COMS 4231: Analysis of Algorithms I, Fall 2018

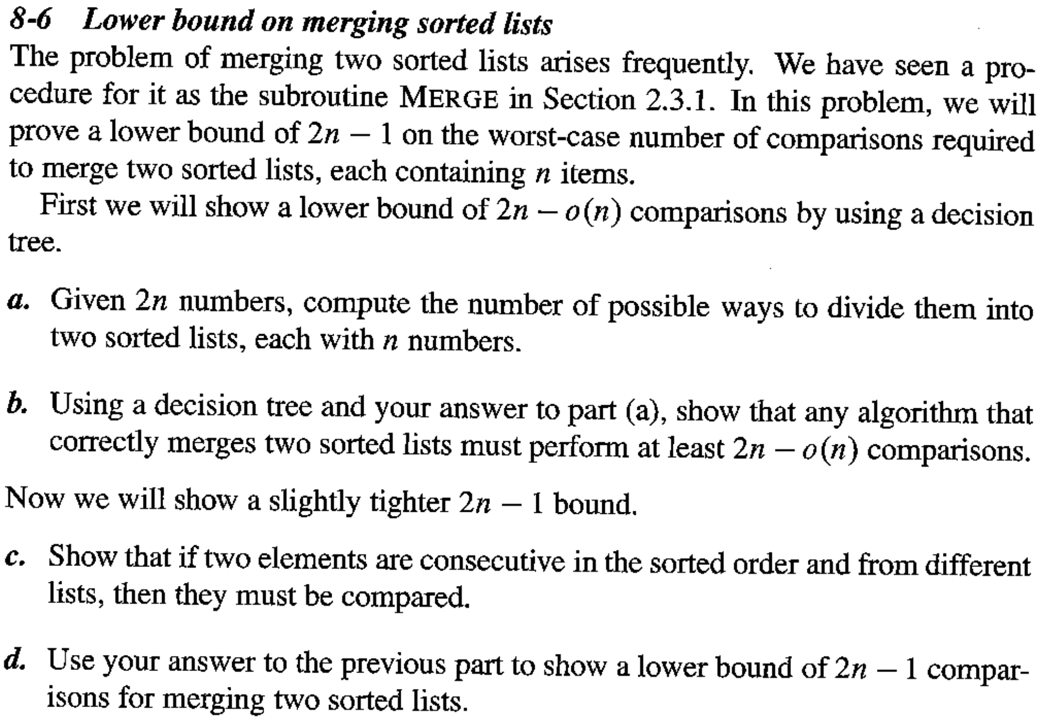
Problem Set 3, due Friday October 19, 11:59pm on Courseworks.

Please follow the homework submission guidelines posted on courseworks.

In all problems that ask you to give an algorithm, include a justification of the correctness of the algorithm and of its running time.

All time bounds refer to deterministic worst-case complexity, unless specified otherwise.

Problem 1. Problem 8.6 in CLRS, page 208. We reproduce it here for convenience.



8-6 Lower bound on merging sorted lists

The problem of merging two sorted lists arises frequently. We have seen a procedure for it as the subroutine MERGE in Section 2.3.1. In this problem, we will prove a lower bound of $2n - 1$ on the worst-case number of comparisons required to merge two sorted lists, each containing n items.

First we will show a lower bound of $2n - o(n)$ comparisons by using a decision tree.

a. Given $2n$ numbers, compute the number of possible ways to divide them into two sorted lists, each with n numbers.

b. Using a decision tree and your answer to part (a), show that any algorithm that correctly merges two sorted lists must perform at least $2n - o(n)$ comparisons.

Now we will show a slightly tighter $2n - 1$ bound.

c. Show that if two elements are consecutive in the sorted order and from different lists, then they must be compared.

d. Use your answer to the previous part to show a lower bound of $2n - 1$ comparisons for merging two sorted lists.

Problem 2. We are given *n* lists $L_1, L_2, ..., L_n$ of integers in the range 1 to *n*. The sum of the sizes of the lists is *m*. Give an O(*n*+*m*)-time algorithm to sort all the lists; i.e., the algorithm should compute lists $L_1', L_2', ..., L_n'$, where L_i' contains in sorted order the same elements as L_i for each *i*=1,…,*n*.

Note: There are no restrictions on *n, m* or on the numbers in the lists. In particular, the same integer may appear in different lists, or may appear multiple times on the same list. Some lists may be empty or short, some may be long.

Problem 3. Give an O($n$log$k$)-time algorithm to merge k sorted lists into one sorted list, where n is the total number of elements in all the input lists.
(*Hint:* One solution uses a min-heap.)

Problem 4. Give an algorithm which takes as input a binary search tree T with distinct keys and two keys x, y (which may or may not be in T), and outputs in sorted order all the elements of T whose key is in the (closed) interval $[x,y]$, i.e., such that their key satisfies $x \leq$ key $\leq y$. The algorithm should run in time O(h+s) where h is the height of T and s is the number of elements output.

Problem 5. In the *Bin Packing* problem, we want to pack a set of n items with sizes s_1, s_2, …, s_n into bins of capacity B; all sizes satisfy $0< s_i <B$. In other words we want to assign each item to a bin, so that the sum of the sizes of the items assigned to each bin is at most B. The objective is to use as few bins as possible. There are several heuristic algorithms for the Bin Packing problem, which compute reasonable solutions, although not always the optimal solution. One of them is the *Best Fit* heuristic, which works as follows:
Process the items in order 1,…,n. For each item i, put i in a bin that still has enough room for the item and is as full as possible; if none of the bins used so far has enough room for item i, then start a new bin and put i in it.
For example, if we have n=6 items with sizes s_1=4, s_2=6.3, s_3=2.5, s_4=3, s_5=1, s_6=4, and the bin capacity is B=10, then Best Fit will put item 1 in bin 1, item 2 in bin 2, item 3 in bin 2, item 4 in bin 1, item 5 in bin 2, and item 6 in bin 3.
Give an algorithm that implements the Best Fit heuristic and runs in time O($n$log$k$), where k is the number of bins used. The algorithm should compute the mapping from items to bins according to the Best Fit heuristic.

Problem 6. We wish to maintain a collection S of pairs (birthdate, salary) for the employees of a company. Note that different employees may have the same birthdate and/or salary, so S may contain duplicate pairs. Propose a data structure that supports efficient Insertion and Deletion of pairs and in addition allows us to answer efficiently the following queries:
1. *MinBday*: What is the minimum birthdate?
2. *Count*(d): How many employees (tuples) have birthdate $\leq d$?
3. *AvgSal*(d): What is the average salary of employees with birthdate $\leq d$?
Describe each operation and analyze its time complexity. Make your data structure as efficient as you can. You can assume that comparison of dates and salaries as well as arithmetic operations take constant time.