Chapter 1 Section 2 Exercises

1. How many substrings aab are in ww^Rw, where $w = aabbab$.

Solution. Because

$$w^R = babbaa, \qquad ww^Rw = \underline{aab}babbabbaa\underline{aab}bab,$$

there are two substrings aab in ww^Rw.

2. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n.

Proof.

1. **Basis**

   If $n = 0$, then

   $$|u^1| = |u|, \qquad 1 \cdot |u| = |u|.$$

   Therefore,

   $$|u^1| = 1 \cdot |u|.$$

2. **Inductive Assumption**

   Assume that for $i = 0, 1, 2, \cdots, n$

   $$|u^i| = i \cdot |u|.$$

3. **Inductive Step**

   For $i = n + 1$,

   $$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n+1)|u|.$$

Thus, $|u^n| = n|u|$ for all strings u and all n.

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3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$a^R = a,$$

$$(wa)^R = aw^R,$$

for all $a \in \Sigma, w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

Proof.

1. **Basis**

   For $|v| = 1$, suppose that $v = a, a \in \Sigma$

   $$(uv)^R = (ua)^R = au^R = v^R u^R.$$

2. **Inductive Assumption**

   Suppose that for $|v| = 1, 2, \cdots, n$

   $$(uv)^R = v^R u^R.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $v = wa, a \in \Sigma$

   $$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

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4. Prove that $(w^R)^R = w$ for all $w \in \Sigma^*$.

Proof.

1. **Basis**

   For $|w| = 0$, which means that $w = \lambda$,

   $$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

2. **Inductive Assumption**

   Suppose that for $|w| = 1, 2, \cdots, n$

$$(w^R)^R = w.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus, $(w^R)^R = w$ for all $w \in \Sigma^*$.

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5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^*: $abaabaaabaa$, $aaaabaaaa$, $baaaaabaaaab$, $baaaaabaa$? Which strings are in L^4?

Solution.

$$\begin{aligned} abaabaaabaa &= ab|aa|baa|ab|aa \in L^5; \\ aaaabaaaa &= aa|aa|baa|aa \in L^4; \\ baaaaabaaaab &\notin L^*; \\ baaaaabaa &= baa|aa|ab|aa \in L^4. \end{aligned}$$

6. Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe $\overline{L}$.

Solution.

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let L be any language on a nonempty alphabet. Show that L and $\overline{L}$ cannot both be finite.

Proof. Suppose that both L and $\overline{L}$ are finite. Therefore, $L \cup \overline{L}$ is also finite. However, $L \cup \overline{L} = \Sigma^*$ is an infinite set, which leads to a contradiction. Thus, L and $\overline{L}$ cannot both be finite.

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8. Are there languages for which $\overline{L^*} = (\overline{L})^*$?

Solution. No, because

$$\overline{L^*} = \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \cdots),$$
$$(\overline{L})^* = (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \cdots .$$

Thus, $\lambda \notin \overline{L^*}$ and $\lambda \in (\overline{L})^*$. There is no language for which $\overline{L^*} = (\overline{L})^*$.

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

Proof. $\forall\, u \in L_1$ and $\forall\, v \in L_2$, then

$$uv \in L_1 L_2 \quad \Rightarrow \quad (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \quad \Rightarrow \quad v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

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10. Show that $(L^*)^* = L^*$ for all languages.

Proof. It is obvious that

$$(L^*)^n = \{w_1 w_2 \cdots w_n : w_1, w_2, \cdots, w_n \in L^*\},$$

where $n \in \mathbb{N}$. Because

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots$$

Suppose that $w_1 \in L^{m_1}, w_2 \in L^{m_2}, \cdots, w_2 \in L^{m_n}$ where $0 \leq m_1, m_2, \cdots, m_n \leq n$ and $m_1, m_2, \cdots, m_n \in \mathbb{N}$. Therefore,

$$w_1 w_2 \cdots w_n \in L^{m_1 + m_2 + \cdots + m_n}.$$

However,

$$L^{m_1+m_2+\cdots+m_n} \in L^*.$$

Thus, $(L^*)^* = L^*$ for all languages.

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11. Prove or disprove the following claims.

(a) $(L_1 \cup L_2)^R = L_1^R \cup L_2^R$ for all languages L_1 and L_2.

(b) $(L^R)^* = (L^*)^R$ for all languages L.

Proof.

(a)

$$\begin{aligned} L_1 \cup L_2 &= \{w : w \in L_1 \text{ or } w \in L_2\}, \\ (L_1 \cup L_2)^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}, \\ L_1^R \cup L_2^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}. \end{aligned}$$

Thus,

$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R.$$

(b) It is obvious that

$$(L^R)^0 = \lambda = (\lambda)^R = (L^0)^R.$$

$\forall\, n \in \mathbb{N}^+$

$$\begin{aligned} (L^n)^R &= \{(w_1 w_2 \cdots w_n)^R : w_1, w_2, \cdots, w_n \in L\} \\ &= \{w_n^R \cdots w_2^R w_1^R : w_1, w_2, \cdots, w_n \in L\} \\ &= \{w_n^R \cdots w_2^R w_1^R : w_1^R, w_2^R, \cdots, w_n^R \in L^R\} \\ &= (L^R)^n. \end{aligned}$$

Thus,

$$\begin{aligned} (L^R)^* &= (L^R)^0 \cup (L^R)^1 \cup (L^R)^2 \cup \cdots \\ &= (L^0)^R \cup (L^1)^R \cup (L^2)^R \cup \cdots \\ &= (L^*)^R. \end{aligned}$$

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12. Find a grammar for the language $L = \{a^n, \text{where } n \text{ is even}\}$.

Solution. The grammar is $G = (\{S\}, \{a\}, S, P)$ with P given by

$$S \to aaS|\lambda.$$

13. Find a grammar for the language $L = \{a^n, \text{where } n \text{ is even and } n > 3\}$.

Solution. The grammar is $G = (\{S, A\}, \{a\}, S, P)$ with P given by

$$S \to aaaaA,$$

$$A \to aaA|\lambda.$$

14. Find grammars for $\Sigma = \{a, b\}$ that generate the sets of

(a) all strings with exactly two a's.

(b) all strings at least two a's.

Solution.

(a) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to AaAaA,$$

$$A \to bA|\lambda.$$

(b) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to AaAaA,$$

$$A \to aA|bA|\lambda.$$