1) La ecuacion de la orbita del cometa:

$$r = \frac{q}{1+\cos\theta}$$

donde $q = \frac{l^2}{\mu k}$ $\Rightarrow$ $k = GMm_s$

$l = \mu r^2 \dot{\theta} = cte$

$\mu = \frac{m m_s}{m + m_s}$

$$R_T = \frac{q}{1+\cos\alpha}$$

$$\cos\alpha = \frac{q}{RT} - 1$$

$$\alpha = \cos^{-1}\left(\frac{q}{RT} - 1\right)$$

Por lo tanto el inicio y el final del trayecto dentro de la orbita de la tierra

$\theta_0 = 2\pi - \alpha$, $\theta_f = \alpha$

$\Rightarrow dA = \frac{1}{2} r^2 d\theta$

$\frac{dA}{dt} = \frac{l}{2\mu} \rightarrow dA = \frac{l\,dt}{2\mu}$

igualamos $r^2 d\theta = \frac{l}{\mu} dt$

$$\frac{\mu}{l} r^2 d\theta = dt$$

$$\frac{\mu}{l} \frac{q^2}{(1+\cos\theta)^2} d\theta = dt$$

$$\frac{\mu}{l} q^2 \int_{2\pi-\alpha}^{\alpha} \frac{1}{(1+\cos\theta)^2} = t$$

$$\frac{\mu q^2}{l}\left(\frac{\tan^3\left(\frac{\alpha}{2}\right) + 3\tan\left(\frac{\alpha}{2}\right)}{6}\right) - \left(\tan^3\left(\frac{2\pi-\alpha}{2}\right)\right.$$

$$\left. + 3\tan\left(\frac{2\pi-\alpha}{2}\right)\right) = t$$

$$t \text{ en dias} = \frac{c}{86400}$$

P

2) $V(r) = \frac{a}{r} + \frac{b}{r^2}$

a) Determine la orbita de $r = r(\theta)$

con la sustitucon $u = \frac{1}{r}$

$$V(u) = au + bu^2$$

$$\frac{L^2}{\mu}\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\partial V}{\partial u}$$

$$\frac{d^2u}{d\theta^2} + u = \left(-a - 2bu\right)\frac{\mu}{l^2}$$

$$\frac{d^2u}{d\theta^2} + u + \frac{2bu\mu}{l^2} = -\frac{a\mu}{l^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{2b\mu}{l^2}\right)u = -\frac{a\mu}{l^2}$$

$$u'' + \omega_n^2 u = 0 \qquad \omega^2 = 1 + \frac{2b\mu}{l^2}$$

$$u_p = A\cos(\omega\theta - \theta_0)$$

$$u_c = -\frac{a\mu}{l^2 + 2b\mu}$$

$$u(\theta) = -\frac{a\mu}{l^2\omega^2} + A\cos(\omega\theta - \theta_0)$$

$$\frac{1}{r} = -\frac{a\mu}{l^2\omega^2}\left[1 + e\cos(\omega\theta - \theta_0)\right]$$

$$\frac{q}{r} = 1 + e\cos\theta$$

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu V(u)}{l^2} - u^2}}$$

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu(au + bu^2)}{l^2} - u^2}}$$

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu a u}{l^2} - \left(\frac{2\mu b}{l^2}\right)u^2}}$$

$$\int \frac{du}{\sqrt{au^2 + bu + c}} = \frac{1}{\sqrt{-a}} \cos^{-1}\left[-\frac{(b + 2au)}{\sqrt{b^2 - 4ac}}\right]$$

$$a = -\frac{2\mu b}{l^2} \qquad b = -\frac{2\mu a}{l^2} \qquad c = \frac{2\mu E}{l^2}$$

$$b^2 - 4ac = \left(-\frac{2\mu a}{l^2}\right)^2 - 4\left(-\frac{2\mu b}{l^2}\right)\left(\frac{2\mu E}{l^2}\right)$$

$$= \frac{4\mu^2 a^2}{l^4} + \frac{16\mu^2 bE}{l^4} = \left(\frac{2\mu}{l^2}\right)^2 \left(a^2 + 4bE\right)$$

$$b + 2au = -\frac{2\mu a}{l^2} + 2\left(-\frac{2\mu b}{l^2}\right)u$$

$$= -\frac{2\mu a}{l^2} - \frac{4\mu b u}{l^2}$$

$$= -\frac{2\mu a}{l^2}\left(1 + \frac{2bu}{a}\right)$$

$$\theta = \theta_0 - \frac{1}{\sqrt{2\mu b}} \cos^{-1}\left(\frac{\frac{2\mu a}{l^2}\left(1 + \frac{2bu}{a}\right)}{\frac{2\mu}{l^2}\sqrt{a^2 + 4bE}}\right)$$

$$-\frac{\sqrt{a^2+4bE}\,\sqrt{2\mu b}}{l a}\cos(\theta+\theta_0) = 1 + \frac{2bu}{a}$$

$$u = \frac{1}{r} \Rightarrow -\left(\frac{\sqrt{a^2+4bE}\,\sqrt{2\mu b}}{l a}\cos(\theta-\theta_0) + 1\right)\frac{a}{2b} = \frac{1}{r}$$

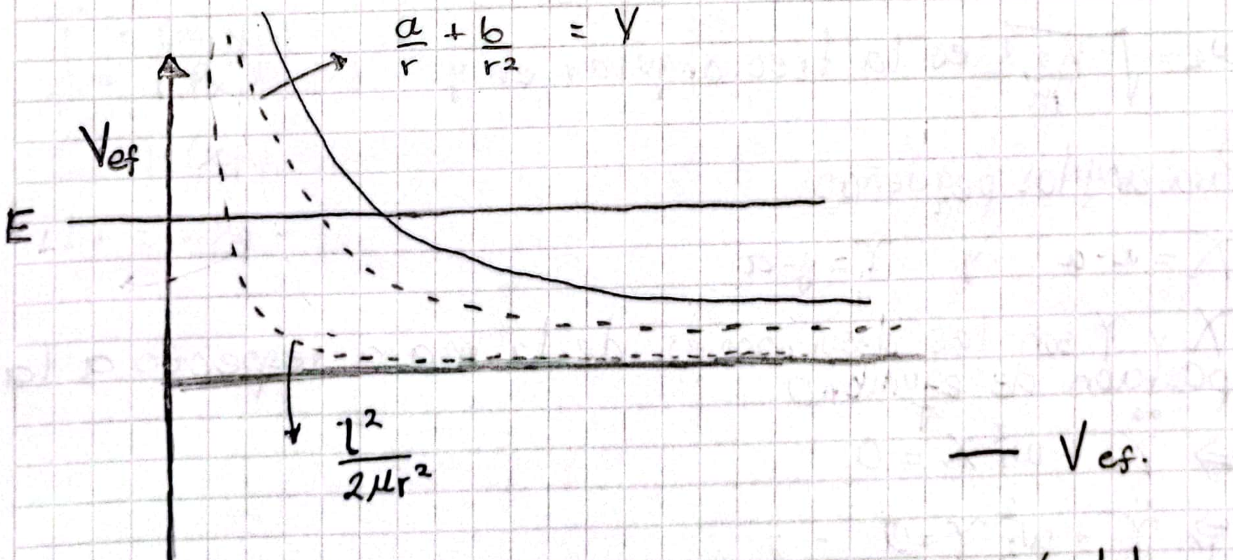
Para $\theta_0 = 0$:

$$\Rightarrow \frac{1}{r} = -\frac{a}{2b}\left(\frac{\sqrt{(a^2+4bE)(2\mu b)}}{l a}\cos(\theta-\theta_0) + 1\right)$$

$$e = \frac{\sqrt{(a^2+4bE)(2\mu b)}}{l a}$$

$$q = -\frac{2b}{a}$$

$$r(\theta) = \frac{q}{1+e\cos\theta} = \frac{-2b}{1+\sqrt{(a^2+4bE)(2\mu b)}\cos\theta}$$



$E > 0 \Rightarrow r_{min} > 0$ y $r_{max} \to \infty$ (orbita abierta)

3. $V(r) = -\frac{K e^{-r/a}}{r}$, donde $k>a$ y $a>0$

a) Suponiendo que está en un plano:

$$\mathcal{L} = \frac{1}{2} m \cdot \vec{v}^2 - \frac{K \cdot e^{-r/a}}{r}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K \cdot e^{-r/a}}{r}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \quad ; \quad \frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \dot{\theta}$$

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{K e^{-\frac{r}{a}}}{r}$$

Como θ es cíclica, $\dot{\theta}$ es constante.

$P_\theta = m r^2 \dot{\theta} =$ constante.

La componente L_z de $\vec{L}$ en coordenadas cartesianas es la misma:

$L_z = m(x\dot{y} - \dot{x}y) \equiv m r^2 \dot{\theta} = P_\theta$

$$\dot{\theta} = \frac{L}{m r^2}$$

Sustituyendo en la energía

$$E = \frac{m \cdot \dot{r}}{2} + \frac{L^2}{2 m r^2} - \frac{K e^{-r/a}}{r}$$

Y el potencial efectivo:

$$V_{ef} = \frac{L^2}{2 m r^2} - \frac{K e^{-\frac{r}{a}}}{r}$$

Usando la condición de estabilidad: $\frac{3 f(r_0)}{r_0} + f'(r_0) < 0$

$$-\frac{\partial V}{\partial r} = f(r) = -K\left(\frac{1}{ar} + \frac{1}{r^2}\right) e^{-\frac{r}{a}}$$

$$\frac{\partial f}{\partial r} = f'(r) = K\left(\frac{2}{r^3} + \frac{2}{a r^2} + \frac{1}{a^2 r}\right) e^{-\frac{r}{a}}$$

La condición de estabilidad se puede escribir como

$$3 + \frac{r_0 f'(r)}{f(r)} > 0$$

$$-\left(\frac{a r_0^5 + 2a^2 r_0^4 + 2a^3 r_0^3}{a^2 r_0^3}\right) \frac{1}{r_0 + a} + 3 > 0$$

$$\cdot \; -\frac{r_0^2 + 2 a r_0 + 2a^2}{a(r_0 + a)} + 3 > 0$$

Usando wolfram alpha:

$$\frac{a^2}{r_0^2} + \frac{a}{r_0} - 1 > 0 \longrightarrow \frac{a}{r_0} \gtrsim 0{,}618$$

b) $T_r = \frac{2\pi}{\omega_r}$

$$\omega_r^2 = \frac{1}{\mu} \left.\frac{\partial^2 V_{ef}}{dr^2}\right|_{r_0} = \frac{1}{m}\left[\frac{3L^2}{m r_0^4} - K\left(\frac{2}{r_0^3} + \frac{2}{a r_0^2} + \frac{1}{a^2 r_0}\right)\right]$$

$$T_r = 2\pi \cdot \left\{\frac{1}{m}\left[\frac{3L^2}{m r_0^4} - K\left(\frac{2}{r_0^3} + \frac{2}{a r_0^2} + \frac{1}{a^2 r_0}\right)\right]\right\}$$

4) $x\hat{i} + y\hat{j} = r\cos\theta\,\hat{i} + r\,\text{sen}\,\theta\,\hat{j}$

$r = a(1+\cos\theta)$

$u = \frac{1}{r} = \frac{1}{a(1+r\cos\theta)}$

$$\frac{l^2}{\mu a}\left(\frac{d}{d\theta}\left(\frac{\text{sen}\,\theta}{(1+r\cos\theta)^2} + \frac{1}{(1+\cos\theta)}\right)\right.$$

$$\frac{\cos\theta + 1 + \text{sen}^2\theta}{(1+\cos\theta)^3} + \frac{1}{(1+\cos\theta)}$$

$$\frac{(\cos\theta + 1 + \text{sen}^2\theta) + (1+\cos\theta)^2}{(1+\cos\theta)^3}$$

$$\cos\theta + 1 + \text{sen}^2\theta + 1 + 2\cos\theta + \cos^2\theta$$

$$\frac{3(1+\cos\theta)}{(1+\cos\theta)^3} = \frac{3}{(1+\cos\theta)^2}$$

$$\frac{a l^2}{\mu r^2} = -\frac{\partial V}{\partial r}$$