1 Switching Regressions

1.1 Estimation via OLS

Suppose the following switching regression notation:

$$Y = Y_1 D + (1 - D) Y_0$$

and consider running a regression of Y on D.

- ▷ Since

$$Y = \underbrace{\mathbb{E}[Y_0]}_{:=\alpha} + \underbrace{(Y_1 - Y_0)}_{:=\beta_i} D + \underbrace{(Y_0 - \mathbb{E}[Y_0])}_{:=\epsilon_i}$$

we have

$$\beta_{OLS} = \frac{Cov(Y, D)}{Var[D]} = \frac{\mathbb{E}[YD] - \mathbb{E}[Y]\mathbb{E}[D]}{\mathbb{E}[D](1 - \mathbb{E}[D])}$$

- ▷ Assuming $D \perp (Y_1, Y_0)$, we have

$$\begin{aligned}
&= \frac{\mathbb{E}[(\alpha_i + \beta_i D + \epsilon_i) D] - \mathbb{E}[\alpha_i + \beta_i D + \epsilon_i]\mathbb{E}[D]}{\mathbb{E}[D](1 - \mathbb{E}[D])} \\
&= \frac{\mathbb{E}[\beta_i]\mathbb{E}[D](1 - \mathbb{E}[D])}{\mathbb{E}[D](1 - \mathbb{E}[D])} = \mathbb{E}[\beta_i] \equiv ATE
\end{aligned}$$

How about when $D = 1\,(Y_1 \geq Y_0)$?

- ▷ Start again with the expression for β_{OLS}:

$$\beta_{OLS} = \frac{Cov(Y, D)}{Var[D]} = \frac{\mathbb{E}[YD] - \mathbb{E}[Y]\mathbb{E}[D]}{\mathbb{E}[D](1 - \mathbb{E}[D])}$$

- ▷ Now it's easier to think of the switching regression notation:

$$\begin{aligned}
&= \frac{\mathbb{E}[Y_1 D + Y_0 D(1 - D)] - \mathbb{E}[Y_1 D + (1 - D) Y_0]\mathbb{E}[D]}{\mathbb{E}[D](1 - \mathbb{E}[D])} \\
&= \frac{\mathbb{E}[Y_1 | D = 1]\mathbb{E}[D] - \mathbb{E}[Y_1 | D = 1]\mathbb{E}[D]^2 - \mathbb{E}[Y_0 | D = 0](1 - \mathbb{E}[D])\mathbb{E}[D]}{\mathbb{E}[D](1 - \mathbb{E}[D])} \\
&= \mathbb{E}[Y_1 | D = 1] - \mathbb{E}[Y_0 | D = 0]
\end{aligned}$$

since

$$\begin{aligned}
\mathbb{E}[Y_1 D] &= \mathbb{E}[DY_1 | D = 1] P(D = 1) + \mathbb{E}[DY_1 | D = 0] P(D = 0) \\
&= \mathbb{E}[Y_1 | D = 1]\mathbb{E}[D]
\end{aligned}$$

In sum, we have the following result:

- ▷ $D \perp (Y_1, Y_0)$: $\beta_{OLS} = \mathbb{E}[Y_1] - \mathbb{E}[Y_0]$
- ▷ $D \not\perp (Y_1, Y_0)$: $\beta_{OLS} = \mathbb{E}[Y_1 | D = 1] - \mathbb{E}[Y_0 | D = 0]$

Another way to notice this is that $\mathbb{E}[Y_1 - Y_0]$ is a mean of a normal distribution, whereas $\mathbb{E}[Y_1 | D = 1]$ and $\mathbb{E}[Y_0 | D = 0]$ are each truncated normal variables, so you are computing two different means and taking its difference.

1.2 Using Propensity Score

Consider the following setup:

$$D = 1\left\{\tilde{u} \leq \nu\left(W\right)\right\}$$

where $W = [X, Z]'$. Applying an increasing transformation to both sides:

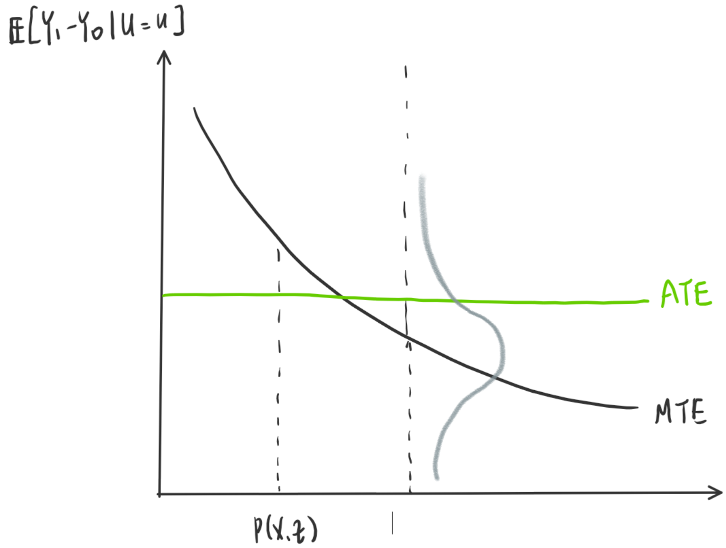
$$= 1\left\{F_{\tilde{U}}\left(\tilde{u}\right) \leq F_{\tilde{u}}\left(\nu\left(X, Z\right)\right)\right\}$$
$$= 1\left\{u \leq p\left(X, Z\right)\right\} = p\left(X, Z\right)$$

where F is a cdf of u. The last equality follows since we showed in the first quarter that $F_U\left(u\right) \sim U\left[0, 1\right]$.

Define the MTE as the following

$$MTE = \mathbb{E}^i\left[Y_1 - Y_0 | U = u\right]$$

Assuming selection on the gains (which makes the MTE downward-sloping), we can draw the following graph:



1.3 LATE Estimators

Define the LATE estimator as the following quantity:

$$LATE = \frac{\mathbb{E}^i\left[Y|Z = z\right] - \mathbb{E}^i\left[Y|Z = z'\right]}{\mathbb{E}^i\left[D|Z = z\right] - \mathbb{E}^i\left[D|Z = z'\right]}$$

The numerator:

1. Plug in the expression for Y:

$$\mathbb{E}^i\left[Y|Z = z\right] = \mathbb{E}^i\left[Y_1 D + Y_0\left(1 - D\right)|Z = z\right]$$

2. Condition on D_z:

$$= \mathbb{E}^i\left[Y_1|Z = z, D_z = 1\right]P\left(D_z = 1\right) + \mathbb{E}^i\left[Y_0|Z = z, D_z = 0\right]P\left(D_z = 0\right)$$

Since we know that

$$\mathbb{E}\left[D_z\right] = \mathbb{E}\left[1\left\{u \leq p\left(z\right)\right\}\right] = p\left(z\right)$$

we can write

$$= \mathbb{E}^i\left[Y_1|Z = z, D_z = 1\right]p\left(z\right) + \mathbb{E}^i\left[Y_0|Z = z, D_z = 0\right]\left(1 - p\left(z\right)\right)$$

3. Assuming $Z \perp Y_1, Y_0$

$$= \mathbb{E}^i \left[Y_1 | D_z = 1\right] p\left(z\right) + \mathbb{E}^i \left[Y_0 | D_z = 0\right] \left(1 - p\left(z\right)\right)$$

4. Since $D_z = 1$ corresponds to $\{U \leq p(z)\}$ and $D_z = 0$ corresponds to $\{U > p(z)\}$:

$$= \int_0^{p(z)} \mathbb{E}^i \left(Y_1 | U = u\right) p\left(z\right) \frac{1}{p\left(z\right)} du + \int_{p(z)}^1 \mathbb{E}^i \left(Y_0 | U = u\right) \left(1 - p\left(z\right)\right) \frac{1}{1 - p\left(z\right)} du$$

$$= \int_0^{p(z)} \mathbb{E}^i \left(Y_1 | U = u\right) du + \int_{p(z)}^1 \mathbb{E}^i \left(Y_0 | U = u\right) du$$

5. Therefore, the numerator is given as

$$\int_{p(z')}^{p(z)} \mathbb{E}^i \left(Y_1 | U = u\right) du - \int_{p(z')}^{p(z)} \mathbb{E}^i \left(Y_0 | U = u\right) du$$
$$= \int_{p(z')}^{p(z)} \mathbb{E}^i \left(Y_1 - Y_0 | U = u\right) du$$

The denominator:

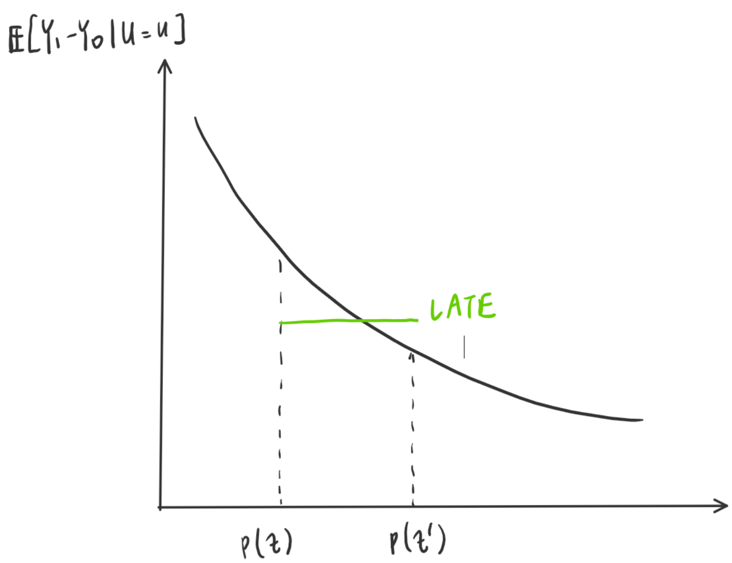
▷ Since we know that

$$\mathbb{E}\left[D_z\right] = \mathbb{E}\left[1\left\{u \leq p\left(z\right)\right\}\right] = p\left(z\right)$$

the denominator is

$$p\left(z\right) - p\left(z'\right)$$

Graphically, we have:



▷ Different instruments subsets different regions and computes the average.

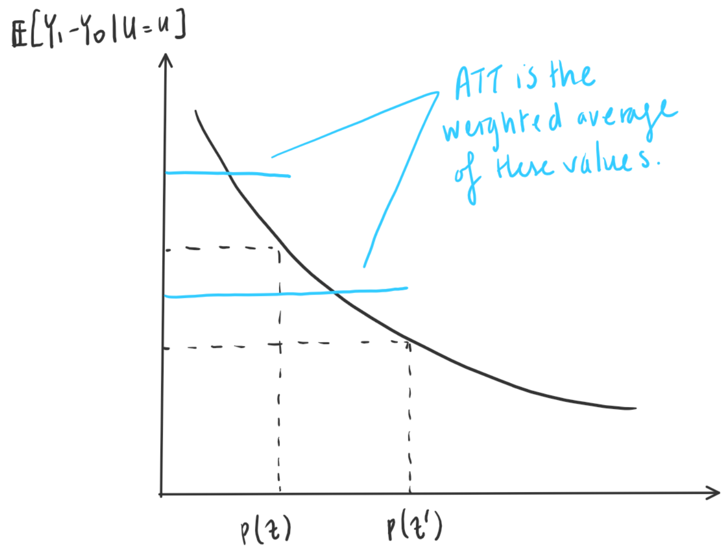
▷ If you tried different instruments and obtained similar values of LATE, then the MTE is likely to be horizontal.

1.4 Average Treatment Effect on the Treated (ATT)

Write:

$$\begin{aligned} ATT &= \mathbb{E}^i\left[Y_1 - Y_0 | D = 1\right] \\ &= \sum_z \mathbb{E}^i\left[Y_1 - Y_0 | D_z = 1\right] P\left(Z = z\right) \end{aligned}$$

Graphically, we have:



- ▷ Each blue line is $\mathbb{E}^i\left[Y_1 - Y_0 | D_z = 1\right]$ for a given value of z since $D_z = 1$ corresponds to values of u less than the propensity score.

1.5 General Equilibrium Effects

Suppose you have two different policies:

$$\begin{aligned} Y &= Y_1 D + Y_0\left(1 - D\right) \\ Y^* &= Y_1 D^* + Y_0\left(1 - D^*\right) \end{aligned}$$

It is possible that the policy affects the outcome varible Y_1.

- ▷ For example, if college tuition goes down, more people will become skilled and the skill premium will go down.
- ▷ If we ignore this, we are abstracting away from the general equilibrium effects.

Then consider:

$$PRTE = \frac{\mathbb{E}^i\left(Y^*\right) - \mathbb{E}^i\left(Y\right)}{\mathbb{E}^i\left(D^*\right) - \mathbb{E}^i\left(D\right)}$$

becomes our quantity of interest.