# When Are Linear Stochastic Bandits Attackable?

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#### Abstract

We study adversarial attacks on linear stochastic bandits, a sequential decision making problem with many important applications in recommender systems, online advertising, medical treatment, and etc. By manipulating the rewards, an adversary aims to control the behaviour of the bandit algorithm. Perhaps surprisingly, we first show that some attack goals can never be achieved. This is in sharp contrast to context-free stochastic bandits, and is intrinsically due to the correlation among arms in linear stochastic bandits. Motivated by this observation, this paper studies the attackability of a k-armed linear bandit environment. We first provide a full necessity and sufficiency characterization of attackability based on the geometry of the context vectors. We then propose a two-stage attack method against LinUCB and Robust Phase Elimination. The method first asserts whether the current environment is attackable, and if Yes, modifies the rewards to force the algorithm to pull a target arm linear times using only a sublinear cost. Numerical experiments further validate the effectiveness and cost-efficiency of the proposed method.

#### 1 Introduction

In a contextual bandit problem, a learner takes sequential actions to interact with an environment to maximize its received cumulative reward. As a natural and important variant, linear stochastic bandits (Auer, 2002; Li et al., 2010; Abbasi-yadkori et al., 2011) assume the expected reward of an arm a is a linear function of its feature vector  $x_a$  and an unknown bandit parameter  $\theta^*$ . A linear bandit algorithm thus adaptively improves its estimation of  $\theta^*$  based on the reward feedback on its pulled arms. Thanks to their sound theoretical guarantees and promising empirical performance, linear stochastic bandits have become a reference solution to many real-world problems, such as content recommendation and online advertisement (Li et al., 2010; Chapelle and Li, 2011; Durand et al., 2018).

Since bandit algorithms adapt their behavior according to its received feedback, such algorithms are susceptible to adversarial attacks, especially the data poisoning attacks. Under such an attack, a malicious adversary observes the pulled arm and its reward feedback, and then modifies the reward to misguide the bandit algorithm to pull a target arm, which is of the adversary's interest. Due to the wide applicability of bandit algorithms in practice, understanding the robustness of such algorithms under data poisoning attacks becomes increasingly important (Jun et al., 2018; Liu and Shroff, 2019; Garcelon et al., 2020).

Most existing studies on adversarial attacks in bandits focused on context-free settings. Jun et al. (2018) and Liu and Shroff (2019) showed that in a context-free stochastic multi-armed bandit

environment, an adversary can force any bandit algorithm to pull a target arm linear times only using a logarithmic cost. Garcelon et al. (2020) studied poisoning attacks on k-armed linear contextual bandits and showed its vulnerability. Linear stochastic bandits are related to context-free stochastic bandits and linear contextual bandits. But surprisingly, there is no known result about attacks on linear stochastic bandit to our best knowledge; in fact, even whether such a learning environment is attackable is unknown. It turns out that there is a reason for this gap — as we will elaborate later, the analysis of attacks to linear stochastic bandits turns out to be more challenging due to correlation among arms.

In this paper, we fill the aforementioned gap by studying data poisoning attacks on k-armed linear stochastic bandits, where an adversary modifies the reward using a sublinear budget to misguide the bandit algorithm to pull a target arm  $\tilde{x}$  linear times. We first prove a linear stochastic bandit environment is not always efficiently attackable<sup>1</sup>, and its attackability is determined by the feasibility of finding a parameter vector  $\tilde{\boldsymbol{\theta}}$ , under which the rewards of all non-target arms are smaller than the reward of target arm  $\tilde{x}$  and the reward of  $\tilde{x}$  remains the same as that in the original environment specified by  $\boldsymbol{\theta}^*$ . Intuitively, to promote the target arm  $\tilde{x}$ , an adversary needs to lower the rewards of non-target arms in the null space of  $\tilde{x}$  by  $\tilde{\boldsymbol{\theta}}$ , which might not be always feasible. We prove the feasibility of the resulting quadratically constrained linear program is both sufficient and necessary for attacking a linear stochastic bandit environment.

Inspired by our attackability analysis, we propose a two-stage attack framework against linear stochastic bandit algorithms and demonstrate its application against LinUCB (Li et al., 2010) and Robust Phase Elimination (Bogunovic et al., 2021): the former is one of the most widely used linear contextual bandit algorithms, and the latter is a robust version designed for settings with adversarial corruptions. In the first stage, our method collects a carefully calibrated amount of rewards on the target arm to assess whether the given environment is attackable. The decision is based on an "empirical" version of our feasibility characterization. If attackable, i.e., there exists a feasible solution  $\tilde{\theta}$ ; in the second stage the method depresses the rewards the bandit algorithm receives on non-target arms based on  $\tilde{\theta}$ , in order to fool the bandit algorithm to recognize the target arm as optimal.

Our main contributions can be summarized as follows:

- We characterize the sufficient and necessary conditions about when a linear bandit environment is attackable as the feasibility of a set of linear constraints. En route to proving the sufficiency, we also provide an oracle attack method that can attack any no-regret learning algorithm given the knowledge of ground-truth bandit parameter  $\theta^*$ . A direct corollary of our characterization is that context-free stochastic MAB is always attackable, recovering the findings in (Jun et al., 2018; Liu and Shroff, 2019).
- We propose a two-stage attack method that works without the knowledge of ground-truth bandit parameter. In the first stage, the algorithm detects the attackability of the environment and estimates the model parameter. In the second stage, it solves for a working solution  $\tilde{\boldsymbol{\theta}}$  and attacks accordingly. Our theoretical analysis shows this attack method is effective against LinUCB (Li et al., 2010) and Robust Phase Elimination (Bogunovic et al., 2021), i.e., pulling the target arm T o(T) times using o(T) budget when the environment is attackable.

<sup>&</sup>lt;sup>1</sup>Throughout this paper, "efficient attack" means fooling bandit algorithm to pull the target arm for linear times with sublinear attack cost. We will use *attackable* and *efficiently attackable* interchangeable, as the adversary normally only has a limited budget and needs to design a cost-efficient strategy.

## 2 Preliminaries

Linear stochastic bandit. We study the problem of adversarial attacks on k-arm linear stochastic bandit, where a bandit algorithm sequentially interacts with an environment for T rounds. In each round t, the algorithm pulls an arm  $a_t$  from a set  $\mathcal{A} = \{x_i\}_{i=1}^k$  with k arms, and receives reward  $r_t$  from the environment. Each arm a is associated with a d-dimension context feature vector  $x_a \in \mathbb{R}^d$  and we assume  $\|x_a\|_2 \leq 1$ . The expected reward of arm a is assumed to be a linear function of both context feature and unknown bandit parameter  $\theta^*$ , i.e.,  $\mathbb{E}[r_a] = x_a^\mathsf{T} \theta^*$ , where  $\theta^* \in \mathbb{R}^d$  and we assume  $\|\theta^*\|_2 \leq 1$ . After pulling arm  $a_t$ , the algorithm observes reward feedback  $r_{a_t,t} = x_{a_t}^\mathsf{T} \theta^* + \eta_t$ , where  $\eta_t$  is an R-sub-Gaussian noise term. The performance of a bandit algorithm is evaluated by its pseudo-regret, which is defined as  $R(T) = \sum_{t=1}^T (x_{a_t}^\mathsf{T} \theta^* - x_{a_t}^\mathsf{T} \theta^*)$ , where  $a^*$  is the best arm according to  $\theta^*$ , i.e.,  $x_{a^*} = \arg\max_{x \in \mathcal{A}} [x^\mathsf{T} \theta^*]$ . Due to the possible correlation among the context vectors, manipulating the reward of an arm will also change the reward estimation of other correlated arms. This is different from the k-arm linear contextual bandits setting considered in (Garcelon et al., 2020), where each arm has its own bandit parameter and the reward estimation is independent among arms. Thus the reward manipulation of an arm will not affect other arms.

LinUCB. LinUCB (Li et al., 2010; Abbasi-yadkori et al., 2011) is a classical algorithm for linear stochastic bandit. It estimates a bandit model parameter  $\hat{\boldsymbol{\theta}}$  using ridge regression, i.e.,  $\hat{\boldsymbol{\theta}}_t = \mathbf{A}_t^{-1} \sum_{i=1}^t x_{a_i} r_i$ , where  $\mathbf{A}_t = \sum_{i=1}^t x_{a_i} x_{a_i}^\mathsf{T} + \lambda \mathbf{I}$  and  $\lambda$  is the coefficient of L2-regularization. We use  $\|x\|_{\mathbf{A}} = \sqrt{x^\mathsf{T}} \mathbf{A} x$  to denote the matrix norm of vector x. Confidence bound about reward estimation on arm x is defined as  $\mathrm{CB}_t(x) = \alpha_t \|x\|_{\mathbf{A}_t^{-1}}$ , where  $\alpha_t$  is a high probability bound of  $\|\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_t\|_{\mathbf{A}_t}$ . In each round t, LinUCB pulls an arm with the highest upper confidence bound, i.e.,  $a_t = \arg\max_a [x_a^\mathsf{T} \hat{\boldsymbol{\theta}}_t + \mathrm{CB}_t(x_a)]$  to balance the explore-exploit trade-off. LinUCB algorithm achieves a sublinear upper regret bound (Chu et al., 2011; Abbasi-yadkori et al., 2011), i.e.,  $R(T) = \tilde{O}(\sqrt{T})$  ignoring the logarithmic term.

Threat model. The goal of an adversary is to fool the linear stochastic bandit algorithm to pull the target arm  $\tilde{x} \in \mathcal{A}$  for T - o(T) times. Like most recent works in this space (Jun et al., 2018; Liu and Shroff, 2019; Garcelon et al., 2020; Zhang et al., 2020), we consider the data poisoning attack on the rewards: after arm  $a_t$  is pulled by the bandit algorithm, the adversary modifies the original reward  $r_{a_t}$  from the environment by  $\Delta r_t$  to become  $\tilde{r}_{a_t}$ , i.e.,  $\tilde{r}_{a_t} = r_{a_t} + \Delta r_t$ , and provides the manipulated reward  $\tilde{r}_{a_t}$  to the algorithm. Naturally, the adversary should achieve its attack goal with minimum attack cost  $C(T) = \sum_{t=1}^{T} |\mathbb{E} [\Delta r_t]|$ . An attack strategy is considered efficient, if it achieves a sublinear cost, i.e., C(T) = o(T). Note that the expectation of reward manipulation  $\Delta r_t$  is taken with respect to only the sub-Gaussian noise in the rewards.

# 3 The Attackability of Linear Stochastic Bandits

We study the attackability of a linear stochastic bandit environment. At a first glance, one might wonder whether attackability is the property of a bandit algorithm rather than a property of the environment, since if an algorithm can be attacked, we should have "blamed" the algorithm for not being robust, rather than blaming the environment. A key finding of this work is that attackability is also a property of the linear stochastic bandit environment; and in other words, not all environments are attackable.

**Definition 1** (Attackability of a k-Arm Linear Stochastic Bandit Environment). A k-arm linear stochastic bandit environment  $\langle \mathcal{A} = \{x_i\}_{i=1}^k, \boldsymbol{\theta}^* \rangle$  is attackable with respect to (w.r.t.) target arm

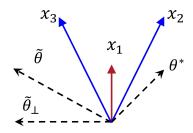


Figure 1: Illustration of attackability.

 $\tilde{x} \in \mathcal{A}$  if for any no-regret learning algorithm, there exists an attack method that uses o(T) attacking budget but fools the algorithm to pull arm  $\tilde{x}$  at least T - o(T) times for any T large enough.

It is worthwhile to further digest the above definition of attackability. First, this definition is all about the bandit environment  $\langle \mathcal{A}, \boldsymbol{\theta}^* \rangle$  and the target arm  $\tilde{x}$ , but independent of any specific bandit algorithms. Second, if an attack method can only fool a bandit algorithm to pull the target arm  $\tilde{x}$  under (only) a few different time horizons T, it does not count as a successful attack — a successful attack has to succeed for infinitely many time horizons. Third, by reversing the order of quantifiers, we obtain an assertion that a bandit environment is not attackable w.r.t. the target arm  $\tilde{x}$  if there exists some no-regret learning algorithm such that no attack method can use o(T) attack budget to fool the algorithm to pull arm  $\tilde{x}$  at least T - o(T) times for any T large enough. The following simple yet insightful example illustrates that there are indeed linear stochastic bandit environment setups where some no-regret learning algorithm cannot be attacked.

Example 1 (An unattackable environment). Figure 1 shows a three-arm environment with  $\mathcal{A} = \{x_1 = (0,1), x_2 = (1,2), x_3 = (-1,2)\}$ . Suppose the target arm  $\tilde{x} = x_1$  and the ground-truth bandit parameter  $\boldsymbol{\theta}^* = (1,1)$ . The expected true rewards of the arms are  $r_1^* = 1, r_2^* = 3, r_3^* = 1$  and  $x_2$  is the best arm in this environment. We give an intuitive argument here that this environment with target arm  $\tilde{x}$  is not attackable, while its formal proof is an instantiation of our Theorem 1. Specifically, we argue that LinUCB cannot be attacked in the above environment (our argument shall generalize to any linear-regression based no-regret algorithms). Suppose, for the sake of contradiction, that there exists an efficient attack which fools LinUCB to pull  $x_1$  T - o(T) times. LinUCB then must estimate  $\boldsymbol{\theta}^*$  in the  $x_1$ 's direction almost accurately as T becomes large, since the  $\Omega(T)$  amount of true reward feedback in  $x_1$  direction will ultimately dominate the o(T) adversarial contamination. This suggests that the estimated parameter  $\hat{\boldsymbol{\theta}}_t$  will be close to  $\rightarrow (\alpha, 1)$  for some  $\alpha$ . Since  $(\alpha, 1)^T(x_2 + x_3) = 4$ , implying that either  $x_2$  or  $x_3$  will have its estimated reward larger than 2 (i.e., strictly larger than  $x_1$ 's estimated reward) for any  $\alpha$ . This shows that  $x_1$  cannot be the best arm in LinUCB's estimation, which causes a contradiction. Therefore, we can safely conclude that this environment cannot be attacked given o(T) budget.

Note that when  $\mathcal{A} = \{x_1, x_2\}$ , the environment becomes attackable: as shown in the figure, a feasible attack strategy is to perturb reward according to  $\tilde{\boldsymbol{\theta}} = (-2, 1)$ . The key idea is that in the null space of  $x_1$ ,  $\tilde{\boldsymbol{\theta}}_{\perp}$  reduces the reward of  $x_2$  to make  $x_1$  the best arm but without changing the reward of  $x_1$  from the environment. The presence of arm  $x_3$  prevents the existence of such a  $\tilde{\boldsymbol{\theta}}_{\perp}$  (and therefore  $\tilde{\boldsymbol{\theta}}$ ) and makes the environment unattackable.

The above example motivates us to study when a linear stochastic bandit environment is attackable. After all, only when we face an unattackable environment, we can ensure the existence of certain no-regret algorithms that would be immune to some particular type of adversarial attacks.

We now proceed to give a complete characterization about what bandit environments are attackable. As Example 1 shows, the attackability of a bandit algorithm depends on the arm set  $\mathcal{A} = \{x_i\}_{i=1}^k$ , the target arm  $\tilde{x}$ , and the underlying bandit parameter  $\boldsymbol{\theta}^*$ . For clarity of presentation, in this section, we shall always assume that the adversary knows exactly the ground-truth bandit parameter  $\boldsymbol{\theta}^*$  and thus the true expected reward of each arm. This is also called the *oracle attack* in previous works (Jun et al., 2018; Liu and Shroff, 2019; Rakhsha et al., 2020). However, in the next section, we will show that this assumption is not needed: when the bandit environment is indeed attackable, we can design provably successful attacks even if the adversary does not know the underlying bandit parameter  $\boldsymbol{\theta}^*$ .

We need the following convenient notation to state our results. Let

$$\boldsymbol{\theta}_{\parallel}^* = \frac{\tilde{x}^{\mathsf{T}} \boldsymbol{\theta}^*}{\|\tilde{x}\|_2^2} \tilde{x} \tag{1}$$

denote the projection of ground-truth bandit parameter  $\theta^*$  onto the  $\tilde{x}$  direction. Since the attackability depends on the target arm  $\tilde{x}$  as well, we shall include the target arm  $\tilde{x}$  as part of the bandit environment. The following theorem provides a clean characterization of attackability.

**Theorem 1** (Characterization of Attackability). A bandit environment  $\langle \mathcal{A} = \{x_i\}_{i=1}^k, \boldsymbol{\theta}^*, \tilde{x} \rangle$  is attackable if and only if the optimal objective  $\epsilon^*$  of the following quadratically constrained linear program (QCLP) satisfies  $\epsilon^* > 0$ .

maximize 
$$\epsilon$$
  
subject to  $\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} \geq \epsilon + x_{a}^{\mathsf{T}}(\boldsymbol{\theta}_{\parallel}^{*} + \tilde{\boldsymbol{\theta}}_{\perp}), \text{ for } x_{a} \neq \tilde{x}.$   
 $\tilde{x}^{\mathsf{T}}\tilde{\boldsymbol{\theta}}_{\perp} = 0$   
 $\|\boldsymbol{\theta}_{\parallel}^{*} + \tilde{\boldsymbol{\theta}}_{\perp}\|_{2} \leq 1$  (2)

where  $\epsilon \in \mathbb{R}$  and  $\tilde{\boldsymbol{\theta}}_{\perp} \in \mathbb{R}^d$  are variables.

Since the conceptual message of Theorem 1 significantly differs from previous studies on adversarial attacks in bandit algorithms, we would like to elaborate on its implications. Specifically, our characterization of the attackability is a property of the bandit environment but not a property of some bandit algorithms. We, for the first time, point out some learning environment is just intrinsically vulnerable such that any no regret learning algorithm can be attacked (as later shown in our oracle attack). The significance is that only when an environment is not attackable, i.e., QCLP (2) has optimal objective  $\epsilon^* \leq 0$ , it is possible to design good robust algorithms for such an environment. And only in this situation, if the regret is still large we can then "blame" the algorithm for not being robust enough, since by the definition of (un)attackability there does exist robust no-regret algorithms. Notably, almost all previous works have focused on the vulnerability of bandit algorithms, e.g., by designing attacks against UCB,  $\epsilon$ -Greedy (Jun et al., 2018), LinUCB (Garcelon et al., 2020), as it just so happens that they were already studied under an attackable environment (see our Corollary 1). Additionally, as we will show next, our proof techniques are also significantly different from existing ones, since what's central to our proof is to argue that when QCLP (2) is not satisfied, then there will exist a robust algorithm that cannot be efficiently attacked by any adversary. This can be viewed as analyzing the robustness properties of certain bandit algorithms when  $\epsilon^* < 0$  in QCLP (2).

Since QCLP (2) and its solutions will show up very often in our later discussions, we provide the following definition for reference convenience.

#### Algorithm 1 Oracle Null Space Attack

```
1: Inputs: T, \theta^*
 2: Initialize:
     if Optimal objective \epsilon^* of QCLP (2) > 0 then
                                                                                                                          // Attackability Test
          Find the optimal solution \hat{\boldsymbol{\theta}}_{\perp}
          Set \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_{\parallel}^* + \tilde{\boldsymbol{\theta}}_{\perp}
 5:
 6: else
 7:
          return Not attackable
 8: end if
 9: for t = 1 to T do
          Bandit algorithm pulls arm a_t
10:
          Attacker observes the corresponding reward r_t = x_{a_t}^\mathsf{T} \theta^* + \eta_t from the environment
11:
12:
          if x_{a_t} \neq \tilde{x} then
                Set \tilde{r}_t = x_{a_t}^\mathsf{T} \tilde{\boldsymbol{\theta}} + \tilde{\eta}_t
                                                                                                                                            // Attack
13:
14:
                Set \tilde{r}_t = r_t
15:
          end if
16:
          Bandit algorithm observes modified reward \tilde{r}_t
17:
```

**Definition 2** (Attackability Index and Certificate). The optimal objective  $\epsilon^*$  of QCLP (2) is called the attackability index and the optimal solution  $\tilde{\boldsymbol{\theta}}_{\perp}$  to QCLP (2) is called the attackability certificate.

Notably, both the index  $\epsilon^*$  and certificate  $\tilde{\boldsymbol{\theta}}_{\perp}$  are intrinsic to the bandit environment  $\langle \mathcal{A} = \{x_i\}_{i=1}^k, \boldsymbol{\theta}^*, \tilde{x} \rangle$ , and are irrelevant to any bandit algorithms used. As we will see in the next section when designing attack algorithms without the knowledge of  $\boldsymbol{\theta}^*$ , the attackability index  $\epsilon^*$  will determine how difficult it is to attack the environment.

*Proof of Theorem 1.* **Proof of sufficiency.** For sufficiency proof, we show that there exists an efficient attack strategy if QCLP (2) holds.

Suppose the attackability index  $\epsilon^* > 0$  and let  $\tilde{\boldsymbol{\theta}}_{\perp}$  be a certificate. In Algorithm 1, we design the **oracle null space attack** based on the knowledge of bandit parameter  $\boldsymbol{\theta}^*$ . Let  $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_{\parallel}^* + \tilde{\boldsymbol{\theta}}_{\perp}$  where  $\boldsymbol{\theta}_{\parallel}^*$  is defined in Eq (1). The adversary perturbs the reward of any non-target arm  $x_a \neq \tilde{x}$  by  $\tilde{r}_{a,t} = x_a^{\mathsf{T}} \tilde{\boldsymbol{\theta}} + \tilde{\eta}_t$ , whereas the reward of the target arm  $\tilde{x}$  is not touched. In other words, the adversary misleads the algorithm to believe that  $\tilde{\boldsymbol{\theta}}$  is the ground-truth parameter: we should note both  $\tilde{\boldsymbol{\theta}}$  and  $\boldsymbol{\theta}^*$  generate the same expected reward on  $\tilde{x}$ , i.e.,  $\tilde{x}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} = \tilde{x}^{\mathsf{T}} \boldsymbol{\theta}_{\parallel}^* = \tilde{x}^{\mathsf{T}} \boldsymbol{\theta}^*$ . To make the attack appear less "suspicious", a sub-Gaussian noise  $\tilde{\eta}_t$  is added to the perturbed reward to make it stochastic. The key idea is that the attacker does not need to perturb the reward of target arm because the original reward is the same as perturbed reward in expectation. Instead, the attacker only perturbs the reward in the null space of  $\tilde{x}$  according to  $\tilde{\boldsymbol{\theta}}_{\perp}$ , which guarantees the cost-efficiency of the attack.

Since the perturbed rewards observed by the bandit algorithm are generated by  $\tilde{\boldsymbol{\theta}}$ , the target arm  $\tilde{x}$  is the optimal arm in this environment due to the attackability index  $\epsilon^*$  being strictly positive Any no-regret bandit algorithm will only pull the non-target arms o(T) times and pull target arm

<sup>&</sup>lt;sup>2</sup>We sometimes omit "attackability" when it is clear from the context, and simply mention *index* and *certificate*.

T - o(T) times. Thus the attack is successful. Moreover, the cost of oracle attack is o(T) because the attacker only perturbs rewards on the non-target arms for o(T) times, and the expected cost on each attack is bounded by a constant (because of the finite norm of  $x_a$  and  $\theta^*$ ).

**Proof of necessity.** We discuss the proof sketch here and leave the detailed proof in the appendix. We shall prove that if  $\epsilon^* \leq 0$ , the bandit environment is not attackable. By our definition of (un)attackability, we will need to identify at least one no-regret bandit algorithm such that no attack strategy can succeed in attacking it. In particular, we will show that LinUCB is robust to any attack strategy with o(T) budget when  $\epsilon^* \leq 0$ .

LinUCB maintains a model estimate  $\hat{\boldsymbol{\theta}}_t$  at round t using the attacked rewards  $\{\tilde{r}_{1:t}\}$ . We consider LinUCB with the choice of L2-regularization parameter  $\lambda$  that guarantees  $\|\hat{\theta}_t\|_2 < 1$  in order to satisfy the last constraint in QCLP (2). We decompose  $\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t,\parallel} + \hat{\boldsymbol{\theta}}_{t,\perp}$ , where  $\tilde{x} \perp \hat{\boldsymbol{\theta}}_{t,\perp}$  and  $\tilde{x} \parallel \hat{\boldsymbol{\theta}}_{t,\parallel}$ . Suppose, for the sake of contradiction, LinUCB is attackable when  $\epsilon^* \leq 0$ . According to Definition 1, the target arm  $\tilde{x}$  will be pulled T - o(T) times for infinitely many different time horizons T. Note that the following inequalities hold when  $\tilde{x}$  has the unique largest UCB score (and thus is pulled with probability 1):

$$\tilde{x}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{t,\parallel} + \mathrm{CB}_{t}(\tilde{x}) > x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{t,\parallel} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{t,\perp} + \mathrm{CB}_{t}(x_{a}), \forall x_{a} \neq \tilde{x}$$
(3)

By attackability, we know that the above inequality will hold for infinitely many t's. As  $t \to \infty$ , we have  $CB_t(\tilde{x}) \to 0$ , and  $CB_t(x_a)$  is strictly positive. Moreover, the estimation of  $\hat{\boldsymbol{\theta}}_{t,\parallel}$  will converge to  $\boldsymbol{\theta}_{\parallel}^*$  since  $\tilde{x}$  will be pulled for t - o(t) times. By letting  $t \to \infty$  in both sides of the above inequalities, we have the following conclusion:

$$\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} > x_{a}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{t,\perp}, \forall x_{a} \neq \tilde{x}$$

$$\tag{4}$$

This implies that there must exist a  $\hat{\boldsymbol{\theta}}_{t,\perp}$  that satisfies  $\tilde{x} \perp \hat{\boldsymbol{\theta}}_{t,\perp}$  and makes the objective of QCLP (2) strictly above 0. Therefore, its optimal objective  $\epsilon^*$  must also be strictly positive. This however contradicts our assumption  $\epsilon^* \leq 0$ , implying that LinUCB is not attackable by any attack strategy under such an environment.

We now provide an intuitive explanation about Theorem 1. QCLP (2) is to find a vector  $\tilde{\boldsymbol{\theta}}_{\perp}$  such that: 1) it is orthogonal to  $\tilde{x}$  (hence its subscript); and 2) it maximizes the gap  $\epsilon$  between  $\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^*$  and the largest  $x_a^{\mathsf{T}}(\boldsymbol{\theta}_{\parallel}^* + \tilde{\boldsymbol{\theta}}_{\perp})$  among all  $x_a \neq \tilde{x}$ . Theorem 1 states that the bandit environment is attackable if and only if such a gap (i.e., the attackability index) is strictly larger than 0, i.e., when the geometry of arm context vectors allows the adversary to lower non-target arms' rewards by attacking in the null space of  $\tilde{x}$ . The constraint  $\|\boldsymbol{\theta}_{\parallel}^* + \tilde{\boldsymbol{\theta}}_{\perp}\|_2 \leq 1$  ensures the attacked rewards are in the same scale as the unattacked rewards. Our analysis characterizes the attackability based on the geometry of arm features: when the geometry forbids an adversary from lowering the rewards of non-target arms in the null space of the target arm, the environment is unattackable.

Recent works have shown that any no-regret algorithm for the context-free k-armed setting (where arm set  $\mathcal{A}$  is orthonormal) can be attacked (Liu and Shroff, 2019), i.e., a context-free stochastic bandit environment is always attackable under Definition 1. This finding turns out to be a corollary of Theorem 1.

**Corollary 1.** For standard stochastic bandit setting where arm set A is orthonormal, the environment  $A = \{x_a\}, \theta^*, \tilde{x}\}$  is attackable for any target arm  $\tilde{x}$ .

*Proof.* Since arms are orthogonal to each other, it is easy to see that  $\tilde{\boldsymbol{\theta}}_{\perp} = -C[\sum_{x_a:x_a\neq \tilde{x}} x_a]$  achieves objective value  $C - \tilde{x}^T \boldsymbol{\theta}_{\parallel}^*$  in QCLP (2). Let C be a large enough positive constant such that the objective value is positive gives us a feasible  $\tilde{\boldsymbol{\theta}}_{\perp}$  to QCLP (2), which yields the corollary.

The intuition behind this corollary is that arms in context-free stochastic bandits are independent, and an adversary can lower the rewards of non-target arms to make the target arm optimal. This is also the attack strategy in (Jun et al., 2018; Liu and Shroff, 2019). Garcelon et al. (Garcelon et al., 2020) showed that similar idea works for k-arm linear contextual bandits, where each arm is associated with an unknown bandit parameter and the reward estimations are independent among different arms. Arguably, our setting is more challenging since arms are correlated and the simple attack idea may not be successful as shown in our Example 1.

# 4 Effective Attacks without Knowledge of True Model Parameters

In the previous section, we characterized the attackability of a linear stochastic bandit environment by assuming the knowledge of ground-truth bandit parameter  $\theta^*$ . We now show that such oracle knowledge is actually not needed when designing executable attacks. Towards this end, we propose provably effective attacks against two representative bandit algorithms, the most well-known LinUCB (Abbasi-yadkori et al., 2011) and a state-of-the-art robust linear stochastic bandit algorithm based on robust phase elimination (Bogunovic et al., 2021). Their different levels of robustness lead to different amount of required attack cost, which further illustrates that the attack analysis often goes hand-in-hand with robustness analysis.

Two-stage Null Space Attack. Our proposed attack method is presented in Algorithm 2. The adversary spends the first  $T_1$  rounds as the first stage to attack rewards on all arms by imitating a bandit environment  $\theta_0$ , in which  $\tilde{x}$  is the best arm such that arm  $\tilde{x}$  will be pulled most often by the bandit algorithm. This stage is for the adversary to observe the rewards for  $\tilde{x}$  from the environment, which helps it estimate the parameter  $\boldsymbol{\theta}_{\parallel}^*$ . At round  $T_1$ , the method uses the estimate of  $\boldsymbol{\theta}_{\parallel}^*$ , denoted as  $\tilde{\boldsymbol{\theta}}_{\parallel}$ , to perform the "attackability test" by solving QCLP (2) using  $\tilde{\boldsymbol{\theta}}_{\parallel}$  to obtain an estimated index  $\tilde{\epsilon}^*$  and certificate  $\tilde{\boldsymbol{\theta}}_{\perp}$ . If  $\tilde{\epsilon}^* > 0$ , the method asserts the environment is attackable and starts the second stage of attack. From  $T_1$  to T, the adversary perturbs the reward by  $\tilde{r} = x^{\mathsf{T}}(\tilde{\boldsymbol{\theta}}_{\parallel} + \tilde{\boldsymbol{\theta}}_{\perp})$  (just like the oracle attack but using the estimated  $\hat{\theta}_{\parallel}$ ). When the bandit algorithm pulls the target arm  $\tilde{x}$  for the first time in the second stage, the adversary will compensate its reward as shown in line 19.  $n(\tilde{x})$  is the number of times target arm is pulled before  $T_1$ . The goal is to correct the rewards on  $\tilde{x}$ collected in the first stage to follow  $\theta$  instead of  $\theta_0$ . Note that a larger  $T_1$  brings in more observations on  $\tilde{x}$  to improve the estimate of  $\boldsymbol{\theta}_{\parallel}^*$ ; but it also means a larger attacking cost. The optimal choice of  $T_1$  depends on certain "robustness" property of the bandit algorithm in use. Consequently, it also leads to different amount of attack cost for different algorithms. For example, as we will show below, the attack to Robust Phase Elimination will be more costly than the attack to LinUCB.

Our attackability test might make both false positive and false negative assertions due to the estimation error in  $\tilde{\boldsymbol{\theta}}_{\parallel}$ . But as T become large, the estimate is more accurate and the assertion is correct with high probability (see below). We note that an important step in our attack is that the adversary manipulates the rewards for both the targeted arm and other arms in the second stage, as shown in line 21 of Algorithm 2. This is different from the oracle attack where only the rewards of non-target arms are perturbed. This difference is crucial because it guarantees that the rewards are

#### Algorithm 2 Two-stage Null Space Attack

```
1: Inputs: T, T_1
  2: Initialize:
  3: Compute \theta_0 = \arg \max_{\|\boldsymbol{\theta}\|_2 \le 1} \left[ \tilde{x}^T \boldsymbol{\theta} - \max_{x_a \ne \tilde{x}} x_a^T \boldsymbol{\theta} \right] and let \epsilon_0^* be its optimal objective
  4: if \epsilon_0^* \leq 0 then
                                                                                                                                                 // Initial attackability test
              return Not attackable and stop.
  6: end if
  7: for t = 1 to T_1 do
                                                                                                                                                                       // Attack stage
              Set \tilde{r}_t = x_{a_t}^\mathsf{T} \boldsymbol{\theta}_0 + \tilde{\eta}_t
                                                                                                                                    // Always attack as \tilde{x} is the best
              Bandit algorithm observes modified reward \tilde{r}_t
  9:
10: end for
11: Estimate \tilde{\boldsymbol{\theta}}_{\parallel} = \frac{\sum_{i=1}^{n(\tilde{x})} r_i(\tilde{x})}{n(\tilde{x}) \|\tilde{x}\|_2^2} \tilde{x}
12: Solve QCLP (2) using \tilde{\boldsymbol{\theta}}_{\parallel} to obtain estimated attackability index \tilde{\epsilon}^* and certificate \tilde{\boldsymbol{\theta}}_{\perp}
13: if \tilde{\epsilon}^* < 0 then
                                                                                                                                                              // Attackability test
14:
              return Not attackable, and stop
15: else
                                                                                                                                                                       // Attack stage
              Set \hat{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}}_{\parallel} + \boldsymbol{\theta}_{\perp}
16:
              for t = T_1 + 1 to T do
17:
                     if x_{a_t} = \tilde{x} for the first time then
Set \tilde{r}_t = n(\tilde{x}) \times \tilde{x}^{\mathsf{T}} (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + \tilde{x}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} + \tilde{\eta}_t
18:
                                                                                                                                                                    // Compensate \tilde{x}
19:
                     else
20:
                            Set \tilde{r}_t = x_{a_t}^\mathsf{T} \tilde{\boldsymbol{\theta}} + \tilde{\eta}_t
21:
22:
                     Bandit algorithm observes modified reward \tilde{r}_t
23:
24:
25: end if
```

(almost) always generated by  $\tilde{\boldsymbol{\theta}}$ , which is the key to the attack's success. Specifically, if the adversary does not perturb the rewards of the target arm  $\tilde{x}$  and passes the original rewards generated by  $\boldsymbol{\theta}^*$  to the bandit algorithm, these rewards could be viewed as "corrupted" — the corruption comes from the difference between  $\tilde{\boldsymbol{\theta}}_{\parallel}$  and  $\boldsymbol{\theta}_{\parallel}^*$ , which may accumulate to a large discrepancy over T-o(T) rounds' pulling of the target arm. This discrepancy may harm our attempts on lowering the bandit algorithm's estimated rewards of non-target arms due to its correlation with the feature vectorss of other arms.<sup>3</sup>

**Remark 1.** Note that the rewards from the two stages follow different reward distributions and could be detected, e.g., using some homogeneity test. We acknowledge that a bandit player could realize the attack if equipped with some change detector as part of its procedure. However, attacking such a cautious bandit algorithm is beyond the scope of this paper. Moreover, it is very difficult (if not impossible) to attack linear stochastic bandits with a stationary reward distribution or undetectable perturbation (using a fixed target parameter  $\tilde{\theta}$ ). We could easily find cases that the adversary's parameter  $\tilde{\theta}$  is limited to a few choices and it is almost impossible to directly start the attack with a

<sup>&</sup>lt;sup>3</sup>Previous works (Jun et al., 2018; Liu and Ho, 2018; Garcelon et al., 2020) do not attack the target arm since in their setting, the reward of target arm would not affect reward estimate of non-target arms. Our problem is strictly harder due to the correlation among arms.

valid  $\tilde{\theta}$  without knowing  $\theta^*$ . For example, if we change the third arm in Example 1 to  $x_3 = (-1 + \epsilon, 0)$  where  $\epsilon$  is a small constant, we can see that the valid parameters are in a small range around  $\tilde{\theta} = (-1 - \epsilon, 1)$ . Therefore, in order to attack with a stationary reward distribution, the adversary needs to immediately attack close to  $\tilde{\theta} = (-1 - \epsilon, 1)$ , which we believe is extremely difficult without knowing  $\theta^*$ . Overall, we think designing an attack strategy for a bandit algorithm with reward change detector or showing the inability to attack such cautious algorithms is an important future work of ours.

Attack against LinUCB. We now show how LinUCB algorithm can be attacked by Algorithm 2.

**Theorem 2.** For large enough T, the attack strategy in Algorithm 2 will correctly assert the attackability with probability at least  $1 - \delta$ . Moreover, when the environment is attackable, with probability at least  $1 - 2\delta$  the attack strategy will fool LinUCB to pull non-target arms at most

$$O(d(\sqrt{\log(T/\delta)} + \sqrt{T_1}\log(T_1/\delta))\sqrt{T\log(T/\delta)}/\epsilon^*)$$

rounds and the adversary's cost is at most

$$2T_1 + O(T/\sqrt{T_1}) + O(d(\sqrt{\log(T)/\delta} + \sqrt{T_1}\log(T_1/\delta))\sqrt{T\log(T/\delta)}/\epsilon^*),$$

where the last term is due to the manipulation whenever a non-target arm is pulled at the second stage. Specifically, when  $T_1 = T^{1/2}$ , the strategy gives the minimum attack cost in the order of  $\tilde{O}(T^{3/4})$ , and the non-target arms are pulled at most  $\tilde{O}(T^{3/4})$  rounds.

Proof Sketch. To prove the the assertion is correct with high probability, the key idea is that the estimated  $\tilde{\boldsymbol{\theta}}_{\parallel}$  is close to the true parameter  $\boldsymbol{\theta}_{\parallel}^*$ . We first note that in the first stage, the bandit algorithm will pull the target arm  $\tilde{x}$   $T_1 - O(\sqrt{T_1})$  times, since  $\tilde{x}$  is the best arm according to  $\boldsymbol{\theta}_0$ . According to Hoeffding's inequality, the estimation error  $\|\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*\|_2 \leq \sqrt{\frac{2\log(2/\delta)}{T_1 - O(\sqrt{T_1})}}$ . So with a large enough  $T_1$ , the error of  $\tilde{x}$ 's reward estimation is smaller than  $\epsilon^*$ . Thus solving QCLP (2) with  $\tilde{\boldsymbol{\theta}}_{\parallel}$  and we can correctly assert attackability with positive estimated index  $\tilde{\epsilon}^*$  when the environment is attackable with index  $\epsilon^*$ .

To prove the success and the cost of the attack, the main challenge lies at analyzing the behavior of LinUCB under the reward discrepancy between the two stages, i.e., corrupted rewards in the first stage. Our proof crucially hinges on the following robustness property of LinUCB.

**Lemma 1** (Robustness of ridge regression). Consider LinUCB with ridge regression for linear stochastic bandits under poisoning attack. For any t = 1...T, with probability at least  $1 - \delta$  we have

$$\|\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_t\|_{A_t} = \alpha_t + S_t / \sqrt{\lambda}$$

where  $S_t = \sum_{\tau=1}^t |\Delta_{\tau}|$  is the total corruption until time t, and  $\alpha_t = \sqrt{d \log \left(\frac{1+t/\lambda}{\delta}\right)} + \sqrt{\lambda}$ . Consequently, the regret of LinUCB can be bounded by  $O(d(\sqrt{\log(T/\delta)} + S_t)\sqrt{T \log(T/\delta)})$ .

Based on this lemma, we can derive the regret R(T) of LinUCB with  $\hat{\boldsymbol{\theta}}$  as the true parameter. The total corruption is  $O(d\sqrt{T_1}\log{(T_1/\delta)})$  due to the rewards of non-target arms generated by  $\boldsymbol{\theta}_0$  in the first stage (the rewards of target arm are compensated in line 19). So the non-target arms are

pulled at most  $R(T)/\epsilon^*$  rounds. Substitute the total manipulation back and we have the resulting bound.

The attack cost has three sources: 1) attacks in the first stage bounded by  $2T_1$ ; 2) attacks on the target arm in the second stage; and 3) attacks on non-target arms in the second stage. The second term is in the order of  $O(T/\sqrt{T_1})$  because the cost per-round  $\|\tilde{x}^T(\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*)\|_2$  is in the order of  $O(1/\sqrt{T_1})$ . The third term has the same order as the number of rounds non-target arms are pulled by LinUCB. By setting  $T_1 = T^{1/2}$ , the total cost achieves  $\tilde{O}(T^{3/4})$ .

**Remark 2.** Lemma 1 shows that LinUCB still enjoys sublinear regret for any corruption amount  $S = o(\sqrt{T})$ . This tolerance of  $o(\sqrt{T})$  attack turns out to be the same as the recently proposed robust linear contextual bandit algorithm based on phase elimination in (Bogunovic et al., 2021) (which we examine next). However, the regret term  $S\sqrt{T}$  in LinUCB has a worse dependence on S within the  $S = o(\sqrt{T})$  regime compared to the  $O(S^2)$  regret dependence of the robust algorithm in (Bogunovic et al., 2021).

Attack against Robust Phase Elimination. We now show that Robust Phase Elimination (RobustPhE) can also be attacked by Algorithm 2. Comparing to attacking LinUCB, robustness of this algorithm brings challenge to the first stage as attack cost is more sensitive to the length of this stage.

Corollary 2. For any large enough T, the attack strategy in Algorithm 2 will correctly assert the attackability with high probability. Moreover, when the environment is attackable, with probability at least  $1 - \delta$  the attack strategy will fool RobustPhE to pull non-target arms at most

$$O((d\sqrt{T}\log(T/\delta) + T_1^2)/\epsilon^*)$$

rounds and the adversary spends cost at most

$$2T_1 + O(T/\sqrt{T_1}) + O((d\sqrt{T}\log(T/\delta) + T_1^2)/\epsilon^*)$$

where the last term is due to the manipulation whenever a non-target arm is pulled at the second stage. Specifically,  $T_1 = T^{2/5}$  gives the minimum attack cost order  $\tilde{O}(T^{4/5})$  and the non-target arms are pulled at most  $\tilde{O}(T^{4/5})$  rounds.

Robust Phase Elimination has an additional regret term  $O(S^2)$  for total corruption S (assuming S is unknown to the bandit algorithm). If we view the second stage attack model  $\tilde{\boldsymbol{\theta}}$  as the underlying environment bandit model, rewards generated by  $\boldsymbol{\theta}_0$  in the first stage are corrupted rewards. The  $T_1$  amount of rewards from the first stage means  $T_1$  corruption, which leads to the additional  $T_1^2$  term in the cost and the number of non-target arm pulls compared with Theorem 2. Hence, the adversary can only run fewer iterations in the first stage but spends more budget there. On the other hand, this also favors the design of attack such that line 18-19 in Algorithm 2 is not necessary: the corruption in the first stage can be handled by the robustness of bandit algorithm. Our success of attacking RobustPhE does not violate the robustness claim in the original paper (Bogunovic et al., 2021): RobustPhE could tolerate  $O(\sqrt{T})$  corruption and our attack cost is  $\tilde{O}(T^{4/5})$ .

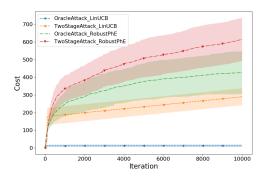


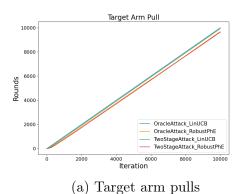
Figure 2: Total cost of the attacks.

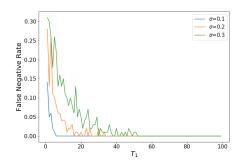
# 5 Experiments

We use simulation-based experiments to validate the effectiveness and cost-efficiency of our proposed attack methods. In our simulations, we generate a size-k arm pool  $\mathcal{A}$ , in which each arm a is associated with a context vector  $x_a \in \mathbb{R}^d$ . Each dimension of  $x_a$  is drawn from a set of zero-mean Gaussian distributions with variances sampled from a uniform distribution U(0,1). Each  $v_a$  is then normalized to  $||x_a||_2 = 1$ . The bandit model parameter  $\theta^*$  is sampled from N(0,1) and normalized to  $||\theta^*||_2 = 1$ . We set d to 10, the standard derivation  $\sigma$  of Gaussian noise  $\eta_t$  to 0.1, and the arm pool size k to 30 in our simulations. We run the experiment for T = 10,000 iterations. We will re-sample the environment  $\langle \mathcal{A}, \theta^*, \tilde{x} \rangle$  until it is attackable, following Theorem 1.

We compare the two proposed attack methods, oracle null space attack and two-stage null space attack, against LinUCB (Li et al., 2010) and Robust Phase Elimination (RobustPhE) (Bogunovic et al., 2021). We report average results of 10 runs where in each run we sample a random attackable environment. Both oracle attack and two-stage attack can effectively fool the two bandit algorithms to pull the target arm linear times and we report this result in appendix. Figure 2 shows the total cost of the attack. We observe that both attack methods are cost-efficient with sublinear total cost, while two-stage attack requires more attack budget. Specifically, we notice that the adversary spends almost linear budget in the first stage. This is because in the first stage the adversary attacks according to parameter  $\theta_0$  which leads to a almost constant cost at every round. In the second stage, the cost is much smaller: the adversary only spends  $O(1/\sqrt{T_1})$  cost when pulling the target arm. This also corresponds to our theoretical analysis that total cost of two-stage attack is  $O(T^{3/4})$ against LinUCB and  $O(T^{4/5})$  against RobustPhE. To attack the same bandit algorithm, the total cost of two-stage attack is larger than oracle attack. The key reason is that when pulling target arm, the oracle attack does not perturb the reward. We see that cost does not increase in oracle attack against LinUCB in the later stage, but the curve of two-stage attack against LinUCB keeps increase over time. We also notice that for the same attack method, attacking RobustPhE requires more budget and the target arm pull is also smaller comparing with attacking LinUCB, due to the robustness of the algorithm.

In Figure 3(a) we report the number of target arm pulls under the two attack methods (i.e., with and without the ground-truth bandit parameter  $\theta^*$ ) against LinUCB and RobustPhE. The settings of the experiment are described in Experiments section. We observe that both oracle attack and two-stage attack can effectively fool the two bandit algorithms to pull the target arm linear times. RobustPhE will pull the target arm less often than LinUCB because of its robustness to reward





(b) False negative rate of attackability test

Figure 3: Additional experiment results

#### manipulation.

In another experiment, we study the false negative rate of the attackability test in Algorithm 2, i.e., how often the adversary mistakenly asserts that an attackable environment is not attackable. As we explained in Proof of Theorem 1, the wrong assertion is because of using estimated  $\theta_{\parallel}$  instead of the ground-truth bandit parameter. In this experiment, we consider an attackable three-arm environment with  $\mathcal{A} = \{x_1 = (0,1), x_2 = (0.11,1.1), x_3 = (-2,0)\}, \ \tilde{x} = x_1 \text{ and } \theta^* = (0,0.5).$  By solving QCLP (2), we have attackability index  $\epsilon^* = 0.005$  and certificate  $\theta_{\perp} = (-0.5, 0)^4$ . We test two-stage null space attack against LinUCB with T = 10,000 and the adversary will test the attackability after the first  $T_1 = T^{1/2} = 100$  rounds. We vary  $T_1$  from 1 to 100 to see how many iterations is sufficient for attackability test. We report averaged results of 100 runs. We also vary the standard derivation  $\sigma$  of Gaussian noise from 0.1 to 0.3. In Figure 3(b), we can see that the false negative rate is almost zero when  $T_1 > 50$ , suggesting  $T_1 = 100$  is sufficient. When  $\sigma = 0.1$  the adversary only needs around 10 rounds to make a correct assertion. We also notice the false negative rate becomes higher under a larger noise scale. As suggested in Lemma 2, the error in  $\theta_{\parallel}$  estimation is larger if noise scale is larger or the number of target arm's rewards  $n(\tilde{x})$  is smaller, which highly depends on  $T_1$ . Larger error means QCLP (2) with  $\theta_{\parallel}$  is more likely to be unfeasible and gives false negative assertion. However,  $T_1 = 100$  is still enough for the attackability test when  $\epsilon^* = 0.005$ .

### 6 Related Work

Adversarial attacks to bandit algorithms was first studied in the stochastic multi-armed bandit setting (Jun et al., 2018; Liu and Shroff, 2019) and recently in linear contextual bandits (Garcelon et al., 2020). These works share a similar attack idea: lowering the rewards of non-target arms while not modifying the reward of target arm. However, as our attackability analysis revealed, this idea may fail for attacking linear stochastic bandit environment since one cannot lower the rewards of non-target arms without modifying the reward of target arm due to their correlation. This insight is a key reason that gives rise to unattackable environments. Ma et al. (2018) also considered the attackability issue of linear bandits, but under the setting of offline data poisoning attack where the adversary has the power to modify the rewards in history. There are also several recent works on

<sup>&</sup>lt;sup>4</sup>We introduce arm  $x_3$  to guarantee the first dimension of  $\hat{\boldsymbol{\theta}}_{\perp}$  cannot be smaller than -0.5. Comparing  $\tilde{x}$  and  $x_2$  and we can see the optimal solution is  $\epsilon^* = 0.005$ .

poisoning attacks against reinforcement learning (Yu and Sra, 2019; Zhang et al., 2020; Rakhsha et al., 2021, 2020).

A parallel line of works focused on improving the robustness of bandit algorithms. Lykouris et al. (2018) proposed a robust MAB algorithm and Gupta et al. (2019) further improved the solution with additive regret dependency on attack budget. They assumed a weaker oblivious adversary who determines the manipulation before the bandit algorithm pulls an arm. Hajiesmaili et al. (2020) studied robust adversarial bandit algorithm. Bogunovic et al. (2021) proposed robust phase elimination algorithm for linear stochastic bandits under a stronger adversary (same as ours), which could tolerate  $O(\sqrt{T})$  corruption when the total corruption is unknown to the algorithm. We showed that our two-stage null space attack could effectively attack this algorithm with  $\tilde{O}(T^{4/5})$  budget.

### 7 Conclusion

In this paper, we studied the problem of data poisoning attacks in k-armed linear stochastic bandits, with the goal of forcing the bandit algorithm to pull the target arm linear times using a sublinear budget. Different from context-free stochastic bandits and k-armed linear contextual bandits where the environment is always attackable, we showed that some linear stochastic bandit environment is not attackable due to the correlation among arms. We characterized the attackability condition as the feasibility of a linear program based on the geometry of the arm features. Our key insight is that giving the ground-truth parameter  $\theta^*$ , the adversary should perform oracle attack that lowers the reward of non-target arms in the null space of target arm's feature  $\tilde{x}$ . Based on this insight, we proposed a two-stage null space attack without the knowledge of  $\theta^*$  and applied it against LinUCB and Robust Phase Elimination. We showed that the proposed attack methods are effective and cost-efficient, both theoretically and empirically. As future work, it is interesting to study the lower bound of attack cost in linear stochastic bandits and also design cost-optimal attack method with a matching upper bound.

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## A Notations

For clarity, we collect the notations used in the paper below.

 $\tilde{x}$  | Context vector of target arm

 $x_a$  | Context vector of arm a

 $\theta^*$  | Unknown model parameter

 $\boldsymbol{\theta}_{\parallel}^{*} \mid \text{Projection of } \boldsymbol{\theta}^{*} \text{ on } \tilde{x}$ 

 $r_t$  | Unattacked reward feedback at time t

 $\eta_t$  | Sub-Gaussian noise in reward, i.e.,  $r_t = x_t^\mathsf{T} \boldsymbol{\theta}^* + \eta_t$ .

 $\tilde{r}_t$  Attacked reward

 $\hat{\theta}_t$  | Parameter estimated by the victim bandit algorithm with attacked rewards  $\{\tilde{r}_{1:t}\}$ 

 $\tilde{\boldsymbol{\theta}}_{\parallel}$  | Parameter parallel to  $\tilde{x}$ , estimated by adversary with unattacked rewards

 $\tilde{m{ heta}}^{"}$  | Paramter of adversary's attack strategy

 $\theta_{\perp}$  | Attackability certificate, the parameter orthogonal to  $\tilde{x}$  solved by QCLP (2)

 $\epsilon^*$  Attackability index, optimal objective of QCLP (2)

 $\tilde{\epsilon}^*$  | Estimated index, optimal objective of QCLP (2) with  $\tilde{\theta}_{\parallel}$  replacing  $\theta_{\parallel}^*$ 

# B Details on Attackability of Linear Stochastic Bandits

We show the necessity proof of Theorem 1 below. Our proof relies on the following results.

Claim 1. Suppose arm x is pulled n times till round t by LinUCB. Its confidence bound  $CB_t(x)$  in LinUCB satisfies

$$CB_t(x) \le O\left(\sqrt{\frac{\log t/\delta}{n}}\right).$$
 (5)

with probability at least  $1 - \delta$ .

*Proof.* In (Abbasi-yadkori et al., 2011), the exploration bonus term is computed as  $CB_t(x) = \alpha_t \|x\|_{\mathbf{A}_t^{-1}}$ . Denote  $\mathbf{A}_t' = n \times xx^{\mathsf{T}}$ . Since  $\mathbf{A}_t = \sum_{i=1}^t x_{a_i} x_{a_i}^{\mathsf{T}} + \lambda \mathbf{I}$ , we have  $\mathbf{A}_t \succ \mathbf{A}_t'$ . We can thus bound  $\|x\|_{\mathbf{A}_t^{-1}}$  by

$$||x||_{\mathbf{A}_{t}^{-1}} \le ||x||_{\mathbf{A}_{t}^{\prime}^{-1}} \le \frac{L}{\sqrt{n}} \tag{6}$$

According to Theorem 2 in (Abbasi-yadkori et al., 2011),

$$\alpha_t = \sqrt{d \log \left(\frac{1 + t/\lambda}{\delta}\right)} + \sqrt{\lambda} S = O(\sqrt{\log t/\delta}).$$

Combining Eq (5) and (6) finishes the proof.

Claim 2. Suppose the non-target arms  $\{x_a \neq \tilde{x}\}$  are pulled o(T) times, the arm  $\tilde{x}$  is pulled T - o(T) times, and the total manipulation is  $C_T$ . With probability at least  $1 - \delta$ , reward estimation error by the attacker satisfies

$$|x^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\parallel} - x^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^*| \le \frac{C_T}{T - o(T)} + \frac{\alpha_t}{\sqrt{T - o(T)}}.$$
 (7)

Proof.

$$\|\hat{\boldsymbol{\theta}}_{T,\parallel} - \boldsymbol{\theta}_{\parallel}^*\|_{2} = \|\frac{\tilde{x}^{\mathsf{T}}(\hat{\boldsymbol{\theta}}_{T} - \boldsymbol{\theta}^*)}{\|\tilde{x}\|_{2}^{2}} \tilde{x}\|_{2}$$

$$= \frac{1}{\|\tilde{x}\|_{2}^{2}} \|\tilde{x}^{\mathsf{T}} \mathbf{A}_{t}^{-1} \left( \sum_{t=1}^{T} x_{t} (\tilde{r}_{t}(x_{t}) - x_{t}^{\mathsf{T}} \boldsymbol{\theta}^*) + \lambda \boldsymbol{\theta}^* \right) \tilde{x}\|_{2}$$

$$\leq \frac{1}{\|\tilde{x}\|_{2}^{2}} \|\tilde{x}^{\mathsf{T}} \mathbf{A}_{t}^{-1} \left( \sum_{t=1}^{T} x_{t} \Delta_{t} + \sum_{t=1}^{T} x_{t} \eta_{t} + \lambda \boldsymbol{\theta}^* \right) \tilde{x}\|_{2}$$

$$\leq \frac{1}{\|\tilde{x}\|_{2}^{2}} \|\tilde{x}^{\mathsf{T}} \mathbf{A}_{t}^{-1} \sum_{t=1}^{T} x_{t} \Delta_{t} \tilde{x}\|_{2} + \frac{1}{\|\tilde{x}\|_{2}^{2}} \|\tilde{x}^{\mathsf{T}} \mathbf{A}_{t}^{-1/2} \alpha_{t} \tilde{x}\|_{2}$$

$$\leq \frac{C_{T}}{T - o(T)} + \frac{\alpha_{t}}{\sqrt{T - o(T)}}$$

where the last step is because there are T - o(T) number of  $\tilde{x}\tilde{x}^{\mathsf{T}}$  in  $A_t$ . We finish the proof with the fact that  $\|\tilde{x}\|_2 \leq 1$ .

Now we are ready to prove that the index  $\epsilon^*$  in QCLP (2) being positive is the necessary condition of an attackable environment.

Proof of necessity of Theorem 1. We shall prove that if  $\epsilon^* \leq 0$ , the bandit environment is not attackable. To prove this, we show that there exists some no-regret bandit algorithm (LinUCB in particular) such that no attacking strategy can succeed. In particular, we will show that LinUCB (with a specific choice of L2-regularization parameter  $\lambda$ ) is robust under any attacking strategy with o(T) budget when  $\epsilon^* \leq 0$ . We prove it by contradiction: assume LinUCB is attackable with o(T) budget when  $\epsilon^* \leq 0$ . According to Definition 1, the target arm  $\tilde{x}$  will be pulled T - o(T) times for infinitely many different time horizons T, and the following inequalities hold when arm  $\tilde{x}$  is pulled by LinUCB:

$$\tilde{x}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\parallel} + \mathrm{CB}_{T}(\tilde{x}) > x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\parallel} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\perp} + \mathrm{CB}_{T}(x_{a}), \forall x_{a} \neq \tilde{x}$$
 (8)

where  $\hat{\boldsymbol{\theta}}_t$  is LinUCB's estimated parameter at round t based on the attacked rewards. We decompose  $\hat{\boldsymbol{\theta}}_T = \hat{\boldsymbol{\theta}}_{T,\parallel} + \hat{\boldsymbol{\theta}}_{T,\perp}$ , where  $\tilde{x} \perp \hat{\boldsymbol{\theta}}_{t,\perp}$  and  $\tilde{x} \parallel \hat{\boldsymbol{\theta}}_{T,\parallel}$ . We will now show that the above inequalities lead to

$$\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} > x_{a}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\perp}, \forall x_{a} \neq \tilde{x}$$

when  $T \to \infty$ .

By Claim 1 we have

$$CB_T(\tilde{x}) \le O\left(\sqrt{\frac{\log T/\delta}{T - o(T)}}\right)$$

We also have

$$CB_T(x_a) = \alpha_T ||x_a||_{\mathbf{A}_T^{-1}} > 0$$

By Claim 2 we have

$$x_a^{\mathsf{T}} \hat{\boldsymbol{\theta}}_{T,\parallel} \ge x_a^{\mathsf{T}} \boldsymbol{\theta}_{\parallel}^* - \frac{C_T}{T - o(T)} - \frac{\alpha_T}{\sqrt{T - o(T)}}$$
$$\tilde{x}^{\mathsf{T}} \hat{\boldsymbol{\theta}}_{T,\parallel} \le \tilde{x}^{\mathsf{T}} \boldsymbol{\theta}_{\parallel}^* + \frac{C_T}{T - o(T)} + \frac{\alpha_T}{\sqrt{T - o(T)}}$$

Substitute them back and we have that with probability at least  $1-3\delta$ ,

$$\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} > x_{a}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\perp} + \mathrm{CB}_{T}(x_{a}) - O\left(\sqrt{\frac{\log T/\delta}{T - o(T)}}\right) - \frac{2C_{T}}{T - o(T)} - \frac{2\alpha_{T}}{\sqrt{T - o(T)}}, \forall x_{a} \neq \tilde{x}$$

Taking  $T \to \infty$  and noticing that  $C_T = o(T)$ , the last three terms on the right-hand side diminish to 0. If  $x_a \not\parallel \tilde{x}$ ,  $CB_T(\tilde{x})$  diminishes faster than  $CB_T(x_a)$  since arm  $\tilde{x}$  is pulled linear times and we have,

$$\tilde{x}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} > x_{a}^{\mathsf{T}}\boldsymbol{\theta}_{\parallel}^{*} + x_{a}^{\mathsf{T}}\hat{\boldsymbol{\theta}}_{T,\perp}, \forall x_{a} \neq \tilde{x}$$

$$\tag{9}$$

Now we consider the special case that there exists  $x_a \parallel \tilde{x}$  and  $\tilde{x}$  is pulled a linear number of times, and show that the above inequality is still strict. Let  $x_a = c\tilde{x}$ . If |c| > 1, we have  $CB_T(x_a) - CB_T(\tilde{x}) = (c-1)CB_T(\tilde{x}) > 0$ . If |c| < 1, since  $\tilde{x}$  is pulled linear times for any large t with sublinear cost, then  $\tilde{x}^T\hat{\theta}_{t,\parallel} > 0$ ; otherwise the cost would be linear. We directly have  $\tilde{x}^T\hat{\theta}_{t,\parallel} = x_a^T\hat{\theta}_{t,\parallel} + (1-c)\tilde{x}^T\hat{\theta}_{t,\parallel} > x_a^T\hat{\theta}_{t,\parallel}$ . This leads to  $\tilde{x}^T\theta_{\parallel}^* > x_a^T\theta_{\parallel}^*$  (inequality (9)) since  $x_a \perp \hat{\theta}_{T,\perp}$ .

This implies that there must exist a  $\hat{\boldsymbol{\theta}}_{T,\perp}$  that satisfies inequality (9) (the first constraint of QCLP (2)),  $\tilde{x} \perp \hat{\boldsymbol{\theta}}_{T,\perp}$  (the second constraint of QCLP (2)), and makes the objective of QCLP (2) larger than 0. To satisfy the last constraint, we consider LinUCB with the choice of L2-regularization parameter  $\lambda$  that guarantees  $\|\hat{\boldsymbol{\theta}}_t\|_2 < 1$  given the data for large enough T and any t < T. Note that ridge regression is equivalent to minimizing square loss under some constraint  $\|\hat{\boldsymbol{\theta}}_t\|_2 \leq c$ , and there always exists a one-to-one correspondence between  $\lambda$  and c (one simple way to show the correspondence is using KKT conditions). Therefore, we are guaranteed to find a  $\lambda$  that gives us  $c = 1 - \delta$  where  $\delta$  is an arbitrarily small constant. Then we know that  $\hat{\boldsymbol{\theta}}_{T,\perp}$  satisfies  $\|\hat{\boldsymbol{\theta}}_T = \hat{\boldsymbol{\theta}}_{T,\parallel} + \hat{\boldsymbol{\theta}}_{T,\perp}\|_2 < 1$ . We prove the last constraint  $\|\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_{\parallel}^* + \hat{\boldsymbol{\theta}}_{T,\perp}\|_2 \leq 1$  by the fact that  $\|\hat{\boldsymbol{\theta}}_{T,\parallel} + \hat{\boldsymbol{\theta}}_{T,\perp}\|_2 < 1$  and  $\|\boldsymbol{\theta}_{\parallel}^* - \hat{\boldsymbol{\theta}}_{T,\parallel}\|_2$  is arbitrarily small for large enough T according to Claim 2.

Now we proved that there exists a  $\hat{\theta}_{T,\perp}$  that satisfies all the constraints in QCLP (2) with positive objective, which means the optimal objective  $\epsilon^*$  must also be positive. This however contradicts our assumption  $\epsilon^* \leq 0$ , implying that such LinUCB is not attackable by any attack strategy.

# C Details on Effective Attacks Without Knowledge of Model Parameters

We now prove the theorems of using Two-stage Null Space Attack (Algorithm 2) against LinUCB and Robust Phase Elimination.

#### C.1 Proof of Theorem 2

*Proof.* We first prove that for a large enough T, Algorithm 2 will correctly assert the attackability with probability at least  $1 - \delta$ . We rely on the following lemma to show  $\tilde{\boldsymbol{\theta}}_{\parallel}$  estimated in step 11 of Algorithm 2 is close to the true parameter  $\boldsymbol{\theta}_{\parallel}^*$ .

**Lemma 2** (Estimation error of  $\tilde{\boldsymbol{\theta}}_{\parallel}$ ). Algorithm 2 estimates  $\boldsymbol{\theta}_{\parallel}^{*}$  by

$$\tilde{\boldsymbol{\theta}}_{\parallel} = \frac{\sum_{i=1}^{n(\tilde{x})} r_i(\tilde{x})}{n(\tilde{x}) \|\tilde{x}\|_2^2} \tilde{x}.$$
 (10)

With probability at least  $1 - \delta$ , the estimation error is bounded by

$$\|\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*\|_2 \le \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}} \tag{11}$$

where the rewards have  $\sigma$ -sub-Gaussian noise.

*Proof.*  $\boldsymbol{\theta}_{\parallel}^{*}$  is the projected vector of  $\boldsymbol{\theta}^{*}$  onto  $\tilde{x}$ , which is

$$\boldsymbol{\theta}_{\parallel}^* = rac{ ilde{x}^{\mathsf{T}} \boldsymbol{\theta}^*}{\| ilde{x}\|_2^2} ilde{x}$$

as defined in Eq (1). Though we need to estimate the vector  $\tilde{\boldsymbol{\theta}}_{\parallel} \in \mathbb{R}^d$ , we actually only need to estimate the scale of it by  $\hat{l} = \frac{\sum_{i=1}^{n(\tilde{x})} r_i(\tilde{x})}{n(\tilde{x}) \|\tilde{x}\|_2^2}$ , since the direction is known to be  $\tilde{x}$ . Based on Hoeffding's inequality, the estimation error of averaged rewards on  $\tilde{x}$  is bounded by

$$P\left(\left|\frac{\sum_{i=1}^{n(\tilde{x})} r_i(\tilde{x})}{n(\tilde{x})} - r^*(\tilde{x})\right| \ge \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n(\tilde{x})}}\right) \le \delta$$
 (12)

where  $r^*(\tilde{x}) = \tilde{x}^\mathsf{T} \boldsymbol{\theta}^*$ . Considering  $\|\tilde{x}\|_2^2 = 1$  and we finish the proof.

In the first stage, the bandit algorithm will pull the target arm  $\tilde{x}$  for  $T_1 - O(\sqrt{T_1})$  times, since  $\tilde{x}$  is the best arm according to  $\theta_0$ . According to Lemma 2, with probability at least  $1 - \delta$  the estimation error is bounded as

$$\|\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*\|_2 \le \sqrt{\frac{2\sigma^2 \log(1/\delta)}{T_1 - O(\sqrt{T_1})}}.$$

As a result, with a large enough  $T_1$ , the error of  $\tilde{x}$ 's reward estimation satisfies

$$\|\tilde{x}^\mathsf{T}\tilde{\boldsymbol{\theta}}_{\parallel} - \tilde{x}^\mathsf{T}\boldsymbol{\theta}_{\parallel}^*\| \leq \|\tilde{x}\|_2 \|\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*\|_2 \leq \sqrt{\frac{2\sigma^2 \log(1/\delta)}{T_1 - O(\sqrt{T_1})}} \leq \epsilon^*.$$

Thus solving QCLP (2) with  $\tilde{\boldsymbol{\theta}}_{\parallel}$  replacing  $\boldsymbol{\theta}_{\parallel}^{*}$  and we could correctly assert attackability with an estimated positive index  $\tilde{\epsilon}^{*}$  when the environment is attackable with index  $\epsilon^{*}$ .

**Remark 3.** From the analysis above, we notice that the adversary requires sufficiently large  $T_1$  to collect enough rewards on the target arm, in order to make the correct attackability assertion. When  $T_1$  is not large enough, the attackability test may have false positive or false negative conclusions. We empirically test the error rate and report the results in Additional Experiments section.

We now prove the correctness and total cost of the proposed attack. The analysis relies on the "robustness" property of LinUCB stated in Lemma 1, which is proved below.

Proof of Lemma 1. Based on the closed form solution of ridge regression, we have

$$\hat{\boldsymbol{\theta}}_t = \boldsymbol{\theta}^* - \lambda \mathbf{A}_t^{-1} \boldsymbol{\theta}^* + \mathbf{A}_t^{-1} \sum_{\tau=1}^t x_{a_\tau} [\eta_\tau + \Delta_\tau]$$

Therefore, using ideas from (Abbasi-yadkori et al., 2011), we can have

$$\|\hat{\boldsymbol{\theta}}_{t} - \boldsymbol{\theta}^{*}\|_{\mathbf{A}_{t}} \leq \lambda^{1/2} \|\boldsymbol{\theta}^{*}\|_{2} + \|\sum_{\tau=1}^{t} x_{a_{\tau}} \eta_{\tau}\|_{\mathbf{A}_{t}^{-1}} + \|\sum_{\tau=1}^{t} x_{a_{\tau}} \Delta_{\tau}\|_{\mathbf{A}_{t}^{-1}}$$

$$\leq \alpha_{t} + \|\sum_{\tau=1}^{t} x_{a_{\tau}} \Delta_{\tau}\|_{\mathbf{A}_{t}^{-1}}$$

$$\leq \alpha_{t} + \|\sum_{\tau=1}^{t} x_{a_{\tau}} \Delta_{\tau}\|_{2} / \sqrt{\lambda}$$

$$\leq \alpha_{t} + S_{t} / \sqrt{\lambda},$$

where the second step is based on the definition of  $\alpha_t$ , the third step is because of  $\mathbf{A}_t \succeq \lambda \mathbf{I}$  and the last step is because of  $||x_a||_2 \leq 1$ .

Let us first analyze the attack in the first stage. Denote  $R_T(\theta)$  as the regret of LinUCB until round T, where  $\theta$  is the ground-truth parameter. We know from (Abbasi-yadkori et al., 2011) that if the rewards are all generated by  $\theta$  then with probability at least  $1 - \delta$  we have

$$R_T(\boldsymbol{\theta}) = \alpha_T \sqrt{dT \log\left(\frac{1 + T/\lambda}{\delta}\right)} = O(d\sqrt{T}\log(T/\delta))$$
 (13)

where  $\alpha_t = \sqrt{d \log \left(\frac{1+t/\lambda}{\delta}\right)} + \sqrt{\lambda}$ . Then the attack in the first  $T_1$  rounds based on  $\boldsymbol{\theta}_0$  should make the bandit algorithm pull  $\tilde{x}$  at least  $T_1 - R_{T_1}(\boldsymbol{\theta}_0)/\epsilon_0^*$  times. According to Lemma 2, with probability at least  $1 - \delta$  parameter estimation error is bounded by

$$\|\tilde{\boldsymbol{\theta}}_{\parallel,T_1} - \boldsymbol{\theta}_{\parallel}^*\|_2 \le 1/\sqrt{T_1 - R_{T_1}(\boldsymbol{\theta}_0)/\epsilon_0^*} \le 2/\sqrt{T_1}$$
(14)

for large enough  $T_1$ .

Now we prove the attack is successful with high probability. Consider the regret of the LinUCB against  $\tilde{\boldsymbol{\theta}}$  as the ground-truth parameter. Since the rewards at the second stage are all generated by  $\tilde{\boldsymbol{\theta}}$ , the estimation error in  $\hat{\boldsymbol{\theta}}_t - \tilde{\boldsymbol{\theta}}$  has two sources, one is the sub-Gaussian noise and the other is the rewards on non-target arms in the first stage generated by  $\boldsymbol{\theta}_0$  (the rewards on the target arm are corrected to  $\tilde{\boldsymbol{\theta}}$  in step 18-19 in Algorithm 2). According to Lemma 1, with probability at least  $1-2\delta$ , we have

$$\|\hat{\boldsymbol{\theta}}_t - \tilde{\boldsymbol{\theta}}\|_{\mathbf{A}_t} \le \alpha_t + R_{T_1}(\boldsymbol{\theta}_0)/\sqrt{\lambda}, t > T_1.$$

To show the number of rounds pulling non-target arms, we first look at the regret against  $\tilde{\boldsymbol{\theta}}$ , i.e.,  $R_T(\tilde{\boldsymbol{\theta}})$ .

$$R_{T}(\tilde{\boldsymbol{\theta}}) \leq \sum_{t=1}^{T} \left( \tilde{x}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} - x_{a_{t}}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} \right)$$

$$\leq \sum_{t=1}^{T} \left( \tilde{x}^{\mathsf{T}} \hat{\boldsymbol{\theta}}_{t} + CB_{t}(\tilde{x}) - x_{a_{t}}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} \right)$$

$$\leq \sum_{t=1}^{T} \left( x_{a_{t}}^{\mathsf{T}} \hat{\boldsymbol{\theta}}_{t} + CB_{t}(x_{a_{t}}) - x_{a_{t}}^{\mathsf{T}} \tilde{\boldsymbol{\theta}} \right)$$

$$\leq \sum_{t=1}^{T} 2CB_{t}(x_{a_{t}})$$

$$\leq 2\sqrt{T \sum_{t=1}^{T} CB_{t}^{2}(x_{a_{t}})}$$

$$\leq 2\|\hat{\boldsymbol{\theta}}_{T} - \tilde{\boldsymbol{\theta}}\|_{\mathbf{A}_{T}} \sqrt{T \sum_{t=1}^{T} \|x\|_{\mathbf{A}_{t}^{-1}}^{2}}$$

$$\leq 2(\alpha_{T} + R_{T_{1}}(\boldsymbol{\theta}_{0})/\sqrt{\lambda}) \sqrt{dT \log\left(\frac{1 + T/\lambda}{\delta}\right)}$$

holds with probability at least  $1 - 2\delta$ . And LinUCB will pull non-target arms at most  $R_T(\tilde{\boldsymbol{\theta}})/\epsilon^*$  times, which can be bounded by

$$R_{T}(\tilde{\boldsymbol{\theta}})/\epsilon^{*} \leq 2\left(\alpha_{T} + R_{T_{1}}(\boldsymbol{\theta}_{0})/\sqrt{\lambda}\right)\sqrt{dT\log\left(\frac{1+T/\lambda}{\delta}\right)}/\epsilon^{*}$$

$$\leq 2\left(\sqrt{dT\log\left(\frac{1+T/\lambda}{\delta}\right)} + \sqrt{\lambda} + R_{T_{1}}(\boldsymbol{\theta}_{0})/\sqrt{\lambda}\right)\sqrt{dT\log\left(\frac{1+T/\lambda}{\delta}\right)}/\epsilon^{*}$$

and is in the order of

$$O\left(d\left(\sqrt{\log(T/\delta)} + \sqrt{T_1}\log\left(T_1/\delta\right)\right)\sqrt{T\log(T/\delta)}/\epsilon^*\right)$$
(15)

The  $\sqrt{T_1} \log (T_1/\delta)$  term is due to the "corrupted" rewards of non-target arms observed in the first stage.

Now we prove the total cost C(T). Note that we consider expectation over per-step cost, i.e.,  $C(T) = \sum_{t=1}^{T} |\mathbb{E}[\Delta r_t]|$  so that we do not need to consider the sub-Gaussian noise in the reward. The attack cost has three sources: 1) attacks in the first stage bounded by  $2T_1$ , because each attack costs at most 2 and the first stage lasts  $T_1$  rounds; 2) attacks on the target arm in the second stage; 3) attacks on non-target arms in the second stage.

In the second stage, when attacking the target arm  $\tilde{x}$ , the cost of every attack is

$$|\mathbb{E}[\tilde{r}(\tilde{x})] - \mathbb{E}[r^*(\tilde{x})]| = |\tilde{x}^{\mathsf{T}}(\tilde{\boldsymbol{\theta}}_{\parallel} - \boldsymbol{\theta}_{\parallel}^*)| \leq 2/\sqrt{T_1}$$

because of Eq (14), and the total cost is at most  $4T/\sqrt{T_1}$ . The cost of attacking non-target arms is at most  $2R_T(\tilde{\boldsymbol{\theta}})/\epsilon^*$ . So with probability at least  $1-2\delta$  the total cost is

$$2T_1 + 4T/\sqrt{T_1} + O(d(\sqrt{\log(T)/\delta} + \sqrt{T_1}\log(T_1/\delta))\sqrt{T\log(T/\delta)}/\epsilon^*)$$
(16)

Setting  $T_1 = T^{1/2}$  gives us the minimum attack cost  $\tilde{O}(T^{3/4})$  according to Eq (16) and the number of rounds pulling non-target arms in  $\tilde{O}(T^{3/4})$  according to Eq (15).

**Remark 4.** We define the attack cost as  $C(T) = \sum_{t=1}^{T} |\mathbb{E}[\Delta r_t]|$  so that the sub-Gaussian noise in rewards can be averaged out  $(\Delta_t = \tilde{r}_t - r_t \text{ and the unperturbed reward } r_t \text{ is defined to have sub-Gaussian noise})$ . In terms of theoretical analysis, if we consider  $\mathbb{E}[|\Delta r_t|]$  then the sub-Gaussian noise itself could lead to a linear cost in a gap-independent proof framework, since gap-independent analysis will sum over T terms and each term has the sub-Gaussian noise.

## C.2 Proof of Corollary 2

The proof is similar to the proof of Theorem 2, and thus we only explain the difference here.

Instead of using Lemma 1 to analyze the impact of corrupted rewards generated by  $\theta_0$  (against  $\tilde{\theta}$ ) collected in the first stage, we know RobustPhE has an additional regret term  $O(S^2)$  for total corruption S (assuming S is unknown to the bandit algorithm). Since the bandit algorithm observes  $T_1$  rewards in the first stage,  $S \leq 2T_1$  and the regret  $R_T(\tilde{\theta})$  is  $O(d\sqrt{T}\log(T/\delta) + T_1^2)$ . Therefore, we have with probability at least  $1 - \delta$ , the attack strategy will fool RobustPhE to pull non-target arms at most  $O((d\sqrt{T}\log(T/\delta) + T_1^2)/\epsilon^*)$  rounds. Again, with the new  $R_T(\tilde{\theta})$  for RobustPhE, the total cost is

$$2T_1 + 4T/\sqrt{T_1} + O((d\sqrt{T}\log(T/\delta) + T_1^2)/\epsilon^*).$$

Setting  $T_1 = T^{2/5}$  gives us the minimum attack cost  $\tilde{O}(T^{4/5})$ , and the non-target arms are pulled at most  $\tilde{O}(T^{4/5})$  rounds.

**Remark 5.** Note that we bound the total corruption by  $T_1$ , which means the adversary does not need to compensate the rewards on the target arm as shown in step 18-19 in Algorithm 2. The robustness of RobustPhE allows us to carry over the rewards in the first stage while LinUCB does not.