Chapter 1 Section 1 Exercises

1. With $S_1 = \{ 2, 3, 5, 7 \},\ S_2 = \{ 2, 4, 5, 8, 9 \}$, and $U = \{ 1 : 10 \}$, compute $\overline{S}_1 \cup S_2$.

Solution.

$$\overline{S}_1 = \{ 1, 4, 6, 8, 9, 10 \} \quad \Rightarrow \quad \overline{S}_1 \cup S_2 = \{ 1, 2, 4, 5, 6, 8, 9, 10 \}.$$

2. With $S_1 = \{ 2, 3, 5, 7 \},\ S_2 = \{ 2, 4, 5, 8, 9 \}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned} S_1 \times S_2 = \{ & (2,2), (2,4), (2,5), (2,8), (2,9), \\ & (3,2), (3,4), (3,5), (3,8), (3,9), \\ & (5,2), (5,4), (5,5), (5,8), (5,9), \\ & (7,2), (7,4), (7,5), (7,8), (7,9) \}. \\ S_2 \times S_1 = \{ & (2,2), (2,3), (2,5), (2,7), \\ & (4,2), (4,3), (4,5), (4,7), \\ & (5,2), (5,3), (5,5), (5,7), \\ & (8,2), (8,3), (8,5), (8,7), \\ & (9,2), (9,3), (9,5), (9,7) \}. \end{aligned}$$

3. For $S = \{ 2, 5, 6, 8 \}$ and $T = \{ 2, 4, 6, 8 \}$, compute $|S \cap T| + |S \cup T|$.

Solution.

$$S \cap T = \{ 2, 6, 8 \}, \quad S \cup T = \{ 2, 4, 5, 6, 8 \} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \varnothing.$$

Therefore, S and T are disjoint.

5. Show that for all sets S and T, $S - T = S \cap \overline{T}$.

Proof.

$$\begin{aligned} & S - T = \{ x : x \in S \text{ and } x \notin T \} \\ \Longleftrightarrow \quad & S - T = \{ x : x \in S \text{ and } x \in \overline{T} \} \\ \Longleftrightarrow \quad & S - T = S \cap \overline{T}. \end{aligned}$$

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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$
$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

Proof.

$$S_1 \cup S_2 = \{\, x : x \in S_1 \text{ or } x \in S_2 \,\} \quad \Longleftrightarrow \quad \overline{S_1 \cup S_2} = \{\, x : x \notin S_1 \text{ and } x \notin S_2 \,\}.$$
$$\overline{S_1} = \{\, x : x \notin S_1 \,\}, \quad \overline{S_2} = \{\, x : x \notin S_2 \,\} \quad \Longleftrightarrow \quad \overline{S_1} \cap \overline{S_2} = \{\, x : x \notin S_1 \text{ and } x \notin S_2 \,\}.$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$S_1 \cap S_2 = \{\, x : x \in S_1 \text{ and } x \in S_2 \,\} \quad \Longleftrightarrow \quad \overline{S_1 \cap S_2} = \{\, x : x \notin S_1 \text{ or } x \notin S_2 \,\}.$$
$$\overline{S_1} = \{\, x : x \notin S_1 \,\}, \quad \overline{S_2} = \{\, x : x \notin S_2 \,\} \quad \Longleftrightarrow \quad \overline{S_1} \cup \overline{S_2} = \{\, x : x \notin S_1 \text{ or } x \notin S_2 \,\}.$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

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7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.
Proof.

$$\begin{aligned} & S_1 \subseteq S_2 \\ \Longleftrightarrow \quad & x \in S_1 \quad \Rightarrow \quad x \in S_2 \\ \Longleftrightarrow \quad & x \notin S_2 \quad \Rightarrow \quad x \notin S_1 \\ \Longleftrightarrow \quad & x \in \overline{S_2} \quad \Rightarrow \quad x \notin \overline{S_1} \\ \Longleftrightarrow \quad & \overline{S_2} \subseteq \overline{S_1}. \end{aligned}$$

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8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.
Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$

$$\left.\begin{aligned} S_1 = S_2 \quad &\Rightarrow \quad S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \quad &\Rightarrow \quad S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{aligned}\right\} \quad \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$$

2. $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2.$
   Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,
   - $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2.$

- $\exists\, x \in S_2$ and $x \notin S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.

The result contradicts with the permise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2$.

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

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9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.
Proof.

1. **Basis**

   If $|S| = 1$, assume that $S = \{\ a\ \}$. Then

   $$2^S = \{\ \varnothing,\ \{\ a\ \}\ \}.$$

   Therefore, $|2^S| = 2^{|S|} = 2$.

2. **Inductive Assumption**

   Assume that $|2^S| = 2^{|S|}$, for $|S| = 1,\ 2,\ \cdots,\ n$.

3. **Inductive Step**

   For $|S| = n+1$, assume that $S = \{\ a_1,\ a_2,\ \cdots,\ a_n,\ a_{n+1}\ \}$. Let $T = \{\ a_1,\ a_2,\ \cdots,\ a_n\ \}$, then

   $$2^T = \{\ T_1,\ T_2,\ \cdots,\ T_{2^n}\ \}.$$

   For $\forall\ i = 1,\ 2,\ \cdots,\ 2^n$ where $i \in \mathbb{N}^*$

   $$\left.\begin{array}{r} T_i \subseteq T \\ T \subseteq S \end{array}\right\} \quad \Longleftrightarrow \quad T_i \subseteq S.$$

   However,

   $$S - T = \{\ a_{n+1}\ \} \quad \Rightarrow \quad a_{n+1} \notin T \quad \Rightarrow \quad a_{n+1} \notin T_i.$$

   In addition

   $$\left.\begin{array}{rcr} & & T_i \subseteq S \\ a_{n+1} \in S_i & \Longleftrightarrow & \{\ a_{n+1}\ \} \subseteq S \end{array}\right\} \quad \Longleftrightarrow \quad T_i \cup \{\ a_{n+1}\ \} \subseteq S.$$

   Let

   $$T_i^{'} = T_i \cup \{\ a_{n+1}\ \}, \qquad U = \{\ T_1^{'},\ T_2^{'},\ \cdots,\ T_{2^n}^{'}\ \}.$$

   Now, for $\forall\ S_i \subseteq S$

   - If $a_{n+1} \notin S_i$, then $S_i \subseteq T$, so $S_i \in 2^T$.
   - If $a_{n+1} \in S_i$, then $S_i - \{\ a_{n+1}\ \} \subseteq T$, so $S_i - \{\ a_{n+1}\ \} \in 2^T$. Assume that

   $$S_i - \{\ a_{n+1}\ \} = T_j \quad \Rightarrow \quad S_i = T_j \cup \{\ a_{n+1}\ \} \quad \Rightarrow \quad S_i \in U.$$

   Moreover, 2^T and U are disjoint. Therefore,

   $$2^S = 2^T \cup U, \qquad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$.

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10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

Proof. Assume that

$$S_1 = \{\ a_1,\ a_2,\ \cdots\ ,\ a_n\ \}, \qquad S_2 = \{\ b_1,\ b_2,\ \cdots\ ,\ b_m\ \}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{\ a_1,\ a_2,\ \cdots\ ,\ a_n,\ b_1,\ b_2,\ \cdots\ ,\ b_m\ \}.$$

   Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1,\ c_2,\ \cdots\ ,\ c_k \in S_1 \text{ and } c_1,\ c_2,\ \cdots\ ,\ c_k \in S_2.$$

   where $k \leqslant n,\ k \leqslant m,\ k \in \mathbb{N}^*$. Assume that

$$b_{i_1} = c_1,\ b_{i_2} = c_2,\ \cdots\ ,\ b_{i_k} = c_k.$$

   Now

$$S_1 \cup S_2 = \{\ a_1,\ a_2,\ \cdots\ ,\ a_n,\ b_1,\ b_2,\ \cdots\ ,\ b_{i_1-1},\ b_{i_1+1}\ \cdots\ ,\ b_{i_k-1},\ b_{i_k+1}\ \cdots\ ,\ b_m\ \}.$$

   Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

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11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.
Proof. Assume that

$$S_1 = \{\ a_1,\ a_2,\ \cdots\ ,\ a_n\ \}, \qquad S_2 = \{\ b_1,\ b_2,\ \cdots\ ,\ b_m\ \}.$$

Therefore,

$$\begin{aligned} S_1 \times S_2 = \{\ & (a_1, b_1), (a_2, b_1),\ \cdots\ ,(a_n, b_1), \\ & (a_1, b_2), (a_2, b_2),\ \cdots\ ,(a_n, b_2), \\ & \qquad\qquad \vdots \\ & (a_1, b_m),(a_2, b_m), \cdots\ ,(a_n, b_m)\ \}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

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12. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.
Proof.

1. **Reflexivity**
$$|S_1| = |S_1| \text{ for all } S_1. \quad \Longleftrightarrow \quad S_1 \equiv S_1 \text{ for all } S_1.$$

2. **Symmetry**
$$\text{if } |S_1| = |S_2|, \text{ then } |S_2| = |S_1|. \quad \Longleftrightarrow \quad \text{if } S_1 \equiv S_2, \text{ then } S_2 \equiv S_1.$$

3. **Transitivity**
$$\begin{gathered} \text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|, \text{ then } |S_1| = |S_3|. \\ \Updownarrow \\ \text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3, \text{ then } S_1 \equiv S_3. \end{gathered}$$

Therefore, this is an equivalence relation.

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