

# 1 Mathematical Preliminaries and Notation

## Sets

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A **set** is a collection of elements, without any structure other than membership.

The usual set operations are **union** ( $\cup$ ), **intersection** ( $\cap$ ), **difference** ( $-$ ) and **complementation** defined as

$$\begin{aligned} S_1 \cup S_2 &= \{ x : x \in S_1 \text{ or } x \in S_2 \}, \\ S_1 \cap S_2 &= \{ x : x \in S_1 \text{ and } x \in S_2 \}, \\ S_1 - S_2 &= \{ x : x \in S_1 \text{ and } x \notin S_2 \}, \\ \overline{S} &= \{ x : x \in U \text{ and } x \notin S \}. \end{aligned}$$

### DeMorgan's laws

$$\begin{aligned} \overline{S_1 \cup S_2} &= \overline{S_1} \cap \overline{S_2}, \\ \overline{S_1 \cap S_2} &= \overline{S_1} \cup \overline{S_2}. \end{aligned}$$

A set  $S_1$  is said to be a **subset** of  $S$  if every element of  $S_1$  is also an element of  $S$ . We write this as

$$S_1 \subseteq S.$$

If  $S_1 \subseteq S$ , but  $S$  contains an element not in  $S_1$ , we say that  $S_1$  is a **proper subset** of  $S$ ; we write this as

$$S_1 \subset S.$$

If  $S_1$  and  $S_2$  have no common element, then the sets are said to be **disjoint**. We write this as

$$S_1 \cap S_2 = \emptyset.$$

A set is said to be finite if it contains a **finite** number of elements; otherwise it is **infinite**.

The set of all subsets of a set  $S$  is called the **powerset** of  $S$  and is denoted by  $2^S$ . If  $S$  is finite, then

$$|2^S| = 2^{|S|}.$$

The sets whose elements are ordered sequences of elements from other sets are said to be the **Cartesian product** of other sets. For the Cartesian product of  $n$  sets, which itself is a set of ordered pairs, we write

$$S = S_1 \times S_2 \times \cdots \times S_n = \{ (x_1, x_2, \cdots, x_n) : x_i \in S_i \}.$$

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Suppose that  $S_1, S_2, \cdots, S_n$  are subsets of a given set  $S$  and that the following holds:

1. The subsets  $S_1, S_2, \cdots, S_n$  are mutually disjoint;
2.  $S_1 \cup S_2 \cup \cdots \cup S_n = S$ ;
3. none of the  $S_i$  is empty.

Then  $S_1, S_2, \cdots, S_n$  is called a **partition** of  $S$ .

# Functions and Relations

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A **function** is a rule that assigns to elements of one set a unique element of another set. If  $f$  denotes a function, then the first set is called the **domain** of  $f$ , and the second set is its **range**. We write

$$f : S_1 \rightarrow S_2$$

to indicate that the domain of  $f$  is a subset of  $S_1$  and that the range of  $f$  is a subset of  $S_2$ . If the domain of  $f$  is all of  $S_1$ , we say that  $f$  is a **total function** on  $S_1$ ; otherwise  $f$  is said to be a **partial function**.

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Let  $f(n)$  and  $g(n)$  be functions whose domain is a subset of the positive integers. We say that

1.  $f$  has **order at most**  $g$  if there exists a positive constant  $c$  such that for all sufficiently large  $n$

$$f(n) \leq c|g(n)| \quad \xrightarrow{\text{expressed as}} \quad f(n) = O(g(n)).$$

2.  $f$  has **order at least**  $g$  if there exists a positive constant  $c$  such that for all sufficiently large  $n$

$$f(n) \geq c|g(n)| \quad \xrightarrow{\text{expressed as}} \quad f(n) = \Omega(g(n)).$$

3.  $f$  and  $g$  have the **same order of magnitude** if there exist constant  $c_1$  and  $c_2$  such that for all sufficiently large  $n$

$$c_1|g(n)| \leq |f(n)| \leq c_2|g(n)| \quad \xrightarrow{\text{expressed as}} \quad f(n) = \Theta(g(n)).$$

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Some functions can be represented by a set of pairs

$$\{ (x_1, y_1), (x_2, y_2), \dots \}.$$

where  $x_i$  is an element in the domain of the function, and  $y_i$  is the corresponding value in its range. For such a set to define a function, each  $x_i$  can occur at most once as the first element of a pair. If this is not satisfied, the set is called a **relation**.

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**Equivalence** is a generalization of the concept of equality (identity). A relation denoted by  $\equiv$  is considered an equivalence if it satisfies three rules:

1. The reflexivity rule

$$x \equiv x \text{ for all } x;$$

2. The symmetry rule

$$\text{if } x \equiv y, \text{ then } y \equiv x;$$

3. The transitivity rule

$$\text{if } x \equiv y \text{ and } y \equiv z, \text{ then } x \equiv z.$$

If  $S$  is a set on which we have a defined equivalence relation, then we can use this equivalence to partition the set into **equivalence classes**.

# Graphs and Trees

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A graph is a construct consisting of two finite sets, the set  $V = \{ v_1, v_2, \dots, v_n \}$  of **vertices** and the set  $E = \{ e_1, e_2, \dots, e_m \}$  of **edges**. Each edge is a pair of vertices from  $V$ , for instance

$$e_i = (v_j, v_k)$$

is an edge from  $v_j$  to  $v_k$ . We say that the edge  $e_i$  is an outgoing edge for  $v_j$  and an incoming edge for  $v_k$ .

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1. A sequence of edges  $(v_i, v_j), (v_j, v_k), \dots, (v_m, v_n)$  is said to be a **walk** from  $v_i$  to  $v_n$ ;
  2. The length of a walk is the total number of edges traversed in going from the initial vertex to the final one;
  3. A walk in which no edge is repeated is said to be a **path**;
  4. A path is **simple** if no vertex is repeated;
  5. A walk from  $v_i$  to itself with no repeated edges is called a **cycle** with **base**  $v_i$ ;
  6. An edge from a vertex to itself is called a **loop**.
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A tree is a directed graph that has no cycles and that has one distinct vertex, called the **root**, such that there is exactly one path from the root to every other vertex.

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1. The vertices which have no outgoing edges are called the **leaves** of the tree;
2. If there is an edge from  $v_i$  to  $v_j$ , then  $v_i$  is said to be the **parent** of  $v_j$ , and  $v_j$  the **child** of  $v_i$ ;
3. The **level** associated with each vertex is the number of edges in the path from the root to the vertex;
4. The **height** of the tree is the largest level number of any vertex;
5. In **ordered trees**, an ordering with the nodes is associated with the nodes at each level.

# Graphs and Trees

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## Proof by induction

Induction is a technique by which the truth of a number of statements can be inferred from the truth of a few specific instances. Suppose we have a sequence of statements  $P_1, P_2, \dots$  we want to prove to be true. Furthermore, suppose also that the following holds:

1. For some  $k \geq 1$ , we know that  $P_1, P_2, \dots, P_k$  are true.
2. The problem is such that for any  $n \geq k$ , the truths of  $P_1, P_2, \dots, P_n$  imply the truth of  $P_{n+1}$ .

We can then use induction to show that every statement in this sequence is true.

1. The starting statements  $P_1, P_2, \dots, P_k$  are called the **basis** of the induction.
  2. The step connecting  $P_n$  with  $P_{n+1}$  is called the **inductive step**.
  3. The inductive step is generally made easier by the **inductive assumption** that  $P_1, P_2, \dots, P_n$  are true, then argue that the truth of these statements guarantees the truth of  $P_{n+1}$ .
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## Proof by contradiction

Suppose we want to prove that some statement  $P$  is true. We then assume, for the moment, that  $P$  is false and see where that assumption leads us. If we arrive at a conclusion that we know is incorrect, we can lay the blame on the starting assumption and conclude that  $P$  must be true.