Economics 331 Prof. Stokey

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RECURSIVE COMPETITIVE EQUILIBRIUM

We will use the neoclassical growth model with taxes to illustrate the concept of recursive competitive equilibrium (RCE).

There is a continuum of identical households, of total measure one. For simplicity, assume no population growth. The Representative Household (RH) has preferences

$$U\left(\{c_t, h_t\}\right) = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \alpha \ln\left(1 - h_t\right)\right], \qquad \alpha > 0,\ \beta \in (0,1), \tag{1}$$

over consumption c_t and leisure $1 - h_t$, where both are measured per capita. The representative firm (RF) has the (stationary) CRS technology

$$y_t = \left(Ah_t\right)^{1-\theta} k_t^{\theta}, \qquad A > 0,\ \theta \in (0,1), \qquad \text{all } t. \tag{2}$$

Output can be used for consumption or investment, and capital depreciates at the constant rate $\delta > 0$, so the resource constraint is

$$k_{t+1} = (1-\delta)\, k_t + y_t - c_t, \qquad \text{all } t, \tag{3}$$

where k_t, y_t are also measured per capita.

1. Undistorted economies: the Planner's problem

In the absence of taxes, external effects, or other distortions, the competitive equilibrium is Pareto efficient. Since households are identical, in equilibrium all get the same allocation and utility. Hence the competitive equilibrium solves a Social Planner's problem, which means it maximizes the utility of the RH. We can solve this problem either in sequence form or recursively.

A. Sequence form for the Planner's problem.—

The Planner's problem in sequence form is to choose $\{(c_t, h_t, k_{t+1})\}_{t=0}^{\infty}$ to maximize (1) subject to the constraints in (2) and (3), or

$$k_{t+t} = (1-\delta)\, k_t + (Ah_t)^{1-\theta}\, k_t^{\theta} - c_t, \qquad \text{all } t, \tag{4}$$

given the initial capital stock $k_0 > 0$. To characterize the optimum, put a Lagrange multiplier λ_t or $\beta^t \lambda_t$ on each constraint in (4), and use the resulting first-order conditions. A transversality condition is also required.

This problem can also be interpreted as letting each household operate its own "backyard" firm. This interpretation works because the technology has CRS and all households are identical.

With this interpretation we could also write the household's budget constraint as

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta)\, k_t - (Ah_t)^{1-\theta}\, k_t^{\theta}\right] \leq 0,$$

where $\{p_t\}_{t=0}^{\infty}$, with $p_0 = 1$, are the prices of goods at t in terms of goods at date zero. The term in square brackets is the household's borrowing at date t (saving if it is negative) from a fictional "bank." Its net borrowing must be nonnegative over its lifetime. In equilibrium, since all households are identical, borrowing must be zero at every date, which determines $\{p_t\}_{t=0}^{\infty}$. Note that the one-period real interest rate is

$$1 + \hat{r}_t = p_t / p_{t+1}, \qquad \text{all } t.$$

The p_t's are related to the λ_t's.

You should already know how to analyze this problem. In particular, you should be able to calculate the (unique) interior state capital stock k^{ss}, characterize the dynamics for capital if $k_0 \neq k^{ss}$, describe how consumption and labor supply each period are related to the current capital stock, and describe how the solution changes with the parameters α, β, θ and δ.

B. Recursive approach for the Planner's problem.—

The Planner's problem can also be analyzed with a Bellman equation. To verify that the problem is concave, define the one-period return function

$$F(k,k') \equiv \max_{h\in[0,1]} \left\{ \ln\left[(1-\delta)\,k + (Ah)^{1-\theta}\,k^{\theta} - k' \right] + \alpha \ln(1-h) \right\}, \tag{5}$$

for $k > 0$ and $k' \in \Gamma(k) \equiv [0, (1-\delta)\,k + A^{1-\theta}k^{\theta}]$. Since the term on the right is continuous in h, and the feasible set for h is compact (closed and bounded), the maximum is attained. And since the RHS is strictly concave in h, the maximizer is unique.

By the Theorem of the Maximum, F is continuous, and clearly F is strictly increasing in k and strictly decreasing in k'.

Concavity of F To see that F is concave. Choose k_i and $k_i' \in \Gamma(k_i)$, $i = 1, 2$, and let h_i be the maximizer for (k_i, k_i') . Choose $\lambda \in (0,1)$, and define the weighted averages $x_\lambda = \lambda x_1 + (1-\lambda)\,x_2$, for $x = k, k', h$. Clearly $k_\lambda' \in \Gamma(k_\lambda)$.

Define the function

$$\phi(k,k',h) \equiv \ln\left[(1-\delta)\,k + (Ah)^{1-\theta}\,k^{\theta} - k' \right] + \alpha \ln(1-h)$$

and note that it is strictly concave in (k, k', h) . To show that F is concave, note that

$$\begin{aligned} F\left(k_\lambda, k_\lambda'\right) &\geq \lambda F(k_1, k_1') + (1-\lambda)\,F(k_2, k_2') \\ &= \lambda \phi(k_1, k_1', h_1) + (1-\lambda)\,\phi(k_2, k_2', h_2), \end{aligned}$$

with equality if and only if $(k_1, k_1', h_1) = (k_2, k_2', h_2)$, which in turn holds if and only if $(k_1, k_1') = (k_2, k_2')$.

Hence the Planner's problem can be studied with the Bellman equation

$$v(k) = \max_{k'\in\Gamma(k)} \left[F(k,k') + \beta v(k') \right],$$

where F is define in (5) or, equivalently, as

$$v(k) = \max_{c,k'\geq 0,\ h\in[0,1]} \left[\ln c + \alpha \ln (1-h) + \beta v(k')\right] \tag{6}$$
$$\text{s.t.} \qquad k' = (1-\delta)\, k + (Ah)^{1-\theta} k^{\theta} - c.$$

2. Economies with distortions: competitive equilibrium approach

To study the effects of taxes, a different approach is needed. With taxes (or external effects, or money, or other distortions) the competitive equilibrium must be analyzed directly.

The economy has one aggregate state variable, the aggregate (per capita) capital stock k_t. The state variable for the individual household is its own (per capita) capital stock, K_t. All households are identical, and total population is unity, so in equilibrium $K_t = k_t$. The individual household makes its consumption, labor supply, and capital accumulation decisions, taking as given the future evolution of the wage rate, the return to capital, its lump-sum transfer, and anything else that is a function of k_t.

A. Sequence form for CE problem.—

For the CE approach, introduce the prices $\{(w_t, R_t, p_t)\}_{t=0}^{\infty}$, where w_t, R_t are factor returns at t, measured in terms of goods at t, and p_t is as above. It is also convenient to let $\{(c_t, h_t)\}_{t=0}^{\infty}$ denote the economy-wide average for per capita consumption and labor supply.

The representative firm has the production technology in (2). It maximizes profits, taking prices as given. Perfect competition among firms implies factors are paid their marginal products,

$$w_t = (1-\theta)\, A^{1-\theta} (k_t/h_t)^{\theta}, \tag{7}$$
$$R_t = \theta A^{1-\theta} (k_t/h_t)^{\theta-1}, \qquad \text{all } t, \tag{8}$$

and CRS implies that factor payments exhaust output.

Market clearing for capital and labor are already built in: the RF employs all the factor inputs the RH supplies. Market clearing for goods each period requires

$$c_t + k_{t+1} - (1-\delta)\,k_t = (Ah_t)^{1-\theta}\,k_t^{\theta}, \qquad \text{all } t. \tag{9}$$

The individual household's problem is to choose $\{(C_t, H_t, K_{t+1})\}_{t=0}^{\infty}$ to maximize (1), given factor returns $\{(w_t, R_t)\}_{t=0}^{\infty}$, subject to the period budget constraints

$$C_t + K_{t+1} - (1-\delta)\,K_t - R_t K_t - w_t H_t \leq 0, \qquad \text{all } t, \tag{10}$$

given $K_0 = k_0 > 0$. Alternatively, one can think about complete markets at $t = 0$ and write the lifetime budget constraint for the individual household, given prices $\{p_t\}_{t=0}^{\infty}$ for claims to dated consumption good. In this case the budget constraint is

$$\sum_{t=0}^{\infty} p_t \left[C_t + K_{t+1} - (1-\delta)\,K_t - R_t K_t - w_t H_t\right] \leq 0. \tag{11}$$

To characterize the CE, write the first-order conditions for the individual household; impose $C_t = c_t$, $H_t = h_t$, $K_{t+1} = k_{t+1}$, all t, and the other equilibrium conditions; and use the resulting equations to characterize $\{(c_t, h_t, k_{t+1})\}_{t=0}^{\infty}$. The solution coincides with the one in Section 1A above.

To introduce a flat-tax on capital income, the proceeds of which are rebated lump sum to the household, let $\{(\tau_{Kt}, T_t)\}_{t=0}^{\infty}$ denote the sequence of tax rates and transfers. The household takes these as given. Modify the budget constraint in (11) to get

$$\sum_{t=0}^{\infty} p_t \left[C_t + K_{t+1} - K_t - (1-\tau_{Kt})\,(R_t - \delta)\,K_t - w_t H_t - T_t\right] \leq 0. \tag{12}$$

Budget balance for the government requires

$$T_t = \tau_{Kt}\,(R_t - \delta)\,k_t, \qquad \text{all } t. \tag{13}$$

The key element here is that the individual household takes $\{T_t\}_{t=0}^{\infty}$ as exogenous when making its decisions about consumption, labor supply, and capital accumulation.

B. Recursive form for CE problem

To study this economy as a (stationary) recursive competitive equilibrium, the tax rate must be constant over time, $\tau_{kt} = \tau_k$, all t.

Consider the problem of the individual household. It takes as given the law of motion for the aggregate capital stock, as well as functions that express the wage rate, the return to capital, aggregate consumption, the transfer, etc. as functions of the aggregate capital stock. Call these

$$\begin{aligned} k_{t+1} &= g(k_t), \qquad w_t = \widetilde{w}(k_t), \qquad R_t = \widetilde{R}(k_t), \\ h_t &= \widetilde{h}(k_t), \qquad c_t = \widetilde{c}(k_t), \qquad T_t = \widetilde{T}(k_t), \qquad \text{etc.} \end{aligned}$$

Fix these functions.

Then the Bellman equation for the individual household is

$$V(K,k) = \max_{C,K' \geq 0,\ H \in [0,1]} \left[\ln C + \alpha \ln(1-H) + \beta V(K', g(k)) \right]$$

$$\text{s.t.} \qquad C + K' - K - (1-\tau_k)\left[\widetilde{R}(k) - \delta\right] K - \widetilde{w}(k) H - \widetilde{T}(k) \leq 0.$$

The individual household's one-period return function can be written as in (5) as $F(K, K')$, so it has the properties described there. Let $G(K,k)$, $\widetilde{C}(K,k)$, $\widetilde{H}(K,k)$ be the optimal policy functions for the household's problem. Note that they depend on the functions $g, \widetilde{w}$, and so on conjectured by the individual household to describe the behavior of other households.

The FOCs and envelop condition for the individual household are

$$\begin{aligned} \frac{1}{\widetilde{C}} &= \lambda(K,k), && MUC \\ \frac{\alpha}{1-\widetilde{H}} &= \lambda(K,k)\widetilde{w}(k), && MUL \\ \beta V_K(G(K,k), g(k)) &= \lambda(K,k), && MVK' \end{aligned}$$

$$V_K(K,k) = \left[1 + (1-\tau_k)\left(\widetilde{R}(k) - \delta\right)\right] \lambda(K,k), \qquad MVK$$

where the multiplier $\lambda(K,k)$ on the resource constraint is the marginal value (in utility) of goods in the current period. The direct effect of the capital tax is to reduce the marginal value of capital in the current period. Of course, the tax also affects the functions $\lambda, \widetilde{w}, \widetilde{R}$, etc.

In equilibrium, the optimal policy functions for the individual household must coincide with the (conjectured) aggregates for the economy. That is, equilibrium requires

$$G(k,k) = g(k), \qquad \widetilde{C}(k,k) = \widetilde{c}(K), \qquad \widetilde{H}(k,k) = \widetilde{h}(k), \qquad \text{all } k.$$

In addition, the factor returns and lump sum transfer satisfy (7), (8), and (13), so

$$\begin{aligned}
\widetilde{w}(k) &= (1-\theta)\, A^{1-\theta} \left(\frac{k}{\widetilde{h}(k)}\right)^{\theta}, && (14)\\
\widetilde{R}(k) &= (1-\theta)\, A^{1-\theta} \left(\frac{k}{\widetilde{h}(k)}\right)^{\theta-1}, &&\\
\widetilde{T}(k) &= \tau_k \left[\widetilde{R}(k) - \delta\right] k, \qquad \text{all } k. &&
\end{aligned}$$

It is convenient to define

$$v_k(k) \equiv V_K(k,k), \tag{15}$$

the marginal value of (own) capital to the individual household, when its own capital stock is the same as that of other households, $K = k$.

Use v_k and the equilibrium conditions, and eliminate λ, T to get

$$\begin{aligned}
\frac{1}{\widetilde{c}} &= \beta v_k \left[g(k)\right], && (16)\\
\frac{\alpha}{1-\widetilde{h}} &= \frac{\widetilde{w}(k)}{\widetilde{c}(k)}, && (17)\\
v_k(k) &= \left[1 + (1-\tau_k)\left(\widetilde{R}(k) - \delta\right)\right] \beta v_k \left[g(k)\right], && (18)
\end{aligned}$$

$$\widetilde{c} + g(k) - (1-\delta)\, k = \left(A\widetilde{h}\right)^{1-\theta} k^{\theta}, \tag{19}$$

where the factor returns $\widetilde{w}, \widetilde{R}$ are functions of $k/\widetilde{h}(k)$, as in (14). These four equations characterize the functions $\widetilde{c}, \widetilde{h}, g, v_k$, describing consumption, labor supply, and capital accumulation, as well as the marginal value of capital.

Steady state.—

At the steady state $g(k^{ss}) = k^{ss}$. Hence from (18),

$$\rho = (1 - \tau_k) \left[\widetilde{R}(k^{ss}) - \delta \right],$$

where $\beta \equiv 1/(1+\rho)$. This condition determines the SS capital-labor ratio k^{ss}/h^{ss}. From the second condition, intratemporal optimization,

$$\frac{1}{c^{ss}} = \frac{\alpha}{1 - h^{ss}} \frac{1}{w^{ss}},$$

where w^{ss} is determined by the capital-labor ratio, and the resource constraint is

$$c^{ss} + \delta k^{ss} = (Ah^{ss})^{1-\theta} (k^{ss})^{\theta}.$$

The latter two equations determine k^{ss} and h^{ss} separately, as well as c^{ss}.

Transitional dynamics.—

Transitional dynamics can be studied by linearizing around the steady state. For simplicity, in this section we will make labor supply inelastic, setting $h \equiv 1$. In this case we can write the individual household's Bellman equation as

$$V(K,k) \equiv \max_{K'} \left\{ F(K, K'; k) + \beta V[K', g(k)] \right\},$$

where

$$F(K, K'; k) \equiv \ln \left[w(k) + T(k) + K + (1 - \tau_k) \left[\widetilde{R}(k) - \delta \right] K - K' \right].$$

The FOC and EC are

$$\begin{aligned} F_2 \left[K, G(K,k); k \right] + \beta V_1[G(K,k), g(k)] &= 0, \\ V_1(K,k) &= F_1[K, G(K,k); k]. \end{aligned}$$

Imposing the equilibrium conditions gives a second-order difference equation in k_t,

$$F_2(k_t, k_{t+1}; k_t) + \beta F_1(k_{t+1}, k_{t+2}; k_{t+1}) = 0. \tag{20}$$

Fix a tax rate $\tau \geq 0$, and consider a reduction in the capital tax from $\tau + \varepsilon$ to τ.

a. Define $z_t = k_t - k^{ss}(\tau)$ as the deviation of the capital stock from the new steady state. Linearize (20) around the new steady state, and write the first-order difference equation for the "stacked" system (z_{t+1}, z_t).

b. Describe the characteristic roots for this system.

The effects of a tax change.—

Consider an economy that changes its tax on capital income. What are the long and short-run effects on employment (hours per household), the wage rate, and earnings?

The long run effect can be determined by comparing the two steady states. Use (16)-(19), specialized to the SS, with the two values for τ_k.