1 PS8 Q2 (Bilateral Trade with Discrete Types)

Consider a bilateral trading situation similar to Myerson and Satterthwaite (1983), except that the distributions of types are discrete. Specifically, suppose $\theta_s \in \{1, 4\}$ with equal probability, and $\theta_b \in \{0, 3\}$ with equal probability. Hence, trade is efficient for some pairs of types, but not all.

Problem 1.1. Do the conditions of the Myerson-Satterthwaite impossibility theorem apply to this situation? Explain.

Solution. We have

$$\bar{\theta}_s = 4 > 0 = \underline{\theta}_b$$

and

$$\underline{\theta}_s = 1 < 3 = \bar{\theta}_b$$

and so the conditions of the Myerson-Sattethwaite impossibility theorem apply. Thus, any ex-post efficient mechanism that is budget balanced and IC is not IR.

Problem 1.2. If your answer to (a) is "No," then attempt to provide a simple counter example in which efficient trade can be implemented. If your answer to (a) is "Yes," then for which type pairs is trade necessarily inefficient?

Solution. We can see that the only case where $\theta_b > \theta_s$ is $\theta_b = 3$ and $\theta_s = 1$. Therefore, for the pairs $(3, 4)$, $(0, 1)$, $(0, 4)$ trade is necessarily inefficient.

2 PS8 Q4 (Required Subsidy for Efficient Bilateral Trade) (Xindi)

Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

Problem 2.1. A necessary condition for IC (as shown in class) requires

$$\mathbb{E}_{\theta_B,\theta_S}\left[\phi\left(\theta_B,\theta_S\right)\left[\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)-\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\right]\right]\geq 0 \tag{1}$$

Using the fact that types are uniformly distributed on $[0,2]$, show that, in any mechanism, conditional on trade, the expected difference between θ_B and θ_S must be at least 1.

Solution. By revelation principle, we know that any mechanism corresponds to a truth-telling direct mechanism that satisfies (1). So, by (1) and LIE, we have

$$\begin{aligned}
0 &\leq \mathbb{E}_{\theta_B,\theta_S}\left[\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)-\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\mid\phi\left(\theta_B,\theta_S\right)=1\right]\mathbb{P}(\phi(\theta_B,\theta_S)=1)\\
&=\mathbb{E}_{\theta_B,\theta_S}\left[\left(\theta_B-\frac{1-\frac{\theta_B}{2}}{\frac{1}{2}}\right)-\left(\theta_S+\frac{\frac{\theta_S}{2}}{\frac{1}{2}}\right)\mid\phi\left(\theta_B,\theta_S\right)=1\right]\\
&=\mathbb{E}_{\theta_B,\theta_S}\left[\left(\theta_B-2+\theta_B\right)-\left(\theta_S+\theta_S\right)\mid\phi\left(\theta_B,\theta_S\right)=1\right]\\
&=\mathbb{E}_{\theta_B,\theta_S}\left[2\theta_B-2\theta_S\mid\phi\left(\theta_B,\theta_S\right)=1\right]-2\\
\Rightarrow 1 &\leq \mathbb{E}_{\theta_B,\theta_S}\left[\theta_B-\theta_S\mid\phi\left(\theta_B,\theta_S\right)=1\right]
\end{aligned}$$

Problem 2.2. Compute the expected difference between θ_B and θ_S, conditional on efficient trade occurring (i.e., conditional on $\theta_B\geq\theta_S$). Using (a), conclude directly that an ex post efficient, budget-balanced, IC mechanism cannot satisfy IR.

Solution. Consider

$$\begin{aligned}
&\mathbb{E}_{\theta_B,\theta_S}\left[\theta_B-\theta_S\mid\theta_B\geq\theta_S\right]\\
=&\int_0^2\int_0^{\theta_B}\left(\theta_B-\theta_S\right)f\left(\theta_B,\theta_S\mid\theta_B\geq\theta_S\right)d\theta_S d\theta_B\\
=&\int_0^2\int_0^{\theta_B}\left(\theta_B-\theta_S\right)\frac{1}{2}d\theta_S d\theta_B\\
=&\frac{1}{2}\int_0^2\left[\left(\theta_B^2-0\right)-\left(\frac{1}{2}\theta_B^2-0\right)\right]d\theta_B\\
=&\frac{1}{4}\int_0^2\theta_B^2 d\theta_B\\
=&\frac{2}{3}
\end{aligned}$$

From the result in (a), we learned that any mechanism that is budget balanced, IC and IR, it must be that

$$\mathbb{E}_{\theta_B,\theta_S}\left[\theta_B-\theta_S\mid\phi\left(\theta_B,\theta_S\right)=1\right]\geq 1$$

However, we have shown that, under ex post efficient mechanism,

$$\mathbb{E}_{\theta_B,\theta_S}\left[\theta_B-\theta_S \mid \phi^\star\left(\theta_B,\theta_S\right)=1\right]=\mathbb{E}_{\theta_B,\theta_S}\left[\theta_B-\theta_S \mid \theta_B \geq \theta_S\right]=\frac{2}{3}$$

which implies that

$$U_B\left(\underline{\theta}_B\right)+U_S\left(\overline{\theta}_S\right)<0$$

i.e. the mechanism is not IR.

Problem 2.3. Compute the value of

$$\mathbb{E}_{\theta_B,\theta_S}\left[\phi\left(\theta_B,\theta_S\right)\left[\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)-\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\right]\right]$$

when ϕ is ex post efficient. Using your answers to (a) and (b), explain why this number is a measure of the minimum amount of external subsidy which must be introduced to obtain ex post efficient trade in an IC and IR mechanism.

Solution. Under the ex post efficient mechanism and uniform distribution, we have

$$
\begin{aligned}
&\mathbb{E}_{\theta_B,\theta_S}\left[\phi\left(\theta_B,\theta_S\right)\left[\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)-\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\right]\right]\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\overline{\theta}_S}\left[\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)-\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\right]\phi\left(\theta_B,\theta_S\right)f_B\left(\theta_B\right)f_S\left(\theta_S\right)d\theta_Bd\theta_S\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\overline{\theta}_S}\left(\theta_B-\frac{1-F_B\left(\theta_B\right)}{f_B\left(\theta_B\right)}\right)\phi\left(\theta_B,\theta_S\right)f_B\left(\theta_B\right)f_S\left(\theta_S\right)d\theta_Bd\theta_S\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\overline{\theta}_S}\left(\theta_S+\frac{F_S\left(\theta_S\right)}{f_S\left(\theta_S\right)}\right)\phi\left(\theta_B,\theta_S\right)f_B\left(\theta_B\right)f_S\left(\theta_S\right)d\theta_Bd\theta_S\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\overline{\theta}_S}\left(\theta_Bf_B\left(\theta_B\right)+F_B\left(\theta_B\right)-1\right)\phi\left(\theta_B,\theta_S\right)f_S\left(\theta_S\right)d\theta_Bd\theta_S\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\overline{\theta}_S}\left(\theta_Sf_S\left(\theta_S\right)+F_S\left(\theta_S\right)\right)\phi\left(\theta_B,\theta_S\right)f_B\left(\theta_B\right)d\theta_Bd\theta_S\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\min\left\{\theta_B,\overline{\theta}_S\right\}}\left(\theta_Bf_B\left(\theta_B\right)+F_B\left(\theta_B\right)-1\right)f_S\left(\theta_S\right)d\theta_Bd\theta_S\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\int_{\underline{\theta}_S}^{\min\left\{\theta_B,\overline{\theta}_S\right\}}\left(\theta_Sf_S\left(\theta_S\right)+F_S\left(\theta_S\right)\right)f_B\left(\theta_B\right)d\theta_Bd\theta_S\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(\theta_Bf_B\left(\theta_B\right)+F_B\left(\theta_B\right)-1\right)\left(\int_{\underline{\theta}_S}^{\min\left\{\theta_B,\overline{\theta}_S\right\}}f_S\left(\theta_S\right)d\theta_S\right)d\theta_B\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(\int_{\underline{\theta}_S}^{\min\left\{\theta_B,\overline{\theta}_S\right\}}\left(\theta_Sf_S\left(\theta_S\right)+F_S\left(\theta_S\right)\right)d\theta_S\right)f_B\left(\theta_B\right)d\theta_B\\
&=\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(\theta_Bf_B\left(\theta_B\right)+F_B\left(\theta_B\right)-1\right)F_S\left(\theta_B\right)d\theta_B\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(\theta_SF_S\left(\theta_S\right)\Big|_{\underline{\theta}_S}^{\min\left\{\theta_B,\overline{\theta}_S\right\}}\right)f_B\left(\theta_B\right)d\theta_B\\
&=-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(1-F_B\left(\theta_B\right)\right)F_S\left(\theta_B\right)d\theta_B+\int_{\underline{\theta}_B}^{\overline{\theta}_B}\theta_Bf_B\left(\theta_B\right)F_S\left(\theta_B\right)d\theta_B\\
&\quad-\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(\min\left\{\theta_BF_S\left(\theta_B\right),\overline{\theta}_S\right\}\right)f_B\left(\theta_B\right)d\theta_B
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(\theta_B F_S\left(\theta_B\right) - \min\left\{\theta_B F_S\left(\theta_B\right), \overline{\theta}_S\right\}\right) f_B\left(\theta_B\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \int_{\overline{\theta}_S}^{\overline{\theta}_B} \left(\theta_B - \overline{\theta}_S\right) f_B\left(\theta_B\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \left(\theta_B - \overline{\theta}_S\right) F_B\left(\theta_B\right)\Big|_{\overline{\theta}_S}^{\overline{\theta}_B} - \int_{\overline{\theta}_S}^{\overline{\theta}_B} F_B\left(\theta_B\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \left(\overline{\theta}_B - \overline{\theta}_S\right) - \underbrace{\left(\overline{\theta}_S - \overline{\theta}_S\right) F_B\left(\overline{\theta}_S\right)}_{=0} - \int_{\overline{\theta}_S}^{\overline{\theta}_B} F_B\left(\theta_B\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \left(\overline{\theta}_B - \overline{\theta}_S\right) - \int_{\overline{\theta}_S}^{\overline{\theta}_B} F_B\left(\theta_B\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B - \int_{\overline{\theta}_S}^{\overline{\theta}_B} \left(F_B\left(\theta_B\right) - 1\right) d\theta_B \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B - \int_{\overline{\theta}_S}^{\overline{\theta}_B} \left(F_B\left(\theta_B\right) - 1\right) F_S\left(\theta_B\right) d\theta_B \\
&= -\left[\int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B + \int_{\overline{\theta}_B}^{\overline{\theta}_S} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B\right] \\
&= -\int_{\underline{\theta}_B}^{\overline{\theta}_S} \left(1 - F_B\left(\theta_B\right)\right) F_S\left(\theta_B\right) d\theta_B \\
&= -\int_0^2 \left(1 - \frac{\theta_B}{2}\right) \frac{\theta_B}{2} d\theta_B \\
&= -\int_0^2 \left(\frac{\theta_B}{2} - \frac{\theta_B^2}{4}\right) d\theta_B \\
&= -\left(\frac{1}{4}\theta_B^2 - \frac{1}{12}\theta_B^3\Big|_0^2\right) \\
&= -\left(1 - \frac{2}{3}\right) \\
&= -\frac{1}{3}
\end{aligned}
$$

Notice that the result $-\frac{1}{3}$ obtained here is exactly equal to the difference in (a) and (b). Result in (a) that

$$\mathbb{E}_{\theta_B, \theta_S}\left[\theta_B - \theta_S \mid \phi\left(\theta_B, \theta_S\right) = 1\right] \geq 1$$

means that for the trade to happen in any budget balanced, IC and IR mechanism we must have a difference no less than 1. Result in (b) that

$$\mathbb{E}_{\theta_B, \theta_S}\left[\theta_B - \theta_S \mid \phi^{\star}\left(\theta_B, \theta_S\right) = 1\right] = \frac{2}{3}$$

means that for the trade to happen in the ex post efficient mechanism, we have the difference equal to $\frac{2}{3}$, which is smaller than 1 by $\frac{1}{3}$. So, in order to make the ex post efficient mechanism IC and IR, we have to subsidize it by the minimum difference $\frac{1}{3}$ to make it satisfy the result in (a).

3 PS8 Q6 (Maximizing Welfare in Bilateral Trade)

Consider the setting of Myerson and Satterthwaite (1983) and assume that both the buyer's and seller's values are uniformly (and independently) distributed on $[0, 2]$.

Problem 3.1. Solve for the welfare-maximizing trading allocation direct mechanism $\{\phi(\cdot), t(\cdot)\}$, that is incentive compatible, individually rational and ex post budget-balanced (i.e. the seller receives exactly the payment that the buyer pays – there are no additional subsidies). For this question, it is sufficient to characterize $\phi(\cdot)$, you do not need to characterize $t(\cdot)$.

Solution. The problem is now to solve

$$\max_{\phi(\cdot,\cdot)} \mathbb{E}_{\theta_B,\theta_S}\left[\phi(\theta_B,\theta_S)(\theta_B-\theta_S)\right]$$

subject to $\{\phi(\cdot,\cdot), t(\cdot,\cdot)\}$ is incentive compatible and individually rational. To solve for this, we need to invoke a theorem. In particular, we know that for any $\phi(\cdot,\cdot)$ such that $\bar{\phi}_B(\cdot)$ is non-decreasing and $\bar{\phi}_S(\cdot)$ is non-decreasing, if

$$\mathbb{E}_{\theta_B,\theta_S}\left\{\phi(\theta_B,\theta_S)\left[\left(\theta_B-\frac{1-F_B(\theta_B)}{f_B(\theta_B)}\right)-\left(\theta_S+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right]\right\}\geq 0$$

then there exists a $t(\cdot,\cdot)$ such that $\{\phi(\cdot,\cdot), t(\cdot,\cdot)\}$ is incentive compatible (IC) and individually rational (IR). Using this theorem, we can write the lagrangian as

$$\begin{aligned}
L &= \mathbb{E}_{\theta_S,\theta_B}\left[\phi(\theta_B,\theta_S)(\theta_B-\theta_S)\right]+\lambda\mathbb{E}_{\theta_B,\theta_S}\left\{\phi(\theta_B,\theta_S)\left[\left(\theta_B-\frac{1-F_B(\theta_B)}{f_B(\theta_B)}\right)-\left(\theta_S+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right]\right\}\\
&= \mathbb{E}_{\theta_S,\theta_B}\left\{\phi(\theta_B,\theta_S)\left(\theta_B-\theta_S+\lambda\left[\left(\theta_B-\frac{1-F_B(\theta_B)}{f_B(\theta_B)}\right)-\left(\theta_S+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right]\right)\right\}\\
&= \mathbb{E}_{\theta_S,\theta_B}\left\{\phi(\theta_B,\theta_S)\left((\theta_B-\theta_S)-\frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)}+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right)\right\}
\end{aligned}$$

it's then immediate from this expression that the solution to the maximization problem is

$$\phi(\theta_B,\theta_S)=\begin{cases}1 & \theta_B-\theta_S\geq\frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)}+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\\ 0 & \theta_B-\theta_S<\frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)}+\frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\end{cases}$$

specializing to a uniform distribution over $[0, 2]$ we then have the functional form

$$\phi(\theta_B,\theta_S)=\begin{cases}1 & \theta_B-\theta_S\geq\frac{2\lambda}{1+2\lambda}\\ 0 & \theta_B-\theta_S<\frac{2\lambda}{1+2\lambda}\end{cases}$$

Problem 3.2. How does the efficiency gap in (a) compare to the one you found in exercise 5 in the linear equilibrium to the Chatterjee-Samuelson bilateral trading game?

Solution. In Chatterjee-Samuelson we found that the efficiency gap in the linear equilibrium was $\alpha = \frac{1}{2}$. For the purposes of comparison, I will specialize to the case of a $U[0,2]$ distribution as in question 5. This yields an efficiency gap of

$$\begin{aligned}
\theta_B - \theta_S &\geq \frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)} + \frac{F_S(\theta_S)}{f_S(\theta_S)}\right) \\
\theta_B - \theta_S &\geq \frac{\lambda}{1+\lambda}\left(\frac{1-\left(\frac{1}{2}\theta_B\right)}{\frac{1}{2}} + \frac{\frac{1}{2}\theta_S}{\frac{1}{2}}\right) \\
\theta_B - \theta_S &\geq \frac{\lambda}{1+\lambda}\left(2-\theta_B+\theta_s\right) \\
(1+\lambda)(\theta_B-\theta_S) &\geq 2\lambda - \lambda(\theta_B-\theta_S) \\
\theta_B - \theta_S &\geq \frac{2\lambda}{1+2\lambda}
\end{aligned}$$

From this, we can write

$$\begin{aligned}
&\mathbb{E}_{\theta_S,\theta_B}\left\{\phi(\theta_B,\theta_S)\left((\theta_B-\theta_S) - \frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)} + \frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right)\right\} \\
&\quad= \mathbb{E}_{\theta_S,\theta_B}\left\{\mathbb{I}\left[\theta_B-\theta_S \geq \frac{2\lambda}{1+2\lambda}\right]\left((\theta_B-\theta_S) - \frac{\lambda}{1+\lambda}\left(\frac{1-F_B(\theta_B)}{f_B(\theta_B)} + \frac{F_S(\theta_S)}{f_S(\theta_S)}\right)\right)\right\} \\
&\quad= \mathbb{E}_{\theta_S,\theta_B}\left\{\mathbb{I}\left[\theta_B-\theta_S \geq \frac{2\lambda}{1+2\lambda}\right](\theta_B-\theta_S-1)\right\} \\
&\quad= \int_{\frac{2\lambda}{1+2\lambda}}^{2}\int_0^{\theta_B-\frac{2\lambda}{1+2\lambda}} (\theta_B-\theta_s-1)\, f(\theta_S)\, d\theta_S f(\theta_B)\, d\theta_B \\
&\quad= \frac{2(\lambda+1)^2(2\lambda-1)}{3(2\lambda+1)^3}
\end{aligned}$$

which has roots -1 and, in particular, $\frac{1}{2}$. Using $\lambda = \frac{1}{2}$, we then have that

$$\begin{aligned}
\theta_B - \theta_S &\geq \frac{2\lambda}{1+2\lambda} \\
&= \frac{2\times\frac{1}{2}}{1+2\times\frac{1}{2}} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}$$

which is identical to the wedge in the case of the Chatterjee-Samuelson bilateral trading game with a linear equilibrium.