Delayed Feedback in Episodic Reinforcement Learning

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Abstract

There are many provably efficient algorithms for episodic reinforcement learning. However, these algorithms are built under the assumption that the sequences of states, actions and rewards associated with each episode arrive immediately, allowing policy updates after every interaction with the environment. This assumption is often unrealistic in practice, particularly in areas such as healthcare and online recommendation. In this paper, we study the impact of delayed feedback on several provably efficient algorithms for regret minimisation in episodic reinforcement learning. Firstly, we consider updating the policy as soon as new feedback becomes available. Using this updating scheme, we show that the regret increases by an additive term involving the number of states, actions, episode length and the expected delay. This additive term changes depending on the optimistic algorithm of choice. We also show that updating the policy less frequently can lead to an improved dependency of the regret on the delays.

1 Introduction

Reinforcement Learning (RL) considers an agent interacting with an unknown environment in a sequence of steps. At each of these steps, the agent must select an action. Consequently, the environment generates a reward and presents the agent with a new situation, also called a state. The goal of an agent is to maximise the cumulative sum of these rewards by learning the best action to take in each state. In this paper, we consider the episodic reinforcement learning problem. Here, the agent interacts with the environment in a sequence of episodes. Each episode consists of a fixed number of steps where the agent sees a state, selects an action, observes some reward and transitions to the next state. This feedback helps the agent to learn to take better decisions in the future.

There exist many theoretically successful algorithms for episodic reinforcement learning [Bartlett and Tewari, 2009, Jaksch et al., 2010, Filippi et al., 2010, Fruit et al., 2020, Azar et al., 2017, Dann et al., 2017]. The majority of these are built under the assumption that the feedback is immediate or, at the very least, received at the end of each episode. This allows the agent to quickly learn about the quality of their decisions and adapt to take better actions.

However, in many practical settings, feedback is not immediate. Instead, it is received only after some delay. This can occur due to the nature of the environment or for computational reasons. Examples of the former arise in healthcare, finance and online recommender systems. Here, the outcome of a treatment protocol, particular investment strategy or sequence of recommendations often returns at some unknown time in the future. Autonomous vehicles or wearable technology is an example of the latter, where heavy computation may occur on a separate machine meaning that the agent only receives the feedback at some later point in time. Currently, there is little theoretical understanding of the impact of delays on the regret of reinforcement learning algorithms.

1.1 Contributions

In this paper, we study the impact of stochastic delays on the regret of optimistic reinforcement learning algorithms in episodic Markov Decision Processes (MDPs). We propose two procedures for dealing with delayed feedback, which we call active and lazy updating.

In the active updating procedure, a base algorithm updates the policy as soon as feedback is received. We show that this updating scheme leads to an additive increase in regret for a broad class of algorithms (the so-called 'optimistic' model-based algorithms). This additive penalty depends on the expected delay and the base algorithm of choice.

The lazy updating protocol uses the *doubling trick* and waits for the observed visits to a state-action-step to double before computing a new policy. We show that this leads to an additive increase in regret that depends on the expected delay and is independent of the chosen optimistic model-based algorithm, in some cases leading to improved results.

1.2 Related Work

Delays in Bandits Stochastically delayed feedback has received much attention in the simpler multi-armed bandit and contextual bandit problems [Agarwal and Duchi, 2011, Joulani et al., 2013, Vernade et al., 2017, Manegueu et al., 2020, Dudik et al., 2011, Vernade et al., 2020]. The results in the multi-armed bandit setting show delayed feedback causes an additive penalty in the regret that scales with the number of actions and the expected delay. Thus, we can expect a similar result in the harder episodic RL problem. However, dealing with delays in RL is more challenging than in the (contextual) bandit setting. Indeed in RL, since the environment is stochastic and changes depending on the actions taken, we cannot guarantee to observe a particular state nor the effects of taking a particular action. Consequently, this means that approaches such as the queuing technique of Joulani et al. [2013] cannot easily be applied.

Delays in RL Despite being of practical importance, there is limited literature on stochastically delayed feedback in RL. Previous work has considered constant delays in observing the current state in an MDP [Katsikopoulos and Engelbrecht, 2003]. However, the challenges in this setting are somewhat different to that of delayed feedback. More recently, Lancewicki et al. [2021] considered delayed feedback in adversarial MDPs. They consider adversarial delays and develop algorithms based on policy optimisation whose regret depends on the sum of the delays, the number of states and the number of steps per episode. For stochastic MDPs and adversarial delays, they also state that the regret will be increased by an additive term in the regret bound of order: $H^4SA\tau_{\rm max}$ when all delays are less than $\tau_{\rm max}$ almost surely. In our paper, we consider stochastic delays and show that it is only the expected delay, which is often considerably smaller than the maximal delay, that impacts the regret.

2 Preliminaries

In this paper, we consider the task of learning to act optimally in an unknown episodic finite-horizon Markov Decision Process, EFH-MDP. We focus on the case where there is a delay between playing an episode and observing the sequence of states, actions and rewards sampled by the *agent*; we refer to this sequence as feedback throughout this paper.

2.1 Episodic Finite-Horizon MDPs

An EFH-MDP is formalisable as a quintuple: $M = (\mathcal{S}, \mathcal{A}, H, P, R)$. Here, \mathcal{S} is the set of states, \mathcal{A} is the set of actions, H is the horizon and gives the number of steps per episode, $P = \{P_h(\cdot|s,a)\}_{h,s,a}$ is the set of probability distributions over the next state and $R = \{R_h(s,a)\}_{h,s,a}$ is the set of reward functions. For conciseness, we assume that the reward function is known, deterministic and bounded between zero and one for all state-action-step triples. 1

In the episodic reinforcement learning problem, an algorithm interacts with an MDP in a sequence of episodes: k = 1, 2, ..., K. We denote the set of episodes by: $[K] = \{1, 2, ..., K\}$; a convention that we adopt for sets of integers. In each of the episodes, the learner takes a fixed number of steps, H. At the start of the k^{th} episode, the environment samples an initial state: s_1^k . The learner can use all available information to select a sequence of H actions from this initial state.

We consider algorithms that compute a deterministic policy $\pi_k: \mathcal{S} \times [H] \to \mathcal{A}$ at the start of each episode $k \in [K]$. One can show that if an optimal policy exists, there is a deterministic optimal policy [Puterman, 1994]. The agent samples feedback from the environment by: selecting an action, $a_h^k = \pi_k(s_h^k, h)$; receiving a reward, $r_h^k = R_h(s_h^k, a_h^k)$; and transitioning to the next state, $s_{h+1}^k \sim P_h(\cdot|s_h^k, a_h^k)$; for each $h=1,\cdots,H$.

The feedback of the k^{th} episode is given by $\{(s_h^k, a_h^k, r_h^k, s_{h+1}^k)\}_{h=1}^H$. In the standard episodic reinforcement learning setting, this feedback is received immediately. In this paper, we will consider the setting where this feedback is received after some stochastic delay which will be formally introduced in Section 3.

¹The main challenge in model-based reinforcement learning lies in estimating the transition function. Thus, an extension to unknown bounded stochastic rewards is relatively straightforward.

We measure the quality of a policy, π , using the value function, which is the expected return at the end of the episode from the current step, given the current state:

$$V_{h}^{\pi}(s) = \mathbb{E}_{\pi} \left[\sum_{h'=h}^{H} r_{h'}^{k} \middle| s_{h'}^{k} = s \right] . \tag{1}$$

Further, we denote the optimal value function by: $V_h^*(s) = \max_{\pi} \{V_h^{\pi}(s)\}$, which gives the maximum expected return $\forall (s,h) \in \mathcal{S} \times [H]$.

When evaluating a learning algorithm, it is common to use regret. The regret measures the expected loss in rewards over a fixed number of K episodes as a result of following (possibly) sub-optimal policies π_1, \ldots, π_K :

$$\mathfrak{R}_{T} = \sum_{k=1}^{K} V_{1}^{*} \left(s_{1}^{k} \right) - V_{1}^{\pi_{k}} \left(s_{1}^{k} \right) = \sum_{k=1}^{K} \Delta_{1}^{k} . \tag{2}$$

where T = KH denote the total number of steps taken in the EFH-MDP. Domingues et al. [2020] show that the lower bound for the regret in the standard episodic reinforcement learning setting with stage-dependent transitions is: $\Omega(H\sqrt{SAT})$.

2.2 Regret Minimisation in RL

Many provably efficient algorithms exist for learning EFH-MDPs when feedback is immediate. In this paper, we focus on optimistic model-based reinforcement learning algorithms. These algorithms maintain estimators of the transition probabilities for each $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$:

$$\hat{P}_{kh}\left(s'|s,a\right) = \frac{\sum_{i:i < k} \mathbb{1}\left\{\left(s_h^i, a_h^i, s_{h+1}^i\right) = (s, a, s')\right\}}{1 \lor N_{kh}\left(s, a\right)}$$

where

$$N_{kh}(s,a) = \sum_{i:i < k} \mathbb{1}\left\{ \left(s_h^i, a_h^i \right) = (s,a) \right\}$$
 (3)

is the total number of visits to the given state-action-step triple before the k^{th} episode. In this paper, we consider the model-based algorithms that compute an optimistic value function of the form:

$$\tilde{V}_{h}^{\pi_{k}} = \min\{H - h + 1, R_{h} + \langle \hat{P}_{kh}, \tilde{V}_{h+1}^{\pi} \rangle + \beta_{kh}\}$$
 (4)

with high probability, where $\beta_{kh}(s,a)$ is an exploration bonus whose general form is given by:

$$\beta_{kh}(s, a) = \min \left\{ \beta_{kh}^{*}(s, a), H - h \right\}$$

$$\beta_{kh}^{*}(s, a) = \frac{C_{kh}}{\sqrt{1 \vee N_{kh}(s, a)}} + \frac{B_{kh}}{1 \vee N_{kh}(s, a)}$$
(5)

Here, C_{kh} and B_{kh} are algorithm-dependent quantities. Naturally, the value-optimistic algorithms compute value functions of this form. Examples of value-optimistic algorithms include: UBEV, UCBVI-CH, UCBVI-BF [Dann et al., 2017, Azar et al., 2017]. However, recent work proved that model-optimistic algorithms share this form too [Neu and Pike-Burke, 2020]. Some model-optimistic algorithms include: UCRL2, KL-UCRL and UCRL2B [Jaksch et al., 2010, Filippi et al., 2010, Fruit et al., 2020]. For UCRL2, $C_{kh} = H\sqrt{S\log(SAT/\delta)}$ and $B_{kh} = 0$ [Neu and Pike-Burke, 2020].

Throughout this paper, we make use of the following assumption:

Assumption 1. The exploration bonus upper bounds the estimation error: $\beta_{kh}(s,a) \ge \langle (\hat{P}_{kh} - P_h)(\cdot|s,a), \tilde{V}_{h+1}^{\pi_k}(\cdot) \rangle$ for all $(k,h) \in \times[K] \times [H]$.

Using Assumption 1, we can upper bound the regret by [Neu and Pike-Burke, 2020]:

$$\mathfrak{R}_T \le 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + 2\sum_{k=1}^K \sum_{h=1}^H \beta_{kh}\left(s_h^k, a_h^k\right) \tag{6}$$

with probability $1 - \delta'$. See Proposition 1 of Appendix A.1 for proof of this claim.

²Throughout: $x \lor y = \max\{x, y\}$ and $x \land y \min\{x, y\}$.

2.3 The Importance of Counts

From Equation (6), it is clear that upper bounding the sum of the bonuses leads to an upper bound on the regret. Bounding the sum of bonuses involves upper bounding the sum of: $1/\sqrt{N_{kh}(s,a)}$ and $1/N_{kh}(s,a)$ over all steps and episodes. After rearranging these summations to get indicator functions in the numerators, standard techniques show that:

$$\sum_{s,a,h} \sum_{k=1}^{K} \frac{1 \left\{ s_h^k = s, a_h^k = a \right\}}{\sqrt{1 \vee N_{kh}(s, a)}} \le 2\sqrt{HSAT}$$
 (7)

$$\sum_{s,a,b} \sum_{k=1}^{K} \frac{1 \left\{ s_h^k = s, a_h^k = a \right\}}{1 \vee N_{kh} \left(s, a \right)} \le HSA \log (8T) \tag{8}$$

The result follows from realising that the term in the denominator increases by one whenever the indicator in the numerator is equal to one. Appendix A.2 gives full details of the bounds in (7) and (8). Consequently, any algorithm that satisfies Assumption 1 has a regret bound of the form: $C\sqrt{HSAT} + BHSA\log(T)$.

3 Delayed Feedback

To model problems where feedback returns at some unknown time in the future, we introduce a random variable for the delay between playing the k^{th} episode and observing the corresponding feedback, which we denote by τ_k . Throughout this paper, we make the following assumption on the delays:

Assumption 2. The delays are positive, independent and identically distributed random variables with finite expectation. Denoting the delay of the k^{th} episode by τ_k , we require: $\tau_k \stackrel{iid}{\sim} f_{\tau}(\cdot)$, $\mathbb{P}(\tau < 0) = 0$ and $\mathbb{E}[\tau] < \infty$.

As a consequence of the delays, the feedback associated with an episode does not return immediately. Instead, it returns at some unknown time in the future: $k + \tau_k$. Therefore, the base algorithm only makes use of the feedback associated with the k^{th} episode at the start of the episode: $\lceil k + \tau_k \rceil + 1$.

When working with delayed feedback in RL, it is helpful to introduce the observed and missing visitation counters, and their relationship with the total visitation counter:

$$N'_{kh}(s,a) = \sum_{i:i+\tau_i < k} \mathbb{1}\left\{ \left(s_h^i, a_h^i \right) = (s,a) \right\}$$
 (9)

$$N_{kh}''(s,a) = \sum_{i:i+\tau_i > k} \mathbb{1}\left\{ \left(s_h^i, a_h^i\right) = (s,a) \right\}$$
 (10)

$$\underbrace{N_{kh}(s,a)}_{\text{total visits}} = \underbrace{N'_{kh}(s,a)}_{\text{observed visits}} + \underbrace{N''_{kh}(s,a)}_{\text{missing visits}}$$
(11)

Analogously to (9) we define

$$N'_{kh}(s, a, s') = \sum_{i: i+\tau_i < k} \mathbb{1}\left\{\left(s_h^i, a_h^i, s_{h+1}^i\right) = (s, a, s')\right\}$$
(12)

allowing for the estimation of transition probabilities. Instead of the total-visitation count, model-based optimistic algorithms can only compute their bonuses using the observed visitation counter. The corresponding value functions are still optimistic, but they will contract to the optimal value function more slowly.

From Equation (6), it is clear that we can control the regret by controlling the bonuses. With delayed feedback, bounding the bonus terms is more difficult since the observed visitation count can remain constant across numerous episodes. Therefore, (7) and (8) do not upper bound the equivalent summations involving the observed visitation counters. One approach to delaying with delayed feedback would be to maintain numerous versions of the base algorithm and bound their regret separately using the standard techniques [Lancewicki et al., 2021]. To do so would require one to maintain $\tau_{\rm max}+1$ versions of the algorithm, where $\tau \leq \tau_{\rm max}$ almost surely. Consequently, the regret of this approach would would scale the regret of the base algorithm by the maximal delay. In addition to being highly inefficient from a space-complexity perspective, in cases where the delay distribution has infinite support or the bound on its distribution is large, this approach can have regret that is linear in the total number of steps, T. Further, $\tau_{\rm max}$ can be large in comparison to the expected delay. In Sections 4 and 5, we show that it is possible to obtain regret bounds scaling with the expected delay rather than the maximal delay.

3.1 Bounding the Missing Episodes

In Sections 4 and 5, we propose two procedures for dealing with delayed feedback in episodic RL: one based on updating as soon as new data becomes available and the other based on updating less frequently. In either case, we can bound the number of missing visits to each state-action-step triple by the number of missing episodes. The reasoning behind this due to the agent playing only one state-action pair per step in each episode.

Lemma 1. Let $S_k = \sum_{i=1}^{k-1} \mathbb{1}\{i + \tau_i \ge k\}$, where $\tau_1, \tau_2, \cdots \tau_{k-1} \sim f_{\tau}(\cdot)$ are independent and identically distributed random variables with finite expected value. Further, let:

$$F_k^{\tau} = \left\{ S_k \ge \mathbb{E}\left[\tau\right] + \log\left(\frac{K\pi}{6\delta'}\right) + \sqrt{2\mathbb{E}\left[\tau\right]\log\left(\frac{K\pi}{6\delta'}\right)} \right\}$$

be the failure event for a single k. Then, $\mathbb{P}(F_{\tau}) = \mathbb{P}(\bigcup_{k=1}^{\infty} F_k^{\tau}) \leq \delta'$.

Proof. The proof follows from applying Bernstein's inequality to the sum of indicator random variables. See Lemma 1 of Appendix B for a full proof. \Box

A direct consequence of this lemma is an upper bound on the number of missing episodes:

$$S_k \le \psi_K^{\tau} \coloneqq \mathbb{E}\left[\tau\right] + \log\left(\frac{K\pi}{6\delta'}\right) + \sqrt{2\mathbb{E}\left[\tau\right]\log\left(\frac{K\pi}{6\delta'}\right)}$$

which holds across all $k \in \mathbb{Z}^+$ with probability $1 - \delta'$.

4 Active Updating

We first consider the effect of delays on the regret when we update the policy as soon as new data becomes available. This procedure is outlined in Algorithm 1. We assume we have some base algorithm which takes all available data and outputs an optimistic policy.

Algorithm 1: Active Updating

```
Initialise visitation counter: N'_{1h}\left(s,a\right)=0
2 Initialise transition counter: N'_{1h}\left(s,a,s'\right)=0
3 Initialise the policy: \pi_1=Base\left(\left\{N'_{1h}\right\}_h\right)
4 for k=1,2,\cdots,K do
5 | if \exists i:i+\tau_i=k-1 then
6 | Update visitation counters: (9) and (12).
7 | Compute policy: \pi_k=Base\left(\left\{N'_{kh}\right\}_h\right)
8 | else
9 | Reuse old policy: \pi_k=\pi_{k-1}
Sample an episode using policy: \pi_k.
```

Theorem 1 (Active Updating). *Under Assumption 1, with probability* $1 - \delta$, *the regret of any model-based algorithm under delayed feedback is upper bounded by:*

$$\tilde{\mathcal{O}}\left(C\sqrt{HSAT} + BHSA + \max\left\{C, B\right\} HSA\mathbb{E}\left[\tau\right]\right)\,,$$

where C and B are universal upper bounds on the algorithm-dependent quantities in the exploration bonus of (5).

The second column of Table 1 presents regret bounds for various optimistic algorithms using active updating under delayed feedback that fit into our framework.

4.1 Bounding the Observed Visitation Counter

From Equation (6), it is clear that we must bound the summation of the bonuses. In the standard setting, one can use the fact that the total visitation counter increases by one between successive plays of a state-action-step triple. However, this is not the case in the delayed feedback setting. The following lemma provides an upper bound on summations in the form of the bonuses calculated using the observed visitation counter, albeit without the terms in the numerator.

Lemma 2. Let $Z_T^p = \sum_{k=1}^K \sum_{h=1}^H 1/(N'_{kh}(s_h^k, a_h^k))^p$. Then, with probability $1 - \delta'$:

$$Z_T^p \leq \begin{cases} 4\sqrt{HSAT} + 3HSA\psi_K^\tau & \text{if } p = \frac{1}{2} \\ 2HSA\log\left(8T\right) + HSA\psi_K^\tau\log(16\psi_K^\tau) & \text{if } p = 1 \end{cases}$$

Proof. Unless otherwise stated, we let: $N_{kh}(s,a) = 1 \vee N_{kh}(s,a)$ and $N'_{kh}(s,a) = 1 \vee N'_{kh}(s,a)$ for notational convenience. First, we use the relationships between the observed, missing and total visitation counters to split the summation into two parts in a similar manner to Lancewicki et al. [2021]:

$$Z_{T}^{p} = \sum_{k=1}^{K} \sum_{h=1}^{H} \left(\frac{1}{N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p} = \sum_{k,h} \left(\frac{1}{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p} \left(\frac{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right)}{N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p}$$

From Equation (11), $N_{kh}(s, a) = N'_{kh}(s, a) + N''_{kh}(s, a)$, for any $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$. Consequently,

$$Z_T^p \leq \underbrace{\sum_{k,h} \left(\frac{1}{N_{kh}\left(s_h^k, a_h^k\right)}\right)^p}_{A} + \underbrace{\sum_{k,h} \left(\frac{1}{N_{kh}\left(s_h^k, a_h^k\right)} \frac{N_{kh}''\left(s_h^k, a_h^k\right)}{N_{kh}'\left(s_h^k, a_h^k\right)}\right)^p}_{B},$$

since $(1+x)^p \le 1+x^p$ for p=1/2 and p=1 and any x>0. Term A is the summation of the total visitation counter seen in the non-delayed setting and so can be bounded using e.g. Equations (7) and (8).

Bounding B requires more care, as it involves the observed and missing visitation counters. Recall that the algorithm plays one state-action pair at each step in every episode. Thus, the missing visitation counter is upper bounded by the number of missing episodes: $N_{kh}''(s,a) \leq S_k$. Lemma 1 bounds the number of missing episodes: with probability $1-\delta', S_k \leq \psi_K^{\tau}$ across all $k \in \mathbb{Z}^+$. Splitting B using the observed visitation counts and the upper bound on S_k gives:

$$B \leq \sum_{k,h} \left(\frac{\mathbb{1}\left\{ N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right) \geq \psi_{K}^{\tau} \right\} \psi_{K}^{\tau}}{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right) N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p} + \sum_{k,h} \left(\frac{\mathbb{1}\left\{ N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right) \leq \psi_{K}^{\tau} \right\} \psi_{K}^{\tau}}{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right) N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p} \right)$$

$$\leq \underbrace{\sum_{k,h} \left(\frac{\mathbb{1}\left\{ N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right) \geq \psi_{K}^{\tau} \right\}}{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p}}_{B,1} + \underbrace{\sum_{k,h} \left(\frac{\mathbb{1}\left\{ N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right) \leq \psi_{K}^{\tau} \right\} \psi_{K}^{\tau}}{N_{kh}\left(s_{h}^{k}, a_{h}^{k}\right) N_{kh}'\left(s_{h}^{k}, a_{h}^{k}\right)} \right)^{p}} \right)}_{B,2}$$

The last inequality follows since for the first sum, $N'_{kh}(s,a) \geq \psi_K^{\tau}$.

Clearly, $B.1 \leq A$, as it is a summation over a subset of all the episodes. Using (11), it is possible to rewrite the indicator in the remaining term as: $\mathbb{1}\{N_{kh}(s,a)-N''_{kh}(s,a)\leq \psi_K^{\tau}\}$, for any $(s,a,h)\in \mathcal{S}\times\mathcal{A}\times[H]$. Further, $N''_{kh}(s,a)\leq \psi_K^{\tau}$ and $N'_{kh}(s,a)\geq 1$. Therefore,

$$B.2 \le (\psi_K^{\tau})^p \sum_{k,h} \left(\frac{1 \left\{ N_{kh} \left(s_h^k, a_h^k \right) \le 2 \psi_K^{\tau} \right\}}{N_{kh} \left(s_h^k, a_h^k \right)} \right)^p \le (\psi_K^{\tau})^p \sum_{s,a,h} \sum_{n=0}^{2 \psi_K^{\tau}} \frac{1}{(1 \lor n)^p}$$

Lemma 6 of Appendix A.2 gives an upper bound of: $\sum_{n=0}^{N} 1/(1 \vee n)^p$. Summing this upper bound over all stateaction-step triples gives:

$$B.2 \leq \begin{cases} 3HSA\psi_K^\tau & \text{if } p = \frac{1}{2} \\ HSA\psi_K^\tau \log\left(16\psi_K^\tau\right) & \text{if } p = 1 \end{cases}$$

Therefore:

$$Z_T^p \leq 2A + B.2 \leq \begin{cases} 4\sqrt{HSAT} + 3HSA\psi_K^\tau & \text{if } p = \frac{1}{2} \\ HSA\left(2\log\left(8T\right) + \psi_K^\tau\log\left(16\psi_K^\tau\right)\right) & \text{if } p = 1 \end{cases}$$

as required.

4.2 Proof of Theorem 1

Using Lemma 1 and 2, we can bound the regret of optimistic model-based algorithms using active updating to handle the delayed feedbacks.

Proof. We split the regret into two sets of episodes based on whether their feedback is returned before the beginning of the final episode:

$$\mathfrak{R}_T = \sum_{k=1}^K \Delta_1^k \mathbb{1} \{k + \tau_k \ge K\} + \Delta_1^k \mathbb{1} \{k + \tau_k < K\} .$$

Trivially, the regret of each of the first set of episodes can be bounded by the horizon, H. Lemma 1 gives an upper bound on the number of missing episodes. Therefore, with probability $1 - \delta'$:

$$\sum_{k=1}^{K} \Delta_{1}^{k} \mathbb{1} \{k + \tau_{k} \ge K\} \le H \psi_{K}^{\tau}.$$

For the second set of episodes, we bound the regret by summing the bonuses. Using Equation (6):

$$\sum_{k=1}^{K} \Delta_{1}^{k} \mathbb{1} \left\{ k + \tau_{k} < K \right\} \leq 2H \sqrt{T \log \left(\frac{K\pi}{6\delta'} \right)} + 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \beta_{kh} \left(s_{h}^{k}, a_{h}^{k} \right)$$

Therefore, all that remains is to bound the sum of the exploration bonuses. To do so, we let $C \ge C_{kh}$ and $B \ge B_{kh}$ be universal upper bounds on the numerators in the exploration bonus. Then, the sum of the exploration bonuses has the following upper bound:

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \left(\frac{C}{\sqrt{1 \vee N'_{kh} \left(s_{h}^{k}, a_{h}^{k} \right)}} + \frac{B}{1 \vee N'_{kh} \left(s_{h}^{k}, a_{h}^{k} \right)} \right)$$

An application of Lemma 2 shows that the regret from summing the bonuses is upper bounded by:

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \beta_{kh} \left(s_h^k, a_h^k \right) \le 4C\sqrt{HSAT} + 6CHSA\psi_K^{\tau} + 2BHSA\log\left(8T\right) + BHSA\psi_K^{\tau}\log\left(16\psi_K^{\tau}\right)$$

Bringing everything together gives an upper bound on the regret:

$$\mathfrak{R}_T \leq 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + 8C\sqrt{HSAT} + 4BHSA\log\left(8T\right) + 12CHSA\psi_K^\tau + 2BHSA\psi_K^\tau\log\left(16\psi_K^\tau\right)$$

Rescaling $\delta' = \delta/(9+1)$ to accommodate for the additional failure event from Lemma 1 gives the final result.

Algorithm	\mathfrak{R}_T (Active Updating)	\mathfrak{R}_T (Lazy Updating)
UBEV	$\sqrt{H^3SATL} + H^2SAL^{1/2}\mathbb{E}\left[au ight]$	$\sqrt{H^3SATL} + H^2SA\mathbb{E}\left[au\right]L_K$
UCRL2	$\sqrt{H^3S^2ATL} + H^2S^{3/2}AL^{1/2}\mathbb{E}\left[\tau\right]$	$\sqrt{H^3S^2ATL} + H^2SA\mathbb{E}\left[au ight]L_K$
KL-UCRL	$\sqrt{H^3S^2ATL} + H^2S^{3/2}AL^{1/2}\mathbb{E}\left[au ight]$	$\sqrt{H^3S^2ATL} + H^2SA\mathbb{E}\left[au ight]L_K$
χ^2 -UCRL	$\sqrt{H^3S^2ATL^2} + H^2S^2AL_T + H^2S^2A\mathbb{E}\left[\tau\right]$	$\sqrt{H^3S^2ATL^2} + H^3S^3A^2L_K + H^2SA\mathbb{E}\left[\tau\right]L_K$
UCRL2B	$\sqrt{H^3\Gamma SATL^2} + H^2S^2ALL_T + H^2S^2AL\mathbb{E}\left[\tau\right]$	$\sqrt{H^3\Gamma SATL^2} + H^3S^3A^2LL_K + H^2SA\mathbb{E}\left[\tau\right]L_K$

Table 1: A selection of algorithms that fit into our framework and their regret bounds under delayed feedback. Here, $\Gamma \leq S$ is a uniform upper bound on the number of reachable states at a subsequent step. Further, $L_T = \log(8T)$, $L_K = \log(K/SA)$, and $L = \log(S, A, H, K, \delta')$ arises from the failure events of the chosen base algorithm.

5 Lazy Updating

In Section 4, we derived regret bounds for a wide range of model-based algorithms using optimism. Although the active updating scheme does maintain good theoretical guarantees, a slower approach to updating can lead to improved regret bounds in some cases. Briefly, we consider waiting until the observed visits to a state-action-step double before updating the policy. Algorithm 2 presents the pseudo-code for this slower approach to updating.

Theorem 2 (Lazy Updating). Under Assumption 1, for $T \ge HSA$, with probability $1 - \delta$, the regret of any model-based algorithm under delayed feedback has the following upper bound:

$$\mathcal{O}\left(C\sqrt{HSAT}+B\left(HSA\right)^{2}L_{K}+H^{2}SAL_{K}\mathbb{E}\left[\tau\right]\right)$$

where $L_K = \log(K/SA)$, C and B are universal upper bounds on the algorithm-dependent quantities in the exploration bonus of (5).

Theorem 2 shows that the delay dependency is independent of the base algorithm. The third column of Table 1 gives regret bounds for various algorithms using lazy updating to handle delayed feedback.

Algorithm 2: Lazy Updating

```
Set k=1
2 Set j=1 // initialise first epoch
3 Set k_j=k // starting time of first epoch
4 Initialise visitation counter: N'_{k_jh}\left(s,a\right)=0
5 Initialise transition counter: N'_{k_jh}\left(s,a,s'\right)=0
6 Initialise the policy: \pi_k=Base\left(\{N'_{k_jh}\}_h\right)
7 for k=1,2,\cdots,K do
8 Update the within epoch counter: (13)
9 if \exists s,a,h:N'_{k_jh}(s,a)=n^k_{jh}(s,a) then
10 Start new epoch: j=j+1; k_j=k
11 Update visitation counters: (9) and (12)
12 Compute policy: \pi_j=Base\left(\{N'_{kh}\}_h\right)
13 Sample an episode using policy: \pi_k=\pi_j.
```

5.1 The Doubling Trick

Instead of updating as soon as the algorithm sees feedback, we consider waiting until the observed visits to a state-action-step triple have doubled. Each update marks the start of an epoch, which we denote by: $j=1,2,\cdots,J$. At the start of each epoch, the base algorithm computes a policy and uses it to sample feedback from the MDP. Therefore, each epoch is just a set of episodes where the algorithm interacts with the environment using the same policy.

Let k_j denote the episode where the j^{th} epoch starts. Then, the number of visits to each state-action-step triple that are observed while playing the j^{th} epoch is simply:

$$n_{jh}^{k}(s, a) := \sum_{i=1}^{k-1} \mathbb{1}\left\{s, a, h, k_{j} \le i + \tau_{i} < k\right\}$$
 (13)

where $\mathbb{1}\{s,a,h\} = \mathbb{1}\{s_h^k = s, a_h^k = a\}$. Epoch j+1 starts whenever the observed visits to a state-action-step triple have doubled since the start of the j^{th} epoch:

$$k_{j+1} = \underset{k>k_{j}}{\operatorname{arg\,min}} \left\{ \exists \, s, a, h : n_{jh}^{k} \left(s, a \right) = 1 \lor N_{k_{j}h}' \left(s, a \right) \right\} .$$

Lemma 3. For $K \geq SA$, Algorithm 2 ensures that the number of epochs has the following upper bound:

$$J \le 3HSA\log\left(\frac{4K}{SA}\right) .$$

Proof. See Lemma 3 of Appendix B.

5.2 Proof of Theorem 2

Using Lemma 3 and the arguments in its proof ensures that the number of epochs is logarithmic in the number of episodes and that we can make use of Lemma 19 of Jaksch et al. [2010] to bound the sum of the bonuses. Therefore, we can bound the regret of model-based algorithms using lazy updating to handle the delayed feedback.

Proof. Similarly to the first step in Section 4.2, we can bound the regret of the episodes that are not returned by the beginning of the final episode can be bounded by: $H\psi_K^{\tau}$. Now, we focus on the episodes that are returned before beginning the final episode. Lemma 1 gives the following recursive regret decomposition for any model-based algorithm:

$$\mathfrak{R}_T \le 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + 2\sum_{k=1}^K \sum_{h=1}^H \beta_{kh}\left(s_h^k, a_h^k\right)$$

where we omit the use of the indicator function: $\mathbb{1}\{k+\tau_k < K\}$, for notational convenience. Therefore, all that remains is to bound the sum of the exploration bonuses. To do so, we utilise the fact that epochs are disjoint sets of episodes and that the algorithm uses the same bonus within each epoch, as the counts do not change:

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \beta_{kh} \left(s_h^k, a_h^k \right) = \sum_{j=1}^{J} \sum_{k=k_j}^{k_{j+1}-1} \sum_{h=1}^{H} \beta_{k_j h} \left(s_h^k, a_h^k \right) = \sum_{s, a, h} \sum_{j=1}^{J} \beta_{k_j h} \left(s, a \right) \sum_{k=k_j}^{k_{j+1}-1} \mathbbm{1} \left\{ s, a, h \right\} \; .$$

The regret associated with the bonuses can then be split in two terms based on whether the state-action-step triples played in the j^{th} epoch are observed before starting the next epoch:

$$\underbrace{\sum_{j=1}^{J} \sum_{s,a,h} \beta_{k_{j}h}\left(s,a\right) \sum_{k=k_{j}}^{k_{j+1}-1} \mathbb{1}\left\{s,a,h,k+\tau_{k} < k_{j+1}\right\}}_{A} + \underbrace{\sum_{j=1}^{J} \sum_{s,a,h} \beta_{k_{j}h}\left(s,a\right) \sum_{k=k_{j}}^{k_{j+1}-1} \mathbb{1}\left\{s,a,h,k+\tau_{k} \geq k_{j+1}\right\}}_{B}.$$

Focusing on the former and extending the inner-most summation to all episodes before the start of the next epoch gives:

$$A \le \sum_{j=1}^{J} \sum_{s,a,h} n_{jh}^{k_{j+1}}(s,a) \, \beta_{k_{j}h}(s,a)$$

Recalling the form of the bonuses gives:

$$A \leq \sum_{s,a,h} \sum_{j=1}^{J} \left(\frac{C \, n_{jh}^{k_{j+1}} \left(s,a \right)}{\sqrt{N_{k_{j}h}' \left(s,a \right)}} + \frac{B \, n_{jh}^{k_{j+1}} \left(s,a \right)}{N_{k_{j}h}' \left(s,a \right)} \right)$$

$$\leq C \sum_{s,a,h} \sum_{j=1}^{J} \frac{n_{jh}^{k_{j+1}} \left(s,a \right)}{\sqrt{N_{k_{j}h}' \left(s,a \right)}} + HSAJB$$

$$\leq C \sum_{s,a,h} \sqrt{N_{h}' \left(s,a \right)} + HSAJB$$

$$\leq C \sqrt{HSAT} + HSAJB$$

since $n_{jh}^{k_{j+1}}(s,a) \leq N'_{k_jh}(s,a)$, due to the doubling scheme. The penultimate inequality follows from Lemma 19 of Jaksch et al. [2010], which is restated in Appendix B.1 (Lemma 7).

From Equation (4), $\beta_{k_jh}(s,a) \leq H$. Utilising this and the fact that the policy plays only one state-action pair at each step of an episode allows the following bound on the remaining term:

$$B \le H \sum_{j=1}^{J} \sum_{k=k_{j}}^{k_{j+1}-1} \sum_{s,a,h} \mathbb{1} \left\{ s, a, h, k + \tau_{k} \ge k_{j+1} \right\}$$

$$\leq H \sum_{j=1}^{J} \sum_{k=k_{j}}^{k_{j+1}-1} \mathbb{1} \left\{ k + \tau_{k} \geq k_{j+1} \right\}$$

$$\leq H \sum_{j=1}^{J} S_{k_{j+1}} \leq H J \psi_{K}^{\tau}$$

where the last inequality used Lemma 3. Therefore,

$$\mathfrak{R}_T \le 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + C\sqrt{HSAT} + B\left(HSA\right)^2 L_K + 3H^2SAL_K\mathbb{E}\left[\tau\right]$$

where $L_k = \log(K/SA)$. Rescaling $\delta' = \delta/(9+1)$ to accommodate for the additional failure event from Lemma 1 gives the final result.

6 Discussion

Table 1 presents a selection of algorithms that fit into our framework and their regret bounds under delayed feedback using both active and lazy updating. Here, we see that acting in delayed environments causes an additive increase in regret for all optimistic algorithms considered. For active updating, this additive increase scales with: $\max\{C,B\}HSA\mathbb{E}[\tau]$, where C and B are quantities depending on the base algorithm of choice. Typically, C and B involve the horizon, H, and the number of states, S. For lazy updating, this additive increase scales with: $H^2SA\log(K/SA)\mathbb{E}[\tau]$, regardless of the chosen base algorithm. In either case, the results mirror what is seen in the bandit setting, most algorithms incur an additive regret penalty of $A\mathbb{E}[\tau]$ [Joulani et al., 2013].

For many algorithms such as UBEV, UCRL2, KL-UCRL, B=0. In such a case, lazy updating is preferable if: $H \log(K/SA) \leq C$, where C usually contains a logarithmic term at least as large as $\sqrt{\log(SAT/\delta)}$. Otherwise, active updating is preferable. When considering active updating, algorithms with tighter regret bounds in the non-delayed setting exhibit better dependency on the expected delay. Further, UBEV, UCRL2 and KL-UCRL recover their standard regret bounds in the non-delayed setting (when $\tau_k=0$ for all $k\in[K]$) using either active or lazy updating.

Typically, B>0 for algorithms using variance reduction techniques, for example χ^2 -UCRL and UCRL2B. For these algorithms, from Table 1, it is clear that it is always better to use active updating. Thus, we focus our discussion on this updating scheme. Neither χ^2 -UCRL nor UCRL2B recover their regret bounds in the non-delayed feedback setting, due to bounding the empirical variance of the optimistic value function in each episode by H. This was necessary to avoid a multiplicative increase in regret due to the delays. Further, introducing the empirical variance comes at the expense of some lower-order terms: $HS\log(H,S,A,T,\delta')/N'_{kh}(s,a)$. Lemma 2 shows that these lower-order terms can be problematic, as they now depend on the expected delay. In particular, they lead to a term of order $H^2S^2A\mathbb{E}[\tau]$ in the regret bound. We note that the impact of these lower order terms when using non-standard counters has also been observed in settings such as differentially private reinforcement learning [Garcelon et al., 2021].

Unfortunately, UCBVI [Azar et al., 2017] does not fit into our framework. However, even if it did, we argue that it would not lead to improved results. Indeed, the algorithm makes use of Bernstein's inequality to handle the estimation error, resulting in the need to bound an additional term of the form: $H^2S/N'_{kh}(s,a)$. Lemma 2 shows that this term alone introduces the following dependency on the delay: $H^3S^2A\mathbb{E}[\tau]$, which is worse than any of the other algorithms we consider in this paper. UBEV [Dann et al., 2017], for example, has a delay dependency of: $H^2SA\mathbb{E}[\tau]$; an improvement of HS. Moreover, due to bounding the empirical variance, UCBVI and UBEV will have the same leading order term in their regret bound. Therefore, we do not believe that using UCBVI in the delayed setting will lead to an improvement over UBEV, an algorithm which does fit into our framework.

Lastly, we point out that although we consider model-based algorithms in this paper, we believe that lazy-updating could be used for a wider class of algorithms. Analyzing the effect that this would have on the regret of such algorithms remains an open question.

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A Standard Results

In this section of the appendix, we present numerous results from the standard immediate feedback setting. We split the results into two subsections: those relating to the recursive regret decomposition of (6) and those relating to bounding the summation of the bonuses.

A.1 Recursive Regret Decomposition

Lemma 4. For any $h \in [H]$, the difference between the optimistic and true value function under the policy of the k^{th} episode can be written as:

$$\tilde{\Delta}_{h}^{k} = \tilde{\Delta}_{h+1}^{k} + \beta_{kh}(s, a) + \zeta_{h}^{k}(s, a) + \left\langle \left(\hat{P}_{kh} - P_{h}\right)(\cdot|s, a), \tilde{V}_{h+1}^{\pi_{k}}(\cdot) \right\rangle$$

where $\zeta_h^k(s,a) = \langle P_h(\cdot|s,a), \tilde{\Delta}_{h+1}^k(\cdot) \rangle - \tilde{\Delta}_h^k$ is a martingale difference with $\gamma = \max\{\zeta_h^k\} - \min\{\zeta_h^k\} = 2H$, and $s = s_h^k$ and $a = a_h^k$.

Proof. By definition, the difference between the two value functions is simply:

$$\begin{split} \tilde{\Delta}_{h}^{k} &= \beta_{kh}\left(s,a\right) + \left\langle \hat{P}_{kh}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right)\right\rangle - \left\langle P_{h}\left(\cdot|s,a\right), V_{h+1}^{\pi_{k}}\left(\cdot\right)\right\rangle \\ &= \beta_{kh}\left(s,a\right) + \left\langle \hat{P}_{kh}\left(\cdot|s,a\right) - P_{h}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right)\right\rangle + \left\langle P_{h}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) - V_{h+1}^{\pi_{k}}\left(\cdot\right)\right\rangle \end{split}$$

To obtain the recursive decomposition seen in the lemma, we add and subtract the difference between the value functions at the next step:

$$\begin{split} \tilde{\Delta}_{h}^{k} &= \tilde{\Delta}_{h+1}^{k} + \beta_{kh}\left(s,a\right) + \left\langle \hat{P}_{kh}\left(\cdot|s,a\right) - P_{h}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) \right\rangle + \left\langle P_{h}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) - V_{h+1}^{\pi_{k}}\left(\cdot\right) \right\rangle - \tilde{\Delta}_{h+1}^{k} \\ &= \tilde{\Delta}_{h+1}^{k} + \beta_{kh}\left(s,a\right) + \left\langle \hat{P}_{kh}\left(\cdot|s,a\right) - P_{h}\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) \right\rangle + \left\langle P_{h}\left(\cdot|s,a\right), \tilde{\Delta}_{h+1}^{k}\left(\cdot\right) \right\rangle - \tilde{\Delta}_{h+1}^{k} \\ &= \tilde{\Delta}_{h+1}^{k} + \beta_{kh}\left(s,a\right) + \zeta_{h}^{k}\left(s,a\right) + \left\langle \left(\hat{P}_{kh} - P_{h}\right)\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) \right\rangle \end{split}$$

where $\zeta_h^k(s,a) = \langle P_h(\cdot|s,a), \tilde{\Delta}_{h+1}^k(\cdot) \rangle - \tilde{\Delta}_{h+1}^k$, giving the required recursive regret decomposition. Define the filtration $\mathcal{F}_h^k = \left\{s_h^k, a_h^k\right\} \cup \left\{\mathcal{H}_i\right\}_{i:i+\tau_i < k}$. Clearly, $\zeta_h^k(s,a)$ is a martingale difference, because:

$$\mathbb{E}\left[\tilde{\Delta}_{h+1}^{k}|\mathcal{F}_{kh}\right] = \mathbb{E}\left[\tilde{V}_{h+1}^{\pi_{k}}\left(s_{h+1}^{k}\right) - V_{h+1}^{\pi_{k}}\left(s_{h+1}^{k}\right)|\mathcal{F}_{kh}\right] = \left\langle P_{h}\left(\cdot|s,a\right),\tilde{\Delta}_{h+1}^{k}\left(\cdot\right)\right\rangle$$

meaning its expectation is zero, and

$$\gamma = \max\{\zeta_h^k\} - \min\{\zeta_h^k\} = 2H$$

which completes the proof of the lemma.

Lemma 5. Let ζ_h^k be a martingale difference with respect to the filtration: $\mathcal{F}_h^k = \{s_h^k, a_h^k\} \cup \{\mathcal{H}_i\}_{i:i+\tau_i < k}$. Further, let $\gamma = \max\{\zeta_h^k\} - \min\{\zeta_h^k\}$ and define:

$$F_k^{\zeta} \coloneqq \left\{ \sum_{i=1}^k \sum_{h=1}^H \zeta_h^k \ge \gamma \sqrt{T \log \left(\frac{K\pi}{6\delta'}\right)} \right\}$$

Then, for any $\delta' > 0$, it holds that: $\mathbb{P}\left(\bigcup_{k=1}^{\infty} F_k^{\zeta}(\gamma)\right) \leq \delta'$

Proof. By Azuma-Hoeffding, we have that:

$$\mathbb{P}\left(\sum_{i=1}^{k} \sum_{h=1}^{H} \zeta_h^i \ge \epsilon\right) \le \exp\left(-\frac{2\epsilon^2}{Hk\gamma^2}\right) = \frac{6\delta'}{(k\pi)^2}$$

Rearranging the above reveals that:

$$\epsilon = \sqrt{\frac{Hk\gamma^2}{2}\log\left(\frac{(k\pi)^2}{6\delta'}\right)} \leq \sqrt{Hk\gamma^2\log\left(\frac{k\pi}{6\delta'}\right)} \leq \gamma\sqrt{HK\log\left(\frac{K\pi}{6\delta'}\right)} = \gamma\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)}$$

Therefore, we have that:

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty}F_{k}^{\zeta}\right)\leq\sum_{k=1}^{\infty}\mathbb{P}\left(F_{k}^{\zeta}\right)=\frac{6\delta'}{\pi^{2}}\sum_{k=1}^{\infty}\frac{1}{k^{2}}=\delta'$$

as required.

Proposition 1. *Under Assumption 1, with probability* $1-\delta'$ *, the regret of any model-based algorithm has the following upper bound:*

$$\Re_T \le 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + 2\sum_{k=1}^K \sum_{h=1}^H \beta_{kh}\left(s_h^k, a_h^k\right)$$

where $s = s_h^k$ and $a = a_h^k$.

Proof. By definition, we have that:

$$\mathfrak{R}_{T} = \sum_{k=1}^{K} \Delta_{1}^{k} = \sum_{k=1}^{K} V_{1}^{*} \left(s_{1}^{k} \right) - V_{1}^{\pi_{k}} \left(s_{1}^{k} \right)$$

With high probability, the algorithm computes an optimistic value function at the beginning of each episode: $\tilde{V}_h^{\pi_k}(s) \geq V_h^*(s)$ for all $(s,h) \in \mathcal{S} \times [H]$. Lemma 13 of Neu and Pike-Burke [2020] presents proof of this claim. Therefore, we have that:

$$\mathfrak{R}_{T} \leq \sum_{k=1}^{K} \tilde{\Delta}_{1}^{k} = \sum_{k=1}^{K} \tilde{V}_{1}^{\pi_{k}} \left(s_{1}^{k}\right) - V_{1}^{\pi_{k}} \left(s_{1}^{k}\right)$$

Now, consider the difference between the optimistic and true value function under the policy of the $k^{\rm th}$ episode from the $h^{\rm th}$ step. From Lemma 4, we have that:

$$\tilde{\Delta}_{h}^{k} = \tilde{\Delta}_{h+1}^{k} + \beta_{kh}\left(s,a\right) + \zeta_{h}^{k}\left(s,a\right) + \left\langle \left(\hat{P}_{kh} - P_{h}\right)\left(\cdot|s,a\right), \tilde{V}_{h+1}^{\pi_{k}}\left(\cdot\right) \right\rangle \leq \tilde{\Delta}_{h+1}^{k} + 2\beta_{kh}\left(s,a\right) + \zeta_{h}^{k}\left(s,a\right) + \zeta_{h}^{k}\left(s,a\right)$$

where $\zeta_h^k(s,a) = \langle P_h(\cdot|s,a), \tilde{\Delta}_{h+1}^k(\cdot) \rangle - \tilde{\Delta}_h^k$ is a martingale difference and Δ denotes the set of valid transitions for the state-action pair at the h^{th} step of the k^{th} episode. Now, to prove the statement of the lemma, we must show that the difference between the optimistic and true value function from the h^{th} step under the policy of the k^{th} episode can be written as:

$$\tilde{\Delta}_{h'}^{k} \leq \sum_{h=h'}^{H} 2\beta_{kh} \left(s, a \right) + \zeta_{h}^{k} \left(s, a \right)$$

which we do by induction. Recall that: $\tilde{V}_{H+1}^{\pi_k} = V_{H+1}^* = V_{H+1}^{\pi_k} = \vec{0}$. For h' = H, the statement holds: $\tilde{\Delta}_H^k = \tilde{\Delta}_{H+1}^k + 2\beta_{kh}(s,a) + \zeta_H^k(s,a) = 2\beta_{kh}(s,a) + \zeta_H^k(s,a)$. Now, assuming the statement holds for h = h' + 1, it is clear that:

$$\tilde{\Delta}_{h'}^{k} = \tilde{\Delta}_{h'+1}^{k} + 2\beta_{kh'}^{+}(s,a) + \zeta_{h'}^{k}(s,a) = \sum_{h=h'+1}^{H} \left(2\beta_{kh}(s,a) + \zeta_{h}^{k}(s,a) \right) + 2\beta_{kh'}^{+} + \zeta_{h'}^{k} = \sum_{h=h'}^{H} 2\beta_{kh}(s,a) + \zeta_{h}^{k}(s,a)$$

Therefore, setting h' = 1 in the above and summing over all episodes gives:

$$\mathfrak{R}_{T} \leq \sum_{k=1}^{K} \sum_{h=1}^{H} 2\beta_{kh}\left(s,a\right) + \zeta_{h}^{k}\left(s,a\right) = 2\sum_{k=1}^{K} \sum_{h=1}^{H} \beta_{kh}\left(s,a\right) + \sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_{h}^{k}\left(s,a\right)$$

Lemma 4 shows that $\zeta_h^k(s,a)$ is a martingale difference with: $\gamma \geq 2H$. From Lemma 5, which bounds the sum of martingale differences, it is clear that for all $k \in [K]$:

$$\mathfrak{R}_T \le 2H\sqrt{T\log\left(\frac{K\pi}{6\delta'}\right)} + 2\sum_{k=1}^K \sum_{h=1}^H \beta_h^k(s,a)$$

with probability $1 - \delta'$, as required.

A.2 Importance of Counts

In Section 2.3, we bound the summations involving the total visitation counter. To do so, we rewrite the summation by introducing an indicator:

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{\left(1 \vee N_{kh} \left(s_{h}^{k}, a_{h}^{k}\right)\right)^{p}} = \sum_{s, a, h} \sum_{k=1}^{K} \frac{1 \left\{s_{h}^{k} = s, a_{h}^{k} = a\right\}}{\left(1 \vee N_{kh} \left(s, a\right)\right)^{p}} = \sum_{s, a, h} \sum_{n=0}^{N_{h}(s, a)} \frac{1}{\left(1 \vee n\right)^{p}}$$

which shares the same form as the terms found in the bonuses, albeit without the terms in the numerator. Here, $N_h(s,a)$ denotes the total number of visits to the given state-action-step after completing the final episode. Thus, summing the final total visitation counts simply gives the total number of steps: $\sum_{s,a,h} N_h(s,a) = T = KH$. The lemma below gives the following upper bound on the inner summation.

Lemma 6. Let $Z_n^p = \sum_{n=0}^N 1/(1 \vee n)^p$. Then, Z_n^p has the following upper bound:

$$Z_n^p \le \begin{cases} 2\sqrt{N} & \text{if } p \in \frac{1}{2} \\ \log(8N) & \text{if } p = 1 \end{cases}$$

for p = 1/2 and p = 1.

Proof. Removing the first two terms from the summation and upper bounding the remaining terms by an integral gives:

$$\begin{split} Z_n^p &= 2 + \sum_{n=2}^N \frac{1}{n^p} \leq 2 + \int_1^N \frac{1}{n^p} dn \leq 2 + \begin{cases} 2\sqrt{N} - 2 & \text{if } p \in \frac{1}{2} \\ \log\left(N\right) & \text{if } p = 1 \end{cases} \\ &\leq \begin{cases} 2\sqrt{N} & \text{if } p \in \frac{1}{2} \\ \log\left(8N\right) & \text{if } p = 1 \end{cases} \end{split}$$

as required.

Using Lemma 6 and Cauchy-Schwarz gives the results states in (7) and (8):

$$\sum_{s,a,h} \sum_{k=1}^{K} \frac{\mathbb{1}\left\{s_{h}^{k} = s, a_{h}^{k} = a\right\}}{\sqrt{1 \vee N_{kh}\left(s,a\right)}} \leq 2\sqrt{HSAT}$$

$$\sum_{s,a,h} \sum_{k=1}^{K} \frac{\mathbb{1}\left\{s_{h}^{k} = s, a_{h}^{k} = a\right\}}{1 \vee N_{kh}\left(s,a\right)} \leq HSA\log\left(8T\right)$$

B Proof of Main Results

Here, we present proof of the results stated in the paper. Throughout, we assume that the algorithm defines the following failure event:

$$F_{k}^{p} = \left\{ \exists \, s, a, h : D\left(\hat{P}_{kh}\left(\cdot | s, a\right), P_{h}\left(\cdot | s, a\right)\right) \ge \epsilon_{kh}^{p}\left(s, a\right) \right\}$$

which holds across all episodes with probability δ' . Outside the failure event, with probability $1-\delta'$, the divergence between the empirical and actual transition density of the h^{th} step at the start of the k^{th} episode is therefore, at most: $D(\hat{P}_{kh}(\cdot|s,a),P_h(\cdot|s,a)) \leq \epsilon_{kh}^p(s,a)$. Using $\epsilon_{kh}^p(s,a)$ as the maximum divergence allows for the construction of the following plausible set:

$$\mathcal{P}_{kh} = \left\{ \tilde{P}_h\left(\cdot|s,a\right) \in \Delta : D\left(\tilde{P}_h\left(\cdot|s,a\right), \hat{P}_{kh}\left(\cdot|s,a\right)\right) \le \epsilon_{kh}^p\left(s,a\right) \right\}$$

for each $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$. Here, Δ denotes the set of valid transition densities. From here, it is possible to derive the bonus by finding the conjugate of the divergence:

$$\beta_{kh}^{+}\left(s,a\right) = \max_{\tilde{P}_{h}\left(\cdot|s,a\right) \in \Delta} \left\{ \left\langle \tilde{V}, \tilde{P}_{h}\left(\cdot|s,a\right) - \hat{P}_{h}\left(\cdot|s,a\right) \right\rangle \right\}$$

$$\beta_{kh}^{-}\left(s,a\right) = \max_{\tilde{P}_{h}\left(\cdot|s,a\right) \in \Delta} \left\{ \left\langle -\tilde{V}, \tilde{P}_{h}\left(\cdot|s,a\right) - \hat{P}_{h}\left(\cdot|s,a\right) \right\rangle \right\}$$
$$\beta_{kh}^{*}\left(s,a\right) \ge \max \left\{ \beta_{kh}^{+}\left(s,a\right), \beta_{kh}^{-}\left(s,a\right) \right\}$$

by introducing a Lagrange multiplier. For a derivation of the bonuses associated with each divergence, we refer the read to Appendix A.5 of Neu and Pike-Burke [2020].

In Appendix B.1, we present proofs of the lemmas and theorems relating to lazy updating, respectively. Below is the failure event we use to control the number of missing episodes.

Lemma 1. Let $S_k = \sum_{i=1}^{k-1} \mathbb{1}\{i + \tau_i \ge k\}$, where $\tau_1, \tau_2, \cdots, \tau_{k-1} \sim f_{\tau}(\cdot)$ are independent and identically distributed random variables with finite expected value. Further, let:

$$F_k^{\tau} = \left\{ S_k \ge \mathbb{E}\left[\tau\right] + \log\left(\frac{K\pi}{6\delta'}\right) + \sqrt{2\mathbb{E}\left[\tau\right]\log\left(\frac{K\pi}{6\delta'}\right)} \right\}$$

be the failure event for a single k. Then, $\mathbb{P}(F_{\tau}) = \mathbb{P}(\bigcup_{k=1}^{\infty} F_k^{\tau}) \leq \delta'$.

Proof. By definition, the summation involves a sequence of independent indicator random variables. Considering its expectation reveals that:

$$\mathbb{E}\left[S_{k}\right] = \sum_{i=1}^{k-1} \mathbb{E}\left[\mathbb{1}\left\{i + \tau_{i} \geq k\right\}\right] = \sum_{i=1}^{k-1} \mathbb{P}\left[\mathbb{1}\left\{i + \tau_{i} \geq k\right\}\right] = \sum_{i=1}^{k-1} \mathbb{P}\left[\tau_{k-i} > i\right] = \sum_{i=0}^{k-2} \mathbb{P}\left[\tau_{k-i+1} > i\right]$$

$$\leq \sum_{i=0}^{\infty} \mathbb{P}\left[\tau > i\right] = \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{P}\left[\tau = j\right] = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \mathbb{P}\left[\tau = j\right] = \sum_{j=1}^{\infty} j \,\mathbb{P}\left[\tau = j\right]$$

$$= \mathbb{E}\left[\tau\right].$$

Next, looking at its variance reveals that:

$$\begin{aligned} \operatorname{Var}\left(S_{k}\right) &= \sum_{i=1}^{k-1} \operatorname{Var}\left(\mathbbm{1}\left\{i + \tau_{i} \geq k\right\}\right) = \sum_{i=1}^{k-1} \mathbb{E}\left[\left(\mathbbm{1}\left\{i + \tau_{i} \geq k\right\} - \mathbb{E}\left[\mathbbm{1}\left\{i + \tau_{i} \geq k\right\}\right]\right)^{2}\right] \\ &\leq \sum_{i=1}^{k-1} \mathbb{E}\left[\mathbbm{1}\left\{i + \tau_{i} \geq k\right\}^{2}\right] = \sum_{i=1}^{k-1} \mathbb{E}\left[\mathbbm{1}\left\{i + \tau_{i} \geq k\right\}\right] = \mathbb{E}\left[S_{k}\right] \\ &\leq \mathbb{E}\left[\tau\right] \end{aligned}$$

By Bernstein's inequality, we have that:

$$\mathbb{P}\left(S_k - \mathbb{E}\left[S_k\right] \ge \epsilon\right) \le \exp\left(-\frac{\epsilon^2}{\operatorname{Var}\left(S_k\right) + \frac{\epsilon}{3}}\right) = \frac{6\delta'}{\left(k\pi\right)^2}$$

Rearranging the above reveals that:

$$\epsilon \leq \frac{1}{3} \log \left(\frac{\left(k\pi\right)^2}{6\delta'} \right) + \sqrt{\operatorname{Var}\left(S_k\right) \log \left(\frac{\left(k\pi\right)^2}{6\delta'} \right)} \leq \frac{2}{3} \log \left(\frac{k\pi}{6\delta'} \right) + \sqrt{2\mathbb{E}\left[\tau\right] \log \left(\frac{k\pi}{6\delta'} \right)}$$

Since $k \leq K$, we have that:

$$\mathbb{P}\left(F_k^{\tau}\right) = \mathbb{P}\left(S_k - \mathbb{E}\left[\tau\right] \ge \frac{2}{3}\log\left(\frac{K\pi}{6\delta'}\right) + \sqrt{2\mathbb{E}\left[\tau\right]\log\left(\frac{K\pi}{6\delta'}\right)}\right) \le \frac{6\delta'}{(k\pi)^2}$$

By Boole's inequality, we have that:

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} F_k^{\tau}\right) \leq \sum_{k=1}^{\infty} \mathbb{P}\left(F_k^{\tau}\right) = \frac{6\delta'}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \delta'$$

as required.

B.1 Proofs for Lazy Updating

Lemma 3. For $K \geq SA$, Algorithm 2 ensures that the number of epochs has the following upper bound:

$$J \le 3HSA\log\left(\frac{4K}{SA}\right) .$$

Proof. First, we show that summing the within epoch visitation counter over all prior epochs is equal to the observed visitation counter at the start of the $(j+1)^{th}$ epoch. By definition, the observed visitation counter at the start of this episode is given by:

$$N'_{k_{j+1}h}(s,a) = \sum_{k=1}^{k_{j+1}-1} \mathbb{1} \left\{ s_h^k = s, a_h^k = a, k + \tau_k < k_{j+1} \right\}$$

$$= \sum_{i=1}^{j} \sum_{k=1}^{k_{i+1}-1} \mathbb{1} \left\{ s_h^k = s, a_h^k = a, k_i \le k + \tau_k < k_{i+1} \right\}$$

$$= \sum_{i=1}^{j} n_{ih}(s,a)$$

where the second line follows from epochs forming a disjoint set of episodes, and the third follows from the definition of the within epoch visitation counter.

Let J(s,a,h) denote the number of times the algorithm starts a new epoch due to the given state-action-step triple doubling and consider only these epochs. Denote the total visitation counter and observed visitation counter after the completing the final episode by: $N_h(s,a)$ and $N_h'(s,a)$, respectively. Then, we have that:

$$N_{h}(s,a) \ge N'_{h}(s,a) = \sum_{j=1}^{J} n_{jh}(s,a) \ge 1 + \sum_{j:n_{jh}(s,a)=N'_{k_{j}h}(s,a)} N'_{k_{j}h}(s,a) = 1 + \sum_{j=1}^{J(s,a,h)} 2^{j-1} \ge 2^{J(s,a,h)} - 1$$

Summing over all state-action-step triples and applying Jensen's inequality gives:

$$T \ge \sum_{s,a,h} 2^{J(s,a,h)} - 1 \ge HSA\left(2^{\frac{\sum_{s,a,h} J(s,a,h)}{HSA}} - 1\right) \ge HSA\left(2^{\frac{J-1}{HSA}-1} - 1\right)$$

Rearranging gives:

$$2^{\frac{J-1}{HSA}-1} \le 1 + \frac{T}{HSA} \le \frac{2T}{HSA} \implies \frac{J-1}{HSA} \le 1 + \log_2\left(\frac{2T}{HSA}\right)$$

for all $T \geq HSA$. Further rearrangement reveals:

$$\begin{split} J &\leq 1 + HSA + HSA \log_2 \left(\frac{2T}{HSA}\right) \leq HSA + HSA \log_2 \left(\frac{4T}{HSA}\right) \\ &\leq 2HSA \log_2 \left(\frac{4T}{HSA}\right) = 2HSA \log_2 \left(\frac{4K}{SA}\right) \\ &\leq 3HSA \log \left(\frac{4K}{SA}\right) \end{split}$$

as required.

Lemma 7. For any sequence of numbers z_1, z_2, \dots, z_n with $z_k \leq Z_k = \max\{1, \sum_{i=1}^{k-1} z_i\}$:

$$\sum_{k=1}^{n} \frac{z_k}{Z_k} \le \sqrt{Z_n}$$

Proof. See Lemma 19 of Jaksch et al. [2010] for proof.