Class note II

Overlapping Generations Economy, Social Security and the 1st Welfare Theorem

In this note we will study a version of an Overlapping Generations Economy (OLG) without production. The note is divided in four parts:

- The first part analyzes an endowment OLG economy. It describes in detail the model and it is argued, by way of an example, that the First Welfare Theorem, as stated so far, does not always hold. Importantly, if this is the case, the equilibrium interest rate turns out to be negative.
- Since the competitive equilibrium need not be efficient, it is maybe feasible for the government to improve upon the CE allocation. In the second part of the note we analyze the introduction of a pay-as-you-go social security system. It is shown that in the cases where the CE fails to be efficient, an appropriate social security system could be welfare improving.

- The third part of the note discusses why the First Welfare Theorem fails to hold in general. It is shown that the key step in the proof of the 1st Welfare Thm that breaks down is when we try to compute the value the aggregate consumption plan.

- The last part of the note introduces population and productivity growth. As in the standard model, the CE may fail to be efficient. In the former, the condition for inefficiency is a negative interest rate. In a growing economy, the condition for inefficiency is an interest rate small enough (we will make this clear below).

- Problems sets and a separate note will extend the analysis to heterogeneity within generations, long lived generations, capital accumulation, uncertainty (after other notes) and a set of fiscal polices to discuss “Ricardian" equivalence propositions.

Overlapping Generations Economy

We will consider an economy with a unique consumption good in each time period $t = 1, 2, ...$, so that $L = R^{\infty}$.

This economy is populated by agents who live and consume for only two periods. We index time by $t = 1, 2, ...$ and generations by the date of their birth. Thus the set of agent is $I = \{0, 1, 2, ...\}$. Agents born at dates $i \geq 1$ have utility function

$$u^i\left(x_1, x_2, x_3, ...\right) = v^i\left(x_i, x_{i+1}\right)$$

so that they only care about consumption in dates i and $i + 1$.

This implies that in any CE or in any efficient allocation for all agents born at $t = 0, 1, 2$,...

$$x_t^i = 0 \text{ for all } t \neq i \text{ or } t \neq i + 1$$

Agent born at date 0 are different, they are already old at time $t = 1$.Their utility function is given by

$$u^0 (x_1, x_2, x_3, ...) = x_1^0$$

so they just like to consume goods at date $t = 1$. Clearly in any CE or in any efficient allocation, $x_t^0 = 0$ for all $t = 2, 3....$

The agents born at date $i \geq 1$ have positive endowments at times i and $i + 1$.In particular we assume that

$$e^i = \left(e_1^i, e_2^i, ..., e_i^i, e_{i+1}^i, ...\right)$$

with

$$\begin{aligned} e_i^i &> 0, \; e_{i+1}^i > 0 \\ e_t^i &= 0 \text{ for all } t \neq i \text{ or } t \neq i+1 \end{aligned}$$

Generation 0 is also different. They only have a time $t = 1$ endowment, so that

$$e_1^0 > 0 \text{ and } e_t^0 = 0 \text{ for all } t = 2, 3....$$

To simplify we will first consider a pure exchange economy, so that there are no firms. In this case feasible allocations satisfy

$$\sum_{i \in I} x^i = \sum_{i \in I} e^i$$

which, since at each time period t only agents born at t and $t-1$ want to consume, can be written as

$$x_i^{i-1} + x_i^i = e_i^{i-1} + e_i^i \equiv \bar{e}_i$$

for all dates $i = 1, 2, ...$where we refer to $\bar{e}_t$ as the time t aggregate endowment.

Competitive Equilibrium

The price vector p is an element of R^{∞}, so that

$$p = (p_1, p_2, p_3, ...)$$

The agent problem is

$$\max_{x} u^i(x)$$

subject to

$$px \leq pe^i$$

which, since generation i neither consume nor has endowments at time $t \neq i$ or $t \neq i+1$, can be specialized as

$$\max_{x_i, x_{i+1}} v^i(x_i, x_{i+1})$$

subject to

$$p_i \; x_i \; + \; p_{i+1} \; x_{i+1} = p_i \; e_i^i \; + \; p_{i+1} \; e_{i+1}^i$$

and for generation $i = 0$ as

$$\max_{x_1} \; x_1 \text{ subject to } p_1 \; x_1 = p_1 \; e_1^0.$$

Proposition. The only competitive equilibrium has

$$x^i = e^i$$

i.e. there is no trade in equilibrium.

Proof. To see why, notice that since agent $i = 0$ only cares about goods dated at $t = 0$ and has only endowment at time $t = 0$, then for any price vector p

$$x_1^0 = e_1^0$$

Using market clearing for $t = 1$ gives

$$x_1^1 = \bar{e}_1 - x_1^0 = \bar{e}_1 - e_1^0 = e_1^1$$

Then, by examining the budget constraint of generation $t = 1$,

$$x_2^1 = e_2^1$$

and using this into the market clearing for $t = 2$, we conclude that

$$x_2^2 = \bar{e}_2 - x_2^1 = \bar{e}_2 - e_2^1 = e_2^2$$

Continuing in this way (formally, using induction) we show that $x^i = e^i$. QED

Equilibrium prices.

We now describe the equilibrium price vector for this economy. As always, we can normalize one price, so we set

$$p_1 = 1$$

Using the previous proposition, we know that the unique equilibrium is given by $x^i = e^i$. Since the first order conditions are necessary for the agent maximization problem,

$$\frac{p_{i+1}}{p_i} = \frac{v_2^i\left(e_i^i, e_{i+1}^i\right)}{v_1^i\left(e_i^i, e_{i+1}^i\right)}$$

must hold for all $i \geq 1$. Notice that these relative prices have the interpretation of interest rates.

Let r_t be the time t net interest rate satisfying

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

for all t. From our previous condition we have

$$r_t = \frac{v_1^t\left(e_t^t, e_{i+1}^t\right)}{v_2^t\left(e_t^t, e_{t+1}^t\right)} - 1$$

for all $t > 1$ and

$$p_t = \frac{1}{(1+r_1)(1+r_2)\cdots(1+r_{t-1})}.$$

Now we turn to analyze whether competitive equilibrium allocations are Pareto Optimal or not.

To simplify the analysis let's specialize the model to have v^i the same for all generations, and equal to

$$v^i\left(c_y, c_0\right) = (1-\beta)\log c_y + \beta \log c_0$$

for some constant $\beta \in (0,1)$.

For the endowments we assume that

$$e_i^i = 1 - \alpha \text{ and } e_{i+1}^i = \alpha$$

for all $i \geq 1$ and $e_1^0 = \alpha$ for some $\alpha \in (0,1)$.

Notice that in this case the aggregate endowment is constant, i.e. $\bar{e}_t = 1$ for all $t \geq 1$.

Equilibrium prices.

Let's first describe the equilibrium price vector for this particular economy. Using the previous result we have

$$\frac{1}{1+r_i} = \frac{p_{i+1}}{p_i} = \frac{v_2^i\left(e_i^i, e_{i+1}^i\right)}{v_1^i\left(e_i^i, e_{i+1}^i\right)} = \frac{\beta}{(1-\beta)}\frac{1-\alpha}{\alpha}$$

so that

$$r_t \equiv \bar{r} = \frac{(1-\beta)}{\beta}\frac{\alpha}{1-\alpha} - 1 = \frac{\alpha-\beta}{\beta(1-\alpha)}$$

or

$$p_t = \left[\frac{\beta}{(1-\beta)}\frac{1-\alpha}{\alpha}\right]^{t-1} \text{ for } t \geq 1.$$

The quantity $e_t^t - x_t^t$ are the savings of the generation t while they are young. For this particular preferences, savings are increasing in the interest rate.

Question. Show that the optimal saving $s(r;\alpha,\beta)$

$$s(\bar{r};\alpha,\beta) = \arg\max_{s} \ (1-\beta)\log(1-\alpha-s) + \beta\log(\alpha + s(1+\bar{r}))$$

are given by

$$s(\bar{r};\alpha,\beta) = (1-\alpha) - (1-\beta)\left[(1-\alpha) + \frac{\alpha}{1+\bar{r}}\right]$$

and hence are increasing in $\bar{r}$, increasing in β and decreasing in α.

By the previous proposition, in equilibrium, real interest rates are such that the savings of the young are zero; we will use this to give intuition for why the equilibrium interest rate is increasing in α and decreasing in β.

For higher preference parameter β, young agents prefer to consume more when old and less when young, so agents wants to save more at the same interest rate. Thus, the equilibrium interest rate $\bar{r}$ has to be lower.

For higher parameter α, the endowment when old is higher and the endowment when young is smaller, so agents prefer to save less at the same interest rate.

Thus the equilibrium interest rate $\bar{r}$ has to be higher. Indeed, notice that $\bar{r} > 0$ if $\alpha > \beta$ and $\bar{r} \leq 0$ if $\alpha \leq \beta$.

Symmetric allocations.

It will be convenient to consider the set of symmetric allocations, i.e. those allocations where the consumption of an agent when young (or when old) does not depend on her generation.

These allocations are describe by two numbers c_y, c_o :

$$\begin{aligned} x_i^i &= c_y \text{ for all } i \geq 1 \\ x_{i+1}^i &= c_o \text{ for all } i \geq 0 \end{aligned}$$

A symmetric allocation described by (c_y, c_o) is feasible if

$$c_y + c_o = 1$$

Best symmetric allocation

We will solve for the best feasible symmetric allocation, where best is for the point of view of the young. In particular, consider the problem

$$\max_{c_y, c_o} \; v(c_y, c_o) = \max_{c_y, c_o} \; (1-\beta)\log c_y + \beta \log c_o$$

subject to

$$c_y + c_o = 1$$

Its sufficient first order condition is given by

$$\frac{\beta}{1-\beta}\frac{c_y}{c_o} = 1$$

so the solution of this f.o.c. that also is feasible, i.e. the solution of the problem is

$$c_y = 1-\beta, \; c_o = \beta.$$

The best symmetric allocation depends on β in this way because for higher preference parameter β agents give less weight to consumption when young and more weight to consumption when old.

Comparing the CE allocation with the best symmetric allocation.

We will compare the utility of the unique competitive equilibrium allocation

$$\bar{c}_i^i = 1 - \alpha, \quad \bar{c}_{i+1}^i = \alpha \text{ for } i \geq 1 \text{ and } \bar{c}_1^0 = \alpha$$

with the one for the best symmetric allocation

$$c_i^{*i} = 1 - \beta, \;\; c_{i+1}^{*i} = \beta \text{ for } i \geq 1 \text{ and } c_1^{*0} = \beta$$

Notice that, since the CE allocation has $x^i = e^i$, and since that allocation is a feasible symmetric allocation, then, unless $c^* = \bar{c}$ –which happens only when $\alpha = \beta$– the best symmetric feasible allocation is strictly preferred by the agents of generations $i = 1, 2, ...$. It only remains to compare the utility of the initial old, i.e. generation $i = 0$, between the best symmetric and CE allocations.

– Case 1. $\beta > \alpha$. In this case, the initial old $i = 0$ strictly prefer the best symmetric allocation, and hence, all agents are strictly better off. Thus, the CE allocation is Pareto-dominated by the best symmetric feasible allocation. Notice that in this case the competitive equilibrium has negative interest rate $\bar{r}$.

– Case 2. $\beta = \alpha$. In this case the best symmetric allocation and the CE are exactly the same. Notice that in this case the competitive equilibrium has a zero interest rate $\bar{r}$.

– Case 3. $\beta < \alpha$. In this case, the initial old $i = 0$ strictly prefer the CE to the best symmetric allocation. As in case 1, all the other generations strictly prefer the best symmetric allocation. Thus, the CE and the best symmetric allocation are not comparable, in the sense that not all agents prefer one allocation to the other. Notice that in this case the competitive equilibrium has a strictly positive interest rate $\bar{r}$.

Social Security

Consider a tax policy indexed by a single parameter τ.

This policy tax young agents of each generation by τ and gives this tax as a subsidy to the old currently alive. By construction this policy is budget feasible for the government for any τ since taxes equal subsidies in each period. This policy generates the following after tax endowments

$$\begin{aligned} e_i^i &= (1-\alpha) - \tau \text{ and } e_{i+1}^i = \alpha + \tau \text{ for all } i \geq 1 \\ e_1^0 &= \alpha + \tau \end{aligned}$$

For positive τ this tax policy resembles a pay-as-you-go social security system.

Notice that by suitable choice of τ we can make the after-tax endowments equal to the best symmetric allocations, the required τ is

$$\tau = \beta - \alpha$$

Question.

Suppose you propose a pay-as-you-go social security system for the OLG economy as described above.

Under what parameter values will this policy will produce a competitive equilibrium that Pareto dominates the CE without the policy?

What interest rate the CE without policy should have so that the introduction of social security produces a CE that Pareto dominates the one without policy?

1st Welfare Theorem

Case 1 above shows a counter-example to the first welfare theorem.

Formally, this is due to a special feature of the OLG model: there are infinitely many agents and infinitely many goods. Under these circumstances, the 1st welfare theorem, as the example illustrates, may not hold.

We will see that the key condition required for the 1st welfare theorem to hold when there are ∞ many agents and ∞ many goods is that value of the aggregate consumption is finite, i.e. that

$$p \sum_{i \in I} \bar{x}^i < \infty$$

If this condition is met, the 1st welfare theorem holds. If it is not met, as in case 1 of the example above, the 1st welfare theorem does not apply.

In our example we have

$$\sum_{i=0}^{\infty} \bar{x}^i = (1, 1, 1, 1, ...)'$$

and since p is given by

$$p = \left(1, \frac{1}{1+\bar{r}}, \left(\frac{1}{1+\bar{r}}\right)^2, \left(\frac{1}{1+\bar{r}}\right)^3, ...\right)$$

so it depends on whether the real interest rate

$$\bar{r} = \frac{\alpha - \beta}{\beta\,(1-\alpha)}$$

is positive or negative.

Thus we have

$$
\begin{aligned}
p\sum_{i=0}^{\infty}\bar{x}^i &= \lim_{T\to\infty}\left(1+\frac{1}{1+\bar{r}}+\left(\frac{1}{1+\bar{r}}\right)^2+\left(\frac{1}{1+\bar{r}}\right)^3+\ldots\right) \\
&= \lim_{T\to\infty}\frac{1+\bar{r}-\left(\frac{1}{1+\bar{r}}\right)^T}{\bar{r}}
\end{aligned}
$$

This sum diverges to $+\infty$ for cases 1 ($\beta > \alpha, \bar{r} < 0$) and 2 ($\beta = \alpha, \bar{r} = 0$).

Instead this sum converges to a finite limit in case 3 ($\beta < \alpha, \bar{r} > 0$).

To see why the proof of the 1st welfare theorem requires that

$$p \sum_{i \in I} x^i < \infty$$

recall that (see the corresponding class note) its proof proceeds by contradiction. In particular, the proof of the 1st welfare theorem starts by assuming that there is a feasible allocation $\{x^i\}$ that Pareto dominates $\{\bar{x}^i\}$ and then one shows that

$$\begin{aligned} p\, x^i &\geq p\, \bar{x}^i \text{ for all } i \in I \text{ and} \\ p\, x^{i'} &> p\, \bar{x}^{i'} \text{ for some } i' \in I. \end{aligned}$$

If, additionally,

$$p \sum_{i \in I} \bar{x}^i < \infty$$

one concludes that

$$p \sum_{i \in I} x^i = \sum_{i \in I} p x^i > \sum_{i \in I} p \bar{x}^i = p \sum_{i \in I} \bar{x}^i$$

and finds a contradiction with feasibility.

If otherwise

$$p \sum_{i \in I} \bar{x}^i = \infty$$

then the inequality

$$p \sum_{i \in I} x^i > p \sum_{i \in I} \bar{x}^i$$

cannot be established.

One will be comparing to non-finite quantities, i.e. it will by saying "$\infty > \infty$".

To see why is the combination of infinitely many consumers and infinitely many goods the key consider the following two examples. To simplify we will restrict the discussion to pure exchange economies (i.e. no production).

Question. Assume that there are finitely many goods, i.e. $L = R^m$ for finite m, but the set of agents I is infinite. Assume that aggregate endowment is bounded above in each component

$$\bar{e}_l \equiv \sum_{i \in I} e_l^i < \infty$$

for each $l = 1, 2, ..., m$. Show that under these assumptions in any CE $p, \ \left\{ \bar{x}^i \right\},$

$$p \sum_{i \in I} \bar{x}^i < \infty.$$

Hint: write

$$p \sum_{i \in I} \bar{x}^i = p_1 \sum_{i \in I} \bar{x}_1^i + p_2 \sum_{i \in I} \bar{x}_2^i + ... + \ p_m \sum_{i \in I} \bar{x}_m^i$$

Question. Assume that there are finitely many agents, i.e. that the set I is finite, but that $L = R^\infty$ so that there are infinitely many commodities.

Assume that all u^i are strictly increasing.

Show that under these assumptions in any CE p, $\left\{\bar{x}^i\right\}$,

$$p \sum_{i \in I} \bar{x}^i < \infty.$$

(Hint: could $p \sum_{i \in I} \bar{x}^i$ be infinite and $p\bar{x}^i$ finite for all i? Could $p\bar{x}^i$ be infinite?)

Growing economy

We will now consider an economy with population and productivity growth. Let N_t the number of young agents at time t. Let n be the growth rate of population, so that

$$N_{t+1} = (1 + n)\, N_t \text{ for } t \geq 1 \text{ and } N_0 = 1.$$

Let g denote the growth rate of productivity of the endowments of each cohort, so that

$$e_{t+1}^{t+1} = (1 + g)\, e_t^t \text{ and } e_{t+2}^{t+1} = (1 + g)\, e_{t+1}^t$$

so that

$$\begin{aligned} e_t^t &= (1 + g)^t\, (1 - \alpha) \\ e_{t+1}^t &= (1 + g)^t\, \alpha \end{aligned}$$

for all $t \geq 1$.

Question. Solve the optimal savings for each young of generation t

$$s_t(\bar{r}; \alpha, \beta, g) = \arg\max_s$$
$$(1-\beta)\log\Big((1-\alpha)(1+g)^t - s\Big) + \beta \log\Big(\alpha(1+g)^t + s(1+\bar{r})\Big)$$

Show that

$$s_t(\bar{r}; \alpha, \beta, g) = s(\bar{r}; \alpha, \beta)(1+g)^t$$

Question. Argue that the only CE of this economy is one where $x^i = e^i$, i.e. there is no trade. (Hint: repeat the argument from the previous proposition)

Question. Find the equilibrium interest rate $\bar{r}$ for the growing economy as a function of the parameters α, β, g and n.

(Hint: find r_t that makes the aggregate savings of generation t zero, i.e. that solves $N_t \; s_t(r_t; \alpha, \beta, g) = 0$). Is there any difference with the case with no growth (i.e. $g = n = 0$), why or why not?

Define the feasible symmetric allocations as those solving

$$N_t c_y^t + N_{t-1} c_o^t = N_t (1-\alpha)(1+g)^t + N_{t-1} \alpha (1+g)^{t-1}$$

where each agent born at time t and young at t consumes

$$c_y^t = \hat{c}_y (1+g)^t ,$$

and each agent born at time $t-1$ and old at t consumes

$$c_o^t = \hat{c}_o (1+g)^{t-1} .$$

Notice that this constraint can be written as

$$\hat{c}_y (1+g)(1+n) + \hat{c}_o = (1-\alpha)(1+g)(1+n) + \alpha$$

Question.

Solve for the best feasible symmetric allocation $\hat{c}_y$ and $\hat{c}_o$ that solves

$$\max_{\hat{c}_y,\hat{c}_o} \; (1-\beta)\log\left[\hat{c}_y\,(1+g)^t\right] + \beta\log\left[\hat{c}_o\,(1+g)^t\right]$$

subject to

$$\hat{c}_y\,(1+g)\,(1+n) + \hat{c}_o = (1-\alpha)\,(1+g)\,(1+n) + \alpha\ .$$

Show that the solution is

$$\begin{aligned} \hat{c}_y &= (1-\beta)\,\frac{[(1-\alpha)\,(1+g)\,(1+n)+\alpha]}{(1+g)\,(1+n)} \\ \hat{c}_o &= \beta\;[(1-\alpha)\,(1+g)\,(1+n)+\alpha] \end{aligned}$$

Question.

Compare the CE with the best symmetric allocation as a function of the parameters α, β, g and n. For what configuration of parameters α, β, g, n does the best symmetric allocation Pareto dominates the CE allocation?

(Hint: make sure that every agent, including the old born at $t = 0$ prefers the best symmetric allocation.) What interest rate corresponds to the parameters for which the best symmetric allocation Pareto dominates the CE allocation? (Hint: compare $1 + \bar{r}$ with $(1 + g)(1 + n)$)

Question.

Suppose that the parameters α, β, g, n are such that the best feasible allocation Pareto dominates the CE. Suppose that this is to be implemented as a pay-as-you-go social security system. Let τ_t be the tax to an agent born at date t when young, and let $\hat{\tau}$

$$\tau_t = \hat{\tau}\,(1+g)^t \text{ for each } t \geq 1.$$

Assume that all the revenues for this tax levied at time t are used to pay subsided to agents born at $t-1$ that are old at time t. Show that each agent born at $t-1$ that is old at t receives a subsidy, normalized by her productivity $(1+g)^{t-1}$, equal to

$$\hat{\tau}\,(1+g)\,(1+n)$$

Derive a formula for $\hat{\tau}$ as a function of the parameters α, β, g and n.