ECN 820A: Homework 1

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1. (a)
$$P(\text{A or B or Both}) = P(A) + P(B) - P(A \cap B)$$
   (b)
$$P(\text{A or B but not both Both}) = P(A) + P(B) - 2P(A \cap B)$$
   (c)
$$P(\text{At least one of A or B}) = P(A) + P(B) - P(A \cap B)$$
   (d)
$$\begin{aligned} P(\text{A or B or Both}) &= P(A) + P(B) - 2P(A \cap B) + (1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A \cap B) \end{aligned}$$

$\blacksquare$

2. (a) To be a sigma algebra $\mathcal{B}$ needs to fulfill 3 criteria:
   i. $S \in \mathcal{B}$: This is trivial by the definition of $\mathcal{B}$
   ii. $\forall\, A \in \mathcal{B},\, A^c \in \mathcal{B}$: $S^c = \varnothing \in \mathcal{B}$ and $\varnothing^c = S \in \mathcal{B}$
   iii. $\forall\, i \in \mathbb{N}, A_i \in \mathcal{B}, \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{B}$: $S \cup S = S$, $S \cup \varnothing = S$, $\varnothing \cup \varnothing = \varnothing$, so $\bigcup\limits_{i=1}^{\infty} A_i = S \vee \varnothing \in \mathcal{B}$

   Hence $\mathcal{B}$ is a σ-algebra

   (b) $\mathcal{B} = 2^S$
   i. $S \in \mathcal{B}$: This is trivial by the definition of $\mathcal{B}$
   ii. $\forall\, A \in \mathcal{B},\, A^c \in \mathcal{B}$: $\forall\, A \in \mathcal{B},\, A \subseteq S$, so we know that $A^c = S \setminus A \subseteq S \in \mathcal{B}$. So $A^c \in \mathcal{B}$
   iii. $\forall\, i \in \mathbb{N}, A_i \in \mathcal{B}, \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{B}$: $\forall A_i \in \mathcal{B},\, A_i \subseteq S$, so $\bigcup\limits_{i=1}^{\infty} A_i \subseteq S \in \mathcal{B}$

   (c) Let $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$
   i. $S \in \mathcal{A}$: $S \in \mathcal{B} \wedge S \in \mathcal{C} \Rightarrow S \in \mathcal{A}$
   ii. $\forall\, A \in \mathcal{A},\, A^c \in \mathcal{A}$: $A \in \mathcal{A} \Rightarrow A \in \mathcal{B} \wedge A \in \mathcal{C}$. This implies that
$$A^c = \mathcal{A} \setminus \{A\} = (\mathcal{B} \cap \mathcal{C}) \setminus \{A\} = [\underbrace{(\mathcal{B} \setminus \{A\})}_{\in \mathcal{B}} \cap \underbrace{(\mathcal{C} \setminus \{A\})}_{\in \mathcal{C}}] \in \mathcal{A}$$

iii. $\forall i \in \mathbb{N}, A_i \in \mathcal{A}, \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$: Suppose otherwise that $\exists n \in \mathbb{N}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Then we know that, by construction, $A_n \in \mathcal{B}$ or $A_n \in \mathcal{C}$ but not both. But then $A \in \mathcal{A}$, which is a contradiction. Hence $\mathcal{A}$ satisfies the third axiom.

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3. Let $C_i = \{H_i, T_i\},\ i \in \{1, 2, \ldots, 12\}$ denote the ith toss. Assume that the coin is fair so that $P(H_i) = P(C_i) = \frac{1}{2}$

   (a) $P(\bigcap_{i=1}^{12} H_i) = \prod_{i=1}^{12} P(H_i) = \frac{1}{2^{12}} = \frac{1}{4096}$

   (b) This is equivalent to 1 minus the probability of no one getting 12 heads in a row.
   $P($At least one person in the group of n people gets 12 heads in a row$)$
   $= 1 - \prod^{n}(1 - P(\bigcap_{i=1}^{12} H_i)) = 1 - (1 - \frac{1}{2^{12}})^n$

   (c) Solve n for $(1 - \frac{1}{2^{12}})^n < 0.6$, $n >= 2093$

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4. (a) **TRUE**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) + P(B) - P(A \cup B)}{1} = P(A) + 1 - 1 = P(A)$$

   (b) **FALSE**

$$A \subset B \Rightarrow P(A) < P(B)$$
$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1 \not< 1$$

   (c) **TRUE**

$$A \subset B \Rightarrow P(A) < P(B)$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

   (d) **TRUE**

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ \Rightarrow P(A|A \cup B) &= \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P((A \cap A) \cup (A \cap B))}{P(A) + P(B)} \\ &= \frac{P(A \cup \varnothing}{P(A) + P(B)} = \frac{P(A)}{P(A) + P(B)} \end{aligned}$$

(e) **TRUE**

$$\begin{aligned}P(A|B\cap C)P(B|C)P(C) &= \frac{P(A\cap(B\cap C))}{P(B\cap C)}\cdot\frac{P(B\cap C)}{P(C)}\cdot P(C)\\ &= \frac{P(A\cap B\cap C)}{\cancel{P(B\cap C)}}\cdot\frac{\cancel{P(B\cap C)}}{\cancel{P(C)}}\cdot\cancel{P(C)} = P(A\cap B\cap C)\end{aligned}$$

5. (a) Since $\{B_1, B_2, \ldots, B_j, \ldots\}$ is a partition of Ω, we know that (1) $\bigcup\limits_{i=1}^{n} B_j = \Omega$, and (2) $\forall\, i \neq j$, $B_i \cap B_j = \varnothing$. Let $A \subseteq \Omega$, we need to show that (i) $\bigcup\limits_{i=1}^{n}(A\cap B_j) = A$, and (ii) $\forall\, i\neq j$, $(A\cap B_i)\cap(A\cap B_j) = \varnothing$

(i)

$$\bigcup_{i=1}^{n}(A\cap B_j) = A\cap\bigcup_{i=1}^{n}(B_j) = A\cap\Omega = A$$

(ii)

$$\forall\, i\neq j,\ (A\cap B_i)\cap(A\cap B_j) = A\cap(B_i\cap B_j) = A\cap\varnothing = \varnothing$$

So $\{A\cap B_j \mid j = 1, 2, \ldots\}$ is a partition of A.

(b) Show that $P(A\cap B) = P(A)P(B) \Rightarrow P(A^c\cap B^c) = P(A^c)P(B^c)$

$$\begin{aligned}P(A^c\cap B^c) &= P((A\cup B)^c) = 1 - P(A\cup B) = 1-(P(A)+P(B)-P(A\cap B))\\ &= 1-P(A)-P(B)+P(A)P(B) = (1-P(A))(1-P(B)) = P(A^c)P(B^c)\end{aligned}$$

(c) **This is FALSE**

Let A and C be independent, $B\subset C$, and $0 < p < P(C) = q < 1$, then we have:

$$P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{P(A)\cancel{P(C)}}{\cancel{P(C)}} = P(A) = p$$

$$P(B|C) = \frac{P(B\cap C)}{P(C)} = \frac{P(B)}{P(C)} = \frac{p}{q} \neq p$$

(d)

$$\begin{aligned}P(A_1\cup A_2\cup A_3|B) &= \frac{P((A_1\cup A_2\cup A_3)\cap B)}{P(B)} = \frac{P((A_1\cap B)\cup(A_2\cap B)\cup(A_3\cap B))}{P(B)}\\ &= \frac{\overbrace{P(A_1\cap B)+(A_2\cap B)+P(A_3\cap B)}^{\text{Since } A_1, A_2, A_3 \text{ are mutually exclusive}}}{P(B)}\\ &= \frac{P(A_1)}{P(B)}+\frac{P(A_2)}{P(B)}+\frac{P(A_3)}{P(B)}\\ &= P(A_1|B)+P(A_2|B)+P(A_3|B)\end{aligned}$$

6. Since $P(A|C) = 0.1 > 0$, we know that $P(A \cap C) \neq \varnothing$. Moreover, we know that $P(A \cup C) \leq 1$, hence

$$\begin{aligned}0.1 \cdot P(C) = P(A \cap C) = P(A) + P(C) - P(A \cup C) &\geq P(C) - 0.6 \\ 0.1P(C) &\geq P(C) - 0.6 \\ 0.6 &\geq 0.9P(C) \\ P(C) &\leq \frac{2}{3}\end{aligned}$$

Since $P(C)$ has an upper bound of $\frac{2}{3}$, $P(C)$ cannot be 0.7.

7. There are 366 possible birthdays. The probability that at least two people have the same birthday is 1 minus the probability that no one has the same birthday. So

$$\begin{aligned}&P(\text{At least two people have the same birthday}) \\ &\quad = 1 - P(\text{There are exactly 40 unique birthdays in this class}) \\ &\quad = 1 - (\underbrace{P_{40}^{366}}_{\text{Permutations of picking 40 unqiue birthdays}} \div \underbrace{366^{40}}_{\text{Permutations of picking 40 days}}) \\ &\quad = 1 - (\frac{366!}{326!} \cdot \frac{1}{366^{40}}) \approx 1 - 0.109 \\ &\approx 0.891\end{aligned}$$

8.

$$\begin{aligned}&P(\text{2R3B in bowl B}|\text{Blue is drawn from bowl B}) \\ &\quad = \frac{P((\text{2R3B in bowl B}) \cap (\text{Blue is drawn from bowl B}))}{P(\text{Blue is drawn from bowl B})} \\ &\quad = \frac{P((\text{Blue is drawn from bowl B})|(\text{2R3B in bowl B})) \cdot P(\text{2R3B in bowl B})}{P(\text{Blue is drawn from bowl B})}\end{aligned}$$

We know that

$$P(\text{2R3B in bowl B}) = \frac{C_2^6 \cdot C_3^4}{C_5^{10}} = \frac{15 \cdot 4}{2 \cdot 3 \cdot 2 \cdot 7 \cdot 3} = \frac{60}{252} = \frac{5}{21}$$

and

$$P((\text{Blue is drawn from bowl B})|(\text{2R3B in bowl B})) = \frac{3}{5}$$

$$\begin{aligned}
&P(\text{Blue is drawn from Bowl B}) = \\
&\quad P((\text{Blue is drawn from bowl B}) \cap (\text{4R1B in bowl B})) \\
&\quad + P((\text{Blue is drawn from bowl B}) \cap (\text{3R2B in bowl B})) \\
&\quad + P((\text{Blue is drawn from bowl B}) \cap (\text{2R3B in bowl B})) \\
&\quad + P((\text{Blue is drawn from bowl B}) \cap (\text{1R4B in bowl B})) \\
&\quad = \frac{1}{5} \cdot \frac{C_4^6 \cdot C_1^4}{C_5^{10}} + \frac{2}{5} \cdot \frac{C_3^6 \cdot C_2^4}{C_5^{10}} + \frac{3}{5} \cdot \frac{C_2^6 \cdot C_3^4}{C_5^{10}} + \frac{4}{5} \cdot \frac{C_1^6 \cdot C_4^4}{C_5^{10}} \\
&\quad = \frac{1}{5} \cdot \frac{60}{252} + \frac{2}{5} \cdot \frac{120}{252} + \frac{3}{5} \cdot \frac{60}{252} + \frac{4}{5} \cdot \frac{6}{252} = \frac{60 + 240 + 180 + 24}{5 \cdot 252} = \frac{504}{5 \cdot 252} = \frac{2}{5}
\end{aligned}$$

So

$$P((\text{Blue is drawn from bowl B}) \cap (\text{2R3B in bowl B})) = \frac{3}{5} \cdot \frac{5}{21} = \frac{1}{7}$$

and

$$\begin{aligned}
&P(\text{2R3B in bowl B}|\text{Blue is drawn from bowl B}) \\
&= \frac{1}{7} \div \frac{2}{5} = \frac{5}{14}
\end{aligned}$$

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9. (a)

$$\begin{aligned}
P(\text{At least one coin is tail}) &= 1 - P(\text{All coins heads}) - 1 - \frac{1}{4} \\
P(\text{At least one coin is tail} \cap \text{Both coins tails}) &= P(\text{Both coins tail}) = \frac{1}{4} \\
P(\text{Both coins tails} \mid \text{At least one coin is tail}) &= \frac{P(\text{At least one coin is tail} \cap \text{Both coins tails})}{P(\text{At least one coin is tail})} \\
&= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}
\end{aligned}$$

(b) The desired probability is:

$$\frac{C_2^4}{C_2^{50}} = \frac{4!}{2!2!} \cdot \frac{2!48!}{50!} = \frac{3 \cdot 3 \cdot 2}{50 \cdot 49} = \frac{6}{1225}$$

(c)

$$\begin{aligned}
P(A) &= \frac{C_2^{49}}{C_2^{50}} \cdot 1 + \frac{C_1^{49}}{C_2^{50}} \cdot \frac{1}{2} = \frac{48}{50} + \frac{49 \cdot 2}{50 \cdot 49} \cdot \frac{1}{2} = \frac{49}{50} \\
P(A \cap B) &= \frac{C_1^{49}}{C_2^{50}} \cdot \frac{1}{2} = \frac{1}{50} \\
P(B|A) &= P(A \cap B) \cdot \frac{1}{P(A)} = \frac{1}{49}
\end{aligned}$$

10. (a) (i) Since we need to see 2 specific letters, we have $2 \leq L$. The longest is the case where all other letters are drawn first, so we have $2 \leq L \leq 25$

(ii) Let $\alpha, \alpha', \alpha'' \in \{A, B, C\}$, $\alpha \neq \alpha' \neq \alpha''$For each value l, it must be that the permutation of a complete draw would consists of

$$\underbrace{\ldots, \alpha, \ldots, \alpha'}_{l}\underbrace{\ldots, \alpha'', \ldots}_{26-l}$$

Hence for each value l, there are $l-1$ places where α could be, and the rest of the letters follow the permutation P^{23}_{l-2}. So we have:

$$P(L = l) = \frac{(l-1)\frac{23!}{(25-l)!}}{\sum\limits_{a=2}^{a=25} \frac{(a-1)23!}{(25-a)!}}$$

(b) This is the probability that for the first k draws, we only draw from cards that is not A nor B nor C, and then immediately after k, we draw 2 of these 3 cards that are consecutive. So the probability is:

$$\underbrace{\frac{23}{26} \cdot \frac{22}{25} \cdot \frac{21}{24} \cdot \ldots \cdot \frac{23-k}{26-k}}_{\text{the first k draws}} \cdot \underbrace{\frac{4}{(25-k)(24-k)}}_{\text{drawing consecutive cards}}$$