Chapter 1 Section 1 Exercises

1. With $S_1 = \{2,3,5,7\},\ S_2 = \{2,4,5,8,9\}$, and $U = \{1:10\}$, compute $\overline{S}_1 \cup S_2$.

Solution.

$$\overline{S}_1 = \{1,4,6,8,9,10\} \quad \Rightarrow \quad \overline{S}_1 \cup S_2 = \{1,2,4,5,6,8,9,10\}.$$

2. With $S_1 = \{2,3,5,7\},\ S_2 = \{2,4,5,8,9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned}
S_1 \times S_2 = \{&(2,2),(2,4),(2,5),(2,8),(2,9),\\
&(3,2),(3,4),(3,5),(3,8),(3,9),\\
&(5,2),(5,4),(5,5),(5,8),(5,9),\\
&(7,2),(7,4),(7,5),(7,8),(7,9)\}.\\
S_2 \times S_1 = \{&(2,2),(2,3),(2,5),(2,7),\\
&(4,2),(4,3),(4,5),(4,7),\\
&(5,2),(5,3),(5,5),(5,7),\\
&(8,2),(8,3),(8,5),(8,7),\\
&(9,2),(9,3),(9,5),(9,7)\}.
\end{aligned}$$

3. For $S = \{2,5,6,8\}$ and $T = \{2,4,6,8\}$, compute $|S \cap T| + |S \cup T|$.

Solution.

$$S \cap T = \{2,6,8\}, \quad S \cup T = \{2,4,5,6,8\} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \varnothing.$$

Therefore, S and T are disjoint.

5. Show that for all sets S and T, $S - T = S \cap \overline{T}$.

Proof.

$$\begin{aligned}
& S - T = \{x : x \in S \text{ and } x \notin T\}\\
\Longleftrightarrow \quad & S - T = \{x : x \in S \text{ and } x \in \overline{T}\}\\
\Longleftrightarrow \quad & S - T = S \cap \overline{T}.
\end{aligned}$$

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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$
$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

Proof.

$$\begin{aligned} S_1 \cup S_2 = \{x : x \in S_1 \text{ or } x \in S_2\} \quad &\Rightarrow \quad \overline{S_1 \cup S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}. \\ \overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad &\Rightarrow \quad \overline{S_1} \cap \overline{S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}. \end{aligned}$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$\begin{aligned} S_1 \cap S_2 = \{x : x \in S_1 \text{ and } x \in S_2\} \quad &\Rightarrow \quad \overline{S_1 \cap S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}. \\ \overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \quad &\Rightarrow \quad \overline{S_1} \cup \overline{S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}. \end{aligned}$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

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7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.
Proof.

$$\begin{aligned} & S_1 \subseteq S_2 \\ \Rightarrow \quad & (\in S_1 \quad \Rightarrow \quad x \in S_2) \\ \Rightarrow \quad & (x \notin S_2 \quad \Rightarrow \quad x \notin S_1) \\ \Rightarrow \quad & (x \in \overline{S_2} \quad \Rightarrow \quad x \notin \overline{S_1}) \\ \Rightarrow \quad & \overline{S_2} \subseteq \overline{S_1}. \end{aligned}$$

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8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.
Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$

$$\left. \begin{aligned} S_1 = S_2 \quad &\Rightarrow \quad S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \quad &\Rightarrow \quad S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{aligned} \right\} \Rightarrow \quad S_1 \cup S_2 = S_1 \cap S_2.$$

2. $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2.$

   Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,

- $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.
- $\exists\, x \in S_2$ and $x \notin S_1 \quad \Rightarrow \quad x \in S_1 \cup S_2$ and $x \notin S_1 \cap S_2 \quad \Rightarrow \quad S_1 \cup S_2 \neq S_1 \cap S_2$.

The result contradicts with the permise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \quad \Rightarrow \quad S_1 = S_2$.

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

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9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.
Proof.

1. **Basis**
   If $|S| = 1$, assume that $S = \{a\}$. Then
   $$2^S = \{\varnothing, \{a\}\}.$$
   Therefore, $|2^S| = 2^{|S|} = 2$.
2. **Inductive Assumption**
   Assume that $|2^S| = 2^{|S|}$, for $|S| = 1, 2, \cdots, n$.
3. **Inductive Step**
   For $|S| = n + 1$, assume that $S = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$. Let $T = \{a_1, a_2, \cdots, a_n\}$, then
   $$2^T = \{T_1, T_2, \cdots, T_{2^n}\}.$$
   For $\forall\, i = 1, 2, \cdots, 2^n$ where $i \in \mathbb{N}^*$
   $$\left.\begin{array}{c} T_i \subseteq T \\ T \subseteq S \end{array}\right\} \Rightarrow \quad T_i \subseteq S.$$
   However,
   $$S - T = \{a_{n+1}\} \quad \Rightarrow \quad a_{n+1} \notin T \quad \Rightarrow \quad a_{n+1} \notin T_i.$$
   In addition
   $$\left.\begin{array}{c} T_i \subseteq S \\ a_{n+1} \in S_i \quad \Rightarrow \quad \{a_{n+1}\} \subseteq S \end{array}\right\} \Rightarrow \quad T_i \cup \{a_{n+1}\} \subseteq S.$$
   Let
   $$T_i^{'} = T_i \cup \{a_{n+1}\}, \qquad U = \{T_1^{'}, T_2^{'}, \cdots, T_{2^n}^{'}\}.$$
   Now, for $\forall\, S_i \subseteq S$
   - If $a_{n+1} \notin S_i$, then $S_i \subseteq T$, so $S_i \in 2^T$.
   - If $a_{n+1} \in S_i$, then $S_i - \{a_{n+1}\} \subseteq T$, so $S_i - \{a_{n+1}\} \in 2^T$. Assume that
     $$S_i - \{a_{n+1}\} = T_j \quad \Rightarrow \quad S_i = T_j \cup \{a_{n+1}\} \quad \Rightarrow \quad S_i \in U.$$

Moreover, 2^T and U are disjoint. Therefore,

$$2^S = 2^T \cup U, \qquad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$.

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10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

Proof. Assume that

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1, c_2, \cdots, c_k \in S_1 \text{ and } c_1, c_2, \cdots, c_k \in S_2.$$

where $k \leqslant n,\ k \leqslant m,\ k \in \mathbb{N}^*$. Assume that

$$b_{i_1} = c_1,\ b_{i_2} = c_2,\ \cdots,\ b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_{i_1-1}, b_{i_1+1} \cdots, b_{i_k-1}, b_{i_k+1} \cdots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leqslant n + m.$$

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11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.
Proof. Assume that

$$S_1 = \{a_1, a_2, \cdots, a_n\}, \qquad S_2 = \{b_1, b_2, \cdots, b_m\}.$$

Therefore,

$$\begin{aligned} S_1 \times S_2 = \{&(a_1, b_1), (a_2, b_1), \cdots, (a_n, b_1), \\ &(a_1, b_2), (a_2, b_2), \cdots, (a_n, b_2), \\ &\vdots \\ &(a_1, b_m), (a_2, b_m), \cdots, (a_n, b_m)\}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

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12. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.
Proof.

1. **Reflexivity**
$$|S_1| = |S_1| \text{ for all } S_1. \quad \Rightarrow \quad S_1 \equiv S_1 \text{ for all } S_1.$$

2. **Symmetry**
$$\text{if } |S_1| = |S_2|, \text{ then } |S_2| = |S_1|. \quad \Rightarrow \quad \text{if } S_1 \equiv S_2, \text{ then } S_2 \equiv S_1.$$

3. **Transitivity**
$$\text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|, \text{ then } |S_1| = |S_3|.$$
$$\Downarrow$$
$$\text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3, \text{ then } S_1 \equiv S_3.$$

Therefore, this is an equivalence relation.

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13. Occassionally, we need to use the union and intersection symbols in a manner analogous to the summation sign $\sum$. We define

$$\bigcup_{p \in \{i,j,k,\cdots\}} S_p = S_i \cup S_j \cup S_k \cdots$$

with an analogous notation for the intersection of several sets.
With this notation, the gereral DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when P is a finite set.
Proof.

1. **Basis**

   For $|P| = 2$, according to DeMorgan's laws
$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}, \qquad \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

2. **Inductive Assumption**

   For $|P| = 2, 3, \cdots, n$ where $n \in \mathbb{N}^*$
$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}, \qquad \overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

3. **Inductive Step**

   For $|P| = n + 1$ where $n \in \mathbb{N}^*$, $\forall\, i \in P,\ |P - \{i\}| = n$,

$$\overline{\bigcup_{p\in P} S_p} = \overline{(\bigcup_{p\in P-\{i\}} S_p) \cup S_i} = \overline{(\bigcup_{p\in P-\{i\}} S_p)} \cap \overline{S_i} = (\bigcap_{p\in P-\{i\}} \overline{S_p}) \cap \overline{S_i} = \bigcap_{p\in P} \overline{S_p},$$

$$\overline{\bigcap_{p\in P} S_p} = \overline{(\bigcap_{p\in P-\{i\}} S_p) \cap S_i} = \overline{(\bigcap_{p\in P-\{i\}} S_p)} \cup \overline{S_i} = (\bigcup_{p\in P-\{i\}} \overline{S_p}) \cup \overline{S_i} = \bigcup_{p\in P} \overline{S_p}.$$

Therefore, for $|P| = 2, 3, \cdots$

$$\overline{\bigcup_{p\in P} S_p} = \bigcap_{p\in P} \overline{S_p}, \qquad \overline{\bigcap_{p\in P} S_p} = \bigcup_{p\in P} \overline{S_p}.$$

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14. Show that

$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

Proof. According to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \quad \Rightarrow \quad \overline{\overline{S_1 \cup S_2}} = \overline{\overline{S_1} \cap \overline{S_2}} \quad \Rightarrow \quad S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

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15. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

Proof.

1. $S_1 = S_2 \quad \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$

$$S_1 = S_2 \quad \Rightarrow \left\{ \begin{array}{c} S_1 \cap \overline{S_2} = S_1 \cap \overline{S_1} = \varnothing \\ \overline{S_1} \cap S_2 = \overline{S_1} \cap S_2 = \varnothing \end{array} \right\} \Rightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

2. $S_1 = S_2 \quad \Leftarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$

   Assume that $S_1 \neq S_2$,

   - $\exists\, x \in S_1$ and $x \notin S_2 \quad \Rightarrow \quad x \in S_1 \cap \overline{S_2} \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$
   - $\exists\, x \notin S_1$ and $x \in S_2 \quad \Rightarrow \quad x \in \overline{S_1} \cap S_2 \quad \Rightarrow \quad x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$

   Therefore, $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \varnothing$, which is a contradiction. Thus $S_1 = S_2$.

To sum up,

$$S_1 = S_2 \quad \Longleftrightarrow \quad (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

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