow \frac{\partial h^*(k';k,z)}{\partial k'} &> 0.
\end{aligned}$$

Rearranging (2.9) gives

$$h = \left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}} \tag{2.10}$$

$$\Rightarrow c = \theta e^z (1-\alpha) A k^\alpha \left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{-\frac{\alpha+\gamma}{1-\alpha}}$$

$$\begin{aligned}
\Rightarrow \ln c = \ln \theta e^z (1-\alpha) A + \alpha \ln k - \frac{\alpha+\gamma}{1-\alpha} \ln(c-(1-\delta)k+k') \\
+ \frac{\alpha+\gamma}{1-\alpha} \ln(e^z A k^\alpha)
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\frac{1}{c}\frac{\partial c(k')}{\partial k'} &= -\frac{\alpha+\gamma}{1-\alpha} \frac{\frac{\partial c(k')}{\partial k'} + 1}{c-(1-\delta)k+k'} \\
\left(\frac{1}{c} + \frac{\alpha+\gamma}{1-\alpha}\right)\frac{\partial c(k')}{\partial k'} &= -\frac{\alpha+\gamma}{1-\alpha}\frac{1}{y}
\end{aligned}$$

$$\frac{\partial c^*(k';k,z)}{\partial k'} = -\frac{\frac{\alpha+\gamma}{1-\alpha}}{\frac{y}{c} + y\frac{\alpha+\gamma}{1-\alpha}} < 0, \tag{2.12}$$

where we note that $c + k' - (1-\delta)k$ is simply consumption plus investment, which equals output. Hence, $c^*(k';k,z)$ is decreasing in k'. Since

$$u_c = \frac{\theta}{c}$$

with the assumed form of utility, we realise that u_c is increasing in k'. In other words, we would have the same figure as in the case of inelastic labour.

What happens to $g(k)$ (i.e. capital next period) if current capital stock, k, increases from k to $\hat{k}$? Differentiating (2.11) with respect to k gives

$$\begin{aligned}
\frac{1}{c}\frac{\partial c^*(k';k,z)}{\partial k} &= \alpha\frac{1}{k} - \frac{\alpha+\gamma}{1-\gamma}\frac{\frac{\partial c^*(k';k,z)}{\partial k} - (1-\delta)}{c-(1-\delta)k+k'} + \frac{\alpha+\gamma}{1-\alpha}\frac{\alpha e^z A k^{\alpha-1}}{e^z A k^\alpha} \\
\left(\frac{1}{c} + \frac{\alpha+\gamma}{1-\gamma}\frac{1}{c-(1-\delta)k+k'}\right)\frac{\partial c^*(k';k,z)}{\partial k} &= \alpha\frac{1}{k} + \frac{\alpha+\gamma}{1-\gamma}\frac{1-\delta}{c-(1-\delta)k+k'} + \frac{\alpha+\gamma}{1-\alpha}\frac{\alpha}{k} \\
\frac{\partial c^*(k';k,z)}{\partial k} &= \frac{\alpha\frac{y}{k} + \frac{\alpha+\gamma}{1-\gamma}\left(1-\delta+\alpha\frac{y}{k}\right)}{\frac{y}{c} + \frac{\alpha+\gamma}{1-\gamma}} > 0,
\end{aligned}$$

which tells us that $c^*(k';k,z)$ is increasing in k. That $c^*(k';k,z)$ is increasing in z is immediate from (2.11). Thus, the comparative static is the same as before. That is, a higher capital stock increases both consumption and savings.

Example 2.1. Suppose now that

$$u(c, 1-h) = \theta \ln c + (1-\theta) \ln (1-h).$$

How would the analysis change?

$$u_c = \frac{\theta}{c},$$
$$u_\ell = \frac{1-\theta}{1-h}$$

So (2.7) becomes

$$\frac{\frac{1-\theta}{1-h}}{\theta / c} = e^z (1-\alpha) A \left(\frac{k}{h}\right)^{\alpha}$$
$$\Rightarrow c = \frac{\theta}{1-\theta} e^z (1-\alpha) A \left(\frac{k}{h}\right)^{\alpha} (1-h). \tag{2.13}$$

Equating the this with the feasibility condition gives

$$\begin{aligned}
(1-\delta) k - k' &= \frac{\theta}{1-\theta} e^z (1-\alpha) A \left(\frac{k}{h}\right)^{\alpha} (1-h) - e^z A k^{\alpha} h^{1-\alpha} \\
&= e^z A k^{\alpha} \left[\frac{\theta}{1-\theta} (1-\alpha) h^{-\alpha} - \frac{\theta}{1-\theta} h^{1-\alpha} - 1 h^{1-\alpha}\right] \\
&= e^z A k^{\alpha} \left[\frac{\theta}{1-\theta} (1-\alpha) h^{-\alpha} - \frac{1-\alpha\theta}{1-\theta} h^{1-\alpha}\right] \\
\Rightarrow -1 &= e^z A k^{\alpha} \left[-\alpha \frac{\theta}{1-\theta} (1-\alpha) h^{-(1+\alpha)} \frac{\partial h(k')}{\partial k'} - \frac{(1-\alpha\theta)(1-\alpha)}{1-\theta} h^{-\alpha} \frac{\partial h(k')}{\partial k'}\right] \\
&= -e^z A k^{\alpha} \frac{1-\alpha}{1-\theta} \left(\alpha\theta h^{-(1+\alpha)} + (1-\alpha\theta) h^{-\alpha}\right) \frac{\partial h(k')}{\partial k'} \\
&= -e^z A k^{\alpha} \frac{1-\alpha}{1-\theta} h^{-\alpha} \left(1 - \alpha\theta \left(\underbrace{1 - \frac{1}{h}}_{\leq 0}\right)\right) \frac{\partial h(k')}{\partial k'}
\end{aligned}$$

$$\Rightarrow \frac{\partial h^*(k'; k, z)}{\partial k'} > 0.$$

Moreover, substituting (2.10) into (2.13) gives

$$c = \frac{\theta}{1-\theta} e^z (1-\alpha) A k^{\alpha} \left(\frac{c - (1-\delta) k + k'}{e^z A k^{\alpha}}\right)^{-\frac{\alpha}{1-\alpha}} \left(1 - \left(\frac{c - (1-\delta) k + k'}{e^z A k^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right)$$

$$\begin{aligned}
\ln c = \ln \frac{\theta}{1-\theta} e^z (1-\alpha) A + \alpha \ln k - \frac{\alpha}{1-\alpha} \ln (c - (1-\delta) k + k') \\
+ \frac{\alpha}{1-\alpha} (\ln (e^z A) + \ln k^{\alpha}) + \ln \left[1 - \left(\frac{c - (1-\delta) k + k'}{e^z A k^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right].
\end{aligned} \tag{2.14}$$

Differentiating with respect to k' yields

$$
\begin{aligned}
\frac{1}{c}\frac{\partial c\left(k'\right)}{\partial k'} &= -\frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k'}+1}{c-\left(1-\delta\right)k+k'} - \frac{\frac{1}{1-\alpha}\frac{1}{e^z A k^\alpha}\left(\frac{\partial c\left(k'\right)}{\partial k'}+1\right)\left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}-1}}{1-\left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}}} \\
&= -\frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k'}+1}{y} - \frac{\frac{1}{1-\alpha}\frac{h^{1-\alpha}}{y}\left(\frac{\partial c\left(k'\right)}{\partial k'}+1\right)\left(\frac{y}{e^z A k^\alpha}\right)^{\frac{\alpha}{1-\alpha}}}{1-\left(\frac{y}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}}} \\
&= -\frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k'}+1}{y} - \frac{\frac{1}{1-\alpha}\frac{h^{1-\alpha}}{y}\left(\frac{\partial c\left(k'\right)}{\partial k'}+1\right)h^\alpha}{1-h} \\
&= -\frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k'}+1}{y} - \frac{\frac{1}{1-\alpha}\left(\frac{\partial c\left(k'\right)}{\partial k'}+1\right)\frac{h}{y}}{1-h} \\
\Rightarrow \left(\frac{1}{1-\alpha}\frac{1}{y}\left(1+\frac{h}{1-h}\right)+\frac{1}{c}\right)\frac{\partial c\left(k'\right)}{\partial k'} &= -\frac{\alpha}{1-\alpha}\frac{1}{y} - \frac{1}{1-\alpha}\frac{h}{1-h}\frac{1}{y} \\
&= -\frac{1}{1-\alpha}\frac{1}{y}\left(\alpha+\frac{h}{1-h}\right) \\
\Rightarrow \frac{\partial c^*\left(k';k,z\right)}{\partial k'} &= -\frac{\frac{1}{1-\alpha}\left(\alpha+\frac{h}{1-h}\right)}{\frac{1}{1-\alpha}\left(1+\frac{h}{1-h}\right)+\frac{y}{c}} < 0,
\end{aligned}
$$

where we used

$$
\begin{aligned}
\frac{y}{h^{1-\alpha}} &= e^z A k^\alpha, \\
\left(\frac{c-\left(1-\delta\right)k+k'}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}} &= \left(\frac{y}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}} = h.
\end{aligned}
$$

So we again have that $c^*\left(k';k,z\right)$ is decreasing in k'.

Finally, differentiating (2.14) with respect to k gives

$$
\begin{aligned}
\frac{1}{c}\frac{\partial c\left(k'\right)}{\partial k} &= \frac{\alpha}{k} - \frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k}-\left(1-\delta\right)}{c-\left(1-\delta\right)k+k'} + \frac{\alpha^2}{1-\alpha}\frac{1}{k} - \frac{\frac{\frac{\partial c\left(k'\right)}{\partial k}-(1-\delta)}{e^z A k^\alpha}\left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{\frac{\alpha}{1-\alpha}}}{1-\left(\frac{c-(1-\delta)k+k'}{e^z A k^\alpha}\right)^{\frac{1}{1-\alpha}}} \\
&= \frac{\alpha}{k}\left(1+\frac{\alpha}{1-\alpha}\right) - \frac{\alpha}{1-\alpha}\frac{\frac{\partial c\left(k'\right)}{\partial k}-\left(1-\delta\right)}{y} - \frac{\left(\frac{\partial c\left(k'\right)}{\partial k}-\left(1-\delta\right)\right)h^\alpha\frac{h^{1-\alpha}}{y}}{1-h} \\
\left(\frac{\alpha}{1-\alpha}\frac{1}{y}+\frac{h}{1-h}\frac{1}{y}+\frac{1}{c}\right)\frac{\partial c\left(k'\right)}{\partial k} &= \frac{\alpha}{1-\alpha}\frac{1}{k}+\frac{1-\delta}{y}\frac{1}{1-h} \\
\frac{\partial c\left(k'\right)}{\partial k} &= \frac{\frac{\alpha}{1-\alpha}\frac{y}{k}+\frac{1-\delta}{1-h}}{\frac{\alpha}{1-\alpha}+\frac{h}{1-h}+\frac{y}{c}} > 0,
\end{aligned}
$$

which is the same as before.

2.2.5 Simulation

Step 1. Find the long-run expected value of the shock, $\bar{z}$. Here, for convenience, we can let $\bar{z} = 0$ so $e^{\bar{z}} = 1$. We can think of this as a normalisation since we already have A in the production function.

Step 2. Solve for the deterministic steady state with $\bar{z} = 0$. Let $\left(\bar{c}, \bar{h}, \bar{k}\right)$ denote the steady-state values.

Step 3. Writing the Euler equation using (2.2), (2.4) and (2.5):

$$u_c\left(c, 1-h\right) = \beta \mathbb{E}_{z'}\left[u_c\left(c', 1-h'\right)\left(e^{z'} F_k\left(k', h'\right) + 1 - \delta\right)|z\right].$$

We also have the resource constraint

$$c + k' - e^z F\left(k, h\right) - \left(1-\delta\right) k = 0,$$

and the first-order conditions for $\left(c^*, h^*\right)$,(2.7),

$$\frac{u_\ell\left(c, 1-h\right)}{u_c\left(c, 1-h\right)} = e^z F_h\left(k, h\right).$$

Linearise these equations around the steady-state values $\left(\bar{c}, \bar{h}, \bar{k}\right)$. The linearised form would have derivatives such as $u_{c\ell}$, u_{cc}, F_{kk} etc.

Step 4. Conjecture a solution of the form (guess and verify):

$$\begin{aligned}
\hat{c} &= a_1 \hat{k} + a_2 \hat{z}, \\
\hat{h} &= b_1 \hat{k} + b_2 \hat{z}, \\
\hat{k} &= c_1 \hat{k} + c_2 \hat{z},
\end{aligned}$$

where the hat denotes deviation from the steady state (e.g. $\hat{c} = c - \bar{c}$).

Step 5. Estimate γ, θ, α, A and the stochastic process. Recall that

$$y_t = e^{z_t} A k_t^\alpha h_t^{1-\alpha}.$$

We can obtain data for y_t, k_t, h_t and α—since our model does not have growth, we would need to de-trend the data (e.g. use a Hodrick-Prescott filter). We can then back out z_t as the Solow residual. In general, z_t loos like an AR(1) process; i.e.

$$z_t = \rho z_{t-1} + \varepsilon,$$

where $\varepsilon \sim N\left(0, \sigma^2\right)$ and $\rho \approx 0.95$.

Remark 2.1. The model simulation produces investment far more volatile than observed in the data. One solution is to add convex costs (i.e. adjustment costs).

Remark 2.2. We would like labour supply to be elastic (i.e. we would like $\gamma \approx 0$ so that u is linear in h). Alternatively, we can consider "lotteries" with indivisible labour (Gary Hansen).

3 Asset pricing

3.1 The set up

General equilibrium exchange economy model with representative household, with a "Lucas tree" that produce stochastic dividends. Since agents are identical, there will be no trading in equilibrium—we wish to know how to price assets.

3.1.1 Givens

The "givens".

- ▷ J states and N assets.
- ▷ $\mathbf{1} = (1, 1, \ldots, 1)'$ endowment of assets (per capita). This is the only endowments agents receive, which can be used to purchase consumption goods and/or assets.
- ▷ $Q = [q_{jk}]_{J\times J}$ is the transition matrix that describes the probability of moving from state j today to state k tomorrow.
- ▷ $Y = [y_{jn}]_{J\times N}$ is the dividends matrix, where the jth row, $\mathbf{y}_{j\cdot}$, gives a vector of dividends for all the N assets in state j, and the nth column, $\mathbf{y}_{\cdot n}$, gives a vector of dividends across all J states for an asset n.
- ▷ $P = [p_{jn}]_{J\times N}$ is the price matrix (in terms of contemporaneous consumption good)
- ▷ $\mathcal{Z} = [1-\varepsilon, 1+\varepsilon]^N$ is the (bounded) space of allowable portfolios, where $\varepsilon > 0$. It takes this form because we know that, in equilibrium, consumers will hold $\mathbf{1}$ as their portfolio so, to obtain differentiability, we only need some values around $\mathbf{1}$.
- ▷ $\mathbf{z} = (z_1, z_2, \ldots, z_N)' \in \mathcal{Z}$ is a portfolio.
- ▷ Preferences:

$$u(\mathbf{c}) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta u(c_t)\right], \ \beta \in (0,1),$$

  where $u(\cdot)$ is strictly increasing, strictly concave and once differentiable.

Remark 3.1. We assume that the sequence of states follows a first-order Markov process. In general, this *does not* imply that dividends follow a first-order Markov process. This is because two or more states could have the same dividends but different probabilities for the next-period state. In this case, observing only the dividends does not allow us to identify the current state!

Assumptions.

- ▷ (non-degenerate assets) for each asset n, $y_{jn} > 0$ for some j; i.e. every asset pays some dividends in some state.
- ▷ (non-degenerate states), for each state j, $y_{jn} > 0$ for some n; i.e. in every state, some assets pays dividends.
- ▷ $\mathbf{y}_{j\cdot}\mathbf{1} > 0$ for all j; i.e. if the agent holds the endowment $\mathbf{1}$, his income will be strictly positive.

- $\triangleright$ Q has one ergodic set and no transient states. This guarantees that $\mathbf{p}_{j\cdot} \gg \mathbf{0}$ (i.e. every asset pays in some state and every state has positive probability associated) for any j since it implies
$$\frac{1}{T}\sum_{s=1}^{T} Q^s \gg \mathbf{0}$$
for some T and the row converge to some constant vector as $T \to \infty$.

3.1.2 Timing

The critical assumption here is that agents receive dividends *before* the trade assets. Thus, the price of the assets only includes the future expected dividends, and *not* the current-period dividends.

(i) At the beginning of period, the consumer holds $\mathbf{z}$ as his portfolio.

(ii) The state of the world j is realised and informed to the agents. This determines the probability distribution for the next period's state ($\mathbf{q}_{j\cdot}$).

(iii) Agents receive their share of dividends: sum of dividends are given by $\mathbf{y}_{j\cdot}\mathbf{z}$ (remember, all dividends are paid in terms of contemporaneous consumption good).

(iv) The asset market "opens" and agents trade (if they wish to) assets for current consumption good based on $\mathbf{p}_{j\cdot} = (p_{j1}, p_{j2}, \ldots, p_{jN})$.

(v) After trading, the agent hold a portfolio $\mathbf{z}'$ and he consumes c.

We wish to evaluate the agent's action in the asset market—the timing of the Bellman equation will be in between steps (ii) and (iii).

3.1.3 Equilibrium condition

Our goal is to find prices P such that the agent chooses to walk out of the asset market with the portfolio that he walked in with; i.e. $\mathbf{z} = \mathbf{1}$ in all periods.

3.2 Bellman equation

Why Bellman? In sequence form, the solution would be of the form $c_t(h^t, z_0)$, where h^t is the history of all shocks in the past. This is cumbersome to work with. We therefore impose some recursive structure to remove history dependence allowing us to use the recursive form.

The consumer's problem is to choose next period portfolio $\mathbf{x} \in \mathcal{Z}$ while satisfying the budget constraint, which is given by
$$c_j + \mathbf{p}_{j\cdot}\mathbf{x} = \mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}\mathbf{z},$$
where $\mathbf{p}_{j\cdot}\mathbf{x}$ is the cost of purchasing portfolio $\mathbf{x}$, $\mathbf{y}_{j\cdot}\mathbf{z}$ denotes the dividend income in the current period and $\mathbf{p}_{j\cdot}\mathbf{z}$ is the value of endowment portfolio.

The state consists of a pair $(\mathbf{z}, j)$. We can therefore write the household's feasibility set as
$$\Gamma(\mathbf{z}, j) = \{\mathbf{x} \in \mathcal{Z} : c_j = \mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}(\mathbf{z} - \mathbf{x}) \geq 0\},$$
where we imposed that consumption should be nonnegative.

The Bellman equation for the household can then be defined as

$$\begin{aligned} v^P(\mathbf{z},j) &= \max_{c,\mathbf{x}\in\Gamma(\mathbf{z},j)} \left\{u(c)+\beta\mathbb{E}_Q\left[v^P(\mathbf{x},k)\,|j\right]\right\} \\ &= \max_{\mathbf{x}\in\Gamma(\mathbf{z},j)} \left\{u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\mathbf{x})\right)+\beta\mathbb{E}_Q\left[v^P(\mathbf{x},k)\,|j\right]\right\} \\ &= \max_{\mathbf{x}\in\Gamma(\mathbf{z},j)} \left\{u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\mathbf{x})\right)+\beta\sum_{k=1}^{J} q_{jk} v^P(\mathbf{x},k)\right\} \qquad (3.1) \end{aligned}$$

where the superscript P expresses the fact that household take as given the prices (as in the RCE case).

3.2.1 Is it well defined?

We want to ensure that the Contraction Mapping Theorem is applicable. Recall that $j \in \mathcal{J}$ is the discrete exogenous state variable where $\mathcal{J}$ is finite, and $\mathbf{z}$ is the (continuous) endogenous state variable in $\mathcal{Z}$.

- ▷ For any $j \in \mathcal{J}$, $\Gamma(\mathbf{z},j)$ is nonempty (we can always choose $\mathbf{x}=\mathbf{z}$), varies continuously with $\mathbf{z}$, and compact-valued (closed and bounded for any given $(\mathbf{z},j)$). It is also convex in $\mathbf{z}$ as a correspondence.[5]
- ▷ The state space $\mathcal{Z}\times\mathcal{J}$ is finite since we bounded $\mathcal{Z}$ and we have a finite number of states. This implies that the Bellman operator $T\tilde{v}^P$ is bounded.
- ▷ u varies continuously with $\mathbf{x}$ and $\mathbf{z}$ any any $j \in \mathcal{J}$. Since $\mathcal{J}$ is discrete, the Feller property is trivially satisfied.
- ▷ The right-hand side of (3.1) satisfies the Blackwell's sufficient condition;[6] hence $T\tilde{v}$ is a contraction.
- ▷ By the Contraction Mapping Theorem, we conclude that there exists a (continuous) unique v^P satisfying (3.1).

[5] Let $\mathbf{z}_a \neq \mathbf{z}_b$, and for any $\theta\in(0,1)$, define $\mathbf{z}_\theta=\theta\mathbf{z}_a+(1-\theta)\mathbf{z}_b$. Then,

$$\begin{aligned} \Gamma(\mathbf{z}_\theta,j) &= \{\mathbf{x}\in\mathcal{Z} : c=\mathbf{y}_j.\mathbf{z}_\theta+\mathbf{p}_{j\cdot}(\mathbf{z}_\theta-\mathbf{x})\geq 0\} \\ &= \{\mathbf{x}\in\mathcal{Z} : c=\theta(\mathbf{y}_j.\mathbf{z}_a+\mathbf{p}_{j\cdot}(\mathbf{z}_a-\mathbf{x}))+(1-\theta)(\mathbf{y}_j.\mathbf{z}_b+\mathbf{p}_{j\cdot}(\mathbf{z}_b-\mathbf{x}))\geq 0\}. \end{aligned}$$

Suppose $\mathbf{x}_a\in\Gamma(\mathbf{z}_a,j)$ and $\mathbf{x}_b\in\Gamma(\mathbf{z}_b,j)$, then clearly,

$$\mathbf{x}_\theta=\theta\mathbf{x}_a+(1-\theta)\mathbf{x}_b\in\Gamma(\mathbf{z}_\theta,j).$$

[6] Define the Bellman operator as

$$\left(T\tilde{v}^P\right)(\mathbf{z},j) := \max_{\mathbf{x}\in\Gamma(\mathbf{z},j)}\left\{u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\mathbf{x})\right)+\beta\mathbb{E}_Q\left[\tilde{v}^P(\mathbf{x},k)\,|j\right]\right\}$$

Monotone. Let $\tilde{v}^P(\mathbf{z},j)\geq\hat{v}^P(\mathbf{z},j)$ for all $(\mathbf{z},j)\in\mathcal{Z}\times\mathcal{J}$. Let $\hat{g}(\mathbf{z},j)$ denote the optimal policy with $\hat{v}^P$ and $\tilde{g}(\mathbf{z},j)$ denote the optimal policy with $\tilde{v}^P$. Then,

$$\begin{aligned} \left(T\hat{v}^P\right)(\mathbf{z},j) &= u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\hat{g}(\mathbf{z},j))\right)+\beta\mathbb{E}_Q\left[\hat{v}^P(\hat{g}(\mathbf{z},j),k)\,|j\right] \\ &\leq u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\hat{g}(\mathbf{z},j))\right)+\beta\mathbb{E}_Q\left[\tilde{v}^P(\hat{g}(\mathbf{z},j),k)\,|j\right] \\ &\leq u\left(\mathbf{y}_j.\mathbf{z}+\mathbf{p}_{j\cdot}(\mathbf{z}-\tilde{g}(\mathbf{z},j))\right)+\beta\mathbb{E}_Q\left[\tilde{v}^P(\tilde{g}(\mathbf{z},j),k)\,|j\right] \\ &= \left(T\tilde{v}^P\right)(\mathbf{z},j). \end{aligned}$$

The Bellman equation in (3.1) is well-defined.

Can we conclude that $v^P(\cdot)$ is (strictly) increasing in $(z_1, z_2, \ldots, z_N)$?

3.2.2 Properties of v^P

Monotonicity in z For each $(\mathbf{x}, j) \in \mathcal{Z} \times \mathcal{J}$, since $\mathbf{p}_{j\cdot} \gg \mathbf{0}$ for all $j \in \mathcal{J}$, the period-return function, $u(\cdot, \mathbf{x}, j)$ is strictly increasing in $\mathbf{z}$, and for each $j \in \mathcal{J}$, $\Gamma(\cdot, j)$ is monotone (if $\mathbf{z}$ is higher, then the agent can afford more). Thus, for each $j \in \mathcal{J}$, $v^P(\cdot, j)$ is strictly increasing. If $\mathbf{p}_{j\cdot} \geq \mathbf{0}$ for all $j \in \mathcal{J}$, then $v^P(\cdot, j)$ is weakly increasing.

Seems to be using concept of strongly increasing, rather than strictly increasing.

Concavity We know that, for each $j \in \mathcal{J}$, $\Gamma(\mathbf{z}, j)$ is convex in $\mathbf{z}$ and $u(\cdot, \cdot)$ is concave in $(\mathbf{z}, \mathbf{x})$. Thus, $v^P(\cdot, j)$ is (weakly) concave. However, we would not expect it to be strictly concave as there could be assets with linearly dependent payoffs so a convex combination would not yield a strictly higher utility. Thus, we cannot conclude that the optimal policy correspondence is single-valued here.

Differentiability Recall that the problem with differentiability is that we may have kinks in the value function. However, we know that v^P is continuous and weakly concave. So we would expect v^P to be differentiable in $\mathbf{z}$ if it is interior.

Theorem 3.1. *Let $G^P(\mathbf{z}, j)$ be the optimal policy correspondence for v^P. Fix $\hat{\mathbf{z}} \in \operatorname{int}(\mathcal{Z})$. If $G^P(\hat{\mathbf{z}}, j)$ has at least one element $\mathbf{x}^* \in \operatorname{int}\Gamma(\hat{\mathbf{z}}, j)$, then $v^P(\mathbf{z}, j)$ is differentiable at $\hat{\mathbf{z}}$.*

Proof. We need to modify the proof from before slightly. If $\mathbf{x}^* \in \operatorname{int}\Gamma(\hat{\mathbf{z}}, j)$, it follows that $\mathbf{x}^* \in \operatorname{int}\Gamma(\mathbf{z}, j)$ for $\mathbf{z}$ in some neighbourhood of $\hat{\mathbf{z}}$. In other words, $\mathbf{x}^*$ is feasible for some initial portfolio/endowment $\mathbf{z}$. Define

$$w(\mathbf{z}) = u(\mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}(\mathbf{z} - \mathbf{x}^*)) + \beta\mathbb{E}\left[v^P(\mathbf{x}^*, k) \,|j\right].$$

By construction, $W(\mathbf{z})$ attains its maximum when $\mathbf{z} = \hat{\mathbf{z}}$. Hence,

$$v^P(\hat{\mathbf{z}}, j) = w(\hat{\mathbf{z}}) \geq w(\mathbf{z})$$

and the equality holds if and only if $\mathbf{z} = \hat{\mathbf{z}}$. Since u is differentiable, then W is differentiable in $\mathbf{z}$. ■

Note that since $\mathbf{1} \in \operatorname{int}(\mathcal{Z})$, the argument above holds at the point of interest.

Discounting. For any $a \geq 0$,

$$\begin{aligned}
\left(T\left(\tilde{v}^P + a\right)\right)(\mathbf{z}, j) &= \max_{\mathbf{x}\in\Gamma(\mathbf{z},j)} \left\{u(\mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}(\mathbf{z} - \mathbf{x})) + \beta\mathbb{E}_Q\left[\left(\tilde{v}^P(\mathbf{x}, k) + a\right)|j\right]\right\} \\
&= \beta a + \max_{\mathbf{x}\in\Gamma(\mathbf{z},j)} \left\{u(\mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}(\mathbf{z} - \mathbf{x})) + \beta\mathbb{E}_Q\left[\tilde{v}^P(\mathbf{x}, k)\,|j\right]\right\} \\
&= \beta a + \left(T\tilde{v}^P\right)(\mathbf{z}, j).
\end{aligned}$$

3.3 Solving for the recursive equilibrium

First-order condition with respect x_n is given by

$$u'(c_j)\, p_{jn} = \beta \sum_{k=1}^{j} q_{jk} v_n^P(\mathbf{x}^*, k)\,,\ \forall n, j, \tag{3.2}$$

where v_n^P is the derivative of the value function with respect to x_n. The envelope condition (with respect to z_n) is given by

$$v_n^P(\mathbf{z}, j) = u'(c_j)\,(y_{jn} + p_{jn})\,,\ \forall n, j, \tag{3.3}$$

where $c_j = \mathbf{y}_{j\cdot}\mathbf{z} + \mathbf{p}_{j\cdot}(\mathbf{z} - \mathbf{x}^*)$. The latter condition says that the marginal value of having more of asset n is equal to the marginal increase in the form of dividends (y_{jn}) and the value of the endowment (p_{jn}), evaluated at the current marginal utility.

3.3.1 Definition of a recursive equilibrium

A recursive equilibrium in this context is a price matrix P^e associated with the value function v^e and an optimal policy correspondence G^e such that the following market clearing conditions hold:

$$\mathbf{x}^* = \mathbf{1} \in G^e(\mathbf{1}, j)\,,\ \forall j \in \mathcal{J}.$$

That is, if the agent has $\mathbf{z} = \mathbf{1}$ as his endowment at the beginning of the period, it is optimal for him to choose the next period portfolio $\mathbf{x}^* = \mathbf{1}$ (note, it could also be optimal to choose other portfolios). Of course, the equilibrium conditions also include the optimality conditions—i.e. the first-order and the envelope conditions.

3.3.2 Solving for P^e

Suppose now that $P = P^e$. Define c_j^e as the level of consumption in state j given portfolio of $\mathbf{x}^* = \mathbf{z} = \mathbf{1}$:.

$$\begin{aligned} c_j^e &= \mathbf{y}_{j\cdot}\mathbf{1} + \mathbf{p}_{j\cdot}^e(\mathbf{1} - \mathbf{1}) \\ &= \mathbf{y}_{j\cdot}\mathbf{1},\ \forall j \in \mathcal{J}. \end{aligned}$$

Then, the first-order condition (3.2) becomes

$$u'\left(c_j^e\right) p_{jn}^e = \beta \sum_{k=1}^{J} q_{jk} v_n^e(\mathbf{1}, k)\,,\ \forall j, n.$$

The envelope condition (3.3) evaluated at $\mathbf{z} = \mathbf{x}^* = \mathbf{1}$ and $j = k$ is

$$v_n^e(\mathbf{1}, k) = u'(c_k^e)\,(y_{kn} + p_{kn}^e)\,,\ \forall n.$$

Combining the two together yields the Euler equation:

$$u'\left(c_j^e\right) p_{jn}^e = \beta \sum_{k=1}^{J} q_{jk} u'(c_k^e)\,(y_{kn} + p_{kn}^e)\,,\ \forall j, n. \tag{3.4}$$

This gives $J \times N$ equations in $J \times N$ unknowns (i.e. $[p^e_{ij}]$). We see that the price of asset n, $\mathbf{p}_{\cdot n} = (p_{1n}, p_{2n}, \dots, p_{Jn})$ depends on

(i) $u'\left(c^e_j\right), \forall j$; i.e. risk aversion and variability of consumption.

(ii) variability of assets n's dividends, $\mathbf{y}_{\cdot n} = (y_{1n}, y_{2n}, \dots, y_{Jn})$;

(iii) "persistence" in the state Q.

(iv) covariance of dividends and u'.

(v) dividends of other assets $\mathbf{y}_{\cdot m}$ do not affect price of asset n directly, but only through the indirect effect via $u'(c^e_k)$.

Our strategy to solve this is to compute prices in utility terms, and then back out prices of assets denominated in contemporaneous consumption units. Define

$$
\begin{aligned}
a_j &\coloneqq u'\left(c^e_j\right), \ \forall j, \\
h_{jn} &\coloneqq \beta \sum_{k=1}^{J} q_{jk} a_k y_{kn}, \ \forall j, n, && (3.5) \\
\varphi^e_{jn} &\coloneqq a_j p^e_{jn}, \ \forall j, n, && (3.6)
\end{aligned}
$$

where h_{jn} is the discounted expected value of next-period dividends in utility terms and φ_{jn} is the price of asset n in state j in utility terms. We can then write (3.4) as

$$
\varphi^e_{jn} = h_{jn} + \beta \sum_{k=1}^{J} q_{jk} \varphi^e_{kn}, \ \forall j, n \tag{3.7}
$$

which can be written in vector form (stacking across the j's)

$$
\begin{aligned}
\boldsymbol{\varphi}^e_{\cdot n} &= \mathbf{h}_{\cdot n} + \beta \sum_{k=1}^{J} \mathbf{q}_{\cdot k} \varphi^e_{kn} \\
&= \mathbf{h}_{\cdot n} + \beta Q \boldsymbol{\varphi}^e_{\cdot n}, \ \forall n
\end{aligned}
$$

For each asset n, the condition above defines an operator mapping $\mathbf{f}_{\cdot n}$ to $T\mathbf{f}_{\cdot n}$:

$$
T\mathbf{f}_{\cdot n} = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n}.
$$

Note, in particular, that this is a contraction.[7] Hence, we can interpret $\boldsymbol{\varphi}^e_{\cdot n}$ as a fixed point. This also allows us to use the Contraction Mapping Theorem. To emphasise, we can write above as

$$
T\mathbf{f}_{[j]n} = \mathbf{h}_{\cdot n} + \beta \mathbb{E}\left[\mathbf{f}_{[j']n} | j\right].
$$

[7] Check Blackwell's sufficient conditions.
(Monotone) If $\mathbf{f}_{\cdot n} \geq \mathbf{g}_{\cdot n}$, then

$$
T\mathbf{f}_{\cdot n} = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n} \geq \mathbf{h}_{\cdot n} + \beta Q \mathbf{g}_{\cdot n} = T\mathbf{g}_{\cdot n}.
$$

(Discounting) Let $a \geq 0$, then

$$
\begin{aligned}
T\left(\mathbf{f}_{\cdot n} + a\right) &= \mathbf{h}_{\cdot n} + \beta Q \left(\mathbf{f}_{\cdot n} + a\mathbf{I}\right) = \mathbf{h}_{\cdot n} + \beta Q \mathbf{f}_{\cdot n} + a\beta Q \mathbf{I} \\
&= T\mathbf{f}_{\cdot n} + a\beta Q \mathbf{I} \leq T\mathbf{f}_{\cdot n} + a\beta \mathbf{I},
\end{aligned}
$$

where the last (in)equality holds because the rows of Q sum to one by definition.

For example, to show that the fixed point $\boldsymbol{\varphi}_{[j]n}$ is increasing in j, we can show that: (i) the states are ordered in an increasing manner of the dividends (i.e. $y_{1n} < y_{2n} < \cdots < y_{Jn}$); (ii) Q is monotone (in this case, this is equivalent to the states being ordered by first-order stochastic dominance); and (iii) $h_{1n} < h_{2n} < \cdots < h_{Jn}$; then we may conclude that the fixed point is also increasing in j.

Since $\boldsymbol{\varphi}^e_{\cdot n}$ is in units of (contemporaneous) consumption good, we can write (3.6) in matrix form:

$$\Phi^e = AP^e$$
$$\begin{bmatrix} \varphi^e_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{J1} & \cdots & \varphi^e_{JN} \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_J \end{bmatrix} \begin{bmatrix} p^e_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{J1} & \cdots & p^e_{JN} \end{bmatrix},$$

where A is a diagonal matrix of a_j's, Φ^e is the matrix of prices denominated in utilities. We can then obtain P^e as

$$P^e = A^{-1}\Phi^e. \tag{3.8}$$

Notice that since A is a diagonal matrix with strictly positive elements (utility functions are strictly increasing), A^{-1} is well-defined.

3.4 Asset pricing

Note that (3.8) contains marginal utility on both sides so it is cumbersome to work with. We can do better. Recall the Euler equation (3.4):

$$u'\left(c^e_j\right) p^e_{jn} = \beta \sum_{k=1}^{J} q_{jk} u'\left(c^e_k\right)\left(y_{kn} + p^e_{kn}\right), \forall j, n.$$

We can express this in matrix form:

$$AP^e = \beta QA\left(Y + P^e\right) = \beta QAY + \beta QAP^e.$$

Repeated substitution yields

$$\begin{aligned} AP^e &= \beta QAY + \beta Q\left(\beta QAY + \beta QAP^e\right) \\ &= \left(\beta Q + \beta^2 Q^2\right) AY + \beta^2 Q^2 AP^e \\ &= \left(\beta Q + \beta^2 Q^2 + \cdots + \beta^T Q^T\right) AY + \beta^T Q^T AP^e \\ &\to \left(\beta Q + \beta^2 Q^2 + \cdots\right) AY \\ \Rightarrow P^e &= A^{-1}\left(\beta Q + \beta^2 Q^2 + \cdots\right) AY, \end{aligned}$$

where we used the fact that

$$\lim_{T\to\infty} \beta^T Q^T AP^e = \mathbf{0}.$$

The limit is zero since we assumed Q is stationary; i.e. $\lim_{T\to\infty} Q^T$ is a constant.

We could also write

$$\begin{aligned} (A - \beta QA) P^e &= \beta QAY \\ \Leftrightarrow (\mathbb{I} - \beta Q) AP^e &= \beta QAY \\ \Leftrightarrow P^e &= A^{-1} (\mathbb{I} - \beta Q)^{-1} \beta QAY. \end{aligned}$$

Proposition 3.1. *There exists a unique recursive competitive equilibrium, and the prices P^e given as*

$$P^e = A^{-1} \left(\beta Q + \beta^2 Q^2 + \cdots\right) AY. \tag{3.9}$$

Define a portfolio Λ_0 as an $N \times 1$ vector given by $\Lambda_0 = (\Lambda_{01}, \Lambda_{02}, \ldots, \Lambda_{0N})'$.

Example 3.1. A set of payoffs has payoffs that are *linearly dependent* if there exists Λ_0 such that

$$Y\Lambda_0 = \mathbf{0}_{J\times 1}.$$

Its price is given by

$$P^e\Lambda_0 = A^{-1} \left(\beta Q + \beta^2 Q^2 + \cdots\right) AY\Lambda_0 = \mathbf{0}.$$

3.4.1 Complete market

Let $\mathbf{e}_j = (0, \ldots, 0, 1, 0 \ldots, 0)'$ be an $J \times 1$ vector with 1 in its jth element with zeros elsewhere. Suppose there exists a portfolio $\Lambda_{\cdot j}$ ($N \times 1$ vector) such that

$$Y\Lambda_{\cdot j} = \mathbf{e}_j,$$

then its price is given by

$$\left(\Pi^e_{\cdot j}\right)_{J\times 1} = \mathbf{P}^e \Lambda_{\cdot j},$$

where each row j gives the price of the portfolio in state j.

Proposition 3.2. *It is possible to find such $\Lambda_{\cdot j}$ for all j if and only if Y has rank J.*

Since Y is a $J \times N$ matrix, for Y to have rank J, it must be that $J \geq N$.

Suppose we introduce a new asset (small enough so as to leave consumption/marginal utilities approximately unchanged). Suppose also the there exists $\Lambda_{\cdot j}$ for all j. Then the price of a (perpetual) state contingent claim is

$$\Pi^e = [\pi_{ij}]_{J\times J} = (P^e)_{J\times N} (\Lambda)_{N\times J},$$

where

$$Y\Lambda = \mathbb{I}_{J\times J}.$$

We can use this price matrix Π^e to price the new asset. If the new asset has dividends given by $\hat{\mathbf{y}} = (\hat{y}_1, \ldots, \hat{y}_J)'$, then its price vector is given by

$$\hat{\mathbf{p}}_{J\times 1} = \Pi^e \hat{\mathbf{y}},$$

where $\hat{p}_j$ is the price of the asset in state j.

3.4.2 Examples

Example 3.2. Suppose $N = 1$ and $J = 1$ where the sole asset is a safe asset (pays dividend of one unit in every state), then

$$\begin{aligned} p^e &= \frac{1}{u'(c)} \left(\beta + \beta^2 + \cdots\right) u'(c)\, y \\ &= \frac{\beta}{1-\beta} y = \frac{y}{\rho} \end{aligned}$$

where $\beta = 1/(1+\rho)$.

Example 3.3. $N = 1$, $J > 1$ with iid states (which, in turn, implies iid dividends). Then, state today does not inform us about future payoffs; i.e. $q_{jk} = q_k$ for all j. (3.5) becomes

$$h_j = \beta \sum_{k=1}^{J} q_k a_k y_k = \bar{h}, \ \forall j.$$

That is, the expected value of next-period dividends as well as marginal utility are the same across all states. This follows because the distribution of next-period dividends are identical across states. Then, (3.7) can be written as

$$\varphi_j^e = \bar{h} + \beta \sum_{k=1}^{J} q_k \varphi_k^e = \bar{\varphi}, \ \forall j,$$

where we note again that φ_j^e does not vary across states. Since $\sum_{k=1}^{J} q_k = 1$, above simplifies to

$$\begin{aligned} \bar{\varphi} &= \bar{h} + \beta \bar{\varphi} \\ &= \frac{\bar{h}}{1-\beta}. \end{aligned}$$

Hence, price of the asset in utility terms are constant across all states. However, if we express prices in units of the consumption good using (3.6),

$$p_j = \frac{\bar{\varphi}}{u'(y_j)}.$$

So that the price of assets in units of the consumption good fluctuates with y_j. In particular, states associated with high levels of dividends imply low marginal utility and therefore high price. Price variability depends the dividend y_j and how much u' varies with c_j (i.e. curvature).

Example 3.4. Suppose J and N are large, and that returns on assets are symmetric and dividends are iid across assets and over time. Recall that equilibrium consumption is given by

$$c_j = \sum_{n=1}^{N} y_{jn}, \ \forall j.$$

That assets are symmetric means that the distribution of $y_{\cdot n}$ and $y_{\cdot n'}$ are the same. Specifically, we can think of the dividends in each state for security n to have been drawn from the same distribution of possible dividend values. Since j is large, this means that each security n would have the "same" expected value of dividends (averaging across states). It also means—along with the fact that dividends across assets are iid—that the expected value of dividends in each state would also be the same across states. Together, they imply that consumption will be approximately constant across all states; i.e.

$$c_j \approx \bar{c}, \forall j.$$

Loosely speaking, this is an "application" of law of large numbers where we let N tend to a finite number that is large. Now, consider 2 assets n and n' which may have different payoff in the current period. Suppose

$$y_n < y_{n'}.$$

The two assets will have the same price since current dividends are not included in prices; i.e.

$$p_{jn} = p_{jn'},\ \forall j, \forall n, n'.$$

Example 3.5. Suppose $U(c) = \ln c$, that there is one asset $N = 1$ with iid dividends, and $J > 1$. Then, $a_j = u'(c_j^e) = 1/c_j^e = 1/y_j$, so

$$\begin{aligned} a_j p_j^e &= \beta \sum_{k=1}^{J} q_k a_k \left(y_k + p_k^e\right) \ \because q_{jk} = q_k,\ \forall j \\ \Rightarrow p_j^e &= \beta \sum_{k=1}^{J} q_k \frac{1/y_k}{1/y_j} \left(y_k + p_k^e\right) = \beta \sum_{k=1}^{J} q_k \frac{y_j}{y_k} \left(y_k + p_k^e\right) \\ &= \beta \sum_{k=1}^{J} q_k \left(y_j + \frac{y_j}{y_k} p_k^e\right). \end{aligned}$$

Guess that the price is of the form $p_j^e = \alpha y_j$. Then, using the fact that $\sum_{k=1}^{J} q_k = 1$,

$$\begin{aligned} \alpha y_j &= \beta \sum_{k=1}^{J} q_k \left(y_j + \frac{y_j}{y_k} \alpha y_k\right) = \beta y_j \sum_{k=1}^{J} q_k \left(1+\alpha\right) = \beta y_j \left(1+\alpha\right) \\ \Rightarrow \alpha &= \frac{\beta}{1-\beta}. \end{aligned}$$

So we can see that our guess was correct. We therefore have

$$p_j^e = \frac{\beta}{1-\beta} y_j.$$

Suppose now that transition matrix is arbitrary, then

$$p_j^e = \beta \sum_{k=1}^{J} q_{jk} \left(y_j + \frac{y_j}{y_k} p_k^e\right).$$

The same guess works!

$$\begin{aligned} \alpha y_j &= \beta \sum_{k=1}^{J} q_{jk} \left(y_j + \frac{y_j}{y_k} \alpha y_k\right) = \beta y_j \sum_{k=1}^{J} q_{jk} \left(1+\alpha\right) = \beta y_j \left(1+\alpha\right) \\ \Rightarrow \alpha &= \frac{\beta}{1-\beta} \\ \Rightarrow p_j^e &= \frac{\beta}{1-\beta} y_j. \end{aligned} \tag{3.10}$$

That is, in the case of log-utility, the asset price is independent Q; e.g. it is independent of the persistence in states.

Suppose now we introduce a second asset that pays $y_{2j} = \varepsilon > 0$ in every state where ε is small relative to y_{1j}, for all j. Now, we have

$$c_j^e = y_{1j} + \varepsilon, \ \forall j.$$

The pricing formula is then

$$\begin{aligned}
a_j p_{j2}^e &= \beta \sum_{k=1}^{J} q_{jk} a_k \left(y_{k2} + p_{k2}^e\right) \\
\Rightarrow p_{j2}^e &= \beta \sum_{k=1}^{J} q_{jk} \frac{y_{1j} + \varepsilon}{y_{1k} + \varepsilon} \left(\varepsilon + p_{k2}^e\right) \\
&\approx \beta \sum_{k=1}^{J} q_{jk} \frac{y_{1j}}{y_{1k}} \left(\varepsilon + p_{k2}^e\right),
\end{aligned}$$

where we assumed that marginal utilities remain unchanged with the addition of this asset; i.e.

$$\frac{y_{1j} + \varepsilon}{y_{1k} + \varepsilon} \approx \frac{y_{1j}}{y_{1k}}, \ \forall j.$$

Guess the solution of the form $p_{j2}^e = \alpha y_{1j}$, then

$$\begin{aligned}
\alpha y_{1j} &= \beta \sum_{k=1}^{J} q_{jk} \frac{y_{1j}}{y_{1k}} \left(\varepsilon + \alpha y_{1k}\right) = \beta y_{1j} \sum_{k=1}^{J} q_{jk} \left(\frac{\varepsilon}{y_{1k}} + \alpha\right) \\
\Rightarrow \alpha &= \beta\varepsilon \sum_{k=1}^{J} \frac{q_{jk}}{y_{1k}} + \alpha\beta \sum_{k=1}^{J} q_{jk} = \frac{\beta\varepsilon}{1-\beta} \sum_{k=1}^{J} q_{jk} \frac{1}{y_{1k}} \\
\Rightarrow p_{j2}^e &= \frac{\beta\varepsilon}{1-\beta} \sum_{k=1}^{J} q_{jk} \frac{y_{1j}}{y_{1k}}.
\end{aligned}$$

Note that if we had instead "guessed", $p_{j2}^e = \alpha y_{1j} + \varepsilon$, then

$$\begin{aligned}
\alpha y_{1j} + \varepsilon &= \beta \sum_{k=1}^{J} q_{jk} \frac{y_{1j}}{y_{1k}} \left(\varepsilon + \alpha y_{1k} + \varepsilon\right) = \beta y_{1j} \sum_{k=1}^{J} q_{jk} \left(\frac{2\varepsilon}{y_{1k}} + \alpha\right) \\
\Rightarrow \alpha + \frac{\varepsilon}{y_{1j}} &= \beta 2\varepsilon \sum_{k=1}^{J} \frac{q_{jk}}{y_{1k}} + \alpha\beta \sum_{k=1}^{J} q_{jk} \\
\Rightarrow \alpha &= \frac{\beta 2\varepsilon}{1-\beta} \sum_{k=1}^{J} q_{jk} \frac{1}{y_{1k}} - \frac{\varepsilon}{y_{1j}\left(1-\beta\right)} \\
\Rightarrow p_{j2}^e &= \left(\frac{\beta 2\varepsilon}{1-\beta} \sum_{k=1}^{J} q_{jk} \frac{1}{y_{1k}} - \frac{\varepsilon}{y_{1j}\left(1-\beta\right)}\right) y_{1j} + \varepsilon \\
&= \frac{\beta 2\varepsilon}{1-\beta} \sum_{k=1}^{J} q_{jk} \frac{y_{1j}}{y_{1k}} + \frac{\beta}{1-\beta}\varepsilon.
\end{aligned}$$

Because ε has coefficient that is not one, our guess was wrong.

4 Consumption and savings

4.1 Aiyagari's model

The key feature here is that markets are incomplete and that households face labour supply shocks that cannot be fully insured (i.e. no state-dependent bonds). Thus, unlike the RBC models which are driven by aggregate shocks (all individual shocks were insured by the complete market assumption), what drives the result in Aiyagari's model is the idiosyncratic shocks, as well as the borrowing constraints that the households face.

4.1.1 The set up

Labour

- ▷ Inelastically supplied by continuum of households of a unit measure.
- ▷ Labour supplied by each household, ℓ, follows a first-order Markov process.
- ▷ $G(\ell';\ell)$ is the transition CDF with support $[\underline{\ell},\overline{\ell}]$ with $\underline{\ell}>0$ (density is denoted as $g(\ell';\ell)$). Let $\varphi_t(\ell)$ be the marginal density, then:[8]

$$\begin{aligned}
\varphi_{t+1}(\ell') &= \int_{\underline{\ell}}^{\overline{\ell}} g(\ell';\ell)\,\varphi_t(\ell)\,d\ell \\
\Rightarrow \Phi_{t+1}\left(\hat{\ell}\right) \equiv \int_{\underline{\ell}}^{\hat{\ell}} \varphi_{t+1}(\ell')\,d\ell' &= \int_{\underline{\ell}}^{\hat{\ell}} \left(\int_{\underline{\ell}}^{\overline{\ell}} g(\ell';\ell)\,\varphi_t(\ell)\,d\ell \right) d\ell' \\
&= \int_{\underline{\ell}}^{\overline{\ell}} \int_{\underline{\ell}}^{\hat{\ell}} g(\ell';\ell)\,d\ell'\varphi_t(\ell)\,d\ell \\
&= \int_{\underline{\ell}}^{\overline{\ell}} G\left(\hat{\ell};\ell\right)\varphi_t(\ell)\,d\ell
\end{aligned}$$

- ▷ G has the Feller property and has a unique invariant distribution.
- ▷ $\Phi(\ell)$ denotes the invariant CDF with density φ (i.e. $\varphi_{t+1}=\varphi_t=\varphi$):

$$\Phi\left(\hat{\ell}\right) = \int_{\underline{\ell}}^{\hat{\ell}} \varphi(\ell')\,d\ell' = \int_{\underline{\ell}}^{\overline{\ell}} G\left(\hat{\ell};\ell\right)\varphi(\ell)\,d\ell,$$

  which can be interpreted as (i) an individual's long-run average labour supply; or (ii) cross-section distribution at a point in time.

- ▷ Normalise so that aggregate labour supply is one:

$$\int_{\underline{\ell}}^{\overline{\ell}} \ell\varphi(\ell)\,d\ell = 1.$$

[8]The first line is Bayes' rule: $g(\ell';\ell)$ is the conditional density so multiplying by the marginal density $\varphi_t(\ell)$ gives the joint density. Integrating over ℓ then gives the marginal density again.

Preferences

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right],\ \beta\in(0,1)\,,$$

where u is strictly increasing, strictly concave, continuously differentiable and $u'(0)=\infty$.

Technology

$$Y=F\left(K,L\right)=\hat{F}\left(K,L\right)-\delta K,$$

where F is the neoclassical net-of-depreciation production function that exhibits CRS. Firms are perfectly competitive so capital and labour are paid their marginal products:

$$w=F_L\left(K,L\right)=F_L\left(K,1\right),$$
$$r=F_K\left(K,L\right)=F_K\left(K,1\right),$$

where we used the fact that F is CRS and we normalised aggregated labour supply to be one. K is the average level of assets held across households (given a unit measure of households).

4.1.2 Strategy

(i) Characterise the optimal decisions of a household for some given (r,w), focusing on the decisions to accumulate assets.

(ii) Characterise the joint distribution of labour supply and assets across households.

(iii) Calculate average assets, k^{av}.

(iv) Check market clearing; i.e. ask if

$$r=F_K\left(k^{av},1\right),\ w=F_K\left(k^{av},1\right)?$$

If yes, we're done. If not, go back to step 1 with a different choice of (r,w).

4.1.3 Household's problem

Let $(a,\ell)\in A\times[\underline{\ell},\overline{\ell}]$ denote a state. We will define A later.

The budget constraint for the household is

$$a'+c=(1+r)\,a+w\ell.$$

Income consists of savings (which are allowed to be negative here) from the previous period, $(1+r)\,a$, and labour income (which is stochastic here), $w\ell$. Total income can be spent on consumption, c, and savings for the next period, a'.

The Bellman equation is

$$v\left(a,\ell\right)=\max_{a'\in\Gamma(a,\ell)}\left\{u\left((1+r)\,a+w\ell-a'\right)+\beta\int_{\underline{\ell}}^{\overline{\ell}}v\left(a',\ell'\right)dG\left(\ell',\ell\right)\right\},$$
$$\Gamma\left(a,\ell\right)=\left[B,(1+r)\,a+w\ell\right],$$

where B is the borrowing limit.

Borrowing limit The upper bound for a' is clearly the income that the household has. What about the lower bound? We want to allow a' to be negative (so that households can borrow) but rule out bankruptcy. So, we set the borrowing limit B to be such that, even if the household received the worst labour shock, $\underline{\ell}$ (i.e. lowest level of labour income), it can pay the interest on its debt; i.e.

$$w\underline{\ell} \geq -rB \Leftrightarrow B \geq -\frac{w\underline{\ell}}{r}.$$

Notice that this is implied by requiring consumption to be nonnegative since

$$w\underline{\ell} \geq -rB \Leftrightarrow \underbrace{(1+r)\,B + w\underline{\ell} - B}_{=c} \geq 0$$

We define the borrowing limit as

$$B := -\frac{w\underline{\ell}}{r},$$

which is often called the *natural borrowing limit*.

Time preference vs interest rate Suppose that an agent has asset a today. If he saves everything, then the present value from doing so is

$$\frac{a\,(1+r)}{1+\rho}.$$

Suppose that $r \geq \rho$, then

$$r \geq \rho \Leftrightarrow \frac{a\,(1+r)}{1+\rho} \geq a;$$

i.e. the present value from saving a is greater. In this case, the household will always want to accumulate more assets. To make the problem economically interesting (so that assets do not diverge), we impose that

$$r < \rho.$$

Ensuring that there exists some incentive to accumulate assets We want to ensure that at least some agents in the economy have the incentive to accumulate assets.

The first-order condition to the household problem is

$$u'\left((1+r)\,a + w\ell - a'\right) \gtreqless \beta \mathbb{E}_{\ell'}\left[v_a\left(a', \ell'\right)|\ell\right],$$

where the inequality holds with $\geq$ if $a' = B$ and $\leq$ if $a' = (1+r)\,a + w\ell$. Given the Inada condition on u—i.e. $u'(0) = +\infty$—the second case (which implies zero consumption) can be ruled out.

The Envelope condition is

$$v_a\left(a, \ell\right) = (1+r)\,u'\left((1+r)\,a + w\ell - a'\right).$$

Combining gives the Euler equation:

$$u'\left((1+r)\,a + w\ell - a'\right) \geq \frac{1+r}{1+\rho}\mathbb{E}_{\ell'}\left[u'\left((1+r)\,a' + w\ell' - a''\right)|\ell\right].$$

The left-hand side is the marginal utility from consumption, and the right-hand side is the expected (discounted) marginal utility from savings. Thus, in order for the household to accumulate wealth, we need the right-hand side to be bigger than the left-hand side for some agents.

So consider a household with no wealth (i.e. $a = a' = a'' = 0$) and who receives labour supply shock ℓ. To ensure that such household saves, we need

$$u'(w\ell) < \frac{1+r}{1+\rho}\mathbb{E}_{\ell'}\left[u'(w\ell')\,|\ell\right].$$

As a sufficient condition, we can just ensure that it holds for the ℓ that minimises the left-hand side; i.e. $\ell = \overline{\ell}$. So, the requirement is

$$u'\left(w\overline{\ell}\right) < \frac{1+r}{1+\rho}\mathbb{E}_{\ell'}\left[u'(w\ell')\,|\overline{\ell}\right].$$

This says that if a household holds no asset, and the household receives the "highest" labour supply shock, then it would have the incentive to save. This is a restriction that has implications on the preferences, as well as the Markov process (there needs to be "enough" variance in the shocks).

State space We need the state space to be compact for the problem to be well defined. Specifically, we impose that there exists some $\bar{a} > 0$ such that a household with $a = \bar{a}$ has no incentive to accumulate more assets. The potential problem is that, for sufficiently risk-averse preferences, such $\bar{a}$ might not exist.

Why "opposite" directions?

The restriction we place on u is that utility exhibits *decreasing absolute risk aversion* (DARA); i.e.

$$R^a(c) = \frac{-u''(c)}{c'(c)}$$

is decreasing in c so that

$$\lim_{c\to\infty} R^a(c) = 0.$$

For example, CRRA utility satisfies this property.

This assumptions ensures that there exists $\bar{a}$ such that

$$u'\left(w\overline{\ell} + r\bar{a}\right) \geq \frac{1+r}{1+\rho}u'\left(w\underline{\ell} + r\bar{a}\right). \tag{4.1}$$

so that once the household holds $\bar{a}$ assets, then it has no incentive to accumulate more assets. Notice that we set $\ell = \overline{\ell}$ on the left-hand side to make the left-hand as small as possible, and set $\ell = \underline{\ell}$ on the right-hand side to ensure that the right-hand side is as large as possible (recall that u' is a strictly decreasing function given that u is strictly concave). This guarantees that there is no incentive to accumulate assets given $a = \bar{a}$ for any realisations of ℓ.

We can now define the state space as

$$A = [B, \bar{a}].$$

Alternative Bellman equation We can set up the Bellman equation just after the point the household has observed its total cash-on-hand; i.e.

$$y = (1+r)a + w\ell.$$

In this case, the Bellman equation becomes

$$\hat{v}(y,\ell) = \max_{a' \in \hat{\Gamma}(y)} \left\{ u'(y-a') + \beta \mathbb{E}_{\ell'}\left[\hat{v}\left((1+r)a' + w\ell', \ell'\right)|\ell\right]\right\},$$
$$\hat{\Gamma}(y) = [B, y].$$

Notice that we can drop ℓ as a state variable if the shocks were i.i.d.

4.1.4 Solving the household's problem

Standard arguments show that

- ▷ v is strictly increasing in a (the period-return function is strictly increasing in a and the feasibility set is monotone in a);
- ▷ v is strictly increasing in ℓ if G is monotone (the period-return function is strictly increasing in ℓ, the feasibility set is monotone in ℓ);
- ▷ v is strictly concave in a (the period-return function is strictly concave and the feasibility set is convex);
- ▷ v is differentiable in a if $a' \in \text{int}\Gamma(a,\ell)$.

The optimal policy is characterised by the first-order condition:

$$u'\left((1+r)a + w\ell - a'\right) \geq \beta \mathbb{E}_{\ell'}\left[v_a(a',\ell')|\ell\right],$$

where the inequality holds with $\geq$ if $a' = B$ (recall we ruled out the other boundary case, $a' = (1+r)a + w\ell$, given Inada condition that $u'(0) = +\infty$).

We also have the envelope condition:

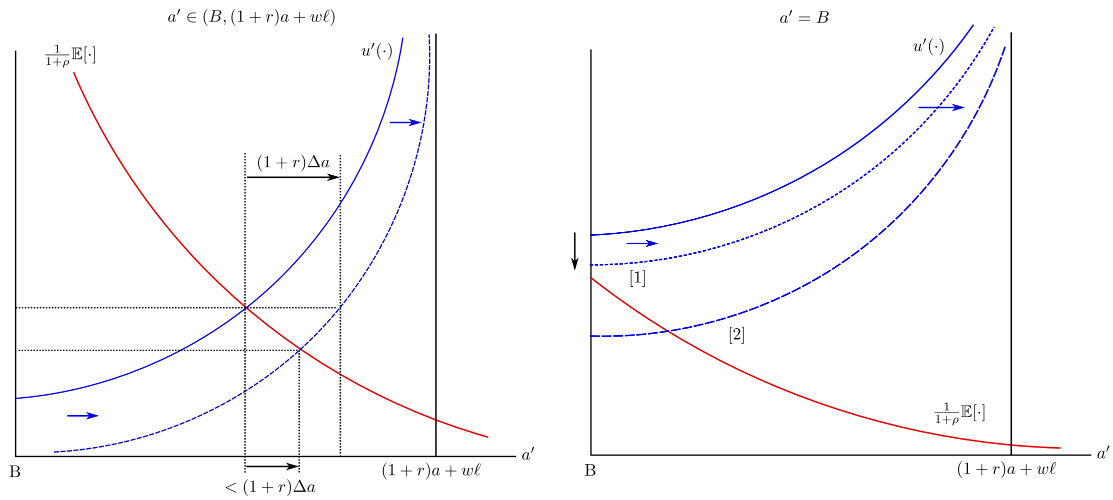
$$v_a(a,\ell) = (1+r)u'\left((1+r)a + w\ell - a'\right).$$

As usual, we can plot the two sides of the equation:

- ▷ LHS: Since u is strictly concave, u' is strictly decreasing. Thus, it is strictly increasing in a' and strictly decreasing in a. Also, as $a' \to (1+r)a + w\ell$, then $u'(\cdot) \to +\infty$ due to the Inada condition.
- ▷ RHS: Since v is strictly concave, v_a is strictly decreasing in a.

There are two cases to consider: $a' \in (B, \bar{a})$ and $a' = B$.

The upper bound should move when a increases!



Effect of increase in a A higher a moves the LHS curve down and to the right. In the interior case, optimal policy a' is greater, although some of the increase in a translates to higher consumption. In the boundary case, optimal policy may not change (if the boundary condition is not broken) but consumption nevertheless increases. In case the LHS curve shifts to [1], then households consumes the entire increase in a. However, if the shift is to [2], then household splits the increase in a between consumption and savings.

Letting Δa denote the change in a, we can see from the figure that the increase in a', $\Delta a'$, is always less; i.e.

$$\Delta a' < (1+r)\,\Delta a.$$

This means that the slope of the optimal policy is less than $1+r$. So there are two cases:

- ▷ $\Delta a' < \Delta a < (1+r)\,\Delta a$ in which case the optimal policy has slope less than one;
- ▷ $\Delta a < \Delta a' < (1+r)\,\Delta a$ in which case the optimal policy ha slope between one and $(1+r)$.

The latter case holds if u' is sufficiently flat (i.e. utility is close to linear, which, in turn, implies that RHS is also flat). The significance of this case is explained next.

Effect of increase in ℓ A change in ℓ shifts both the LHS and RHS curves. Although we know that the LHS moves down and to the right, we do not know how the RHS moves. Therefore, the effect of the increase in ℓ on a' is ambiguous. If we assume i.i.d. shocks, then the RHS curve becomes independent of ℓ and so the comparative static becomes isomorphic to the case for a.

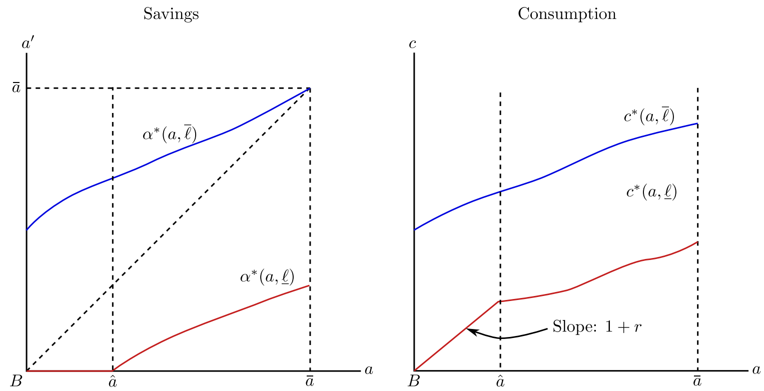
4.1.5 Optimal policy

Let $\alpha^*(a,\ell)$ and $c^*(a,\ell)$ denote the optimal policies for asset and consumption respectively.

Slope of $\alpha^*\left(\cdot,\bar{\ell}\right)$ is less than 1 First, assume that the slope of $\alpha^*\left(\cdot,\bar{\ell}\right)$ is less than 1; i.e. the household saves less than one-to-one when a increases ($\Delta a' < \Delta a$). Recall $\bar{a}$ denotes the level of assets above which the agent has no more incentive to accumulate assets. Here, we defined this as

$$\bar{a} = \alpha^*\left(\bar{a},\bar{\ell}\right);$$

i.e. we define $\bar{a}$ relative to $\bar{\ell}$ because we assume that $\alpha^*(\cdot, \ell)$ is ordered by ℓ.

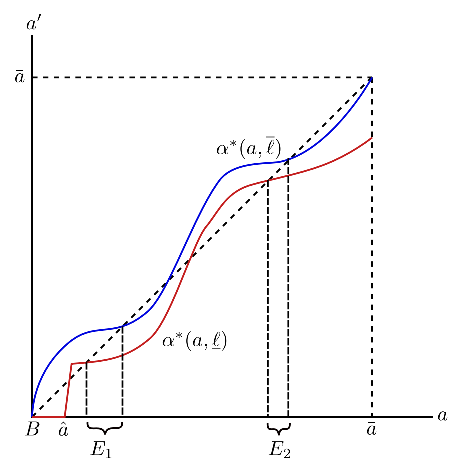


Observe that:

- ▷ Recall that we defined $\bar{a}$ as the point level of assets that no one would have incentive to save more—when ℓ is ordered, then this implies that $\alpha^*\left(a, \bar{\ell}\right)$ at $a = \bar{a}$ must intersect the 45 degree line so that $a' = a$; i.e. it is optimal not to save more.
- ▷ The flat portion of $\alpha^*(\alpha, \underline{\ell})$ represents boundary case in which the household would like to borrow but cannot, due to the borrowing constraint.
- ▷ The slope of $c^*(\alpha, \ell)$ is initially linear until $\tilde{a}$, the point at which the household begins accumulating assets. The slope before this point is $(1 + r)$ since a unit of a gives $(1 + r)$ units of consumption (see the budget constraint).

The ergodic set in this case is the entire state space $[B, \bar{a}]$. This means that, even in the long-run, there will be a positive "mass" of individuals who hold $a = 0$ assets.

Slope of $\alpha^*\left(\cdot, \bar{\ell}\right)$ is greater than 1 This means that $\Delta a' > \Delta a$. In this case, $\alpha^*\left(\cdot, \bar{\ell}\right)$ may cross the 45 degree line multiple times. The figure below shows the case in which there are 2 ergodic sets, which can be interpreted as representing a two-class society: E_2 is the rich, and E_1 is the poor. Observe that household in the transient states (T's) will diverge to the one of the surrounding ergodic set. More generally, there may be more ergodic and transient states (alternating) until $\bar{a}$.



Note that if the curve lies above the 45 degree line, then $a' > a$ so that households are accumulating assets (i.e. saving). On the other hand, if the curve lies below the 45 degree line, then $a' < a$ so that households are dissaving. The ergodic set is given by the interval defined by the point at which $\alpha^*\left(a, \bar{\ell}\right)$ crosses the 45 degree line from above to below, and the point at which $\alpha^*\left(a, \underline{\ell}\right)$ does the same. To the left (right) of these points, households are savings (dissaving) so that we are lead back to the same point.

Notice, in particular, that the interval $[B, \hat{a}]$ is not an ergodic set—hence, in this case, in the long run, everyone will hold some assets and we would not have any mass on $a = 0$ as in the previous case.

4.1.6 $B = 0$ and iid shocks

Now suppose that $B = 0$ and that labour shocks are i.i.d. As we discussed above, in this case,

$$\begin{aligned}\hat{v}(y, \ell) &= \max_{a' \in \hat{\Gamma}(y)} \left\{u'(y - a') + \beta \mathbb{E}_{\ell'}\left[\hat{v}\left((1+r)a' + w\ell', \ell'\right)\right]\right\}, \\ \hat{\Gamma}(y) &= [B, y].\end{aligned}$$

where $y = (1+r)a + w\ell$ and we may drop ℓ as the state variable. In other words, savings depends only on cash-in-hand, y. We can write therefore write the optimal policy as

$$\alpha^*(a, \ell) = \sigma\left((1+r)a + w\ell\right),$$

where σ is nondecreasing (strictly increasing if $\sigma > 0$ since α^* is strictly increasing in both a and ℓ in the i.i.d. case with $B = 0$).

Fix some asset level $a' = z$, then

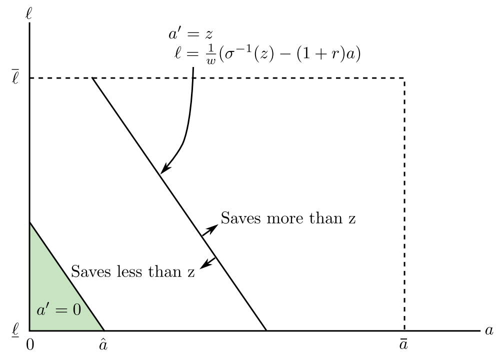
sigma is the optimal policy

$$\begin{aligned}\sigma\left((1+r)a + w\ell\right) \leq z &\Leftrightarrow (1+r)a + w\ell \leq \sigma^{-1}(z) \\ &\Leftrightarrow \ell \leq \frac{1}{w}\left[\sigma^{-1}(z) - (1+r)a\right].\end{aligned}$$

For $z > 0$, we may define σ^{-1} in the usual fashion since σ is strictly increasing. However, z, in general is zero, for a range of values of cash-in-hand. Thus, for $z = 0$, define σ^{-1} to be the largest value for $(1+r)a + w\ell$ that leads to $\sigma = 0$; i.e.

$$\sigma^{-1}(0) = \lim_{z \downarrow 0} \sigma^{-1}(z).$$

We can now write the "indifference curves" that depicts the combinations of a and ℓ that leads to the same value of savings $a' = z$. Households with pairs of (a, ℓ) that lie above the indifference curve will save more than z; and those that lie below the indifference curve will save less than z. For $a' = z = 0$, observe that there will be mass of households.



Recall $\Phi(\ell)$ is the stationary distribution of labour supply and $\varphi(\ell)$ is its density. Let $M(a)$ denote the stationary distribution of assets. For i.i.d. shocks, the product of the two gives the the joint distribution of (a, ℓ). To solve for $M(a)$, we first compute the probability that a' is less than some $z \in [0, \overline{a}]$:

$$\begin{aligned}
\mathbb{P}(a' \leq z) &= \mathbb{P}(\sigma((1+r)a + w\ell) \leq z) \\
&= \mathbb{P}\left((1+r)a + w\ell \leq \sigma^{-1}(z)\right) \\
&= \mathbb{P}\left(a \leq \frac{\sigma^{-1}(z) - w\ell}{1+r}\right) \\
&= \int_{\underline{\ell}}^{\bar{\ell}} M\left(\frac{\sigma^{-1}(z) - w\ell}{1+r}\right)\varphi(\ell)\, d\ell,
\end{aligned}$$

where the integral converts the distribution (a, ℓ) into a marginal distribution—the integral "sums" the probability that next period asset is less than z given a specific value of ℓ across all possible values of ℓ (since we may reach $a' \leq z$ from many different levels of ℓ).

$M(a)$ can be characterised as a solution to fixed-point problems for each value of z:

$$M(z) = \int_{\underline{\ell}}^{\bar{\ell}} M\left(\frac{\sigma^{-1}(z) - w\ell}{1+r}\right)\varphi(\ell)\, d\ell, \ \forall z \in [0, \bar{a}].$$

With $M(a)$ in hand, we can calculate the average capital:

$$k^{av} = \int_0^{\bar{a}} a dM(a).$$

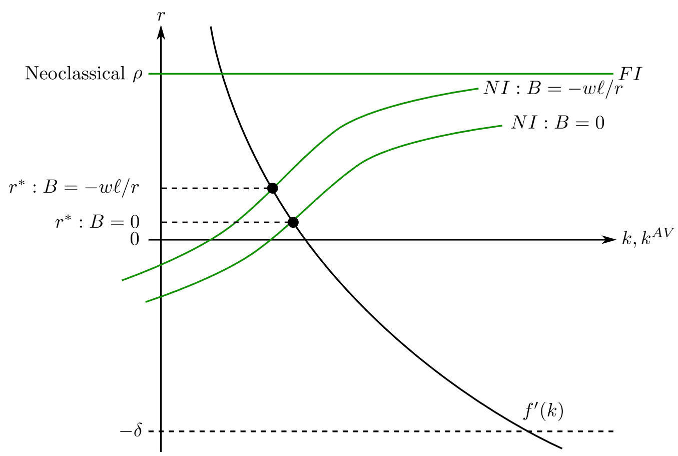
Then, we ask if the following conditions hold:

$$\begin{aligned}
f'(k^{av}) &= r, \\
f(k^{av}) - k^{av} f'(k^{av}) &= w;
\end{aligned}$$

i.e. whether marginal product of capital is equated with the interest rate and whether the marginal product of labour is equated with the wage rate.

In the full insurance/complete market case, the supply of capital is horizontal at ρ. In the incomplete market case, we see that the interest rate will be lower, and lower still for the case when $B = 0$ relative to $B = -w\ell/r$. Note that supply of assets need not be increasing. An increase in k has both an income effect (from a higher w) and a substitution effect (from the lower interest).

How do we know r^* is lower than ρ?


r
Neoclassical ρ
FI
$NI : B = -w\ell/r$
$NI : B = 0$
$r^* : B = -w\ell/r$
$r^* : B = 0$
0
k, k^{AV}
$f'(k)$
$-\delta$

5 Menu cost models

One-time decision to shut down a plant can be interpreted as a decision to exercise the one-time (i.e. discrete) option to shutdown.

5.1 Option pricing: Deterministic

Consider a firm's decision to shutdown its plant given that it faces falling profits over time. The plant has a salvage value (e.g. the value of the scraps) which is fixed at $S > 0$. In such a problem, working in continuous time is simpler as it avoids the issue of under/over-shooting.

Let $X(t)$ be the state variable, which can be interpreted as (log of) capital or demand. The law of motion for $X(t)$ is

$$X(t) = x_0 - \delta t, \tag{5.1}$$

where $\delta > 0$ is the rate at which $X(t)$ falls over time.[9] Let $\pi(x)$ denote the profit function that is strictly increasing in the state variable x, and the interest rate is denoted as $r > 0$.

The firm's problem is to decide when to shutdown the plant.

We explore three approaches to solving this problem:

(i) optimise by choosing a date T to shut down;

(ii) optimise by choosing a threshold of state variable b^* to shut down;

(iii) Bellman equation.

5.1.1 Approach 1: Solving for threshold rule in terms of date

In this approach, we wish to optimise by choosing a date T to shut down the plant. The present value of profits for having the plant open from date 0 to T is $\int_0^T e^{-rt}\pi(X(t))\,dt$. At date T, when the plant is shut down, the firm receives the salvage value S, and its discounted value (as of period 0) is $e^{-rT}S$. Thus, we can write the problem as

$$V(x_0) = \max_{T\geq 0}\left\{\int_0^T e^{-rt}\pi(X(t))\,dt + e^{-rT}S\right\} \tag{5.2}$$

subject to the law of motion (5.1).

To make the problem economically interesting, we make the following set of assumptions.

Assumption 1. *Assume that $r, \delta, S > 0$ and that $\pi(\cdot)$ is bounded, continuous and strictly increasing with*

$$\lim_{x\to-\infty}\pi(x) < rS < \lim_{x\to+\infty}\pi(x). \tag{5.3}$$

That $\delta > 0$ ensures that profits are falling over time (if not, then it will never be optimal to shut down the plant). $S > 0$ means that there is some positive value from selling the scraps. To interpret (5.3), it helps to write the first-order condition of (5.2):

$$e^{-rT^*}\pi(X(T^*)) - re^{-rT^*}S \leq 0 \text{ with equality if } T^* > 0,$$

[9] For example, if we interpret X as log of capital (i.e. $X(t) = \ln K(t)$), then

$$X(t) = X_0 - \delta t \Leftrightarrow K(t) = K(0)\exp[-\delta t].$$

which simplifies to

$$\pi\left(X\left(T^*\right)\right) \leq rS \text{ with equality if } T^* > 0.$$

Thus, (5.3) (together with the assumption that π is continuous and strictly increasing) ensures that the intersection between $\pi\left(X\left(t\right)\right)$ and rS is unique. We can therefore interpret Assumption 1 as ensuring that the optimal policy (i.e. the threshold rule based on T^*) is unique and finite.

Define

$$b^* \coloneqq X\left(T^*\right) = x_0 - \delta T^*,$$

which is the value of the state at the optimal shutdown date. Using b^*, we can express the optimal shutdown rule, T^*, as

$$T^* = \begin{cases} \frac{x_0 - b^*}{\delta} & \text{if } x_0 \geq b^* \\ 0 & \text{if } x_0 < b^* \end{cases}. \tag{5.4}$$

We may also rewrite the the value function $V\left(x_0\right)$ as

$$V\left(x_0\right) = \begin{cases} \left\{\int_0^{(x_0-b^*)/\delta} e^{-rt}\pi\left(X\left(t\right)\right)dt + e^{-r(x_0-b^*)/\delta}S\right\} & \text{if } x_0 \geq b^* \\ S & \text{if } x_0 < b^* \end{cases}.$$

5.1.2 Approach 2: Solving for threshold rule in terms of state variable

Under approach 1, we characterised the optimal shut-down date T^*. However, the form of the optimal policy, (5.4), suggests that we can also formulate the problem in terms of a critical value b for the state variable. We use the substitution rule to convert the integral with respect to time that we derived under approach 1 into an integral with respect to state. Specifically, we use the substitution rule to alter (5.2) so that we maximise with respect to the threshold b (corresponding to $t = T$) given initial state x (corresponding to $t = 0$).

Substitution we would like to make is:

$$\xi \coloneqq X\left(t\right) = x_0 - \delta t \Leftrightarrow t = \frac{x_0 - \xi}{\delta}.$$

Note that

$$d\xi = -\delta dt \Leftrightarrow dt = -\frac{1}{\delta}d\xi,$$

and

$$X\left(0\right) = x_0,$$
$$b \coloneqq X\left(T\right) = x_0 - \delta T \Leftrightarrow T = \frac{x_0 - b}{\delta}.$$

Therefore, (5.2) becomes:[10]

$$\begin{aligned} V(x_0) &= \max_{b \le x_0} F(x_0, b) \\ &= \max_{b \le x_0} \left\{ -\frac{1}{\delta} \int_{x_0}^{b} e^{-\frac{r}{\delta}(x_0-\xi)} \pi(x) \, d\xi + e^{-\frac{r}{\delta}(x_0-b)} S \right\} \\ &= \max_{b \le x_0} \left\{ \frac{1}{\delta} \int_{b}^{x_0} e^{-\frac{r}{\delta}(x_0-\xi)} \pi(x) \, d\xi + e^{-\frac{r}{\delta}(x_0-b)} S \right\}. \end{aligned} \tag{5.5}$$

The first-order condition is given by

$$\frac{\partial F(x,b)}{\partial b} = -\frac{1}{\delta} e^{-\frac{r}{\delta}(x_0-b)} (\pi(b) - rS) \ge 0 \text{ with equality if } b < x_0. \tag{5.6}$$

The optimal solution is then

$$b = \min\{x_0, b^*\},$$

where b^* solves $\pi(b^*) = rS$. The value function $V(x_0)$ is:

$$V(x_0) = \begin{cases} \left\{ \frac{1}{\delta} \int_{b^*}^{x_0} e^{-r(x_0-b^*)/\delta} \pi(\xi) \, d\xi + e^{-r(x_0-b^*)/\delta} S \right\} & \text{if } x_0 \ge b^* \\ S & \text{if } x_0 < b^* \end{cases}.$$

Since $\delta > 0$, the state variable is falling over time, and together with the assumption that $\pi(x)$ is strictly increasing, profits must be falling over time/state. Thus, the optimal policy is a threshold policy as was before (but this time in terms of the state variable, rather than the date). For the threshold b, call the interval $(-\infty, b]$ the shut-down (or action) region and the interval $(b, +\infty)$ the continuation (or inaction) region.

[10] Recall the substitution rule:

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) \, dx = \int_{\alpha}^{\beta} f(\varphi(z)) \varphi'(z) \, dz.$$

Here,

$$\begin{aligned} \varphi(z) &= \frac{x-z}{\delta}, \\ f(t) &= e^{-rt} \pi(X(t)). \end{aligned}$$

Hence,

$$\begin{aligned} \varphi(\beta) = T &\Leftrightarrow \frac{x-\beta}{\delta} = T \\ &\Leftrightarrow \beta = x - \delta T = b, \\ \varphi(\alpha) = 0 &\Leftrightarrow \frac{x-\alpha}{\delta} = 0 \\ &\Leftrightarrow \alpha = x, \\ \varphi'(z) &= -\frac{1}{\delta}, \\ f(\varphi(z)) &= e^{-r(x-z)/\delta} \pi\left(X\left(\frac{x-z}{\delta}\right)\right) \\ &= e^{-r(x-z)/\delta} \pi(z). \end{aligned}$$

5.1.3 Approach 3

When deriving the expected discounted value from using an arbitrary threshold rule b, given the initial state x_0—i.e. $F(x,b)$ in (5.5)—under approach 2, we began with the integral with respect to time that we derived under approach 1 and converted it into the integral with respect to state. Here, we will show that the function $F(x,b)$ satisfies a Bellman equation in the continuation region (together with a boundary condition). Under approach 2, we showed that the optimal threshold b^* can be characterised with the derivative of $F(x,b)$ with respect to b, fixing state x, being equal to zero, (5.6). We show here that b^* can also be characterised with the derivative of $F(x,b)$ with respect to x, evaluated at $x=b$, fixing b.

Given initial state x_0, fix $b < x_0$. Then, we can choose a time interval $\varepsilon > 0$ sufficiently small so that the plant operates for at least ε units of time. The value from operating for ε units of time is

$$\begin{aligned} F(x_0,b) &= \int_0^{\varepsilon} e^{-rt}\pi(X(t))\,dt + e^{-r\varepsilon}F(X(\varepsilon),b) \\ [1] &\approx \pi(x_0)\,\varepsilon + e^{-r\varepsilon}F(x_0-\delta\varepsilon,b) \\ [2] &\approx \pi(x_0)\,\varepsilon + \frac{1}{1+r\varepsilon}F(x_0-\delta\varepsilon,b) \\ [3] &\approx \pi(x_0)\,\varepsilon + \frac{1}{1+r\varepsilon}\left[F(x_0,b) - \delta F_x(x_0,b)\,\varepsilon\right], \end{aligned}$$

where [1] uses the fact that ε is small so the integral is approximately equal to $\pi(x_0)\,\varepsilon$; [2] uses the fact that $e^{-r\varepsilon} \approx (1+r\varepsilon)^{-1}$; and [3] uses first-order Taylor expansion of $F(X(\varepsilon),b)$ around $\varepsilon = 0$. Multiplying through by $(1+r\varepsilon)$,

$$\begin{aligned} (1+r\varepsilon)\,F(x_0,b) &\approx \pi(x_0)\,\varepsilon + \underbrace{r\pi(x_0)\,\varepsilon^2}_{\approx 0} + F(x_0,b) - \delta F_x(x_0,b)\,\varepsilon \\ \Rightarrow r\varepsilon F(x_0,b) &\approx \pi(x_0)\,\varepsilon - \delta F_x(x_0,b)\,\varepsilon \\ \Rightarrow rF(x,b) &= \pi(x) - \delta F_x(x,b), \end{aligned} \tag{5.7}$$

where, in the last line, we removed the subscript 0. This is a (continuous-time version of) Bellman equation. The left-hand side can be interpreted as the current return, and the right-hand side as the current profit flow less capital loss from falling X.

Observe that (5.7) is a first-order ordinary differential equation (ODE) since b is fixed. We can solve for $F(x,b)$ given a functional form for $\pi(\cdot)$ and a boundary condition, which, in this case, is given by

$$\lim_{x \downarrow b} F(x,b) = S. \tag{5.8}$$

That is, when the state variable falls to the threshold value, then the value of F must be the salvage value (since we are using a threshold rule). We can show that integrating (5.7) and using the boundary condition gives the function (5.5). Rearranging gives

$$F_x(x_0,b) = \frac{1}{\delta}\pi(x_0) - \frac{r}{\delta}F(x_0,b).$$

Guess that the solution is given by

$$F(x_0,b) = \exp\left(-\frac{r}{\delta}x_0\right)c(x_0).$$

Then,

$$\begin{aligned}F_x(x_0,b) &= -\frac{r}{\delta}\exp\left(-\frac{r}{\delta}x_0\right)c(x_0)+\exp\left(-\frac{r}{\delta}x_0\right)c_x(x_0)\\ &= -\frac{r}{\delta}F(x_0,b)+\exp\left(-\frac{r}{\delta}x_0\right)c_x(x_0).\end{aligned}$$

Then, we need that

$$\begin{aligned}\exp\left(-\frac{r}{\delta}x_0\right)c_x(x_0) &= \frac{1}{\delta}\pi(x_0)\\ \Leftrightarrow c_x(x_0) &= \frac{1}{\delta}\pi(x_0)\exp\left(\frac{r}{\delta}x_0\right)\\ \Rightarrow \int_b^{x_0} c_x(\xi)\,d\xi &= \int_b^{x_0}\frac{1}{\delta}\pi(\xi)\exp\left(\frac{r}{\delta}\xi\right)d\xi\\ \Rightarrow c(x_0) &= \int_b^{x_0}\frac{1}{\delta}\pi(\xi)\exp\left(\frac{r}{\delta}\xi\right)d\xi + c(b).\end{aligned}$$

Then,

$$\begin{aligned}F(x_0,b) &= \exp\left(-\frac{r}{\delta}x_0\right)\left(\int_b^{x_0}\frac{1}{\delta}\pi(\xi)\exp\left(\frac{r}{\delta}\xi\right)d\xi + c(b)\right)\\ &= \int_b^{x_0}\frac{1}{\delta}\pi(\xi)\exp\left(\frac{r}{\delta}(\xi-x_0)\right)d\xi+\exp\left(-\frac{r}{\delta}x_0\right)c(b)\\ &= \frac{1}{\delta}\int_b^{x_0}\pi(\xi)e^{-r(x_0-\xi)/\delta}d\xi+\exp\left(-\frac{r}{\delta}x_0\right)c(b).\end{aligned}$$

To solve for $c(b)$, we use the boundary condition (5.8).

$$\begin{aligned}S = F(b,b) &= \frac{1}{\delta}\int_b^{b}\pi(\xi)e^{-r(b-\xi)/\delta}d\xi+\exp\left(-\frac{r}{\delta}x_0\right)c(b)\\ &= \exp\left(-\frac{r}{\delta}b\right)c(b)\\ \Leftrightarrow c(b) &= \exp\left(\frac{r}{\delta}b\right)S.\end{aligned}$$

Hence,

$$\begin{aligned}F(x_0,b) &= \frac{1}{\delta}\int_b^{x_0}\pi(\xi)e^{-r(x_0-\xi)/\delta}d\xi+\exp\left(-\frac{r}{\delta}x_0\right)\exp\left(\frac{r}{\delta}b\right)S\\ &= \frac{1}{\delta}\int_b^{x_0}\pi(\xi)e^{-r(x_0-\xi)/\delta}d\xi+e^{-r(x_0-b)/\delta}S,\end{aligned}$$

which is exactly $F(x_0,b)$ defined in (5.5).

Consider $F(b,b)=S$. Suppose we increase the (initial) state to $x=b+\varepsilon$, then the firm will collect profits for the duration that it takes for the state to fall from $b+\varepsilon$ to b, given by $\varepsilon/\delta>0$. The rate of profit is $\pi(b)$ so the gain in profits is approximately $\pi(b)\varepsilon$. However, moving the state also means that the firm obtains the salvage value later; i.e. it will be discounted for the time interval that it takes the state to fall from $b+\varepsilon$ to b. The rate of loss is the interest rate (same as the discount rate) r, so the loss here is rS.[11] Thus, the net gain from the increase in the state is

$$\pi(b)\varepsilon - rS. \tag{5.9}$$

Suppose instead that we decrease the threshold to $\tilde{b} = b - \varepsilon$. The firm again collects some profits for ε/δ units of time. The gain from this is, again, $\pi(b)\varepsilon$. Since it takes more time for the firm to receive the salvage value, there is a loss given, again, by rS. The net gain from the decrease in the threshold is the same as (5.9). Thus, at $x = b$, a positive perturbation of the state and a negative perturbation (of the same magnitude) on the threshold have identical effects;[12] i.e.

$$F_x(b,b) = -F_b(b,b), \ \forall b. \tag{5.10}$$

Recall that, under approach 2, the optimal choice of threshold, b^*, satisfies

$$F_b(x,b^*) = \frac{\partial F(x,b^*)}{\partial b} = 0, \ \forall x \geq b^*.$$

Then, using (5.10), b^* also satisfies

$$-F_b(b^*,b^*) = F_x(b^*,b^*) = \left.\frac{\partial F(x,b^*)}{\partial x}\right|_{x=b^*} = 0.$$

In contrast to (5.6), which involves a derivative with respect to the threshold b holding the state x fixed, we see from above that we can instead use the derivative with respect to the state x, evaluated at $x = b^*$, holding the threshold fixed.

[11] We do not multiply this by ε since the salvage value is earned at a particular point in time, as opposed to profits, which is a flow over time.

[12] If $x > b$, the threshold b is reached at date $t = (x-b)/\delta > 0$ and the changes in the firm's profits must be discounted by $e^{-r(x-b)/\delta}$. To see this, evaluate (5.7) at $x = b$,

$$rS = rF(b,b) = \pi(b) - \delta F_x(b,b) = \pi(b) - \delta F_x(b,b).$$

So we can write the first-order condition (5.6) as

$$F_b(b,b) = -e^{-r(x_0-b)/\delta} F_x(b,b) = 0, \ \forall x_0 > b.$$

Proposition 5.1. *Recall (5.5)*

$$V(x) = \max_{b \leq x} F(x, b)$$

and denote the optimal threshold from solving the problem b^*. Then, $V(x)$ satisfies:

(i) (Bellman equation)

$$\begin{aligned} rV(x) &= \pi(x) - \delta V'(x), \ \forall x \geq b^*, \\ V(x) &= S(b^*), \ \forall x < b^*. \end{aligned}$$

(ii) (Value matching) $\lim_{x \downarrow b^*} V(x) = S(b^*) = \lim_{x \uparrow b^*} V(x)$*;*

(iii) (Smooth pasting) $\lim_{x \downarrow b^*} V'(b^*) = 0 = \lim_{x \uparrow b^*} V'(b^*)$.[a]

[a] This condition comes from the fact that

$$\left.\frac{\partial F(x, b^*)}{\partial x}\right|_{x=b^*} = 0 \Leftrightarrow V'(x)\big|_{x=b^*} = 0.$$

Proof. From before, we know that there is a (one-dimensional) family of solutions to the Bellman equation(s), given by

$$F(x, b) = \begin{cases} \left\{\frac{1}{\delta}\int_b^{x_0} e^{-r(x_0-b)/\delta}\pi(\xi)\, d\xi + e^{-r(x_0-b)/\delta} S\right\} & \text{if } x_0 \geq b \\ S & \text{if } x_0 < b \end{cases}.$$

The function is continuous at $x = b$, in particular, it is continuous at $x = b^*$. Hence, value matching is satisfied.

The optimal threshold b^* satisfies

$$F_x(x, b^*) = \frac{\partial F(x, b^*)}{\partial b} = 0, \ \forall x \geq b^*.$$

Since $F_x(b, b) = -F_b(b, b)$, in particular,

$$F_x(b^*, b^*) = -F_b(b^*, b^*) = 0,$$

which establishes the smooth pasting condition. ■

Remark 5.1. The Bellman equation is a first-order ODE which has a one-dimensional family of solutions of the form

$$V(x) = V_p(x) + C_1 e^{Rx},$$

where V_p is any particular solution, and R is the root of the homogeneous equation (i.e. solution to $rV(x) = -\delta V'(x)$). We can let $V_p(x)$ to be the value of operating forever, and choose C_1 to satisfy the value matching condition. If we do this, then we can interpret $C_1 e^{Rx}$ as the option value of shutting down.

To see this, note that the value of operating the firm forever is given by

$$V_p(x_0) = \int_0^\infty e^{-rt}\pi(t)\, dt.$$

Using the substitution rule, we can write

$$V_p\left(x_0\right) = \frac{1}{\delta}\int_{-\infty}^{x_0} e^{-r(x_0-\xi)/\delta}\pi\left(\xi\right)d\xi. \tag{5.11}$$

For this to be a particular solution, it must satisfy the ODE (5.7). Using Liebniz's rule,

$$\begin{aligned}
V_p'\left(x_0\right) &= \frac{1}{\delta}\pi\left(x_0\right) - \frac{r}{\delta^2}\int_0^{x_0} e^{-r(x_0-\xi)/\delta}\pi\left(\xi\right)d\xi \\
&= \frac{1}{\delta}\pi\left(x_0\right) - \frac{r}{\delta}V_p\left(x_0\right) \\
\Leftrightarrow rV_p\left(x_0\right) &= \pi\left(x_0\right) - \delta V_p'\left(x_0\right)
\end{aligned}$$

so that we verified that (5.11) satisfies (5.7). The solution to the homogeneous equation, denoted $V_h\left(x_0\right)$, is

$$\begin{aligned}
rV_h\left(x_0\right) &= -\delta V_h'\left(x_0\right) \\
\Leftrightarrow -\frac{r}{\delta} &= \frac{V_h'\left(x_0\right)}{V_h\left(x_0\right)} \\
\Rightarrow \int -\frac{r}{\delta}dx &= \int \frac{V_h'\left(x\right)}{V_h\left(x\right)}dx \\
\Rightarrow -\frac{r}{\delta}x &= \ln V_h\left(x\right) + C \\
\Rightarrow V_h\left(x\right) &= e^{-C}e^{-rx/\delta} \\
&= C_1 e^{-rx/\delta},
\end{aligned}$$

where $C_1 = e^{-C}$ is an arbitrary constant. Then,

$$\begin{aligned}
V\left(x_0\right) &= V_p\left(x_0\right) + C_1 e^{Rx_0} \\
&= \frac{1}{\delta}\int_{-\infty}^{x_0} e^{-r(x_0-\xi)/\delta}\pi\left(\xi\right)d\xi + C_1 e^{-rx_0/\delta},
\end{aligned}$$

To solve for C_1, we use value matching, which gives us that

$$\begin{aligned}
S = V\left(b^*\right) &= \frac{1}{\delta}\int_{-\infty}^{b^*} e^{-r(b^*-\xi)/\delta}\pi\left(\xi\right)d\xi + C_1 e^{-rb^*/\delta} \\
&= e^{-rb^*/\delta}\frac{1}{\delta}\int_{-\infty}^{b^*} e^{r\xi/\delta}\pi\left(\xi\right)d\xi + C_1 e^{-rb^*/\delta} \\
\Leftrightarrow C_1 &= Se^{rb^*/\delta} - \frac{1}{\delta}\int_{-\infty}^{b^*} e^{r\xi/\delta}\pi\left(\xi\right)d\xi \\
&= e^{rb^*/\delta}\left(S - V_p\left(b^*\right)\right)
\end{aligned}$$

Then,[13]

$$V(x_0) = V_p(x_0) + e^{-r(x_0-b^*)/\delta}(S - V_p(b^*)),$$

where $S - V_p(b^*)$ is the value of the option, so that the second-term is the present value of the value of the option (and recall that $V_p(x_0)$ is the value of operating forever).

To pin down b^*, we use the smooth pasting condition:

$$\begin{aligned} 0 &= V'(b^*) \\ &= V_p'(b^*) - \frac{r}{\delta}(S - V_p(b^*)), \end{aligned}$$

where

$$\begin{aligned} V_p'(b^*) = \left.\frac{\partial V_p(x)}{\partial x}\right|_{x=b^*} &= \left.\frac{\partial}{\partial x}\left(\frac{1}{\delta}\int_{-\infty}^{x} e^{-r(x-\xi)/\delta}\pi(\xi)\,d\xi\right)\right|_{x=b^*} \\ &= \frac{1}{\delta}\pi(b^*) - \frac{r}{\delta}\left(\frac{1}{\delta}\int_{-\infty}^{b^*} e^{-r(b^*-\xi)/\delta}\pi(\xi)\,d\xi\right) \\ &= \frac{1}{\delta}\pi(b^*) - \frac{r}{\delta}V_p(b^*). \end{aligned}$$

Hence,

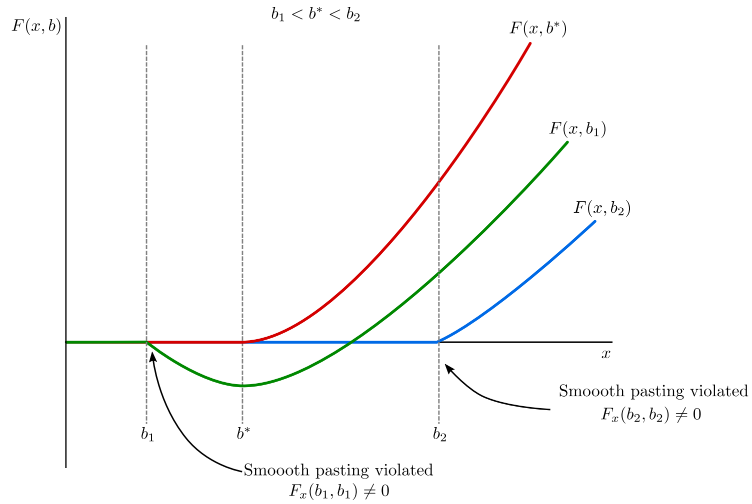
$$\begin{aligned} \frac{1}{\delta}\pi(b^*) - \frac{r}{\delta}V_p(b^*) &= \frac{r}{\delta}(S - V_p(b^*)) \\ \Rightarrow \pi(b^*) &= rS, \end{aligned}$$

which is the first-order condition we saw under approach 2.

The figure below shows the function $F(x, b)$ looks for various values of b.

[13] To verify that this equation indeed holds:

$$\begin{aligned} V(x_0) &= V_p(x_0) + e^{-r\left(x_0-b^*\right)/\delta}(S - V_p(b^*)) \\ &= \frac{1}{\delta}\int_0^{x_0} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi + Se^{-r\left(x_0-b^*\right)/\delta} - \frac{1}{\delta}\int_0^{b^*} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi \\ &= \frac{1}{\delta}\int_0^{b^*} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi + \frac{1}{\delta}\int_{b^*}^{x_0} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi \\ &\quad + Se^{-r\left(x_0-b^*\right)/\delta} - \frac{1}{\delta}\int_0^{b^*} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi \\ &= \frac{1}{\delta}\int_{b^*}^{x_0} e^{-r\left(b^*-\xi\right)/\delta}\pi(\xi)\,d\xi + Se^{-r\left(x_0-b^*\right)/\delta}. \end{aligned}$$



If the threshold level is too low, such as in the case of $b_1 < b^*$, increasing the state initially decreases $F(x,b)$ because present value of $\pi(x)$ is low relative to the present value of salvage value. If the threshold level is too high, such as in the case of $b_2 > b^*$, then increasing the state beyond b_2 leads to higher profit. Observe that, by construction, value matching implies that the curve joins S at the threshold. However, smooth pasting is violated at threshold levels different from the optimal threshold.

5.1.4 Exercise

Let r, δ ad π satisfy Assumption 1. Assume, in addition, that

(i) $S(x)$ is bounded and continuously differentiable

(ii) the function

$$\phi(x) := \pi(x) - \delta S'(x) - rS(x)$$

is strictly increasing in x with

$$\lim_{x \to -\infty} \phi(x) < 0 < \lim_{x \to +\infty} \phi(x).$$

Use all three approaches to characterise the value function and optimal policy for the shut-down problem

$$V(x) := \sup_{T \geq 0} \left\{ \int_0^T e^{-rt} \pi(X(t))\, dt + e^{-rT} S(X(T)) \right\}, \tag{5.12}$$

where $X(t)$ is as in (5.1) with $X(0) = x$. What is the interpretation of the inequality restriction in (ii)?

Approach 1 Here, we wish to find the optimal date of shut down by solving (5.12). The first-order condition is

$$e^{-rT} \pi(X(T)) - re^{-rT} S(X(T)) + e^{-rT} S'(X(T))\, X'(T) \leq 0$$

with equality if $T > 0$. Since $X'(T) = -\delta$, we can rearrange above as

$$e^{-rT} [\pi(X(T)) - \delta S'(X(T)) - rS(X(T))] = e^{-rT} \phi(X(T)) \leq 0$$

with equality if $T > 0$. We can immediately see that the assumption (ii) implies that there is a unique solution $X(T)$ that satisfies above with equality. Denoting such T as T^*, define $b^* = x - \delta T^*$. Then, the optimal shut-down date is given as

$$T^* = \begin{cases} 0 & \text{if } x_0 \le b^* \\ \frac{x_0 - b^*}{\delta} & \text{if } x_0 > b^*. \end{cases}.$$

Approach 2 We now wish to solve for the optimal threshold b, instead of T. We use the substitution rule as before which gives

$$\begin{aligned} V(x) &= \max_{b \le x} \; F(x,b) \\ &= \max_{b \le x} \left\{ \frac{1}{\delta} \int_b^x e^{-r(x-\xi)/\delta} \pi(\xi)\, d\xi + e^{-r(x-b)/\delta} S(b) \right\}, \end{aligned}$$

where the only difference is that the salvage value now depends on the state in which the plant is shut down.

The first-order condition is given by

$$\frac{\partial F(x,b)}{\partial b} = -\frac{1}{\delta} e^{-r(b-\xi)/\delta} \underbrace{(\pi(b) - rS(b) - \delta S'(b))}_{=-\phi(b)} \ge 0 \tag{5.13}$$

with equality if $b < x$. The optimal solution is, again,

$$b = \min\{x, b^*\},$$

where b^* solves $\phi(b^*) = 0$.

Approach 3 The Bellman equation remains unchanged:

$$rF(x,b) = \pi(x) - \delta F_x(x,b)$$

but the boundary condition changes now since S depends on the state:

$$\lim_{x \downarrow b} F(x,b) = S(b).$$

Evaluating the Bellman equation at $x = b$;

$$rS(b) = rF(b,b) = \pi(b) - \delta F_x(b,b)$$

so the first-order condition (5.13) is

$$F_b(b,b) = \frac{1}{\delta} e^{-r(b-\xi)/\delta} \delta \left(-F_x(b,b) + S'(b)\right) = 0$$

for all $x > b$. So we now have that

$$-F_b(b^*, b^*) = F_x(b^*, b^*) - S'(b) = \underbrace{\left.\frac{\partial F(x,b^*)}{\partial x}\right|_{x=b^*}}_{=V'(x)|_{x=b^*}} - S'(b) = 0.$$

Hence, the "smooth pasting" condition now becomes

$$V'(b^*) = S'(b).$$

The proposition below summarises the results.

Proposition 5.2. *Recall (5.5)*

$$V(x) = \max_{b \le x} F(x, b)$$

and denote the optimal threshold from solving the problem b^*. Then, $V(x)$ satisfies:

(i) (Bellman equation)

$$rV(x) = \pi(x) - \delta V'(x), \forall x \ge b^*,$$
$$V(x) = S(b^*), \forall x < b^*.$$

(ii) (Value matching) $\lim_{x \downarrow b^*} V(x) = S(b^*) = \lim_{x \uparrow b^*} V(x)$*;*

(iii) (Smooth pasting) $\lim_{x \downarrow b^*} V'(b^*) = S'(b^*) = \lim_{x \uparrow b^*} V'(b^*)$*.*

Remark. Observe that smooth pasting requires the slope of V at $x = b^*$ to equal $S'(b^*)$ in this case. This is because the the limit for the value matching condition, $S(b^*)$, is no longer a constant (i.e. not a horizontal line in $(x, V(\cdot))$ space). So to ensure that $V(x)$ when $x \ge b^*$ is smoothly attached to $V(x)$ with $x < b^*$, we require the slope of $V'(b^*)$ to equal the slope of $S(b^*)$.

5.2 Option pricing: Stochastic

Now suppose that the state variable $X(t)$ is a Brownian motion with parameters (μ, σ^2) (with $\mu = -\delta < 0$); i.e.

$$X(t) = X(0) + \mu t + \sigma W(t), \forall t,$$

where W is a Weiner process (see Appendix for definition).

An optimal policy for the stochastic model has two properties.

- ▷ First, since the problem is stationary (i.e. the future does not depend on the past), if an optimal policy requires (allows) shutting down in state b, it requires (allows) shutting down the *first* time the process reaches state b. This allows us to limit our attention to threshold policies. For any b, let $T(b)$ be the *stopping time* defined as the first time $X(t)$ takes the value b (more aptly, this should be called the stopping rule since we cannot know the time at which $X(t)$ hits b).
- ▷ Second, as in the deterministic case, an optimal policy cannot involve waiting for the state to increase and then shutting down since profits are strictly increasing in the state.

Thus, the optimal policy is characterised by a single threshold b, a lower bound on the region where the plant continues operating, which divides the state space into an action region, $(-\infty, b]$, and an inaction region, $(b, +\infty)$.

The firm's problem is then

$$v(x) = \sup_{b \le x} \mathbb{E}_x \left[\int_0^{T(b)} e^{-rt} \pi(X(t))\, dt + e^{-rT(b)} S \right],$$

where $\mathbb{E}_x$ denotes an expectation conditional on the initial state $X(0) = x$.

We first consider the expected return from an arbitrary policy. Let $f(x, b)$ be the total expected discounted return, given the initial state x, and operating until the first time the state reaches $b \leq x$, and then shutting down. If $x > b$, this policy implies that plant is operated (at least for a while). Let ε be a sufficiently small time interval so that the probability of shutting down before h is negligible. Then, following the same step as in the deterministic case:

$$\begin{aligned} f(x,b) &= \mathbb{E}_x\left[\int_0^{\varepsilon} e^{-rt}\pi(X(t))\,dt + e^{-r\varepsilon} f(X(\varepsilon), b)\right] \\ &\approx \pi(x)\varepsilon + \frac{1}{1+r\varepsilon}\mathbb{E}_x\left[f(x+dx, b)\right] \end{aligned} \tag{5.14}$$

We then apply Ito's Lemma (with $\varepsilon = dt$, $dx = \mu\varepsilon + \sigma dW$, where $dW = \epsilon\sqrt{dt}$ and $\epsilon \sim N(0,1)$), which allows us to ignore terms with higher order than ε and $(dW)^2$ (See technical appendix for more):

$$\begin{aligned} f(x+dx,b) &= f(x,b) + f_x(x,b)\,dx + \frac{1}{2} f_{xx}(x,b)(dx)^2 \\ df = f(x+dx,b) - f(x,b) &= f_x(x,b)\,dx + \frac{1}{2} f_{xx}(x,b)(dx)^2 \\ &= f_x(x,b)(\mu\varepsilon + \sigma dW) + \frac{1}{2} f_{xx}(x,b)(\mu\varepsilon + \sigma dW)^2 \\ &= f_x(x,b)(\mu\varepsilon + \sigma dW) + \frac{1}{2} f_{xx}(x,b)\left(\mu^2(\varepsilon)^2 + 2\mu\sigma\varepsilon dW + \sigma^2(dW)^2\right) \\ &= f_x(x,b)(\mu\varepsilon + \sigma dW) + \frac{1}{2}\sigma^2 f_{xx}(x,b)(dW)^2 \\ \Rightarrow \mathbb{E}_x[df] &= \left(f_x(x,b)\mu + \frac{1}{2}\sigma^2 f_{xx}(x,b)\right)\varepsilon \\ \Rightarrow \mathbb{E}_x[f(x+dx,b)] &= f(x,b) + \left(f_x(x,b)\mu + \frac{1}{2}\sigma^2 f_{xx}(x,b)\right)\varepsilon \end{aligned}$$

so that

$$f(x,b) = \pi(x)\varepsilon + \frac{1}{1+r\varepsilon}\left(f(x,b) + f_x(x,b)\mu + \frac{1}{2}\sigma^2 f_{xx}(x,b)\right)\varepsilon.$$

Following the same steps as in the deterministic case,

$$\begin{aligned} (1+r\varepsilon) f(x,b) &\approx \pi(x)\varepsilon + \underbrace{\pi(x) r\varepsilon^2}_{\approx 0} + f(x,b) + \mu f_x(x,b)\varepsilon + \frac{1}{2} f_{xx}(x,b)\sigma^2\varepsilon \\ \Rightarrow r\varepsilon f(x,b) &\approx \pi(x)\varepsilon + \mu f_x(x,b)\varepsilon + \frac{1}{2} f_{xx}(x,b)\sigma^2\varepsilon \\ \Rightarrow r f(x,b) &= \pi(x) + \mu f_x(x,b) + \frac{1}{2} f_{xx}(x,b)\sigma^2, \end{aligned} \tag{5.15}$$

which holds for all $x > b$. This is the counterpart to equation (5.7). Observe that the expression is now a second-order linear ODE with constant coefficients (given fixed b).

Define

$$v_p(x) := \mathbb{E}_x\left[\int_0^{\infty} e^{-rt}\pi(X(t))\,dt\right], \forall x$$

as the expected discounted value if the plant is operated forever. Under Assumption 1, π is bounded and $r > 0$, so v_p is bounded.

Exercise 5.1. Show that the function v_p is a particular solution of 5.15.

Solution. [???]

Exercise 5.2. Show that the functions $h_i(x) = e^{R_i,x}$ for $i = 1, 2$ are homogeneous solutions of 5.15, where R_1 and R_2 are the roots of the characteristic equation:

$$rf(x,b) = \mu f_x(x,b) + \frac{1}{2} f_{xx}(x,b)\sigma^2.$$

Show that the two roots are of opposite sign.

Solution. Rearranging gives

$$\frac{1}{2}\sigma^2 f_{xx}(x,b) + \mu f_x(x,b) - rf(x,b) = 0.$$

Guess the solution $f(x,b) = e^{Rx}$, so

$$\begin{aligned} f(x,b) &= e^{Rx}, \\ f_x(x,b) &= Re^{Rx}, \\ f_{xx}(x,b) &= R^2 e^{Rx}. \end{aligned}$$

Substituting yields

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 R^2 e^{Rx} + \mu R e^{Rx} - re^{Rx} \\ &= e^{Rx}\left(\frac{1}{2}\sigma^2 R^2 + \mu R - r\right), \end{aligned}$$

where the terms inside the bracket is the characteristic equation. Solving for R:

$$\begin{aligned} R_1 &= \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} < 0 \\ R_2 &= \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} > 0, \end{aligned}$$

where the inequalities follow because $2\sigma^2 r > 0$. So the solution to the homogeneous equation is

$$C_1 e^{R_1 x} + C_2 e^{R_2 x},$$

where C_1 and C_2 are arbitrary constants.

Any solution of (5.15) can be written as

$$f(x,b) = v_p(x) + C_1 e^{R_1 x} + C_2 e^{R_2 x}. \tag{5.16}$$

We will use (two) boundary conditions to solve for C_1 and C_2.

▷ For the upper boundary, we use the fact that, as $x \to \infty$, the stopping time $T(b) \to +\infty$. Hence, for any b,

$$\lim_{x\to\infty} [f(x,b) - v_p(x)] = 0. \tag{5.17}$$

Since $R_2 > 0$, for (5.16) to to satisfy this condition, it must be that $C_2 = 0$.

▷ For the lower boundary, we use the same condition as in the deterministic case; i.e. for any $x \leq b$, the policy involves stopping immediately so

$$f(x,b) = S, x \leq b. \tag{5.18}$$

This means, in particular, that at $x = b$,

$$\begin{aligned} S = f(b,b) &= v_p(b) + C_1 e^{R_1 b} \\ \Leftrightarrow C_1 &= e^{-R_1 b}(S - v_p(b)). \end{aligned}$$

Hence, for any b,

$$f(x,b) = \begin{cases} v_p(x) + e^{R_1(x-b)}(S - v_p(b)) & \text{if } x \geq b \\ S & \text{if } x < b \end{cases}.$$

As in the deterministic case, when the firm shuts down, it receives the salvage value S but loses the remaining profit stream $v_p(b)$. Hence, for any fixed b, the value of a firm that uses b for its threshold can be written as the value of operating forever, the term $v_p(x)$, plus the expected discounted net gain from shutting down.

To obtain the optimal threshold b^*, we solve the following maximisation problem (for given initial state x):

$$\begin{aligned} v(x) &= \max_{b \leq x}\ f(x,b) \\ &= v_p(x) + \max_{b \leq x}\ e^{R_1(x-b)}(S - v_p(b)). \end{aligned}$$

The first-order condition is

$$\frac{\partial f(x,b)}{\partial b} \leq 0 \text{ with equality if } x \geq b,$$

where

$$\begin{aligned} \frac{\partial f(x,b)}{\partial b} &= -R_1 e^{R_1(x-b)}(S - v_p(b)) - e^{R_1(x-b)} v_p'(b) \\ &= -e^{R_1(x-b)}\left[\left(R_1(S - v_p(b)) + v_p'(b)\right)\right]. \end{aligned}$$

Thus, the first-order condition can be written as

$$R_1(S - v_p(b)) \leq -v_p'(b) \text{ with equality if } x \geq b.$$

As before, we can interpret

$$e^{R_1(x-b^*)}(S - v_p(b^*)) = \max_{b \leq x}\ e^{R_1(x-b)}(S - v_p(b))$$

as the value of the option to shutdown, or, equivalently, the value of exercising control. For $x \geq b^*$, it is the price of the option in an (appropriate) market setting. The term $S - v_p(b)$ is positive, reflecting the fact that the firm has the option to remain open; i.e. a necessary condition for exercising control is that doing so raises the total return.

As in the deterministic case, it can be shown that

$$-f_b(b,b) = f_x(b,b), \forall b.$$

The interpretation is the same as before. Moving the state slightly in one way has a symmetric effect as moving the threshold in the other way (starting from the threshold).

It can also be shown that

$$f_b(x,b^*) = 0, \forall x \geq b^* \Leftrightarrow f_x(b^*,b^*) = 0.$$

Hence, the condition $f_x(b^*,b^*) = 0$ provides a convenient method for characterising b^*. Specifically, among all the functions $f(x,b)$ satisfying the boundary conditions (5.17) and (5.18), the optimised value function $v = f(\cdot,b^*)$ is the one that has the additional property that $f_x(b^*,b^*) = 0$. We record the results below.

Proposition 5.3. *Let r, $\mu = -\delta$, S and π satisfy Assumption 1, and let X be a Brownian motion with parameters μ and $\sigma^2 > 0$. The optimal threshold b^* and optimised value function v have the following properties:*

(i) (Bellman equation)

$$\begin{aligned} rv(x) &= \pi(x) + \mu v'(x) + \frac{1}{2}\sigma^2 v''(x), \forall x \geq b^*, \\ v(x) &= S, \forall x < b^*. \end{aligned}$$

(ii) (No bubble condition) $\lim_{x\to\infty}[v(x) - v_p(x)] = 0$.

(iii) (Value matching) $\lim_{x\downarrow b^*} v(x) = S = \lim_{x\uparrow b^*} v(x)$*;*

(iv) (Smooth pasting) $\lim_{x\downarrow b^*} v'(b^*) = 0 = \lim_{x\uparrow b^*} v'(b^*)$.

Remark 5.2. For a heuristic argument that leads to smooth pasting condition, suppose the threshold b has been chosen and the state is $x = b$, and consider the expected returns from two strategies: (i) shutting down immediately; and (ii) continuing to operate for a short interval of time ε and then deciding what to do. The return from the first strategy is $\Pi_1 = S$. The return from the latter can be calculated using a random walk approximation.

The payoff $\pi(b)$ is collected over the time interval ε, and then a decision is made. The increment to the state over the duration is

$$dX(\varepsilon) = \pm\sigma\sqrt{\varepsilon}$$

and the probability of an upward jump is

$$p = \frac{1}{2}\left[1 + \frac{\mu\sqrt{\varepsilon}}{\sigma}\right].$$

Assume that, at the endof the time interval ε, the firm keeps the plan open if X has increased and shuts down if X has decreased. Then, the expected return from the second strategy is

$$\Pi_2 = \pi(b)\varepsilon + (1 - r\varepsilon)\left[pf\left(b + \sigma\sqrt{\varepsilon}, b\right) + (1-p)S\right].$$

Use a Taylor series expansion to evaluate f to find that

$$\begin{aligned} pf\left(b+\sigma\sqrt{\varepsilon},b\right)+(1-p)\,S &\approx p\left[f\left(b,b\right)+f_x\left(b,b\right)\sigma\sqrt{\varepsilon}\right]+(1-p)\,S \\ &= S+pf_x\left(b,b\right)\sigma\sqrt{\varepsilon}, \end{aligned}$$

where we use $f\left(b,b\right)=S$. Hence, the difference between the two payoffs is

$$\begin{aligned} \Pi_2-\Pi_1 &= \pi\left(b\right)\varepsilon+(1-r\varepsilon)\left[S+pf_x\left(b,b\right)\sigma\sqrt{\varepsilon}\right]-S \\ &\approx \left(\pi\left(b\right)-rS\right)\varepsilon+pf_x\left(b,b\right)\sigma\sqrt{\varepsilon}-\underbrace{r\varepsilon pf_x\left(b,b\right)\sigma\sqrt{\varepsilon}}_{\approx 0} \\ &\approx \frac{1}{2}f_x\left(b,b\right)\sigma\sqrt{\varepsilon}. \end{aligned}$$

If the threshold b is optimal, the firm is indifferent between the two strategies; i.e.

$$f_x\left(b^*,b^*\right)=0,$$

which is the smooth pasting condition.

Exercise 5.3. Describe the qualitative effects of small changes in he parameters μ, σ^2, S and r on: (i) the optimal threshold b^*; (ii) the maximised value function v; and (iii) the expected length of time until the option is exercised, conditional on a fixed initial condition $x_0>b$.

Exercise 5.4. Consider an unemployed worker who continuously receives wage offers. Suppose that the stochastic process $\{X\left(t\right)\}$ describes these offers. The worker can accept a job at any time. When he does accept an offer, his wage remains constant forever after. The worker receives an unemployment benefit s (a flow) as long as he remains unemployed. The work lives forever and is interested in maximising the expected discounted value of his lifetime income, discounted at the constant rate $r>0$.

(i) Describe his optimal strategy when $X\left(t\right)$ is a Brownian motion with parameters $\left(\mu,\sigma^2\right)$. What restrictions (if any) are needed to make the problem well behaved?

(ii) Repeat (a) assuming that $X\left(t\right)$ is an Ornstein-Uhlenbeck process with parameters $\left(\alpha,\sigma^2\right)$.

Solution 1. Part (a) The value from never accepting an offer is the present value of receiving s forever; i.e.

$$\int_0^\infty e^{-rt}sdt=s\left[-\frac{1}{r}e^{-rt}\right]_0^\infty=\frac{s}{r}.$$

he optimal strategy would be a rule such that if he observes $X\left(t\right)$ above a certain threshold b, he will accept the job offer; i.e. the inaction region is $(-\infty,b]$ and the action region is $(0,+\infty)$. Given the stationary nature of the problem, we would expect that the worker will accept the first wage offer that is higher than s.

We first show that this problem permits the Bellman equation form. Let $f\left(x,b\right)$ denote the continuation value when the worker faces wage offer of x with a threshold x. If he chooses to accept, he receives x forever, which has present value x/r. If he rejects, then he receives s (flow) and the continuation value with $x=dx$. We suppose that $x<b$ and consider a time interval ε

sufficiently small that there is positive probability that the worker will continue to search. Then,

$$\begin{aligned} f(x,b) &= \mathbb{E}_x\left[\int_0^{\varepsilon} s dt + e^{-r\varepsilon} f(X(\varepsilon), b)\right] \\ &\approx s\varepsilon + \frac{1}{1+r\varepsilon}\mathbb{E}_x\left[f(x+dx,b)\right]. \end{aligned}$$

Observe that this is the same expression as (5.14) so this can be expressed as (5.15). Using Proposition 5.3:

- ▷ Bellman equation:

$$\begin{aligned} rv(x) &= s + \mu v'(x) + \frac{1}{2}\sigma^2 v''(x), \forall x \leq b \\ v(x) &= \frac{x}{r}, \forall x > b. \end{aligned} \tag{5.19}$$

- ▷ No bubble condition: $\lim_{x\to-\infty} f(x,b^*) = \frac{s}{r}$ (this is to bound the case when the worker searches forever);

- ▷ Value matching: $\lim_{x\to b^*} v(x) = \frac{b^*}{r}$ (the value at the threshold should equal to present value of reaching b^* forever);

- ▷ Smooth pasting: $\lim_{x\to b^*} v'(b^*) = 0$ (since the value at the threshold of v is independent of x (i.e. b^*/r is a horizontal line in $(x, v(\cdot))$ space).

Let $v_p(x)$ denote the value of searching forever; i.e. value of receiving s forever:

$$v_p(x) = \int_0^{\infty} e^{-rt} s dt = \frac{s}{r}.$$

Observe that this equation satisfies (5.19). We now look for the homogeneous solution; i.e. $v_h(x)$ that solves

$$rv(x) = \mu v'(x) + \frac{1}{2}\sigma^2 v''(x).$$

We already solved this above. The characteristic equation has two roots:

$$\begin{aligned} R_1 &= \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} < 0 \\ R_2 &= \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} > 0, \end{aligned}$$

and the solution to the homogeneous equation is

$$C_1 e^{R_1 x} + C_2 e^{R_2 x}.$$

So any solution for (5.19) can be written as

$$v(x) = \frac{s}{r} + C_1 e^{R_1 x} + C_2 e^{R_2 x}.$$

Given the no bubble condition, since we are concerned about the case as $x \to -\infty$, and $R_1 < 0$, we have to set $C_1 = 0$. Then, observe that

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{s}{r} + C_2 e^{R_2 x} = \frac{s}{r}$$

so that the no bubble condition is satisfies. To pin down C_2, we use the value-matching condition; i.e.

$$\begin{aligned}\lim_{x\to b^*} \frac{s}{r} + C_2 e^{R_2 x} &= \frac{b^*}{r}\\ \Rightarrow C_2 &= e^{-R_2 b^*}\left(\frac{b^*}{r} - \frac{s}{r}\right).\end{aligned}$$

Thus, the solution can now be written as

$$v(x) = \frac{1}{r}\left[s + e^{R_2(x-b^*)}(b^* - s)\right].$$

To pin down b^*, we use smooth pasting.

$$\begin{aligned}\lim_{x\to b^*} v'(x) &= \frac{R_2}{r}(b^* - s) = 0\\ &\Rightarrow b^* = s.\end{aligned}$$

Part (b). Now the Bellman equation is given by

$$rv(x) = s - \alpha x v'(x) + \frac{1}{2}\sigma^2 v''(x), \forall x \le b.$$

The solution is of the form

$$v(x) = h(x)e^{\eta x^2},$$

where

Now what?

$$\begin{aligned}\eta &= \frac{\alpha}{\sigma^2},\\ (\alpha - r)h(x) &= \alpha x h'(x) + \frac{1}{2}\sigma^2 h''(x).\end{aligned}$$

5.2.1 Technical appendix: Brownian motion

Definition 5.1. (Weiner process/standard Brownian motion) is a stochastic process W having: (i) continuous sample paths; (ii) stationary independent increments, and (iii) $W(t) \sim N(0,t)$ for all t.

If $W(t)$ is a Weiner process, then over any time interval Δt, the corresponding random change is normally distributed with mean zero and variance Δt; i.e.

$$\Delta W = \epsilon_t\sqrt{\Delta t},\ \epsilon_t \sim N(0,1).$$

As $\Delta t \to 0$, write $dW = \epsilon_t\sqrt{dt}$ and

$$\begin{aligned}\mathbb{E}[dW] &= \mathbb{E}\left[\epsilon_t\sqrt{dt}\right] = 0,\\ \mathbb{E}\left[(dW)^2\right] &= \mathbb{E}\left[\epsilon_t^2 dt\right] = dt.\end{aligned}$$

Definition 5.2. (Brownian motion) A stochastic process X is a Brownian motion with drift μ and variance σ^2 if

$$X(t) = X(0) + \mu t + \sigma W(t), \forall t, \tag{5.20}$$

where $W(t)$ is a Weiner process.

Observe that

$$\begin{aligned}\mathbb{E}\left[X_t - X(0)\right] &= \mu t,\\ \mathbb{E}\left[(X_t - X(0))^2\right] &= \mathbb{E}\left[\sigma^2 W(t)^2\right] = \sigma^2 t.\end{aligned}$$

Any continuous stochastic process $\{X(t)\}_{t=0}^{\infty}$ that has stationary independent increments is a Brownian motion. The Brownian motion equation (5.20), can be written in differential form:

$$dX(t) = \mu dt + \sigma dW(t), \forall t,$$

where $W(t)$ is a Weiner process (above is equivalent to (5.20) given the initial condition).

Definition 5.3. (Geometric Brownian motion) Geometric Brownian motion is a stochastic process $X(t)$ such that

$$dX = \mu X dt + \sigma X dW.$$

Observe that dividing the expression by X makes the right-hand side the same as in (5.20).

Definition 5.4. (Ornstein-Uhlenbeck process) An Ornstein-Uhlenbeck process is stochastic process that is mean reverting with $\mu = -\alpha x$ and $\alpha, \sigma^2 > 0$. The stationary distribution is normal with mean zero and variance $\gamma = \sigma^2/2\alpha$.

5.2.2 Technical appendix: Ito's Lemma

Recall the expression for a Brownian motion:

$$dX(t) = \mu dt + \sigma dW(t), \forall t,$$

where $W(t)$ is a Weiner process. Let $F(t,x)$ be a function that is differentiable at least once in t and twice differentiable in x. The "total differential" of $F(t,x)$, call it dF, can be approximated with a Taylor series expansion:

$$\begin{aligned}dF &= F_t dt + F_x dX + \frac{1}{2}F_{xx}(dX)^2 + \cdots\\ &= F_t dt + F_x(\mu dt + \sigma dW) + \frac{1}{2}F_{xx}(\mu dt + \sigma dW)^2 + \cdots\\ &= F_t dt + F_x(\mu dt + \sigma dW) + \frac{1}{2}F_{xx}\left(\mu^2 (dt)^2 + 2\mu\sigma dt dW + \sigma^2 (dW)^2\right) + \cdots.\end{aligned}$$

We then drop terms of order higher than dt or $(dW)^2$ and obtain

$$dF = F_t dt + \mu F_x dt + \sigma F_x dW + \frac{1}{2}\sigma^2 F_{xx}(dW)^2.$$

Since $\mathbb{E}[dW]=0$ and $\mathbb{E}\left[(dW)^2\right]=dt$,

$$\begin{aligned}
\mathbb{E}[dF] &= \left(F_t+\mu F_x+\frac{1}{2}\sigma^2F_{xx}\right)dt,\\
\operatorname{Var}[dF] &= \mathbb{E}\left[(dF-\mathbb{E}[df])^2\right]\\
&= \mathbb{E}\left[\left(\sigma F_x dW+\frac{1}{2}\sigma^2F_{xx}\left((dW)^2-dt\right)\right)^2\right]\\
&= \sigma^2F_x^2\mathbb{E}\left[(dW)^2\right]+\sigma F_x\sigma^2F_{xx}\mathbb{E}\left[dW\left((dW)^2-dt\right)\right]\\
&\quad+\frac{1}{4}\sigma^4F_{xx}^2\mathbb{E}\left[\left((dW)^2-dt\right)^2\right]\\
&= \sigma^2F_x^2dt+\sigma F_x\sigma^2F_{xx}\left(\underbrace{\mathbb{E}\left[(dW)^3\right]}_{=0}-\underbrace{\mathbb{E}[dW\,dt]}_{=0}\right)+\frac{1}{4}\sigma^4F_{xx}^2\underbrace{\mathbb{E}\left[\underbrace{(\varepsilon dt-dt)^2}_{(\varepsilon-1)^2(dt)^2}\right]}_{\approx 0}\\
&= \sigma^2F_x^2dt.
\end{aligned}$$

Suppose $F(t,x)=e^{-rt}f(x)$, where $r\geq 0$ is the discount rate. Then,

$$\mathbb{E}\left[d\left(e^{-rf}f\right)\right]=e^{-rt}\left(-rf+\mu f'+\frac{1}{2}\sigma^2f''\right)dt,$$

where f, f' and f'' are evaluated at $X(t)$.

Theorem 5.1. *(Ito's lemma). Let $F:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be once continuously differentiable in its first argument and twice continuously differentiable in its second, and let X be the diffusion*

$$\begin{aligned}
X(t,\omega)=X(0,\omega)&+\int_0^t\mu(s,X(s,\omega))\,ds\\
&+\int_0^t\sigma(s,X(s,\omega))\,dW(s,\omega),\forall t,\omega.
\end{aligned}$$

Then,

$$\begin{aligned}
F(t,X(t,\omega))=F(0,X(0,\omega))&+\int_0^tF_t(s,X)\,ds\\
&+\int_0^tF_x(s,X)\,\mu(s,X)\,ds\\
&+\int_0^tF_x(s,X)\,\sigma(s,X)\,dW(s,\omega)\\
&+\frac{1}{2}\int_0^tF_{xx}(s,X)\,\sigma^2(s,X)\,ds,\forall t,\omega,
\end{aligned}\tag{5.21}$$

where the arguments of $X(s,\omega)$ have been suppressed.

Remark 5.3. If μ and σ are stationary and $F(t,x)=e^{-rt}f(x)$, where $r\geq 0$ is a constant discount rate, then (5.21) takes the form

$$\begin{aligned}
e^{-rt} f\left(X\left(t\right)\right) = f\left(X\left(0\right)\right) - r \int_0^t e^{-rs} f\left(X\right) ds \\
+ \int_0^t e^{-rs} f'\left(X\right) \mu\left(X\right) ds + + \int_0^t e^{-rs} f'\left(X\right) \sigma\left(X\right) dW \\
+ \frac{1}{2} \int_0^t e^{-rs} f''\left(X\right) \sigma^2 ds.
\end{aligned}$$

5.3 A menu cost model

In the option problem, that the option can be exercised only once acts as a fixed cost. In other settings, however, actions can be taken many times, but only with explicit (fixed) cost of adjustment. The individual/firm in such settings must decide on when to take action as well as how much action to take, while being forward looking.

In the menu cost model, the profit flow of a firm depends on the price of its own product relative to a general price index. The latter follows a geometric Brownian motion, and the firm's problem is to choose a policy for changing the nominal price of its own product. Changing the price entails a fixed cost (e.g. time cost of decision making/cost of printing) but no variable costs. The optimal policy (under some restrictions) has the following form: there is an inaction region (b, B) and a return point $S \in (b, B)$. While the relative price remains inside the inaction region, the firm does nothing; however, when the relative price leaves the region, the firm adjusts its nominal price so that the relative price is equal to the return value S.

Consider a firm whose profit flow at any date t depends on the ratio of its own nominal price $P\left(t\right)$ to an aggregate (industry/economy-wide) price index, $\bar{P}\left(t\right)$, where the latter is a geometric Brownian motion. We work in log form—let $p\left(t\right) = \log P\left(t\right)$ be the log of the firm's nominal price, and $\bar{p}\left(t\right) = \log \bar{P}\left(t\right)$ be the log of the aggregate price index. Geometric Brownian motion with parameters $\left(-\mu, \sigma^2\right)$ is given by

$$d\bar{P} = -\mu \bar{P} dt + \sigma \bar{P} dW.$$

Total differentiating $\bar{p} = \log \bar{P}\left(t\right)$ gives $d\bar{P}/\bar{P}\left(t\right)$, where

$$\frac{d\bar{P}}{\bar{P}\left(t\right)} = -\mu dt + \sigma dW.$$

That is, $\bar{p}$ is a Brownian motion (with given initial value $\bar{p}_0$).

The initial value of the firm's log nominal price is given as p_0. The firm can change its price at any time but doing so entails a fixed cost of $c > 0$ (constant over time and in real terms). Since the profit flow at any date depends only on the firm's relative price, we define the log of the firm's relative price at date t as

$$\begin{aligned}
Z\left(t\right) &\coloneqq p\left(t\right) - \bar{p}\left(t\right), \forall t \geq 0 \\
z_0 &\coloneqq p_0 - \bar{p}_0.
\end{aligned}$$

When the firm adjusts its nominal price, it effectively chooses $z\left(t\right)$.

An impulse control is a pair $\left(T_i, z_i\right)$ where T_i denotes the stopping time such that the firm's relative price $Z\left(t\right)$ jumps to the targeted level z_i. During the subsequent (random) time interval

$[T_i, T_{i+1})$, the increments to $Z(t)$ mirror—with a sign change—the increments to $\bar{p}(t)$. Thus, given the initial value $Z(0) = z_0$, an impulse control policy $\gamma = \{(T_i, z_i)\}_{i=1}^{\infty}$, the stochastic process for the firm's relative price is

$$Z(t) = z_i - (\bar{p}(t) - \bar{p}(T_i)), t \in [T_i, T_{i+1}), \forall i,$$

where $T_0 = 0$. Since $\bar{p}(t)$ is a Brownian motion, over each interval (T_i, T_{i+1}), the relative price $Z(t)$ also behaves like a Brownian motion. Given parameters $\left(-\mu, \sigma^2\right)$ of the process $\bar{p}$, Z has parameters $\left(\mu, \sigma^2\right)$ since

$$dZ = -d\bar{p} = \mu dt + \sigma dW.$$

The profit flow of the firm, $\pi(z)$, is a stationary function of its relative price z, and profits are discounted at a constant interest rate r. We impose the following restrictions to ensure that the problem is well behaved.

Assumption 2. *Assume that $r, c, \sigma^2 > 0$ and that $\pi(\cdot)$ is continuous and strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, +\infty)$.*

In particular, the assumptions ensure that π is single peaked so that the optimal policy is unique and involves no more than one inaction region. That the peak is at $z = 0$ should be seen as a simplifying normalisation.

Given an initial state z_0 and an impulse control policy γ, let $H(z_0; \gamma)$ be the expected discounted value of returns net of adjustments costs:

$$H(z_0; \gamma) \coloneqq \mathbb{E}_{z_0}\left[\int_{T_0}^{T_1} e^{-rt}\pi[Z(t)]\,dt + \sum_{i=1}^{\infty}\left(e^{-rT_i}c + \int_{T_i}^{T_{i+1}} e^{-rt}\pi[Z(t)]\,dt\right)\right].$$

Under the stated assumptions, the integrals in H are well defined and the expected value exists (although it may be $-\infty$).

Let $v(z)$ be the maximised value for expected discounted returns, given the initial state z:

$$v(z) \coloneqq \sup_{\gamma \in \Gamma} H(z; \gamma), \forall z,$$

where Γ is the set of all impulse control policies. Under Assumption 2, v is bounded. Since $\pi(z)$ is maximised at $z = 0$, the upper bound for H is given by a stream of profits $\pi(0)$ for ever; i.e.

$$\int_0^{\infty} e^{-rt}\pi(0)\,dt = \pi(0)\left[-\frac{1}{r}e^{-rt}\right]_0^{\infty} = \frac{\pi(0)}{r}.$$

For a lower bound, choose any three values $a < s < A$, and calculate the expected returns from this policy: immediately adjust relative price to s, and afterwards, re-adjust to s whenever the relative price leaves the interval (a, A). Under such policy, profits are bounded below between $\pi(a)$ and $\pi(A)$ so that the expected value is bounded.

Since v is bounded, we can express the firm's problem in a recursive manner. Since the environment is stationary (because $\bar{p}$ is a Brownian motion), we can write the value function as

$$v(z) = \sup_{T \geq 0, s} \mathbb{E}_z\left[\int_0^T e^{-rt}\pi(Z(t))\,dt + e^{-rT}(v(s) - c)\right], \tag{5.22}$$

where T is a stopping time.

We now impose assumptions to ensure that there is a unique inaction region. Let $v_p(z)$ be the expected returns over an infinite horizon if no control is exercised.

$$v_p(z) \coloneqq \mathbb{E}_z\left[\int_0^\infty e^{-rt}\pi(Z(t))\,dt\right], \forall z. \tag{5.23}$$

The next result provides a necessary and sufficient conditions for $v = v_p$ for no control to be an optimal policy.

Proposition 5.4. *Under assumption 2, $v = v_p$ if and only if $v_p > -\infty$ and*

$$\inf_z v_p(z) \geq \sup_z v_p(z) - c. \tag{5.24}$$

Proof. Suppose that $v_p > -\infty$ and (5.24) holds. Then

$$v(z) = \sup_{T\geq 0, x} \mathbb{E}_z\left[\int_0^T e^{-rt}\pi(Z(t))\,dt + e^{-rT}(v_p(s) - c)\right].$$

Now, the expected returns up to the stopping time T can be written as the value from never exercising control from beginning, $v_p(z)$, less the value of never exercising control from period T onwards, discounted back to today, $e^{-rT}v_p(Z(T))$. Of course, expectation of this must equal the (expected) present value of profits between period 0 to T; i.e.

$$\int_0^T e^{-rt}\pi(Z(t))\,dt = v_p(z) - \mathbb{E}_z\left[e^{-rT}v_p(Z(T))\right].$$

Thus,

$$\begin{aligned}
v(z) &= \sup_{T\geq 0, x} \mathbb{E}_z\left[\left(v_p(z) - \mathbb{E}_z\left[e^{-rT}v_p(Z(T))\right]\right) + e^{-rT}(v_p(s) - c)\right] \\
&= v_p(z) + \sup_{T\geq 0, s} \mathbb{E}_z\left[e^{-rT}(v_p(s) - c - v_p(Z(T)))\right] \\
&= v_p(z) + \sup_{T\geq 0} \mathbb{E}_z\left[e^{-rT}\left(\sup_s v_p(s) - c - v_p(Z(T))\right)\right].
\end{aligned}$$

By assumption, we have

$$\sup_s v_p(s) - c \leq \inf_z v_p(z) \leq v_p(Z(T))$$

so $\sup_s v_p(s) - c - v_p(Z(T)) < 0$. Thus, the optimal is to set $T = 0$. That is,

$$v(z) = v_p(z).$$

Conversely, suppose $v = v_p$. Then the result holds because v_p is bounded, satisfies (5.22) and $T = 0$ is feasible. ∎

even with no stopping?

The idea is that, if (5.24) holds, then the function v_p is very flat, varying by less than c between its minimum and its maximum. Thus, the fixed cost is too large to justify even a single adjustment. We impose the following assumptions.

Assumption 3. *Either* $v_p = -\infty$ *or*

$$c < \sup_z v_p(z) - \inf_z v_p(z).$$

The next result establishes that under Assumptions 2 and 3, there exists unique critical values b^*, B^*, with $b^* < 0 < B^*$, that characterise the optimal stopping times; i.e. there is a unique inaction region given by(b^*, B^*).

Proposition 5.5. *Under Assumptions 2 and 3, there exists* $b^* < 0 < B^*$ *with* $|b^*| < \infty$, $B^* < \infty$, *or both, such that: the unique optimal stopping time in (5.22) is* $T = 0$ *for* $z \notin (b^*, B^*)$ *and* $T = T(b^*) \wedge T(B^*)$ *for* $z \in (b^*, B^*)$.

Proof. Admitted. ■

5.3.1 HJB approach

The HJB equation is derived in the same way as in the option case. Consider the value of a firm with initial state $Z(0) = z$ in the interior of the inaction region. Consider the function v. For a sufficiently small interval of time ε, the value of the firm can be written as

$$\begin{aligned}
v(z) &= \mathbb{E}_z\left[\int_0^{\varepsilon} e^{-rt}\pi(Z(t))\,dt + e^{-r\varepsilon}v(Z(\varepsilon))\right] \\
&\approx \pi(z)\varepsilon + \frac{1}{1+r\varepsilon}\mathbb{E}_z[v(z+dz)] \\
&\approx \pi(z)\varepsilon + \frac{1}{1+r\varepsilon}\left[v(z) + \mu v'(z)\varepsilon + \frac{1}{2}\sigma^2 v''(z)\varepsilon\right] \\
\Rightarrow rv(z) &= \pi(z) + \mu v'(z) + \frac{1}{2}\sigma^2 v''(z),
\end{aligned}$$

where the steps are the same as (5.22). Given that HJB equation is a second-order ODE, we need to two boundary conditions to complete the solution. We also require three additional conditions to determine b^*, B^* and S^*.

Proposition 5.6. *Suppose Assumptions 2 and 3 hold. The optimal stopping rule* $\{b^*, B^*, S^*\}$ *and the optimised function* v *have the following properties.*

(i) (Bellman equations)

$$
\begin{aligned}
rv(z) &= \pi(z) + \mu v'(z) + \frac{1}{2}\sigma^2 v''(z), \forall z \in (b^*, B^*), \\
v(z) &= v(S^*) - c, \forall z \notin (b^*, B^*).
\end{aligned}
$$

(ii) (Optimal return)

$$v'(S^*) = 0.$$

(iii) (Value matching)[a]

$$
\begin{aligned}
\lim_{z \downarrow b^*} v(z) &= v(S^*) - c, \\
\lim_{z \uparrow B*} v(z) &= v(S^*) - c.
\end{aligned}
$$

(iv) (Smooth pasting)

$$
\begin{aligned}
\lim_{z \downarrow b^*} v'(z) &= 0, \\
\lim_{z \uparrow B*} v'(z) &= 0.
\end{aligned}
$$

[a] If $b^* = -\infty$, then $\lim_{z\to-\infty} v(z) = \lim_{z\to-\infty} v_p(z) \geq v(S^*) - c$. If $B^* = \infty$, then $\lim_{z\to+\infty} v(z) = \lim_{z\to+\infty} v_p(z) \geq v(S^*) - c$, where v_p is the value of never exercising control as defined in (5.23).

As before, all solutions for $v(z)$ in the inaction region can be written in the form

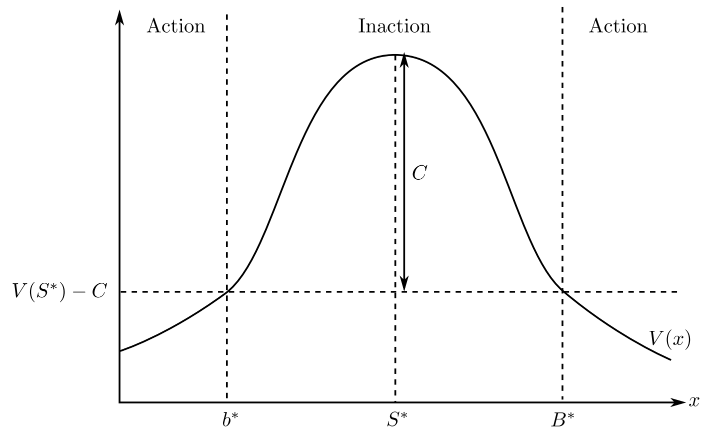
$$v(z) = v_p(z) + d_1(b, B, S)e^{R_1 z} + d_2(b, B, S)e^{R_2 z},$$

where v_p is the value of never exercising control (assumed to be bounded), and $d_1 e^{R_1 z} + d_2^{R_2 z}$ is the solution to the homogenous equation

$$rv(z) = \mu v'(z) + \frac{1}{2}\sigma^2 v''(z).$$

The constants d_1 and d_2 are derived using the boundary conditions.

The figure below shows the action and inactions regions given the optimal policy. Note that since adjustment costs are fixed here, $V(X)$ is symmetric around S^*.


Action
Inaction
Action
C
$V(S^*) - C$
$V(x)$
x
b^*
S^*
B^*

6 Money

6.1 Stylised facts

$M1$ measure of money supply consists of cash and demand deposits (i.e. money held in current/checking accounts). Money multiplier is the ratio of $M1$ to the money base (central bank's money, also called high-powered money, which is the sum of cash and reserves (reserves includes demand deposits)), MB. The stylised facts are from Freeman and Kydland (2000).

(i) $M1$ is positively correlated with output.

(ii) Money multiplier and deposit-to-cash ratio are positively correlated with real output.

(iii) Price level is negatively correlated with real output

The quantity theory of money states that the general price level is proportional to money supply, and refers to the following (accounting identity) equation:

$$MV = PT,$$

where M is money supply, V is the velocity (number of times money exchanges hands), P is the price level and T is the volume of transactions.

The classical view is that, if we change M, then

- ▷ in the short run, only V changes (affects output);
- ▷ in the medium run, T changes;
- ▷ in the long run, P changes.

6.2 Means of payment model

We consider how changes in output (i.e. V) and M feed into T and P.

A representative household receives constant endowment every period. The household is unable to consume endowments directly and must first sell then use the cash to buy the consumption good. The household has two means of payment: cash and demand deposits. There is a short-run government debt (T-bills), which provides a means of saving for households. Thus, the household has two decisions to make: savings vs consumption, and cash vs demand deposits. There is no government spending and the government balances its budget (money issue equals lump-sum transfer to the household).

6.2.1 Household transactions

In each period, the household consumes a continuum of goods that have different values (or types) and consumes each type in fixed (over time) proportions. Specifically, let $f(z)$ denote the number of transactions associated with good of value $z \in \mathbb{R}_{++}$. Then, the total number of transactions that the household carries out to buy goods with values less than or equal to z is given by

$$F(z) := \int_0^z f(x)\,dx$$

In each period, household consume all types of goods, from $z \in (0, \infty]$ so that the total number of transactions period, n, can be defined as

$$n := \lim_{z \to \infty} F(z) = \int_0^\infty f(x)\, dx.$$

The total value associated with purchasing goods of value z is given by $zf(z)$ (i.e. the value of each transaction multiplied by the number of transactions). We can the define $\Omega(z)$ to be the total value of transactions for goods with values smaller than z as

$$\Omega(z) = \int_0^z xf(x)\, dx.$$

We scale this so that

$$\lim_{z \to \infty} \Omega(z) = 1.$$

With this rescaling, we can interpret $\Omega(z)$ as the proportion of household spending on goods whose values are smaller than size z.

6.2.2 Means of payment

The household has two means of payment.

Cash that pays no interest and is subject to loss/theft at rate $\eta \geq 0$ per period.

Demand deposit pays interest $i^d \geq 0$. Each transaction has a real fixed cost $k^a > 0$ that is independent of the size of the transaction. The settlement of purchases made with demand deposit occurs in the period after the transaction.

The real cost of using cash is the rate of loss/theft, η, and hence the cost is increasing with the total value of transactions. The real cost of purchasing with demand deposits have two components. First, there is the fixed cost component, k^a, that depends on the number of transactions (and not on the total value of those transactions). Second, since settlement of purchases made with demand deposits occur in the period after, demand deposits earn nominal interest i^d for one period, thereby reducing real cost of purchases made today by i^d per value of transaction.

Given the fixed cost associated with demand deposit purchase, there exists γ such that for any $z \leq \gamma$, the household uses cash and for any $z > \gamma$, the household uses demand deposit. Given this threshold γ, $F(\gamma)$ is the number of transactions that are paid with cash and $n - F(\gamma)$ is the number of transactions that are paid with demand deposits. Moreover, $\Omega(\gamma)$ is the proportion of total value of transactions spent on goods with size smaller than γ. $c\Omega(\gamma)$ is the real cost of consumption, where c is the total value of consumption good.

The household therefore wants to minimise the total cost of transaction:

$$\begin{aligned}
\chi(c) &= \min_\gamma \left\{ \eta c \Omega(\gamma) + k^a (n - F(\gamma)) - i^d c (1 - \Omega(\gamma)) \right\} \\
&= \min_\gamma \left\{ k^a (n - F(\gamma)) + c \left[\eta \Omega(\gamma) - i^d (1 - \Omega(\gamma)) \right] \right\},
\end{aligned}$$

where $\eta c \Omega(\gamma)$ is the cost of using cash to pay for goods with size less than γ. $k^a (n - F(\gamma))$ is the fixed cost associated with the use of demand deposits on goods with size larger than γ. Finally, $-i^d c (1 - \Omega(\gamma))$ is the nominal interest earned from the use of demand deposits.

The first-order condition is

$$-k^a F'(\gamma) + c\left(\eta + i^d\right)\Omega'(\gamma) = 0,$$

where

$$\begin{aligned} F'(\gamma) &= \frac{dF(\gamma)}{d\gamma} = f(\gamma), \\ \Omega'(\gamma) &= \frac{d\Omega(\gamma)}{d\gamma} = \frac{d}{d\gamma}\int_0^\gamma xf(x)\,dx = \gamma f(\gamma). \end{aligned}$$

The first-order condition can then be written as

$$\begin{aligned} 0 &= -k^a f(\gamma) + c\left(\eta + i^d c\right)\gamma f(\gamma) \\ \Rightarrow \gamma &= \frac{1}{c}\frac{k^a}{\eta + i^d}, \end{aligned} \tag{6.1}$$

which gives the threshold value. Observe that, if c increases, then γ decreases.

6.2.3 Government policy

The government chooses the aggregate supply of cash, $\bar{M}_t$, and nominal debt, $\bar{B}_t$. Then, it has two policy instruments:

$$\begin{aligned} \xi_t &= \frac{\bar{B}_t}{\bar{M}_t}, \\ \omega_t &= \frac{\bar{M}_t}{\bar{M}_{t-1}}, \end{aligned}$$

where ξ_t is the share of debt to money supply, and ω_t is the growth rate of money supply.

6.2.4 Recursive competitive equilibrium

The aggregate state is $s = (y, \omega, \xi)$, where y is output. We assume a first-order Markov process for S with transition function $G(s'|s)$. We will look for a recursive competitive equilibrium and derive $P\left(s_t, \bar{M}_t\right)$ is the price level, $Q_D\left(S_t, \bar{M}_t\right)$ is the price of demand deposits, and $Q_B\left(S_t, \bar{M}_t\right)$ is the price of government debt (all in nominal terms).

6.2.5 Government budget constraint

To avoid the need to replenish money supply for the lost cash, we assume that cash is stolen and spent (but we do not account for utility from spending the stolen money). The government can finance the lump-sum transfer to the household, τ_t, by: (i) issuing new money, $\bar{M}_t - \bar{M}_{t-1}$; or (ii) issuing nominal government bonds at price Q_B. We assume one-period government bonds. Therefore, the government budget constraint is

$$\begin{aligned}
\tau_t &= \left[Q_B\left(s_t, \bar{M}_t\right)\bar{B}_t - \bar{B}_{t-1}\right] + \left(\bar{M}_t - \bar{M}_{t-1}\right) \\
&= \left[Q_B\left(s_t, \bar{M}_t\right)\frac{\bar{B}_t}{\bar{M}_{t-1}} - \frac{\bar{B}_{t-1}}{\bar{M}_{t-1}}\right]\bar{M}_{t-1} + \left(\frac{\bar{M}_t}{\bar{M}_{t-1}} - 1\right)\bar{M}_{t-1} \\
&= \left[Q_B\left(s_t, \bar{M}_t\right)\frac{\bar{B}_t}{\bar{M}_t}\frac{\bar{M}_t}{\bar{M}_{t-1}} - \frac{\bar{B}_{t-1}}{\bar{M}_{t-1}}\right]\bar{M}_{t-1} + \left(\omega_t - 1\right)\bar{M}_{t-1} \\
&= \left[Q_B\left(s_t, \bar{M}_t\right)\xi_t\omega_t - \xi_{t-1}\right]\bar{M}_{t-1} + \left(\omega_t - 1\right)\bar{M}_{t-1}.
\end{aligned}$$

In fact, we will assume away revenues from the government bond (Stokey: "it doesn't add much"). This means that consumers save through nominal bonds with zero net supply in each period. (Nominal bonds pay one unit of cash tomorrow.) Under this assumption, the government budget constraint simplifies to

$$\tau_t = \bar{M}_t - \bar{M}_{t-1} = \left(\omega_t - 1\right)\bar{M}_{t-1}.$$

Add discussion about: consol, indexed, nominal bonds; and one-period vs perpetual bonds

6.2.6 Household's problem

We incorporate the banking sector into the household. In particular, we require fraction θ of purchases via demand deposit to be paid as cash in the period of purchase, and the remainder to be paid in the next period. We think of the fraction θ as cash reserves against which the demand deposit is held. Let A_t denote the current asset holding of the household.

The household's problem is the following:

$$\begin{aligned}
v\left(A_t; s_t, \bar{M}_t\right) = \max_{c_t, \gamma_t, M_t, Z_t, N_t} \ & u\left(c_t\right) + \beta\mathbb{E}\left[v\left(A_{t+1}; s_{t+1}, \bar{M}_{t+1}\right)\middle| s_t\right] \\
s.t. \quad & M_t + Z_t + Q_N\left(s_t, \bar{M}_t\right)N_t \leq A_t && (6.2) \\
& P_t c_t \Omega\left(\gamma_t\right) \leq \left(1-\eta\right)M_t, && (6.3) \\
& \theta P_t c_t\left(1 - \Omega\left(\gamma_t\right)\right) \leq Z_t, && (6.4) \\
& A_{t+1} = \left(1-\eta\right)M_t + Z_t + N_t + P_t\left(y - c_t - k^d\left(n - F\left(\gamma_t\right)\right)\right) + \left(\bar{M}_{t+1} - \bar{M}_t\right). && (6.5)
\end{aligned}$$

The household chooses: (i) consumption in the period, c_t; (ii) the threshold γ_t (household buys goods with size greater than γ_t with demand deposits, and less than γ_t with cash); (iii) amount of cash to hold in the period, M_t; (iv) demand deposit held as cash as reserves, Z_t; and (v) quantity of nominal bonds to purchase, N_t.

We can consider each constraint in turn:

- ▷ (6.2). This is the nominal account constraint which says that the total assets carried over from the previous period, A_t, can be used as (i) money held as cash today that can be spent, M_t; (ii) cash held as reserves for demand deposit, Z_t, and (iii) the cost of purchasing N_t nominal bonds, $Q_N\left(s_t, \bar{M}_t\right)N_t$.
- ▷ (6.3). This says that the total cost of purchasing goods with cash must be less than the disposable cash holding (note η of cash is "lost").
- ▷ (6.4). The constraint ensures that the household meets the reserve requirement (which is proportion θ of the deposit demand).

- ▷ (6.5). This gives the beginning-of-period asset holding for the next period. Since the nominal bonds pays out one unit of cash, purchasing N_t of bonds in period t yields N_t units of cash next period. M_t and Z_t do not earn interest so they have the same nominal value in the next period. $y - c_t - k^d (n - F(\gamma_t))$ is the real net income, and multiplying this by P_t gives the nominal value in period t. Again, this earns no interest. Finally, in period $t+1$, the household receives a transfer $\bar{M}_{t+1} - \bar{M}_t$ (the amount of new money issued) from the government.

We restrict monetary policy to be one in which ω is chosen (and not, for example, price level).[14] We will say that a recursive competitive equilibrium displays the quantity theory (QT) if

$$\begin{aligned} \frac{P\left(s_t, \bar{M}_t\right)}{\bar{M}_t} &= p\left(s_t\right), \ \forall s_t, \bar{M}_t, t, \\ Q_N\left(s_t, \bar{M}_t\right) &= q_N\left(s_t\right), \ \forall s_t, \bar{M}_t, t. \end{aligned}$$

(If we suppose that money supply $\bar{M}_t$ is, say \$100, and the price of consumption good is \$50, then $p = 0.5$.) The first expression implies that the price level P is proportion to money supply $\bar{M}_t$, which is what quantity theory of money requires.

To denote the problem in terms of $p(s_t)$, we normalise the other variables in a similar fashion:

$$a_t = \frac{A_t}{\bar{M}_t}, \ m_t = \frac{M_t}{\bar{M}_t}, \ z_t = \frac{Z_t}{\bar{M}_t}, \ n_t = \frac{N_t}{\bar{M}_t}.$$

Since the constraints are homogeneous of degree zero in $\bar{M}_t$, the value function is also homogeneous of degree zero in $\bar{M}_t$; i.e. (with slight abuse of notation)

$$v\left(A_t; S_t, \bar{M}_t\right) = v\left(a_t; s_t, 1\right) = v\left(a_t, s_t\right).$$

We can then rewrite the household's problem as

$$\begin{aligned} v(a,s) = \max_{c,\gamma,m,z,n} \quad & u(c) + \beta \mathbb{E}\left[v\left(a', s'\right) | s\right] \\ s.t. \quad & m + z + q_N(s)\, n_t \le a_t && (6.6) \\ & p(s)\, c\Omega(\gamma) \le (1-\eta)\, m, && (6.7) \\ & \theta p(s)\, c\left(1 - \Omega(\gamma)\right) \le z, && (6.8) \\ & a' = \frac{1}{\omega'}\left[(1-\eta)\, m + z + n + p(s)\left(y - c - k^d\left(\bar{n} - F(\gamma)\right)\right) + \left(\omega' - 1\right)\right], && (6.9) \end{aligned}$$

where we imposed the quantity theory and denote the number of transactions as $\bar{n}$ to avoid confusion. The last constraint is derived as

$$\begin{aligned} \frac{A_{t+1}}{M_{t+1}} \frac{M_{t+1}}{M_t} &= (1-\eta)\, m + z + n + p\left(y - c - k^d\left(\bar{n} - F(\gamma)\right)\right) + \left(\frac{\bar{M}_{t+1}}{\bar{M}_t} - 1\right) \\ \Leftrightarrow a'\omega' &= (1-\eta)\, m + z + n + p\left(y - c - k^d\left(\bar{n} - F(\gamma)\right)\right) + \left(\omega' - 1\right). \end{aligned}$$

Substituting for a' using (6.9) into the objective function and letting λ, μ_1, μ_2 denote the Lagrange multipliers on the constraints (6.6), (6.7) and (6.8) respectively, we can write the Lagrangian

[14] If ω is Markov, note that price level will not be Markov.

as

$$\mathcal{L} = u(c) + \beta \mathbb{E}\left[v\left(\frac{1}{\omega'}\left[(1-\eta)m + z + n + p(s)\left(y - c - k^d(\bar{n} - F(\gamma))\right) + (\omega' - 1)\right], s' \right) \Bigg| s \right]$$
$$+ \lambda\left[a - m - z - q_N(s) n_t\right]$$
$$+ \mu_1\left[(1-\eta)m - p(s) c\Omega(\gamma)\right]$$
$$+ \mu_2\left[z - \theta p(s) c (1 - \Omega(\gamma))\right].$$

Consider first the first-order condition with respect to n:

$$q_N(s)\lambda(s) = \beta \mathbb{E}\left[\frac{1}{\omega'} v_a(a', s') \Bigg| s \right],$$

where we've made explicit that λ depends on the current state s. The envelope condition gives the shadow value of asset in the current period.

$$v_a(a, s) = \lambda(s).$$

Combining the two gives the price of the bonds, which is inverse of the gross nominal interest rate:

$$q_N(s) = \beta \mathbb{E}\left[\frac{1}{\omega'} \frac{\lambda(s')}{\lambda(s)} \Bigg| s \right] \equiv \frac{1}{1 + i_N(s)}. \tag{6.10}$$

Observe that, in equilibrium, $q_N(s)$ does not appear in the budget constraint since $n = 0$ in equilibrium. Therefore, $q_N(s)$ does not affect equilibrium quantities of prices.

The other first-order conditions are:

$$\begin{aligned}
\{c\} \qquad & u'(c) = p(s)\left[\mu_1 \Omega(\gamma) + \mu_2 \theta (1 - \Omega(\gamma))\right] + p(s)\beta \mathbb{E}\left[\frac{1}{\omega'} v_a(a', s') \Bigg| s \right] \\
& \qquad = p(s)\left[\mu_1 \Omega(\gamma) + \mu_2 \theta (1 - \Omega(\gamma)) + q_N(s)\lambda(s)\right], \\
\{\gamma\} \qquad & 0 = \mu_1 p(s) c \Omega'(\gamma) - \mu_2 \theta p(s) c \Omega'(\gamma) - k^d F'(\gamma) p(s) \beta \mathbb{E}\left[\frac{1}{\omega'} v_a(a', s') \Bigg| s \right] \\
& \qquad = \gamma f(\gamma) c (\mu_1 - \mu_2 \theta) - k^d f(\gamma) q_N(s) \lambda(s) \\
& \gamma c (\mu_1 - \mu_2 \theta) = k^d q_N(s) \lambda(s), \\
\{m\} \qquad & \mu_1 (1 - \eta) - \lambda(s) = -(1 - \eta)\beta \mathbb{E}\left[\frac{1}{\omega'} v_a(a', s') \Bigg| s \right] = -(1 - \eta) q_N(s) \lambda(s) \\
& \mu_1 = \lambda(s)\left[\frac{1}{1 - \eta} - q_N(s) \right], \\
\{z\} \qquad & \mu_2 - \lambda(s) = -\beta \mathbb{E}\left[\frac{1}{\omega'} v_a(a', s') \Bigg| s \right] = -q_N(s)\lambda(s) \\
& \mu_2 = \lambda(s)\left(1 - q_N(s)\right),
\end{aligned}$$

where we used the fact that

$$\Omega'(\gamma) = \frac{d}{d\gamma}\left(\int_0^\gamma z f(z)\, dz \right) = \gamma f(\gamma).$$

Substituting out μ_1 and μ_2 yields

$$\begin{aligned}
u'(c) &= p(s)\left[\lambda(s)\left[\frac{1}{1-\eta}-q_N(s)\right]\Omega(\gamma)+\lambda(s)(1-q_N(s))\theta(1-\Omega(\gamma))+q_N(s)\lambda(s)\right] \\
&= p(s)\lambda(s)\left[\frac{\Omega(\gamma)}{1-\eta}+(1-\Omega(\gamma))q_N(s)+(1-q_N(s))\theta(1-\Omega(\gamma))\right] \\
&= p(s)\lambda(s)\left[\frac{\Omega(\gamma)}{1-\eta}+((1-q_N(s))\theta+q_N(s))(1-\Omega(\gamma))\right] \\
&= p(s)\lambda(s)\left[\frac{\Omega(\gamma)}{1-\eta}+(\theta+(1-\theta)q_N(s))(1-\Omega(\gamma))\right],
\end{aligned}$$

and

$$\begin{aligned}
k^d q_N(s)\lambda(s) &= \gamma c\left(\lambda(s)\left[\frac{1}{1-\eta}-q_N(s)\right]-\lambda(s)(1-q_N(s))\theta\right) \\
&= \gamma c\lambda(s)\left(\frac{1}{1-\eta}-q_N(s)-(1-q_N(s))\theta\right) \\
\Rightarrow k^d q_N(s) &= \gamma c\left(\frac{1}{1-\eta}-(\theta+(1-\theta)q_N(s))\right).
\end{aligned}$$

Remark 6.1. To see how the last expression relates to (6.1), using (6.10),

$$k^d\frac{1}{1+i_N(s)}=\gamma c\left(\frac{1}{1-\eta}-\left(\theta+\frac{1-\theta}{1+i_N(s)}\right)\right).$$

If η and i_N are small, then

$$\gamma=\frac{1}{c}\frac{k^d}{\eta+i_n}\frac{1}{1+i_n},$$

which is similar to (6.1) except for the multiplication by the q_N.

6.2.7 Recursive competitive equilibrium

For the bond market to clear, we require the stock of nominal bonds to be zero (recall no government bonds so nominal bonds must be in net zero supply)

$$n=0.$$

and for the goods market to clear, we require

$$y=c+k^d(n-F(\gamma))+\eta\frac{m}{p};$$

i.e. output is split between consumption, the (real) costs of transacting with demand deposits and cash. The last term $\eta m/p$ relates to the stolen cash. Here, we are assuming that the stolen cash is, in fact, spent (although purchases by the stolen cash does not enter into the utility function).

Recall that $\bar{M}$ is the money supply and $M+Z$ is the money demand. Hence, in equilibrium, $\bar{M}=M+Z$, which is equivalent to saying that

$$m+z=1.$$

Substituting this into (6.9) together with the goods and bond market clearing conditions gives us

$$
\begin{aligned}
a' &= \frac{1}{\omega'}\left[(1-\eta)\,m + z + \underbrace{n}_{=0} + p\,(s)\underbrace{\left(y - c - k^d\left(\bar{n} - F\left(\gamma\right)\right)\right)}_{=\eta m/p(s)} + (\omega' - 1)\right] \\
\Leftrightarrow a'\omega' &= \underbrace{m + z}_{=1} + \omega' - 1 \\
\Leftrightarrow a' &= 1.
\end{aligned}
$$

A recursive competitive equilibrium consists of functions $(p\,(s)\,, q_N\,(s))$ and $c\,(a,s)$, $\gamma\,(a,s)$, $m\,(a,s)$, $z\,(a,s)$, $n\,(a,s)$, λ, μ_1, μ_2 such that, given (p, q_N), the functions solve

(i) the household's problem;

(ii) for $a = 1$, market clearing conditions hold.

To solve for the RCE, recall that we have the following equations:

$$q_N = \beta\mathbb{E}\left[\frac{1}{\omega'}\frac{\lambda\,(s')}{\lambda\,(s)}|s\right] = \frac{1}{1+i_N}, \tag{6.11}$$

$$m + z = 1, \tag{6.12}$$

$$pc\Omega\,(\gamma) = (1-\eta)\,m, \tag{6.13}$$

$$\theta pc\,(1 - \Omega\,(\gamma)) = z, \tag{6.14}$$

$$u'\,(c) = p\lambda\,(s)\left[\frac{\Omega\,(\gamma)}{1-\eta} + (\theta + (1-\theta)\,q_N)\,(1 - \Omega\,(\gamma))\right], \tag{6.15}$$

$$k^d q_N\,(s) = \gamma c\left(\frac{1}{1-\eta} - (\theta + (1-\theta)\,q_N\,(s))\right), \tag{6.16}$$

$$y = c + k^d\,(n - F\,(\gamma)) + \eta\frac{m}{p}, \tag{6.17}$$

$$m + z = 1, \tag{6.18}$$

where c, γ, m, z, n, λ, p and q_N are all functions of $s = (y, \omega)$. Our goal is to express these as functions of γ.

The equations (6.11) to (6.16) come from the household's problem, and equations (6.17) and (6.18) are the market clearing conditions. Observe that (6.12) and (6.18) are the same due to Walras' law, which says that one equation is always redundant.[15]

We take q_N as given and first use (6.13) and (6.14) to obtain

$$\theta\,(1-\eta)\,\frac{m}{1-m} = \frac{\Omega\,(\gamma)}{1-\Omega\,(\gamma)}, \tag{6.19}$$

where we used the fact that $z = 1 - m$. Observe that the left-hand side is strictly increasing in m (derivative given by $(1-m)^{-2} > 0$) and the right-hand side is strictly increasing in γ (recall Ω is strictly increasing in γ). This means that there is a one-to-one mapping between γ and $m\,(\gamma)$.[16]

[15] Note that the budget constraint for the government is already built into (6.17).

[16] Since $\lim_{\gamma\to\infty}\Omega\,(\gamma) = 1$, there is no division by zero problem on the right-hand side of (6.19) so long as γ is finite. Similarly, since $m + z = 1$ in equilibrium, and $\theta > 0$ implies $z > 0$, $m \neq 1$ so again, we do not have a division by zero problem on the left-hand side of (6.19).

We can replace (6.14) with (6.19). Now take (6.13) and (6.17), which can be rearranged as

$$c = \frac{1-\eta}{\Omega(\gamma)}\frac{m}{p},$$
$$c = y - k^d(n - F(\gamma)) - \eta\frac{m}{p}.$$

Eliminating m/p yields

$$\begin{aligned} c &= y - k^d(n - F(\gamma)) - \frac{\eta}{1-\eta}\Omega(\gamma)c \\ &= \frac{y - k^d(n-F(\gamma))}{1+\frac{\eta}{1-\eta}\Omega(\gamma)}, \quad (6.20) \\ \frac{m}{p} &= \frac{\Omega(\gamma)}{1-\eta}\frac{y-k^d(n-F(\gamma))}{1+\frac{\eta}{1-\eta}\Omega(\gamma)} = \Omega(\gamma)\frac{y-k^d(n-F(\gamma))}{1-\eta(1-\Omega(\gamma))}. \quad (6.21) \end{aligned}$$

We may therefore replace (6.13) with (6.20), and (6.17) with (6.21).

Note that (6.16) is a hyperbola in (c,γ) space:

$$c = \frac{k^d q_N(s)}{\left(\frac{1}{1-\eta} - (\theta + (1-\theta)q_N(s))\right)}\frac{1}{\gamma}, \quad (6.22)$$

and (6.20), for small values of η, is increasing in γ. Thus, there is a unique intersection that determines c for a particular choice of γ. Given c, we can pin down m/p using (6.21), m using (6.19), and λ using (6.15).

It remains to pin down q_N. But notice that q_N depends on the expectation of future money growth rate as well as state. In order to analyse further, we must introduce further specialisation. From now on, we assume that $\eta = 0$.[17]

Example 6.1. Suppose y and ω are constant. We first "guess" that λ is constant. Then, (6.10) implies

$$q_N = \frac{\beta}{\omega}. \quad (6.23)$$

Define the money growth rate as Δ so that $\omega = 1+\Delta$. Then,

$$q_N = \frac{1}{1+\rho}\frac{1}{1+\Delta} = \frac{1}{1+i_N}.$$

For Δ and ρ small, then

$$i_N \approx \rho + \Delta;$$

i.e. the nominal interest rate is the sum of the real interest rate, ρ, and the rate of money growth, Δ. Recall that m/p is constant here and that $m = M/\bar{M}$ and $p = P/\bar{M}$. So inflation rate must equal the money growth rate:

$$\pi = \frac{P(\bar{M}') - P(\bar{M})}{P(\bar{M})} = \frac{\bar{M}' - \bar{M}}{\bar{M}} = \omega - 1 = \Delta.$$

Thus, we see that the money growth rate in this case is equal to the inflation rate.

[17] Recall that η is the real cost associated with using cash. The assumption that $\eta > 0$ ensures that the model can handle small deflation. If $\eta = 0$, and there is deflation, holding cash is strictly better than holding demand deposit.

Now suppose that money growth rate increases, i.e. ω increases. Then, from (6.23), we see that q_N decreases. This leaves (6.20) unchanged while (6.22) shifts down. Hence, we realise that both γ and c falls. That real consumption falling reflects the fact that k^d has to be paid more often.

Observe that for nominal interest to be nonnegative. i.e. $i_N \geq 0$, we must have

$$\begin{aligned} \frac{\beta}{\omega} \leq 1 \Leftrightarrow \beta \leq \omega \\ \Leftrightarrow \frac{1}{1+\rho} \leq 1+\Delta. \end{aligned}$$

Since $\rho \in (0,1)$, we can have some small deflation (i.e. $\Delta < 0$) while maintaining nonnegative nominal interest rate.

Friedman rule says that, in order to maximise welfare, the policy maker should make ω as small possible, which is to say that the policy maker should set Δ so that the inequality above holds with equality, which also means deflation and that $i_N = 0$. The idea is that the household should be made indifferent between holding cash and nominal bonds so that they do not economise over cash (since economising implies that households incurs the cost k^d).

Finally, to verify that λ is constant observe from (6.22) that c is constant here. Then, since m and m/p and are constant (see (6.19) and (6.21)), p must be constant. Finally, it follows from (6.15) that λ must also be constant.

Example 6.2. Maintain the assumption that y is constant but now assume that ω is iid. We guess again that λ is constant so that

$$q_N = \beta \mathbb{E}\left[\frac{1}{w'}\right];$$

i.e. q_N is constant. Therefore, all of the observations we made in the previous example carries through in this case.

Example 6.3. Maintain the assumption that y is constant and suppose that ω is a Markov. This implies that q_N is no longer constant. Thus, the previous "trick" no longer works. To proceed, we simplify the model to a cash-in-advance model; i.e.

$$\gamma = +\infty \Rightarrow F(\gamma) = n,\ \Omega(\gamma) = 1,$$

and we also drop the first-order condition for γ. That $\gamma = +\infty$ means that the household uses cash for all purchases. Since the household can only use cash carried over from the previous period, the model reduces to a cash-in-advance model. Goods market clearing implies $c = y$, and money market clearing implies $m = a = a' = 1$ (since $z = 0$ here). So, together with the cash-in-advance constraint,

$$p = \frac{1}{c} = \frac{1}{y}$$

We also assume that preferences are CRRA so

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},\ \sigma > 0 \Rightarrow u'(c) = c^{-\sigma}.$$

Then,

$$\lambda = \frac{1}{p} u'(c) = y^{1-\sigma}.$$

$$pc\Omega(\gamma) = (1-\eta)\, m$$
$$pc = m$$

Since $m = 1$, (6.13) reduces to $pc = m$, and so

$$pc = py = 1,$$

The nominal interest rate is given by

$$q_N = \beta \mathbb{E}\left[\frac{1}{\omega'} \left(\frac{y'}{y} \right)^{1-\sigma} \middle| s \right].$$

With log preferences (i.e. $\sigma = 1$), we get back to the previous example in which interest rate depends only on the expected money growth rate (though the expectation depends on the current state in this set up). More generally, the impact of changes in real income depends on whether σ, the coefficient of relative risk aversion, is greater or smaller than one. This is because changes in real income have two effects: it leads to higher consumption which depresses the real interest rate; but, it also lowers price level (i.e. more inflation). ?

Remark 6.2. Over the period 1929–33, i.e. the Great Depression, the cumulative decline were

$$\Delta\text{real GDP} + \Delta\text{price level} = (-34\%) + (-24\%) = -58\% = \Delta\text{nominal GDP}.$$

Bank deposit also fell by -48% as a result of “bank runs” (despite reserves increasing slightly). It appears that there is a strong connection between the decline in bank deposit and the decline in GDP—one solution is to provide deposit insurance, another is to prevent withdrawals from deposit accounts. In 2008, although deposit did not fall (they were insured), money market mutual funds played the role of bank deposit—the Fed eventually provided insurance on such funds which mitigated the “bank run” on the funds.