

# Chapter 1      Section 2      Exercises

1. How many substrings  $aab$  are in  $ww^Rw$ , where  $w = aabbab$ .

**Solution.** Because

$$w^R = babbaa, \quad ww^Rw = \underline{aabb}abbabbaa\underline{aabb}ab,$$

there are two substrings  $aab$  in  $ww^Rw$ .

2. Use induction on  $n$  to show that  $|u^n| = n|u|$  for all strings  $u$  and all  $n$ .

**Proof.**

## 1. Basis

If  $n = 0$ , then

$$|u^1| = |u|, \quad 1 \cdot |u| = |u|.$$

Therefore,

$$|u^1| = 1 \cdot |u|.$$

## 2. Inductive Assumption

Assume that for  $i = 0, 1, 2, \dots, n$

$$|u^i| = i \cdot |u|.$$

## 3. Inductive Step

For  $i = n + 1$ ,

$$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n + 1)|u|.$$

Thus,  $|u^n| = n|u|$  for all strings  $u$  and all  $n$ .

■

3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$\begin{aligned} a^R &= a, \\ (wa)^R &= aw^R, \end{aligned}$$

for all  $a \in \Sigma, w \in \Sigma^*$ . Use this to prove that

$$(uv)^R = v^R u^R,$$

for all  $u, v \in \Sigma^+$ .

**Proof.**

### 1. Basis

For  $|v| = 1$ , suppose that  $v = a, a \in \Sigma$

$$(uv)^R = (ua)^R = au^R = v^R u^R.$$

### 2. Inductive Assumption

Suppose that for  $|v| = 1, 2, \dots, n$

$$(uv)^R = v^R u^R.$$

### 3. Inductive Step

For  $i = n + 1$ , suppose that  $v = wa, a \in \Sigma$

$$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all  $u, v \in \Sigma^+$ .

■

**4. Prove that  $(w^R)^R = w$  for all  $w \in \Sigma^*$ .**

**Proof.**

### 1. Basis

For  $|w| = 0$ , which means that  $w = \lambda$ ,

$$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

## 2. Inductive Assumption

Suppose that for  $|w| = 1, 2, \dots, n$

$$(w^R)^R = w.$$

## 3. Inductive Step

For  $i = n + 1$ , suppose that  $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus,  $(w^R)^R = w$  for all  $w \in \Sigma^*$ .

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5. Let  $L = \{ab, aa, baa\}$ . Which of the following strings are in  $L^*$ :  $abaabaaabaa$ ,  $aaaabaaaa$ ,  $baaaaabaaaab$ ,  $baaaaabaa$ ? Which strings are in  $L^4$ ?

**Solution.**

$$abaabaaabaa = ab|aa|baa|ab|aa \in L^5;$$

$$aaaabaaaa = aa|aa|baa|aa \in L^4;$$

$$baaaaabaaaab \notin L^*;$$

$$baaaaabaa = baa|aa|ab|aa \in L^4.$$

6. Let  $\Sigma = \{a, b\}$  and  $L = \{aa, bb\}$ . Use set notation to describe  $\overline{L}$ .

**Solution.**

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let  $L$  be any language on a nonempty alphabet. Show that  $L$  and  $\overline{L}$  cannot both be finite.

**Proof.** Suppose that both  $L$  and  $\overline{L}$  are finite. Therefore,  $L \cup \overline{L}$  is also finite. However,  $L \cup \overline{L} = \Sigma^*$  is an infinite set, which leads to a contradiction. Thus,  $L$  and  $\overline{L}$  cannot both be finite.

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8. Are there languages for which  $\overline{L^*} = (\overline{L})^*$ ?

**Solution.** No, because

$$\begin{aligned}\overline{L^*} &= \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \dots), \\ (\overline{L})^* &= (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \dots.\end{aligned}$$

Thus,  $\lambda \notin \overline{L^*}$  and  $\lambda \in (\overline{L})^*$ . There is no language for which  $\overline{L^*} = (\overline{L})^*$ .

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages  $L_1$  and  $L_2$ .

**Proof.**  $\forall u \in L_1$  and  $\forall v \in L_2$ , then

$$uv \in L_1 L_2 \Rightarrow (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \Rightarrow v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages  $L_1$  and  $L_2$ .

■

10. Show that  $(L^*)^* = L^*$  for all languages.

**Proof.** It is obvious that

$$(L^*)^n = \{w_1 w_2 \cdots w_n : w_1, w_2, \dots, w_n \in L^*\},$$

where  $n \in \mathbb{N}$ . Because

$$L^* = L^0 \cup L^1 \cup L^2 \cup \dots$$

Suppose that  $w_1 \in L^{m_1}, w_2 \in L^{m_2}, \dots, w_n \in L^{m_n}$  where  $0 \leq m_1, m_2, \dots, m_n \leq n$  and  $m_1, m_2, \dots, m_n \in \mathbb{N}$ . Therefore,

$$w_1 w_2 \cdots w_n \in L^{m_1 + m_2 + \dots + m_n}.$$

However,

$$L^{m_1+m_2+\dots+m_n} \in L^*.$$

Thus,  $(L^*)^* = L^*$  for all languages.

■

**11.** Prove or disprove the following claims.

(a)  $(L_1 \cup L_2)^R = L_1^R \cup L_2^R$  for all languages  $L_1$  and  $L_2$ .

(b)  $(L^R)^* = (L^*)^R$  for all languages  $L$ .

**Proof.**

(a)

$$\begin{aligned} L_1 \cup L_2 &= \{w : w \in L_1 \text{ or } w \in L_2\}, \\ (L_1 \cup L_2)^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}, \\ L_1^R \cup L_2^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}. \end{aligned}$$

Thus,

$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R.$$

(b) It is obvious that

$$(L^R)^0 = \lambda = (\lambda)^R = (L^0)^R.$$

$$\forall n \in \mathbb{N}^+$$

$$\begin{aligned} (L^n)^R &= \{(w_1 w_2 \dots w_n)^R : w_1, w_2, \dots, w_n \in L\} \\ &= \{w_n^R \dots w_2^R w_1^R : w_1, w_2, \dots, w_n \in L\} \\ &= \{w_n^R \dots w_2^R w_1^R : w_1^R, w_2^R, \dots, w_n^R \in L^R\} \\ &= (L^R)^n. \end{aligned}$$

Thus,

$$\begin{aligned} (L^R)^* &= (L^R)^0 \cup (L^R)^1 \cup (L^R)^2 \cup \dots \\ &= (L^0)^R \cup (L^1)^R \cup (L^2)^R \cup \dots \\ &= (L^*)^R. \end{aligned}$$



12. Find a grammar for the language  $L = \{a^n, \text{ where } n \text{ is even}\}$ .

**Solution.** The grammar is  $G = (\{S\}, \{a\}, S, P)$  with  $P$  given by

$$S \rightarrow aaS|\lambda.$$

13. Find a grammar for the language  $L = \{a^n, \text{ where } n \text{ is even and } n > 3\}$ .

**Solution.** The grammar is  $G = (\{S, A\}, \{a\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaaA,$$

$$A \rightarrow aaA|\lambda.$$

14. Find grammars for  $\Sigma = \{a, b\}$  that generate the sets of

- (a) all strings with exactly two  $a$ 's.
- (b) all strings with at least two  $a$ 's.
- (c) all strings with no more than three  $a$ 's.
- (d) all strings with at least three  $a$ 's.
- (e) all strings that start with  $a$  and end with  $b$ .
- (f) all strings with an even number of  $b$ 's.

In each case, give convincing arguments that the grammar you give does indeed generate the indicated language.

**Solution.**

- (a) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow AaAaA,$$

$$A \rightarrow bA|\lambda.$$

(b) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$\begin{aligned} S &\rightarrow AaAaA, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(c) The grammar is  $G = (\{S, A, B, C\}, \{a, b\}, S, P)$  with  $P$  given by

$$\begin{aligned} S &\rightarrow aA|bS|\lambda, \\ A &\rightarrow aB|bA|\lambda, \\ B &\rightarrow aC|bB|\lambda, \\ C &\rightarrow bC|\lambda. \end{aligned}$$

(d) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$\begin{aligned} S &\rightarrow AaAaAaA, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(e) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$\begin{aligned} S &\rightarrow aAb, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(f) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$\begin{aligned} S &\rightarrow SbSbS|A|\lambda, \\ A &\rightarrow aA|\lambda. \end{aligned}$$

**15.** Give a simple description of the language generated by the grammar with productions

$$\begin{aligned} S &\rightarrow aaA, \\ A &\rightarrow bS, \\ S &\rightarrow \lambda. \end{aligned}$$

**Solution.** The language generated by this grammar is

$$L = \{(aab)^n : n \geq 0\}.$$

16. What language does the grammar with these productions generate?

$$S \rightarrow Aa,$$

$$A \rightarrow B,$$

$$B \rightarrow Aa.$$

**Solution.** The language generated by this grammar is

$$L = \{Aa^n : n \geq 1\}.$$

17. Let  $\Sigma = \{a, b\}$ . For each of the following languages, find a grammar that generates it.

(a)  $L_1 = \{a^n b^m : n \geq 0, m < n\}.$

(b)  $L_2 = \{a^{3n} b^{2n} : n \geq 2\}.$

(c)  $L_3 = \{a^{n+3} b^n : n \geq 2\}.$

(d)  $L_4 = \{a^n b^{n-2} : n \geq 3\}.$

(e)  $L_1 L_2.$

(f)  $L_1 \cup L_2.$

(g)  $L_1^3.$

(h)  $L_1^*.$

(i)  $L_1 - \overline{L_4}$

**Solution.**

(a) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(b) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaaaaAbbbb,$$

$$A \rightarrow aaaAbb|\lambda.$$



(c) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaaaAbb,$$

$$A \rightarrow aAb|\lambda.$$

(d) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaAb,$$

$$A \rightarrow aAb|\lambda.$$

(e) The grammar is  $G = (\{S, A, B\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aAaaaaaBbbbb|aaaaaBbbbb,$$

$$A \rightarrow aA|aAb|\lambda,$$

$$B \rightarrow aaaBbb|\lambda.$$

(f) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(g) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aAaAaA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(h) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aAS|\lambda,$$

$$A \rightarrow aA|aAb|\lambda.$$

(i) The grammar is  $G = (\{S, A\}, \{a, b\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaAb,$$

$$A \rightarrow aAb|\lambda.$$

**18.** Find grammars for the following languages on  $\Sigma = \{a\}$ .

(a)  $L = \{w : |w| \bmod 3 > 0\}.$

(b)  $L = \{w : |w| \bmod 3 = 2\}.$

(c)  $w = \{|w| \bmod 5 = 0\}.$

**Solution.**

(a) The grammar is  $G = (\{S, A\}, \{a\}, S, P)$  with  $P$  given by

$$S \rightarrow aA|aaA,$$

$$A \rightarrow aaaA|\lambda.$$

(b) The grammar is  $G = (\{S, A\}, \{a\}, S, P)$  with  $P$  given by

$$S \rightarrow aaA,$$

$$A \rightarrow aaaA|\lambda.$$

(c) The grammar is  $G = (\{S\}, \{a\}, S, P)$  with  $P$  given by

$$S \rightarrow aaaaaS|\lambda.$$