MOMENTS ESTIMATORS AND OMNIBUS CHI-SQUARE TESTS FOR SOME USUAL PROBABILITY LAWS

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ABSTRACT. For many probability laws, in parametric models, the estimation of the parameters can be done in the frame of the maximum likelihood method, or in the frame of moment estimation methods, or by using the plug-in method, etc. Usually, for estimating more than one parameter, the same frame is used. We focus on the moment estimation method in this paper. We use the instrumental tool of the functional empirical process (fep) in Lo (2016) to show how it is practical to derive, almost algebraically, the joint distribution Gaussian law and to derive omnibus chi-square asymptotic laws from it. We choose four distributions to illustrate the method (Gamma law, beta law, Uniform law and Fisher law) and completely describe the asymptotic laws of the moment estimators whenever possible. Simulations studies are performed to investigate for each case the smallest sizes for which the obtained statistical tests are recommendable. Generally, the omnibus chi-square test proposed here work fine with sample sizes around fifty.

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1. Introduction

Parameter estimations are important steps in parametric statistical modeling. Estimators of parameters can be derived from the maximum likelihood approach, the plug-in methods, the moment methods, etc. Of course, by far Maximum Likelihood Estimators (MLE) are preferred because of the statistical meaning of its derivation. Moments estimators (**ME**)'s and (*MLE*)'s may exists without having closed-form expressions. When a *MLE* does no have a closed-from estimator, the *ME* is a backup solution for authors who wish to a clear idea of the estimation and a quick and more controlled ways of computation. Finding *ME*'s is an important step in modeling. However, deriving the related statistical tests is needed for accepting or rejecting hypotheses.

For more that two parameters, it is more practicable to join the individual normal asymptotic laws for each parameters into one chi-square asymptotic laws which is qualified as omnibus following the Jarque-Berra chi-square asymptotic law.

This motivates us to investigate asymptotic laws of *ME*'s estimators of as much as possible of usual and non-usual statistical laws. The found law should be validated by simulation studies before being proposed to potential users.

For a large review of asymptotic estimations and statistical tests, we refer to Van der Vaart (2000), Billingsley (1968), etc. Especially, for methods including functional empirical process, van der Vaart and Wellner (1996) is recommended.

However the main tool used here, but not limited to, is the function empirical process (fep) transformed into a instrument tools in Lo (2016) (see below). let us begin by giving a few words on that tool and next basic notation.

In this paper, we used the *fep* tool to show direct and efficient ways for deriving asymptotic statistical tests for moment estimators for a selected set of four probability laws. Four these laws, all the computations are given in details. Computer codes for simulations are also provided. We could have treated more statistical distributions. However, we wanted this paper to be a model for researchers who need asymptotic statistical tests. Later, we

expect to compose a handbook which includes a great number of laws.

The paper is organized as follows. We will close this introductory section by describing the *lofep* tool in Subsection 1.1 and, in Section 1.2 by showing how to derive omnibus chi-square tests from the Gaussian asymptotic theorems for distributions of more than two parameters. In Section 2, we expose the asymptotic laws of the moments estimators of *gamma*, *uniform*, *beta* and *Fisher* distributions. The proofs and the implementation of the *fep* tool on these distributions are stated in Section 4. In Section 3, we proceed to a simulation study on the asymptotic results and show that the omnibus chi-square tests work fine for small samples. The codes used for the simulations are stated in an appendix from page 27. The paper ends with conclusions and perspectives in Section 5.

1.1. A brief reminder of the *fep*. Let Z_1 , Z_2 , ... be a sequence of independent copies of a random variable Z defined on the same probability space with values on some metric space (S,d). Define for each $n \ge 1$, the functional empirical process by

$$\mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (f(Z_i) - \mathbb{E}f(Z_i)),$$

where f is a real and measurable function defined on \mathbb{R} such that

(1.1)
$$\mathbb{V}_{Z}(f) = \int \left(f(x) - \mathbb{P}_{Z}(f)\right)^{2} dP_{Z}(x) < \infty,$$

which entails

(1.2)
$$\mathbb{P}_{Z}(|f|) = \int |f(x)| dP_{Z}(x) < \infty.$$

Denote by $\mathcal{F}(S)$ - \mathcal{F} for short -the class of real-valued measurable functions that are defined on S such that (1.1) holds. The space \mathcal{F} , when endowed with the addition and the external multiplication by real scalars, is a linear space. Next, it remarkable that \mathbb{G}_n is linear on \mathcal{F} , that is for f and g in \mathcal{F} and for $(a,b) \in \mathbb{R}^2$, we have

$$a\mathbb{G}_n(f) + b\mathbb{G}_n(g) = \mathbb{G}_n(af + bg).$$

We have this result

Lemma 1. Given the notation above, then for any finite number of elements $f_1, ..., f_k$ of $S, k \ge 1$, we have

$$^{t}(\mathbb{G}_{n}(f_{1}),...,\mathbb{G}_{n}(f_{k})) \rightsquigarrow \mathcal{N}_{k}(0,\Gamma(f_{i},f_{j})_{1\leq i,j\leq k}),$$

where

$$\Gamma(f_i, f_j) = \int (f_i - \mathbb{P}_Z(f_i)) (f_j - \mathbb{P}_Z(f_j)) d\mathbb{P}_Z(x), 1 \le i, j \le k.$$

Proof. It is enough to use the Cramér-Wold Criterion (see for example Billingsley (1968), page 45), that is to show that for any a = t $(a_1, ..., a_k) \in \mathbb{R}^k$, by denoting $T_n = t$ $(\mathbb{G}_n(f_1), ..., \mathbb{G}_n(f_k))$, we have $a_i, T_i > t \to t$ follows the $\mathcal{N}_k(0, \Gamma(f_i, f_j)_{1 \le i,j \le k})$ law and $a_i < 0, t > t$ stands for the usual product scalar in \mathbb{R}^k . But, by the standard central limit theorem in \mathbb{R} , we have

$$\langle a, T_n \rangle = \mathbb{G}_n \left(\sum_{i=1}^k a_i f_i \right) = \sum_{i=1}^k a_i \mathbb{G}_n \left(f_i \right) \leadsto N(0, \sigma_{\infty}^2),$$

where, for $g = \sum_{1 \le i \le k} a_i f_i$,

$$\sigma_{\infty}^2 = \int (g(x) - \mathbb{P}_Z(g))^2 d\mathbb{P}_Z(x)$$

and this easily gives

$$\sigma_{\infty}^2 = \sum_{1 \le i, j \le k} a_i a_j \Gamma(f_i, f_j),$$

so that $N(0,\sigma_{\infty}^2)$ is the law of < a,T> . The proof is finished.

1.2. **Main notations in the** *fep*. In the context of this paper, we use univariate samples $X, X_1, X_2, \dots, X_n, n \geq$, with common cumulative distribution function (cdf) $F_X = F$, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will usually need the cumulants m_k and the centered moments μ_k defined by

$$m_k = \mathbb{E}X^k, \quad \mu_k = \mathbb{E}(X - \mathbb{E}(X))^k, \quad k \ge 1$$

and their plug-in estimators

$$\overline{X}_{k,n} = \frac{1}{n} \sum_{j=1}^{n} X_j^k, \ \overline{\mu}_{k,n} = \frac{1}{n} \sum_{j=1}^{n} (X_j - \overline{X}_n)^k, \ k \ge 1$$

with the special case of the empirical mean $\overline{X}_{k,n} = \overline{X}_n$. Also the standard variance

$$S_n^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j - \overline{X}_n)^2$$

will be preferred to the plug-in estimator $\overline{\mu}_{2,n}$ of $\sigma^2 = \mathbb{V}ar(X)$. We suppose that any moment of order $k \geq 1$ exists whenever it is used.

The moment method in a parametric estimation related to the studied random variable having $\ell \geq 1$ parameters $(\theta_1, \cdots, \theta_\ell)$ and which generates the sample $\{X_1, \cdots, X_n\}$ consisted in simultaneously solving ℓ equations, each of these equations $r \in \{1, \cdots, \ell\}$ being the equality between a cumulant or a moment of order k_r and the corresponding plug-in estimator of the same order h_r where all order k_j are pairwise distinct. In general, it is simpler to take equations between the ℓ^{th} first cumulants or moments. The solution, whenever exists and statistics of the empirical cumulant or moments,

$$\hat{\theta}_n = (\hat{\theta}_{1,n}, \cdots, \hat{\theta}_{\ell,n})$$

is the vector moment estimator (ME).

Once the ME's are found, we will need the joint asymptotic law of the vector $\hat{\theta}_n$. The tool of the fep will greatly help in that target. We will go beyond and derive chi-square tests as much as possible.

The rest of the paper is organized as follows...

1.3. **Chi-square law derivation.** We are going to show how to derive asymptotic chi-square laws from moment estimators for at least two parameters. In each case below, we treat a two-parameter estimation problem. Suppose that the two parameters are denoted by a and b and their moment estimators are denoted by \hat{a}_n and \hat{b}_n , $n \geq 2$. We will get in each case a first law in the form: as $n \to +\infty$,

(1.3)
$$\left(\sqrt{n}(\hat{a}_n - a), \sqrt{n}(\hat{b}_n - b)\right)^T \rightsquigarrow Z, Z \sim \mathcal{N}_2(0, \Sigma),$$

where $\sigma_1^2 = \Sigma_{1,1}$, $\sigma_2^2 = \Sigma_{2,2}$ and $\sigma_{12} = \Sigma_{1,2}$. From usual properties of Gaussian vectors, we have that, whenever $det(\Sigma) = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2 \neq 0$,

$$Z^T \Sigma^{-1} Z \sim \chi_2^2$$
.

(See for example Lo (2018), Proposition 12, page 150). By the continuous mapping theorem (see for example Lo *et al.* (2016), Proposition 03, page 34), we will have, as $n \to +\infty$,

$$Q_n = \left(\sqrt{n}(\hat{a}_n - a), \sqrt{n}(\hat{b}_n - b)\right) \Sigma^{-1} \left(\sqrt{n}(\hat{a}_n - a), \sqrt{n}(\hat{b}_n - b)\right)^T \rightsquigarrow \chi_2^2,$$

which, as $n \to +\infty$, leads to

(1.4)
$$Q_n = \frac{n}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \left[\sigma_2^2 (\hat{a}_n - a)^2 + \sigma_1^2 (\hat{b}_n - b)^2 - 2\sigma_{12} (\hat{a}_n - a)(\hat{b}_n - b) \right] \rightsquigarrow \chi_2^2.$$

So, below, for each treated case, we will state two results according to (1.3) and (1.4).

2. Asymptotics related to moments estimators

In that section, we are going to treat the following probability laws:

$$(1)X \sim \gamma(a,b), (2)X \sim \beta(a,b), (3)X \sim \mathcal{U}(a,b) \text{ and } (4)X \sim \mathcal{F}(a,b)$$

These results are meant to be interesting examples for other cases not handing here. We stress that the techniques in Lo (2016) will be extensively used in the following.

2.1. Gamma laws $\gamma(a,b)$ **of parameters** a>0, b>0. The gamma law $\gamma(a,b)$ has the probability density function pdf

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) 1_{(x \ge 0)} \text{ with } \Gamma(a) = \int_a^{+\infty} x^{a-1} \exp(-x) dx.$$

The k-th cumulant ($k \ge 1$) is given by

$$\mathbb{E}X = \frac{a}{b}$$
 and $\mathbb{E}X^k = \frac{a}{b^k} \prod_{i=1}^{k-1} (a+j)$ for $k \ge 2$,

and the variance is

$$\sigma^2 = \mathbb{V}ar(X) = \frac{a}{b^2}.$$

The moment estimators \hat{a}_n and \hat{b}_n are solution of the equations $a/b = \overline{X}_n$ and $ab^{-2} = S_n^2$. We get

$$(\hat{a}_n, \hat{b}_n) = \begin{pmatrix} \overline{X}_n^2, \overline{X}_n \\ \overline{S}_n^2, \overline{S}_n^2 \end{pmatrix}.$$

Here are the results for the γ -law of parameters a > 0 and b > 0.

Theorem 1. We have

$$\sqrt{n}(\hat{a}_n - a, \hat{b}_n - b) \rightsquigarrow \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma_{1,1} = \mathbb{V}ar(H(X)), \quad \Sigma_{2,2} = \mathbb{V}ar(L(X)), \quad \Sigma_{1,2} = \mathbb{C}ov(H(X), L(X))$$

and

$$H = \frac{2\mu (\sigma^2 + 1)}{\sigma^4} h_1 + \frac{\mu^2}{\sigma^4} h_2,$$

$$L = \left(\frac{\sigma^2 + 2\mu}{\sigma^4}\right) h_1 - \frac{\mu}{\sigma^4} h_2.$$

We also have

(2.1)
$$Q_n = \frac{n}{\det(\Sigma)} \left[\Sigma_{2,2} (\hat{a}_n - a)^2 + \Sigma_{1,1} (\hat{b}_n - b)^2 - 2\Sigma_{1,2} (\hat{a}_n - a) (\hat{b}_n - b) \right] \rightsquigarrow \chi_2^2.$$

2.2. Beta law $\beta(a,b)$ of parameters a>0, b>0. The Beta law has the following probability distribution function

$$f(x) = \frac{x^{a-1} (1-x)^{b-1}}{B(a,b)}, x > 0.$$

Where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The expectation is given by

$$\mathbb{E}\left(X\right) = \frac{a}{a+b}$$

and the second moment order cumulant is given by

$$\mathbb{E}\left(X^{2}\right) = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

The moment estimators \hat{a}_n and \hat{b}_n are solutions of the equations $a/(a+b) = \overline{X}_n$ and $a(a+1)/(a+b)(a+b+1) = \overline{X}_n^2$. We get

$$(\hat{a}_n, \hat{b}_n) = \left(\frac{\overline{X}_n \left(\overline{X}_n - \overline{X}_n^2\right)}{\overline{X}_n^2 - \overline{X}_n^2}, \frac{\left(1 - \overline{X}_n\right) \left(\overline{X}_n - \overline{X}_n^2\right)}{\overline{X}_n^2 - \overline{X}_n^2}\right)$$

Here are the results for the β -law of parameters a > 0 and b > 0.

Theorem 2. We have

$$\sqrt{n}(\hat{a}_n - a, \hat{b}_n - b) \rightsquigarrow \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma_{1,1} = \mathbb{V}ar(H(X)), \quad \Sigma_{2,2} = \mathbb{V}ar(L(X)), \quad \Sigma_{1,2} = \mathbb{C}ov(H(X), L(X)),$$

$$H = \frac{\sigma^2 (2\mu - m_2) + 2\mu^2 (\mu - m_2)}{\sigma^4} h_1 - \frac{\sigma^2 \mu + \mu (\mu - m_2)}{\sigma^4} h_2$$

and

$$L = \frac{\sigma^2 (m_2 - 2\mu + 1) + 2\mu (1 - \mu)(\mu - m_2)}{\sigma^4} h_1 + \frac{(\mu - 1) (\sigma^2 + \mu - m_2)}{\sigma^4} h_2.$$

We also have the following asymptotic χ^2 result

(2.2)
$$Q_n = \frac{n}{\det(\Sigma)} \left[\Sigma_{2,2} (\hat{a}_n - a)^2 + \Sigma_{1,1} (\hat{b}_n - b)^2 - 2\Sigma_{1,2} (\hat{a}_n - a) (\hat{b}_n - b) \right] \rightsquigarrow \chi_2^2.$$

2.3. The Uniform law of parameters $\mathcal{U}(a,b)$, a>0 and b>a. The probability distribution function of the uniform law is given by

$$f(x) = \frac{1}{b-a}, x \in [a, b].$$

Its expectation is

$$\mathbb{E}(X) = \frac{a+b}{2}.$$

The variance is

$$\mathbb{V}ar\left(X\right) = \frac{\left(b - a\right)^2}{12}.$$

The moment estimators are the solutions of the equations

$$(\hat{a}_n, \hat{b}_n) = \left(\overline{X}_n - \lambda \left(S_n^2\right)^{1/2}, \overline{X}_n + \lambda \left(S_n^2\right)^{1/2}\right)$$

where $\lambda = 12^{1/2}/2$.

Here are the results for the Uniform-law.

Theorem 3. We have

$$\sqrt{n}(\hat{a}_n - a, \hat{b}_n - b) \rightsquigarrow \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma_{1,1} = \mathbb{V}ar(H(X)), \quad \Sigma_{2,2} = \mathbb{V}ar(L(X)), \quad \Sigma_{1,2} = \mathbb{C}ov(H(X), L(X)),$$

where

$$H = \left(1 + \frac{\lambda \mu}{\sigma}\right) h_1 - \frac{\lambda}{2\sigma} h_2$$

and

$$L = \left(1 - \frac{\lambda \mu}{\sigma}\right) h_1 + \frac{\lambda}{2\sigma} h_2,$$

where $\lambda = 12^{1/2}/2$.

We also have

(2.3)
$$Q_n = \frac{n}{\det(\Sigma)} \left[\Sigma_{2,2} (\hat{a}_n - a)^2 + \Sigma_{1,1} (\hat{b}_n - b)^2 - 2\Sigma_{1,2} (\hat{a}_n - a) (\hat{b}_n - b) \right] \rightsquigarrow \chi_2^2.$$

2.4. **Fisher law** $\mathcal{F}(a,b)$ **of parameters** a>0 **and** b>0. For a Fisher law with a and b degrees of freedom, the parameters are supposed to be integers. But in the general case, the probability density function has the same form and is associated to the quotient of two independent random variables Z_1/Z_2 , where $Z_1 \sim \gamma(a,1/2)/a$ and $Z_2 \sim \gamma(b,1/2)/b$, where a and b are positive. The pdf is expressed as follows:

$$f(x) = \frac{a^{1/2}b^{1/2}\Gamma\left(\frac{a+b}{2}\right)x^{a/2-1}}{\Gamma(a/2)\Gamma(b/2)(b+ax)^{(a+b)/2}}, x > 0.$$

The expectation is given by

$$\mathbb{E}(X) = \frac{b}{b-2} = \overline{X_n}.$$

The variance is given by

$$Var(X) = \frac{2b^2(a+b-2)}{a(b-2)^2(b-4)} = S_n^2, \ b > 4.$$

The moment estimators are solutions of the equations

$$(\hat{a}_n, \hat{b}_n) = \left(\frac{2\overline{X}_n^2}{S^2(2 - \overline{X}_n) - \overline{X}_n^2(\overline{X}_n - 1)}, \frac{2\overline{X}_n}{\overline{X}_n - 1}\right).$$

Here are the results for the Fisher-law.

Theorem 4. We have

$$\sqrt{n}(\hat{a}_n - a, \hat{b}_n - b) \rightsquigarrow \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma_{1,1} = \mathbb{V}ar(H(X)), \quad \Sigma_{2,2} = \mathbb{V}ar(L(X)), \quad \Sigma_{1,2} = \mathbb{C}ov(H(X), L(X)),$$

where

$$H = \frac{2\mu(2-\mu)}{\beta}h_1 - \frac{2\mu^2(2-\mu)}{\beta^2}h_2$$

with

$$\beta = \sigma^2 + 2\mu(2 - \mu) - \mu(3\mu - 2)$$

and where

$$L = -\frac{2}{(\mu - 1)^2} h_1.$$

We also have

(2.4)
$$Q_n = \frac{n}{\det(\Sigma)} \left[\Sigma_{2,2} (\hat{a}_n - a)^2 + \Sigma_{1,1} (\hat{b}_n - b)^2 - 2\Sigma_{1,2} (\hat{a}_n - a) (\hat{b}_n - b) \right] \rightsquigarrow \chi_2^2.$$

3. SIMULATIONS

We are going to describe our simulation works for one of studied distributions. Next we will explain their outputs and their interpretations. Finally, we will display results for all cases. Important scripts will be posted in the appendix 27.

3.1. **Simulation works.** In all cases, we estimate two parameters. In the case of the $\gamma(a,b)$ law, the moment estimators are denoted by *achap* and and *bchap*. We will have three parts.

A- Computing the exact moments and other coefficients of the estimators.

- (a) Before proceeding to the Monte-Carlo method, we have to computed the function H and L, demoted as bigH and bigL.
- (b) We proceed to numerical methods for computing $\mathbb{E}H(X)$, $sigmaHexaC = \mathbb{V}ar(E)H(X)$, $\mathbb{E}L(X)$, $sigmaLexaC = \mathbb{V}ar(E)L(X)$ and the exact co-variance $SigmaHLexa = \mathbb{C}ov(H(X), L(X))$, where X stands for random variable with the studied law (here a $\gamma(a,b)$ law). The trapezoidal method algorithm is used for all integral computations here.

In page 27, the related script is given under the title **A1** - **Computing exact coefficients** . Table gives exact values of the variances for different

Table 1. Over/under estimation of variances and covariances of estimators with respect to the true values for (a,b)=(2,3)

(a,b)	(2,3)	(3,10)	(10,3)
sigmaHexaC	7.78	107.08	63.08
	7.37	76.58	16.99
sigmaHLexa	40.3	7.985.01	1006.95
Correlation	69.76%	98.11%	93.95%

pairs (a, b).

B- Monte-Carlo estimation.

- (a) Fix a sample size $n \ge 2$. Fix values to a and b.
- (b) Fix the number of repetitions B=1000 (big enough to ensure the stability of outcomes).
- (c) At each repetition $j \in \{1, \dots, B\}$, we generate an sample of X of size n. Next
 - (1) $DA[j] = \sqrt{n}(achap a)$
 - (2) $DB[j] = \sqrt{n(bchap a)}$
 - (3) VH[j] = sd(bigH(X))
 - (4) VL[j] = sd(bigL(X))
 - (5) VHL[j] = cov(bigH(X), bigL(X))

In page 28, the related script is under the title **A2- Script Monte Carlo works**.

C- Computing the empirical moments and other coefficients.

- (a) Now, we have: (1) an estimate of $\Sigma_{1,1}$ by the square of the average of the vector VH, (2) an estimate of $\Sigma_{2,2}$ by the square of the average of the vector VH and (3) an estimate of $\Sigma_{1,2}$ by the average of the vector VHL. We denote them as SigmaHEMP, SigmaLEMP and SigmaHLEMP.
- (b) We also have: (1) an estimate of $\Sigma_{1,1}$ by empirical variance of DA, (2) an estimate of $\Sigma_{2,2}$ by empirical variance of DB and (3) an estimate of $\Sigma_{1,2}$ by

Table 2. Over/under estimation of variances and covariances of estimators with respect to the true values for (a,b)=(2,3)

size	n=50	n=100	n=200	n=1000
Qsig-1emp	98.98%	98.12%	98.47%	98.49%
Qsig-2emp	72.00%	74.59%	74.85%	75.56%
Qsig-12emp	81%	81.05%	82.39%	81.89%
Qsig-1samp	44.03%	43.34%	43.57%	44.93%
Qsig-2samp	79.11%	76.62%	76.58%	78.81%
Qsig-12samp	46.34%	44.57%	45.04%	48.23%

Table 3. Over/under estimation of variances and covariances of estimators with respect to the true values for (a,b)=(3,10)

size	n=50	n=100	n=200	n=1000
Qsig-1emp	96.87%	97.59%	98.46%	98.19%
Qsig-2emp	99.44%	99.95%	100.49%	100.32%
Qsig-12emp	98.72%	99.33%	100.46%	100.00%
Qsig-1samp	4.96%	4.46%	4.43%	4.40%
Qsig-2samp	25.03%	22.34%	21.98%	22.12%
Qsig-12samp	1.199%	0.96%	0.93%	0.94%

by empirical covariance between DA and DB. We denote them as sigmaH-SAMP, sigmaLSAMP and sigmaHLSAMP.

In page 29, the related script for computing sigmaHEMP, sigmaLEMP, sigmaHLEMP, sigmaHSAMP, sigmaLSAMP and sigmaHLSAMP is given under the title A3- Over/under estimations of variances and covariances from the script A2- Script Monte Carlo works in page 28.

In Tables 2, 3 and 4 display the quotients of empirical coefficients over the true coefficients, allowing to over or under-estimation, for three values of pairs (a, b).

D- Statistical tests for Computing the empirical moments and other coefficients.

(1) Performance of the point estimation. From the script **A2-Script Monte Carlo works** in page 28, we can compute the mean error (ME), the mean absolute error (MAE) and the square-root of the mean square error (MSE) of

Table 4. Over/under estimation of variances and covariances of estimators with respect to the true values for (a,b)=(10,3)

size	n=50	n=100	n=200	n=1000
Qsig-1emp	91.58%	92.92%	94.29%	94.25%
Qsig-2emp	90.25%	91.77%	93.30%	94.29%
Qsig-12emp	85.72%	86.95%	88.93%	88.14%
Qsig-1samp	25.09%	23.26%	23.79%	23.54%
Qsig-2samp	30.30%	28.15%	28.19%	28.37%
Qsig-12samp	7.43%	6.404%	6.56%	6.54%

Error type	n=25	n=50	n=75	n=100	n=200	n=300	n=1000
ME (A)	1.09	0.53	0.33	0.13	0.13	0.08	0.02
MAE (A)	2.88	1.99	1.54	0.92	0.92	0.76	0.411
\sqrt{MSE} (A)	3.88	2.49	1.99	1.163	1.17	0.95	0.5
ME (B)	0.42	0.21	0.12	0.05	0.048	0.029	0.009
MAE (B)	1.07	0.72	0.57	0.34	0.034	0.28	0.15
\sqrt{MSE} (B)	1.44	0.92	0.73	0.43	0.43	0.35	0.19

Table 5. Evolution of the errors on estimation of a=10 and b=3 in the sample sizes

the point estimations on a and b the R codes mean(DACHAP-a), mean(DBCHAP-b), mean(abs(DACHAP-a)). mean(abs(DBCHAP-b)), sd(DACHAP-a) and sd(DBCHAP-b)

- b). We report their values in Table 5
- (2) Statistical tests on a. We have three tools

$$DA_{1}[j] = \sqrt{\frac{n}{SigmaHexaC}}(achap - a) \approx \mathcal{N}(0, 1),$$

$$DA_{2}[j] = \sqrt{\frac{n}{SigmaHEMP}}(achap - a) \approx \mathcal{N}(0, 1),$$

$$DA_{3}[j] = \sqrt{\frac{n}{SigmaHSAMP}}(achap - a) \approx \mathcal{N}(0, 1).$$

(3) Statistical tests on b. We have three tools

cases	n=50	n=100	n=150	n=200	n=1000
Exact	0%	0%	0%	0%	0%
Empirical	0%	0%	0%	0%	0%
Sample	5.4%	5.3%	5.18%	5.09%	5%
Exact	0.01%	0%	0%	0%	0%
Empirical	0.02%	0%	0%	0%	0%
Sample	5.43%	5.49%	5.12%	5.28%	5.28%

Table 6. Empirical p-values for statistical test of a and b (a = 10 and b = 3)

$$DB_{1}[j] = \sqrt{\frac{n}{SigmaLexaC}}(bchap - b) \approx \mathcal{N}(0, 1),$$

$$DB_{2}[j] = \sqrt{\frac{n}{SigmaLEMP}}(bchap - b) \approx \mathcal{N}(0, 1),$$

$$DB_{3}[j] = \sqrt{\frac{n}{SigmaLSAMP}}(bchap - b) \approx \mathcal{N}(0, 1).$$

In page 29, the scripts **A4- Computations of the p-values**, for each parameter a and b, we compute the empirical p-values for each sequence, as the frequency of element of the sequence exceeding 1.96. The test is satisfactory if that p value is less of around 5%. The different p-values for a=2 and b=3 are given for different values of p in Table 6.

To test the quality of the normal approximations, we display the QQ-plots and the Parzen estimators graphs for each parameter in Fig 1 (QQ-plots and Parzen estimators related to the parameter a for n=50, according to the type of estimation of the coefficients), in Fig 2 (QQ-plots and Parzen estimators related to the parameter b for n=50, according to the type of estimation of the coefficients), in Fig 3 (QQ-plots and Parzen estimators related to the parameter a for n=300, according to the type of estimation of the coefficients) and in Fig 4 (QQ-plots and Parzen estimators related to the parameter b for n=300, according to the type of estimation of the coefficients) in Appendix C (Page 32).

(E) Omnibus test. We mean by omnibus test that the combine both test into a chi-square test as in Part (b) of each of Theorems 4, 3, 2. Depending on the use of exact values, empirical values or sample values of the

Table 7. QNE, QNEMP and QNSAMP according to the size n for a=10 and b=3

cases	n=50	n=100	n=200	n=300	n=1000
QNE	1.2%	0.1%	0.02%	0.001%	0.01%
QNEMP	0%	0%	0%	0%	0.8%
QNSAMP	0.6%	0.1%	0.3%	0.01%	0.2%

variance of co-variances, we have three statistics that can be used each for the chi-square test:

$$\begin{aligned} \mathbf{QExa}[j] &= \frac{n}{detHL} \bigg(sigmaLExaC(achap[j]-a)^2 \\ &+ sigmaHexaC(bchap[j]-b)^2 \\ &- 2 \times sigmaHLexa(achap[j]-a) \times (bchap[j]-b) \bigg) \sim \chi_1^2 \\ \mathbf{QEMPDB}[j] &= \frac{n}{detHLEMP} \bigg((sigmaLEMP^2)(achap[j]-a)^2 \\ &+ (sigmaHEMP^2)(bchap[j]-b)^2 \\ &- 2 \times sigmaHLEMP(achap[j]-a) \times (bchap[j]-b) \bigg) \sim \chi_1^2 \\ \mathbf{QEMPDB}[j] &= \frac{n}{detHLSAMP} \bigg((sigmaLSAMP^2)(achap[j]-a)^2 \\ &+ (sigmaHSAMP^2)(bchap[j]-b)^2 \\ &- 2 \times sigmaHLSAMP(achap[j]-a) \times (bchap[j]-b) \bigg) \sim \chi_1^2 \end{aligned}$$

Table 7 provides the p-value related to the omnibus test for different sizes according to the estimations of the coefficients used. The related test is given in the script under the title *A5- p-values for the omnibus statistical test* in page 30.

(D) - Conclusions of recommendations from simulations. The simulation studies show that the omnibus statistical test is very good even for sizes as small as n=50 for all estimations of the coefficients in the test statistics. When we do Gaussian separate tests for a and b, the outcomes are remarkable in the use the variance and covariance of $\sqrt{n}(\hat{a}_n-a)$ and

 $\sqrt{n}(\hat{b}_n - b)$. This is observable in the QQ-plots, the Parzen graphs and in the p-values of the tests. The separate tests seem to recommend the tests when n is bigger than 100. But, definitively, the omnibus works fine for small sizes as n = 11 with p-values 5.6%, 0%, 1.7%.

We strongly suggest to no use the tests with empirical estimations of the variance and covariance which lead to severe under or over estimation.

4. Proofs of Theorems

Here, we provide the computations for each treated probability law.

4.1. Gamma Law $\gamma(a,b)$ of parameters a>0, b>0. We have

(4.1)
$$\overline{X}_n = \mu + n^{-1/2} G_n(h_1)$$

and

(4.2)
$$S_n^2 = \frac{1}{n-1} \left[\sum_{j=1}^n X_j^2 - n \overline{X}_n^2 \right]$$

(4.3)
$$= \frac{n}{n-1} \left[\frac{1}{n} \sum_{j=1}^{n} X_j^2 - \overline{X}_n^2 \right]$$

$$= \frac{n}{n-1} \left[\overline{X_n^2} - \overline{X}_n^2 \right].$$

By the delta method,

(4.5)
$$\overline{X}_n^2 = \mu^2 + n^{-1/2} G_n (2\mu h_1) + O_p (n^{-1/2})$$

and

(4.6)
$$\overline{X_n^2} = \frac{1}{n} \sum_{j=1}^n X_j^2 = m_2 + n^{-1/2} G_n(h_2).$$

We know that

$$\frac{n}{n-1} = \left(\frac{n-1}{n}\right)^{-1} = (1-1/n)^{-1} = 1 + O_p(1).$$

Hence

(4.7)
$$S_n^2 = \sigma^2 + n^{-1/2} G_n \left(h_2 - 2\mu h_1 \right) + O_p \left(n^{-1/2} \right).$$

Let us handle \hat{b}_n . We have

$$\hat{b}_n = \frac{\overline{X_n}}{S_n^2}.$$

By equations (4.1), (4.7) and by lemma 11 in Lo (2016), we have

$$\hat{b_n} = \frac{\mu}{\sigma^2} + n^{-1/2} G_n \left(\frac{h_1}{\sigma^2} - \frac{\mu}{\sigma^4} \left(h_2 - 2\mu h_1 \right) \right) + O_p \left(n^{-1/2} \right)$$

$$= \frac{\mu}{\sigma^2} + n^{-1/2} G_n \left(L \right),$$

where

$$L = \frac{h_1}{\sigma^2} - \frac{\mu}{\sigma^4} (h_2 - 2\mu h_1) = \left(\frac{\sigma^2 + 2\mu h_1}{\sigma^4}\right) h_1 - \frac{\mu}{\sigma^4} h_2$$

Let us treat \hat{a}_n . By combining equations (4.5), (4.7) and lemma 11 in Lo (2016), we get

$$\hat{a}_n = \frac{\mu}{\sigma^2} + n^{-1/2} G_n(H) ,$$

where

$$H = \frac{2\mu}{\sigma^2} h_1 - \frac{\mu^2}{\sigma^4} (h_2 - 2\mu h_1)$$
$$= \frac{2\mu\sigma^2 + 2\mu}{\sigma^4} h_1 - \frac{\mu^2}{\sigma^4} h_2$$
$$= \frac{2\mu (\sigma^2 + 1)}{\sigma^4} h_1 - \frac{\mu^2}{\sigma^4} h_2.$$

4.2. Beta Law $\beta(a,b)$ **of parameters** a > 0, b > 0. The moment estimators \hat{a}_n and \hat{b}_n are solutions of the equations

$$\hat{a}_n = \frac{\overline{X}_n \left(\overline{X}_n - \overline{X}_n^2 \right)}{\overline{X}_n^2 - \overline{X}_n^2}$$

and

$$\hat{b}_n = \frac{\left(1 - \overline{X}_n\right) \left(\overline{X}_n - \overline{X}_n^2\right)}{\overline{X}_n^2 - \overline{X}_n^2}.$$

We have

(4.8)
$$\overline{X}_n = \mu + n^{-1/2} G_n(h_1)$$

and

(4.9)
$$\overline{X_n^2} = m_2 + n^{-1/2} G_n(h_2).$$

By the delta method, we have

(4.10)
$$\overline{X_n}^2 = \mu^2 + n^{-1/2} G_n(2\mu h_1) + O_p(n^{-1/2})$$

and

(4.11)
$$1 - \overline{X}_n = 1 - \mu - n^{-1/2} G_n(h_1) = 1 - \mu + n^{-1/2} G_n(-h_1).$$

By combining equations (4.12) and (4.13), we have

$$A_n = \overline{X}_n - \overline{X}_n^2 = \mu - m_2 + n^{-1/2} G_n (h_1 - h_2).$$

By combining equations (4.13) and (4.10), we have

$$B_n = \overline{X_n^2} - \overline{X_n^2} = m_2 - \mu^2 + n^{-1/2} G_n(h_2) - n^{1/2} G_n(2\mu h_1)$$

= $m_2 - \mu^2 + n^{-1/2} G_n(h_2 - 2\mu h_1) + O_p(n^{-1/2}).$

Hence

$$\hat{a}_n = \frac{C_n}{B_n}$$

with

$$C_n = \overline{X}_n \times A_n$$

= $\mu (\mu - m_2) + n^{-1/2} G_n ((\mu h_1 - \mu h_2) + (\mu - m_2) h_1) + O_p (n^{-1/2}).$

Then

$$C_n = \mu (\mu - m_2) + n^{-1/2} G_n (H_2) + O_p (n^{-1/2}),$$

where

$$H_2 = (\mu h_1 - \mu h_2) + (\mu - m_2) h_1$$

= $(2\mu - m_2) h_1 - \mu h_2$.

Hence

$$\hat{a}_n = \frac{\mu \left(\mu - m_2\right) + n^{-1/2} G_n \left(H_2\right) + O_p \left(n^{-1/2}\right)}{m_2 - \mu^2 + n^{-1/2} G_n \left(h_2 - 2\mu h_1\right) + O_p \left(n^{-1/2}\right)}.$$

By lemma 11 in Lo (2016), we get

$$\hat{a}_n = \frac{\mu (\mu - m_2)}{m_2 - \mu^2} + n^{-1/2} G_n (H) + O_p (n^{-1/2}),$$

Where

$$H = \frac{\sigma^2 (2\mu - m_2) + 2\mu^2 (\mu - m_2)}{\sigma^4} h_1 - \frac{\sigma^2 \mu + \mu (\mu - m_2)}{\sigma^4} h_2.$$

Let us handle now \hat{b}_n .

Remind that

$$\hat{b}_n = \frac{\left(1 - \overline{X}_n\right)\left(\overline{X}_n - X_n^2\right)}{\overline{X}_n^2 - \overline{X}_n^2} = \frac{B_1(n)}{B_2(n)}.$$

Thanks to the previous calculus, we have

$$\hat{b}_n = \frac{(1-\mu)(\mu - m_2)}{\sigma^2} + n^{-1/2}G_n(L) + O_p(n^{-1/2}),$$

where

$$L = \frac{\sigma^2 (m_2 - 2\mu + 1) + 2\mu (1 - \mu)(\mu - m_2)}{\sigma^4} h_1 + \frac{(\mu - 1) (\sigma^2 + \mu - m_2 - \mu)}{\sigma^4} h_2.$$

4.3. Uniform Law $\mathcal{U}(a,b)$, of parameters a>0 and b>a. We have

(4.12)
$$\overline{X}_n = \mu + n^{-1/2} G_n(h_1)$$

and

(4.13)
$$\overline{X_n^2} = m_2 + n^{-1/2} G_n(h_2).$$

By combining equations (4.12) and (4.13), we have

$$S_n^2 = \sigma^2 + n^{-1/2} G_n \left(h_2 - 2\mu h_1 \right).$$

By the delta method, we have

$$(S_n^2)^{1/2} = \sigma + n^{-1/2} G_n \left(\frac{1}{2\sigma} (h_2 - 2\mu h_1) \right) + O_p (n^{-1/2}).$$

So

$$a_{n} = \mu - \lambda \sigma + n^{-1/2} G_{n} \left(\left(h_{1} - \frac{\lambda}{2\sigma} \left(h_{2} - 2\mu h_{1} \right) \right) \right) + O_{p} \left(n^{-1/2} \right)$$
$$= \mu - \lambda \sigma + n^{-1/2} G_{n} \left(L \right) + O_{p} \left(n^{-1/2} \right),$$

with

$$L = \left(1 + \frac{\lambda \mu}{\sigma}\right) h_1 - \frac{\lambda}{2\sigma} h_2.$$

By using the same technique, we have

$$\hat{b}_n = \mu - \lambda \sigma + n^{-1/2} G_n(H) + O_p(n^{-1/2}),$$

where

$$H = \left(1 - \frac{\lambda \mu}{\sigma}\right) h_1 + \frac{\lambda}{2\sigma} h_2.$$

4.4. **Fisher Law** $\mathcal{F}(a,b)$ **of parameters** a **and** b**.** The moment estimators are defined below. The first moment estimator is

(4.14)
$$\hat{b} = \frac{2\overline{X}_n}{\overline{X}_n - 1}$$

$$= \frac{2\mu + n^{-1/2}\mathbb{G}_n(2h_1)}{\mu - 1 + n^{-1/2}\mathbb{G}_n(-h_1)}$$

$$= \frac{2\mu}{\mu - 1} + n^{-1/2}\mathbb{G}_n(L) + o_{\mathbb{P}}(n^{-1/2}),$$

where

$$L = \frac{2h_1}{\mu - 1} - \frac{2\mu}{(\mu - 1)^2} h_1$$
$$= -\frac{2}{(\mu - 1)^2} h_1.$$

The second estimator is given by

$$a = \frac{2b^2(b-2)}{S^2(b-2)^2(b-4) - 2b^4}.$$

By equation (4.14), we know that

$$b = \frac{2\overline{X}_n}{\overline{X}_n - 1}.$$

So we have,

$$b-2 = \frac{2\overline{X}_n}{\overline{X}_n - 1} - 2 = \frac{2}{\overline{X}_n - 1},$$

$$b-4 = \frac{2\overline{X}_n}{\overline{X}_n - 1} - 4 = \frac{-2\overline{X}_n + 4}{\overline{X}_n - 1}.$$

Hence

$$\hat{a} = \frac{16\overline{X}_n^2}{8S^2(2 - \overline{X}_n) - 8\overline{X}_n^2(\overline{X}_n - 1)}$$
$$= \frac{2\overline{X}_n^2}{S^2(2 - \overline{X}_n) - \overline{X}_n^2(\overline{X}_n - 1)}.$$

From (4.14), we have

(4.15)
$$S_n^2 = \sigma^2 + n^{-1/2} \mathbb{G}_n(h_2 - 2\mu h_1) + o_{\mathbb{P}(n^{-1/2})}.$$

Further, we have

(4.16)
$$2 - \overline{X}_n = 2 - \mu + n^{-1/2} \mathbb{G}_n(-h_1)$$

(4.17)
$$S_n^2(2-\overline{X}_n) = \sigma^2(2-\mu) + n^{-1/2}\mathbb{G}_n(L_1) + o_{\mathbb{P}(n^{-1/2})},$$

where

$$L_1 = -\sigma^2 h_1 + (2 - \mu)(h_2 - 2\mu h_1)$$

= $(\sigma^2 + 2\mu(2 - \mu))h_1 + (2 - \mu)h_2$

and

$$\overline{X}_n^2 = \mu^2 + n^{-1/2} \mathbb{G}_n(2\mu h_1) + o_{\mathbb{P}(n^{-1/2})}.$$

We also have

(4.18)
$$\overline{X}_n - 1 = \mu - 1 + \mathbb{G}_n(h_1)$$

(4.19)
$$\overline{X}_n^2(\overline{X}_n - 1) = \mu^2(\mu - 1) + n^{-1/2}\mathbb{G}_n(L_2),$$

where

$$L_2 = \mu^2 h_1 + (\mu - 1)(2\mu h_1) = \mu(3\mu - 2)h_1.$$

So the denominator (4.17)/(4.19) is expanded as

$$denom = (2 - \mu)\sigma^2 - \mu^2(\mu - 1) + n^{-1/2}\mathbb{G}_n(L_3) + o_{\mathbb{P}(n^{-1/2})},$$

where

$$L_3 = [\sigma^2 + 2\mu(2-\mu) - \mu(3\mu - 2)]h_1 + (2-\mu)h_2.$$

Hence

$$\hat{a} = \frac{2\mu^2 + n^{-1/2}\mathbb{G}_n(4\mu h_1) + o_{\Im(n^{-1/2})}}{Denom}$$

$$= \frac{2\mu^2}{(2-\mu)\sigma^2 - \mu^2(\mu-1) + n^{-1/2}\mathbb{G}_n(H) + o_{\Im(n^{-1/2})}},$$

where

$$H = \frac{2\mu}{(2-\mu)\sigma^2 - \mu^2(\mu-1)} h_1 - \frac{2\mu^2}{((2-\mu)\sigma^2 - \mu^2(\mu-1))^2} L_3.$$

Let us set

$$\alpha = (2 - \mu)\sigma^2 - \mu^2(\mu - 1)$$

$$\beta = \sigma^2 + 2\mu(2 - \mu) - \mu(3\mu - 2).$$

Then

$$L_3 = \beta h_1 + (2 - \mu)h_2.$$

Hence

$$H = \left(\frac{4}{\beta}\mu - \frac{2\mu^2}{\beta}\right)h_1 - \frac{2\mu^2(2-\mu)}{\beta^2}h_2$$
$$= \frac{2\mu(2-\mu)}{\beta}h_1 - \frac{2\mu^2(2-\mu)}{\beta^2}h_2. \quad \blacksquare$$

5. Conclusions and perspectives

Moment estimators for four statistical distributions been studied through their asymptotic Gaussian laws with the help of the *fep* tool. Chi-square omnibus tests have been derived for each distribution. The results have been simulated and the chi-square tests revealed themselves efficient for small sample sizes. The R codes of the simulations are attached to the paper in an appendix. The main perspective is to develop a full chapter in with the study of a large number of distributions.

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Appendix: Scripts.

A1 - Computing exact coefficients

```
a=2: b=3
(ml=a/(a+b))
(m2=a*(a+1)/((a+b)*(a+b+1)))
mu=a/b
sigmac=a/(b^2)
(a1=2*mu*(sigmac+1)/(sigmac^2))
(a2=-mu^2/(sigmac^2))
b1 = (sigmac + 2*mu) / (sigmac^2)
b2=-mu/(sigmac^2)
bigH <- function(x) al*x+a2*x^2
bigL <- function(x) b1*x+b2*x^2
# calcul variance exacte avec l'algorithme Integrations
#appel de la fonction gamma
CC=(b^a)/gamma(a)
#Variance exacte
\#bigHFV \leftarrow function(x) x^(a-1)*exp(-b*x)*bigH(x)*CC
\#bigLFV \leftarrow function(x) x^(a-1)*exp(-b*x)*bigL(x)*CC
\#bigHFVC \leftarrow function(x) x^(a-1)*exp(-b*x)*(bigH(x)^2)*CC
\#bigLFVC \leftarrow function(x) x^(a-1)*exp(-b*x)*(bigL(x)^2)*CC
bigHInv <- function(u) bigH(qgamma(u,a,b))
bigHInvC <- function(u) bigH(qgamma(u,a,b))^2
bigLInv <- function(u) bigL(qgamma(u,a,b))</pre>
bigLInvC \leftarrow function(u) bigL(qgamma(u,a,b))^2
bigHLInv<- function(u) bigH(qgamma(u,a,b))*bigL(qgamma(u,a,b))
(moHl=imhIntegD1 (0, 1, bigHInv, 100, 0.0001))
(m2H1=imhIntegD1(0,1,bigHInvC,100,0.0001))
(sigmaHexaC=m2H1 - (moH1^2))
(moL1=imhIntegD1(0,1,bigLInv,100,0.0001))
(m2L1=imhIntegD1(0,1,bigLInvC,100,0.0001))
(sigmaLexaC=m2L1 - (moL1^2))
(moHL=imhIntegD1(0,1,bigHLInv,100,0.0001))
(sigmaHLexa=moHL-(moH1*moL1))
(detHL=(sigmaHexaC*sigmaLexaC) -(sigmaHLexa^2))
```

A2- Script Monte Carlo works.

```
n=50
B=1000
DACHAP<-numeric (B)
DBCHAP<-numeric (B)
sigmaVL<-numeric(B)
sigmaVH<-numeric (B)
sigmaVHL<-numeric (B)
for(j in 1:B){
        X=rgamma(n,a,b)
        mlX=mean(X)
        varX=var(X)
         (achap = ((mlX)^2)/varX)
         (bchap = ((mlX)) / varX)
        sigmaVL[j]=sd(bigL(X))
        sigmaVH[j]=sd(bigH(X))
        sigmaVHL[j] = cov(bigH(X), bigL(X))
        DACHAP[j]=achap
        DBCHAP[j]=bchap
}
DA=sqrt(n)*(DACHAP-a)
DB=sqrt(n)*(DBCHAP-b)
```

A3- Over/under estimations of variances and covariances

```
sigmaHEMP/sqrt(sigmaHexaC)
sigmaLEMP/sqrt(sigmaLexaC)
sigmaHLEMP/sigmaHLexa
sigmaHSAMP/sqrt(sigmaHexaC)
sigmaLSAMP/sqrt(sigmaLexaC)
sigmaHLSAMP/sigmaHexa
```

A4- Computations of the p-values for statistical tests pour a et b.

```
imhPvalue (DA/sqrt (sigmaHexaC), B, 1.96)
imhPvalue (DA/sigmaHEMP, B, 1.96)
imhPvalue (DA/sigmaHSAMP, B, 1.96)
imhPvalue (DB/sqrt (sigmaLexaC), B, 1.96)
imhPvalue (DB/sigmaLEMP, B, 1.96)
imhPvalue (DB/sigmaLSAMP, B, 1.96)
```

See Function imhPvalue, under the title: B1-(a) Empirical tests from approximated Gaussian data, below

A5- p-values for the omnibus statistical test.

```
#detHL calcule dans la partie
#\textbf{Al - Computing exact coefficients}
#Estimation du determinannt Empirique et Sample
(detHLEMP=(sigmaHEMP^2)*(sigmaLEMP^2)-(sigma12EMP^2))
(detHLSAMP=((sigmaHSAMP^2)*(sigmaLSAMP^2)) -(sigma12SAMP^2))
#Calcul du khi-deux avec les parametres exacts
QNE=(sigmaLexaC*(achap-a)^2)+(sigmaHexaC*(bchap-b)^2)
QNE=QNE-2*sigmaHLexa*(achap-a)*(bchap-b)
QNE=QNE/(detHL)
#Calcul du khi-deux avec les parametres empiriques
QNEMP = (sigmaLEMP^2 * (achap-a)^2) + (sigmaHEMP^2 * (bchap-b)^2)
QNEMP=QNEMP-2*sigma12EMP*(achap-a)*(bchap-b)
QNEMP=QNEMP/(detHLEMP)
#Calcul du khi-deux avec les parametres estimes sur les statistiques
QNSAMP = (sigmaLSAMP^2 * (achap-a)^2) + (sigmaHSAMP^2 * (bchap-b)^2)
QNSAMP=QNSAMP-2*sigma12SAMP*(achap-a)*(bchap-b)
QNSAMP=QNSAMP/(detHLSAMP)
#mean p-values using function in:
#B2-(b) Empirical tests from approximated Chi-square data
pchisq (QNE, 2)
pchisq (QNEMP, 2)
pchisq (QNSAMP, 2)
```

B- Auxiliary Functions.

B1-(a) Empirical tests from approximated Gaussian data.

B2-(b) Empirical tests from approximated Chi-square data.

```
 \begin{array}{l} size=1000 \\ df=2 \text{ // degrees of freedom} \\ Z=rchisq(size\,,\,df) \\ th=0.05 \\ j=0 \\ pv=0 \\ \\ imhPvalueChisq <- function(Z,size\,,df,th) \{ \\ pv=0 \\ for(j in 1:size) \{ \\ if(Z[j] > qchisq(1-th\,,df)) \{ \\ pv=pv+1 \\ \} \\ \\ pv=100*pv/size \\ return(pv) \\ \end{array}
```

Appendix C

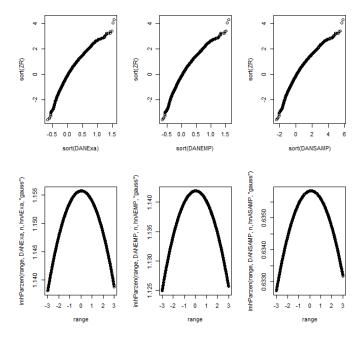


Figure 1. QQ-plots and Parzen estimators related to the parameter a for n=50, according to the type of estimation of the coefficients

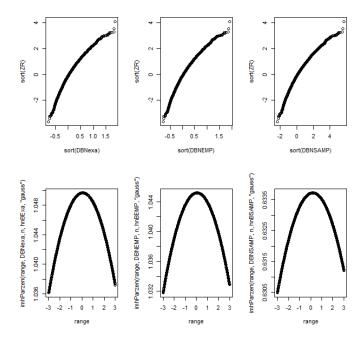


Figure 2. QQ-plots and Parzen estimators related to the parameter a for n=50, according to the type of estimation of the coefficients

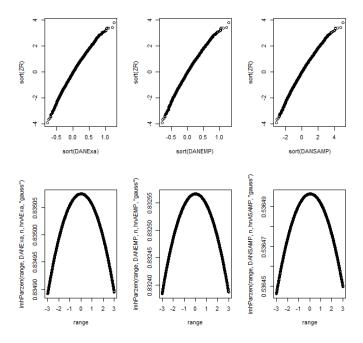


Figure 3. QQ-plots and Parzen estimators related to the parameter a for n=300, according to the type of estimation of the coefficients

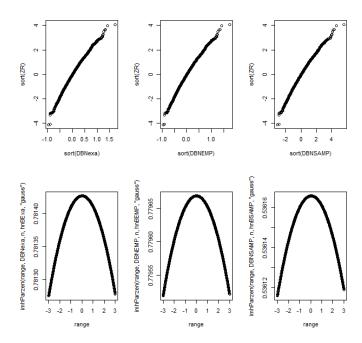


Figure 4. QQ-plots and Parzen estimators related to the parameter a for n=300, according to the type of estimation of the coefficients