

Chapter 1 Section 1 Exercises

1. With $S_1 = \{ 2, 3, 5, 7 \}$, $S_2 = \{ 2, 4, 5, 8, 9 \}$, and $U = \{ 1 : 10 \}$, compute $\overline{S_1} \cup S_2$.

Solution.

$$\overline{S_1} = \{ 1, 4, 6, 8, 9, 10 \} \quad \Rightarrow \quad \overline{S_1} \cup S_2 = \{ 1, 2, 4, 5, 6, 8, 9, 10 \}.$$

2. With $S_1 = \{ 2, 3, 5, 7 \}$, $S_2 = \{ 2, 4, 5, 8, 9 \}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned} S_1 \times S_2 &= \{ (2, 2), (2, 4), (2, 5), (2, 8), (2, 9), \\ &\quad (3, 2), (3, 4), (3, 5), (3, 8), (3, 9), \\ &\quad (5, 2), (5, 4), (5, 5), (5, 8), (5, 9), \\ &\quad (7, 2), (7, 4), (7, 5), (7, 8), (7, 9) \}. \\ S_2 \times S_1 &= \{ (2, 2), (2, 3), (2, 5), (2, 7), \\ &\quad (4, 2), (4, 3), (4, 5), (4, 7), \\ &\quad (5, 2), (5, 3), (5, 5), (5, 7), \\ &\quad (8, 2), (8, 3), (8, 5), (8, 7), \\ &\quad (9, 2), (9, 3), (9, 5), (9, 7) \}. \end{aligned}$$

3. For $S = \{ 2, 5, 6, 8 \}$ and $T = \{ 2, 4, 6, 8 \}$, compute $|S \cap T| + |S \cup T|$.

Solution.

$$S \cap T = \{ 2, 6, 8 \}, \quad S \cup T = \{ 2, 4, 5, 6, 8 \} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \emptyset.$$

Therefore, S and T are disjoint.

5. Show that for all sets S and T , $S - T = S \cap \overline{T}$.

Proof.

$$\begin{aligned} S - T &= \{ x : x \in S \text{ and } x \notin T \} \\ \iff S - T &= \{ x : x \in S \text{ and } x \in \overline{T} \} \\ \iff S - T &= S \cap \overline{T}. \end{aligned}$$

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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

Proof.

$$S_1 \cup S_2 = \{ x : x \in S_1 \text{ or } x \in S_2 \} \iff \overline{S_1 \cup S_2} = \{ x : x \notin S_1 \text{ and } x \notin S_2 \}.$$

$$\overline{S_1} = \{ x : x \notin S_1 \}, \quad \overline{S_2} = \{ x : x \notin S_2 \} \iff \overline{S_1} \cap \overline{S_2} = \{ x : x \notin S_1 \text{ and } x \notin S_2 \}.$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$S_1 \cap S_2 = \{ x : x \in S_1 \text{ and } x \in S_2 \} \iff \overline{S_1 \cap S_2} = \{ x : x \notin S_1 \text{ or } x \notin S_2 \}.$$

$$\overline{S_1} = \{ x : x \notin S_1 \}, \quad \overline{S_2} = \{ x : x \notin S_2 \} \iff \overline{S_1} \cup \overline{S_2} = \{ x : x \notin S_1 \text{ or } x \notin S_2 \}.$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

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7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.

Proof.

$$S_1 \subseteq S_2$$

$$\iff x \in S_1 \Rightarrow x \in S_2$$

$$\iff x \notin S_2 \Rightarrow x \notin S_1$$

$$\iff x \in \overline{S_2} \Rightarrow x \in \overline{S_1}$$

$$\iff \overline{S_2} \subseteq \overline{S_1}.$$

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8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

Proof.

$$1. S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$\left. \begin{array}{l} S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \Rightarrow S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{array} \right\} \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$2. S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2.$$

Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,

$$\bullet \exists x \in S_1 \text{ and } x \notin S_2 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$$

$$\bullet \exists x \in S_2 \text{ and } x \notin S_1 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$$

The result contradicts with the permise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2$.

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

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9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.

Proof.

1. Basis

If $|S| = 1$, assume that $S = \{ a \}$. Then

$$2^S = \{ \emptyset, \{ a \} \}.$$

Therefore, $|2^S| = 2^{|S|} = 2$.

2. Inductive Assumption

Assume that $|2^S| = 2^{|S|}$, for $|S| = 1, 2, \dots, n$.

3. Inductive Step

For $|S| = n+1$, assume that $S = \{ a_1, a_2, \dots, a_n, a_{n+1} \}$. Let $T = \{ a_1, a_2, \dots, a_n \}$, then

$$2^T = \{ T_1, T_2, \dots, T_{2^n} \}.$$

For $\forall i = 1, 2, \dots, 2^n$ where $i \in \mathbb{N}^*$

$$\left. \begin{array}{l} T_i \subseteq T \\ T \subseteq S \end{array} \right\} \iff T_i \subseteq S.$$

However,

$$S - T = \{ a_{n+1} \} \Rightarrow a_{n+1} \notin T \Rightarrow a_{n+1} \notin T_i.$$

In addition

$$a_{n+1} \in S_i \iff \left. \begin{array}{l} T_i \subseteq S \\ \{ a_{n+1} \} \subseteq S \end{array} \right\} \iff T_i \cup \{ a_{n+1} \} \subseteq S.$$

Let

$$T'_i = T_i \cup \{ a_{n+1} \}, \quad U = \{ T'_1, T'_2, \dots, T'_{2^n} \}.$$

Now, for $\forall S_i \subseteq S$

- If $a_{n+1} \notin S_i$, then $S_i \subseteq T$, so $S_i \in 2^T$.
- If $a_{n+1} \in S_i$, then $S_i - \{ a_{n+1} \} \subseteq T$, so $S_i - \{ a_{n+1} \} \in 2^T$. Assume that

$$S_i - \{ a_{n+1} \} = T_j \Rightarrow S_i = T_j \cup \{ a_{n+1} \} \Rightarrow S_i \in U.$$

Moreover, 2^T and U are disjoint. Therefore,

$$2^S = 2^T \cup U, \quad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$. ■

10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

Proof. Assume that

$$S_1 = \{ a_1, a_2, \dots, a_n \}, \quad S_2 = \{ b_1, b_2, \dots, b_m \}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{ a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1, c_2, \dots, c_k \in S_1 \text{ and } c_1, c_2, \dots, c_k \in S_2.$$

where $k \leq n$, $k \leq m$, $k \in \mathbb{N}^*$. Assume that

$$b_{i_1} = c_1, b_{i_2} = c_2, \dots, b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{ a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{i_1-1}, b_{i_1+1}, \dots, b_{i_k-1}, b_{i_k+1}, \dots, b_m \}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then ■

$$|S_1 \cup S_2| \leq n + m.$$