

Chapter 1 Section 1 Exercises

1. With $S_1 = \{2, 3, 5, 7\}$, $S_2 = \{2, 4, 5, 8, 9\}$, and $U = \{1 : 10\}$, compute $\overline{S_1} \cup S_2$.

Solution.

$$\overline{S_1} = \{1, 4, 6, 8, 9, 10\} \quad \Rightarrow \quad \overline{S_1} \cup S_2 = \{1, 2, 4, 5, 6, 8, 9, 10\}.$$

2. With $S_1 = \{2, 3, 5, 7\}$, $S_2 = \{2, 4, 5, 8, 9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

Solution.

$$\begin{aligned} S_1 \times S_2 = \{ & (2, 2), (2, 4), (2, 5), (2, 8), (2, 9), \\ & (3, 2), (3, 4), (3, 5), (3, 8), (3, 9), \\ & (5, 2), (5, 4), (5, 5), (5, 8), (5, 9), \\ & (7, 2), (7, 4), (7, 5), (7, 8), (7, 9) \}. \end{aligned}$$

$$\begin{aligned} S_2 \times S_1 = \{ & (2, 2), (2, 3), (2, 5), (2, 7), \\ & (4, 2), (4, 3), (4, 5), (4, 7), \\ & (5, 2), (5, 3), (5, 5), (5, 7), \\ & (8, 2), (8, 3), (8, 5), (8, 7), \\ & (9, 2), (9, 3), (9, 5), (9, 7) \}. \end{aligned}$$

3. For $S = \{2, 5, 6, 8\}$ and $T = \{2, 4, 6, 8\}$, compute $|S \cap T| + |S \cup T|$.

Solution.

$$S \cap T = \{2, 6, 8\}, \quad S \cup T = \{2, 4, 5, 6, 8\} \quad \Rightarrow \quad |S \cap T| + |S \cup T| = 3 + 5 = 8.$$

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

Solution.

$$|S \cup T| = |S| + |T| - |S \cap T| = |S| + |T| \quad \Rightarrow \quad |S \cap T| = 0 \quad \Rightarrow \quad S \cap T = \emptyset.$$

Therefore, S and T are disjoint.

5. Show that for all sets S and T , $S - T = S \cap \overline{T}$.

Proof.

$$\begin{aligned} S - T &= \{x : x \in S \text{ and } x \notin T\} \\ \iff S - T &= \{x : x \in S \text{ and } x \in \overline{T}\} \\ \iff S - T &= S \cap \overline{T}. \end{aligned}$$

■

6. Prove DeMorgan's laws,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2},$$

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

by showing that if an element x is in the set on one side of the equality, then it must also be in the set on the other side of the equality.

Proof.

$$S_1 \cup S_2 = \{x : x \in S_1 \text{ or } x \in S_2\} \Rightarrow \overline{S_1 \cup S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$

$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \Rightarrow \overline{S_1} \cap \overline{S_2} = \{x : x \notin S_1 \text{ and } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}.$$

$$S_1 \cap S_2 = \{x : x \in S_1 \text{ and } x \in S_2\} \Rightarrow \overline{S_1 \cap S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$

$$\overline{S_1} = \{x : x \notin S_1\}, \quad \overline{S_2} = \{x : x \notin S_2\} \Rightarrow \overline{S_1} \cup \overline{S_2} = \{x : x \notin S_1 \text{ or } x \notin S_2\}.$$

Therefore,

$$\overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

■

7. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.

Proof.

$$S_1 \subseteq S_2$$

$$\Rightarrow (\in S_1 \Rightarrow x \in S_2)$$

$$\Rightarrow (x \notin S_2 \Rightarrow x \notin S_1)$$

$$\Rightarrow (x \in \overline{S_2} \Rightarrow x \in \overline{S_1})$$

$$\Rightarrow \overline{S_2} \subseteq \overline{S_1}.$$

■

8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

Proof.

$$1. S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$\left. \begin{array}{l} S_1 = S_2 \Rightarrow S_1 \cup S_2 = S_1 \cup S_1 = S_1 \\ S_1 = S_2 \Rightarrow S_1 \cap S_2 = S_1 \cap S_1 = S_1 \end{array} \right\} \Rightarrow S_1 \cup S_2 = S_1 \cap S_2.$$

$$2. S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2.$$

Assume that $S_1 \cup S_2 = S_1 \cap S_2$ and $S_1 \neq S_2$,

- $\exists x \in S_1 \text{ and } x \notin S_2 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$
- $\exists x \in S_2 \text{ and } x \notin S_1 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin S_1 \cap S_2 \Rightarrow S_1 \cup S_2 \neq S_1 \cap S_2.$

The result contradicts with the premise. Therefore, $S_1 \cup S_2 = S_1 \cap S_2 \Rightarrow S_1 = S_2.$

To sum up, $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2.$

■

9. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}.$

Proof.

1. Basis

If $|S| = 0, S = \emptyset.$ Then

$$2^S = \{\emptyset\}.$$

Therefore, $|2^S| = 2^{|S|} = 1.$

If $|S| = 1,$ assume that $S = \{a\}.$ Then

$$2^S = \{\emptyset, \{a\}\}.$$

Therefore, $|2^S| = 2^{|S|} = 2.$

2. Inductive Assumption

Assume that $|2^S| = 2^{|S|},$ for $|S| = 1, 2, \dots, n.$

3. Inductive Step

For $|S| = n + 1,$ assume that $S = \{a_1, a_2, \dots, a_n, a_{n+1}\}.$ Let $T = \{a_1, a_2, \dots, a_n\},$ then

$$2^T = \{T_1, T_2, \dots, T_{2^n}\}.$$

For $\forall i = 1, 2, \dots, 2^n$ where $i \in \mathbb{N}^*$

$$\left. \begin{array}{l} T_i \subseteq T \\ T \subseteq S \end{array} \right\} \Rightarrow T_i \subseteq S.$$

However,

$$S - T = \{a_{n+1}\} \Rightarrow a_{n+1} \notin T \Rightarrow a_{n+1} \notin T_i.$$

In addition

$$\left. \begin{array}{l} T_i \subseteq S \\ a_{n+1} \in S_i \Rightarrow \{a_{n+1}\} \subseteq S \end{array} \right\} \Rightarrow T_i \cup \{a_{n+1}\} \subseteq S.$$

Let

$$T'_i = T_i \cup \{a_{n+1}\}, \quad U = \{T'_1, T'_2, \dots, T'_{2^n}\}.$$

Now, for $\forall S_i \subseteq S$

- If $a_{n+1} \notin S_i,$ then $S_i \subseteq T,$ so $S_i \in 2^T.$

- If $a_{n+1} \in S_i$, then $S_i - \{a_{n+1}\} \subseteq T$, so $S_i - \{a_{n+1}\} \in 2^T$. Assume that

$$S_i - \{a_{n+1}\} = T_j \Rightarrow S_i = T_j \cup \{a_{n+1}\} \Rightarrow S_i \in U.$$

Moreover, 2^T and U are disjoint. Therefore,

$$2^S = 2^T \cup U, \quad |2^S| = |2^T| \cup |U| = 2^n + 2^n = 2^{n+1} = 2^{|S|}.$$

To sum up, if S is a finite set, then $|2^S| = 2^{|S|}$.

■

10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

Proof. Assume that

$$S_1 = \{a_1, a_2, \dots, a_n\}, \quad S_2 = \{b_1, b_2, \dots, b_m\}.$$

1. S_1 and S_2 are disjoint. Then

$$S_1 \cup S_2 = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m.$$

2. S_1 and S_2 are not disjoint. Assume that

$$c_1, c_2, \dots, c_k \in S_1 \text{ and } c_1, c_2, \dots, c_k \in S_2.$$

where $k \leq n$, $k \leq m$, $k \in \mathbb{N}^*$. Assume that

$$b_{i_1} = c_1, b_{i_2} = c_2, \dots, b_{i_k} = c_k.$$

Now

$$S_1 \cup S_2 = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{i_1-1}, b_{i_1+1}, \dots, b_{i_k-1}, b_{i_k+1}, \dots, b_m\}.$$

Therefore,

$$|S_1 \cup S_2| = n + m - k < n + m.$$

To sum up, if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

■

11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.

Proof. Assume that $S_1 = \emptyset$ or $S_2 = \emptyset$, then

$$S_1 \times S_2 = \emptyset \Rightarrow |S_1 \times S_2| = 0, |S_1||S_2| = 0 \times 0 = 0 \Rightarrow |S_1 \times S_2| = |S_1||S_2|.$$

Assume that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$,

$$S_1 = \{a_1, a_2, \dots, a_n\}, \quad S_2 = \{b_1, b_2, \dots, b_m\}.$$

where $n, m \in \mathbb{N}^*$.

Therefore,

$$\begin{aligned} S_1 \times S_2 = \{ & (a_1, b_1), (a_2, b_1), \dots, (a_n, b_1), \\ & (a_1, b_2), (a_2, b_2), \dots, (a_n, b_2), \\ & \vdots \\ & (a_1, b_m), (a_2, b_m), \dots, (a_n, b_m) \}. \end{aligned}$$

Thus,

$$|S_1 \times S_2| = nm = |S_1||S_2|.$$

■

12. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.

Proof.

1. **Reflexivity**

$$|S_1| = |S_1| \text{ for all } S_1. \Rightarrow S_1 \equiv S_1 \text{ for all } S_1.$$

2. **Symmetry**

$$\text{if } |S_1| = |S_2|, \text{ then } |S_2| = |S_1|. \Rightarrow \text{if } S_1 \equiv S_2, \text{ then } S_2 \equiv S_1.$$

3. **Transitivity**

$$\text{if } |S_1| = |S_2| \text{ and } |S_2| = |S_3|, \text{ then } |S_1| = |S_3|.$$

$$\Downarrow$$

$$\text{if } S_1 \equiv S_2 \text{ and } S_2 \equiv S_3, \text{ then } S_1 \equiv S_3.$$

Therefore, this is an equivalence relation.

■

13. Occasionally, we need to use the union and intersection symbols in a manner analogous to the summation sign \sum . We define

$$\bigcup_{p \in \{i, j, k, \dots\}} S_p = S_i \cup S_j \cup S_k \dots$$

with an analogous notation for the intersection of several sets.

With this notation, the general DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when P is a finite set.

Proof.

1. Basis

For $|P| = 2$, according to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}, \quad \overline{S_1 \cap S_2} = \overline{S_1} \cup \overline{S_2}.$$

2. Inductive Assumption

For $|P| = 2, 3, \dots, n$ where $n \in \mathbb{N}^*$

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}, \quad \overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

3. Inductive Step

For $|P| = n + 1$ where $n \in \mathbb{N}^*, \forall i \in P, |P - \{i\}| = n$,

$$\begin{aligned} \overline{\bigcup_{p \in P} S_p} &= \overline{\left(\bigcup_{p \in P - \{i\}} S_p \right) \cup S_i} = \overline{\left(\bigcup_{p \in P - \{i\}} S_p \right) \cap \overline{S_i}} = \left(\bigcap_{p \in P - \{i\}} \overline{S_p} \right) \cap \overline{S_i} = \bigcap_{p \in P} \overline{S_p}, \\ \overline{\bigcap_{p \in P} S_p} &= \overline{\left(\bigcap_{p \in P - \{i\}} S_p \right) \cap S_i} = \overline{\left(\bigcap_{p \in P - \{i\}} S_p \right) \cup \overline{S_i}} = \left(\bigcup_{p \in P - \{i\}} \overline{S_p} \right) \cup \overline{S_i} = \bigcup_{p \in P} \overline{S_p}. \end{aligned}$$

Therefore, for $|P| = 2, 3, \dots$

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}, \quad \overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

■

14. Show that

$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

Proof. According to DeMorgan's laws

$$\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2} \Rightarrow \overline{\overline{S_1 \cup S_2}} = \overline{\overline{S_1} \cap \overline{S_2}} \Rightarrow S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

■

15. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

Proof.

$$1. S_1 = S_2 \Rightarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

$$S_1 = S_2 \Rightarrow \left\{ \begin{array}{l} S_1 \cap \overline{S_2} = S_1 \cap \overline{S_1} = \emptyset \\ \overline{S_1} \cap S_2 = \overline{S_1} \cap S_2 = \emptyset \end{array} \right\} \Rightarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

$$2. S_1 = S_2 \Leftarrow (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

Assume that $S_1 \neq S_2$,

- $\exists x \in S_1 \text{ and } x \notin S_2 \Rightarrow x \in S_1 \cap \overline{S_2} \Rightarrow x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$
- $\exists x \notin S_1 \text{ and } x \in S_2 \Rightarrow x \in \overline{S_1} \cap S_2 \Rightarrow x \in (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2).$

Therefore, $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \emptyset$, which is a contradiction. Thus $S_1 = S_2$.

To sum up,

$$S_1 = S_2 \iff (S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \emptyset.$$

■

16. Show that

$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

Proof.

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup \overline{\overline{S_2}}) \\ &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2). \end{aligned}$$

1. If $x \in S_2$

$$x \in S_2 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \Rightarrow x \in (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

2. If $x \notin S_2$ and $x \in S_1$

$$x \notin S_2 \text{ and } x \in S_1 \Rightarrow x \in S_1 \cup S_2 \text{ and } x \notin \overline{S_1} \cup S_2 \Rightarrow x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

3. If $x \notin S_2$ and $x \notin S_1$

$$x \notin S_2 \text{ and } x \notin S_1 \Rightarrow x \notin S_1 \cup S_2 \text{ and } x \in \overline{S_1} \cup S_2 \Rightarrow x \notin (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2).$$

To sum up

$$\begin{aligned} S_1 \cup S_2 - (S_1 \cap \overline{S_2}) &= (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ &= S_2. \end{aligned}$$

■

17. Show that the distributive law

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.

Proof.

1. If $x \notin S_1$

$$x \notin S_1 \Rightarrow \begin{cases} x \notin S_1 \cap (S_2 \cup S_3) \\ x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \Rightarrow x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

2. If $x \in S_1$, $x \notin S_2$ and $x \notin S_3$

$$\begin{aligned} x \in S_1, x \notin S_2 \text{ and } x \notin S_3 &\Rightarrow x \notin S_2 \cup S_3 \Rightarrow x \notin S_1 \cap (S_2 \cup S_3). \\ x \in S_1, x \notin S_2 \text{ and } x \notin S_3 &\Rightarrow x \notin S_1 \cap S_2 \text{ and } x \notin S_1 \cap S_3 \\ &\Rightarrow x \notin (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

3. If $x \in S_1$ and $x \in S_2$

$$x \in S_1 \text{ and } x \in S_2 \Rightarrow \begin{cases} x \in S_1 \text{ and } x \in S_2 \cup S_3 \Rightarrow x \in S_1 \cap (S_2 \cup S_3) \\ x \in S_1 \cap S_2 \Rightarrow x \in (S_1 \cap S_2) \cup (S_1 \cap S_3) \end{cases}$$

4. If $x \in S_1$, $x \notin S_2$ and $x \in S_3$

$$\begin{aligned} x \in S_1, x \notin S_2 \text{ and } x \in S_3 &\Rightarrow x \in S_1 \text{ and } x \in S_2 \cup S_3 \\ &\Rightarrow x \in S_1 \cap (S_2 \cup S_3). \\ x \in S_1, x \notin S_2 \text{ and } x \in S_3 &\Rightarrow x \in S_1 \cap S_3 \\ &\Rightarrow x \in (S_1 \cap S_2) \cup (S_1 \cap S_3). \end{aligned}$$

To sum up

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3).$$

■

18. Show that

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Proof. Assume that $S_1 = \emptyset$, then

$$\left. \begin{aligned} S_1 \times (S_2 \cup S_3) &= \emptyset \\ S_1 \times S_2 = \emptyset, S_1 \times S_3 &= \emptyset \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = \emptyset \end{aligned} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Assume that $S_2 = \emptyset$, then

$$\left. \begin{aligned} S_2 \cup S_3 &= S_3 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_3 \\ S_1 \times S_2 = \emptyset &\Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_3 \end{aligned} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Assume that $S_3 = \emptyset$, then

$$\left. \begin{array}{l} S_2 \cup S_3 = S_2 \Rightarrow S_1 \times (S_2 \cup S_3) = S_1 \times S_2 \\ S_1 \times S_3 = \emptyset \Rightarrow (S_1 \times S_2) \cup (S_1 \times S_3) = S_1 \times S_2 \end{array} \right\} \Rightarrow S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

Assume that $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_3 \neq \emptyset$

$$S_1 = \{a_1, a_2, \dots, a_p\}, \quad S_2 = \{b_1, b_2, \dots, b_q\}, \quad S_3 = \{c_1, c_2, \dots, c_r\}.$$

where $p, q, r \in \mathbb{N}^*$.

Then

$$\begin{aligned} S_2 \cup S_3 &= \{b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_r\}. \\ S_1 \times (S_2 \cup S_3) &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q), \\ &\quad (a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \\ S_1 \times S_2 &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q)\} \\ S_1 \times S_3 &= \{(a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \\ (S_1 \times S_2) \cup (S_1 \times S_3) &= \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_q), \\ &\quad (a_2, b_1), (a_2, b_2), \dots, (a_2, b_q), \\ &\quad \vdots \\ &\quad (a_p, b_1), (a_p, b_2), \dots, (a_p, b_q), \\ &\quad (a_1, c_1), (a_1, c_2), \dots, (a_1, c_r), \\ &\quad (a_2, c_1), (a_2, c_2), \dots, (a_2, c_r), \\ &\quad \vdots \\ &\quad (a_p, c_1), (a_p, c_2), \dots, (a_p, c_r)\} \end{aligned}$$

Therefore,

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3).$$

■

19. Give conditions on S_1 and S_2 necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$

Solution.

$$S_1 \cap S_2 = \emptyset \iff S_1 = (S_1 \cup S_2) - S_2.$$

$$1. S_1 \cap S_2 = \emptyset \Rightarrow S_1 = (S_1 \cup S_2) - S_2$$

$$\left. \begin{array}{l} S_1 \cap S_2 = \emptyset \\ S_1 = S_1 \cap U = S_1 \cap (S_2 \cup \overline{S_2}) = (S_1 \cap S_2) \cup (S_1 \cap \overline{S_2}) \end{array} \right\} \Rightarrow S_1 = S_1 \cap \overline{S_2},$$

$$(S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 = (S_1 \cup S_2) - S_2.$$

$$2. S_1 \cap S_2 = \emptyset \Leftarrow S_1 = (S_1 \cup S_2) - S_2$$

$$S_1 = (S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S_2}) \cup (S_2 \cap \overline{S_2}) = S_1 \cap \overline{S_2},$$

Therefore,

$$S_1 \cap S_2 = (S_1 \cap \overline{S_2}) \cap S_2 = S_1 \cap (\overline{S_2} \cap S_2) = S_1 \cap \emptyset = S_1.$$

To sum up,

$$S_1 = (S_1 \cup S_2) - S_2.$$

20. Use the equivalence defined in Example 1.4 to partition the set $\{2, 4, 5, 6, 9, 22, 24, 25, 31, 37\}$ into equivalence classes.

Solution. Because

$$2 \bmod 3 = 5 \bmod 3 = 2,$$

$$4 \bmod 3 = 22 \bmod 3 = 25 \bmod 3 = 31 \bmod 3 = 37 \bmod 3 = 1,$$

$$6 \bmod 3 = 9 \bmod 3 = 24 \bmod 3 = 0.$$

The equivalence classes are

$$\{2, 5\}, \quad \{4, 22, 25, 31, 37\}, \quad \{6, 9, 24\}.$$

21. Show that if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.

Proof. Because $f(n) = O(g(n))$, $\exists c_1 > 0, N_1 \in \mathbb{N}^*$ such that $\forall n > N_1$

$$f(n) \leq c_1 |g(n)|.$$

Because $g(n) = O(f(n))$, $\exists c_2 > 0, N_2 \in \mathbb{N}^*$ such that $\forall n > N_2$

$$g(n) \leq c_2 |f(n)|.$$

Let $N = \max\{N_1, N_2\}$, assume that $\forall n > N$

$$f(n) \geq 0, \quad g(n) \geq 0.$$

Therefore

$$\frac{1}{c_2} |g(n)| \leq |f(n)| \leq c_1 |g(n)| \Rightarrow f(n) = \Theta(g(n)).$$

■

22. Show that $2^n = O(3^n)$, but $2^n \neq \Theta(3^n)$.

Proof. $\exists c_1 = 1 > 0$ such that for all $n \geq 1$

$$2^n \leq c_1 |3^n| = 3^n.$$

Therefore,

$$2^n = O(3^n).$$

However, $\forall c_2 > 0$, $\exists N = \lceil \log_{\frac{2}{3}} c_2 \rceil + 1$, if $n > N$

$$c_2 |3^n| = c_2 3^n > |2^n| = 2^n.$$

Therefore,

$$2^n \neq \Theta(3^n).$$

■

23. Show that the following order-of-magnitude results hold.

$$1. \ n^2 + 5 \log n = O(n^2).$$

$$2. \ 3^n = O(n!).$$

$$3. \ n! = O(n^n).$$

Proof.

$$1. \ n^2 + 5 \log n = O(n^2).$$

Let

$$f(n) = n^2 + 5 \log n, \quad g(n) = n^2.$$

Let $c = 2$, then $h(n) = f(n) - c|g(n)| = 5 \log n - n^2$.

$$h'(n) = \frac{5}{n} - 2n \Rightarrow h'(n) \text{ is a monotonically decreasing function.}$$

If $n \geq 2$, $h'(n) < 0$, so if $n \geq 2$, $h(n)$ is a monotonically decreasing function. Because $h(2) = 5 \log 2 - 4 < 0$, if $n \geq 2$, $h(n) < 0$. $\exists c = 2 > 0$ such that for all $n \geq 2$

$$h(n) = f(n) - c|g(n)| = 5 \log n - n^2 < 0 \Rightarrow f(n) \leq c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow n^2 + 5 \log n = O(n^2).$$

$$2. \ 3^n = O(n!).$$

Let

$$f(n) = 3^n, \quad g(n) = n!.$$

$\exists c = 9 > 0$ such that for all $n \geq 3$

$$f(n) - c|g(n)| = 3^n - 9n! = 3^n \left(1 - \frac{9n!}{3^n}\right) = 3^n \left(1 - 2 \prod_{i=3}^n \frac{i}{3}\right) < 0 \Rightarrow f(n) < c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow 3^n = O(n!).$$

3. $n! = O(n^n)$.

Let

$$f(n) = n!, \quad g(n) = n^n.$$

$\exists c = 1 > 0$ such that for all $n \geq 1$

$$f(n) - c|g(n)| = n! - n^n = n!(1 - \prod_{i=1}^n \frac{n}{i}) \leq 0 \Rightarrow f(n) \leq c|g(n)|.$$

Thus

$$f(n) = O(g(n)) \Rightarrow n! = O(n^n).$$

■

24. Show that $\frac{n^3 - 2n}{n + 1} = \Theta(n^2)$.

Proof. Let

$$f(n) = \frac{n^3 - 2n}{n + 1}, \quad g(n) = n^2, \quad c_1 = \frac{1}{3}, \quad c_2 = 1.$$

Because

$$f(n) = \frac{n^3 - 2n}{n + 1} = \frac{n(n^2 - 2)}{n + 1}.$$

If $n \geq 2$

$$|f(n)| = \left| \frac{n(n^2 - 2)}{n + 1} \right| \geq 0 \Rightarrow |f(n)| = f(n).$$

Now

$$|f(n)| - c_1|g_n| = \left| \frac{n^3 - 2n}{n + 1} \right| - \frac{1}{3}|n^2| = \frac{n^3 - 2n}{n + 1} - \frac{1}{3}n^2 = \frac{n(2n + 3)(n - 2)}{3(n + 1)}.$$

If $n \geq 2$

$$|f(n)| - c_1|g_n| = \frac{n(2n + 3)(n - 2)}{3(n + 1)} \geq 0 \Rightarrow |f(n)| \geq c_1|g_n|.$$

Then

$$|f(n)| - c_2|g_n| = \left| \frac{n^3 - 2n}{n + 1} \right| - |n^2| = \frac{n^3 - 2n}{n + 1} - n^2 = -\frac{n(n + 2)}{n + 1}.$$

If $n > 0$

$$|f(n)| - c_2|g_n| = -\frac{n(n + 2)}{n + 1} < 0 \Rightarrow |f(n)| < c_2|g_n|.$$

To sum up, if $n \geq 2$

$$c_1|g_n| \leq |f(n)| < c_2|g_n| \Rightarrow f(n) = \Theta(g(n)) \Rightarrow \frac{n^3 - 2n}{n + 1} = \Theta(n^2).$$

■

25. Show that $\frac{n^3}{\log(n+1)} = O(n^3)$ but not $O(n^2)$.

Proof. $\forall x > 0$, assume that the base of $\log(x+1)$ is 2. Let

$$f(x) = \log(x+1) - 1.$$

Then

$$\frac{df(x)}{dx} = \frac{1}{(x+1)\ln 2} > 0.$$

Therefore, if $x > 0$, $f(x)$ is a strictly monotonically increasing function, then

$$f(1) = \log(1+1) - 1 = \log 2 - 1 = 0.$$

If $x \in \mathbb{N}^*$, $x \geq 1$. Let $x = n$

$$f(n) \geq f(1) = 0 \Rightarrow \log(n+1) - 1 \geq 0 \Rightarrow \frac{n^3}{\log(n+1)} \leq n^3.$$

Let $c_1 = 1$

$$\frac{n^3}{\log(n+1)} - c_1|n^3| = \frac{n^3}{\log(n+1)} - n^3 \leq 0 \Rightarrow \frac{n^3}{\log(n+1)} = O(n^3).$$

$\forall x \geq 1$, assume that the base of $\log(x+1)$ is 2. Let

$$g(x) = \sqrt{x} - \log(x+1).$$

Then

$$\frac{dg(x)}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{(x+1)\ln 2} > \frac{1}{2\sqrt{x}} - \frac{1}{x\ln 2} > \frac{1}{2\sqrt{x}} - \frac{2}{x} = \frac{\sqrt{x}-4}{2x}.$$

If $x > 16$,

$$\frac{dg(x)}{dx} = \frac{\sqrt{x}-4}{2x} > 0.$$

Therefore, if $x > 16$, $g(x)$ is a strictly monotonically increasing function, then

$$g(19) = \sqrt{19} - \log(20) > 0.$$

Thus, if $x > 19$,

$$g(x) = \sqrt{x} - \log(x+1) > 0 \Rightarrow -\log(x+1) > -\sqrt{x}.$$

$\forall c_2 > 0$, $\exists N = \max\{20, [c_2^2] + 1\}$ such that

$$\begin{aligned} \frac{N^3}{\log(N+1)} - c_2|N^2| &= \frac{N^2}{\log(N+1)}[N - c_2\log(N+1)] \\ &> \frac{N^2}{\log(N+1)}(N - c_2\sqrt{N}) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\frac{n^3}{\log(n+1)} \neq O(n^2).$$

26. What is wrong with the following argument?

$$x = O(n^4), \quad y = O(n^2), \quad \text{therefore} \quad \frac{x}{y} = O(n^2).$$

Proof. Let

$$f_1(n) = n^3, \quad f_2(n) = 1, \quad g_1(n) = n^4, \quad g_2(n) = n^2.$$

$\exists c_1 = 1, c_2 = 1$ such that $\forall n \in \mathbb{N}^*$

$$f_1(n) - c_1|g_1(n)| = n^3 - n^4 \leq 0, \quad f_2(n) - c_2|g_2(n)| = 1 - n^2 \leq 0.$$

Therefore

$$f_1(n) = O(g_1(n)) = O(n^4), \quad f_2(n) = O(g_2(n)) = O(n^2).$$

Let

$$x = f_1(n) = n^3, \quad y = f_2(n) = 1. \quad \Rightarrow \quad \frac{x}{y} = n^3.$$

However $\forall c > 0, \exists N = [c] + 1$ such that

$$\frac{x}{y_{n=N}} - c|N^2| = N^3 - cN^2 = N^2(N - c) > 0.$$

Thus

$$\frac{x}{y} \neq O(n^2).$$

27. What is wrong with the following argument?

$$x = \Theta(n^4), \quad y = \Theta(n^2), \quad \text{therefore} \quad \frac{x}{y} = \Theta(n^2).$$

Proof. This statement is correct. Assume that $\exists c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$ and $\exists N_1 > 0, N_2 > 0$ where $N_1, N_2 \in \mathbb{N}^*$ such that if $n > N_1$

$$c_1|n^4| \leq |x| \leq c_2|n^4|$$

and if $n > N_2$

$$c_3|n^2| \leq |y| \leq c_4|n^2| \quad \Rightarrow \quad \frac{1}{c_4|n^2|} \leq \frac{1}{|y|} \leq \frac{1}{c_3|n^2|}.$$

Therefore let $N = \max\{N_1, N_2\}$, if $n > N$

$$\frac{c_1|n^4|}{c_4|n^2|} \leq \frac{|x|}{|y|} \leq \frac{c_2|n^4|}{c_3|n^2|} \quad \Rightarrow \quad \frac{c_1}{c_4}|n^2| \leq \left| \frac{x}{y} \right| \leq \frac{c_2}{c_3}|n^2|.$$

Thus

$$\frac{x}{y} = \Theta(n^2).$$

28. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Proof. Because $f(n) = O(g(n))$, $\exists c_1 > 0, N_1 \in \mathbb{N}^*$ such that $\forall n > N_1$

$$f(n) \leq c_1 |g(n)|.$$

Because $g(n) = O(h(n))$, $\exists c_2 > 0, N_2 \in \mathbb{N}^*$ such that $\forall n > N_2$

$$g(n) \leq c_2 |h(n)|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall n > N$

$$g(n) \geq 0.$$

Then $\forall n > N$

$$f(n) \leq c_1 |g(n)| = c_1 g(n) \leq c_1 c_2 |h(n)|.$$

Therefore

$$f(n) = O(h(n)).$$

■

29. Show that if $f(n) = O(n^2)$ and $g(n) = O(n^3)$, then

$$f(n) + g(n) = O(n^3)$$

and

$$f(n)g(n) = O(n^5).$$

Proof. Because $f(n) = O(n^2)$, assume that $\exists c_1 > 0, N_1 \in \mathbb{N}^*$ such that $\forall n > N_1$

$$f(n) \leq c_1 |n^2|.$$

Because $g(n) = O(n^3)$, assume that $\exists c_2 > 0, N_2 \in \mathbb{N}^*$ such that $\forall n > N_2$

$$g(n) \leq c_2 |n^3|.$$

Let $N = \{N_1, N_2\}$. Assume that $\forall n > N$

$$g(n) \geq 0.$$

Then $\forall n > N$

$$f(n) + g(n) \leq c_1 |n^2| + c_2 |n^3| \leq (c_1 + c_2) |n^3|,$$

$$f(n)g(n) \leq c_1 |n^2| \cdot c_2 |n^3| = c_1 c_2 |n^5|.$$

Therefore

$$f(n) + g(n) = O(n^3), \quad f(n)g(n) = O(n^5).$$

■