

Chapter 1 Section 2 Exercises

1. How many substrings aab are in ww^Rw , where $w = aabbab$.

Solution. Because

$$w^R = babbaa, \quad ww^Rw = \underline{aabb}abbabbaa\underline{aabb}ab,$$

there are two substrings aab in ww^Rw .

2. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n .

Proof.

1. Basis

If $n = 0$, then

$$|u^1| = |u|, \quad 1 \cdot |u| = |u|.$$

Therefore,

$$|u^1| = 1 \cdot |u|.$$

2. Inductive Assumption

Assume that for $i = 0, 1, 2, \dots, n$

$$|u^i| = i \cdot |u|.$$

3. Inductive Step

For $i = n + 1$,

$$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n + 1)|u|.$$

Thus, $|u^n| = n|u|$ for all strings u and all n .

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3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$\begin{aligned} a^R &= a, \\ (wa)^R &= aw^R, \end{aligned}$$

for all $a \in \Sigma, w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

Proof.

1. Basis

For $|v| = 1$, suppose that $v = a, a \in \Sigma$

$$(uv)^R = (ua)^R = au^R = v^R u^R.$$

2. Inductive Assumption

Suppose that for $|v| = 1, 2, \dots, n$

$$(uv)^R = v^R u^R.$$

3. Inductive Step

For $i = n + 1$, suppose that $v = wa, a \in \Sigma$

$$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

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4. Prove that $(w^R)^R = w$ for all $w \in \Sigma^*$.

Proof.

1. Basis

For $|w| = 0$, which means that $w = \lambda$,

$$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

2. Inductive Assumption

Suppose that for $|w| = 1, 2, \dots, n$

$$(w^R)^R = w.$$

3. Inductive Step

For $i = n + 1$, suppose that $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus, $(w^R)^R = w$ for all $w \in \Sigma^*$.

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5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^* : $abaabaaabaa, aaaabaaaa, baaaaabaaaab, baaaaabaa$? Which strings are in L^4 ?

Solution.

$$abaabaaabaa = ab|aa|baa|ab|aa \in L^5;$$

$$aaaabaaaa = aa|aa|baa|aa \in L^4;$$

$$baaaaabaaaab \notin L^*;$$

$$baaaaabaa = baa|aa|ab|aa \in L^4.$$

6. Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe \overline{L} .

Solution.

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let L be any language on a nonempty alphabet. Show that L and \overline{L} cannot both be finite.

Proof. Suppose that both L and \overline{L} are finite. Therefore, $L \cup \overline{L}$ is also finite. However, $L \cup \overline{L} = \Sigma^*$ is an infinite set, which leads to a contradiction. Thus, L and \overline{L} cannot both be finite.

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8. Are there languages for which $\overline{L^*} = (\overline{L})^*$?

Solution. No, because

$$\begin{aligned}\overline{L^*} &= \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \dots), \\ (\overline{L})^* &= (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \dots.\end{aligned}$$

Thus, $\lambda \notin \overline{L^*}$ and $\lambda \in (\overline{L})^*$. There is no language for which $\overline{L^*} = (\overline{L})^*$.

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2 .

Proof. $\forall u \in L_1$ and $\forall v \in L_2$, then

$$uv \in L_1 L_2 \Rightarrow (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \Rightarrow v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2 . ■

10. Show that $(L^*)^* = L^*$ for all languages.

Proof. It is obvious that

$$(L^*)^n = \{w_1 w_2 \cdots w_n : w_1, w_2, \dots, w_n \in L^*\},$$

where $n \in \mathbb{N}$. Because

$$L^* = L^0 \cup L^1 \cup L^2 \cup \dots$$

Suppose that $w_1 \in L^{m_1}, w_2 \in L^{m_2}, \dots, w_n \in L^{m_n}$ where $0 \leq m_1, m_2, \dots, m_n \leq n$ and $m_1, m_2, \dots, m_n \in \mathbb{N}$. Therefore,

$$w_1 w_2 \cdots w_n \in L^{m_1 + m_2 + \dots + m_n}.$$

However,

$$L^{m_1+m_2+\dots+m_n} \in L^*.$$

Thus, $(L^*)^* = L^*$ for all languages.

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11. Prove or disprove the following claims.

(a) $(L_1 \cup L_2)^R = L_1^R \cup L_2^R$ for all languages L_1 and L_2 .

(b) $(L^R)^* = (L^*)^R$ for all languages L .

Proof.

(a)

$$\begin{aligned} L_1 \cup L_2 &= \{w : w \in L_1 \text{ or } w \in L_2\}, \\ (L_1 \cup L_2)^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}, \\ L_1^R \cup L_2^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}. \end{aligned}$$

Thus,

$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R.$$

(b) It is obvious that

$$(L^R)^0 = \lambda = (\lambda)^R = (L^0)^R.$$

$$\forall n \in \mathbb{N}^+$$

$$\begin{aligned} (L^n)^R &= \{(w_1 w_2 \dots w_n)^R : w_1, w_2, \dots, w_n \in L\} \\ &= \{w_n^R \dots w_2^R w_1^R : w_1, w_2, \dots, w_n \in L\} \\ &= \{w_n^R \dots w_2^R w_1^R : w_1^R, w_2^R, \dots, w_n^R \in L^R\} \\ &= (L^R)^n. \end{aligned}$$

Thus,

$$\begin{aligned} (L^R)^* &= (L^R)^0 \cup (L^R)^1 \cup (L^R)^2 \cup \dots \\ &= (L^0)^R \cup (L^1)^R \cup (L^2)^R \cup \dots \\ &= (L^*)^R. \end{aligned}$$



12. Find a grammar for the language $L = \{a^n, \text{ where } n \text{ is even}\}$.

Solution. The grammar is $G = (\{S\}, \{a\}, S, P)$ with P given by

$$S \rightarrow aaS | \lambda.$$

13. Find a grammar for the language $L = \{a^n, \text{ where } n \text{ is even and } n > 3\}$.

Solution. The grammar is $G = (\{S, A\}, \{a\}, S, P)$ with P given by

$$S \rightarrow aaaaA,$$

$$A \rightarrow aaA | \lambda.$$

14. Find grammars for $\Sigma = \{a, b\}$ that generate the sets of

- (a) all strings with exactly two a 's.
- (b) all strings with at least two a 's.
- (c) all strings with no more than three a 's.
- (d) all strings with at least three a 's.
- (e) all strings that start with a and end with b .
- (f) all strings with an even number of b 's.

In each case, give convincing arguments that the grammar you give does indeed generate the indicated language.

Solution.

- (a) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow AaAaA,$$

$$A \rightarrow bA | \lambda.$$

(b) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$\begin{aligned} S &\rightarrow AaAaA, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(c) The grammar is $G = (\{S, A, B, C\}, \{a, b\}, S, P)$ with P given by

$$\begin{aligned} S &\rightarrow aA|bS|\lambda, \\ A &\rightarrow aB|bA|\lambda, \\ B &\rightarrow aC|bB|\lambda, \\ C &\rightarrow bC|\lambda. \end{aligned}$$

(d) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$\begin{aligned} S &\rightarrow AaAaAaA, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(e) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$\begin{aligned} S &\rightarrow aAb, \\ A &\rightarrow aA|bA|\lambda. \end{aligned}$$

(f) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$\begin{aligned} S &\rightarrow SbSbS|A|\lambda, \\ A &\rightarrow aA|\lambda. \end{aligned}$$

15. Give a simple description of the language generated by the grammar with productions

$$\begin{aligned} S &\rightarrow aaA, \\ A &\rightarrow bS, \\ S &\rightarrow \lambda. \end{aligned}$$

Solution. The language generated by this grammar is

$$L = \{(aab)^n : n \geq 0\}.$$

16. What language does the grammar with these productions generate?

$$S \rightarrow Aa,$$

$$A \rightarrow B,$$

$$B \rightarrow Aa.$$

Solution. The language generated by this grammar is

$$L = \{Aa^n : n \geq 1\}.$$

17. Let $\Sigma = \{a, b\}$. For each of the following languages, find a grammar that generates it.

(a) $L_1 = \{a^n b^m : n \geq 0, m < n\}.$

(b) $L_2 = \{a^{3n} b^{2n} : n \geq 2\}.$

(c) $L_3 = \{a^{n+3} b^n : n \geq 2\}.$

(d) $L_4 = \{a^n b^{n-2} : n \geq 3\}.$

(e) $L_1 L_2.$

(f) $L_1 \cup L_2.$

(g) $L_1^3.$

(h) $L_1^*.$

(i) $L_1 - \overline{L_4}$

Solution.

(a) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(b) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aaaaaaAbbbb,$$

$$A \rightarrow aaaAbb|\lambda.$$

(c) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aaaaaAbb,$$

$$A \rightarrow aAb|\lambda.$$

(d) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aaaAb,$$

$$A \rightarrow aAb|\lambda.$$

(e) The grammar is $G = (\{S, A, B\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aAaaaaaBbbbb|aaaaaBbbbb,$$

$$A \rightarrow aA|aAb|\lambda,$$

$$B \rightarrow aaaBbb|\lambda.$$

(f) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(g) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aAaAaA,$$

$$A \rightarrow aA|aAb|\lambda.$$

(h) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aAS|\lambda,$$

$$A \rightarrow aA|aAb|\lambda.$$

(i) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aaaAb,$$

$$A \rightarrow aAb|\lambda.$$

18. Find grammars for the following languages on $\Sigma = \{a\}$.

(a) $L = \{w : |w| \bmod 3 > 0\}.$

(b) $L = \{w : |w| \bmod 3 = 2\}.$

(c) $w = \{|w| \bmod 5 = 0\}.$

Solution.

(a) The grammar is $G = (\{S, A\}, \{a\}, S, P)$ with P given by

$$S \rightarrow aA|aaA,$$

$$A \rightarrow aaaA|\lambda.$$

(b) The grammar is $G = (\{S, A\}, \{a\}, S, P)$ with P given by

$$S \rightarrow aaA,$$

$$A \rightarrow aaaA|\lambda.$$

(c) The grammar is $G = (\{S\}, \{a\}, S, P)$ with P given by

$$S \rightarrow aaaaaS|\lambda.$$

19. Find a grammar that generates the language

$$L = \{ww^R : w \in \{a, b\}^+\}.$$

Give a complete justification for your answer.

Solution. The grammar G is $G = (\{S\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aSa|bSb|\lambda.$$

20. Find three strings in the language generated by

$$S \rightarrow aSa|bSa|a.$$

Solution. Three strings are shown below:

$$w_1 = a, \quad w_2 = aab, \quad w_3 = baa.$$

21. Consider the grammar $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \rightarrow aAb|\lambda,$$

$$A \rightarrow aAb|\lambda.$$

Prove that

$$L(G) = \{a^n b^n : n \geq 0\}.$$

Proof.

(a) We should prove that every $w \in L(G)$ can be derived from S using G .

1. Basis

If $n = 0$, then $S \Rightarrow \lambda$.

If $n = 1$, then $S \Rightarrow aAb \Rightarrow ab$.

2. Inductive Assumption

Assume that for $i = 1, 2, \dots, n$, then $S \xRightarrow{*} a^i A b^i \Rightarrow a^i b^i$.

3. Inductive Step

For $i = n + 1$, $S \xRightarrow{*} a^n A b^n \Rightarrow a^{n+1} A b^{n+1} \Rightarrow a^{n+1} b^{n+1}$.

Thus, $w \in L(G)$ can be derived from S using G .

(b) We should prove that every string so derived is in $L(G)$. Suppose that there are n steps from S to a string.

1. Basis

If $n = 1$, then $S \Rightarrow \lambda \in L(G)$.

If $n = 2$, then $S \Rightarrow aAb \Rightarrow ab \in L(G)$.

2. Inductive Assumption

Assume that for $i = 2, \dots, n$, then $S \xRightarrow{*} a^{i-1} A b^{i-1} \Rightarrow a^i b^i \in L(G)$.

3. Inductive Step

For $i = n + 1$, $S \xRightarrow{*} a^n A b^n \Rightarrow a^n b^n$.

Thus, every string so derived is in $L(G)$.

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22. Show that the grammar $G_1 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow SS|SSS|aSb|bSa|\lambda,$$

is equivalent to the grammar $G_2 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow SS|aSb|bSa|\lambda.$$

Proof.

(a) For every string obtained by the grammar G_1 , if there is any step of

$$S \Rightarrow SSS$$

in the derivation process, we replace it with two steps of

$$S \Rightarrow SS \Rightarrow SSS$$

with the grammar G_2 . For other steps in the derivation process, we copy them as the same steps with the grammar G_2 . Therefore, every string obtained by the grammar G_1 can also be obtained by the same steps in the grammar G_2 .

(b) Every string obtained by the grammar G_2 can also be obtained by the same steps in the grammar G_1 .

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23. Show that the grammar $G_1 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow aSb|ab|\lambda,$$

is equivalent to the grammar $G_2 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow aaSbb|aSb|ab|\lambda.$$

Proof.

(a) Every string obtained by the grammar G_1 can also be obtained by the same steps in the grammar G_2 .

(b) For every string obtained by the grammar G_2 , if there is any step of

$$S \Rightarrow aaSbb$$

in the derivation process, we replace it with two steps of

$$S \Rightarrow aSb \Rightarrow aaSbb$$

with the grammar G_1 . For other steps in the derivation process, we copy them as the same steps with the grammar G_1 . Therefore, every string obtained by the grammar G_2 can also be obtained by the same steps in the grammar G_1 . ■

24. Show that the grammar $G_1 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow aSb|bSa|SS|a,$$

is equivalent to the grammar $G_2 = (\{S\}, \{a, b\}, S, P)$ with productions

$$S \rightarrow aSb|bSa|a.$$

Proof. The grammar G_1 can derive the string aa :

$$S \Rightarrow SS \Rightarrow aS \Rightarrow aa.$$

However, the string aa can not be derived by the grammar G_2 . Because every step of the grammar will produce the same number of a and b or one a to end the derivation process. The string derived by the grammar G_2 can not have two more a than b . ■