Chapter 7

Renormalization group and critical phenomena *(last version: 31 May 2012)*

Contents

In the preceding chapters, we have seen examples of interacting systems where mean-field theory is a good starting point. In many cases however, fluctuations about the mean-field approximation are important and cannot be neglected. In low dimensions, they often play a crucial role and tend to suppress long-range order. In a more subtle way, they can also affect the correlation functions near a second-order phase transition and invalidate the mean-field predictions regarding their long-distance and long-time behavior.

In this chapter, we give a general introduction to the theory of second-order phase transitions (i.e. transitions with a continuous order parameter and a diverging correlation length). After a brief introduction to critical phenomena (Sec. 7.1), we review Landau's mean-field theory for a (classical) $(\boldsymbol{\varphi}^2)^2$ theory with $\mathrm{O}(N)$ symmetry (Sec. 7.2). By studying fluctuations about the mean-field approximation, we find that Landau's theory becomes inapplicable near the transition for dimensions below the upper critical dimension $d_c^+ = 4$, while long-range order is suppressed at and below the lower critical dimension $d_c^- = 2$ (Sec. 7.3). We then discuss second-order phase transitions in the framework of the scaling hypothesis and derive relations between critical exponents (scaling laws). The renormalization group (RG) - with the important notions of RG flows, fixed points and critical exponents - is discussed in Sec. 7.5.

We show how the RG naturally leads to universality and scaling. The critical exponents are computed perturbatively near the upper critical dimension (Sec. 7.6), and near the lower critical dimension in the framework of the non-linear sigma model (Sec. 7.7). The Berezinskii-Kosterlitz-Thouless transition is discussed in section. 7.8. In section 7.9, we give a brief introduction to the functional RG, the Wilson-Polchinski equation and its solution in the local potential approximation.[1] We conclude the chapter by a discussion of (zero-temperature) quantum phase transitions; we consider the quantum $(\boldsymbol{\varphi}^2)^2$ theory, the quantum non-linear sigma model and the dilute Bose gas (Sec. 7.10). Details about the perturbative calculation of critical exponents and the large-N limit of the (quantum) $(\boldsymbol{\varphi}^2)^2$ theory or non-linear sigma model can be found in the appendices.

7.1 Introduction to critical phenomena

Let us consider a system with the partition function

$$Z(K) = \mathrm{Tr}\, e^{-\beta \hat{H}}, \tag{7.1}$$

where $K = \{K_i\}$ denotes a set of parameters or "coupling constants" (external fields, microscopic parameters, etc.) of the Hamiltonian $\hat{H}$, as well as temperature. We assume that the thermodynamic limit exists, i.e. that the limit

$$f(K) = \lim_{V\to\infty} \frac{F(K)}{V} = -\lim_{V\to\infty} \frac{1}{\beta V} \ln Z(K) \tag{7.2}$$

is defined.[2] When this is not the case, surface effects remain important in the limit $V \to \infty$ and it is not possible to define a bulk free energy density $f(K)$.

A region in the $K = \{K_i\}$ space where $f(K)$ is analytic defines a phase of the system. Phase transitions correspond to non-analyticities of $f(K)$.[3] In general, the free energy of a finite-size system is analytic. We are interested in the case where non-analyticities arise from the thermodynamic limit $V \to \infty$ in which the number of degrees of freedom becomes infinite, and focus on thermal phase transitions (i.e. transitions driven by thermal fluctuations). Quantum fluctuations do not play an important role in the low-energy behavior of a system near a finite-temperature phase transition and we consider classical models in the most part of this chapter. Quantum phase transitions, i.e. zero-temperature phase transitions driven by quantum fluctuations, are discussed in section 7.10 and in Appendix 7.D.

7.1.1 Spontaneous symmetry breaking

There are different kinds of phase transitions. Some, such as the liquid-gas transition or the Mott transition in solids,[4] do not break any symmetry. Others, e.g. the

[1] The functional RG will be discussed at length in chapter 8.
[2] A necessary condition for the thermodynamic limit to exist is that the interactions are sufficiently short range.
[3] This definition is in fact ambiguous since it is sometimes possible to go from one phase to the other without crossing a phase boundary. This is possible when, as in the liquid-gas transition, both phases have the same symmetry.
[4] The Mott transition is a metal-insulator transition induced by the Coulomb repulsion.

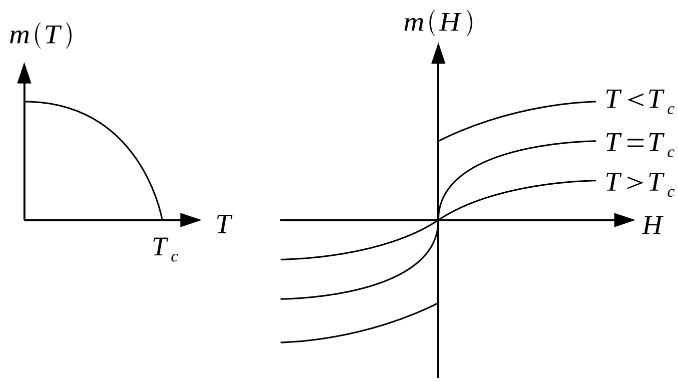

Figure 7.1: Magnetization density of an easy-axis ferromagnet *vs* temperature in zero field (left panel), and *vs* magnetic field below, above and at the transition temperature (right panel).

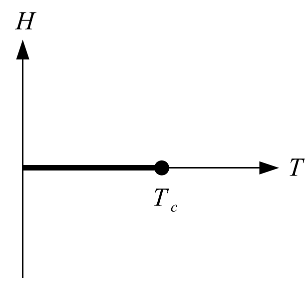

Figure 7.2: Phase diagram of an easy-axis ferromagnet. The discontinuous transition line (thick line) between states with magnetization density $m(T)$ and $-m(T)$ terminates at the critical point $(T_c, H = 0)$.

ferromagnetic transition in a magnetic system (chapter **??**) or the superfluid transition in a Bose gas (chapter 6), are associated to spontaneous symmetry breaking. We are interested in the case where one of the phases has the full symmetry of the Hamiltonian, while the other one has a reduced symmetry. It is then possible to introduce an order parameter which vanishes in the "disordered" (symmetric) phase and takes a nonzero value in the "ordered" phase.[5] Since the state of the system is determined by the minimum of the free energy $F = E - TS$, the disordered phase usually corresponds to the high-temperature phase (where entropy effects dominate) and the ordered phase to the low-temperature phase.

In the following, we mostly use the language of the ferromagnetic transition and, unless otherwise specified, consider an easy-axis ferromagnet (the easy axis determines the direction of the magnetization). While the system is paramagnetic at high temperatures, below the transition temperature T_c there appears a spontaneous magnetization M in zero magnetic field (Fig. 7.1). To understand the nature of the

[5]In the liquid-gas transition, although there is no spontaneous symmetry breaking, it is possible to define an order parameter, namely the density of the fluid (or the difference between the density of the fluid and that of the gas).

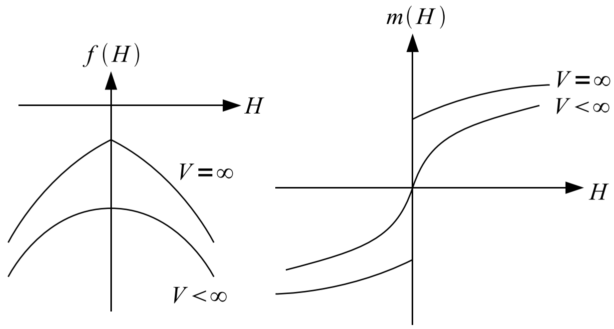

Figure 7.3: Free energy density $f(H) = F(H)/V$ and magnetization density $m(H) = M(H)/V$ of an easy-axis ferromagnet as a function of the magnetic field for finite and infinite systems.

broken symmetry, we assume that the ferromagnet consists of a collection of spins $S_{\mathbf{r}}$ located at the sites $\mathbf{r}$ of a lattice and parallel to the easy axis. Because of time reversal symmetry in the absence of an external field, the Hamiltonian $\mathcal{H}(S_{\mathbf{r}})$ is invariant under spin inversion $S_{\mathbf{r}} \to -S_{\mathbf{r}}$. In the ferromagnetic phase, spin inversion is spontaneously broken. On the other hand, if the ferromagnet has no easy axis but is isotropic, then the Hamiltonian is invariant under a simultaneous rotation of all the spins, $\mathcal{H}(\mathbf{S}_{\mathbf{r}}) = \mathcal{H}(R(\mathbf{S}_{\mathbf{r}}))$, where R is an arbitrary rotation matrix acting in spin space and the spin variables are now three-dimensional vectors. In the ferromagnetic phase, spin rotation invariance is spontaneously broken: $\mathbf{m} = \langle \mathbf{S}_{\mathbf{r}} \rangle \neq 0$. In this case, the broken symmetry is continuous. We shall see that the nature (discrete or continuous) of the spontaneously broken symmetry plays a crucial role in the low-energy properties of the system.

The phase diagram of a ferromagnet is shown in figure 7.2. There is a discontinuity in the magnetization density m as the magnetic field H goes trough zero for $T < T_c$ (the magnetic field is assumed to be along the easy axis of the ferromagnet). This discontinuity terminates at the "critical" point $(T_c, H = 0)$.[6]

Because of time reversal invariance, the free energy does not change if we reverse the direction of the field, $F(H) = F(-H)$. This implies

$$m(H) = -\frac{\partial f(H)}{\partial H} = -\frac{\partial f(-H)}{\partial H} = \frac{\partial f(-H)}{\partial(-H)} = -m(-H), \tag{7.3}$$

so that it seems that the zero-field magnetization density $m(H=0) = M(H=0)/V$ must vanish. This argument however requires $f(H) = F(H)/V$ to be analytic at $H = 0$, i.e. $\partial f(H)/\partial H$ to be smooth at $H = 0$. While this is true if the volume is finite, this is violated in the infinite volume limit when $T < T_c$:

$$\lim_{V\to\infty} \lim_{H\to 0} \frac{1}{V} \frac{\partial F(H)}{\partial H} = 0, \tag{7.4}$$

but

$$\lim_{H\to 0} \lim_{V\to\infty} \frac{1}{V} \frac{\partial F(H)}{\partial H} \neq 0, \tag{7.5}$$

[6]See footnote 10 page 449.

as illustrated in figure 7.3. Spontaneous broken symmetry is possible only in the thermodynamic limit where the free energy density $f(H)$ becomes non-analytic at $H = 0$.

To illustrate this point, let us consider spins $S_{\mathbf{r}}$ located at the sites of a lattice (with N the number of sites). The probability to find the system in the state $\{S_{\mathbf{r}}\}$ is given by the Boltzmann distribution

$$P(\{S_{\mathbf{r}}\}) = \frac{e^{-\beta\mathcal{H}(\{S_{\mathbf{r}}\})}}{Z}. \tag{7.6}$$

If $P(\{S_{\mathbf{r}}\})$ is invariant under $S_{\mathbf{r}} \to -S_{\mathbf{r}}$ (time-reversal invariance), then $\langle S_{\mathbf{r}}\rangle = \mathrm{Tr}P(\{S_{\mathbf{r}}\})S_{\mathbf{r}} = 0$ and it seems that spontaneous symmetry breaking is impossible. At low temperature, for a ferromagnetic coupling between spins, the latter are either "up" ($\langle S_{\mathbf{r}}\rangle = +m$) or "down" ($\langle S_{\mathbf{r}}\rangle = -m$). These two configurations are related by time reversal symmetry and their probabilities are equal: $P_\oplus = P_\ominus$. Now apply an external positive field H ($H > 0$). Due to the coupling term $-H\sum_{\mathbf{r}} S_{\mathbf{r}}$,

$$\frac{P_\ominus}{P_\oplus} = e^{-2\beta NHm} \tag{7.7}$$

and

$$\lim_{N\to\infty} \frac{P_\ominus}{P_\oplus} = 0 \tag{7.8}$$

Thus the presence of an infinitesimal field $H \to 0^+$, together with the thermodynamic limit $N \to \infty$, is sufficient to select the configuration $\oplus$. The configuration $\ominus$ is inaccessible. Equivalently, we could set $H = 0$ and use a restricted ensemble where the configuration $\ominus$, and more generally all microstates with a negative magnetization, is not allowed. The fact that some part of the phase space is forbidden is known as ergodicity breaking.[7]

Gibbs free energy

The stability of the system requires the isothermal susceptibility

$$\chi = \frac{\partial m}{\partial H} = -\frac{1}{V}\frac{\partial^2 F(H)}{\partial H^2} \tag{7.9}$$

to be positive. Thus $F''(H) \leq 0$ and the free energy is a convex function of the magnetic field. This allows us to invert the relation $m = -\frac{1}{V}\frac{\partial F}{\partial H}$ and introduce the Gibbs free energy

$$G(m) = F(H) + VHm, \tag{7.10}$$

defined as the Legendre transform of $F(H)$. $G(m)$ satisfies the equation of state

$$\frac{\partial G(m)}{\partial m} = VH. \tag{7.11}$$

From figure 7.3, one easily deduces the general form of the Gibbs free energy (Fig. 7.4). The convexity of $F(H)$ ensures that $G(m)$ is also a convex function. In the ordered phase, the (absolute value of the) magnetization is always larger than the zero-field result $m_0 = -\partial f/\partial H|_{H=0^+}$. The region $]-m_0, m_0[$ is not physically accessible.

[7] A detailed discussion of ergodicity breaking may be found in Ref. [19].

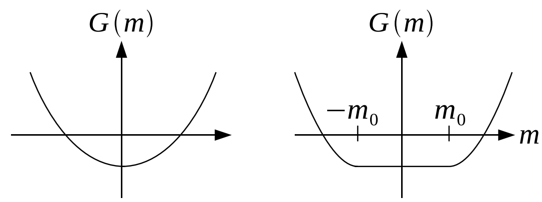

Figure 7.4: Gibbs free energy $G(m)$: high-temperature (left) and low-temperature (right) phase.

7.1.2 Landau's classification of phase transitions

Landau distinguishes between discontinuous (or first-order) and continuous (or second-order) phase transitions.[8] In a first-order transition, the order parameter is discontinuous at the transition (Fig. 7.5). The correlation length ξ is finite,[9] and the two phases (ordered and disordered) coexist at the transition temperature T_c. In a second-order phase transition, the order parameter is continuous (Fig. 7.5) and the correlation length diverges. Fluctuations become correlated over all distances, which forces the whole system to be in a unique phase. The two phases of either side of the transition must therefore become identical at T_c; as the correlation length diverges, the order parameter in the ordered phase goes smoothly to zero. The ferromagnetic transition we have discussed above is an example of a second-order phase transition.

When the correlation length is finite, we can view the system as a collection of subsystems of size ξ^d (with d the space dimension) with no mutual interaction. By the central limit theorem, we expect the fluctuations at large distances ($\gg \xi$) to have a Gaussian probability distribution. By contrast, all degrees of freedom become correlated at a second-order phase transition where ξ diverges (Sec. 7.1.4). We will see that standard perturbation theories break down in the vicinity of a second-order phase transition unless the dimension is high enough or the interactions sufficiently long range (Secs. 7.2.1 and 7.3). The temperature regime where the mean-field or Gaussian theories (Secs. 7.2 and 7.3) are not valid any more is called the critical regime.[10]

On the other hand, the divergence of the correlation length at a second-order phase transition renders microscopic details irrelevant for the long-distance properties. As a consequence, near the critical point, the singular part of the free energy and the asymptotic behavior of the correlation functions depend only on general properties such as the space dimension, the dimension of the order parameter or the symmetry and range of the interactions. This essential property of second-order phase transitions is called universality. We shall see that universal properties of a system near a second-order phase transition can be accurately described within an effective theory involving

[8] Landau's classification differs from Ehrenfest's classification where a transition is said to be nth order if all $(n-1)$th-order derivatives of the free energy are continuous while there is at least one nth-order derivative which is discontinuous. (Ehrenfest did not realize that some thermodynamic quantities (e.g. the specific heat) can diverge rather than exhibiting a simple discontinuity.)

[9] The correlation length ξ is a measure of the distance over which correlations are important; it will be precisely defined in Sec. 7.1.4.

[10] Second-order phase transitions are often called critical phenomena; the transition temperature is then referred to as the critical temperature/point.

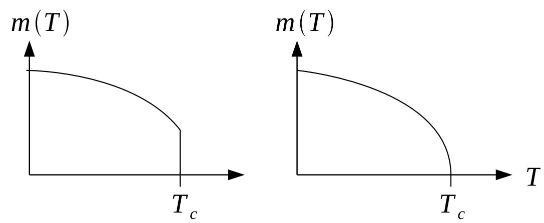

Figure 7.5: Temperature dependence of the order parameter $m(T)$ in a first-order (left) and second-order (right) phase transition.

only long-distance fluctuations.

Unless otherwise specified, we will only consider second-order (continuous) phase transitions in the following.

7.1.3 Critical behavior

The singular behavior at the critical point is characterized by a set of critical exponents (Table 7.1). Below T_c, the magnetization density $m(T,H)$ varies as[11]

$$m(T,0) \sim (-t)^\beta \qquad (T \to T_c^-), \tag{7.12}$$

where

$$t = \frac{T - T_c}{T_c} \tag{7.13}$$

is the reduced temperature. At the transition temperature,

$$m(T_c, H) \sim H^{1/\delta} \qquad (H \to 0), \tag{7.14}$$

which defines the exponent δ. Near the critical point, when the system is about to spontaneously order, the susceptibility (i.e. the response to an external magnetic) becomes very large and diverges at T_c,

$$\chi = \frac{\partial m}{\partial H}\bigg|_{H=0} \sim \begin{cases} t^{-\gamma} & (T \to T_c^+), \\ (-t)^{-\gamma'} & (T \to T_c^-), \end{cases} \tag{7.15}$$

where the two exponents γ and γ' refer to the high- and low-temperature phases, respectively. The critical behavior is also characterized by a divergence of the specific heat

$$C_V \sim \begin{cases} t^{-\alpha} & (T \to T_c^+), \\ (-t)^{-\alpha'} & (T \to T_c^-), \end{cases} \tag{7.16}$$

with an exponent α or α'. In most cases, the exponents on both sides of the transition coincide: $\gamma = \gamma'$ and $\alpha = \alpha'$ (Sec. 7.4.2). The divergence of the correlation length and the singular behavior of the correlation function are discussed in the next section.

[11]Note that we use the same notation for the critical exponent defined in (7.12) and the inverse temperature $\beta = 1/T$.

Order parameter	$m(T,0) \sim (-t)^\beta \qquad (T \to T_c^-)$ $m(T_c,H) \sim H^{1/\delta} \qquad (H \to 0)$
Susceptibility	$\chi = \dfrac{\partial m}{\partial H}\Big\vert_{H=0} \sim \begin{cases} t^{-\gamma} & (T \to T_c^+) \\ (-t)^{-\gamma'} & (T \to T_c^-) \end{cases}$
Specific heat	$C_V \sim \begin{cases} t^{-\alpha} & (T \to T_c^+) \\ (-t)^{-\alpha'} & (T \to T_c^-) \end{cases}$
Correlation length	$\xi \sim \begin{cases} t^{-\nu} & (T \to T_c^+) \\ (-t)^{-\nu'} & (T \to T_c^-) \end{cases}$
Correlation function	$G(\mathbf{r}) \sim \dfrac{1}{\vert\mathbf{r}\vert^{d-2+\eta}} \qquad (T = T_c)$ $G(\mathbf{p}) \sim \dfrac{1}{\vert\mathbf{p}\vert^{2-\eta}} \qquad (T = T_c)$

Table 7.1: Critical exponents at a second-order (continuous) phase transition.

7.1.4 Long-range order

A nonzero order parameter implies not only spontaneous broken symmetry but also long-range order. In most cases of interest, the order parameter is the mean value of an observable $\varphi(\mathbf{r})$. Besides $m = \langle\varphi(\mathbf{r})\rangle$, one can also consider the correlation function of the φ field. In the disordered phase, the order parameter vanishes and the correlation function decays exponentially at large distances,

$$C(\mathbf{r}-\mathbf{r}') = \langle\varphi(\mathbf{r})\varphi(\mathbf{r}')\rangle \sim \exp\left(-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}\right). \tag{7.17}$$

Equation (7.17) defines the correlation length ξ. For $|\mathbf{r}-\mathbf{r}'| \to \infty$, there should be no correlation between the magnetization densities at point $\mathbf{r}$ and $\mathbf{r}'$, so that

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} C(\mathbf{r}-\mathbf{r}') = \langle\varphi(\mathbf{r})\rangle\langle\varphi(\mathbf{r}')\rangle = 0, \tag{7.18}$$

in agreement with (7.17). In the ordered phase, one finds instead

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} C(\mathbf{r}-\mathbf{r}') = \langle\varphi(\mathbf{r})\rangle\langle\varphi(\mathbf{r}')\rangle = m^2. \tag{7.19}$$

Equation (7.19) is the mathematical definition of long-range order; it suggests that ξ diverges as $T \to T_c^+$. This can be shown to be a consequence of the (classical) fluctuation-dissipation theorem (3.59),[12]

$$\chi = \beta \int d^d r \langle\varphi(\mathbf{r})\varphi(0)\rangle \leq \beta \int d^d r\, e^{-|\mathbf{r}|/\xi} \sim \beta\xi^d. \tag{7.20}$$

Since the susceptibility χ diverges at the transition [Eq. (7.15)], $\xi \sim t^{-\nu}$ must also diverge when $T \to T_c^+$, which defines the exponent ν.

[12] The correlation function $C(\mathbf{r}) \sim e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ generally decays faster then $e^{-|\mathbf{r}|/\xi}$.

In the ordered phase, it is convenient to work with the connected correlation function

$$G(\mathbf{r}-\mathbf{r}') = \langle(\varphi(\mathbf{r})-m)(\varphi(\mathbf{r}')-m)\rangle = C(\mathbf{r}-\mathbf{r}') - m^2 \sim e^{-|\mathbf{r}-\mathbf{r}'|/\xi}, \qquad (7.21)$$

which defines the correlation length ξ for $T < T_c$. Both $\chi = \beta G(\mathbf{p}=0) \sim (-t)^{-\gamma'}$ and $\xi \sim (-t)^{-\nu'}$ diverge when $T \to T_c^-$.[13] In almost all cases, the correlation length critical exponents ν and ν' are equal (and $\gamma' = \gamma$) (Sec. 7.4.2).

> **N-component order parameter.** For a N-component order parameter $\mathbf{m} = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$ (e.g. $N = 3$ for a Heisenberg ferromagnet), the connected correlation function is defined by
>
> $$G_{ij}(\mathbf{r}-\mathbf{r}') = \langle(\varphi_i(\mathbf{r})-m_i)(\varphi_j(\mathbf{r}')-m_j)\rangle = C_{ij}(\mathbf{r}-\mathbf{r}') - m_i m_j. \qquad (7.22)$$
>
> If the system is isotropic (O(N) symmetry), the Fourier transformed correlation function takes the form
>
> $$G_{ij}(\mathbf{p}) = \frac{m_i m_j}{m^2} G_\parallel(\mathbf{p}) + \left(\delta_{i,j} - \frac{m_i m_j}{m^2}\right) G_\perp(\mathbf{p}), \qquad (7.23)$$
>
> where the longitudinal and transverse components, $G_\parallel$ and $G_\perp$, are functions of $|\mathbf{p}|$. We shall see later on that when $N \geq 2$ and $d \leq 4$ neither $G_\parallel$ nor $G_\perp$ decays exponentially in space below T_c,[14] so that the susceptibility χ diverges in the whole low-temperature phase and it is not possible to define a correlation length. It is nevertheless possible to define a characteristic length, the Josephson length ξ_J, which diverges as $T \to T_c^-$ with an exponent $\nu' = \nu$ (Sec. 7.7.2).

Scale invariance

At the critical point ($T = T_c$), the correlation length ξ is infinite and the correlation function decays as a power law,

$$G(\mathbf{r}) \sim \frac{1}{|\mathbf{r}|^{d-2+\eta}}, \qquad (7.24)$$

where η is called the anomalous dimension (this terminology is explained in Sec. 7.4).[15] Equation (7.24) will be derived in section 7.4. Under a change of scale, $G(\mathbf{r})$ behaves as

$$G(\mathbf{r}/s) = s^{d-2+\eta} G(\mathbf{r}) \qquad (7.25)$$

and is therefore invariant (apart from a multiplication by a factor $s^{d-2+\eta}$). A critical system exhibits scale invariance or self-similarity. The concept of scale invariance at the critical point is central in the renormalization-group approach (Sec. 7.5).

[13] The divergence of ξ in the ordered phase as $T \to T_c^-$ is obtained from the same argument as that used for the ordered phase with the correlation function C replaced by its connected part G.

[14] Mean-field and Gaussian fluctuation theories predict $G_\parallel(\mathbf{r})$ to decay exponentially below T_c for any value of N. For $N \geq 2$ and $d \leq 4$, this result is an artifact coming from neglecting the coupling between transverse and longitudinal fluctuations (see Sec. 7.7.3).

[15] In Fourier space, Eq. (7.24) gives $G(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta}$.

7.2 Landau's theory of phase transitions

7.2.1 Landau's theory as a mean-field theory

Microscopic Landau's theory

In a microscopic description, one would like to start from the partition function $Z = \mathrm{Tr}\, e^{-\beta\hat H}$. While the (quantum) Hamiltonian $\hat H$ is certainly necessary to understand the microscopic origin of a phase transition, it is not crucial to understand the role of thermal or quantum fluctuations in the vicinity of the phase transition. It appears more appropriate to have a simplified description which emphasizes the role of the order parameter field $\boldsymbol{\varphi}(\mathbf{r})$ associated to the order parameter $\mathbf{m}(\mathbf{r}) = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$. $\boldsymbol{\varphi}(\mathbf{r})$ is a quantum field or a function of the classical dynamical variables. The order parameter field is usually defined at a mesoscopic scale Λ^{-1} which is much larger than the lattice spacing (assuming that the microscopic model is defined on a lattice) but small wrt macroscopic scales.[16]

The process of obtaining a low-energy effective description by averaging over many unit cells is called "coarse graining". The averaging procedure ensures that $\boldsymbol{\varphi}(\mathbf{r})$ is a continuous variable, and the partition function can be expressed as a functional integral,[17,18]

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}]}, \tag{7.26}$$

where the momentum of the Fourier transformed field $\boldsymbol{\varphi}(\mathbf{p})$ satisfied $|\mathbf{p}| \leq \Lambda$. Formally, we can obtain the low-energy effective action $S[\boldsymbol{\varphi}]$ from a partial trace over all microscopic configurations compatible with a given configuration of the coarse grained field $\boldsymbol{\varphi}(\mathbf{r})$. In general the partial integration involves a finite number of degrees of freedom and $S[\boldsymbol{\varphi}]$ is an analytic function of $\boldsymbol{\varphi}$. In some cases, it is possible to explicitly derive $S[\boldsymbol{\varphi}]$ from a microscopic action as in the example of classical spin models discussed in section 7.2.2. In many cases however, we simply include in $S[\boldsymbol{\varphi}]$ all terms allowed by symmetry within a derivative expansion. For example, for a N-component field, the simplest action with $\mathrm{O}(N)$ symmetry reads

$$S[\boldsymbol{\varphi}] = \int d^d r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!}\left(\boldsymbol{\varphi}^2\right)^2 \right\}, \tag{7.27}$$

corresponding to the so-called $(\boldsymbol{\varphi}^2)^2$ theory (or linear $\mathrm{O}(N)$ model).

Such an approach is clearly inappropriate to compute the transition temperature (which depends on microscopic parameters of the system) but is sufficient to understand the behavior of the system near the critical point and in particular the universal properties (Sec. 7.5).

We can compute the partition function (7.26) using the perturbative methods introduced in chapter 1. The simplest approach is to perform a mean-field (or saddle-point) approximation,

$$Z_{\mathrm{MF}} \simeq e^{-S[\boldsymbol{\varphi}]}, \tag{7.28}$$

[16] In practice, Λ^{-1} is often identified to the lattice spacing.

[17] The action $S[\boldsymbol{\varphi}]$ is sometimes written as $\beta H[\boldsymbol{\varphi}]$, or merely $H[\boldsymbol{\varphi}]$, where $H[\boldsymbol{\varphi}]$ is the effective Hamiltonian. It is often (somewhat improperly) referred to as the "microscopic" action.

[18] In a quantum description, the field $\boldsymbol{\varphi}$ would also depend on an imaginary time $\tau \in [0,\beta]$. A classical description is nevertheless sufficient to study the critical behavior at a finite-temperature phase transition (Sec. 7.10).

where the field $\boldsymbol{\varphi}$ is determined from the saddle-point equation $\delta S/\delta\boldsymbol{\varphi} = 0$. The free energy is simply given by

$$F = -\frac{1}{\beta}\ln Z_{\rm MF} = \frac{1}{\beta}S[\mathbf{m}], \tag{7.29}$$

where the actual value of the order parameter $\mathbf{m} = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$ is obtained by minimizing the action.

Phenomenological Landau's theory

It is possible to rephrase the preceding discussion from a more phenomenological point of view with no reference to any microscopic model. In the phenomenological Landau theory one postulates that the free energy density $f = F/V$ is an analytic function of the order parameter, whose absolute minimum specifies the state of the system. Near the critical point of a second-order phase transition, the order parameter is small and f can be expanded in a power series. The form of f and its expansion must be consistent with the symmetries of the system.

Let us consider the case of a N-component real order parameter $\mathbf{m}(\mathbf{r})$ and assume the system to be O(N) symmetric: the free energy density is invariant if we uniformly rotate the order parameter. For a uniform order parameter and in the absence of magnetic field, f must be a function of the O(N) invariant $\mathbf{m}^2$. Near the phase transition, we write the free energy density as

$$\beta f = \beta f_0 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 - \mathbf{h}\cdot\mathbf{m}, \tag{7.30}$$

where the last term, which breaks the O(N) symmetry, is due to the coupling to an external field $\mathbf{H} = \mathbf{h}/\beta$ (i.e. the external magnetic field in the case of a ferromagnet). When the order parameter is inhomogeneous, we must include in f a term corresponding to the energy coast due to deviations from spatial uniformity. For a slowly varying order parameter, this leads to the Ginzburg-Landau free energy

$$F[\mathbf{m}] = \int d^dr\, f(\mathbf{m}, \boldsymbol{\nabla}\mathbf{m}), \tag{7.31}$$

where

$$\beta f = \beta f_0 + \frac{1}{2}(\boldsymbol{\nabla}\mathbf{m})^2 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 - \mathbf{h}\cdot\mathbf{m}. \tag{7.32}$$

We will see below that higher-order terms such as $(\mathbf{m}^2)^3$ or $(\mathbf{m}\cdot\boldsymbol{\nabla}\mathbf{m})^2$ are negligible near the phase transition. It is always possible to rescale the order parameter to set the coefficient of $(\boldsymbol{\nabla}\mathbf{m})^2$ equal to $1/2$. The free energy functional (7.31) is identical to that obtained from a saddle-point approximation of the $(\boldsymbol{\varphi}^2)^2$ theory [Eqs. (7.27,7.29)] and is to be identified with the coarse-grained action $S[\mathbf{m}]$ of the order parameter field.

f_0 is the free energy density in the disordered phase in the absence of magnetic field. Since it is expected to vary smoothly with temperature, it is usually omitted. We assume

$$r_0 = \bar{r}_0(T - T_{c0}) \tag{7.33}$$

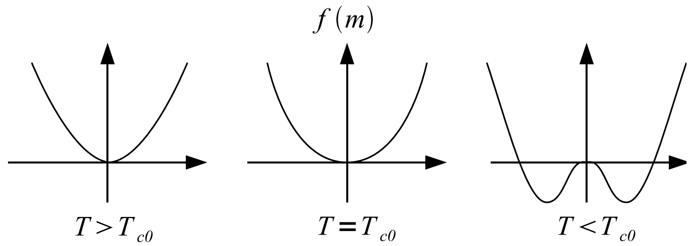

Figure 7.6: Free energy density $f(m)$ in zero field ($\mathbf{h} = 0$) near a second-order phase transition [Eq. (7.30)].

and neglect the temperature dependence of u_0 which, as we shall see, is unimportant near the phase transition. The stability of the system requires $u_0 > 0$ (otherwise the free energy is minimized for $|\mathbf{m}| \to \infty$).

The minimization of the free energy is straightforward (Fig. 7.6). In the presence of a uniform field along the $\mathbf{e}_1$ axis ($\mathbf{h} = h\mathbf{e}_1$), the magnetization density $\mathbf{m} = m\mathbf{e}_1$ satisfies

$$\frac{\partial \beta f}{\partial m} = r_0 m + \frac{u_0}{6} m^3 - h = 0. \tag{7.34}$$

For a vanishing field, one finds

$$m = \begin{cases} 0 & \text{if} \quad T \geq T_{c0}, \\ \sqrt{\dfrac{-6r_0}{u_0}} & \text{if} \quad T \leq T_{c0}, \end{cases} \tag{7.35}$$

i.e. a phase transition at the temperature T_{c0} (referred to as the mean-field transition temperature). It usually differs from the actual transition temperature T_c. Right at the transition and in the presence of an external field, the order parameter takes the value

$$m(T_{c0}, h) = \left(\frac{6h}{u_0} \right)^{1/3}. \tag{7.36}$$

From (7.32) and (7.35), we deduce the singular part of the free energy,

$$f = \begin{cases} -\dfrac{3}{2} \dfrac{T\bar{r}_0^2}{u_0} (T - T_{c0})^2 & \text{if} \quad T \leq T_{c0}, \\ 0 & \text{if} \quad T \geq T_{c0}. \end{cases} \tag{7.37}$$

The regular part of the free energy comes from the term f_0 in (7.32), which we have omitted in our discussion.

Equations (7.35) and (7.36) yield the critical exponents $\beta = 1/2$ and $\delta = 3$. Differentiating (7.34) with respect to $H = \beta^{-1} h$, one obtains the uniform susceptibility

$$\chi = \beta \frac{\partial m}{\partial h} = \frac{\beta}{r_0 + \frac{u_0}{2} m^2} = \begin{cases} \dfrac{\beta}{r_0} & \text{if} \quad T > T_{c0}, \\ \dfrac{\beta}{2|r_0|} & \text{if} \quad T < T_{c0}, \end{cases} \tag{7.38}$$

and a critical exponent $\gamma = \gamma' = 1$. The singular part of the specific heat per unit volume,[19]

$$c_V = -T\frac{\partial^2 f}{\partial T^2} = \left\{ \begin{array}{ll} 0 & \text{if} \quad T > T_{c0}, \\ 3\dfrac{\bar{r}_0^2}{u_0}T^2 & \text{if} \quad T < T_{c0}, \end{array} \right. \tag{7.39}$$

is discontinuous at the transition.

Let us now we consider the equation

$$0 = \frac{\delta\beta F[\mathbf{m}]}{\delta m_i(\mathbf{r})} = r_0 m_i(\mathbf{r}) - \boldsymbol{\nabla}^2 m_i(\mathbf{r}) + \frac{u_0}{6}\mathbf{m}(\mathbf{r})^2 m_i(\mathbf{r}) - h_i(\mathbf{r}) \tag{7.40}$$

in an arbitrary field $\mathbf{h}(\mathbf{r})$. From (7.40), we see that for r_0 and h small,

$$|\mathbf{m}| \sim |r_0|^{1/2}, \quad |h| \sim |r_0|^{3/2}, \quad \frac{|\nabla \mathbf{m}|}{|\mathbf{m}|} \sim |r_0|^{1/2}, \tag{7.41}$$

so that all terms in (7.32) are of order r_0^2. Terms not included are of higher order and can be neglected: $(\mathbf{m}^2)^3, (\mathbf{m}\cdot\boldsymbol{\nabla}\mathbf{m})^2 \sim r_0^3$, $(\mathbf{m}^2)^4 \sim r_0^4$, etc.

To compute the susceptibility

$$\chi_{ij}(\mathbf{r}-\mathbf{r}') = \frac{\delta m_i(\mathbf{r})}{\delta H_j(\mathbf{r}')}\bigg|_{\mathbf{H}=0} = \beta\frac{\delta m_i(\mathbf{r})}{\delta h_j(\mathbf{r}')}\bigg|_{\mathbf{h}=0}, \tag{7.42}$$

we take the functional derivative $\delta/\delta h_j(\mathbf{r}')$ in (7.40) and set $\mathbf{h}(\mathbf{r}) = 0$ (i.e. $\mathbf{m}(\mathbf{r}) = m\mathbf{e}_1$),

$$\left(r_0 - \boldsymbol{\nabla}^2 + \frac{u_0}{6}m^2 + \delta_{i,1}\frac{u_0}{3}m^2\right)\chi_{ij}(\mathbf{r}-\mathbf{r}') = \beta\delta_{i,j}\delta(\mathbf{r}-\mathbf{r}'). \tag{7.43}$$

In Fourier space, we finally obtain

$$\chi_\parallel(\mathbf{p}) = \chi_\perp(\mathbf{p}) = \frac{\beta}{\mathbf{p}^2 + r_0} \quad \text{if} \quad T > T_{c0} \tag{7.44}$$

and

$$\left\{ \begin{array}{l} \chi_\parallel(\mathbf{p}) = \dfrac{\beta}{\mathbf{p}^2 + 2|r_0|} \\ \chi_\perp(\mathbf{p}) = \dfrac{\beta}{\mathbf{p}^2} \end{array} \right. \quad \text{if} \quad T < T_{c0}, \tag{7.45}$$

where $\chi_\parallel$ and $\chi_\perp$ are the longitudinal and transverse components of the susceptibility (see Eq. (7.23)).[20] Equations (7.44) and (7.45) imply $\chi_\parallel(\mathbf{r}) \sim e^{-|\mathbf{r}|/\xi}$ with[21]

$$\xi = \left\{ \begin{array}{ll} r_0^{-1/2} & \text{if} \quad T > T_{c0}, \\ |2r_0|^{-1/2} & \text{if} \quad T < T_{c0}, \end{array} \right. \tag{7.46}$$

and therefore a critical exponent $\nu = \nu' = 1/2$. The connected correlation functions $G_\parallel(\mathbf{p})$ and $G_\perp(\mathbf{p})$ of the order parameter field $\boldsymbol{\varphi}$ can be deduced from (7.44) and (7.45) by using the (classical) fluctuation-dissipation theorem $\chi_{ij}(\mathbf{p}) = \beta G_{ij}(\mathbf{p})$ [Eq. (3.59)].

[19] The regular part of the specific heat comes from the term f_0 in (7.32).

[20] Since the magnetization is along the $\mathbf{e}_1$ axis, $\chi_\parallel = \chi_{11}$ and $\chi_\perp = \chi_{22}$.

[21] See footnote 14 page 452.

	Landau approximation	Gaussian model
$\nu = \nu'$	1/2	1/2
β	1/2	1/2
$\gamma = \gamma'$	1	1
δ	3	3
$\alpha = \alpha'$	c_V discontinuous	$2 - d/2$
η	0	0

Table 7.2: Critical exponents of the $(\boldsymbol{\varphi}^2)^2$ theory with $\mathrm{O}(N)$ symmetry [Eq. (7.27)] in the Landau approximation (Sec. 7.2) and in the Gaussian approximation for $d \leq 4$ (Sec 7.3). (For $d > 4$, the Gaussian approximation predicts the specific heat to be discontinuous.)

We deduce that the anomalous dimension η vanishes: $G(\mathbf{p}, T_c) = 1/\mathbf{p}^2$. The critical exponents obtained within the Landau approximation are sometimes referred to as "classical" exponents (Table 7.2). The $1/\mathbf{p}^2$ divergence of the transverse susceptibility for $\mathbf{p} \to 0$ is a manifestation of Goldstone theorem; it will be further discussed in section 7.3.2.

Effective action $\Gamma[\mathbf{m}]$ within the Landau approximation

It is sometimes convenient to work with the effective action $\Gamma[\mathbf{m}]$ (or the Gibbs free energy $G[\mathbf{m}] = \beta^{-1}\Gamma[\mathbf{m}]$) rather than the Helmholtz free energy $F[\mathbf{h}]$ or the microscopic action $S[\boldsymbol{\varphi}]$.[22] When the system is coupled to an external field, the partition function reads

$$Z[\mathbf{h}] = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}] + \int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}} \tag{7.47}$$

and the order parameter is given by

$$m_i(\mathbf{r}) = \langle \varphi_i(\mathbf{r}) \rangle = \frac{\delta \ln Z[\mathbf{h}]}{\delta h_i(\mathbf{r})}. \tag{7.48}$$

The effective action is defined as the Legendre transform

$$\Gamma[\mathbf{m}] = -\ln Z[\mathbf{h}] + \int d^d r\, \mathbf{h}\cdot\mathbf{m}, \tag{7.49}$$

where $\mathbf{h} \equiv \mathbf{h}[\mathbf{m}]$ is obtained by inverting (7.48). It satisfies the equation of state

$$\frac{\delta \Gamma[\mathbf{m}]}{\delta m_i(\mathbf{r})} = h_i(\mathbf{r}). \tag{7.50}$$

In the mean-field approximation, $\ln Z[\mathbf{h}] = -S[\boldsymbol{\varphi}] + \int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}$ satisfies the saddle-point equation

$$\frac{\delta S[\boldsymbol{\varphi}]}{\delta \varphi_i(\mathbf{r})} - h_i(\mathbf{r}) = 0. \tag{7.51}$$

[22] The effective action $\Gamma[\mathbf{m}]$ is introduced in Sec. 1.6.2.

We deduce

$$\begin{aligned} m_i(\mathbf{r}) &= \frac{\delta \ln Z[\mathbf{h}]}{\delta h_i(\mathbf{r})} \\ &= -\int d^d r' \sum_j \frac{\delta S[\boldsymbol{\varphi}]}{\delta \varphi_j(\mathbf{r}')} \frac{\delta \varphi_j(\mathbf{r}')}{\delta h_i(\mathbf{r})} + \int d^d r' \sum_j h_j(\mathbf{r}') \frac{\delta \varphi_j(\mathbf{r}')}{\delta h_i(\mathbf{r})} + \varphi_i(\mathbf{r}) \\ &= \varphi_i(\mathbf{r}), \end{aligned} \tag{7.52}$$

so that the effective action

$$\Gamma[\mathbf{m}] = S[\mathbf{m}] \tag{7.53}$$

reduces to the microscopic action within the Landau (mean-field) approximation, in agreement with the general discussion of section 1.7.2. The zero-field (connected) correlation function $G_{ij}(\mathbf{r}-\mathbf{r}') = \langle \varphi_i(\mathbf{r})\varphi_j(\mathbf{r}')\rangle - \langle \varphi_i(\mathbf{r})\rangle\langle \varphi_j(\mathbf{r}')\rangle$ [Eq. (7.21)] is given by the inverse of the two-point vertex

$$\Gamma^{(2)}_{ij}(\mathbf{r}-\mathbf{r}') = \frac{\delta^{(2)}\Gamma[\mathbf{m}]}{\delta m_i(\mathbf{r})\delta m_j(\mathbf{r}')}\bigg|_{\mathbf{m}(\mathbf{r})=\mathbf{m}} = \frac{\delta^{(2)} S[\mathbf{m}]}{\delta m_i(\mathbf{r})\delta m_j(\mathbf{r}')}\bigg|_{\mathbf{m}(\mathbf{r})=\mathbf{m}} \tag{7.54}$$

(see Sec. 1.6.2), i.e.

$$\begin{aligned} \Gamma_\parallel(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{u_0}{2}\mathbf{m}^2, \\ \Gamma_\perp(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{u_0}{6}\mathbf{m}^2, \end{aligned} \tag{7.55}$$

where $\mathbf{m}$ is determined from the stationarity condition $\delta\Gamma[\mathbf{m}]/\delta\mathbf{m} = 0$ [Eq. (7.50)]. We thus recover the mean-field propagator $G_{ij} = T\chi_{ij}$ [Eqs. (7.44) and (7.45)].

Universality

Within mean-field theory, the critical exponents of the $(\boldsymbol{\varphi}^2)^2$ theory (7.27) are independent of the values of the coupling constants $\bar{r}_0$ and u_0. Two apparently different systems share the same set of critical exponents: this is called universality. Two systems with the same critical exponents are said to be in the same universality class. We shall see that the universality predicted by the Landau theory is too strong. The critical exponents take their mean-field ("classical") value only above the upper critical dimension d_c^+. When $d < d_c^+$, they generally depend on the dimension d, the number N of components of the order parameter as well as the symmetries and range of the interactions. The explanation of universality and the computation of the critical exponents is one of the great successes of the renormalization group (Sec. 7.5).

Breakdown of mean-field theory

Mean-field theory is a good approximation if fluctuations of the order parameter about its mean value are small. Since we have been able, using the fluctuation-dissipation theorem, to obtain the correlation functions from the mean-field theory, we can check the internal consistency of the theory. Let us consider a coherence volume $V \sim \xi^d \sim |r_0|^{-d/2}$ in which the fluctuations are correlated. The average magnetization is

$$M \sim Vm \sim \xi^d \sqrt{\frac{6|r_0|}{u_0}} \sim \xi^{d-1} \tag{7.56}$$

and the fluctuation

$$\Delta M^2 \sim \int d^d r\, d^d r' \langle(\varphi(\mathbf{r}) - m)(\varphi(\mathbf{r}') - m)\rangle \sim V \int d^d r\, G(\mathbf{r}) \sim \xi^{d+2} \tag{7.57}$$

(for simplicity we assume a scalar order parameter, i.e. $N = 1$) can be related to the correlation function[23]

$$G(\mathbf{r}) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \sim \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{d-2}} \qquad (|\mathbf{r}| \lesssim \xi). \tag{7.58}$$

We therefore obtain

$$\frac{\Delta M^2}{M^2} \sim \xi^{4-d}. \tag{7.59}$$

When $d > 4$, the rhs of (7.59) vanishes as $T \to T_{c0}^-$ and $\xi \to \infty$, and the mean-field theory appears internally consistent. When $d < 4$, the mean-field theory is clearly not reliable in the vicinity of the critical point. The dimension $d_c^+ = 4$ where the mean-field theory breaks down is called the upper critical dimension. The criterion $\Delta M^2 \sim M^2$ giving the size of the critical region is known as the Ginzburg criterion; it will be discussed in more detail in section 7.3.4.

7.2.2 $(\varphi^2)^2$ theory for classical spin models.

Let us first consider the Hamiltonian

$$\beta H = -\frac{1}{2} \sum_{i,j} \sigma_i K_{ij} \sigma_j - \sum_i h_i \sigma_i \tag{7.60}$$

of the Ising model, where $\sigma_i = \pm 1$ is a classical spin variable and i denotes a site of a d-dimensional cubic lattice. h_i/β is an external magnetic field and $K_{ij} = \beta J_{ij}$. We assume the exchange coupling constant J_{ij} to be equal to J for nearest-neighbor spins and to vanish otherwise. The mean-field Hamiltonian is obtained by writing $\sigma_i = (\sigma_i - m_i) + m_i$, with $m_i = \langle \sigma_i \rangle$, and linearizing wrt the fluctuation term $\sigma_i - m_i$,

$$\beta H_{\rm MF} = -\sum_i \Big(h_i + \sum_j K_{ij} m_j \Big) \sigma_i + \frac{1}{2} \sum_{i,j} m_i K_{ij} m_j. \tag{7.61}$$

A self-consistent equation for m_i is obtained by computing the mean value $\langle \sigma_i \rangle$ with (7.61),

$$m_i = \frac{\mathrm{Tr}\big(\sigma_i e^{-\beta H_{\rm MF}}\big)}{\mathrm{Tr}\big(e^{-\beta H_{\rm MF}}\big)} = \tanh\Big(h_i + \sum_j K_{ij} m_j \Big). \tag{7.62}$$

[23] Correlation functions in real space are discussed in Sec. 7.3.1.

In the absence of external field ($h_i = 0$) and for a uniform order parameter $m_i = m$, we obtain

$$m = \tanh(2dKm), \tag{7.63}$$

where $K = \beta J$. Equation (7.63) admits a nonzero solution when $2dK > 1$, i.e. below the mean-field transition temperature $T_{c0} = 2dJ$.

To rewrite the Ising model as a functional integral over a continuous field φ_i, we use the identity

$$e^{\frac{1}{2}\sum_{i,j}\sigma_i K_{ij}\sigma_j} = \int_{-\infty}^{\infty}\prod_i d\varphi_i\, e^{-\frac{1}{2}\sum_{i,j}\varphi_i K^{-1}_{ij}\varphi_j + \sum_i \varphi_i\sigma_i} \tag{7.64}$$

(Hubbard-Stratonovich transformation) and rewrite the partition function as

$$Z = \sum_{\{\sigma_i\}}\int_{-\infty}^{\infty}\prod_i d\varphi_i\, e^{-\frac{1}{2}\sum_{i,j}(\varphi_i - h_i)K^{-1}_{ij}(\varphi_j - h_j) + \sum_i \varphi_i\sigma_i} \tag{7.65}$$

(we have shifted the field $\varphi_i \to \varphi_i - h_i$). The sum over the discrete variables σ_i can be done,

$$\sum_{\{\sigma_i\}} e^{\sum_i \varphi_i\sigma_i} = \prod_i 2\cosh(\varphi_i), \tag{7.66}$$

which leads to

$$Z = \int_{-\infty}^{\infty}\prod_i d\varphi_i\, e^{-\frac{1}{2}\sum_{i,j}(\varphi_i - h_i)K^{-1}_{ij}(\varphi_j - h_j) + \sum_i \ln(2\cosh\varphi_i)}. \tag{7.67}$$

The Fourier transform of K_{ij} is given by

$$K(\mathbf{p}) = 2K\sum_{\nu=1}^{d}\cos(q_\nu) = 2K\left(d - \frac{\mathbf{p}^2}{2}\right) + \mathcal{O}(p_\nu^4), \tag{7.68}$$

so that

$$K^{-1}(\mathbf{p}) = (2Kd)^{-1}\left(1 + \frac{\mathbf{p}^2}{2d}\right) + \mathcal{O}(p_\nu^4), \tag{7.69}$$

where we have taken the lattice spacing as the unit length. Assuming the field φ to be small (which allows us to expand $\ln(2\cosh\varphi_i)$) and slowly varying (which justifies the continuum limit), we obtain the action

$$S[\varphi] = \int d^dr\left[\left(\frac{1}{4Kd} - \frac{1}{2}\right)\varphi^2 + \frac{1}{8Kd^2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{12}\varphi^4\right] \tag{7.70}$$

in zero magnetic field. With an appropriate rescaling of the field, equation (7.70) takes the form (7.27) with a momentum cutoff $\Lambda \sim \pi$, $N = 1$, and

$$\frac{r_0}{2} \simeq \frac{d}{T_{c0}}(T - T_{c0}), \qquad \frac{u_0}{4!} \simeq \frac{4J^2d^4}{3T_{c0}^2} = \frac{d^2}{3} \tag{7.71}$$

for T near the mean-field transition temperature T_{c0}. It should be noted that the derivation of (7.70) is questionable since the matrix K_{ij} has no inverse. This difficulty is circumvented in (7.68) and (7.69) by considering the small $\mathbf{p}$ limit of $K(\mathbf{p})$.

This derivation can easily be generalized to a spin Hamiltonian $H = \frac{1}{2}\sum_{i,j} J_{i,j}\mathbf{S}_i \cdot \mathbf{S}_j$ where $\mathbf{S}_i$ is a N-component spin with $\mathbf{S}_i^2 = 1$. $N = 2$ ($N = 3$) corresponds to the classical XY (Heisenberg) model. The long-distance physics of this model is described by the action (7.27) of the $(\boldsymbol{\varphi}^2)^2$ theory with $\boldsymbol{\varphi}$ a N-component field.

The $(\boldsymbol{\varphi}^2)^2$ theory is not strictly equivalent to the original spin model since we have neglected $(\boldsymbol{\varphi}^2)^3$, $(\boldsymbol{\varphi}^2)^4$, etc., as well as higher-order derivative terms. However these terms can be ignored at the mean-field level in the close vicinity of the phase transition (Sec. 7.2.1). Furthermore, they do not affect the long-distance (universal) physics. In particular, the value of the critical exponents is the same in the spin model and in the $(\boldsymbol{\varphi}^2)^2$ theory. In the renormalization group sense, both models belong to the same universality class (Sec. 7.5). The utility of the $(\boldsymbol{\varphi}^2)^2$ theory is that it allows us to use powerful field theoretical methods to study the critical behavior (Secs. 7.6 and 7.B).[24]

7.2.3 First-order phase transitions

The Landau theory can also be used to study first-order phase transitions although the order parameter is not necessarily small at the transition (unless the transition is weakly first order). Let us consider the free energy density

$$\beta f = \beta f_0 + \frac{r_0}{2}m^2 - u_3 m^3 + \frac{u_0}{4!}m^4 \tag{7.72}$$

for a scalar order parameter, where $r_0 = \bar{r}_0(T - T_0)$ and $u_0, u_3 > 0$ are independent of temperature.[25] The cubic term in (7.72) makes the transition first order: the order parameter is discontinuous at the transition (Fig. 7.7). The phase of the system is determined by requiring the free energy to be minimum. In the high-temperature phase, $m = 0$ and $f = f_0$. The first-order transition temperature T_c and the value of the order parameter m_c at T_c are obtained by requiring that $f(m_c)$ is a minimum of $f(m)$ and that the ordered and disordered phases have the same free energy,

$$\begin{aligned} \beta f'(m) &= m\left(r_0 - 3u_3 m + \frac{u_0}{6}m^2\right) = 0, \\ \beta f(m) &= m^2\left(\frac{r_0}{2} - u_3 m + \frac{u_0}{4!}m^2\right) = 0, \end{aligned} \tag{7.73}$$

where we have assumed that $f_0(T_c) = 0$. This yields

$$\begin{aligned} r_{0c} &= \bar{r}_0(T_c - T_0) = 12\frac{u_3^2}{u_0}, \\ m_c &= 12\frac{u_3}{u_0}. \end{aligned} \tag{7.74}$$

In the disordered phase, the entropy density $s = S/V = -\partial f/\partial T$ is simply given by $s = -\partial f_0/\partial T$. In the ordered phase, an elementary calculation gives

$$s = -\frac{\partial f_0}{\partial T} - \frac{\bar{r}_0}{2}m^2 T. \tag{7.75}$$

[24] See also chapter 8.

[25] Note that if the order parameter vanishes in the disordered phase, then $\partial f/\partial m|_{m=0} = 0$ and there is no linear term in the free energy density $f(m)$.

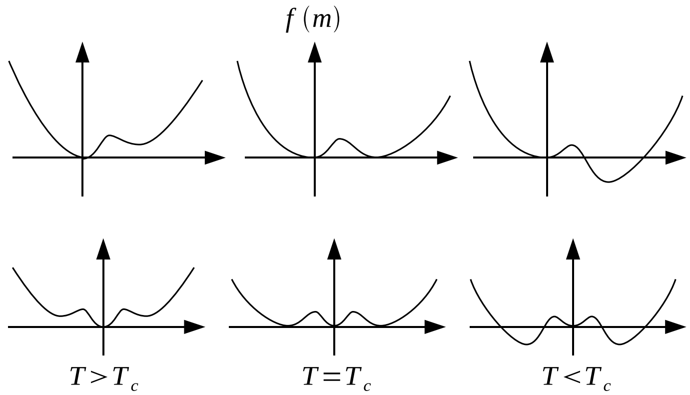

Figure 7.7: Free energy density $f(m)$ near a first-order phase transition. $f(m)$ is defined by (7.72) (top) and (7.77) (bottom).

The entropy density is therefore discontinuous at the transition from the low- to the high-temperature phase,

$$\Delta s = 2T_c\bar{r}_0 \left(\frac{6u_3}{u_0}\right)^2 , \tag{7.76}$$

which corresponds to a latent heat $Q/V = T_c\Delta s$ per unit volume.

Another exemple of first-order phase transition is provided by the free energy density

$$\beta f = \beta f_0 + \frac{r_0}{2}\mathbf{m}^2 + \frac{u_0}{4!}\left(\mathbf{m}^2\right)^2 + u_6\left(\mathbf{m}^2\right)^3 . \tag{7.77}$$

When $u_0 > 0$, the phase transition is second order and the sixth-order term can be neglected when the order parameter is small. If $u_0 < 0$, the sixth-order term is necessary to maintain stability. In this case, the transition is first order (Fig. 7.7). The transition temperature T_c and the value of the order parameter at T_c can be calculated as in the preceding example,

$$r_{0c} = \bar{r}_0(T_c - T_0) = \begin{cases} 0 & \text{if} \quad u_0 \geq 0, \\ \dfrac{1}{2u_6}\left(\dfrac{u_0}{4!}\right)^2 & \text{if} \quad u_0 \leq 0, \end{cases} \tag{7.78}$$

and

$$\mathbf{m}_c^2 = \begin{cases} 0 & \text{if} \quad u_0 \geq 0, \\ \dfrac{|u_0|}{48u_6} & \text{if} \quad u_0 \leq 0. \end{cases} \tag{7.79}$$

The phase diagram in the plane (r_0, u_0) is shown in figure 7.8. The line of second-order transitions ($u_0 > 0$) and that of first-order transitions ($u_0 < 0$) meet at the tricritical point $r_0 = u_0 = 0$.

7.3 Gaussian model

The simplest improvement of Landau's theory consists in taking into account Gaussian fluctuations of the field $\boldsymbol{\varphi}$ about its saddle-point value. For $T > T_{c0}$, this amounts to

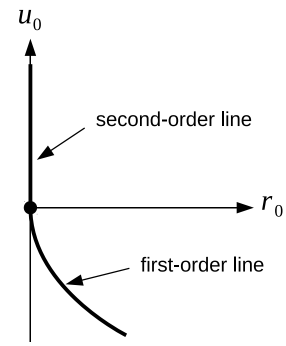

Figure 7.8: Phase diagram obtained from the free energy density (7.77).

neglecting the interacting part of the action. For $T < T_{c0}$, we write the field as

$$\boldsymbol{\varphi}(\mathbf{r}) = \left(\sqrt{\frac{-6r_0}{u_0}} + \varphi'_1(\mathbf{r})\right)\mathbf{e}_1 + \sum_{i=2}^{N} \varphi'_i(\mathbf{r})\mathbf{e}_i \tag{7.80}$$

(with $\mathbf{e}_1$ the direction of the order parameter) and expand the action to quadratic order in $\boldsymbol{\varphi}'$. This gives

$$S[\boldsymbol{\varphi}] = \begin{cases} \frac{1}{2}\sum_{\mathbf{p},i} |\varphi_i(\mathbf{p})|^2(\mathbf{p}^2 + r_0) & \text{if} \quad T > T_{c0}, \\ S_{\rm MF} + \frac{1}{2}\sum_{\mathbf{p}} \left[|\varphi'_1(\mathbf{p})|^2(\mathbf{p}^2 - 2r_0) + \sum_{i=2}^{N} |\varphi'_i(\mathbf{p})|^2 \mathbf{p}^2 \right] & \text{if} \quad T < T_{c0}, \end{cases} \tag{7.81}$$

for the $(\boldsymbol{\varphi}^2)^2$ theory (7.27).

7.3.1 Correlation functions

The propagator $G_{ij} = T\chi_{ij}$ deduced from the action (7.81) agrees with (7.44) and (7.45). The longitudinal propagator reads[26]

$$G_\parallel(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + \xi^{-2}}, \tag{7.82}$$

with $\xi = r_0^{-1/2}$ ($|2r_0|^{-1/2}$) if $T > T_{c0}$ ($T < T_{c0}$). In the disordered phase, $G_\perp = G_\parallel$, while

$$G_\perp(\mathbf{p}) = \frac{1}{\mathbf{p}^2} \tag{7.83}$$

in the ordered phase. The Gaussian fluctuations do not change the value of the order parameter $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ and the transition temperature is still given by T_{c0}. The critical exponents $\beta, \delta, \gamma, \nu$ and the anomalous dimension η keep their mean-field value (Table 7.1).

[26] See footnote 14 page 452.

Correlation functions in direct space ($d \geq 2$).[27] To obtain $G_\parallel(\mathbf{r})$ and $G_\perp(\mathbf{r})$, we need to Fourier transform $1/\mathbf{p}^2$ and $1/(\mathbf{p}^2 + \xi^{-2})$. The Fourier transform $G(\mathbf{r})$ of $1/\mathbf{p}^2$ satisfies

$$-\boldsymbol{\nabla}^2 G(\mathbf{r}) = \delta(\mathbf{r}). \tag{7.84}$$

Since $G(\mathbf{r}) = G(|\mathbf{r}|)$, $\boldsymbol{\nabla} G(\mathbf{r}) = G'(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|}$ and equation (7.84) can be solved by considering[28]

$$\int_V d^d r\, \delta(\mathbf{r}) = -\int_V d^d r\, \boldsymbol{\nabla}^2 G(\mathbf{r}) = -\int_S d\mathbf{S}\cdot\boldsymbol{\nabla} G(\mathbf{r}) = -|\mathbf{r}|^{d-1} S_d G'(|\mathbf{r}|), \tag{7.85}$$

where V and S are the volume and the surface of the d-dimensional sphere of radius $|\mathbf{r}|$ centered at the origin. $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the unit sphere in d dimensions. From (7.85), we deduce $G'(|\mathbf{r}|) = -|\mathbf{r}|^{1-d}/S_d$ and

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2} = \begin{cases} \dfrac{1}{(d-2)S_d|\mathbf{r}|^{d-2}} & \text{if} \quad d > 2, \\ -\dfrac{1}{2\pi}\ln|\mathbf{r}| + \text{const} & \text{if} \quad d = 2, \end{cases} \tag{7.86}$$

where we have used $\lim_{|\mathbf{r}|\to\infty} G(\mathbf{r}) = 0$ when $d > 2$.

We now consider the Fourier transform $G(\mathbf{r}) = G(|\mathbf{r}|)$ of $1/(\mathbf{p}^2 + \xi^{-2})$. It satisfies

$$\left(-\boldsymbol{\nabla}^2 + \xi^{-2}\right) G(\mathbf{r}) = \left(-\frac{\partial^2}{\partial|\mathbf{r}|^2} - \frac{d-1}{|\mathbf{r}|}\frac{\partial}{\partial|\mathbf{r}|} + \xi^{-2}\right) G(|\mathbf{r}|) = \delta(\mathbf{r}). \tag{7.87}$$

Let us try a solution $G(|\mathbf{r}|) \propto e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ which decays exponentially at large distance. For $|\mathbf{r}| \neq 0$, equation (7.87) is satisfied if

$$\frac{p(p+1)}{|\mathbf{r}|^2} + \frac{2p}{|\mathbf{r}|\xi} + \frac{1}{\xi^2} - \frac{d-1}{|\mathbf{r}|}\left(\frac{p}{|\mathbf{r}|} + \frac{1}{\xi}\right) - \frac{1}{\xi^2} = 0. \tag{7.88}$$

The choice of ξ as the decay length ensures that the constant term vanishes. For $|\mathbf{r}| \ll \xi$, the $1/|\mathbf{r}|^2$ terms are the most important and we must have $p(p+1) - p(d-1) = 0$, i.e. $p = d-2$ or $p = 0$. Since for $|\mathbf{r}| \ll \xi$ we must reproduce the result (7.86), we deduce $p = d-2$ and

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \simeq \frac{e^{-|\mathbf{r}|/\xi}}{(d-2)S_d|\mathbf{r}|^{d-2}} \quad \text{if} \quad |\mathbf{r}| \ll \xi \quad \text{and} \quad d > 2. \tag{7.89}$$

The $|\mathbf{r}|$ dependence is logarithmic when $d = 2$ [Eq. (7.86)] and the assumption $G(|\mathbf{r}|) \propto e^{-|\mathbf{r}|/\xi}/|\mathbf{r}|^p$ is not correct. For $|\mathbf{r}| \gg \xi$, the $1/|\mathbf{r}|\xi$ terms dominate in (7.88) and therefore $2p - (d-1) = 0$. We thus obtain

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + \xi^{-2}} \sim \frac{\xi^{(3-d)/2}}{(d-2)S_d} \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{(d-1)/2}} \quad \text{if} \quad |\mathbf{r}| \gg \xi, \tag{7.90}$$

where the prefactor is fixed from the condition that (7.89) and (7.90) match for $|\mathbf{r}| \sim \xi$.

[27] In one dimensional, a straightforward calculation gives $\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqx}}{q^2+\xi^{-2}} = \frac{\xi}{2} e^{-|x|/\xi}$.

[28] Eq. (7.84) is Poisson's equation satisfied by the potential $G(|\mathbf{r}|)$ created by a charge located at $\mathbf{r} = 0$ and (7.85) is nothing but Gauss' theorem.

These results can also be obtained from the exact solution

$$G(\mathbf{r}) = (2\pi)^{-d/2} \frac{\xi^{(2-d)/2}}{|\mathbf{r}|^{(d-2)/2}} K_{(d-2)/2}(|\mathbf{r}|/\xi), \tag{7.91}$$

where K_n is the modified Bessel function of the second kind. Equations (7.89) and (7.90) are then simply obtained from the following asymptotic properties,

$$\begin{aligned} K_n(x) &\simeq \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \qquad (x \gg |n^2 - 1/4|), \\ K_n(x) &\simeq \frac{\Gamma(n)}{2} \left(\frac{x}{2}\right)^{-n} \qquad (x \ll \sqrt{n+1} \text{ and } n > 0), \\ K_0(x) &\simeq -\ln\left(\frac{x}{2}\right) - \gamma \qquad (x \ll 1), \end{aligned} \tag{7.92}$$

where $\gamma \simeq 0.5772$ is the Euler constant and $\Gamma(z)$ the Gamma function. The last equation gives the behavior of the two-dimensional correlation function in the limit $|\mathbf{r}|/\xi \ll 1$ in agreement with (7.86). For $d = 3$, one deduces from (7.92) with $n = 1/2$ that

$$G(\mathbf{r}) = \frac{e^{-|\mathbf{r}|/\xi}}{4\pi|\mathbf{r}|}. \tag{7.93}$$

7.3.2 Goldstone's theorem

In the broken-symmetry phase, $G_\perp(\mathbf{p}) = 1/\mathbf{p}^2$ and the uniform transverse susceptibility $\chi_\perp = \beta G_\perp(\mathbf{p} = 0)$ is infinite: it requires an infinitesimal field to rotate the direction of the magnetization. This result is due to the existence of soft modes, i.e. field configurations $\boldsymbol{\varphi}(\mathbf{p})$ whose action $S[\boldsymbol{\varphi}]$ vanishes in the long-wavelength limit $\mathbf{p} \to 0$. It can also be seen as a manifestation of Goldstone's theorem for quantum systems discussed in section 3.6.3: a spontaneously broken continuous symmetry implies the existence of a low-energy mode whose energy $\omega_\mathbf{p}$ vanishes for $\mathbf{p} \to 0$.[29] For the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry, there are $N - 1$ Goldstone modes.[30] These modes play a crucial role since they often give the main contribution to the observables of the system. When $N = 1$, the broken symmetry is discrete and there are no Goldstone modes (as expected since it is not possible to produce slowly varying rotations from one state to an equivalent one).

Another important consequence of a spontaneously broken continuous symmetry is the emergence of rigidity. Let us write the field as $\boldsymbol{\varphi}(\mathbf{r}) = m(\mathbf{e}_1 + \delta\tilde{\varphi}_\parallel(\mathbf{r})\mathbf{e}_1 + \delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{r}))$, where $\delta\tilde{\varphi}_\parallel$ denotes a longitudinal fluctuation and $\delta\tilde{\boldsymbol{\varphi}}_\perp \perp \mathbf{e}_1$ a transverse fluctuation, i.e., in the limit $\delta\tilde{\boldsymbol{\varphi}}_\perp \to 0$, a fluctuation of the direction of the magnetization. According to (7.81), the action corresponding to transverse fluctuations reads

$$S[\delta\tilde{\boldsymbol{\varphi}}_\perp] = \frac{\rho_s}{2} \sum_\mathbf{p} \delta\tilde{\boldsymbol{\varphi}}_\perp(-\mathbf{p}) G_\perp^{-1}(\mathbf{p}) \delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{p}) = \frac{\rho_s}{2} \int d^d r\, (\boldsymbol{\nabla}\delta\tilde{\boldsymbol{\varphi}}_\perp)^2 \tag{7.94}$$

[29] Goldstone's theorem requires the interactions to be short range (Sec. 3.6.3).

[30] More generally, let us consider a model with a symmetry group $\mathcal{G}$, and call $\mathcal{H}$ the subgroup of $\mathcal{G}$ which leaves the order parameter $\mathbf{m} = \langle\boldsymbol{\varphi}(\mathbf{r})\rangle$ invariant. Then there are $g - h$ Goldstone modes, where g and h are the number of generators of the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$. For the $(\boldsymbol{\varphi}^2)^2$ theory, $\mathcal{G} = O(N)$, $\mathcal{H} = O(N-1)$, $g = \frac{1}{2}N(N-1)$ and $h = \frac{1}{2}(N-1)(N-2)$, so that there are $g - h = N - 1$ Goldstone modes.

(with $\rho_s = m^2$), which implies that the transverse correlation function is given by

$$G_\perp(\mathbf{p}) = \frac{m^2}{\rho_s \mathbf{p}^2}, \tag{7.95}$$

in agreement with the general discussion of section 3.6.3 on spontaneous symmetry breaking and Goldstone's theorem (see Eq. (3.254)). Equation (7.94) shows that any spatial variation of the order parameter in the direction perpendicular to the ordering raises the energy of the system. In the mean-field (Landau) approximation, the stiffness $\rho_s = m^2$ is simply given by the square of the order parameter. The stiffness plays an essential role in superfluid systems where it determines the superfluid density (chapter 6). We shall see in section 7.8 that in certain two-dimensional systems, the stiffness can be finite although there is no broken symmetry.

7.3.3 Mermin-Wagner theorem – Lower critical dimension

For the ordered phase to be stable, the fluctuations of the $\boldsymbol{\varphi}$ field must be finite. For $N \geq 2$, we can study the stability by looking at the most dangerous modes, namely the transverse fluctuations $\delta\boldsymbol{\varphi}_\perp = m\delta\tilde{\boldsymbol{\varphi}}_\perp$,

$$\langle \delta\boldsymbol{\varphi}_\perp(\mathbf{r})^2 \rangle = (N-1) \int \frac{d^d p}{(2\pi)^d} G_\perp(\mathbf{p}) = (N-1) \frac{S_d}{(2\pi)^d} \int_0^\Lambda \frac{d|\mathbf{p}|}{|\mathbf{p}|^{3-d}}. \tag{7.96}$$

When $d \leq 2$, the integral is infrared divergent and the assumption of spontaneous symmetry breaking cannot be maintained: thermally excited transverse fluctuations destroy long-range order. We recover the Mermin-Wagner theorem discussed in section 3.6.4: at finite temperature, a continuous symmetry cannot be broken in dimension $d \leq 2$. The dimension at and below which fluctuations prevent long-range order is called the lower critical dimension d_c^-. For the $(\boldsymbol{\varphi}^2)^2$ theory with $N \geq 2$, $d_c^- = 2$. The Mermin-Wagner theorem does not, however, preclude a phase transition (without long-range order) in two-dimensional systems with a continuous symmetry. The most famous example of such a transition is the Berezinskii-Kosterlitz-Thouless transition in the two-dimensional XY model (Sec. 7.8).

When $N = 1$, the broken symmetry is discrete and there are no Goldstone modes. It is nevertheless possible to determine the lower critical dimension from a simple argument based on the Ising model. We assume a d-dimensional hypercubic lattice, a coupling constant J between nearest-neighbor spins (Sec. 7.2.2), and periodic boundary conditions. In the ground state, all the spins are up. Let us now consider a configuration with an island of down spins with a linear size L equal to a fraction of the linear size N of the system. The energy of such a configuration is $E \sim 2JL^{d-1}$, while the entropy (related to the number of ways to locate the island in the system) $S \sim \ln(N-L) \sim \ln L$. In one dimension, the entropy term always dominates in the thermodynamic limit $L, N \to \infty$, and we expect the formation of islands of down spins to lower the free energy, which makes the magnetization vanish. In contrast, for $d > 1$, the energy term dominates and the presence of islands increases the free energy.[31] We then expect the magnetization to remain finite at sufficiently low tem-

[31] This argument does not say anything about islands with typical sizes $L \ll N$ (e.g. a single down spin). For $d > 1$, a small density of such defects does not destroy long-range order.

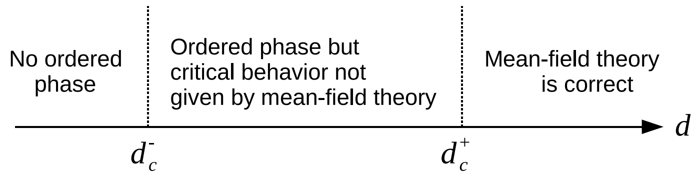

Figure 7.9: Long-range order and critical behavior *vs* lower (d_c^-) and upper (d_c^+) critical dimensions.

peratures.[32] We conclude that the lower critical dimension of the Ising model (and the $(\boldsymbol{\varphi}^2)^2$ theory with $N=1$) is $d_c^- = 1$.

7.3.4 Breakdown of mean-field theory – Upper critical dimension

In the preceding section, we have seen that the mean-field theory breaks down at the lower critical dimension d_c^- since it erroneously predicts long-range order for $d \leq d_c^-$. In this section, we show that mean-field theory also breaks down below the upper critical dimension d_c^+ ($d_c^+ > d_c^-$) in a slightly more subtle way. Although long-range order is not suppressed for $d_c^- < d < d_c^+$, the critical behavior in the vicinity of a second-order phase transition is not given by mean-field theory (as already anticipated in section 7.2.1) (Fig. 7.9).

Fluctuation corrections to the specific heat – Ginzburg criterion

In the Gaussian approximation, we can integrate out the $\boldsymbol{\varphi}$ field and compute the free energy. In the disordered phase, one finds

$$Z = \prod_{\mathbf{p}} \left(r_0 + \mathbf{p}^2\right)^{-N/2} \tag{7.97}$$

(see equation (7.81)), where we have neglected an unimportant multiplicative constant. The (most) singular part of the specific heat (per unit volume) $c_V = -T\frac{\partial^2 f}{\partial T^2}$ reads

$$c_V^{\rm sing} = \frac{NT^2}{2V}\sum_{\mathbf{p}} \frac{\bar r_0^2}{(\mathbf{p}^2+r_0)^2} = \frac{N}{2}T^2\bar r_0^2 K_d \int_0^\Lambda d|\mathbf{p}| \frac{|\mathbf{p}|^{d-1}}{(\mathbf{p}^2+r_0)^2}, \tag{7.98}$$

where $K_d = S_d/(2\pi)^d$. Setting $x = |\mathbf{p}|\xi = |\mathbf{p}|r_0^{-1/2}$, this result can be rewritten as

$$c_V^{\rm sing} = \frac{N}{2}T^2\bar r_0^2 K_d \xi^{4-d} I(\Lambda\xi), \tag{7.99}$$

where

$$I(\Lambda\xi) = \int_0^{\Lambda\xi} dx \frac{x^{d-1}}{(1+x^2)^2}. \tag{7.100}$$

[32] The existence of a finite temperature phase transition in the two-dimensional Ising model is confirmed by Onsager's exact solution [39].

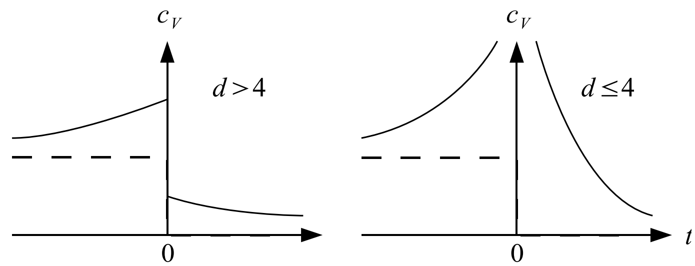

Figure 7.10: Specific heat below and above $d=4$ in the Gaussian model. The dashed line shows the result of the mean-field approximation. The smooth contribution coming from f_0 in (7.30) is not shown.

If $d>4$, the integral I is dominated by the upper cutoff, $I \simeq (\Lambda\xi)^{d-4}/(d-4)$, while $I \sim \ln(\Lambda\xi)$ for $d=4$. If $d<4$, the integral is independent of $\Lambda\xi$. We therefore obtain

$$c_V^{\rm sing} \simeq \frac{N}{2}(T\bar r_0)^2 K_d \times \begin{cases} \dfrac{\Lambda^{d-4}}{d-4} & \text{if} \quad d>4, \\ \ln(\Lambda\xi) & \text{if} \quad d=4, \\ I\xi^{4-d} & \text{if} \quad d<4. \end{cases} \tag{7.101}$$

When $d\leq 4$, the specific heat diverges as $T\to T_{c0}^+$,

$$c_V^{\rm sing} \sim \xi^{4-d} \sim (T-T_{c0})^{-\alpha}, \tag{7.102}$$

with a critical exponent $\alpha = 2-d/2$ (the divergence is logarithmic for $d=4$).

A similar analysis can be made below the transition temperature T_{c0}. From (7.81), we obtain

$$Z = Z_{\rm MF} \prod_{\mathbf p} \left(\mathbf p^2 - 2r_0\right)^{-1/2} \left(\mathbf p^2\right)^{-(N-1)/2}. \tag{7.103}$$

We deduce the singular part of the specific heat

$$\begin{aligned} c_V^{\rm sing} &= c_{V,\rm MF}^{\rm sing} + 2(T\bar r_0)^2 K_d \xi^{4-d} I(\Lambda\xi), \\ &= c_{V,\rm MF}^{\rm sing} + 2(T\bar r_0)^2 K_d \times \begin{cases} \dfrac{\Lambda^{d-4}}{d-4} & \text{if} \quad d>4, \\ \ln(\Lambda\xi) & \text{if} \quad d=4, \\ I\xi^{4-d} & \text{if} \quad d<4, \end{cases} \end{aligned} \tag{7.104}$$

where $\xi = |2r_0|^{-1/2}$. While for $d>4$ the specific heat remains discontinuous at the transition, the fluctuation corrections to the mean-field result yields a divergence when $d\leq 4$ (Fig. 7.10). As far as the critical behavior is concerned, this is the only change wrt mean-field theory, since all other critical exponents are unchanged (Table 7.2). The divergence of the specific heat implies that the mean-field results are not reliable below $d=4$ in the vicinity of the phase transition: fluctuations dominate the thermodynamics and the predictions of Landau's theory are not valid. We therefore recover our previous result that the upper critical dimension is $d_c^+ = 4$ for the $(\boldsymbol\varphi^2)^2$ theory (Sec. 7.2.1).

Below the upper critical dimension, the mean-field theory remains valid sufficiently far away from the transition. One can estimate the temperature at which the mean-field theory breaks down by comparing the mean-field discontinuity to the fluctuation correction, $\Delta c_{V,\mathrm{MF}} \sim c_{V,\mathrm{fl}}$, i.e.

$$\Delta c_{V,\mathrm{MF}} \sim \frac{N}{2} K_d \xi_0^{-d} |t|^{d/2-2} \tag{7.105}$$

(the factor $N/2$ is absent for $t<0$), where t is defined in (7.13). $\xi_0 \sim (\bar{r}_0 T_{c0})^{-1/2}$ is a microscopic length, of the order of the correlation length far away from the transition. Thus, for the mean-field theory to be valid below the upper critical dimension, the temperature must satisfy the Ginzburg criterion

$$|t| \gg t_G \sim \left(\frac{N K_d}{2\xi_0^d \Delta c_{V,\mathrm{MF}}} \right)^{1/(2-d/2)} . \tag{7.106}$$

$t_G = |T_G - T_c|/T_c$ is related to the Ginzburg temperature T_G, the temperature corresponding to the onset of the critical regime where fluctuations become important.[33]

In practice, the mean-field theory can be valid very close to the transition temperature if ξ_0 is large. This is the case in systems with long-range forces or in conventional superconductors where $\xi_0 \sim 1000$ Angströms corresponds to the BCS coherence length (Sec. 6.5) and is sufficiently large to make the critical regime unobservable. In many systems however, ξ_0 is of the order of a few Angströms, and the crossover from mean-field to critical behavior is observable.

One can also define a Ginzburg length $\xi_G = \xi(t_G) \sim \xi_0 t_G^{-1/2}$, i.e.

$$\xi_G \sim \xi_0 \left(\frac{2\xi_0^d \Delta c_{V,\mathrm{MF}}}{N K_d} \right)^{1/(4-d)} \sim \left(\frac{6}{N K_d u_0} \right)^{1/(4-d)} . \tag{7.107}$$

Below the upper critical dimension d_c^+, ξ becomes larger than ξ_G when $|t| \lesssim t_G$ (critical regime). We shall see in section 7.6 that for a critical system ($T = T_c$), ξ_G separates a high-momentum (perturbative) regime $|\mathbf{p}| \gtrsim \xi_G^{-1}$ where the Gaussian model is essentially correct from a low-momentum (critical) regime where the propagator acquires an anomalous dimension.

The fact that Gaussian fluctuations yield strong corrections to the mean-field results calls into question the validity of the Gaussian model itself when $d < 4$. One could try to perform a systematic loop expansion about the mean-field solution (see Sec. 1.7.2) and see whether it converges or not. Simple dimensional analysis is sufficient to answer this question. In the high-temperature phase, the loop expansion is merely an expansion wrt u_0. The actual expansion parameter can be identified y dimensional analysis. Since $S[\boldsymbol{\varphi}]$ is dimensionless, we must have $[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2] = d$, and therefore $[\boldsymbol{\varphi}] = (d-2)/2$.[34] Similarly, one finds $[r_0] = 2$ and $[u_0] = 4-d$.[35] In terms

[33] *Stricto sensu* there are two Ginzburg temperatures, one (T_G^+) above and one (T_G^-) below T_c. Fluctuations are important in the temperature range $[T_G^-, T_G^+]$. In Sec. 7.7.3, we shall see that for $N \geq 2$ (continuous broken symmetry) the Gaussian approximation breaks down in the whole temperature phase.

[34] The notation $[A] = n$ means that the quantity A is expressed in units of L^{-n} (with L the unit length). For example, $[\mathbf{r}] = -1$ and $[\mathbf{p}] = 1$.

[35] $[r_0] = 2$ and $[u_0] = 4-d$ are referred to as the naive scaling dimensions of r_0 and u_0 (see Sec. 7.4).

of the dimensionless variables

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{\xi}, \quad \tilde{\boldsymbol{\varphi}}(\tilde{\mathbf{r}}) = \xi^{(d-2)/2}\boldsymbol{\varphi}(\mathbf{r}), \quad \tilde{u}_0 = \xi^{4-d}u_0 \tag{7.108}$$

(with $\xi = r_0^{-1/2}$), the action becomes

$$S[\tilde{\boldsymbol{\varphi}}] = \int d^d\tilde{r} \left[\frac{1}{2}(\boldsymbol{\nabla}_{\tilde{\mathbf{r}}}\tilde{\boldsymbol{\varphi}})^2 + \frac{1}{2}\tilde{\boldsymbol{\varphi}}^2 + \frac{\tilde{u}_0}{4!}\left(\tilde{\boldsymbol{\varphi}}^2\right)^2 \right]. \tag{7.109}$$

In dimension $d > 4$, since $\tilde{u}_0 \to 0$ when $T \to T_{c0}$, it is reasonable to expect mean-field theory to become increasingly accurate as the transition is approached. For $d < 4$ on the other hand, $\tilde{u}_0$ diverges as $T \to T_{c0}$ and perturbation theory becomes meaningless. If we estimate that perturbation theory breaks down when $\tilde{u}_0 \sim (\xi/\xi_G)^{4-d}$ becomes of order 1, we recover the Ginzburg criterion (7.106).

One-loop correction to the two-point vertex $\Gamma^{(2)}(\mathbf{p})$

Let us consider the lowest-order (one-loop) correction to the two-point vertex $\Gamma^{(2)}(\mathbf{p}) = G(\mathbf{p})^{-1}$ [Eq. (7.55)]. In the high-temperature phase ($m = 0$), one finds

$$\begin{aligned} \Gamma^{(2)}(\mathbf{p}) &= \mathbf{p}^2 + r_0 + \frac{N+2}{6}u_0 \int \frac{d^dq}{(2\pi)^d} G(\mathbf{q}) \\ &= \mathbf{p}^2 + r_0 + \frac{N+2}{6}K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0}, \end{aligned} \tag{7.110}$$

where $G(\mathbf{p}) = \Gamma^{(2)}(\mathbf{p})^{-1}$ is the propagator. We deduce

$$\begin{aligned} \Gamma^{(2)}(\mathbf{p} = 0) = r &= r_0 + \frac{N+2}{6}K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0} \\ &\simeq r_0 + \frac{N+2}{6}K_d u_0 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r}, \end{aligned} \tag{7.111}$$

where the replacement of r_0 by r in the last term introduces corrections of order u_0^2 which are beyond the one-loop accuracy. The critical temperature is obtained from the condition

$$0 = r = \bar{r}_0(T_c - T_{c0}) + \frac{N+2}{6}K_d u_0 \int_0^\Lambda d|\mathbf{q}|\, |\mathbf{q}|^{d-3}. \tag{7.112}$$

For $d > 2$, the integral in the rhs is convergent and the perturbative calculation of the shift $T_c - T_{c0}$ of the transition temperature makes sense. Subtracting (7.112) from (7.111) to eliminate T_{c0}, we obtain

$$r = \bar{r}_0(T - T_c) - \frac{N+2}{6}K_d u_0 r \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2(\mathbf{q}^2 + r)}. \tag{7.113}$$

If $d > 4$, the integral converges even when $r = 0$. In the limit $r \to 0$, we then obtain

$$r = \bar{r}_0(T - T_c) - Cr = \frac{\bar{r}_0(T - T_c)}{1 + C}, \tag{7.114}$$

with C a constant. Since $\chi^{-1} \sim r \sim T - T_c$, the susceptibility critical exponent keeps its mean-field value $\gamma = 1$. On the other hand, the integral in (7.113) diverges when $d \leq 4$ and $r \to 0$, thus signaling the breakdown of the perturbative expansion. For $d < 4$, we can write

$$r = \bar{r}_0(T - T_c) - \frac{N+2}{6} u_0 \tilde{K}_d r (\sqrt{r})^{d-4}, \tag{7.115}$$

where

$$\tilde{K}_d = K_d \int_0^{\Lambda/\sqrt{r}} dx \frac{x^{d-1}}{x^2(x^2+1)} \simeq K_d \int_0^{\infty} dx \frac{x^{d-1}}{x^2(x^2+1)} = -\frac{\pi}{2} \frac{K_d}{\sin(\pi d/2)}. \tag{7.116}$$

Since the integral is convergent for $|x| \to \infty$, we have taken the limit $\Lambda \to \infty$. Equation (7.115) is incompatible with $r \sim T - T_c$ and the value of the critical exponent γ cannot be equal to the mean-field prediction $\gamma = 1$. We can recover the Ginzburg criterion (7.106) for the validity of the mean-field approximation by demanding the last term in (7.115) to be a small correction,

$$\frac{N+2}{6} u_0 \tilde{K}_d (\sqrt{r})^{d-4} \simeq \frac{N+2}{6} u_0 \tilde{K}_d [\bar{r}_0(T - T_c)]^{d/2-2} \ll 1. \tag{7.117}$$

7.4 The scaling hypothesis

An essential feature of second-order phase transitions is the divergence of the correlation length $\xi \sim |t|^{-\nu}$. The scaling hypothesis states that this divergence is responsible for the singular dependence on $t = |T - T_c|/T_c$ of physical quantities. It leads to "scaling laws", i.e. relations between critical exponents (Table 7.3).

If ξ were the only relevant length scale *stricto sensu*, the singular behavior could be simply obtained from (naive) dimensional analysis. For a physical quantity X with naive scaling dimension $d_X^0 = [X]$ (also called engineering dimension),[36] the singular part would indeed be given by $X \sim \xi^{-d_X^0}$. We shall see below that the singular behavior s determined by a scaling dimension d_X which may differ from d_X^0. The difference between d_X and d_X^0 determines the "anomalous" dimension of X. For the latter to be nonzero without violating dimensional analysis, another characteristic length a must necessarily be involved (besides the correlation length ξ) to yield the (dimensionally correct) result $X \sim \xi^{d_X} a^{d_X^0 - d_X}$.[37]

7.4.1 Scaling form of the correlation function

Let us first consider the propagator $G(\mathbf{p}) = \langle \varphi(\mathbf{p}) \varphi(-\mathbf{p}) \rangle$ in the φ^4 theory of a scalar field ($N = 1$). From (naive) dimensional analysis, $[G(\mathbf{p})] = -2$, one can write the singular part of $G(\mathbf{p})$ in the scaling form

$$G(\mathbf{p}) = \frac{1}{\mathbf{p}^2} g_1(|\mathbf{p}|\xi, a/\xi), \tag{7.118}$$

[36] Recall that a quantity X has (naive) scaling dimension $d_X^0 = [X]$ if it is expressed in physical units of $L^{-d_X^0}$ (with L the unit length).

[37] For a thorough discussion of anomalous dimensions, see [8].

where a is a characteristic length which does not diverge at the transition.[38] The following discussion can be straightforwardly generalized to the case where the function g_1 depends on several lengths $a_1, a_2...$ We assume that in the limit $|\mathbf{p}|\xi \to \infty$, $|\mathbf{p}|a \to 0$ and $a/\xi \to 0$, g_1 behaves as

$$g_1(|\mathbf{p}|\xi, a/\xi) \sim (|\mathbf{p}|\xi)^{x_1}(a/\xi)^{x_2} \qquad (\xi \to \infty). \tag{7.119}$$

For $G(\mathbf{p})$ to be defined and nonzero at the critical point, we must have $x_1 = x_2 \equiv \eta$. We deduce

$$G(\mathbf{p}, T_c) \sim \frac{a^\eta}{|\mathbf{p}|^{2-\eta}}. \tag{7.120}$$

More generally, in the vicinity of the critical point, one can write

$$\begin{aligned} G(\mathbf{p}) &= \frac{1}{|\mathbf{p}|^{2-\eta}} g_2(|\mathbf{p}|\xi, a/\xi) \\ &= \frac{1}{|\mathbf{p}|^{2-\eta}} \left[g_2(|\mathbf{p}|\xi, 0) + \text{higher powers of } \xi^{-1} \right], \end{aligned} \tag{7.121}$$

where the scaling function g_2 has a well-defined limit when $\xi \to \infty$ ($g_2 \sim a^\eta$). Consider now the change $\mathbf{p} \to s\mathbf{p}$, $\xi \to \xi/s$ and $a \to a/s$. According to (7.118), this gives $G(s\mathbf{p}, \xi/s, a/s) = s^{-2} G(\mathbf{p}, \xi, a)$ in agreement with the naive scaling dimension $[G(\mathbf{p})] = -2$ of the propagator. If we apply the same transformation with a fixed (or with $a/\xi \equiv 0$), we obtain

$$G(s\mathbf{p}, \xi/s) = s^{-2+\eta} G(\mathbf{p}, \xi), \tag{7.122}$$

and it appears that the field has acquired an anomalous dimension,

$$[\varphi(\mathbf{p})] = -1 + \eta/2, \quad \text{i.e.} \quad d_\varphi = [\varphi(\mathbf{r})] = \frac{d}{2} - 1 + \frac{\eta}{2} = d_\varphi^0 + \frac{\eta}{2}. \tag{7.123}$$

$d_\varphi^0 = d/2 - 1$ is often referred to as the canonical dimension of the φ field and η as its anomalous dimension. The correlation function can also be written in the form

$$G(\mathbf{p}, \xi) = T\chi g_3(|\mathbf{p}|\xi), \tag{7.124}$$

where $\chi = T^{-1} G(\mathbf{p} = 0, \xi)$ is the susceptibility. The function $g_3(x)$ is then a universal scaling function (independent of the parameters of the model). In particular, $g_3(0) = 1$.

If we consider the propagator to be a function of $(\mathbf{p}, t)$, instead of $(\mathbf{p}, \xi)$, one has

$$G(\mathbf{p}, t) = \frac{1}{|\mathbf{p}|^{2-\eta}} g_\pm(|\mathbf{p}||t|^{-\nu}), \qquad G(s\mathbf{p}, s^{1/\nu} t) = s^{-2+\eta} G(\mathbf{p}, t), \tag{7.125}$$

or, in real space,

$$G(\mathbf{r}, t) = \frac{1}{|\mathbf{r}|^{d-2+\eta}} g_\pm(|\mathbf{r}||t|^{\nu}), \qquad G(\mathbf{r}/s, s^{1/\nu} t) = s^{d-2+\eta} G(\mathbf{r}, t). \tag{7.126}$$

[38] In general, a is either the Ginzburg length ξ_G (Sec. 7.3.4) or the lattice spacing (i.e. the inverse of the upper momentum cutoff Λ of the theory).

Fisher	$\gamma = \nu(2-\eta)$
Rushbrooke	$\alpha + 2\beta + \gamma = 2$
Widom	$\beta(\delta - 1) = \gamma$
Josephson	$\alpha = 2 - \nu d$

Table 7.3: Scaling laws [Eqs. (7.128,7.131,7.136,7.138)]. All exponents are the same above and below T_c.

In the presence of a magnetic field, equations (7.126) become

$$\begin{aligned} G(\mathbf{r}, t, h) &= s^{-d+2-\eta} G(\mathbf{r}/s, s^{1/\nu} t, s^{d_h} h) \\ &= \frac{1}{|\mathbf{r}|^{d-2+\eta}} g_\pm \left(|\mathbf{r}||t|^\nu, \frac{h}{|t|^\Delta} \right), \end{aligned} \tag{7.127}$$

where d_h denotes the scaling dimension of the field (see Sec. 7.4.2) and $\Delta = \nu d_h$ is sometimes referred to as the gap exponent. We have allowed different scaling functions, g_+ and g_-, above and below the critical temperature T_c, and assumed $\nu = \nu'$ (this assumption will be justified below). Equations (7.125-7.127) show that the correlation function satisfies a generalized homogeneity relation.[39] Using (7.125) with $\mathbf{p} = 0$ and $s = |t|^{-\nu}$, one obtains $\chi \sim G(\mathbf{p} = 0, t) \sim |t|^{\nu(\eta-2)}$, i.e.

$$\gamma = \gamma' = \nu(2 - \eta). \tag{7.128}$$

This result,[40] which relates the critical exponents γ, ν and η, is called a scaling law.

Scaling of the stiffness

The preceding discussion applies with no change to the disordered phase of the O(N)-symmetric $(\boldsymbol{\varphi}^2)^2$ theory where $G_{ij} = \delta_{i,j} G$. The ordered phase is characterized by long-range transverse correlations and a finite stiffness ρ_s [Eq. (7.95)]. ρ_s vanishes as $t \to 0^-$ with an exponent x which can be obtained from dimensional analysis. From (7.94), since the action is dimensionless while $[\delta\tilde{\boldsymbol{\varphi}}_\perp(\mathbf{r})] = 0$, we deduce that

$$[\rho_s] = d - 2, \tag{7.129}$$

i.e.

$$\rho_s \sim (-t)^{\nu(d-2)} \sim (-t)^{2\beta - \nu\eta} \tag{7.130}$$

for $t \to 0^-$, where the last result is obtained using the scaling law (7.131) derived in section 7.4.2. When the anomalous dimension vanishes, one obtains $\rho_s \sim (-t)^{2\beta}$ as in Landau's theory ($\rho_s = m^2$).

[39] A function $f(x)$ is said to be homogeneous if it satisfies $f(sx) = s^k f(x)$ for any rescaling factor s. More generally, a function $f(x_1, \cdots, x_n)$ is homogeneous if $f(s^{k_1} x_1, \cdots, s^{k_n} x_n) = s^k f(x_1, \cdots, x_n)$.

[40] Similarly, from (7.125) with $t = 0$ and $s = 1/|\mathbf{p}|$, one obtains $G(\mathbf{p}, t = 0) \sim 1/|\mathbf{p}|^{2-\eta}$ in agreement with (7.120).

7.4.2 Scaling form of the free energy density

The magnetization density m being the average value of the field, one expects $m \sim \xi^{-d_\varphi} \sim (-t)^{\nu d_\varphi}$ for $t < 0$,[41] i.e.

$$\beta = \nu d_\varphi = \frac{\nu}{2}(d - 2 + \eta). \tag{7.131}$$

Since $\ln Z$ is dimensionless and $[V] = -d$, the free energy density $f = -T \ln Z/V$ does not carry an anomalous dimension and has scaling dimension d. Its singular part satisfies $f \sim \xi^{-d}$.[42] From

$$m = -\frac{\partial f}{\partial H} = -\frac{1}{T}\frac{\partial f}{\partial h}, \tag{7.132}$$

we deduce that the magnetic field has scaling dimension

$$d_h = d - d_\varphi = \frac{1}{2}(d + 2 - \eta). \tag{7.133}$$

In practice, the anomalous dimension η is small and d_h is positive. We can write the singular part of the free energy density in the scaling form

$$f = \xi^{-d} g_\pm(h\xi^{d_h}). \tag{7.134}$$

In zero field, $f = \xi^{-d} g_\pm(0) \sim |t|^{d\nu}$, and we obtain

$$c_V = T\frac{\partial^2 f}{\partial T^2} \sim |t|^{d\nu - 2}, \tag{7.135}$$

i.e.

$$\alpha = \alpha' = 2 - \nu d. \tag{7.136}$$

For the magnetization $m \sim \xi^{d_h - d} g'_\pm(h\xi^{d_h})$ to be defined at T_c, we must have $g'_\pm(x) \sim x^{(d-d_h)/d_h} = x^{d_\varphi/d_h}$ for $x \to \infty$. This implies

$$m(T_c) \sim h^{d_\varphi/d_h}, \tag{7.137}$$

and a critical exponent

$$\delta = \frac{d_h}{d_\varphi} = \frac{d + 2 - \eta}{d - 2 + \eta}. \tag{7.138}$$

The scaling form (7.134) of the free energy density is often written as

$$f(t, h) = |t|^{2-\alpha} g_\pm\left(\frac{h}{|t|^\Delta}\right), \tag{7.139}$$

where $\Delta = \nu d_h = 2 - \alpha - \beta$ is the gap exponent. The equation of state then takes the form

$$m = -\frac{1}{T}\frac{\partial f}{\partial h} = -\frac{|t|^\beta}{T} g'_\pm\left(\frac{h}{|t|^\Delta}\right). \tag{7.140}$$

[41] Since m vanishes in the high-temperature phase, it has no regular part for $t \to 0^-$. The same is true for the stiffness ρ_s.

[42] It would be more correct to write $f \sim T\xi^{-d}$. However, the factor T is not singular and can be omitted.

It seems that we could postulate a more general form,

$$f(t,h) = |t|^{2-\alpha_\pm} g_\pm \left(\frac{h}{|t|^{\Delta_\pm}} \right), \tag{7.141}$$

with different exponents for $t > 0$ and $t < 0$. For $h \neq 0$, i.e. away from the critical point $t = h = 0$, the free energy density must be analytic in t,

$$f(t,h) = f_0(h) + t f_1(h) + \mathcal{O}(t^2) \qquad (h \neq 0). \tag{7.142}$$

Expanding (7.141) for $t \to 0$, we obtain

$$f(t,h) = |t|^{2-\alpha_\pm} \left[A_\pm \left(\frac{h}{|t|^{\Delta_\pm}} \right)^{p_\pm} + B_\pm \left(\frac{h}{|t|^{\Delta_\pm}} \right)^{q_\pm} + \cdots \right], \tag{7.143}$$

where $p_\pm, q_\pm$ are the leading powers in the expansion of $g_\pm$ for large arguments ($\Delta_\pm$ is positive). Equations (7.142) and (7.143) require $p_\pm \Delta_\pm = 2 - \alpha_\pm$ and $q_\pm \Delta_\pm = 1 - \alpha_\pm$, so that

$$f(t,h) = A_\pm h^{(2-\alpha_\pm)/\Delta_\pm} + B_\pm h^{(1-\alpha_\pm)/\Delta_\pm} |t| + \mathcal{O}(t^2). \tag{7.144}$$

Continuity at $t = 0$ forces $(2-\alpha_+)/\Delta_+ = (2-\alpha_-)/\Delta_-$ and $(1-\alpha_+)/\Delta_+ = (1-\alpha_-)/\Delta_-$, which implies $\alpha_+ = \alpha_-$ and $\Delta_+ = \Delta_-$. We conclude that the critical exponents are necessary the same for $t > 0$ and $t < 0$, which justifies the assumption $\nu = \nu'$ used earlier.

Scaling laws are summarized in Table 7.3. Relations involving d are called hyperscaling relations. They are satisfied by the mean-field exponents only at the upper critical dimension d_c^+.[43] Quite generally hyperscaling applies to transitions that are fluctuation dominated (i.e. non mean-field). There are then only two independent critical exponents, e.g. ν and η.

Scaling functions in mean-field theory. For $N = 1$, we can rewrite the correlation function obtained within Landau's theory [Eqs. (7.44,7.45)] in the scaling form

$$G(\mathbf{p}) = T\chi g_3(|\mathbf{p}|\xi), \tag{7.145}$$

with $\chi = T^{-1}\xi^2$ and $g_3(x) = (1+x^2)^{-1}$. This Lorentzian form of the correlation function was first proposed by Ornstein and Zernicke. The free energy density

$$f(r_0,h) = \begin{cases} -\dfrac{3}{2}\dfrac{Tr_0^2}{u_0} & \text{if} \quad h = 0 \text{ and } r_0 \leq 0, \\ -3\dfrac{6^{1/3}}{4}\dfrac{Th^{4/3}}{u_0^{1/3}} & \text{if} \quad r_0 = 0, \end{cases} \tag{7.146}$$

can be written in the scaling form

$$f(r_0,h) = r_0^2 g\left(\frac{h}{|r_0|^{3/2}} \right), \tag{7.147}$$

where the scaling function g satisfies

$$\begin{aligned} \lim_{x\to 0} g(x) &= -\frac{3T}{2u_0}, \\ \lim_{x\to\infty} g(x) &= -3\frac{6^{1/3}}{4}\frac{Tx^{4/3}}{u_0^{1/3}}. \end{aligned} \tag{7.148}$$

[43] Above d_c^+, hyperscaling relations break down because of the existence of a dangerously irrelevant variable in the renormalization-group sense. See Sec. 7.6.1 for an example.

7.5 The renormalization group

In the previous sections, we have seen that standard perturbation theory and mean-field approaches break down below the upper critical dimension. In this section, we discuss an alternative method, Wilson's renormalization group (RG). Instead of considering all degrees of freedom on the same footing, one first integrates out short-distance (or high-energy) degrees of freedom. This leads to an effective theory for the long-distance (low-energy) degrees of freedom. This approach is interesting in particular (but not only) for the study of critical phenomena where (at least for universal quantities) an effective description based on the low-energy degrees of freedom should be sufficient.

7.5.1 Renormalization-group transformations

A transformation whereby a subset of (short-distance) degrees of freedom is integrated out is called a RG transformation. For definiteness we consider a field theory with a N-component real field $\boldsymbol{\varphi}(\mathbf{r})$ as in the $(\boldsymbol{\varphi}^2)^2$ theory discussed in previous sections and assume an ultraviolet momentum cutoff Λ. We denote by $S[\boldsymbol{\varphi};K]$ the action with $K=\{K_i\}$ a set of coupling constants. A RG transformation consists in two steps:

- *Mode elimination.*[44] In the first step, one eliminates the short-distance (or high-energy) degrees of freedom. This is achieved by writing the field as
$$\boldsymbol{\varphi}(\mathbf{r}) = \boldsymbol{\varphi}_<(\mathbf{r}) + \boldsymbol{\varphi}_>(\mathbf{r}), \tag{7.149}$$
where $\boldsymbol{\varphi}_<(\mathbf{r})$ has Fourier components in the range $0\leq|\mathbf{p}|\leq\Lambda/s$ and $\boldsymbol{\varphi}_>(\mathbf{r})$ in $\Lambda/s\leq|\mathbf{p}|\leq\Lambda$ (with $s>1$). One then integrates out the "fast" modes $\boldsymbol{\varphi}_>$ to obtain an effective action for the "slow" modes $\boldsymbol{\varphi}_<$. The action $S[\boldsymbol{\varphi}_<;K_<]$ governing the dynamics of the slow modes is defined by a new set $K_<$ of coupling constants. In general, the functional form of the action is not preserved, and the set $K_<$ is larger than K.[45]

- *Rescaling.* In the second step of the RG transformation, one rescales momenta and coordinates,
$$\mathbf{p}' = s\mathbf{p} \quad \text{and} \quad \mathbf{r}' = \mathbf{r}/s, \tag{7.150}$$
thus restoring the momentum cutoff to its original value ($0\leq|\mathbf{p}'|\leq\Lambda$). One also defines a rescaled field,
$$\boldsymbol{\varphi}'(\mathbf{r}') = \lambda_s(K)\boldsymbol{\varphi}_<(\mathbf{r}), \tag{7.151}$$
or
$$\boldsymbol{\varphi}'(\mathbf{p}') = \frac{1}{\sqrt{V'}}\int d^dr' e^{-i\mathbf{p}'\cdot\mathbf{r}'}\boldsymbol{\varphi}'(\mathbf{r}') = s^{-d/2}\lambda_s(K)\boldsymbol{\varphi}_<(\mathbf{p}). \tag{7.152}$$
Equation (7.152) defines a linear RG transformation, since the new field depends linearly on the old one. We will see below how to determine the value of the rescaling parameter $\lambda_s(K)$, which is conveniently written as
$$\lambda_s(K) = s^{d_\varphi^0}\sqrt{Z_s(K)}, \tag{7.153}$$

[44] Also called decimation or coarse graining.

[45] For instance, if the initial action corresponds to a $(\boldsymbol{\varphi}^2)^2$ theory, the renormalized action is likely to include a $(\boldsymbol{\varphi}^2)^3$ term, etc.

where d^0_φ is the canonical dimension of the field $\boldsymbol{\varphi}(\mathbf{r})$ (Sec. 7.4). $Z_s(K)$ determines the anomalous dimension η when the system is critical (Sec. 7.5.2).[46] The rescaling (7.150) and (7.152) transforms $K_<$ into a new set K' of coupling constants.

These two steps can be summarized by[47]

$$e^{-S[\boldsymbol{\varphi}';K']} = \left\{ \int_{\Lambda/s\leq|\mathbf{p}|\leq\Lambda} \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi};K]} \right\}_{\boldsymbol{\varphi}(\mathbf{p})\to s^{d/2}\lambda_s(K)^{-1}\boldsymbol{\varphi}'(\mathbf{p}')} . \tag{7.154}$$

The coupling constants K'_i are naturally associated to the momentum scale Λ/s and are often referred to as the coupling constants at the scale Λ/s.

The momentum-shell RG we have described so far is not the only possible RG procedure. In particular, for some models such as classical spin models, it is possible to implement a real-space RG following Kadanoff's idea of block spins. This approach has played a very important role in the genesis of Wilson's RG. Quite generally, we can therefore view a RG transformation as a transformation

$$K(s) = R_s(K) \tag{7.155}$$

($s > 1$) acting in the space of possible actions $\{S[\boldsymbol{\varphi}; K]\}$ (or Hamiltonians $\{H(K)\}$). Since two successive transformations R_{s_1} and R_{s_2} should be equivalent to $R_{s_1 s_2}$, the actions $S[\boldsymbol{\varphi}; R_{s_1}(R_{s_2}(K))]$ and $S[\boldsymbol{\varphi}; R_{s_1 s_2}(K)]$ should agree up to a global rescaling of the fields. One can always choose the rescaling parameter $\lambda_s(K)$ so that

$$R_{s_1 s_2} = R_{s_1} R_{s_2}. \tag{7.156}$$

The RG transformations $\{R_s\}$ then form a semi-group. This term refers to the action of the RG transformations in the space of field configurations. As some short-scale information is lost in the mode elimination, the procedure cannot be inverted. There is however no problem in inverting the transformation $K \mapsto K(s) = R_s(K)$ in the space of the parameters of the action.

Infinitesimal RG transformations

In practice, one often chooses $s = e^{dl}$ with $dl \to 0$ and integrates out fields with momenta in the infinitesimal shell $\Lambda(1 - dl) \leq |\mathbf{p}| \leq \Lambda$. After l/dl infinitesimal RG transformations, momenta have been rescaled by a factor $s = \lim_{dl\to 0}(1+dl)^{l/dl} = e^l$. With $s_1 = 1 + \epsilon$ and $s_2 = s$ ($\epsilon \to 0$), equation (7.156) gives the differential RG transformation

$$s\frac{\partial K(s)}{\partial s} = \beta\big(K(s)\big), \tag{7.157}$$

where the beta function

$$\beta\big(K(s)\big) = \left.\frac{\partial R_{s'}(K(s))}{\partial s'}\right|_{s'=1} \tag{7.158}$$

[46] Z_s is the inverse of the so-called wave-function renormalization factor.

[47] We ignore any additive contributions to the action $S[\boldsymbol{\varphi}'; K']$ coming from the mode elimination or the Jacobian due to the change of variables $\boldsymbol{\varphi} \to \boldsymbol{\varphi}'$. These matter only when considering the free energy and will be discussed at the end of section 7.5.3.

is a function of $K(s)$ only (and not of both $K(s)$ and s). It is sometimes convenient to consider $K(s)$ as a function $K(l)$ of the variable $l = \ln s$ (and similarly for the field rescaling factor $Z_l \equiv Z_s(K)$). We can define a "running" (l-dependent) anomalous dimension η_l by

$$\eta_l = \partial_l \ln Z_l \tag{7.159}$$

i.e.

$$Z_{l+dl} = Z_l e^{\eta_l dl} \quad \text{or} \quad Z_l = \exp\left(\int_0^l dl' \eta_{l'}\right), \tag{7.160}$$

so that $\lambda_{l+dl} = \lambda_l e^{(d_\varphi^0 + \eta_l/2)dl}$. η_l is a function of the running coupling constants $K_i(l)$. We shall see below how it is related to the actual anomalous dimension η when the system is critical.

Advantages of the RG approach

There are several advantages in computing the partition function by means of RG transformations rather than e.g. standard perturbation theory:

- Since a RG transformation involves only a finite number of degrees of freedom, no singularity (divergence) is expected. Singular behavior can arise only after an infinite number of iterations in which all degrees of freedom in the thermodynamic limit have been integrated.

- The RG turns out to be an efficient computational tool which often goes well beyond standard perturbation theory. Suppose for instance that we compute the change $\frac{d}{ds}K_i(s)$ of the coupling constants to a given order in a series expansion wrt $K_j(s)$. Solving the flow equations $\beta_i(K(s)) = s\frac{d}{ds}K_i(s) = \frac{d}{dl}K_i(l)$ partially resums the perturbation series to infinite order. For this reason, perturbative RG approaches (i.e. based on a perturbative calculation of the beta functions $\beta_i(K)$) are sometimes referred to as RG improved perturbation theories. However, the computation of the beta functions need not be based on perturbation theory, and the RG provides a natural framework to set up non-perturbative approaches. This will be illustrated in section 7.9 and more generally in chapter 8.

- A RG transformation is not a mere scale transformation as the coarse graining (mode elimination) changes the coupling constants of the action. By iterating the RG transformations, one generates a trajectory $K(s)$ in the coupling constant space. The set of all such trajectories, obtained from different initial conditions $K(s = 1)$ generates a RG flow. In practice, one generally finds that the trajectories flow into fixed points $K^* = R_s(K^*)$ of the RG transformation. We shall see in the following sections that the fixed points govern the long-distance physics and explain scaling and universality observed in the vicinity of a second-order phase transition.

7.5.2 Fixed points

In a RG transformation R_s, the correlation lengthtransforms as $\xi(K(s)) = \xi(K)/s$.[48] At a fixed point $K^* = R_s(K^*)$ of the transformation, we must have $\xi(K^*) = \xi(K^*)/s$ which implies that $\xi(K^*)$ can only be zero or infinity. We refer to a fixed point with $\xi = \infty$ as a critical fixed point, and a fixed point with $\xi = 0$ as a trivial fixed point. Critical fixed points describe the singular behavior at a second-order phase transition (Sec. 7.5.3), whereas trivial fixed points describe the various phases of the system (see below).

The set of initial conditions K which flow to a given fixed point is called the basin of attraction of that fixed point. The basin of attraction of a critical fixed point is often called the critical manifold or critical surface. Let us consider a physical system represented by a point K in the coupling constant space. A point $K(s) = R_s(K)$ of the RG trajectory originating from K has a correlation length $\xi(K(s)) = \xi(K)/s < \xi(K)$. If the point K belongs to the critical surface, $\lim_{s\to\infty} R_s(K) = K^*$ and in turn $\lim_{s\to\infty} \xi(K(s)) = \infty$, which implies that $\xi(K) = \infty$. We conclude that all points on the critical surface have infinite correlation length.

Local behavior of RG flows near a fixed point

Near a fixed point K^* the RG transformation $K' = R_s(K)$ can be linearized,

$$\begin{aligned} K'_i &\simeq K^*_i + \sum_j \frac{\partial K'_i}{\partial K_j}\bigg|_{K^*} (K_j - K^*_j) \\ &= K^*_i + \sum_j T_{ij}(s)(K_j - K^*_j), \end{aligned} \tag{7.161}$$

where

$$T_{ij}(s) = \frac{\partial K'_i}{\partial K_j}\bigg|_{K^*}. \tag{7.162}$$

The matrix $T_{ij}(s)$ is real, but in general not symmetric and therefore not necessary diagonalizable.[49] We nevertheless assume that the right eigenstates $\mathbf{e}^{(\alpha)}$ form a complete basis with real eigenvalues $\lambda_s^{(\alpha)}$,

$$\sum_j T_{ij}(s) e_j^{(\alpha)} = \lambda_s^{(\alpha)} e_i^{(\alpha)}. \tag{7.163}$$

The group property (7.156) implies that the matrices $T(s)$ for different s commute. It is thus possible to diagonalize them simultaneously in a basis $\{\mathbf{e}^{(\alpha)}\}$ which does not depend on s. Equation (7.156) also implies that $\lambda_s^{(\alpha)} = s^{y_\alpha}$. Writing $\delta K_i = \sum_\alpha t_\alpha e_i^{(\alpha)}$, we can express the action near the fixed point as

$$S[\boldsymbol{\varphi}; K] = S[\boldsymbol{\varphi}; K^*] + \sum_i \delta K_i A_i[\boldsymbol{\varphi}] = S[\boldsymbol{\varphi}; K^*] + \sum_\alpha t_\alpha O_\alpha[\boldsymbol{\varphi}], \tag{7.164}$$

[48] In the low-temperature phase of the O(N) model with $N \geq 2$, ξ should be understood as the Josephson length ξ_J. See Secs. 7.1.4 and 7.7.2.

[49] If $T_{ij}(s)$ is not symmetric, there is no guaranty that the eigenvalues are real and that the right or left eigenstates form a complete basis.

where $O_\alpha = \sum_i e_i^{(\alpha)} A_i$. The t_α's are called the scaling fields (or scaling variables) and the O_α's the scaling operators (or scaling directions). In the linearized RG transformation $T(s)$, the scaling field t_α is multiplied by s^{y_α}. We are then led to distinguish three cases:

1. $y_\alpha > 0$: the scaling field increases with s. t_α is called a relevant scaling field.

2. $y_\alpha = 0$: the scaling field does not change as s varies and is called a marginal field. To determine its behavior, one must go beyond the linear approximation. If t_α turns out to be (ir)relevant, it is said to be marginally (ir)relevant. Marginal scaling fields are responsible for logarithmic corrections to scaling and are important at the upper and lower critical dimensions (Secs. 7.6.2 and 7.7.2).

3. $y_\alpha < 0$: the scaling field decreases as s increases and is called an irrelevant field.

It should be remembered that the terms relevant, marginal and irrelevant should always be specified with respect to a particular fixed point. To complete the description of the linearized RG transformation $T(s)$, one should determine the rescaling parameter λ_s [Eq. (7.152)]. Since a field rescaling multiplies K_i by some power of $\lambda_s(K)$,[50] the only possible form compatible with $\lambda_s^{(\alpha)} = s^{y_\alpha}$ is $\lambda_s = s^{d_\varphi}$. With $s = e^l$, we obtain $\lambda_l = e^{ld_\varphi^0}\sqrt{Z_l} = e^{ld_\varphi}$, which implies that the running anomalous dimension $\eta_l \equiv \eta$ is independent of l and $d_\varphi = d_\varphi^0 + \eta/2$. We will see below that d_φ is nothing but the dimension of the field and η the anomalous dimension (Sec. 7.4.1).

Thus if we start with a set of coupling constants near the fixed point K^* but not in the basin of attraction, the flow along the relevant directions $\mathbf{e}^{(\alpha)}$ ($y_\alpha > 0$) will go away from the fixed point. The irrelevant directions $\mathbf{e}^{(\alpha)}$ ($y_\alpha < 0$) correspond to direction of the flow into the fixed point (for an example, see Fig. 7.11 below). If there are N relevant scaling fields $t_1, \cdots, t_N$, we need to fix $t_1 = \cdots = t_N = 0$ to be in the basin of attraction in the linear approximation. This condition defines the plan tangent to the basin of attraction at the fixed point.

Global properties of RG flows

The global behavior of the RG flow determines the phase diagram of the system. In general, any point in the coupling constant space flows to some fixed point. The state of the system described by this fixed point represents the phase at the original point in the coupling constant space. The phase diagram is therefore determined by the global topology of the RG flow.[51]

The distinction between relevant and irrelevant scaling fields, as well as between critical ($\xi = \infty$) and trivial ($\xi = 0$) fixed points, implies a classification of different types of fixed points. Let us briefly list the most important ones:[52]

- *Stable fixed points (or sinks)* have only irrelevant scaling fields and trajectories can only flow into them. Sinks correspond to stable bulk phases ($\xi = 0$), and the nature of the coupling constants at the fixed point characterize the phase. All points in the basin of attraction of the sink correspond to physical systems in the same phase.

[50] For instance, if K_i appears in the action as $K_i\varphi_{j_1}\cdots\varphi_{j_n}$, $K_i \to \lambda_s^{-n} K_i$ when $\varphi_j \to \lambda_s^{-1}\varphi_j$.
[51] This point will be illustrated in Sec. 7.6 and following ones.
[52] The following list is not exhaustive. For a more complete discussion, see Ref. [8].

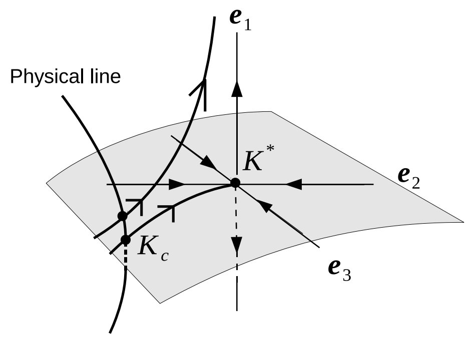

Figure 7.11: RG flow near a critical fixed point (for simplicity, only one of the two relevant directions is shown). Two RG trajectories are shown (thick solid lines): one on the critical surface (gray area) flowing into the fixed point, the other one near the critical surface. The physical line meets the critical surface at the critical point K_c.

- *Unstable fixed points* have only relevant scaling fields and all trajectories flow away from them. These fixed points have no direct physical meaning but play a role in the global topology of the RG flow.

- There are also *fixed points with both relevant and irrelevant scaling fields.* Of particular interest are the fixed points with two relevant scaling fields and an infinite correlation length (critical fixed point). In this case, two variables (e.g. temperature and magnetic field) must be tuned to reach the fixed point. Trajectories which start slightly off the critical surface initially flow towards the fixed point, but are ultimately repelled from the fixed point along the two relevant directions to flow into a stable fixed point (sink) corresponding to the phase of the system (Fig. 7.11). Such a critical fixed point corresponds to a phase transition between two stable phases of matter and the RG flow in its vicinity determines the critical behavior at the phase transition (Sec. 7.5.3). When the temperature changes, K varies along a line in the space of coupling constants, the physical line, which (in zero field) meets the critical surface at the critical point K_c ($T = T_c$).

  Critical fixed points with more than two relevant variables are generically called multicritical fixed points. Tricritical fixed points have three relevant variables; three parameters (e.g. temperature, magnetic field and pressure) must be tuned to hold the system at the critical point.

In some cases, it is possible to obtain a continuum of fixed points in the coupling constant space. A well-known example is the line of fixed points in the RG flow of the two-dimensional XY model (Sec. 7.8). RG flows can also exhibit more exotic behavior such as limit cycles or even chaotic behavior.

7.5.3 Universality and scaling

Let us consider a critical fixed point K^*. We denote by h and t_1 the two relevant scaling fields, and by $t_2, \cdots$, irrelevant scaling fields. We assume that there are no marginal fields.[53] If t_1 and h are initially very small, then K first moves towards K^* and remains a long "time" near K^* before eventually going away along the relevant directions. The critical behavior emerges from the long time where the flow is determined by the vicinity of the fixed point. Now the fixed point $K^* = R_s(K^*)$ and the linear approximation to R_s (i.e. the eigenvalues s^{y_α} and the eigendirections $\mathbf{e}^{(\alpha)}$) are properties of the RG transformation itself. Thus the dynamics near the fixed point is independent of the initial conditions of the RG trajectory. All systems represented by a point K near the critical surface of the fixed point K^* therefore exhibit the same critical behavior (universality). We show below how, by considering a RG transformation near a critical point, we can justify the results obtained from the scaling hypothesis (Sec. 7.4).

Correlation length

Let us first consider the correlation length ξ in the absence of magnetic field, and assume that the relevant field t_1 vanishes linearly with $T - T_c$: $t_1 \sim t = (T - T_c)/T_c$. In a RG transformation, $\xi(K) = s\xi(K')$, so that

$$\xi(t_1, t_2, \cdots) = s\xi(s^{y_1}t_1, s^{y_2}t_2, \cdots) \tag{7.165}$$

in the region near K^* where the transformation can be linearized,[54] with $y_1 > 0$ and $0 > y_2 > y_3 > \cdots$. Setting $s \sim |t_1|^{-1/y_1}$, we obtain[55]

$$\xi(t_1, t_2, \cdots) \sim |t_1|^{-1/y_1}\xi(\pm 1, |t_1|^{-y_2/y_1}t_2, \cdots). \tag{7.166}$$

In the critical region, defined by $|t_1|^{-y_2/y_1}t_2 \ll 1$, all irrelevant scaling fields can be set to zero to leading order and we deduce $\xi \sim |t_1|^{-1/y_1} \sim |T - T_c|^{-1/y_1}$.[56] The correlation length diverges at the transition with the exponent $\nu = 1/y_1$. The correction-to-scaling exponent $\omega = -y_2$, defined as the absolute value of the largest negative eigenvalues y_i, not only determines the size of the critical region but also gives the leading correction to the critical behavior $\xi \sim |t|^{-1/y_1}$. From (7.166), provided that $\xi(\pm 1, t_2', \cdots)$ is analytic in t_2',[56] we deduce

$$\xi \sim |t|^{-1/y_1}\big(1 + A_\pm |t|^{\omega/y_1} + \cdots\big), \tag{7.167}$$

where $A_\pm$ is a non-universal constant.

If the point K representing the system is too far away from the fixed point K^* for the RG transformation to be linearized, one must first consider a RG transformation $K(\tilde{s}) = R_{\tilde{s}}(K)$ which brings $K(\tilde{s}) = \tilde{K}$ near the fixed point where its subsequent evolution can be obtained from the linearized RG transformation. The previous argument then gives $\xi(\tilde{t}_1) = s\xi(s^{1/\nu}\tilde{t}_1) \sim |\tilde{t}_1|^{-\nu}$ in the critical region. But $\tilde{t}_1$, which vanishes for $t = 0$, is expected to be proportional to $t = (T - T_c)/T_c$, and therefore $\xi(t_1) = \tilde{s}\xi(\tilde{t}_1) \sim |t|^{-\nu}$.

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[53] Marginal fields are considered in Secs. 7.6.2 and 7.7.2.

[54] An (approximate) condition is that $|t_i|$ and $s^{y_i}|t_i|$ are at most of order 1.

[55] Here and in the following, we do not write the microscopic length scale that is needed to make dimensional sense of these equations.

[56] We can set $t_2' = |t_1|^{-y_2/y_1}t_2 = 0$ in (7.166) only if ξ is an analytic function of t_2. If not, t_2 is called a dangerously irrelevant variable (see Sec. 7.6.1 for an example).

The scaling dimension d_h of the other relevant field (the magnetic field) is easily obtained from the following argument. Since the magnetic field contributes to the action a term $-\int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}(\mathbf{r})$, it couples only to the $\mathbf{p}=0$ mode and is not affected by the partial integration of fields with momenta $\Lambda/s \le |\mathbf{p}| \le \Lambda$. The renormalization of $\mathbf{h}$ is therefore entirely due to rescaling of momenta (or lengths) and fields,

$$\int d^d r\, \mathbf{h}\cdot\boldsymbol{\varphi}(\mathbf{r}) = s^d \lambda_s(K)^{-1} \int d^d r'\, \mathbf{h}\cdot\boldsymbol{\varphi}'(\mathbf{r}'), \tag{7.168}$$

i.e. $h' = s^{d_h} h$ ($h = |\mathbf{h}|$) with

$$d_h = d - d_\varphi = \frac{1}{2}(d+2-\eta) \tag{7.169}$$

(we use $\lambda_s(K) = s^{d_\varphi}$ in the critical regime). In practice, the smallness of η ensures that d_h is positive and $\mathbf{h}$ a relevant scaling field.

Order parameter

The RG transformation $K \mapsto K'$ [Eq. (7.154)] implies

$$\begin{aligned} m_i(K) &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}]\, \varphi_i(\mathbf{r})\, e^{-S[\boldsymbol{\varphi};K]} \\ &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}']\, \lambda_s(K)^{-1} \varphi'_i(\mathbf{r}/s)\, e^{-S[\boldsymbol{\varphi}';K']} \\ &= \lambda_s(K)^{-1} m_i(K'), \end{aligned} \tag{7.170}$$

where $m_i(K')$ is the mean value of $\varphi'_i(\mathbf{r}') = \varphi'_i(\mathbf{r}/s)$ computed with the action $S[\boldsymbol{\varphi}';K']$. Neglecting irrelevant scaling fields (assuming that there is no dangerously irrelevant variable[56]), we obtain

$$m(t,h) = s^{-d_\varphi} m(s^{1/\nu} t, s^{d_h} h) \tag{7.171}$$

in the critical regime. For $t=0$ and $h \neq 0$, we obtain $m(0,h) \sim h^{d_\varphi/d_h}$, i.e. a critical exponent

$$\delta = \frac{d_h}{d_\varphi} = \frac{d+2-\eta}{d-2+\eta}. \tag{7.172}$$

For $h=0$ and $t<0$, $m(t,0) \sim (-t)^{\nu d_\varphi}$ so that

$$\beta = \nu d_\varphi = \frac{\nu}{2}(d-2+\eta). \tag{7.173}$$

Correlation function

The two-point correlation function satisfies

$$\begin{aligned} G_{ij}(\mathbf{p};K) &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}]\, \varphi_i(\mathbf{p})\varphi_j(-\mathbf{p})\, e^{-S[\boldsymbol{\varphi};K]} \\ &= \frac{1}{Z}\int \mathcal{D}[\boldsymbol{\varphi}']\, s^d \lambda_s(K)^{-2} \varphi'_i(s\mathbf{p})\varphi'_j(-s\mathbf{p})\, e^{-S[\boldsymbol{\varphi}';K']} \\ &= s^d \lambda_s(K)^{-2} G_{ij}(s\mathbf{p};K') \end{aligned} \tag{7.174}$$

when $|\mathbf{p}| \leq \Lambda/s$, and in turn

$$\begin{aligned} G_{ij}(\mathbf{r}/s; K') &= \int_{|\mathbf{p}'|\leq\Lambda} \frac{d^d p'}{(2\pi)^d} e^{i\mathbf{p}'\cdot\mathbf{r}/s} G_{ij}(\mathbf{p}'; K') \\ &= \int_{|\mathbf{p}|\leq\Lambda/s} \frac{d^d p}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \lambda_s(K)^2 G_{ij}(\mathbf{p}; K) \\ &\simeq \lambda_s(K)^2 G_{ij}(\mathbf{r}; K) \qquad \text{for} \qquad |\mathbf{r}| \gg \frac{s}{\Lambda}. \end{aligned} \tag{7.175}$$

For a linearized RG transformation, equation (7.175) implies

$$G_{ij}(\mathbf{r}; t_1, t_2, \cdots) = s^{-2d_\varphi} G_{ij}(\mathbf{r}/s; s^{y_1} t_1, s^{y_2} t_2, \cdots). \tag{7.176}$$

Thus G_{ij} satisfies a generalized homogeneity relation in agreement with the scaling hypothesis (Sec. 7.4). On the critical surface ($t_1 = 0$), setting $s = |\mathbf{r}|$, we obtain[57]

$$G_{ij}(\mathbf{r}; 0, t_2, \cdots) = |\mathbf{r}|^{-2d_\varphi} G_{ij}(\mathbf{r}/|\mathbf{r}|; 0, |\mathbf{r}|^{y_2} t_2, \cdots). \tag{7.177}$$

If $|\mathbf{r}|^{y_2}|t_2| \ll 1$ then all irrelevant scaling fields can be neglected and one obtains

$$G_{ij}(\mathbf{r}; 0, t_2, \cdots) \sim \frac{1}{|\mathbf{r}|^{2d_\varphi}} = \frac{1}{|\mathbf{r}|^{d-2+\eta}}. \tag{7.178}$$

The more negative y_2, the larger the critical region (in coordinate space) $|\mathbf{r}|^{y_2}|t_2| \ll 1$ where (7.178) holds. Away from the critical surface ($t_1 \neq 0$), setting $s = |t_1|^{-1/y_1}$ in (7.176), one finds

$$G_{ij}(\mathbf{r}; t_1, t_2, \cdots) = |t|^{2d_\varphi/y_1} G_{ij}(|t_1|^{1/y_1}\mathbf{r}; \pm 1, |t_1|^{-y_2/y_1} t_2, \cdots). \tag{7.179}$$

In the limit where $|t_1|^{-y_2/y_1} t_2$ can be set to zero, we obtain the scaling form

$$G_{ij}(\mathbf{r}; t_1, t_2, \cdots) = \frac{1}{|\mathbf{r}|^{d-2+\eta}} f^{\pm}_{ij}\left(\frac{\mathbf{r}}{\xi}\right), \tag{7.180}$$

where $\xi \sim |t_1|^{-y_1} \sim |t|^{-\nu}$ is the correlation length (in agreement with (7.126)).

A similar analysis can be made for the Fourier transformed correlation function, starting from

$$G_{ij}(\mathbf{p}; t_1, t_2, \cdots) = s^{d-2d_\varphi} G_{ij}(s\mathbf{p}; s^{y_1} t_1, s^{y_2} t_1, \cdots), \tag{7.181}$$

where $d - 2d_\varphi = 2 - \eta$. At the critical point $t_1 = 0$,

$$G_{ij}(\mathbf{p}; 0, t_2, \cdots) = |\mathbf{p}|^{-2+\eta} G_{ij}(\mathbf{p}/|\mathbf{p}|; 0, |\mathbf{p}|^{-y_2} t_2, \cdots). \tag{7.182}$$

In the critical regime (in momentum space) $|\mathbf{p}|^{-y_2}|t_2| \ll 1$, one obtains

$$G_{ij}(\mathbf{p}; 0, t_2, \cdots) \sim \frac{1}{|\mathbf{p}|^{2-\eta}}. \tag{7.183}$$

[57] Because of space isotropy, $G_{ij}(\mathbf{r}/|\mathbf{r}|; 0, |\mathbf{r}|^{y_2} t_2, \cdots)$ is independent of $\mathbf{r}/|\mathbf{r}|$.

On the other hand, for $\mathbf{p} = 0$,

$$G_{ij}(0; t_1, t_2, \cdots) = |t_1|^{-(d-2d_\varphi)/y_1} G_{ij}(0; 1, |t_1|^{-y_2/y_1} t_2, \cdots), \tag{7.184}$$

so that the susceptibility $\chi \sim G_{ij}(\mathbf{p} = 0) \sim |t|^{-\gamma}$ diverges with an exponent[58]

$$\gamma = \frac{d - 2d_\varphi}{y_1} = \nu(2 - \eta). \tag{7.185}$$

Finally, in the presence of an external field,

$$G_{ij}(\mathbf{r}, t, h) = s^{-2d_\varphi} G_{ij}(\mathbf{r}/s, s^{1/\nu} t, s^{d_h} h) \tag{7.186}$$

in the critical regime, where we use the scaling variable t rather than t_1. With $s = |\mathbf{r}|$, we obtain

$$G_{ij}(\mathbf{r}, t, h) = \frac{1}{|\mathbf{r}|^{d-2+\eta}} g_{ij}^{\pm} \left(\frac{\mathbf{r}}{\xi}, \frac{h}{|t|^\Delta} \right), \tag{7.187}$$

where $\Delta = \nu d_h$ is the gap exponent introduced in section 7.4.2.

Free energy

To obtain the critical exponent α we must consider the free energy. So far we did not keep track of any possible additive contribution to the action produced by the RG transformation. Let us follow the convention that the action vanishes for $\boldsymbol{\varphi} = 0$ and write additive contributions explicitly. The first step in the RG transformation (coarse graining) will in general yield an additive contribution to the action,

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}; K]} = \int \mathcal{D}[\boldsymbol{\varphi}_<]\, e^{-S[\boldsymbol{\varphi}_<; K_<] - \beta V A(K,s)}. \tag{7.188}$$

When rescaling the momenta and fields, $\boldsymbol{\varphi}'(\mathbf{p}') = s^{-d/2} \lambda_s(K) \boldsymbol{\varphi}_<(\mathbf{p})$, one should take into account the Jacobian,

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-S[\boldsymbol{\varphi}'; K'] - \beta V A(K,s) - \beta V B(K,s)}, \tag{7.189}$$

where

$$e^{-\beta V B(K,s)} = \prod_{\substack{\mathbf{p} \\ (0 \leq |\mathbf{p}| \leq \Lambda/s)}} \left[\lambda_s(K)^{-1} s^{d/2} \right]^N. \tag{7.190}$$

If we denote by $f(K)$ and $f(K')$ the free energy densities associated to $S[\boldsymbol{\varphi}; K]$ and $S[\boldsymbol{\varphi}'; K']$,

$$\begin{aligned} e^{-\beta V f(K)} &= \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-S[\boldsymbol{\varphi}; K]}, \\ e^{-\beta V' f(K')} &= \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-S[\boldsymbol{\varphi}'; K']} \end{aligned} \tag{7.191}$$

[58] If $N \geq 2$, this result holds only in the high-temperature phase, since the susceptibility $\chi = \beta G_\parallel(\mathbf{p} = 0)$ diverges in the whole low-temperature phase (see Secs. 7.1.4 and 7.7.2.

$(V' = s^{-d}V)$, we then obtain

$$f(K) = s^{-d} f(K') + A(K,s) + B(K,s), \tag{7.192}$$

i.e.

$$f(t,h) = s^{-d} f(s^{1/\nu}t, s^{d_h}h) + A(t,s) + B(t,s) \tag{7.193}$$

if we ignore the irrelevant fields. Note that $A+B$ cannot depend on h, since the latter couples only to the uniform part $\boldsymbol{\varphi}(\mathbf{p}=0)$ of the field. With $s = |t|^{-\nu} \sim \xi$, we deduce

$$f(t,h) = |t|^{d\nu} g_\pm \left(\frac{h}{|t|^\Delta} \right) + A(t,|t|^{-\nu}) + B(t,|t|^{-\nu}). \tag{7.194}$$

If we could discard the term $A+B$, we would have derived the scaling form of the free energy. There is however no reason for $A(t,|t|^{-\nu})$ (which represents the free energy density of the modes $|\mathbf{p}| \gtrsim \xi^{-1}$) and $B(t,|t|^{-\nu})$ to be less singular than the free energy density $|t|^{d\nu} g_\pm$ of the modes $|\mathbf{p}| \lesssim \xi^{-1}$. A detailed calculation (see below) shows that

$$f(t,h) = |t|^{d\nu} g_\pm \left(\frac{h}{|t|^\Delta} \right) + |t|^{d\nu} \tilde{g}_\pm, \tag{7.195}$$

from which we deduce the specific heat per unit volume

$$c_V = -T \frac{\partial^2 f}{\partial T^2} \sim |t|^{-\alpha} \tag{7.196}$$

in zero field, with $\alpha = 2 - d\nu$.

It should be noted that A or B type terms did not appear in the calculation of $m(t,h)$ or $G_{ij}(\mathbf{p},t,h)$. The reason is that $G_{ij}(\mathbf{p},t,h)$ (for $|\mathbf{p}| < \Lambda/s$) and $m_i(t,h) = \langle \varphi_i(\mathbf{r}) \rangle = V^{-1/2} \langle \varphi_i(\mathbf{p}=0) \rangle$ do not directly involve the fast modes $|\mathbf{p}| \geq \Lambda/s$ and are affected by the RG transformation only through the renormalization of the coupling constants $K' = R_s(K)$. By contrast, the free energy involves all Fourier components in a direct manner.

To derive (7.195), let us first consider a RG transformation with $s = e^{\Delta l}$,

$$f(K) = e^{-d\Delta l} f\big(R_{\Delta l}(K)\big) + \tilde{A}(K, \Delta l), \tag{7.197}$$

where $R_{\Delta l} = R_{s=e^{\Delta l}}$ and $\tilde{A}(K,\Delta l) = A(K, e^{\Delta l}) + B(K, e^{\Delta l})$. Using

$$f\big(R_{\Delta l}(K)\big) = e^{-d\Delta l} f\big(R_{\Delta l}^2(K)\big) + \tilde{A}\big(R_{\Delta l}(K), \Delta l\big), \tag{7.198}$$

we obtain

$$f(K) = e^{-d\Delta l} \Big[e^{-d\Delta l} f\big(R_{\Delta l}^2(K)\big) + \tilde{A}\big(R_{\Delta l}(K), \Delta l\big) \Big] + \tilde{A}(K, \Delta l). \tag{7.199}$$

After $l/\Delta l$ iterations, we have

$$f(K) = e^{-dl} f\big(R_l(K)\big) + \sum_{m=0}^{l/\Delta l - 1} e^{-md\Delta l} \tilde{A}\big(R_{\Delta l}^m(K), \Delta l\big). \tag{7.200}$$

Comparing with (7.192), we deduce

$$A(K,e^l)+B(K,e^l)=\sum_{m=0}^{l/\Delta l-1} e^{-md\Delta l}\tilde A\big(R^m_{\Delta l}(K),\Delta l\big)$$
$$=\int_0^l dl'\, e^{-l'd}\lim_{\Delta l\to 0}\frac{1}{\Delta l}\tilde A\big(R_{l'}(K),\Delta l\big), \tag{7.201}$$

where we have taken the limit $\Delta l\to 0$. Since $A(K,1)+B(K,1)=0$,

$$C(K)=\lim_{\Delta l\to 0}\frac{1}{\Delta l}\tilde A(K,\Delta l)=\partial_s[A(K,s)+B(K,s)]\big|_{s=1} \tag{7.202}$$

and equation (7.201) can be rewritten as

$$A(K,s)+B(K,s)=\int_0^{\ln s}dl'\,e^{-l'd}C\big(R_{l'}(K)\big)=\int_0^s\frac{ds'}{s'}s'^{-d}C\big(R_{s'}(K)\big). \tag{7.203}$$

Equation (7.203) implies

$$s\frac{\partial}{\partial s}[A(t,s)+B(t,s)]=s^{-d}C\big(R_s(K)\big)=s^{-d}C(s^{1/\nu}t) \tag{7.204}$$

and, setting $s=|t|^{-\nu}\sim\xi$,

$$\xi\frac{\partial}{\partial\xi}(A+B)=|t|^{d\nu}C(\pm1)\sim\xi^{-d}. \tag{7.205}$$

We finally obtain

$$\frac{\partial}{\partial t}(A+B)=\frac{1}{\xi}\frac{\partial\xi}{\partial t}\xi\frac{\partial}{\partial\xi}(A+B)\sim|t|^{d\nu-1} \tag{7.206}$$

and

$$A(t,|t|^{-\nu})+B(t,|t|^{-\nu})\sim|t|^{d\nu}. \tag{7.207}$$

Together with (7.194), this proves equation (7.195).

7.6 Perturbative renormalization group

In this section we show how the critical exponents can be computed perturbatively wrt $\epsilon=4-d$ near 4 dimensions (see also Appendix 7.A). We start with the RG solution of the Gaussian model before considering the $(\boldsymbol{\varphi}^2)^2$ theory.

7.6.1 RG solution of the Gaussian model

RG equation

In the high-temperature phase, the Gaussian model is defined by the action

$$S[\boldsymbol{\varphi}]=\frac{1}{2}\int d^dr\,\left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2+r_0\boldsymbol{\varphi}^2\right]=\frac{1}{2}\sum_{\mathbf{p},i}\varphi_i(-\mathbf{p})(\mathbf{p}^2+r_0)\varphi_i(\mathbf{p}), \tag{7.208}$$

where $r_0=\bar r_0(T-T_{c0})$. The integration of fields with momenta $\Lambda/s\le|\mathbf{p}|\le\Lambda$ yields a constant contribution to the action (i.e. a contribution to the free energy) which

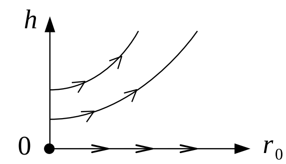


Figure 7.12: RG flow for the Gaussian model. The Gaussian fixed point is located at the origin $r_0 = h = 0$.

we ignore in the following. Rescaling the momenta, $\mathbf{p} \to \mathbf{p}/s$, to restore the original value of the cutoff, we obtain

$$S[\boldsymbol{\varphi}] = \frac{1}{2} \sum_{\mathbf{p},i} \varphi_i(-\mathbf{p})(s^{-2}\mathbf{p}^2 + r_0)\varphi_i(\mathbf{p}). \tag{7.209}$$

We then rescale the field, $\boldsymbol{\varphi}(\mathbf{p}) \to s^{d/2-d^0_\varphi}\boldsymbol{\varphi}(\mathbf{p})$ with $d^0_\varphi = \frac{d-2}{2}$ the naive scaling dimension of $\boldsymbol{\varphi}$, to restore the coefficient 1/2 of the $(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$ term,

$$S[\boldsymbol{\varphi}] = \frac{1}{2} \sum_{\mathbf{p},i} \varphi_i(-\mathbf{p})(\mathbf{p}^2 + s^2 r_0)\varphi_i(\mathbf{p}). \tag{7.210}$$

We deduce the RG equation

$$r_0' = s^2 r_0. \tag{7.211}$$

In the presence of an (external) magnetic field $\mathbf{h}$, we must include in the action the term $-\int d^d r \mathbf{h} \cdot \boldsymbol{\varphi}(\mathbf{r})$. The RG equation satisfied by the field is given by (7.169),

$$h' = s^{d^0_h} h, \tag{7.212}$$

where $d^0_h = d - d^0_\varphi = d/2 + 1$. Equations (7.211,7.212), together with the vanishing anomalous dimension η ($d_\varphi = d^0_\varphi$), could have been anticipated on purely dimensional ground. The RG transformation for the Gaussian model is a mere scale transformation.

Fixed points

For $h = 0$, equation (7.211) admits two fixed points:

- a critical fixed point $r_0^* = 0$ (to be referred to as the Gaussian fixed point) obtained for $T = T_{c0}$ and corresponding to the scale invariant action $S[\boldsymbol{\varphi}] = \frac{1}{2}\int d^d r(\boldsymbol{\nabla}\boldsymbol{\varphi}^2)$ ($\xi = \infty$). r_0 is a relevant scaling field with eigenvalue $y = 2$, which yields $\nu = 1/y = 1/2$.

- a high-temperature fixed point $r_0^* \to \infty$ with the action $S[\boldsymbol{\varphi}] = \frac{r_0^*}{2}\int d^d r \boldsymbol{\varphi}^2$ corresponding to a vanishing correlation length $\xi = 1/\sqrt{r_0^*} \to 0$.

The RG flow is shown in figure 7.12.

Critical exponents

One can obtain the correlation length from $\xi(r_0) = s\xi(r_0') = s\xi(s^2 r_0)$. With $s = r_0^{-1/2}$, one obtains $\xi(r_0) = r_0^{-1/2}\xi(1) \propto r_0^{-1/2}$, i.e. $\nu = 1/2$. Similarly, the susceptibility exponent is derived from the scaling law

$$G_{ii}(\mathbf{p}, r_0) = s^{d-2d_\varphi^0} G_{ii}(s\mathbf{p}, r_0') = s^2 G_{ii}(s\mathbf{p}, s^2 r_0) = r_0^{-1} G_{ii}(\mathbf{p} r_0^{-1/2}, 1), \tag{7.213}$$

where the last result is obtained with $s = r_0^{-1/2}$. We deduce that the susceptibility $\chi = \beta G_{ii}(\mathbf{p} = 0, r_0) \sim 1/r_0$ diverges with the exponent $\gamma = 1$ when $r_0 \to 0$.

Stability of the Gaussian fixed point

The Gaussian model ignores the $u_0(\boldsymbol{\varphi}^2)^2$ term of the O(N) model (7.27) as well as other terms which would then be generated by the RG procedure, e.g. $u_6(\boldsymbol{\varphi}^2)^3$, $v_0\boldsymbol{\varphi}^2(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$, etc. At the Gaussian fixed point, the field has scaling dimension $d_\varphi^0 = \frac{d-2}{2}$, and we deduce

$$[u_0] = 4 - d, \quad [u_6] = 6 - 2d, \quad [v_0] = 2 - d, \quad \text{etc.} \tag{7.214}$$

In a RG transformation,

$$\begin{aligned} u_0' &= s^{4-d} u_0 + \cdots \\ u_6' &= s^{6-2d} u_6 + \cdots \\ v_0' &= s^{2-d} v_0 + \cdots \end{aligned} \tag{7.215}$$

For small coupling constants, the leading terms in (7.215) come from the rescaling of momenta and fields, and reflect the canonical dimensions (7.214) (i.e. the scaling dimensions at the Gaussian fixed point). The ellipses stand for higher-order (at least quadratic) terms which are generated in the coarse graining step of the RG transformation when the starting action is not quadratic. Since all canonical dimensions, except $[r_0] = 2$, are negative for $d > 4$, the Gaussian fixed point $r_0 = u_0 = u_6 = v_0 = \cdots = 0$ is stable (i.e. has only one relevant direction besides the magnetic field) and the critical exponents take their classical values. Note that this conclusion (which merely follows from a dimensional analysis of the action) has already been reached in section 7.3.4. In section 7.6.2, we shall see that at the upper critical dimension $d_c^+ = 4$, the Gaussian fixed point is still stable but the critical behavior is modified by logarithmic corrections.

A dangerously irrelevant variable in Landau's theory

From the analysis of the Gaussian model, we conclude that the hyperscaling law

$$\delta = \frac{d_h^0}{d_\varphi^0} = \frac{d+2}{d-2} \tag{7.216}$$

agrees with the mean-field result $\delta = 3$ only for $d = 4$ whereas one expects it to be also valid for $d > 4$ since the Gaussian fixed point is then stable. This discrepancy is due to the fact that u_0 is a dangerously irrelevant variable for $d > 4$; although irrelevant, it cannot be ignored. Dangerously irrelevant variables lead to a breakdown of hyperscaling relations above the upper critical dimension d_c^+.

We observe that the partition function with $u_0 = 0$ is not defined for $T < T_{c0}$ so that in this case u_0 has to be included into the analysis one way or the other. Let us try to understand the role of the irrelevant variable u_0 (we now assume $d > 4$) in the RG framework.

The singular part of the free energy satisfies the scaling law

$$f(t, h, u_0) = s^{-d} f(s^{1/\nu} t, s^{d_h^0} h, s^\epsilon u_0) \tag{7.217}$$

(with $\nu = 1/2$), where $t \sim T - T_{c0}$ and $\epsilon = 4 - d$ ($\epsilon < 0$ for $d > 4$). We deduce

$$m(t, h, u_0) = -\frac{1}{T}\frac{\partial f}{\partial h} = s^{-d_\varphi^0} m(s^{1/\nu} t, s^{d_h^0} h, s^\epsilon u_0). \tag{7.218}$$

For $h = 0$ and with $s = |t|^{-\nu}$, we obtain

$$m(t, 0, u_0) = |t|^{\nu d_\varphi^0} m(\pm 1, 0, |t|^{-\nu\epsilon} u_0). \tag{7.219}$$

Since $\epsilon < 0$, it is tempting to set $|t|^{-\nu\epsilon} u_0 = 0$ when $|t| \to 0$. If we do so, we obtain a critical exponent $\beta = \nu d_\varphi^0 = \frac{d-2}{4}$ in disagreement with the exact result $\beta = 1/2$ for $d > 4$. The reason for this disagreement can be understood from Landau's theory. The mean-field result $m = \sqrt{-6 r_0/u_0}$ clearly shows that we cannot set $u_0 = 0$ in the scaling law (7.219). The correct result is obtained if we write

$$m(t, 0, u_0) = |t|^{\nu d_\varphi^0} g_\pm(|t|^{-\nu\epsilon} u_0) \tag{7.220}$$

with $g_-(x) \sim x^{-1/2}$ for $x \to 0$. Then

$$m(t, 0, u_0) \sim \frac{|t|^{\nu(d_\varphi^0 + \frac{\epsilon}{2})}}{\sqrt{u_0}} \quad \text{for} \quad t < 0 \tag{7.221}$$

so that

$$\beta = \nu(d_\varphi^0 + \frac{\epsilon}{2}) = \nu = \frac{1}{2}. \tag{7.222}$$

The fact that the function $g_-(x)$ is not analytic for $x \to 0$ shows that u_0 is a dangerously irrelevant variable; although irrelevant it cannot be set to zero.

The value of δ can be obtained in a similar way. For $t = 0$,

$$\begin{aligned} m(0, h, u_0) &= s^{-d_\varphi^0} m(0, s^{d_h^0} h, s^\epsilon u_0) \\ &= h^{d_\varphi^0/d_h^0} m(0, 1, h^{-\epsilon/d_h^0} u_0) \\ &\equiv h^{d_\varphi^0/d_h^0} g(h^{-\epsilon/d_h^0} u_0). \end{aligned} \tag{7.223}$$

Again it is not possible to set $h^{-\epsilon/d_h^0} u_0 = 0$ in the scaling function $g(x)$ even though $h^{-\epsilon/d_h^0} u_0 \to 0$ for $h \to 0$. Landau's theory gives $g(x) \sim x^{-1/3}$ for $x \to 0$ so that

$$m(0, h, u_0) \sim h^{(d_\varphi^0 + \frac{\epsilon}{3})/d_h^0}, \tag{7.224}$$

i.e.

$$\delta = \frac{d_h^0}{d_\varphi^0 + \frac{\epsilon}{3}} = 3, \tag{7.225}$$

in agreement with the mean-field result.

7.6.2 The ϵ expansion

We now consider the $(\boldsymbol{\varphi}^2)^2$ theory (7.27). For $d < 4$, u_0 becomes a relevant variable and the Gaussian fixed point $r_0 = u_0 = 0$ does not describe the phase transition. We expect the transition to be described by another fixed point with only one relevant scaling field (besides the magnetic field). We will see that the fixed point value u_0^* is of order $\epsilon = 4 - d$ for d near 4. This enables us to compute the critical exponents below the upper critical dimension $d_c^+ = 4$ within a systematic ϵ expansion. Furthermore, to order ϵ, it is sufficient to compute the RG equations to one-loop order.

One-loop RG equations

Following the general RG procedure (Sec. 7.5.1), we split the field $\boldsymbol{\varphi}(\mathbf{r}) = \boldsymbol{\varphi}_<(\mathbf{r}) + \boldsymbol{\varphi}_>(\mathbf{r})$ into slow and fast modes and rewrite the action as

$$S[\boldsymbol{\varphi}_< + \boldsymbol{\varphi}_>] = S_0[\boldsymbol{\varphi}_<] + S_{\rm int}[\boldsymbol{\varphi}_<] + S_0[\boldsymbol{\varphi}_>] + S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]. \tag{7.226}$$

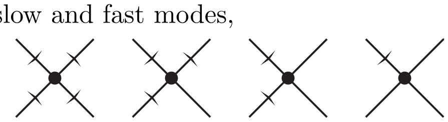
The first two terms in the rhs denote the action in the absence of fast modes ($\boldsymbol{\varphi}_> = 0$), while the last term denotes the interacting part of the action involving fast modes ($S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_> = 0] = 0$). The integration over the fast modes can be done using the linked cluster theorem (Sec. 1.5),

$$\begin{aligned} \int \mathcal{D}[\boldsymbol{\varphi}_>] \exp\bigl\{-S_0[\boldsymbol{\varphi}_>] - S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]\bigr\} & \\ &= Z_{0,>} \exp\left\{ \sum_{n=1}^\infty \frac{(-1)^n}{n!} \langle S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]^n \rangle_{0,>,c} \right\} \\ &= Z_{0,>} \exp\left\{ \sum \text{connected graphs} \right\}, \end{aligned} \tag{7.227}$$

where

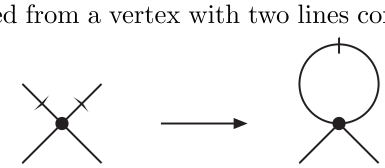
$$\langle \cdots \rangle_{0,>} = \frac{1}{Z_{0,>}} \int \mathcal{D}[\boldsymbol{\varphi}_>] \cdots e^{-S_0[\boldsymbol{\varphi}_>]} \quad \text{and} \quad Z_{0,>} = \int \mathcal{D}[\boldsymbol{\varphi}_>]\, e^{-S_0[\boldsymbol{\varphi}_>]}. \tag{7.228}$$

The notation $\langle \cdots \rangle_{0,>,c}$ means that only the connected graphs are to be considered. The action $S_{\rm int}[\boldsymbol{\varphi}_<, \boldsymbol{\varphi}_>]$ contains several types of vertex depending on the number of legs corresponding to slow and fast modes,



where a slashed line indicates a fast mode. The cumulants $\langle \cdots \rangle_{0,>,c}$ either contribute to the free energy or renormalize the action of the slow modes.

Let us first discuss the case $N = 1$. The one-loop correction to the self-energy of the slow modes is obtained from a vertex with two lines corresponding to fast modes,



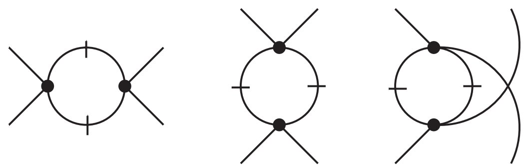
i.e.

$$d\Sigma(\mathbf{p}) = \frac{u_0}{2} \fint_{\mathbf{q}} G_0(\mathbf{q}), \tag{7.229}$$

where $G_0(\mathbf{q}) = (\mathbf{q}^2 + r_0)^{-1}$. We use the notation $\fint_{\mathbf{q}}$ to indicate that the momentum integration is restricted to fast modes $\Lambda/s \leq |\mathbf{q}| \leq \Lambda$. Similarly, the one-loop correction to the (bare) 4-point vertex

$$\Gamma^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{u_0}{V} \delta_{\sum_i \mathbf{p}_i, 0} \tag{7.230}$$

of the slow modes is represented diagrammatically by



i.e.

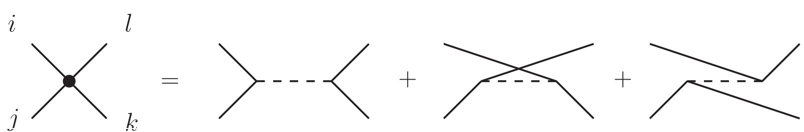
$$d\Gamma^{(4)}(0,0,0,0) = -\frac{3}{2V} u_0^2 \fint_{\mathbf{q}} G_0^2(\mathbf{q}) \tag{7.231}$$

for vanishing momenta $\mathbf{p}_1 = \cdots = \mathbf{p}_4 = 0$. Equations (7.229) and (7.231) follow from standard diagrammatic rules (Sec. 1.5). The momentum dependence of $d\Gamma^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ generates new terms in the action, e.g. $\boldsymbol{\varphi}^2(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$, which do not modify the critical exponents to $\mathcal{O}(\epsilon)$ and can therefore be ignored (see the discussion below).

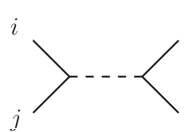
For the $O(N)$ model, the bare 4-point vertex is given by

$$\begin{aligned} \Gamma^{(4)}_{ijkl}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) &= \frac{\delta^{(4)} S_{\rm int}[\boldsymbol{\varphi}]}{\delta\varphi_i(-\mathbf{p}_1)\delta\varphi_j(-\mathbf{p}_2)\delta\varphi_k(-\mathbf{p}_3)\delta\varphi_l(-\mathbf{p}_4)} \\ &= \frac{u_0}{3V} \delta_{\sum_i \mathbf{p}_i, 0} \left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}\right). \end{aligned} \tag{7.232}$$

It convenient to represent this (fully symmetrized) vertex as



where



$$= \frac{u_0}{3} \delta_{i,j}\delta_{k,l}$$

The one-loop corrections to the self-energy and the 4-point vertex are shown in figure 7.13. This leads to

$$d\Sigma(\mathbf{p}) = \frac{N+2}{6} u_0 \fint_{\mathbf{q}} G_0(\mathbf{q}) \tag{7.233}$$

and

$$d\Gamma^{(4)}_{ijkl}(0,0,0,0) = -\left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}\right) \frac{N+8}{18V} u_0^2 \fint_{\mathbf{q}} G_0^2(\mathbf{q}). \tag{7.234}$$

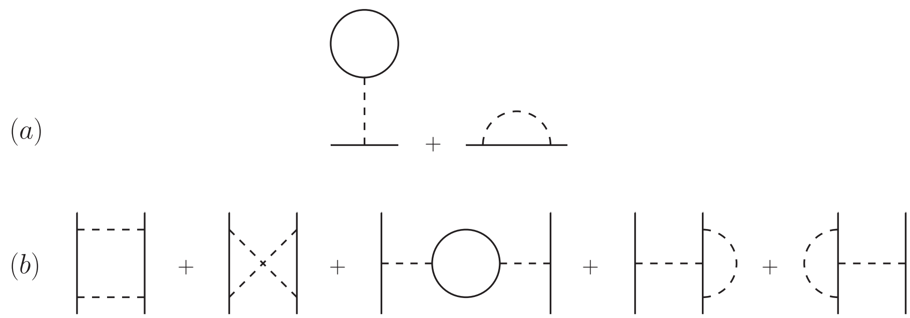


Figure 7.13: One-loop corrections to the self-energy (a) and the 4-point vertex (b).

For $N = 1$, one recovers the previous results (7.229) and (7.231). The one-loop correction due to the fast modes leads to a change of the parameters r_0 and u_0 of the action of the slow modes,

$$\begin{aligned} r_0' &= r_0 + \frac{N+2}{6} u_0 \int_{\mathbf{q}} G_0(\mathbf{q}), \\ u_0' &= u_0 - \frac{N+8}{6} u_0^2 \int_{\mathbf{q}} G_0^2(\mathbf{q}). \end{aligned} \tag{7.235}$$

To complete the RG procedure we must rescale momenta ($\mathbf{p} \to \mathbf{p}/s$) and fields. Since the $(\boldsymbol{\nabla}\boldsymbol{\varphi})^2$ term is not renormalized (the self-energy correction $d\Sigma(\mathbf{p})$ is momentum independent), the field rescaling is simply $\boldsymbol{\varphi}(\mathbf{p}) \to s^{d/2-d_\varphi^0}\boldsymbol{\varphi}(\mathbf{p})$. Finally, it is also convenient to multiply r_0 by Λ^{-2} and u_0 by Λ^{d-4} to obtain coupling constants in dimensionless units. This yields the flow equations

$$\begin{aligned} \tilde r_0' &= s^2 \left[\tilde r_0 + \frac{N+2}{6} \tilde u_0 \Lambda^{2-d} \int_{\mathbf{q}} G_0(\mathbf{q}) \right], \\ \tilde u_0' &= s^\epsilon \left[\tilde u_0 - \frac{N+8}{6} \tilde u_0^2 \Lambda^\epsilon \int_{\mathbf{q}} G_0^2(\mathbf{q}) \right] \end{aligned} \tag{7.236}$$

for the dimensionless variables $\tilde r_0$ and $\tilde u_0$.[59] Using

$$\begin{aligned} \int_{\mathbf{q}} G_0(\mathbf{q}) &= K_d \int_{(1-dl)\Lambda}^{\Lambda} d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{\mathbf{q}^2 + r_0} = K_d \frac{\Lambda^{d-2}}{1+\tilde r_0} dl, \\ \int_{\mathbf{q}} G_0^2(\mathbf{q}) &= K_d \int_{(1-dl)\Lambda}^{\Lambda} d|\mathbf{q}| \frac{|\mathbf{q}|^{d-1}}{(\mathbf{q}^2 + r_0)^2} = K_d \frac{\Lambda^{-\epsilon}}{(1+\tilde r_0)^2} dl, \end{aligned} \tag{7.237}$$

[59] Even without the last rescaling (which implies that $\tilde r_0$ and $\tilde u_0$ are expressed in dimensionless physical units), r_0 and u_0 are "dimensionless" to the extent where they are expressed in units of $(\Lambda'/\Lambda)^{[r_0]}$ and $(\Lambda'/\Lambda)^{[u_0]}$, respectively, i.e. in units of the running cutoff $\Lambda' = \Lambda/s$. This property, which is a consequence of the rescaling of momenta and fields, is crucial for the existence of a fixed point of the RG equations at criticality.

for $s = e^{dl}$ (with $dl \to 0$), we finally obtain the equations

$$\begin{aligned} \frac{d\tilde r_0}{dl} &= 2\tilde r_0 + \frac{N+2}{6} K_d \frac{\tilde u_0}{1+\tilde r_0}, \\ \frac{d\tilde u_0}{dl} &= \epsilon \tilde u_0 - \frac{N+8}{6} K_d \frac{\tilde u_0^2}{(1+\tilde r_0)^2} \end{aligned} \tag{7.238}$$

satisfied by $\tilde r_0(l)$ and $\tilde u_0(l)$.

Fixed points and critical exponents

The RG equations (7.238) admit two fixed points: the Gaussian fixed point $\tilde r_0^* = \tilde u_0^* = 0$ obtained in section 7.6.1 and the Wilson-Fisher fixed point

$$\begin{aligned} \tilde r_0^* &= -\frac{1}{2}\frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2), \\ \tilde u_0^* &= \frac{6}{N+8}\frac{\epsilon}{K_4} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{7.239}$$

where $K_4 = 1/8\pi^2$. To obtain the critical exponents associated to these fixed points, we need to find the eigenvalues $e^{y_1 dl}$ and $e^{y_2 dl}$ of the linearized RG transformation $T(dl)$ defined by

$$\begin{pmatrix} \delta\tilde r_0(l+dl) \\ \delta\tilde u_0(l+dl) \end{pmatrix} = T(dl) \begin{pmatrix} \delta\tilde r_0(l) \\ \delta\tilde u_0(l) \end{pmatrix}, \tag{7.240}$$

where $\delta\tilde r_0 = \tilde r_0 - \tilde r_0^*$ and $\delta\tilde u_0 = \tilde u_0 - \tilde u_0^*$. Equation (7.240) can be rewritten as

$$\frac{d}{dl}\begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix} = \frac{T(dl)-1}{dl}\begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix}, \tag{7.241}$$

where the matrix $\frac{T(dl)-1}{dl}$ has eigenvalues y_1 and y_2 for $dl \to 0$. From the linearized RG equations

$$\frac{d}{dl}\begin{pmatrix} \tilde r_0 \\ \tilde u_0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{N+2}{6}K_d \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} \tilde r_0 \\ \tilde u_0 \end{pmatrix} \tag{7.242}$$

about the Gaussian fixed point, we obtain the eigenvalues $y_1 = 2$ and $y_2 = \epsilon$ and the corresponding eigenvectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} -1 \\ \frac{12}{(N+2)K_4} \end{pmatrix}, \tag{7.243}$$

to $\mathcal{O}(\epsilon^0)$. The linearized RG equations about the Wilson-Fisher fixed point read

$$\frac{d}{dl}\begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix} = \begin{pmatrix} 2 - \frac{N+2}{N+8}\epsilon & \frac{N+2}{6}\frac{K_d}{1+\tilde r_0^*} \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} \delta\tilde r_0 \\ \delta\tilde u_0 \end{pmatrix}, \tag{7.244}$$

to order ϵ. We thus obtain the eigenvalues $y_1 = 2 - \frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2)$ and $y_2 = -\epsilon + \mathcal{O}(\epsilon^2)$. To $\mathcal{O}(\epsilon^0)$, the eigenvectors are given by (7.243). The Wilson-Fisher fixed point is in the direction $\mathbf{e}_2$ from the origin (Gaussian fixed point) since

$$\begin{pmatrix} \tilde r_0^* \\ \tilde u_0^* \end{pmatrix}_{\mathrm{WF}} = \frac{1}{2}\frac{N+2}{N+8}\epsilon\mathbf{e}_2 + \mathcal{O}(\epsilon^2). \tag{7.245}$$

For $d > 4$, the Gaussian fixed point has one relevant direction ($\mathbf{e}_1$) and governs the critical behavior. The correlation length critical exponent takes the classical value $\nu = 1/y_1 = 1/2$. On the other hand, the Wilson-Fisher fixed point is not physical since $\tilde{u}_0^* < 0$ for $\epsilon < 0$, and has two relevant directions (besides the magnetic field). For $d < 4$, the Gaussian fixed point has two relevant directions. The critical behavior is governed by the Wilson-Fisher fixed point which has only one relevant direction. The critical exponent ν is given by

$$\nu = \frac{1}{y_1} = \frac{1}{2} + \frac{N+2}{N+8}\frac{\epsilon}{4} + \mathcal{O}(\epsilon^2), \tag{7.246}$$

whereas the correction-to-scaling exponent (Sec. 7.5.3) takes the value

$$\omega = -y_2 = \epsilon + \mathcal{O}(\epsilon^2). \tag{7.247}$$

The anomalous dimension η vanishes to $\mathcal{O}(\epsilon)$ since the field has been trivially rescaled in the RG procedure. All other exponents can then be deduced from the scaling laws derived in sections 7.4 and 7.5.3,

$$\begin{aligned}
\gamma &= 1 + \frac{N+2}{N+8}\frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \\
\beta &= \frac{1}{2} - \frac{3}{2}\frac{\epsilon}{N+8} + \mathcal{O}(\epsilon^2), \\
\delta &= 3 + \epsilon + \mathcal{O}(\epsilon^2), \\
\alpha &= \frac{4-N}{N+8}\frac{\epsilon}{2} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{7.248}$$

If u_0 vanishes, the phase transition is governed by the Gaussian fixed point even when $d < 4$. If u_0 is finite but small and $T = T_c$, the RG trajectory spends a lot of "time" near the Gaussian fixed point before eventually flowing into the Wilson-Fisher fixed point. We can describe this crossover behavior with two relevant scaling fields $\tilde{t}_1$ and $\tilde{t}_2$ at the Gaussian fixed point, defined by

$$\begin{pmatrix} \tilde{r}_0 \\ \tilde{u}_0 \end{pmatrix} = \tilde{t}_1\mathbf{e}_1 + \tilde{t}_2\mathbf{e}_2 = \left(\tilde{r}_0 + \frac{(N+2)K_4}{12}\tilde{u}_0\right)\mathbf{e}_1 + \frac{(N+2)K_4}{12}\tilde{u}_0\mathbf{e}_2. \tag{7.249}$$

The singular part of the free energy satisfies[60]

$$f(t_1, u_0) = s^{-d} f(s^{y_1} t_1, s^{y_2} u_0) = |t_1|^{d/y_1} g_\pm\left(u_0|t_1|^{-\phi}\right) \tag{7.250}$$

(we consider $\tilde{u}_0$ rather than $\tilde{t}_2$), where $y_1 = 1/2$ and $y_2 = \epsilon$. The crossover exponent ϕ is given by

$$\phi = \frac{y_2}{y_1} > 0. \tag{7.251}$$

Thus we expect to see effective Gaussian behavior when $u_0|t_1|^{-\phi} \ll 1$ and critical behavior (governed by the Wilson-Fisher fixed point) when $u_0|t_1|^{-\phi} \gg 1$.

To ensure that the preceding discussion is correct, we have to verify that the omitted coupling constants do not modify the critical exponents to $\mathcal{O}(\epsilon)$.[61] Even if

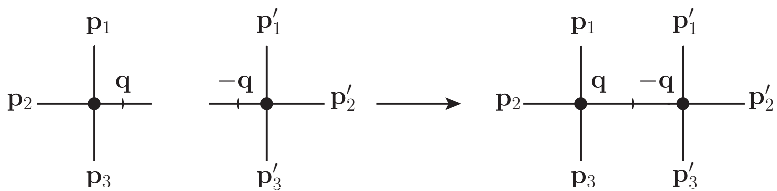
[60] Eq. (7.250) can be alternatively written as $f(\tilde{t}_1(1), \tilde{u}_0(1)) = s^{-d} f(\tilde{t}_1(s), \tilde{u}_0(s))$.
[61] See Ref. [3] for a thorough discussion.

we start from the $(\boldsymbol{\varphi}^2)^2$ theory, the RG procedure will generate all coupling constants allowed by symmetry. An arbitrary coupling $\tilde{\omega}$, different from $\tilde{r}_0$ and $\tilde{u}_0$, satisfies the RG equation

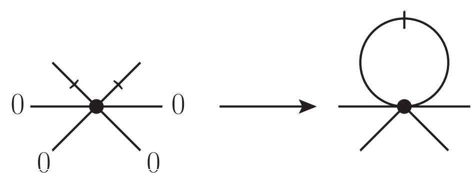
$$\frac{d\tilde{w}}{dl} = \big(d_w^0 + \mathcal{O}(\epsilon^2)\big)\tilde{w} + \mathcal{O}(\tilde{u}_0^2, \tilde{u}_0\tilde{w}, \tilde{w}^2, \cdots), \tag{7.252}$$

where d_w^0 is the engineering dimension of $\tilde{w}$ while the $\mathcal{O}(\epsilon^2)$ term comes from the anomalous dimension η of the $\boldsymbol{\varphi}$ field. There is no term linear in $\tilde{u}_0$ in (7.252) since the only $\mathcal{O}(\tilde{u}_0)$ terms enter the RG equation of $\tilde{r}_0$ and $\tilde{u}_0$.[62] The fixed point value $\tilde{w}^*$ is of order ϵ^2 or higher. Thus the Wilson-Fisher fixed point is characterized by an infinite number of nonzero coupling constants, but only $\tilde{r}_0^*$ and $\tilde{u}_0^*$ are of order ϵ.

We must now ask whether the $\mathcal{O}(\epsilon^2)$ couplings can change the values of the critical exponents to order ϵ. Only terms of order ϵ are important for the equation fixing $\tilde{r}_0^*$, so that $\tilde{w} = \mathcal{O}(\epsilon^2)$ is negligible. By contrast, if a $\mathcal{O}(\epsilon^2)$ term enters the equation $d\tilde{u}_0/dl$ linearly, $\tilde{u}_0^*$ will change to order ϵ and in turn the critical exponents. Thus the $u_6(\boldsymbol{\varphi}^2)^3$ term, which contributes to $d\tilde{u}_0/dl$, is potentially dangerous. The leading contribution to $d\tilde{u}_6/dl$ comes from[63]



where $\mathbf{q} = -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3$, $|\mathbf{q}| \in [\Lambda(1-dl), \Lambda]$ and $|\mathbf{p}_i|, |\mathbf{p}'_i| < \Lambda(1-dl)$ $(i = 1,2,3)$. Now the $\tilde{u}_6$ term contribution to $d\tilde{u}_0/dl$ must have $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}'_1 = \mathbf{p}'_2 = 0$:



But the condition $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}'_1 = \mathbf{p}'_2 = 0$ implies that $\mathbf{p}_3 = -\mathbf{p}'_3 = -\mathbf{q}$, which is in contradiction with $|\mathbf{p}_3|, |\mathbf{p}'_3| < \Lambda(1-dl)$ and $|\mathbf{q}| \in [\Lambda(1-dl), \Lambda]$. Thus $\tilde{u}_6$ does not contribute an $\mathcal{O}(\epsilon^2)$ term to the equation for $\tilde{u}_0^*$, and the critical exponents to order ϵ are unchanged.

Flow diagrams

The analysis of the RG equations (7.238) is not restricted to the determination of fixed points and critical exponents. In figure 7.14 we show typical solutions for generic initial conditions near the critical surface (i.e. $T \simeq T_c$) for $d = 3$ and $N = 1$ (boldly extrapolating the result of the ϵ-expansion to $\epsilon = 1$). The top plots correspond to $T > T_c$ and the bottom ones to $T < T_c$. In both cases, one can identify a critical regime (in momentum space) where $\tilde{r}_0$ and $\tilde{u}_0$ are nearly equal to their fixed point

[62] The only diagrams of order $\tilde{u}_0$ are the one-loop self-energy diagram (which contributes to $d\tilde{r}_0/dl$) and the bare vertex $\tilde{u}_0$ (which gives the term linear in $\tilde{u}_0$ in $d\tilde{u}_0/dl$).

[63] There is also a contribution to $\tilde{u}_6$ coming from the $\tilde{u}_8(\boldsymbol{\varphi}^2)^4$ term which turns out to be $\mathcal{O}(\epsilon^3)$ for $\tilde{u}_8 \simeq \tilde{u}_8^*$.

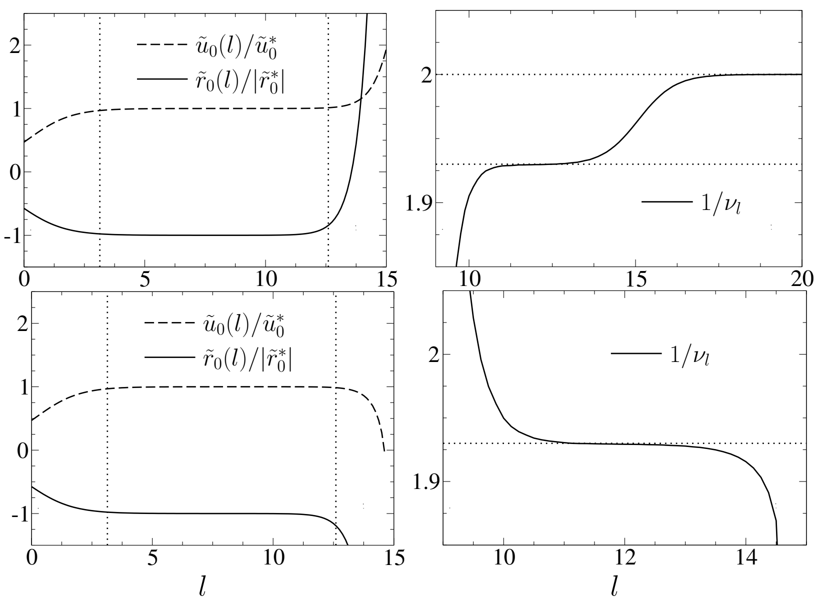


Figure 7.14: Solution of the RG equations (7.238) for initial conditions near the critical surface ($d = 3$, $N = 1$, $\Lambda = 1$, $\tilde{r}_0^* \simeq -0.1$ and $\tilde{u}_0^* \simeq 10.66$): $T > T_c$ (top) and $T < T_c$ (bottom). The vertical dotted lines show the Ginzburg scale $l_G = \ln(\Lambda\xi_G) \simeq 3$ and the correlation length scale $l_\xi = \ln(\Lambda\xi) \simeq 12$.

values $\tilde{r}_0^*$ and $\tilde{u}_0^*$. Note that $\tilde{r}_0$ can take negative values as long as $G_0(\mathbf{p}) = (\mathbf{p}^2 + r_0)^{-1}$ remains positive for $|\mathbf{p}| = \Lambda(l)$, i.e. $1 + \tilde{r}_0 > 0$. The critical regime begins when $\Lambda(l) \sim \xi_G^{-1} \sim u_0^{1/(d-4)}$, where ξ_G is the Ginzburg length introduced in section 7.3.4. It ends when $\Lambda(l) \sim \xi^{-1}$ where ξ is the correlation length. The critical regime is preceded by a perturbative regime $\Lambda(l) \gg \xi_G^{-1}$ where the Gaussian approximation is essentially correct (Sec. 7.3.4).[64]

Since the relevant scaling field $\tilde{t}_1 \propto e^{l/\nu}$ grows exponentially near the fixed point (which implies $\delta\tilde{r}_0 \propto e^{l/\nu}$ and $\delta\tilde{u}_0 \propto e^{l/\nu}$), it is possible to obtain the value of the critical exponent ν from the behavior of $\tilde{r}_0$ and $\tilde{u}_0$ in the critical regime. To this end it is convenient to define a "running" critical exponent ν_l by

$$\frac{1}{\nu_l} = \frac{d}{dl} \ln|\tilde{r}_0 - \tilde{r}_0^*|, \tag{7.253}$$

in the same way as we have defined a running anomalous dimension η_l [Eq. (7.159)]. Near the end of the critical regime, we observe that ν_l takes a constant value which can be identified to the actual value of the exponent ν; the obtained value clearly differs from the mean-field value $\nu = 1/2$ (Fig. 7.14). In principle, the anomalous dimension η can be obtained by a similar method, since the (running) anomalous dimension η_l coincides with η when $\xi^{-1} \ll \Lambda(l) \ll \xi_G^{-1}$. It is however necessary to

[64]Strictly speaking, $\tilde{u}_0(l)$ must be small for the perturbative regime to exist. The fact that $\tilde{u}_0^* = \mathcal{O}(\epsilon)$ for $d = 4 - \epsilon$ does not say anything about the value of $\tilde{u}_0(l)$ for $\Lambda(l) \gg \xi_G^{-1}$.

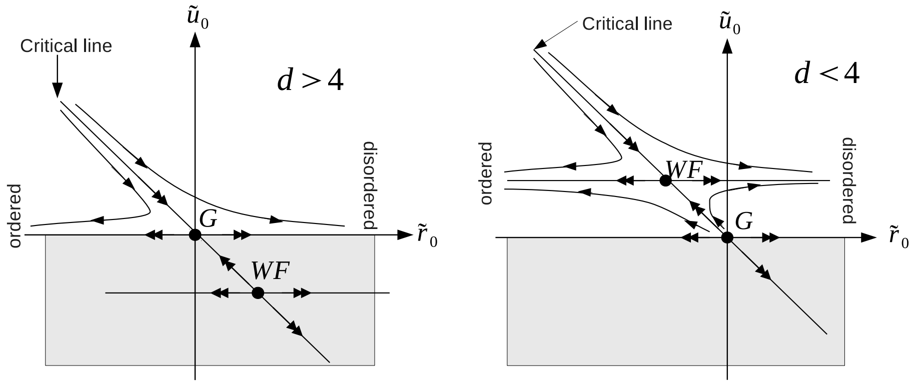


Figure 7.15: Schematic flow diagrams in the $(\tilde{r}_0, \tilde{u}_0)$ plane obtained from the linearized one-loop RG equations for $d > 4$ (left) and $d < 4$ (right). G indicates the Gaussian fixed point and WF the Wilson-Fisher fixed point. The gray areas correspond to the nonphysical region $\tilde{u}_0 \leq 0$.

go beyond the one-loop RG equations to obtain a nonzero anomalous dimension (see chapter 8).

When $T > T_c$, the flow for $\Lambda(l) \ll \xi^{-1}$ is characteristic of the disordered phase. $\tilde{r}_0$ takes a large value ($\tilde{r}_0 \gg 1$) which suppresses the perturbative corrections to the coupling constants. The flow of $\tilde{r}_0$ and $\tilde{u}_0$ is then purely dimensional, i.e. $\tilde{r}_0 \propto e^{2l}$ and $\tilde{u}_0 \propto e^{\epsilon l}$, while the "dimensionful" variables satisfies $dr_0/dl \simeq du_0/dl \simeq 0$ [Eq. (7.235)]. ν_l takes the mean-field value 1/2. This is a general feature of Wilson's RG: a "gapped" fluctuation mode, with propagator $G(\mathbf{p}) \sim (\mathbf{p}^2 + \xi^{-2})^{-1}$, does not contribute to the RG flow once $\Lambda(l) \ll \xi^{-1}$.

When $T < T_c$, the critical regime is followed by a decrease of $\tilde{r}_0$ until $1 + \tilde{r}_0 = 0$, which corresponds to a pole in the propagator $G_0(\mathbf{q}) = (\mathbf{q}^2 + r_0)^{-1}$ for $|\mathbf{q}| = \Lambda$. One could try to circumvent this difficulty by expanding the action about one of its degenerate minima. For $N \geq 2$ however, one would have to deal with the nontrivial physics of the low-temperature phase due to the (gapless) Goldstone modes associated to the spontaneous broken symmetry, an impossible task within the framework of the perturbative RG we have discussed so far. The ordered phase of the $(\boldsymbol{\varphi}^2)^2$ theory with $N \geq 2$ will be studied in section 7.7 from the non-linear sigma model (see also Secs. 7.B and 7.C).

By solving the RG equations (7.238) for various initial conditions, we obtain the flow diagrams shown in figure 7.15 for $d > 4$ and $d < 4$. The critical line (or, more precisely, the line tangent to the critical line at the fixed point) is determined by

$$\tilde{t}_1 = \delta\tilde{r}_0 + \frac{N+2}{12} K_4 \delta\tilde{u}_0 = 0, \tag{7.254}$$

where the relevant scaling field $\tilde{t}_1$ is defined by

$$\begin{pmatrix} \delta\tilde{r}_0 \\ \delta\tilde{u}_0 \end{pmatrix} = \tilde{t}_1 \mathbf{e}_1 + \tilde{t}_2 \mathbf{e}_2. \tag{7.255}$$

We see that fluctuations lead to a reduction of the transition temperature compared to the mean-field result. For a given value of $\tilde{u}_0(0)$, we need to choose $\tilde{r}_0(0) < 0$, i.e. $T_c < T_{c0}$, to be on the critical line (Fig. 7.15).

The upper critical dimension

At the upper critical dimension $d_c^+ = 4$, $\tilde{u}_0$ is a marginal variable. To obtain the critical behavior, one must go beyond the linear approximation. We shall see that the Gaussian fixed point governs the critical behavior ($\tilde{u}_0$ is marginally irrelevant at the Gaussian fixed point) but the mean-field predictions are modified by logarithmic corrections.

Expanding the one-loop RG equations (7.238) to quadratic order in $\tilde{r}_0$ and $\tilde{u}_0$, we obtain

$$\begin{aligned} \frac{d\tilde{t}_1}{dl} &= 2\tilde{t}_1 - \frac{N+2}{6} K_4 \tilde{t}_1 \tilde{u}_0 + \mathcal{O}(\tilde{u}_0^2), \\ \frac{d\tilde{u}_0}{dl} &= -\frac{N+8}{6} K_4 \tilde{u}_0^2, \end{aligned} \tag{7.256}$$

where $\tilde{t}_1$ is the relevant scaling field defined in (7.255). The second equation gives

$$\tilde{u}_0(l) = \frac{\tilde{u}_0(0)}{1 + \frac{N+8}{6} K_4 \tilde{u}_0(0) l}. \tag{7.257}$$

$\tilde{u}_0(l)$ vanishes for $l \to \infty$, but only logarithmically wrt the running momentum cutoff $\Lambda(l) = \Lambda e^{-l}$. From

$$\frac{d \ln \tilde{t}_1}{dl} = 2 - \frac{N+2}{6} K_4 \tilde{u}_0, \tag{7.258}$$

we then deduce

$$\tilde{t}_1(l) \propto \tilde{t}_1(0) e^{2l} l^{-\frac{N+2}{N+8}} \tag{7.259}$$

when $\frac{N+8}{6} K_4 \tilde{u}_0(0) l \gg 1$.

To obtain the correlation length (or the Josephson length ξ_J for $N \geq 2$ and $T < T_c$) we use $\xi(\tilde{t}_1(0), \tilde{u}_0(0)) = e^l \xi(\tilde{t}_1(l), \tilde{u}_0(l))$ and choose l such that $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0(l) \ll 1$, i.e.

$$e^{2l} \sim \frac{1}{|\tilde{t}_1(0)|} \big| \ln |\tilde{t}_1(0)| \big|^{\frac{N+2}{N+8}}. \tag{7.260}$$

We then have $\xi(\tilde{t}_1(l), \tilde{u}_0(l)) \sim \xi(1, 0) \sim 1$ and therefore

$$\xi \sim \frac{1}{\sqrt{|T - T_c|}} \big| \ln |T - T_c| \big|^{\frac{N+2}{2(N+8)}}, \tag{7.261}$$

where we have used $\tilde{t}_1(0) \propto T - T_c$. Thus the marginal variable $\tilde{u}_0$ leads to a logarithmic correction to the mean-field result $\xi \sim 1/\sqrt{|T - T_c|}$.

Similarly, we can compute the uniform susceptibility[65] $\chi = \beta G_{ii}(\mathbf{p} = 0)$ starting from

$$\chi(\tilde{t}_1(0), \tilde{u}_0(0)) = e^{(d - 2d_\varphi^0) l} \chi(\tilde{t}_1(l), \tilde{u}_0(l)), \tag{7.262}$$

[65] The uniform susceptibility is defined in the ordered phase only for $N = 1$ (see Sec. 7.7.3).

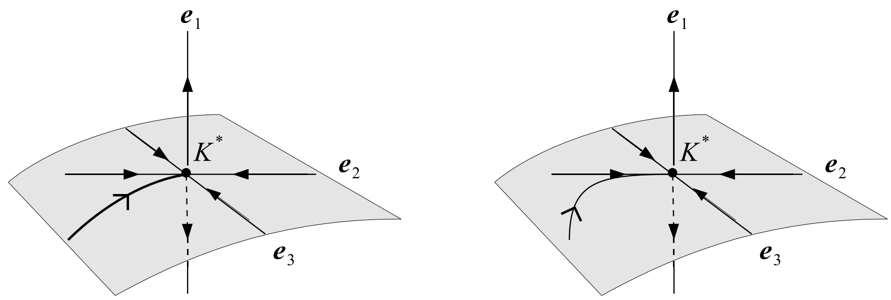


Figure 7.16: RG trajectories on the critical surface ($\mathbf{e}_1$ is a relevant direction): without marginal variable, $y_2, y_3 < 0$ (left), and with a marginally irrelevant variable, $y_2 = 0$ and $y_3 < 0$ (right).

where $d - 2d_\varphi^0 = 2$ ($\eta = 0$). When $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0 \ll 1$, the rhs in (7.262) can be calculated by perturbation theory, which yields

$$\chi \sim \frac{1}{|T - T_c|} \big| \ln |T - T_c| \big|^{\frac{N+2}{N+8}}. \tag{7.263}$$

Again, we obtain a logarithmic correction to the mean-field result $\chi \sim 1/|T - T_c|$. Note that there is no logarithmic ($\ln |\mathbf{p}|$) correction to the critical correlation function $G_{ii}(\mathbf{p}) \sim 1/\mathbf{p}^2$ since the anomalous dimension η vanishes for $d = 4$ (see Appendix 7.A).

To obtain the singular part of the free energy, we use

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) = e^{-4l} f(\tilde{t}_1(l), \tilde{u}_0(l)). \tag{7.264}$$

Let us assume that the system is in the low-temperature phase. For $|\tilde{t}_1(l)| \sim 1$ and $\tilde{u}_0(l) \ll 1$, the rhs in (7.264) can be calculated within mean-field theory,[66]

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) \sim -e^{-4l} \frac{\tilde{r}_0(l)^2}{\tilde{u}_0(l)} \sim -|\tilde{t}_1(0)|^2 l^{1-2\frac{N+2}{N+8}}, \tag{7.265}$$

where we have used $\tilde{r}_0(l) \sim \tilde{t}_1(l) \sim 1$, $\tilde{u}_0(l) \sim 1/l \ll 1$ and (7.259). With l given by (7.260), we finally obtain

$$f(\tilde{t}_1(0), \tilde{u}_0(0)) \sim -|\tilde{t}_1(0)|^2 \big| \ln |\tilde{t}_1(0)| \big|^{\frac{4-N}{N+8}}. \tag{7.266}$$

Since $\tilde{t}_1(0) \propto T - T_c$, we deduce the (most) singular part of the specific heat,[67]

$$C_V \sim \big| \ln |T - T_c| \big|^{\frac{4-N}{N+8}}. \tag{7.267}$$

Thus the singular part of C_V diverges for $N < 4$ but vanishes for $N > 4$. These results significantly differ from the Gaussian model predictions $C_V \sim \big| \ln |T - T_c| \big|$ (Sec. 7.3.4).

[66] The singular dependence of $f(\tilde{t}_1(l), \tilde{u}_0(l))$ on $\tilde{u}_0(l)$ is due to $\tilde{u}_0(l)$ being a dangerously irrelevant variable in the low-temperature phase (see Sec. 7.6.1).

[67] This result can also be obtained from the high-temperature phase using $f(\tilde{t}_1(l), \tilde{u}_0(l)) \simeq f(\tilde{t}_1(l))$ for $\tilde{u}_0(l) \to 0$ and $s^{(4-d)} \equiv \ln s$ for $d \to 4$ ($s = e^l$).

Exponent	Gaussian model	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^5)$	Numerics
ν	1/2	0.583	0.6290(25)	0.6302(1)
β	1/2	0.333	0.3257(25)	
γ	1	1.167	1.2380(50)	
δ	3	4		
α	1/2	0.167		
η	0	0	0.0360(50)	0.0368(2)

Table 7.4: Critical exponents obtained from the ϵ expansion [40] and numerical methods (Monte Carlo and high-temperature series) [41] ($d = 3$ and $N = 1$).

Logarithmic corrections are a generic consequence of a marginally irrelevant variable. Figure 7.16 shows two trajectories on the critical surface with and without a marginal variable. In the absence of a marginal variable, the trajectory rapidly converges to the fixed point. On the other hand, when there is a marginally irrelevant variable, the trajectory first moves closer to the corresponding axis (because the irrelevant variables rapidly decrease to zero) before converging to the fixed point due to the slow vanishing of the marginally irrelevant variable. When the relevant field $\tilde{t}_1$ is nonzero but small, the trajectory eventually runs away from the fixed point. Nevertheless, the marginal irrelevant variable still controls the (slow) approach to the fixed point and leads to logarithmic corrections.

Utility of the ϵ expansion

The ϵ expansion to first order does not yield reliable estimates of the critical exponents of three-dimensional systems. Its great virtue is to provide a technically easy way of determining what kind of universality classes one can expect. Although the value of the critical exponents changes when one goes away from the upper critical dimension, the topology of the flow diagram does not. One can therefore investigate the phase transitions and their universality classes in various models. Furthermore, the ϵ expansion can also be applied to the analysis of systems near the lower critical dimension (see Sec. 7.7).

Calculating the critical exponents to order ϵ^2 and beyond in the Wilson approach is quite complicated as one must keep track of additional coupling constants besides $\tilde{r}_0$ and $\tilde{u}_0$. In practice, higher-order calculations are carried out using field theoretical perturbative methods (see Appendix 7.A for an introduction to this type of approach). The expansion has been pushed to $\mathcal{O}(\epsilon^5)$. Although the series in ϵ is only asymptotic, it can be evaluated by the Borel summation method. Results for three-dimensional systems are in very good agreement with numerical approaches. Table 7.4 shows the critical exponents obtained from various methods for $d = 3$ and $N = 1$.[68] Note that the crude estimates obtained from the $\mathcal{O}(\epsilon)$ expansion with $\epsilon = 1$ are closer to the

[68] In chapter 8 we shall see that the non-perturbative RG is also a powerful tool to compute the critical exponents with high precision.

exact results than those of the Gaussian model.

7.7 The non-linear sigma model

In section 7.2.2, we have argued that the critical behavior of the lattice classical spin model $H = J\sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \mathbf{S}_\mathbf{r}\cdot\mathbf{S}_{\mathbf{r}'}$ ($\mathbf{S}_\mathbf{r}^2 = 1$) with O($N$) symmetry is described by a $(\boldsymbol{\varphi}^2)^2$ field theory. In this section, we study this model at low temperature in the ordered phase by expanding about the classical configuration $\mathbf{S}_\mathbf{r} = (1, 0, \cdots, 0)$. We assume $N \geq 2$. For $T < T_c$, the spontaneous symmetry breaking gives rise to a nontrivial physics due to the (gapless) Goldstone modes. We shall see that, rather surprisingly, the low temperature expansion allows us to study the critical behavior near $d = 2$ when $N > 2$.

We thus start from the partition function

$$Z = \int \mathcal{D}[\mathbf{n}] \prod_\mathbf{r} \delta(\mathbf{n}_\mathbf{r}^2 - 1) \exp\biggl\{ -\frac{1}{2g} \sum_{\mathbf{r},\mu} (D_\mu \mathbf{n}_\mathbf{r})^2 + \frac{\mathbf{h}}{g}\cdot \sum_\mathbf{r} \mathbf{n}_\mathbf{r} \biggr\} \tag{7.268}$$

of the non-linear sigma model on a d-dimensional hypercubic lattice, where

$$D_\mu \mathbf{n}_\mathbf{r} = \frac{\mathbf{n}_{\mathbf{r}+\boldsymbol{\mu}} - \mathbf{n}_\mathbf{r}}{a} \tag{7.269}$$

(with a the lattice spacing) denotes a discrete derivative, $g = T/Ja^2$, $\mathbf{h}$ is an external field and $\mu = 1, \cdots, d$. In the continuum limit, the partition function of the non-linear sigma model is often written as

$$Z = \int \mathcal{D}[\mathbf{n}]\, \delta(\mathbf{n}^2 - 1) \exp\left\{ -\frac{1}{2g} \int d^d r (\boldsymbol{\nabla}\mathbf{n})^2 + \frac{\mathbf{h}}{g} \cdot \int d^d r\, \mathbf{n} \right\}. \tag{7.270}$$

However, a proper handling of the measure requires to define the model on a lattice and take the continuum limit only at a later stage (see below).

7.7.1 Perturbative expansion

In the limit of a vanishing coupling constant, $g \to 0$, the dominant field configuration is $\mathbf{n}_\mathbf{r} = (1, 0, \cdots, 0)$ if the field $\mathbf{h} = (h, 0, \cdots, 0)$ ($h > 0$). To study the fluctuations about this configuration, it is convenient to use the parametrization

$$\mathbf{n}_\mathbf{r} = (\sigma_\mathbf{r}, \boldsymbol{\pi}_\mathbf{r}), \tag{7.271}$$

where $\sigma_\mathbf{r}$ denotes the component of $\mathbf{n}_\mathbf{r}$ along $\mathbf{h}$ and $\boldsymbol{\pi}_\mathbf{r}$ is a $(N-1)$-component field perpendicular to $\mathbf{h}$. For small fluctuations, $|\boldsymbol{\pi}_\mathbf{r}| \ll 1$ and $\sigma_\mathbf{r} > 0$, we can express $\sigma_\mathbf{r} = (1 - \boldsymbol{\pi}_\mathbf{r})^{1/2}$ in terms of $\boldsymbol{\pi}_\mathbf{r}$ using $\mathbf{n}_\mathbf{r}^2 = 1$. It is then possible to derive an effective action for the $\boldsymbol{\pi}$ field by integrating out $\sigma_{\mathbf{r}}$,

$$\begin{aligned} Z &= \int \mathcal{D}[\sigma, \boldsymbol{\pi}] \prod_{\mathbf{r}} \delta(\sigma_r^2 + \boldsymbol{\pi}_{\mathbf{r}}^2 - 1) \exp\biggl\{ -\frac{1}{2g} \sum_{\mathbf{r},\mu} \left[(D_\mu \sigma_{\mathbf{r}})^2 + (D_\mu \boldsymbol{\pi}_{\mathbf{r}})^2 \right] + \frac{h}{g} \sum_{\mathbf{r}} \sigma_{\mathbf{r}} \biggr\} \\ &= \int \mathcal{D}[\boldsymbol{\pi}] \exp\biggl\{ -\frac{1}{2g} \sum_{\mathbf{r},\mu} \Bigl[(D_\mu \boldsymbol{\pi}_{\mathbf{r}})^2 + (D_\mu \sqrt{1 - \boldsymbol{\pi}_{\mathbf{r}}^2})^2 \Bigr] \\ &\qquad\qquad + \frac{h}{g} \sum_{\mathbf{r}} \sqrt{1 - \boldsymbol{\pi}_{\mathbf{r}}^2} - \frac{1}{2} \sum_{\mathbf{r}} \ln(1 - \boldsymbol{\pi}_{\mathbf{r}}^2) \biggr\}, \end{aligned} \tag{7.272}$$

where the integration over $\boldsymbol{\pi}_{\mathbf{r}}$ is restricted to $|\boldsymbol{\pi}_{\mathbf{r}}| \leq 1$. The last term in (7.272) comes from the measure $\mathcal{D}[\mathbf{n}] \prod_{\mathbf{r}} \delta(\mathbf{n}_{\mathbf{r}}^2 - 1)$ in the functional integral (7.268).[69]

We are now in a position to take the continuum limit $a \to 0$ whereby $\mathbf{r}$ becomes a continuous position variable and $D_\mu \boldsymbol{\pi}_{\mathbf{r}} = \partial_\mu \boldsymbol{\pi}$ a standard derivative. The term coming from the measure becomes

$$\frac{\rho}{2} \int d^d r \ln(1 - \boldsymbol{\pi}^2) \qquad \text{where} \qquad \rho = \frac{1}{V} \sum_{\mathbf{p} \in \mathrm{BZ}} = a^{-d} \tag{7.273}$$

is the number of degrees of freedom per unit volume. In the continuum limit, the Brillouin zone (BZ) is replaced by a spherical region of radius Λ with the same volume,

$$\rho = a^{-d} \equiv \int \frac{d^d p}{(2\pi)^d} \Theta(\Lambda - |\mathbf{p}|) = \frac{K_d}{d} \Lambda^d. \tag{7.274}$$

The final form of the action therefore reads

$$\begin{aligned} S[\boldsymbol{\pi}] = \frac{1}{2g} &\int d^d r \left[(\boldsymbol{\nabla} \boldsymbol{\pi})^2 + (\boldsymbol{\nabla} \sqrt{1 - \boldsymbol{\pi}^2})^2 \right] \\ &- \frac{h}{g} \int d^d r \sqrt{1 - \boldsymbol{\pi}^2} + \frac{\rho}{2} \int d^d r \ln(1 - \boldsymbol{\pi}^2) \end{aligned} \tag{7.275}$$

(we set the lattice spacing a equal to unity), where the $\boldsymbol{\pi}$ field satisfies the constraint $|\boldsymbol{\pi}| \leq 1$.

The minimum of the action is reached for $\boldsymbol{\pi} = 0$. For small g, we expect the dominant fluctuations to satisfy $|\boldsymbol{\pi}| \sim \sqrt{g}$. Field configurations with $|\boldsymbol{\pi}| \sim 1$ give exponentially small contributions (of order $\exp(-\mathrm{const}/g)$) to the partition function that can be neglected in the perturbative approach ($g \ll 1$). This allows us to ignore the constraint $|\boldsymbol{\pi}| \leq 1$ and freely integrate over π_μ from $-\infty$ to ∞. The expansion wrt g is then similar to a loop expansion (Sec. 1.7), the only difference being that the term coming from the measure is not multiplied by $1/g$. To leading order (using $|\boldsymbol{\pi}| \sim \sqrt{g}$),

$$S_0[\boldsymbol{\pi}] = \frac{1}{2g} \int d^d r \left[(\boldsymbol{\nabla} \boldsymbol{\pi})^2 + h \boldsymbol{\pi}^2 \right], \tag{7.276}$$

and the propagator of the $\boldsymbol{\pi}$ field reads

$$G_0(\mathbf{p}) = \langle \pi_\mu(\mathbf{p}) \pi_\mu(-\mathbf{p}) \rangle = \frac{g}{\mathbf{p}^2 + h}. \tag{7.277}$$

[69] We have used $\int d\sigma\, \delta(\sigma^2 + \boldsymbol{\pi}^2 - 1) = \int d\sigma \frac{1}{2\sigma} \delta(\sigma - \sqrt{1 - \boldsymbol{\pi}^2}) \propto \exp\left\{ -\frac{1}{2} \ln(1 - \boldsymbol{\pi}^2) \right\}$.

Alternatively, one can obtain the action (7.276) by introducing a rescaled field $\boldsymbol{\pi}_r = g^{-1/2}\boldsymbol{\pi}$ and then setting $g=0$. For $h=0$, the state with $\mathbf{n}=(1,0,\cdots,0)$ spontaneously breaks the O(N) symmetry, and $S_0[\boldsymbol{\pi}]$ is nothing but the action of the $N-1$ Goldstone modes (to leading order). A nonzero field h explicitly breaks the O(N) symmetry and gives a "mass" term to the Goldstone modes. From (7.276) and (7.277), we deduce the (bare) stiffness of the non-linear sigma model,[70]

$$\rho_s^0 = \frac{1}{g}. \tag{7.278}$$

In the RG language, the action (7.276) corresponds to the Gaussian fixed point $g=0$. This fixed point is stable if the O(N) symmetry remains spontaneously broken for $g=0^+$ and $h\to 0$. We can repeat the argument of section 7.3.3 to show that this should be the case for $d>2$. The reduction of the order parameter by fluctuations is given by

$$\begin{aligned}\langle\sigma(\mathbf{r})\rangle &= 1-\frac{1}{2}\langle\boldsymbol{\pi}(\mathbf{r})^2\rangle + \mathcal{O}(g^2) \\ &= 1-\frac{(N-1)g}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{\mathbf{p}^2}+\mathcal{O}(g^2)\end{aligned} \tag{7.279}$$

for $h=0$. The momentum integral in (7.279) is infrared divergent for $d\leq 2$, so that long-range order cannot exist in that case (Mermin-Wagner theorem; Sec. 7.3.3). On the other hand we expect long-range order for $d>2$ when g is small enough. This suggests that higher-order vertices neglected in (7.276) are irrelevant at the Gaussian fixed point when $d>2$.

Alternatively, one can check the stability of the Gaussian fixed point using dimensional analysis. By expanding the various terms in (7.275), we find two types of vertices: $(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2(\boldsymbol{\pi}^2)^n$ and $(\boldsymbol{\pi}^2)^n$. At the Gaussian fixed point, the $\boldsymbol{\pi}$ field has scaling dimension[71]

$$[\boldsymbol{\pi}] = \frac{d-2}{2}. \tag{7.280}$$

A vertex with $2n$ fields and $2r$ derivatives has then dimension

$$y_{nr} = d-n(d-2)-2r. \tag{7.281}$$

The vertices $(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2(\boldsymbol{\pi}^2)^n$ are relevant for $d<2$ and irrelevant for $d>2$. The vertices $(\boldsymbol{\pi}^2)^n$ are relevant for $d<2$. Surprisingly, a finite number of them are relevant also for $d>2$, which seems to contradict our expectation that the Gaussian fixed point should be stable for $d>2$. In fact, the vertices coming from the measure maintain the O(N) symmetry of the action and ensure that the propagator of the $\boldsymbol{\pi}$ field remains gapless to all orders in perturbation theory. Their relevance does not, as we shall see, invalidate the conclusion that the Gaussian fixed point is stable for $d>2$.

[70] Eq. (7.278) follows from $\rho_s^0=\sigma^2/g$ with $\sigma=1$ the order parameter in the classical configuration $\mathbf{n}_\mathbf{r}=(1,0,\cdots,0)$.

[71] To define the scaling dimension of the field, it is convenient to use the rescaled field $\boldsymbol{\pi}_r = g^{-1/2}\boldsymbol{\pi}$. Since $S[\boldsymbol{\pi}_r]=\frac{1}{2}\int d^dr\left[(\boldsymbol{\nabla}\boldsymbol{\pi}_r)^2+h\boldsymbol{\pi}_r^2\right]$, $[\boldsymbol{\pi}_r]=d/2-1$.

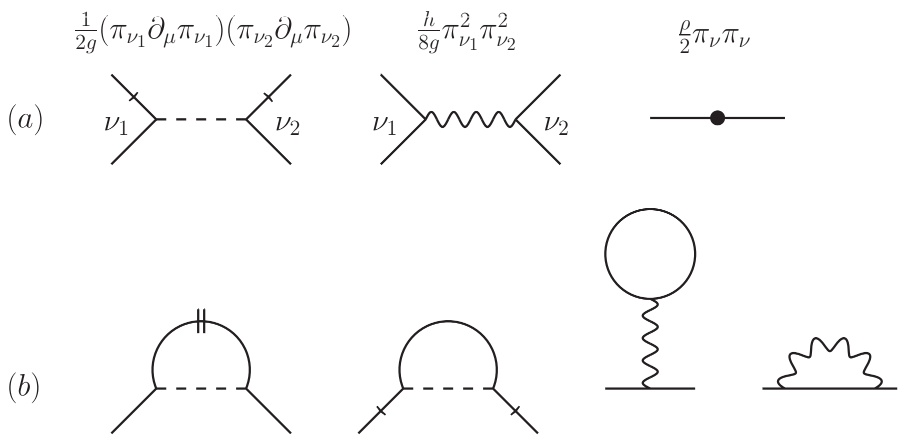


Figure 7.17: (a) Diagrammatic representation of the three terms of $S_1[\boldsymbol{\pi}]$. The slashed lines represent the spatial derivative ∂_μ. (b) One-loop diagrams obtained from the diagrams shown in (a).

7.7.2 RG approach

To implement the RG procedure, we proceed as in the case of the (linear) O(N) model (Sec. 7.6.2). We split the field $\boldsymbol{\pi}(\mathbf{r}) = \boldsymbol{\pi}^<(\mathbf{r}) + \boldsymbol{\pi}^>(\mathbf{r})$ into slow ($|\mathbf{p}| \leq \Lambda/s$) and fast ($\Lambda/s \leq |\mathbf{p}| \leq \Lambda$) modes and integrate out the latter. Since the renormalization procedure maintains the O(N) symmetry for $h = 0$, the action of the slow modes must be of the form $S[\mathbf{n}] = \frac{1}{2g'}\int d^dr(\boldsymbol{\nabla}\mathbf{n})^2 - \frac{\mathbf{h}'}{g'}\int d^dr\,\mathbf{n}$ up to higher-order (irrelevant) terms in the derivative expansion. It therefore depends only on the two renormalized coupling constants g' and h' as well as on a field renormalization factor necessary to keep the normalization $\mathbf{n}^2 = 1$. This implies that the vertices in the action (7.275) are not independent. For instance, the ratio of the coefficients of $(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2$ and $(\boldsymbol{\nabla}\boldsymbol{\pi})^2$ is one. If this ratio were different, the action would not have O(N) symmetry. Thus we need only consider RG equations for the two independent coupling constants g and h/g.

One-loop RG equations

To obtain the RG equations to lowest order, it is sufficient to consider the $\mathcal{O}(g)$ correction to S_0,

$$S_1[\boldsymbol{\pi}] = \frac{1}{2g}\int d^dr\Big[(\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{\pi})^2 + \frac{h}{4}(\boldsymbol{\pi}^2)^2\Big] - \frac{\rho}{2}\int d^dr\,\boldsymbol{\pi}^2. \tag{7.282}$$

The three terms in S_1 are diagrammatically represented in figure 7.17a. Integrating out the fast modes, one finds

$$\begin{aligned} Z &= \int\mathcal{D}[\boldsymbol{\pi}^<]\,e^{-S_0[\boldsymbol{\pi}^<]}\int\mathcal{D}[\boldsymbol{\pi}^>]\,e^{-S_0[\boldsymbol{\pi}^>]-S_1[\boldsymbol{\pi}^<+\boldsymbol{\pi}^>]+\mathcal{O}(g^2)} \\ &= \int\mathcal{D}[\boldsymbol{\pi}^<]\,e^{-S_0[\boldsymbol{\pi}^<]-\langle S_1[\boldsymbol{\pi}^<+\boldsymbol{\pi}^>]\rangle_{0,>}+\mathcal{O}(g^2)}, \end{aligned} \tag{7.283}$$

where

$$\begin{aligned}\langle S_1[\boldsymbol{\pi}^< + \boldsymbol{\pi}^>]\rangle_{0,>} &= \frac{1}{2gV} \sum_{\mathbf{p},\nu} \pi_\nu^<(-\mathbf{p})\pi_\nu^<(\mathbf{p}) {\sum_{\mathbf{p}'}}' \left[\mathbf{p}^2 + \mathbf{p}'^2\right] G_0(\mathbf{p}') \\ &\quad + \frac{h}{4gV} \sum_{\mathbf{p},\nu} [(N-1)+2] \pi_\nu^<(-\mathbf{p})\pi_\nu^<(\mathbf{p}) {\sum_{\mathbf{p}'}}' G_0(\mathbf{p}') \\ &\quad - \frac{1}{2}\rho^> \sum_{\mathbf{p},\nu} \pi_\nu^<(-\mathbf{p})\pi_\nu^<(\mathbf{p}). \end{aligned} \tag{7.284}$$

The notation ${\sum}'_{\mathbf{p}'}$ means the sum over $\mathbf{p}'$ is restricted to the fast modes $\Lambda/s \leq |\mathbf{p}'| \leq \Lambda$. The first two terms are represented by the diagrams of figure 7.17b. The last term in (7.284) is obtained from $\frac{\rho}{2} \int d^d r\, (\boldsymbol{\pi}^<)^2$ by writing $\rho = \rho^< + \rho^>$ with

$$\rho^> = \frac{1}{V} {\sum_{\mathbf{p}'}}' \equiv \fint_{\mathbf{p}'} \qquad (V \to \infty) \tag{7.285}$$

and including the contribution $-\frac{1}{2}\rho^> \int d^d r\, (\boldsymbol{\pi}^<)^2$ in the action of the slow modes. We therefore obtain the following action for the slow modes,

$$S[\boldsymbol{\pi}^<] = \frac{1}{2g}(1+dI) \int d^d r (\boldsymbol{\nabla}\boldsymbol{\pi}^<)^2 + \frac{h}{2g}\left(1 + \frac{N-1}{2} dI\right) \int d^d r (\boldsymbol{\pi}^<)^2, \tag{7.286}$$

where

$$dI = \fint_{\mathbf{p}} G_0(\mathbf{p}) \tag{7.287}$$

and we have used

$$\fint_{\mathbf{p}} \mathbf{p}^2 G_0(\mathbf{p}) = g\rho^> - h dI. \tag{7.288}$$

Rescaling momenta and field, i.e. $\mathbf{r} \to s\mathbf{r}$ and $\boldsymbol{\pi}^< \to \lambda^{-1}\boldsymbol{\pi}^<$, we reproduce the original action $S_0[\boldsymbol{\pi}^<] + \mathcal{O}(g)$ but with renormalized coupling constants,

$$\begin{aligned} \frac{1}{g'} &= \frac{1}{g}(1+dI)\lambda^{-2}s^{d-2}, \\ \frac{h'}{g'} &= \frac{h}{g}\left(1 + \frac{N-1}{2} dI\right)\lambda^{-2}s^d. \end{aligned} \tag{7.289}$$

To obtain the field rescaling factor λ, one could compute the renormalized coefficient of $(\boldsymbol{\pi}^< \cdot \boldsymbol{\nabla}\boldsymbol{\pi}^<)^2$. Because of the O($N$) symmetry, this coefficient should be equal to the coefficient of $(\boldsymbol{\nabla}\boldsymbol{\pi}^<)^2$. Alternatively, one can notice that h/g scales trivially. The O(N) symmetry implies that the renormalization of h/g does not depend of the direction of the field $\mathbf{h}$. If one couples $\mathbf{h}$ to π_μ (rather than to σ), one obtains the following term in the action,

$$-\frac{h}{g} \int d^d r\, \pi_\mu(\mathbf{r}). \tag{7.290}$$

Since h/g couples to $\pi_\mu^<(\mathbf{p}=0)$, it is not affected by the integration of $\boldsymbol{\pi}^>$ and its renormalization is entirely due to the rescaling of momenta and field,

$$\frac{h'}{g'} = \frac{h}{g} s^d \lambda^{-1}. \tag{7.291}$$

Comparing (7.291) with (7.289), we deduce

$$\begin{aligned} \lambda &= 1 + \frac{N-1}{2} dI, \\ g' &= g \frac{\left(1+\frac{N-1}{2}dI\right)^2}{1+dI} s^{2-d}. \end{aligned} \tag{7.292}$$

Taking $s = e^{dl}$ ($dl \to 0$) and using

$$dI = gK_d\Lambda^{d-2}dl \tag{7.293}$$

for $h=0$, we can transform the RG equations (7.292) into a differential equation for the coupling constant $g(l)$. Introducing a dimensionless coupling constant $\tilde{g} = \Lambda^{2-d}g$, we finally obtain

$$\frac{d\tilde{g}}{dl} = -\epsilon\tilde{g} + (N-2)K_d\tilde{g}^2 + \mathcal{O}(\tilde{g}^3), \tag{7.294}$$

where $\epsilon = d-2$.

It is possible to deduce the (running) dimension $d_\pi(l)$ of the field from the rescaling factor λ [Eq. (7.292)]. Because the propagator $G_0(\mathbf{p})$ depends explicitly on the coupling constant g, one should however disentangle the contribution to λ coming from the running of $\tilde{g}$. Let us consider the rescaled fields $\boldsymbol{\pi}_r = g(l)^{-1/2}\boldsymbol{\pi}$ and $\boldsymbol{\pi}'_r = g(l+dl)^{-1/2}\boldsymbol{\pi}'$, where $\boldsymbol{\pi}' = \lambda\boldsymbol{\pi}$. The scaling dimension $d_\pi = [\boldsymbol{\pi}_r]$ can be deduced from the behavior of the $\boldsymbol{\pi}_r$ field in the renormalization process, i.e.

$$\boldsymbol{\pi}'_r = s^{d_\pi(l)dl}\boldsymbol{\pi}_r = \lambda\left(\frac{g(l)}{g(l+dl)}\right)^{1/2}\boldsymbol{\pi}_r \tag{7.295}$$

(see Sec. 7.5.1) with

$$\lambda = 1 + \frac{N-1}{2}\tilde{g}(l)K_d dl + \mathcal{O}(dl^2). \tag{7.296}$$

Equation (7.295) gives

$$d_\pi(l) = \frac{d-2}{2} + \frac{K_d}{2}\tilde{g}(l) \tag{7.297}$$

and in turn the (running) anomalous dimension

$$\eta(l) = K_d\tilde{g}(l). \tag{7.298}$$

Fixed points and critical exponents

As expected, $\tilde{g}$ is irrelevant at the Gaussian fixed point $\tilde{g}=0$ for $d>2$ and relevant for $d\le 2$ (Fig. 7.18). In the following, we assume $N>2$; the case $N=2$ will be discussed later.

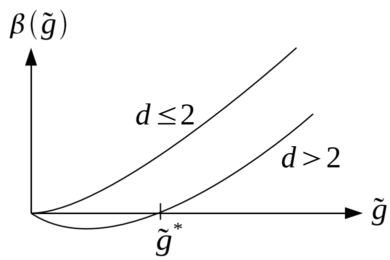


Figure 7.18: β function $\beta(\tilde{g}) = d\tilde{g}/dl$ of the non-linear sigma model for $d \leq 2$ and $d > 2$.

For $d \leq 2$, the growing of the coupling constant invalidates the perturbation approach and indicates that the system is disordered for any finite value of $\tilde{g}$ in agreement with the Mermin-Wagner theorem. Let us consider the two-dimensional case where $\tilde{g} = g$ is marginally relevant,

$$g(l) = \frac{g(0)}{1 - \frac{N-2}{2\pi} g(0) l}. \tag{7.299}$$

We can obtain the correlation length from

$$\xi(g(0)) = \xi(g(l)) e^l = \xi(g(l)) \exp\left\{ \int_{g(0)}^{g(l)} \frac{dg}{\beta(g)} \right\} \tag{7.300}$$

where $\beta(g) = dg/dl$ is β function. For $g(l) \lesssim 1$, we can use (7.294) to obtain

$$\xi(g(0)) = \xi(g(l)) \exp\left\{ \frac{2\pi}{N-2} \left(\frac{1}{g(0)} - \frac{1}{g(l)} \right) \right\} \tag{7.301}$$

For $g(l) \sim 1$, we expect $\xi(g(l)) \sim \Lambda^{-1}$, so that

$$\xi \sim \Lambda^{-1} \exp\left(\frac{2\pi}{(N-2)g} \right) = \Lambda^{-1} \exp\left(\frac{2\pi\rho_s^0}{(N-2)} \right) \tag{7.302}$$

for $g \equiv g(0) \ll 1$. The correlation length diverges exponentially with $1/g$.

For $d > 2$, there is a critical fixed point $\tilde{g}^*$ located away from the origin corresponding to a second-order phase transition between an ordered phase (described by the Gaussian fixed point $\tilde{g} = 0$) and a disordered phase (Fig. 7.18). Near the lower critical dimension $d_c^- = 2$, this fixed point and the associated critical exponents can be obtained within an ϵ expansion ($\epsilon = d - 2$). To leading order in ϵ, one finds[72]

$$\begin{aligned} \tilde{g}^* &= \frac{2\pi}{N-2}\epsilon + \mathcal{O}(\epsilon^2), \\ \eta &= \frac{\tilde{g}^*}{2\pi} + \mathcal{O}(\epsilon^2) = \frac{\epsilon}{N-2} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{7.303}$$

[72]Note that the one-loop RG equations (giving $\beta(\tilde{g})$ to $\mathcal{O}(\tilde{g}^2)$) are sufficient to obtain the fixed-point value $\tilde{g}^*$ and the anomalous dimension η to $\mathcal{O}(\epsilon)$.

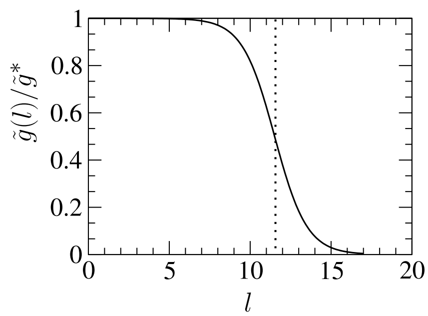


Figure 7.19: Solution of the RG equation (7.294) for $\tilde{g}(0)$ smaller but very close to the critical point $\tilde{g}^*$ ($d = 3$ and $N = 3$). The dotted vertical line indicates the Josephson scale $l_J = \ln(\xi_J \Lambda)$ separating the critical regime $l \ll l_J$ from the Goldstone regime $l \gg l_J$.

We can determine a characteristic length ξ from (7.300). If $\tilde{g}(0)$ is close to $\tilde{g}^*$, we linearize the β function,

$$\frac{d\tilde{g}}{dl} = (\tilde{g} - \tilde{g}^*)\beta'(\tilde{g}^*) + \mathcal{O}\left((\tilde{g} - \tilde{g}^*)^2\right), \tag{7.304}$$

to obtain

$$\xi = \xi(l) \left| \frac{\tilde{g}(l) - \tilde{g}^*}{\tilde{g}(0) - \tilde{g}^*} \right|^{1/\beta'(\tilde{g}^*)}, \tag{7.305}$$

which holds provided that $|\tilde{g}(l) - \tilde{g}^*| \lesssim 1$ (thus allowing the linearized form (7.304) to be used). To eliminate the dependence on l in (7.305), we note that we expect $\xi(l) \sim \Lambda^{-1}$ for $\tilde{g}(l) \sim 2\tilde{g}^*$ (the factor 2 is somewhat arbitrary here) in the disordered phase, and $\xi(l) \sim \Lambda^{-1}$ for $\tilde{g}(l) \sim \tilde{g}^*/2$ in the ordered phase. We conclude that

$$\xi \sim |\tilde{g} - \tilde{g}^*|^{-\nu} \tag{7.306}$$

($\tilde{g} \equiv \tilde{g}(0)$) diverges at the transition with an exponent

$$\nu = \frac{1}{\beta'(\tilde{g}^*)} = \frac{1}{\epsilon}. \tag{7.307}$$

Josephson length and stiffness

While the characteristic length ξ is naturally identified to the correlation length in the disordered phase, its physical meaning is less clear in the ordered phase. Let us consider the coupling constant $\tilde{g}(l)$ in the ordered phase. When $\tilde{g}(0)$ is close to $\tilde{g}^*$, one can identify a critical regime $l \ll l_J$ where $\tilde{g}(l)$ is very close to the critical value $\tilde{g}^*$ and $\eta_l \simeq \eta$ (Fig. 7.19). For $l \gg l_J$, $\tilde{g}(l)$ goes to zero and the long-distance physics is governed by the Gaussian fixed point $\tilde{g} = 0$. Thus the Josephson length $\xi_J \simeq \Lambda^{-1} e^{l_J}$ separates a critical regime and a "Goldstone" regime where the Goldstone modes are effectively non-interacting.

Since the stiffness has scaling dimension $[\rho_s] = d - 2$ (Sec. 7.4.1), the renormalized stiffness satisfies

$$\rho_s(\tilde{g}(l)) = \rho_s(\tilde{g}(0)) e^{(d-2)l}. \tag{7.308}$$

In the low-temperature phase, the coupling constant $\tilde{g}(l)$ goes to zero for $l \to \infty$ and we can read off the renormalized stiffness directly from the renormalized action,

$$\rho_s(\tilde{g}(l)) \simeq \rho_s^0(\tilde{g}(l)) = \frac{\Lambda^{d-2}}{\tilde{g}(l)} \quad (l \to \infty), \tag{7.309}$$

so that

$$\rho_s \equiv \rho_s(\tilde{g}(0)) = \Lambda^{d-2} \lim_{l\to\infty} \frac{e^{-(d-2)l}}{\tilde{g}(l)}. \tag{7.310}$$

Using the solution

$$\tilde{g}(l) = \frac{\tilde{g}^*}{1 - e^{(d-2)l}\left(1 - \frac{\tilde{g}^*}{\tilde{g}(0)}\right)} \tag{7.311}$$

of the RG equation (7.294), we finally obtain

$$\rho_s = \rho_s^0 \left(1 - \frac{\tilde{g}(0)}{\tilde{g}^*}\right). \tag{7.312}$$

At the transition ($\tilde{g}(0) \to \tilde{g}^*$), the stiffness vanishes with an exponent $\nu(d-2) = 1$.

In the Goldstone regime, one can also compute the connected propagator of the longitudinal field σ using $\sigma \simeq 1 - \boldsymbol{\pi}^2/2$ for small transverse fluctuations,

$$\begin{aligned} \langle\sigma(\mathbf{r})\sigma(0)\rangle_c &\simeq \frac{1}{4}\langle\boldsymbol{\pi}(\mathbf{r})^2\boldsymbol{\pi}(0)^2\rangle_c \simeq \frac{1}{4}\sum_{\nu,\nu'}\langle\pi_\nu^2(\mathbf{r})\pi_{\nu'}^2(0)\rangle_c \simeq \frac{N-1}{2}\langle\pi_\nu(\mathbf{r})\pi_\nu(0)\rangle^2 \\ &\sim \frac{1}{|\mathbf{r}|^{2d-4}}, \end{aligned} \tag{7.313}$$

where $\langle\pi_\nu(\mathbf{r})\pi_\nu(0)\rangle \sim 1/|\mathbf{r}|^{d-2}$ is the Goldstone modes propagator in the limit $g \to 0$ obtained from the Fourier transform of $1/\mathbf{p}^2$. The last result in (7.313) is obtained using Wick's theorem (which holds at the Gaussian fixed point). We therefore deduce that the longitudinal propagator

$$\langle\sigma(\mathbf{p})\sigma(-\mathbf{p})\rangle_c \sim \left\{ \begin{array}{ll} \ln|\mathbf{p}| & \text{for} \quad d=4, \\ \dfrac{1}{|\mathbf{p}|^{4-d}} & \text{for} \quad d<4, \end{array} \right. \tag{7.314}$$

is singular for $\mathbf{p} \to 0$ below four dimensions. (Equations (7.314) hold for $|\mathbf{p}| \ll \xi_J^{-1}$.) This result should be contrasted with the predictions of the Gaussian approximation (which neglects the coupling between transverse and longitudinal fluctuations) to the $(\boldsymbol{\varphi}^2)^2$ theory, according to which the longitudinal fluctuation mode is gapped and the longitudinal propagator finite in the limit $\mathbf{p} \to 0$ [Eq. (7.82)]. The longitudinal propagator singularity $\sim 1/|\mathbf{p}|^{4-d}$ is weaker than that ($\sim 1/|\mathbf{p}|^2$) of the transverse propagator for $d > d_c^-$. Both singularities would coincide at the lower critical dimension $d_c^- = 2$ if long-range order were not suppressed by fluctuations thus making the propagator of the $\mathbf{n}$ field gapped and O(N) symmetric.

Non-linear sigma model *vs* $(\boldsymbol{\varphi}^2)^2$ theory

The $\epsilon = d-2$ expansion of the non-linear sigma model near the lower critical dimension bears some similarities with the $\epsilon = 4-d$ expansion of the (linear) O(N) model near the upper critical dimension (section 7.6.2). In both approaches, we find a second-order phase transition and are able to compute the critical exponents to order ϵ using a perturbative (one-loop) RG. In the linear O(N) model near criticality, we found two characteristic lengths, the correlation length (or the Josephson length in the ordered phase) ξ and the Ginzburg length ξ_G. These two characteristic lengths, which determine the critical regime $\xi^{-1} \ll |\mathbf{p}| \ll \xi_G^{-1}$ in momentum space, are related to the two (bare) parameters, r_0 and u_0, of the model. By contrast, there is only one coupling constant in the non-linear sigma model and therefore one characteristic length, the correlation length (or the Josephson length) ξ, the critical regime being defined by $\xi^{-1} \ll |\,p| \leq \Lambda$.

Since the $(\boldsymbol{\varphi}^2)^2$ theory and the non-linear sigma model derive from the same spin model $H = J\sum_{\langle\mathbf{r},\mathbf{r}'\rangle} \mathbf{S}_\mathbf{r}\cdot\mathbf{S}_{\mathbf{r}'}$, they should belong to the same universality class. There are two limits where this can be shown explicitly. i) In the large N limit: to all orders of the $1/N$ expansion, the correlation functions in the critical regime have the same asymptotic behavior in both models (Appendix 7.B). ii) Near two dimensions: when $d \to d_c^-$, amplitude fluctuations play no role in the critical behavior of the $(\boldsymbol{\varphi}^2)^2$ theory which then reduces to that of the non-linear sigma model discussed in this section (chapter 8).

However, in contrast to the $(\boldsymbol{\varphi}^2)^2$ theory, the non-linear sigma model does not yield estimates of the critical exponents in three dimensions, the $\epsilon = d-2$ expansion being not Borel summable.

The O(2) non-linear sigma model

If $N = 2$, the one-loop correction to the β function vanishes and $\beta(\tilde g) = d\tilde g/dl$ reduces to the purely dimensional term $-\epsilon\tilde g$ (coming from the rescaling of momenta and fields) [Eq. (7.294)]. With the parametrization $\mathbf{n} = (\cos\theta, \sin\theta)$, one can write the non-linear sigma model as a free field theory,

$$S[\theta] = \frac{1}{2g}\int d^dr (\boldsymbol{\nabla}\theta)^2. \tag{7.315}$$

The result $\beta(\tilde g) = -\epsilon\tilde g$ is therefore exact (to all orders in the loop expansion) for the O(2) model.

For $d > 2$, the continuous symmetry remains broken for any value of g (the fixed point $\tilde g^* = 2\pi\epsilon/(N-2) \to \infty$ for $N \to 2^+$), with a mean value of the field given by $|\langle\mathbf{n}(\mathbf{r})\rangle| = \langle e^{i\theta(\mathbf{r})}\rangle$, i.e.

$$|\langle\mathbf{n}(\mathbf{r})\rangle| = \exp\left(-\frac{1}{2}\langle\theta(\mathbf{r})^2\rangle\right) = \exp\left(-\frac{1}{2}\int_\mathbf{p}\frac{g}{\mathbf{p}^2}\right). \tag{7.316}$$

Spin-wave excitations alone are not able to disorder the system. There is however no doubt that the original lattice model (7.268) (XY model) is disordered at sufficiently high temperature (i.e. sufficiently large g).[73] This apparent paradox can be explained

[73]See, e.g. , the argument given in Sec. 7.8.

by the observation that the non-linear sigma model ignores the fact that θ is a cyclic variable, which can be justified only at small g (up to exponentially small corrections in $1/g$).

In two dimensions, the β function of the O(2) non-linear sigma model vanishes identically. Mermin-Wagner theorem forbids long-range order. However, the fact that the expression $g^* = 2\pi\epsilon/(N-2)$ becomes undetermined when $\epsilon \to 0$ and $N \to 2$ suggests that this case might be special (see Sec. 7.8).

7.7.3 Low-temperature limit of the $(\boldsymbol{\varphi}^2)^2$ theory

In section 7.3, we have studied the $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry in the Gaussian approximation. This approximation breaks down in the vicinity of the phase transition because of critical fluctuations. We shall see below that the Gaussian approximation breaks down in the whole low-temperature phase for $N \geq 2$ due to the presence of Goldstone modes. It is however possible to circumvent the difficulties of the perturbation theory by considering the "good" hydrodynamic variables, namely the amplitude and the direction of the N-component field $\boldsymbol{\varphi}$. While amplitude fluctuations are gapped, the effective theory describing the low-energy direction fluctuations is a non-linear sigma model.

Breakdown of perturbation theory

Within the mean-field (or saddle-point) approximation, the order parameter $\boldsymbol{\varphi}_0 = \langle \boldsymbol{\varphi}(\mathbf{r}) \rangle$ has an amplitude $\varphi_0 = (-6r_0/u_0)^{1/2}$ in the low-temperature phase ($r_0 < 0$). By expanding the action to quadratic order about the mean-field solution, one finds the (zero-loop) self-energy

$$\Sigma^{(0)}_{ii}(\mathbf{p}) = \begin{cases} -3r_0 & \text{if} \quad i = 1, \\ -r_0 & \text{if} \quad i \neq 1, \end{cases} \tag{7.317}$$

and the longitudinal and transverse propagators

$$\begin{aligned} G^{(0)}_{\parallel}(\mathbf{p}) &= G^{(0)}_{11}(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + 2|r_0|}, \\ G^{(0)}_{\perp}(\mathbf{p}) &= G^{(0)}_{22}(\mathbf{p}) = \frac{1}{\mathbf{p}^2} \end{aligned} \tag{7.318}$$

(see Sec. 7.3.1). We assume the order parameter $\boldsymbol{\varphi}_0$ to be parallel to the direction $(1, 0, \cdots, 0)$. In agreement with Goldstone's theorem, the transverse propagator is gapless whereas the longitudinal susceptibility $G_{\parallel}(\mathbf{p} = 0) = 1/|2r_0|$ is finite.

Let us now consider the one-loop correction $\Sigma^{(1)}$ to the self-energy shown in figure 7.20. While the first diagram is finite, the second one gives a diverging contribution to Σ_{11} in the infrared limit $\mathbf{p} \to 0$ when $d \leq 4$. The divergence arises when both internal lines correspond to transverse fluctuations (which is possible only for Σ_{11}). Retaining only the divergent contribution, we obtain

$$\Sigma^{(1)}_{11}(\mathbf{p}) \simeq -\frac{N-1}{18} u_0^2 \varphi_0^2 \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 (\mathbf{p} + \mathbf{q})^2}. \tag{7.319}$$

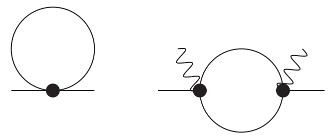


Figure 7.20: One-loop correction $\Sigma^{(1)}$ to the self-energy. The dots represent the bare interaction, the zigzag lines the order parameter φ_0, and the solid lines the connected propagator $G^{(0)}$.

The momentum integration in (7.319) gives[74]

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2} = \begin{cases} A_d|\mathbf{p}|^{d-4} & \text{if} \quad d<4, \\ A_4[1+\ln(\Lambda/|\mathbf{p}|)] & \text{if} \quad d=4, \end{cases} \tag{7.320}$$

for $|\mathbf{p}| \ll \Lambda$, where [42]

$$A_d = \begin{cases} -\dfrac{2^{1-d}\pi^{1-d/2}}{\sin(\pi d/2)} \dfrac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if} \quad d<4, \\ \dfrac{1}{8\pi^2} & \text{if} \quad d=4. \end{cases} \tag{7.321}$$

The one-loop correction (7.319) diverges for $\mathbf{p} \to 0$ and the perturbation expansion about the Gaussian approximation breaks down. By comparing the one-loop correction to the zero-loop result, i.e. $|\Sigma^{(1)}_{11}(\mathbf{p})| \sim \Sigma^{(0)}_{11}(\mathbf{p})$, one can extract the characteristic momentum scale

$$p_G \sim \begin{cases} [A_d(N-1)u_0]^{1/(4-d)} & \text{if} \quad d<4, \\ \Lambda \exp\left(\frac{-1}{A_4(N-1)u_0}\right) & \text{if} \quad d=4. \end{cases} \tag{7.322}$$

p_G^{-1} is nothing but the Ginzburg length ξ_G introduced in section 7.3.4. In the critical regime, p_G is the momentum scale associated to the breakdown of the Gaussian approximation and the onset of critical fluctuations (Sec. 7.6.2). Here we see that in the whole low-temperature phase, the Ginzburg momentum scale p_G signals the breakdown of the Gaussian approximation: while the Gaussian or perturbative approach remains valid for $|\mathbf{p}| \gg p_G$, the limit $|\mathbf{p}| \ll p_G$ cannot be studied perturbatively. We shall see below that, when the system is away from the critical regime, the breakdown of perturbation theory is due to the coupling between transverse and longitudinal fluctuations.

Amplitude-direction representation

To distinguish between amplitude and direction fluctuations in the ordered phase, we use the parametrization

$$\boldsymbol{\varphi}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{n}(\mathbf{r}), \tag{7.323}$$

with $\mathbf{n}(\mathbf{r})^2 = 1$. The partition function becomes

$$Z = \int \mathcal{D}[\rho,\mathbf{n}] \prod_{\mathbf{r}} \rho^{N-1}(\mathbf{r})\, e^{-S[\rho,\mathbf{n}]}, \tag{7.324}$$

[74] For simplicity, in the following we approximate $1+\ln(\Lambda/|\mathbf{p}|) \simeq \ln(\Lambda/|\mathbf{p}|)$ for $|\mathbf{p}| \ll \Lambda$.

where the action is given by

$$S[\rho, \mathbf{n}] = \int d^d r \left[\frac{1}{2}\rho^2 (\boldsymbol{\nabla}\mathbf{n})^2 + \frac{1}{2}(\boldsymbol{\nabla}\rho)^2 + \frac{r_0}{2}\rho^2 + \frac{u_0}{4!}\rho^4 \right]. \tag{7.325}$$

In the low-temperature phase ($r_0 < 0$), the mean-field theory yields a finite order parameter $\rho_0 = (-6r_0/u_0)^{1/2}$. To quadratic order in the fluctuations $\rho' = \rho - \rho_0$, we obtain the action

$$S[\rho', \mathbf{n}] = \int d^d r \left[\frac{\rho_0^2}{2}(\boldsymbol{\nabla}\mathbf{n})^2 + \frac{1}{2}(\boldsymbol{\nabla}\rho')^2 + |r_0|\rho'^2 \right] \tag{7.326}$$

and deduce that the amplitude fluctuations are gapped,

$$\langle \rho'(\mathbf{p})\rho'(-\mathbf{p})\rangle = \frac{1}{\mathbf{p}^2 + p_c^2}. \tag{7.327}$$

If we are interested only in momenta $|\mathbf{p}| \ll p_c = \sqrt{2|r_0|}$, to first approximation we can ignore the higher-order terms in ρ' that were neglected in (7.326), since they would only lead to a finite renormalization of the coefficients of the action $S[\rho', \mathbf{n}]$.

Equation (7.326) shows that in the "hydrodynamic" regime $|\mathbf{p}| \ll p_c$ direction fluctuations are described by a non-linear sigma model. It is convenient to use the parametrization $\mathbf{n} = (\sigma, \boldsymbol{\pi})$ introduced in section 7.7.1. Integrating over σ, one then obtains

$$S[\rho', \boldsymbol{\pi}] = \int d^d r \left[\frac{1}{2}(\boldsymbol{\nabla}\rho')^2 + |r_0|\rho'^2 + \frac{1}{2}\rho_0^2(\boldsymbol{\nabla}\boldsymbol{\pi})^2 \right] \tag{7.328}$$

for small transverse fluctuations $\boldsymbol{\pi}$ (i.e. to leading order in the coupling constant $1/\rho_0^2$ of the non-linear sigma model). From (7.328), we deduce the propagator of the $\boldsymbol{\pi}$ field,

$$\langle \pi_i(\mathbf{p})\pi_j(-\mathbf{p})\rangle = \frac{\delta_{i,j}}{\rho_0^2 \mathbf{p}^2}. \tag{7.329}$$

Again we note that the terms neglected in (7.328) would only lead to a finite renormalization of the (bare) stiffness ρ_0^2 of the non-linear sigma model at sufficiently low temperature. In fact, equation (7.328) gives an exact description of the low-energy behavior $|\mathbf{p}| \ll p_c$ if one replaces ρ_0^2 by the exact stiffness and $p_c^{-1} = (2|r_0|)^{-1/2}$ by the exact correlation length of the ρ' field.

We are now in a position to compute the longitudinal and transverse propagators using

$$\begin{aligned} \varphi_\parallel &= \rho\sigma = \rho\sqrt{1-\boldsymbol{\pi}^2} \simeq \rho_0 + \rho' - \frac{1}{2}\rho_0\boldsymbol{\pi}^2, \\ \boldsymbol{\varphi}_\perp &= \rho\boldsymbol{\pi} \simeq \rho_0\boldsymbol{\pi}. \end{aligned} \tag{7.330}$$

Since the long-distance physics is governed by transverse fluctuations, we have retained in (7.330) the leading contributions in $\boldsymbol{\pi}$. Making use of (7.329), one readily obtains

$$G_\perp(\mathbf{p}) \simeq \rho_0^2 \langle \pi_i(\mathbf{p})\pi_i(-\mathbf{p})\rangle = \frac{1}{\mathbf{p}^2}. \tag{7.331}$$

The longitudinal propagator is given by

$$\begin{aligned} G_\parallel(\mathbf{r}) &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{1}{4}\rho_0^2 \langle \boldsymbol{\pi}(\mathbf{r})^2 \boldsymbol{\pi}(0)^2\rangle_c \\ &= \langle \rho'(\mathbf{r})\rho'(0)\rangle + \frac{N-1}{2\rho_0^2} G_\perp(\mathbf{r})^2, \end{aligned} \tag{7.332}$$

where $\langle\cdots\rangle_c$ stands for the connected part of $\langle\cdots\rangle$. The second line is obtained using Wick's theorem. In Fourier space, this gives

$$G_\parallel(\mathbf{p}) = \frac{1}{\mathbf{p}^2+p_c^2} + \frac{N-1}{2\rho_0^2}\int_\mathbf{q} \frac{1}{\mathbf{q}^2(\mathbf{p}+\mathbf{q})^2}, \tag{7.333}$$

where the momentum integral is given by (7.320) for $|\mathbf{p}| \ll \Lambda$ and $d \leq 4$. By comparing the two terms in the rhs of (7.333), we recover the Ginzburg momentum scale (7.322). For $|\mathbf{p}| \gg p_G$ (Gaussian regime), the longitudinal propagator $G_\parallel(\mathbf{p}) \simeq 1/(\mathbf{p}^2+p_c^2)$ is dominated by amplitude fluctuations and we reproduce the result of the Gaussian approximation. On the other hand, for $|\mathbf{p}| \ll p_G$ (Goldstone regime), $G_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ is dominated by direction fluctuations and diverges for $\mathbf{p}\to 0$ (the divergence is logarithmic for $d=4$).

The divergence of the longitudinal propagator is a direct consequence of the coupling between longitudinal and transverse fluctuations. In the long-distance limit, amplitude fluctuations become frozen so that $|\boldsymbol{\varphi}| = \rho \simeq \rho_0$. This implies that the longitudinal and transverse components $\varphi_\parallel$ and $\boldsymbol{\varphi}_\perp$ cannot be considered independently as in the Gaussian approximation (Sec. 7.3) but satisfy the constraint $\varphi_\parallel^2 + \boldsymbol{\varphi}_\perp^2 \simeq \rho_0^2$. To leading order, $\varphi_\parallel \simeq \rho_0(1-\frac{\pi^2}{2})^{1/2}$ and $G_\parallel(\mathbf{r}) \sim G_\perp(\mathbf{r})^2$ [Eq. (7.332)], i.e. $G_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ for $d \leq 4$ (the divergence is logarithmic for $d=4$).

Equations (7.331) and (7.333) imply that in the limit $\mathbf{p}\to 0$ the self-energies are given by

$$\begin{aligned} \Sigma_{11}(\mathbf{p}) &= -r_0 + C_1|\mathbf{p}|^{4-d} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{22}(\mathbf{p}) &= -r_0 + \mathcal{O}(\mathbf{p}^2), \end{aligned} \tag{7.334}$$

for $d<4$, and

$$\begin{aligned} \Sigma_{11}(\mathbf{p}) &= -r_0 + \frac{C_1}{\ln(\Lambda/|\mathbf{p}|)} + \mathcal{O}(\mathbf{p}^2), \\ \Sigma_{22}(\mathbf{p}) &= -r_0 + \mathcal{O}(\mathbf{p}^2), \end{aligned} \tag{7.335}$$

for $d=4$. $\Sigma_{11}(\mathbf{p})$ contains a non-analytic term that is dominant for $\mathbf{p}\to 0$, in marked contrast with the prediction of the Gaussian approximation [Eq. (7.317)].

7.8 Berezinskii-Kosterlitz-Thouless phase transition

7.9 Functional renormalization group

In this section, we discuss a RG approach which, contrary to the study of sections 7.6 and 7.7, is not based on perturbation theory. This approach relies on an exact RG equation for the action which can be approximately solved within a derivative expansion. It deals with functions rather than a limited set of coupling constants and is intrinsically non-perturbative (no small parameter is assumed).

7.9.1 Wilson-Polchinski equation

We consider the partition function

$$Z = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_\Lambda^{-1}\cdot\boldsymbol{\varphi} - V_\Lambda[\boldsymbol{\varphi}]}, \tag{7.336}$$

where we use the notation

$$\begin{aligned} \boldsymbol{\varphi}\cdot C_\Lambda^{-1}\cdot\boldsymbol{\varphi} &= \int d^d r \int d^d r' \sum_i \varphi_i(\mathbf{r}) C_\Lambda^{-1}(\mathbf{r}-\mathbf{r}')\varphi_i(\mathbf{r}') \\ &= \sum_{\mathbf{p},i} \varphi_i(-\mathbf{p}) C_\Lambda^{-1}(\mathbf{p})\varphi_i(\mathbf{p}). \end{aligned} \tag{7.337}$$

Λ is an arbitrary momentum cutoff and

$$C_\Lambda(\mathbf{p}) = \frac{1}{\mathbf{p}^2} K\left(\frac{\mathbf{p}^2}{\Lambda^2}\right) \tag{7.338}$$

a cutoff function which ensures that only modes with momenta $|\mathbf{p}| \lesssim \Lambda$ are included in the partition function. In the perturbative RG (section 7.6), we took a sharp cutoff $K(x) = \Theta(1-x)$. In this section, we consider a smooth cutoff as shown in figure 7.21. $K(0) = 1$ and $K(x)$ decays rapidly (typically exponentially) for $x \gg 1$.[75] If we take Λ_0 as the initial value of the cutoff, then

$$S_{\Lambda_0}[\boldsymbol{\varphi}] = \frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda_0}^{-1}\cdot\boldsymbol{\varphi} + V_{\Lambda_0}[\boldsymbol{\varphi}] \tag{7.339}$$

is the (bare) microscopic action.

As Λ is reduced to Λ', fluctuation modes with momenta $\Lambda' \lesssim |\mathbf{p}| \lesssim \Lambda$ are integrated out, which changes the form of the action $S_\Lambda[\boldsymbol{\varphi}]$. Instead of considering a finite number of coupling constants as in section 7.6 (by expanding $S_\Lambda[\boldsymbol{\varphi}]$ in powers of $\boldsymbol{\varphi}$), we want to determine the full action $S_{\Lambda'}$. Let us show that $V_{\Lambda'}$ is related to V_Λ by

$$e^{-V_{\Lambda'}[\boldsymbol{\varphi}]} = \int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}' - V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}']}, \tag{7.340}$$

where

$$D_{\Lambda,\Lambda'} = C_\Lambda - C_{\Lambda'}. \tag{7.341}$$

Here and in the following we ignore any multiplicative constant which only affects the

[75] The sharp cutoff version of the Wilson-Polchinski equation is known as the Wegner-Houghton equation [35].

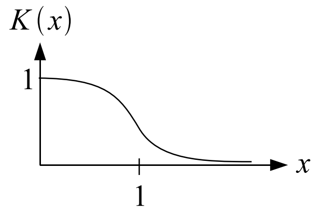


Figure 7.21: Cutoff function $K(x)$ [Eq. (7.338)].

field-independent part of $V_\Lambda[\boldsymbol{\varphi}]$. From (7.340), we deduce

$$\begin{aligned}\int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}-V_{\Lambda'}[\boldsymbol{\varphi}]} &= \int \mathcal{D}[\boldsymbol{\varphi},\boldsymbol{\varphi}']\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}-\frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}'-V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}']} \\ &= \int \mathcal{D}[\boldsymbol{\varphi},\boldsymbol{\varphi}']\, e^{-\frac{1}{2}(\boldsymbol{\varphi}-\boldsymbol{\varphi}')\cdot C_{\Lambda'}^{-1}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}')-\frac{1}{2}\boldsymbol{\varphi}'\cdot D_{\Lambda,\Lambda'}^{-1}\cdot\boldsymbol{\varphi}'-V_\Lambda[\boldsymbol{\varphi}]}. \end{aligned} \tag{7.342}$$

The integral over $\boldsymbol{\varphi}'$ gives

$$\int \mathcal{D}[\boldsymbol{\varphi}']\, e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot(C_{\Lambda'}^{-1}+D_{\Lambda,\Lambda'}^{-1})\cdot\boldsymbol{\varphi}'+\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}'} = e^{\frac{1}{2}\boldsymbol{\varphi}\cdot C_{\Lambda'}^{-1}(C_{\Lambda'}^{-1}+D_{\Lambda,\Lambda'}^{-1})^{-1}C_{\Lambda'}^{-1}\cdot\boldsymbol{\varphi}}, \tag{7.343}$$

with

$$C_{\Lambda'}^{-1}(C_{\Lambda'}^{-1}+D_{\Lambda,\Lambda'}^{-1})^{-1}C_{\Lambda'}^{-1} = C_{\Lambda'}^{-1} - C_\Lambda^{-1}, \tag{7.344}$$

and we recognize in (7.342) the partition function, which proves equation (7.340).

We can write equation (7.340) in differential form by choosing $\Lambda' = \Lambda + d\Lambda$. Then

$$D_{\Lambda,\Lambda'} = -\frac{dC_\Lambda}{d\Lambda}d\Lambda + \mathcal{O}(d\Lambda^2) \tag{7.345}$$

and the fields $\boldsymbol{\varphi}'$ contributing to the functional integral (7.340) are $\mathcal{O}(\sqrt{d\Lambda})$. In the limit $d\Lambda \to 0$, it is sufficient to expand $V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}']$ to second order in $\boldsymbol{\varphi}'$,

$$\begin{aligned} V_\Lambda[\boldsymbol{\varphi}+\boldsymbol{\varphi}'] = V_\Lambda[\boldsymbol{\varphi}] &+ \int d^dr \sum_i \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})}\varphi_i'(\mathbf{r}) \\ &+ \frac{1}{2}\int d^dr \int d^dr' \sum_{i,j} \frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_j(\mathbf{r}')}\varphi_i'(\mathbf{r})\varphi_j'(\mathbf{r}'). \end{aligned} \tag{7.346}$$

Integrating out φ' in (7.340) within a cumulant expansion gives

$$\begin{aligned} \exp\{-V_{\Lambda+d\Lambda}[\boldsymbol{\varphi}]\} = \exp\bigg\{ &- V_\Lambda[\boldsymbol{\varphi}] - \frac{1}{2}\int d^dr \int d^dr' \sum_i \bigg(\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_i(\mathbf{r}')} \\ &- \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r}')}\bigg) D_{\Lambda,\Lambda+d\Lambda}(\mathbf{r}-\mathbf{r}')\bigg\}, \end{aligned} \tag{7.347}$$

where we have used

$$\langle \varphi'_i(\mathbf{r})\varphi'_j(\mathbf{r}')\rangle = \delta_{i,j} D_{\Lambda,\Lambda+d\Lambda}(\mathbf{r}-\mathbf{r}') = -\delta_{i,j}\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}-\mathbf{r}')d\Lambda. \tag{7.348}$$

$C_\Lambda(\mathbf{r})$ denotes the Fourier transform of $C_\Lambda(\mathbf{p})$. We thus obtain the differential equation

$$\begin{aligned}\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= -\frac{1}{2}\int d^dr \int d^dr' \sum_i \frac{dC_\Lambda}{d\Lambda}(\mathbf{r}-\mathbf{r}')\left(\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})\delta\varphi_i(\mathbf{r}')} - \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r})}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\varphi_i(\mathbf{r}')}\right) \\ &\equiv -\frac{1}{2}\mathrm{Tr}\frac{dC_\Lambda}{d\Lambda}\left(\frac{\delta^{(2)}V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}} - \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right).\end{aligned} \tag{7.349}$$

With elementary algebra, equation (7.349) can be rewritten as a differential equation for the action (Wilson-Polchinski equation),

$$\begin{aligned}\frac{dS_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} =& -\frac{1}{2}\mathrm{Tr}\frac{dC_\Lambda}{d\Lambda}\left(\frac{\delta^{(2)}S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}} - \frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right) \\ & - \boldsymbol{\varphi}\cdot\frac{d\ln C_\Lambda}{d\Lambda}\cdot\frac{\delta S_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}.\end{aligned} \tag{7.350}$$

To complete the RG procedure, we still have to rescale momenta and fields. We postpone this step to section 7.9.2.

An alternative derivation of the Wilson-Polchinski equation. Let us consider the functional $W_\Lambda[\mathbf{h}]$ defined by

$$e^{W_\Lambda[\mathbf{h}]} = \int \mathcal{D}[\boldsymbol{\varphi}]\, e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi} - V_{\Lambda_0}[\boldsymbol{\varphi}] + \mathbf{h}\cdot\boldsymbol{\varphi}}. \tag{7.351}$$

Using

$$\begin{aligned}\frac{dW_\Lambda[\mathbf{h}]}{d\Lambda}e^{W_\Lambda[\mathbf{h}]} &= -\frac{1}{2}\int\mathcal{D}[\boldsymbol{\varphi}]\,\boldsymbol{\varphi}\cdot\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\cdot\boldsymbol{\varphi}\,e^{-\frac{1}{2}\boldsymbol{\varphi}D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi} - V_{\Lambda_0}[\boldsymbol{\varphi}]+\mathbf{h}\cdot\boldsymbol{\varphi}} \\ &= -\frac{1}{2}\mathrm{Tr}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\frac{\delta^{(2)}e^{W_\Lambda[\mathbf{h}]}}{\delta\mathbf{h}\delta\mathbf{h}},\end{aligned} \tag{7.352}$$

we obtain the flow equation

$$\frac{dW_\Lambda[\mathbf{h}]}{d\Lambda} = -\frac{1}{2}\mathrm{Tr}\frac{D^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)}W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}} + \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\right). \tag{7.353}$$

One can then relate W_Λ to V_Λ using (7.340),

$$\begin{aligned}e^{-V_\Lambda[\boldsymbol{\varphi}]} &= \int\mathcal{D}[\boldsymbol{\varphi}']\,e^{-\frac{1}{2}\boldsymbol{\varphi}'\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}' - V_{\Lambda_0}[\boldsymbol{\varphi}+\boldsymbol{\varphi}']} \\ &= \int\mathcal{D}[\boldsymbol{\varphi}']\,e^{-\frac{1}{2}(\boldsymbol{\varphi}'-\boldsymbol{\varphi})\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot(\boldsymbol{\varphi}'-\boldsymbol{\varphi}) - V_{\Lambda_0}[\boldsymbol{\varphi}']} \\ &= e^{-\frac{1}{2}\boldsymbol{\varphi}\cdot D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi} + W_\Lambda[D^{-1}_{\Lambda_0,\Lambda}\cdot\boldsymbol{\varphi}]}.\end{aligned} \tag{7.354}$$

We can now use (7.353) to derive a flow equation for V_Λ,

$$\begin{aligned}\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= \frac{1}{2}\boldsymbol{\varphi}\cdot\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\cdot\boldsymbol{\varphi} - \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}}\cdot\frac{d}{d\Lambda}(D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}) \\ &\quad + \frac{1}{2}\mathrm{Tr}\frac{D^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)}W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}} + \frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\right)_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}}. \end{aligned} \tag{7.355}$$

From the relation (7.354) between V_Λ and W_Λ, we obtain

$$\begin{aligned}\frac{\delta W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}} &= \boldsymbol{\varphi} - D_{\Lambda_0,\Lambda}\cdot\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}, \\ \frac{\delta^{(2)} W_\Lambda[\mathbf{h}]}{\delta\mathbf{h}\delta\mathbf{h}}\bigg|_{\mathbf{h}=D^{-1}_{\Lambda_0,\Lambda}\boldsymbol{\varphi}} &= D_{\Lambda_0,\Lambda} - D_{\Lambda_0,\Lambda}\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}}D_{\Lambda_0,\Lambda}, \end{aligned} \tag{7.356}$$

and in turn

$$\begin{aligned}\frac{dV_\Lambda[\boldsymbol{\varphi}]}{d\Lambda} &= -\frac{1}{2}\mathrm{Tr}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}D_{\Lambda_0,\Lambda}\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}}D_{\Lambda_0,\Lambda} \\ &\quad + \frac{1}{2}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\cdot D_{\Lambda_0,\Lambda}\frac{dD^{-1}_{\Lambda_0,\Lambda}}{d\Lambda}D_{\Lambda_0,\Lambda}\cdot\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}} \\ &= \frac{1}{2}\mathrm{Tr}\frac{dD_{\Lambda_0,\Lambda}}{d\Lambda}\left(\frac{\delta^{(2)} V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}\delta\boldsymbol{\varphi}} - \frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\frac{\delta V_\Lambda[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}}\right), \end{aligned} \tag{7.357}$$

which is nothing but equation (7.349) since $\frac{dD_{\Lambda_0,\Lambda}}{d\Lambda} = -\frac{dC_\Lambda}{d\Lambda}$.

7.9.2 Local potential approximation

The Wilson-Polchinski equation cannot be solved exactly. A possible approximation relies on a field expansion of the functional $V_\Lambda[\varphi]$,

$$V_\Lambda[\varphi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^dr_1\cdots d^dr_n\, V^{(n)}_\Lambda(\mathbf{r}_1,\cdots,\mathbf{r}_n)\varphi(\mathbf{r}_1)\cdots\varphi(\mathbf{r}_n) \tag{7.358}$$

(in this section we consider a scalar field), truncated to a given order. Such an expansion is reminiscent of the perturbative RG studied in section 7.6. To reproduce the $\mathcal{O}(\epsilon)$ results, it is sufficient to retrain $V^{(2)}_\Lambda$, $V^{(4)}_\Lambda$ and $V^{(6)}_\Lambda$ in the expansion (7.358).

The functional Wilson-Polchinski equation suggests a different kind of approximation, namely a derivative expansion. To leading order, the action reads

$$S_\Lambda[\varphi] = \frac{1}{2}\varphi\cdot C^{-1}_\Lambda\cdot\varphi + \int d^dr\, U_\Lambda\big(\varphi(\mathbf{r})\big). \tag{7.359}$$

The approximation (7.359) is called is the local potential approximation (LPA). In the LPA, the action is entirely determined by the function U_Λ, whose RG equation follows from the Wilson-Polchinski equation. Using

$$\begin{aligned}\frac{\delta V_\Lambda[\varphi]}{\delta\varphi(\mathbf{r})} &= \frac{\partial U_\Lambda\big(\varphi(\mathbf{r})\big)}{\partial\varphi(\mathbf{r})}, \\ \frac{\delta^{(2)} V_\Lambda[\varphi]}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')} &= \frac{\partial^2 U_\Lambda\big(\varphi(\mathbf{r})\big)}{\partial\varphi(\mathbf{r})^2}\delta(\mathbf{r}-\mathbf{r}'), \end{aligned} \tag{7.360}$$

one finds

$$\frac{d}{d\Lambda}\int d^dr\, U_\Lambda\big(\varphi(\mathbf{r})\big) = -\frac{1}{2}\int d^dr\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0)U''_\Lambda\big(\varphi(\mathbf{r})\big) + \frac{1}{2}\int d^drd^dr'\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}-\mathbf{r}')U'_\Lambda\big(\varphi(\mathbf{r})\big)U'_\Lambda\big(\varphi(\mathbf{r}')\big). \tag{7.361}$$

If we consider this equation for a uniform field $\varphi(\mathbf{r})=\varphi$, one obtains a differential equation for the function $U_\Lambda(\varphi)$,

$$\frac{d}{d\Lambda}U_\Lambda = -\frac{1}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0)U''_\Lambda + \frac{1}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{p}=0){U'_\Lambda}^2. \tag{7.362}$$

Using (7.338), one finally obtains

$$\frac{d}{d\Lambda}U_\Lambda = -\Lambda^{d-3}I_1U''_\Lambda + \Lambda^{-3}I_0{U'_\Lambda}^2, \tag{7.363}$$

where

$$\begin{aligned} I_0 &= \frac{\Lambda^3}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{p}=0) = -K'(0), \\ I_1 &= \frac{\Lambda^{3-d}}{2}\frac{dC_\Lambda}{d\Lambda}(\mathbf{r}=0) = -\frac{K_d}{2}\int_0^\infty dx\, x^{d/2-1}K'(x) \end{aligned} \tag{7.364}$$

are Λ-independent parameters depending on the cutoff function K. To obtain a fixed point solution of (7.363), one must first eliminate the explicit dependence on Λ. This corresponds to the momentum and field rescaling step in the perturbative RG of section 7.6. Since $[\varphi]=\frac{d-2}{2}$ and $[U(\varphi)]=d$,[76] one is lead to introduce the dimensionless variables

$$\tilde\varphi = \Lambda^{(2-d)/2}\varphi, \qquad \tilde U_\Lambda(\tilde\varphi) = \Lambda^{-d}U_\Lambda(\varphi), \tag{7.365}$$

which then yields the RG equation

$$\Lambda\frac{d}{d\Lambda}\tilde U_\Lambda = -d\tilde U_\Lambda + \left(\frac{d}{2}-1\right)\tilde\varphi\tilde U'_\Lambda - I_1\tilde U''_\Lambda + I_0{\tilde U'_\Lambda}{}^2. \tag{7.366}$$

The constant I_0 and I_1 can be eliminated by a trivial rescaling, $\tilde\varphi\to\sqrt{I_1}\tilde\varphi$ and $\tilde U_\Lambda\to(I_1/I_0)\tilde U_\Lambda$, leading to

$$\partial_l\tilde U_\Lambda = d\tilde U_\Lambda + \left(1-\frac{d}{2}\right)\tilde\varphi\tilde U'_\Lambda + \tilde U''_\Lambda - {\tilde U'_\Lambda}{}^2, \tag{7.367}$$

with $\Lambda=\Lambda_0e^{-l}$. Finally, to get rid of the field independent part of $\tilde U_\Lambda$, it is convenient to consider the function $f=\tilde U'_\Lambda$,

$$\dot f = f'' - 2ff' + \left(1+\frac{d}{2}\right)f + \left(1-\frac{d}{2}\right)xf', \tag{7.368}$$

where $\dot f=\partial_l f$, $f'=\partial_x f$ and we denote the dimensionless field $\tilde\varphi$ by x.

We are now in a position to look for the fixed point solutions $\dot f^*=0$ and the corresponding critical exponents. The RG equation (7.368) admits the trivial fixed point $f^*=0$ that we first discuss before considering nontrivial fixed points.

[76]The scaling dimension of φ field is obtained by noting that $[C_\Lambda(\mathbf{p})]=-2$.

The Gaussian fixed point in the LPA

The solution $f^*=0$ corresponds to a vanishing function $\tilde U^*_\Lambda$ and is therefore associated to the Gaussian fixed point.[77] To obtain the critical exponents, we linearize the flow equation about the solution $f^*=0$,

$$\dot f = f'' + \left(1+\frac{d}{2}\right) f + \left(1-\frac{d}{2}\right) x f'. \tag{7.369}$$

To solve this equation, we write

$$f_l(x) = \alpha h(\beta x) e^{\lambda l}, \tag{7.370}$$

with

$$\alpha = \frac{4}{d-2} \quad \text{and} \quad \beta = \left(\frac{d-2}{4}\right)^{1/2} \tag{7.371}$$

(we assume $d>2$), which leads to

$$h''(y) - 2yh'(y) + \frac{2}{d-2}(2+d-2\lambda)h(y) = 0 \tag{7.372}$$

with $y=\beta x$. This equation is known to have polynomial solutions,[78] given by the Hermite polynomials $h(y)=\hat H_{2k-1}(y) = 2^{k-1/2} H_{2k-1}(y)$ of degree $2k-1$, only for the set of discrete values of λ satisfying

$$2k-1 = \frac{d+2-2\lambda_k}{d-2} \quad \text{i.e.} \quad \lambda_k = d - k(d-2) \qquad (k=1,2,3,\cdots). \tag{7.373}$$

Even degree Hermite's polynomials are not allowed since the function $f(x)$ is odd ($\tilde U_\Lambda(\tilde\varphi)$ is even). Note that the λ_k's coincide with the scaling dimension $[v_{2k}]$ of the vertex $v_{2k}\int d^dr\,(\varphi)^{2k}$ about the Gaussian fixed point.

When $d>4$, all eigenvalues λ_k are negative except $\lambda_1=2$. The corresponding relevant eigenvector is given by $\hat H_1(y)\propto y$ (which corresponds to a φ^2 term in U_Λ). For $d<4$, the eigenvalue $\lambda_2 = 4-d$ becomes relevant[79] and we expect the phase transition to be described by a nontrivial fixed point with a single relevant field. For the φ^4 theory, we shall see below that this fixed point is the Wilson-Fisher fixed point found in the perturbative RG (Sec. 7.6).

Non-trivial fixed points in the LPA

Non-Gaussian fixed points cannot be found analytically and one must solve the fixed point equation

$$0 = f^{*\prime\prime} - 2f^*f^{*\prime} + \left(1+\frac{d}{2}\right) f^* + \left(1-\frac{d}{2}\right) x f^{*\prime} \tag{7.374}$$

[77] The condition $f^*=0$ implies $\tilde U^*_\Lambda=$ const. Eq. (7.367) with $\partial_l \tilde U_\Lambda = 0$ shows that the only solution is $\tilde U^*_\Lambda = 0$.

[78] It can be shown that non-polynomial solutions imply a continuum of eigenvalues and are therefore non physical [34].

[79] To see whether the field associated to the eigenvalue $\lambda_2 = 4-d$ is relevant or irrelevant in four dimensions, one must go beyond the linear approximation (7.369). We do not discuss the case $d=4$ here.

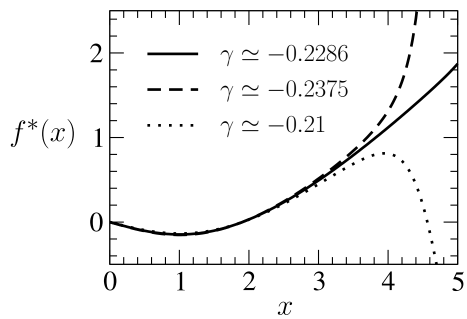


Figure 7.22: Fixed point solution $f^*(x)$ obtained by the shooting method for $d = 3$ vs $\gamma = f^{*\prime}(0)$. The actual (regular) solution is obtained for $\gamma \simeq -0.2286$.

numerically. Since equation (7.374) is a second-order differential equation, a solution is *a priori* parametrized by two arbitrary constants. But since $U_\Lambda(\varphi)$ is even, $f^*(0) = 0$, and there is only one free parameter, e.g. $\gamma = f^{*\prime}(0)$. However, most solutions are singular at some x_c,

$$f^*(x) \sim \frac{1}{x - x_c} \quad \text{for} \quad x \to x_c. \tag{7.375}$$

By requiring $f^*(x)$ to be defined for all x, the numerical solution of (7.374) shows that only a finite set of values of γ is obtained. In practice, the fixed point solution $f^*(x)$ is therefore determined by fine tuning $\gamma = f^{*\prime}(0)$ until a regular solution is obtained (shooting method). Note that one can easily determine the large x behavior of $f^*(x)$,

$$f^*(x) \simeq x + Cx^{(d-2)/(d+2)} \quad \text{for} \quad x \to \infty, \tag{7.376}$$

where C is a constant.

For $d > 4$ only the Gaussian fixed point $f^*(x) = 0$ is found. For $3 \leq d < 4$, a nontrivial fixed point (the Wilson-Fisher fixed point) is found for a nonzero value of γ ($\gamma \simeq -0.2286$ for $d = 3$). Figure 7.22 shows $f^*(x)$ obtained by the shooting method. Note that the vanishing of $f^*(x)$ (i.e. the minimum of $\tilde{U}(x)$) at some $x_0 > 0$ is not in contradiction with the system being critical. Going back to dimensionful variables, on finds that the potential $U^*(\varphi)$ has a minimum at $\varphi_0 \propto \Lambda^{(d-2)/2} x_0 \to 0$ for $\Lambda \to 0$.

A new nontrivial fixed point emanates from the Gaussian fixed point each time that one of the eigenvalues λ_k [Eq. (7.373)] vanishes, which occurs at the dimensional thresholds

$$d_k = \frac{2k}{k-1} \quad (k \geq 2). \tag{7.377}$$

Once a fixed point is identified, one can determine the critical exponents by linearizing the flow equation about f^*. Setting

$$f_l(x) = f^*(x) + e^{\lambda l} g(x), \tag{7.378}$$

one finds

$$\lambda g = g'' + \left(1 + \frac{d}{2}\right) g + \left(1 - \frac{d}{2}\right) x g' - 2f^* g' - 2f^{*\prime} g \tag{7.379}$$

to linear order in g. Again one expects solutions to be labeled by two parameters. However, one can choose $g(0) = 0$ (since $f_l(x)$ is odd) and $g'(0) = 1$ (arbitrary normalization of the eigenvectors). The solution is then unique for a given λ. Now it is easy to see that for large x,

$$g(x) \sim x^{(d-2-2\lambda)/(d+2)} \quad \text{or} \quad g(x) \sim e^{\frac{d+2}{4}x^2}. \tag{7.380}$$

Discarding the solutions with exponential asymptotic behavior,[78] one finds a regular solution only for a countable set of λ's. For $3 \leq d < 4$, the Wilson-Fisher fixed point possesses only one positive eigenvalue $\lambda_1 = 1/\nu$. The less negative eigenvalue λ_2 determines the correction-to-scaling exponent $\omega = -\lambda_2$ (Sec. 7.5.3). For $d = 3$, this gives

$$\begin{aligned} \nu &\simeq 0.6496, \\ \omega &\simeq 0.6557, \end{aligned} \tag{7.381}$$

with $\eta = 0$ in the LPA. These results improve over the $\mathcal{O}(\epsilon)$ perturbative results obtained in section 7.6. To compete with the best estimates obtained from the ϵ expansion (table 7.4), one must however go beyond the LPA.

It is also possible to determine the critical exponents by solving the flow equation (7.368) for a system near criticality.[80] For the φ^4 theory, the initial condition is $f_{l=0}(x) = \tilde r_0 x + \frac{\tilde u_0}{6}x^3$ and the critical point can be reached by varying $\tilde r_0$ (with $\tilde u_0$ fixed). When $\tilde r_0$ is near the critical value $\tilde r_{0c}$, the solution takes the form

$$f_l(x) \simeq f^*(x) + g_1(x)e^{-\omega l} + g_2(x)e^{l/\nu} \tag{7.382}$$

for large l. For $\tilde r_0 = \tilde r_{0c}$, g_2 vanishes and the approach to the fixed point is controlled by the rate $-\lambda_2 = \omega$. For $\tilde r_0$ slightly detuned from $\tilde r_{0c}$, $f_l(x)$ first moves closer to $f^*(x)$ but the flow eventually goes away from the fixed point with a rate given by $\lambda_1 = 1/\nu$.

Beyond the LPA

The LPA is the leading order of a derivative expansion. To next order,

$$V_\Lambda[\varphi] = \int d^dr \left\{ U_\Lambda\big(\varphi(\mathbf{r})\big) + \frac{1}{2}\left[Z_\Lambda\big(\varphi(\mathbf{r})\big) - 1\right]\big(\boldsymbol{\nabla}\varphi(\mathbf{r})\big)^2 \right\} \tag{7.383}$$

and

$$S_\Lambda[\varphi] = \int d^dr \left\{ U_\Lambda\big(\varphi(\mathbf{r})\big) + \frac{1}{2}Z_\Lambda\big(\varphi(\mathbf{r})\big)\big(\boldsymbol{\nabla}\varphi(\mathbf{r})\big)^2 \right\} + \mathcal{O}\big((\boldsymbol{\nabla}\varphi)^4\big), \tag{7.384}$$

where we have used (7.338) with $K(0) = 1$. The action is now determined by two functions, U_Λ and Z_Λ, whose RG equations can be obtained by inserting (7.383) into the Wilson-Polchinski equation (7.349). Contrary to the LPA, these equations are not independent of the cutoff function K [Eq. (7.338)], but depend on two K-dependent parameters A and B.[81] To eliminate the cutoff dependence, one could try to compute

[80] This method will be further discussed in chapter 8.

[81] More generally, the cutoff dependence can be absorbed into $2k$ parameters at the kth order of the derivative expansion [34].

the anomalous dimension as well as other critical exponents for various functions K and then use a principle of minimum sensitivity to choose the "best" function: if K depends on several parameters α_i, then the best value of the anomalous dimension must satisfy $d\eta/d\alpha_i = 0$, i.e. $d\eta/dA = d\eta/dB = 0$. Unfortunately, it appears that such a principle of minimum sensitivity cannot be used since η depends approximately linearly on B. This shortcoming seriously limits the use of the Wilson-Polchinski approach beyond the LPA. It should also be noted that the correlation functions of the "fast" modes (with momenta larger than Λ) cannot be computed from the action S_Λ. To obtain correlation functions with arbitrary momenta, it is necessary to introduce a spatially-varying external field $\mathbf{h}(\mathbf{r})$ in the action S_Λ and compute the resulting flow, which is technically difficult. In chapter 8, we shall see how the non-perturbative RG enables to circumvent the difficulties of the Wilson-Polchinski approach.

7.10 Quantum phase transitions

7.10.1 Zero-temperature phase transitions[82]

Classical *vs* quantum phase transitions

In classical statistical mechanics, thermodynamics and dynamics decouple. The reason is that the particles' positions and momenta are independent variables in the partition function of a classical system. One can integrate out momenta (which yields a non-singular additive contribution to the free energy) and write the partition function in terms of the position variables only. To study the dynamics of the system, the knowledge of the partition function is therefore not sufficient; one also needs an equation of a motion (kinetic equation). Near a phase transition, in addition to the diverging length scale set by ξ, there is a diverging time scale $\tau_c \sim \xi^z$. Its divergence, which is controlled by the dynamical critical exponent z, results in the phenomenon of critical slowing down, i.e. the very slow relaxation towards equilibrium. z is independent of other critical exponents (ν, γ, etc.).

In quantum systems, coordinate and momentum variables are non-commuting operators so that statics and dynamics are not independent. The existence of $\hbar$ implies that any energy scale E that enters thermodynamics necessarily determines a time scale $\hbar/E$. One can also note that the quantum partition function $Z = \mathrm{Tr}\, e^{-\beta\hat H}$ ($\beta = 1/k_BT$) involves the trace of the evolution operator $\hat U(t) = e^{-\frac{i}{\hbar}\hat H t}$ for an imaginary time $t = -i\hbar\beta$ (chapter 1).

It can nevertheless be argued that quantum mechanics is irrelevant for the study of critical fluctuations at a finite temperature phase transition.[83] Since the characteristic frequency $\omega_c = \tau_c^{-1}$ vanishes at the transition, sufficiently close to the transition temperature T_c, the condition $\hbar\omega_c \ll k_BT_c$ ensures that critical fluctuations behave classically. In this sense, any finite-temperature phase transition is classical.

This argument shows that zero-temperature phase transitions are qualitatively different and their critical fluctuations must be treated quantum mechanically. Quantum

[82] We restore $\hbar$ and k_B in this section.

[83] Of course, in most cases, quantum mechanics remains relevant at microscopic scales and for the very existence of the transition.

phase transitions occur by changing a non-thermal control parameter of the system such as pressure, magnetic field, chemical composition, etc. At zero temperature, the free energy

$$F(K) = - \lim_{\beta\to\infty} \frac{1}{\beta} \ln Z(K) = E_0(K) \tag{7.385}$$

reduces to the ground-state energy $E_0(K)$ of the system. A quantum phase transition is therefore due to a non-analyticity of $E_0(K)$ as a function of the parameters $K = \{K_i\}$ of the Hamiltonian. If the singularity arises from a level crossing, then the transition is first order. Although such a transition can occur in a finite size system, a genuine phase transition takes place in the thermodynamic limit. A continuous quantum phase transition always involves an infinite number of many-body states and can occur only in the thermodynamic limit. The corresponding critical point is referred to as a quantum critical point.

Continuous quantum phase transitions

In many cases the critical behavior at a quantum critical point can be studied from a low-energy effective action,[84]

$$S[\varphi] = \int_0^{\beta\hbar} d\tau \int d^d r\, \mathcal{L}(\varphi, \boldsymbol{\nabla}\varphi, \partial_\tau\varphi), \tag{7.386}$$

where the Lagrangian density $\mathcal{L}$ is a function of the order parameter field $\varphi_j(\mathbf{r},\tau)$ and its space and time derivatives (j denotes an internal index). $\mathcal{L}$ can sometimes be derived from a microscopic model; in many cases however it is simply deduced from symmetries and other general considerations. Equation (7.386) shows that a quantum system behaves like a $(d+1)$-dimensional system whose $(d+1)$th dimension (the imaginary-time dimension) has a "length" $L_\tau = \beta\hbar = \hbar/k_BT$.

The simplest example of a low-energy effective action is given by the quantum $(\boldsymbol{\varphi}^2)^2$ theory, defined by the action

$$S[\boldsymbol{\varphi}] = \int_0^{\beta\hbar} d\tau \int d^d r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{c^{-2}}{2}(\partial_\tau\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!}(\boldsymbol{\varphi}^2)^2 \right\}, \tag{7.387}$$

where $\boldsymbol{\varphi}(\mathbf{r},\tau)$ is an N-component real field. It is a straightforward generalization to the quantum case of the classical O(N) model [Eq. (7.27)]. Contrary to the classical case however, the coupling constants r_0 and u_0 are assumed to be temperature independent. Two other examples will be studied in the following: the quantum nonlinear sigma model and the quantum $O(2)$ model with Galilean symmetry describing interacting bosons (also studied in chapter 6).

A continuous (or second order) quantum phase transition occurring at $T = 0$ is characterized by two diverging scales, a correlation length ξ and a correlation time ξ_τ in the imaginary-time dimension. Let us call δ the non-thermal parameter which

[84] In some cases, it is not possible to define a local order parameter and the phase transition is not associated to a spontaneous symmetry breaking (topological phase transitions). In other cases, although one can identify a local order parameter, the presence of gapless degrees of freedom that are not related to the critical fluctuations prevents a (non-singular) derivative expansion of the action $S[\boldsymbol{\varphi}]$ as in (7.386).

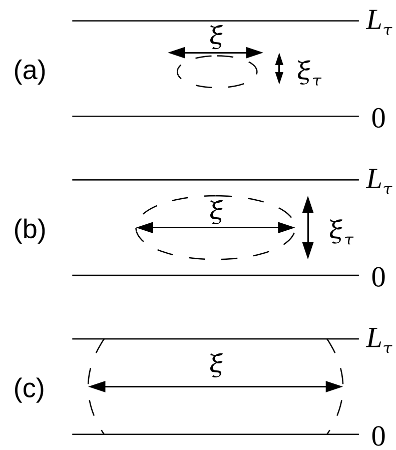


Figure 7.23: Growing correlation volume $\xi^d\xi_\tau$ as $|\delta| \to 0$. (a) $\xi_\tau \ll L_\tau$: the temperature has little effect; the behavior of the system is essentially quantum. (b) For $\xi_\tau \sim L_\tau$, there is a crossover from a quantum to a classical fluctuation regime. (c) ξ_τ is limited by L_τ and the system behaves classically (thermal fluctuations).

controls the phase transition. We assume the ordered phase to correspond to $\delta < 0$ and the disordered phase to $\delta > 0$. Both ξ and ξ_τ diverge,

$$\begin{aligned} \xi &\sim |\delta|^{-\nu}, \\ \xi_\tau &\sim \xi^z \sim |\delta|^{-\nu z}, \end{aligned} \tag{7.388}$$

when $\delta \to 0$. These asymptotic forms define the correlation length exponent ν and the dynamical critical exponent z. z is related to the anisotropy between space and time at the quantum critical point.

Quantum-classical crossover

Remarkably the only effect of finite temperature is to make the temporal dimension finite, $\tau \in [0, L_\tau]$ (the couplings K_i are independent of temperature), i.e. to induce a dimensional crossover from $d+1$ to d dimensions as we consider the system to lower and lower momentum or energy scales. This crossover drives the system away from the $T = 0$ critical point. However, if the system is sufficiently close to the quantum critical point (low temperature and small $|\delta|$), the latter controls the crossover; in the RG language, this is due to the long "time" spent by the RG trajectories in the vicinity of the quantum critical point. For this reason, the quantum critical point manifests itself not only in the $T = 0$ physical properties but also at low (but finite) temperatures.

Let us consider a system in the vicinity of the quantum critical point with $T = 0$ correlation lengths ξ and ξ_τ. As long as ξ_τ is smaller than L_τ, the system does not realize that the temperature is finite. The characteristic fluctuation frequencies are quantum: $\omega \gg k_BT$. As $|\delta|$ is reduced, ξ_τ grows and eventually becomes of the order of L_τ. At this point, the system knows that the temperature is finite and the actual correlation time ξ_τ becomes limited by the size L_τ in the temporal dimension.

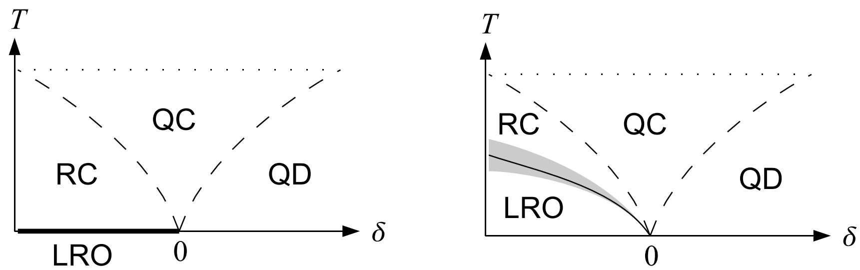


Figure 7.24: Phase diagram in the vicinity of a quantum critical point ($\delta = 0, T = 0$). The dashed lines are crossover lines $T \sim |\delta|^{\nu z}$ obtained from the criterion $L_\tau = \hbar\beta \sim \xi_\tau$ (with ξ_τ the $T = 0$ correlation time). The dotted lines mark the onset of the high-T region where the physics is not controlled by the quantum critical point anymore. RC: renormalized classical regime, QC: quantum critical regime, QD: quantum disordered regime. (Left) Long-range order occurs only at $T = 0$. (Right) Long-range order occurs at $T \geq 0$. The shaded area shows the region of classical fluctuations in the vicinity of the transition line $T_c(\delta)$.

The system then behaves as a d-dimensional classical system (with no fluctuations in the imaginary-time dimension) (Fig. 7.23). The $T = 0$ crossover length ξ for which $\xi_\tau \sim L_\tau$ defines the quantum-classical crossover length L_φ. Since $\xi_\tau \sim \xi^z$, $L_\varphi \sim L_\tau^{1/z} \sim 1/T^{1/z}$. L_φ corresponds to the maximum range of quantum fluctuations and has the meaning of a dephasing length. To this length scale is associated a characteristic time scale, the phase coherence time τ_φ. Loosely speaking, τ_φ is the time over which the wavefunction of the many-body state retains memory of its phase. Once dephasing has taken place and the phase coherence is lost, the behavior of the system becomes classical. τ_φ is infinite at zero temperature since the quantum system has perfect phase coherence in this limit. The phase coherence time plays a crucial role in the structure of the dynamic correlations at finite temperature. The computation of the latter is in general difficult, as it requires to analytically continue an imaginary-time correlation function to real time: $\chi^R(\omega) = \chi(i\omega_n \to \omega + i0^+)$ (chapter 3). There is no guarantee that an approximation scheme which works in imaginary time is also valid in real time. The computation of the real-time dynamics near a quantum critical point is beyond the scope of this chapter and we refer to Ref. [44] for a thorough discussion.

Phase diagram in the vicinity of a quantum critical point

Two cases need to be distinguished depending on whether long-range order can exist at finite temperatures. When d is equal to or below the lower critical dimension of the classical limit of $S[\varphi]$ (obtained by ignoring the time dependence of $\varphi_j(\mathbf{r}, \tau)$), then order can exist only at $T = 0$ and for $\delta < 0$. One can nevertheless distinguish three finite-temperature regimes depending on whether the behavior of the system is dominated by thermal or quantum fluctuations (Fig. 7.24). Let us first consider the case $\delta > 0$ where the ground state is disordered by quantum fluctuations. At sufficiently small temperatures, when $L_\tau > \xi_\tau$ (here ξ_τ denotes the $T = 0$ correlation time), the system does not realize that L_τ is finite and temperature has little effect. This regime, where the system is primarily disordered by quantum fluctuations, is called the quantum disordered regime. On the other hand, when $\xi_\tau > L_\tau$, the system is driven away from criticality by thermal fluctuations. The physics is then controlled primarily by the quantum critical point ($\delta = 0$ and $T = 0$) and its thermal excitations. This results in unconventional power-law temperature dependencies of physical observables. This regime is called the quantum critical regime. The case $\delta < 0$ can be analyzed similarly. At zero temperature, the characteristic scales ξ and ξ_τ are associated to long-range order. In particular, when the broken symmetry is continuous, $\xi \equiv \xi_J$ is the Josephson length beyond which fluctuations are dominated by the Goldstone modes (Goldstone regime). As ξ_τ becomes of the order of L_τ, there is a crossover from a $(d+1)$- to a d-dimensional fluctuation regime where long-range order is destroyed by thermal fluctuations. This regime is called the renormalized classical regime (because quantum fluctuations renormalize the parameters of the effective model describing thermal fluctuations).

In the second case, where long-range order is possible at finite temperature, there is a low-temperature ordered phase for $\delta < 0$ and the quantum critical point can be viewed as the end point of a line $T_c(\delta)$ of finite-temperature transitions (Fig. 7.24). Near $T_c(\delta)$, there is a regime of classical critical fluctuations. We shall see in the following how the generic phase diagram of figure 7.24 can be derived in particular models (Secs. 7.10.4 and 7.D).

7.10.2 Scaling at a quantum phase transition

A quantum system being similar to a $(d+1)$-dimensional classical system, we can derive the scaling laws near a quantum critical point along the lines of section 7.4. Since $\xi_\tau \sim \xi^z$, we deduce

$$[\tau] = -z. \tag{7.389}$$

It follows that the field has scaling dimension

$$d_\varphi = \frac{d+z-2+\eta}{2} \tag{7.390}$$

in the case of the quantum $(\boldsymbol{\varphi}^2)^2$ theory. This result is obtained by requiring the spatial derivative term of the action (7.387) to be dimensionless. The engineering dimension of the field is $d_\varphi^0 = \frac{d+z-2}{2}$ and the difference $d_\varphi - d_\varphi^0 = \eta/2$ defines the anomalous dimension η.

Scaling form of the free energy

The singular part of the $T = 0$ free energy density

$$f = -\lim_{\beta,V\to\infty} \frac{1}{\beta V} \ln Z \tag{7.391}$$

reads

$$f \sim \frac{1}{\xi^d \xi_\tau} g_\pm(h\xi^{d_h}), \tag{7.392}$$

where $\mathbf{h}$ is a uniform and static external field which couples linearly to the $\boldsymbol{\varphi}$, and the $+/-$ index refers to the disordered ($\delta > 0$) and ordered ($\delta < 0$) phases. This relation implies the homogeneity relation

$$f(\delta, h) = s^{-(d+z)} f(s^{1/\nu}\delta, s^{d_h} h). \tag{7.393}$$

One can define a critical exponent α (analog to the specific heat critical exponent of a thermal phase transition) from

$$\frac{\partial^2 f}{\partial \delta^2} \sim |\delta|^{-\alpha}. \tag{7.394}$$

From (7.393), one obtains

$$\alpha = 2 - \nu(d+z). \tag{7.395}$$

The magnitude of the order parameter $\mathbf{m} = \langle \boldsymbol{\varphi}(\mathbf{r}, \tau) \rangle$ in zero field is given by

$$m = -\left.\frac{\partial f}{\partial h}\right|_{h=0} \sim \frac{\xi^{d_h}}{\xi^d \xi_\tau} g'_\pm(0) \sim \xi^{-d-z+d_h}. \tag{7.396}$$

Since $m \sim \xi^{-d_\varphi} \sim (-\delta)^{\nu d_\varphi}$ for $\delta < 0$, we obtain

$$d_\varphi = d + z - d_h, \quad \text{i.e.} \quad d_h = \frac{d+z+2-\eta}{2} \tag{7.397}$$

and

$$\beta = \nu d_\varphi = \frac{\nu}{2}(d+z-2+\eta). \tag{7.398}$$

In a nonzero field,

$$m(\delta, h) \sim \xi^{d_h - d - z} g'_\pm(h\xi^{d_h}). \tag{7.399}$$

For $m(0,h)$ to be defined, we must have $g'_\pm(x) \sim x^\zeta$ for $x \to 0$, with $\zeta d_h = -d_h + d + z = d_\varphi$. Thus $m(0,h) \sim h^{1/\delta}$ with[85]

$$\delta = \frac{d_h}{d_\varphi} = \frac{d+z+2-\eta}{d+z-2+\eta}. \tag{7.400}$$

Scaling form of the propagator

Let us consider the retarded propagator $G^R(\mathbf{p}, \omega)$ of the φ field. Its singular part satisfies

$$G^R(\mathbf{p}, \omega; \delta) = \xi^{2-\eta} g_{1\pm}(|\mathbf{p}|\xi, \omega\xi_\tau) = \frac{1}{|\mathbf{p}|^{2-\eta}} g_{2\pm}(|\mathbf{p}|\xi, \omega\xi_\tau) \tag{7.401}$$

in zero external field (we consider a scalar field), where $g_{1\pm}$ and $g_{2\pm}$ are different scaling functions. Equation (7.401) follows from $[\varphi(\mathbf{p}, i\omega_n)] = -\frac{d+z}{2} + d_\varphi = -1 + \eta/2$. At the critical point ($\delta = 0$),

$$G^R(\mathbf{p}, \omega) = \frac{1}{|\mathbf{p}|^{2-\eta}} g_{3\pm}\left(\frac{|\mathbf{p}|^z}{\omega}\right). \tag{7.402}$$

For the singular part of the susceptibility $\chi = G^R(\mathbf{p} = 0, \omega = 0)$, we obtain $\chi \sim \xi^{2-\eta}$, so that

$$\gamma = \nu(2-\eta). \tag{7.403}$$

[85] In Eq. (7.400), δ stands for the critical exponent (not to be confused with the parameter δ that controls the quantum phase transition).

Scaling of the stiffness

The preceding discussion applies with no change to the disordered phase of the quantum $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry where $G_{ij} = \delta_{i,j} G$. As in the classical case, the ordered phase is characterized by long-range order, $\langle \boldsymbol{\varphi}(\mathbf{r},\tau)\rangle = m\mathbf{e}_1$, and a finite stiffness ρ_s. We follow section 7.3.2 and write the field as

$$\boldsymbol{\varphi}(\mathbf{r},\tau) = m[\mathbf{e}_1 + \delta\tilde{\varphi}_\parallel(\mathbf{r},\tau)\mathbf{e}_1 + \tilde{\boldsymbol{\varphi}}_\perp(\mathbf{r},\tau)], \tag{7.404}$$

where $\delta\tilde{\varphi}_\parallel$ denotes a longitudinal fluctuation and $\tilde{\boldsymbol{\varphi}}_\perp$ a transverse fluctuation. The low-energy action of transverse fluctuations reads

$$S[\tilde{\boldsymbol{\varphi}}_\perp] = \frac{\rho_s}{2} \int_0^\beta d\tau \int d^d r \, \left[(\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}_\perp)^2 + c^{-2}(\partial_\tau \boldsymbol{\varphi})^2 \right], \tag{7.405}$$

where c is the velocity of the Goldstone modes. Since both $S[\tilde{\boldsymbol{\varphi}}_\perp]$ and $\tilde{\boldsymbol{\varphi}}_\perp$ are dimensionless, we find[86]

$$[\rho_s] = d + z - 2. \tag{7.406}$$

At the phase transition ($\delta \to 0^-$), the stiffness must therefore vanish as $\rho_s \sim \xi^{-d-z+2} \sim (-\delta)^{\nu(d+z-2)}$.

Effect of finite temperature

Since the only effect of finite temperature is to make the temporal dimension finite ($0 \leq \tau \leq L_\tau$), we can use a finite-size scaling analysis. The temperature is an inverse time and has scaling dimension

$$[T] = z. \tag{7.407}$$

It is therefore a relevant variable at the zero-temperature quantum critical point.

The singular part of the free energy can be written as

$$f(\delta, h, T) = \frac{1}{\xi^d \xi_\tau} g_{1\pm}\left(h\xi^{d_h}, \xi_\tau/L_\tau\right), \tag{7.408}$$

where g_{1+} (g_{1-}) is a scaling function for $\delta > 0$ ($\delta < 0$). f satisfies the homogeneity relation

$$f(\delta, h, T) = s^{-(d+z)} f(s^{1/\nu}\delta, s^{d_h} h, s^z T). \tag{7.409}$$

With $s = |\delta|^{-\nu} \sim \xi$, we obtain

$$f(\delta, T) = \xi^{-(d+z)} g_{2\pm}(T\xi^z) = T^{(d+z)/z} g_{3\pm}\left(\frac{|\delta|}{T^{1/\nu z}}\right) \tag{7.410}$$

for $h = 0$. In the quantum critical regime ($|\delta| \ll T^{1/\nu z}$), this implies $f \sim T^{(d+z)/z}$ and a specific heat $C_V \sim T^{d/z}$.

A similar analysis can be made for the retarded propagator,

$$\begin{aligned} G^R(\mathbf{p}, \omega; \delta, T) &= \xi^{2-\eta} g_{1\pm}(|\mathbf{p}|\xi, \omega\xi^z, T\xi^z) \\ &= T^{-(2-\eta)/z} g_{2\pm}(|\mathbf{p}|T^{-1/z}, \omega/T, T\xi^z), \end{aligned} \tag{7.411}$$

[86] Alternatively, one can consider a time-independent twist of the order parameter, $\langle \boldsymbol{\varphi}(\mathbf{r},\tau)\rangle = m(1 - \cos\theta(\mathbf{r}))\mathbf{e}_1 + m\sin\theta(\mathbf{r})\mathbf{e}_2$, which gives rise to an increase in the energy of the system: $\delta E = \frac{\rho_s}{2}\int d^d r \, (\boldsymbol{\nabla}\theta)^2$. Since $[\delta E] = z$, we reproduce (7.406).

so that

$$G^R(\mathbf{p},\omega;\delta,T) = s^{2-\eta} G(s\mathbf{p}, s^z\omega; s^{1/\nu}\delta, s^z T). \tag{7.412}$$

In particular, for the susceptibility $\chi(\delta,T) = G^R(\mathbf{p}=\omega=0;\delta,T)$, we find

$$\chi(\delta,T) = s^{2-\eta}\chi(s^{1/\nu}\delta, s^z T) = |\delta|^{-\nu(2-\eta)}\chi(\pm 1, |\delta|^{-\nu z}T), \tag{7.413}$$

where the last result is obtained with $s = |\delta|^{-\nu}$. At $T=0$, the susceptibility diverges when $\delta \to 0$ with the exponent $\gamma = \nu(2-\eta)$. It can also diverge for $T>0$ and $\delta<0$ if $\chi(\pm 1, x)$ diverges for some value of x. In this case, there is a finite-temperature phase transition at a critical temperature

$$T_c(\delta) \sim (-\delta)^{\nu z}. \tag{7.414}$$

7.10.3 The quantum $(\boldsymbol{\varphi}^2)^2$ theory

We consider the action (7.387) with a cutoff Λ in Fourier space, $(\mathbf{p}^2 + \omega_n^2/c^2)^{1/2} \leq \Lambda$, which respects the Lorentz invariance of the zero-temperature theory.[87]

For $T=0$ ($\beta\to\infty$), the quantum $(\boldsymbol{\varphi}^2)^2$ theory is equivalent to the $(d+1)$-dimensional classical model.[88] We deduce that the lower and upper critical dimensions are $d_c^- = 1$ and $d_c^+ = 3$, respectively. For $d>1$, there is a quantum phase transition (obtained for instance by varying r_0 with u_0 fixed) between a disordered and an ordered phase. The transition is mean-field like for $d\geq 3$ (with logarithmic corrections for $d=3$).

For $T \geq c\Lambda$, only the classical part $\boldsymbol{\varphi}(\mathbf{r}, i\omega_{n=0})$ of the field contributes to the partition function and the quantum model becomes classical with action

$$S[\boldsymbol{\varphi}] = \beta\int d^d r \left\{ \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!}(\boldsymbol{\varphi}^2)^2 \right\}. \tag{7.415}$$

By an appropriate rescaling of the field, $\boldsymbol{\varphi} \to \sqrt{T}\boldsymbol{\varphi}$, one recovers the action of the d-dimensional classical $(\boldsymbol{\varphi}^2)^2$ theory studied previously with coupling constants r_0 and $u_0 T$.

Gaussian approximation

At the mean-field level, the O(N) symmetry is spontaneously broken when $r_0 < 0$, with an order parameter $\boldsymbol{\varphi}_0 = \langle \boldsymbol{\varphi}(\mathbf{r},\tau)\rangle$ satisfying

$$\boldsymbol{\varphi}_0^2 = -\frac{6r_0}{u_0}. \tag{7.416}$$

The fact that the phase diagram is independent of temperature is clearly an artifact of the mean-field approximation. By expanding the action to quadratic order

[87] At finite temperature, since the Matsubara frequencies $\omega_n = 2\pi T n$ are discrete, it is preferable to take a smooth cutoff rather than the hard one defined in the text. It is also possible (and sometimes technically advantageous) to choose a non Lorentz-invariant cutoff, for example a cutoff acting only on momenta (e.g. $|\mathbf{p}| \leq \Lambda$).

[88] We can eliminate the anisotropy between time and space by using the $(d+1)$-dimensional coordinate $\mathbf{x} = (\mathbf{r}, c\tau)$ and the rescaled field $\boldsymbol{\varphi}/\sqrt{c}$. This yields a $(d+1)$-dimensional classical O(N) model with coupling constants r_0 and cu_0.

in the fluctuations $\boldsymbol{\varphi} - \boldsymbol{\varphi}_0$ about the mean-field solution, we find the (connected) propagator[89]

$$\begin{aligned} G_{ij}(\mathbf{p}, i\omega_n) &= \langle \varphi_i(\mathbf{p}, i\omega_n)\varphi_j(-\mathbf{p}, -i\omega_n)\rangle - \langle \varphi_i(\mathbf{p}, i\omega_n)\rangle\langle\varphi_j(-\mathbf{p}, -i\omega_n)\rangle \\ &= \frac{\delta_{i,j}}{\mathbf{p}^2 + \omega_n^2/c^2 + r_0} \end{aligned} \tag{7.417}$$

in the disordered phase ($r_0 \geq 0$). The spectrum is determined by the poles of the retarded propagator $G^R_{ij}(\mathbf{p}, \omega) = G_{ij}(\mathbf{p}, i\omega_n \to \omega + i0^+)$. We thus find N gapped modes with dispersion $\omega = c(\mathbf{p}^2 + \xi^{-2})^{1/2}$ where the correlation length is given by $\xi = r_0^{-1/2}$ and the gap by $\Delta = c/\xi$. At the phase transition ($r_0 \to 0^+$), the correlation length diverges with the exponent $\nu = 1/2$, the anomalous dimension η vanishes, and the dynamical exponent z is equal to one. In the ordered phase ($r_0 \leq 0$), the longitudinal and transverse parts of the propagator are given by[90]

$$\begin{aligned} G_\parallel(\mathbf{p}, i\omega_n) &= \frac{1}{\mathbf{p}^2 + \omega_n^2/c^2 + 2|r_0|}, \\ G_\perp(\mathbf{p}, i\omega_n) &= \frac{1}{\mathbf{p}^2 + \omega_n^2/c^2}. \end{aligned} \tag{7.418}$$

The longitudinal mode is gapped whereas the $N-1$ transverse modes are gapless in agreement with Goldstone's theorem. The transverse propagator can be written in the form

$$G_\perp(\mathbf{p}, i\omega_n) = \frac{\varphi_0^2}{\rho_s(\mathbf{p}^2 + \omega_n^2/c^2)} \tag{7.419}$$

(see Sec. 3.6.3), where $\rho_s = \varphi_0^2$ is the stiffness in the Gaussian (or mean-field) approximation.

For the ordered phase to be stable, fluctuations must be finite. Stability against transverse fluctuations $\delta\boldsymbol{\varphi}_\perp(\mathbf{r}, \tau)$ can be verified by considering

$$\begin{aligned} \langle \delta\boldsymbol{\varphi}_\perp(\mathbf{r}, \tau)^2\rangle &= \frac{(N-1)}{\beta}\sum_{\omega_n}\int \frac{d^dp}{(2\pi)^d} G_\perp(\mathbf{p}, i\omega_n) \\ &= \frac{(N-1)}{\beta}\sum_{\omega_n}\int \frac{d^dp}{(2\pi)^d}\frac{c^2}{\omega_n^2 + c^2\mathbf{p}^2}. \end{aligned} \tag{7.420}$$

At zero temperature, the rhs can be written as a $(d+1)$-dimensional integral which is infrared divergent for $d \leq 1$ in agreement with the result $d_c^- = 1$. At finite temperature, the term with $\omega_{n=0} = 0$ is divergent when $d \leq 2$, which identifies $d_c^- = 2$ as the lower critical dimension in agreement with the Mermin-Wagner theorem (Secs. 3.6.3 and 7.3.3).

As in the classical case, the Gaussian approximation suffers from a number of drawbacks. In particular, the predicted critical behavior of the zero-temperature quantum phase transition is not correct below the upper critical dimension. Moreover, the Gaussian approximation breaks down at low energy in the whole ordered phase

[89] The calculation is similar to that of Sec. 7.3.1.

[90] Recall that in the ordered phase the propagator can be expressed as $G_{ij}(\mathbf{p}, i\omega_n) = \hat\varphi_{0,i}\hat\varphi_{0,j}G_\parallel(\mathbf{p}, i\omega_n) + (\delta_{i,j} - \hat\varphi_{0,i}\hat\varphi_{0,j})G_\perp(\mathbf{p}, i\omega_n)$ where $\hat{\boldsymbol{\varphi}}_0 = \boldsymbol{\varphi}_0/|\boldsymbol{\varphi}_0|$.

when $N \geq 2$. The coupling between transverse and longitudinal fluctuations cannot be ignored and gives rise to a divergence of the longitudinal correlation function. Because of the Lorentz invariance of the action (7.387) at zero temperature, the arguments of section 7.7.3 can be straightforwardly generalized and lead to[91]

$$G_\parallel(\mathbf{p}, i\omega) \sim \begin{cases} \dfrac{1}{(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}} & \text{if} \quad d < 3, \\ \ln\left(\dfrac{c\Lambda}{\sqrt{\omega^2 + c^2\mathbf{p}^2}}\right) & \text{if} \quad d = 3, \end{cases} \tag{7.421}$$

for $\mathbf{p}, \omega \to 0$, in contrast to the gapped behavior predicted by the Gaussian approximation. (Note that for $T \to 0$ the Matsubara frequency ω_n becomes a continuous variable ω.)

Renormalization-group approach

A possible way to go beyond the Gaussian approximation is to use the RG approach. Let us briefly discuss the zero-temperature quantum phase transition. Of course, one could directly use the analogy with the $(d+1)$-dimensional classical model and the results of section 7.6. It is however instructive to go through the RG procedure. We split the field $\boldsymbol{\varphi}(\mathbf{r}, \tau) = \boldsymbol{\varphi}_<(\mathbf{r}, \tau) + \boldsymbol{\varphi}_>(\mathbf{r}, \tau)$ into slow $((\mathbf{p}^2 + \omega^2/c^2)^{1/2} \leq \Lambda/s)$ and fast $(\Lambda/s \leq (\mathbf{p}^2 + \omega^2/c^2)^{1/2} \leq \Lambda)$ modes and integrate out the latter. To one-loop order, the action of the slow modes is determined by renormalized coupling constants,

$$\begin{aligned} r_0' &= r_0 + \frac{N+2}{6} u_0 \fint_p G_0(p), \\ u_0' &= u_0 - \frac{N+8}{6} u_0^2 \fint_p G_0^2(p), \end{aligned} \tag{7.422}$$

where $G_0(p) = (\mathbf{p}^2 + \omega^2/c^2 + r_0)^{-1}$ and $p = (\mathbf{p}, i\omega)$. We use the notation $\fint_p$ to indicate that the frequency-momentum integral is restricted to fast modes. Note that the Lorentz invariance of the theory ensures that the velocity is not renormalized at zero temperature. The next step is to rescale coordinate and time variables, $\mathbf{r} \to s\mathbf{r}$ and $\tau \to s^z\tau$, and fields, $\boldsymbol{\varphi}(\mathbf{r}, \tau) \to s^{d_\varphi}\boldsymbol{\varphi}(\mathbf{r}, \tau)$, where $d_\varphi = \frac{1}{2}(d + z - 2 + \eta)$ is the scaling dimension of the field. $z = 1$ because of Lorentz invariance and $\eta = 0$ at one-loop order. This yields the renormalized coupling constants

$$\begin{aligned} r_0' &= e^{2dl}\left[r_0 + \frac{N+2}{6} u_0 \fint_p G_0(p)\right], \\ u_0' &= e^{(3-d)dl}\left[u_0 - \frac{N+8}{6} u_0^2 \fint_p G_0^2(p)\right], \end{aligned} \tag{7.423}$$

where we have taken $s = e^{dl}$ $(dl \to 0)$. Using

$$\begin{aligned} \fint_p G_0(p) &= cK_{d+1}\frac{\Lambda^{d+1}}{\Lambda^2 + r_0} dl, \\ \fint_p G_0^2(p) &= cK_{d+1}\frac{\Lambda^{d+1}}{(\Lambda^2 + r_0)^2} dl, \end{aligned} \tag{7.424}$$

[91] Eq. (7.421) is obtained from the result of Sec. 7.7.3, $G_\parallel(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$, by replacing $\mathbf{p}^2$ by $\mathbf{p}^2 + \omega^2/c^2$ and d by $d+1$.

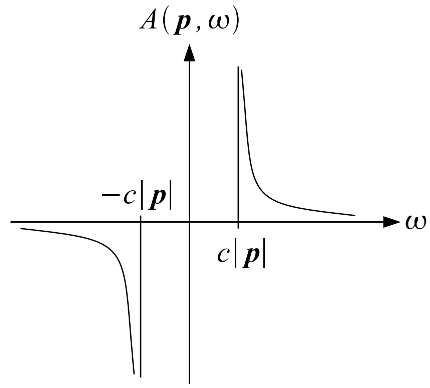


Figure 7.25: Spectral function $A(\mathbf{p},\omega) = \Im[G^R(\mathbf{p},\omega)]$ at a quantum critical point [Eq. (7.428)].

we obtain the RG equations

$$\begin{aligned} \frac{d\tilde{r}_0}{dl} &= 2\tilde{r}_0 + cK_{d+1}\frac{N+2}{6}\frac{\tilde{u}_0}{1+\tilde{r}_0}, \\ \frac{d\tilde{u}_0}{dl} &= (3-d)\tilde{r}_0 - cK_{d+1}\frac{N+8}{6}\frac{\tilde{u}_0^2}{(1+\tilde{r}_0)^2} \end{aligned} \tag{7.425}$$

for the dimensionless coupling constants $\tilde{r}_0 = \Lambda^{-2}r_0$ and $\tilde{u}_0 = \Lambda^{d-3}u_0$. As anticipated, we reproduce the one-loop RG equations for the coupling constants $(\tilde{r}_0, c\tilde{u}_0)$ of the $(d+1)$-dimensional classical model.[92]

To study the effect of a finite temperature, one could derive the one-loop RG equation at finite T. However, because the one-loop RG equations break down in the ordered phase for length scales beyond the correlation length (the Josephson length when $N \geq 2$) (Sec. 7.6), it is difficult to obtain a clear picture of the long-distance physics near the zero-temperature quantum critical point. It turns out to be more instructive to consider the quantum non-linear sigma model (Sec. 7.10.4).

Spectral function at the quantum critical point. For the quantum $(\boldsymbol{\varphi}^2)^2$ theory, Lorentz invariance implies that $z=1$ at the quantum critical point. The transition is in the universality class of the classical $(d+1)$-dimensional $(\boldsymbol{\varphi}^2)^2$ theory. At the critical point the propagator takes the form

$$G(\mathbf{p}, i\omega) \sim \frac{1}{(\omega^2 + c^2\mathbf{p}^2)^{1-\eta/2}}, \qquad G(\mathbf{r},\tau) \sim \frac{1}{(\mathbf{r}^2 + c^2\tau^2)^{(d+1)-2+\eta}}, \tag{7.426}$$

where c is the velocity of the critical fluctuations (equal to the velocity in (7.387) because of Lorentz invariance). The retarded propagator, obtained from the analytic continuation $i\omega \to \omega + i0^+$, reads

$$G^R(\mathbf{p},\omega) \sim \frac{1}{[(\omega+i0^+)^2 - c^2\mathbf{p}^2]^{1-\eta/2}}. \tag{7.427}$$

[92]See footnote 88 page 531.

With no anomalous dimension, $G^R(\mathbf{p},\omega)$ would have a pole at $\omega = \pm c|\mathbf{p}|$ which would manifest itself as a Dirac peak in the spectral function $A(\mathbf{p},\omega) = \Im[G^R(\mathbf{p},\omega)]$. Since η is nonzero, the pole is suppressed and replaced by a branch cut. This gives rises to a critical continuum of excitations,

$$A(\mathbf{p},\omega) \sim \mathrm{sgn}(\omega) \frac{\Theta(|\omega| - c|\mathbf{p}|)}{(\omega^2 - c^2\mathbf{p}^2)^{1-\eta/2}}, \tag{7.428}$$

as shown in figure 7.25. Thus the fluctuation modes become overdamped at the critical point.

7.10.4 The quantum non-linear sigma model

The quantum non-linear sigma model is defined by the partition function

$$\begin{aligned} Z = \int \mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2 - 1) \exp\biggl\{ -\frac{1}{2g} \int_0^\beta d\tau \int d^d r \left[(\boldsymbol{\nabla}\mathbf{n})^2 + c^{-2}(\partial_\tau \mathbf{n})^2 \right] \\ + \frac{1}{g} \int_0^\beta d\tau \int d^d r \, \mathbf{h}\cdot\mathbf{n} \biggr\}, \end{aligned} \tag{7.429}$$

where $\mathbf{n}(\mathbf{r},\tau)$ is a N-component field satisfying $\mathbf{n}(\mathbf{r},\tau)^2 = 1$. We assume $N > 2$. $\mathbf{h}(\mathbf{r},\tau) = \mathbf{h}$ is a uniform and time-independent external field. The coupling constant g is related to the (bare) stiffness $\rho_s^0 = 1/g$ and c is a velocity. The model is regularized by an ultraviolet momentum cutoff Λ (no cutoff is imposed on frequencies). The precise meaning of the measure $\mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2 - 1)$ is explained below.

When $T = 0$, the quantum non-linear sigma model is equivalent to a classical non-linear sigma model in dimension $d + 1$.[93] In the high-temperature limit $2\pi T \gg c\Lambda$, only the zero-frequency mode $\mathbf{n}(\mathbf{r}) \equiv \mathbf{n}(\mathbf{r}, i\omega_{n=0})$ contributes to the action (7.429) and we obtain an effective classical d-dimensional non-linear sigma model

$$S[\mathbf{n}] = \frac{1}{2t} \int d^d r (\boldsymbol{\nabla}\mathbf{n})^2 - \frac{1}{t} \int d^d r \, \mathbf{h}\cdot\mathbf{n} \tag{7.430}$$

with coupling constant $t = gT = T/\rho_s^0$.

As pointed out in section 7.7, to properly define the measure in (7.429), one must define the non-linear sigma model on a lattice. For the quantum model, one has to discretize both space and time, so that the partition function reads

$$Z = \int \mathcal{D}[\mathbf{n}] \prod_{\mathbf{r},\tau} \delta(\mathbf{n}_\mathbf{r}^2 - 1) \exp\{-S[\mathbf{n}]\} \tag{7.431}$$

(we do not write explicitly the τ dependence of the field), where $S[\mathbf{n}]$ is a discretized version of the action appearing in (7.429). We can now proceed as in section 7.7 and use the parametrization $\mathbf{n}_\mathbf{r} = (\sigma_\mathbf{r}, \boldsymbol{\pi}_\mathbf{r})$ with $\sigma_\mathbf{r}$ the component of $\mathbf{n}_\mathbf{r}$ in the direction

[93] If we use the $(d+1)$-dimensional "coordinate" variable $\mathbf{x} = (\mathbf{r}, c\tau)$, we obtain an isotropic classical model with coupling constant gc.

of the field $\mathbf{h}$,

$$\begin{aligned} Z &= \int \mathcal{D}[\sigma,\boldsymbol{\pi}] \prod_{\mathbf{r},\tau} \delta(\sigma_{\mathbf{r}}^2 + \boldsymbol{\pi}_{\mathbf{r}}^2 - 1) \exp\{-S[\sigma,\boldsymbol{\pi}]\} \\ &= \int \mathcal{D}[\sigma,\boldsymbol{\pi}] \prod_{\mathbf{r},\tau} \delta(\sigma_{\mathbf{r}} - \sqrt{1-\boldsymbol{\pi}^2}) \exp\Big\{-S[\sigma,\boldsymbol{\pi}] - \frac{1}{2}\sum_{\mathbf{r},\tau} \ln(1-\boldsymbol{\pi}_{\mathbf{r}}^2)\Big\}. \end{aligned} \tag{7.432}$$

In the continuum limit, the action $S[\sigma,\boldsymbol{\pi}]$ can be read off from (7.429) while the last term in (7.432) becomes

$$\frac{\rho}{2}\int_0^\beta d\tau \int d^dr \ln(1-\boldsymbol{\pi}^2) \quad \text{where} \quad \rho = a^{-d}\tau_c^{-1} = \frac{1}{V}\sum_{\mathbf{p}\in\mathrm{BZ}} \frac{1}{\beta} \sum_{\substack{\omega_n \\ (|\omega_n|\leq\omega_c)}} , \tag{7.433}$$

where a and τ_c are the lattice spacings in the space and time directions, respectively. As for the classical non-linear sigma model, we replace the Brillouin zone (BZ) in momentum space by a spherical region of radius Λ, and send the cutoff frequency $\omega_c = \pi/\tau_c$ to infinity in the final results. It would be more appropriate to use the regularization $\sqrt{\mathbf{p}^2+\omega_n^2/c^2} \leq \Lambda$ in order to satisfy the Lorentz invariance of the quantum non-linear sigma model at zero temperature. We shall see that to one-loop order, the RG equations do not violate the Lorentz invariance even when one uses a cutoff acting only on momenta.[94]

From now on, the analysis of the quantum non-linear sigma model is very similar to that of its classical counterpart (Sec. 7.7). The final form of the action reads

$$\begin{aligned} S[\boldsymbol{\pi}] = {} & \frac{1}{2g}\int_0^\beta d\tau \int d^dr \Big[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + c^{-2}(\partial_\tau\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-\boldsymbol{\pi}^2})^2 + c^{-2}(\partial_\tau\sqrt{1-\boldsymbol{\pi}^2})^2\Big] \\ & - \frac{h}{g}\int_0^\beta d\tau \int d^dr \sqrt{1-\boldsymbol{\pi}_{\mathbf{r}}^2} + \frac{\rho}{2}\int_0^\beta d\tau \int d^dr \ln(1-\boldsymbol{\pi}^2). \end{aligned} \tag{7.434}$$

To leading order in g,

$$S_0[\boldsymbol{\pi}] = \frac{1}{2g}\int_0^\beta d\tau \int d^dr \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + c^{-2}(\partial_\tau\boldsymbol{\pi})^2 + h\boldsymbol{\pi}^2\right], \tag{7.435}$$

and the corresponding propagator of the π_i field is

$$G_0(p) = \frac{g}{\omega_n^2/c^2 + \mathbf{p}^2 + h}, \tag{7.436}$$

where $p = (\mathbf{p}, i\omega_n)$. For a vanishing external field, the $\mathrm{O}(N)$ symmetry of the model is spontaneously broken and the fields π_i are the $N-1$ associated Goldstone modes, with dispersion $\omega = c|\mathbf{p}|$. These gapless modes yield a reduction of the order parameter $\langle\sigma\rangle = \langle\sqrt{1-\boldsymbol{\pi}^2}\rangle$,

$$\langle\sigma\rangle \simeq 1 - \frac{1}{2}\langle\boldsymbol{\pi}^2\rangle = 1 - \frac{N-1}{2\beta}\sum_{\omega_n}\int_{\mathbf{p}} \frac{gc^2}{\omega_n^2 + c^2\mathbf{p}^2}. \tag{7.437}$$

[94] At higher orders, one only expects a finite renormalization of the velocity.

At zero temperature, the last term in (7.437) becomes a $(d+1)$-dimensional integral which is infrared divergent in dimensions $d \leq 1$. This shows that long-range order can exist only for $d > 1$. At finite temperature, the momentum integral in (7.437) is infrared divergent when $\omega_n = 0$ for $d \leq 2$, from which we conclude that long-range order cannot exist in and below two dimensions (Mermin-Wagner theorem).

RG approach

To go beyond the leading order in g, we use the RG approach.[95] The procedure is strictly analogous to that followed in the classical case (Sec. 7.7.2) except we now have to keep track of the frequency dependence of the propagator. We start from the $\mathcal{O}(g)$ correction to S_0,

$$\begin{aligned} S_1[\boldsymbol{\pi}] = \frac{1}{2g} \int_0^\beta d\tau \int d^d r \Big[(\boldsymbol{\pi} \cdot \boldsymbol{\nabla} \boldsymbol{\pi})^2 + c^{-2} (\boldsymbol{\pi} \cdot \partial_\tau \boldsymbol{\pi})^2 + \frac{h}{4} (\boldsymbol{\pi}^2)^2 \Big] \\ - \frac{\rho}{2} \int_0^\beta d\tau \int d^d r \, \boldsymbol{\pi}^2. \end{aligned} \tag{7.438}$$

We split the field $\boldsymbol{\pi} = \boldsymbol{\pi}^< + \boldsymbol{\pi}^>$ into slow ($|\mathbf{p}| \leq \Lambda/s$) and fast ($\Lambda/s \leq |\mathbf{p}| \leq \Lambda$) modes and integrate out the latter. This yields the following action for the slow modes,

$$\begin{aligned} S[\boldsymbol{\pi}^<] = \frac{1}{2g} (1 + dI) \int_0^\beta d\tau \int d^d r \left[(\boldsymbol{\nabla} \boldsymbol{\pi}^<)^2 + c^{-2} (\partial_\tau \boldsymbol{\pi}^<)^2 \right] \\ + \frac{h}{2g} \left(1 + \frac{N-1}{2} dI \right) \int_0^\beta d\tau \int d^d r (\boldsymbol{\pi}^<)^2, \end{aligned} \tag{7.439}$$

where

$$dI = \frac{1}{\beta} \sum_{\substack{\omega_n \\ (|\omega_n| \leq \omega_c)}} \fint_{\mathbf{p}} G_0(p) \tag{7.440}$$

(the notation $\fint_{\mathbf{p}}$ is defined in Sec. 7.6.2). We have used

$$\frac{1}{\beta} \sum_{\substack{\omega_n \\ (|\omega_n| \leq \omega_c)}} \fint_{\mathbf{p}} \left(\mathbf{p}^2 + \frac{\omega_n^2}{c^2} \right) G_0(p) = g\rho^> - h dI, \tag{7.441}$$

where

$$\rho^> = \frac{1}{\beta} \sum_{\substack{\omega_n \\ (|\omega_n| \leq \omega_c)}} \fint_{\mathbf{p}} . \tag{7.442}$$

Rescaling coordinate and time variables and fields, i.e. $\mathbf{r} \to s\mathbf{r}$, $\tau \to s\tau$ and $\boldsymbol{\pi}^< \to \lambda^{-1} \boldsymbol{\pi}^<$, we reproduce the original action but with renormalized parameters,

$$\begin{aligned} \frac{1}{g'} &= \frac{1}{g} (1 + dI) \lambda^{-2} s^{d-1}, \\ \frac{h'}{g'} &= \frac{h}{g} \left(1 + \frac{N-1}{2} dI \right) \lambda^{-2} s^{d+1}, \\ \beta' &= \frac{\beta}{s}. \end{aligned} \tag{7.443}$$

[95] Another possibility is to consider the large-N limit; see Appendix 7.D.

Note that the velocity c of the Goldstone modes is not renormalized to this order. The value of λ is obtained by noting that h/g scales trivially (see Sec. 7.7),

$$\frac{h'}{g'} = \frac{h}{g} s^{d+1} \lambda^{-1}, \tag{7.444}$$

so that

$$\lambda = 1 + \frac{N-1}{2} dI. \tag{7.445}$$

Taking $s = e^{dl}$ ($dl \to 0$) and using

$$\frac{dI}{dl} = K_d \Lambda^d \frac{1}{\beta} \sum_{\omega_n} \frac{gc^2}{\omega_n^2 + c^2\Lambda^2} = \frac{K_d \Lambda^{d-1}}{2} gc \coth\left(\frac{c\Lambda}{2T}\right) \tag{7.446}$$

for $h = 0$ and $\omega_c \to \infty$, we obtain the following one-loop RG equations

$$\begin{aligned} \frac{d\tilde{g}}{dl} &= (1-d)\tilde{g} + (N-2)\frac{K_d}{2}\tilde{g}^2 \coth\left(\frac{1}{2\tilde{T}}\right), \\ \frac{d\tilde{T}}{dl} &= \tilde{T}, \end{aligned} \tag{7.447}$$

where we have introduced the dimensionless variables

$$\begin{aligned} \tilde{g} &= c\Lambda^{d-1} g, \\ \tilde{T} &= \frac{T}{c\Lambda}. \end{aligned} \tag{7.448}$$

We can distinguish two regimes (recall the discussion at the beginning of the section about the quantum and classical limits of the quantum non-linear sigma model). If $\tilde{T} \ll 1$, fluctuations are essentially quantum and governed by a classical $(d+1)$-dimensional non-linear sigma model with coupling constant $\tilde{g}$ satisfying the RG equation[96]

$$\frac{d\tilde{g}}{dl} = (1-d)\tilde{g} + (N-2)\frac{K_d}{2}\tilde{g}^2. \tag{7.449}$$

If $\tilde{T} \gg 1$, fluctuations are thermal and the system is described by a classical non-linear sigma model with dimensionless coupling constant $\tilde{t} = \tilde{g}\tilde{T}$ satisfying the RG equation

$$\frac{d\tilde{t}}{dl} = (2-d)\tilde{t} + (N-2)K_d\tilde{t}^2 \tag{7.450}$$

of the classical d-dimensional non-linear sigma model. It is therefore convenient to rewrite the RG equations (7.447) in terms of $\tilde{g}$ and $\tilde{t}$,

$$\begin{aligned} \frac{d\tilde{g}}{dl} &= (1-d)\tilde{g} + (N-2)\frac{K_d}{2}\tilde{g}^2 \coth\left(\frac{\tilde{g}}{2\tilde{t}}\right), \\ \frac{d\tilde{t}}{dl} &= (2-d)\tilde{t} + (N-2)\frac{K_d}{2}\tilde{g}\tilde{t} \coth\left(\frac{\tilde{g}}{2\tilde{t}}\right). \end{aligned} \tag{7.451}$$

[96] The $\mathcal{O}(\tilde{g}^2)$ term is slightly different from that in (7.294) because the cutoff used in the quantum model breaks the Lorentz invariance.

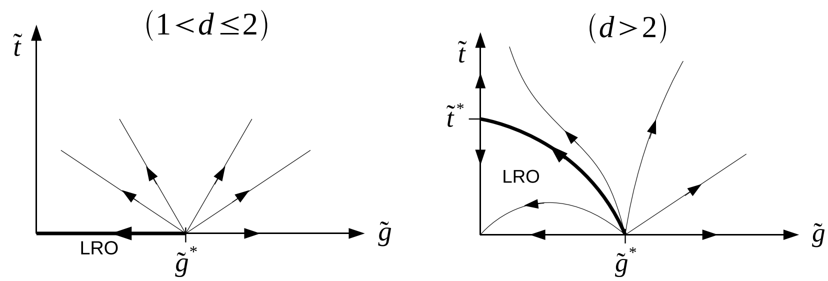


Figure 7.26: Flow diagram for the quantum non-linear sigma model for $1 < d \leq 2$ and $d > 2$.

For $d > 1$, we find a zero-temperature fixed point $(\tilde{t} = 0, \tilde{g}^*)$, with

$$\tilde{g}^* = \frac{d-1}{N-2} \frac{2}{K_d}, \tag{7.452}$$

describing a quantum phase transition with the exponents of the classical non-linear sigma model in $d+1$ dimensions. When $d > 2$ there is also a finite-temperature fixed point $(\tilde{t}^*, \tilde{g} = 0)$ with

$$\tilde{t}^* = \frac{d-2}{N-2} \frac{1}{K_d}. \tag{7.453}$$

Finally, there is a third fixed point $(\tilde{t} = 0, \tilde{g} = 0)$ describing the ordered phase. $\tilde{t}$ is a relevant perturbation in $d \leq 2$, but is irrelevant in dimension $d > 2$ where the fixed point corresponds to a stable phase. The flow diagrams for $1 < d \leq 2$ and $d > 2$ are shown in figure 7.26.

Zero-temperature stiffness

In the ordered phase near the transition ($\tilde{g}(0) < \tilde{g}^*$), the Josephson length ξ_J separates the long-wavelength Goldstone modes from the critical fluctuations at shorter length scales (Sec. 7.7.2). It satisfies the RG equation

$$\xi_J(\tilde{g}(0)) = \xi_J(\tilde{g}(l)) e^l. \tag{7.454}$$

Since the stiffness has scaling dimension $[\rho_s] = d + z - 2 = d - 1$ ($z = 1$ because of the Lorentz invariance of the model) while $[c] = 0$, we expect the stiffness to be related to the Josephson length by $\rho_s = c^x \xi_J^{1-d}$. The exponent x can be fixed from the physical units of ξ_J, ρ_s and c,

$$\begin{aligned} \xi_J &= \text{length}, \\ \rho_s &= (\text{length})^{2-d}/\text{time}, \\ c &= \text{length}/\text{time}. \end{aligned} \tag{7.455}$$

This yields $x = 1$ and in turn

$$\rho_s = \frac{c}{\xi_J^{d-1}}. \tag{7.456}$$

We deduce that the spin stiffness satisfies the RG equation

$$\rho_s(\tilde{g}(0)) = \rho_s(\tilde{g}(l))e^{-(d-1)l}. \tag{7.457}$$

For $l \to \infty$, the coupling constant $\tilde{g}(l) \to 0$ in the ordered phase and we can read off the corresponding stiffness directly from the renormalized action,

$$\rho_s(\tilde{g}(l)) \simeq \rho_s^0(\tilde{g}(l)) = \frac{c\Lambda^{d-1}}{\tilde{g}(l)}, \tag{7.458}$$

so that the zero-temperature stiffness can be defined by

$$\rho_s \equiv \rho_s(\tilde{g}(0)) = c\Lambda^{d-1} \lim_{l\to\infty} \frac{e^{-(d-1)l}}{\tilde{g}(l)}. \tag{7.459}$$

Solving the RG equation (7.449), we obtain

$$\tilde{g}(l) = \frac{\tilde{g}^*}{1 - e^{(d-1)l}\left(1 - \frac{\tilde{g}^*}{\tilde{g}(0)}\right)} \tag{7.460}$$

with $\tilde{g}^*$ given by (7.452), and therefore

$$\rho_s = c\Lambda^{d-1}\left(\frac{1}{\tilde{g}(0)} - \frac{1}{\tilde{g}^*}\right) = \rho_s^0\left(1 - \frac{\tilde{g}(0)}{\tilde{g}^*}\right). \tag{7.461}$$

We conclude that the Josephson length $\xi_J = (c/\rho_s)^{1/(d-1)}$ diverges when $\tilde{g}(0) \to \tilde{g}^*$ with a critical exponent $\nu = 1/(d-1)$.

Gap in the $T = 0$ disordered phase

Let us now compute the correlation length ξ and the gap m_0 in the zero-temperature disordered phase. When $\tilde{g}(0) > \tilde{g}^*$, $\tilde{g}(l)$ grows under renormalization and the perturbative RG approach breaks down. We can nevertheless estimate the correlation length from the criterion $\xi(l) = \xi e^{-l} \sim \Lambda^{-1}$ for $\tilde{g}(l) = 2\tilde{g}^*$ (the factor 2 is arbitrary here; a different factor (of order 1) would lead to similar results). From (7.460), we find

$$e^{-(d-1)l} = 2\frac{\tilde{g}(0) - \tilde{g}^*}{\tilde{g}(0)} \tag{7.462}$$

when $\tilde{g}(l) = 2\tilde{g}^*$, so that

$$\xi \sim \Lambda^{-1}\left(\frac{\tilde{g}(0)}{\tilde{g}(0) - \tilde{g}^*}\right)^{1/(d-1)} \tag{7.463}$$

and $\nu = 1/(d-1)$. A finite correction length is the signature of a gap in the excitation spectrum. In fact, the Lorentz invariance of the model implies that the excitation spectrum must be of the form $\omega = c\sqrt{\mathbf{p}^2 + \xi^{-2}}$, i.e. $m_0 = c/\xi$. Thus we obtain

$$m_0 \sim c\Lambda\left(\frac{\tilde{g}(0) - \tilde{g}^*}{\tilde{g}(0)}\right)^{1/(d-1)} \tag{7.464}$$

and conclude that $z\nu = 1/(d-1)$, i.e. $\nu = 1/(d-1)$ since $z = 1$.

Finite temperature: $\tilde{g}(0) < \tilde{g}^*$

We are now in a position to discuss the behavior of the system at finite temperature. Let us first consider the case $\tilde{g}(0) < \tilde{g}^*$, where $\tilde{g}(l)$ decreases when l increases, whereas $\tilde{T}(l)$ increases; the system is disordered by thermal fluctuations. There is a quantum-classical crossover at the scale l_x defined by $\tilde{T}(l_x) = (T/c\Lambda)e^{l_x} = 1$. For $l \gtrsim l_x$, fluctuations become classical and are described by a classical non-linear sigma model with effective coupling constant $\tilde{t}(l_x) = \tilde{T}(l_x)\tilde{g}(l_x) = \tilde{g}(l_x)$ and momentum cutoff $\Lambda_x = \Lambda e^{-l_x} \sim T/c$. Using (7.460), we obtain

$$\tilde{t}(l_x) = \frac{\tilde{g}^*}{1 - \left(\frac{c\Lambda}{T}\right)^{d-1}\left(1 - \frac{\tilde{g}^*}{\tilde{g}(0)}\right)} = \frac{\tilde{g}^*}{1 + \tilde{g}^* \rho_s \frac{c^{d-2}}{T^{d-1}}}, \tag{7.465}$$

where ρ_s is the zero-temperature stiffness.

In two dimensions, the classical non-linear sigma model is always in the disordered phase. In the renormalized classical regime, defined by $T \ll \rho_s$, the effective coupling constant $\tilde{t}(l_x) = T/\rho_s$ is small (note that $\tilde{g}^* = \mathcal{O}(1)$), and we can use (7.302) to estimate the correlation length,

$$\xi \sim \Lambda_x^{-1} \exp\left(\frac{2\pi}{(N-2)\tilde{t}(l_x)}\right) \sim \frac{c}{T} \exp\left(\frac{2\pi\rho_s}{(N-2)T}\right). \tag{7.466}$$

When $T \gg \rho_s$, the effective classical non-linear sigma model describing the momentum scales $|\mathbf{p}| \lesssim \Lambda_x$ is in the strong-coupling regime ($\tilde{t}(l_x) = \tilde{g}^* = \mathcal{O}(1)$) so that

$$\xi \sim \Lambda_x^{-1} \sim \frac{c}{T}, \tag{7.467}$$

which corresponds to the quantum critical regime.

For $d > 2$, the classical d-dimensional non-linear sigma model has a fixed point $\tilde{t}^*$ [Eq. (7.453)]. The system is ordered if $\tilde{t}(l_x) < \tilde{t}^*$ and disordered (by thermal fluctuations) if $\tilde{t}(l_x) > \tilde{t}^*$. If $\rho_s c^{d-2}/T^{d-1} \ll 1$, $\tilde{t}(l_x) = \tilde{g}^* \sim \mathcal{O}(1)$ and the system is disordered with a correlation length $\xi \sim \Lambda_x^{-1} \sim c/T$ (quantum critical regime). If $\rho_s c^{d-2}/T^{d-1} \gg 1$, there is a crossover to a thermal fluctuation regime at length scales larger than Λ_x^{-1}, and the system is in the ordered phase since $\tilde{t}(l_x) \ll \tilde{g}^* \sim \mathcal{O}(1)$. We deduce that the transition temperature to the ordered phase behaves as

$$T_c \sim \left(c^{d-2}\rho_s\right)^{1/(d-1)} \sim \left(1 - \frac{\tilde{g}(0)}{\tilde{g}^*}\right)^{1/(d-1)} \tag{7.468}$$

in agreement with the general result (7.414).[97] For $T \simeq T_c$, there is a classical critical regime and the correlation length $\xi \sim |T - T_c|^{-\nu}$ diverges with the exponent $\nu = 1/(d-2)$ of the classical d-dimensional non-linear sigma model.

Finite temperature: $\tilde{g}(0) > \tilde{g}^*$

The case $\tilde{g}(0) > \tilde{g}^*$, where both $\tilde{g}(l)$ and $\tilde{T}(l)$ increase with l, can be analyzed similarly. The system is disordered by quantum fluctuations if the condition $\tilde{g}(l) \simeq 2\tilde{g}^*$ (or

[97] We expect the crossover temperature T_x ($T_x > T_c$) between the quantum critical regime and the renormalized classical regime to be close to T_c, but our analysis here is too crude to distinguish between T_x and Tc.

$c/m_0 \sim \Lambda^{-1}e^l$) is reached before $\tilde{T}(l) = 1$ (or $c/T \sim \Lambda^{-1}e^l$), i.e. if $m_0 \gg T$. In this case, thermal fluctuations have little effect (quantum disordered regime). When $T \gg m_0$, there is a crossover to a classical regime described by a non-linear sigma model with coupling constant $\tilde{t}(l_x) = \tilde{T}(l_x)\tilde{g}(l_x) = \tilde{g}(l_x)$ and momentum cutoff $\Lambda_x \sim T/c$. Since $\tilde{g}^* < \tilde{g}(l_x) < 2\tilde{g}^*$, $\tilde{g}(l_x) \sim \mathcal{O}(\tilde{g}^*) \sim \mathcal{O}(1)$; the effective coupling constant is of order one so that ξ is again determined by the thermal length c/T [Eq. (7.467)] and we recover the quantum critical regime.

We conclude that the phase diagram of the quantum non-linear sigma model agrees with the generic phase diagram shown in figure 7.24.

7.10.5 The dilute Bose gas

The dilute Bose gas provides another example of a quantum phase transition. Interacting bosons are described by the action

$$S[\psi^*, \psi] = \int_0^\beta d\tau \int d^d r \left\{ \psi^* \left(\partial_\tau - \mu - \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi + \frac{g}{2} (\psi^* \psi)^2 \right\}, \tag{7.469}$$

where ψ is a complex field. We assume the interaction to be local in space (with $g > 0$) and we regularize the model by a cutoff Λ in momentum space. By varying the chemical potential at zero temperature, we can induce a quantum phase transition between a state with no particles ($\mu \leq 0$) and a state with a finite density of particles ($\mu > 0$). Because of the Galilean invariance of the action (7.469) when $\beta \to \infty$, the state with a positive chemical potential is always superfluid at zero temperature with the superfluid density given by the density of the fluid (chapter 6).

In the absence of interactions ($g = 0$) the quantum critical point $\mu = 0$ separates the state with no particles (vacuum) from a Bose-Einstein condensate. (For $\mu > 0$, the system is however not stable since without interactions nothing prevents an infinite density of bosons to condense into the zero-momentum state.) Straightforward dimensional analysis gives

$$\begin{aligned} [\psi] &= \frac{1}{2}(d + z - 2) = \frac{d}{2}, \\ z &= 2, \end{aligned} \tag{7.470}$$

where z is the dynamical critical exponent. It follows that

$$\begin{aligned} [\mu] &= 2, \\ [g] &= 2 - d. \end{aligned} \tag{7.471}$$

The Gaussian fixed point ($\mu = 0, g = 0$) is therefore stable for $d \geq 2$; the upper critical dimension is $d_c^+ = 2$ in agreement with the result $z = 2$ ($d_c^+ + z = 4$).

To understand the critical behavior below two dimensions, a RG calculation is necessary. Let us consider the one-loop RG equations at zero temperature. We proceed as in the preceding sections; we split the field into slow ($|\mathbf{p}| \leq \Lambda/s$) and fast ($\Lambda/s \leq |\mathbf{p}| \leq \Lambda$) modes and integrate out the latter. The renormalization of μ and g at one-loop order is given by the diagrams of figure 7.27. Since the quantum critical

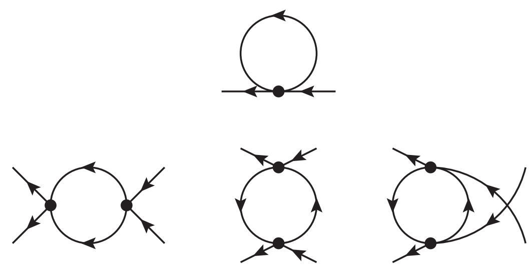


Figure 7.27: One-loop diagrams contributing to the self-energy (a) and the two-particle vertex (b).

point obviously corresponds to a vanishing chemical potential, we restrict our analysis to $\mu = 0$. The self-energy diagram in figure 7.27a vanishes, since

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_0(p) e^{i\omega 0^+} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega 0^+}}{i\omega - \epsilon_{\mathbf{p}}} = -n_B(\epsilon_{\mathbf{p}}) = 0 \tag{7.472}$$

when $T = 0$. Here $G_0(p) = (i\omega - \epsilon_{\mathbf{p}})^{-1}$ is the free boson propagator for $\mu = 0$, with $p = (\mathbf{p}, i\omega)$ and $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. The last two diagrams of figure 7.27b vanish by virtue of

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_0(p)^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(i\omega - \epsilon_{\mathbf{p}})^2} = 0, \tag{7.473}$$

and only the first diagram gives a correction to the interaction constant g,

$$g' = g - g^2 \fint_p G_0(p) G_0(-p). \tag{7.474}$$

We use the notation $\fint_p$ to indicate that the integration is restricted to fast modes. The next step is to rescale coordinate and time variables, $\mathbf{r} \to s\mathbf{r}$ and $\tau \to s^z \tau$, and fields, $\psi \to s^{-d_\psi} \psi$, where $d_\psi = \frac{1}{2}(d + z - 2 + \eta)$ is the scaling dimension of the field. $z = 2$ and $\eta = 0$ at one-loop order. This yields the RG equation

$$g(l + dl) = e^{(2-d)dl} \left[g(l) - g(l)^2 \fint_p G_0(p) G_0(-p) \right], \tag{7.475}$$

where we have taken $s = e^{dl}$. Using

$$\fint_p G_0(p) G_0(-p) = m K_d \Lambda^{d-2} dl, \tag{7.476}$$

we finally obtain

$$\frac{d\tilde{g}}{dl} = (2 - d)\tilde{g} - \frac{1}{2}\tilde{g}^2, \tag{7.477}$$

where $\tilde{g}(l) = 2mK_d \Lambda^{d-2} g(l)$. For $d \geq 2$, $\tilde{g}$ vanishes for $l \to \infty$ and we recover the Gaussian fixed point. For $d < 2$, the critical behavior is controlled by the nontrivial fixed point $\tilde{g}^* = 2(2 - d)$. $\tilde{g}$ is an irrelevant variable at the fixed point and μ is therefore the only relevant variable.

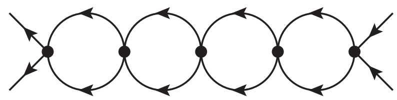


Figure 7.28: Bubble diagrams contributing to the two-particle vertex in vacuum.

It turns out that the one-loop RG equations are exact. The reason is that for $\mu \leq 0$, the ground state is the state with no particles (so that the quantum critical point is not a many-body problem). Thus a particle introduced in the system propagates freely and $G_0(p)$ is the exact propagator. Since the Green function

$$G_0(\mathbf{p},\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} G_0(p) = -\Theta(\tau) e^{-\tau\epsilon_{\mathbf{p}}} \tag{7.478}$$

is causal, it is indeed easy to see that all self-energy diagrams necessarily vanish. Thus the results $[\psi] = d/2$, $[\mu] = 2$, $z = 2$ and $\eta = 0$ are exact. Although the quantum critical point is not Gaussian for $d < 2$ ($\tilde{g}^* \neq 0$), the quantum critical behavior remains trivial.

Similarly, the two-particle vertex $\Gamma^{(4)}$ can be computed exactly since this amounts to solving the two-body problem in vacuum. The only diagrams contributing to $\Gamma^{(4)}$ are the bubble diagrams shown in figure 7.28. If we integrate out fields with momenta in the range $\Lambda(l) \leq |\mathbf{p}| \leq \Lambda$, we obtain the effective interaction

$$g(l) = g - g^2\Pi(l) + g^3\Pi(l)^2 - \cdots = \frac{g}{1 + g\Pi(l)} \tag{7.479}$$

for the remaining fields ($|\mathbf{p}| \leq \Lambda(l)$), where

$$\Pi(l) = \int_p \Theta[|\mathbf{p}| - \Lambda(l)] G_0(p) G_0(-p) = m\frac{K_d}{d-2}\left[\Lambda^{d-2} - \Lambda(l)^{d-2}\right]. \tag{7.480}$$

With $\Lambda(l) = \Lambda e^{-l}$, equation (7.480) reproduces the RG equation (7.477).

7.A Perturbative calculation of critical exponents

It is difficult to push the RG calculation of critical exponents beyond order ϵ using the approach of section 7.6. Although the perturbation theory breaks down near the critical point (Sec. 7.3.4), it turns out to be an efficient tool to compute the critical exponents if one admits that the correlation functions take the form predicted by the RG. In this section, we compute the anomalous dimension η to $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(1/N)$ using the perturbative approach.

7.A.1 ϵ expansion

Let us consider the two-point vertex

$$\Gamma^{(2)}_{ii}(\mathbf{p}) = \mathbf{p}^2 + r_0 + \Sigma(\mathbf{p}) \tag{7.481}$$

at the critical temperature T_c (we assume a $(\boldsymbol{\varphi}^2)^2$ theory with O(N) symmetry). The RG predicts

$$\begin{aligned}\Gamma^{(2)}_{ii}(\mathbf{p}) &\propto |\mathbf{p}|^{2-\eta}\left[1+\mathcal{O}(|\mathbf{p}|^{-y_2})\right] \\ &\propto \mathbf{p}^2\left[1-\eta\ln|\mathbf{p}|+\mathcal{O}(|\mathbf{p}|^{-y_2})+\cdots\right]\end{aligned} \tag{7.482}$$

(see Eq. (7.182)), where $y_2<0$ refers to the dominant irrelevant field t_2. Thus it seems that we can extract the anomalous dimension η from the coefficient of $\mathbf{p}^2\ln|\mathbf{p}|$ in the small $\mathbf{p}$ expansion of $\Gamma^{(2)}(\mathbf{p})$. But for $d<4$, $y_2=-\epsilon+\mathcal{O}(\epsilon^2)$ (Sec. 7.6) and the $\mathcal{O}(|\mathbf{p}|^{-y_2})$ term will also appear as a series in $\ln|\mathbf{p}|$, i.e. $|\mathbf{p}|^{-y_2}=1+\epsilon\ln|\mathbf{p}|+\cdots$, when expanded in powers of ϵ. If we compute $\Gamma^{(2)}(\mathbf{p})$ in powers of ϵ, we will not only obtain powers of $\ln|\mathbf{p}|$ coming from $|\mathbf{p}|^{2-\eta}$ but also powers of $\ln|\mathbf{p}|$ coming from $|\mathbf{p}|^{-y_2}$. If, however, we are able to set the scaling field t_2 to zero, then we can directly deduce η (as well as other critical exponents) from the perturbative calculation of the two-point vertex. This can be done by choosing a particular value $u_0(\epsilon)$ of the coupling constant u_0.

$u_0(\epsilon)$ can be determined by considering the 4-point vertex $\Gamma^{(4)}$. Near the fixed point,

$$\Gamma^{(4)}_{ijkl}(0,K')=s^{d-4d_\varphi}\Gamma^{(4)}_{ijkl}(0,K), \tag{7.483}$$

i.e.

$$\Gamma^{(4)}_{ijkl}(0,t_1',t_2')=s^{d-4d_\varphi}\Gamma^{(4)}_{ijkl}(0,t_1,t_2) \tag{7.484}$$

if we retain only the leading irrelevant field t_2. The first argument of $\Gamma^{(4)}_{ijkl}$ indicates that all momenta are set to zero. Equation (7.484) can be rewritten as

$$\Gamma^{(4)}_{ijkl}(0,t_1,t_2)=s^{d-4+2\eta}\Gamma^{(4)}_{ijkl}(0,s^{1/\nu}t_1,s^{y_2}t_2). \tag{7.485}$$

Instead of t_1, we take the scaling field $r=\chi^{-1}\sim t^\gamma$ and restrict ourselves to the high-temperature phase ($t\geq 0$),

$$\Gamma^{(4)}_{ijkl}(0,r,t_2)=s^{-\epsilon+2\eta}\Gamma^{(4)}_{ijkl}(0,s^{1/\nu}r^{1/\gamma},s^{y_2}t_2). \tag{7.486}$$

With $s=r^{-\nu/\gamma}$, equation (7.486) gives

$$\Gamma^{(4)}_{ijkl}(0,r,t_2)=r^{(\epsilon-2\eta)\nu/\gamma}\Gamma^{(4)}_{ijkl}(0,1,r^{-y_2\nu/\gamma}t_2). \tag{7.487}$$

To $\mathcal{O}(\epsilon)$, $\eta=0$ and $\nu=\gamma/2$, so that

$$\begin{aligned}\Gamma^{(4)}_{ijkl}(0,r,t_2)&=r^{\epsilon/2}\Gamma^{(4)}_{ijkl}(0,1,r^{-y_2/2}t_2) \\ &=A_{ijkl}\left(1+\frac{\epsilon}{2}\ln r+B_{ijkl}t_2\frac{\epsilon}{2}\ln r+\mathcal{O}(\epsilon^2)\right).\end{aligned} \tag{7.488}$$

Let us compare this expression with the one-loop result

$$\begin{aligned}\Gamma^{(4)}_{ijkl}(0)&=(\delta_{i,j}\delta_{k,l}+\delta_{i,k}\delta_{j,l}+\delta_{i,l}\delta_{j,k})\frac{1}{V}\left[\frac{u_0}{3}-u_0^2\frac{N+8}{18}\int_{\mathbf{q}}\frac{1}{(\mathbf{q}^2+r_0)^2}\right] \\ &\simeq(\delta_{i,j}\delta_{k,l}+\delta_{i,k}\delta_{j,l}+\delta_{i,l}\delta_{j,k})\frac{u_0}{3V}\left[1-u_0\frac{N+8}{12}K_4\ln\frac{\Lambda^2}{r}\right]\end{aligned} \tag{7.489}$$

for $r \ll \Lambda$. Since u_0 will eventually be of order ϵ, we have set $d=4$ and replaced r_0 by r. Equation (7.489) is then correct to order ϵ^2. Comparing (7.488) and (7.489), we see that t_2 vanishes if u_0 takes the value[98]

$$u_0(\epsilon) = \frac{6}{(N+8)K_4}\epsilon + \mathcal{O}(\epsilon^2). \tag{7.490}$$

We are now in a position to compute the critical exponents from the perturbation theory. To compute γ to $\mathcal{O}(\epsilon)$ we use equation (7.113),

$$\begin{aligned} r &= \bar{r}_0(T-T_c) - \frac{N+2}{6}K_4 u_0 r \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|}{\mathbf{q}^2+r} + \mathcal{O}(\epsilon^2) \\ &= \bar{r}_0(T-T_c) - \frac{N+2}{12}K_4 u_0 r \ln\frac{\Lambda^2}{r} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{7.491}$$

i.e.

$$t \sim \bar{r}_0(T-T_c) = r\left(1 - \frac{N+2}{12}K_4 u_0 r \ln\frac{r}{\Lambda^2}\right) + \mathcal{O}(\epsilon^2), \tag{7.492}$$

where $u_0 \equiv u_0(\epsilon)$. This expression must be compared with

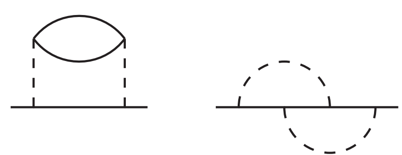
$$t \sim r^{1/\gamma} = r\left[1 + \left(\frac{1}{\gamma} - 1\right)\ln r\right] + \mathcal{O}(\epsilon^2). \tag{7.493}$$

We deduce

$$\gamma = 1 + \frac{N+2}{N+8}\frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \tag{7.494}$$

in agreement with the result obtained in section 7.6.

To compute the anomalous dimension η to $\mathcal{O}(\epsilon^2)$, we must consider the two-loop contributions



to the self-energy:

$$\Sigma(\mathbf{p}) = -\frac{N+2}{18}u_0^2 \int d^d r\, e^{-i\mathbf{p}\cdot\mathbf{r}} G_0^3(\mathbf{r}) \tag{7.495}$$

(other two-loop diagrams, as well as one-loop diagrams, give self-energy corrections independent of the external momentum). This expression cannot be directly used since the bare propagator $G_0(\mathbf{p}) = (\mathbf{p}^2 + r_0)^{-1}$ has a finite correlation length $r_0^{-1/2}$ when $T = T_c$ ($T_c \neq T_{c0}$). To circumvent this difficulty, we include $\Sigma(0)$ in the "bare" propagator, i.e. we take $G_0(\mathbf{p}) = [\mathbf{p}^2 + r_0 + \Sigma(0)]^{-1}$.[99] At the critical point, $\Gamma^{(2)}(\mathbf{p}=0) = r_0 + \Sigma(0) = 0$, and the propagator $G_0(\mathbf{p})$ has now an infinite correlation length. The two-point vertex reads

$$\Gamma_{ii}^{(2)}(\mathbf{p}) = \mathbf{p}^2 + \Sigma(\mathbf{p}) - \Sigma(0) \tag{7.496}$$

[98] To make dimensional sense of (7.488), we must interpret $\ln r$ as $\ln\frac{r}{a\Lambda^2}$. The expression of $u_0(\epsilon)$ is independent of the constant a in the limit $r \to 0$.

[99] For a justification of this procedure (in particular regarding combinatorial factors in the Feynman diagrams), see Sec. 1.6.3.

when $T = T_c$, where $\Sigma(\mathbf{p})$ is given by (7.495) to two-loop order.

To obtain the $\mathbf{p}^2 \ln|\mathbf{p}|$ term in $\Sigma(\mathbf{p})$, it is sufficient to expand

$$e^{-i\mathbf{p}\cdot\mathbf{r}} = 1 - i\mathbf{p}\cdot\mathbf{r} - \frac{1}{2}(\mathbf{p}\cdot\mathbf{r})^2 + \cdots \tag{7.497}$$

in (7.495). This gives

$$\begin{aligned}\int d^4r\,(e^{-i\mathbf{p}\cdot\mathbf{r}} - 1)G_0^3(\mathbf{r}) &= -\frac{\mathbf{p}^2}{8(2S_4)^3}\int d^4r\,|\mathbf{r}|^{-4} + \mathcal{O}(|\mathbf{p}|^4) \\ &= -\frac{\mathbf{p}^2}{64S_4^2}\int d|\mathbf{r}|\,|\mathbf{r}|^{-1} + \mathcal{O}(|\mathbf{p}|^4)\end{aligned} \tag{7.498}$$

($S_4 = 2\pi^2$), where we have used the expression (7.86) of $G_0(\mathbf{r}) = \int_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{r}}\mathbf{p}^{-2}$. We have also set $d = 4$, since we are interested in the result to $\mathcal{O}(\epsilon^2)$ while $u_0 \equiv u_0(\epsilon) = \mathcal{O}(\epsilon)$ in (7.495). The lower and upper limits in the last integral of (7.498) are approximately given by Λ^{-1} and $1/|\mathbf{p}|$, so that

$$\Gamma_{ii}^{(2)}(\mathbf{p}) = \mathbf{p}^2 - u_0^2\frac{N+2}{18}\frac{\mathbf{p}^2}{64S_4^2}(\ln|\mathbf{p}| + \text{const}) + \mathcal{O}(|\mathbf{p}|^4). \tag{7.499}$$

Comparing this result with (7.482) (without the $\mathcal{O}(|\mathbf{p}|^{-y_2}$ term) and using (7.490), we finally obtain

$$\eta = \frac{1}{2}\frac{N+2}{(N+8)^2}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{7.500}$$

7.A.2 $1/N$ expansion

Perturbation theory can also be used to calculate the critical exponents within a $1/N$ expansion. We start from the action

$$S[\boldsymbol{\varphi}] = \int d^dr\left\{\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2 + \frac{u_0}{4!N}\left(\boldsymbol{\varphi}^2\right)^2\right\} \tag{7.501}$$

where the factor $1/N$ is introduced to obtain a meaningful limit $N \to \infty$. The $1/N$ expansion is not a mere expansion in u_0 since for each closed loop there is a factor N coming from the sum over internal O(N) index.

Leading order

To leading order, the self-energy is given by the "Hartree" approximation (Fig. 7.29),

$$\Sigma = \frac{u_0}{6}\int_\mathbf{q}\frac{1}{\mathbf{q}^2 + r_0 + \Sigma}. \tag{7.502}$$

The two-point vertex is then given by

$$\Gamma_{ii}^{(2)}(\mathbf{p}) = \mathbf{p}^2 + r_0 + \Sigma = \mathbf{p}^2 + r, \tag{7.503}$$

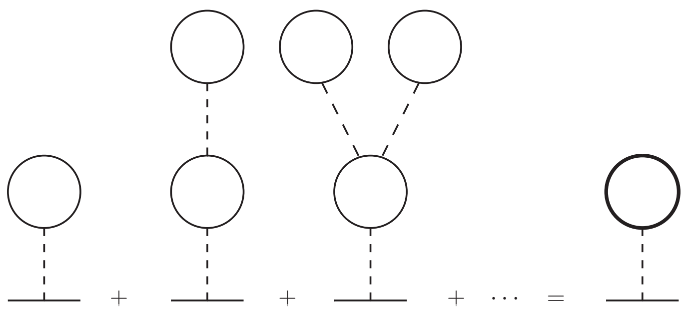


Figure 7.29: Self-energy Σ to leading order in the limit $N \to \infty$. The thick solid line stands for $G = (G_0^{-1} + \Sigma)^{-1}$.

where $r = r_0 + \Sigma$. The critical temperature is determined by $r = 0$,

$$0 = \bar{r}_0(T - T_{c0}) + \frac{u_0}{6} \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2}, \tag{7.504}$$

which allows us to express r as

$$\begin{aligned} r &= \bar{r}_0(T - T_c) + \frac{u_0}{6} \int_{\mathbf{q}} \left(\frac{1}{\mathbf{q}^2 + r} - \frac{1}{\mathbf{q}^2} \right) \\ &= \bar{r}_0(T - T_c) + \frac{u_0 r}{6} \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)}. \end{aligned} \tag{7.505}$$

For $d > 4$, the integral converges when $r \to 0$ so that $r \sim T - T_c$ when $T \to T_c$, i.e. $\gamma = 1$, in agreement with the mean-field result. For $d = 4$, one finds

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)} = K_4 \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|}{\mathbf{q}^2 + r} \simeq \frac{K_4}{2} \ln \frac{\Lambda^2}{r} \tag{7.506}$$

for $\Lambda/\sqrt{r} \gg 1$. From (7.505), we then deduce

$$\bar{r}_0(T - T_c) \simeq \frac{u_0}{12} K_4 r \ln \frac{\Lambda^2}{r}, \tag{7.507}$$

i.e.

$$r \simeq \frac{12}{u_0 K_4} \frac{\bar{r}_0(T - T_c)}{\ln \left(\frac{u_0 K_4 \Lambda^2}{12 \bar{r}_0 (T - T_c)} \right)} \tag{7.508}$$

for $T \to T_c$. As expected at the upper critical dimension, there are logarithmic corrections to the mean-field result $r \sim T - T_c$.

For $d < 4$, we use

$$\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + r)} = K_d \int_0^\Lambda d|\mathbf{q}| \frac{|\mathbf{q}|^{d-3}}{\mathbf{q}^2 + r} = r^{d/2-2} \tilde{K}_d, \tag{7.509}$$

where $\tilde{K}_d$ is defined in (7.116). From (7.505), we then deduce

$$\bar{r}_0(T - T_c) \simeq \frac{u_0}{6} r^{d/2-1} \tilde{K}_d \tag{7.510}$$

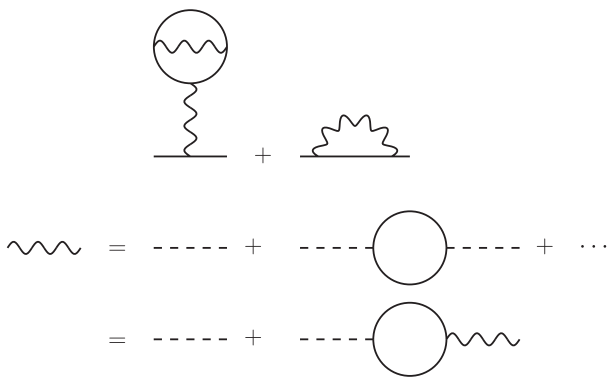


Figure 7.30: Self-energy to order $1/N$.

for $T \to T_c$ and $d < 4$, i.e.

$$r \sim (T - T_c)^\gamma \tag{7.511}$$

with the susceptibility critical exponent

$$\gamma = \frac{2}{d-2} + \mathcal{O}(N^{-1}). \tag{7.512}$$

Since $\eta = 0$ (the self-energy is momentum independent), we obtain all other critical exponents using the scaling laws (Sec. 7.4),

$$\begin{aligned} \nu &= \frac{1}{d-2} + \mathcal{O}(N^{-1}), \\ \beta &= \frac{1}{2} + \mathcal{O}(N^{-1}), \\ \alpha &= \frac{d-4}{d-2} + \mathcal{O}(N^{-1}), \\ \delta &= \frac{d+2}{d-2} + \mathcal{O}(N^{-1}). \end{aligned} \tag{7.513}$$

$1/N$ corrections

The self-energy to order $1/N$ is shown in figure 7.30. As in the case of the ϵ expansion, we include $\Sigma(0)$ in the "bare" propagator $G(\mathbf{p}) = (\mathbf{p}^2 + r)^{-1}$. The effective vertex u (the wavy line in Fig. 7.30) is defined by[100]

$$u(\mathbf{p}) = \frac{u_0}{1 + \frac{u_0}{6}\Pi(\mathbf{p})}, \qquad \Pi(\mathbf{p}) = \int_{\mathbf{q}} G(\mathbf{q})G(\mathbf{p}+\mathbf{q}). \tag{7.514}$$

[100]This result is easily obtained by considering the expansion for the 4-point vertex,

$$\begin{aligned} \Gamma^{(4)}_{ijkl}(\mathbf{p}_1, \cdots, \mathbf{p}_4) = {}& \delta_{\sum_i \mathbf{p}_i, 0} \left(\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}\right) \\ & \times \left[\frac{u_0}{3N} - \frac{N}{2}\left(\frac{u_0}{3N}\right)^2 \int_{\mathbf{q}} G(\mathbf{q})G(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q}) + \cdots \right]. \end{aligned}$$

Since the first diagram in figure 7.30 is independent of $\mathbf{p}$, we obtain

$$\Sigma(\mathbf{p}) - \Sigma(0) = \frac{1}{3N} \int_{\mathbf{q}} u(\mathbf{q}) \left[G(\mathbf{p}+\mathbf{q}) - G(\mathbf{q}) \right]. \tag{7.515}$$

At the critical point ($r = 0$), we can use

$$\Pi(\mathbf{p}) = \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2 (\mathbf{p}+\mathbf{q})^2} \simeq A_d |\mathbf{p}|^{d-4} \tag{7.516}$$

in the limit $|\mathbf{p}| \ll \Lambda$ and for $d < 4$ (see Eq. (7.320)). Thus, for $\mathbf{p} \to 0$,

$$u(\mathbf{p}) \simeq \frac{6}{\Pi(\mathbf{p})} \simeq \frac{6}{A_d |\mathbf{p}|^{d-4}} \tag{7.517}$$

and

$$\begin{aligned} \Sigma(\mathbf{p}) - \Sigma(0) &\simeq \frac{2}{N A_d} \int_{\mathbf{q}} |\mathbf{q}|^{4-d} \left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2} \right] \\ &\simeq A \mathbf{p}^2 \ln \left(\frac{\Lambda}{|\mathbf{p}|} \right). \end{aligned} \tag{7.518}$$

A can be obtained by expanding (7.518) for small $\mathbf{p}$ since the $\mathbf{p}^2 \ln \Lambda$ term is due to large values of $\mathbf{q}$. Using

$$\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2} = -2 \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^4} - \frac{\mathbf{p}^2}{|\mathbf{q}|^4} + 4 \frac{(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{q}|^6} + \mathcal{O}(|\mathbf{p}|^3), \tag{7.519}$$

we obtain

$$\int_{\mathbf{q}} |\mathbf{q}|^{4-d} \left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2} \right] \simeq \mathbf{p}^2 \frac{4-d}{d} \int_{\mathbf{q}} |\mathbf{q}|^{-d} = \mathbf{p}^2 \frac{4-d}{d} K_d \ln \Lambda \tag{7.520}$$

and

$$\Gamma^{(2)}_{ii}(\mathbf{p}) = \mathbf{p}^2 + \frac{2}{N A_d} \frac{4-d}{d} K_d \mathbf{p}^2 \ln \frac{\Lambda}{|\mathbf{p}|}, \tag{7.521}$$

which leads to

$$\eta = \frac{2}{N A_d} \frac{4-d}{d} K_d + \mathcal{O}(N^{-2}). \tag{7.522}$$

7.B The $(\boldsymbol{\varphi}^2)^2$ theory in the large-N limit

In this section we reconsider the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory [Eq. (7.501)] and show how we can extend the results of section 7.A.2 to the low-temperature phase.

Using

$$\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\lambda \, e^{-i\frac{\lambda}{2}(\boldsymbol{\varphi}^2 - \rho)} = \int_{-\infty}^{\infty} d\rho \, \delta(\boldsymbol{\varphi}^2 - \rho) = 1 \tag{7.523}$$

(we ignore any multiplicative constant), we rewrite the partition function as

$$\begin{aligned} Z &= \int \mathcal{D}[\boldsymbol{\varphi}, \rho, \lambda] \exp\left\{ - \int d^d r \left[\frac{1}{2} (\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + \frac{r_0}{2}\rho + \frac{u_0}{4! N}\rho^2 + i\frac{\lambda}{2}(\boldsymbol{\varphi}^2 - \rho) \right] \right\} \\ &= \int \mathcal{D}[\boldsymbol{\varphi}, \lambda] \exp\left\{ -\frac{1}{2} \int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2 + i\lambda\boldsymbol{\varphi}^2 \right] + \frac{3N}{2u_0} \int d^d r \, (i\lambda - r_0)^2 \right\}, \end{aligned} \tag{7.524}$$

where the second line is obtained by integrating out the ρ field. We now split $\boldsymbol{\varphi} = (\sigma, \boldsymbol{\pi})$ into a scalar field σ and a $(N-1)$-component field $\boldsymbol{\pi}$. As in the non-linear sigma model (Sec. 7.7), this parametrization will allow spontaneous symmetry breaking. The integration over the $\boldsymbol{\pi}$ field gives

$$\int \mathcal{D}[\boldsymbol{\pi}] \exp\left\{-\frac{1}{2}\int d^d r \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + i\lambda\boldsymbol{\pi}^2\right]\right\} = \left(\det g^{-1}\right)^{-(N-1)/2}, \tag{7.525}$$

where

$$g^{-1}(\mathbf{r}, \mathbf{r}') = -\boldsymbol{\nabla}^2\delta(\mathbf{r}-\mathbf{r}') + i\lambda(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') \tag{7.526}$$

is the propagator of the π_i field in the presence of a fluctuating λ field. We thus obtain the partition function

$$\begin{aligned} Z = \int \mathcal{D}[\sigma, \lambda] \exp\bigg\{ & -\frac{1}{2}\int d^d r \left[(\boldsymbol{\nabla}\sigma)^2 + i\lambda\sigma^2\right] \\ & + \frac{3N}{2u_0}\int d^d r\,(i\lambda - r_0)^2 - \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}\bigg\}. \end{aligned} \tag{7.527}$$

If we rescale the σ field, $\sigma \to \sqrt{N}\sigma$,[101] then the action becomes proportional to N in the limit $N\to\infty$ and the saddle-point approximation becomes exact. For uniform fields $\sigma(\mathbf{r}) = \sigma$ and $\lambda(\mathbf{r}) = \lambda$, the action is given by

$$\frac{1}{V}S[\sigma, \lambda] = \frac{i}{2}\lambda\sigma^2 - \frac{3N}{2u_0}(i\lambda - r_0)^2 + \frac{N}{2V}\mathrm{Tr}\ln g^{-1} \tag{7.528}$$

(we use $N-1 \simeq N$ for large N), with $g^{-1}(\mathbf{p}) = \mathbf{p}^2 + i\lambda$ in Fourier space. Since σ and λ do not fluctuate when $N\to\infty$, $g(\mathbf{p})$ is the propagator of the field π_i.[102] From (7.528), we deduce the saddle-point equations

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= \frac{6N}{u_0}(m^2 - r_0) - N\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2 + m^2}, \end{aligned} \tag{7.529}$$

where we use the notation $m^2 = i\lambda$ ($i\lambda$ is real at the saddle point). These equations show that the component σ of the $\boldsymbol{\varphi}$ field which was singled out plays the role of an order parameter. In the disordered phase, $\sigma = 0$ and $m \neq 0$ (the $N-1$ π_i field are gapped). The saddle-point equation for m^2 reproduces our previous result (7.502) with $\Sigma = m^2 - r_0$. In the ordered phase, σ is nonzero and the propagator $g(\mathbf{p}) = 1/\mathbf{p}^2$ is gapless, thus identifying the π_i fields as the $N-1$ Goldstone modes associated to spontaneous rotation symmetry breaking.

7.B.1 Correlation functions in the low-temperature phase

In the broken-symmetry phase, $m = 0$ and the saddle-point equation for the order parameter reads

$$\sigma^2 = -\frac{6N}{u_0}(r_0 - r_{0c}), \tag{7.530}$$

[101] In the following, we work with the "unrescaled" field which is therefore $\mathcal{O}(\sqrt{N})$.

[102] If σ and λ were fluctuating, one would have to integrate them out to obtain the propagator of the $\boldsymbol{\pi}$ field.

where

$$r_{0c} = -\frac{u_0}{6}\int_{\mathbf{p}} \frac{1}{\mathbf{p}^2} = -\frac{u_0}{6}\frac{K_d\Lambda^{d-2}}{d-2} \tag{7.531}$$

is the critical value of r_0 defining the critical temperature: $r_{0c} = \bar{r}_0(T_c - T_{c0})$.[103] The correction to the mean-field result $\sigma^2 = -6Nr_0/u_0$ in (7.530) is due to the Goldstone modes. Since $\sigma \sim \sqrt{T_c - T}$, the critical exponent β is equal to $1/2$ in agreement with (7.513).

The action being obtained from a saddle-point approximation, the effective action $\Gamma[\sigma,\lambda]$ is simply given by the action $S[\sigma,\lambda]$ defined by (7.527).[104,105] We deduce

$$\begin{aligned}\Gamma^{(2)}(\mathbf{r}-\mathbf{r}') &= \begin{pmatrix} \Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\sigma\lambda}(\mathbf{r}-\mathbf{r}') \\ \Gamma^{(2)}_{\lambda\sigma}(\mathbf{r}-\mathbf{r}') & \Gamma^{(2)}_{\lambda\lambda}(\mathbf{r}-\mathbf{r}') \end{pmatrix} \\ &= \begin{pmatrix} -\boldsymbol{\nabla}^2\delta(\mathbf{r}-\mathbf{r}') & i\sigma\delta(\mathbf{r}-\mathbf{r}') \\ i\sigma\delta(\mathbf{r}-\mathbf{r}') & \frac{N}{2}\Pi(\mathbf{r}-\mathbf{r}') + \frac{3N}{u_0}\delta(\mathbf{r}-\mathbf{r}') \end{pmatrix}, \end{aligned} \tag{7.532}$$

where

$$\Pi(\mathbf{r}-\mathbf{r}') = g(\mathbf{r}-\mathbf{r}')g(\mathbf{r}'-\mathbf{r}) \tag{7.533}$$

and we use the notation $\Gamma^{(2)}_{\sigma\sigma}(\mathbf{r}-\mathbf{r}') = \delta^{(2)}\Gamma/\delta\sigma(\mathbf{r})\delta\sigma(\mathbf{r}')$, etc. The two-point vertex $\Gamma^{(2)}$ is computed for the saddle-point values of the σ and λ fields. In Fourier space, we obtain

$$\Gamma^{(2)}(\mathbf{p}) = \begin{pmatrix} \mathbf{p}^2 & i\sigma \\ i\sigma & \frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0} \end{pmatrix}, \tag{7.534}$$

where $\Pi(\mathbf{p}) = \int_{\mathbf{q}} g(\mathbf{q})g(\mathbf{p}+\mathbf{q})$. The propagator $G = \Gamma^{(2)-1}$ takes the form

$$G(\mathbf{p}) = \frac{1}{\det\Gamma^{(2)}(\mathbf{p})}\begin{pmatrix} \frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0} & -i\sigma \\ -i\sigma & \mathbf{p}^2 \end{pmatrix}, \tag{7.535}$$

with

$$\det\Gamma^{(2)}(\mathbf{p}) = \mathbf{p}^2\left[\frac{N}{2}\Pi(\mathbf{p}) + \frac{3N}{u_0}\right] + \sigma^2. \tag{7.536}$$

The last equation, together with the small $\mathbf{p}$ behavior (7.516) of $\Pi(\mathbf{p})$, leads us to introduce three characteristic momentum scales,

$$\begin{aligned} p_G &= \left(\frac{u_0A_d}{6}\right)^{1/(4-d)}, \\ p_J &= \left(\frac{2\sigma^2}{NA_d}\right)^{1/(d-2)} = \left[\frac{12}{u_0A_d}(r_{0c}-r_0)\right]^{1/(d-2)}, \\ p_c &= \left(\frac{u_0\sigma^2}{3N}\right)^{1/2} = [2(r_{0c}-r_0)]^{1/2}, \end{aligned} \tag{7.537}$$

[103] We assume $r_0 = \bar{r}_0(T - T_{c0})$.

[104] The equality between the effective action Γ and the "microscopic" action S within a saddle-point approximation has been shown in Sec. 7.2.1 when discussing the Landau theory.

[105] To simplify the notations, we note σ and λ the arguments of both the action S and the effective action Γ.

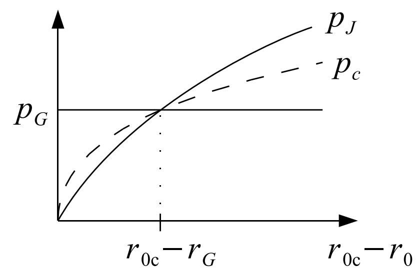


Figure 7.31: Characteristic momentum scales p_G, p_J and p_c [Eq. (7.537)] *vs* $r_{0c}-r_0 = \bar r_0(T_c-T)$ for fixed u_0.

which will be referred to as the Ginzburg scale, the Josephson scale and the correlation scale, respectively (p_G and p_c were previously defined in section 7.7.3 while the Josephson length $\xi_J = p_J^{-1}$ was discussed in Sec. 7.7.2). Here and in the following we assume $d<4$ and postpone the case $d=4$ to section 7.B.3. These momentum scales are not independent since

$$p_c^2 = p_G^2 \left(\frac{p_J}{p_G}\right)^{d-2} . \tag{7.538}$$

If we vary r_0 (i.e. the temperature) with u_0 fixed, we find that the three characteristic scales (7.537) are equal for a temperature T_G defined by

$$r_{0c} - r_{0G} = \bar r_0 (T_c - T_G) = \frac{1}{2}\left(\frac{u_0 A_d}{6}\right)^{2/(4-d)} \tag{7.539}$$

(see Fig. 7.31). Equation (7.539) is similar to the Ginzburg criterion (7.117) obtained from the one-loop calculation of the two-point vertex in the disordered phase. As we show below, T_G separates a critical regime from a non-critical regime in the ordered phase.

In the critical regime ($T_c - T \ll T_c - T_G$ or $p_J \ll p_G$), using $p_J \ll p_c \ll p_G$ one finds

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{p_J^{2-d}}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_J, \\ \dfrac{1}{\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \end{cases} \tag{7.540}$$

while in the non-critical regime ($T_c - T_G \ll T_c - T$ or $p_G \ll p_c$),

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{1}{p_c^2}\left(\dfrac{p_G}{|\mathbf{p}|}\right)^{4-d} & \text{if} \quad |\mathbf{p}| \ll p_G, \\ \dfrac{1}{\mathbf{p}^2+p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G. \end{cases} \tag{7.541}$$

In the non-critical regime, we recover the results of section 7.7.3. We find two characteristic momentum scales (p_G and p_c) and two regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime ($|\mathbf{p}| \lesssim p_G$) characterized by a diverging longitudinal propagator $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ in the limit $\mathbf{p}\to 0$, ii) a Gaussian regime ($|\mathbf{p}| \gtrsim p_G$) where the

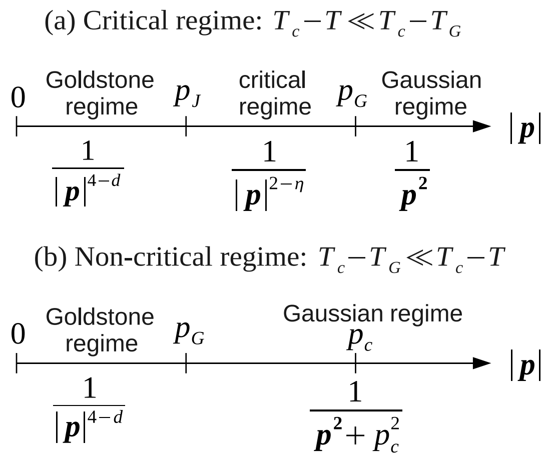


Figure 7.32: Momentum dependence of the longitudinal correlation function $G_{\sigma\sigma}(\mathbf{p})$ in the critical and non-critical regimes of the low-temperature phase ($2 < d < 4$). Note that $\eta = 0$ in the limit $N \to \infty$.

Gaussian approximation (and the perturbative approach) is essentially correct. The critical regime is characterized by two momentum scales (p_J and p_G). The Josephson length diverges at the phase transition with the exponent $\nu = 1/(d-2)$. The same exponent was found for the divergence of the correlation length ξ in the disordered phase [Eq. (7.513)]. There are three regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime ($|\mathbf{p}| \lesssim p_J$) with a diverging longitudinal propagator, ii) a critical regime ($p_J \lesssim |\mathbf{p}| \lesssim p_G$) with a vanishing anomalous dimension η (η is $\mathcal{O}(1/N)$ in the large-N limit, see Sec. 7.A.2), iii) a Gaussian regime ($p_G \lesssim |\mathbf{p}|$). These results are summarized in figure 7.32.

The non-linear sigma model

At the critical point in the limit $N \to \infty$, from the preceding results we deduce

$$\begin{aligned} G_{\lambda\lambda}(\mathbf{p}) &= \frac{2}{N A_d |\mathbf{p}|^{d-4}}, \\ G_{\sigma\sigma}(\mathbf{p}) &= \frac{1}{\mathbf{p}^2}, \end{aligned} \tag{7.542}$$

so that $[\sigma] = (d-2)/2$ and $[\lambda] = 2$. It follows that the perturbation $\int d^d r\, \lambda^2$ is irrelevant for $d < 4$. Thus, if we shift the λ field by its fixed-point value, $i\lambda \to i\lambda + m^2$, we can omit the λ^2 term in the action,

$$\begin{aligned} S[\sigma, \lambda] = \frac{1}{2} \int d^d r\, \left[(\boldsymbol{\nabla}\sigma)^2 + (m^2 + i\lambda)\sigma^2 \right] \\ - \frac{3N}{u_0} \int d^d r\, i\lambda(m^2 - r_0) + \frac{N-1}{2} \mathrm{Tr} \ln g^{-1}. \end{aligned} \tag{7.543}$$

We can now reintroduce the $\boldsymbol{\pi}$ field using

$$\exp\left\{-\frac{N-1}{2}\mathrm{Tr}\ln g^{-1}\right\} = \int\mathcal{D}[\boldsymbol{\pi}]\exp\left\{-\frac{1}{2}\int d^dr\,\left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2+(i\lambda+m^2)\boldsymbol{\pi}^2\right]\right\} \tag{7.544}$$

to obtain

$$S[\boldsymbol{\varphi},\lambda] = \frac{1}{2}\int d^dr\,\left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^2+i\lambda\boldsymbol{\varphi}^2\right] - \frac{3N}{u_0}\int d^dr\,(i\lambda-m^2)(m^2-r_0), \tag{7.545}$$

where we have shifted λ back to its original value. Integrating over λ, we eventually obtain

$$Z = \int\mathcal{D}[\boldsymbol{\varphi}]\,\delta(\boldsymbol{\varphi}^2-\boldsymbol{\varphi}_0^2)\exp\left\{-\frac{1}{2}\int d^dr\,(\boldsymbol{\nabla}\boldsymbol{\varphi})^2\right\} \tag{7.546}$$

with $\boldsymbol{\varphi}_0^2=\frac{6N}{u_0}(m^2-r_0)$, which is nothing but the action of the non-linear sigma model (see Sec. 7.7). We conclude that in the limit $N\to\infty$, the correlation functions of the $(\boldsymbol{\varphi}^2)^2$ theory and the non-linear sigma model have the same asymptotic long-distance behavior at the critical point. It can be shown that this equivalence holds to all orders in the $1/N$ expansion [42]. The non-linear sigma model in the large-N limit is studied in section 7.C.

7.B.2 Gibbs free energy

Let us consider the system in the presence of an external field,

$$Z[h] = \int\mathcal{D}[\sigma,\lambda]\,e^{-S[\sigma,\lambda]+\int d^dr\,h\sigma}, \tag{7.547}$$

where the action $S[\sigma,\lambda]$ is defined by (7.527). The Gibbs free energy reads

$$\Gamma[M] = -\ln Z[h] + \int d^dr\,hM, \tag{7.548}$$

where h is related to the order parameter $M(\mathbf{r})=\langle\sigma(\mathbf{r})\rangle$ by

$$M(\mathbf{r}) = \frac{\delta\ln Z[h]}{\delta h(\mathbf{r})}. \tag{7.549}$$

In the large-N limit, the saddle-point approximation in (7.547) is exact so that $\Gamma[M]=S[M,\lambda]$. For a uniform order parameter,

$$\frac{1}{V}\Gamma(M) = \frac{1}{2}m^2M^2 - \frac{3N}{2u_0}(m^2-r_0)^2 + \frac{N}{2}\int_{\mathbf{p}}\ln(\mathbf{p}^2+m^2). \tag{7.550}$$

The momentum integral in (7.550) diverges for small m and should be regularized, e.g. by considering

$$D(m^2) = \int_{\mathbf{p}}[\ln(\mathbf{p}^2+m^2)-\ln(\mathbf{p}^2)], \tag{7.551}$$

which amounts to removing an infinite normalization constant in the partition function. For $m \to 0$ and $d < 4$,[106]

$$D(m^2) = -2\frac{\tilde{K}_d}{d} m^d + K_d \frac{\Lambda^{d-2}}{d-2} m^2 + \frac{K_d}{2} \frac{\Lambda^{d-4}}{4-d} m^4, \tag{7.552}$$

where $\tilde{K}_d$ is defined in (7.116). For $d < 4$, we can neglect m^4 wrt m^d for small m and we obtain[107]

$$\frac{1}{NV}\Gamma(M) = -\frac{3r_0^2}{2u_0} + \frac{3}{u_0}(r_0 - r_{0c})m^2 + \frac{m^2 M^2}{2N} - \frac{\tilde{K}_d}{d} m^d. \tag{7.553}$$

The value of m^2 is obtained by requiring that $\Gamma[M]$, as $-\ln Z[h]$, is extremum.[108] The condition $\frac{\partial}{\partial m^2}\Gamma[M] = 0$ gives

$$m^2 = \left[\frac{1}{\tilde{K}_d} \left(\frac{6\tau}{u_0} + \frac{M^2}{N} \right) \right]^{2/(d-2)} \tag{7.554}$$

with $\tau = r_0 - r_{0c} = \bar{r}_0(T - T_c)$. This expression makes sense only if $\frac{6\tau}{u_0} + \frac{M^2}{N} > 0$. If not, the extremum is reached for $m^2 = 0$.

In the high-temperature phase, $\tau \geq 0$ and $m^2 \neq 0$ for any value of M. This yields

$$\frac{1}{NV}\Gamma(M) = -\frac{3r_0^2}{2u_0} + \frac{d-2}{2d} \tilde{K}_d^{2/(2-d)} \left(\frac{6\tau}{u_0} + \frac{M^2}{N} \right)^{d/(d-2)}. \tag{7.555}$$

The minimum of the Gibbs free energy is reached for $M = 0$. We deduce the specific heat

$$C_V = -T \frac{\partial^2}{\partial T^2} \beta^{-1} \Gamma(M = 0) \sim \tau^{-\alpha}, \tag{7.556}$$

with a critical exponent $\alpha = (d-4)/(d-2)$. At the transition ($\tau = 0$),

$$\frac{1}{NV}\Gamma(M) \sim M^{2d/(d-2)} + \text{const} \quad \text{and} \quad h = \frac{1}{V}\frac{\partial \Gamma}{\partial M} \sim M^\delta \tag{7.557}$$

with a critical exponent $\delta = (d+2)/(d-2)$. The values of α and δ agree with (7.513).

In the low-temperature phase, $\tau \leq 0$ and m^2 can be zero or nonzero depending on the value of M. Thus

$$\frac{1}{NV}\Gamma(M) = \begin{cases} -\dfrac{3r_0^2}{2u_0} & \text{if} \quad |M| \leq M_0, \\ -\dfrac{3r_0^2}{2u_0} + \dfrac{d-2}{2d} \tilde{K}_d^{2/(2-d)} \left(\dfrac{M^2 - M_0^2}{N} \right)^{d/(d-2)} & \text{if} \quad |M| \geq M_0, \end{cases} \tag{7.558}$$

[106] Eq. (7.552) is obtained from $D(0) = 0$, $D'(m^2) = \frac{\Lambda^d}{dm^2}\, {}_2F_1\left(1, \frac{d}{2}, \frac{d}{2}+1, -\frac{\Lambda^2}{m^2}\right)$, and the expansion of the hypergeometric function ${}_2F_1$ for small m^2.

[107] We use $\frac{1}{2}\frac{K_d}{d-2}\Lambda^{d-2} = -\frac{3r_{0c}}{u_0}$.

[108] This follows from the property $\frac{\partial \Gamma[M, m^2]}{\partial m^2}\big|_M = -\frac{\partial \ln Z[h, m^2]}{\partial m^2}\big|_h$, which is a direct consequence of the definition (7.548) of the effective action.

where

$$M_0 = \sqrt{\frac{6N|\tau|}{u_0}} \tag{7.559}$$

is equal to the saddle point value of the σ field [Eq. (7.530)]. Thus the large-N approach gives a convex Gibbs free energy both in the high- and low-temperature phases (see Fig. 7.4 in Sec. 7.1.1).

7.B.3 The upper critical dimension

The results obtained in sections 7.B.1 and 7.B.3 for $2 < d < 4$ can easily be extended to the case $d = d_c^+ = 4$. Using the small $\mathbf{p}$ behavior (7.516) of $\Pi(\mathbf{p})$ when $d = 4$, the three characteristic momentum scales introduced in section 7.B.1 become

$$\begin{aligned} p_G &= \Lambda \exp\left(\frac{-6}{u_0 A_4}\right), \\ p_J &= \left[\frac{24(r_{0c} - r_0)}{u_0 A_4 \ln\left(\frac{u_0 A_4 \Lambda^2}{24(r_{0c}-r_0)}\right)}\right]^{1/2}, \\ p_c &= [2(r_{0c} - r_0)]^{1/2}. \end{aligned} \tag{7.560}$$

As expected, we find a mean-field like divergence of the Josephson length ($\xi_J = p_J^{-1} \sim (T_c - T)^{-1/2}$) with logarithmic corrections. The three momentum scales (7.560) satisfy

$$p_J^2 \simeq p_c^2 \frac{\ln(\Lambda/p_G)}{\ln(\Lambda/p_c)}. \tag{7.561}$$

and therefore coincide at the Ginzburg temperature T_G defined by

$$r_{0c} - r_{0G} = \bar{r}_0(T_c - T_G) \simeq \frac{\Lambda^2}{2} \exp\left(\frac{-12}{u_0 A_4}\right). \tag{7.562}$$

In the critical regime ($T_c - T \ll T_c - T_G$ or $p_J \ll p_G$), one finds

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{\ln(\Lambda/|\mathbf{p}|)}{p_J^2 \ln(\Lambda/p_J)} & \text{if} \quad |\mathbf{p}| \ll p_J, \\ \dfrac{1}{\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \end{cases} \tag{7.563}$$

while in the non-critical regime ($T_c - T_G \ll T_c - T$ or $p_G \ll p_c$),

$$G_{\sigma\sigma}(\mathbf{p}) = \begin{cases} \dfrac{\ln(\Lambda/|\mathbf{p}|)}{p_c^2 \ln(\Lambda/p_G)} & \text{if} \quad |\mathbf{p}| \ll p_G, \\ \dfrac{1}{\mathbf{p}^2 + p_c^2} & \text{if} \quad |\mathbf{p}| \gg p_G. \end{cases} \tag{7.564}$$

The behavior of the longitudinal correlation function is similar to the case $d < 4$ (Fig. 7.32) except that the divergence $\sim 1/|\mathbf{p}|^{4-d}$ in the Goldstone regime is now logarithmic, and the anomalous dimension η vanishes (Sec. 7.A).

To compute the Gibbs free energy $\Gamma(M)$ [Eq. (7.550)], we use

$$D(m^2) = \frac{K_4}{2}\left[m^2\Lambda^2 + m^4\left(\ln\frac{m}{\Lambda} - \frac{1}{4}\right)\right] + \mathcal{O}(m^6). \tag{7.565}$$

Minimizing $\Gamma(M)$ wrt m^2, we then obtain

$$m^2 \simeq \begin{cases} \dfrac{2X}{K_4 \ln\left(\frac{K_4\Lambda^2}{2X}\right)} & \text{if} \quad X \geq 0, \\ 0 & \text{if} \quad X \leq 0, \end{cases} \tag{7.566}$$

where

$$X = \frac{M^2}{N} + \frac{6\tau}{u_0} \tag{7.567}$$

and $\tau = r_0 - r_{0c} = \bar{r}_0(T - T_c)$. In the high-temperature phase, this leads to

$$\frac{1}{NV}\Gamma(M) = -\frac{3r_0^2}{2u_0} + \frac{X^2}{2K_4\ln\left(\frac{K_4\Lambda^2}{2X}\right)}. \tag{7.568}$$

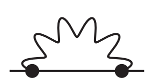
The minimum is reached for $M = 0$ and the singular part of $\Gamma(M = 0)$ is given by $\tau^2/\ln\tau$. The singular part of the specific heat in the large-N limit, $C_V \sim 1/|\ln\tau|$, agrees with the general result (7.267) of section 7.6.2. In the low-temperature phase, we find

$$\frac{1}{NV}\Gamma(M) = \begin{cases} -\dfrac{3r_0^2}{2u_0} & \text{if} \quad |M| \leq M_0, \\ -\dfrac{3r_0^2}{2u_0} + \dfrac{1}{2K_4N^2}\dfrac{(M^2 - M_0^2)^2}{\ln\left(\frac{\Lambda^2 K_4 N}{2(M^2 - M_0^2)}\right)} & \text{if} \quad |M| \geq M_0, \end{cases} \tag{7.569}$$

where M_0 is equal to the saddle-point value of the σ field [Eq. (7.559)].

7.B.4 $1/N$ correction

The $\mathcal{O}(1/N)$ correction to the propagator $G_{\sigma\sigma}$ comes from the one-loop self-energy diagram



where the dot stands for the vertex $i\lambda\sigma^2$ (see Eq. (7.527)). $G_{\sigma\sigma}$ (solid line) and $G_{\lambda\lambda}$ (wavy line) are the propagators in the limit $N \to \infty$. Thus,

$$\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p}) = \mathbf{p}^2 + \int_{\mathbf{q}} G_{\lambda\lambda}(\mathbf{q})\left[G_{\sigma\sigma}(\mathbf{p}+\mathbf{q}) - G_{\sigma\sigma}(\mathbf{q})\right] + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{7.570}$$

at the critical point ($T = T_c$). As in Sec. 7.A.1, the subtraction of $G_{\sigma\sigma}(\mathbf{q})$ in (7.570) ensures that $\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p} = 0) = 0$ when $T = T_c$. Using (7.542) we finally obtain

$$\Gamma^{(2)}_{\sigma\sigma}(\mathbf{p}) = \mathbf{p}^2 + \frac{2}{NA_d}\int_{\mathbf{q}} \frac{1}{|\mathbf{q}|^{d-4}}\left[\frac{1}{(\mathbf{p}+\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{7.571}$$

which agrees with our previous result (7.518) and yields the result (7.522) for the anomalous dimension η to $\mathcal{O}(1/N)$.

7.C The non-linear sigma model in the large-N limit

In section 7.B, we have studied the $(\boldsymbol{\varphi}^2)^2$ theory in the large-N limit and shown that the critical behavior is the same as that of the non-linear sigma model. In this section, we directly consider the non-linear sigma model in the large-N limit (along similar lines). We start from the partition function

$$Z = \int \mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2 - 1) \exp\left\{ -\frac{N}{2g} \int d^d r \, (\boldsymbol{\nabla}\mathbf{n})^2 \right\} \tag{7.572}$$

with an implicit ultraviolet cutoff Λ on the momenta. The factor N in (7.572) is introduced to yield a meaningful limit $N \to \infty$. As in section 7.7 we write the field $\mathbf{n} = (\sigma, \boldsymbol{\pi})$ in terms of a scalar field σ and a $(N-1)$-component field $\boldsymbol{\pi}$. Introducing a Lagrange multiplier field λ to impose the constraint $\mathbf{n}^2 = 1$, we obtain

$$\begin{aligned} Z &= \int \mathcal{D}[\sigma, \boldsymbol{\pi}, \lambda] \exp\left\{ -i \int d^d r \frac{\lambda}{2}(\sigma^2 + \boldsymbol{\pi}^2 - 1) - \frac{N}{2g} \int d^d r \, \left[(\boldsymbol{\nabla}\sigma)^2 + (\boldsymbol{\nabla}\boldsymbol{\pi})^2 \right] \right\} \\ &= \int \mathcal{D}[\sigma, \lambda] \exp\left\{ -i \int d^d r \frac{\lambda}{2}(\sigma^2 - 1) - \frac{N}{2g} \int d^d r \, (\boldsymbol{\nabla}\sigma)^2 - \frac{N-1}{2} \mathrm{Tr} \ln g_\pi^{-1} \right\}, \end{aligned} \tag{7.573}$$

where $g_\pi^{-1}(\mathbf{r}, \mathbf{r}') = -\frac{N}{g} \boldsymbol{\nabla}^2 \delta(\mathbf{r} - \mathbf{r}') + i\lambda(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')$. The second line in (7.573) is obtained by integrating out the $\boldsymbol{\pi}$ field. If we take the limit $N \to \infty$ (and rescale the field $\lambda \to N\lambda$), the action becomes proportional to N and the saddle-point approximation is then exact. For uniform fields σ and λ, the saddle-point equations read

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= 1 - g \int_{\mathbf{p}} \frac{1}{\mathbf{p}^2 + m^2}, \end{aligned} \tag{7.574}$$

where $m^2 = i\lambda g/N$. The critical coupling constant separating the low- and high-temperature phases is given by

$$g_c = \frac{d-2}{K_d \Lambda^{d-2}}. \tag{7.575}$$

g_c vanishes in two dimensions and the system is always in the disordered phase in agreement with the Mermin-Wagner theorem.

7.C.1 The high-temperature phase

In the high-temperature phase, $\sigma = 0$ and m is determined by the equation

$$\frac{1}{g_c} - \frac{1}{g} = m^{d-2} K_d \int_0^{\Lambda/m} dx \frac{x^{d-3}}{x^2 + 1}. \tag{7.576}$$

For $d < 4$, one can take the limit $\Lambda/m \to \infty$ since the integral is convergent. This gives a correlation length

$$\xi = m^{-1} \sim (g - g_c)^{-1/(d-2)} \tag{7.577}$$

for $g \to g_c$ so that $\nu = 1/(d-2)$. In two dimensions, the correlation length is determined from

$$\frac{1}{g} = \int_{\mathbf{p}} \frac{1}{\mathbf{p}^2 + m^2} \simeq \frac{1}{2\pi} \ln\left(\frac{\Lambda}{m}\right), \tag{7.578}$$

which gives

$$\xi \sim \Lambda^{-1} \exp\left(\frac{2\pi}{g}\right). \tag{7.579}$$

ξ diverges exponentially for small g in agreement with the RG analysis of section 7.7.2 in the large-N limit.[109]

Upper critical dimension

At the upper critical dimension $d_c^+ = 4$,

$$\xi^{-2} = m^2 \sim \frac{2\tau/g}{K_4 \ln\left(\frac{\Lambda^2 K_4 g}{\tau}\right)}, \tag{7.580}$$

where $\tau = g/g_c - 1$. The mean-field expectation $\xi \sim 1/\sqrt{\tau}$ is modified by logarithmic corrections.

7.C.2 The low-temperature phase

In the low-temperature phase, $m = 0$ and the order parameter is determined by

$$\sigma^2 = 1 - g \int_{\mathbf{p}} \frac{1}{\mathbf{p}^2} = 1 - \frac{g}{g_c}, \tag{7.581}$$

which gives a critical exponent $\beta = 1/2$.

The propagator of $\boldsymbol{\pi}$ field is given by $g(\mathbf{p}) = g/N\mathbf{p}^2$. To obtain the propagator of the longitudinal field σ, we proceed as in section 7.B.1. In the limit $N \to \infty$, the effective action $\Gamma[\sigma, \lambda] = S[\sigma, \lambda]$ and

$$\Gamma^{(2)}(\mathbf{p}) = \begin{pmatrix} \dfrac{N\mathbf{p}^2}{g} & i\sigma \\ i\sigma & \frac{N}{2}\Pi(\mathbf{p}) \end{pmatrix}, \tag{7.582}$$

where $\Pi(\mathbf{p}) = \int_{\mathbf{q}} g_\pi(\mathbf{q}) g_\pi(\mathbf{p}+\mathbf{q})$. We deduce

$$\det \Gamma^{(2)}(\mathbf{p}) = \frac{1}{2} A_d g \left(|\mathbf{p}|^{d-2} + p_J^{d-2}\right) \tag{7.583}$$

for $|\mathbf{p}| \ll \Lambda$ and $d < 4$. The Josephson momentum scale is defined by

$$p_J = \left(\frac{2\sigma^2}{A_d g}\right)^{1/(d-2)} = \left[\frac{2}{A_d}\left(\frac{1}{g} - \frac{1}{g_c}\right)\right]^{1/(d-2)} \tag{7.584}$$

[109] Without the factor N in (7.572), we would obtain $\xi \sim \Lambda^{-1} \exp\left(\frac{2\pi}{Ng}\right)$ in agreement with (7.302) for $N \to \infty$.

and vanishes with the critical exponent $\nu = 1/(d-2)$. By inverting $\Gamma^{(2)}(\mathbf{p})$, we obtain the propagator

$$G_{\sigma\sigma}(\mathbf{p}) \simeq \frac{g|\mathbf{p}|^{d-4}/N}{|\mathbf{p}|^{d-2} + p_J^{d-2}} = \begin{cases} \dfrac{g}{N\mathbf{p}^2} & \text{if} \quad |\mathbf{p}| \gg p_J, \\ \dfrac{g}{N}\dfrac{p_J^{2-d}}{|\mathbf{p}|^{4-d}} & \text{if} \quad |\mathbf{p}| \ll p_J. \end{cases} \tag{7.585}$$

If g is sufficiently close to g_c, then $p_J \ll \Lambda$ and the system is in the critical regime. One can then distinguish two regimes for the behavior of $G_{\sigma\sigma}(\mathbf{p})$: i) a Goldstone regime $|\mathbf{p}| \ll p_J$ characterized by a diverging longitudinal susceptibility $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$, ii) a critical regime $|\mathbf{p}| \gg p_J$ where $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{2-\eta}$ (with $\eta = \mathcal{O}(1/N)$). On the other hand, if $p_J \gtrsim \Lambda$, the system is in a non-critical regime and the longitudinal propagator exhibits the behavior $G_{\sigma\sigma}(\mathbf{p}) \sim 1/|\mathbf{p}|^{4-d}$ for any value of the momentum. These results are similar to those obtained from the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory. Note however that the Ginzburg momentum scale p_G does not show up in the non-linear sigma model. The same conclusion was reached from the RG analysis of section 7.7.2.

7.C.3 Gibbs free energy

The Gibbs free energy can be obtained as in section 7.B.2. In the large-N limit and for a uniform order parameter M,

$$\frac{1}{NV}\Gamma(M) = \frac{m^2}{2g}(M^2 - 1) + \frac{1}{2}\int_{\mathbf{p}} \left[\ln\left(\frac{\mathbf{p}^2 + m^2}{g} \right) - \left(\frac{\mathbf{p}^2}{g} \right) \right], \tag{7.586}$$

where the last (M independent) term is introduced to make the Gibbs free energy finite. Using (7.552) for $d < 4$, one obtains

$$\frac{1}{NV}\Gamma(M) = \frac{m^2}{2g}(M^2 - 1) + \frac{m^2}{2}\left(\frac{1}{g_c} - \frac{1}{g} \right) - \frac{\tilde{K}_d}{d} m^d \tag{7.587}$$

for small m, where $\tilde{K}_d$ is defined by (7.116). By requiring $\Gamma(M)$ to be minimum wrt m^2, one deduces

$$m^2 = \begin{cases} \left(\frac{M^2 + \tau}{g\tilde{K}_d} \right)^{2/(d-2)} & \text{if} \quad M^2 + \tau \geq 0, \\ 0 & \text{if} \quad M^2 + \tau \leq 0. \end{cases} \tag{7.588}$$

This yields

$$\frac{1}{NV}\Gamma(M) = \frac{d-2}{2d}\tilde{K}_d^{-2/(d-2)} \left(\frac{M^2 + \tau}{g} \right)^{d/(d-2)} \tag{7.589}$$

in the high-temperature phase ($\tau > 0$) and

$$\frac{1}{NV}\Gamma(M) = \Theta(M^2 - M_0^2)\frac{d-2}{2d}\tilde{K}_d^{-2/(d-2)} \left(\frac{M^2 - M_0^2}{g} \right)^{d/(d-2)} \tag{7.590}$$

in the low-temperature phase ($\tau < 0$), where $M_0 = \sqrt{-\tau}$ is equal to the saddle-point value of the σ field [Eq. (7.581)]. The results (7.589) and (7.590) are similar to those obtained in the large-N limit of the $(\boldsymbol{\varphi}^2)^2$ theory (Sec. 7.B.2).

7.D The quantum non-linear sigma model in the large-N limit

In this section we consider the large-N limit of the quantum non-linear sigma model defined by the partition function

$$Z = \int \mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2-1)\exp\biggl\{-\frac{N}{2g}\int_0^\beta d\tau \int d^dr\, \left[(\boldsymbol{\nabla}\mathbf{n})^2 + c^{-2}(\partial_\tau \mathbf{n})^2\right]\biggr\} \tag{7.591}$$

with an implicit ultraviolet cutoff Λ on the momenta. We introduce a Lagrange multiplier field λ to impose the constraint $\mathbf{n}^2=1$ and use the parametrization $\mathbf{n} = (\sigma, \boldsymbol{\pi})$ (Sec. 7.7). Integrating out the $\boldsymbol{\pi}$ field, we obtain

$$\begin{aligned} Z = \int \mathcal{D}[\sigma,\lambda]\exp\biggl\{ & -i\int_0^\beta d\tau \int d^dr\, \frac{\lambda}{2}(\sigma^2-1) \\ & - \frac{N}{2g}\int_0^\beta d\tau\int d^dr\, \left[(\boldsymbol{\nabla}\sigma)^2 + c^{-2}(\partial_\tau\sigma)^2\right] - \frac{N-1}{2}\mathrm{Tr}\ln g_\pi^{-1}\biggr\}, \end{aligned} \tag{7.592}$$

where

$$g_\pi^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau') = \left[-\frac{N}{g}(\boldsymbol{\nabla}^2 + c^{-2}\partial_\tau^2) + i\lambda(\mathbf{r},\tau)\right]\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau') \tag{7.593}$$

is the inverse propagator of the π_i field. In the limit $N\to\infty$, the saddle-point approximation becomes exact. For uniform and time-independent fields, the saddle-point equations read

$$\begin{aligned} \sigma m^2 &= 0, \\ \sigma^2 &= 1 - gc^2\int_p \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2}, \end{aligned} \tag{7.594}$$

where $m^2 = i\lambda gc^2/N$. We use the notation $\int_p = \frac{1}{\beta}\sum_{\omega_n}\int_{\mathbf{p}}$ with $p = (\mathbf{p}, i\omega_n)$.

7.D.1 Zero temperature

When $T=0$, the quantum non-linear sigma model is analog to the classical non-linear sigma model in dimension $d+1$ and its analysis closely follows that of section 7.C.[110] There is a quantum phase transition between a disordered phase ($\sigma=0$ and $m^2>0$) and an ordered phase ($m^2=0$ and $\sigma>0$) for a critical value g_c of the coupling, obtained by setting $\sigma=0$ and $m=0$ in (7.594),

$$\frac{1}{g_c} = c^2\int_p \frac{1}{\omega^2 + c^2\mathbf{p}^2} = \frac{c}{2}\frac{K_d\Lambda^{d-1}}{d-1}. \tag{7.595}$$

[110] For this reason, we shall not repeat many of the technical details already discussed in Sec. 7.C.

Disordered phase

In the disordered phase $g \geq g_c$, the zero-temperature gap m_0 is determined by the equation

$$\begin{aligned} \frac{1}{g_c} - \frac{1}{g} &= \frac{m_0^2}{c} \int_{\mathbf{P}} \frac{1}{\mathbf{P}^2(\mathbf{P}^2 + m_0^2/c^2)} \\ &= c \left(\frac{m_0}{c} \right)^{d-1} K_{d+1} \int_0^\infty dx \frac{x^{d-2}}{x^2+1}, \end{aligned} \tag{7.596}$$

where $\mathbf{P} = (\mathbf{p}, \omega/c)$ is a $(d+1)$-dimensional vector. We have sent the cutoff to infinity since the integral over $\mathbf{P}$ converges (assuming $d < 3$). We find a gap

$$m_0 \sim \left(\frac{1}{g_c} - \frac{1}{g} \right)^{1/(d-1)} \tag{7.597}$$

and a correlation length

$$\xi = \frac{c}{m_0} \sim \left(\frac{1}{g_c} - \frac{1}{g} \right)^{-1/(d-1)}. \tag{7.598}$$

Since $m_0 \sim (g - g_c)^{\nu z}$ and $\xi \sim (g - g_c)^{-\nu}$, we deduce that

$$\nu = \frac{1}{d-1} \tag{7.599}$$

while the dynamical critical exponent z is equal to one. As expected, ν is equal to the critical exponent of the classical non-linear sigma model in dimensions $d + z = d+1$. When $d = 3$ (upper critical dimension), a similar calculation as in the classical case shows that the mean-field result $m_0 \sim (g - g_c)^{1/2}$ is modified by logarithmic corrections.

Thus in the zero-temperature disordered phase, we can define a characteristic energy scale $\Delta_+ = m_0$ and a characteristic length scale $\xi = c/\Delta_+$.

Ordered phase

In the ordered phase ($m_0 = 0$), the value of the order parameter is determined by

$$\sigma^2 = 1 - gc^2 \int_p \frac{1}{\omega^2 + c^2 \mathbf{p}^2} = 1 - \frac{g}{g_c} \tag{7.600}$$

and the critical exponent β takes the value 1/2. The transverse susceptibility is given by

$$\chi_\perp(p) = g_\pi(p) = \frac{g/N}{\mathbf{p}^2 + \omega^2/c^2}, \tag{7.601}$$

which allows us to identify the stiffness

$$\rho_s = \frac{N\sigma^2}{g} = N \left(\frac{1}{g} - \frac{1}{g_c} \right) \tag{7.602}$$

(see Sec. 3.6.3). The latter vanishes linearly with $g_c - g$ in agreement with the general result $\rho_s \sim (g_c - g)^{\nu(d+z-2)}$, i.e. $[\rho_s] = d + z - 2$ (Sec. 7.10.2).

In section 7.10.4 we have seen that the stiffness defines a characteristic length scale, the Josephson length

$$\xi_J = \left(\frac{cN}{\rho_s} \right)^{1/(d-1)}, \tag{7.603}$$

and therefore a characteristic energy scale

$$\Delta_- = \frac{c}{\xi_J} = \left(\frac{\rho_s}{N} \right)^{1/(d-1)} c^{(d-2)/(d-1)}. \tag{7.604}$$

To understand the physical meaning of Δ_- and ξ_J, we compute the longitudinal propagator $G_{\sigma\sigma}(p)$. Since the effective action $\Gamma[\sigma, \lambda]$ is simply given by the action $S[\sigma, \lambda]$ in the large-N limit, we deduce

$$\Gamma^{(2)}(p) = \begin{pmatrix} \frac{N}{gc^2}(\omega^2 + c^2\mathbf{p}^2) & i\sigma \\ i\sigma & \frac{N}{2}\Pi(p) \end{pmatrix} \tag{7.605}$$

(compare with Eq. (7.582)), where

$$\begin{aligned} \Pi(p) &= \int_q g_\pi(q) g_\pi(p+q) \\ &\simeq \frac{g^2}{N^2} c^{4-d} A_{d+1} (\omega^2 + c^2\mathbf{p}^2)^{(d-3)/2} \end{aligned} \tag{7.606}$$

for $|\mathbf{p}|, |\omega|/c \ll \Lambda$ and $d < 3$. Thus, in the low-energy limit,

$$\begin{aligned} \det \Gamma^{(2)}(p) &= \frac{1}{2} g c^{2-d} A_{d+1} (\omega^2 + c^2\mathbf{p}^2)^{(d-1)/2} + \sigma^2 \\ &= \frac{1}{2} g c^{2-d} A_{d+1} \left[(\omega^2 + c^2\mathbf{p}^2)^{(d-1)/2} + \tilde{\Delta}_-^{d-1} \right], \end{aligned} \tag{7.607}$$

where

$$\tilde{\Delta}_- = \left(\frac{2}{A_{d+1}} \right)^{1/(d-1)} \Delta_-. \tag{7.608}$$

By inverting $\Gamma^{(2)}$, we obtain the longitudinal propagator

$$\begin{aligned} G_{\sigma\sigma}(p) &= \frac{gc^2}{N} \frac{(\omega^2 + c^2\mathbf{p}^2)^{(d-3)/2}}{(\omega^2 + c^2\mathbf{p}^2)^{(d-1)/2} + \tilde{\Delta}_-^{d-1}} \\ &\simeq \begin{cases} \dfrac{gc^2 \tilde{\Delta}_-^{1-d}/N}{(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}} & \text{if} \quad |\omega|, c|\mathbf{p}| \ll \tilde{\Delta}_-, \\ \dfrac{gc^2/N}{\omega^2 + c^2\mathbf{p}^2} & \text{if} \quad |\omega| \text{ or } c|\mathbf{p}| \gg \tilde{\Delta}_-. \end{cases} \end{aligned} \tag{7.609}$$

As for the classical non-linear sigma model (Sec. 7.C.2), we can therefore distinguish two regimes: i) a Goldstone regime $|\omega|, c|\mathbf{p}| \ll \tilde{\Delta}_-$ characterized by a diverging longitudinal susceptibility $G_{\sigma\sigma}(p) \sim 1/((\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$, ii) a critical regime $|\omega| \gg \tilde{\Delta}_-$ or $c|\mathbf{p}| \gg \tilde{\Delta}_-$ where $G_{\sigma\sigma}(p) \sim 1/(\omega^2 + c^2\mathbf{p}^2)^{1-\eta/2}$ (with $\eta = \mathcal{O}(1/N)$). Thus $\xi_J = c/\Delta_- \sim c/\tilde{\Delta}_-$ is nothing but the Josephson length.

7.D.2 Finite temperatures in two dimensions

At finite temperatures, the second of the saddle-point equations (7.594) reads

$$\sigma^2 = 1 - \frac{gc^2}{\beta} \sum_{\omega_n} \int_{\mathbf{p}} \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2}. \tag{7.610}$$

For $\omega_n = 0$ the momentum integral in (7.610) is infrared divergent if $m = 0$ and $d \leq 2$. We conclude that there is no ordered phase at finite temperatures for $d \leq 2$ (Mermin-Wagner's theorem). The saddle-point equations must be solved with $\sigma = 0$ and $m^2 > 0$. Performing the Matsubara sum in (7.610), we obtain

$$\begin{aligned}\frac{1}{g} &= \frac{c^2}{2} \int_{\mathbf{p}} \frac{\text{cotanh}\left(\frac{\beta}{2}\sqrt{c^2\mathbf{p}^2 + m^2}\right)}{\sqrt{c^2\mathbf{p}^2 + m^2}} \\ &\simeq \frac{T}{2\pi} \left[\ln\sinh\left(\frac{c\Lambda}{2T}\right) - \ln\sinh\left(\frac{m}{2T}\right) \right]\end{aligned} \tag{7.611}$$

for $m \ll c\Lambda$. This gives

$$\begin{aligned}m &\simeq 2T\,\text{asinh}\left[\frac{1}{2}\exp\left(\frac{c\Lambda}{2\pi} - \frac{2\pi}{gT}\right)\right] \\ &\simeq 2T\,\text{asinh}\left\{\frac{1}{2}\exp\left[\frac{2\pi}{T}\left(\frac{1}{g_c} - \frac{1}{g}\right)\right]\right\}\end{aligned} \tag{7.612}$$

for $m, T \ll c\Lambda$, where g_c is the critical value of the coupling constant at the $T = 0$ quantum phase transition [Eq. (7.595)]. We show below that equation (7.612) agrees with the general phase diagram of figure (7.24) (left panel).

Ordered side $g \leq g_c$

When $g \leq g_c$, the system is ordered at $T = 0$. At finite temperatures, the gap m is determined by

$$m = 2T\,\text{asinh}\left[\frac{1}{2}\exp\left(-\frac{2\pi\Delta_-}{T}\right)\right], \tag{7.613}$$

i.e. by the ratio $c/T\xi_J = \Delta_-/T$ between the thermal length c/T and the Josephson length ξ_J. If $T \ll \Delta_-$, we recover the result of the classical two-dimensional non-linear sigma model,

$$m \simeq Te^{-\frac{2\pi\Delta_-}{T}}, \quad \text{i.e.} \quad \xi(T) = \frac{c}{m(T)} \simeq \frac{c}{T}e^{\frac{2\pi\Delta_-}{T}}, \tag{7.614}$$

but with a stiffness $\rho_s = N\Delta_-$ renormalized by quantum fluctuations [Eq. (7.602)]. This is the renormalized classical regime (Sec. 7.10.1). The classical behavior of the system is due to the thermal occupation number

$$n_B(c\xi^{-1}) = \frac{1}{e^{\beta c/\xi} - 1} \simeq \frac{T\xi}{c} \simeq e^{\frac{2\pi\Delta_-}{T}} \gg 1 \tag{7.615}$$

being much larger than one (c/ξ is the typical excitation energy). On the other hand, when $T \gg \Delta_-$,

$$m \sim T, \quad \text{i.e.} \quad \xi \sim \frac{c}{T}, \tag{7.616}$$

the correlation length is fixed by the thermal length c/T and the system is in the quantum critical regime.

Disordered side $g \geq g_c$

When $g \geq g_c$, the system is disordered by quantum fluctuations at $T = 0$. The finite temperature gap m is determined by

$$m \simeq 2T \operatorname{asinh}\left[\frac{1}{2} \exp\left(\frac{m_0}{2T}\right)\right]. \tag{7.617}$$

For $T \ll \Delta_+ = m_0$, the finite temperature gap $m \simeq m_0$ is nearly equal to its $T = 0$ value. The temperature has no significant effect; the system is primarily disordered by quantum fluctuations (quantum disordered regime). On the other hand, when $T \gg \Delta_+$, $m \sim T$ and the system is in the quantum critical regime.

7.D.3 Finite temperatures in dimension $d > 2$

When d is above the lower critical dimension, the system can order at finite temperatures. Let us first consider the case $g \geq g_c$ where the system is disordered at any temperature. The finite temperature gap m is determined by

$$\frac{1}{g} = \frac{c^2}{\beta} \sum_{\omega_n} \int_{\mathbf{p}} \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2}. \tag{7.618}$$

Using the analog equation at $T = 0$, we deduce

$$\begin{aligned} 0 &= \frac{1}{\beta} \sum_{\omega_n} \int_{\mathbf{p}} \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2} - \int_{\mathbf{p}} \int_{\omega} \frac{1}{\omega^2 + c^2\mathbf{p}^2 + m_0^2} \\ &= \frac{1}{\beta} \sum_{\omega_n} \int_{\mathbf{p}} \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2} - \frac{1}{c} \int_{\mathbf{P}} \frac{1}{\mathbf{P}^2 + m_0^2/c^2}, \end{aligned} \tag{7.619}$$

where $\mathbf{P} = (\mathbf{p}, \omega/c)$ is a $(d+1)$-dimensional vector, $\int_{\mathbf{P}} = \int \frac{d^{d+1}P}{(2\pi)^{d+1}}$ and $\int_\omega = \int \frac{d\omega}{2\pi}$. We rewrite this equation as

$$\begin{aligned} &\int_{\mathbf{p}} \left\{ \frac{1}{\beta} \sum_{\omega_n} \frac{1}{\omega_n^2 + c^2\mathbf{p}^2 + m^2} - \int_{\omega} \frac{1}{\omega^2 + c^2\mathbf{p}^2 + m^2} \right\} \\ &+ \frac{1}{c} \int_{\mathbf{P}} \left\{ \frac{1}{\mathbf{P}^2 + m^2/c^2} - \frac{1}{\mathbf{P}^2 + m_0^2/c^2} \right\} = 0. \end{aligned} \tag{7.620}$$

Using

$$\frac{1}{\beta} \sum_{\omega_n} \frac{1}{\omega_n^2 + a^2} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + a^2} = \frac{1}{|a|} \frac{1}{e^{\beta|a|} - 1}, \tag{7.621}$$

we obtain

$$\begin{aligned} &K_d \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|^{d-1}}{\sqrt{c^2\mathbf{p}^2 + m^2}} \frac{1}{e^{\beta\sqrt{c^2\mathbf{p}^2+m^2}} - 1} \\ &+ \frac{K_{d+1}}{c} \int_0^\infty d|\mathbf{P}|\, |\mathbf{P}|^d \left(\frac{1}{\mathbf{P}^2 + m^2/c^2} - \frac{1}{\mathbf{P}^2 + m_0^2/c^2} \right) = 0. \end{aligned} \tag{7.622}$$

Here we assume $d<3$ and send the cutoff Λ to infinity since the integrals are convergent. For $d=3$, there are cutoff-dependent logarithmic corrections we do not discuss. Performing the integral over $|\mathbf{P}|$ and setting $x=c|\mathbf{p}|/T$, we finally obtain

$$K_d T^{d-1}\int_0^\infty dx \frac{x^{d-1}}{\sqrt{x^2+m^2/T^2}}\frac{1}{e^{\sqrt{x^2+m^2/T^2}}-1} - X_{d+1}\left(m^{d-1}-m_0^{d-1}\right)=0, \quad (7.623)$$

where

$$X_d = K_d\int_0^\infty dx \frac{x^{d-3}}{x^2+1} = \frac{2^{-d}\pi^{1-d/2}}{\Gamma(d/2)\sin\left(\frac{\pi}{2}(d-2)\right)} \quad (7.624)$$

for $2<d<4$. The solution is of the form

$$\frac{m}{T}=F_+\left(\frac{m_0}{T}\right), \quad (7.625)$$

where the function $F_+(s)$ is determined by[111]

$$K_d\int_0^\infty dx \frac{x^{d-1}}{\sqrt{x^2+F_+^2(s)}}\frac{1}{e^{\sqrt{x^2+F_+^2(s)}}-1} - X_{d+1}\left[F_+(s)^{d-1}-s^{d-1}\right]=0. \quad (7.626)$$

Let us now consider the case $g\leq g_c$ and assume $\sigma=0$ (which must hold at sufficiently high temperatures). The $T=0$ stiffness (7.602) can be expressed as

$$\frac{\rho_s}{N}=\frac{c^2}{\beta}\sum_{\omega_n}\int_{\mathbf{p}}\frac{1}{\omega_n^2+c^2\mathbf{p}^2+m^2} - c\int_{\mathbf{P}}\frac{1}{\mathbf{P}^2}. \quad (7.627)$$

This equation can be rewritten as

$$\begin{aligned}\frac{\rho_s}{Nc^2} = &\int_{\mathbf{p}}\left\{\frac{1}{\beta}\sum_{\omega_n}\frac{1}{\omega_n^2+c^2\mathbf{p}^2+m^2} - \int_\omega \frac{1}{\omega^2+c^2\mathbf{p}^2+m^2}\right\} \\ &+\frac{1}{c}\int_{\mathbf{P}}\left\{\frac{1}{\mathbf{P}^2+m^2/c^2}-\frac{1}{\mathbf{P}^2}\right\}. \end{aligned} \quad (7.628)$$

As above, the solution is of the form

$$\frac{m}{T}=F_-\left(\frac{\Delta_-}{T}\right), \quad (7.629)$$

where[111]

$$K_d\int_0^\infty dx \frac{x^{d-1}}{\sqrt{x^2+F_-^2(s)}}\frac{1}{e^{\sqrt{x^2+F_-^2(s)}}-1} - X_{d+1}F_-(s)^{d-1} = s^{d-1}. \quad (7.630)$$

As in the two-dimensional case, we can distinguish three regimes. The quantum disordered regime is defined by $g\geq g_c$ and $T\ll\Delta_+$. From (7.626), it is clear that for $s\to\infty$, $F_+(s)\sim s$ up to exponentially small terms $\mathcal{O}(e^{-s})$, i.e.

$$m=m_0+\mathcal{O}(e^{-m_0/T}). \quad (7.631)$$

[111]For $d=2$, one easily obtains $F_+(s)=2\,\mathrm{asinh}\left(\frac{1}{2}e^{s/2}\right)$ and $F_-(s)=2\,\mathrm{asinh}\left(\frac{1}{2}e^{-2\pi s}\right)$, which reproduces the results of Sec. 7.D.2 [Eqs. (7.613,7.617)].

The system is primarily disordered by quantum fluctuations and thermal fluctuations have little effect. In the quantum critical regime $T \gg \Delta_\pm$, $m \simeq TF_+(0) = TF_-(0)$, where $F_\pm(0)$ is a pure number. The correlation length is

$$\xi = \frac{c}{m} \simeq \frac{1}{F_\pm(0)} \frac{c}{T}. \tag{7.632}$$

The renormalized classical regime, dominated by thermal fluctuations, is defined by $g \leq g_c$ and $T \ll \Delta_-$. As long as T is large enough so that σ vanishes, then $F_-(s)$ is determined by (7.630). If F_- vanishes, the integral over x diverges for $d > 2$ and the saddle-point equations cannot be solved with $\sigma = 0$. There is therefore a finite-temperature phase transition at a critical temperature determined by $F_-(s) = 0$, i.e.

$$\begin{aligned} T_c &= \frac{\Delta_-}{s_c}, \\ s_c^{d-1} &= K_d \int_0^\infty dx \frac{x^{d-1}}{x(e^x - 1)} = K_d \Gamma(d-1)\zeta(d-1), \end{aligned} \tag{7.633}$$

where $\zeta(z)$ is the Riemann Zeta function. T_c vanishes when $d \to 2$ as expected. These results are in agreement with the general phase diagram of figure 7.24 (right panel).

Guide to the bibliography

- In addition to Wilson's original papers [1, 2], there are many reviews [3, 4, 5, 6] and books [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] with a detailed presentation of the renormalization group (Ref. [7] contains a general introduction to phase transitions). Many of these references discuss the $\epsilon = 4 - d$ expansion as well as the $1/N$ expansion.

- The field theoretical approach to critical phenomena is discussed in Refs. [11, 17, 18].

- For a discussion of the non-linear sigma model, see Refs. [7, 15, 17, 42, 18, 28, 29, 30, 31, 32].

- The Wilson-Polchinski equation [3, 33] and the functional renormalization group are discussed in Refs. [34, 42]. (See also [35].)

- A pedagogic introduction to quantum phase transitions can be found in Ref. [43]. For a more advanced presentation, see Ref. [44]. The quantum non-linear sigma model is studied in Refs. [44, 45]. For a discussion of the dilute Bose gas, see Refs. [44, 46, 47]. Quantum phase transitions in fermion systems are discussed in Refs. [48, 49, 50, 51].

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