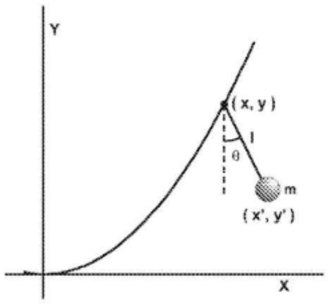
25.- *El punto de suspensión de un péndulo está obligado a moverse a lo largo de la parábola $y = ax^2$. Encontrar el Hamiltoniano.*



$$x' = x + l\,\mathrm{sen}\theta \quad ; \quad y' = y - l\cos\theta \qquad \rightarrow y = \alpha x^2$$

$$\dot{x}' = \dot{x} + l\cos\theta\,\dot{\theta} \quad ; \quad \dot{y}' = \dot{y} + l\,\mathrm{sen}\theta\,\dot{\theta} \qquad \dot{y} = 2\alpha x\dot{x}$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2) - mgy'$$

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{x}^2 + 4\alpha^2x^2\dot{x}^2 + 2l\dot{x}\dot{\theta}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)\right) - mg(\alpha x^2 - l\cos\theta)$$

$$\mathcal{H} = \sum p_i\dot{x}_i - \mathcal{L}$$

$$p_x = m\dot{x} + 4m\alpha^2x^2\dot{x} + ml\dot{\theta}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)$$

$$\dot{x} = \frac{p_x - ml\dot{\theta}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)}{m + 4m\alpha^2x^2}$$

$$p_\theta = ml^2\dot{\theta} + 2l\dot{x}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)m$$

$$\dot{\theta} = \frac{p_\theta - 2l\dot{x}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)m}{ml^2}$$

inyectando $\dot{\theta}$ en $\dot{x}$

$$\dot{x} = \frac{p_x - ml\left(\frac{p_\theta - 2l\dot{x}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)m}{ml^2}\right)(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)}{m + 4m\alpha^2x^2}$$

$$\dot{x} = \frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)} + \frac{2l\dot{x}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2}{lm(1 + 4\alpha^2x^2)}$$

$$\dot{x} - \frac{2l\dot{x}(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}{lm(1 + 4\alpha^2x^2)} = \frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}$$

$$\dot{x} = \left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)$$

$$\dot{\theta} = \frac{p_\theta - 2lm\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)}{ml^2}$$

$$\begin{aligned}
\mathcal{H} = \Bigg(& p_x\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right) + \\
& p_\theta\left(\frac{p_\theta - 2lm\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)}{ml^2}\right) \\
& - \frac{1}{2}m\Bigg[\frac{\left(p_\theta - 2lm\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)\right)^2}{m^2l^2} \\
& + \left(\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)\right)^2 \\
& + 2l(\cos\theta + 2x\,\mathrm{sen}\theta)\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)\left(\frac{p_\theta - 2lm\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2}\right)\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)}{ml^2}\right) \\
& + 4\alpha^2x^2\left(\frac{lm(1 + 4\alpha^2x^2)}{1 - 2l(\cos\theta + 2\alpha x\,\mathrm{sen}\theta)^2 m}\right)^2\left(\frac{p_x}{m + 4m\alpha^2x^2} - \frac{p_\theta(\cos\theta + 2\alpha\,\mathrm{sen}\theta)}{lm(1 + 4\alpha^2x^2)}\right)^2\Bigg] + mg(\alpha x^2 - l\cos\theta)
\end{aligned}$$

2) Sea el Hamiltoniano

$$H = \frac{p^2}{2m} - A\left(\frac{p}{m}\cos\gamma t + \gamma q \,\text{Sen}\,\gamma t\right) + \frac{1}{2}kq^2$$

donde A, γ, k son constantes

a) Hallar el lagrangeano L

$$L = \sum p_i \dot{q}_i - H$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} - \frac{A\cos(\gamma t)}{m}$$

$$m\dot{q} + \frac{A\cos(\gamma t)}{m} = p$$

$$L = (m\dot{q} + A\cos(\gamma t))\dot{q} - H(q, \dot{q})$$

$$H(q,\dot{q}) = \frac{(m\dot{q} + A\cos(\gamma t))^2}{2m} - A\left(\left(\dot{q} + \frac{A}{m}\cos(\gamma t)\right)\cos(\gamma t)\right.$$

$$\left. + \gamma q \,\text{Sen}\,\gamma t\right) + \frac{1}{2}kq^2$$

$$\frac{(m\dot{q} + A\cos\gamma t)^2}{2m} = \frac{m^2\dot{q}^2 + 2m\dot{q}A\cos(\gamma t) + A^2\cos^2(\gamma t)}{2m}$$

$$= \frac{m\dot{q}^2}{2} + A\dot{q}\cos(\gamma t) + \frac{A^2\cos^2(\gamma t)}{2m}$$

$$\Rightarrow H = \frac{m\dot{q}^2}{2} + A\dot{q}\cos(\gamma t) + \frac{A^2\cos^2(\gamma t)}{2m} - A\dot{q}\cos(\gamma t) - \frac{A^2\cos^2(\gamma t)}{m}$$

$$- A\gamma q\,\text{Sen}(\gamma t) + \frac{1}{2}kq^2$$

$$\Rightarrow H = \frac{m\dot{q}^2}{2} - \frac{A^2\cos^2(\gamma t)}{2m} - A\gamma q\,\text{Sen}(\gamma t) + \frac{1}{2}kq^2$$

$$L = m\dot{q}^2 + A\dot{q}\cos(\gamma t) - \left(\frac{m\dot{q}^2}{2} - \frac{A^2\cos^2(\gamma t)}{2m} - A\gamma q\,\text{Sen}(\gamma t) + \frac{1}{2}kq^2\right)$$

$$L = \frac{m\dot{q}^2}{2} + A\dot{q}\cos(\gamma t) + \frac{A^2\cos^2(\gamma t)}{2m} + A\gamma q\,\text{Sen}(\gamma t) - \frac{1}{2}kq^2$$

3 Sabemos que

$$H = q + te^{p}$$

y realizamos la transformación

$$Q = q + te^{p} \quad , \quad P = p$$

Para que la transformación sea canónica se debe cumplir que

- $\{Q, Q\} = 0$ → $\frac{\partial Q}{\partial q} = 1$ → $\frac{\partial Q}{\partial p} = e^{p}$
- $\{P, P\} = 0$ → $\frac{\partial P}{\partial q} = 0$ → $\frac{\partial P}{\partial p} = 1$
- $\{Q, P\} = 1$

- $\{Q, Q\} = 1\, e^{P} - 1\, e^{P} = 0$
- $\{P, P\} = 1\, 0 - 0\, 1 = 0$
- $\{Q, P\} = 1\, 1 - e^{P}\, 0 = 1$

Sabiendo que sí es una transformación canónica, hallamos la función generatriz

La función $F(q, P)$ cumple que:

$$Q = \frac{\partial F}{\partial P} \quad , \quad p = \frac{\partial F}{\partial q}$$

$$F = qP + e^{P}$$

Para hallar el nuevo hamiltoniano tenemos que

$$H = q + t e^{p}$$

Para expresarlo en terminos de Q y P

$$q = Q - e^{P} \quad , \quad P = p$$

Así,

$$H' = Q - e^{P} + t e^{P}$$

$$H' = Q + e^{P}(t-1)$$

Ahora, para resolver las ecuaciones de movimiento

$$\dot{Q} = \frac{\partial H'}{\partial P} = (t-1)e^{P} \quad (1)$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -1 \quad (2)$$

Ahora, resolviendo para P tenemos que

$$P(t) = -t + C \quad (3)$$

y reemplazando en (1)

$$\dot{Q} = (t-1)e^{c-t}$$

Integrando

$$Q = e^{c}\int (t-1)\, e^{-t}\, dt$$

$u = (t-1)$ $\quad dv = e^{-t}\, dt$
$du = dt$ $\quad v = -e^{-t}$

Entonces

$$\int (t-1)e^{-t}\, dt = e^{-t}(1-t) + \int e^{-t}\, dt$$

$$Q = e^{c}\left(e^{-t}(\cancel{1}-t) - \cancel{e^{-t}}\right)$$

$$Q(t) = -e^{c-t} + D$$

Así tenemos nuestras ecuaciones de movimiento

$$\boxed{\begin{aligned} P(t) &= -t + C \\ Q(t) &= -e^{c-t} + D \end{aligned}}$$

4 Tenemos el hamiltoniano

$$H = \frac{1}{2}\left(qp^{3} + \frac{q}{p}\right)$$

Para escribir las ecuaciones de movimiento usamos las ecuaciones de Hamilton

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\frac{\partial H}{\partial p} = \dot{q} = \frac{1}{2}\left(3qp^{2} - \frac{q}{p^{2}}\right)$$

$$-\frac{\partial H}{\partial q} = \dot{p} = -\frac{1}{2}\left(p^{3} + \frac{1}{p}\right)$$