Chapter 1 Section 1 Exercises

1. With $S_1 = \{2,3,5,7\},\ S_2 = \{2,4,5,8,9\}$, and $U = \{1:10\}$, compute $\overline{S}_1 \cup S_2$.

Solution.

$$\overline{S}_1 = \{1,4,6,8,9,10\} \quad \Rightarrow \quad \overline{S}_1 \cup S_2 = \{1,2,4,5,6,8,9,10\}.$$

2. With $S_1 = \{2,3,5,7\},\ S_2 = \{2,4,5,8,9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned}
S_1 \times S_2 = \{&(2,2),(2,4),(2,5),(2,8),(2,9),\\
&(3,2),(3,4),(3,5),(3,8),(3,9),\\
&(5,2),(5,4),(5,5),(5,8),(5,9),\\
&(7,2),(7,4),(7,5),(7,8),(7,9)\}.\\
S_2 \times S_1 = \{&(2,2),(2,3),(2,5),(2,7),\\
&(4,2),(4,3),(4,5),(4,7),\\
&(5,2),(5,3),(5,5),(5,7),\\
&(8,2),(8,3),(8,5),(8,7),\\
&(9,2),(9,3),(9,5),(9,7)\}.
\end{aligned}$$

3. For $S = \{2,5,6,8\}$ and $T = \{2,4,6,8\}$, compute $|S \cap T| + |S \cup T|$.

Solution.

$$S \cap T = \{2,6,8\}, \quad S \cup T = \{2,4,5,6,8\} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \varnothing.$$

Therefore, S and T are disjoint.

5. Show that for all sets S and T, $S - T = S \cap \overline{T}$.

Proof.

$$\begin{aligned}
& S - T = \{x : x \in S \text{ and } x \notin T\}\\
\Longleftrightarrow \quad & S - T = \{x : x \in S \text{ and } x \in \overline{T}\}\\
\Longleftrightarrow \quad & S - T = S \cap \overline{T}.
\end{aligned}$$

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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$
$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

Proof.

$$\begin{aligned} S_1 \cup S_2 = \{x : x \in S_1 \text{ or } x \in S_2\} \quad &\Rightarrow \quad \overline{S_1 \cup S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}. \\ \overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad &\Rightarrow \quad \overline{S_1} \cap \overline{S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}. \end{aligned}$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$\begin{aligned} S_1 \cap S_2 = \{x : x \in S_1 \text{ and } x \in S_2\} \quad &\Rightarrow \quad \overline{S_1 \cap S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}. \\ \overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad &\Rightarrow \quad \overline{S_1} \cup \overline{S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}. \end{aligned}$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

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7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.
Proof.

$$\begin{aligned} & S_1 \subseteq S_2 \\ \Rightarrow \quad & (\in S_1 \quad \Rightarrow \quad x \in S_2) \\ \Rightarrow \quad & (x \notin S_2 \quad \Rightarrow \quad x \notin S_1) \\ \Rightarrow \quad & (x \in \overline{S_2} \quad \Rightarrow \quad x \notin \overline{S_1}) \\ \Rightarrow \quad & \overline{S_2} \subseteq \overline{S_1}. \end{aligned}$$

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8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.
Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$

$$\left.\begin{aligned} S_1 = S_2 \quad &\Rightarrow \quad S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \quad &\Rightarrow \quad S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{aligned}\right\} \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$$

2. $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2.$

   Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,

- $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.
- $\exists\, x \in S_2$ and $x \notin S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.

The result contradicts with the permise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2$.

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

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9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.
Proof.

1. **Basis**

   If $|S| = 0$, $S = \varnothing$. Then
   $$2^S = \{\varnothing\}.$$
   Therefore, $|2^S| = 2^{|S|} = 1$.

   If $|S| = 1$, assume that $S = \{a\}$. Then
   $$2^S = \{\varnothing, \{a\}\}.$$
   Therefore, $|2^S| = 2^{|S|} = 2$.

2. **Inductive Assumption**

   Assume that $|2^S| = 2^{|S|}$, for $|S| = 1, 2, \cdots, n$.

3. **Inductive Step**

   For $|S| = n + 1$, assume that $S = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$. Let $T = \{a_1, a_2, \cdots, a_n\}$, then
   $$2^T = \{T_1, T_2, \cdots, T_{2^n}\}.$$
   For $\forall\, i = 1, 2, \cdots, 2^n$ where $i \in \mathbb{N}^*$
   $$\left.\begin{array}{c} T_i \subseteq T \\ T \subseteq S \end{array}\right\} \Rightarrow \quad T_i \subseteq S.$$
   However,
   $$S - T = \{a_{n+1}\} \quad \Rightarrow \quad a_{n+1} \notin T \quad \Rightarrow \quad a_{n+1} \notin T_i.$$
   In addition
   $$\left.\begin{array}{c} T_i \subseteq S \\ a_{n+1} \in S_i \quad \Rightarrow \quad \{a_{n+1}\} \subseteq S \end{array}\right\} \Rightarrow \quad T_i \cup \{a_{n+1}\} \subseteq S.$$
   Let
   $$T_i^{'} = T_i \cup \{a_{n+1}\}, \qquad U = \{T_1^{'}, T_2^{'}, \cdots, T_{2^n}^{'}\}.$$
   Now, for $\forall\, S_i \subseteq S$

   - If $a_{n+1} \notin S_i$, then $S_i \subseteq T$, so $S_i \in 2^T$.

- If $a_{n+1} \in S_i$, then $S_i - \{a_{n+1}\} \subseteq T$, so $S_i - \{a_{n+1}\} \in 2^T$. Assume that

$$S_i - \{a_{n+1}\} = T_j \quad \Rightarrow \quad S_i = T_j \cup \{a_{n+1}\} \quad \Rightarrow \quad S_i \in U.$$

Moreover, 2^T and U are disjoint. Therefore,

$$2^S = 2^T \cup U, \qquad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$.

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10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

Proof. Assume that

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1, c_2, \cdots, c_k \in S_1 \text{ and } c_1, c_2, \cdots, c_k \in S_2.$$

where $k \leqslant n,\ k \leqslant m,\ k \in \mathbb{N}^*$. Assume that

$$b_{i_1} = c_1,\ b_{i_2} = c_2,\ \cdots,\ b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_{i_1-1}, b_{i_1+1} \cdots, b_{i_k-1}, b_{i_k+1} \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

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11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.
Proof. Assume that $S_1 = \varnothing$ or $S_2 = \varnothing$, then

$$S_1 \times S_2 = \varnothing \quad \Rightarrow \quad |S_1 \times S_2| = 0,\ |S_1||S_2| = 0 \times 0 = 0 \quad \Rightarrow \quad |S_1 \times S_2| = |S_1||S_2|.$$

Assume that $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$,

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

where $n, m \in \mathbb{N}^*$.
Therefore,

$$\begin{aligned} S_1 \times S_2 = \{&(a_1, b_1),\ (a_2, b_1),\ \cdots, (a_n, b_1), \\ &(a_1, b_2),\ (a_2, b_2),\ \cdots, (a_n, b_2), \\ &\qquad\qquad\vdots \\ &(a_1, b_m), (a_2, b_m), \cdots, (a_n, b_m)\}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

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12. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.
Proof.

1. **Reflexivity**
$$|S_1| = |S_1| \text{ for all } S_1. \quad \Rightarrow \quad S_1 \equiv S_1 \text{ for all } S_1.$$
2. **Symmetry**
$$\text{if } |S_1| = |S_2|,\ \text{then } |S_2| = |S_1|. \quad \Rightarrow \quad \text{if } S_1 \equiv S_2,\ \text{then } S_2 \equiv S_1.$$
3. **Transitivity**
$$\begin{gathered} \text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|,\ \text{then } |S_1| = |S_3|. \\ \Downarrow \\ \text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3,\ \text{then } S_1 \equiv S_3. \end{gathered}$$

Therefore, this is an equivalence relation.

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13. Occassionally, we need to use the union and intersection symbols in a manner analogous to the summation sign $\sum$. We define

$$\bigcup_{p \in \{i,j,k,\cdots\}} S_p = S_i \cup S_j \cup S_k \cdots$$

with an analogous notation for the intersection of several sets.

With this notation, the gereral DeMorgan's laws are written as

$$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$

Prove these identities when P is a finite set.
Proof.

1. **Basis**
   For $|P| = 2$, according to DeMorgan's laws
   $$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}, \qquad \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$
2. **Inductive Assumption**
   For $|P| = 2, 3, \cdots, n$ where $n \in \mathbb{N}^*$
   $$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}, \qquad \overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$
3. **Inductive Step**
   For $|P| = n + 1$ where $n \in \mathbb{N}^*$, $\forall\, i \in P,\ |P - \{i\}| = n$,
   $$\overline{\bigcup_{p\in P} S_p} = \overline{(\bigcup_{p\in P-\{i\}} S_p) \cup S_i} = \overline{(\bigcup_{p\in P-\{i\}} S_p)} \cap \overline{S_i} = (\bigcap_{p\in P-\{i\}} \overline{S_p}) \cap \overline{S_i} = \bigcap_{p\in P} \overline{S_p},$$
   $$\overline{\bigcap_{p\in P} S_p} = \overline{(\bigcap_{p\in P-\{i\}} S_p) \cap S_i} = \overline{(\bigcap_{p\in P-\{i\}} S_p)} \cup \overline{S_i} = (\bigcup_{p\in P-\{i\}} \overline{S_p}) \cup \overline{S_i} = \bigcup_{p\in P} \overline{S_p}.$$

Therefore, for $|P| = 2, 3, \cdots$

$$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}, \qquad \overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$

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14. Show that

$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

Proof. According to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \quad \Rightarrow \quad \overline{\overline{S_1 \cup S_2}} = \overline{\overline{S_1} \cap \overline{S_2}} \quad \Rightarrow \quad S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

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15. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing$.

$$S_1 = S_2 \quad \Rightarrow \left\{ \begin{array}{c} S_1 \cap \overline{S_2} = S_1 \cap \overline{S_1} = \varnothing \\ \overline{S_1} \cap S_2 = \overline{S_1} \cap S_2 = \varnothing \end{array} \right\} \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

2. $S_1 = S_2 \quad \Leftarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing$.
   Assume that $S_1 \neq S_2$,
   - $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cap \overline{S_2} \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2)$.
   - $\exists\, x \notin S_1$ and $x \in S_2 \quad \Rightarrow \quad x \in \overline{S_1} \cap S_2 \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2)$.

   Therefore, $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \varnothing$, which is a contradiction. Thus $S_1 = S_2$.

To sum up,

$$S_1 = S_2 \quad \Longleftrightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

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16. Show that

$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

Proof.

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup \overline{\overline{S_2}}) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2). \end{aligned}$$

1. If $x \in S_2$

$$x \in S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \in (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

2. If $x \notin S_2$ and $x \in S_1$

$$x \notin S_2 \text{ and } x \in S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2 \text{ and } x \notin \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

3. If $x \notin S_2$ and $x \notin S_1$

$$x \notin S_2 \text{ and } x \notin S_1 \quad \Rightarrow \quad x \notin S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \quad \Rightarrow \quad x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

To sum up

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= S_2. \end{aligned}$$

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17. Show that the distributive law

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.
Proof.

1. If $x \notin S_1$

$$x \notin S_1 \quad \Rightarrow \begin{cases} x \notin S_1 \cap (S_2 \cup S_3) \\ x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \quad \Rightarrow \quad x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

2. If $x \in S_1,\ x \notin S_2$ and $x \notin S_3$

$$\begin{aligned} x \in S_1,\ x \notin S_2 \text{ and } x \notin S_3 \quad &\Rightarrow \quad x \notin S_2 \cup S_3 \quad \Rightarrow \quad x \notin S_1 \cap (S_2 \cup S_3). \\ x \in S_1,\ x \notin S_2 \text{ and } x \notin S_3 \quad &\Rightarrow \quad x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \\ &\Rightarrow \quad x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

3. If $x \in S_1$ and $x \in S_2$

$$x \in S_1 \text{ and } x \in S_2 \quad \Rightarrow \begin{cases} x \in S_1 \text{ and } x \in S_2 \cup S_3 \quad \Rightarrow \quad x \in S_1 \cap (S_2 \cup S_3) \\ x \in S_1 \cap S_2 \quad \Rightarrow \quad x \in (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

4. If $x \in S_1,\ x \notin S_2$ and $x \in S_3$

$$\begin{aligned} x \in S_1,\ x \notin S_2 \text{ and } x \in S_3 \quad &\Rightarrow \quad x \in S_1 \text{ and } x \in S_2 \cup S_3 \\ &\Rightarrow \quad x \in S_1 \cap (S_2 \cup S_3). \\ x \in S_1,\ x \notin S_2 \text{ and } x \in S_3 \quad &\Rightarrow \quad x \in S_1 \cap S_3 \\ &\Rightarrow \quad x \in (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

To sum up

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3).$$

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18. Show that

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Proof. Assume that $S_1 = \varnothing$, then

$$\left.\begin{array}{c} S_1 \times (S_2 \cup S_3) = \varnothing \\ S_1 \times S_2 = \varnothing,\ S_1 \times S_3 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = \varnothing \end{array}\right\} \Rightarrow \begin{array}{c} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{array}$$

Assume that $S_2 = \varnothing$, then

$$\left.\begin{array}{c} S_2 \cup S_3 = S_3 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_3 \\ S_1 \times S_2 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_3 \end{array}\right\} \Rightarrow \begin{array}{c} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{array}$$

Assume that $S_3 = \varnothing$, then

$$\left.\begin{aligned} S_2 \cup S_3 = S_2 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_2 \\ S_1 \times S_3 = \varnothing \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_2 \end{aligned}\right\} \Rightarrow \begin{gathered} S_1 \times (S_2 \cup S_3) = \\ (S_1 \times S_2) \cup (S_1 \times S_3). \end{gathered}$$

Assume that $S_1 \neq \varnothing,\ S_2 \neq \varnothing,\ S_3 \neq \varnothing$

$$S_1 = \{a_1, a_2, \cdots, a_p\}, \qquad S_2 = \{b_1, b_2, \cdots, b_q\}, \qquad S_3 = \{c_1, c_2, \cdots, c_r\}.$$

where $p, q, r \in \mathbb{N}^*$.
Then

$$S_2 \cup S_3 = \{b_1, b_2, \cdots, b_q, c_1, c_2, \cdots, c_r\}.$$

$$\begin{aligned} S_1 \times (S_2 \cup S_3) = \{&(a_1, b_1),(a_1, b_2), \cdots, (a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots, (a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots, (a_p, b_q), \\ &(a_1, c_1),(a_1, c_2), \cdots, (a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots, (a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots, (a_p, c_r)\} \end{aligned}$$

$$\begin{aligned} S_1 \times S_2 = \{&(a_1, b_1),(a_1, b_2), \cdots, (a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots, (a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots, (a_p, b_q)\} \end{aligned} \qquad \begin{aligned} S_1 \times S_3 = \{&(a_1, c_1),(a_1, c_2), \cdots, (a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots, (a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots, (a_p, c_r)\} \end{aligned}$$

$$\begin{aligned} (S_1 \times S_2) \cup (S_1 \times S_3) = \{&(a_1, b_1),(a_1, b_2), \cdots, (a_1, b_q), \\ &(a_2, b_1),(a_2, b_2), \cdots, (a_2, b_q), \\ &\vdots \\ &(a_p, b_1),(a_p, b_2), \cdots, (a_p, b_q), \\ &(a_1, c_1),(a_1, c_2), \cdots, (a_1, c_r), \\ &(a_2, c_1),(a_2, c_2), \cdots, (a_2, c_r), \\ &\vdots \\ &(a_p, c_1),(a_p, c_2), \cdots, (a_p, c_r)\} \end{aligned}$$

Therefore,

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

■

19. Give conditions on S_1 and S_2 necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$

Solution.

$$S_1 \cap S_2 = \varnothing \quad \Longleftrightarrow \quad S_1 = (S_1 \cup S_2) - S_2.$$

1. $S_1 \cap S_2 = \varnothing \quad \Rightarrow \quad S_1 = (S_1 \cup S_2) - S_2$

$$\left.\begin{array}{c} S_1 \cap S_2 = \varnothing \\ S_1 = S_1 \cap U = S_1 \cap (S_2 \cup \overline{S_2}) = (S_1 \cap S_2) \cup (S_1 \cap \overline{S_2}) \end{array}\right\} \Rightarrow S_1 = S_1 \cap \overline{S_2},$$

$$(S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 = (S_1 \cup S_2) - S_2.$$

2. $S_1 \cap S_2 = \varnothing \quad \Leftarrow \quad S_1 = (S_1 \cup S_2) - S_2$

$$S_1 = (S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 \cap S_2 = (S_1 \cap \overline{S_2}) \cap S_2 = S_1 \cap (\overline{S_2} \cap S_2) = S_1 \cap \varnothing = S_1.$$

To sum up,

$$S_1 = (S_1 \cup S_2) - S_2.$$

20. Use the equivalence defined in Example 1.4 to partition the set $\{2, 4, 5, 6, 9, 22, 24, 25, 31, 37\}$ into equivalence classes.
Solution. Because

$$\begin{aligned} &2 \text{ mod } 3 = 5 \text{ mod } 3 = 2, \\ &4 \text{ mod } 3 = 22 \text{ mod } 3 = 25 \text{ mod } 3 = 31 \text{ mod } 3 = 37 \text{ mod } 3 = 1, \\ &6 \text{ mod } 3 = 9 \text{ mod } 3 = 24 \text{ mod } 3 = 0. \end{aligned}$$

The equivalence classes are

$$\{2, 5\}, \qquad \{4, 22, 25, 31, 37\}, \qquad \{6, 9, 24\}.$$

21. Show that if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.
Proof. Because $f(n) = O(g(n))$, $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^*$ such that $\forall\, n > N_1$

$$f(n) \leqslant c_1 |g(n_1)|.$$

Because $g(n) = O(f(n))$, $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^*$ such that $\forall\, n > N_2$

$$g(n) \leqslant c_2 |f(n_2)|.$$

Let $N = \max\{N_1, N_2\}$, assume that $\forall\, n > N$

$$f(n) \geqslant 0, \qquad g(n) \geqslant 0.$$

Therefore

$$\frac{1}{c_2}|g(n)| \leqslant |f(n)| \leqslant c_1 |g(n)| \quad \Rightarrow \quad f(n) = \Theta(g(n)).$$

■

22. Show that $2^n = O(3^n)$, but $2^n \neq \Theta(3^n)$.
Proof. $\exists\, c_1 = 1 > 0$ such that for all $n \geqslant 1$

$$2^n \leqslant c_1|3^n| = 3^n.$$

Therefore,

$$2^n = O(3^n).$$

However, $\forall\, c_2 > 0,\ \exists\, N = [\log_{\frac{2}{3}} c_2] + 1$, if $n > N$

$$c_2|3^n| = c_2 3^n > |2^n| = 2^n.$$

Therefore,

$$2^n \neq \Theta(3^n).$$

■

23. Show that the following order-of-magnitude results hold.

1. $n^2 + 5\log n = O(n^2)$.
2. $3^n = O(n!)$.
3. $n! = O(n^n)$.

Proof.

1. $n^2 + 5\log n = O(n^2)$.
   Let
   $$f(n) = n^2 + 5\log n, \qquad g(n) = n^2.$$
   Let $c = 2$, then $h(n) = f(n) - c|g(n)| = 5\log n - n^2$.
   $$h^{'}(n) = \frac{5}{n} - 2n \quad \Rightarrow \quad h^{'}(n) \text{ is a monotonically decreasing function.}$$
   If $n \geqslant 2, h^{'}(n) < 0$, so if $n \geqslant 2,\ h(n)$ is a monotonically decreasing function. Because $h(2) = 5\log 2 - 4 < 0$, if $n \geqslant 2,\ h(n) < 0$. $\exists\, c = 2 > 0$ such that for all $n \geqslant 2$
   $$h(n) = f(n) - c|g(n)| = 5\log n - n^2 < 0 \quad \Rightarrow \quad f(n) \leqslant c|g(n)|.$$
   Thus
   $$f(n) = O(g(n)) \quad \Rightarrow \quad n^2 + 5\log n = O(n^2).$$
2. $3^n = O(n!)$.
   Let
   $$f(n) = 3^n, \qquad g(n) = n!.$$
   $\exists\, c = 9 > 0$ such that for all $n \geqslant 3$
   $$f(n) - c|g(n)| = 3^n - 9n! = 3^n(1 - \frac{9n!}{3^n}) = 3^n(1 - 2\prod_{i=3}^{n}\frac{i}{3}) < 0 \quad \Rightarrow \quad f(n) < c|g(n)|.$$
   Thus
   $$f(n) = O(g(n)) \quad \Rightarrow \quad 3^n = O(n!).$$

3. $n! = O(n^n)$.

Let

$$f(n) = n!, \qquad g(n) = n^n.$$

$\exists\, c = 1 > 0$ such that for all $n \geqslant 1$

$$f(n) - c|g(n)| = n! - n^n = n!(1 - \prod_{i=1}^{n} \frac{n}{i}) \leqslant 0 \quad \Rightarrow \quad f(n) \leqslant c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \quad \Rightarrow \quad n! = O(n^n).$$

■

24. Show that $\dfrac{n^3 - 2n}{n+1} = \Theta(n^2)$.

Proof. Let

$$f(n) = \frac{n^3 - 2n}{n+1}, \qquad g(n) = n^2, \qquad c_1 = \frac{1}{3}, \qquad c_2 = 1.$$

Because

$$f(n) = \frac{n^3 - 2n}{n+1} = \frac{n(n^2 - 2)}{n+1}.$$

If $n \geqslant 2$

$$|f(n)| = \left|\frac{n(n^2 - 2)}{n+1}\right| \geqslant 0 \quad \Rightarrow \quad |f(n)| = f(n).$$

Now

$$|f(n)| - c_1|g_n| = \left|\frac{n^3 - 2n}{n+1}\right| - \frac{1}{3}|n^2| = \frac{n^3 - 2n}{n+1} - \frac{1}{3}n^2 = \frac{n(2n+3)(n-2)}{3(n+1)}.$$

If $n \geqslant 2$

$$|f(n)| - c_1|g_n| = \frac{n(2n+3)(n-2)}{3(n+1)} \geqslant 0 \quad \Rightarrow \quad |f(n)| \geqslant c_1|g_n|.$$

Then

$$|f(n)| - c_2|g_n| = \left|\frac{n^3 - 2n}{n+1}\right| - |n^2| = \frac{n^3 - 2n}{n+1} - n^2 = -\frac{n(n+2)}{n+1}.$$

If $n > 0$

$$|f(n)| - c_2|g_n| = -\frac{n(n+2)}{n+1} < 0 \quad \Rightarrow \quad |f(n)| < c_2|g_n|.$$

To sum up, if $n \geqslant 2$

$$c_1|g_n| \leqslant |f(n)| < c_2|g_n| \quad \Rightarrow \quad f(n) = \Theta(g(n)) \quad \Rightarrow \quad \frac{n^3 - 2n}{n+1} = \Theta(n^2).$$

■

25. Show that $\dfrac{n^3}{\log(n+1)} = O(n^3)$ but not $O(n^2)$.

Proof. $\forall\, x > 0$, assume that the base of $\log(x+1)$ is 2. Let

$$f(x) = \log(x+1) - 1.$$

Then

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{(x+1)\ln 2} > 0.$$

Therefore, if $x > 0$, $f(x)$ is a strictly monotonically increasing function, then

$$f(1) = \log(1+1) - 1 = \log 2 - 1 = 0.$$

If $x \in \mathbb{N}^*$, $x \geqslant 1$. Let $x = n$

$$f(n) \geqslant f(1) = 0 \quad \Rightarrow \quad \log(n+1) - 1 \geqslant 0 \quad \Rightarrow \quad \frac{n^3}{\log(n+1)} \leqslant n^3.$$

Let $c_1 = 1$

$$\frac{n^3}{\log(n+1)} - c_1|n^3| = \frac{n^3}{\log(n+1)} - n^3 \leqslant 0 \quad \Rightarrow \quad \frac{n^3}{\log(n+1)} = O(n^3).$$

$\forall\, x \geqslant 1$, assume that the base of $\log(x+1)$ is 2. Let

$$g(x) = \sqrt{x} - \log(x+1).$$

Then

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} - \frac{1}{(x+1)\ln 2} > \frac{1}{2\sqrt{x}} - \frac{1}{x\ln 2} > \frac{1}{2\sqrt{x}} - \frac{2}{x} = \frac{\sqrt{x}-4}{2x}.$$

If $x > 16$,

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{\sqrt{x}-4}{2x} > 0.$$

Therefore, if $x > 16$, $g(x)$ is a strictly monotonically increasing function, then

$$g(19) = \sqrt{19} - \log(20) > 0.$$

Thus, if $x > 19$,

$$g(x) = \sqrt{x} - \log(x+1) > 0 \quad \Rightarrow \quad -\log(x+1) > -\sqrt{x}.$$

$\forall\, c_2 > 0,\ \exists\, N = \max\{20,\ [c_2^2] + 1\}$ such that

$$\begin{aligned}\frac{N^3}{\log(N+1)} - c_2|N^2| &= \frac{N^2}{\log(N+1)}[N - c_2\log(N+1)] \\ &> \frac{N^2}{\log(N+1)}(N - c_2\sqrt{N}) \\ &\geqslant 0.\end{aligned}$$

Therefore,

$$\frac{n^3}{\log(n+1)} \neq O(n^2).$$

$\blacksquare$

26. What is wrong with the following argument?

$$x = O(n^4), \quad y = O(n^2), \quad \text{therefore} \quad \frac{x}{y} = O(n^2).$$

Proof. Let

$$f_1(n) = n^3, \qquad f_2(n) = 1, \qquad g_1(n) = n^4, \qquad g_2(n) = n^2.$$

$\exists\, c_1 = 1,\ c_2 = 1$ such that $\forall\, n \in \mathbb{N}^*$

$$f_1(n) - c_1|g_1(n)| = n^3 - n^4 \leqslant 0, \qquad f_2(n) - c_2|g_2(n)| = 1 - n^2 \leqslant 0.$$

Therefore

$$f_1(n) = O(g_1(n)) = O(n^4), \qquad f_2(n) = O(g_2(n)) = O(n^2).$$

Let

$$x = f_1(n) = n^3, \qquad y = f_2(n) = 1. \quad \Rightarrow \quad \frac{x}{y} = n^3.$$

However $\forall\, c > 0,\ \exists\, N = [c] + 1$ such that

$$\frac{x}{y}_{n=N} - c|N^2| = N^3 - cN^2 = N^2(N - c) > 0.$$

Thus

$$\frac{x}{y} \neq O(n^2).$$

$\blacksquare$

27. What is wrong with the following argument?

$$x = \Theta(n^4), \quad y = \Theta(n^2), \quad \text{therefore} \quad \frac{x}{y} = \Theta(n^2).$$

Proof. This statement is correct. Assume that $\exists\ c_1 > 0,\ c_2 > 0,\ c_3 > 0,\ c_4 > 0$ and $\exists\ N_1 > 0,\ N_2 > 0$ where $N_1, N_2 \in \mathbb{N}^*$ such that if $n > N_1$

$$c_1|n^4| \leqslant |x| \leqslant c_2|n^4|$$

and if $n > N_2$

$$c_3|n^2| \leqslant |y| \leqslant c_4|n^2| \quad \Rightarrow \quad \frac{1}{c_4|n^2|} \leqslant \frac{1}{|y|} \leqslant \frac{1}{c_3|n^2|}.$$

Therefore let $N = \max\{N_1, N_2\}$, if $n > N$

$$\frac{c_1|n^4|}{c_4|n^2|} \leqslant \frac{|x|}{|y|} \leqslant \frac{c_2|n^4|}{c_3|n^2|} \quad \Rightarrow \quad \frac{c_1}{c_4}|n^2| \leqslant \left|\frac{x}{y}\right| \leqslant \frac{c_2}{c_3}|n^2|.$$

Thus

$$\frac{x}{y} = \Theta(n^2).$$

$\blacksquare$

28. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
Proof. Because $f(n) = O(g(n))$, $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^*$ such that $\forall\, n > N_1$

$$f(n) \leqslant c_1|g(n)|.$$

Because $g(n) = O(h(n))$, $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^*$ such that $\forall\, n > N_2$

$$g(n) \leqslant c_2|h(n)|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall\, n > N$

$$g(n) \geqslant 0.$$

Then $\forall\, n > N$

$$f(n) \leqslant c_1|g(n)| = c_1 g(n) \leqslant c_1 c_2 |h(n)|.$$

Therefore

$$f(n) = O(h(n)).$$

■

29. Show that if $f(n) = O(n^2)$ and $g(n) = O(n^3)$, then

$$f(n) + g(n) = O(n^3)$$

and

$$f(n)g(n) = O(n^5).$$

Proof. Because $f(n) = O(n^2)$, assume that $\exists\, c_1 > 0,\ N_1 \in \mathbb{N}^*$ such that $\forall\, n > N_1$

$$f(n) \leqslant c_1|n^2|.$$

Because $g(n) = O(n^3)$, assume that $\exists\, c_2 > 0,\ N_2 \in \mathbb{N}^*$ such that $\forall\, n > N_2$

$$g(n) \leqslant c_2|n^3|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall\, n > N$

$$g(n) \geqslant 0.$$

Then $\forall\, n > N$

$$f(n) + g(n) \leqslant c_1|n^2| + c_2|n^3| \leqslant (c_1 + c_2)|n^3|,$$
$$f(n)g(n) \leqslant c_1|n^2| \cdot c_2|n^3| = c_1 c_2 |n^5|.$$

Therefore

$$f(n) + g(n) = O(n^3), \qquad f(n)g(n) = O(n^5).$$

■

30. Assume that $f(n) = 2n^2 + n$ and $g(n) = O(n^2)$. What is wrong with the following argument?

$$f(n) = O(n^2) + O(n),$$

so that

$$f(n) - g(n) = O(n^2) + O(n) - O(n^2).$$

Therefore,

$$f(n) - g(n) = O(n).$$

Proof. Assume that

$$g(n) = n^2.$$

$\exists\, c_1 = 2,\ N_1 = 1,\ \forall\, n > N_1$

$$g(n) - c_1|n^2| = n^2 - 2n^2 = -n^2 < 0 \quad \Rightarrow \quad g(n) \leqslant 2|n^2| \quad \Rightarrow \quad g(n) = O(n^2).$$

Let

$$h(n) = f(n) - g(n) = 2n^2 + n - n^2 = n^2 + n.$$

However, $\forall\, c_2 > 0,\ \exists\, N_2 = [c_2] + 1$ such that $\forall\, n > N_2$

$$h(n) - c_2|n| = n^2 + n - c_2 n = (n - c_2 + 1)n > 0.$$

Therefore

$$h(n) = f(n) - g(n) \neq O(n).$$

$\blacksquare$

31. Show that if $f(n) = \Theta(\log_2 n)$, then $f(n) = \Theta(\log_{10} n)$.
Proof. Because $f(n) = \Theta(\log_2 n)$, assume that $\exists\, c_1 > 0,\ c_2 > 0,\ N_1 \in \mathbb{N}^*$ such that $\forall\, n > N_1$

$$c_1|\log_2 n| \leqslant |f(n)| \leqslant c_2|\log_2 n|.$$

According to The Change-of-Base Formula

$$\log_2 n = \frac{\log_{10} n}{\log_{10} 2} \quad \Rightarrow \quad |\log_2 n| = \left|\frac{\log_{10} n}{\log_{10} 2}\right| = \frac{|\log_{10} n|}{|\log_{10} 2|} = \frac{|\log_{10} n|}{\log_{10} 2}.$$

Therefore

$$\frac{c_1}{\log_{10} 2}|\log_{10} n| \leqslant |f(n)| \leqslant \frac{c_2}{\log_{10} 2}|\log_{10} n| \quad \Rightarrow \quad f(n) = \Theta(\log_{10} n).$$

$\blacksquare$