

# Chapter 1      Section 1      Exercises

1. With  $S_1 = \{2, 3, 5, 7\}$ ,  $S_2 = \{2, 4, 5, 8, 9\}$ , and  $U = \{1 : 10\}$ , compute  $\overline{S_1} \cup S_2$ .

**Solution.**

$$\overline{S_1} = \{1, 4, 6, 8, 9, 10\} \quad \Rightarrow \quad \overline{S_1} \cup S_2 = \{1, 2, 4, 5, 6, 8, 9, 10\}.$$

2. With  $S_1 = \{2, 3, 5, 7\}$ ,  $S_2 = \{2, 4, 5, 8, 9\}$ , compute  $S_1 \times S_2$  and  $S_2 \times S_1$ .

**Solution.**

$$\begin{aligned} S_1 \times S_2 = \{ & (2, 2), (2, 4), (2, 5), (2, 8), (2, 9), \\ & (3, 2), (3, 4), (3, 5), (3, 8), (3, 9), \\ & (5, 2), (5, 4), (5, 5), (5, 8), (5, 9), \\ & (7, 2), (7, 4), (7, 5), (7, 8), (7, 9) \}. \\ S_2 \times S_1 = \{ & (2, 2), (2, 3), (2, 5), (2, 7), \\ & (4, 2), (4, 3), (4, 5), (4, 7), \\ & (5, 2), (5, 3), (5, 5), (5, 7), \\ & (8, 2), (8, 3), (8, 5), (8, 7), \\ & (9, 2), (9, 3), (9, 5), (9, 7) \}. \end{aligned}$$

3. For  $S = \{2, 5, 6, 8\}$  and  $T = \{2, 4, 6, 8\}$ , compute  $|S \cap T| + |S \cup T|$ .

**Solution.**

$$S \cap T = \{2, 6, 8\}, \quad S \cup T = \{2, 4, 5, 6, 8\} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets  $S$  and  $T$  must hold so that  $|S \cup T| = |S| + |T|$ .

**Solution.**

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \emptyset.$$

Therefore,  $S$  and  $T$  are disjoint.

5. Show that for all sets  $S$  and  $T$ ,  $S - T = S \cap \overline{T}$ .

**Proof.**

$$\begin{aligned} S - T &= \{x : x \in S \text{ and } x \notin T\} \\ \iff S - T &= \{x : x \in S \text{ and } x \in \overline{T}\} \\ \iff S - T &= S \cap \overline{T}. \end{aligned}$$

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6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element  $x$  is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

**Proof.**

$$S_1 \cup S_2 = \{x : x \in S_1 \text{ or } x \in S_2\} \Rightarrow \overline{S_1 \cup S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$

$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \Rightarrow \overline{S_1} \cap \overline{S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$S_1 \cap S_2 = \{x : x \in S_1 \text{ and } x \in S_2\} \Rightarrow \overline{S_1 \cap S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$

$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \Rightarrow \overline{S_1} \cup \overline{S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

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7. Show that if  $S_1 \subseteq S_2$ , then  $\overline{S_2} \subseteq \overline{S_1}$ .

**Proof.**

$$S_1 \subseteq S_2$$

$$\Rightarrow (\in S_1 \Rightarrow x \in S_2)$$

$$\Rightarrow (x \notin S_2 \Rightarrow x \notin S_1)$$

$$\Rightarrow (x \in \overline{S_2} \Rightarrow x \in \overline{S_1})$$

$$\Rightarrow \overline{S_2} \subseteq \overline{S_1}.$$

■

8. Show that  $S_1 = S_2$  if and only if  $S_1 \cup S_2 = S_1 \cap S_2$ .

**Proof.**

$$1. S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$\left. \begin{array}{l} S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \Rightarrow S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{array} \right\} \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$2. S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2.$$

Assume that  $S_1 \cup S_2 = S_1 \cap S_2$  and  $S_1 \neq S_2$ ,

- $\exists x \in S_1 \text{ and } x \notin S_2 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$
- $\exists x \in S_2 \text{ and } x \notin S_1 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$

The result contradicts with the premise. Therefore,  $S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2.$

To sum up,  $S_1 = S_2$  if and only if  $S_1 \cup S_2 = S_1 \cap S_2.$

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**9.** Use induction on the size of  $S$  to show that if  $S$  is a finite set, then  $|2^S| = 2^{|S|}.$

**Proof.**

**1. Basis**

If  $|S| = 0, S = \emptyset.$  Then

$$2^S = \{\emptyset\}.$$

Therefore,  $|2^S| = 2^{|S|} = 1.$

If  $|S| = 1,$  assume that  $S = \{a\}.$  Then

$$2^S = \{\emptyset, \{a\}\}.$$

Therefore,  $|2^S| = 2^{|S|} = 2.$

**2. Inductive Assumption**

Assume that  $|2^S| = 2^{|S|},$  for  $|S| = 1, 2, \dots, n.$

**3. Inductive Step**

For  $|S| = n + 1,$  assume that  $S = \{a_1, a_2, \dots, a_n, a_{n+1}\}.$  Let  $T = \{a_1, a_2, \dots, a_n\},$  then

$$2^T = \{T_1, T_2, \dots, T_{2^n}\}.$$

For  $\forall i = 1, 2, \dots, 2^n$  where  $i \in \mathbb{N}^*$

$$\left. \begin{array}{l} T_i \subseteq T \\ T \subseteq S \end{array} \right\} \Rightarrow T_i \subseteq S.$$

However,

$$S - T = \{a_{n+1}\} \Rightarrow a_{n+1} \notin T \Rightarrow a_{n+1} \notin T_i.$$

In addition

$$\left. \begin{array}{l} T_i \subseteq S \\ a_{n+1} \in S_i \Rightarrow \{a_{n+1}\} \subseteq S \end{array} \right\} \Rightarrow T_i \cup \{a_{n+1}\} \subseteq S.$$

Let

$$T'_i = T_i \cup \{a_{n+1}\}, \quad U = \{T'_1, T'_2, \dots, T'_{2^n}\}.$$

Now, for  $\forall S_i \subseteq S$

- If  $a_{n+1} \notin S_i,$  then  $S_i \subseteq T,$  so  $S_i \in 2^T.$

- If  $a_{n+1} \in S_i$ , then  $S_i - \{a_{n+1}\} \subseteq T$ , so  $S_i - \{a_{n+1}\} \in 2^T$ . Assume that

$$S_i - \{a_{n+1}\} = T_j \Rightarrow S_i = T_j \cup \{a_{n+1}\} \Rightarrow S_i \in U.$$

Moreover,  $2^T$  and  $U$  are disjoint. Therefore,

$$2^S = 2^T \cup U, \quad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if  $S$  is a finite set, then  $|2^S| = 2^{|S|}$ .

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**10.** Show that if  $S_1$  and  $S_2$  are finite sets with  $|S_1| = n$  and  $|S_2| = m$ , then

$$|S_1 \cup S_2| \leq n + m.$$

**Proof.** Assume that

$$S_1 = \{a_1, a_2, \dots, a_n\}, \quad S_2 = \{b_1, b_2, \dots, b_m\}.$$

1.  $S_1$  and  $S_2$  are disjoint. Then

$$S_1 \cup S_2 = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2.  $S_1$  and  $S_2$  are not disjoint. Assume that

$$c_1, c_2, \dots, c_k \in S_1 \text{ and } c_1, c_2, \dots, c_k \in S_2.$$

where  $k \leq n$ ,  $k \leq m$ ,  $k \in \mathbb{N}^*$ . Assume that

$$b_{i_1} = c_1, b_{i_2} = c_2, \dots, b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{i_1-1}, b_{i_1+1}, \dots, b_{i_k-1}, b_{i_k+1}, \dots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if  $S_1$  and  $S_2$  are finite sets with  $|S_1| = n$  and  $|S_2| = m$ , then

$$|S_1 \cup S_2| \leq n + m.$$

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**11.** If  $S_1$  and  $S_2$  are finite sets, show that  $|S_1 \times S_2| = |S_1||S_2|$ .

**Proof.** Assume that  $S_1 = \emptyset$  or  $S_2 = \emptyset$ , then

$$S_1 \times S_2 = \emptyset \Rightarrow |S_1 \times S_2| = 0, |S_1||S_2| = 0 \times 0 = 0 \Rightarrow |S_1 \times S_2| = |S_1||S_2|.$$

Assume that  $S_1 \neq \emptyset$  and  $S_2 \neq \emptyset$ ,

$$S_1 = \{a_1, a_2, \dots, a_n\}, \quad S_2 = \{b_1, b_2, \dots, b_m\}.$$

where  $n, m \in \mathbb{N}^*$ .

Therefore,

$$\begin{aligned} S_1 \times S_2 = \{ & (a_1, b_1), (a_2, b_1), \dots, (a_n, b_1), \\ & (a_1, b_2), (a_2, b_2), \dots, (a_n, b_2), \\ & \vdots \\ & (a_1, b_m), (a_2, b_m), \dots, (a_n, b_m) \}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

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**12.** Consider the relation between two sets defined by  $S_1 \equiv S_2$  if and only if  $|S_1| = |S_2|$ . Show that this is an equivalence relation.

**Proof.**

1. **Reflexivity**

$$|S_1| = |S_1| \text{ for all } S_1. \Rightarrow S_1 \equiv S_1 \text{ for all } S_1.$$

2. **Symmetry**

$$\text{if } |S_1| = |S_2|, \text{ then } |S_2| = |S_1|. \Rightarrow \text{if } S_1 \equiv S_2, \text{ then } S_2 \equiv S_1.$$

3. **Transitivity**

$$\text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|, \text{ then } |S_1| = |S_3|.$$

$$\Downarrow$$

$$\text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3, \text{ then } S_1 \equiv S_3.$$

Therefore, this is an equivalence relation.

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**13.** Occasionally, we need to use the union and intersection symbols in a manner analogous to the summation sign  $\sum$ . We define

$$\bigcup_{p \in \{i, j, k, \dots\}} S_p = S_i \cup S_j \cup S_k \dots$$

with an analogous notation for the intersection of several sets.

With this notation, the general DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when  $P$  is a finite set.

**Proof.**

### 1. Basis

For  $|P| = 2$ , according to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}, \quad \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

### 2. Inductive Assumption

For  $|P| = 2, 3, \dots, n$  where  $n \in \mathbb{N}^*$

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}, \quad \overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

### 3. Inductive Step

For  $|P| = n + 1$  where  $n \in \mathbb{N}^*$ ,  $\forall i \in P$ ,  $|P - \{i\}| = n$ ,

$$\begin{aligned} \overline{\bigcup_{p \in P} S_p} &= \overline{\left( \bigcup_{p \in P - \{i\}} S_p \right) \cup S_i} = \overline{\left( \bigcup_{p \in P - \{i\}} S_p \right) \cap \overline{S_i}} = \left( \bigcap_{p \in P - \{i\}} \overline{S_p} \right) \cap \overline{S_i} = \bigcap_{p \in P} \overline{S_p}, \\ \overline{\bigcap_{p \in P} S_p} &= \overline{\left( \bigcap_{p \in P - \{i\}} S_p \right) \cap S_i} = \overline{\left( \bigcap_{p \in P - \{i\}} S_p \right) \cup \overline{S_i}} = \left( \bigcup_{p \in P - \{i\}} \overline{S_p} \right) \cup \overline{S_i} = \bigcup_{p \in P} \overline{S_p}. \end{aligned}$$

Therefore, for  $|P| = 2, 3, \dots$

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}, \quad \overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

■

### 14. Show that

$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

**Proof.** According to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \Rightarrow \overline{\overline{S_1 \cup S_2}} = \overline{\overline{S_1} \cap \overline{S_2}} \Rightarrow S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

■

### 15. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

**Proof.**

$$1. S_1 = S_2 \Rightarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

$$S_1 = S_2 \Rightarrow \left\{ \begin{array}{l} S_1 \cap \overline{S_2} = S_1 \cap \overline{S_1} = \emptyset \\ \overline{S_1} \cap S_2 = \overline{S_1} \cap S_2 = \emptyset \end{array} \right\} \Rightarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

$$2. S_1 = S_2 \Leftarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

Assume that  $S_1 \neq S_2$ ,

- $\exists x \in S_1$  and  $x \notin S_2 \Rightarrow x \in S_1 \cap \overline{S_2} \Rightarrow x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$
- $\exists x \notin S_1$  and  $x \in S_2 \Rightarrow x \in \overline{S_1} \cap S_2 \Rightarrow x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$

Therefore,  $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \emptyset$ , which is a contradiction. Thus  $S_1 = S_2$ .

To sum up,

$$S_1 = S_2 \iff (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

■

**16. Show that**

$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

***Proof.***

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup \overline{\overline{S_2}}) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2). \end{aligned}$$

1. If  $x \in S_2$

$$x \in S_2 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \Rightarrow x \in (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

2. If  $x \notin S_2$  and  $x \in S_1$

$$x \notin S_2 \text{ and } x \in S_1 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin \overline{S_1} \cup S_2 \Rightarrow x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

3. If  $x \notin S_2$  and  $x \notin S_1$

$$x \notin S_2 \text{ and } x \notin S_1 \Rightarrow x \notin S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \Rightarrow x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

To sum up

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= S_2. \end{aligned}$$

■

17. Show that the distributive law

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.

**Proof.**

1. If  $x \notin S_1$

$$x \notin S_1 \Rightarrow \begin{cases} x \notin S_1 \cap (S_2 \cup S_3) \\ x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \Rightarrow x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

2. If  $x \in S_1$ ,  $x \notin S_2$  and  $x \notin S_3$

$$\begin{aligned} x \in S_1, x \notin S_2 \text{ and } x \notin S_3 &\Rightarrow x \notin S_2 \cup S_3 \Rightarrow x \notin S_1 \cap (S_2 \cup S_3). \\ x \in S_1, x \notin S_2 \text{ and } x \notin S_3 &\Rightarrow x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \\ &\Rightarrow x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

3. If  $x \in S_1$  and  $x \in S_2$

$$x \in S_1 \text{ and } x \in S_2 \Rightarrow \begin{cases} x \in S_1 \text{ and } x \in S_2 \cup S_3 \Rightarrow x \in S_1 \cap (S_2 \cup S_3) \\ x \in S_1 \cap S_2 \Rightarrow x \in (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

4. If  $x \in S_1$ ,  $x \notin S_2$  and  $x \in S_3$

$$\begin{aligned} x \in S_1, x \notin S_2 \text{ and } x \in S_3 &\Rightarrow x \in S_1 \text{ and } x \in S_2 \cup S_3 \\ &\Rightarrow x \in S_1 \cap (S_2 \cup S_3). \\ x \in S_1, x \notin S_2 \text{ and } x \in S_3 &\Rightarrow x \in S_1 \cap S_3 \\ &\Rightarrow x \in (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

To sum up

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3).$$

■

18. Show that

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

**Proof.** Assume that  $S_1 = \emptyset$ , then

$$\left. \begin{aligned} S_1 \times (S_2 \cup S_3) &= \emptyset \\ S_1 \times S_2 = \emptyset, S_1 \times S_3 = \emptyset &\Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = \emptyset \end{aligned} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Assume that  $S_2 = \emptyset$ , then

$$\left. \begin{aligned} S_2 \cup S_3 = S_3 &\Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_3 \\ S_1 \times S_2 = \emptyset &\Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_3 \end{aligned} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$



Assume that  $S_3 = \emptyset$ , then

$$\left. \begin{array}{l} S_2 \cup S_3 = S_2 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_2 \\ S_1 \times S_3 = \emptyset \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_2 \end{array} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Assume that  $S_1 \neq \emptyset$ ,  $S_2 \neq \emptyset$ ,  $S_3 \neq \emptyset$

$$S_1 = \{a_1, a_2, \dots, a_p\}, \quad S_2 = \{b_1, b_2, \dots, b_q\}, \quad S_3 = \{c_1, c_2, \dots, c_r\}.$$

where  $p, q, r \in \mathbb{N}^*$ .

Then

$$\begin{aligned} S_2 \cup S_3 &= \{b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_r\}. \\ S_1 \times (S_2 \cup S_3) &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q), \\ &\quad (a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \\ S_1 \times S_2 &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q)\} \\ S_1 \times S_3 &= \{(a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \\ (S_1 \times S_2) \cup (S_1 \times S_3) &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q), \\ &\quad (a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \end{aligned}$$

Therefore,

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

■

**19.** Give conditions on  $S_1$  and  $S_2$  necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$

**Solution.**

$$S_1 \cap S_2 = \emptyset \iff S_1 = (S_1 \cup S_2) - S_2.$$

$$1. S_1 \cap S_2 = \emptyset \Rightarrow S_1 = (S_1 \cup S_2) - S_2$$

$$\left. \begin{array}{l} S_1 \cap S_2 = \emptyset \\ S_1 = S_1 \cap U = S_1 \cap (S_2 \cup \overline{S_2}) = (S_1 \cap S_2) \cup (S_1 \cap \overline{S_2}) \end{array} \right\} \Rightarrow S_1 = S_1 \cap \overline{S_2},$$

$$(S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 = (S_1 \cup S_2) - S_2.$$

$$2. S_1 \cap S_2 = \emptyset \Leftarrow S_1 = (S_1 \cup S_2) - S_2$$

$$S_1 = (S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 \cap S_2 = (S_1 \cap \overline{S_2}) \cap S_2 = S_1 \cap (\overline{S_2} \cap S_2) = S_1 \cap \emptyset = S_1.$$

To sum up,

$$S_1 = (S_1 \cup S_2) - S_2.$$

**20.** Use the equivalence defined in Example 1.4 to partition the set  $\{2, 4, 5, 6, 9, 22, 24, 25, 31, 37\}$  into equivalence classes.

**Solution.** Because

$$2 \bmod 3 = 5 \bmod 3 = 2,$$

$$4 \bmod 3 = 22 \bmod 3 = 25 \bmod 3 = 31 \bmod 3 = 37 \bmod 3 = 1,$$

$$6 \bmod 3 = 9 \bmod 3 = 24 \bmod 3 = 0.$$

The equivalence classes are

$$\{2, 5\}, \quad \{4, 22, 25, 31, 37\}, \quad \{6, 9, 24\}.$$

**21.** Show that if  $f(n) = O(g(n))$  and  $g(n) = O(f(n))$ , then  $f(n) = \Theta(g(n))$ .

**Proof.** Assume that

$$f(n) \geq 0, \quad g(n) \geq 0.$$

Because  $f(n) = O(g(n))$ ,  $\exists c_1 > 0$  such that for all sufficiently large  $n_1$

$$|f(n_1)| = f(n) \leq c_1 |g(n_1)|.$$

Because  $g(n) = O(f(n))$ ,  $\exists c_2 > 0$  such that for all sufficiently large  $n_2$

$$|g(n_2)| = g(n) \leq c_2 |f(n_2)|.$$

Let

$$n = \max\{n_1, n_2\}.$$

Therefore

$$\frac{1}{c_2} |g(n)| \leq |f(n)| \leq c_1 |g(n)| \Rightarrow f(n) = \Theta(g(n)).$$

■

**22.** Show that  $2^n = O(3^n)$ , but  $2^n \neq \Theta(3^n)$ .

**Proof.**  $\exists c_1 = 1 > 0$  such that for all  $n \geq 1$

$$2^n \leq c_1 |3^n| = 3^n.$$

Therefore,

$$2^n = O(3^n).$$

However,  $\forall c_2 > 0$ ,  $\exists N = \lceil \log_{\frac{2}{3}} c_2 \rceil + 1$ , if  $n > N$

$$c_2 |3^n| = c_2 3^n > |2^n| = 2^n.$$

Therefore,

$$2^n \neq \Theta(3^n).$$

■

**23.** Show that the following order-of-magnitude results hold.

$$1. \ n^2 + 5 \log n = O(n^2).$$

$$2. \ 3^n = O(n!).$$

$$3. \ n! = O(n^n).$$

**Proof.**

$$1. \ n^2 + 5 \log n = O(n^2).$$

Let

$$f(n) = n^2 + 5 \log n, \quad g(n) = n^2.$$

Let  $c = 2$ , then  $h(n) = f(n) - c|g(n)| = 5 \log n - n^2$ .

$$h'(n) = \frac{5}{n} - 2n \Rightarrow h'(n) \text{ is a monotonically decreasing function.}$$

If  $n \geq 2$ ,  $h'(n) < 0$ , so if  $n \geq 2$ ,  $h(n)$  is a monotonically decreasing function. Because  $h(2) = 5 \log 2 - 4 < 0$ , if  $n \geq 2$ ,  $h(n) < 0$ .  $\exists c = 2 > 0$  such that for all  $n \geq 2$

$$h(n) = f(n) - c|g(n)| = 5 \log n - n^2 < 0 \Rightarrow f(n) \leq c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow n^2 + 5 \log n = O(n^2).$$

$$2. \ 3^n = O(n!).$$

Let

$$f(n) = 3^n, \quad g(n) = n!.$$

$\exists c = 9 > 0$  such that for all  $n \geq 3$

$$f(n) - c|g(n)| = 3^n - 9n! = 3^n \left(1 - \frac{9n!}{3^n}\right) = 3^n \left(1 - 2 \prod_{i=3}^n \frac{i}{3}\right) < 0 \Rightarrow f(n) < c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow 3^n = O(n!).$$

3.  $n! = O(n^n)$ .

Let

$$f(n) = n!, \quad g(n) = n^n.$$

$\exists c = 1 > 0$  such that for all  $n \geq 1$

$$f(n) - c|g(n)| = n! - n^n = n!(1 - \prod_{i=1}^n \frac{n}{i}) \leq 0 \Rightarrow f(n) \leq c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow n! = O(n^n).$$

■

24. Show that  $\frac{n^3 - 2n}{n + 1} = \Theta(n^2)$ .

**Proof.** Let

$$f(n) = \frac{n^3 - 2n}{n + 1}, \quad g(n) = n^2, \quad c_1 = \frac{1}{3}, \quad c_2 = 1.$$

Because

$$f(n) = \frac{n^3 - 2n}{n + 1} = \frac{n(n^2 - 2)}{n + 1}.$$

If  $n \geq 2$

$$|f(n)| = \left| \frac{n(n^2 - 2)}{n + 1} \right| \geq 0 \Rightarrow |f(n)| = f(n).$$

Now

$$|f(n)| - c_1|g_n| = \left| \frac{n^3 - 2n}{n + 1} \right| - \frac{1}{3}|n^2| = \frac{n^3 - 2n}{n + 1} - \frac{1}{3}n^2 = \frac{n(2n + 3)(n - 2)}{3(n + 1)}.$$

If  $n \geq 2$

$$|f(n)| - c_1|g_n| = \frac{n(2n + 3)(n - 2)}{3(n + 1)} \geq 0 \Rightarrow |f(n)| \geq c_1|g_n|.$$

Then

$$|f(n)| - c_2|g_n| = \left| \frac{n^3 - 2n}{n + 1} \right| - |n^2| = \frac{n^3 - 2n}{n + 1} - n^2 = -\frac{n(n + 2)}{n + 1}.$$

If  $n > 0$

$$|f(n)| - c_2|g_n| = -\frac{n(n + 2)}{n + 1} < 0 \Rightarrow |f(n)| < c_2|g_n|.$$

To sum up, if  $n \geq 2$

$$c_1|g_n| \leq |f(n)| < c_2|g_n| \Rightarrow f(n) = \Theta(g(n)) \Rightarrow \frac{n^3 - 2n}{n + 1} = \Theta(n^2).$$

■

**25.** Show that  $\frac{n^3}{\log(n+1)} = O(n^3)$  but not  $O(n^2)$ .

**Proof.**  $\forall x > 0$ , assume that the base of  $\log(x+1)$  is 2. Let

$$f(x) = \log(x+1) - 1.$$

Then

$$\frac{df(x)}{dx} = \frac{1}{(x+1)\ln 2} > 0.$$

Therefore, if  $x > 0$ ,  $f(x)$  is a strictly monotonically increasing function, then

$$f(1) = \log(1+1) - 1 = \log 2 - 1 = 0.$$

If  $x \in \mathbb{N}^*$ ,  $x \geq 1$ . Let  $x = n$

$$f(n) \geq f(1) = 0 \Rightarrow \log(n+1) - 1 \geq 0 \Rightarrow \frac{n^3}{\log(n+1)} \leq n^3.$$

Let  $c_1 = 1$

$$\frac{n^3}{\log(n+1)} - c_1|n^3| = \frac{n^3}{\log(n+1)} - n^3 \leq 0 \Rightarrow \frac{n^3}{\log(n+1)} = O(n^3).$$

$\forall x \geq 1$ , assume that the base of  $\log(x+1)$  is 2. Let

$$g(x) = \sqrt{x} - \log(x+1).$$

Then

$$\frac{dg(x)}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{(x+1)\ln 2} > \frac{1}{2\sqrt{x}} - \frac{1}{x\ln 2} > \frac{1}{2\sqrt{x}} - \frac{2}{x} = \frac{\sqrt{x}-4}{2x}.$$

If  $x > 16$ ,

$$\frac{dg(x)}{dx} = \frac{\sqrt{x}-4}{2x} > 0.$$

Therefore, if  $x > 16$ ,  $g(x)$  is a strictly monotonically increasing function, then

$$g(19) = \sqrt{19} - \log(20) > 0.$$

Thus, if  $x > 19$ ,

$$g(x) = \sqrt{x} - \log(x+1) > 0 \Rightarrow -\log(x+1) > -\sqrt{x}.$$

$\forall c_2 > 0$ ,  $\exists N = \max\{20, [c_2^2] + 1\}$  such that

$$\begin{aligned} \frac{N^3}{\log(N+1)} - c_2|N^2| &= \frac{N^2}{\log(N+1)}[N - c_2\log(N+1)] \\ &> \frac{N^2}{\log(N+1)}(N - c_2\sqrt{N}) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\frac{n^3}{\log(n+1)} \neq O(n^2).$$

26. What is wrong with the following argument?

$$x = O(n^4), \quad y = O(n^2), \quad \text{therefore} \quad \frac{x}{y} = O(n^2).$$

**Proof.** Let

$$f_1(n) = n^3, \quad f_2(n) = 1, \quad g_1(n) = n^4, \quad g_2(n) = n^2.$$

$\exists c_1 = 1, c_2 = 1$  such that  $\forall n \in \mathbb{N}^*$

$$f_1(n) - c_1|g_1(n)| = n^3 - n^4 \leq 0, \quad f_2(n) - c_2|g_2(n)| = 1 - n^2 \leq 0.$$

Therefore

$$f_1(n) = O(g_1(n)) = O(n^4), \quad f_2(n) = O(g_2(n)) = O(n^2).$$

Let

$$x = f_1(n) = n^3, \quad y = f_2(n) = 1. \quad \Rightarrow \quad \frac{x}{y} = n^3.$$

However  $\forall c > 0, \exists N = [c] + 1$  such that

$$\frac{x}{y_{n=N}} - c|N^2| = N^3 - cN^2 = N^2(N - c) > 0.$$

Thus

$$\frac{x}{y} \neq O(n^2).$$

27. What is wrong with the following argument?

$$x = \Theta(n^4), \quad y = \Theta(n^2), \quad \text{therefore} \quad \frac{x}{y} = \Theta(n^2).$$

**Proof.** This statement is correct. Assume that  $\exists c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$  and  $\exists N_1 > 0, N_2 > 0$  where  $N_1, N_2 \in \mathbb{N}^*$  such that if  $n > N_1$

$$c_1|n^4| \leq |x| \leq c_2|n^4|$$

and if  $n > N_2$

$$c_3|n^2| \leq |y| \leq c_4|n^2| \quad \Rightarrow \quad \frac{1}{c_4|n^2|} \leq \frac{1}{|y|} \leq \frac{1}{c_3|n^2|}.$$

Therefore let  $N = \max\{N_1, N_2\}$ , if  $n > N$

$$\frac{c_1|n^4|}{c_4|n^2|} \leq \frac{|x|}{|y|} \leq \frac{c_2|n^4|}{c_3|n^2|} \quad \Rightarrow \quad \frac{c_1}{c_4}|n^2| \leq \left| \frac{x}{y} \right| \leq \frac{c_2}{c_3}|n^2|.$$

Thus

$$\frac{x}{y} = \Theta(n^2).$$