Theory Income, Fall 2018

Lecture Note 8

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Stability of Optimal Trajectories

Given an initial state x_0, the solution to a dynamic programming problem completely determines the evolution of the state through time. The purpose of this note is to study the local dynamics and stability of the optimal decision rules in discrete time models.

This notes follow RMED, “Recursive Methods in Economic Dynamics", by Stokey and Lucas with Prescott, Chapter 6.

Local stability of the optimal decision rules (discrete time, one dimensional case).

Consider the Euler equations in the one dimensional discrete time case and let $x_{t+1} = g(x_t)$ be the optimal decision rule. For an interior solution we must have:

$$F_y(x, g(x)) + \beta F_x(g(x), g(g(x))) = 0$$

We want to find out $g'(x^*)$ where x^* is the steady state, i.e. x^* solves:

$$F_y(x^*, x^*) + \beta F_x(x^*, x^*) = 0$$

Notice that there is no time; we replaced x(t+1) with g!

The interest on $g'(x^*)$ is that it allows us to find out the dynamics of x_t close to a steady state. To see this, use a first order Taylor approximation:

$$x_{t+1} = g(x_t) \cong g(x^*) + g'(x^*)(x_t - x^*)$$

so that

$$x_{t+1} - x^* \cong g'(x^*)(x_t - x^*)$$

which gives local stability if $|g'(x^*)| < 1$.

* For vector-cases, the result below applies to the eigenvalues of the partial matrix.

Differentiating the Euler equation with respect to x we obtain

$$\begin{aligned} 0 &= F_{yx}(x, g(x)) + F_{yy}(x, g(x))\, g'(x) \\ &\quad + \beta F_{xx}(g(x), g(g(x)))\, g'(x) + \beta F_{xy}(g(x), g(g(x)))\, g'(g(x))\, g'(x) \end{aligned}$$

Evaluating this derivative at the steady state x^* : (making life easier!)

$$\begin{aligned} 0 &= F_{yx}(x^*, x^*) + F_{yy}(x^*, x^*)\, g'(x^*) + \\ &\quad + \beta F_{xx}(x^*, x^*)\, g'(x^*) + \beta F_{xy}(x^*, x^*)\, [g'(x^*)]^2 \end{aligned} \tag{1}$$

which is a quadratic equation in $g'(x^*)$, and hence it has two values (candidates) for $g'(x^*)$. Indeed we can see that if $\lambda = g'(x^*)$ is a solution of (1), so it must be $1/(\lambda\beta)$. This results is referred as saying that the roots of (1) come in "almost reciprocal pairs".

one is < 1, the another is >1 where both are real. Take the one < 1.

If both of them are > 1, we can't choose from the two. (POSSIBLE, UNSTABLE)
If both of them are < 1, that's a bummer too. (IMPOSSIBLE)

To see this, let λ be a root, and assume that $\lambda \neq 0$, otherwise the result is immediate, then

$$\begin{aligned} 0 &= F_{yx}(x^*, x^*) + [F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)]\lambda + \beta F_{xy}(x^*, x^*)\lambda^2 \\ &= \lambda^2 \beta \left\{ F_{yx}(x^*, x^*) \frac{1}{\lambda^2 \beta} + [F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)] \frac{1}{\beta \lambda} + F_{xy}(x^*, x^*) \right\} \end{aligned}$$

and since

$$F_{yx}(x^*, x^*) = F_{xy}(x^*, x^*)$$

we have:

$$\begin{aligned} 0 &= F_{yx}(x^*, x^*) \left[\frac{1}{\lambda^2 \beta} \right] + [F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)] \left[\frac{1}{\beta \lambda} \right] + F_{xy}(x^*, x^*) \\ &= \beta F_{yx}(x^*, x^*) \left[\frac{1}{\beta \lambda} \right]^2 + [F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)] \left[\frac{1}{\beta \lambda} \right] + F_{xy}(x^*, x^*) \end{aligned}$$

Thus if one root $|\lambda_1| < 1$, then the other root, λ_2, must be bigger than one in absolute value:

$$|\lambda_2| = \left|\frac{1}{\beta\lambda_1}\right| = \frac{1}{\beta\,|\lambda_1|} > 1$$

From here we conclude the following. Let x_0 be close to the steady state x^*, so that a linear approximation of g is accurate.

In continuous times, you can't have a negative slope (you can have a jump only in discrete steps.

Assume we have found that the smaller root has absolute value less than one. Now consider the following sequence of $\{x_{t+1}\}$:

$$x_{t+1} = x^* + g'(x^*)(x_t - x^*) \quad \text{for } t \geq 0 \tag{2}$$

This sequence, by construction, satisfies the Euler Equations.

Since $|g'(x^*)| < 1$, it converges to the steady state x^*, and hence it satisfies the transversality condition.

Thus, if the problem is convex, we have found a solution. If, on the other hand, both roots are bigger than one in absolute value, then we do not know which one describes $g'(x^*)$, but we do know that that steady state is not locally stable.

Exercise: Zero Root

- Assume that $F_{xy}(x^*, x^*) = 0$. Show that $\lambda = 0$ solve the equation.
- What is the interpretation in terms of the dynamics of the optimal decision rule?
- What is the economic intuition of this resutl?

- The argument in the previous paragraph is heuristic, since the Euler Equations are satisfied only approximately for the sequence (2).
- Nevertheless, this approximate solution, by virtue of the implicit function theorem, can be used to construct an exact solution of the Euler equations that converges to the steady state in a neighborhood of x^*.
- Chapter 6 of RMED by SL&P contains a rigorous treatment of this topic. Moreover, this chapter contains a generalization of these results to the n dimensional case.

==*Exercise*==*: Quadratic case one-dimensional* Assume that $F(x, y)$ is quadratic.
then Fxy is constant.

Show that in this case (2) gives the exact solution to the problem.

Show then, that in the quadratic case, the decision rules are linear.
g'(x) is constant.

==*Exercise:*== *Quadratic case n-dimensional* Assume that $F(x, y)$ is quadratic.

What dimensions (and what interpretation) do $g(x)$ and $g'(x)$ have?

What dimensions (and what interpretation) do $F_{xx}(\bar{x}, \bar{x})$, $F_{xy}(\bar{x}, \bar{x})$ and $F_{yy}(\bar{x}, \bar{x})$ have?

Neoclassical growth model

Let's analyze $g'(k^*)$ for the neoclassical growth model.

We want to construct the quadratic form that has $g'(k^*)$ as the smallest root.

In this case we have:

$$F(x, y) = U(f(x) - y)$$

where recall that f is the production function next of depreciation.

So we will take all derivatives and evaluate them at the steady state values.

In NCG, we have infinite elastic supply so demand will only tell us the position of K*.

$$F(x,y) = U(f(x) - y)$$

so

$$\begin{aligned}
F_x(x,y) &= U'(f(x) - y) f'(x) \\
F_y(x,y) &= -U'(f(x) - y) \\
F_{xx}(x,y) &= U''(f(x) - y) f'(x)^2 + U'(f(x) - y) f''(x) \\
F_{yy}(x,y) &= U''(f(x) - y) \\
F_{xy}(x,y) &= -U''(f(x) - y) f'(x)
\end{aligned}$$

Recall that the steady state k^* solves $1 = \beta f'(k^*)$. Evaluating all these derivatives at steady state values we have:

$$\begin{aligned}
0 &= F_{xy} + [F_{yy} + \beta F_{xx}] g' + (g')^2 \beta F_{xy} \\
&= -U'' f' + \left[U'' + \beta U'' f'^2 + \beta U' f''\right] g' - (g')^2 \beta U'' f' \\
&= -U'' \left[1/\beta - \left[1 + 1/\beta + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right] g' + (g')^2\right]
\end{aligned}$$

Exercise. Plot the quadratic function

$$Q(\lambda) = 1/\beta - \left[1 + 1/\beta + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right]\lambda + \lambda^2.$$

Notice that

$$\begin{aligned}
Q(0) &= \frac{1}{\beta} > 0 \\
Q(1) &= -\left(\frac{f''}{f'} / \frac{U''}{U'}\right) < 0 \\
Q'(\lambda^*) &= 0 \Rightarrow \lambda^* = \frac{1}{2}\left[1 + 1/\beta + \left(\frac{f''}{f'} / \frac{U''}{U'}\right)\right] > 1 \\
Q(1/\beta) &= -\left(\frac{f''}{f'} / \frac{U''}{U'}\right)\frac{1}{\beta} < 0 \\
Q(\lambda) &> 0 \text{ for } \lambda \text{ large}
\end{aligned}$$

So,

$$\begin{aligned}
0 &= Q(\lambda_1) = Q(\lambda_2) \\
0 &< \lambda_1 < 1 < 1/\beta < \lambda_2
\end{aligned}$$

Exercise. How does the smallest root $\lambda_1 = g'(k^*)$ changes with $\frac{f''}{f'} / \frac{U''}{U'}$?

1. What does -U''/U' measures? What is the economic interpretation?

2. What does f''/f' measures? What is the economic intuition for this?

3. Draw a graph of two marginal productivities, intersection both at $1/\beta$ at the same value of $\bar{k}$. One is steeper than the other. Take a value of $k_0 < \bar{k}$. Compute (shade) the area under each of the two between k_0 and $\bar{k}$. What is the interpretation of this difference?

4. What is the economic interpretation of the ratio $\left(-\frac{f''}{f'}\right) / \left(-\frac{U''}{U'}\right)$?

5. Is this result consistent with the result obtained in the case of a linear utility function?

6. Is this result consistent with the result obtained in the case of a linear production function? (savings problem)

Stability of linear dynamical systems of higher dimensions

Let $x \in R^n$ and the function $m : R^n \to R^n$ define a dynamical system:

$$x_{t+1} = m(x_t) \text{ for } t \geq 0$$

Let x^* be a steady state, i.e. :

$$x^* = m(x^*)$$

Consider a first order approximation of m around x^* :

$$x_{t+1} = m(x^*) + m'(x^*)(x_t - x^*) .$$

From now on we will analyze this linear difference equation. This analysis is valid globally (i.e. for all R^n) if the system is indeed linear. Alternatively it is valid on a neighborhood of the steady state.

The previous linear difference equation can be written as:

$$y_{t+1} = Ay_t$$

where

$$y_t = x_t - x^*$$

and where A is the matrix with the Jacobian $m'(x^*)$. With this change of variables, the steady state of the system is $y^* = 0$.

Diagonalizing the matrix A we obtain:

$$A = P^{-1} \Lambda P$$

where Λ is a diagonal matrix with the eigenvalues of A, denoted by λ_i, possibly complex, on its diagonal.

The matrix P contains the eigenvectors of A, and, as the notation already uses, it is invertible.

[Generically all matrices admits this decomposition. In the exceptional case of repeated eigenvalues Λ can be chosen to be lower diagonal, which is enough for our purposes. To simplify the analysis we will ignore this case. See RMED, Chapter 6, for a full discussion.].

We can now write our linear system as:

$$P y_{t+1} = \Lambda \; P y_t \text{ for } t \geq 0$$

or defining z as a linear combination of the deviations from steady state using the eigenvectors P:

$$z_t \equiv P y_t \equiv P \left(x_t - x^* \right).$$

Notice that since P is invertible, this is a one-to-one mapping, - i.e. each z corresponds to a unique x and vice-versa. We can then write the dynamical system as

$$z_{t+1} = \Lambda z_t \text{ for } t \geq 0$$

or equivalently:

$$z_{it+1} = \lambda_i \; z_{it}$$

for $i = 1, 2, ..n$ and for all $t \geq 0$. This is progress because we can solve this system element by element. Its solution is:

$$z_{it} = \lambda^t z_{i0}$$

for all $t \geq 0$ and $i = 1, 2, .., n$.

This discussion leads directly to the following important result, whose simple proof is left as an exercise. To simplify let's first consider the case where all the λ_i's are real.

Exercise. Let λ_i be such that for $i = 1, 2, ..., m$ we have $|\lambda_i| < 1$ and for $i = m+1, m+2, ..., n$ we have $|\lambda_i| \geq 1$. Thus the eigenvalues of A are

ordered so that the first m are smaller than one. Consider the sequence

$$x_{t+1} = x^* + A(x_t - x^*) \text{ for } t \geq 0$$

for some initial condition x_0. Show that

$$\lim_{t\to\infty} x_t = x^*,$$

if an only if the initial condition x_0 satisfies:

$$x_0 = P^{-1}\hat{z}_0 + x^*,$$

where $\hat{z}_0$ is a vector with its $n - m$ last coordinates equal to zero, i.e.

$$\hat{z}_{i0} = 0 \text{ for } i = m+1, m+2, ..., n$$

and where the remaining elements of $\hat{z}_0$ are arbitrary.

This result states that, if the sequence generated by the dynamical system is to converge to the steady state, then it must be that the initial conditions x_0 belong to a particular linear subspace.

The dimension of this subspace is equal to the number of eigenvalues that are bigger than one in absolute value, $n - m$ in the notation used above.

Exercise. How should the previous statement be modified if λ_i can be complex?

Exercise. Consider a second order differential equation, i.e. one with:

$$x_{t+2} = A_1 x_{t+1} + A_2 x_t$$

with $x_t \in R^n$ and with initial conditions x_0 and x_{-1}. Define a new variable X_t in R^{2n} as follows:

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}$$

Use X_t to define a first order linear difference equation that is equivalent to the previous second order difference equation. This first order difference equation is of the form:

$$X_{t+1} = J\, X_t$$

where the matrix $2n \times 2n$ matrix J has four $n \times n$ blocks. Display an expression for each of these blocks, as a function of A_1 and A_2:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

Exercise: One dimensional programming problem

- Write EE in ss deviations as: $x_{t+2} - \bar{x} = a_1 \left(x_{t+1} - \bar{x}\right) + a_2 \left(x_t - \bar{x}\right)$.
- What condition is required to be able to write the EE explicitly? (i.e. to solve for x_{t+2} as function of x_{t+1} and x_t)
- Write it as a two dimensional vector, first order difference equation. The matrix A should contain the two scalars a_1 and a_2. Use as a vector $X_{t+1} \equiv [x_{t+2} - \bar{x}, x_{t+1} - \bar{x}]'$.
- Diagonalize A. How do the two eigenvalues of A relate to 1 (unity) ?
- Use the eigenvector associated with the unstable eigenvalue (larger than one) to define $x_1 - \bar{x}$ as a function of $x_0 - \bar{x}$, so that $x_t \to \bar{x}$ in the dynamical system.
- In general, what is the relationship between $x_{t+1} - \bar{x}$ and $x_t - \bar{x}$?