Chapter 1 Section 1 Exercises

1. With $S_1 = \{2, 3, 5, 7\},\ S_2 = \{2, 4, 5, 8, 9\}$, and $U = \{1 : 10\}$, compute $\overline{S}_1 \cup S_2$.

Solution.

$$\overline{S}_1 = \{1, 4, 6, 8, 9, 10\} \quad \Rightarrow \quad \overline{S}_1 \cup S_2 = \{1, 2, 4, 5, 6, 8, 9, 10\}.$$

2. With $S_1 = \{2, 3, 5, 7\},\ S_2 = \{2, 4, 5, 8, 9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned}
S_1 \times S_2 = \{&(2,2), (2,4), (2,5), (2,8), (2,9), \\
&(3,2), (3,4), (3,5), (3,8), (3,9), \\
&(5,2), (5,4), (5,5), (5,8), (5,9), \\
&(7,2), (7,4), (7,5), (7,8), (7,9)\}. \\
S_2 \times S_1 = \{&(2,2), (2,3), (2,5), (2,7), \\
&(4,2), (4,3), (4,5), (4,7), \\
&(5,2), (5,3), (5,5), (5,7), \\
&(8,2), (8,3), (8,5), (8,7), \\
&(9,2), (9,3), (9,5), (9,7)\}.
\end{aligned}$$

3. For $S = \{2, 5, 6, 8\}$ and $T = \{2, 4, 6, 8\}$, compute $|S \cap T| + |S \cup T|$.

Solution.

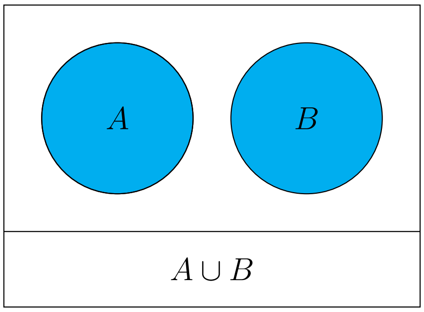
$$S \cap T = \{2, 6, 8\}, \quad S \cup T = \{2, 4, 5, 6, 8\} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \varnothing.$$

Therefore, S and T are disjoint. The Venn diagram is shown as follows:



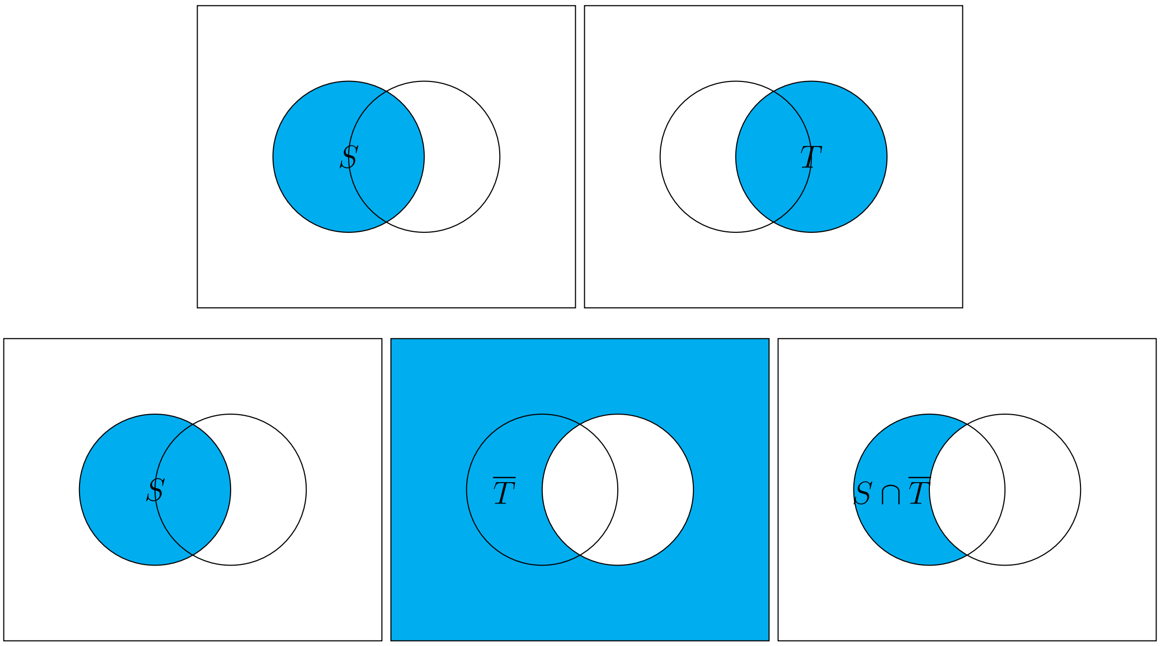
$A \cup B$

5. Show that for all sets S and T, $S - T = S \cap \overline{T}$.

Proof.

$$
\begin{aligned}
& S - T = \{x : x \in S \text{ and } x \notin T\} \\
\iff \quad & S - T = \{x : x \in S \text{ and } x \in \overline{T}\} \\
\iff \quad & S - T = S \cap \overline{T}.
\end{aligned}
$$

The Venn diagrams are shown as follows:



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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

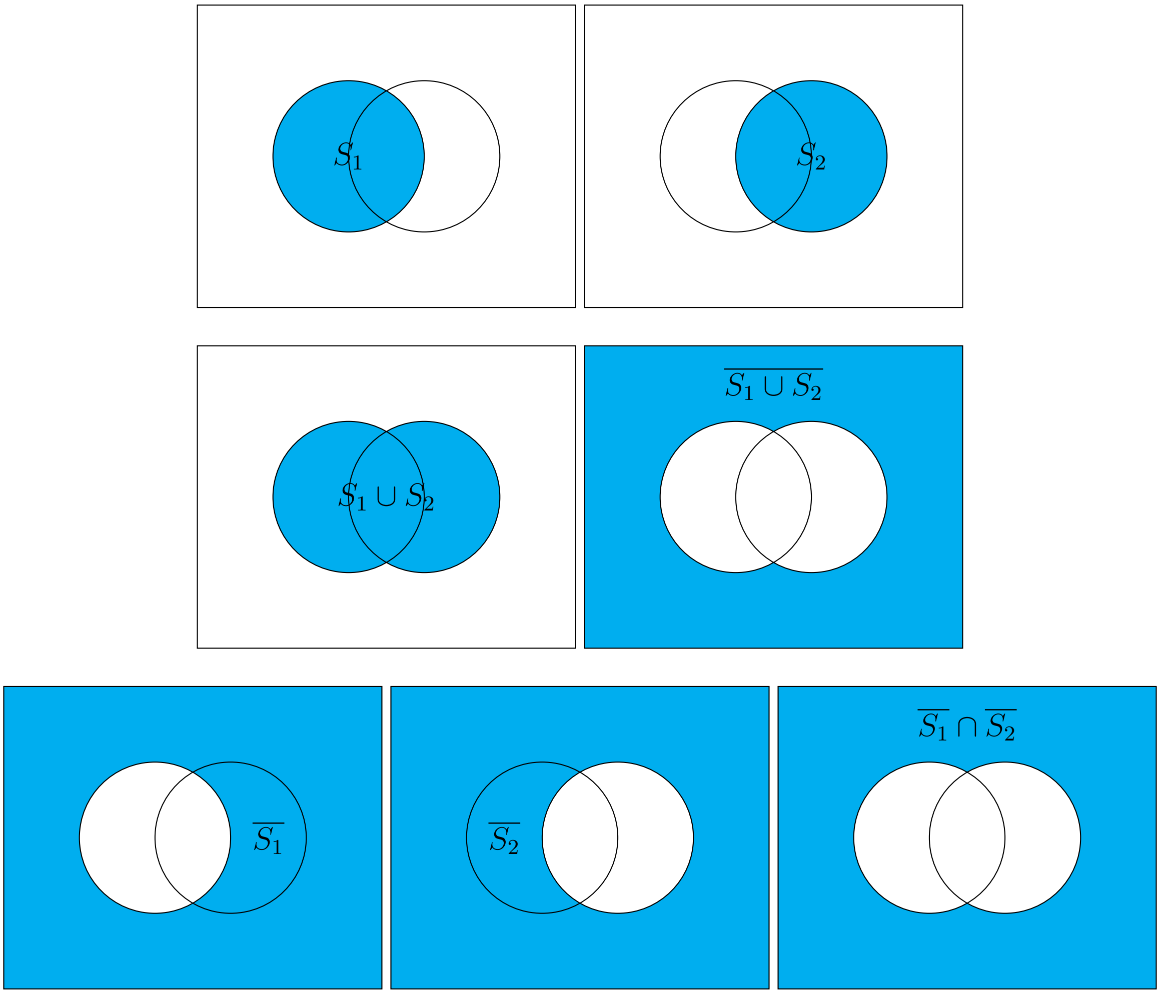
Proof.

$$S_1 \cup S_2 = \{x : x \in S_1 \text{ or } x \in S_2\} \quad \Rightarrow \quad \overline{S_1 \cup S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$
$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad \Rightarrow \quad \overline{S_1} \cap \overline{S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

The Venn diagrams are shown as follows:

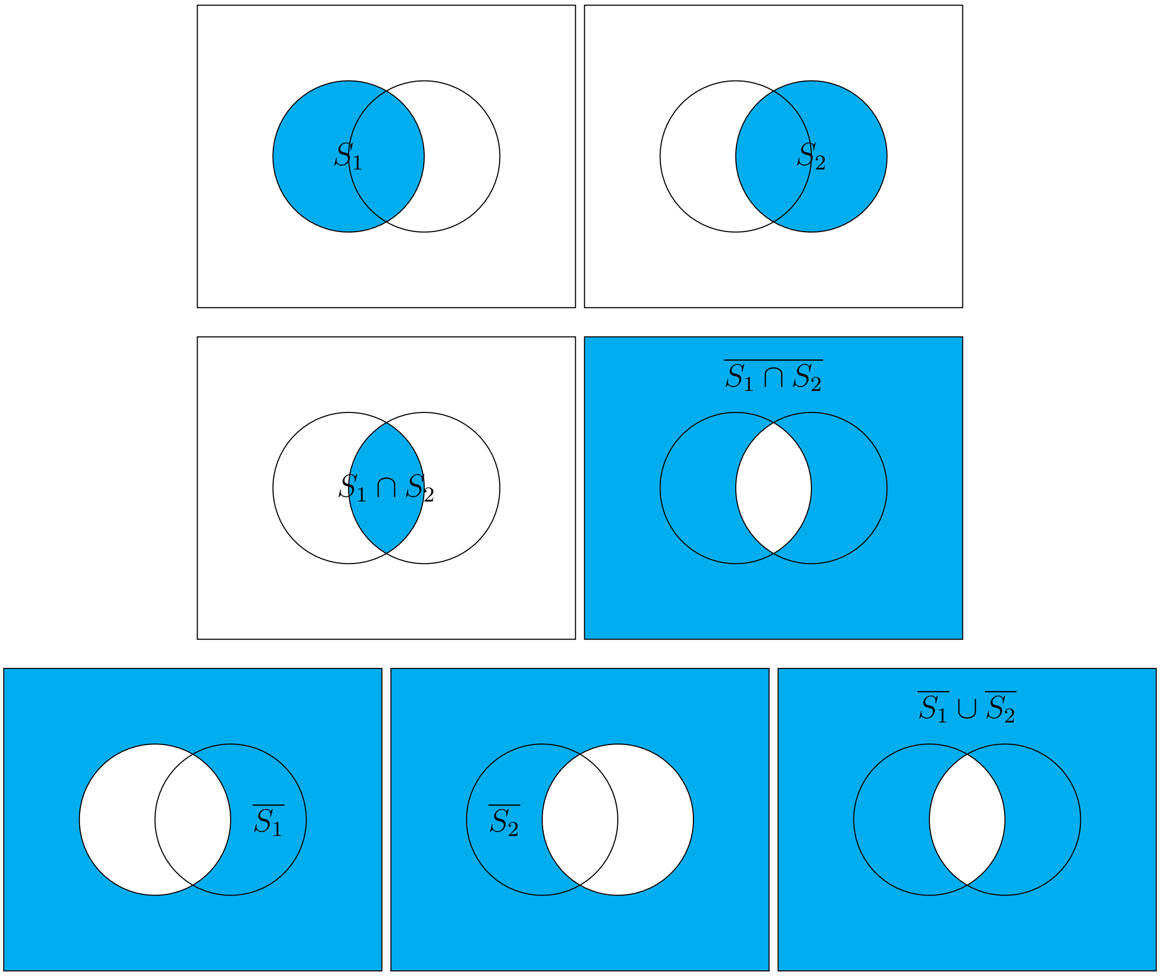


$$S_1 \cap S_2 = \{x : x \in S_1 \text{ and } x \in S_2\} \quad \Rightarrow \quad \overline{S_1 \cap S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$
$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad \Rightarrow \quad \overline{S_1} \cup \overline{S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

The Venn diagrams are shown as follows:



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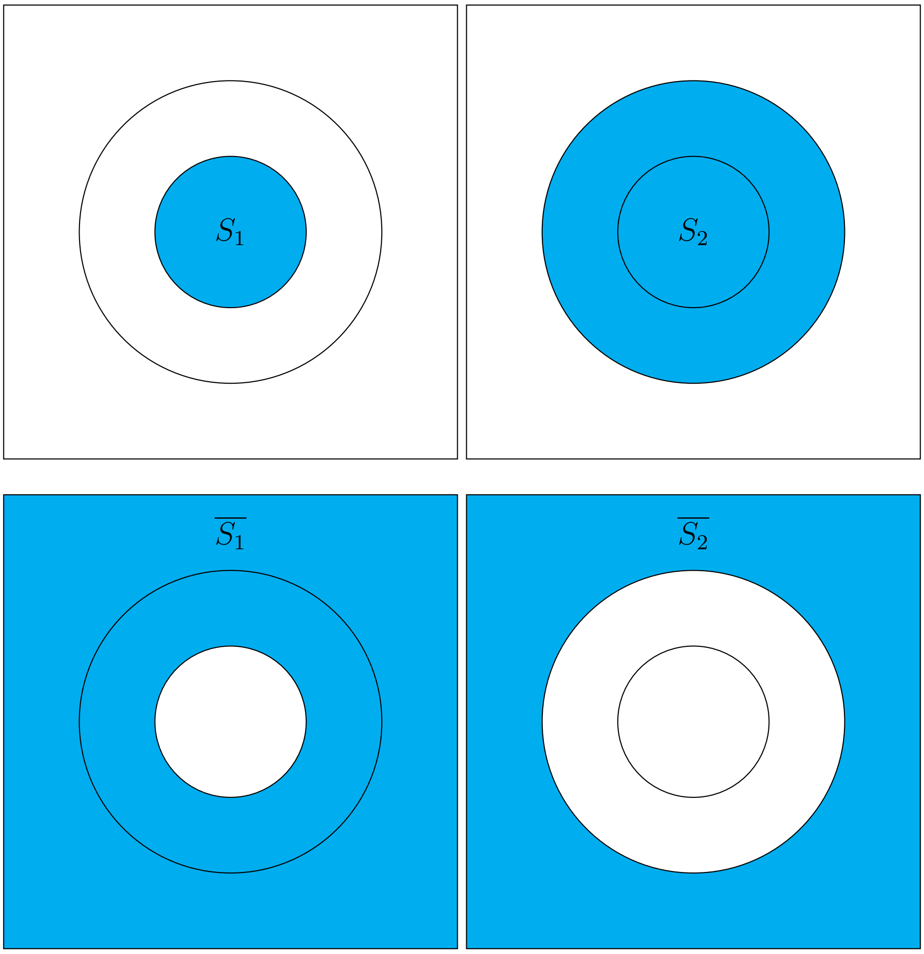
7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.

Proof.

$$\begin{array}{lllll} & S_1 \subseteq S_2 & & & \\ \Rightarrow & (x \in S_1 & \Rightarrow & x \in S_2) \\ \Rightarrow & (x \notin S_2 & \Rightarrow & x \notin S_1) \\ \Rightarrow & (x \in \overline{S_2} & \Rightarrow & x \notin \overline{S_1}) \\ \Rightarrow & \overline{S_2} \subseteq \overline{S_1}. & & \end{array}$$

The Venn diagrams are shown as follows:



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8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2$.

$$\left.\begin{array}{lll} S_1 = S_2 & \Rightarrow & S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 & \Rightarrow & S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{array}\right\} \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$$

2. $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2$.

   Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,

   - $\exists x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.
   - $\exists x \in S_2$ and $x \notin S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.

   The result contradicts with the permise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2$.

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

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9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.

Proof.

1. **Basis**

   If $|S| = 0$, $S = \varnothing$. Then
   $$2^S = \{\varnothing\}.$$
   Therefore, $|2^S| = 2^{|S|} = 1$.

   If $|S| = 1$, assume that $S = \{a\}$. Then
   $$2^S = \{\varnothing, \{a\}\}.$$
   Therefore, $|2^S| = 2^{|S|} = 2$.

2. **Inductive Assumption**

   Assume that $|2^S| = 2^{|S|}$, for $|S| = 1, 2, \cdots, n$.

3. **Inductive Step**

   For $|S| = n + 1$, assume that $S = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$. Let $T = \{a_1, a_2, \cdots, a_n\}$, then
   $$2^T = \{T_1, T_2, \cdots, T_{2^n}\}.$$
   For $\forall\, i = 1, 2, \cdots, 2^n$ where $i \in \mathbb{N}^+$
   $$\left.\begin{array}{c} T_i \subseteq T \\ T \subseteq S \end{array}\right\} \Rightarrow \quad T_i \subseteq S.$$
   However,
   $$S - T = \{a_{n+1}\} \quad \Rightarrow \quad a_{n+1} \notin T \quad \Rightarrow \quad a_{n+1} \notin T_i.$$
   In addition
   $$\left.\begin{array}{c} T_i \subseteq S \\ a_{n+1} \in S_i \quad \Rightarrow \quad \{a_{n+1}\} \subseteq S \end{array}\right\} \Rightarrow \quad T_i \cup \{a_{n+1}\} \subseteq S.$$

Let

$$T_i' = T_i \cup \{a_{n+1}\}, \qquad U = \{T_1', T_2', \cdots, T_{2^n}'\}.$$

Now, for $\forall\, S_i \subseteq S$

- If $a_{n+1} \notin S_i$, then $S_i \subseteq T$, so $S_i \in 2^T$.
- If $a_{n+1} \in S_i$, then $S_i - \{a_{n+1}\} \subseteq T$, so $S_i - \{a_{n+1}\} \in 2^T$. Assume that

$$S_i - \{a_{n+1}\} = T_j \quad \Rightarrow \quad S_i = T_j \cup \{a_{n+1}\} \quad \Rightarrow \quad S_i \in U.$$

Moreover, 2^T and U are disjoint. Therefore,

$$2^S = 2^T \cup U, \qquad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$.

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10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

Proof. Assume that

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1, c_2, \cdots, c_k \in S_1 \text{ and } c_1, c_2, \cdots, c_k \in S_2.$$

where $k \leq n,\ k \leq m,\ k \in \mathbb{N}^+$. Assume that

$$b_{i_1} = c_1,\ b_{i_2} = c_2,\ \cdots,\ b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_{i_1-1}, b_{i_1+1} \cdots, b_{i_k-1}, b_{i_k+1} \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

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11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.

Proof. Assume that $S_1 = \varnothing$ or $S_2 = \varnothing$, then

$$S_1 \times S_2 = \varnothing \quad \Rightarrow \quad |S_1 \times S_2| = 0,\ |S_1||S_2| = 0 \times 0 = 0 \quad \Rightarrow \quad |S_1 \times S_2| = |S_1||S_2|.$$

Assume that $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$,

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

where $n, m \in \mathbb{N}^+$.

Therefore,

$$\begin{aligned} S_1 \times S_2 = \{&(a_1, b_1),\ (a_2, b_1),\ \cdots, (a_n, b_1), \\ &(a_1, b_2),\ (a_2, b_2),\ \cdots, (a_n, b_2), \\ &\vdots \\ &(a_1, b_m), (a_2, b_m), \cdots, (a_n, b_m)\}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

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12. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.

Proof.

1. **Reflexivity**

$$|S_1| = |S_1| \text{ for all } S_1. \quad \Rightarrow \quad S_1 \equiv S_1 \text{ for all } S_1.$$

2. **Symmetry**

$$\text{if } |S_1| = |S_2|, \text{ then } |S_2| = |S_1|. \quad \Rightarrow \quad \text{if } S_1 \equiv S_2, \text{ then } S_2 \equiv S_1.$$

3. **Transitivity**

$$\text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|, \text{ then } |S_1| = |S_3|.$$
$$\Downarrow$$
$$\text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3, \text{ then } S_1 \equiv S_3.$$

Therefore, this is an equivalence relation.

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13. Occassionally, we need to use the union and intersection symbols in a manner analogous to the summation sign $\sum$. We define

$$\bigcup_{p \in \{i,j,k,\cdots\}} S_p = S_i \cup S_j \cup S_k \cdots$$

with an analogous notation for the intersection of several sets.
With this notation, the gereral DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when P is a finite set.
Proof.

1. **Basis**

   For $|P| = 2$, according to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}, \qquad \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

2. **Inductive Assumption**

   For $|P| = 2, 3, \cdots, n$ where $n \in \mathbb{N}^+$

$$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}, \qquad \overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$

3. **Inductive Step**

   For $|P| = n+1$ where $n \in \mathbb{N}^+, \forall\, i \in P,\ |P - \{i\}| = n$,

$$\overline{\bigcup_{p\in P} S_p} = \overline{(\bigcup_{p\in P-\{i\}} S_p) \cup S_i} = \overline{\bigcup_{p\in P-\{i\}} S_p} \cap \overline{S_i} = (\bigcap_{p\in P-\{i\}} \overline{S_p}) \cap \overline{S_i} = \bigcap_{p\in P} \overline{S_p},$$
$$\overline{\bigcap_{p\in P} S_p} = \overline{(\bigcap_{p\in P-\{i\}} S_p) \cap S_i} = \overline{\bigcap_{p\in P-\{i\}} S_p} \cup \overline{S_i} = (\bigcup_{p\in P-\{i\}} \overline{S_p}) \cup \overline{S_i} = \bigcup_{p\in P} \overline{S_p}.$$

Therefore, for $|P| = 2, 3, \cdots$

$$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}, \qquad \overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$

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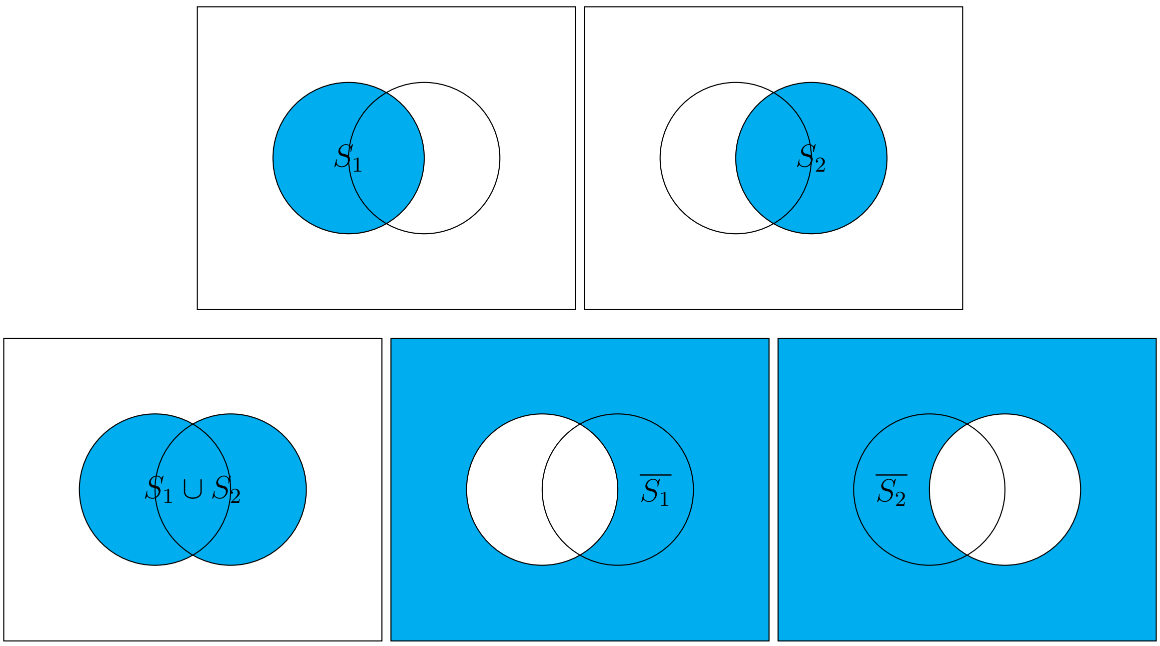
14. Show that

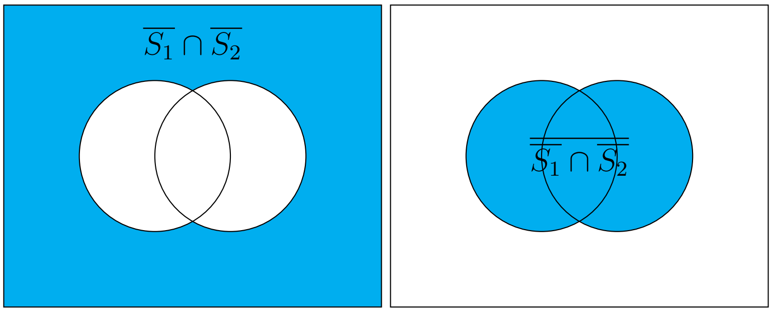
$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

Proof. According to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \quad \Rightarrow \quad \overline{\overline{S_1 \cup S_2}} = \overline{\overline{S_1} \cap \overline{S_2}} \quad \Rightarrow \quad S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

The Venn diagrams are shown as follows:





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15. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$

$$S_1 = S_2 \quad \Rightarrow \left\{ \begin{array}{c} S_1 \cap \overline{S_2} = S_1 \cap \overline{S_1} = \varnothing \\ \overline{S_1} \cap S_2 = \overline{S_1} \cap S_2 = \varnothing \end{array} \right\} \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

2. $S_1 = S_2 \quad \Leftarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$

   Assume that $S_1 \neq S_2$,

   - $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cap \overline{S_2} \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$
   - $\exists\, x \notin S_1$ and $x \in S_2 \quad \Rightarrow \quad x \in \overline{S_1} \cap S_2 \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$

   Therefore, $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \varnothing$, which is a contradiction. Thus $S_1 = S_2$.

To sum up,

$$S_1 = S_2 \quad \Longleftrightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

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16. Show that

$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

Proof.

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup \overline{\overline{S_2}}) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2). \end{aligned}$$

1. If $x \in S_2$

$$x \in S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \in (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

2. If $x \notin S_2$ and $x \in S_1$

$$x \notin S_2 \text{ and } x \in S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2 \text{ and } x \notin \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

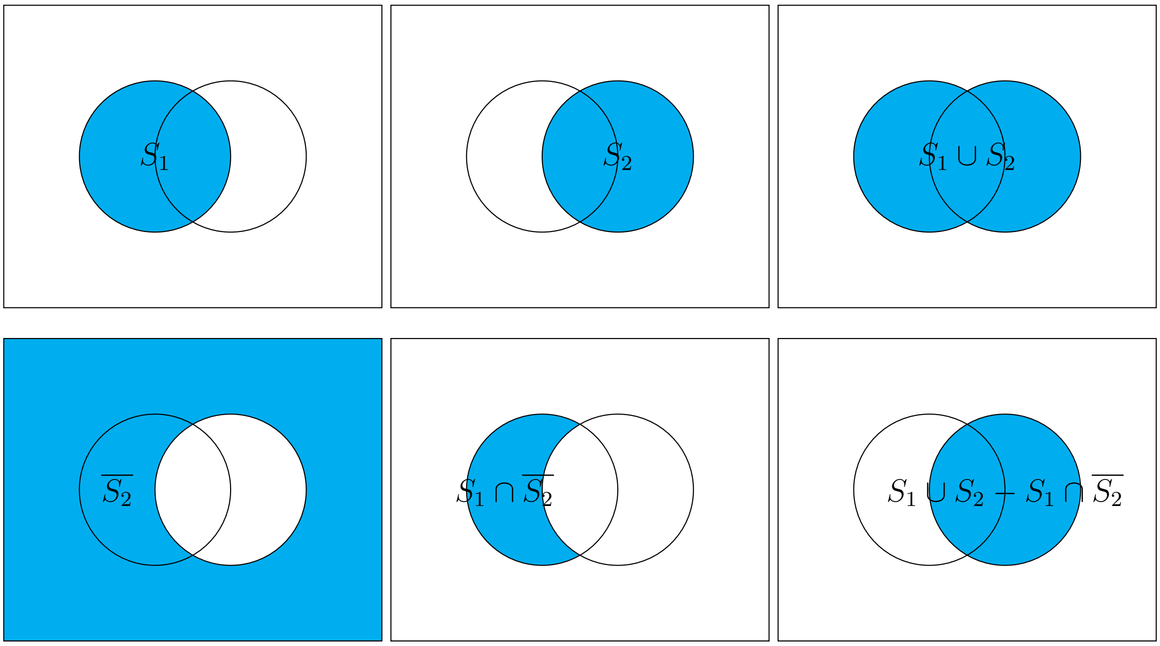
3. If $x \notin S_2$ and $x \notin S_1$

$$x \notin S_2 \text{ and } x \notin S_1 \quad \Rightarrow \quad x \notin S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

To sum up

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= S_2. \end{aligned}$$

The Venn diagrams are shown as follows:



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17. Show that the distributive law

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.

Proof.

1. If $x \notin S_1$

$$x \notin S_1 \quad \Rightarrow \begin{cases} x \notin S_1 \cap (S_2 \cup S_3) \\ x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \quad \Rightarrow \quad x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

2. If $x \in S_1,\ x \notin S_2$ and $x \notin S_3$

$$\begin{aligned} x \in S_1,\ x \notin S_2 \text{ and } x \notin S_3 \quad &\Rightarrow \quad x \notin S_2 \cup S_3 \quad \Rightarrow \quad x \notin S_1 \cap (S_2 \cup S_3). \\ x \in S_1,\ x \notin S_2 \text{ and } x \notin S_3 \quad &\Rightarrow \quad x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \\ &\Rightarrow \quad x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

3. If $x \in S_1$ and $x \in S_2$

$$x \in S_1 \text{ and } x \in S_2 \quad \Rightarrow \begin{cases} x \in S_1 \text{ and } x \in S_2 \cup S_3 \quad \Rightarrow \quad x \in S_1 \cap (S_2 \cup S_3) \\ x \in S_1 \cap S_2 \quad \Rightarrow \quad x \in (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

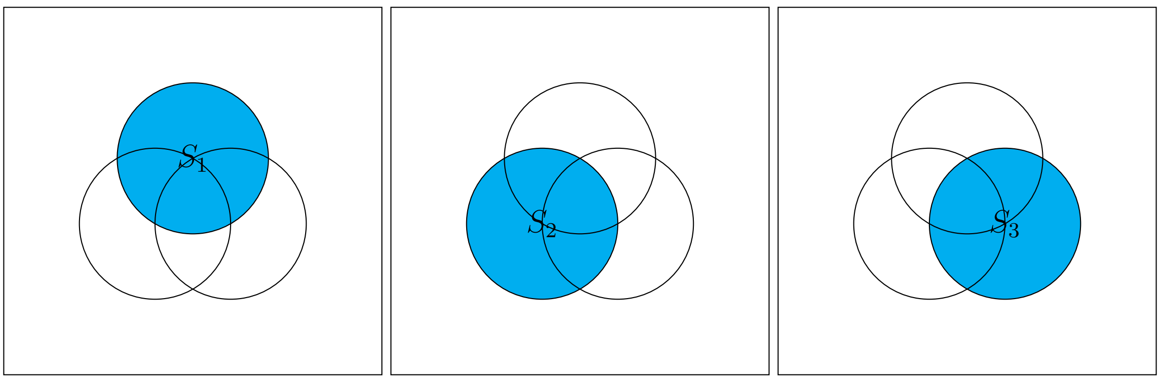
4. If $x \in S_1,\ x \notin S_2$ and $x \in S_3$

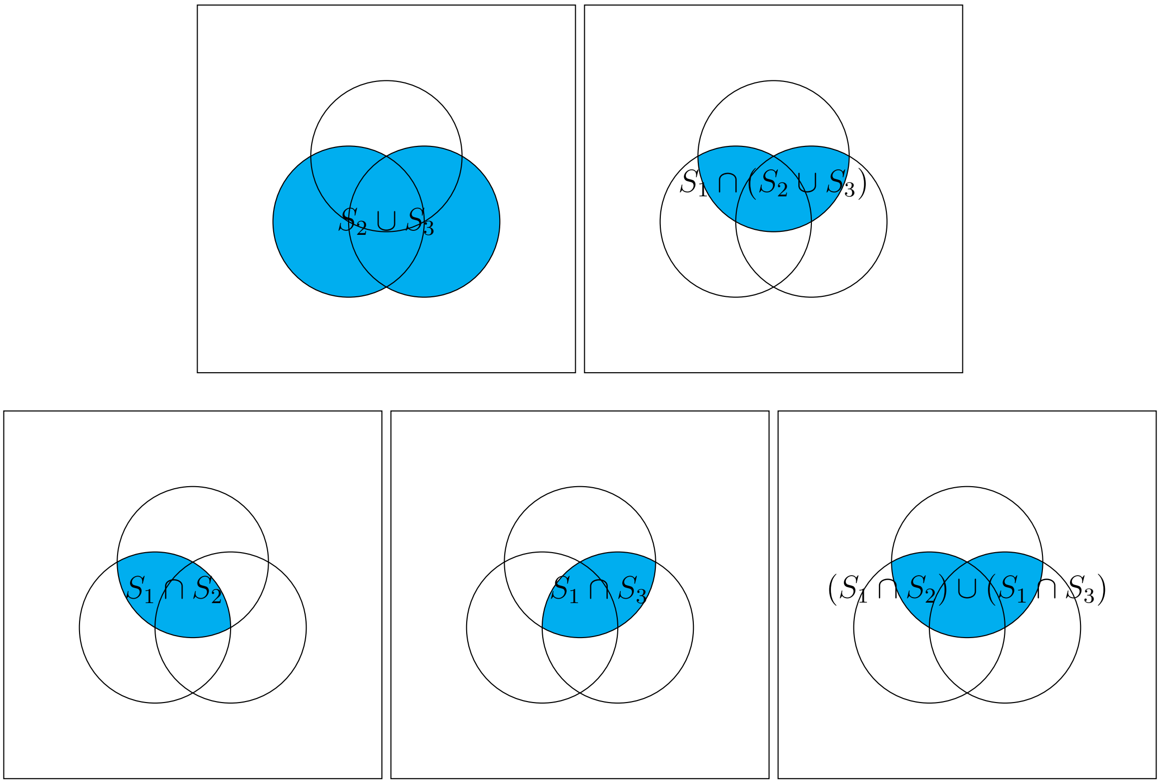
$$\begin{aligned} x \in S_1,\ x \notin S_2 \text{ and } x \in S_3 \quad &\Rightarrow \quad x \in S_1 \text{ and } x \in S_2 \cup S_3 \\ &\Rightarrow \quad x \in S_1 \cap (S_2 \cup S_3). \\ x \in S_1,\ x \notin S_2 \text{ and } x \in S_3 \quad &\Rightarrow \quad x \in S_1 \cap S_3 \\ &\Rightarrow \quad x \in (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

To sum up

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3).$$

The Venn diagrams are shown as follows:





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18. Show that

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Proof. Assume that $S_1 = \varnothing$, then

$$\left.\begin{array}{c} S_1 \times (S_2 \cup S_3) = \varnothing \\ S_1 \times S_2 = \varnothing,\ S_1 \times S_3 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = \varnothing \end{array}\right\} \Rightarrow \begin{array}{c} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{array}$$

Assume that $S_2 = \varnothing$, then

$$\left.\begin{array}{c} S_2 \cup S_3 = S_3 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_3 \\ S_1 \times S_2 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_3 \end{array}\right\} \Rightarrow \begin{array}{c} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{array}$$

Assume that $S_3 = \varnothing$, then

$$\left.\begin{array}{c} S_2 \cup S_3 = S_2 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_2 \\ S_1 \times S_3 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_2 \end{array}\right\} \Rightarrow \begin{array}{c} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{array}$$

Assume that $S_1 \neq \varnothing,\ S_2 \neq \varnothing,\ S_3 \neq \varnothing$

$$S_1 = \{a_1, a_2, \cdots, a_p\}, \qquad S_2 = \{b_1, b_2, \cdots, b_q\}, \qquad S_3 = \{c_1, c_2, \cdots, c_r\}.$$

where $p, q, r \in \mathbb{N}^+$.

Then

$$S_2 \cup S_3 = \{b_1, b_2, \cdots, b_q, c_1, c_2, \cdots, c_r\}.$$

$$\begin{aligned} S_1 \times (S_2 \cup S_3) = \{&(a_1, b_1),(a_1, b_2), \cdots,(a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots,(a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots,(a_p, b_q), \\ &(a_1, c_1),(a_1, c_2), \cdots,(a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots,(a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots,(a_p, c_r)\} \end{aligned}$$

$$\begin{aligned} S_1 \times S_2 = \{&(a_1, b_1),(a_1, b_2), \cdots,(a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots,(a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots,(a_p, b_q)\} \end{aligned} \qquad \begin{aligned} S_1 \times S_3 = \{&(a_1, c_1),(a_1, c_2), \cdots,(a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots,(a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots,(a_p, c_r)\} \end{aligned}$$

$$\begin{aligned} (S_1 \times S_2) \cup (S_1 \times S_3) = \{&(a_1, b_1),(a_1, b_2), \cdots,(a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots,(a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots,(a_p, b_q), \\ &(a_1, c_1),(a_1, c_2), \cdots,(a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots,(a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots,(a_p, c_r)\} \end{aligned}$$

Therefore,

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

$\blacksquare$

19. Give conditions on S_1 and S_2 necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$

Solution.

$$S_1 \cap S_2 = \varnothing \quad \iff \quad S_1 = (S_1 \cup S_2) - S_2.$$

1. $S_1 \cap S_2 = \varnothing \quad \Rightarrow \quad S_1 = (S_1 \cup S_2) - S_2$

$$\left.\begin{aligned} S_1 \cap S_2 = \varnothing \\ S_1 = S_1 \cap U = S_1 \cap (S_2 \cup \overline{S_2}) = (S_1 \cap S_2) \cup (S_1 \cap \overline{S_2}) \end{aligned}\right\} \Rightarrow S_1 = S_1 \cap \overline{S_2},$$

$$(S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 = (S_1 \cup S_2) - S_2.$$

2. $S_1 \cap S_2 = \varnothing \quad \Leftarrow \quad S_1 = (S_1 \cup S_2) - S_2$

$$S_1 = (S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 \cap S_2 = (S_1 \cap \overline{S_2}) \cap S_2 = S_1 \cap (\overline{S_2} \cap S_2) = S_1 \cap \varnothing = S_1.$$

To sum up,

$$S_1 = (S_1 \cup S_2) - S_2.$$

20. Use the equivalence defined in Example 1.4 to partition the set {2, 4, 5, 6, 9, 22, 24, 25, 31, 37} into equivalence classes.

Solution. Because

$$\begin{aligned} &2 \bmod 3 = 5 \bmod 3 = 2, \\ &4 \bmod 3 = 22 \bmod 3 = 25 \bmod 3 = 31 \bmod 3 = 37 \bmod 3 = 1, \\ &6 \bmod 3 = 9 \bmod 3 = 24 \bmod 3 = 0. \end{aligned}$$

The equivalence classes are

$$\{2, 5\}, \qquad \{4, 22, 25, 31, 37\}, \qquad \{6, 9, 24\}.$$

21. Show that if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.

Proof. Because $f(n) = O(g(n))$, $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^+$ such that $\forall\, n > N_1$

$$f(n) \le c_1 |g(n_1)|.$$

Because $g(n) = O(f(n))$, $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^+$ such that $\forall\, n > N_2$

$$g(n) \le c_2|f(n_2)|.$$

Let $N = \max\{N_1, N_2\}$, assume that $\forall\, n > N$

$$f(n) \ge 0, \qquad g(n) \ge 0.$$

Therefore,

$$\frac{1}{c_2}|g(n)| \le |f(n)| \le c_1|g(n)| \quad \Rightarrow \quad f(n) = \Theta(g(n)).$$

$\blacksquare$

22. Show that $2^n = O(3^n)$, but $2^n \ne \Theta(3^n)$.

Proof. $\exists\, c_1 = 1 > 0$ such that for all $n \ge 1$

$$2^n \le c_1|3^n| = 3^n.$$

Therefore,

$$2^n = O(3^n).$$

However, $\forall\, c_2 > 0,\ \exists\, N = [\log_{\frac{2}{3}} c_2] + 1$, if $n > N$

$$c_2|3^n| = c_2 3^n > |2^n| = 2^n.$$

Therefore,

$$2^n \ne \Theta(3^n).$$

$\blacksquare$

23. Show that the following order-of-magnitude results hold.

1. $n^2 + 5\log n = O(n^2)$.
2. $3^n = O(n!)$.
3. $n! = O(n^n)$.

Proof.

1. $n^2 + 5\log n = O(n^2)$.

   Let
   $$f(n) = n^2 + 5\log n, \qquad g(n) = n^2.$$
   Let $c = 2$, then $h(n) = f(n) - c|g(n)| = 5\log n - n^2$.
   $$h^{'}(n) = \frac{5}{n} - 2n \quad \Rightarrow \quad h^{'}(n) \text{ is a monotonically decreasing function.}$$
   If $n \geq 2, h^{'}(n) < 0$, so if $n \geq 2,\ h(n)$ is a monotonically decreasing function. Because $h(2) = 5\log 2 - 4 < 0$, if $n \geq 2,\ h(n) < 0$. $\exists\, c = 2 > 0$ such that for all $n \geq 2$
   $$h(n) = f(n) - c|g(n)| = 5\log n - n^2 < 0 \quad \Rightarrow \quad f(n) \leq c|g(n)|.$$
   Thus
   $$f(n) = O(g(n)) \quad \Rightarrow \quad n^2 + 5\log n = O(n^2).$$

2. $3^n = O(n!)$.

   Let
   $$f(n) = 3^n, \qquad g(n) = n!.$$
   $\exists\, c = 9 > 0$ such that for all $n \geq 3$
   $$f(n) - c|g(n)| = 3^n - 9n! = 3^n(1 - \frac{9n!}{3^n}) = 3^n(1 - 2\prod_{i=3}^{n}\frac{i}{3}) < 0 \quad \Rightarrow \quad f(n) < c|g(n)|.$$
   Thus
   $$f(n) = O(g(n)) \quad \Rightarrow \quad 3^n = O(n!).$$

3. $n! = O(n^n)$.

   Let
   $$f(n) = n!, \qquad g(n) = n^n.$$
   $\exists\, c = 1 > 0$ such that for all $n \geq 1$
   $$f(n) - c|g(n)| = n! - n^n = n!(1 - \prod_{i=1}^{n}\frac{n}{i}) \leq 0 \quad \Rightarrow \quad f(n) \leq c|g(n)|.$$
   Thus
   $$f(n) = O(g(n)) \quad \Rightarrow \quad n! = O(n^n).$$

■

24. Show that $\dfrac{n^3-2n}{n+1} = \Theta(n^2)$.

Proof. Let

$$f(n) = \frac{n^3-2n}{n+1}, \qquad g(n) = n^2, \qquad c_1 = \frac{1}{3}, \qquad c_2 = 1.$$

Because

$$f(n) = \frac{n^3-2n}{n+1} = \frac{n(n^2-2)}{n+1}.$$

If $n \geq 2$

$$|f(n)| = \left|\frac{n(n^2-2)}{n+1}\right| \geq 0 \quad \Rightarrow \quad |f(n)| = f(n).$$

Now

$$|f(n)| - c_1|g_n| = \left|\frac{n^3-2n}{n+1}\right| - \frac{1}{3}|n^2| = \frac{n^3-2n}{n+1} - \frac{1}{3}n^2 = \frac{n(2n+3)(n-2)}{3(n+1)}.$$

If $n \geq 2$

$$|f(n)| - c_1|g_n| = \frac{n(2n+3)(n-2)}{3(n+1)} \geq 0 \quad \Rightarrow \quad |f(n)| \geq c_1|g_n|.$$

Then

$$|f(n)| - c_2|g_n| = \left|\frac{n^3-2n}{n+1}\right| - |n^2| = \frac{n^3-2n}{n+1} - n^2 = -\frac{n(n+2)}{n+1}.$$

If $n > 0$

$$|f(n)| - c_2|g_n| = -\frac{n(n+2)}{n+1} < 0 \quad \Rightarrow \quad |f(n)| < c_2|g_n|.$$

To sum up, if $n \geq 2$

$$c_1|g_n| \leq |f(n)| < c_2|g_n| \quad \Rightarrow \quad f(n) = \Theta(g(n)) \quad \Rightarrow \quad \frac{n^3-2n}{n+1} = \Theta(n^2).$$

■

25. Show that $\dfrac{n^3}{\log(n+1)} = O(n^3)$ but not $O(n^2)$.

Proof. $\forall\, x > 0$, assume that the base of $\log(x+1)$ is 2. Let

$$f(x) = \log(x+1) - 1.$$

Then

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{(x+1)\ln 2} > 0.$$

Therefore, if $x > 0$, $f(x)$ is a strictly monotonically increasing function, then

$$f(1) = \log(1+1) - 1 = \log 2 - 1 = 0.$$

If $x \in \mathbb{N}^+$, $x \geq 1$. Let $x = n$

$$f(n) \geq f(1) = 0 \quad \Rightarrow \quad \log(n+1) - 1 \geq 0 \quad \Rightarrow \quad \frac{n^3}{\log(n+1)} \leq n^3.$$

Let $c_1 = 1$

$$\frac{n^3}{\log(n+1)} - c_1|n^3| = \frac{n^3}{\log(n+1)} - n^3 \leq 0 \quad \Rightarrow \quad \frac{n^3}{\log(n+1)} = O(n^3).$$

$\forall\, x \geq 1$, assume that the base of $\log(x+1)$ is 2. Let

$$g(x) = \sqrt{x} - \log(x+1).$$

Then

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} - \frac{1}{(x+1)\ln 2} > \frac{1}{2\sqrt{x}} - \frac{1}{x\ln 2} > \frac{1}{2\sqrt{x}} - \frac{2}{x} = \frac{\sqrt{x}-4}{2x}.$$

If $x > 16$,

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{\sqrt{x}-4}{2x} > 0.$$

Therefore, if $x > 16$, $g(x)$ is a strictly monotonically increasing function, then

$$g(19) = \sqrt{19} - \log(20) > 0.$$

Thus, if $x > 19$,

$$g(x) = \sqrt{x} - \log(x+1) > 0 \quad \Rightarrow \quad -\log(x+1) > -\sqrt{x}.$$

$\forall\, c_2 > 0,\ \exists\, N = \max\{20,\ [c_2^2] + 1\}$ such that

$$\begin{aligned}
\frac{N^3}{\log(N+1)} - c_2|N^2| &= \frac{N^2}{\log(N+1)}[N - c_2\log(N+1)] \\
&> \frac{N^2}{\log(N+1)}(N - c_2\sqrt{N}) \\
&\geq 0.
\end{aligned}$$

Therefore,

$$\frac{n^3}{\log(n+1)} \neq O(n^2).$$

■

26. What is wrong with the following argument?

$$x = O(n^4), \quad y = O(n^2), \quad \text{therefore} \quad \frac{x}{y} = O(n^2).$$

Proof. Let

$$f_1(n) = n^3, \qquad f_2(n) = 1, \qquad g_1(n) = n^4, \qquad g_2(n) = n^2.$$

$\exists\, c_1 = 1,\ c_2 = 1$ such that $\forall\, n \in \mathbb{N}^+$

$$f_1(n) - c_1|g_1(n)| = n^3 - n^4 \leq 0, \qquad f_2(n) - c_2|g_2(n)| = 1 - n^2 \leq 0.$$

Therefore

$$f_1(n) = O(g_1(n)) = O(n^4), \qquad f_2(n) = O(g_2(n)) = O(n^2).$$

Let

$$x = f_1(n) = n^3, \qquad y = f_2(n) = 1. \quad \Rightarrow \quad \frac{x}{y} = n^3.$$

However $\forall\, c > 0,\ \exists\, N = [c] + 1$ such that

$$\frac{x}{y}_{n=N} - c|N^2| = N^3 - cN^2 = N^2(N - c) > 0.$$

Thus

$$\frac{x}{y} \neq O(n^2).$$

■

27. What is wrong with the following argument?

$$x = \Theta(n^4), \quad y = \Theta(n^2), \quad \text{therefore} \quad \frac{x}{y} = \Theta(n^2).$$

Proof. This statement is correct. Assume that $\exists\ c_1 > 0,\ c_2 > 0,\ c_3 > 0,\ c_4 > 0$ and $\exists\ N_1 > 0,\ N_2 > 0$ where $N_1, N_2 \in \mathbb{N}^+$ such that if $n > N_1$

$$c_1|n^4| \leq |x| \leq c_2|n^4|$$

and if $n > N_2$

$$c_3|n^2| \leq |y| \leq c_4|n^2| \quad \Rightarrow \quad \frac{1}{c_4|n^2|} \leq \frac{1}{|y|} \leq \frac{1}{c_3|n^2|}.$$

Therefore let $N = \max\{N_1, N_2\}$, if $n > N$

$$\frac{c_1|n^4|}{c_4|n^2|} \leq \frac{|x|}{|y|} \leq \frac{c_2|n^4|}{c_3|n^2|} \quad \Rightarrow \quad \frac{c_1}{c_4}|n^2| \leq \left|\frac{x}{y}\right| \leq \frac{c_2}{c_3}|n^2|.$$

Thus

$$\frac{x}{y} = \Theta(n^2).$$

■

28. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Proof. Because $f(n) = O(g(n))$, $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^+$ such that $\forall\, n > N_1$

$$f(n) \leq c_1|g(n)|.$$

Because $g(n) = O(h(n))$, $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^+$ such that $\forall\, n > N_2$

$$g(n) \leq c_2|h(n)|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall\, n > N$

$$g(n) \geq 0.$$

Then $\forall\, n > N$

$$f(n) \leq c_1|g(n)| = c_1 g(n) \leq c_1 c_2|h(n)|.$$

Therefore

$$f(n) = O(h(n)).$$

■

29. Show that if $f(n) = O(n^2)$ and $g(n) = O(n^3)$, then

$$f(n) + g(n) = O(n^3)$$

and

$$f(n)g(n) = O(n^5).$$

Proof. Because $f(n) = O(n^2)$, assume that $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^+$ such that $\forall\, n > N_1$

$$f(n) \leq c_1|n^2|.$$

Because $g(n) = O(n^3)$, assume that $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^+$ such that $\forall\, n > N_2$

$$g(n) \leq c_2|n^3|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall\, n > N$

$$g(n) \geq 0.$$

Then $\forall\, n > N$

$$f(n) + g(n) \le c_1|n^2| + c_2|n^3| \le (c_1 + c_2)|n^3|,$$
$$f(n)g(n) \le c_1|n^2| \cdot c_2|n^3| = c_1c_2|n^5|.$$

Therefore

$$f(n) + g(n) = O(n^3), \qquad f(n)g(n) = O(n^5).$$

■

30. Assume that $f(n) = 2n^2 + n$ and $g(n) = O(n^2)$. What is wrong with the following argument?

$$f(n) = O(n^2) + O(n),$$

so that

$$f(n) - g(n) = O(n^2) + O(n) - O(n^2).$$

Therefore,

$$f(n) - g(n) = O(n).$$

Proof. Assume that

$$g(n) = n^2.$$

$\exists\, c_1 = 2,\ N_1 = 1,\ \forall\, n > N_1$

$$g(n) - c_1|n^2| = n^2 - 2n^2 = -n^2 < 0 \quad \Rightarrow \quad g(n) \le 2|n^2| \quad \Rightarrow \quad g(n) = O(n^2).$$

Let

$$h(n) = f(n) - g(n) = 2n^2 + n - n^2 = n^2 + n.$$

However, $\forall\, c_2 > 0,\ \exists\, N_2 = [c_2] + 1$ such that $\forall\, n > N_2$

$$h(n) - c_2|n| = n^2 + n - c_2n = (n - c_2 + 1)n > 0.$$

Therefore

$$h(n) = f(n) - g(n) \neq O(n).$$

■

31. Show that if $f(n) = \Theta(\log_2 n)$, then $f(n) = \Theta(\log_{10} n)$.

Proof. Because $f(n) = \Theta(\log_2 n)$, assume that $\exists\, c_1 > 0,\ c_2 > 0,\ N_1 \in \mathbb{N}^+$ such that $\forall\, n > N_1$

$$c_1|\log_2 n| \le |f(n)| \le c_2|\log_2 n|.$$

According to The Change-of-Base Formula

$$\log_2 n = \frac{\log_{10} n}{\log_{10} 2} \quad \Rightarrow \quad |\log_2 n| = \left|\frac{\log_{10} n}{\log_{10} 2}\right| = \frac{|\log_{10} n|}{|\log_{10} 2|} = \frac{|\log_{10} n|}{\log_{10} 2}.$$

Therefore

$$\frac{c_1}{\log_{10} 2}|\log_{10} n| \le |f(n)| \le \frac{c_2}{\log_{10} 2}|\log_{10} n| \quad \Rightarrow \quad f(n) = \Theta(\log_{10} n).$$

■

34. Let $G = (V, E)$ be any graph. Prove the following claim: If there is any walk between $v_i \in V$ and $v_j \in V$, then there must be a simple path of length no larger than $|V| - 1$ between these two vertices.

Proof. Assume that if there is any walk between $v_i \in V$ and $v_j \in V$, then every simple path between these two vertices has more than $|V| - 1$ length. Therefore, these simple paths have at least $|V|$ length. However, there are only $|V|$ vertices in the graph, at least one vertex has been repeated in the path, which contradicts with the statement that these paths are simple paths. Thus if there is any walk between $v_i \in V$ and $v_j \in V$, then there must be a simple path of length no larger than $|V| - 1$ between these two vertices.

■

35. Consider graphs in which there is at most one edge between any two vertices. Show that under this condition a graph with n vertices has at most n^2 edges.

Proof.

1. **Basis**

   If $|V| = 1$, there is at most one edge from the only one vertex to itself.

2. **Inductive Assumption**

   Assume that under this condition a graph with n vertices has at most n^2 edges, for $|V| = 1, 2, \cdots, n$.

3. **Inductive Step**

   For $|V| = n+1$, there are $(n+1)$ edges will be increased in E. Thus there are at most

   $$n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$$

   edges in this graph.

■

36. Show that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Proof.

1. **Basis**

   If $n = 0$, then

   $$2^0 = 1, \quad 2^{0+1} - 1 = 2 - 1 = 1 \quad \Rightarrow \quad 2^0 = 2^{0+1} - 1.$$

2. **Inductive Assumption**

   Assume that for $j = 0, 1, 2, \cdots, n$

   $$\sum_{i=0}^{j} 2^i = 2^{j+1} - 1.$$

3. **Inductive Step**

   For $j = n+1$,

   $$\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^{n} 2^i + 2^{n+1} = 2^{n+1} + 2^{n+1} - 1 = 2^{n+2} + 1.$$

■

37. Show that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

Proof.

1. **Basis**

   If $n = 1$, then

   $$\frac{1}{1^2} = 1, \quad 2 - \frac{1}{1} = 1 \quad \Rightarrow \quad \frac{1}{1^2} \leq 2 - \frac{1}{1}.$$

2. **Inductive Assumption**

   Assume that for $j = 1, 2, \cdots, n$

$$\sum_{i=1}^{j} \frac{1}{i^2} \leq 2 - \frac{1}{j}.$$

3. **Inductive Step**

   For $j = n+1$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \frac{1}{(n+1)^2} + \sum_{i=1}^{n} \frac{1}{i^2} \leq \frac{1}{(n+1)^2} + 2 - \frac{1}{n} = 2 - \frac{n(n+1)+1}{n(n+1)^2} < 2 - \frac{1}{n+1}.$$

■

38. Prove that for all $n \geq 4$ the inequality $2^n < n!$ holds.

Proof.

1. **Basis**

   If $n = 4$, then $2^4 = 16 < 4! = 24$.

2. **Inductive Assumption**

   Assume that for $i = 4, 5, 6, \cdots, n$

$$2^i < i!.$$

3. **Inductive Step**

   For $i = n+1$

$$2^{n+1} = 2 \cdot 2^n < 2 \cdot n! < (n+1) \cdot n! = (n+1)!.$$

■

39. The *Fibonacci sequence* is defined recursively by

$$f(n+2) = f(n+1) + f(n), \qquad n = 1, 2, \cdots,$$

with $f(1) = 1, f(2) = 1$. Show that

(a) $f(n) = O(2^n)$,

(b) $f(n) = \Omega(1.5^n)$.

Proof.

(a)

1. **Basis**

   If $n = 1, 2$, then $f(1) = 1 < 2^1 = 2$, $f(2) = 1 < 2^2 = 4$.

2. **Inductive Assumption**

   Assume that for $i = 1, 2, \cdots, n, n+1$

$$f(i) < 2^i, \qquad f(i+1) < 2^{i+1}.$$

3. **Inductive Step**

   For $i = n + 1$

$$f(n+2) = f(n+1) + f(n) < 2^{n+1} + 2^n < 2^{n+1} + 2^{n+1} = 2^{n+2}.$$

Therefore, $\exists\, c = 1 > 0$ such that for all $n \geq 1$

$$f(n) \leq c|2^n| = 2^n.$$

Thus,

$$f(n) = O(2^n).$$

(b)

1. **Basis**

   If $n = 11, 12$, then $f(11) = 89 > 1.5^{11}$, $f(12) = 144 > 1.5^{12}$.

2. **Inductive Assumption**

   Assume that for $i = 11, 12, \cdots, n, n+1$

$$f(i) > 1.5^i, \qquad f(i+1) > 1.5^{i+1}.$$

3. **Inductive Step**

   For $i = n + 1$

$$f(n+2) = f(n+1) + f(n) > 1.5^{n+1} + 1.5^n > 1.5^{n+1} + 0.5 \times 1.5^{n+1} = 1.5^{n+2}.$$

Therefore, $\exists\, c = 1 > 0$ such that for all $n \geq 11$

$$|f(n)| = f(n) \geq c|1.5^n| = 1.5^n.$$

Thus,

$$f(n) = \Omega(1.5^n).$$

$\blacksquare$

40. Show that $\sqrt{8}$ is not a rational number.

Proof. Suppose that $\sqrt{8}$ is a rational number. Let

$$\sqrt{8} = \frac{n}{m},$$

where n and m are integers without a common factor, so $8m^2 = n^2$. Therefore, n^2 is even, and n is also even. Let $n = 2k$ so that $8m^2 = 4k^2$, $2m^2 = k^2$. Therefore, k^2 is even, and k is also even. Let $k = 2t$ so that $2m^2 = 4t^2$, $m^2 = 2t^2$. Therefore, m^2 is even, and m is also even. This contradicts our assumption that n and m have no common factors. Thus, $\sqrt{8}$ is not a rational number.

$\blacksquare$

41. Show that $2 - \sqrt{2}$ is irrational.

Proof. Suppose that $2 - \sqrt{2}$ is a rational number. Let

$$2 - \sqrt{2} = \frac{n}{m},$$

where n and m are integers without a common factor, so

$$\sqrt{2} = 2 - \frac{n}{m} = \frac{2m - n}{n}.$$

Because $2m - n$ is also an integer, and $\sqrt{2}$ is also a rational number. This contradicts our conclusion that $\sqrt{2}$ is irrational. Therefore, $2 - \sqrt{2}$ is irrational.

$\blacksquare$

42. Show that $\sqrt{3}$ is irrational.

Proof. Suppose that $\sqrt{3}$ is a rational number. Let

$$\sqrt{3} = \frac{n}{m},$$

where n and m are integers without a common factor. Therefore,

$$3 = \frac{n^2}{m^2}, \qquad 3m^2 = n^2,$$

so n has a factor of 3. Let $n = 3k$, so that $3m^2 = 9k^2$, $m^2 = 3k^2$. Now m also has a factor of 3. This contradicts our assumption that n and m have no common factors. Thus, $\sqrt{3}$ is irrational.

$\blacksquare$

43. Prove of disprove the following statements.

(a) The sum of a rational and an irrational number must be irrational.

(b) The sum of two positive irrational numbers must be irrational.

(c) The product of a nonzero rational and an irrational number must be irrational.

Solution.

(a) This statement is true. The proof are shown as follows:

Suppose that a is a rational number, b is an irrational number and $a+b$ is a rational number. Let

$$a = \frac{p_1}{q_1}, \qquad a + b = \frac{p_2}{q_2}.$$

Then

$$b = \frac{p_2}{q_2} - a = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2}.$$

Therefore, b is also a rational number, which is a contradiction. Thus, the sum of a rational and an irrational number must be irrational.

(b) This statement is false. The proof are shown as follows:

Let

$$a = 2 + \sqrt{2}, \qquad b = 2 - \sqrt{2}.$$

It is obvious that a and b are two irrational numbers. Then

$$ab = (2 + \sqrt{2})(2 - \sqrt{2}) = 4 - 2 = 2.$$

Therefore, the sum of two positive irrational numbers may not be irrational.

(c) This statement is true. The proof are shown as follows:

Suppose that a is a nonzero rational number, b is an irrational number and ab is a rational number. Let

$$a = \frac{p_1}{q_1} \quad (a \neq 0), \qquad ab = \frac{p_2}{q_2}.$$

Then

$$b = \frac{ab}{a} = \frac{p_2}{q_2} \cdot \frac{q_1}{p_1} = \frac{p_2 q_1}{p_1 q_2}.$$

Therefore, b is also a rational number, which is a contradiction. Thus, the product of a nonzero rational and an irrational number must be irrational.

44. Show that every positive integer can be expressed as the product of prime numbers.

Proof. Suppose that the positive integer n is the smallest number which can not be expressed as the product of prime numbers. If n is a prime number, then this contradicts our assumption that n can not be expressed as the product of prime numbers; if n is not a prime number, then n can be expressed as a series of numbers

$$n = a_1 a_2 \cdots a_k$$

where $k, a_1, a_2, \cdots, a_k \in \mathbb{N}^+$. It is obvious that

$$a_1, a_2, \cdots, a_k < n.$$

If $a_1, a_2, \cdots, a_k$ are all prime numbers, then this contradicts our assumption that n can not be expressed as the product of prime numbers. Suppose that there is $\forall\, a \in \{a_1, a_2, \cdots, a_k\}$ is not a prime number. If a can be expressed as the product of prime numbers, then this contradicts our assumption that n can not be expressed as the product of prime numbers; if a can not be expressed as the product of prime numbers, then a is a smaller number than n which can not be expressed as the product of prime numbers, so this this contradicts our assumption that n is the smallest number which can not be expressed as the product of prime numbers. To sum up, every positive integer can be expressed as the product of prime numbers.

■

45. Prove that the set of all prime numbers is infinite.

Proof. Suppose that the set of all prime numbers is finite, and the largest prime number is a. Let b be the product of all the prime numbers

$$b = 2 \times 3 \times \cdots \times a.$$

Then let

$$c = b + 1.$$

With the Exercise 44, every positive integer can be expressed as the product of prime numbers. However, c can not be divided by every prime number. Therefore, c is also a prime number, which contradicts our assumption that a is the largest prime number. Thus, the set of all prime numbers is infinite.

■