# 0 Introduction

This document is a summary of the material covered in ECE 316 at the University of Waterloo in Summer 2016.

# 1 Summary of Random Variables

## 1.1 Expected Value

TEST: The expected value of a random variable $X$ is the weighted average value of $X$ over the probability space. It is defined by the following:

### 1.1.1 Expected Value of DRV's

$$EX = \sum_x xp(x)$$

### 1.1.2 Expected Value of CRV's

$$EX = \int_{-\infty}^{\infty} xf(x)dx$$

### 1.1.3 Properties of Expectation

**Linearity**

$$E[aX + bY] = aE[X] + bE[Y] \qquad \text{where } a \text{ and } b \text{ are constants, and } X \text{ and } Y \text{ are random variables}$$

## 1.2 Variance

The variance is related to the square of the difference between the output of a random variable and the weighted average, $\mu$. It is a measure of the spread of values of a random variable.

### 1.2.1 Variance of DRV's

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X-\mu)^2] \\
&= \sum_x (x-\mu)^2 p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2) p(x) \\
&= \sum_x x^2 p(x) - 2\mu \sum_x xp(x) + \mu^2 \sum_x p(x) \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}$$

IE:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

### 1.2.2 Variance of CRV's

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X-\mu)^2] \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 p(x) \\
&= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) p(x) \\
&= \int_{-\infty}^{\infty} x^2 p(x) - 2\mu \int_{-\infty}^{\infty} xp(x) + \mu^2 \int_{-\infty}^{\infty} p(x) \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}$$

IE:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

## 1.3 Common Distributions and their Properties

### 1.3.1 Bernoulli

The Bernoulli distribution is a distribution in which there are 2 outcomes: 1, with probability p; and 0 otherwise.

$$p(x) = \begin{cases} 1 & \text{with probability p} \\ 0 & \text{otherwise} \end{cases}$$

**Expectation**

$$EX = p$$

**Variance**

$$\text{Var}(x) = E[X^2] - (EX)^2 \quad = p - p^2$$

### 1.3.2 Binomial

The Binomial distribution is the distribution of the number of success in n independent Bernoulli trials with probability p.

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad i = 0, 1, \ldots, n$$

This pmf can be thought of as the number of ways to distribute $i$ successes $\binom{n}{i}$, times the probability of $i$ successes ($p^i$), times the probability of $(n-i)$ failures ($(1-p)^{n-i}$):

**Expectation** By linearity of expectation of $n$ bernoulli trials with probability $p$:

$$EX = np$$

**Variance**

$$\begin{aligned}\text{Var}(x) &= E[X^2] - (EX)^2 \\ &= np[(n-1)p + 1] - n^2p^2 \\ &= np(1-p)\end{aligned}$$

### 1.3.3 Poisson

The poisson distribution is the binomial distribution taken to its limits where

$$\lim_{n\to\infty} \lim_{p\to 0} np = \lambda$$

$$p(i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

**Expectation**

$$EX = \lambda$$

**Variance**

$$\begin{aligned}\text{Var}(x) &= E[X^2] - (EX)^2 \\ &= \lambda(\lambda + 1) - \lambda^2 \\ &= \lambda\end{aligned}$$

### 1.3.4 Geometric

The geometric distribution represents the number of independent bernoulli trials of probability $p$ until a success occurs:

$$P(X = n) = (1-p)^{n-1} p \qquad n = 1, 2, \ldots$$

**Expectation** The derivation won't be shown here, but can be justified intuitively. The expected value is just the average number of trial until you get a success. An example of this is that you will roll a 1 on a die every 6 rolls on average.

$$EX = 1/p$$

**Variance**

$$
\begin{aligned}
\mathrm{Var}(x) &= E[X^2] - (EX)^2 \\
&= \frac{q+1}{p^2} - \frac{1}{p^2} \qquad = \frac{q}{p^2} = \frac{1-p}{p^2}
\end{aligned}
$$

### 1.3.5 Hypergeometric

**Story** You have an urn with $b$ black balls and $w$ white balls. The random variable $X$ represents the number of white balls grabbed when $n$ balls are randomly grabbed with equal probability, and without replacement.

$$
P(X = i) = \frac{\binom{w}{i}\binom{b}{n-i}}{\binom{w+b}{n}} \qquad i = 0, 1, \ldots, n
$$

**Expectation**

$$
EX = \frac{nm}{N}
$$

**Variance** where $N = w + b$, and $p = m/N$

$$
\begin{aligned}
\mathrm{Var}(x) &= E[X^2] - (EX)^2 \\
&= \frac{nw}{N}[\frac{(n-1)(w-1)}{N-1} + 1 - \frac{nw}{N}] \\
&= np[(n-1)p - (n-1)\frac{1-p}{N-1} + 1 - np] \\
&= np(1-p)(1 - \frac{n-1}{N-1}) \\
&\approx np(1-p)
\end{aligned}
$$

### 1.3.6 Negative Binomial

$X$ is the number of independent bernoulli trials until $r$ successes.

$$
P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \qquad n = r, r+1, \ldots
$$

**Expectation**

$$
EX = \frac{r}{p}
$$

**Variance**

$$
\begin{aligned}
\mathrm{Var}(x) &= E[X^2] - (EX)^2 \\
&= \frac{r}{p}(\frac{r+1}{p} - 1)(\frac{r}{p^2})^2 \\
&= \frac{r(1-p)}{p^2}
\end{aligned}
$$

### 1.3.7 Zeta

$$P(X = k) = \frac{C}{k^{\alpha+1}} \qquad k = 1, 2, \ldots$$

It is related to the Riemann zeta function:

$$\zeta(s) = 1 + (\frac{1}{2})^s + (\frac{1}{3})^s + \cdots + (\frac{1}{k})^s + \ldots$$

## 2 Combinatorial Analysis

Test text

# 3 Axioms of Probability

## 3.1 Sample Space and Events

Union, intersection, definition of sample sapce, etc

## 3.2 Axioms of Probability

Yay axioms!

# 4 Conditional Probability and Independece

# 5 Random Variables

# 6 Continuous Random Variables

# 7 Jointly Distributed Random Variables

# 8 Properties of Expectation