Chapter 1 Section 2 Exercises

1. How many substrings aab are in ww^Rw, where $w = aabbab$.

Solution. Because

$$w^R = babbaa, \qquad ww^Rw = \underline{aab}babbabbaa\underline{aab}bab,$$

there are two substrings aab in ww^Rw.

2. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n.

Proof.

1. **Basis**

   If $n = 0$, then

   $$|u^1| = |u|, \qquad 1 \cdot |u| = |u|.$$

   Therefore,

   $$|u^1| = 1 \cdot |u|.$$

2. **Inductive Assumption**

   Assume that for $i = 0, 1, 2, \cdots, n$

   $$|u^i| = i \cdot |u|.$$

3. **Inductive Step**

   For $i = n + 1$,

   $$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n+1)|u|.$$

Thus, $|u^n| = n|u|$ for all strings u and all n.

■

3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$a^R = a,$$

$$(wa)^R = aw^R,$$

for all $a \in \Sigma, w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

Proof.

1. **Basis**

   For $|v| = 1$, suppose that $v = a, a \in \Sigma$

   $$(uv)^R = (ua)^R = au^R = v^R u^R.$$

2. **Inductive Assumption**

   Suppose that for $|v| = 1, 2, \cdots, n$

   $$(uv)^R = v^R u^R.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $v = wa, a \in \Sigma$

   $$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

■

4. Prove that $(w^R)^R = w$ for all $w \in \Sigma^*$.

Proof.

1. **Basis**

   For $|w| = 0$, which means that $w = \lambda$,

   $$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

2. **Inductive Assumption**

   Suppose that for $|w| = 1, 2, \cdots, n$

$$(w^R)^R = w.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus, $(w^R)^R = w$ for all $w \in \Sigma^*$.

■

5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^*: $abaabaaabaa$, $aaaabaaaa$, $baaaaabaaaab$, $baaaaabaa$? Which strings are in L^4?

Solution.

$$abaabaaabaa = ab|aa|baa|ab|aa \in L^5;$$
$$aaaabaaaa = aa|aa|baa|aa \in L^4;$$
$$baaaaabaaaab \notin L^*;$$
$$baaaaabaa = baa|aa|ab|aa \in L^4.$$

6. Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe $\overline{L}$.

Solution.

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let L be any language on a nonempty alphabet. Show that L and $\overline{L}$ cannot both be finite.

Proof. Suppose that both L and $\overline{L}$ are finite. Therefore, $L \cup \overline{L}$ is also finite. However, $L \cup \overline{L} = \Sigma^*$ is an infinite set, which leads to a contradiction. Thus, L and $\overline{L}$ cannot both be finite.

■

8. Are there languages for which $\overline{L^*} = (\overline{L})^*$?

Solution. No, because

$$\begin{aligned}\overline{L^*} &= \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \cdots), \\ (\overline{L})^* &= (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \cdots .\end{aligned}$$

Thus, $\lambda \notin \overline{L^*}$ and $\lambda \in (\overline{L})^*$. There is no language for which $\overline{L^*} = (\overline{L})^*$.

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

Proof. $\forall\, u \in L_1$ and $\forall\, v \in L_2$, then

$$uv \in L_1 L_2 \quad \Rightarrow \quad (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \quad \Rightarrow \quad v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

■

10. Show that $(L^*)^* = L^*$ for all languages.

Proof. $\forall\, i \in \mathbb{N}^+$, we need to prove that $(L^*)^i = L^*$.

1. **Basis**

   For $i = 0$, which means that $(L^*)^0 = \lambda$, so $(L^*)^0 = \lambda$.

   For $i = 1$, $(L^*)^1 = L^*$.

   For $i = 2$,

$$\begin{aligned}(L^*)^2 &= L^* L^* \\ &= L^*(L^0 \cup L^1 \cup L^2 \cup \cdots) \\ &= L^* L^0 \cup L^* L^1 \cup L^* L^2 \cup \cdots \\ &= L^*.\end{aligned}$$

2. **Inductive Assumption**

   Suppose that for $i = 1, 2, \cdots, n$
$$(L^*)^i = L^*.$$

3. **Inductive Step**

   For $i = n + 1$,
$$(L^*)^{n+1} = (L^*)^n L^* = L^* L^* = (L^*)^2 = L^*.$$

To sum up,

$$(L^*)^* = (L^*)^0 \cup (L^*)^1 \cup (L^*)^2 \cup \cdots = \lambda \cup L^* \cup L^* \cup \cdots = L^*.$$

■

11. Prove or disprove the following claims.

(a) $(L_1 \cup L_2)^R = L_1^R \cup L_2^R$ for all languages L_1 and L_2.

(b) $(L^R)^* = (L^*)^R$ for all languages L.

Proof.

(a)

$$L_1 \cup L_2 = \{w : w \in L_1 \text{ or } w \in L_2\},$$
$$(L_1 \cup L_2)^R = \{w^R : w \in L_1 \text{ or } w \in L_2\},$$
$$L_1^R \cup L_2^R = \{w^R : w \in L_1 \text{ or } w \in L_2\}.$$

Thus,
$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R.$$

(b)

■