Price Theory, III **Professor Lars Stole**
Economics 33003 **Spring 2019**

Assignment 4
(Due Friday, May 3, prior to the start of the Review session)

Problem 1 Consider a principal-agent setting with moral hazard and a finite number of outputs and efforts. There are n possible outputs, $x_i \in \mathcal{X} = \{x_1, \ldots, x_n\}$, $x_n > x_{n-1} > \cdots > x_1$. There are m possible effort levels, $e_j \in \mathcal{E} = \{e_1, \ldots, e_m\}$, $e_m > e_{m-1} > \cdots > e_1$. The probability of output x_i given effort e_j is $\phi_i(e) = \phi(x_i|e_j) > k > 0$ (i.e., bounded away from zero). Assume that $\phi_i(\cdot)$ satisfies MLRP: i.e., for all $e > \tilde{e}$,

$$\frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}, \quad \text{is increasing in } i.$$

Assume that the principal is risk neutral and the agent is risk averse with preferences $U = u(w) - \psi(e)$, where $u(\cdot)$ is increasing and concave, and $\psi(\cdot)$ is increasing and convex. Assume the agent's outside option is $\underline{U} = 0$ and $\psi(e_1) = 0$. Finally, the principal's wage schedule must satisfy $w_i = w(x_i) \in [\underline{w}, \overline{w}]$, but assume that these constraints are slack at the optimum and ignore them.

(a). State the principal's program for the optimal e and $(w_1, \ldots, w_n)$, and write the Lagrangian for the program using $\{\mu_j\}_j$ as multipliers for the IC constraints and λ as the multiplier for the IR constraint.

(b). Prove that the agent's IR constraint is binding (i.e., $\lambda > 0$) in the solution to the program.

(c). Suppose that the optimal choice of effort is $e^* = e_m$, the maximum effort level. Prove that the optimal wage schedule which induces e^m is monotonic, $w_1^* \leq w_2^* \cdots \leq w_n^*$ with strict inequality for some i (i.e., $w_i^* < w_{i+1}^*$).

(d). Prove that the MLRP implies first-order stochastic dominance for the case of a discrete distribution, $(\phi_1(e), \ldots, \phi_n(e))$.

(e). Suppose that the optimal choice of effort $e^* > e_1$. Prove that there must exist some i such that $w_i^* < w_{i+1}^*$ (i.e., prove that $w_1^* \geq w_2^* \geq \cdots \geq w_n^*$ cannot be optimal). [Hint: use the fact in (d) above.]

Problem 2 Consider the following moral hazard problem. The worker can choose one of two (unobservable) effort levels, $e \in \{1, 2\}$. If the worker exerts effort e and receives wage w, her payoff is $\sqrt{w} - e$. The outside option of the worker is zero. The owner of the firm is risk neutral. There are two possible realizations of the output, $x_1 = 1$ and $x_2 = 16$.

(a). Suppose that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	α	0
$x_2 = 16$	$1 - \alpha$	1

where $\alpha \in (0,1)$. Characterize the set of those α's for which the owner of the firm can achieve the same payoff as if the effort of the worker was observable and contractible.

(b). Suppose now that the stochastic relationship between effort and output is described by the following probability matrix

	$e = 1$	$e = 2$
$x_1 = 1$	1	β
$x_2 = 16$	0	$1 - \beta$

where $\beta \in (0,1)$. Again, characterize the set of those β's for which the owner of the firm can achieve the same payoff as if the effort of the worker was contractible.

(c). Suppose that the stochastic relationship between effort and output is described by the probability matrix given in part (b) and $\beta = \frac{1}{4}$. Characterize the optimal contract for the owner of the firm.

(d). Suppose that the production technology is the same as in part (c). Assume that the owner of the firm can invest in a monitoring technology which perfectly reveals the effort of the worker. The outcome of this monitoring is contractible. The owner has to make this (observable) investment decision prior to offering a contract and it costs p. For what values of p should the owner invest in monitoring?

Problem 3 Consider the a setting like in Holmstrom and Milgrom, in which a risk neutral principal contracts with a risk-averse (CARA utility with r) agent. Specifically, suppose that a school wishes to design an optimal compensation contract for its teachers. Teachers engage in two tasks:

$$x_1 = e_1 + \varepsilon_1,$$

$$x_2 = e_2 + \varepsilon_2,$$

where ε_i is independently distributed according to $N(0, \sigma_i^2)$. The cost of the teacher's effort is $\psi(e_1 + e_2)$ where

$$\psi(e) = \begin{cases} 0 & \text{if } e \leq \hat{e} \\ \frac{1}{2}(e^2 - \hat{e}^2) & \text{if } e \geq \hat{e}. \end{cases}$$

Thus, $\hat{e} > 0$ represents an amount of free labor that a teacher will supply.

Suppose that task 1 (teaching math and reading skills) is easy to measure via test scores in math and reading, but task 2 (teaching creativity) is almost impossible to measure. Hence, σ_1^2 is small and σ_2^2 is large. Additionally, suppose that the school cares about the benefits of e_1 and e_2 for its students and these benefits are characterized by the function, $B(e_1, e_2) \geq 0$. $B(\cdot, \cdot)$ is strictly concave, and strictly increasing in both arguments for $e_1 > 0$ and $e_2 > 0$; but assume that $B(e_1, 0) = B(0, e_2) = 0$ for all e_1 and e_2. That is, teaching math and reading is pointless without teaching a little bit of creativity, and vice versa. For reasons given in Holmstrom and Milgrom (1987), the school chooses a linear compensation function,

$$w(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 + \beta.$$

The school maximizes the expectation of $B(e_1, e_2) - w(x_1, x_2)$ subject to incentive compatibility and the teacher's outside option, $\underline{U}$. The teachers are solely motivated by the wage contract and their cost of aggregate effort. They do not care about B or how effort is allocated across tasks, except insofar as it impacts the wage. If they are indifferent between two allocations of effort, however, we assume that they will choose the one preferable to the school.

(a). Suppose in the extreme that $\sigma_2^2 = \infty$; i.e., e_2 cannot be measured. What is the optimal incentive scheme for the school to offer the teachers?

(b). Suppose that σ_2^2 is finite. State the principal's optimal program. Prove that in any solution, $\alpha_1 = \alpha_2 = \alpha$, and prove that $\alpha \notin (0, \hat{e}]$.

Problem 4 Consider the modified linear managerial-incentive-scheme problem, where the managers effort, e, affects current profits, $x_1 = e + \varepsilon_1$, and future profits, $x_2 = e + \varepsilon_2$, where ε_i are i.i.d. with normal distribution $N(0, \sigma_\varepsilon^2)$. Note that the single effort equally impacts the short run (period 1) and the long run (period 2). The manager retires at the end of the first period, and the managers compensation cannot be based on x_2. However, the company can issue stock that she must hold for one year after retirement. The price of the stock one year after retirement is $p = x_1 + x_2 + \eta$, where η is distributed normally, $N(0, \sigma_\eta^2)$, and η is independently distributed from ε_t. The firm maximizes the expectation of $x_1 + x_2 - w - sp$, where s are the shares of stock given to the manager. There is no time discounting and the manager only cares about the total value of compensation once the stock is sold one year after retirement. The manager's utility is CARA with risk parameter, r, and her monetary cost of effort is $\frac{1}{2}e^2$. Her outside option is $\underline{U}$. Assume for reasons given in Holmstrom and Milgrom (1987) that the optimal contract is linear. Derive the optimal compensation contract for period 1 output

$$w(x_1) = \alpha x_1 + \beta,$$

and derive the optimal amount of stock, s, to give to the manager. Explain the differences in the optimal α and s. In particular, what happens if the stock market price is a perfect aggregator of $x_1 + x_2$ with $\sigma_\eta^2 = 0$?

Problem 5 Consider the case of a single firm contracting with two agents, each with a CARA parameter of r in the standard linear-contracts framework of Holmstrom and Milgrom. The outputs of the two agents are $x_1 = e_1 + \varepsilon_1$ and $x_2 = e_2 + \varepsilon_2$, where e_i is agent i's effort and ε_i is a normally distributed noise term, $\varepsilon_i \sim \mathcal{N}(0, \sigma_i^2)$. The cost of effort to each agent i is $\frac{1}{2}e_i^2$. Finally, the measurement errors across the two agents may be positively correlated, with $\mathrm{Cov}(\varepsilon_1, \varepsilon_2) = \sigma_{12} = \rho\sigma_1\sigma_2$, $\rho \geq 0$. Assume for reasons given in Holmström and Milgrom that optimal contracts are linear.

(a). Consider the benchmark where a firm is restricted to allowing only an individual worker's own output to effect compensation:

$$w_i(x_1, x_2) = \alpha_i^i x_i + \beta^i.$$

Superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i).

(b). Now suppose that the firm can use relative performance evaluation. That is

$$w_1(x_1, x_2) \equiv \alpha_1^1 x_1 + \alpha_2^1 x_2 + \beta^1,$$

$$w_2(x_1, x_2) \equiv \alpha_1^2 x_1 + \alpha_2^2 x_2 + \beta^2.$$

Again, superscripts refer to agents; subscripts refer to the output variables. Solve for the optimal contract parameters (you can ignore β^i). Explain how and why α_i^i varies with respect to ρ. Does it matter if the correlation is positive or negative?

Hint: if ε_1 and ε_2 are normal random variables with covariance σ_{12}, then

$$E[-e^{-r(a\varepsilon_1 + b\varepsilon_2)}] = -e^{-\frac{r^2}{2}(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_{12})}.$$

(c). Continue to assume that the firm can use relative performance evaluation, as in (b), but that now $\rho = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$. In addition to choosing e_2 to increase output, agent two can also engage in a second "helpful" activity, h_2. This activity does not affect the output level directly, but rather reduces the effort cost of the other agent. The interpretation is that agent 2 can help agent 1 (but not the other way round). The cost functions of the agents are given by:

$$\psi_1(e_1, h_2) = \frac{1}{2}(e_1 - h_2)^2,$$

$$\psi_2(e_2, h_2) = \frac{1}{2}e_2^2 + \frac{1}{2}h_2^2.$$

Agent 1 chooses his effort level e_1 only after he has observed the level of help h_2.

Compute the optimal incentive scheme and effort levels. Explain your result.

Problem 6 (MWG, 14.B.6) Amend the two-effort-level model with a risk-neutral principal as follows: Suppose now that effort has distinct effects on revenues, R, and costs, C, where $x = R - C$. Let $f_R(R, a)$ and $f_C(C, a)$ denote the density functions of R and C conditional on a, and assume that, conditional on a, R and C are independently distributed. Assume $R \in [R_0, R_1]$, $C \in [C_0, C_1]$, and that for all a, $f_R(R, a) > 0$ for all $R \in [R_0, R_1]$ and $f_C(C, a) > 0$ for all $C \in [C_0, C_1]$.

The two effort choices are now $\{a_R, a_C\}$, where a_R is an effort choice that devotes more time to revenue enhancement and less to cost reduction, and the opposite is true for a_C. In particular, assume that $F_R(R, a_R) < F_R(R, a_C)$ for all $R \in (R_0, R_1)$ and that $F_C(C, a_C) > F_C(C, a_R)$ for all $C \in (C_0, C_1)$. Moreover, assume that the monotone likelihood ratio property holds for each of these variables in the following form: $f_R(R, a_R)/f_R(R, a_C)$ is increasing in R, and $f_C(C, a_R)/f_C(C, a_C)$ is increasing in C. Finally, the agent prefers revenue enhancement over cost reduction: that is, $\psi(a_C) > \psi(a_R)$.

(a). Suppose that the owner wants to implement effort choice a_C and that both R and C are observable. Derive the first-order condition for the optimal compensation scheme $w(R, C)$. How does it depend on R and C?

(b). How would your answer to (a) change if the agent could always unobservably reduce the revenues of the firm (in a way that is of no direct benefit to him)?

(c). What if, in addition, costs are now unobservable by a court (so that compensation can be made contingent only on revenues)?

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Answers to Assignment 3

1 (a). Principal solves

$$\max_{e,(w_1,\ldots,w_n)} \sum_i (x_i - w_i)\phi_i(e),$$

subject to

$$\sum_i u(w_i)\phi_i(e) - \psi(e) \geq 0, \quad \text{(IR)}$$

$$\sum_i u(w_i)(\phi_i(e) - \phi_i(e_j)) - \psi(e) + \psi(e_j) \geq 0, \quad \text{(IC)}.$$

Forming the Lagrangian,

$$\mathcal{L} = \sum_i (x_i - w_i)\phi_i(e) + \sum_{j=1}^{m} \mu_j \left(-\psi(e) + \psi(e_j) + \sum_i u(w_i)(\phi_i(e) - \phi_i(e_j)) \right) + \lambda \left(-\psi(e) + \sum_i u(w_i)\phi_i(e) \right).$$

(b). Suppose that the constraint was not binding:

$$\sum_i u(w_i^*)\phi_i(e) - \psi(e) > 0.$$

The principal could choose a new wage schedule, $(w_1^\varepsilon, w_2^\varepsilon, \ldots, w_n^\varepsilon)$ such that for each i

$$u(w_i^\varepsilon) = u(w_i^*) - \varepsilon.$$

The new wage schedule has lower expected cost to the principal. For ε sufficiently small, the new wage schedule is satisfies IR; the IC constraints are unaffected by the variation. Hence, the variation is feasible and increases profit – a contradiction.

(c). Differentiating the Lagrangian in (a) with respect to w_i, we have

$$\frac{1}{u'(w_i)} = \lambda + \sum_{j=1}^{m-1} \mu_j \left(\frac{\phi_i(e_m) - \phi_i(e_j)}{\phi_i(e_m)} \right).$$

We know that $\mu_j \geq 0$ for all j. Because $e_m > e_j$ for all $j < m$, MLRP implies each likelihood ratio in the parentheses is increasing in i. Because $1/u'(w_i)$ is increasing in w_i, we conclude that $w_1^* \leq \cdots \leq w_n^*$. Note that if $w_1^* = w_n^*$ (i.e., the wage schedule was constant) then the agent's utility payoff $\sum_i u(w_i^*)\phi_i(e)$ is independent of e, and thus to save on the cost of effort the agent would choose e_1 – a contradiction to $e^* = e_m$. Thus, there must exist at least one i such that $w_i^* < w_{i+1}^*$.

(d). Define

$$LR_i(e, \tilde{e}) = \frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}.$$

Note that because ϕ sums to 1 regardless of e,

$$\sum_{i=1}^{n} \phi_i(e) - \phi_i(\tilde{e}) = 0.$$

Rewriting this in terms of the likelihood ratio, we have

$$\sum_{i=1}^{n} LR_i(e,\tilde{e})\phi_i(e) = 0.$$

MLRP implies that $LR_i(e, \tilde{x})$ is increasing in i if $e > \tilde{e}$. Because it's expectation is zero, it must be that for any $k < n$,

$$\sum_{i=1}^{k} LR_i(e,\tilde{e})\phi_i(e) < 0.$$

Rewriting, this is equivalent to

$$\sum_{i=1}^{k} \frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}\phi_i(e) < 0,$$

or

$$\sum_{i=1}^{k} (\phi_i(e) - \phi_i(\tilde{e})) < 0.$$

Defining $\Phi_i(e) = \sum_{i=1}^{k} \phi_i(e)$ as the CDF for $(\phi_1(e), \ldots \phi_n(e))$, we have

$$\Phi_k(e) < \Phi_k(\tilde{e}) \text{ for all } k < n.$$

But this is the discrete form of first-order stochastic dominance.

(e). Suppose to the contrary that $w_1^* \geq w_2^* \geq \cdot \geq w_n^*$. In this case, $u(w(x_i))$ is a nonincreasing function in x_i. As such, FOSD implies for any $e > \tilde{e}$

$$\sum_{i=1}^{n} u(w_i)\phi_i(e) \leq \sum_{i=1}^{n} u(w_i)\phi_i(\tilde{e}).$$

Thus, the agent will choose $e = e_1$ which contradicts the assumption that $e^* > e_1$.

2 (a). If e is contractible, the firm can set $w = 1$ to induce $e = 1$ and get a profit of $15(1-\alpha)$, or set $w = 4$ to induce $e = 2$ and get a profit of 12. He will choose the former iff $\alpha \leq \frac{1}{7}5$.

When e is not contractible, if the firm wishes to induce $e = 1$, he can set $w_1 = w_2 = 1$. The worker will choose $e = 1$, which is less costly than $e = 2$, since his wage is constant. The firm always pays a wage of 1. Therefore, if $\alpha \leq \frac{1}{7}5$, the firm's second-best profit is as high as its first-best.

Now suppose $\alpha > \frac{1}{7}5$. For the firm to achieve its first-best outcome without contracting on effort, the worker should choose $e = 2$ (i.e. IR and IC should be satisfied), and the firm should pay an expected wage of 4. Hence, it must be that $w_2 = 4$. It is then optimal (in terms of enforcing IC) to set $w_1 = 0$, since this is the harshest punishment possible for shirking. (Note that it is possible to do this without worrying about IR because x_1 never occurs in equilibrium, when $e = 2$.) With this wage schedule, IC holds when $\alpha \geq \frac{1}{2}$.

Therefore, the answer is $\alpha \in (0, \frac{1}{5}] \cup [\frac{1}{2}, 1)$.

(b). Similarly to (a), when $\beta \geq \frac{4}{5}$, $e = 1$ is the first-best effort and this can be achieved as a second-best outcome with $w_1 = w_2 = 1$.

In fact, this is the only case when first-best can be achieved. If the firm wishes to induce $e = 2$, IR requires that the worker's expected utility from wage must equal the cost of effort, 4. But to obtain first-best profit, the firm must pay an expected wage of 4. This is only possible if $w_1 = w_2 = 4$. However, under this wage schedule, the worker will not choose $e = 2$.

(c). Implementing $e = 1$ gives the firm a profit of 0. If the firm wishes to implement $e = 2$, the firm wishes to minimize the expected wage subject to IC and IR, i.e.

$$\begin{aligned} &\min \frac{1}{4}w_1 + \frac{3}{4}w_2 \\ \text{subject to} &\frac{1}{4}\sqrt{w_1} + \frac{3}{4}\sqrt{w_2} - 2 \geq \sqrt{w_1} - 1 \\ &\frac{1}{4}\sqrt{w_1} + \frac{3}{4}\sqrt{w_2} - 2 \geq 0 \end{aligned}$$

One way to solve this problem is to use the Kuhn-Tucker conditions. Here, we instead take the approach of arguing that the two constraints must bind. If IR does not bind, we can increase both wages slightly. If IC is slack, we can decrease the wage difference in such a way that we keep the worker's expected utility the same but lower the expected wage. Therefore, we can solve the two equalities to get $w_1 = 1$ and $w_2 = \frac{49}{9}$. This generates a profit of $\frac{95}{12} > 0$, so this is the optimal contract.

(d). The firm's profit when he can contract on effort is $\frac{33}{4}$. He will invest in the technology if

$$\frac{33}{4} - p \geq \frac{95}{12},$$

i.e. when $p \leq \frac{1}{3}$.

When effort is not contractible, the firm must give up on optimal risk-sharing by giving the worker incentives to exert effort. The size of this risk premium that the firm must give up is the value of the monitoring technology.

3 (a). First, notice that if the school offers a wage independent of x_1 and x_2, then the teachers will still expend $\hat{e}$ in effort. They can therefore be directed to allocate $\hat{e}$ optimally across the two tasks and the school can achieve

$$\max_{e_1} \; B(e_1, \hat{e} - e_1).$$

By assumption, we know this is positive because once $e_2 > 0$, B is strictly increasing in its arguments.

Because $\sigma_2^2 = \infty$, it must be that $\alpha_2 = 0$. Now consider what happens if $\alpha_1 > 0$. In this case, the teacher places all of his efforts on task 1. Consequently, $B = 0$.

(b). We apply the tools from Holmstrom and Milgrom. The CE for the teacher is

$$CE = \alpha_1 e_1 + \alpha_2 e_2 - \psi(e_1 + e_2) - \frac{r}{2}(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2) + \beta.$$

Substituting $CE = \underline{U}$ into the school's objective function

$$B(e_1, e_2) - \alpha_1 e_1 - \alpha_2 e_2 - \beta,$$

we have

$$B(e_1, e_2) - \psi(e_1 + e_2) - \frac{r}{2}(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2).$$

Now think about the teacher's choice of effort. If $\alpha_1 > \alpha_2$, then all effort will be put into task 1 (since the derivative of CE with respect to α_1 will always be greater than that with respect to α_2). If $\alpha_2 > \alpha_1$, then all effort will be invested in task 2. In either case, $B = 0$. Thus, an optimal scheme requires $\alpha_1 = \alpha_2 = \alpha$. For any α, the teacher is indifferent about how he divides his aggregate effort across tasks, and so is willing to maximize B for the school. Given α, aggregate effort will satisfy $e = \max\{\alpha, \hat{e}\}$. (That is, either $\alpha < \hat{e}$, in which case the teacher supplies aggregate effort $e = \hat{e}$, or $\alpha > \hat{e}$, in which case the teacher's first-order condition determines $e = \alpha$.) Now we can rewrite the principal's program: choose $\alpha \geq 0$ to maximize

$$\max_{e_1} B(e_1, \max\{\alpha, \hat{e}\} - e_1) - \psi(\alpha) - \frac{r\alpha^2}{2}(\sigma_1^2 + \sigma_2^2).$$

If $\alpha = 0$, then the value of the program is

$$\max_{e_1} B(e_1, \hat{e} - e_1).$$

If $\alpha \in (0, \hat{e}]$, the benefit is unchanged from $\alpha = 0$, but the added risk raises the expected wage the school must pay. Hence, $\alpha \notin (0, \hat{e}]$. The only remaining possibility is that the program may yield a higher value for a choice of $\alpha > \hat{e}$. Whether or not it does depends upon the parameters of the model.

Also, for σ_2^2 sufficiently large, $\alpha = 0$. This can be seen by graphing $B(e_1, \max\{\alpha, \hat{e}\} - e_1)$ and $\psi(\alpha) + \frac{r\alpha^2}{2}(\sigma_1^2 + \sigma_2^2)$ as functions of α.

4 The manager's final compensation is

$$\alpha(e + \varepsilon_1) + \beta + s(2e + \varepsilon_1 + \varepsilon_2 + \eta) = (\alpha + 2s)e + (\alpha + s)\varepsilon_1 + s\varepsilon_2 + s\eta + \beta.$$

Because $\varepsilon_1, \varepsilon_2$ and eta are all independently distributed, this generates a CE of

$$CE(\alpha, \beta, s) = (\alpha + 2s)e - \frac{1}{2}e^2 - \frac{r}{2}\left((\alpha + s)^2\sigma_\varepsilon^2 + s^2\sigma_\varepsilon^2 + s^2\sigma_\eta^2\right) + \beta.$$

Maximizing with respect to e yields the necessary and sufficient condition that

$$e = (\alpha + 2s).$$

The firm maximizes

$$2e - \alpha e - s(2e) - \beta,$$

subject to $e = (\alpha + s)$ and $CE(\alpha, \beta, s) \geq \underline{U}$. Substituting for β and e, we have the reduced program,

$$\max_{\alpha, s} \; 2(\alpha + 2s) - \frac{1}{2}(\alpha + 2s)^2 - \frac{r}{2}\left(((\alpha + s)^2 + s^2)\sigma_\varepsilon^2 + s^2\sigma_\eta^2\right).$$

Differentiating with respect to α and s yields

$$\alpha^* = \frac{2\sigma_\eta^2}{2\sigma_\varepsilon^2 + \sigma_\eta^2 + r\sigma_\varepsilon^2(\sigma_\varepsilon^2 + \sigma_\eta^2)},$$

$$s^* = \frac{2\sigma_\varepsilon^2}{2\sigma_\varepsilon^2 + \sigma_\eta^2 + r\sigma_\varepsilon^2(\sigma_\varepsilon^2 + \sigma_\eta^2)}.$$

Note that if $\sigma_\eta^2 = 0$, $\alpha^* = 0$ and only the stock price is used. Here is one way to understand it. Suppose the firm could contract on x_1 and x_2. Given the variance of each output measure is identical, the firm would choose $\alpha_1 = \alpha_2$. But this contract generates the same payoffs as using the stock price, $x_1 + x_2$, and nothing else. Hence, there is no additional value to using x_1 separately. This is not the same as saying the stock price is sufficient for x_1 with respect to e. But we know (x_1, x_2) is sufficient for x_1 with respect to e, and at the optimal contract the stock price can do as well as seeing x_1 and x_2 separately.

5 (a). Consider agent i. The worker's CE is

$$CE^i(\alpha_i^i, \beta^i) = \alpha_i^i e_i - \frac{1}{2}e_i^2 - \frac{r}{2}(\alpha_i^i)^2\sigma_i^2 + \beta^i.$$

The agent's first-order condition for effort is $e_i = \alpha_i^i$. The firm maximizes $E[x_i - \alpha_i^i x_i - \beta^i]$. Noting that the agent's IR condition requires

$$\beta^i = \underline{U} - (\alpha_i^i e_i - \frac{1}{2}e_i^2 - \frac{r}{2}(\alpha_i^i)^2\sigma_i^2 + \beta^i),$$

we can substitute it into the principal's profit function (and also substitute for e_i using the agent's first-order condition), to obtain

$$(\alpha_i^i) - \frac{1}{2}(\alpha_i^i)^2 - \frac{r}{2}(\alpha_i^i)^2\sigma_i^2 - \underline{U}.$$

Maximizing over α_i^i yields

$$\alpha_i^i - \frac{1}{1 + r\sigma_i^2}.$$

This holds for each agent, $i = 1, 2$.

(b). Now agent i's CE is

$$CE^i(\alpha_1^i, \alpha_2^i\beta^i) = \alpha_i^i e_i + \alpha_j^i e_j - \frac{1}{2}e_i^2 - \frac{r}{2}\left((\alpha_i^i)^2\sigma_i^2 + (\alpha_j^i)^2\sigma_j^2 + 2\alpha_1^i\alpha_2^i\sigma_{12}\right) + \beta^i.$$

The principal's object (including both agents) is to maximize

$$E[x_1 + x_2 - (\alpha_1^1 x_1 + \alpha_2^1 x_2 + \beta^1) - (\alpha_1^2 x_1 + \alpha_2^2 x_2 + \beta^2)].$$

Substituting for β^i (using the result from the agent's CE) and substituting for the agents' first-order conditions, $e_i = \alpha_i^i$, yields

$$\alpha_1^1 - \frac{1}{2}(\alpha_1^1)^2 + \alpha_2^2 - \frac{1}{2}(\alpha_2^2)^2 - \frac{r}{2}\left((\alpha_1^1)^2\sigma_1^2 + (\alpha_2^1)^2\sigma_2^2 + 2\alpha_1^1\alpha_2^1\sigma_{12} + (\alpha_1^2)^2\sigma_1^2 + (\alpha_2^2)^2\sigma_2^2 + 2\alpha_1^2\alpha_2^2\sigma_{12}\right).$$

The first-order conditions for four α's are straightforward (thought tedious) to compute and solve:

$$\begin{aligned}
\alpha_1^1 &= \frac{1}{1+r\sigma_1^2(1-\rho^2)}, \\
\alpha_2^1 &= -\frac{\rho\sigma_2}{(1+r\sigma_1^2(1-\rho^2))\sigma_1} = -\alpha_1^1\frac{\sigma_2}{\sigma_1}\rho, \\
\alpha_2^2 &= \frac{1}{1+r\sigma_2^2(1-\rho^2)}, \\
\alpha_1^2 &= -\frac{\rho\sigma_1}{(1+r\sigma_2^2(1-\rho^2))\sigma_2} = -\alpha_2^2\frac{\sigma_1}{\sigma_2}\rho.
\end{aligned}$$

The stronger the correlation (either negative or positive), the closer α_i^i is to one (i.e., the stronger are the optimal incentives).

(c). Again we compute the agent's CE's and effort levels with these contracts:

$$CE^1 = \alpha_1^1 e_1 + \alpha_2^1 e_2 - \frac{1}{2}(e_1 - h_2)^2 - \frac{r\sigma^2}{2}\left((\alpha_1^1)^2 + (\alpha_2^1)^2\right) + \beta^1.$$

Hence, agent 1 will choose e_1 so that

$$e_1 = \alpha_1^1 + h_2.$$

Now consider agent 2's CE given that $e_1 = \alpha_1^1 + h_2$:

$$CE^2 = \alpha_1^2(\alpha_1^1 + h_2) + \alpha_2^2 e_2 - \frac{1}{2}e_2^2 - \frac{1}{2}h_2^2 - \frac{r\sigma^2}{2}\left((\alpha_1^2)^2 + (\alpha_2^2)^2\right) + \beta^2.$$

Notice that agent 2 will choose $e_2 = \alpha_2^2$ and $h_2 = \alpha_1^2$. This in turn implies that agent 1 will choose $e_1 = \alpha_1^1 + \alpha_1^2$.

Substituting for β^1 and β^2 in the principal's program, we have

$$e_1 - \frac{1}{2}(e_1 - h_2)^2 + e_2 - \frac{1}{2}e_2^2 - \frac{1}{2}h_2^2 - \frac{r\sigma^2}{2}\left((\alpha_1^1)^2 + (\alpha_2^1)^2 + (\alpha_1^2)^2 + (\alpha_2^2)^2\right).$$

Substituting for $e_1 = \alpha_1^1 + \alpha_1^2$, $e_2 = \alpha_2^2$ and $h_2 = \alpha_1^2$ yields

$$(\alpha_1^1 + \alpha_1^2) - \frac{1}{2}(\alpha_1^1)^2 + \alpha_2^2 - \frac{1}{2}(\alpha_2^2)^2 - \frac{1}{2}(\alpha_1^2)^2 - \frac{r\sigma^2}{2}\left((\alpha_1^1)^2 + (\alpha_2^1)^2 + (\alpha_1^2)^2 + (\alpha_2^2)^2\right).$$

Maximizing this over the four α's yields the following FOC's:

$$\begin{aligned}
1 - \alpha_1^1 - r\sigma^2\alpha_1^1 &= 0, \\
-\alpha_2^1 - r\sigma^2\alpha_2^1 &= 0, \\
1 - \alpha_1^2 - r\sigma^2\alpha_1^2 &= 0,
\end{aligned}$$

$$1 - \alpha_2^2 - r\sigma^2\alpha_2^2 = 0.$$

Thus,

$$\alpha_1^1 = \alpha_1^2 = \alpha_2^2 = \frac{1}{1 + r\sigma^2},$$

but

$$\alpha_2^1 = 0.$$

The reason why $\alpha_2^1 = 0$ is that there is no incentive benefit to giving player 1 a fraction of the returns on x_2; $\alpha_2^1 \neq 0$ would introduce costly risk. On the other hand, $\alpha_1^2 > 0$ because the principal wants to incentivize agent 2 to help agent 1 to produce x_1.

6 (a). The principal's problem becomes:

$$\min_{w(R,C)} \int_{R_0}^{R_1} \int_{C_0}^{C_1} w(R,C) f_R(R, a_C) f_C(C, a_C) dC dR,$$

subject to

$$\int_{R_0}^{R_1} \int_{C_0}^{C_1} u(w(R,C)) f_R(R, a_C) f_C(C, a_C) dC dR - \psi(a_C) \geq 0,$$

$$\int_{R_0}^{R_1} \int_{C_0}^{C_1} u(w(R,C)) f_R(R, a_C) f_C(C, a_C) dC dR - \psi(a_C)$$
$$\geq \int_{R_0}^{R_1} \int_{C_0}^{C_1} u(w(R,C)) f_R(R, a_R) f_C(C, a_R) dC dR - \psi(a_R),$$

where the first constraint is the individual rationality (or participation) constraint and the second constraint is the incentive constraint (assuming that a_C is the optimal implemented action). Letting λ and μ be the Kuhn-Tucker multipliers for two constraints, respectively, the first-order conditions imply

$$\frac{1}{u'(w(R,C))} = \lambda + \mu \left[1 - \frac{f_R(R, a_R)}{f_R(R, a_C)} \frac{f_C(C, a_R)}{f_C(C, a_C)}\right].$$

As R increases, $f_R(R, a_R)/f(R, a_C)$ increases, and the concavity of $u(\cdot)$ implies that w should decrease. This makes sense because we want to suppress the incentives of the agent to choose the revenue enhancing effort. Similarly, as C increases, $f_C(C, a_R)/f_C(C, a_C)$ increases, and the concavity of u implies that w should decrease. This again makes sense because we want to strengthen the incentives of the agent to choose the cost-reducing effort.

(b). In this case the principal can no longer use R as a variable in the compensation scheme. The intuition is straightforward: no compensation scheme that induces the agent to choose a_C will have $\frac{\partial w}{\partial R} > 0$; and if $\frac{\partial w}{\partial R} < 0$ for some values of R, the manager will dispose of a portion of the revenues. Therefore, we must have w independent of R. Solving the principal's new program,

$$\min_{w(C)} \int_{C_0}^{C_1} w(C) f_C(C, a_C) dC,$$

subject to

$$\int_{C_0}^{C_1} u(w(C)) f_C(C, a_C) dC - \psi(a_C) \geq 0,$$

$$\int_{C_0}^{C_1} u(w(C)) f_C(C, a_C) dC - \psi(a_C) \geq \int_{C_0}^{C_1} u(w(C)) f_C(C, a_R) dC - \psi(a_R),$$

with the resulting first-order condition

$$\frac{1}{u'(w(C))} = \lambda + \mu \left[1 - \frac{f_C(C, a_R)}{f_C(C, a_C)} \right].$$

Hence, the optimal wage schedule $w(C)$ is decreasing in costs.

(c) In this case no compensation scheme can induce the manager to exert effort level a_C. That is, only a_R is implementable.