Chapter 1 Section 2 Exercises

1. How many substrings aab are in ww^Rw, where $w = aabbab$.

Solution. Because

$$w^R = babbaa, \qquad ww^Rw = \underline{aab}babbabbaa\underline{aab}bab,$$

there are two substrings aab in ww^Rw.

2. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n.

Proof.

1. **Basis**

   If $n = 0$, then
   $$|u^1| = |u|, \qquad 1 \cdot |u| = |u|.$$
   Therefore,
   $$|u^1| = 1 \cdot |u|.$$

2. **Inductive Assumption**

   Assume that for $i = 0, 1, 2, \cdots, n$
   $$|u^i| = i \cdot |u|.$$

3. **Inductive Step**

   For $i = n + 1$,
   $$|u^{n+1}| = |u^n u| = |u^n| + |u| = n \cdot |u| + |u| = (n+1)|u|.$$

Thus, $|u^n| = n|u|$ for all strings u and all n.

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3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$a^R = a,$$
$$(wa)^R = aw^R,$$

for all $a \in \Sigma, w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

Proof.

1. **Basis**

   For $|v| = 1$, suppose that $v = a, a \in \Sigma$

   $$(uv)^R = (ua)^R = au^R = v^R u^R.$$

2. **Inductive Assumption**

   Suppose that for $|v| = 1, 2, \cdots, n$

   $$(uv)^R = v^R u^R.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $v = wa, a \in \Sigma$

   $$(uv)^R = (uwa)^R = a(uw)^R = aw^R u^R = v^R u^R.$$

Thus,

$$(uv)^R = v^R u^R,$$

for all $u, v \in \Sigma^+$.

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4. Prove that $(w^R)^R = w$ for all $w \in \Sigma^*$.

Proof.

1. **Basis**

   For $|w| = 0$, which means that $w = \lambda$,

   $$(\lambda^R)^R = (\lambda)^R = \lambda = w.$$

2. **Inductive Assumption**

   Suppose that for $|w| = 1, 2, \cdots, n$

$$(w^R)^R = w.$$

3. **Inductive Step**

   For $i = n + 1$, suppose that $w = va, a \in \Sigma$

$$(w^R)^R = ((va)^R)^R = (av^R)^R = (v^R)^R a = va = w.$$

Thus, $(w^R)^R = w$ for all $w \in \Sigma^*$.

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5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^*: $abaabaaabaa$, $aaaabaaaa$, $baaaaabaaaab$, $baaaaabaa$? Which strings are in L^4?

Solution.

$$abaabaaabaa = ab|aa|baa|ab|aa \in L^5;$$

$$aaaabaaaa = aa|aa|baa|aa \in L^4;$$

$$baaaaabaaaab \notin L^*;$$

$$baaaaabaa = baa|aa|ab|aa \in L^4.$$

6. Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe $\overline{L}$.

Solution.

$$\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}.$$

7. Let L be any language on a nonempty alphabet. Show that L and $\overline{L}$ cannot both be finite.

Proof. Suppose that both L and $\overline{L}$ are finite. Therefore, $L \cup \overline{L}$ is also finite. However, $L \cup \overline{L} = \Sigma^*$ is an infinite set, which leads to a contradiction. Thus, L and $\overline{L}$ cannot both be finite.

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8. Are there languages for which $\overline{L^*} = (\overline{L})^*$?

Solution. No, because

$$\overline{L^*} = \Sigma^* - (L^0 \cup L^1 \cup L^2 \cup \cdots),$$
$$(\overline{L})^* = (\Sigma^* - L)^0 \cup (\Sigma^* - L)^1 \cup (\Sigma^* - L)^2 \cup \cdots .$$

Thus, $\lambda \notin \overline{L^*}$ and $\lambda \in (\overline{L})^*$. There is no language for which $\overline{L^*} = (\overline{L})^*$.

9. Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

Proof. $\forall\, u \in L_1$ and $\forall\, v \in L_2$, then

$$uv \in L_1 L_2 \quad \Rightarrow \quad (uv)^R \in (L_1 L_2)^R.$$

In addition,

$$(uv)^R = v^R u^R, \quad u^R \in L_1^R, \quad v^R \in L_2^R \quad \Rightarrow \quad v^R u^R \in L_2^R L_1^R.$$

Therefore,

$$(L_1 L_2)^R = L_2^R L_1^R$$

for all languages L_1 and L_2.

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10. Show that $(L^*)^* = L^*$ for all languages.

Proof. It is obvious that

$$(L^*)^n = \{w_1 w_2 \cdots w_n : w_1, w_2, \cdots, w_n \in L^*\},$$

where $n \in \mathbb{N}$. Because

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots$$

Suppose that $w_1 \in L^{m_1}, w_2 \in L^{m_2}, \cdots, w_2 \in L^{m_n}$ where $0 \leq m_1, m_2, \cdots, m_n \leq n$ and $m_1, m_2, \cdots, m_n \in \mathbb{N}$. Therefore,

$$w_1 w_2 \cdots w_n \in L^{m_1 + m_2 + \cdots + m_n}.$$

However,

$$L^{m_1+m_2+\cdots+m_n} \in L^*.$$

Thus, $(L^*)^* = L^*$ for all languages.

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11. Prove or disprove the following claims.

(a) $(L_1 \cup L_2)^R = L_1^R \cup L_2^R$ for all languages L_1 and L_2.

(b) $(L^R)^* = (L^*)^R$ for all languages L.

Proof.

(a)

$$\begin{aligned} L_1 \cup L_2 &= \{w : w \in L_1 \text{ or } w \in L_2\}, \\ (L_1 \cup L_2)^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}, \\ L_1^R \cup L_2^R &= \{w^R : w \in L_1 \text{ or } w \in L_2\}. \end{aligned}$$

Thus,

$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R.$$

(b) It is obvious that

$$(L^R)^0 = \lambda = (\lambda)^R = (L^0)^R.$$

$\forall\, n \in \mathbb{N}^+$

$$\begin{aligned} (L^n)^R &= \{(w_1 w_2 \cdots w_n)^R : w_1, w_2, \cdots, w_n \in L\} \\ &= \{w_n^R \cdots w_2^R w_1^R : w_1, w_2, \cdots, w_n \in L\} \\ &= \{w_n^R \cdots w_2^R w_1^R : w_1^R, w_2^R, \cdots, w_n^R \in L^R\} \\ &= (L^R)^n. \end{aligned}$$

Thus,

$$\begin{aligned} (L^R)^* &= (L^R)^0 \cup (L^R)^1 \cup (L^R)^2 \cup \cdots \\ &= (L^0)^R \cup (L^1)^R \cup (L^2)^R \cup \cdots \\ &= (L^*)^R. \end{aligned}$$

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12. Find a grammar for the language $L = \{a^n, \text{where } n \text{ is even}\}$.

Solution. The grammar is $G = (\{S\}, \{a\}, S, P)$ with P given by

$$S \to aaS|\lambda.$$

13. Find a grammar for the language $L = \{a^n, \text{where } n \text{ is even and } n > 3\}$.

Solution. The grammar is $G = (\{S, A\}, \{a\}, S, P)$ with P given by

$$S \to aaaaA,$$
$$A \to aaA|\lambda.$$

14. Find grammars for $\Sigma = \{a, b\}$ that generate the sets of

(a) all strings with exactly two a's.

(b) all strings with at least two a's.

(c) all strings with no more than three a's.

(d) all strings with at least three a's.

(e) all strings that start with a and end with b.

Solution.

(a) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to AaAaA,$$
$$A \to bA|\lambda.$$

(b) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to AaAaA,$$
$$A \to aA|bA|\lambda.$$

(c) The grammar is $G = (\{S, A, B, C\}, \{a, b\}, S, P)$ with P given by

$$S \to aA|bS|\lambda,$$

$$A \to aB|bA|\lambda,$$

$$B \to aC|bB|\lambda,$$

$$C \to bC|\lambda.$$

(d) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to AaAaAaA,$$

$$A \to aA|bA|\lambda.$$

(e) The grammar is $G = (\{S, A\}, \{a, b\}, S, P)$ with P given by

$$S \to aAb,$$

$$A \to aA|bA|\lambda.$$