Etude mathématique et numérique du groupe de renormalisation non perturbatif

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Modèle d'Ising 2D par BMW

1 Introduction

On part de la fonction de partition

$$\mathcal{Z} \propto \int_{\mathbb{R}} \prod_{\mathbf{r}} \mathrm{d}\varphi_{\mathbf{r}} \, e^{-S_\mu[\varphi]} \tag{1}$$

Avec l'action S s'écrivant :

$$S_\mu[\varphi] = \frac{1}{2} \int_{\mathbf{q}} \varphi(\mathbf{q}) \frac{1}{\lambda_\mu(\mathbf{q})} \varphi(-\mathbf{q}) - \sum_{\mathbf{r}} \ln\left(\cosh(\varphi_{\mathbf{r}})\right) \tag{2}$$

Par le théorème de Parseval, nous réecrivons S sous la forme

$$\begin{aligned} S_\mu[\varphi] = \frac{1}{2} \int_{\mathbf{q}} \varphi(\mathbf{q}) \left[\frac{1}{\lambda_\mu(\mathbf{q})} - \frac{1}{\lambda_\mu(0)} \right] \varphi(-\mathbf{q}) \\ + \sum_{\mathbf{r}} \left[\frac{1}{2\lambda_\mu(0)} \varphi_{\mathbf{r}}^2 - \ln\left(\cosh(\varphi_{\mathbf{r}})\right) \right] \end{aligned} \tag{3}$$

Enfin, soit $\delta \in \mathbb{R}_*^+$, on pose le changement de variable,

$$\varphi \to \delta \sqrt{2\beta J d}\, \varphi \tag{4}$$

On obtient alors

$$S_\mu[\varphi] = \frac{1}{2} \int_{\mathbf{q}} \hat{\varphi}(\mathbf{q}) \varepsilon_0(\mathbf{q}) \hat{\varphi}(-\mathbf{q}) + \sum_{\mathbf{r}} V_0(\varphi(\mathbf{r})) \tag{5}$$

Avec, en posant $\tilde{\mu} = \mu/(Jd)$ et $\tilde{\beta} = \beta J d$,

$$\varepsilon_0(\mathbf{q}) = \delta^2 \frac{1 - \gamma(\mathbf{q})}{(\gamma(\mathbf{q}) + \tilde{\mu})(1 + \tilde{\mu})} \tag{6}$$

$$V_0(\rho) = \delta^2 \frac{1}{1 + \tilde{\mu}} \rho - \ln\left(\cosh\left(2\delta \sqrt{\tilde{\beta}\rho} \right) \right) \tag{7}$$

De plus, on note $\tilde{\beta}_c^{\mathrm{MF}}$ la valeur de $\tilde{\beta}$ en champ moyen à la temperature critique. En faisant un développement limité à l'ordre 1 en ρ nous avons

$$V_0(\rho) = \delta^2 \left(\frac{1}{1 + \tilde{\mu}} - 2\tilde{\beta} \right) \rho + \mathcal{O}(\rho^2) \tag{8}$$

Ainsi, nous obtenons

$$\tilde{\beta}_c^{\mathrm{MF}} \simeq \frac{1}{2(1 + \tilde{\mu})} \tag{9}$$

2 Les équations BMW en ρ dimensionnées

On pose

$$\Gamma_k^{(2)}(p_x, p_y, \rho) = \varepsilon_0(p_x, p_y) + \Delta_k(p_x, p_y, \rho) + \partial_\phi^2 V(\phi) \tag{10}$$

$$W(\phi) = \partial_\phi V(\phi) \quad \text{et} \quad X(\phi) = \partial_\phi^2 V(\phi) \tag{11}$$

Les équations à résoudre numériquement sont

$$\begin{aligned} \partial_t \Delta_k(p_x, p_y, \rho) = &- 2\rho I_3(\rho) u_k^2(\rho) + 2\rho J_3(p_x, p_y, \rho)[u_k(\rho) + \partial_\rho \Delta_k(p_x, p_y, \rho)]^2 \\ &- \frac{1}{2} I_2(\rho) \left[\partial_\rho \Delta_k(p_x, p_y, \rho) + 2\rho \partial_\rho^2 \Delta_k(p_x, p_y, \rho)\right] \end{aligned} \tag{12}$$

$$\partial_t W_k(\rho) = \frac{1}{2} \partial_\rho I_1(\rho) \tag{13}$$

Avec les notations

$$J_n(p_x, p_y, \rho) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y)\, G_k^{n-1}(q_x, q_y, \rho) G_k(p_x + q_x, p_y + q_y, \rho)\, \mathrm{d}q_x\, \mathrm{d}q_y \tag{14}$$

$$I_n(\rho) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y)\, G_k^n(q_x, q_y, \rho)\, \mathrm{d}q_x\, \mathrm{d}q_y \tag{15}$$

$$G_k(q_x, q_y, \rho) = \frac{1}{\varepsilon_0(q_x, q_y) + \Delta_k(q_x, q_y, \rho) + m_k^2(\rho) + \mathcal{R}_k(q_x, q_y)} \tag{16}$$

$$\partial_\rho I_n(\rho) = -n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y) G_k^{n+1}(q_x, q_y, \rho) \left(\partial_\rho \Delta_k(p_x, p_y, \rho) + u_k(\rho)\right) \mathrm{d}q_x\, \mathrm{d}q_y \tag{17}$$

$$m_k^2(\rho) = \partial_\phi^2 V(\phi) = W(\rho) + 2\rho \partial_\rho W(\rho) \tag{18}$$

$$u_k(\rho) = \partial_\rho m_k^2(\rho) = 3\partial_\rho W(\rho) + 2\rho \partial_\rho^2 W(\rho) \tag{19}$$

On pose la fonction

$$\tau(q_x, q_y) = \frac{\varepsilon_0(q_x, q_y)}{2k^2 \|\varepsilon_0\|_\infty} \tag{20}$$

On choisit alors le régulateur

$$\mathcal{R}_k(q_x, q_y) = \frac{\alpha \varepsilon_0(q_x, q_y)}{\exp\left(2\tau(q_x, q_y)\right) - 1} \tag{21}$$

$$\partial_t \mathcal{R}_k(q_x, q_y) = \alpha \varepsilon_0(q_x, q_y) \frac{\tau(q_x, q_y)}{\sinh^2\left(\tau(q_x, q_y)\right)} \tag{22}$$

Et nous pouvons calculer

$$\|\varepsilon_0\|_\infty = \sup_{(p_x, p_y) \in [-\pi, \pi]^2} \varepsilon_0(p_x, p_y) = \frac{2\delta^2}{\mu^2 - 1} \tag{23}$$

3 Les équations BMW en ϕ

3.1 Les équations BMW en ϕ dimensionnées

On rappelle les notations :

$$W(\phi) = \partial_\phi V(\phi) \quad \text{et} \quad X(\phi) = \partial_\phi^2 V(\phi) \tag{24}$$

On doit alors résoudre

$$\begin{aligned} \partial_t \Delta_k(p_x, p_y, \phi) = & J_3(p_x, p_y, \phi)(\partial_\phi \{\Delta_k(p_x, p_y, \phi) + X(\phi)\})^2 \\ & - I_3(\phi)(\partial_\phi X(\phi))^2 - \frac{1}{2} I_2(\phi) \partial_\phi^2 \Delta_k(p_x, p_y, \phi) \end{aligned} \tag{25}$$

$$\partial_t X(\phi) = \frac{1}{2} \partial_\phi^2 I_1(\phi) \tag{26}$$

On garde ici des expressions similaires pour les intégrales que ce que l'on avait en ρ,

$$J_n(p_x, p_y, \phi) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y)\, G_k^{n-1}(q_x, q_y, \phi) G_k(p_x + q_x, p_y + q_y, \phi)\, \mathrm{d}q_x\, \mathrm{d}q_y \tag{27}$$

$$I_n(\phi) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y)\, G_k^n(q_x, q_y, \phi)\, \mathrm{d}q_x\, \mathrm{d}q_y \tag{28}$$

$$G_k(q_x, q_y, \phi) = \frac{1}{\varepsilon_0(q_x, q_y) + \Delta_k(q_x, q_y, \phi) + X(\phi) + \mathcal{R}_k(q_x, q_y)} \tag{29}$$

$$\partial_\phi I_n(\phi) = -n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y) G_k^{n+1}(q_x, q_y, \phi) \left(\partial_\phi \Delta_k(p_x, p_y, \phi) + \partial_\phi X(\phi) \right) \mathrm{d}q_x\, \mathrm{d}q_y \tag{30}$$

$$\begin{aligned} \partial_\phi^2 I_n(\phi) = & - n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y) G_k^{n+1}(q_x, q_y, \phi) \left(\partial_\phi^2 \Delta_k(p_x, p_y, \phi) + \partial_\phi^2 X(\phi) \right) \mathrm{d}q_x\, \mathrm{d}q_y \\ & + n(n+1) \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \partial_t \mathcal{R}_k(q_x, q_y) G_k^{n+2}(q_x, q_y, \phi) (\partial_\phi \Delta_k(p_x, p_y, \phi) + \partial_\phi X(\phi))^2\, \mathrm{d}q_x\, \mathrm{d}q_y \end{aligned} \tag{31}$$

3.2 Les équations BMW en ϕ adimensionnées en impulsion

On note $\tilde{p}_x = k^{-1} p_x$ et $\tilde{p}_y = k^{-1} p_y$. Ainsi que

$$\tilde{\Delta}_k(\tilde{p}_x, \tilde{p}_y, \phi) = \Delta_k(p_x, p_y, \phi)\,;\ \tilde{J}_n(\tilde{p}_x, \tilde{p}_y, \phi) = J_n(p_x, p_y, \phi)\,;\ \tilde{\mathcal{R}}_k(\tilde{p}_x, \tilde{p}_y) = \mathcal{R}_k(p_x, p_y)$$
$$\tilde{\varepsilon}_0(\tilde{p}_x, \tilde{p}_y) = \varepsilon_0(p_x, p_y)\,;\ \tilde{\tau}(\tilde{p}_x, \tilde{p}_y) = \tau(p_x, p_y) = \tilde{\varepsilon}_0(\tilde{p}_x, \tilde{p}_y)/(k^2 \|\varepsilon_0\|_\infty)\,;\ \widetilde{\partial_t \mathcal{R}}_k(\tilde{q}_x, \tilde{q}_y) = \partial_t \mathcal{R}_k(q_x, q_y)$$

Les équations se réécrivent

$$\begin{aligned} \partial_t \tilde{\Delta}_k(\tilde{p}_x, \tilde{p}_y, \phi) = & - I_3(\phi)(\partial_\phi X(\phi))^2 + \tilde{J}_3(\tilde{p}_x, \tilde{p}_y, \phi) \left(\partial_\phi \left\{ \tilde{\Delta}_k(\tilde{p}_x, \tilde{p}_y, \phi) + X(\phi) \right\} \right)^2 \\ & - \frac{1}{2} I_2(\phi) \partial_\phi^2 \tilde{\Delta}_k(\tilde{p}_x, \tilde{p}_y, \phi) + \tilde{p}_x \partial_{\tilde{p}_x} \tilde{\Delta}_k + \tilde{p}_y \partial_{\tilde{p}_y} \tilde{\Delta}_k \end{aligned} \tag{32}$$

$$\partial_t X(\phi) = \frac{1}{2} \partial_\phi^2 I_1(\phi) \tag{33}$$

En effet, l'expression des nouvelles derivées par rapport au temps est :

$$\begin{aligned} \partial_t \Delta_k|_{p_x, p_y, \phi} &= \partial_t \tilde{\Delta}_k|_{\tilde{p}_x, \tilde{p}_y, \phi} + \partial_t \tilde{p}_x|_{p_x} \partial_{\tilde{p}_x} \tilde{\Delta}_k + \partial_t \tilde{p}_y|_{p_y} \partial_{\tilde{p}_y} \tilde{\Delta}_k \\ \partial_t \mathcal{R}_k|_{q_x, q_y} &= \partial_t \tilde{\mathcal{R}}_k|_{\tilde{q}_x, \tilde{q}_y} + \partial_t \tilde{q}_x|_{q_x} \partial_{\tilde{q}_x} \tilde{\mathcal{R}}_k + \partial_t \tilde{q}_y|_{q_y} \partial_{\tilde{q}_y} \tilde{\mathcal{R}}_k \end{aligned} \tag{34}$$

Ce qui donne alors

$$\begin{aligned} \partial_t \Delta_k|_{p_x, p_y, \phi} &= \partial_t \tilde{\Delta}_k|_{\tilde{p}_x, \tilde{p}_y, \phi} - \tilde{p}_x \partial_{\tilde{p}_x} \tilde{\Delta}_k - \tilde{p}_y \partial_{\tilde{p}_y} \tilde{\Delta}_k \\ \partial_t \mathcal{R}_k|_{q_x, q_y} &= \partial_t \tilde{\mathcal{R}}_k|_{\tilde{q}_x, \tilde{q}_y} - \tilde{q}_x \partial_{\tilde{q}_x} \tilde{\mathcal{R}}_k - \tilde{q}_y \partial_{\tilde{q}_y} \tilde{\mathcal{R}}_k \end{aligned} \tag{35}$$

En outre, on peut expliciter le calcul de la dérivée temporelle du régulateur

$$\begin{aligned}\partial_{\tilde{q}_x}\tilde{\mathcal{R}}_k &= \frac{\alpha}{2}\partial_{\tilde{q}_x}\tilde{\varepsilon}_0\left(\frac{1}{\sinh(\tilde{\tau})\exp(\tilde{\tau})} - \frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})}\right)\\ \partial_{\tilde{q}_y}\tilde{\mathcal{R}}_k &= \frac{\alpha}{2}\partial_{\tilde{q}_y}\tilde{\varepsilon}_0\left(\frac{1}{\sinh(\tilde{\tau})\exp(\tilde{\tau})} - \frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})}\right)\end{aligned} \tag{36}$$

$$\partial_t\tilde{\mathcal{R}}_k = \frac{\alpha}{2}\partial_t\tilde{\varepsilon}_0\left(\frac{1}{\sinh(\tilde{\tau})\exp(\tilde{\tau})} - \frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})}\right) + \alpha\tilde{\varepsilon}_0\frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})} \tag{37}$$

Ainsi en assemblant les trois équations précédentes,

$$\partial_t\mathcal{R}_k = \frac{\alpha}{2}\left(\frac{1}{\sinh(\tilde{\tau})\exp(\tilde{\tau})} - \frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})}\right)\left(\partial_t\tilde{\varepsilon}_0 - \partial_{\tilde{q}_x}\tilde{\varepsilon}_0 - \partial_{\tilde{q}_y}\tilde{\varepsilon}_0\right) + \alpha\tilde{\varepsilon}_0\frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})} \tag{38}$$

Avec les expressions des dérivées de la relation de dispersion :

$$\begin{aligned}\partial_{\tilde{q}_x}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) = k\partial_{q_x}\varepsilon_0(q_x,q_y) = 2\delta^2 k\frac{\sin(k\tilde{q}_x)}{(\cos(k\tilde{q}_x)+\cos(k\tilde{q}_y)+2\mu)^2}\\ \partial_{\tilde{q}_y}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) = k\partial_{q_y}\varepsilon_0(q_x,q_y) = 2\delta^2 k\frac{\sin(k\tilde{q}_y)}{(\cos(k\tilde{q}_x)+\cos(k\tilde{q}_y)+2\mu)^2}\end{aligned} \tag{39}$$

$$\partial_t\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) = \partial_t q_x|_{\tilde{q}_x}\partial_{q_x}\varepsilon_0 + \partial_t q_y|_{\tilde{q}_y}\partial_{q_y}\varepsilon_0 = \tilde{q}_x\partial_{\tilde{q}_x}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) + \tilde{q}_y\partial_{\tilde{q}_y}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) \tag{40}$$

$$\partial_t\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) = 2\delta^2 k\frac{\tilde{q}_x\sin(k\tilde{q}_x)+\tilde{q}_y\sin(k\tilde{q}_y)}{(\cos(k\tilde{q}_x)+\cos(k\tilde{q}_y)+2\mu)^2} \tag{41}$$

Ce qui donne, pour la dérivée temporelle du régulateur, en recollant tout,

$$\partial_t\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) - \partial_{\tilde{q}_x}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) - \partial_{\tilde{q}_y}\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y) = 2\delta^2 k\frac{(\tilde{q}_x-1)\sin(k\tilde{q}_x)+(\tilde{q}_y-1)\sin(k\tilde{q}_y)}{(\cos(k\tilde{q}_x)+\cos(k\tilde{q}_y)+2\mu)^2} \tag{42}$$

$$\partial_t\mathcal{R}_k(q_x,q_y)|_{q_x,q_y} = \alpha\delta^2 k\left(\frac{1}{\sinh(\tilde{\tau})\exp(\tilde{\tau})} - \frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})}\right)\frac{(\tilde{q}_x-1)\sin(k\tilde{q}_x)+(\tilde{q}_y-1)\sin(k\tilde{q}_y)}{(\cos(k\tilde{q}_x)+\cos(k\tilde{q}_y)+2\mu)^2} + \alpha\tilde{\varepsilon}_0\frac{\tilde{\tau}}{\sinh^2(\tilde{\tau})} \tag{43}$$

Pour conserver une écriture compacte nous noterons $\widetilde{\partial_t\mathcal{R}_k}(\tilde{q}_x,\tilde{q}_y) = \partial_t\mathcal{R}_k(q_x,q_y)|_{q_x,q_y}$

Les intégrales se calculent selon

$$\tilde{J}_n(\tilde{p}_x,\tilde{p}_y,\phi) = \frac{k^2}{(2\pi)^2}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\widetilde{\partial_t\mathcal{R}_k}(\tilde{q}_x,\tilde{q}_y)\,\tilde{G}_k^{n-1}(\tilde{q}_x,\tilde{q}_y,\phi)\tilde{G}_k(\tilde{p}_x+\tilde{q}_x,\tilde{p}_y+\tilde{q}_y,\phi)\,\mathrm{d}\tilde{q}_x\,\mathrm{d}\tilde{q}_y \tag{44}$$

$$I_n(\phi) = \frac{k^2}{(2\pi)^2}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\widetilde{\partial_t\mathcal{R}_k}(\tilde{q}_x,\tilde{q}_y)\,\tilde{G}_k^{n}(\tilde{q}_x,\tilde{q}_y,\phi)\,\mathrm{d}\tilde{q}_x\,\mathrm{d}\tilde{q}_y \tag{45}$$

$$\tilde{G}_k(\tilde{q}_x,\tilde{q}_y,\phi) = \frac{1}{\tilde{\varepsilon}_0(\tilde{q}_x,\tilde{q}_y)+\tilde{\Delta}_k(\tilde{q}_x,\tilde{q}_y,\phi)+X(\phi)+\tilde{\mathcal{R}}_k(\tilde{q}_x,\tilde{q}_y)} \tag{46}$$

$$\begin{aligned}\partial_\phi^2 I_n(\phi) = &-n\frac{k^2}{(2\pi)^2}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\widetilde{\partial_t\mathcal{R}_k}(\tilde{q}_x,\tilde{q}_y)\tilde{G}_k^{n+1}(\tilde{q}_x,\tilde{q}_y,\phi)\left(\partial_\phi^2\tilde{\Delta}_k(\tilde{p}_x,\tilde{p}_y,\phi)+\partial_\phi^2 X(\phi)\right)\mathrm{d}\tilde{q}_x\,\mathrm{d}\tilde{q}_y\\ &+n(n+1)\frac{k^2}{(2\pi)^2}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\widetilde{\partial_t\mathcal{R}_k}(\tilde{q}_x,\tilde{q}_y)\tilde{G}_k^{n+2}(\tilde{q}_x,\tilde{q}_y,\phi)\left(\partial_\phi\tilde{\Delta}_k(\tilde{p}_x,\tilde{p}_y,\phi)+\partial_\phi X(\phi)\right)^2\mathrm{d}\tilde{q}_x\,\mathrm{d}\tilde{q}_y\end{aligned} \tag{47}$$

4 Calcul du Z_k

On commence par définir

$$\varepsilon_0^0 = \left.\frac{\partial \varepsilon_0}{\partial {p_x}^2}\right|_{p_x=0, p_y=0} \tag{48}$$

Pour calculer Z_k on utilise la définition

$$Z_k = 1 + \frac{1}{\varepsilon_0^0} \left.\frac{\partial \Delta_k}{\partial {p_x}^2}\right|_{p_x=0, p_y=0, \phi=0} \tag{49}$$

Ce qui est équivent à

$$Z_k = 1 + \frac{1}{2\varepsilon_0^0} \left.\frac{\partial^2 \Delta_k}{\partial {p_x}^2}\right|_{p_x=0, p_y=0, \phi=0} \tag{50}$$