Graph Algorithms - III

CSIE 2136 Algorithm Design and Analysis, Fall 2018

https://cool.ntu.edu.tw/courses/61

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Agenda

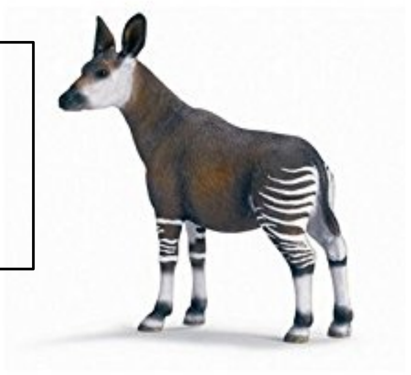
Single-source shortest-path algorithms

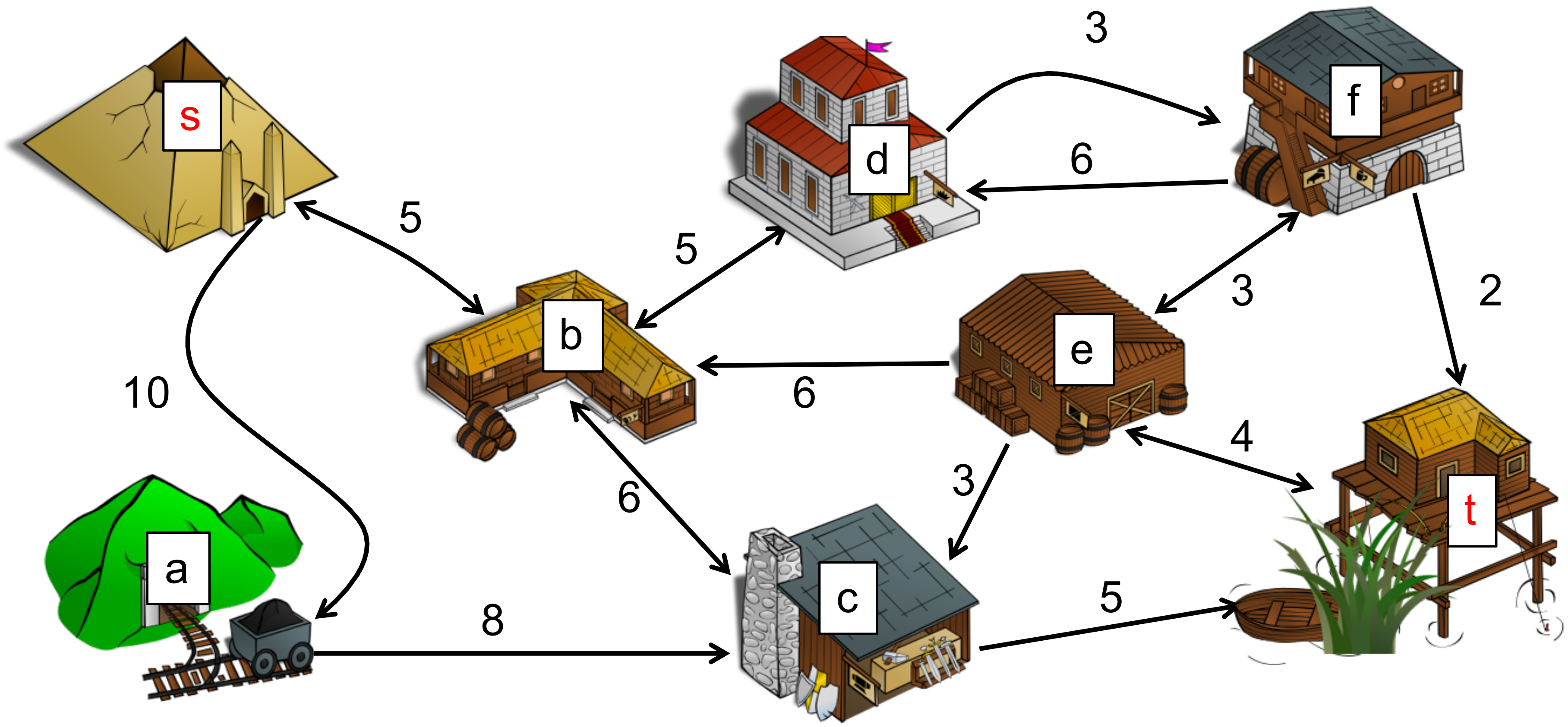
- Dijkstra algorithm
- Bellman-Ford algorithm
- SSSP in DAG

Single-source shortest-path algorithms

Textbook Chapter 24

訓練師在探索神獸級的神奇寶貝時被Okapi 咬傷，每走一公尺就會損血一滴。請找出從金字塔 (s) 到荒野大夫家 (t) 的最短路徑？





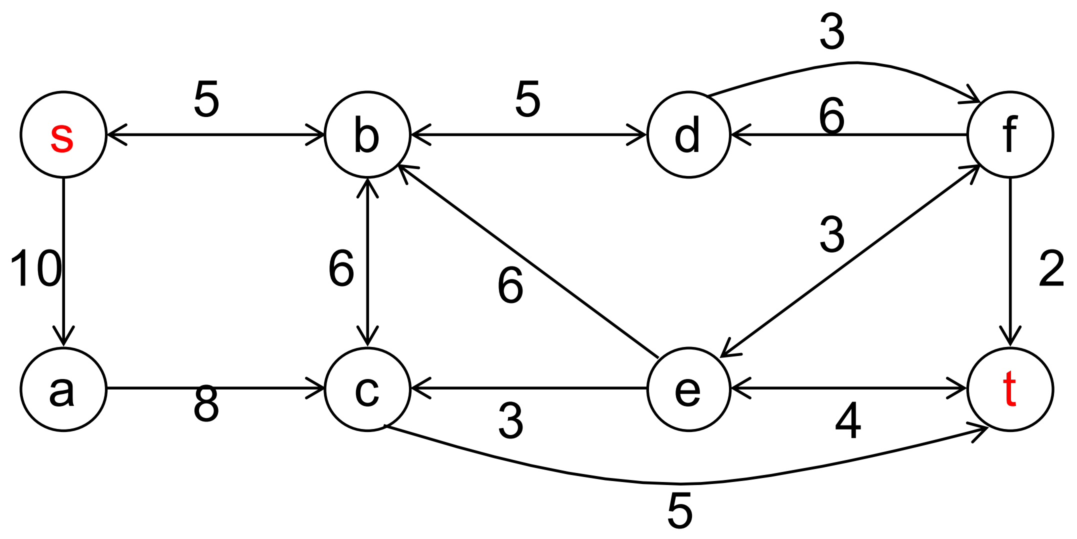
Definitions

Given a weighted, directed graph G = (V, E)

Given a weight function *w* mapping an edge to a weight

- Note that weights are arbitrary numbers, not necessarily distances
- Weight function needs not satisfy triangle inequality (think about airline fares)

Weight of path p = w(p) = sum of weights of edges on p

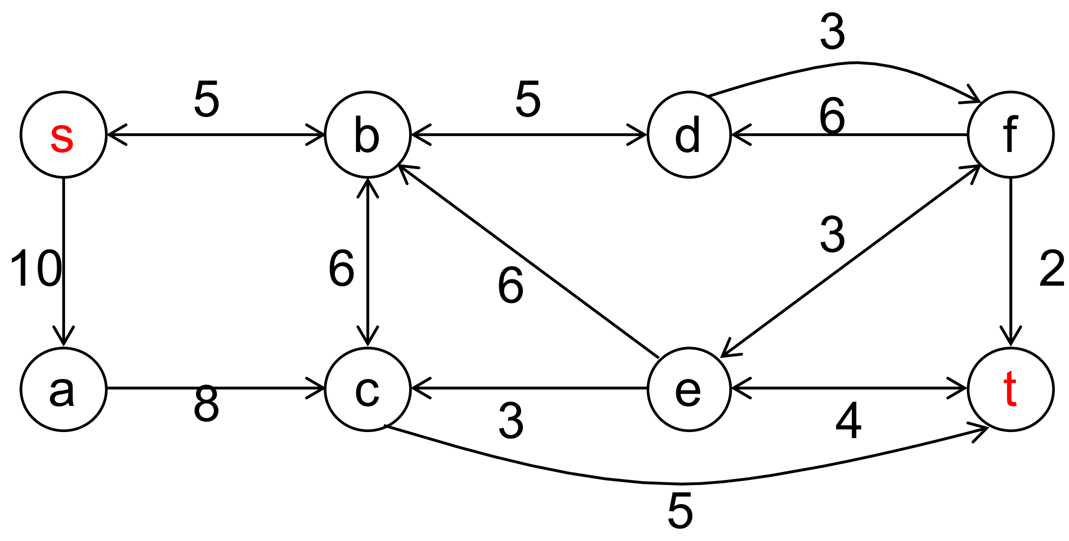


What is the weight of path s->a->c->t?

Definitions

Shortest path weight *δ(s,t)* = minimum weight of path from s to t

A **shortest path** from s to t = any path with weight *δ(s,t)*



δ(s,t) = ?

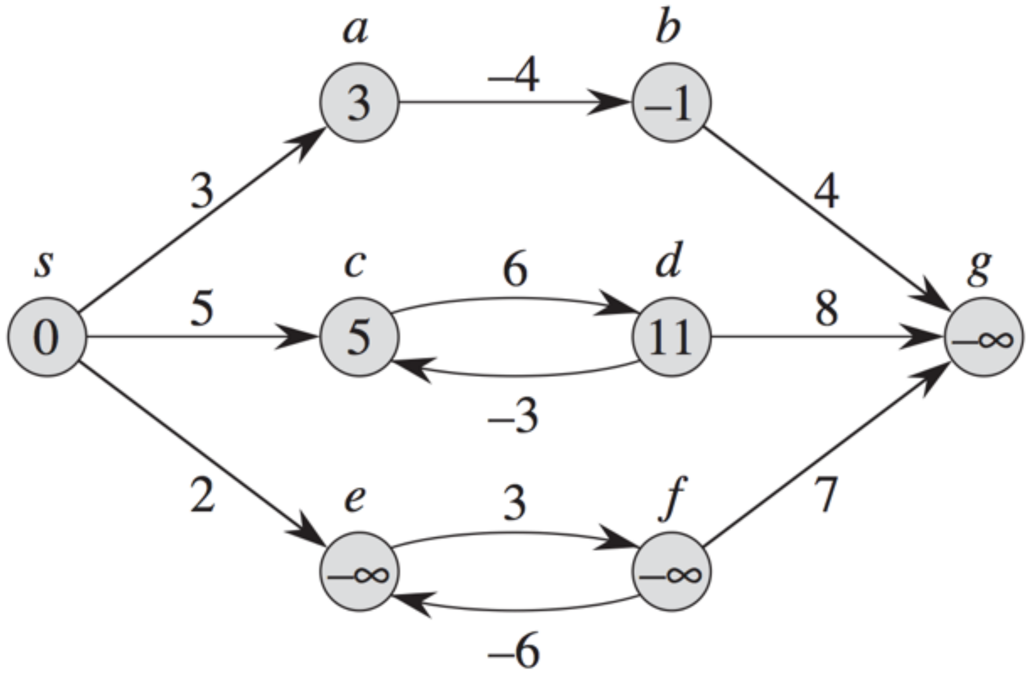
Shortest path from s to t = ?

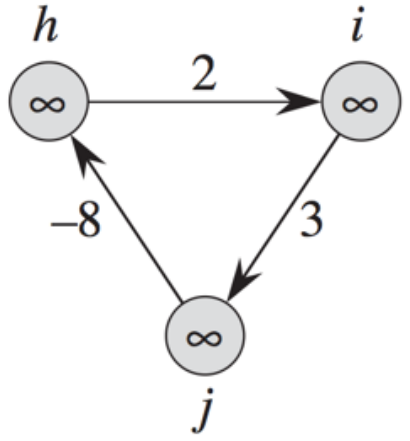
Cycles

Can a shortest path contain a **negative-weight edge**?

Can a shortest path contain a **negative-weight cycle**?

Can a shortest path contain a **cycle**?





Cycles

Can a shortest path contain negative-weight edges?
YES

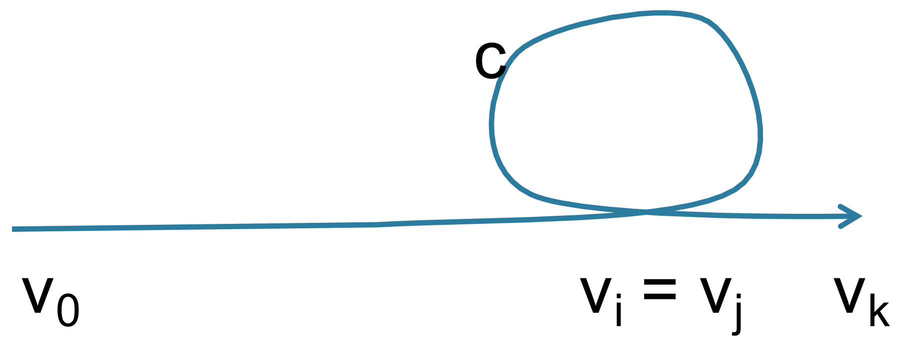
- δ(s,v) remains well defined for all v, if G contains no negative-weight cycles reachable from the source s

Can a shortest path contain negative-weight cycles?
Doesn't make sense

- If there is a negative-weight cycle on some path from s to v, we define δ(s,v) = -∞

Cycles

Can a shortest path contain a positive-weight cycle?



$p = \langle v_0, v_1, \ldots, v_k \rangle$

$c = \langle v_i, v_{i+1}, \ldots, v_j \rangle$

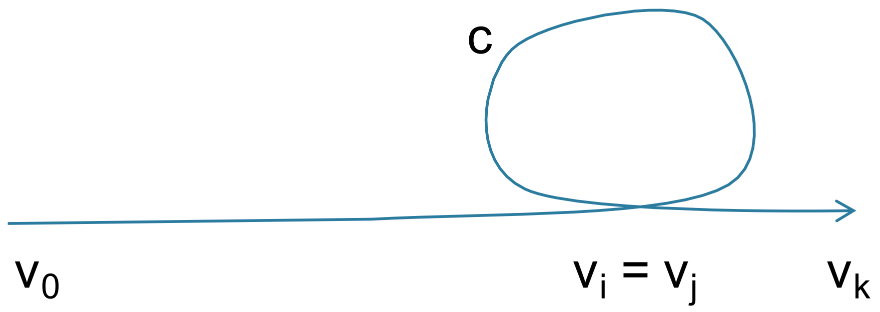
Let $p' = \langle v_0, v_1, \ldots v_i, v_{j+1}, v_{j+2}, \ldots, v_k \rangle$

$w(p') \leq w(p)$ if $w(c) \geq 0$

Cycles

Can a shortest path contain a positive-weight cycle?
NO

- It may contain a zero-weight cycle, but then there must exist a simple path of the same weight.



$p = \langle v_0, v_1, \ldots, v_k \rangle$
$c = \langle v_i, v_{i+1}, \ldots, v_j \rangle$

Let $p' = \langle v_0, v_1, \ldots v_i, v_{j+1}, v_{j+2}, \ldots, v_k \rangle$
$w(p') \leq w(p)$ if $w(c) \geq 0$

Cycles

Hence, we safely assume shortest paths have no cycles

- Define δ(u,v) = ∞ if v is unreachable from u
- Define δ(u,v) = -∞ if there exists a negative cycle on a path from u to v
- No cycle ➔ a shortest path has at most |V| - 1 edges

Variants of shortest-path problems

Our focus of this lecture; it can be applied to solve the following variants

Single-source shortest-path problem: Given a graph G = (V, E) and a *source* vertex s in V, find the minimum cost paths from s to every vertex in V

Single-destination shortest-path problem: Given a graph G = (V, E) and a *destination* vertex t in V, find the minimum cost paths to t from every vertex in V

Single-pair shortest-path problem: Find a shortest path from s to t for *given s and t*

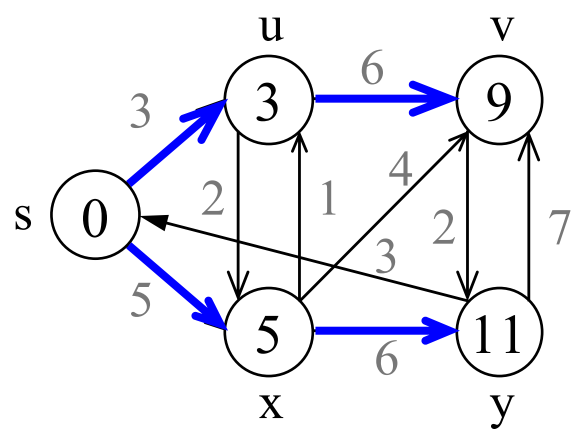
All-pair shortest path problem: Find a shortest path from s to t for *every pair of s and t*

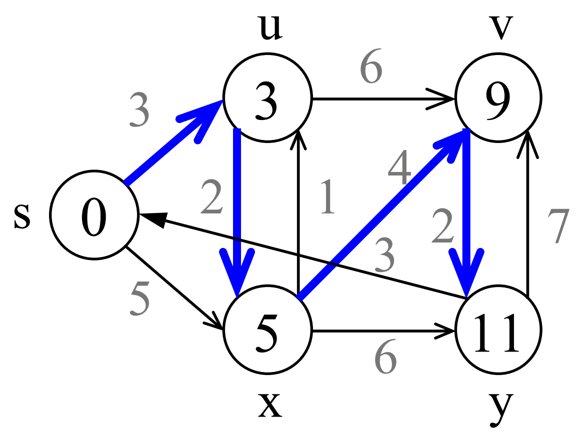
Shortest-paths tree

Let G = (V, E) be a weighted, directed graph with no negative-weight cycles reachable from *s*

A shortest-paths tree G’ = (V’, E’) of *s* is a subgraph of G s.t.:

- V’ is the set of vertices reachable from s in G
- G’ forms a rooted tree with root s
- For all v in V’, the unique simple path from s to v in G’ is a shortest path from s to v in G



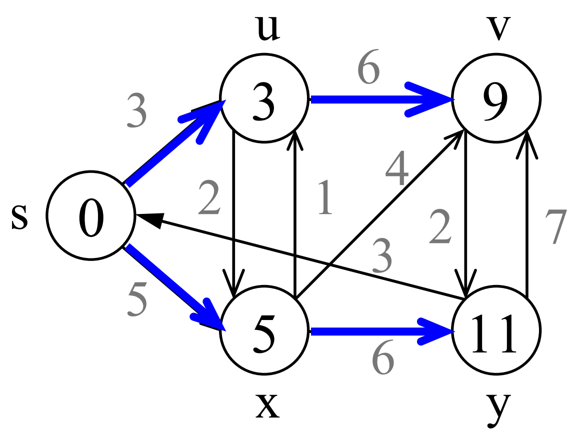


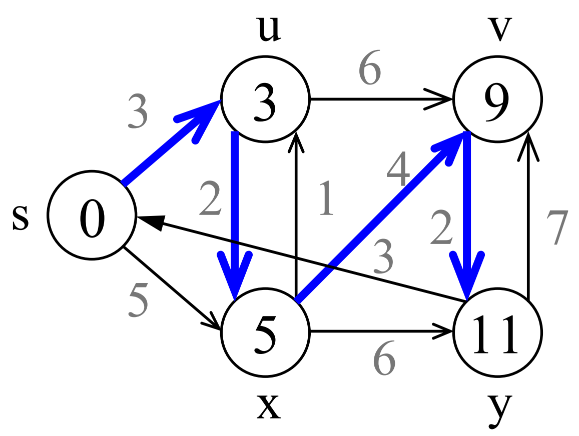
Shortest-paths tree

A shortest-paths tree G' = (V', E') is a subgraph of G s.t.:

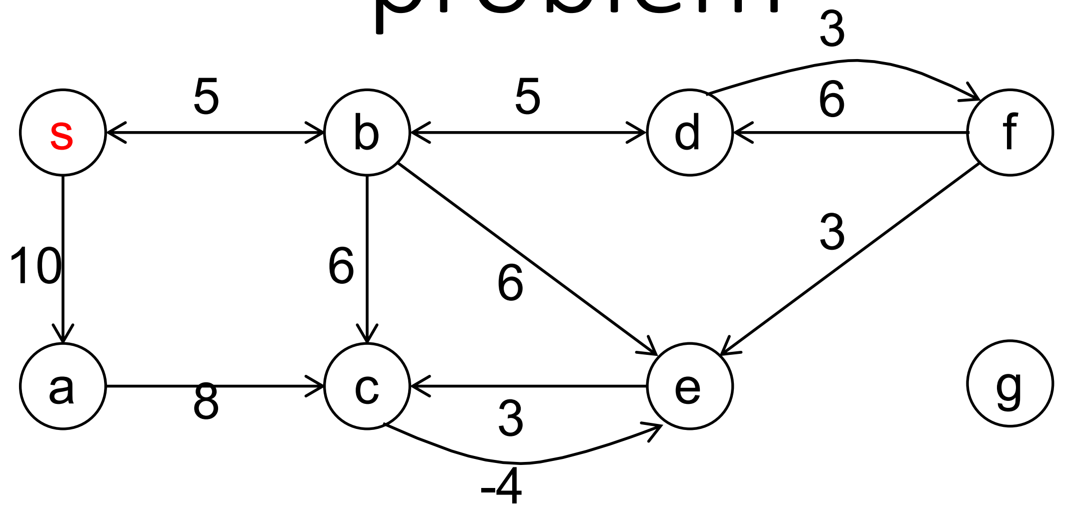
- V' is the set of vertices reachable from s in G
- G' forms a rooted tree with root s
- For all v in V', the unique simple path from s to v in G' is a shortest path from s to v in G

Q: Given G, can there be >1 possible shortest-paths tree? Yes
Q: Given G, can there be >1 possible shortest path weight for a vertex? No
Q: Given G, can we always find one shortest-paths *tree*? Yes. Lemma 24.17





Single-source shortest-path problem

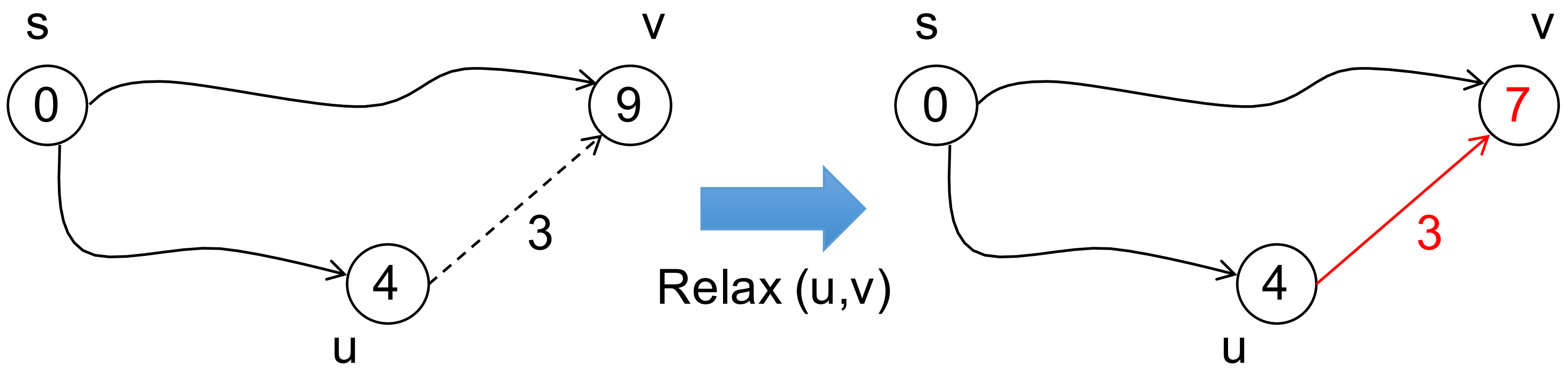


Destination v	Shortest path from s to v	Shortest path weight
a	s a	10
b	s b	5
c	?	
d	s b d	10
e	NIL	-∞
f	?	
g	?	

Edge relaxation

The process of **relaxing** an edge (u,v)
= testing whether the **shortest path weight of v** found so far can be reduced by traveling over u

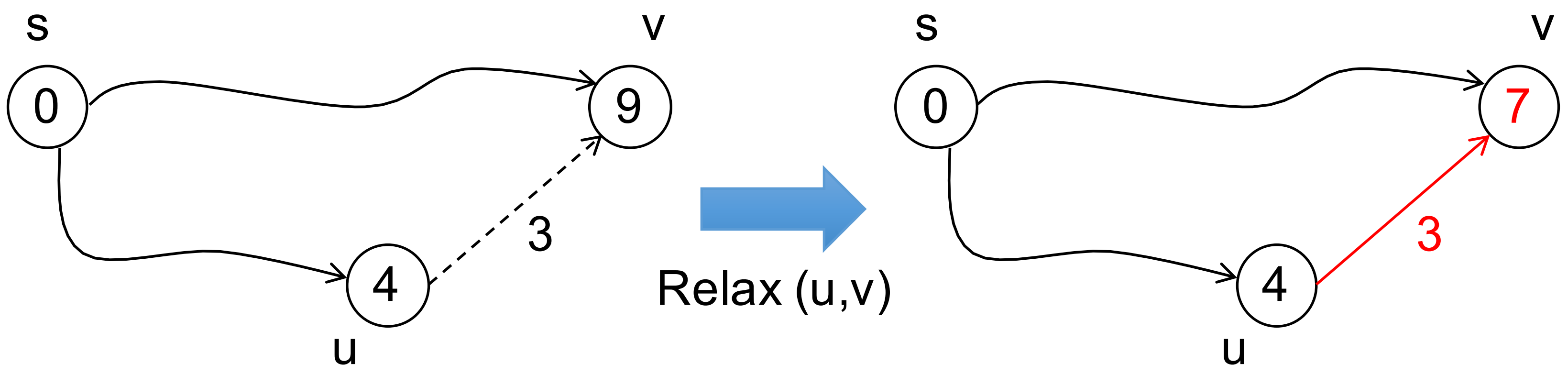
試試看經過u會不會比較好（更短的s-v路徑）



Practice: What if w(u,v) = 10?

Edge relaxation

The process of **relaxing** an edge (u,v) = testing whether the **shortest path weight of v** found so far can be reduced by traveling over u



```
RELAX(u, v, w)
If v.d > u.d + w(u, v)
    v.d = u.d + w(u, v)
    v.π = u
```

v.d = shortest-path estimate

- An **upper bound** on δ(s,v)
- v.d never increases during relaxation

v.π = predecessor attribute

Two classic single source shortest path algorithms

Dijkstra algorithm

- Greedy
- Requiring that all edge weights are **nonnegative**

Bellman-Ford algorithm

- Dynamic programming
- General case, edge weights **may be negative**

- Both on a weighted, directed graph

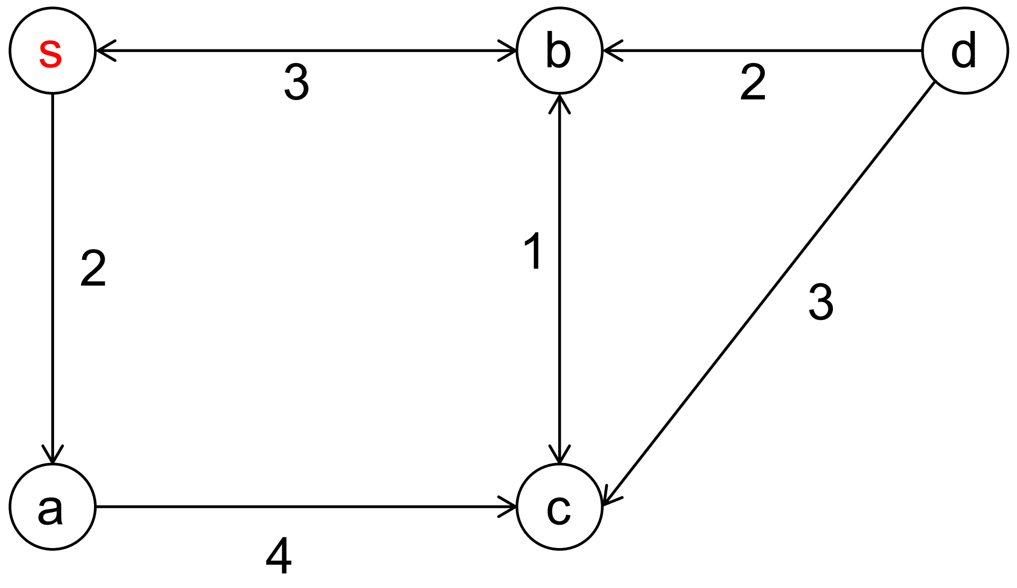
Dijkstra’s algorithm

Textbook Chapter 24.3

Dijkstra's algorithm: intuition

Recall that BFS finds shortest paths on an unweighted graph by expanding the search frontier like ripples.

Can we do the same on weighted graph?



Dijkstra's algorithm: intuition

Recall that BFS finds shortest paths on unweighted graph by expanding the search frontier like ripples

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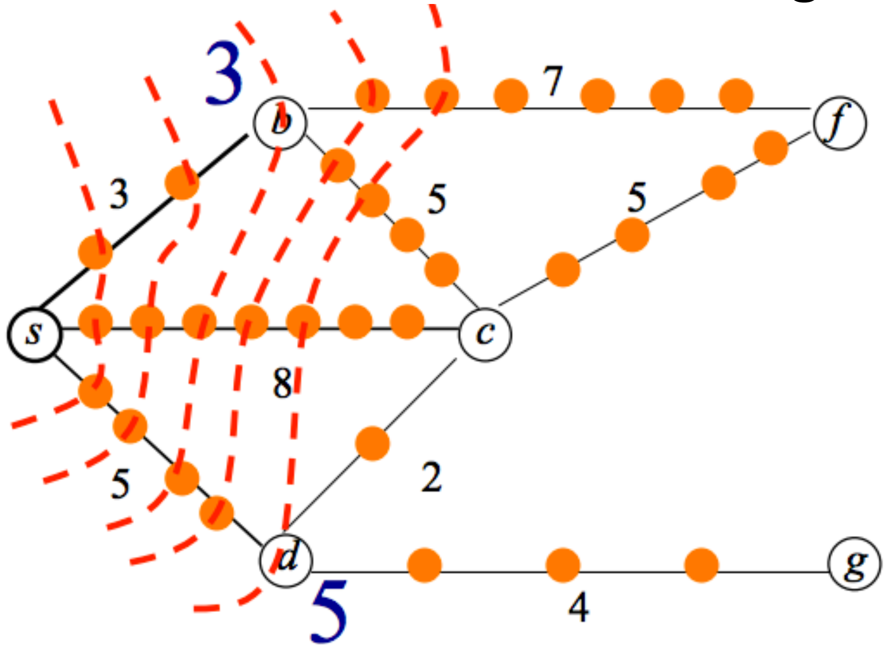


Dijkstra's algorithm speeds up the process by “skipping” layers that do not intersect with any vertex!

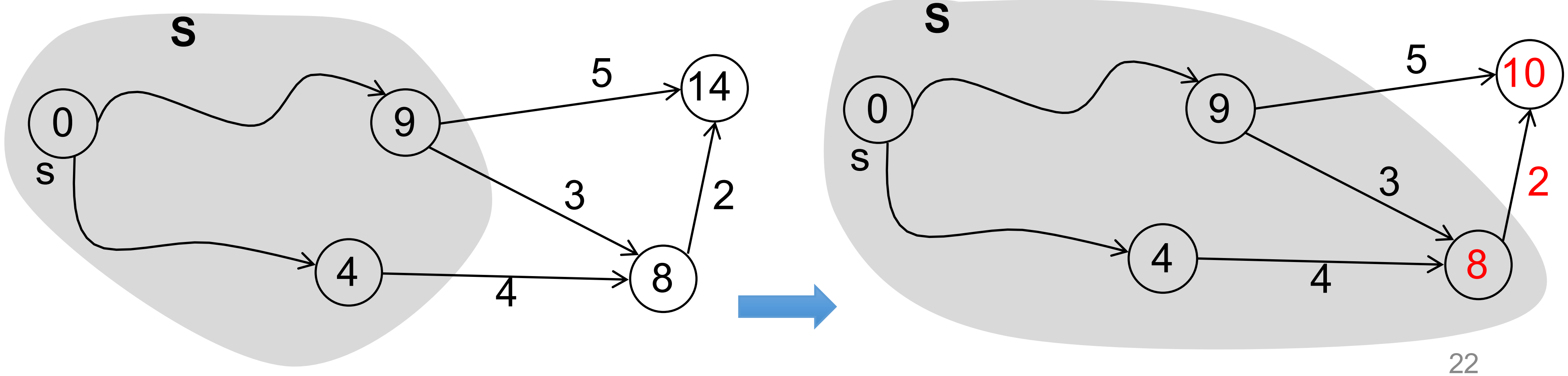
Source: http://courses.csail.mit.edu/6.006/spring11/lectures/lec16.pdf

Dijkstra's algorithm

Dijkstra greedily adds vertices by increasing distance

Maintains a **set of explored vertices** ***S*** whose final shortest-path weights have already been determined

1. Initially, S = {s}, s.d = 0
2. At each step, select unexplored vertex *u* in *V* - *S* with **minimum *u.d***
3. Add *u* to *S*, and **relaxes all edges leaving *u*.** Go back to Step 2.



Implementation of Dijkstra's algorithm

```
DIJKSTRA(G,w,s)
INITIALIZE-SINGLE-SOURCE(G,s)
S= empty
Q= G.v //INSERT
while Q ≠ empty
    u = EXTRACT-MIN(Q)
    S = S∪{u}
    for v in G.adj[u]
        RELAX(u,v,w)
```

```
INITIALIZE-SINGLE-SOURCE(G,s)
for v in G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

```
RELAX(u, v, w)
if v.d > u.d + w(u, v)
    //DECREASE-KEY
    v.d = u.d + w(u, v)
    v.π = u
```

For u in Q = V-S, u.d is the **shortest-path estimate** (i.e., minimum length over all observed s-u paths so far)

For u in S, u.d = length of the shortest path from s to u

Q is a min-priority queue of vertices, keyed by *d* values

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    S = S∪{u}
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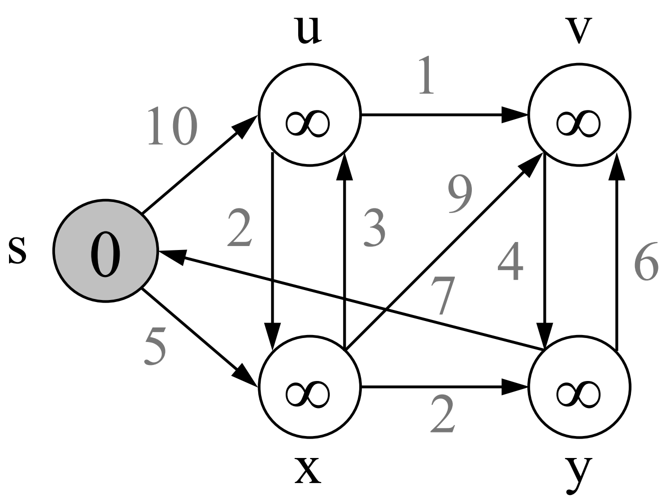
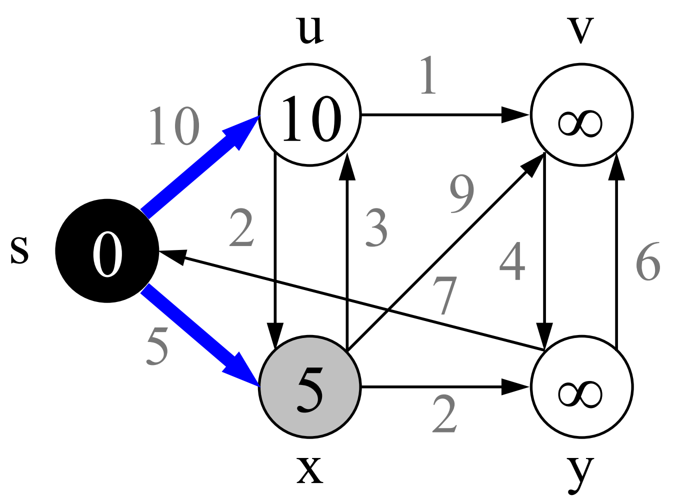
        RELAX(u,v,w)
```

Black: in S
White: in Q
Grey: selected
Blue lines: predecessors

Step 0



Step 1



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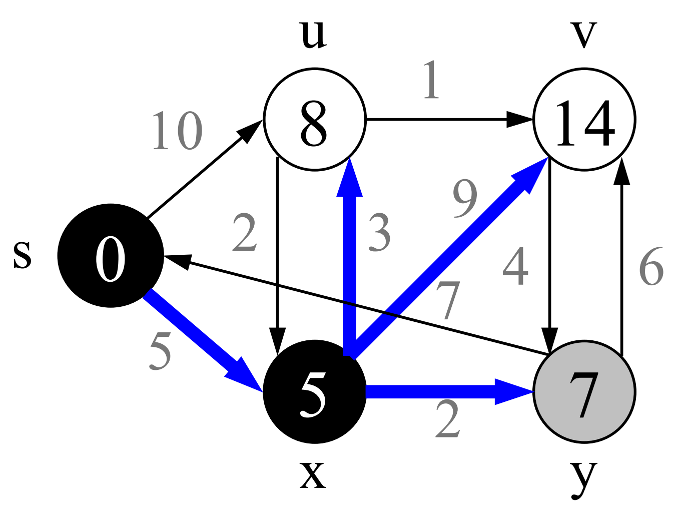
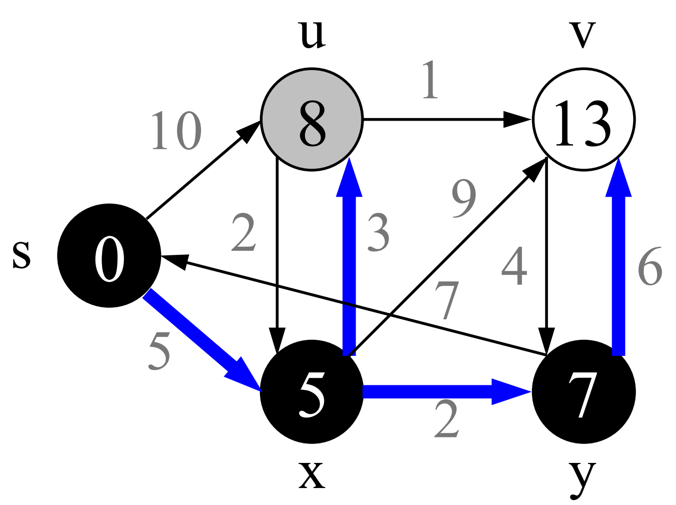
Step 2



Step 3



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    S = S∪{u}
    for v in G.adj[u]
        RELAX(u,v,w)
```

Black: in S
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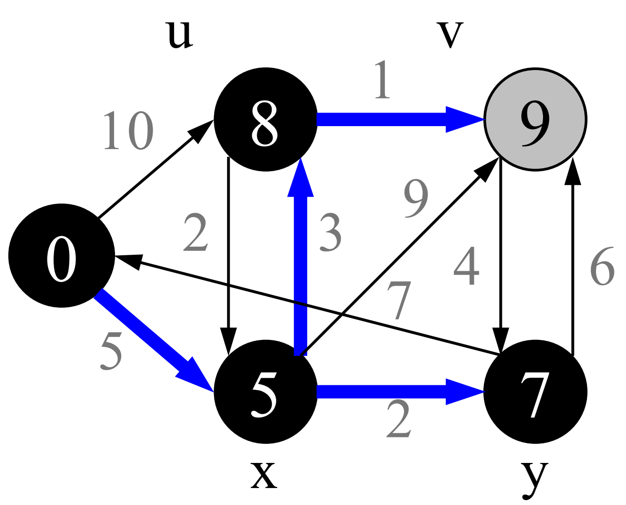
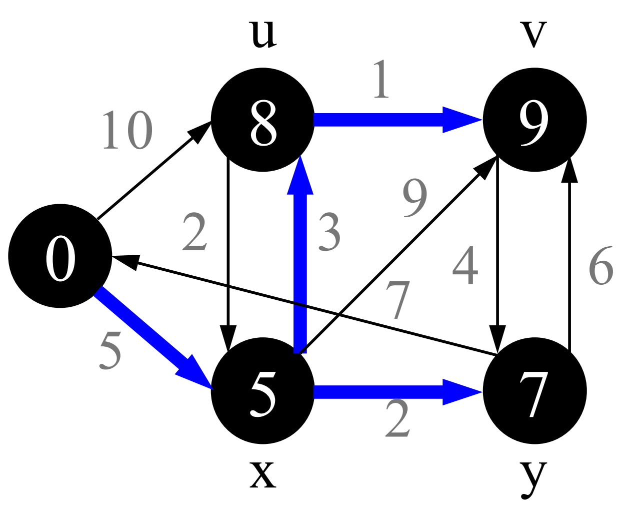
Step 4



Step 5



Running time analysis

Q is a min-priority queue of vertices, keyed by *d* values

- # of INSERT = O(V)
- # of EXTRACT-MIN = O(V)
- # of DECREASE-KEY = O(E)

The running time depends on queue implementation

Implementing the min-priority queue using an array indexed by *v*: $O(V^2+E) = O(V^2)$

- INSERT: O(1)
- EXTRACT-MIN: O(V)
- DECREASE-KEY: O(1)

Proof of correctness

S: the set of explored vertices whose final shortest-path weights have already been determined

- Initially, *S = {s}*, *s.d* = 0
- **Invariant:** for all *u* in S, *u.d* = length of the shortest path from s to u
- Note that for *u* in *V-S*, *u.d* = length of *some* path from *s* to *u*

We want to prove that the loop invariant holds throughout the execution of the algorithm.

Loop invariant: for u in S, u.d = δ(s,u)

Proof by induction on the size of S

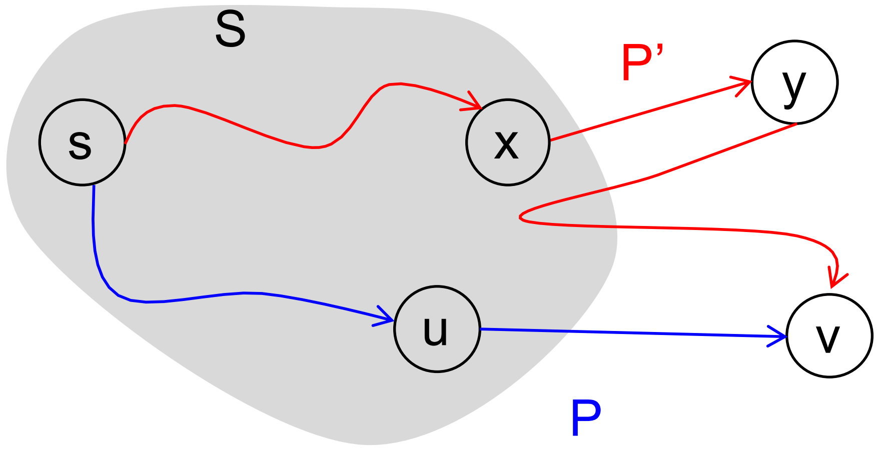
Base case: |S| = 1, correct

Inductive step: Let *v* be the next vertex to be added to S,
u = *v.π*, *P* = shortest path from *s* to *u* + *(u,v)*

=> *v.d* = *w(P)* = ***δ(s,u) + w(u,v)***

Consider any other s-v path P', and Let y be the first vertex on path P' outside S

We want to prove that w(P') ≥ w(P)



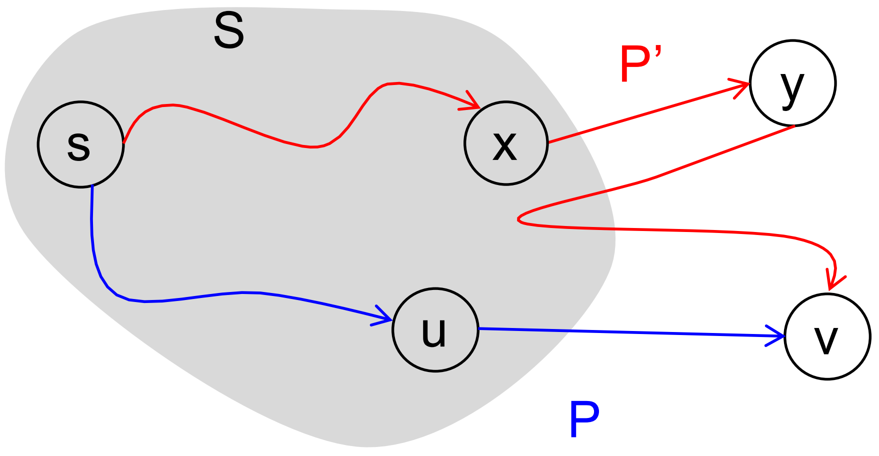
Loop invariant: for u in S, u.d = δ(s,u)

Proof by induction on the size of S (cont'd)

Prove that w(P') ≥ w(P)

1. Because of no negative edges, w(P') ≥ δ(s,x) + w(x,y)
2. By induction hypothesis, δ(s,x) = x.d
3. By construction, y.d ≥ v.d
4. By construction, x.d + w(x,y) ≥ y.d

=> w(P') ≥ δ(s,x) + w(x,y) = x.d + w(x,y) ≥ y.d ≥ v.d = w(P)



Loop invariant: for u in S, u.d = δ(s,u)

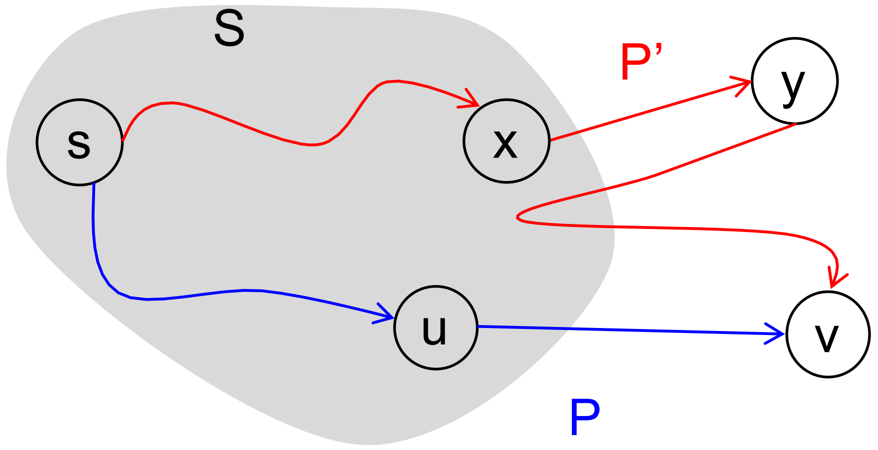
<u>Proof by induction on the size of S (cont'd)</u>

Hence, the greedy choice *v* (and the corresponding path *P* and *u.d*) is at least as good as any other path from *s* to *v*

=> The invariant still holds after adding one more vertex to *S.*

At termination, every vertex is in *S.*

Thus, *u.d = δ(s,v)* for all *u* in *V.*

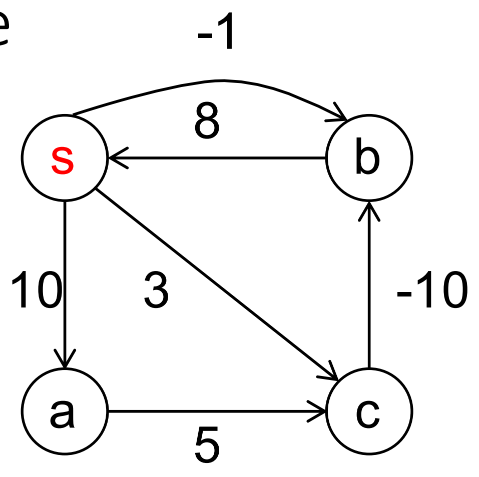


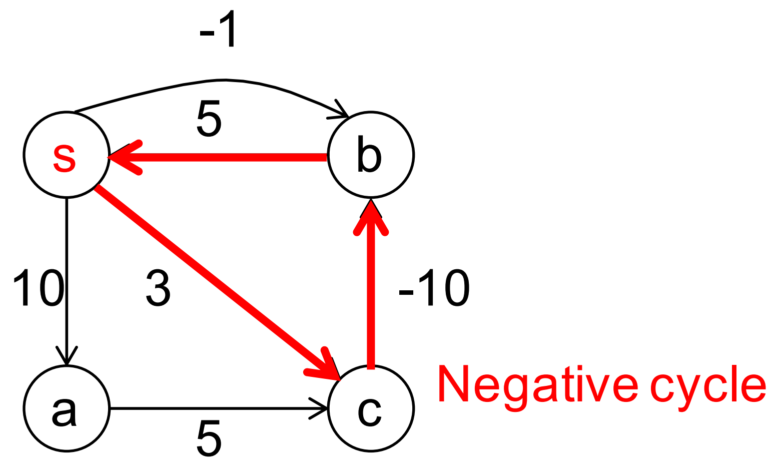
Dijkstra's algorithm may work incorrectly with negative-weight edges.

Let's go back to the proof and see where it breaks!

- This greedy algorithm assumed adding edges always increases path weight, which is not true in case of negative-weight edges

Next topic: a dynamic programming algorithm (Bellman-Ford) either detects negative cycles or returns the shortest-path tree





δ(s,b) = -7
In Dijkstra, *b.d* = -1

δ(s,b) = ??

Priority-first search

See any similarity between BFS, DFS, Prim and Dijkstra?

- They are all greedy algorithms for graph search
- They are each a special case of **priority-first search**

Priority-first search

- Maintain a set of explored vertices *S*
- Grow *S* by exploring **highest-priority edges** with exactly one endpoint leaving S

Priority in each variant

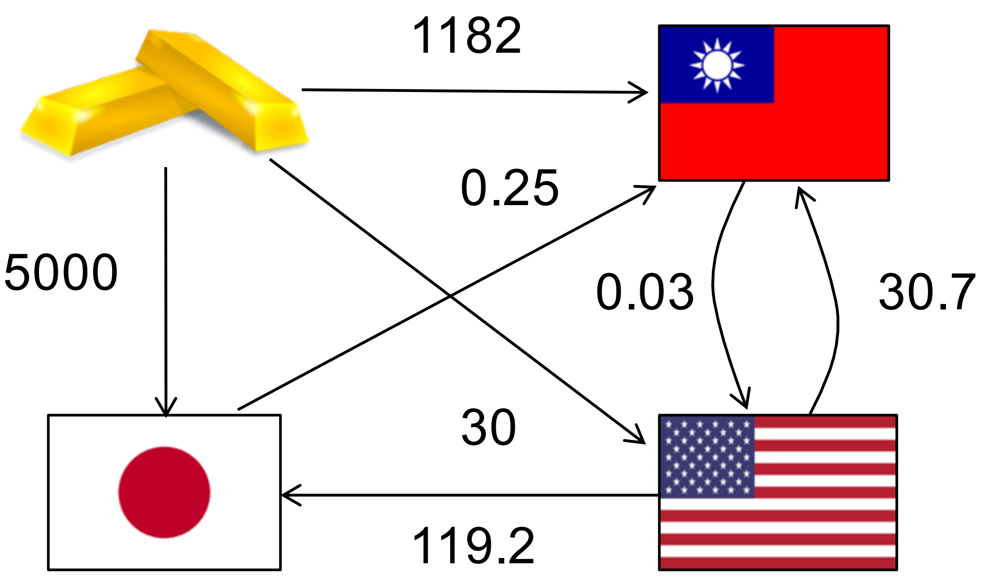
- **BFS:** edges from vertex discovered least recently
- **DFS:** edges from vertex discovered most recently
- **Prim:** edges of minimum weight
- **Dijkstra**: edges to vertex closest to s

Bellman-Ford algorithm

Textbook Chapter 24.1

匯率換算

1公克黃金最多可以換到多少TWD？（假設零手續費）



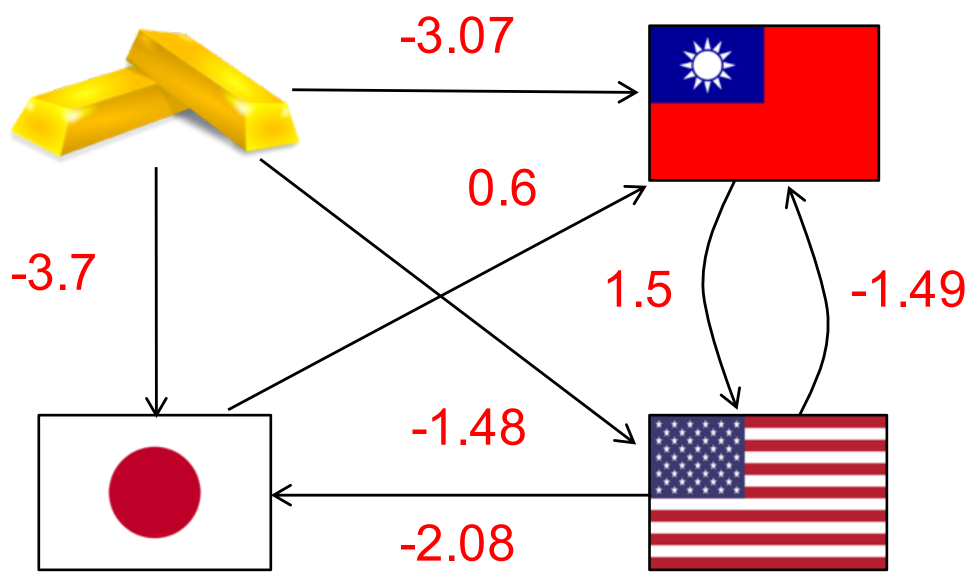
找weight相乘後最大路徑？

如何轉成我們熟悉的最短路徑問題？

匯率換算

1公克黃金最多可以換到多少TWD？（假設零手續費)

Reweighting: w’(e) = - log w(e)

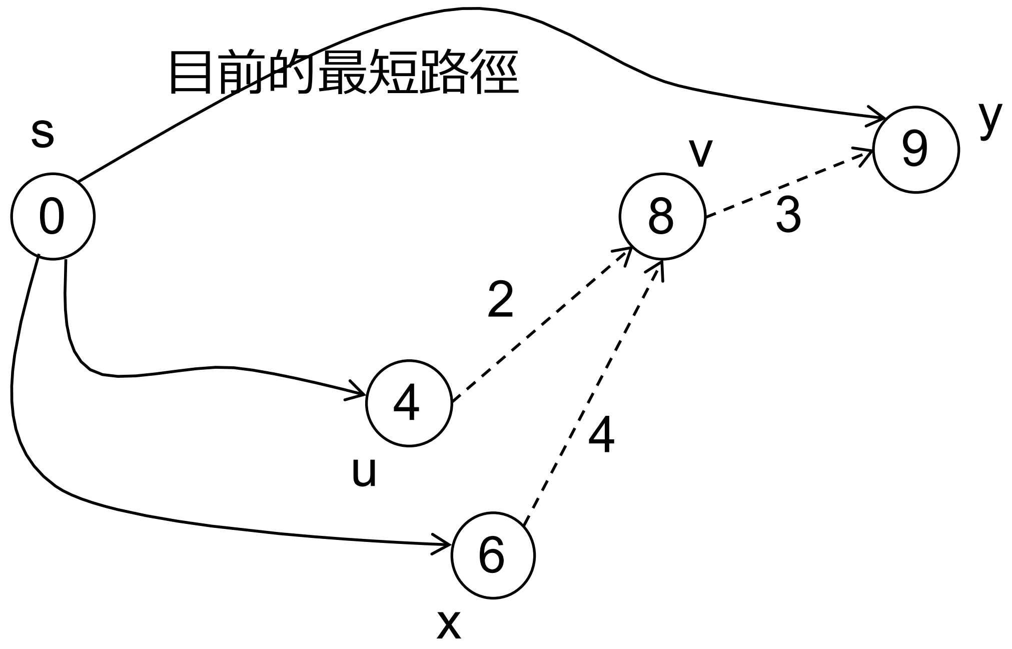


We want to detect the existence of negative cycles (利用匯差賺錢) and find the shortest path (最佳的兌換率)

Bellman-Ford algorithm: intuition

共執行|V|-1回合

- 每一回合中, relax every edge
- 節點v一方面接收從各個“上游”來的最短路徑估計值，試試看改走不同上游會不會比較好，另一方面把自己的估計值傳給“下游”節點



u 和 x 為v的上游
y為v的下游

以v的觀點來看，
relax (u,v), (x,v), (v,y)
順序不重要

Bellman-Ford algorithm: intuition

Bellman-Ford保證在第k回合結束後,**節點v的最短路徑估計值 ≤ 所有邊數至多為k的s-v路徑的最短距離**

=> |V|-1回合結束後，節點v的最短路徑估計值≤所有邊數至多為|V|-1的s-v路徑的最短距離

=>若最短路徑存在，由於最短路徑的邊數不會大於|V|-1，因此Bellman-Ford結束後的確能正確算出最短路徑值

Bellman-Ford algorithm

```
BELLMAN-FORD(G,w,s)
INITIALIZE-SINGLE-SOURCE(G,s)
for i = 1 to |G.V|-1
    for (u,v) in G.E
        RELAX(u,v,w)
for (u,v) in G.E
    if v.d > u.d + w(u,v)
        return FALSE
return TRUE
```

```
INITIALIZE-SINGLE-SOURCE(G,s)
for v in G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

```
RELAX(u, v, w)
If v.d > u.d + w(u, v)
    //DECREASE-KEY
    v.d = u.d + w(u, v)
    v.π = u
```

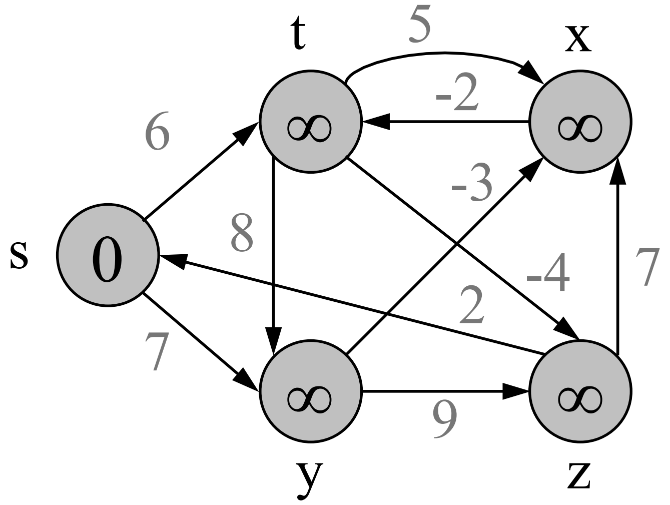
Relax each edge *e*; repeat V-1 times

Detect a negative cycle if any exist

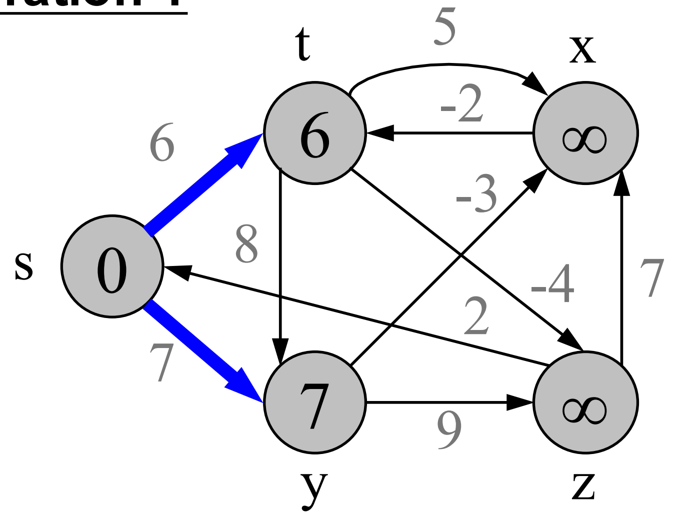
Find shortest simple path if no negative cycle exists

Relaxation sequence in each iteration: (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

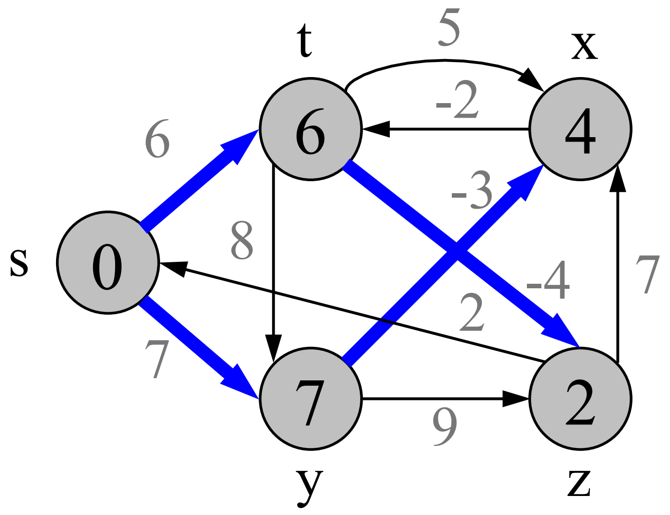
Iteration 0



Iteration 1



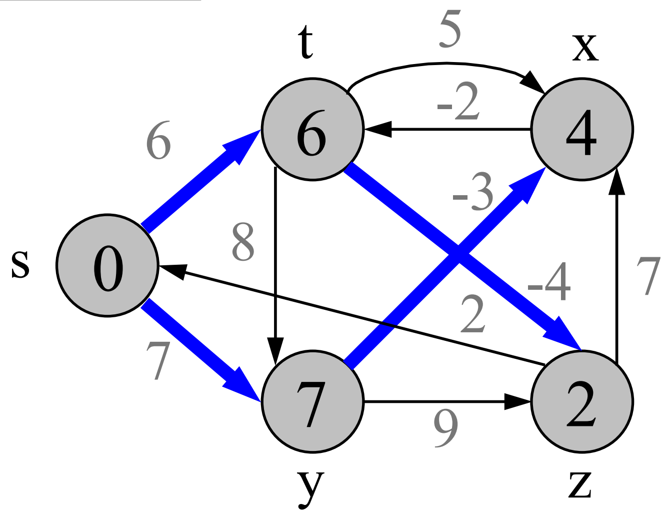
Iteration 2



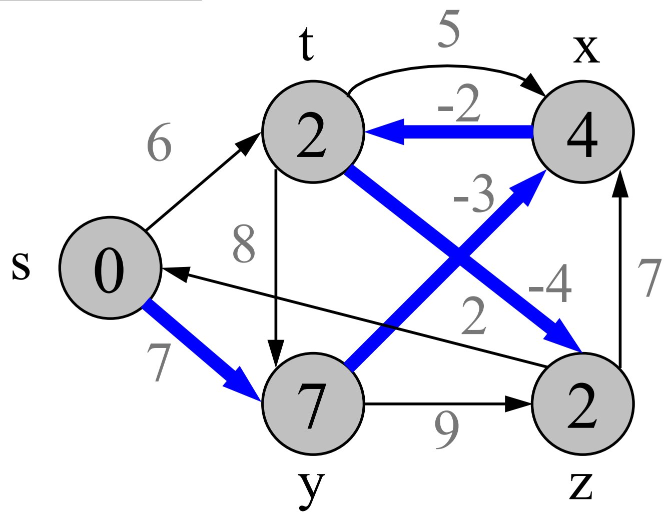
```
BELLMAN-FORD(G,w,s)
INITIALIZE-SINGLE-SOURCE(G,s)
for i = 1 to |G.V|-1
    for (u,v) in G.E
        RELAX(u,v,w)
for (u,v) in G.E
    if v.d > u.d + w(u,v)
        return FALSE
return TRUE
```

Relaxation sequence in each iteration: (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

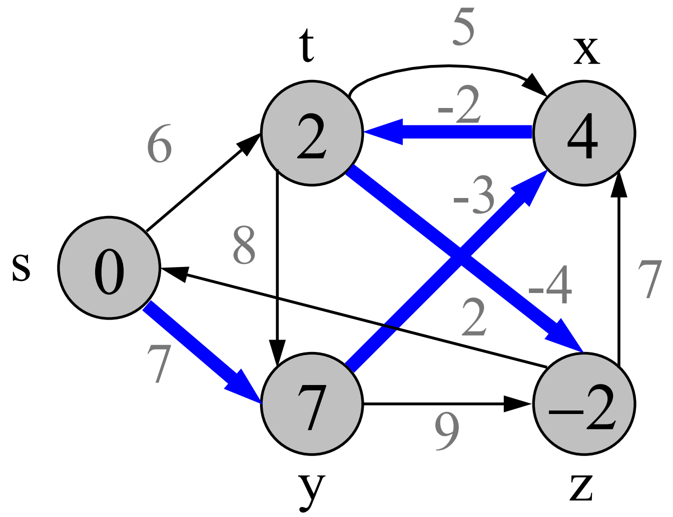
Iteration 2



Iteration 3



Iteration 4



```
BELLMAN-FORD(G,w,s)
INITIALIZE-SINGLE-SOURCE(G,s)
for i = 1 to |G.V|-1
    for (u,v) in G.E
        RELAX(u,v,w)
for (u,v) in G.E
    if v.d > u.d + w(u,v)
        return FALSE
return TRUE
```

Running time analysis

Using adjacency lists,

```
BELLMAN-FORD(G,w,s)
INITIALIZE-SINGLE-SOURCE(G,s)
for i = 1 to |G.V|-1
    for (u,v) in G.E
        RELAX(u,v,w)
for (u,v) in G.E
    if v.d > u.d + w(u,v)
        return FALSE
return TRUE
```

$\Theta(V)$

$\Theta((V\text{-}1)E)$

$\Theta(E)$

Adjacency-list representation = $\Theta(VE)$

Adjacency-matrix representation = ?

- It takes $\Theta(V^2)$ to loop through all edges, thus $\Theta(V^3)$ in total

Correctness of Bellman-Ford

We want to prove the following two statements:

1. Correctly compute δ(s, v) when no negative-weight cycle:
    - After the $|V| - 1$ iterations of relaxation of all edges, it must hold that $v.d = \delta(s, v)$ for all vertices $v \in V$ that are reachable from s
    - For each vertex v in V, there is a path from s to v if and only if the algorithm terminates with $v.d < \infty$
2. Correctly detect the existence of negative cycles
    - Bellman-Ford returns FALSE If G does contain a negative-weight cycle reachable from s

Path-relaxation property

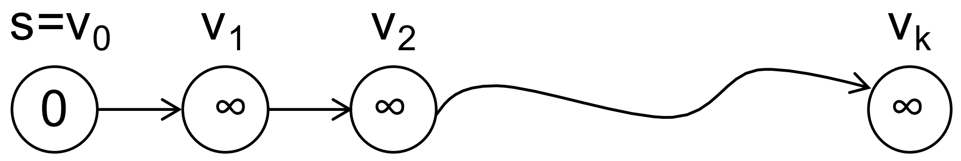
$\boxed{\text{Lemma 24.15}}$

Let $p=\langle v_0,v_1,\ldots,v_k\rangle$ be a shortest path from $s=v_0$ to v_k

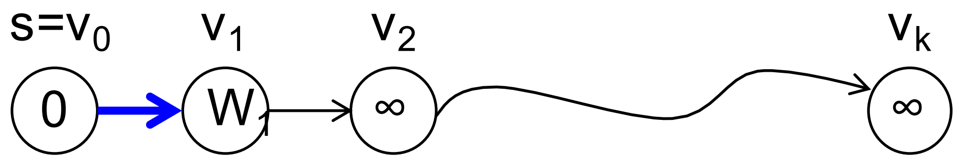
Let $W_i = \Sigma_1^i\, w(v_{i-1}, v_i)$, W_i be the shortest path weight $\delta(s,v_i)$ because of optimal substructure

Path-relaxation property: $v_k.d = \delta(s,v_k)$ after relaxation sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$

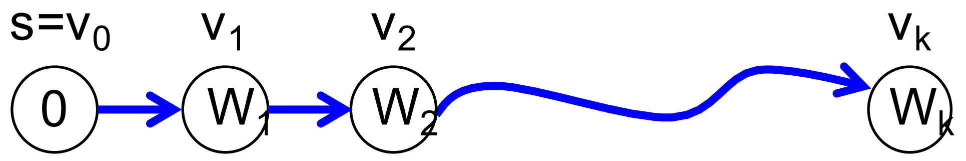
Init



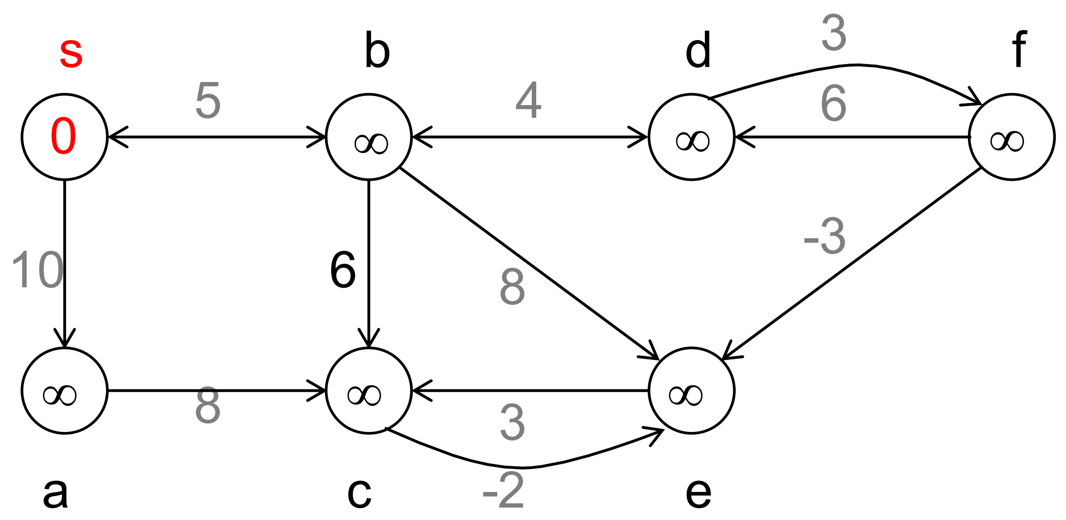
Relax (v_0, v_1)



After relaxation sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$



Note: 此性質對於任何包含這個最短路徑邊的relaxation sequence都成立, e.g., (v_0, v_1), (a, b), (d, c), (v_0, v_1), $(v_1, v_2), \ldots, (v_{k-1}, v_k)$



δ(s,e) = 9

A shortest path from s to e = <s, b, d, f, e>

After relaxing (s, b), (b, d), (d, f), (f, e) in order, what would be the value of e.d?

How about relaxing them in a different order, such as (s, b), (d, f), (b, d), (f, e)?

How about relaxing (s, b), (b, e), (s, a), (b, d), (d, f), (e, c), (f, e)?

Correctness: compute shortest paths

Path-relaxation property: If $p=<v_0,v_1,\ldots,v_k>$ is a shortest path from $s=v_0$ to v_k, then $v_k.d = \delta(s,v_k)$ after relaxation sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$

The relaxation sequence in Bellman Ford must contain all edges in p in order:

- Although the shortest path p from s to v is unknown, we know it has at most V-1 edges if the path exists (i.e., no negative-weight cycles)
- The relaxation sequence must contain all edges in p in order:

(m=|E|)

$$\underbrace{\underbrace{e_1, e_2, \ldots, e_m}_{\text{Must contain 1st edge in p}}; \underbrace{e_1, e_2, \ldots, e_m; \ldots}_{\text{Must contain 2nd edge in p}}\ldots; e_1, e_2, \ldots, e_m}_{\text{Repeated V-1 times, must contain all edges in p in order}}$$

This relaxation sequence contains a shortest path to every vertex!

Correctness: negative cycle detection

Claim: Bellman-Ford returns FALSE If G does contain a negative-weight cycle reachable from s

Proof by contradiction

Suppose Bellman-Ford returns TRUE while G does contain a negative-weight cycle C reachable from s

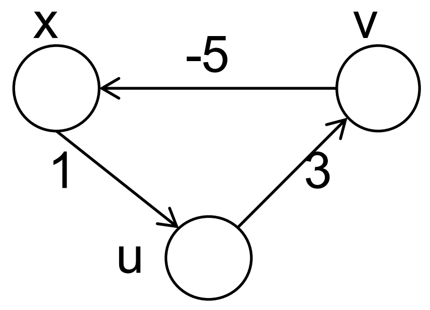
=> $v.d \leq u.d + w(u,v)$ for all (u, v) in C

=> $\Sigma_{v \text{ in } C}\, v.d \leq \Sigma_{u \text{ in } C}\, u.d + \Sigma_{(u,v) \text{ in } C}\, w(u,v)$

=> $0 \leq \Sigma_{(u,v) \text{ in } C}\, w(u,v)$

=> contradiction

Practice: show contradiction using this example

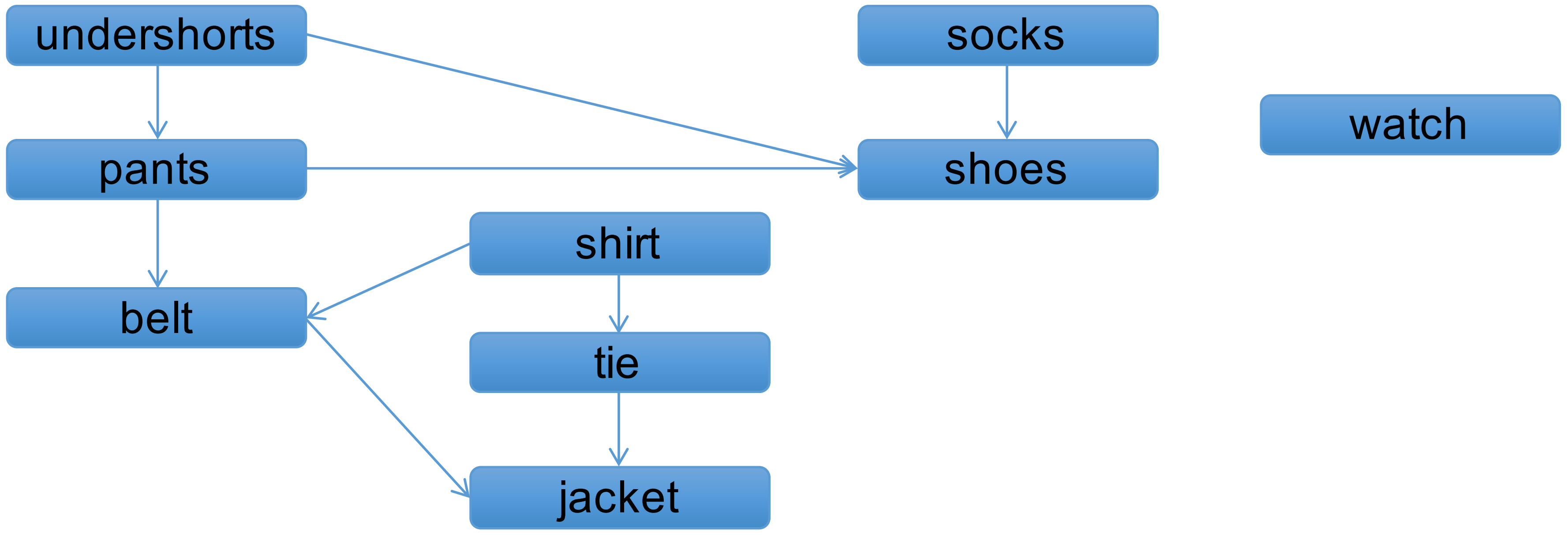


Single-source shortest paths in directed acyclic graphs

Textbook Chapter 24.2

Recall: Directed Acyclic Graphs (DAGs)

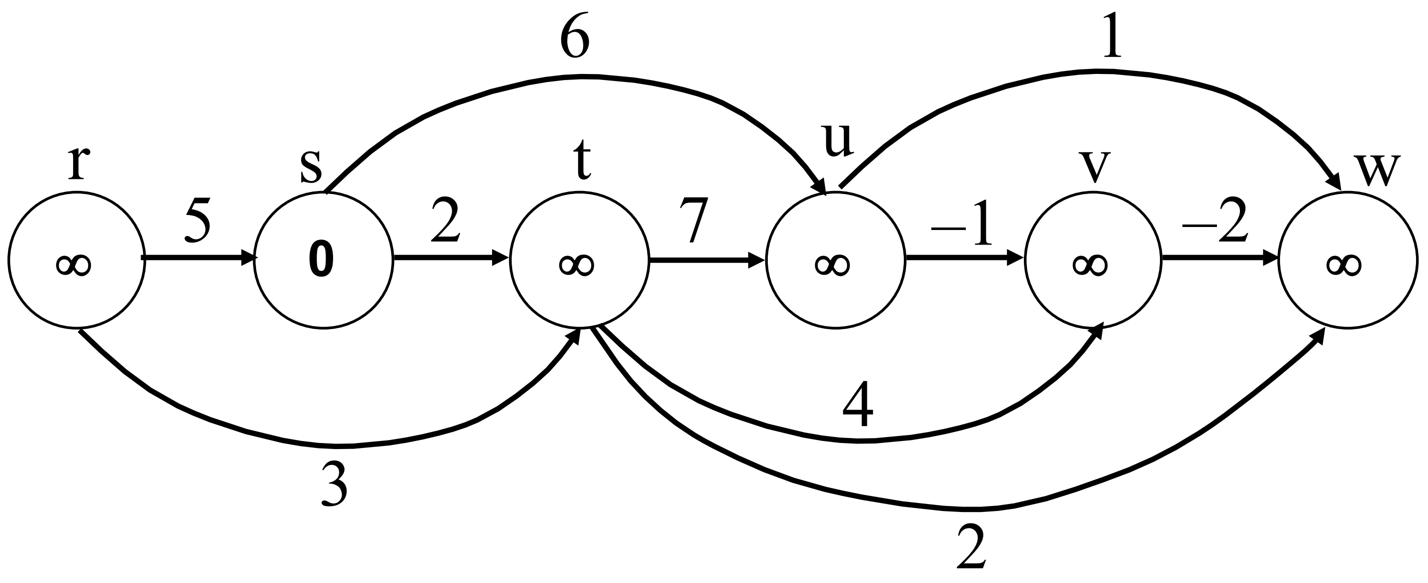
- A DAG is a directed graph with no cycles
- Often used to indicate precedence among events (X must happen before Y)
    - E.g., cooking, taking courses, clothing…



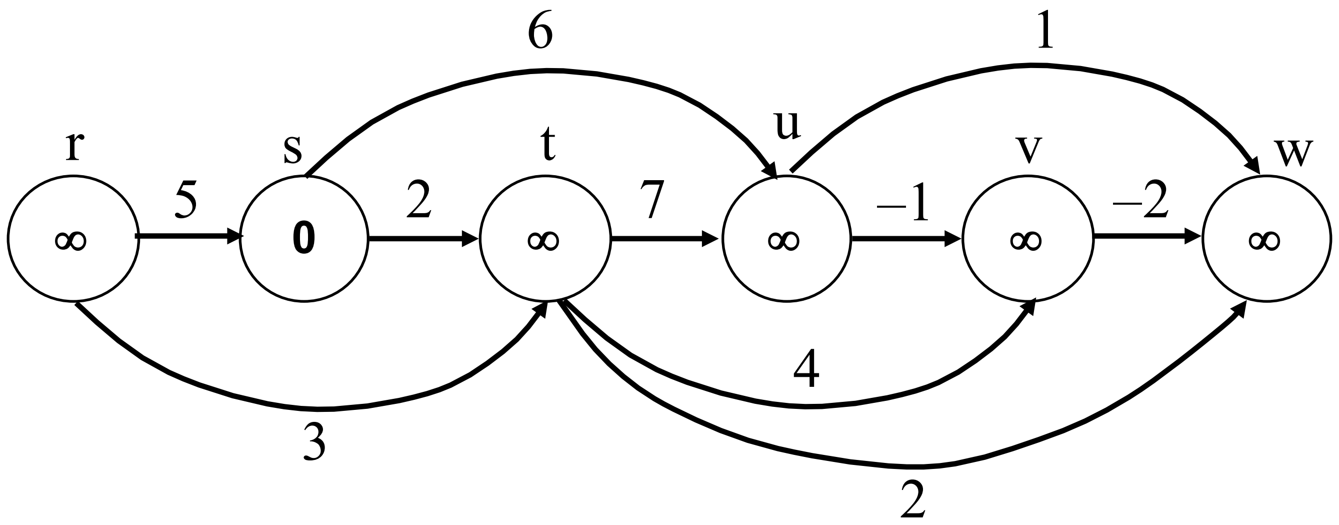
Single-source shortest paths in DAG

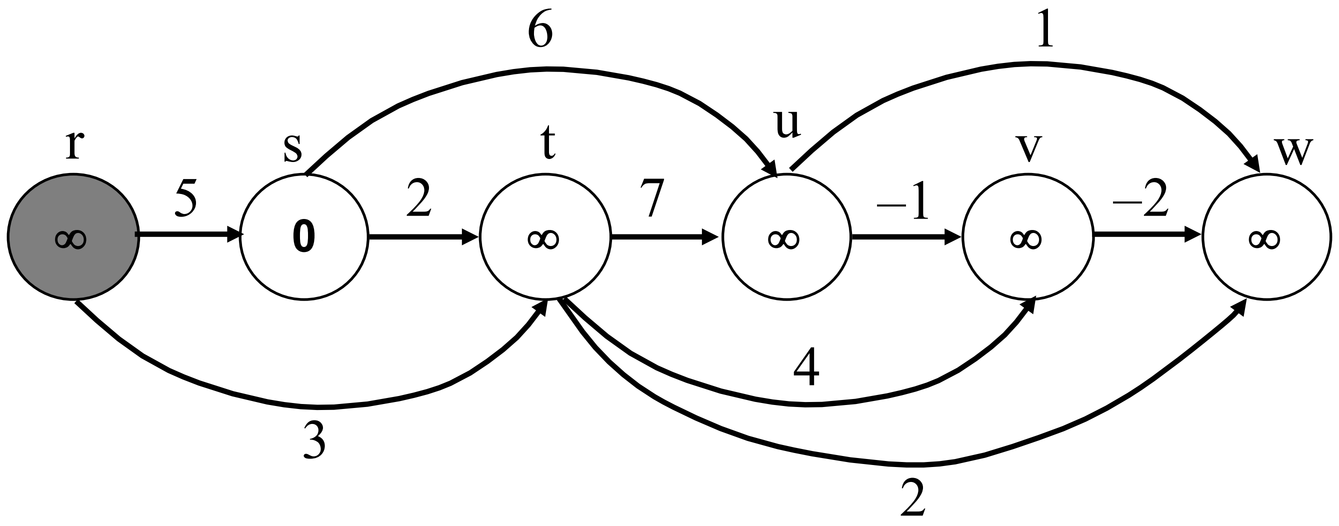
Claim: relaxing the edges in **topologically sorted order** correctly computes the shortest paths

Reason: putting vertices in a topologically sorted order, edges only go from left to right

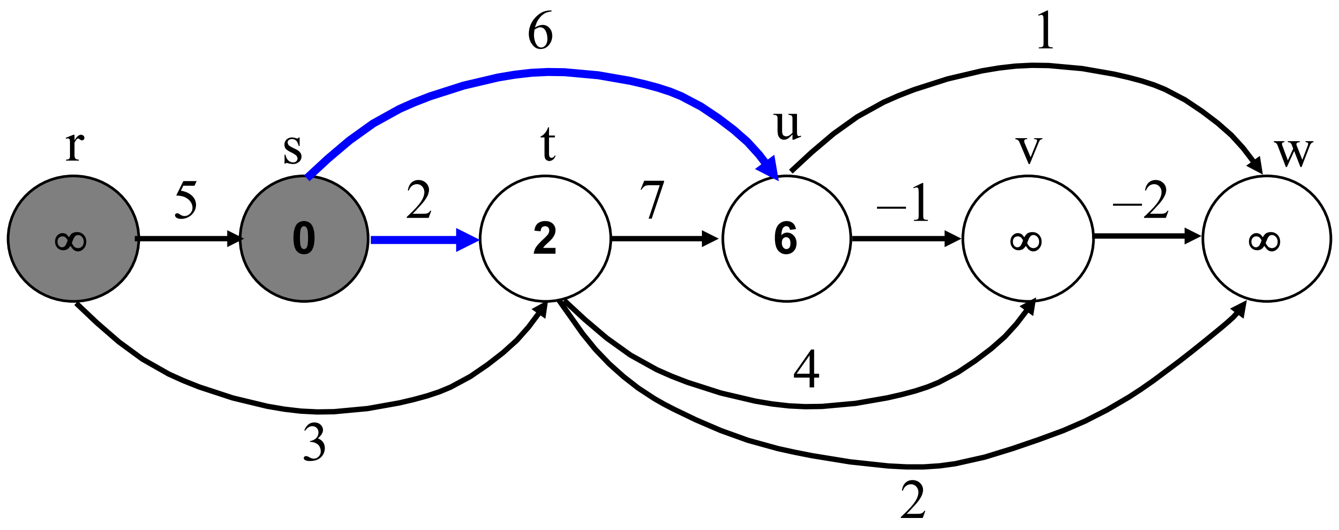


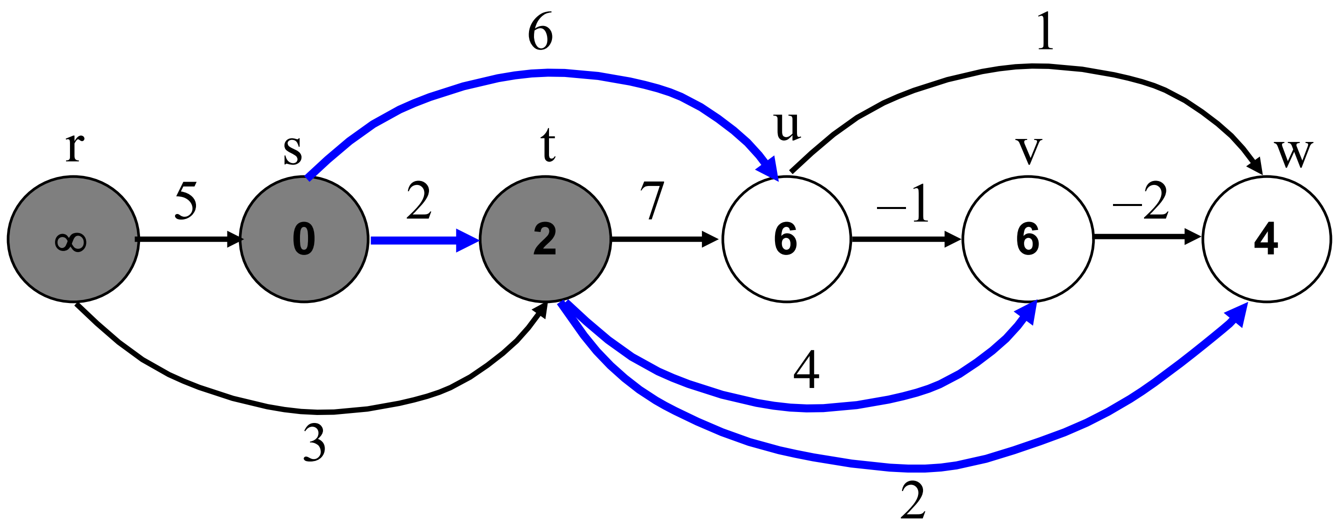
```
DAG-SHORTEST-PATHS(G,w,s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G,s)
for each vertex u, taken in topologically sorted order
    for each vertex v in G.adj[u]
        RELAX(u,v,w)
```



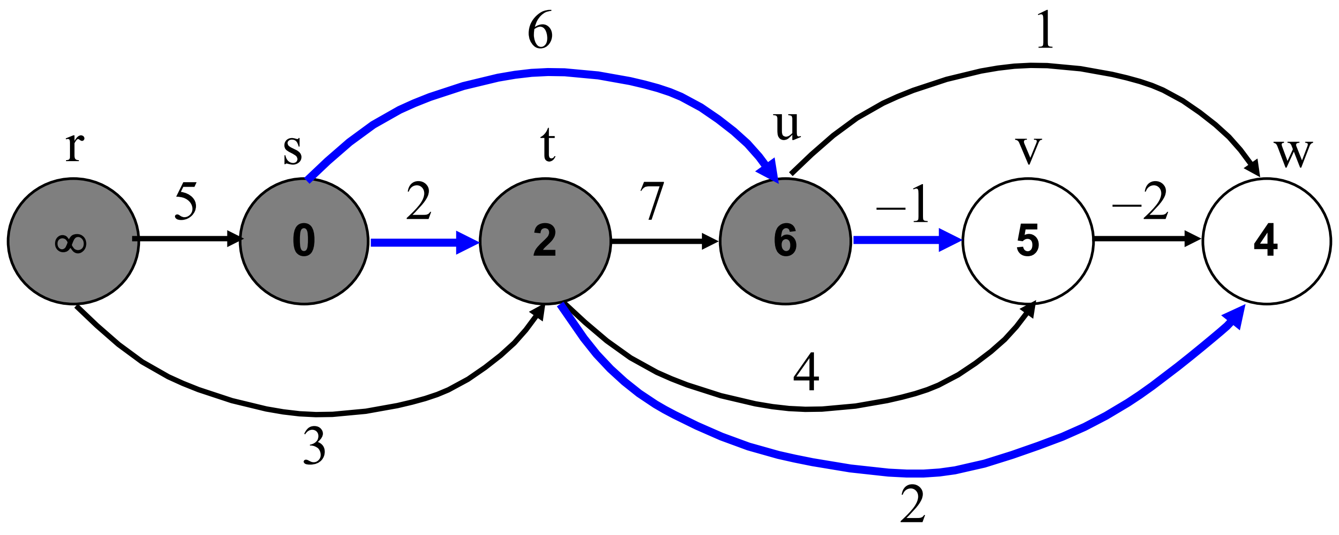


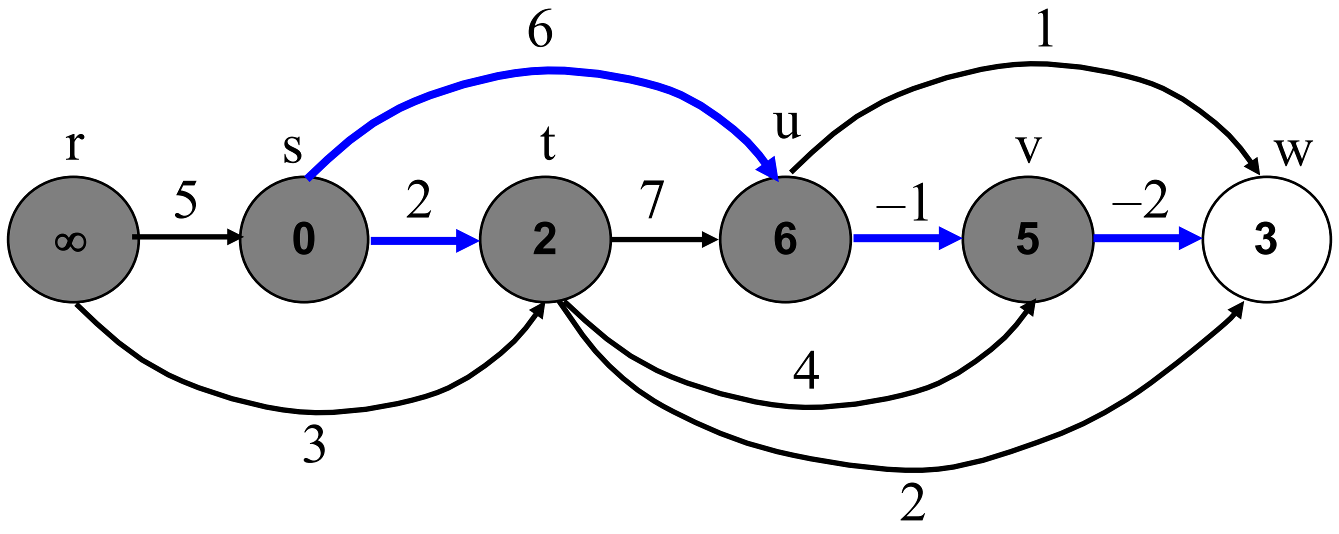
```
DAG-SHORTEST-PATHS(G,w,s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G,s)
for each vertex u, taken in topologically sorted order
    for each vertex v in G.adj[u]
        RELAX(u,v,w)
```





```
DAG-SHORTEST-PATHS(G,w,s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G,s)
for each vertex u, taken in topologically sorted order
    for each vertex v in G.adj[u]
        RELAX(u,v,w)
```

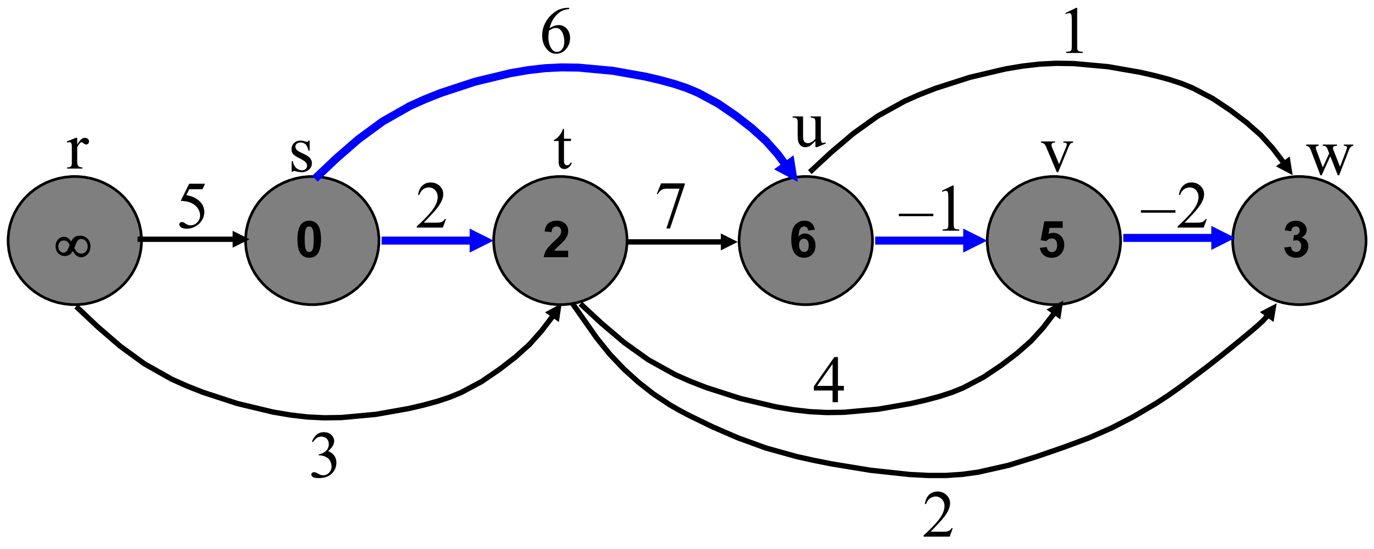




Running time analysis

```
DAG-SHORTEST-PATHS(G,w,s)
topologically sort the vertices of G // Θ(V+E)
INITIALIZE-SINGLE-SOURCE(G,s) // Θ(V)
for each vertex u, taken in topologically sorted order  ⎫
    for each vertex v in G.adj[u]                       ⎬ Θ(V+E)
        RELAX(u,v,w)                                    ⎭
```

=> total running time is Θ(V+ E), same as topological sort



Proof of correctness

Claim: if G = (V, E) is a DAG, then at the termination of `DAG-SHORTEST-PATHS`, v.d = $\delta(s,v)$, for all v

Proof by induction on the position in topological sort order

Inductive hypothesis: if all the vertices before *v* in a topological sort order have been updated, then v.d = $\delta(s,v)$

Base case:

- For all v before s, v.d = ∞ = $\delta(s,v)$
- s.d = 0 = $\delta(s,s)$

Proof of correctness

Claim: if G = (V, E) is a DAG, then at the termination of **`DAG-SHORTEST-PATHS`**, v.d = $\delta(s,v)$, for all v

Proof by induction on the position in topological sort order

Inductive hypothesis: if all the vertices before v in a topological sort order have been updated, then v.d = $\delta(s,v)$

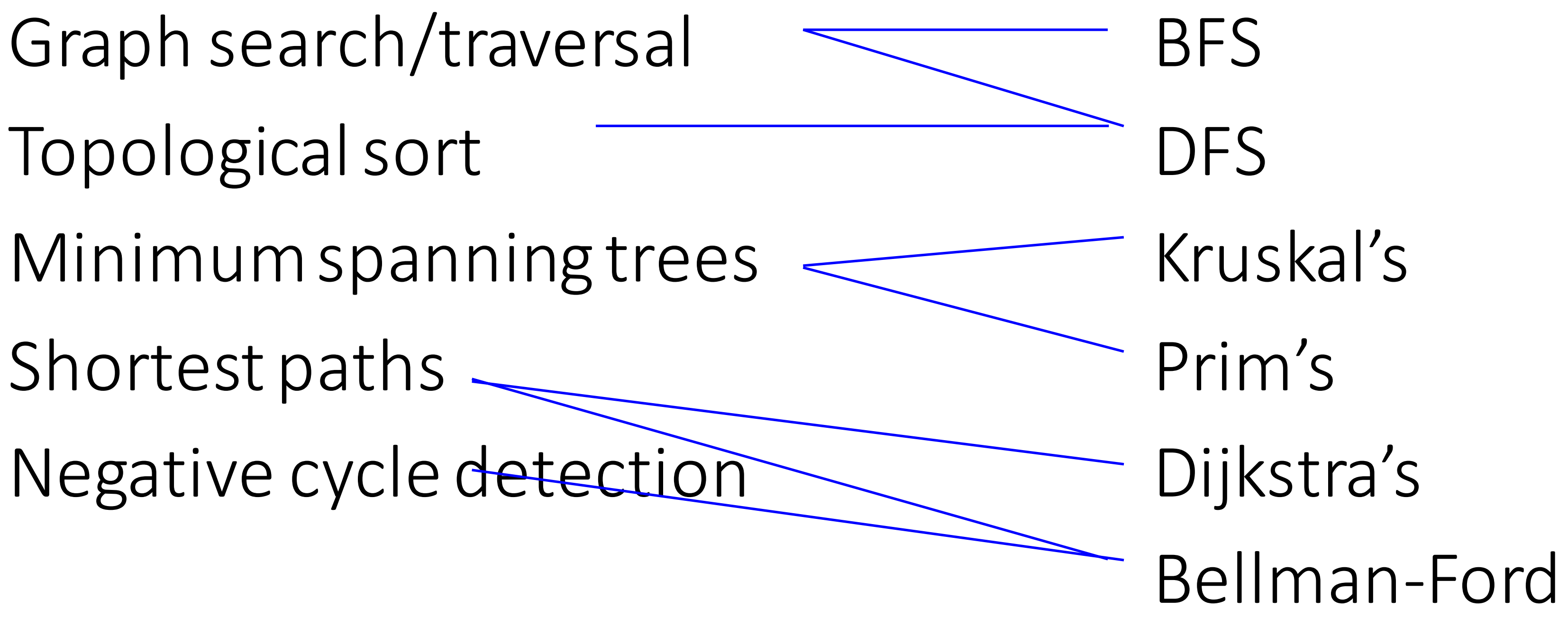
Inductive step:

- Consider a vertex v after s
- By construction, v.d = $\min_{(u,v) \text{ in } E}$ (u.d + w(u,v))
- By inductive hypothesis, u.d + w(u,v) = $\delta(s,u)$ + w(u,v)
- Since some (u,v) must be on the shortest path, by optimal substructure, v.d = $\delta(s,v)$

Summary of single-source shortest-path algorithms

SSSP algorithm	Applicable graph types	Running time
Dijkstra	Nonnegative weights	$\Theta(V^2)$ (array-based)
Topological sort based	DAG	$\Theta(V+E)$
Bellman-Ford	generic	$\Theta(EV)$

Summary of graph algorithms



Application: Internet routing

Vertices = routers, ASes

Edges = network links between routers

Edge weight = delay, bandwidth, cost, hop count, etc.

Link-state (commonly using Dijkstra's algorithm)

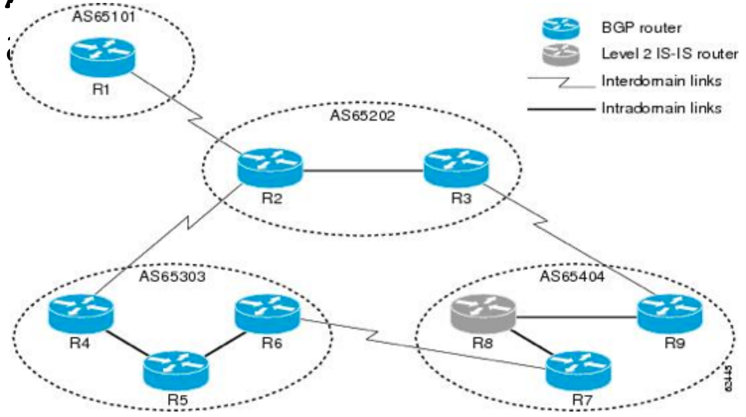
- Nodes flood link state to whole network
- E.g., Open Shortest Path First (OSPF)

Distance-vector (commonly using Bellman-Ford's algorithm)

- Nodes send vectors of destination and distance to neighbors
- E.g., Routing Information Protocol (RIP)

Path-vector (not necessarily shortest paths)

- Nodes advertise the full paths to each destin
- E.g., Border Gateway Routing Protocol (BGP)

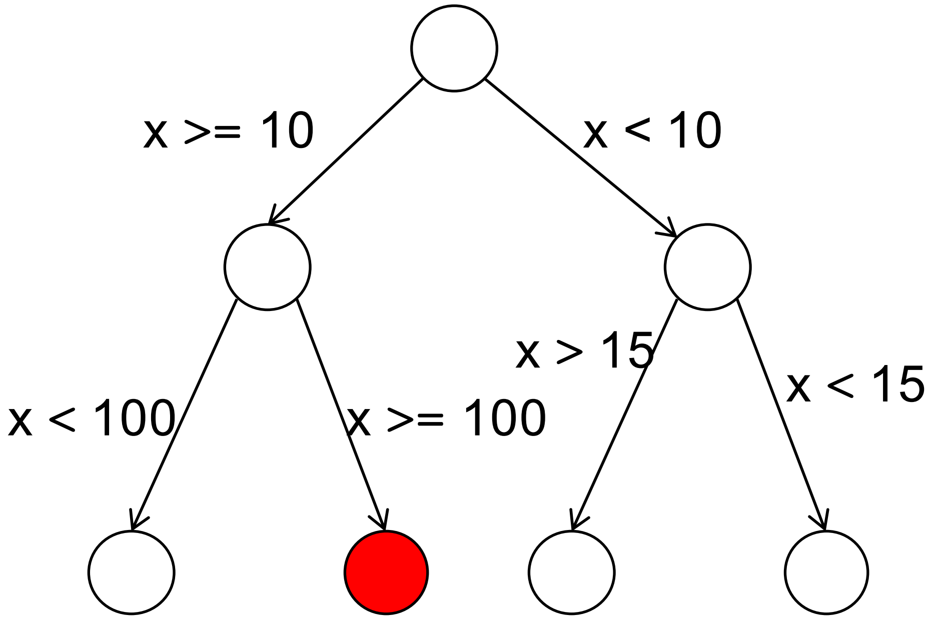


Source: cisco.com

Application: automatic bug finding

Can be seen as a graph search problem on program execution

```
x = int(input())
if x >= 10:
    if x < 100:
        print "Two!"
    else:
        print "Lots!"
else:
    if x > 15:
        print "One!”
    else:
        print ”Three!"
```



How to find the red state?
What should x be to reach the red state?
Is there unreachable state?