Our goal in this section is to define a function and explore multiple ways to describe a function.

Before we get to functions, let's define something called a **relation**. A relation is simply a rule that describes a relationship between two types of things.

For example, let's imagine we are handwashing dishes at a restaurant and we are paid two dollars per dish. This rule is a relation. It relates the number of dishes washed to amount of money paid. If we wash 30 dishes, we receive 60 dollars. In this situation, we would describe the number of dishes washed as the **input** and the amount of money paid as the **output**. Any time there is a rule relating two things, we call it a relation.

Some relations are more complex. For example, let's imagine we have a relation whose input and output are both people. If you input a person, the output will be all of that person's children. For some people, the relation will output nothing because many people have no children. For others, it will output multiple people because some people have multiple children.

A **function** is just a specific type of relation that prevents the situations we just mentioned.

Definition 1. *A **function** is a relation where for each input, there is exactly one output.*

Question 1 *If our function is the "position with respect to time" of some object, then the input is*

Multiple Choice:

(a) *position*

(b) *time* ✓

(c) *none of the above*

and the output is

Multiple Choice:

(a) *position* ✓

(b) *time*

(c) *none of the above*

Question 2 *Which of the following are functions?*

Select All Correct Answers:

(a) *Mapping words to their definition in a dictionary.*

(b) *Mapping the U.S. Government's list of social security numbers of living people to actual living people.* ✓

(c) *Mapping people to their birth date.* ✓

(d) *Mapping mothers to their children.*

Feedback(*`attempt`*):

- *Since words may have more than one definition, "relating words to their definition in a dictionary" is not a function.*
- *Since every social security number corresponds exactly to one person, "relating social security numbers of living people to actual living people" is a function.*
- *Since every person only has one birth date, "relating people to their birth date" is a function.*
- *Since mothers can have more (or less) than one child, "relating mothers to their children" is not a function.*

Whenever we talk about functions, we should explicitly state what type of things the inputs are and what type of things the outputs are. In calculus, functions often define a relation from (a subset of) the real numbers (denoted by $\mathbb{R}$) to (a subset of) the real numbers.

Example 1. *Consider the function that maps from the real numbers to the real numbers by taking a number and mapping it to its cube:*

$$\begin{aligned} 1 &\mapsto 1 \\ -2 &\mapsto -8 \\ 1.5 &\mapsto 3.375 \end{aligned}$$

and so on.

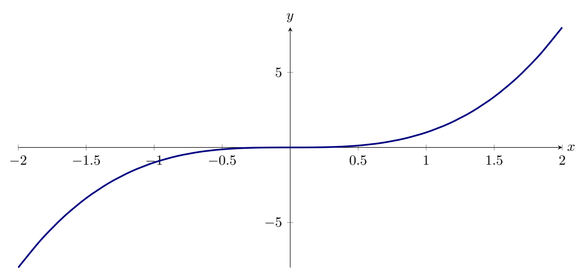
While we can choose to describe this rule with words, like we did here (take the input and cube it to get the output), we would like to come up with some notation which is more concise to write down. The first step is to name our function, which we often call f, and then the notation to say that -2 maps to -8, for example, would be $f(-2) = -8$. Reading this out loud, we say "f of -2 is -8. The input goes into the parentheses, imagine the left side of the equation as f performing its rule (cubing) on -2, and then the right hand side is the output.

Of course, sometimes we want to talk about the rule itself, not acting on a specific number. In this case, we let x represent any real number without being *specific (we call this a* ***variable****) and then the notation becomes* $f(x) = x^3$. *This allows us to talk about the general rule of cubing a number in a concise, symbolic way.*

Warning 1.
- *The parentheses in* $f(x)$ *do* ***NOT*** *represent multiplication. The parentheses in something like* $x^2(x+1)$, *on the other hand,* ***DO*** *represent multiplication, which is why we would be allowed to make a simplification like* $\dfrac{x^2(x+1)}{x+1} = x^2$, *because multiplication and division undo each other.*
- *In this example, we named our function* f, *and we do that in many examples. Keep in mind that for more specific or important functions, we have actual names. Later we will see functions with names like* $\sin, \cos, \ln$ *and those are denoted* $\sin(x)$, $\cos(x)$, $\ln(x)$, *but they are used in exactly the same way as* f *is here.*

Example 2. *Using the same function* $f(x) = x^3$, *notice that we are relating numbers.* $f(1) = 1, f(-2) = -8$, *and so on. If we only care about talking about the input and output, we can use an alternate notation like* $(1, 1), (-2, -8)$, *etc. where the first number is the input and the second is the output. We call this an* ***ordered pair****, ordered because the input must come first, and pair because it is a pair of numbers.*

If we did this for every possible number pair generated by $f(x) = x^3$, *we would get the* ***graph*** *of the function:*



Notice that the graph of the function described above passes the so-called *vertical line test.*

Theorem 1. *The curve* $y = f(x)$ *represents* y *as a function of* x *at* $x = a$, *where* a *is a value in the domain of* f, *if and only if every vertical line* $x = a$ *intersects the curve* $y = f(x)$ *at exactly one point. This is called the* ***vertical line test****.*

If any vertical line drawn at a value, say b, could intersect the curve more than once, then that would mean $x = b$ has two outputs associated with it - that violates the “only one output for each input” rule required to be considered a function.

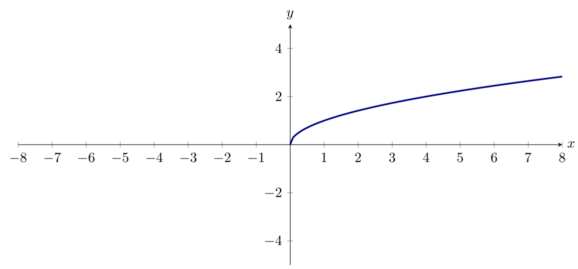
Definition 2. *We call the set of the inputs of a function the **domain**, and we call the set of the outputs of a function the **range**.*

Sometimes the domain and range are the *entire* set of real numbers, denoted by $\mathbb{R}$. In our next examples we show that this is not always the case.

Example 3. *Consider the function that maps non-negative real numbers to their positive square root. This function can be described by the formula*

$$f(x) = \sqrt{x}.$$

The domain is $0 \leq x < \infty$, which we prefer to write as $[0, \infty)$ in interval notation. The range is $[0, \infty)$. Here is a graph of $y = f(x)$:



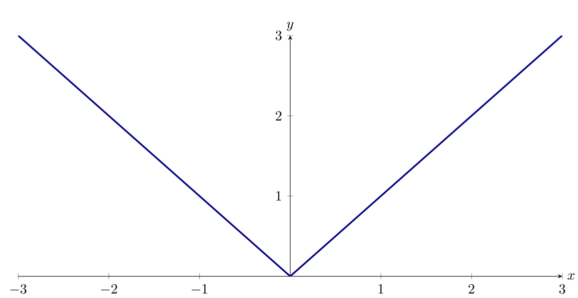
To really tease out the difference between a function and its description, let's consider an example of a function with two different descriptions.

Example 4. *Explain why $\sqrt{x^2} = |x|$.*

Explanation. *Although $\sqrt{x^2}$ may appear to simplify to just x, let's see what happens when we plug in some values.*

$$\begin{aligned}\sqrt{3^2} &= \sqrt{9} \\ &= 3,\end{aligned} \qquad \textit{and} \qquad \begin{aligned}\sqrt{(-3)^2} &= \sqrt{9} \\ &= 3.\end{aligned}$$

Let $f(x) = \sqrt{x^2}$. We see that for any negative x, $x = -|x|$, and, therefore, $f(x) = f(-|x|) = \sqrt{(-|x|)^2} = \sqrt{|x|^2} = \underset{\text{given}}{\boxed{|x|}}$. Hence $\sqrt{x^2} \neq x$. Rather we see that $\sqrt{x^2} = |x|$. The domain of $f(x) = \sqrt{x^2}$ is $(-\infty, \infty)$ and the range is $[0, \infty)$. For your viewing pleasure we've included a graph of $y = f(x)$:



Finally, we will consider a function whose domain is all real numbers except for a single point.

Example 5. *Are*

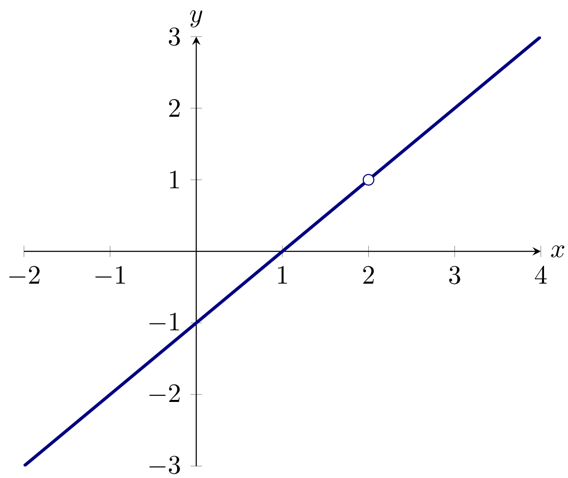
$$f(x) = \frac{x^2 - 3x + 2}{x - 2} = \frac{(x-2)(x-1)}{(x-2)}$$

and

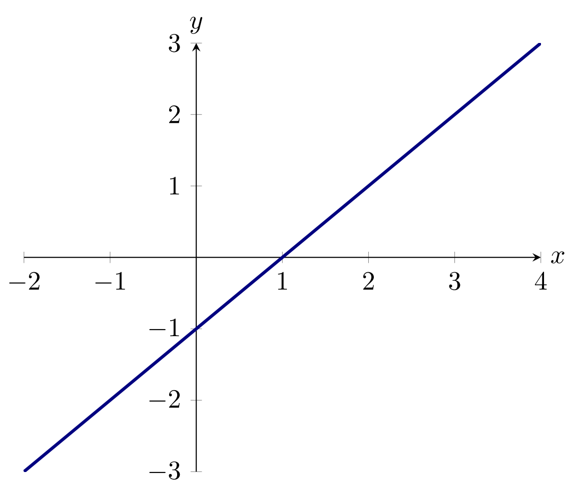
$$g(x) = x - 1$$

the same function?

Explanation. *Let's use a series of steps to think about this question. First, what if we compare graphs? Here we see a graph of f:*



On the other hand, here is a graph of g:



Second, what if we compare the domains? We cannot evaluate f at $x = \underbrace{\boxed{2}}_{\text{given}}$.
This is where f is undefined. On the other hand, there is no value of x where we cannot evaluate g. In other words, the domain of g is $(-\infty, \infty)$.

Since these two functions do not have the same graph, and they do not have the same domain, they must not be the same function.

However, if we look at the two functions everywhere except at $x = 2$, we can say that $f(x) = g(x)$. In other words,

$$f(x) = x - 1 \qquad \textit{when} \qquad x \neq 2.$$

From this example we see that it is critical to consider the domain and range of a function.

Learning Objectives:

The following are **facts** or **definitions** you must memorize from this section:

- A function is a rule such that each input has exactly one output.
- The domain of a function is the set of all inputs.
- The range of a function is the set of all outputs.

The following is **notation** you must be familiar with from this section:

- $f(x) = y$, in this notation, f is the name of the function, x is the input, and y is the output.

The following are **procedures** you must be able to perform from this section:

- Determine whether a given formula or graph represents a function
- Identify the domain of a function from its graph or formula
- Identify the range of a function from its graph or formula
- Plot a point (x, y) in the plane
- Decide whether two functions are equal to each other