

Learning to Recommend Using Non-Uniform Data

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Learning user preferences for products based on their past purchases or reviews is at the cornerstone of modern recommendation engines. One complication in this learning task is that some users are more likely to purchase products or review them, and some products are more likely to be purchased or reviewed by the users. This non-uniform pattern degrades the power of many existing recommendation algorithms, as they assume that the observed data is sampled uniformly at random among user-product pairs. In addition, existing literature on modeling non-uniformity either assume user interests are independent of the products, or lack theoretical understanding. In this paper, we first model the user-product preferences as a partially observed matrix with non-uniform observation pattern. Next, building on the literature about low-rank matrix estimation, we introduce a new weighted trace-norm penalized regression to predict unobserved values of the matrix. We then prove an upper bound for the prediction error of our proposed approach. Our upper bound is a function of a number of parameters that are based on a certain *weight matrix* that depends on the joint distribution of users and products. Utilizing this observation, we introduce a new optimization problem to select a weight matrix that minimizes the upper bound on the prediction error. The final product is a new estimator, NU-Recommend, that outperforms existing methods in both synthetic and real datasets.

Key words: Personalization, Recommender systems, Collaborative filtering, Matrix completion, High-dimensional learning

1. Introduction

Recommendations are now ubiquitous in various settings such as streaming services, e-commerce and social networks. As shown in MacKenzie et al. (2013), 35 percent of what consumers purchase on Amazon and 75 percent of what they watch on Netflix come from product recommendations. Personalized recommendation, an effort to suggest to users different products so as to tailor to their different needs or tastes, has boosted views, number of sold items and overall sales for companies (Jannach and Hegelich 2009). It also vastly increases click-through and conversion rates than the untargeted content such as banner advertisements and top-seller lists (Linden et al. 2003).

Personalized recommender system is powered by methods that can estimate user preferences well. These predictive algorithms analyze information from existing data such as

ratings, purchase history and click-through rates. It has been shown that predicting one user’s preference benefits from leveraging information not just from this single user’s past experience, but also from that of other users, especially those who have similar interest to this single user. Such is the main concept behind a class of commonly used methods called collaborative filtering. Among collaborative filtering algorithms, the popular matrix completion techniques approach the recommendation task as one of recovering a matrix from its partially observed entries. The rows of the matrix correspond to users, the columns correspond to items and the entries contain user preferences.

Most of these matrix completion methods have theoretical guarantees under the assumption that the data are observed uniformly at random. Informally speaking, such assumption entails that all users will equally likely rate, purchase, or click on items and the items will equally likely be rated, purchased, or clicked on by the users. But in reality, such assumption may not hold in most scenarios. For example, in the rating case, some users are more active than others and some items are rated by many people while others are rarely rated. It is mentioned in Srebro and Salakhutdinov (2010) that the uniformly at random assumption is not just a deficiency of the proof techniques. It also leads to a significant deterioration in prediction quality and an increase in the sample complexity. Thus, these aspects point towards a need to include the non-uniform aspect of the sampling scheme when learning user preferences, so as to improve prediction accuracy.

Srebro and Salakhutdinov (2010), Foygel et al. (2011) and Ma and Chen (2019) have demonstrated empirically that if we go beyond the assumption that entries are revealed with the same probability independently, we can substantially improve the accuracy of inferring user preferences. Of course, one needs to consider carefully how to include the non-uniform sampling process in the prediction task. Let us consider an example of a popular matrix completion approach, known as trace-norm penalized regression. The objective of this problem is to predict observed entries close to the observed values and at the same time penalize a convex surrogate of the ‘matrix rank’, i.e., the trace-norm of the matrix. Srebro and Salakhutdinov (2010) suggest to correct the penalty term by marginal distributions of the rows and the columns. However, as pointed out by Foygel et al. (2011), such ‘margin-based’ weighted correction becomes suboptimal when the sampling scheme is not a product distribution which occurs when user preferences is not independent of the products. Indeed, our analysis shows that the margin weighting strategy can be further improved. Foygel et al.

(2011) propose a smoothing variant of the margin weighting strategy. However, their method requires knowledge of the rank of the matrix, which may be unknown in practice. Ma and Chen (2019) channel the inverse propensity weighting technique from the causal inference literature to modify the trace-norm penalized regression. Such weighting strategy addresses the selection bias by constructing a weighted regression based on propensity scores of the observed entries. Ma and Chen (2019) show effectiveness of weighting with the non-uniform propensity score matrix through empirical experiments, but their approach lacks theoretical guarantees.

1.1. Our approach and contributions

This paper proposes a novel algorithm that learns user preferences under a more general non-uniform sampling scheme which allows user preferences to depend on the products. We provide theoretical guarantees for our approach and also show that it outperforms existing benchmarks on synthetic and real data.

We start by introducing a general weighted estimator that includes standard trace-norm penalized regression and marginal weighted-trace-norm penalized regression as special cases. We then prove an upper error bound for this unified weighted objective under general non-uniform sampling scheme. This error bound is on how accurate the estimation of the underlying ground truth preference matrix is, given any arbitrary weighting strategy. Based on the analysis, we further refine the weighted estimator by designing a theory-driven and effective weighting strategy that leverages the distribution of observed user-product pairs. We call this algorithm NU-Recommend (‘NU’ stands for non-uniform). We then demonstrate the superior performance of NU-Recommend by comparing it with the unweighted strategy, the margin weighting strategy and the inverse propensity weighting strategy on both synthetic data and real-world data.

1.2. Other related work

Our work can be viewed as an instance of designing matrix-completion-based collaborative filtering method to learn user preferences, in order to enhance personalized recommender systems (see Aggarwal (2016) for a comprehensive overview of personalized recommendation algorithms).

There are more direct measurements of the effectiveness of recommender systems, but to improve the quality of user preference prediction is believed to be at the foundation of

enhanced recommendations. Indeed, the accuracy of predicting user preferences is very much valued such that the well-known Netflix Data Competition, launched in 2006, awarded the winner one million dollars for their smallest error on estimating how someone is going to rate a movie. Learning user preferences is also the core input for many other operational and marketing activities besides product recommendations, including personalizing search results, designing loyalty programs and delivering one-to-one marketing. We defer to Farias and Li (2010) and references therein for further discussion on this.

The adoption of collaborative filtering to learn users’ preferences dates back to the early stage of Amazon’s effort to build recommendation engines, and it is still highly sought-after (Linden et al. 2003). There is a substantial amount of literature for matrix completion models for collaborative-filtering-based recommender systems (Ramlatchan et al. 2018). Some popular approaches include matrix factorization (Rennie and Srebro 2005, Salakhutdinov and Mnih 2007, Keshavan et al. 2010a,b), rank minimization (Bunea et al. 2011, Klopp 2011) and the aforementioned trace-norm penalized regression (Candes and Recht 2009, Candes and Tao 2010, Candes and Plan 2010). In fact, matrix factorization is one of the two algorithms used by the winner of Netflix Prize, and which Netflix later chose to put into production (Masnick 2012). As mentioned before, one distinct difference between our method and these canonical works is the following: we consider a more general sampling distribution, whereas most of them assumed the restrictive uniform or margin-based sampling scheme while designing their algorithm as well as providing theoretical guarantees.

To provide properties for the general weighted-trace-norm penalized regression that we propose, we deploy techniques from high-dimensional statistics literature. A key challenge in our proof is that we assume the weighting strategy is arbitrary. Negahban and Wainwright (2012) study theoretically the marginal weighting strategy proposed by Srebro and Salakhutdinov (2010) via assuming the sampling scheme is a product distribution. In addition, they directly take the sampling matrix to be the weighting matrix, which is encompassed by our more general setting. Furthermore, our proof techniques are distinct from theirs since they used a more involved ϵ -net argument while we build on ideas developed in Klopp (2014) and Hamidi and Bayati (2019). We note that Hamidi and Bayati (2019) consider the trace-regression problem under a general sampling scheme. However, for the matrix completion problem, their approach is equivalent to the uniform sampling scheme.

We will be working under the regime of missing completely at random (MCAR) data, meaning the missingness pattern is independent of the values of the underlying ground truth preference matrix. There are less restrictive missing data mechanisms such as missing at random (MAR) and missing not at random (MNAR), but we leave such extensions for future research as adding them would make the analysis highly cumbersome. We refer the readers to Fleder and Hosanagar (2007) for a discussion on whether modeling the missing mechanism more generally is necessary, and to Sportisse et al. (2020) for their approach on blending trace-norm penalized regression with MNAR and the reference therein.

1.3. Organization

The rest of this paper is organized as follows. We describe the problem formulation in Section 2. We present the weighted-trace-norm penalized estimator and our main result on the estimator’s theoretical performance in Section 3, as well as the design of our new algorithm ‘NU-Recommend’. In Section 4, we provide a proof sketch for our main theorem. Finally, empirical results on simulated data as well as the MovieLens data and medical test data are presented in Section 5. Proofs are relegated to the appendices.

2. Model and Problem

We use bold capital letters (e.g. \mathbf{B}) for matrices and non-bold capital letters for vectors (e.g., V). For a positive integer m , we denote the set of integers $\{1, \dots, m\}$ by $[m]$. For a matrix \mathbf{B} , \mathbf{B}_{ij} refers to its entry (i, j) . We encode the ground truth user preference data in a $d_r \times d_c$ matrix called \mathbf{B}^* , and the row indices $j \in [d_r]$ of matrix \mathbf{B}^* correspond to users, the column indices $k \in [d_c]$ correspond to items. We will use rating data as a running example. That is, the (j, k) -th element \mathbf{B}_{jk}^* is the rating of user j towards item k . Suppose we have n number of observations. For $i \in [n]$, let (j_i, k_i) be the i -th observed sample and its observed value is y_i . In the rest of this section, we will specify the observation model, the sampling model and also some additional structures we impose on \mathbf{B}^* to make it recoverable.

2.1. Observation Model

We assume that y_i is the noisy realization of entry (j_i, k_i) such that $y_i = \mathbf{B}_{j_i k_i}^* + \epsilon_i$, where ϵ_i is a noise term. Let $Y = [y_1, y_2, \dots, y_n]^\top$ be the n by 1 vector of observed values and $E = [\epsilon_1, \epsilon_2, \dots, \epsilon_n]^\top$ be the vector of independent mean zero noise random variables with variance at most σ^2 . For any positive integer m , $e_1(m), e_2(m), \dots, e_m(m)$ denote the standard

basis vectors for \mathbb{R}^m . Define the design matrix $\mathbf{X}_i := e_{j_i}(d_r)e_{k_i}^\top(d_c)$, then $\mathbf{B}_{j_i k_i}^* = \langle \mathbf{B}^*, \mathbf{X}_i \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the trace inner product of two matrices such that

$$\langle \mathbf{B}_1, \mathbf{B}_2 \rangle := \text{Tr}(\mathbf{B}_1 \mathbf{B}_2^\top).$$

To make the notation more succinct, let us define a sampling operator which takes in a matrix \mathbf{B} and outputs a vector of n dimension (i.e. $\mathfrak{X}: \mathbb{R}^{d_r \times d_c} \rightarrow \mathbb{R}^n$). The operator takes the following form:

$$[\mathfrak{X}(\mathbf{B})]_i := \langle \mathbf{B}, \mathbf{X}_i \rangle.$$

Elements of this vector are the entries of \mathbf{B} at n observed locations. Our data generating model is then given by

$$Y = \mathfrak{X}(\mathbf{B}^*) + E.$$

This model is called the trace regression model (Hastie et al. 2015). Our estimation problem is that of estimating \mathbf{B}^* , having observed Y and the design matrices \mathbf{X}_i , $i \in [n]$.

2.2. General Sampling Model

To model the sampling distribution, let \mathbf{P}^* encode the rating probabilities where \mathbf{P}_{jk}^* denotes the probability that the (j, k) -th entry is observed, i.e. probability that user j will rate item k , and $\sum_{j \in [d_r]} \sum_{k \in [d_c]} \mathbf{P}_{jk}^* = 1$. Then the observation pattern could be viewed as a “noisy realization” of this underlying sampling matrix. The uniform sampling assumption made in the past literature means that \mathbf{P}_{jk}^* are all equal for $j \in [d_r]$ and $k \in [d_c]$. This is equivalent to say that each user will equally likely rate different items and each item will be rated equally likely by different users, which usually is not the case. Instead, we want to consider a general sampling matrix such that entries in \mathbf{P}^* are not necessarily equal to each other. Equivalently, this is saying that each \mathbf{X}_i is sampled independently from a distribution Π on the set of canonical basis matrices for $\mathbb{R}^{d_r \times d_c}$: $\left\{ e_j(d_r)e_k^\top(d_c), 1 \leq j \leq d_r, 1 \leq k \leq d_c \right\}$, such that

$$\mathbf{P}_{jk}^* = \mathbb{P}\left(\mathbf{X} = e_j(d_r)e_k^\top(d_c)\right) = \Pi_{jk}.$$

We further assume that each element is sampled with positive probability and the probabilities are bounded.

ASSUMPTION 1. *There exist positive constants p_{\min} such that*

$$p_{\min} \leq \mathbf{P}_{jk}.$$

Note that $p_{\min} \leq \frac{1}{d_r d_c}$. In the case of uniform sampling scheme, $p_{\min} = \frac{1}{d_r d_c}$.

A less general weighted sampling model considered in literature is to only assume a marginal probability on observing each user and a marginal probability on observing each item, denoted by $R \in \mathbb{R}^{d_r}$ and $C \in \mathbb{R}^{d_c}$ respectively. They are also called row marginal probability and column marginal probability. To be more specific, R_j is the probability that user j is sampled and C_k is the probability that item k is sampled, and $\sum_{j \in [d_r]} R_j = 1$, $\sum_{k \in [d_c]} C_k = 1$. Then the probability of observing entry (j, k) is $R_j C_k$. Note that, if the underlying entry-wise sampling distribution \mathbf{P}^* is RC^\top , i.e. a product distribution, then the marginal model is the same as the general sampling model we have proposed, since $\mathbf{P}_{jk}^* = R_j C_k$, $R_j = \sum_{k \in [d_c]} \mathbf{P}_{jk}^*$ and $C_k = \sum_{j \in [d_r]} \mathbf{P}_{jk}^*$. However, this sampling model is limiting \mathbf{P}^* as a product distribution, i.e. a rank-1 sampling matrix, which is a rather restrictive assumption. Consider the following example when \mathbf{P}^* is not a product distribution: $\mathbf{P}^* = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{bmatrix}$. \mathbf{P}^* is a rank-2 matrix and the entries are not uniformly sampled. The row marginal probability $R = [0.5, 0.5]$ and column marginal probability $C = [0.5, 0.5]$. According to the marginal sampling model, the probability of observing each entry is 0.25, thus each entry is assumed to be uniformly sampled, which is not true. As such, assuming an entry-wise probability covers more broad cases than only assuming the marginal probability.

2.3. Low-rankness

Low-rankness of the preference matrix. In reality, users typically rate only very few items so that there are very few scattered observed entries of this ground truth matrix. That is, the number of observations n is comparably much smaller than the total number of entries $d_r d_c$, which is also the number of parameters needed to be recovered. Therefore, without any additional structure, the estimation problem is *high dimensional*; it is an under-determined system of equations with many solutions which would overfit the observed data and the noise. In many instances, the matrix we wish to recover is known to be low-rank. We recall that, by definition, a matrix of $d_r \times d_c$ has rank r if its rows or columns span an r -dimensional space, and it has roughly $r(d_r + d_c)$ free parameters. We only need a sample size of this order (up to logarithmic factors) to control for the recovery error. Hence a small r , i.e. low-rankness, is very desirable in such high-dimensional estimation problems. Intuitively, the low rank structure entails that only a few factors contribute to a user's tastes or preferences. As such, we have turned the problem into a low-rank matrix completion problem, for which many methods have been designed.

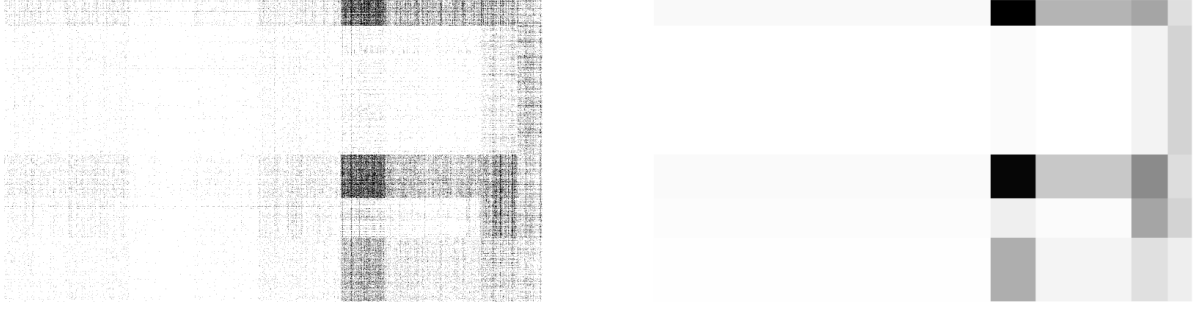


Figure 1 Missingness matrix for MovieLens-100k dataset. Black indicates an entry being observed. On the left is the missingness matrix and on the right is its corresponding block structure identified using spectral biclustering mentioned in Yuval Kluger and Gerst (2003); rows and columns have been rearranged based on the biclustering result.

Low-rankness of the sampling matrix. Sampling-based observation pattern with a low rank structure has been studied recently. In Ma and Chen (2019), they present a missingness matrix of the MovieLens-100k dataset (see Figure 1) where the rows are users, columns are movies, and a black dot indicates an observed entry. Such missingness pattern with few block structures suggests that the observations are generated from a low rank sampling matrix \mathbf{P}^* . This structure can be very useful while estimating the sampling distribution.

2.4. Spikiness

It is also known that low-rank structure alone is not enough to recover the matrix. Consider a matrix that is everywhere else equal to zero except for a single entry that is equal to one. Then it is impossible to recover such spiky matrix unless the entry which contains the single one is observed. Therefore, the recoverability of a matrix also depends on its spikiness. Hamidi and Bayati (2019) define a more natural and less restrictive way to formalize notion of spikiness by the following form:

$$\text{spikiness of } \mathbf{B} := \frac{\mathfrak{N}(\mathbf{B})}{\|\mathbf{B}\|_F},$$

where $\mathfrak{N}(\mathbf{B})$ is some norm of \mathbf{B} that depends on the distribution of the design matrices. We will use this notion of spikiness and provide specific form of $\mathfrak{N}(\cdot)$ in our analysis.

2.5. Our Problem

Our task of estimating \mathbf{B}^* based on the noisy observations Y is combined with the assumptions that the sampling model \mathbf{P}^* is a general one, both \mathbf{B}^* and \mathbf{P}^* are low rank and that

the ground truth matrix \mathbf{B}^* satisfies certain “spikiness” condition. The goal is to construct an estimator $\hat{\mathbf{B}}$ to minimize some loss function with respect to \mathbf{B}^* . A common loss function is the Frobenius error:

$$\|\mathbf{B}^* - \hat{\mathbf{B}}\|_F$$

where $\|\mathbf{B}\|_F := \sqrt{\sum_{(i,j) \in [d_r] \times [d_c]} \mathbf{B}_{ij}^2}$ for a matrix $\mathbf{B} \in \mathbb{R}^{d_r \times d_c}$. However, this error does not reflect the non-uniform sampling nature of the problem since it is putting equal weights to the errors of all entries regardless of each entry’s sampling probability. Instead, we study the convergence properties of the $L^2(\Pi)$ error:

$$\|\mathbf{B}^* - \hat{\mathbf{B}}\|_{L^2(\Pi)}$$

where $\|\mathbf{B}\|_{L^2(\Pi)} := \sqrt{\mathbb{E}[\langle \mathbf{B}, \mathbf{X} \rangle^2]}$, in which the expectation is taken with respect to a probability measure Π on $\mathbb{R}^{d_r \times d_c}$ from which we sample the random matrix \mathbf{X} . This loss function will weigh the error of each entry according to the sampling distribution, and it reflects the test RMSE more readily. Empirically, for the synthetic data, we will present both the relative frobenius error $\frac{\|\mathbf{B}^* - \hat{\mathbf{B}}\|_F}{\|\mathbf{B}^*\|_F}$ and the relative $L^2(\Pi)$ error $\frac{\|\mathbf{B}^* - \hat{\mathbf{B}}\|_{L^2(\Pi)}}{\|\mathbf{B}^*\|_{L^2(\Pi)}}$ for performance comparison; for the real data, since the ground truth matrix \mathbf{B}^* is not known, we will present the test RMSE

$$\sqrt{\frac{1}{n_{te}} \sum_{(j,k) \in \Omega_{te}} (\mathbf{B}_{jk}^* - \hat{\mathbf{B}}_{jk})^2},$$

where n_{te} is the number of test data and Ω_{te} denotes the set of (row, column) indices of test entries.

3. NU-Recommend Algorithm

The weighted matrix completion algorithm NU-Recommend we propose builds on the aforementioned trace-norm penalized regression, which solves the following convex program:

$$\min_{\mathbf{B} \in \mathbb{R}^{d_r \times d_c}} \frac{1}{n} \|\mathbf{Y} - \mathfrak{X}(\mathbf{B})\|_2^2 + \lambda \|\mathbf{B}\|_*. \quad (3.1)$$

The trace norm (also called the nuclear norm) of a matrix $\mathbf{B} \in \mathbb{R}^{d_r \times d_c}$ is defined to be the sum of its singular values, i.e., $\|\mathbf{B}\|_* := \sum_{i \in [r]} \mathbf{D}_{ii}$ where \mathbf{D} is a $r \times r$ diagonal matrix in the singular value decomposition (SVD) of $\mathbf{B} = \mathbf{U} \mathbf{D} \mathbf{V}^\top$ and r denotes the rank of \mathbf{B} . In this formulation, the potential to incorporate information about the sampling distribution is

missing. To address this shortcoming, Srebro and Salakhutdinov (2010) propose the following margin-weighted trace-norm penalized regression:

$$\min_{\mathbf{B} \in \mathbb{R}^{d_r \times d_c}} \frac{1}{n} \|Y - \mathfrak{X}(\mathbf{B})\|_2^2 + \lambda \|\text{diag}(R) \mathbf{B} \text{diag}(C)\|_*, \quad (3.2)$$

where $\text{diag}(R)$ is a diagonal matrix with the row marginal probability R lying on the diagonal and $\text{diag}(C)$ is a diagonal matrix with the column marginal probability C lying on the diagonal. The term inside the nuclear norm $\text{diag}(R) \mathbf{B} \text{diag}(C)$ is equivalent to weighting the matrix \mathbf{B} by a matrix \mathbf{W} element-wise where $\mathbf{W} = RC^\top$, i.e. $\mathbf{W} \circ \mathbf{B}$. Here, \circ denotes the element-wise product (Hadamard product) between two matrices. Negahban and Wainwright (2012) analyze such rank-1 weighting correction under the scenario where the sampling matrix is also the rank-1 matrix RC^\top . Our goal is to generalize the formulation under general sampling schemes to the following convex program that we call the weighted-trace-norm penalized regression, where \mathbf{W} is not specified at this point yet:

$$\min_{\mathbf{B} \in \mathbb{R}^{d_r \times d_c}} \frac{1}{n} \|Y - \mathfrak{X}(\mathbf{B})\|_2^2 + \lambda \|\mathbf{W} \circ \mathbf{B}\|_*. \quad (3.3)$$

Note that if we take \mathbf{W} to be a matrix of all ones, then this formulation reduces to (3.1). So our formulation covers both the regular trace-norm penalized regression and the margin-weighted trace-norm penalized regression.

Under a general sampling scheme \mathbf{P}^* , do we simply match the weight matrix \mathbf{W} to be the sampling matrix \mathbf{P}^* , or is there some other way to design a weight matrix which delivers better performance? The answer becomes obvious through the recovery guarantee of this formulation we present next. But clearly, we want the weight matrix \mathbf{W} to inherit some information about the sampling matrix \mathbf{P}^* and to be not too far off from \mathbf{P}^* . To measure the closeness between \mathbf{W} and \mathbf{P}^* , we introduce the following scalar term l :

DEFINITION 1. We define l , which is a function of a weight matrix \mathbf{W} and the sampling matrix \mathbf{P}^* , to be the smallest constant that satisfies

$$\frac{1}{l} \leq \frac{\mathbf{P}_{ij}^*}{\mathbf{W}_{ij}} \leq l.$$

With this definition, making \mathbf{W} close to \mathbf{P}^* translates into picking a weight matrix \mathbf{W} so that l is not too large.

3.1. Main Result: Recovery Guarantee of Weighted Objective

We assume that $d_r = d_c = d$ to simplify the notation, but the statement also holds for $d_r \neq d_c$. Let $n^* = d^2 \|\sqrt{\mathbf{W}} \circ \mathbf{B}\|_\infty$ and \tilde{r} denote the rank of $\mathbf{N}^* = \sqrt{\mathbf{W}} \circ \mathbf{B}^*$. Our main result states that the weighted objective produces an estimator that is consistent and that the error bound is an increasing function in l , n^* and \tilde{r} .

THEOREM 1. *Let σ^2 be the noise variance. We have:*

$$\|\hat{\mathbf{B}} - \mathbf{B}^*\|_{L^2(\Pi)}^2 \leq C_1(\sigma^2 \vee n^{*2}) \frac{d\rho l^4 \tilde{r}}{np_{\min}},$$

with probability at least $1 - \exp(-C_2\rho)$ for some $\rho \geq \log d$ and constants C_1, C_2 and C_3 , whenever $n \geq C_3(\log \frac{l^2}{p_{\min}d})^2 \frac{\log d}{p_{\min}d}$.

When \mathbf{W} and \mathbf{P} are both equal to the matrix with all entries equal to $1/d^2$, i.e. $p_{\min} = \frac{1}{d^2}$ and $l = 1$, then this result reduces to the bound analyzed by Hamidi and Bayati (2019) for the regular (non-weighted) trace-norm penalized regression under uniform sampling scheme.

3.2. Weight Matrix Construction

As shown in the upper bound, the error indeed increases in l . Although taking $\mathbf{W} = \mathbf{P}^*$ makes $l = 1$, the smallest possible, there are additional terms that depend on \mathbf{W} . One such term is \tilde{r} , which is also a function of \mathbf{W} ; so is the term n^* that controls the spikiness of the weighted ground truth matrix. Motivated by this, we want to minimize these three terms l , \tilde{r} and n^* . From the low rank matrix completion literature, minimizing \tilde{r} can be well approximated by minimizing its convex surrogate $\|\sqrt{\mathbf{W}} \circ \mathbf{B}^*\|_*$. While minimizing this nuclear norm, we add bounds on l and n^* to keep them small. Now by introducing the variable $\mathbf{Q} = \sqrt{\mathbf{W}}$, we propose the following optimization problem where the square root is taken element-wise and l and γ are hyper-parameters:

$$\begin{aligned} & \min_{\mathbf{Q}} \|\mathbf{Q} \circ \mathbf{B}^*\|_* \\ & \text{subject to } \frac{1}{l} \leq \frac{\mathbf{Q}}{\sqrt{\mathbf{P}^*}} \leq l \text{ and } \|\mathbf{Q} \circ \mathbf{B}^*\|_\infty \leq \gamma. \end{aligned} \tag{3.4}$$

Once a solution \mathbf{Q} to the above minimization is found, one can take \mathbf{W} to be the normalized version of \mathbf{Q}^2 , where \mathbf{Q}^2 denotes elementwise square of \mathbf{Q} , i.e.

$$\mathbf{W} = \frac{\mathbf{Q}^2}{\sum_{jk} (\mathbf{Q}_{jk}^2)}.$$

Of course, in reality, we do not have access to the ground truth \mathbf{B}^* and \mathbf{P}^* . Instead, we will use two estimates of $\hat{\mathbf{B}}$ and $\hat{\mathbf{P}}$ and plug them in the following program:

$$\begin{aligned} & \min_{\mathbf{Q}} \|\mathbf{Q} \circ \hat{\mathbf{B}}\|_* \\ & \text{subject to } \frac{1}{l} \leq \frac{\mathbf{Q}}{\sqrt{\hat{\mathbf{P}}}} \leq l \text{ and } \|\mathbf{Q} \circ \hat{\mathbf{B}}\|_\infty \leq \gamma. \end{aligned} \quad (3.5)$$

Our NU-Recommend Algorithm first constructs a matrix \mathbf{W} with Program (3.5) and then plugs it in the weighted objective (3.3) to obtain the estimator $\hat{\mathbf{B}}$.

3.3. Estimation of the Sampling Matrix

As we have specified in our sampling model, the sampling matrix \mathbf{P}^* does not necessarily have equal entries and it has a low rank structure. We can estimate it through the design matrices \mathbf{X}_i , $i \in [n]$, by letting $\mathbf{M} \in \mathbb{R}^{d_r \times d_c}$ denote the observation matrix which records how many times each entry is observed. That is, $\mathbf{M}_{jk} = \sum_{i \in [n]} \mathbb{1}\left\{\mathbf{X}_i = e_j(d_r)e_k^\top(d_c)\right\}$. We introduce two ways to estimate the sampling matrix: rank-1 estimation and Poisson matrix recovery.

3.3.1. Rank-1 estimation

As mentioned in Srebro and Salakhutdinov (2010), if we assume that \mathbf{P}^* is rank-1, then we can first estimate the marginal probability \hat{R} and \hat{C} by the following formula:

$$\hat{R}_j = \sum_{k \in [d_c]} \mathbf{M}_{jk} / n \quad \text{and} \quad \hat{C}_k = \sum_{j \in [d_r]} \mathbf{M}_{jk} / n.$$

Then the rank-1 estimation of \mathbf{P}^* is $\hat{\mathbf{P}} = \hat{R}\hat{C}^\top$. However, this estimation may not be accurate enough as it is simplifying the rank of \mathbf{P}^* to 1. Next, we present a more sophisticated method that does not restrict the rank of \mathbf{P}^* to 1.

3.3.2. Estimation through Poisson matrix recovery

Poisson model. We model the entries in the observation matrix with Poisson model where these entries are Poisson counts of the product of the corresponding entry in the sampling matrix and the number of samples, i.e., $\mathbf{M}_{jk} \sim \text{Poisson}(n \mathbf{P}_{jk}^*)$. By definition of Poisson distribution, this is to say that each entry (j, k) in expectation has $n \mathbf{P}_{jk}^*$ number of observations when the sample size is n and

$$\mathbb{P}(\mathbf{M}_{jk} = k) = \frac{(n \mathbf{P}_{jk}^*)^k e^{-(n \mathbf{P}_{jk}^*)}}{k!}.$$

Poisson matrix recovery. Note that we are trying to recover the low-rank sampling matrix based on the realizations of the entries. Thus, this is a low-rank matrix recovery problem. We adapt a Poisson matrix recovery method called PMLSVT introduced in Cao and Xie (2016), which is a variant of the proximal gradient descent method applied to the following objective:

$$\min_{\mathbf{P}} - \sum_{(i,j) \in \Omega} (\mathbf{M}_{ij} \log \mathbf{P}_{ij} - \mathbf{P}_{ij}) + \lambda \|\mathbf{P}\|_*,$$

where Ω is the set of indices of observed entries. Note that this objective is essentially a nuclear norm penalized maximum log-likelihood estimator for the Poisson model. Algorithm 1 presents the pseudo-code for PMLSVT. In this algorithm, $\mathcal{P}(\mathbf{Z}) = \frac{I}{\text{sum}(\mathbf{Z}_+)} \mathbf{Z}_+$, where $I = n$,

Algorithm 1 Low-rank sampling matrix estimation using PMLSVT

- 1: Initialize: The maximum number of iterations K , parameters η and t . $\mathbf{X} \leftarrow \mathbf{M}$.
 - 2: **for** $\lambda = \lambda_1, \lambda_2, \dots, \lambda_L$ **do**
 - 3: **for** $k = 1, 2, \dots, K$ **do**
 - 4: $\mathbf{C} \leftarrow \mathbf{X} - (1/t) \nabla f(\mathbf{X})$
 - 5: $\mathbf{C} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$ {singular value decomposition}
 - 6: $[\mathbf{\Sigma}]_{ii} \leftarrow ([\mathbf{\Sigma}]_{ii} - \lambda/t)_+, i = 1, \dots, d$
 - 7: $\mathbf{X}' \leftarrow \mathbf{X}$ {record previous step}
 - 8: $\mathbf{X} \leftarrow \mathcal{P}(\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T)$
 - 9: If $f(\mathbf{X}) > f(\mathbf{X}')$, then $t \leftarrow \eta t$, go to 4.
 - 10: If $|f(\mathbf{X}) - f(\mathbf{X}')| < 10^{-5}$ then exit current iteration and go to the next iteration;
 - 11: **end for**
 - 12: $\hat{\mathbf{P}}_\lambda = \mathbf{X} / \text{sum}(\mathbf{X})$.
 - 13: **end for**
-

the number of samples. Cost function $f(\mathbf{X}) = \text{is-non-zero}(\mathbf{M}) \circ \mathbf{X} - \mathbf{M} \circ \log(\mathbf{X})$.

3.4. A Proximal Gradient Descent for the Weighted Objective

Now that we have obtained a weight matrix which makes the upper error bound small, we will be done by plugging it into Program (3.3). Although Program (3.3) is a convex program and can be solved by casting it into a semidefinite programming (SDP) problem, SDP solvers cannot usually solve the problem when d_r and d_c are both much larger than

100, as discussed in Ma et al. (2011). Many iterative methods have been proposed to remedy the large-scale nature of these problems. For example, the algorithm of Ma et al. (2011) is a proximal gradient method. We adapt their algorithm to our weighted objective by change of variable and we expect their proof technique to be easily applicable to this generalization. In this section, we present our iterative solver which is based on the projected gradient descent method. We first introduce a useful transformation that helps apply the projected gradient descent method more readily.

Consider the following mapping: $\mathbf{B} \rightarrow \mathbf{N} := \sqrt{\mathbf{W}} \circ \mathbf{B}$. Consequently, let the modified observation operator be:

$$[\tilde{\mathfrak{X}}(\mathbf{N})]_i = \langle \mathbf{N}, \tilde{\mathbf{X}}_i \rangle \text{ for all } i \in [n],$$

where $\tilde{\mathbf{X}}_i = \sqrt{\mathbf{W}}^{-1} \circ \mathbf{X}_i$. Note that $\tilde{\mathfrak{X}}(\mathbf{N}) = \mathfrak{X}(\mathbf{B})$ by construction.

Then Problem (3.3) can be formulated as the following:

$$\min_{\mathbf{N} \in \mathbb{R}^{d_r \times d_c}} \tilde{\mathcal{L}}(\mathbf{N}) := \frac{1}{n} \|Y - \tilde{\mathfrak{X}}(\mathbf{N})\|_2^2 + \lambda \|\mathbf{N}\|_*. \quad (3.6)$$

Notice that Problem (3.6) resembles the trace-norm penalized regression in Problem (3.1), which can be solved by the projected gradient descent method to deliver estimator $\hat{\mathbf{N}}$. We can then report the estimator for \mathbf{B}^* as $\hat{\mathbf{B}} = \sqrt{\mathbf{W}}^{-1} \circ \hat{\mathbf{N}}$.

To proceed with the projected gradient descent method to solve (3.6), let us denote $g(\mathbf{N}) = \frac{1}{n} \|Y - \tilde{\mathfrak{X}}(\mathbf{N})\|_2^2$ and $h(\mathbf{N}) = \lambda \|\mathbf{N}\|_*$, then $\tilde{\mathcal{L}}(\mathbf{N}) = g(\mathbf{N}) + h(\mathbf{N})$. Note that g is a differentiable function whose gradient is $\nabla g(\mathbf{N}) = \frac{2}{n} \tilde{\mathfrak{X}}^*(\tilde{\mathfrak{X}}(\mathbf{N}) - Y)$, where $\tilde{\mathfrak{X}}^*$ denotes the dual operator.

Recall that the proximal mapping is defined to be:

$$\text{prox}_{h,t}(\mathbf{N}) = \arg \min_{\mathbf{M}} \frac{1}{2t} \|\mathbf{N} - \mathbf{M}\|_2^2 + h(\mathbf{M}),$$

and the generalized gradient of $\tilde{\mathcal{L}}(\mathbf{N})$, denoted as $G_t(\mathbf{N})$, is:

$$G_t(\mathbf{N}) = \frac{\mathbf{N} - \text{prox}_{h,t}(\mathbf{N} - t \nabla g(\mathbf{N}))}{t}.$$

In our case, by Lemma 9 proved in Appendix E, we have

$$\text{prox}_{h,t}(\mathbf{N}) = \arg \min_{\mathbf{M}} \frac{1}{2t} \|\mathbf{N} - \mathbf{M}\|_2^2 + h(\mathbf{M}) = S_{\lambda t}(\mathbf{N})$$

where $S_\lambda(\mathbf{N})$ is the matrix soft-thresholding operator: $S_\lambda(\mathbf{N}) = \mathbf{U} \text{diag}(S_\lambda(\text{diag}(\boldsymbol{\Sigma}))) \mathbf{V}^\top$ and where $\mathbf{N} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^\top$ is the SVD. Now we are ready to write out the proximal-gradient-descent-based Algorithm 2 to solve Problem (3.6).

Algorithm 2 Iterative solver for weighted matrix completion

- 1: Initialize $\mathbf{N} = 0$. Choose step size shrinkage parameter $0 < \beta < 1$, initial step size t_{init} , and tolerance parameter tol .
- 2: Do for tuning parameter $\lambda = \lambda_1 > \lambda_2 > \dots > \lambda_L$:

a) Repeat:

(i) Store previous value: $\mathbf{N}_{old} := \mathbf{N}$.

(ii) Select step size through line search. Initialize $t := t_{init}$. If

$$g(\mathbf{N} - t_{init} G_{t_{init}}(\mathbf{N})) > g(\mathbf{N}) - t_{init} \langle \nabla g(\mathbf{N}), G_{t_{init}}(\mathbf{N}) \rangle + \frac{t_{init}}{2} \|G_{t_{init}}(\mathbf{N})\|_F^2,$$

then, while

$$g(\mathbf{N} - t G_t(\mathbf{N})) > g(\mathbf{N}) - t \langle \nabla g(\mathbf{N}), G_t(\mathbf{N}) \rangle + \frac{t}{2} \|G_t(\mathbf{N})\|_F^2,$$

shrink $t := \beta t$. Else, while

$$g(\mathbf{N} - t G_t(\mathbf{N})) \leq g(\mathbf{N}) - t \langle \nabla g(\mathbf{N}), G_t(\mathbf{N}) \rangle + \frac{t}{2} \|G_t(\mathbf{N})\|_F^2,$$

enlarge $t := \frac{1}{\beta} t$. After exiting this forthtracking while loop, return to the second to the last step size $t := \beta t$.

(iii) Update: $\mathbf{N} := \mathbf{N} - t G_t(\mathbf{N})$.

until stopping criterion is satisfied, i.e. $\|\mathbf{N}_{old} - \mathbf{N}\|_F^2 \leq tol$.

b) Record $\mathbf{N}_\lambda = \mathbf{N}$.

- 3: Output the sequence of solutions $\sqrt{\mathbf{W}}^{-1} \circ \mathbf{N}_{\lambda_1}, \dots, \sqrt{\mathbf{W}}^{-1} \circ \mathbf{N}_{\lambda_L}$.
-

We have shown how to estimate \mathbf{B}^* and \mathbf{P}^* in order to construct the desired weight matrix \mathbf{W} , which is then used in our weighted-trace-norm penalized estimator. We have also outlined a fast iterative method to obtain the estimator. Next, we sketch some proof ideas of the theorem that prompts us to use Program (3.5) to construct the weight matrix.

4. Key Steps of the Analysis of Weighted Matrix Completion

In this section, we outline the proof strategy for Theorem 1. The proof primarily consists of three main steps and the details of each step are provided in the appendices. Here is a high-level overview of them. First, we begin by proving a deterministic upper bound for the estimation error under some conditions on the observation processes. Second, we show

that these conditions in fact hold with high probability. Third, we combine the two previous results, and with some algebra, derive concrete error bounds with appropriate constant terms.

Before we delve into the deterministic result, we need the following assumption that controls the spikiness of the matrix by bounding $\mathfrak{N}(\sqrt{\mathbf{W}} \circ \mathbf{B}^*)$:

ASSUMPTION 2. *Assume that $\mathfrak{N}(\sqrt{\mathbf{W}} \circ \mathbf{B}^*) \leq n^*$ for some $n^* > 0$.*

Step 1 (A deterministic bound). With the mapping we introduce in Section 3.4, we can directly apply Theorem 3.1 from Hamidi and Bayati (2019) to derive the following deterministic result.

PROPOSITION 1 (**Theorem 3.1 from Hamidi and Bayati (2019)**). *Define $\tilde{\eta} = 72\tilde{\mathfrak{r}}$. Assume that, with constants $\tilde{\alpha} = \tilde{\alpha}(\tilde{\mathfrak{X}})$ and $\tilde{\beta} = \tilde{\beta}(\tilde{\mathfrak{X}})$, for all $\Delta \in \mathcal{C}(\tilde{\nu}, \tilde{\eta}) := \{\Delta \in \mathbb{R}^{d \times d} \mid \mathfrak{N}(\Delta) = 1, \|\Delta\|_F \geq \tilde{\nu}, \|\Delta\|_* \leq \sqrt{\tilde{\eta}}\|\Delta\|_F\}$, we have*

$$\frac{\|\tilde{\mathfrak{X}}(\Delta)\|_2^2}{n} \geq \tilde{\alpha}(\tilde{\mathfrak{X}})\|\Delta\|_F^2 - \tilde{\beta}(\tilde{\mathfrak{X}}). \quad (4.1)$$

Additionally, we assume that $\tilde{\lambda}$ is chosen such that

$$\tilde{\lambda} \geq 3\|\tilde{\Sigma}\|_{\text{op}}, \quad (4.2)$$

where $\tilde{\Sigma} := \frac{1}{n} \sum_{i=1}^n \epsilon_i \tilde{\mathbf{X}}_i$. Then for any matrix $\hat{\mathbf{B}}$ such that $\hat{\mathbf{N}} = \sqrt{\mathbf{W}} \circ \hat{\mathbf{B}}$ satisfies $\tilde{\mathcal{L}}(\hat{\mathbf{N}}) \leq \tilde{\mathcal{L}}(\mathbf{N}^)$, we have*

$$\|\hat{\mathbf{N}} - \mathbf{N}^*\|_F^2 \leq \left(\frac{100\tilde{\lambda}^2\tilde{\mathfrak{r}}}{3\tilde{\alpha}^2} + \frac{8n^{*2}\tilde{\beta}}{\tilde{\alpha}} \right) \vee 4n^{*2}\tilde{\nu}.$$

In the above deterministic result, Condition (4.1) and Condition (4.2) are assumptions that will be proved to hold with high probability with one additional distribution assumption on Π :

ASSUMPTION 3. *There exists $\mathfrak{c} > 0$ such that*

$$\mathbb{E} \left[\langle \mathbf{X}, \mathbf{B} \rangle^2 \cdot \mathbb{I}(\|\langle \mathbf{X}, \mathbf{B} \rangle\| \leq \mathfrak{c}) \right] \geq \frac{1}{2} \mathbb{E}[\langle \mathbf{X}, \mathbf{B} \rangle^2]$$

for all \mathbf{B} such that $\mathfrak{N}(\sqrt{\mathbf{W}} \circ \mathbf{B}) \leq 1$, where the expectations are with respect to Π .

Step 2 (A probabilistic bound). Note that Condition (4.1) is called the restricted strong convexity condition (RSC) satisfied by the observation operator $\tilde{\mathfrak{X}}$ over the set $\mathcal{C}(\tilde{\nu}, \tilde{\eta})$. This condition goes back to the work of Negahban and Wainwright (2012), which makes possible the derivation of the non-asymptotic error bounds for matrix estimation problems. Intuitively, if we consider the square loss objective $\frac{1}{2n}\|Y - \tilde{\mathfrak{X}}(\mathbf{N})\|_2^2$, then the Hessian matrix of this function is given by $\tilde{\mathfrak{X}}^* \tilde{\mathfrak{X}}/n$. The RSC condition implies that the quadratic loss is strongly convex in a restricted set \mathcal{C} of directions $\mathbf{\Delta}$. That is to say, the observation operator captures a substantial component of a set of matrices whose spikiness and low-rankness are controlled as in $\mathcal{C}(\tilde{\nu}, \tilde{\eta})$. The next result shows that the RSC condition holds with high probability for the modified observation operator $\tilde{\mathfrak{X}}$:

LEMMA 1 (RSC). *With probability greater than $1 - 2\exp(-\frac{Cn\tilde{\nu}d^2}{\epsilon^2 l})$, the inequality*

$$\frac{\|\tilde{\mathfrak{X}}(\mathbf{\Delta})\|_2^2}{n} \geq \frac{d^2}{4l} \|\mathbf{\Delta}\|_F^2 - 93 \frac{l}{d^2} \tilde{\eta} \epsilon^2 \mathbb{E}[\|\tilde{\mathbf{\Sigma}}_R\|_{\text{op}}]^2$$

holds for all $\mathbf{\Delta} \in \mathcal{C}(\tilde{\nu}, \tilde{\eta})$, where $C > 0$ is an absolute constant, provided that $Cn\tilde{\nu} > l\epsilon^2$, and $\tilde{\mathbf{\Sigma}}_R := \frac{1}{n} \sum_{i=1}^n \zeta_i \tilde{\mathbf{X}}_i$ where $\{\zeta_i\}_{i=1}^n$ is an i.i.d. sequence with Rademacher distribution.

Now, by denoting the threshold for $\tilde{\lambda}$ by $\tilde{\lambda}_1 := C\mathbb{E}[\|\tilde{\mathbf{\Sigma}}_R\|_{\text{op}}]n^*\epsilon$ and defining

$$\tilde{\alpha} := \frac{d^2}{4l}, \quad \tilde{\beta} := 6696\tilde{\eta}\epsilon^2 \frac{l}{d^2} \mathbb{E}[\|\tilde{\mathbf{\Sigma}}_R\|_{\text{op}}]^2, \quad \text{and} \quad \tilde{\nu} := \frac{\tilde{\lambda}_1^2 \tilde{r} l^2}{n^{*2} d^4},$$

and putting together Proposition 1 and Lemma 1, we obtain

$$\|\hat{\mathbf{N}} - \mathbf{N}^*\|_F^2 \leq \left(\frac{1600 \tilde{\lambda}^2 \tilde{r} l^2}{3d^4} + \frac{32 \times 6696 \tilde{\lambda}_1^2 \tilde{r} l^2}{C^2 d^4} \right) \vee \frac{4 \tilde{\lambda}_1^2 \tilde{r} l^2}{d^4} \leq \frac{C' \tilde{\lambda}^2 \tilde{r} l^2}{d^4},$$

for sufficiently large constant $C' > 0$. We note that the only condition of Lemma 1 can be shown to hold by taking C'' such that $C''n\tilde{\lambda}^2 \tilde{r} l > \epsilon^2 n^{*2} d^2$. Now, we can use Assumption 1 to derive the following bound:

$$\|\hat{\mathbf{N}} - \mathbf{N}^*\|_F^2 = \left\| \frac{\sqrt{\mathbf{W}}}{\sqrt{\mathbf{P}}} \circ \sqrt{\mathbf{P}} \circ (\hat{\mathbf{B}} - \mathbf{B}^*) \right\|_F^2 \geq \frac{1}{ld^2} \|\hat{\mathbf{B}} - \mathbf{B}^*\|_{L^2(\Pi)}^2.$$

Therefore, when $\tilde{\lambda} \geq C\mathbb{E}[\|\tilde{\mathbf{\Sigma}}_R\|_{\text{op}}]n^*\epsilon$,

$$\|\hat{\mathbf{B}} - \mathbf{B}^*\|_{L^2(\Pi)}^2 \leq \frac{C' \tilde{\lambda}^2 \tilde{r} l^3}{d^2} \quad (4.3)$$

holds with probability at least $1 - \mathbb{P}(\tilde{\lambda} < 3\|\tilde{\Sigma}\|_{\text{op}}) - 2\exp(-\frac{C''n\tilde{\lambda}^2\tilde{r}l}{\mathfrak{c}^2n^{*2}d^2})$.

To obtain a bound for $\mathbb{P}(\tilde{\lambda} < 3\|\tilde{\Sigma}\|_{\text{op}})$, we rely on a variant of Bernstein tail inequality for the operator norm of matrix martingales, adapted from Lemma 5 in Klopp (2014), also stated in Appendix C for completeness. To specify an explicit choice of $\tilde{\lambda}$ by bounding the operator norm of $\tilde{\Sigma}_R$, we use Lemma 6 from Klopp (2014), also stated in Appendix C for completeness.

Step 3 (A concrete bound). Lastly, we use Orlicz norm to guide selection of $\mathfrak{N}(\mathbf{N}) = d^2\|\mathbf{N}\|_{\infty}$ for any \mathbf{B} and $\mathbf{N} = \sqrt{\mathbf{W}} \circ \mathbf{B}$. So $n^* = 2d^2\|\sqrt{\mathbf{W}} \circ \mathbf{B}^*\|_{\infty}$. Consequently, $\mathfrak{c} = \frac{9}{d}\sqrt{\frac{l}{p_{\min}}}$ fulfills Assumption 3. The details are shown in the appendix. Building on these quantities and integrating Lemma 5 and 6 into the probabilistic bound (4.3), we obtain Theorem 1.

5. Empirical Results

The goal of this section is to compare the performance of our NU-Recommend method with other methods. We perform two sets of experiments, one on synthetic data and the other on real data from two different domains: movie recommendations and lab test recommendations.

5.1. Synthetic Data

With the synthetic dataset, our goal is to see how the NU-Recommend algorithm compares against existing benchmarks when we actually generate ground truth matrix ourselves, which means it could satisfy assumptions such as the low-rankness of the preference matrix and the sampling matrix.

We compare our NU-Recommend method against (i) regular trace-norm penalized regression that goes back to Candes and Plan (2010), abbreviated as the Uniform method, (ii) marginal-weighted trace-norm penalized regression that goes back to Srebro and Salakhutdinov (2010), abbreviated as the Margin method, and (iii) inverse propensity weighting strategy proposed by Ma and Chen (2019) and applied on the regular trace-norm penalized regression, abbreviated as the IPW+Uniform method. We estimate marginal probabilities by their empirical estimates.

Synthetic Data Generation. We assume $d_r = d_c = d = 100$, $r_{\mathbf{B}} = r_{\mathbf{P}} = 20$. This means the ground truth preference matrix \mathbf{B}^* as well as the ground truth sampling matrix \mathbf{P}^* are both 100 by 100 and has rank 20. We use the underlying model $\mathbf{B}^* = \mathbf{U}_{\mathbf{B}} \mathbf{V}_{\mathbf{B}}^{\top}$, where $\mathbf{U}_{\mathbf{B}}$ and $\mathbf{V}_{\mathbf{B}}$ are random matrices of size 100 by 20 with entries drawn independently and uniformly from $[0, 1]$. Similarly, we let the sampling matrix \mathbf{P}^* to be $\mathbf{U}_{\mathbf{P}} \mathbf{V}_{\mathbf{P}}^{\top}$, where $\mathbf{U}_{\mathbf{P}}$ and $\mathbf{V}_{\mathbf{P}}$

are random matrices of size 100 by 20 with entries drawn independently and uniformly from $[0, 1]$. We set the noise variance to be $\sigma^2 = 1$. We try different sample sizes n from $\{1000, 1200, 1400, 1600, 1800, 2000\}$. We generate 100 data sets for each sample size and compare the relative estimation errors (with 2 SE error bars) in both Frobenius norm and $L_2(\Pi)$ norm for all algorithms across these 100 runs. We look at a comprehensive range of λ 's for all algorithms and for each algorithm we pick the lambda that provides the lowest estimation error. We do not perform cross-validation here because we already know the ground truth matrix. Later, when we deal with the real data, we will be using the cross-validation procedure to select a λ . As we have mentioned in Section 3.2, NU-Recommend needs input of $\hat{\mathbf{B}}$ from another estimator and also an estimation on the sampling matrix $\hat{\mathbf{P}}$. We take $\hat{\mathbf{B}}$ from the Margin method as it empirically performs better than the Uniform method. In Section 3.3, we point out that $\hat{\mathbf{P}}$ could be obtained from a Poisson matrix recovery. Alternatively, one can take a simpler route and use the probability marginals. We pick the latter because it is computationally faster.

Results of synthetic experiment. Figure 2 shows that our method NU-Recommend gives the lowest relative matrix estimation error both in Frobenius norm and in $L_2(\Pi)$ norm among all methods. On average, 10.18% better than the IPW, 6.53% better than the Uniform and 2.08% better than the Margin.

5.2. MovieLens 100K

Sub-sample of MovieLens 100K dataset. In addition to synthetic experiments, we also perform experiments using real-world data to show the practicality of our method. Specifically, the low-rank assumption on both \mathbf{B}^* and \mathbf{P}^* may not necessarily hold. The first dataset we use, MovieLens 100K, is a common collaborative filtering benchmark dataset. This dataset contains 100,000 ratings from 943 users on 1682 movies. Each user has rated at least 20 movies. We work with a dense sub-matrix of 235 users, 420 movies, and 40,000 observations for the illustration purpose. This matrix is constructed by taking the observations only from the top 25% users who rated the most and the top 25% movies that are rated the most. With this matrix, we first randomly select 80% samples as evaluation and 20% samples as testing. We further partition the evaluation set to an 80/20 training and validation split and use that to optimize λ . Once the optimal λ is selected, we refit the algorithm using all of the evaluation set, and show results on the test set. We repeat this process by 20 splits of the evaluation set. This leads to 20 RMSE values for each algorithm on the test set. We report these RMSE with 2 standard errors in Figure 3.

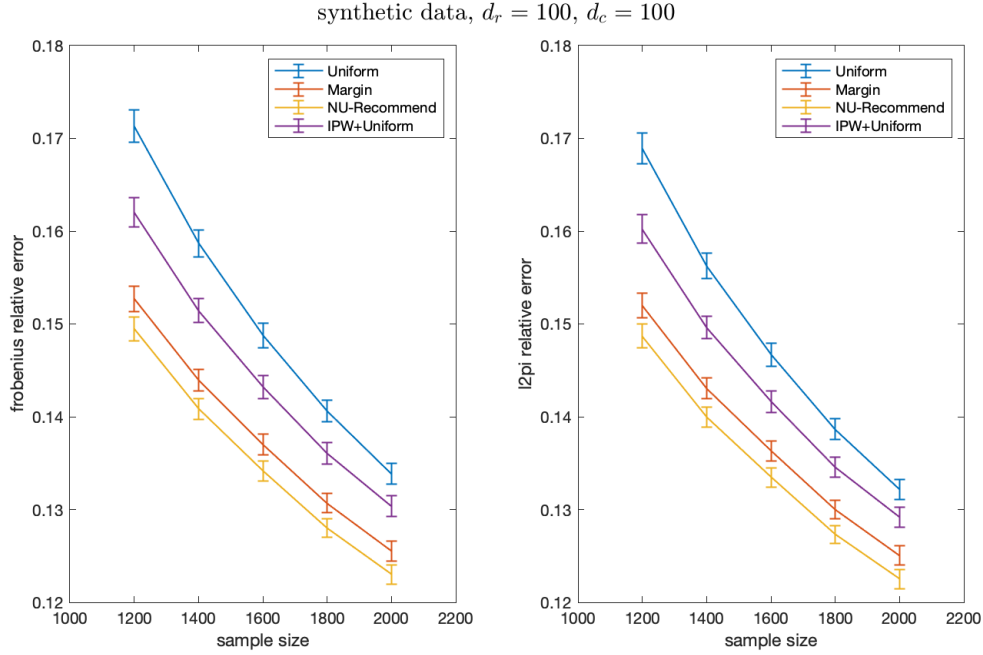


Figure 2 plots of the relative frobenius and $L_2(\Pi)$ errors versus sample sizes 1000, 1200, 1400, 1600, 1800, 2000

Results of MovieLens. Figure 3 shows that our method NU-Recommend gives the lowest test RMSE among all methods. As we can see, NU-Recommend improves RMSE by 1.76% compared with the IPW+Uniform method, by 0.51% compared with the Uniform method, and by 0.45% compared with the Margin method. While these improvements may seem small, they are substantial for this problem. For example, during the final year of the Netflix Prize competition, the best team’s RMSE moved from 0.8627 to 0.8567 (less than 0.7% improvement).

5.3. Lab Test Dataset

Sub-sample of lab test dataset. The other real dataset we use contains data for patient visits at Stanford Hospital and Clinics. In this dataset, we have access to 54653 patients in total and up to 4559 lab tests taken over one year period. If a patient has multiple lab tests in that one year, we will just use the average value. Therefore, we have a matrix of 54653 number of rows and 4559 number of columns. If a patient does not have a test, then the corresponding entry has a missing value. This matrix has many missing entries, hence we restrict the data to those patients who have more than 110 lab test results, and to those lab tests which are done by at least 1000 patients. This leads to 259 patients (rows), 198 tests (columns) and about 40% of the matrix entries are observed. The lab test results are all

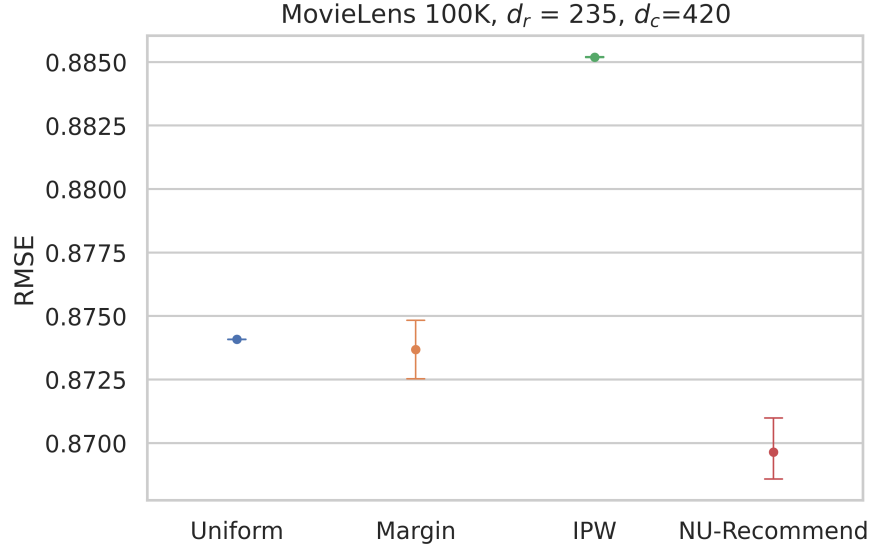


Figure 3 The matrix dimension is 235×420 , NU-Recommend is taken at $L = 3, \gamma = 3$, 20 runs

positive real numbers, and the distribution is skewed, so we take a $\log(1+x)$ transformation on the entries and standardize the columns. We cross-validate for λ the same way as we have done for the MovieLens data. We repeat the process by 10 splits of the evaluation set.

Results of lab test data. We present the results in Figure 4 in the same way as we did for MovieLens. As we can see, our method NU-Recommend gives the lowest test RMSE among all methods. NU-Recommend improves RMSE by 1.16% compared with the IPW+Uniform method, by 0.50%, compared with the Uniform method, and by 0.28%, compared with the Margin method. Overall, these results are consistent with our result on MovieLens Data.

6. Conclusions

This paper introduces a new method to learn user preferences for recommender systems. Motivated by the non-uniform pattern of observing user preferences, we formulated a general weighted-trace-norm penalized regression and examined the upper bound on its estimation error. Experiments on both synthetic data and real data MovieLens and lab test provided empirical evidence that recovering the non-uniformity is beneficial. Furthermore, we showed that, by selecting the weight matrix that minimizes the upper bound, the performance can be further enhanced.

A number of exciting future research are as follows. Our theoretical upper bound is motivated by the oracle inequalities in high-dimensional statistics. It is interesting to see that we

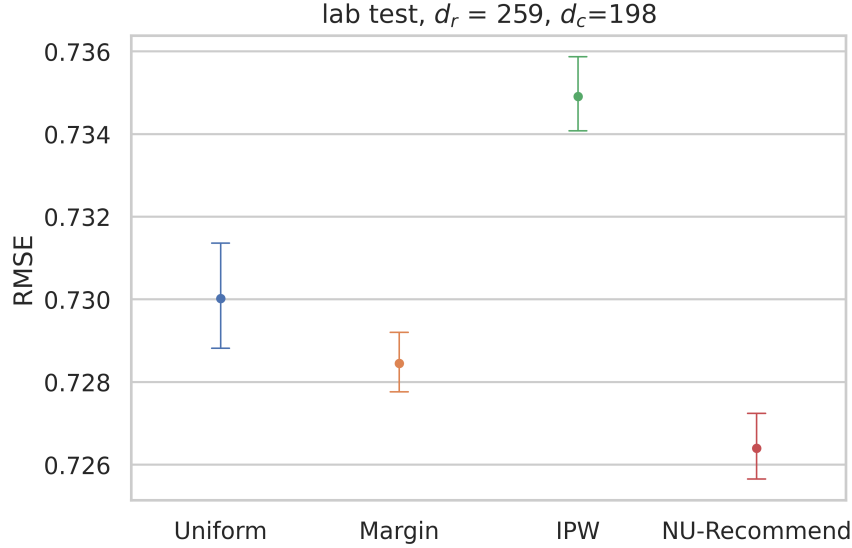


Figure 4 The matrix dimension is 259×198 , NU-Recommend is taken at $L = 3, \gamma = 3$, 10 runs

could take this upper bound and then from it derive a better version of the weighted estimator, which empirically performs very well. We believe this direction could be further explored for the high-dimensional statistics literature. Another practical extension is to explore how to select the weight matrix more efficiently. For some applications, the training process needs to be done faster. With the current convex program solved by semidefinite programming to obtain the desired weight matrix, the scalability of our approach can be compromised. Another important direction is to include side information to aid the sampling matrix estimation. For example, in the movie rating case, the rating data can be much more sparse than the watch data. Thus, to recover the sampling scheme of the rating data, we can instead change to recover that of the watch data. Although the sampling matrix for rating data and watch data may not be exactly the same, we can model the degree to which they reflect each other to make a better informed estimation. Last but not least, as mentioned before, it is worth exploring how to extend the work to the “observational data regime” where the missingness of an entry may depend on the true value of that entry.

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Appendix A: Proof of Proposition 1

First, it follows from $\tilde{\mathcal{L}}(\mathbf{N}^*) > \tilde{\mathcal{L}}(\hat{\mathbf{N}})$ that

$$\frac{1}{n} \|Y - \tilde{\mathbf{X}}(\hat{\mathbf{N}})\|_2^2 + \tilde{\lambda} \|\hat{\mathbf{N}}\|_* \leq \frac{1}{n} \|Y - \tilde{\mathbf{X}}(\mathbf{N}^*)\|_2^2 + \tilde{\lambda} \|\mathbf{N}^*\|_*.$$

By substituting Y with $\tilde{\mathbf{X}}(\mathbf{N}^*) + E$ and doing some algebra, we have

$$\frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_2^2 + 2\langle \tilde{\mathbf{\Sigma}}, \mathbf{N}^* - \hat{\mathbf{N}} \rangle + \tilde{\lambda} \|\hat{\mathbf{N}}\|_* \leq \tilde{\lambda} \|\mathbf{N}^*\|_*.$$

Then, using duality between the operator norm and the trace norm, we get

$$\frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_2^2 + \tilde{\lambda} \|\hat{\mathbf{N}}\|_* \leq 2\|\tilde{\mathbf{\Sigma}}\|_{\text{op}} \cdot \|\mathbf{N}^* - \hat{\mathbf{N}}\|_* + \tilde{\lambda} \|\mathbf{N}^*\|_*.$$

For a given set of vectors S , we denote by \mathbf{P}_S the orthogonal projection on the linear subspace spanned by elements of S (i.e., $\mathbf{P}_S = \sum_{i=1}^k u_i u_i^\top$ if $\{u_1, \dots, u_k\}$ is an orthogonal basis for S). For matrix $\mathbf{N} \in \mathbb{R}^{d_r \times d_c}$, let $S_r(\mathbf{N})$ and $S_c(\mathbf{N})$ be the linear subspace spanned by the left and right orthonormal singular vectors of \mathbf{N} , respectively. Then, for $\mathbf{A} \in \mathbb{R}^{d_r \times d_c}$ define

$$\mathbf{P}_\mathbf{N}^\perp(\mathbf{A}) := \mathbf{P}_{S_r^\perp(\mathbf{N})} \mathbf{A} \mathbf{P}_{S_c^\perp(\mathbf{N})} \quad \text{and} \quad \mathbf{P}_\mathbf{N}(\mathbf{A}) := \mathbf{A} - \mathbf{P}_\mathbf{N}^\perp(\mathbf{A}).$$

We can alternatively express $\mathbf{P}_\mathbf{N}(\mathbf{A})$ as

$$\mathbf{P}_\mathbf{N}(\mathbf{A}) = \mathbf{P}_{S_r(\mathbf{N})} \mathbf{A} + \mathbf{P}_{S_r^\perp(\mathbf{N})} \mathbf{A} \mathbf{P}_{S_c(\mathbf{N})}. \quad (\text{A.1})$$

In particular, since $S_r(\mathbf{N})$ and $S_c(\mathbf{N})$ both have dimension $\text{rank}(\mathbf{N})$, it follows from (A.1) that

$$\text{rank}(\mathbf{P}_\mathbf{N}(\mathbf{A})) \leq 2\text{rank}(\mathbf{N}).$$

Moreover, the definition of $\mathbf{P}_{\mathbf{N}^\perp}$ implies that the left and right singular vectors of $\mathbf{P}_{\mathbf{N}^\perp}(\mathbf{A})$ are orthogonal to those of \mathbf{N} . We thus have

$$\|\mathbf{N} + \mathbf{P}_{\mathbf{N}}^\perp(\mathbf{A})\|_* = \|\mathbf{N}\|_* + \|\mathbf{P}_{\mathbf{N}}^\perp(\mathbf{A})\|_*.$$

By setting $\mathbf{N} := \mathbf{N}^*$ and $\mathbf{A} := \hat{\mathbf{N}} - \mathbf{N}^*$, the above equality entails:

$$\|\mathbf{N}^* + \mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* = \|\mathbf{N}^*\|_* + \|\mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_*.$$

We can then use the above to get the following inequality:

$$\begin{aligned} \|\hat{\mathbf{N}}\|_* &= \|\mathbf{N}^* + \hat{\mathbf{N}} - \mathbf{N}^*\|_* \\ &= \|\mathbf{N}^* + \mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}}) + \mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_* \\ &\geq \|\mathbf{N}^* + \mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* - \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_* \\ &= \|\mathbf{N}^*\|_* + \|\mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* - \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_*. \end{aligned} \tag{A.2}$$

Thus, we get

$$\begin{aligned} \frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_2^2 &\leq 2\|\tilde{\mathbf{\Sigma}}\|_{\text{op}} \cdot \|\mathbf{N}^* - \hat{\mathbf{N}}\|_* + \tilde{\lambda} \|\mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* - \tilde{\lambda} \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_* \\ &\leq (2\|\tilde{\mathbf{\Sigma}}\|_{\text{op}} + \tilde{\lambda}) \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_* + (2\|\tilde{\mathbf{\Sigma}}\|_{\text{op}} - \tilde{\lambda}) \|\mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* \\ &\leq \frac{5}{3} \tilde{\lambda} \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_*. \end{aligned}$$

Now, by the fact that $\text{rank}(\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})) \leq 2\text{rank}(\mathbf{N}^*)$, we can apply Cauchy-Schwartz to singular values of $\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})$ to obtain:

$$\begin{aligned} \frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_2^2 &\leq \frac{5}{3} \tilde{\lambda} \sqrt{2\text{rank}(\mathbf{N}^*)} \|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_F \\ &\leq \frac{5}{3} \tilde{\lambda} \sqrt{2\text{rank}(\mathbf{N}^*)} \|\mathbf{N}^* - \hat{\mathbf{N}}\|_F. \end{aligned}$$

Next, we want to make a connection between $\hat{\mathbf{N}}$ and $\mathcal{C}(\tilde{\nu}, \tilde{\eta})$ with the following lemma.

LEMMA 2. *If $\tilde{\lambda} \geq 3\|\tilde{\mathbf{\Sigma}}\|_{\text{op}}$, then*

$$\|\mathbf{P}_{\mathbf{N}^*}^\perp(\mathbf{N}^* - \hat{\mathbf{N}})\|_* \leq 5\|\mathbf{P}_{\mathbf{N}^*}(\mathbf{N}^* - \hat{\mathbf{N}})\|_*.$$

Proof of Lemma 2. Note that $\|\tilde{\mathbf{X}}(\cdot) - Y\|_2^2$ is a convex function. We can then use the convexity at \mathbf{N}^* to get:

$$\begin{aligned} &\frac{1}{n} \|\tilde{\mathbf{X}}(\hat{\mathbf{N}}) - Y\|_2^2 - \frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^*) - Y\|_2^2 \\ &\geq -\frac{2}{n} \sum_{i=1}^n (y_i - \langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle) \langle \tilde{\mathbf{X}}_i, \hat{\mathbf{N}} - \mathbf{N}^* \rangle \\ &= -2\langle \tilde{\mathbf{\Sigma}}, \hat{\mathbf{N}} - \mathbf{N}^* \rangle \geq -2\|\tilde{\mathbf{\Sigma}}\|_{\text{op}} \|\hat{\mathbf{N}} - \mathbf{N}^*\|_* \\ &\geq -\frac{2}{3} \tilde{\lambda} \|\hat{\mathbf{N}} - \mathbf{N}^*\|_*. \end{aligned}$$

We detail the above procedures below (readers who are familiar with convex functions can omit this):

Let $f(\mathbf{N}) = \frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}) - Y\|_2^2$. Then

$$[\nabla f(\mathbf{N})]_{jk} = \sum_{i=1}^n \frac{2}{n} \tilde{\mathbf{X}}_{i,jk} (\langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle - y_i), \text{ so } \nabla f(\mathbf{N}) = \sum_{i=1}^n \frac{2}{n} \tilde{\mathbf{X}}_i (\langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle - y_i).$$

Now by using convexity at \mathbf{N}^* , we get:

$$\begin{aligned} f(\hat{\mathbf{N}}) - f(\mathbf{N}^*) &\geq \langle \nabla_{\mathbf{N}^*} f(\mathbf{N}^*), \hat{\mathbf{N}} - \mathbf{N}^* \rangle \\ &= \left\langle \frac{2}{n} \sum_{i=1}^n \tilde{\mathbf{X}}_i (\langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle - y_i), \hat{\mathbf{N}} - \mathbf{N}^* \right\rangle \\ &= \frac{2}{n} \sum_{i=1}^n (\langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle - y_i) \langle \hat{\mathbf{N}} - \mathbf{N}^*, \tilde{\mathbf{X}}_i \rangle \\ &= -\frac{2}{n} \sum_{i=1}^n (y_i - \langle \tilde{\mathbf{X}}_i, \mathbf{N}^* \rangle) \langle \tilde{\mathbf{X}}_i, \hat{\mathbf{N}} - \mathbf{N}^* \rangle. \end{aligned}$$

Combining this, we get,

$$\begin{aligned} \frac{2}{3} \tilde{\lambda} \|\hat{\mathbf{N}} - \mathbf{N}^*\|_* &\geq \frac{1}{n} \|\tilde{\mathbf{X}}(\mathbf{N}^*) - Y\|_2^2 - \frac{1}{n} \|\tilde{\mathbf{X}}(\hat{\mathbf{N}}) - Y\|_2^2 \\ &\geq \tilde{\lambda} \|\hat{\mathbf{N}}\|_* - \tilde{\lambda} \|\mathbf{N}^*\|_* \\ &\geq \tilde{\lambda} \|\mathbf{P}_{\mathbf{N}^*}^\perp (\mathbf{N}^* - \hat{\mathbf{N}})\|_* - \tilde{\lambda} \|\mathbf{P}_{\mathbf{N}^*} (\mathbf{N}^* - \hat{\mathbf{N}})\|_*. \end{aligned}$$

Using the triangle inequality, we have

$$\|\mathbf{P}_{\mathbf{N}^*}^\perp (\mathbf{N}^* - \hat{\mathbf{N}})\|_* \leq 5 \|\mathbf{P}_{\mathbf{N}^*} (\mathbf{N}^* - \hat{\mathbf{N}})\|_*.$$

Lemma 2, the triangle inequality and the fact that $\text{rank}(\mathbf{P}_{\mathbf{N}^*}(\hat{\mathbf{N}} - \mathbf{N}^*)) \leq 2\text{rank}(\mathbf{N}^*)$ imply that

$$\begin{aligned} \|\hat{\mathbf{N}} - \mathbf{N}^*\|_* &\leq 6 \|\mathbf{P}_{\mathbf{N}^*} (\mathbf{N}^* - \hat{\mathbf{N}})\|_* \\ &\leq \sqrt{72 \text{rank}(\mathbf{N}^*)} \|\mathbf{P}_{\mathbf{N}^*} (\mathbf{N}^* - \hat{\mathbf{N}})\|_F \\ &\leq \sqrt{72 \text{rank}(\mathbf{N}^*)} \|\mathbf{N}^* - \hat{\mathbf{N}}\|_F. \end{aligned}$$

By the above, we have made a connection with $\mathcal{C}(\tilde{\nu}, \tilde{\eta})$.

Next, define $\hat{n} := \Re(\hat{\mathbf{N}} - \mathbf{N}^*)$ and $\mathbf{A} := \frac{1}{\hat{n}}(\hat{\mathbf{N}} - \mathbf{N}^*)$. We then have that

$$\Re(\mathbf{A}) = 1 \text{ and } \|\mathbf{A}\|_* \leq \sqrt{72 \text{rank}(\mathbf{N}^*)} \|\mathbf{A}\|_F.$$

Now, we consider the following two cases:

Case 1: If $\|\mathbf{A}\|_F^2 < \tilde{\nu}$, then

$$\|\mathbf{N}^* - \hat{\mathbf{N}}\|_F^2 < 4n^* \tilde{\nu}.$$

Case 2: Otherwise, $\mathbf{A} \in \mathcal{C}(\tilde{\nu}, \tilde{\eta})$. We can, now, use the RSC condition to get

$$\tilde{\alpha} \frac{\|\mathbf{N}^* - \hat{\mathbf{N}}\|_F^2}{\hat{n}^2} - \tilde{\beta} \leq \frac{\|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_F^2}{n\hat{n}^2}$$

which leads to

$$\begin{aligned} \tilde{\alpha} \|\mathbf{N}^* - \hat{\mathbf{N}}\|_F^2 - 4n^* \tilde{\beta} &\leq \frac{\|\tilde{\mathbf{X}}(\mathbf{N}^* - \hat{\mathbf{N}})\|_F^2}{n} \\ &\leq \frac{5\tilde{\lambda} \sqrt{2 \text{rank}(\mathbf{N}^*)}}{3} \|\mathbf{N}^* - \hat{\mathbf{N}}\|_F \\ &\leq \frac{50\tilde{\lambda}^2 \text{rank}(\mathbf{N}^*)}{3\tilde{\alpha}} + \frac{\tilde{\alpha}}{2} \|\mathbf{N}^* - \hat{\mathbf{N}}\|_F. \end{aligned}$$

Therefore, we have

$$\|\mathbf{N}^* - \widehat{\mathbf{N}}\|_F^2 \leq \frac{100 \tilde{\lambda}^2 \text{rank}(\mathbf{N}^*)}{3\tilde{\alpha}^2} + \frac{8n^{*2}\tilde{\beta}}{\tilde{\alpha}},$$

which completes the proof of this proposition. Q.E.D.

Appendix B: Proof of Lemma 1

Next, we will show that the RSC condition holds with high probability. Instead of proving Lemma 1 directly, we show a variant of it first:

LEMMA 3 (**Restricted Strong Convexity**). *Define*

$$\mathcal{C}'(\tilde{\theta}, \tilde{\eta}) := \{\Delta \in \mathbb{R}^{d_r \times d_c} | \mathfrak{N}(\Delta) = 1, \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 \geq \tilde{\theta}, \|\Delta\|_* \leq \sqrt{\tilde{\eta}} \|\Delta\|_F\}.$$

If Assumption 3 hold, then the inequality

$$\frac{\|\tilde{\mathfrak{X}}(\Delta)\|_2^2}{n} \geq \frac{1}{4} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 - 93 \frac{l}{d^2} \tilde{\eta} \mathfrak{c}^2 \mathbb{E}[\|\tilde{\Sigma}_R\|_{\text{op}}]^2 \text{ for all } \Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta})$$

holds with probability greater than $1 - 2 \exp(-\frac{Cn\tilde{\theta}}{\mathfrak{c}^2})$ where $C > 0$ is an absolute constant, provided that $Cn\tilde{\theta} > \mathfrak{c}^2$, and $\tilde{\Sigma}_R := \frac{1}{n} \sum_{i=1}^n \zeta_i \tilde{\mathbf{X}}_i$ where $\{\zeta_i\}_{i=1}^n$ is an i.i.d sequence with Rademacher distribution.

Proof of Lemma 3(RSC). Set

$$\beta = 93 \frac{l}{d^2} \tilde{\eta} \mathfrak{c}^2 \mathbb{E}[\|\tilde{\Sigma}_R\|_{\text{op}}]^2.$$

We denote the bad event \mathcal{B} as the following:

$$\mathcal{B} = \left\{ \exists \Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}) \text{ such that } \frac{1}{2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\tilde{\mathfrak{X}}(\Delta)\|_2^2 > \frac{1}{4} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 + \beta \right\}.$$

We thus need to bound the probability of this event. Set $\xi = 6/5$. Then, for $T > 0$, we define

$$\mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T) := \{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}) \mid T \leq \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 < \xi T\}.$$

Clearly, we have

$$\mathcal{C}'(\tilde{\theta}, \tilde{\eta}) = \cup_{k=1}^{\infty} \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, \xi^{k-1}\tilde{\theta}).$$

Now, if the event \mathcal{B} holds for some $\Delta \in \mathcal{C}'(\tilde{\nu}, \tilde{\eta})$, then $\Delta \in \mathcal{C}'(\tilde{\nu}, \tilde{\eta}, \xi^{k-1}\tilde{\nu})$ for some $k \in \mathbb{N}$. In this case, we have

$$\begin{aligned} \frac{1}{2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\tilde{\mathfrak{X}}(\Delta)\|_2^2 &> \frac{1}{4} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 + \beta \\ &\geq \frac{1}{4} \xi^{k-1} \tilde{\theta} + \beta \\ &= \frac{5}{24} \xi^k \tilde{\theta} + \beta. \end{aligned}$$

Next, we define the event \mathcal{B}_k as

$$\mathcal{B}_k = \left\{ \exists \Delta \in \mathcal{C}'(\tilde{\nu}, \tilde{\eta}, \xi^{k-1}\tilde{\nu}) \text{ such that } \frac{1}{2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\tilde{\mathfrak{X}}(\Delta)\|_2^2 > \frac{5}{24} \xi^k \tilde{\theta} + \beta \right\}.$$

It follows that

$$\mathcal{B} \subseteq \cup_{k=1}^{\infty} \mathcal{B}_k.$$

The following lemma helps us control the probability that each of these \mathcal{B}_k 's happen.

LEMMA 4. *Define*

$$Z_T := \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left\{ \frac{1}{2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\tilde{\mathbf{X}}(\Delta)\|_2^2 \right\}.$$

Then, assuming that $(\mathbf{X}_i)_{i=1}^n$ are i.i.d. samples drawn from Π , we get

$$\mathbb{P}(Z_T \geq \frac{5\xi T}{24} + \beta) \leq \exp\left(-\frac{Cn\xi T}{\mathfrak{c}^2}\right), \quad (\text{B.1})$$

for some numerical constant $C > 0$.

Proof of Lemma 4. For a $d_r \times d_c$ matrix Δ , define

$$f(\tilde{\mathbf{X}}; \Delta) := \langle \tilde{\mathbf{X}}, \Delta \rangle^2 \cdot \mathbb{I}(|\langle \tilde{\mathbf{X}}, \Delta \rangle| \leq \mathfrak{c}).$$

Next, letting

$$W_T := \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \frac{1}{n} \sum_{i=1}^n \{\mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)] - f(\tilde{\mathbf{X}}_i; \Delta)\},$$

$$\widetilde{W}_T := \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left| \frac{1}{n} \sum_{i=1}^n \{\mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)] - f(\tilde{\mathbf{X}}_i; \Delta)\} \right|,$$

it follows from Assumption 3 (where \mathfrak{c} is defined) that $Z_T \leq W_T$, and clearly $W_T \leq \widetilde{W}_T$ hence

$$\mathbb{P}(Z_T \geq t) \leq \mathbb{P}(\widetilde{W}_T \geq t),$$

for all t . Therefore, if we prove (B.1) holds when Z_T is replaced with \widetilde{W}_T , we would be done. In the remaining, we will aim to prove this via Massart's inequality. In order to invoke Massart's inequality, we need bounds for $\mathbb{E}[\widetilde{W}_T]$ and $\text{Var}(\widetilde{W}_T)$.

First, we find an upper bound for $\mathbb{E}[\widetilde{W}_T]$. It follows from the symmetrization argument that

$$\mathbb{E}[\widetilde{W}_T] \leq 2\mathbb{E} \left[\sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i f(\tilde{\mathbf{X}}_i; \Delta) \right| \right]. \quad (\text{B.2})$$

Next, we will use the contraction inequality. First, we write $f(\tilde{\mathbf{X}}_i; \Delta) = \alpha_i \langle \tilde{\mathbf{X}}_i; \Delta \rangle$, where $\alpha_i = \langle \tilde{\mathbf{X}}_i; \Delta \rangle \cdot \mathbb{I}(|\langle \tilde{\mathbf{X}}_i; \Delta \rangle| \leq \mathfrak{c})$. By definition, $|\alpha_i| \leq \mathfrak{c}$. Now, for every realization of the random variables $\tilde{\mathbf{X}}_1, \dots, \tilde{\mathbf{X}}_n$ we can obtain

$$\mathbb{E}_\zeta \left[\sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i f(\tilde{\mathbf{X}}_i; \Delta) \right| \right] \leq \mathfrak{c} \mathbb{E}_\zeta \left[\sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i \langle \tilde{\mathbf{X}}_i; \Delta \rangle \right| \right].$$

Now, taking expectation of both sides with respect to $\tilde{\mathbf{X}}_i$'s, using the tower property, and combining with (B.2) we obtain

$$\begin{aligned} \mathbb{E}[\widetilde{W}_T] &\leq 8\mathfrak{c} \mathbb{E} \left[\sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i \langle \tilde{\mathbf{X}}_i; \Delta \rangle \right| \right] \\ &\leq 8\mathfrak{c} \mathbb{E} \left[\|\tilde{\Sigma}_R\|_{\text{op}} \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \|\Delta\|_* \right] \\ &\leq 8\mathfrak{c} \sqrt{\tilde{\eta}} \mathbb{E} \left[\|\tilde{\Sigma}_R\|_{\text{op}} \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \|\Delta\|_F \right] \\ &\leq 8\mathfrak{c} \frac{\sqrt{l}}{d} \sqrt{\tilde{\eta}} \mathbb{E} \left[\|\tilde{\Sigma}_R\|_{\text{op}} \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)} \right] \\ &\leq 8\mathfrak{c} \frac{\sqrt{l}}{d} \sqrt{\tilde{\eta} \xi T} \mathbb{E} [\|\tilde{\Sigma}\|_{\text{op}}]. \end{aligned}$$

In the above, we used definition of $\mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)$ and also the following derivation:

$$\begin{aligned} \|\Delta\|_F^2 &= \|\sqrt{\mathbf{W}} \circ \mathbf{M}\|_F^2 = \|\sqrt{\mathbf{P}} \frac{\sqrt{\mathbf{W}}}{\sqrt{\mathbf{P}}} \circ \mathbf{M}\|_F^2 \\ &\leq \frac{l}{d^2} \|\mathbf{M}\|_{L^2(\Pi)}^2 = \frac{l}{d^2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 \quad \text{for some } \mathbf{M} \text{ such that } \Delta = \sqrt{\mathbf{W}} \circ \mathbf{M}. \end{aligned} \quad (\text{B.3})$$

In the above, we have used Definition 1 and the following property:

$$\begin{aligned} \|\mathbf{A}\|_{L^2(\Pi)}^2 &= \mathbb{E}[\langle \mathbf{X}_i, \mathbf{A} \rangle^2] = \mathbb{E}_\Pi[\mathbb{E}_\xi[\langle \xi_i \cdot e_{r_i} e_{c_i}^T, \mathbf{A} \rangle^2 | r_i, c_i]] = \mathbb{E}_\Pi[\mathbb{E}_\xi[\mathbf{A}_{r_i, c_i}^2 \xi_i^2 | r_i, c_i]] = \mathbb{E}_\Pi[d^2 \mathbf{A}_{r_i, c_i}^2] \\ &= d^2 \sum \mathbf{A}_{r_i, c_i}^2 \mathbf{P}_{r_i, c_i} = d^2 \|\sqrt{\mathbf{P}} \circ \mathbf{A}\|_F^2. \end{aligned}$$

We can, now, use $2ab \leq a^2 + b^2$ to get

$$\mathbb{E}[\tilde{W}_T] \leq \frac{8}{9} \left(\frac{5\xi T}{24} \right) + 87 \frac{l}{d^2} \tilde{\eta} \mathfrak{c}^2 \mathbb{E}[\|\tilde{\Sigma}_R\|_{\text{op}}]^2.$$

Next, we turn to finding an upper bound for the variance of $\frac{1}{n} \sum_{i=1}^n f(\tilde{\mathbf{X}}_i; \Delta) - \mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)]$:

$$\text{Var}(f(\tilde{\mathbf{X}}_i; \Delta) - \mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)]) \leq \mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)^2] \leq \mathfrak{c}^2 \cdot \mathbb{E}[\langle \tilde{\mathbf{X}}_i, \Delta \rangle^2] = \mathfrak{c}^2 \cdot \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2.$$

Therefore, we have that

$$\sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \frac{1}{n} \text{Var}(f(\tilde{\mathbf{X}}_i; \Delta) - \mathbb{E}[f(\tilde{\mathbf{X}}_i; \Delta)]) \leq \frac{\mathfrak{c}^2}{n} \cdot \sup_{\Delta \in \mathcal{C}'(\tilde{\theta}, \tilde{\eta}, T)} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 \leq \frac{\xi T \mathfrak{c}^2}{n}.$$

Finally, noting that $\frac{1}{n} f(\tilde{\mathbf{X}}_i; \Delta) \leq \frac{1}{n} \mathfrak{c}^2$ almost surely, we can use Massart's inequality to conclude that

$$\begin{aligned} \mathbb{P}(\tilde{W}_T \geq \frac{5\xi T}{24} + \beta) &= \mathbb{P}(\tilde{W}_T \geq \frac{5\xi T}{24} + 93 \frac{l}{d^2} \eta \mathfrak{c}^2 \mathbb{E}[\|\tilde{\Sigma}_R\|_{\text{op}}]^2) \\ &\leq \mathbb{P}\left(\tilde{W}_T \geq \frac{18}{17} \mathbb{E}[\tilde{W}_T] + \frac{1}{17} \left(\frac{5\xi T}{24} \right)\right) \leq \exp\left(-\frac{Cn\xi T}{\mathfrak{c}^2}\right), \end{aligned}$$

for some numerical constant $C > 0$.

Lemma 4 entails that

$$\mathbb{P}(\mathcal{B}_k) \leq \exp\left(-\frac{Cn\xi^k \tilde{\theta}}{\mathfrak{c}^2}\right) \leq \exp\left(-\frac{Cnk \log(\xi) \tilde{\theta}}{\mathfrak{c}^2}\right).$$

Therefore, by setting the numerical constant $C > 0$ appropriately, the union bound implies that

$$\mathbb{P}(\mathcal{B}) \leq \sum_{k=1}^{\infty} \mathbb{P}(\mathcal{B}_k) \leq \sum_{k=1}^{\infty} \exp\left(-\frac{Cnk \tilde{\theta}}{\mathfrak{c}^2}\right) = \frac{\exp(-\frac{Cn\tilde{\theta}}{\mathfrak{c}^2})}{1 - \exp(-\frac{Cn\tilde{\theta}}{\mathfrak{c}^2})}.$$

Finally, assuming that $Cn\tilde{\theta} > \mathfrak{c}^2$, we get that

$$\mathbb{P}(\mathcal{B}) \leq 2 \exp\left(-\frac{Cn\tilde{\theta}}{\mathfrak{c}^2}\right),$$

which completes the proof.

Q.E.D.

Note that Lemma 3 states RSC holds for $\mathcal{C}'(\tilde{\theta}, \tilde{\eta})$ which is slightly different than the set $\mathcal{C}(\tilde{\nu}, \tilde{\eta})$ defined earlier. But, using Assumption 1, we can see that

$$\mathcal{C}(\tilde{\nu}, \tilde{\eta}) \subseteq \mathcal{C}'\left(\frac{\tilde{\nu} d^2}{l}, \tilde{\eta}\right).$$

This can be shown as follows. Suppose $\Delta \in \mathcal{C}(\tilde{\nu}, \tilde{\eta})$, then $\|\Delta\|_F^2 \geq \tilde{\nu}$. Inequality (B.3) implies

$$\frac{l}{d^2} \|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 \geq \|\Delta\|_F^2 \geq \tilde{\nu},$$

so $\|\sqrt{\mathbf{W}}^{-1} \circ \Delta\|_{L^2(\Pi)}^2 \geq \frac{\tilde{\nu} d^2}{l}$, therefore, $\Delta \in \mathcal{C}'\left(\frac{\tilde{\nu} d^2}{l}, \tilde{\eta}\right)$. As a result, Lemma 1 follows.

Appendix C: Bounding $\|\tilde{\Sigma}\|_{\text{op}}$ and $\|\tilde{\Sigma}_R\|_{\text{op}}$ with matrix Bernstein inequality

The next proposition is a variant of the Bernstein inequality that will be used to show that Condition (4.2) for $\tilde{\lambda}$ is guaranteed to hold with high probability as well.

PROPOSITION 2. Let $(\mathbf{Z}_i)_{i=1}^n$ be a sequence of $d \times d$ independent random matrices with zero mean, such that

$$\mathbb{E}\left[\frac{\|\mathbf{Z}_i\|_{\text{op}}}{\delta}\right] \leq e \quad \forall i \in [n],$$

and

$$\sigma_{\mathbf{Z}} = \max \left\{ \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathbf{Z}_i \mathbf{Z}_i^T] \right\|_{\text{op}}, \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathbf{Z}_i^T \mathbf{Z}_i] \right\|_{\text{op}} \right\}^{\frac{1}{2}},$$

for some positive values δ and σ_z . Then, there exists numerical constant $C > 0$ such that, for all $t > 0$

$$\left\| \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i \right\| \leq C \max \left\{ \sigma_{\mathbf{Z}} \sqrt{\frac{t + \log(2d)}{n}}, \delta \left(\log \frac{\delta}{\sigma_{\mathbf{Z}}} \right) \frac{t + \log(2d)}{n} \right\},$$

with probability at least $1 - \exp(-t)$.

Next, based on the above proposition, we adapted the following two lemmas from Klopp (2014).

LEMMA 5 (Adapted from Lemma 5 of Klopp (2014)). Let $(\mathbf{Z})_{i=1}^n$ be a sequence of $d \times d$ independent random matrices with zero mean. Suppose that $a \leq \sigma_z \leq b$ and $\mathbb{E}\left[\exp\left(\frac{\|\mathbf{Z}_i\|_{\text{op}}}{\delta}\right)\right] \leq e$ for every $i \in [n]$. Then

$$\mathbb{P}\left(\lambda \leq \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i \right\|_{\text{op}}\right) \leq 2d \exp \left[-C\lambda n \left(\frac{\lambda}{b^2} \wedge \frac{1}{\delta(\log \frac{\delta}{a})} \right) \right]. \quad (\text{C.1})$$

Let $\mathbf{Z}_i = \epsilon_i \tilde{\mathbf{X}}_i$, then the above lemma can be applied to $\tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i$.

Proof of Lemma 5. We can apply Proposition 2 to get: with probability at least $1 - \exp(-t)$,

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i \right\|_{\text{op}} &\leq C \max \left\{ \sigma_z \sqrt{\frac{t + \log(2d)}{n}}, \delta \left(\log \frac{\delta}{\sigma_z} \right) \frac{t + \log(2d)}{n} \right\} \\ &\leq C \max \left\{ \underbrace{b \sqrt{\frac{t + \log(2d)}{n}}}_{\textcircled{1}}, \underbrace{\delta \left(\log \frac{\delta}{a} \right) \frac{t + \log(2d)}{n}}_{\textcircled{2}} \right\} \end{aligned}$$

We can find $t = t^*$ such that $\textcircled{1} = \textcircled{2}$. $\textcircled{1} = \textcircled{2}$ implies that

$$b \sqrt{\frac{t + \log(2d)}{n}} = \delta \left(\log \frac{\delta}{a} \right) \frac{t + \log(2d)}{n}$$

So

$$t^* = -\log(2d) + \frac{b^2 n}{\delta^2 (\log \frac{\delta}{a})^2}.$$

When $t \leq t^*$, we have $\textcircled{1} \leq \textcircled{2}$. Then, with probability at most $\exp(-t)$,

$$\left\| \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i \right\|_{\text{op}} \geq Cb \sqrt{\frac{t + \log(2d)}{n}}.$$

Let $\lambda = Cb \sqrt{\frac{t + \log(2d)}{n}}$, then

$$t = -\log(2d) + \frac{\lambda^2 n}{C^2 b^2} \quad \text{and} \quad \exp(-t) = 2d \exp\left(-\frac{\lambda^2 n}{C^2 b^2}\right).$$

So

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{\text{op}} \geq \lambda\right) \leq 2d \exp\left(-\frac{\lambda^2 n}{C^2 b^2}\right).$$

When $t \geq t^*$, we have $\textcircled{2} \geq \textcircled{1}$.

Then, with probability at most $\exp(-t)$,

$$\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{\text{op}} \geq C\delta\left(\log\frac{\delta}{a}\right)\frac{t + \log(2d)}{n}.$$

Let $\lambda = C\delta\left(\log\frac{\delta}{a}\right)\frac{t + \log(2d)}{n}$, then

$$t = -\log(2d) + \frac{\lambda n}{C\delta\left(\log\frac{\delta}{a}\right)} \quad \text{and} \quad \exp(-t) = 2d \exp\left(-\frac{\lambda n}{C\delta\left(\log\frac{\delta}{a}\right)}\right).$$

So

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{\text{op}} \geq \lambda\right) \leq 2d \exp\left(-\frac{\lambda n}{C\delta\left(\log\frac{\delta}{a}\right)}\right).$$

It follows that

$$\begin{aligned} \mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{\text{op}} \geq \lambda\right) &\leq \left[2d \exp\left(-\frac{\lambda^2 n}{C^2 b^2}\right)\right] \vee \left[2d \exp\left(-\frac{\lambda n}{C\delta\left(\log\frac{\delta}{a}\right)}\right)\right] \\ &= 2d \exp\left[-C\lambda n\left(\frac{\lambda}{b^2} \wedge \frac{1}{\delta\left(\log\frac{\delta}{a}\right)}\right)\right]. \end{aligned}$$

Q.E.D.

LEMMA 6 (Adapted from Lemma 6 of Klopp (2014)). *If Condition (C.1) holds and $n \geq \frac{\delta^2(\log\frac{\delta}{a})^2 \log 2d}{b^2}$, then*

$$\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{\text{op}} \leq C\sqrt{\frac{2e \log(2d)b^2}{n}}.$$

Proof of Lemma 6. Set

$$\nu_1 = \frac{n}{C^2 b^2} \quad \text{and} \quad \nu_2 = \frac{n}{C\delta\left(\log\frac{\delta}{a}\right)}.$$

By Hölder's inequality, we get

$$\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n \zeta_i \mathbf{X}_i\right\| \leq \left(\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n \zeta_i \mathbf{X}_i\right\|^{2\log(d)}\right)^{1/(2\log(d))}.$$

The inequality (C.1) imply that

$$\begin{aligned} &\left(\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n \zeta_i \mathbf{X}_i\right\|^{2\log(d)}\right)^{1/(2\log(d))} \\ &= \left(\int_0^{+\infty} \mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \zeta_i \mathbf{X}_i\right\| > \lambda^{1/(2\log(d))}\right) d\lambda\right)^{1/(2\log(d))} \\ &\leq \left(d \int_0^{+\infty} \exp\{-t^{1/\log(d)} \nu_1\} d\lambda + d \int_0^{+\infty} \exp\{-\lambda^{1/(2\log(d))} \nu_2\} d\lambda\right)^{1/(2\log(d))} \\ &\leq \sqrt{e}(\log(d)\nu_1^{-\log(d)}\Gamma(\log(d)) + 2\log(d)\nu_2^{-2\log(d)}\Gamma(2\log(d)))^{1/(2\log(d))}. \end{aligned} \tag{C.2}$$

The Gamma-function satisfies the following bound: for $x \geq 2$, $\Gamma(x) \leq (\frac{x}{2})^{x-1}$. Plugging this into (C.2), we compute

$$\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \zeta_i \mathbf{X}_i \right\| \leq \sqrt{e} ((\log(d))^{\log(d)} \nu_1^{-\log(d)} 2^{1-\log(d)} + 2(\log(d))^{2\log(d)} \nu_2^{-2\log(d)})^{1/(2\log(d))}.$$

Then

$$n \geq \frac{\delta^2 (\log \frac{\delta}{a})^2 \log(d)}{b^2}$$

implies that $\nu_1 \log(d) \leq \nu_2^2$, which then entails

$$\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \right\|_{\text{op}} \leq C \sqrt{\frac{2e \log(d) b^2}{n}}.$$

Q.E.D.

The next step is to use Proposition 2 for $\mathbf{Z}_i := \epsilon_i \tilde{\mathbf{X}}_i$ to find a tail bound inequality for $\mathbb{P}(\lambda < 3 \|\Sigma\|_{\text{op}})$. Define $\delta = \frac{d\sigma e}{\sqrt{p_{\min}/l(e-1)}}$ and let G_1 and G_2 be two independent standard normal random variables. Then, it follows that

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{\|\mathbf{Z}_i\|_{\text{op}}}{\delta} \right) \right] &= \mathbb{E} \left[\left(\frac{d\sigma |G_1 G_2|}{\delta \sqrt{\mathbf{W}_{r_i c_i}}} \right) | r_i, c_i \right] \leq \mathbb{E} \left[\left(\frac{d\sigma (G_1^2 + G_2^2)}{2\delta \sqrt{\mathbf{W}_{r_i c_i}}} \right) | r_i, c_i \right] \\ &= \mathbb{E} \left[\left(\frac{d\sigma |G_1^2|}{2\delta \sqrt{\mathbf{W}_{r_i c_i}}} \right) | r_i, c_i \right]^2 \leq \mathbb{E} \left[\left(\frac{d\sigma G_1^2}{2\delta \sqrt{p_{\min}/l}} \right) | r_i, c_i \right]^2 \\ &\leq \mathbb{E} \left[\left(\frac{(e-1)G_1^2}{2e} \right) | r_i, c_i \right]^2 = \left[\frac{1}{\sqrt{1 - \frac{e-1}{e}}} \right]^2 = e. \end{aligned}$$

Next, we need to figure out $\mathbb{E}[\mathbf{Z}_i \mathbf{Z}_i^\top]$ and $\mathbb{E}[\mathbf{Z}_i^\top \mathbf{Z}_i]$.

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_i \mathbf{Z}_i^\top] &= \mathbb{E}[\epsilon_i \tilde{\mathbf{X}}_i \epsilon_i \tilde{\mathbf{X}}_i^\top] = \mathbb{E}[\epsilon_i^2 \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top] = \mathbb{E}[\epsilon_i^2 \xi_i^2 (\sqrt{\mathbf{W}}^{-1} \circ e_{r_i} e_{c_i}^\top) (\sqrt{\mathbf{W}}^{-1} \circ e_{r_i} e_{c_i}^\top)^\top | r_i, c_i] \\ &= \mathbb{E}[\epsilon_i^2 \xi_i^2 (\sqrt{\mathbf{W}_{r_i c_i}}^{-1} e_{r_i} e_{c_i}^\top) (\sqrt{\mathbf{W}_{r_i c_i}}^{-1} e_{r_i} e_{c_i}^\top) | r_i, c_i] = \sigma^2 d^2 \mathbb{E}[\mathbf{W}_{r_i c_i}^{-1} e_{r_i} e_{r_i}^\top | r_i, c_i] \\ &= \sigma^2 d^2 \sum_j \sum_k \mathbf{P}_{jk} \mathbf{W}_{jk}^{-1} e_j e_j^\top = \sigma^2 d^2 \text{Diag} \left[\sum_{k=1}^d \frac{\mathbf{P}_{1k}}{\mathbf{W}_{1k}} \quad \sum_{k=1}^d \frac{\mathbf{P}_{2k}}{\mathbf{W}_{2k}} \quad \cdots \quad \sum_{k=1}^d \frac{\mathbf{P}_{dk}}{\mathbf{W}_{dk}} \right] \\ &= \sigma^2 d^2 \text{Diag} \text{ rowsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_i^\top \mathbf{Z}_i] &= \mathbb{E}[\epsilon_i \tilde{\mathbf{X}}_i^\top \epsilon_i \tilde{\mathbf{X}}_i] = \mathbb{E}[\epsilon_i^2 \tilde{\mathbf{X}}_i^\top \tilde{\mathbf{X}}_i] = \mathbb{E}[\epsilon_i^2 \xi_i^2 (\sqrt{\mathbf{W}}^{-1} \circ e_{r_i} e_{c_i}^\top)^\top (\sqrt{\mathbf{W}}^{-1} \circ e_{r_i} e_{c_i}^\top) | r_i, c_i] \\ &= \mathbb{E}[\epsilon_i^2 \xi_i^2 (\sqrt{\mathbf{W}_{r_i c_i}}^{-1} e_{c_i} e_{r_i}^\top) (\sqrt{\mathbf{W}_{r_i c_i}}^{-1} e_{r_i} e_{c_i}^\top) | r_i, c_i] = \sigma^2 d^2 \mathbb{E}[\mathbf{W}_{r_i c_i}^{-1} e_{c_i} e_{c_i}^\top | r_i, c_i] \\ &= \sigma^2 d^2 \sum_j \sum_k \mathbf{P}_{jk} \mathbf{W}_{jk}^{-1} e_k e_k^\top = \sigma^2 d^2 \text{Diag} \left[\sum_{j=1}^d \frac{\mathbf{P}_{j1}}{\mathbf{W}_{j1}} \quad \sum_{j=1}^d \frac{\mathbf{P}_{j2}}{\mathbf{W}_{j2}} \quad \cdots \quad \sum_{j=1}^d \frac{\mathbf{P}_{jd}}{\mathbf{W}_{jd}} \right] \\ &= \sigma^2 d^2 \text{Diag} \text{ columnsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right) \end{aligned}$$

So

$$\sigma_z := \max \left\{ \|\mathbb{E}[\mathbf{Z}_i \mathbf{Z}_i^\top]\|_{\text{op}}, \|\mathbb{E}[\mathbf{Z}_i^\top \mathbf{Z}_i]\|_{\text{op}} \right\}^{1/2}$$

$$\begin{aligned}
&= \sigma d \max \left\{ \sum_{k=1}^d \frac{\mathbf{P}_{1k}}{\mathbf{W}_{1k}}, \sum_{k=1}^d \frac{\mathbf{P}_{2k}}{\mathbf{W}_{2k}}, \dots, \sum_{k=1}^d \frac{\mathbf{P}_{dk}}{\mathbf{W}_{dk}}, \sum_{j=1}^d \frac{\mathbf{P}_{j1}}{\mathbf{W}_{j1}}, \sum_{j=1}^d \frac{\mathbf{P}_{j2}}{\mathbf{W}_{j2}}, \dots, \sum_{j=1}^d \frac{\mathbf{P}_{jd}}{\mathbf{W}_{jd}} \right\}^{1/2} \\
&= \sigma d \max \left\{ \text{rowsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right), \text{colsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right) \right\}^{1/2}.
\end{aligned}$$

Note that,

$$\sigma_z^2 \leq \sigma^2 d^3 l \quad \text{and} \quad \sigma_z^2 \geq \frac{\sigma^2 d^3}{l}.$$

So, in the case

$$\mathbf{Z}_i = \epsilon_i \tilde{\mathbf{X}}_i \quad \text{and} \quad a = \sqrt{\frac{\sigma^2 d^3}{l}} \quad \text{and} \quad b = \sqrt{\sigma^2 d^3 l} \quad \text{and} \quad \delta = \frac{ed\sigma}{(e-1)\sqrt{p_{\min}/l}},$$

we have the following by applying Lemma 5:

$$\begin{aligned}
\mathbb{P} \left(\|\tilde{\Sigma}\|_{\text{op}} \geq \lambda \right) &\leq 2d \exp \left[-Cn\lambda \left(\frac{\lambda}{\sigma^2 d^3 l} \wedge \frac{1}{\frac{d\sigma\sqrt{l}}{\sqrt{p_{\min}}} \frac{e}{e-1} \left(\log \frac{ed\sigma\sqrt{l}\sqrt{l}}{(e-1)\sqrt{p_{\min}\sigma^2 d^3}} \right)} \right) \right] \\
&\leq 2d \exp \left[-\frac{Cn\lambda}{d\sigma} \left(\frac{\lambda}{\sigma d^2 l} \wedge \frac{\sqrt{p_{\min}}}{\sqrt{l} \log(\frac{l^2}{p_{\min} d})} \right) \right].
\end{aligned} \tag{C.3}$$

We can follow the same argument for $\mathbf{Z}_i = \zeta_i \mathbf{X}_i$, where $\delta = \frac{de}{\sqrt{p_{\min}/l(e-1)}}$, and

$$\sigma_z := \max \left\{ \|\mathbb{E}[\mathbf{Z}_i \mathbf{Z}_i^\top]\|_{\text{op}}, \|\mathbb{E}[\mathbf{Z}_i^\top \mathbf{Z}_i]\|_{\text{op}} \right\}^{1/2} = d \max \left\{ \text{rowsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right), \text{colsum} \left(\frac{\mathbf{P}}{\mathbf{W}} \right) \right\}^{1/2}.$$

So

$$\sqrt{\frac{d^3}{l}} \leq \sigma_z \leq \sqrt{d^3 l}.$$

$$\begin{aligned}
\frac{\delta^2 (\log \frac{\delta}{a})^2 \log(2d)}{b^2} &= \frac{d^2 l}{p_{\min}} \left(\frac{e}{e-1} \right)^2 \left(\log \frac{e}{e-1} \sqrt{\frac{d^2 l}{(p_{\min}/l)d^3}} \right)^2 \log(2d) \frac{1}{d^3 l} \\
&\leq C \left(\log \frac{l^2}{p_{\min} d} \right)^2 \frac{\log(2d)}{p_{\min} d}.
\end{aligned}$$

Plugging in b , we have

$$C \sqrt{\frac{2e \log(d) b^2}{n}} \leq C' \sqrt{\frac{\log(d) d^3 l}{n}}.$$

Further, we have

$$n \geq C \left(\log \frac{l^2}{p_{\min} d} \right)^2 \frac{\log(d)}{p_{\min} d} \geq \frac{C}{2} \left(\log \frac{l^2}{p_{\min} d} \right)^2 \frac{\log(2d)}{p_{\min} d},$$

note that we are implicitly using $\log(2d) \leq \log d^2 = 2 \log d$ when $d \geq 2$.

We have

$$\mathbb{E} \|\tilde{\Sigma}_R\|_{\text{op}} \leq C' \sqrt{\frac{\log(d) d^3 l}{n}}. \tag{C.4}$$

Appendix D: Proof of Theorem 1

Following the recipe provided in Hamidi and Bayati (2019), we are left to demonstrate how to choose a norm $\mathfrak{N}(\cdot)$, which is guided by exponential Orlicz norm and obtain the constant n^* and choose an appropriate \mathbf{c} such that Assumption 3 holds.

For simplicity, we assume that $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ for all $i \in [n]$. Let $\mathbf{X}_i = \xi_i \cdot e_{r_i} e_{c_i}^T$ where, for each i , ξ_i is an independent $4d^2$ -sub-Gaussian random variable that is also independent of r_j and $c_j, j \in [n]$. If we set $\xi_i := d$ almost surely, then $\|\xi_i\|_{\psi_2} = d/\sqrt{\log 2} \leq 2d$, and so, satisfies our requirement. This corresponds to the problem studied in Negahban and Wainwright (2012), which notes that ξ_i has no statistical effect. Further, this way, we have $\mathbb{E}[\langle \mathbf{X}_i, \mathbf{B} \rangle] = 0$. Here we show the bounds for the slightly more general case of $\xi_i \sim \mathcal{N}(0, d^2)$.

In order to find a suitable norm $\mathfrak{N}(\cdot)$, we next study $\|\mathbf{N}\|_{\psi_2(\Pi)}$ to see how heavy-tailed $\langle \mathbf{N}, \mathbf{X}_i \rangle$ is, where $\mathbf{N} = \sqrt{\mathbf{W}} \circ \mathbf{B}$ for a given \mathbf{B} . We have

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{|\langle \mathbf{N}, \mathbf{X}_i \rangle|^2}{4d^4 \|\mathbf{N}\|_\infty^2} \right) \right] = \mathbb{E}_\Pi \left[\mathbb{E}_\xi \left[\exp \left(\frac{\xi_i^2 \mathbf{N}_{jk}^2}{4d^4 \|\mathbf{N}\|_\infty^2} \right) \right] \middle| j, k \right] \\ &= \mathbb{E}_\Pi \left[\frac{1}{\sqrt{\left(1 - \frac{\mathbf{N}_{jk}^2}{2d^2 \|\mathbf{N}\|_\infty^2}\right)_+}} \middle| j, k \right] \leq \mathbb{E}_\Pi \left[\frac{1}{\sqrt{\left(1 - \frac{1}{2d^2}\right)_+}} \middle| j, k \right] \\ &\leq \sum \mathbf{P}_{jk} \sqrt{2} \leq \sqrt{2} \leq 2. \end{aligned}$$

Note that we have used Lemma 7 and the following inequality:

$$\frac{\mathbf{N}_{jk}^2}{2d^2 \|\mathbf{N}\|_\infty^2} \leq \frac{1}{2}.$$

Therefore,

$$\|\mathbf{N}\|_{\psi_2(\Pi)} \leq 2d^2 \|\mathbf{N}\|_\infty,$$

which guides selection of $\mathfrak{N}(\mathbf{N}) = d^2 \|\mathbf{N}\|_\infty$ for any \mathbf{B} and $\mathbf{N} = \sqrt{\mathbf{W}} \circ \mathbf{B}$ and $n^* = 2d^2 \|\sqrt{\mathbf{W}} \circ \mathbf{B}^*\|_\infty$. We can now see that $\mathbf{c} = \frac{9}{d} \sqrt{\frac{l}{p_{\min}}}$ fulfills Assumption 3. The reason is, given $\mathfrak{N}(\mathbf{N}) = d^2 \|\sqrt{\mathbf{W}} \circ \mathbf{B}\|_\infty \leq 1$, we can condition on r_i and c_i and use Corollary 2 to obtain

$$\begin{aligned} c_{d,2} &\leq 5 \|\xi_i \mathbf{B}_{r_i, c_i}\|_{\psi_2} = 5 |\mathbf{B}_{r_i, c_i}| \|\xi_i\|_{\psi_2} = 5 \sqrt{\frac{8}{3}} d |\mathbf{B}_{r_i, c_i}| = 5 \sqrt{\frac{8}{3}} d^2 |\mathbf{B}_{r_i, c_i}| \sqrt{\frac{p_{\min}}{l}} \sqrt{\frac{l}{d^2 p_{\min}}} \\ &\leq 5 \sqrt{\frac{8}{3}} d^2 |\mathbf{B}_{r_i, c_i}| \sqrt{\mathbf{W}_{r_i, c_i}} \sqrt{\frac{l}{d^2 p_{\min}}} \leq 5 \sqrt{\frac{8}{3}} d^2 \|\sqrt{\mathbf{W}} \circ \mathbf{B}\|_\infty \sqrt{\frac{l}{d^2 p_{\min}}} \\ &= 5 \sqrt{\frac{8}{3}} \sqrt{\frac{l}{d^2 p_{\min}}} \leq \frac{9}{d} \sqrt{\frac{l}{p_{\min}}} = \mathbf{c}. \end{aligned}$$

So we can obtain

$$\mathbb{E}_\xi \left[\xi_i^2 \mathbf{B}_{r_i, c_i}^2 \cdot \mathbb{I}(|\xi_i \mathbf{B}_{r_i, c_i}| \leq \frac{9}{d} \sqrt{\frac{l}{p_{\min}}}) \middle| r_i, c_i \right] \geq \mathbb{E}_\xi \left[\xi_i^2 \mathbf{B}_{r_i, c_i}^2 \cdot \mathbb{I}(|\xi_i \mathbf{B}_{r_i, c_i}| \leq c_{d,2}) \middle| r_i, c_i \right] \geq \frac{\mathbb{E}[\xi_i^2 \mathbf{B}_{r_i, c_i}^2 | r_i, c_i]}{2}$$

Now we can take the expectation with respect to r_i, c_i and use the tower property to show

$$\mathbb{E} \left[\langle \mathbf{B}, \mathbf{X}_i \rangle^2 \cdot \mathbb{I} \left(|\langle \mathbf{B}, \mathbf{X}_i \rangle| \leq \frac{9}{d} \sqrt{\frac{l}{p_{\min}}} \right) \right] \geq \frac{1}{2} \mathbb{E}[\langle \mathbf{B}, \mathbf{X}_i \rangle^2],$$

which is equivalent to

$$\mathbb{E} \left[\langle \tilde{\mathbf{X}}, \mathbf{N} \rangle^2 \cdot \mathbb{I} \left(\|\langle \tilde{\mathbf{X}}, \mathbf{N} \rangle\| \leq \frac{9}{d} \sqrt{\frac{l}{p_{\min}}} \right) \right] \geq \frac{1}{2} \mathbb{E}[\langle \tilde{\mathbf{X}}, \mathbf{N} \rangle^2].$$

We can now combine Condition (C.3), (C.4) and (4.3) to obtain the following result:

for any $\lambda \geq C_3 n^* \sqrt{\log(d)dl/n}$ and $n \geq C_3 (\log \frac{l^2}{p_{\min}d})^2 \frac{\log d}{p_{\min}d}$, the inequality

$$\|\hat{\mathbf{B}} - \mathbf{B}^*\|_{L^2(\Pi)}^2 \leq C_4 \lambda^2 \tilde{r} l^3,$$

holds with probability at least

$$1 - 2d \exp \left[-\frac{Cn\lambda}{d\sigma} \left(\frac{\lambda}{\sigma l} \wedge \frac{1}{\sqrt{l} \log(\frac{l^2}{d})} \right) \right] - \exp \left(-\frac{C_5 d^2 n \lambda^2 \tilde{r}}{n^{*2}} \right).$$

In particular, setting

$$\lambda = C_6 (\sigma \vee n^*) \sqrt{\frac{\rho l d}{n p_{\min}}}$$

for some $\rho \geq \log d$. We have that

$$\|\hat{\mathbf{B}} - \mathbf{B}^*\|_{L^2(\Pi)}^2 \leq C_1 (\sigma^2 \vee n^{*2}) \frac{d \rho l^4 \tilde{r}}{n p_{\min}},$$

with probability at least $1 - \exp(-C_2 \rho)$ for some $\rho \geq \log d$ and constants C_1, C_2 and C_3 , whenever $n \geq C_3 (\log \frac{l^2}{p_{\min}d})^2 \frac{\log d}{p_{\min}d}$.

Appendix E: Auxiliary proofs

LEMMA 7. Let Z be a $\mathcal{N}(0, \sigma^2)$ random variable. Then, for all $\eta > 0$,

$$\mathbb{E}[e^{\eta Z^2}] = \frac{1}{\sqrt{(1 - 2\sigma^2 \eta)_+}}.$$

LEMMA 8. Let \mathbf{Z} be a non-negative random variable such that $\|\mathbf{Z}\|_{\psi_p} = \nu$ holds for some $p \geq 1$, and assume $c > 0$ is given. Then we have

$$\mathbb{E}[Z^2 \cdot \mathbb{I}(Z \geq c)] \leq (2c^2 + 4c\nu + 4\nu^2) \cdot \exp(-\frac{c^p}{\nu^p}).$$

COROLLARY 1. Let \mathbf{Z} be a random variable satisfying $\|\mathbf{Z}\|_{\psi_p} = \nu$ holds for some $p \geq 1$ and $\mathbb{E}[Z^2] = \sigma^2$. Then, for

$$c_{\sigma,p} := \nu \cdot \max \left\{ 5, [10 \log(\frac{2\nu^2}{\sigma^2})]^{\frac{1}{p}} \right\}, \quad (\text{E.1})$$

we have

$$\mathbb{E}[Z^2 \cdot \mathbb{I}(|Z| \leq c_{\sigma,p})] \geq \frac{\mathbb{E}[Z^2]}{2}.$$

COROLLARY 2. Let Z be a $\mathcal{N}(0, \sigma^2)$ random variable. Then, the constant $c_{\sigma,2}$ defined in (E.1) satisfies $c_{\sigma,p} \leq 5\|Z\|_{\psi_2}$.

The following lemma gives rise to the proximal mapping we came up earlier.

LEMMA 9.

$$\mathcal{S}_\lambda(\mathbf{M}) = \arg \min_{\mathbf{M}} \frac{1}{2} \|\mathbf{M} - \mathbf{N}\|_F^2 + \lambda \|\mathbf{M}\|_*.$$

Proof. (This proof is adapted from Cai et al. (2008)) Since the function

$$h_0(\mathbf{M}) := \frac{1}{2} \|\mathbf{M} - \mathbf{N}\|_F^2 + \lambda \|\mathbf{M}\|_*,$$

is strictly convex, it is easy to see that there exists a unique minimizer, call it $\widehat{\mathbf{M}}$. Now $\widehat{\mathbf{M}}$ minimizes $f(\mathbf{M})$ if and only if 0 is a subgradient of the functional f at the point $\widehat{\mathbf{M}}$, i.e.,

$$\mathbf{0} \in \partial f(\widehat{\mathbf{M}}).$$

We have

$$\partial f(\widehat{\mathbf{M}}) = \lambda \partial \|\widehat{\mathbf{M}}\|_* + \widehat{\mathbf{M}} - \mathbf{N},$$

where $\partial \|\widehat{\mathbf{M}}\|_*$ is the set of subgradients of the trace norm.

It is known that

$$\partial \|\mathbf{M}\|_* = \{\mathbf{U}\mathbf{V}^* + \mathbf{W} : \mathbf{W} \in \mathbb{R}^{n \times m}, \mathbf{U}^* \mathbf{W} = 0, \mathbf{W} \mathbf{V} = 0, \|\mathbf{W}\|_{op} \leq 1\},$$

where $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ is the SVD of \mathbf{M} .

Now we want to show $\widehat{\mathbf{M}} = \mathcal{S}_\lambda(\mathbf{N})$. We first decompose the SVD of \mathbf{N} as:

$$\mathbf{N} = \mathbf{U}_0 \mathbf{\Sigma}_0 \mathbf{V}_0^* + \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^*,$$

where $\mathbf{U}_0, \mathbf{V}_0$ (resp. $\mathbf{U}_1, \mathbf{V}_1$) are the singular vectors associated with singular values greater than λ (resp. smaller than or equal to λ).

With these notation, we have $\mathcal{S}_\lambda(\mathbf{N}) = \mathbf{U}_0(\mathbf{\Sigma}_0 - \lambda \mathbf{I}) \mathbf{V}_0^*$, and therefore, $\mathbf{N} - \mathcal{S}_\lambda(\mathbf{N}) = \lambda(\mathbf{U}_0 \mathbf{V}_0^* + \mathbf{W})$, where $\mathbf{W} = \lambda^{-1} \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^*$.

By definition, $\mathbf{U}_0^* \mathbf{W} = 0, \mathbf{W} \mathbf{V}_0 = 0$ and since the diagonal elements of $\mathbf{\Sigma}_1$ have magnitudes bounded by λ , we also have $\|\mathbf{W}\|_2 \leq 1$. Hence, $\mathbf{N} - \mathcal{S}_\lambda(\mathbf{N}) \in \lambda \partial \|\widehat{\mathbf{M}}\|_*$, which concludes the proof.