Notes on Risk Aversion and Portfolio Choice

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- Definition: Arrow-Pratt coefficient of absolute risk aversion.
- Definition of insurance premium.
- Small risks
  - Computation of ARA coefficient for several $u(\cdot)$.
  - Derivation of insurance premium, Certainty Equivalent.
  - Proportional insurance premium, and relative risk aversion coefficient.
- Arrow-Prat Theorem: insurance premium and curvature of u.
- Portfolio Choice Problems:
- risk aversion and choice of risky vs risk-less asset

- Computing insurance premium for selected examples of utility functions $u(\cdot)$ and risk $\tilde{x}$.
- Quadratic $u(\cdot)$, exact vs approximate risk premium.
- $\tilde{x}$ normal and $u(\cdot)$ with constant absolute risk aversion.
- $log(\tilde{x})$ normal and $u(\cdot)$ with constant relative absolute risk aversion.
- Arrow-Pratt Theorem: complete characterization of when a person with utility $u(\cdot)$ is more risk averse than one with utility $v(\cdot)$.

Arrow-Pratt Coefficient of Absolute Risk Aversion :

$$ra(x) = -\frac{u''(x)}{u'(x)}$$

- This coefficient gives a measure of the curvature of the utility function around the point x.
- The higher this coefficient is the higher is the curvature, and hence the more "risk averse" the agent is.
- We will discuss the precise meaning of this in detail.

Examples:

- Linear utility

$$u(x) = ax + b$$

$$\Rightarrow -\frac{u''(x)}{u'(x)} = -\frac{0}{a} = 0$$

- Log utility

$$u(x) = \ln x$$

$$\Rightarrow -\frac{u''(x)}{u'(x)} = -\frac{-1/x^2}{1/x} = 1/x$$

- CRRA (Constant Relative Risk Aversion)

$$u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$$

$$\Rightarrow \quad -\frac{u''(x)}{u'(x)} = -\frac{-\gamma x^{-\gamma-1}}{x^{-\gamma}} = \gamma x^{-1}$$

Does it look similar to the log utility? Yes for $\gamma = 1$. In fact

$$\begin{aligned}\lim_{\gamma\to 1}\frac{x^{1-\gamma}-1}{1-\gamma} &= \lim_{\gamma\to 1}\frac{e^{(1-\gamma)\ln x}-1}{1-\gamma} \\ &= \lim_{\gamma\to 1}\frac{-(\ln x)\,e^{(1-\gamma)\ln x}}{-1} \\ &= \ln x\end{aligned}$$

- CARA (Constant Absolute Risk Aversion):
  Is there a utility function $u(x)$ such that $-\frac{u''(x)}{u'(x)} = \text{constant}$?

  Yes:

$$
\begin{aligned}
u(x) &= -\frac{1}{a}e^{-ax} \\
\Rightarrow \quad -\frac{u''(x)}{u'(x)} &= -\frac{-ae^{-ax}}{e^{-ax}} = a
\end{aligned}
$$

Risk Aversion and Insurance: The case of small risk.

We define the risk premium p as the maximum amount that an agent is willing to pay to avoid a risk $\tilde{x}$.

$$u(\,E(x) - p\,) \quad = \quad E[\,u(\tilde{x})\,]$$

where p is the premium, $\tilde{x}$ the risk.

insert graph here

- The size of p will depend on the willingness of the agent to bear risk and on the size of the risk. For small risk, there is a simple expression for p

$$p = -\frac{1}{2}\frac{u''(\bar{x})}{u'(\bar{x})}\sigma^2$$

- The term $-u''(\bar{x})/u'(\bar{x})$ measures the risk aversion and the term σ^2 measures the size of the risk.
- The utility function is evaluated at the expected value of the risk, $\bar{x} = E[\tilde{x}]$.
- This expression is valid for a small risk, i.e. one with small variance σ^2

To see this we use:

- a 1st order Taylor expansion of the left hand side and
- a second order Taylor expansion of the right hand side of the equation defining p:

$$u(\bar{x} - p) = E[u(\tilde{x})]$$

or

$$\begin{aligned} u(\bar{x} - p) &\approx u(\bar{x}) - u'(\bar{x})\, p \\ u(\tilde{x}) &\approx u(\bar{x}) + u'(\bar{x})(\tilde{x} - \bar{x}) + \frac{u''(\bar{x})}{2}(\tilde{x} - \bar{x})^2 \\ \bar{x} &= E[\tilde{x}] \end{aligned}$$

So

$$u(\bar{x}) - u'(\bar{x})\,p$$
$$\approx E\left[u(\bar{x}) + u'(\bar{x})(x-\bar{x}) + \frac{u''(\bar{x})}{2}(x-\bar{x})^2\right]$$

and

$$-u'(\bar{x})\,p \approx \frac{u''(\bar{x})}{2}E\left[(x-\bar{x})^2\right]$$

Solving for p we get that

$$p = -\frac{1}{2}\frac{u''(\bar{x})}{u'(\bar{x})}\sigma^2$$

Insuring a small proportional risk

- We examine the insurance premium for the a proportional risk.
- We express the insurance premium as a fraction of the certain non-risky consumption $\bar{x}$.
- The proportional insurance premium ρ solves:

$$u((1-\rho)\bar{x}) = E[u(\bar{x}(1+\varepsilon))]$$

where $E[\varepsilon] = 0$ and $E\left[\varepsilon^2\right] = \sigma_\varepsilon^2$ then

$$\rho = -\frac{1}{2}\frac{u''(\bar{x})\,\bar{x}}{u'(\bar{x})}\sigma_\varepsilon^2$$

- The coefficient of relative risk aversion is then defined as

$$rra(x) = -\frac{u''(x)}{u'(x)}x$$

- As you may have guessed by now CRRA utility functions have $rra(x) = \gamma$ (constant)

To see this notice that we can relate the absolute and proportional risk premium and risk as follows:

$$\begin{aligned} p &= \bar{x}\rho \\ \sigma^2(x) &= (\bar{x})^2 \sigma_\varepsilon^2 \end{aligned}$$

Thus using the expression for p :

$$p = -\frac{u''(\bar{x})}{u'(\bar{x})}\sigma^2(x) = -\frac{u''(\bar{x})}{u'(\bar{x})}(\bar{x})^2 \sigma_\varepsilon^2$$

or

$$\frac{p}{x} = \rho = -\bar{x}\frac{u''(\bar{x})}{u'(\bar{x})}\sigma_\varepsilon^2$$

We can also verify the expression for ρ by using Taylor expansions similar to the ones used above:

$$\begin{aligned} u\left((1-\rho)\bar{x}\right) &\approx u\left(\bar{x}\right) - u'\left(\bar{x}\right)\bar{x}\rho \\ u\left(\bar{x}\left(1+\varepsilon\right)\right) &\approx u\left(\bar{x}\right) + u'\left(\bar{x}\right)\bar{x}\varepsilon + \frac{u''\left(\bar{x}\right)}{2}\bar{x}^2\varepsilon^2 \end{aligned}$$

so

$$\begin{aligned} u\left(\bar{x}\right) - u'\left(\bar{x}\right)\bar{x}\rho &\approx E\left[u\left(\bar{x}\right) + u'\left(\bar{x}\right)\bar{x}\varepsilon + \frac{u''\left(\bar{x}\right)}{2}\bar{x}^2\varepsilon^2\right] \\ -u'\left(\bar{x}\right)\bar{x}\rho &\approx \frac{u''\left(\bar{x}\right)}{2}\bar{x}^2 E\left[\varepsilon^2\right] \end{aligned}$$

and then

$$\rho = -\frac{1}{2}\frac{u''\left(\bar{x}\right)\bar{x}}{u'\left(\bar{x}\right)}\sigma_{\varepsilon}^2$$

Certainty Equivalent

- A concept closely related to the insurance premium is the certainty equivalent of a risk $\tilde{x}$, denoted by $c_e(\tilde{x})$.
- It is the sure amount of consumption that it will be equivalent to a given risk $\tilde{x}$.
- It is defined as follows
$$u(c_e) = E[u(\tilde{x})]$$
- So $c_e = \bar{x} - p$, where $\bar{x} \equiv E[\tilde{x}]$.

Introspection: How risk averse are you?

- Consider the following gamble.
- Suppose you make 1000K a year. Assume you are also faced with the following lottery:
- You win 10K with probability 1/2 and -10K (lose 10K) otherwise.
- What will be the certainly equivalent amount for this lottery? What is the implied relative risk aversion?
- Fabrice (your TA) says he will pay 1000 dollars to avoid this risk (he does not make 1000K though!).
- Thus, his certainty equivalent is 999,000. Based upon this answer his relative risk aversion will be about 20.
- What should be the answer of this question is your TA relative risk aversion is 1? (say log preferences).

To see how we arrive to this answer use the definitions of the certainty equivalent and proportional risk premium:

$$\begin{aligned} c_e &= \bar{x} - p = \bar{x}(1-\rho) \\ \rho &= \frac{\gamma}{2}\sigma_\varepsilon^2 \end{aligned}$$

where we use γ to denote the relative risk aversion. Solving for γ

$$\gamma = \left(1 - \frac{c_e}{\bar{x}}\right) 2/\sigma_\varepsilon^2,$$

and plugging in the corresponding values

$$\begin{aligned} \gamma &= (1-0.999)\,2/\,(0.01)^2 \\ &= 0.001 \times 2/0.0001 = 20 \end{aligned}$$

Insurance Premium for large risks

- So far we have examined insurance premium for small risk.
- Now we will see that, essentially, that insurance premium for large risk are given by the same determinants.
- We do so by computing explicitly insurance premium for especial cases where we can perform the calculations analytically, and by showing a general theorem (Arrow-Pratt)

Example 1: Insurance premium for quadratic utility

$$\begin{aligned} E\left[\tilde{x}\right] &= \mu \\ Var\left[\tilde{x}\right] &= \sigma^2 \end{aligned}$$

and

$$u(x) = x - \frac{\alpha}{2} x^2$$

then

$$p = \frac{1 - \left(1 + \left[\alpha / \left(\mu\alpha - 1\right)\right]^2 \sigma^2\right)^{1/2}}{\alpha / \left(\mu\alpha - 1\right)}$$

or for small σ

$$p = \frac{\alpha}{1 - \mu\alpha} \frac{\sigma^2}{2}$$

where

$$-\frac{u''(\mu)}{u'(\mu)} = \frac{\alpha}{1 - \mu\alpha}.$$

To see this we first solve for p. By definition:

$$\mu - p - \frac{\alpha}{2}(\mu - p)^2 = \mu - \frac{\alpha}{2}\left[\sigma^2 + \mu^2\right]$$

or

$$-p - \frac{\alpha}{2}\left(\mu^2 - 2\mu p + p^2\right) = -\frac{\alpha}{2}\left[\sigma^2 + \mu^2\right]$$

or

$$0 = -p + \frac{\alpha}{2}\sigma^2 / (1 - \mu\alpha) + p^2 \frac{\alpha}{2} / (\mu\alpha - 1)$$

Now we solve the quadratic equation

$$\begin{aligned} 0 &= c + bp + ap^2 \\ p &= \frac{-b + \left(b^2 - 4ac\right)^{1/2}}{2a} \end{aligned}$$

for

$$\begin{aligned} c &= \frac{\alpha}{2}\sigma^2 / \left(1 - \mu\alpha\right) \\ b &= -1 \\ a &= \frac{\alpha}{2} / \left(\mu\alpha - 1\right) \end{aligned}$$

which gives

$$p\left(\sigma^2\right) = \frac{1 - \left(1 + \left[\alpha / \left(\mu\alpha - 1\right)\right]^2 \sigma^2\right)^{1/2}}{\alpha / \left(\mu\alpha - 1\right)}$$

Where we have chosen the root so that $p > 0$.

To see how this looks like for small σ^2, we use a Taylor expansion (the definition of derivative really) and compute

$$p\left(\sigma^2\right) = p\left(0\right) + p'\left(0\right)\sigma^2 + o\left(\sigma^2\right)$$

where $g\left(\sigma^2\right) = o\left(\sigma^2\right)$ means of order smaller than σ^2, or

$$\lim_{\sigma^2 \to 0} \frac{g\left(\sigma^2\right)}{\sigma^2} = 0$$

Notice that

$$p\left(0\right) = 0,$$

and

$$p'\left(\sigma^2\right) = -\frac{\left(1 + \left[\alpha/\left(\mu\alpha - 1\right)\right]^2 \sigma^2\right)^{-1/2}}{2\alpha/\left(\mu\alpha - 1\right)} \left[\alpha/\left(\mu\alpha - 1\right)\right]^2,$$

so... (continue next page)

$$p'\left(\sigma^2\right) = -\frac{\left(1+\left[\alpha/\left(\mu\alpha-1\right)\right]^2\sigma^2\right)^{-1/2}}{2\alpha/\left(\mu\alpha-1\right)}\left[\alpha/\left(\mu\alpha-1\right)\right]^2,$$

so

$$p'\left(0\right) = -\frac{1}{2\alpha/\left(\mu\alpha-1\right)}\left[\alpha/\left(\mu\alpha-1\right)\right]^2 = \frac{\alpha}{2\left(1-\mu\alpha\right)}.$$

Substitute these results and get that

$$\begin{aligned} p\left(\sigma^2\right) &= p\left(0\right)+p'\left(0\right)\sigma^2+o\left(\sigma^2\right) \\ &= \frac{\alpha}{2\left(1-\mu\alpha\right)}\sigma^2+\underbrace{o\left(\sigma^2\right)}_{\textit{of order smaller than } \sigma^2} \end{aligned}$$

Example 2: Insurance premium for utility with constant absolute risk aversion and normal risk

$$\tilde{x} \sim N\left(\mu, \sigma^2\right)$$

$$u(x) = -\frac{1}{\lambda} e^{-\lambda x} \text{ for } \lambda > 0$$

Notice that

$$-\frac{u''(x)}{u'(x)} = \lambda \text{ for all } x$$

Then by solving for p :

$$u(\mu - p) = E\left[u(\tilde{x})\right]$$

we obtain

$$p = \lambda \frac{\sigma^2}{2}$$

This follows because

$$E\left[e^{-\lambda\tilde{x}}\right] = e^{-\lambda\mu+\frac{\lambda^2}{2}\sigma^2}$$

so

$$E\left[u\left(\tilde{x}\right)\right] = -\frac{1}{\lambda}e^{-\lambda\mu+\frac{\lambda^2}{2}\sigma^2}$$

and

$$u\left(\mu - p\right) = -\frac{1}{\lambda}e^{-\lambda(\mu-p)}$$

then equating the two terms

$$e^{-\lambda\mu+\frac{\lambda^2}{2}\sigma^2} = e^{-\lambda(\mu-p)}$$

or

$$e^{\frac{\lambda^2}{2}\sigma^2} = e^{\lambda p} \implies p = \lambda\sigma^2/2$$

If

$$\tilde{x} \sim N\left(\mu, \sigma^2\right)$$

then

$$E\left[e^{-\lambda x}\right] = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\lambda x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2-2\mu x+\mu^2+2\lambda\sigma^2 x}{2\sigma^2}} dx$$

$$= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2-2\left(\mu-\lambda\sigma^2\right)x+\mu^2}{2\sigma^2}} dx$$

$$= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2-2\left(\mu-\lambda\sigma^2\right)x+\left(\mu-\lambda\sigma^2\right)^2+\mu^2-\left(\mu-\lambda\sigma^2\right)^2}{2\sigma^2}} dx$$

$$E\left[e^{-\lambda x}\right] = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{\left(x-\left(\mu-\lambda\sigma^2\right)\right)^2+\mu^2-\left(\mu-\lambda\sigma^2\right)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{\left(x-\left(\mu-\lambda\sigma^2\right)\right)^2}{2\sigma^2}} e^{-\frac{\mu^2-\left(\mu-\lambda\sigma^2\right)^2}{2\sigma^2}} dx$$

$$= e^{-\frac{\mu^2-\left(\mu-\lambda\sigma^2\right)^2}{2\sigma^2}} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{+\infty} e^{-\frac{\left(x-\left(\mu-\lambda\sigma^2\right)\right)^2}{2\sigma^2}} dx$$

$$= e^{-\frac{\mu^2-\left(\mu-\lambda\sigma^2\right)^2}{2\sigma^2}} = e^{-\frac{\mu^2-\mu^2-\lambda^2\sigma^4+2\mu\lambda\sigma^2}{2\sigma^2}} = e^{-\lambda\mu+\frac{\lambda^2}{2}\sigma^2}$$

Example 3: Insurance Premium for utility with constant relative risk aversion and log normal risk

$$
\begin{aligned}
\log \tilde{x} &\sim N\left(\mu, \sigma^2\right) \\
u(x) &= \frac{x^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 0
\end{aligned}
$$

Notice that

$$-x \frac{u''(x)}{u'(x)} = \gamma \text{ for all } x$$

Then solving for ρ

$$u(\ E(x)\ (1-\rho)) = E\left[u(\tilde{x})\right]$$

we obtain

$$
\begin{aligned}
\log(1-\rho) &= -\gamma \frac{\sigma^2}{2} \\
\rho &\simeq \gamma \frac{\sigma^2}{2}
\end{aligned}
$$

This follows because

$$u\left(E(x)\ (1-\rho)\right) = \frac{\left[(1-\rho)\,E(x)\right]^{1-\gamma}}{1-\gamma}$$

$$= \frac{\left[(1-\rho)\,e^{\mu+\sigma^2/2}\right]^{1-\gamma}}{1-\gamma} = \frac{(1-\rho)^{1-\gamma}\,e^{\mu(1-\gamma)+(1-\gamma)\frac{\sigma^2}{2}}}{1-\gamma}.$$

and

$$Eu(\tilde{x}) = \frac{e^{\mu(1-\gamma)+(1-\gamma)^2\frac{\sigma^2}{2}}}{1-\gamma}.$$

This last equality follows because ... (see next page)

This last equality follows because if

$$\begin{aligned} \log \tilde{x} &\sim N\left(\mu, \sigma^2\right) \text{ then} \\ (1-\gamma)\log \tilde{x} &\sim N\left((1-\gamma)\mu, (1-\gamma)^2 \sigma^2\right) \end{aligned}$$

and hence it is also log-normal. Since $\log \tilde{x}^{1-\gamma} = (1-\gamma)\log \tilde{x}$, using the formula obtained above

$$Ex^{1-\gamma} = e^{(1-\gamma)\mu + (1-\gamma)^2\sigma^2/2}$$

Thus

$$Eu(\tilde{x}) = \frac{e^{(1-\gamma)\mu + (1-\gamma)^2\sigma^2/2}}{1-\gamma}.$$

Then, using the expressions for $u\left(E\left(x\right)\left(1-\rho\right)\right)$ and $Eu\left(\tilde{x}\right)$ and solving for $\left(1-\rho\right)$ on

$$u\left(E\left(x\right)\left(1-\rho\right)\right)=Eu\left(\tilde{x}\right)$$

we get

$$\begin{aligned}
\left(1-\rho\right)^{1-\gamma}e^{\mu(1-\gamma)+(1-\gamma)\frac{\sigma^2}{2}} &= e^{\mu(1-\gamma)+(1-\gamma)^2\frac{\sigma^2}{2}} \\
\left(1-\rho\right)^{1-\gamma}e^{(1-\gamma)\frac{\sigma^2}{2}} &= e^{(1-\gamma)^2\frac{\sigma^2}{2}} \\
\left(1-\rho\right)^{1-\gamma}e^{(1-\gamma)\frac{\sigma^2}{2}} &= e^{(1-\gamma)(1-\gamma)\frac{\sigma^2}{2}}
\end{aligned}$$

and taking logs

$$\begin{aligned}
\left(1-\gamma\right)\log\left(1-\rho\right) &= \left(1-\gamma\right)\frac{\sigma^2}{2}\left[\left(1-\gamma\right)-1\right] \\
\log\left(1-\rho\right) &= -\gamma\frac{\sigma^2}{2} \\
\rho &\cong \gamma\frac{\sigma^2}{2}
\end{aligned}$$

where the last line uses that $\log\left(1+y\right)\cong y$ for $y=-\rho$.

Risk Aversion in the large :

The next three statements are equivalent :
(i) there is a function f :

$$\begin{aligned} u(x) &= f(v(x)) \text{ all } x \\ f' &> 0 \text{ and } f'' < 0 \end{aligned}$$

(ii) for all random variables $\tilde{x}$ the insurance premium $p_u(\tilde{x}), p_v(\tilde{x})$ corresponding to the utility functions u and v are :

$$p_u(\tilde{x}) > p_v(\tilde{x})$$

(iii) the absolute risk aversion coefficient of u is higher than the one of v everywhere :

$$-\frac{u''(x)}{u'(x)} > -\frac{v''(x)}{v'(x)} \text{ for all } x$$

Conclusion of Theorem and Examples

- Insurance premium is determined by risk aversion.
- For small risk, there is a simple formula:

$$p = \frac{1}{2} \left(- \frac{u''(\bar{x})}{u'(\bar{x})} \right) \sigma^2$$

- In the general case, more risk aversion implies higher premium, but we don't have a simple formula.
- In the general case, moments of $\tilde{x}$ higher than variance may matter.

Proof of Theorem

Show that diagram "commutes" (3 implications suffice).That is we show that:

A. (i) implies (ii).

B. (ii) implies (iii).

C. (iii) implies (i).

Clearly all the other implications follows from these, by combining them

$$\begin{array}{ccccc} & & (i) & & \\ & \nearrow C & & \searrow A & \\ (iii) & & \leftarrow B & & (ii) \end{array}$$

For instance “(ii) implies (i)" follows from B and C.

A. (i) implies (ii). By Jensen's inequality:

$$Eu(\tilde{x}) = Ef(v(\tilde{x})) < f(Ev(\tilde{x})),$$

and by definition of p_u :

$$Eu(\tilde{x}) = u(\bar{x} - p_u(\tilde{x})),$$

thus combining them

$$u(\bar{x} - p_u(\tilde{x})) < f(Ev(\tilde{x})).$$

By definition of p_v :

$$v(\bar{x} - p_v(\tilde{x})) = Ev(\tilde{x})$$

and applying f to both sides and using $u = f(v)$:

$$u(\bar{x} - p_v(\tilde{x})) = f(v(\bar{x} - p_v(\tilde{x}))) = f(Ev(\tilde{x})).$$

Thus combining the previous equations:

$$u(\bar{x} - p_v(\tilde{x})) = f(Ev(\tilde{x})) > u(\bar{x} - p_u(\tilde{x}))$$

and since u is increasing:

$$p_u(\tilde{x}) > p_v(\tilde{x}).$$

B. (ii) implies (iii).

Consider random variable $\tilde{x}$ with small variance, center around $\bar{x}$.
Since this random variables can be chosen with a very small variance, we can use the result for small risk, which give

$$
\begin{aligned}
p_u(\tilde{x}) &= -\frac{u''(\bar{x})}{u'(\bar{x})}\frac{\sigma^2}{2} \\
p_v(\tilde{x}) &= -\frac{v''(\bar{x})}{v'(\bar{x})}\frac{\sigma^2}{2}
\end{aligned}
$$

Since, by assumption, $p_u(\tilde{x}) > p_v(\tilde{x})$, then it must be that

$$
-\frac{u''(\bar{x})}{u'(\bar{x})} > -\frac{v''(\bar{x})}{v'(\bar{x})}.
$$

C. (iii) implies (i).

Define

$$f(w) = u\left(v^{-1}(w)\right)$$

notice that for all x

$$u\left(v^{-1}(v(x))\right) = u(x)$$

First, notice that with this definition

$$u(x) = f(v(x))$$

since using the top equation for f in the last line:

$$u(x) = u\left(v^{-1}(v(x))\right) = u(x)$$

which hold for all x.

The second step is to show that f so defined is concave and increasing. We show this by differentiating the top equation defining f.

$$f'(w) = u'\left(v^{-1}(w)\right) \frac{1}{v'\left(v^{-1}(w)\right)} > 0$$

and differentiating again:

$$f''(w) = u''\left(v^{-1}(w)\right) \left[\frac{1}{v'\left(v^{-1}(w)\right)}\right]^2 - u'\left(v^{-1}(w)\right) \left[\frac{1}{v'\left(v^{-1}(w)\right)}\right]^2 \frac{v''\left(v^{-1}(w)\right)}{v'\left(v^{-1}(w)\right)}$$

rearranging terms we have

$$f''(w) = -\frac{u'\left(v^{-1}(w)\right) 1}{\left[v'\left(v^{-1}(w)\right)\right]^2} \times \left\{-\frac{u''\left(v^{-1}(w)\right)}{u'\left(v^{-1}(w)\right)} - \left(-\frac{v''\left(v^{-1}(w)\right)}{v'\left(v^{-1}(w)\right)}\right)\right\}$$

which is negative since u has higher absolute risk aversion than v.

Portfolio Choice Problem

- Problem defined by: Returns, N risky asset, riskless asset, initial wealth, portfolio weights
- First and second order conditions.
- One risky asset case: invest in risky asset if and only if its expected return is higher than the risk-free rate.
- One risky asset case: the more risk averse investor invest less in risky asset.
- Case of N assets: quadratic utility and/or normal returns

Portfolio Choice Problem

- We now consider the decision problem of an investor.
- This is a one period problem, i.e. the investor has already decided how much to invest (which we denote by W).
- The only choice is how to invest it, i.e. what assets to buy or sell. The investor has access to a menu of N risky assets and one riskless asset. Each asset has gross return R_i.
- We denote by w_i the fraction of the initial wealth allocated to each asset.
- Once the portfolio decisions are made, the returns are realized and the investor wealth is $\tilde{W}$.
- We assume that the investor chooses the weights w to maximize expected utility.

- Initial wealth W
- Returns $\tilde{R}_i$ on risky assets, $i = 1, \cdots, N$
- Risk free $\bar{\mu}$
- w_i : fraction of wealth invested on asset i
- w_0 : fraction of wealth invested on the risk free asset
- Wealth at the end :

$$\tilde{W} = W \left[\sum_{i=1}^{N} w_i \left(\tilde{R}_i - \bar{\mu} \right) + \bar{\mu} \right]$$

- Next page explain the expression for end of period Wealth.

$$\tilde{W} = w_0 W \bar{\mu} + \sum_{i=1}^{N} w_i W \tilde{R}_i$$
$$= W \left(w_0 \bar{\mu} + \sum_{i=1}^{N} w_i \tilde{R}_i \right)$$

where we a have a constraint that says that we can invest more than what we have, so

$$w_0 + \sum_{i=1}^{N} w_i = 1 \text{ or } w_0 = 1 - \sum_{i=1}^{N} w_i$$

Using this constraint we get that

$$\tilde{W} = W \left[\left(1 - \sum_{i=1}^{N} w_i \right) \bar{\mu} + \sum_{i=1}^{N} w_i \tilde{R}_i \right] = W \left[\bar{\mu} - \sum_{i=1}^{N} w_i \bar{\mu} + \sum_{i=1}^{N} w_i \tilde{R}_i \right]$$
$$= W \left[\bar{\mu} + \sum_{i=1}^{N} \left(w_i \tilde{R}_i - w_i \bar{\mu} \right) \right] = W \left[\sum_{i=1}^{N} w_i \left(\tilde{R}_i - \bar{\mu} \right) + \bar{\mu} \right]$$

Problem:

$$\max_{\{w_i\}} E\left[u\left(\tilde{W}\right)\right]$$

First order conditions :

$$\begin{aligned} & E\left[u'\left(\tilde{W}\right)\left(\tilde{R}_i-\bar{\mu}\right)\right] \\ = \quad & E\left[u'\left(W\left[\sum_{j=1}^{N} w_j\left(\tilde{R}_j-\bar{\mu}\right)+\bar{\mu}\right]\right)\left(\tilde{R}_i-\bar{\mu}\right)\right] \\ = \quad & 0 \end{aligned}$$

for $i = 1, 2, ..., N$.

- We will examine the second order condition for this problem.
- Also sufficient (and necessary) conditions for uniqueness of the solution.

To show that the objective function is concave use the following two properties

- Let $f(x), g(x)$ be concave, then $h(x) = f(x) + g(x)$ is also concave.
- Let $f(x), g(x)$ be concave and $f(x)$ be an increasing function, then $h(x) = f(g(x))$ is also concave.

Recall that if $f(x)$ and $g(x)$ are two C^2 concave functions, then

$$f''(x) < 0,\ g''(x) < 0$$

The proof to the first property is very simple, just differentiate on the definition for h :

$$\begin{aligned} h'(x) &= f'(x) + g'(x) \\ h''(x) &= f''(x) + g''(x) \end{aligned}$$

so h is concave, if both f and g are concave.

Now we look the second property for $h(x) = f(g(x))$.

Direct computation gives that the second derivative of $h(x)$

$$h'(x) = f'(g(x))\, g'(x)$$

$$h''(x) = f''(g(x))\,(g'(x))^2 + f'(g(x))\, g''(x) < 0$$

thus h is concave if f and g are concave and if f is increasing.

So now lets apply these two properties to our portfolio problem.

- Define $W(w_1, w_2|s) \equiv \tilde{W}$ for each s given $w's$.
- Clearly $W(w_1, w_2|s)$ is concave in w_1, w_2 by the first property (see the definition of $W(\cdot|s)$ for each s.
- Since $u(\cdot)$ is increasing and concave then $u(W(w_1, w_2|s))$ by our second property (since it is the composition of two concave functions).
- Finally $E[u(W(w_1, w_2|s))]$ is concave because of the first property (the expectation is the sum of concave functions).

Thus we have shown that the objective function

$$E\left[u\left(\tilde{W}\right)\right] = \sum_{s=1}^{S} \pi(s)\, u(W(w_1, w_2|s))$$

is concave in w.

Consider the one risky asset case, $N = 1$.
Here we only choose one w :

$$\max_w E\left[u\left(W\left[w\left(R-\bar{\mu}\right)+\bar{\mu}\right]\right)\right]$$

The first order condition is:

$$E\left[u'\left(W\left[w\left(R-\bar{\mu}\right)+\bar{\mu}\right]\right)\left(R-\bar{\mu}\right)\right]=0$$

Theorem. Expected return and risk:

The optimal $w^* > 0$ if and only if $E[R] > \bar{\mu}$.

- What does it mean?
- You always invest in a risky asset with higher return than the risk free rate, no matter how risk averse you are.

Proof:

Take the derivative of $E\left[u(\tilde{W})\right]$ with respect to w

$$
\begin{aligned}
f(w) &= \frac{\partial E\left[u(\tilde{W})\right]}{\partial w} \\
&= WE\left[u'\left(W\left[w\left(R-\bar{\mu}\right)+\bar{\mu}\right]\right)\left(R-\bar{\mu}\right)\right]
\end{aligned}
$$

Then notice that $f(w)$ is a strictly decreasing function:

$$
\begin{aligned}
f'(w) &= \frac{\partial^2 E\left[u(\tilde{W})\right]}{\partial w^2} \\
&= W^2 E\left[u''\left(W\left[w\left(R-\bar{\mu}\right)+\bar{\mu}\right]\right)\left(R-\bar{\mu}\right)^2\right] < 0
\end{aligned}
$$

since $u'' < 0$.

Now lets evaluate $f(w)$ at $w = 0$

$$
\begin{aligned}
f(0) &= \left. \frac{\partial E\left[u(\tilde{W})\right]}{\partial w} \right|_{w=0} \\
&= \underbrace{W}_{>0} \underbrace{u'(W\bar{\mu})}_{>0} E\left[(R - \bar{\mu})\right]
\end{aligned}
$$

- $E\left[(R - \bar{\mu})\right] > 0$ that implies that $f(0) > 0$.
- Since $f(w)$ is strictly decreasing then $f(w) > 0, \forall w \leq 0$.
- So the optimal w has to positive.

Risk aversion and w :

- We have seen that even a risk averse agent always invest in a risky asset if $E[R] > \bar{\mu}$.
- But the more risk averse an agent is, then the smaller the size of the investment in the risky asset should be.

Theorem:

- Assume that $E[R] > \bar{\mu}$ and that u is concave ($u'' < 0$).
- Let $N = 1$ (one risky asset).
- If u is more risk averse than v i.e.:

$$-\frac{u''(x)}{u'(x)} > -\frac{v''(x)}{v'(x)}, \forall x > 0$$

- The choices of risky asset w_u and w_v are such :

$$w_u < w_v \ .$$

Proof.

Preliminary result: (we will use this result to prove the main result)

Given that u is more risk averse than v, then for any x, y such that $y > x$

$$\frac{u'(y)}{u'(x)} < \frac{v'(y)}{v'(x)}$$

Try v linear .

Hint : Integrate $\log u'(x)$ between x and y.

Proof of the preliminary result:

$$\begin{aligned}\log u'(y) &= \log u'(x) + \int_x^y \frac{d \log u'(z)}{dz} dz \\ &= \log u'(x) + \int_x^y \frac{u''(z)}{u'(z)} dz\end{aligned}$$

so

$$\begin{aligned}\log \frac{u'(y)}{u'(x)} &= \int_x^y \frac{u''(z)}{u'(z)} dz \\ \text{or } \frac{u'(y)}{u'(x)} &= \exp\left[-\int_x^y \left[\frac{-u''(z)}{u'(z)}\right] dz\right] \text{ and} \\ \frac{v'(y)}{v'(x)} &= \exp\left[-\int_x^y \left[\frac{-v''(z)}{v'(z)}\right] dz\right]\end{aligned}$$

and thus

$$\frac{u'(y)}{u'(x)} < \frac{v'(y)}{v'(x)}$$

Proof to the main Result:

Sketch: Use first order conditions divided by

$$u'\left(W\bar{\mu}\right)$$

and previous property for returns $R > \bar{\mu}$ and $R \leq \bar{\mu}$ separately.

Details for the Proof of the main result:

$$\arg\max_w E\left[u(\tilde{W})\right] = \arg\max_w E\left[\frac{u(\tilde{W})}{u'(W\bar{\mu})}\right]$$

FOC of $\max_w E\left[\frac{u(\tilde{W}_w)}{u'(W\bar{R})}\right]$

$$\frac{\partial E\left[\frac{u(\tilde{W}_w)}{u'(W\bar{\mu})}\right]}{\partial w} = E\left[\frac{u'(\tilde{W}_w)\,(R-\bar{\mu})}{u'(W\bar{\mu})}\right] = 0 \text{ for } w = w_u^*$$

Strict concavity implies

$$\frac{\partial^2 E\left[\frac{u(\tilde{W}_w)}{u'(W\bar{\mu})}\right]}{\partial w^2} = E\left[\frac{u''(\tilde{W}_w)\,(R-\bar{\mu})^2}{u'(W\bar{\mu})}\right] < 0 \text{ for all } w$$

The FOC can be written as

$$
\begin{aligned}
& E\left[\frac{u'(\tilde{W}_w)\,(R-\bar{\mu})}{u'(W\bar{\mu})}\right] \\
= \quad & \sum_{s:R_s>\bar{\mu}} \frac{u'(\tilde{W}_{w,s})}{u'(W\bar{\mu})}\,(R_s-\bar{\mu})\,\pi_s \\
& + \sum_{s:R_s\leq\bar{\mu}} \frac{u'(\tilde{W}_{w,s})}{u'(W\bar{\mu})}\,(R_s-\bar{\mu})\,\pi_s
\end{aligned}
$$

Similarly for v

$$
\begin{aligned}
& E\left[\frac{v'(\tilde{W}_w)\,(R-\bar{\mu})}{v'(W\bar{\mu})}\right] \\
= \quad & \sum_{s:R_s>\bar{\mu}} \frac{v'(\tilde{W}_{w,s})}{v'(W\bar{\mu})}\,(R_s-\bar{\mu})\,\pi_s \\
& + \sum_{s:R_s\leq\bar{\mu}} \frac{v'(\tilde{W}_{w,s})}{v'(W\bar{\mu})}\,(R_s-\bar{\mu})\,\pi_s
\end{aligned}
$$

By the lemma

$$\frac{v'(\tilde{W}_{w,s})}{v'(W\bar{\mu})}(R_s - \bar{\mu}) > \frac{u'(\tilde{W}_{w,s})}{u'(W\bar{\mu})}(R_s - \bar{\mu}) \text{ for } R_s > \bar{\mu},$$

likewise

$$\frac{v'(\tilde{W}_{w,s})}{v'(W\bar{\mu})}(R_s - \bar{\mu}) > \frac{u'(\tilde{W}_{w,s})}{u'(W\bar{\mu})}(R_s - \bar{\mu}) \text{ for } R_s < \bar{\mu}$$

hence

$$E\left[\frac{v'(\tilde{W}_{w_1^*})(R - \bar{\mu})}{v'(W\bar{\mu})}\right] > E\left[\frac{u'(\tilde{W}_{w_1^*})(R - \bar{\mu})}{u'(W\bar{\mu})}\right] = 0$$

QED

- We have analyzed the portfolio decision for a risk averse agent with general (concave) utility u.
- Now we specialize the problem so that agents only care about the expected value and the variance of their consumption or wealth.
- We show that this happens in two important cases:
  - Either u is quadratic and the distribution of the returns is arbitrary.

    How does the absolute risk aversion of u quadratic changes with wealth?
  - Or u is concave but otherwise arbitrary but the returns (and hence wealth) are Normally distributed (or in general symmetrically distributed)

    Example $u(x) = -\exp(-a\,x)$, then $E[u(\tilde{W})] = V(\mu, \sigma^2)$ linear on (μ, σ^2).

    What family of r.v's are closed under addition and symmetrically distributed?.

Quadratic Utility

Assume that

$$u(x) = x - (\alpha/2)x^2, \forall x \in \left(0, \alpha^{-1}\right),$$

where $\alpha > 0$.
Assume that $\tilde{W}$ is has any distribution with support $\left(0, \alpha^{-1}\right)$ and that

$$\begin{aligned} E\left[\tilde{W}\right] &= \mu \\ Var\left(\tilde{W}\right) &= \sigma^2 \end{aligned}$$

Proposition In this case the utility function of the consumer can be expressed as a function of σ and μ, i.e.

$$E\left[u\left(\tilde{W}\right)\right] = V(\mu, \sigma)$$

Proof:
The consumer's objective function is

$$
\begin{aligned}
E\left[u\left(\tilde{W}\right)\right] &= E\left[\tilde{W} - (\alpha/2)\tilde{W}^2\right] \\
&= E\left[\tilde{W}\right] - (\alpha/2)E\left[\tilde{W}^2\right] \\
&= E\left[\tilde{W}\right] - (\alpha/2)\left[Var\left(\tilde{W}\right) + E\left[\tilde{W}\right]^2\right] \\
&= \mu - (\alpha/2)\sigma^2 - (\alpha/2)\mu^2 = V(\mu, \sigma).
\end{aligned}
$$

QED.

- The objective function $V(\mu, \sigma)$, is increasing in μ and decreasing in σ: i.e. $\frac{\partial V}{\partial \mu}(\mu, \sigma) > 0$ and $\frac{\partial V}{\partial \sigma}(\mu, \sigma) < 0$.

- The functional form of utility implies that

$$\frac{\partial V}{\partial \mu}(\mu, \sigma) = 1 - \alpha\mu > 0,$$

  because the assumption $\tilde{W} \in (-\infty, \alpha^{-1})$ implies that $\mu < \alpha^{-1}$. Also,

$$\frac{\partial V}{\partial \sigma}(\mu, \sigma) = -(\alpha/2)\sigma < 0,$$

  since $\sigma > 0$ by definition.

- In the mean/standard-deviation space, indifference curves are upward sloping, because u is increasing and concave (risk aversion).

- Utility increases in the direction North-West.

Normally Distributed Returns

Now we consider the case where wealth $\tilde{W}$ is normally distributed so that $\tilde{W} \sim N\left(\mu, \sigma^2\right)$ and u is concave.

Proposition. In this case the utility function of the consumer can be expressed as a function of σ and μ, i.e.

$$E\left[u\left(\tilde{W}\right)\right] = V\left(\mu, \sigma\right)$$

- Notice that if the returns R of the N assets are normally distributed, then for any weight w the wealth $\tilde{W}$ is also normally distributed.
- This follows because $\tilde{W}$ is a linear combination of Normal random variables.

Proof.

If $W \sim N(\mu, \sigma^2)$ then it can be expressed as $W = \mu + \sigma Z$, where $Z \sim N(0, 1)$. Therefore:

$$\begin{aligned} E\left[u\left(\tilde{W}\right)\right] &= E\left[u\left(\mu + \sigma Z\right)\right] \\ &= \int_{-\infty}^{+\infty} u\left(\mu + \sigma z\right) f\left(z\right) dz = V\left(\mu, \sigma\right). \end{aligned}$$

where

$$f(z) = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{z^2}{2}\right] > 0$$

First lets show that $\frac{\partial V}{\partial \mu}(\mu, \sigma) > 0$:

$$\begin{aligned}
\frac{\partial V}{\partial \mu}(\mu, \sigma) &= \frac{\partial}{\partial \mu} E\left[u\left(\mu + \sigma Z\right)\right] \\
&= E \frac{\partial}{\partial \mu} u\left(\mu + \sigma Z\right) \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial \mu}\left[u\left(\mu + \sigma z\right) f\left(z\right)\right] dz \\
&= \int_{-\infty}^{+\infty} u'\left(\mu + \sigma z\right) f\left(z\right) dz > 0
\end{aligned}$$

since $u' > 0$ and $f(z) > 0$.

Now lets show that $\frac{\partial V}{\partial \sigma}(\mu, \sigma) < 0$:

$$\begin{aligned}
\frac{\partial V}{\partial \sigma}(\mu, \sigma) &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial \sigma}\left[u(\mu + \sigma z) f(z)\right] dz \\
&= \int_{-\infty}^{+\infty} u'(\mu + \sigma z) z f(z) \, dz \\
&= \int_{-\infty}^{0} u'(\mu + \sigma z) z f(z) \, dz \\
&\quad + \int_{0}^{+\infty} u'(\mu + \sigma z) z f(z) \, dz.
\end{aligned}$$

Now notice that, for any function $g(z)$,

$$\int_{-\infty}^{0} g(z)\, dz = \int_{0}^{+\infty} g(-z)\, dz.$$

Think of $g(z) = u(\mu + \sigma z)\, z\, f(z)$.
Then, $g(-z) = u(\mu - \sigma z)(-z) f(-z)$, but by symmetry of the function $f(\cdot)$.
we have that

$$g(-z) = u(\mu - \sigma z)(-z) f(z)$$

So we have that ... (continue next page)

So we have that

$$\int_{-\infty}^{0} u'(\mu + \sigma z)\, zf(z)\, dz = -\int_{0}^{+\infty} u'(\mu - \sigma z)\, zf(z)\, dz,$$

from which it follows that

$$\begin{aligned}\frac{\partial V}{\partial \sigma}(\mu, \sigma) &= \int_{0}^{+\infty} u'(\mu + \sigma z)\, zf(z)\, dz \\ &\quad - \int_{0}^{+\infty} u'(\mu - \sigma z)\, zf(z)\, dz \\ &= \int_{0}^{+\infty} \underbrace{[u'(\mu + \sigma z) - u'(\mu - \sigma z)]}_{<0} \underbrace{z}_{>0} \underbrace{f(z)}_{>0} dz < 0\end{aligned}$$

where $u'(\mu + \sigma z) - u'(\mu - \sigma z) < 0$ follows from the fact that $z > 0$ and u' is decreasing. QED.

- Notice that $\frac{\partial V}{\partial \mu}(\mu, \sigma) > 0$ and $\frac{\partial V}{\partial \sigma}(\mu, \sigma) < 0$ implies that:
- In the mean/standard-deviation space, indifference curves are upward sloping, because $u(\cdot)$ is increasing and concave (risk aversion).
- Utility increases in the direction North-West.