

Chapter 1 Section 2 Exercises

1. How many substrings aab are in ww^Rw , where $w = aabbab$.

Solution. Because

$$w^R = babbaa, \quad ww^Rw = \underline{aabb}abbabbaa\underline{aabb}ab,$$

there are two substrings aab in ww^Rw .

2. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n .

Proof.

1. Basis

If $n = 0$, then

$$|u^1| = |u|, \quad 1 \cdot |u| = |u|.$$

Therefore,

$$|u^1| = 1 \cdot |u|.$$

2. Inductive Assumption

Assume that for $i = 0, 1, 2, \dots, n$

$$|u^i| = i \cdot |u|.$$

3. Inductive Step

For $i = n + 1$,

$$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n + 1)|u|.$$

Thus, $|u^n| = n|u|$ for all strings u and all n .

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3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$\begin{aligned} a^R &= a, \\ (wa)^R &= aw^R, \end{aligned}$$

for all $a \in \Sigma, w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

Proof.

1. Basis

For $|v| = 1$, suppose that $v = a, a \in \Sigma$

$$(uv)^R = (ua)^R = au^R = v^R u^R.$$

2. Inductive Assumption

Suppose that for $|v| = 1, 2, \dots, n$

$$(uv)^R = v^R u^R.$$

3. Inductive Step

For $i = n + 1$, suppose that $v = wa, a \in \Sigma$

$$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

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4. Prove that $(w^R)^R = w$ for all $w \in \Sigma^*$.

Proof.

1. Basis

For $|w| = 0$, which means that $w = \lambda$,

$$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

2. Inductive Assumption

Suppose that for $|w| = 1, 2, \dots, n$

$$(w^R)^R = w.$$

3. Inductive Step

For $i = n + 1$, suppose that $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus, $(w^R)^R = w$ for all $w \in \Sigma^*$.

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5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^* : $abaabaaabaa$, $aaaabaaaa$, $baaaaabaaaab$, $baaaaabaa$? Which strings are in L^4 ?

Solution.

$$abaabaaabaa = ab|aa|baa|ab|aa \in L^5;$$

$$aaaabaaaa = aa|aa|baa|aa \in L^4;$$

$$baaaaabaaaab \notin L^*;$$

$$baaaaabaa = baa|aa|ab|aa \in L^4.$$

6. Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe \overline{L} .

Solution.

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let L be any language on a nonempty alphabet. Show that L and \overline{L} cannot both be finite.

Proof. Suppose that both L and \overline{L} are finite. Therefore, $L \cup \overline{L}$ is also finite. However, $L \cup \overline{L} = \Sigma^*$ is an infinite set, which leads to a contradiction. Thus, L and \overline{L} cannot both be finite.

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8. Are there languages for which $\overline{L^*} = (\overline{L})^*$?

Solution. No, because

$$\begin{aligned}\overline{L^*} &= \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \dots), \\ (\overline{L})^* &= (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \dots.\end{aligned}$$

Thus, $\lambda \notin \overline{L^*}$ and $\lambda \in (\overline{L})^*$. There is no language for which $\overline{L^*} = (\overline{L})^*$.

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2 .

Proof. $\forall u \in L_1$ and $\forall v \in L_2$, then

$$uv \in L_1 L_2 \quad \Rightarrow \quad (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \quad \Rightarrow \quad v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2 .

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