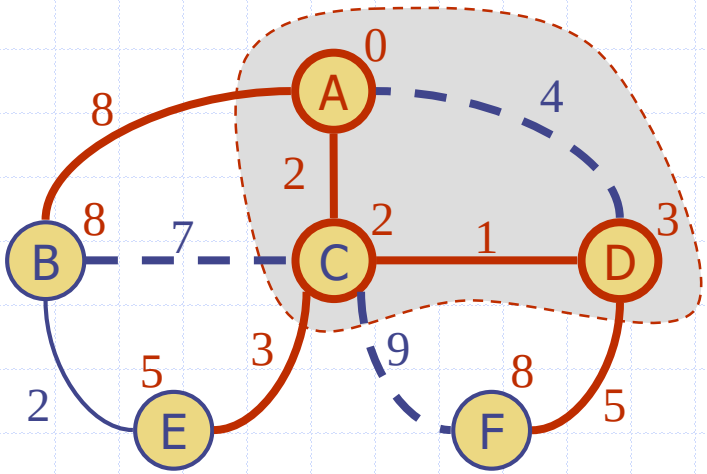
Lesson 11
Algorithms For Weighted Graphs:
Creative Intelligence Manifesting As Material Creation

Wholeness of the Lesson

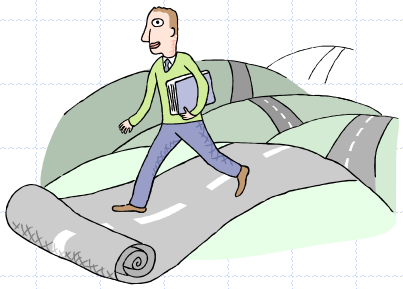
Weighted graphs are graphs that have *weights* or *costs* associated with each edge. Two questions that often need to be answered when working with weighted graphs are (1) What is the least costly path between two given vertices of the graph? (2) What is the least costly subgraph of the given graph which includes all the vertices of the given graph? Dijkstra's Shortest Path Algorithm provides an efficient solution to the first question; Kruskal's Minimum Spanning Tree Algorithm provides an efficient solution to the second. Solutions to optimization problems of all kinds give expression to Nature's tendency to achieve the most possible with the least expenditure of energy. Waking up to one's own deeper values of intelligence has the effect of drawing Nature's style of functioning into our thinking and action so that we automatically achieve goals with less effort.



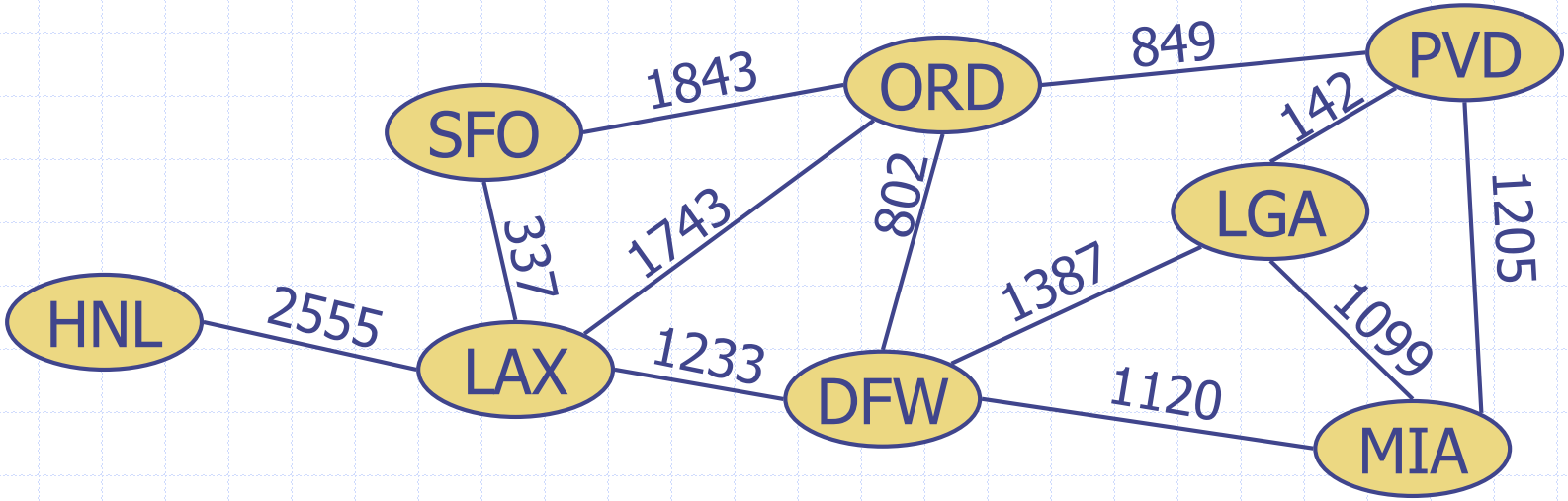
Outline

- Weighted graphs
- Shortest path problem
- Dijkstra's algorithm
- Minimum spanning tree problem
- Kruskal's Algorithm

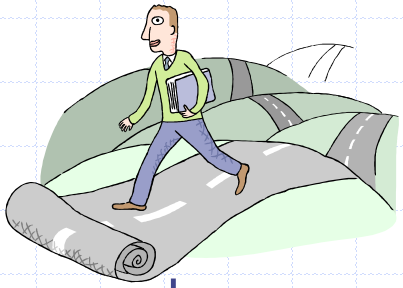
Weighted Graphs



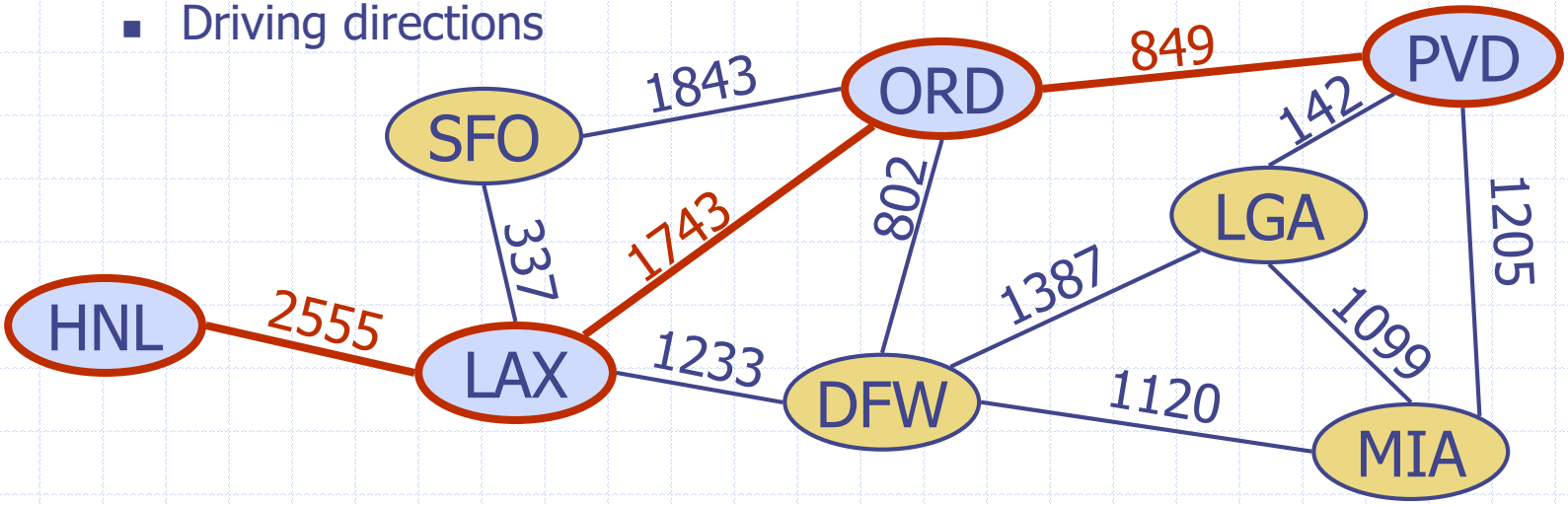
- In a weighted graph, each edge has an associated numerical value, called the weight of the edge (wt: edges → numbers)
- Edge weights may represent distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Path Problem



- Given a connected weighted graph and two vertices s and x, we want to find a path of minimum total weight between s and x.
  - "Length" of a path is the sum of the weights of its edges.
- Example:
  - Shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions



Dijkstra's Algorithm: The Problem

- The *distance* of a vertex v from a vertex s, denoted $d(s,v)$, is the length of a shortest path between s and v
- ***Question:*** Is it always true in a weighted graph that, for any two vertices v, w, $d(v,w) = wt(v,w)$? Prove or give a counterexample.
- Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s
- Assumptions:
  - the graph G = (V,E) is connected
  - the edges are undirected
  - the edge weights are nonnegative

- Starting with weighted graph G = (V,E) and starting vertex s, we wish to compute, for each vertex v, the shortest distance from s to v in G.

- We will store our computed value of the distance from s to any vertex v in a map A:

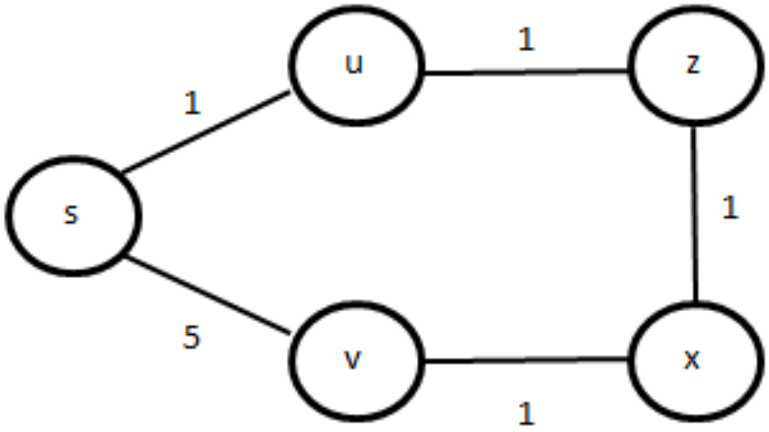
  A[v] = our computed value of distance from s to v

- If our algorithm is right (which we will need to prove) then, after the algorithm has completed execution, we will have A[v] = d(s,v), for each v in V.

The Basic Idea

- *Basic Strategy:* It is reasonable to attempt to compute distances first for vertices close to s, then for vertices farther away.
- Step 1. We set A[s] = 0 (since d(s,s) = 0).
- Step 2. Pick a vertex v adjacent to s so that (s,v) has the least weight among all edges incident to s.
  - It will turn out that for this v, a shortest path from s to v really is wt(s,v). We will set A[v] = wt(s,v).
  - However, it will not be true in general that for any other w adjacent to s, wt(s,w) is a shortest path from s to w

Example



The Logic: Why does Step 2 work? (Pick a vertex u adjacent to s so that (s,u) has least weight among edges incident to s.)

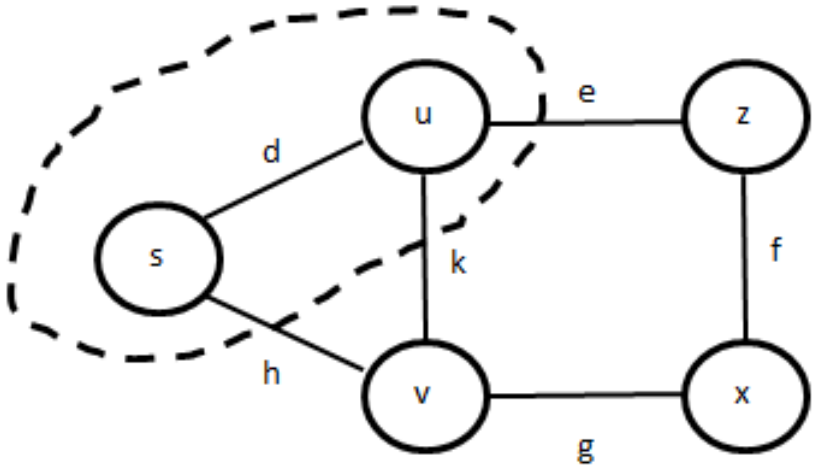
If p : s, w_1, w_2, …, w_k = u is any other path from s to u, then w_1 must itself be u (otherwise the first part of the path – s, w_1 – is already at least long as s, u)

- Step 2: Setting A[u] = wt(s,u) = 1 is correct
  - (s,u) has least weight among all pairs (s,t) where t adjacent to s
- However, setting A[v] = 5 is not correct
  - A less costly path from s to v than s,v is s, u, z, x, v

Basic Idea, continued

- Step 3. After an optimal vertex v adjacent to s has been chosen, there are two possible ways to extend further
  - For some other vertex w adjacent to s, wt(s,w) may turn out to be the shortest path length from s to w, OR
  - For some w adjacent to v, the path s, v, w is shortest possible from s to w.
  - Among these choices, the algorithm will pick the path that has minimal total weight

Example of Step 3



Compute minimum of

wt(s, v)

A[u] + wt(u,z)

A[u] + wt(u,v)

One of these will be a correct minimum path length (either from s to v or from s to z)

Why Step 3 Works

Case I. wt(s,v) is minimum

Any path from s to v begins either as s, u or s, v. By assumption s, v is no longer than s, u, v. The other possibility is s, u, z, x, v, but s, v is (by assumption) no longer than s, u, z.

Case II. A[u] + wt(u,v) is minimum

Case III. A[u] + wt(u,z) is minimum.

Cases II and III are similar. We prove it for Case III: We want to show that s, u, z is a shortest path to z. The other possible choices are s, u, v, x, z and s, v, x, z

Path s, u, v, x, z:
length s,u,z ≤ length of s,u,v (by assumption)

Path s, v, x, z:
length of s, u, z ≤ length s, v (by assumption)

Dijkstra's Algorithm

- We extend Steps 1, 2, 3 till every vertex has been reached
- In general, we build a "cloud" X of vertices, beginning with s and eventually including all of V. Associated with each vertex v in X will be a value A[v] representing the algorithm's current estimate of the shortest path length from s to v
- In each step, we add one new vertex w to X When w is placed in X (as we will show), the value stored in A[w] is precisely the value d(s, w).

Slow Dijkstra's Algorithm

Input: A simple connected undirected weighted graph *G* with nonnegative edge weights, determined by a weight function *wt* (*x,y*), and a starting vertex *s* of *G*.

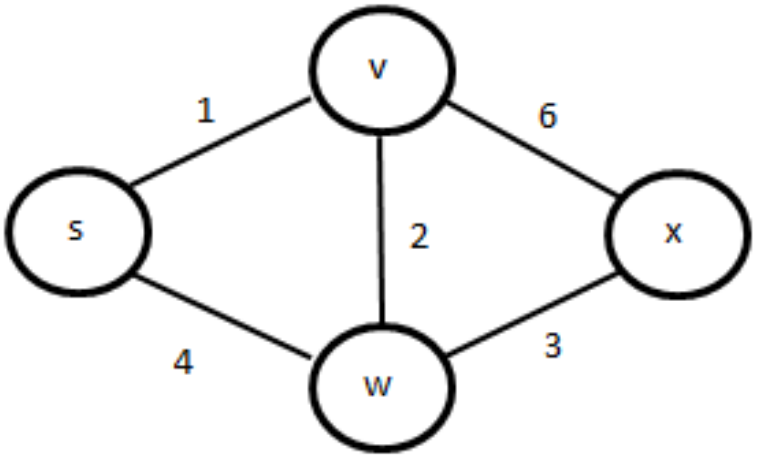
Output: Table A of shortest distances d(s,v) from s to v, for each v in V, so A[v] = d(s,v) for each v
Aux Output: Table B with property that B[v] is a shortest path from s to v.

The Algorithm:

$A[s] \leftarrow 0$.
$X \leftarrow \{s\}$ //Basis step

while $X \neq V$ **do**
{ $POOL \leftarrow \{(v,w) \in E \mid v \in X \text{ and } w \notin X\}$ }
//here, we have no control over how much of E is searched, leading to slow running time
(v',w') ← search POOL for edge (v,w) for which greedy length A[v] + wt(v,w) is minimal
A[w'] ← A[v'] + wt(v',w')
add w' to X

Worked Example: Step 1



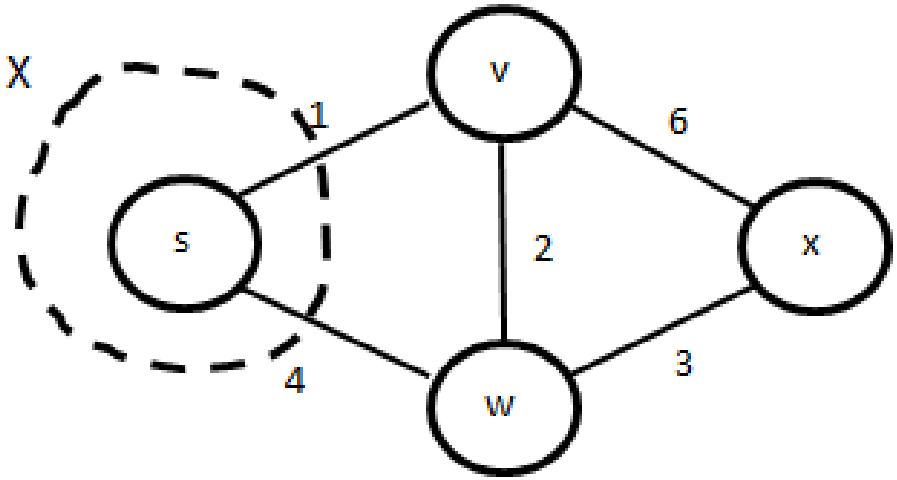
Step 1.

A[s] ← 0

B[s] ← { }

Put s in X

Worked Example: Step 2



Step 2.

X = { s }

POOL = { (s,v), (s,w) }

Find minimum greedy length – min of the following

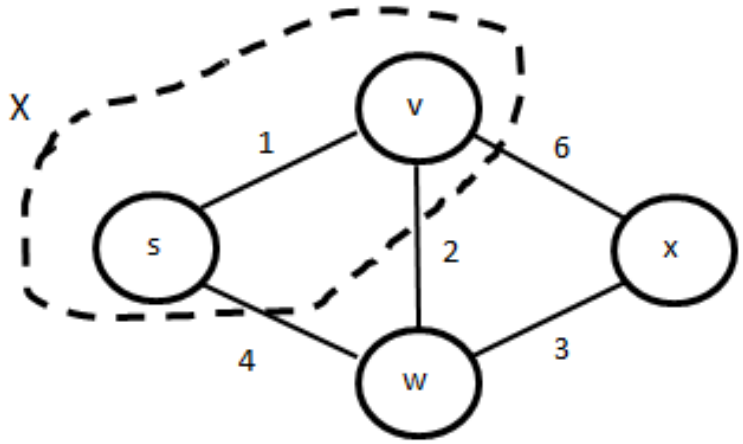
A[s] + wt(s,v) = wt(s,v) = 1 ←

A[s] + wt(s,w) = wt(s,w) = 4

Add v to X and set value of A[v]: A[v] ←1

Auxiliary Storage: B[v] = B[s] U {(s,v)} = {(s,v)}

Worked Example: Step 3



Step 3.

X = { s, v }

POOL = { (s,w), (v,w), (v,x) }

Find minimum greedy length – min of the following

A[s] + wt(s,w) = wt(s,w) = 4

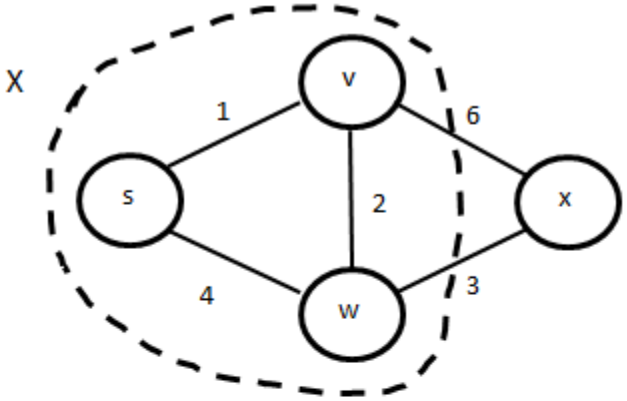
A[v] + wt(v,w) = 1 + wt(v,w) = 1 + 2 = 3 ←

A[v] + wt(v,x) = 1 + wt(v,x) = 1 + 6 = 7

Add w to X and set value of A[w]: A[w] ← 3

Auxiliary Storage: B[w] = B[v] U {(v,w)} = {(s,v), (v,w)}

Worked Example: Step 4



Step 4.

X = { s, v, w }

POOL = { (w, x), (v,x) }

Find minimum greedy length – min of the following

A[v] + wt(v,x) = 1 + wt(v,x) = 1 + 6 = 7

A[w] + wt(w,x) = 3 + wt(w,x) = 3 + 3 = 6 ←

Add x to X and set value of A[x]: A[x] ← 6

Algorithm complete since X = V. Computed values:

A[s] = 0 A[v] = 1 A[w] = 3 A[x] = 6

Computeed values of table B:
B[s] = { }, B[v] = {(s,v)}, B[w] = {(s,v), (v,w)}, B[x] = {(s,v), (v,w), (w,x)}

Slow Dijkstra – Running Time

- Running time without optimizations can be computed by observing that a (potentially) exhaustive search of edges is made in each iteration, leading to a running time of O(mn).
- This can be improved to O(m * log n) if an optimal data structure is used.

Improving Dijkstra

- Since "mins" are needed in each iteration, serve them using a Priority Queue instead of doing an exhaustive search of edges.
- Begin by setting A[v] = ∞ for each v (other than starting vertex s) and progressively refine the value in A[v] as the algorithm proceeds.
- Use the values A[v] as keys in the priority queue, with corresponding value v. At each step, the min in the priority queue represents the smallest among all approximations to a correct distance.

Dijkstra – Using Priority Queue

Input: Weighted undirected graph G=(V,E), starting vertex s

Output: Map A where for each vertex v, A[v] = d(s,v)

```
A[s] ← 0
A[v] ← ∞  (for each vertex v in V where v != s)
Q ← new heap-based priority queue     //each node in Q has value a vertex u and key A[u]
while !Q.isEmpty() do
    (u, A[u]) ← Q.removeMin()  //logically, put u in X; A[u] is now correct
    for each v in Q that is adjacent to u do
        greedyLen ← A[u] + wt(u, v)
        if greedyLen < A[v]
            A[v] ← greedyLen
            Q.updateNode(v, greedyLen)    //update value A[v] for v
return the map A
```

Correctness

Main Idea. First s is brought into X and A[s] is correct. Then A[v] is updated for each v adjacent to s, and the smallest such A[v] becomes the min in the queue. Our earlier argument showed that in this case A[v] is correct and v is correctly pulled into X. Then the value A[w] is updated for every w adjacent to v, and again the min is chosen. We show that, for each such minimum element (v, A[v]), we have A[v] = d(s,v).

Main Lemma

Main Lemma. Suppose G is a weighted graph. Suppose $q : s, \ldots, y, z$ is a true shortest path from s to z in G. Then the path $s, \ldots, y$ is a true shortest path from s to y; the path y, z is a true shortest path from y to z; and $d(s, z) = d(s, y) + \mathrm{wt}(y, z)$.

Proof. Let r and t be the following subpaths of q:

$$r : s, \ldots, y \qquad t : y, z.$$

We claim that $d(s, y)$ is equal to the length of r: If not, then there must be a shorter path r' than r that goes from s to y. But then $r' \cup t$ is a shorter path from s to z than q, which contradicts the fact that q is a true shortest path from s to z. For the same reason, $d(y, z)$ is equal to length of t. In particular, $d(y, z) = \mathrm{wt}(y, z)$.

We have:

$$d(s, z) = \mathrm{length}(q) = \mathrm{length}(r) + \mathrm{length}(t) = d(s, y) + \mathrm{wt}(y, z),$$

as required. □

Correctness Proof (1)

Assume that through the ith iteration, for each vertex v in X (i.e. removed from Q), we have A[v]=d(s,v). We show this holds through the i+1st iteration. Suppose at this stage that w is pulled off the Q (so enters X). We wish to show A[w] = d(s,w); assume that this is not the case, so that A[w] > d(s,w). We will arrive at a contradiction.

Let q: s, . . ., y, z, . . ., w be a true shortest path from s to w; let L denote the length of q. As in previous proof, z is the first vertex in q that is not yet in X. Let q_0 : s, . . . , y, z be the subpath of q terminating in z; let L_0 be the length of q_0. We will show that A[w] ≤ L_0 (and this will give us a contradiction).

Correctness Proof (2)

Claim. A[z] = d(s,z)

Proof. Recall that earlier in the algorithm, when y was removed from Q, since z is adjacent to y, A[z] was possibly updated (after which A[z] = A[y] + wt(y,z)), and could have been updated again after that. It follows that

$$A[z] \leq A[y] + wt(y,z)$$

By assumption, since y is in X, A[y]=d(s,y). Also, since q_0 is a true shortest path to z, by the Lemma, d(s,z) = d(s,y) + wt(y,z).

We have

$$A[z] \leq A[y] + wt(y,z) = d(s,y) + wt(y,z) = d(s,z)$$

However, for all u, A[u] ≥ d(s,u) (approximations decrease toward the true distance). It follows that

$$A[z] = d(s,z)$$

Correctness Proof (3)

Continuation of the Main Proof. Notice that in the i+1st stage, when w is being removed from Q, z is not yet in X, and (w, A[w]) is the minimum element of Q, and so $A[w] \leq A[z]$. Therefore, by the Claim,

$$A[w] \leq A[z] = d(s,z) = L_0 \leq L,$$

as required.

Implementation Issues

- How can we locate a node in the queue that contains a vertex which is adjacent to a given vertex v? And how can we update such a node?

- *Solution*: Use auxiliary map M that matches vertices to nodes in the Queue. Given a vertex v, to locate node that contains a vertex adjacent to v, check adjacency list for next adjacent vertex u, then look up node n that contains u by consulting M. (This requires us to expose the private nodes of Q to the algorithm.)

  To update the key (u, A[u]) in the queue, we enhance the priority queue operations to include

  updateNode(n, key, value) (where n is a node in Q)

  which behaves as follows:

  1. The current key in n is changed to ABSOLUTE_MIN, and upheap is performed (O(log n) time)
  2. Perform removeMin to remove n from Q (O(log n) time)
  3. Insert the new (key,value) using insertItem (O(log n) time)

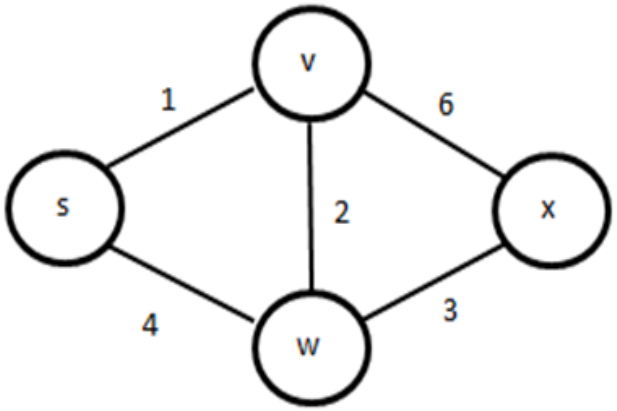
Dijkstra – Using Priority Queue and Node Map

Input: Weighted undirected graph G=(V,E), starting vertex s

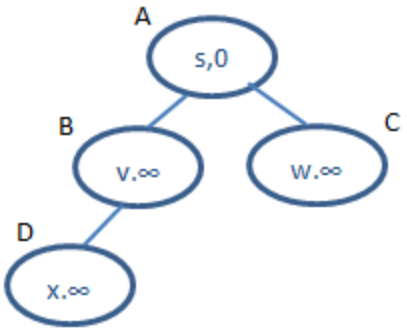
Output: Map A where for each vertex v, A[v] = d(s,v)

```
A[s] ← 0
A[v] ← ∞  (for each vertex v in V where v != s)
Q ← new heap-based priority queue     //each node in Q has value a vertex u and key A[u]
M ← new HashMap   {key/value pair is (v,n), v a vertex, n a node in heap containing v }
while !Q.isEmpty() do
    (u, A[u]) ← Q.removeMin()  //logically, put u in X; A[u] is now correct
    M.remove(u)
    for each v in Q that is adjacent to u do      //update values A[v] for v adjacent to u
        greedyLen= A[u] + wt(u, v)
        if greedyLen < A[v]
            n ← M.get(v)
            A[v] ← greedyLen
            Q.updateNode(n, v, greedyLen)     //update value A[v] for v if better value has been found
return the map A
```

Worked Example (Start)



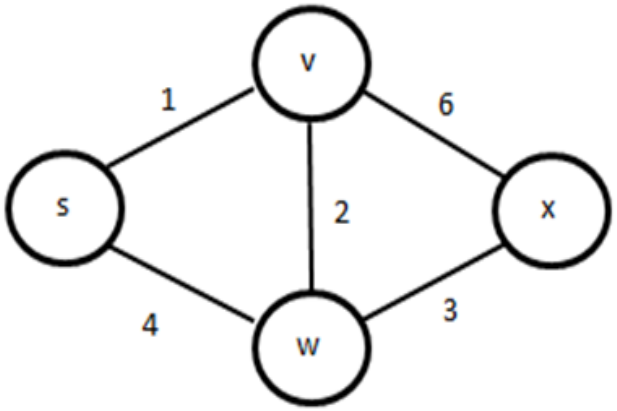
A	Current Value
A[s]	0
A[v]	∞
A[w]	∞
A[x]	∞



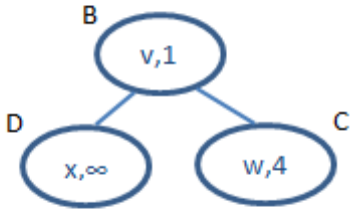
M	
s	A
v	B
w	C
x	D

X = {}

Worked Example (Step 1)



A	Current Value
A[s]	0
A[v]	1
A[w]	4
A[x]	∞

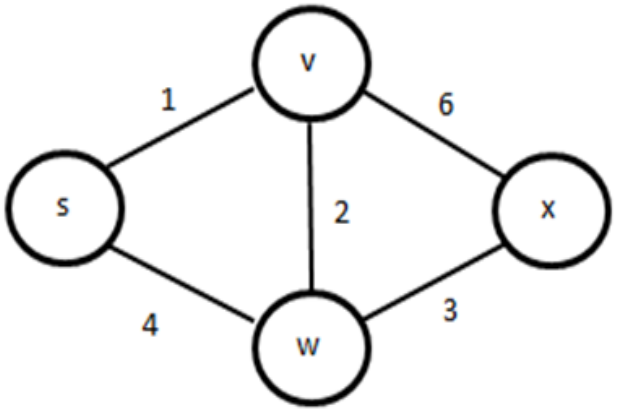


M	
v	B
w	C
x	D

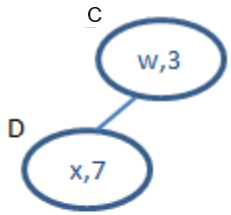
(s,0) ← Q. removeMin
M.remove(s)
Vertices adjacent to s: v, w
  Process v:
    greedyLen = A[s] + wt(s,v) = 0 + 1 = 1
    greedyLen < A[v] ? Yes
    A[v] ← 1
    Q.updateNode(B, v, 1)
  Process w:
    greedyLen = A[s] + wt(s,w) = 0 + 4 = 4
    greedyLen < A[w] ? Yes
    A[w] ← 4
    Q.updateNode(C, w, 4)

X = {s}

Worked Example (Step 2)



A	Current Value
A[s]	0
A[v]	1
A[w]	3
A[x]	7

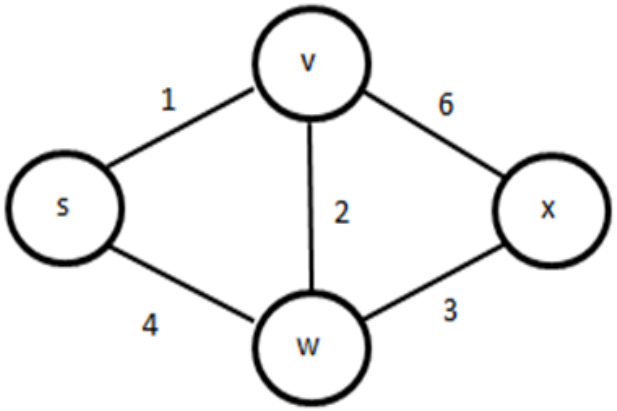


M	
w	C
x	D

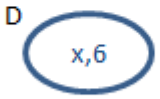
(v,1) ← Q. removeMin
M.remove(v)
Vertices adjacent to v still in Q: x, w
  Process x:
    greedyLen = A[v] + wt(v,x) = 1 + 6 = 7
    greedyLen < A[x] ? Yes
    A[x] ← 7
    Q.updateNode(D, x, 7)
  Process w:
    greedyLen = A[v] + wt(v,w) = 1 + 2 = 3
    greedyLen < A[w] ? Yes
    A[w] ← 3
    Q.updateNode(C, w, 3)

X = {s, v}

Worked Example (Step 3)



A	Current Value
A[s]	0
A[v]	1
A[w]	3
A[x]	6



(w,3) ← Q. removeMin
M.remove(w)
Vertices adjacent to w still in Q: x
  Process x:
    greedyLen = A[w] + wt(w,x) = 3 + 3 = 6
    greedyLen < A[x] ? Yes
    A[x] ← 6
    Q.updateNode(D, x, 6)

M	
x	D

X = {s, v, w}

Final Step:
(x,6) ← Q. removeMin
M.remove(x)
return A // X = {s, v, w, x}

Running time

- Initialize A[v] for all vertices. O(n)
- Build priority queue for all vertices O(nlog n)
- Initialize the hashmap M O(n)
- The while loop removes one min node each time until the priority queue is empty. So the algorithm is going to execute while loop n times.
  - Remove min node and do downheap O(nlog n)
  - Remove this node from the map M (O(1) each time) O(n)
- For each vertex v removed from queue and each w adjacent to v in queue,

  - update w in the queue (if necessary) O(mlog n)

  [requires log(n) for each update, O(deg(v)) times =

  $$\sum_{v \in V} \mathrm{O}(\deg(v) \log(n)) = \mathrm{O}(\log(n)) \sum_{v \in V} \mathrm{O}(\deg(v)) = \mathrm{O}(m \log(n)).$$]
- Since the graph is connected, n is O(m).

Total Running Time: O(m log n)

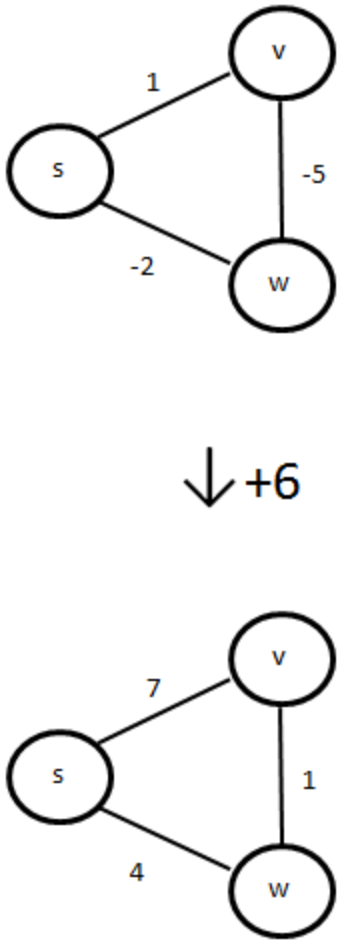
(Note: This improves the O(mn) running time of the slow algorithm.)

Dijkstra - Exercises

- Why is there a requirement that edges have *non-negative* weights? Why can't we add a large positive constant to every edge (to eliminate negative edge weights) and compute shortest paths for the new graph using Dijkstra?

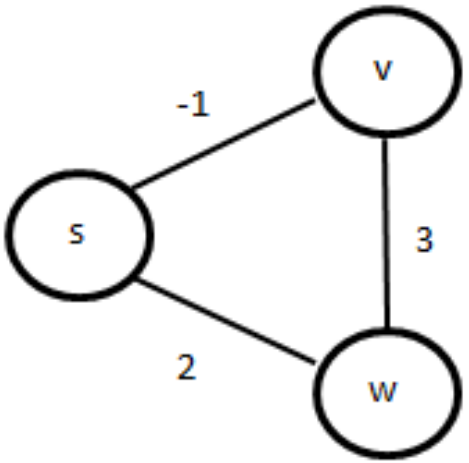
Dijkstra - Exercises

- Why is there a requirement that edges have *non-negative* weights? Why can't we add a large positive constant to every edge (to eliminate negative edge weights) and compute shortest paths for the new graph using Dijkstra?



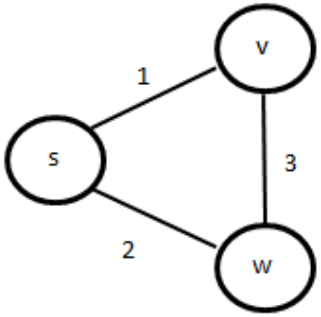
Exercises, continued

Does Dijkstra's Algorithm *sometimes* work correctly when there are negative edge weights? Consider this weighted graph.



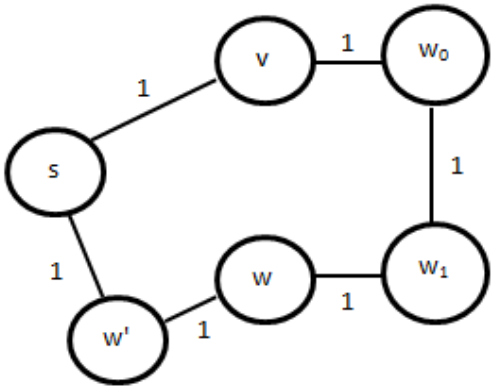
Exercises, continued

Why is Dijkstra's approach to the shortest path problem better than simply using BFS, as described in the previous lesson?



[BFS approach: Making all edge weights = 1 is same as removing all weights. Perform BFS with start vertex s and compute distance to each vertex by returning its *level* in the BFS spanning tree. These computed values should be same as values found using Dijkstra]

↓ BFS Style

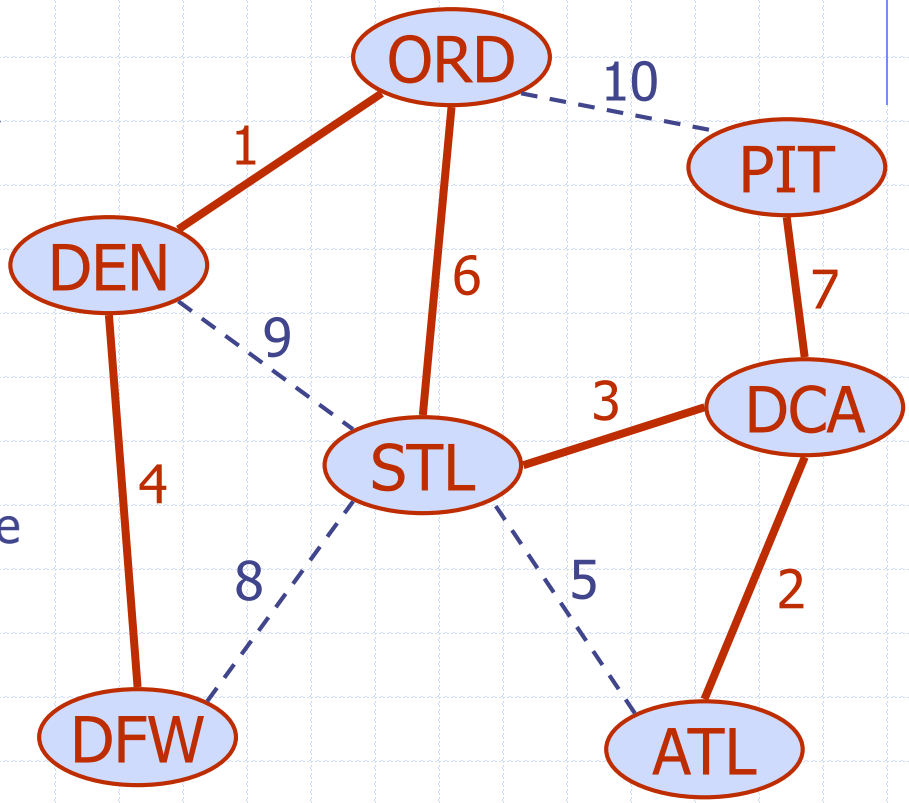


Main Point

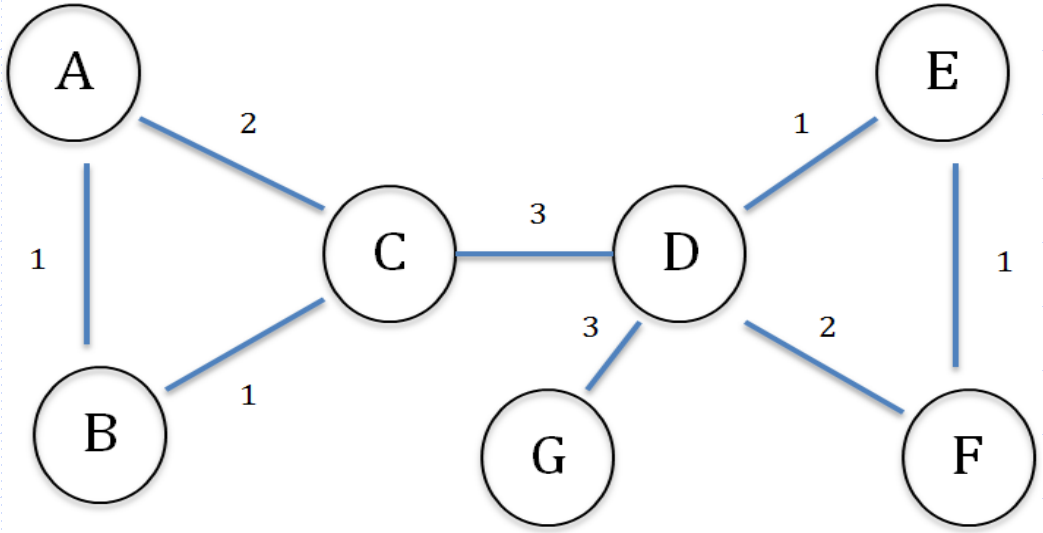
- Dijkstra's algorithm is an example of a *shortest-path algorithm* – an algorithm that efficiently (O(mlog n)) computes the shortest distance between two vertices in a graph.
- Analogously, Nature itself is known to obey the law of least action – Nature does the least possible amount of work to proceed from one location or state to another. Nature's way of achieving this makes use of computational dynamics that involve "no effort" and no steps.

Minimum Spanning Tree Problem

- Spanning subgraph
  - Subgraph of a graph G containing all the vertices of G
- Spanning tree
  - Spanning subgraph that is itself a tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Computer network (minimize cost of cable)
  - Transportation networks (minimize cost of road construction)



Kruskal's Greedy Strategy



Build a collection T of edges by doing the following: At each step, add an edge e to T of least weight subject to the constraint that adding e to T does not create a cycle in T. To answer questions about correctness and running time of this algorithm, we need to specify certain details.

Implementation Questions

1. How do we pick the next edge of least weight at each step?

2. How do we make sure that we do not add an edge to T that produces a cycle?

Solutions

1. We can arrange edges by sorting them by weight (in ascending order), and so we pick edges according to this sorted order. (This is a *greedy strategy*.)

2. We can ensure no cycles are created by building local minimum spanning trees around each vertex (to be explained further in coming slides)

Kruskal's Algorithm

- First step is to sort all edges by weight.
- Second step involves creation of *clusters*
  - Every vertex is initially placed in a trivial *cluster* -- the cluster for a vertex v, denoted C(v), is simply {v}. A cluster represents a local minimum spanning tree.
  - When the next edge (u,v) is considered, C(u) and C(v) are compared -- if different, (u,v) is included as an edge in the final output tree, and C(u) and C(v) are merged.

Kruskal's Algorithm

Input: A simple connected weighted graph $G = (V, E)$ with n vertices and m edges
Output: A minimum spanning tree T of G

The Algorithm:

sort E in increasing order of edge weight

for each vertex v in G, define an elementary cluster $C(v)$ (which will grow) by $C(v) = \{v\}$

$T \leftarrow$ an empty tree // T will eventually become the minimum spanning tree

while T has fewer than $n - 1$ edges **do**

$(u,v) \leftarrow$ next edge

$C(v) \leftarrow$ cluster containing v

$C(u) \leftarrow$ cluster containing u

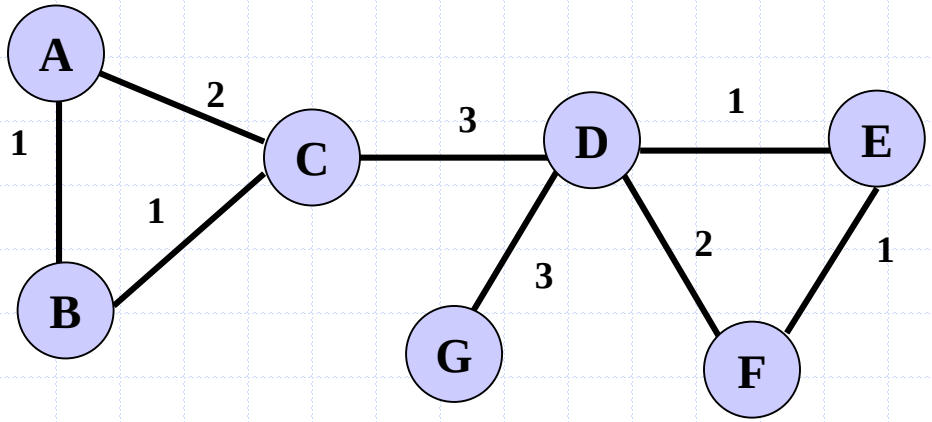
if $C(v) \neq C(u)$ **then**
add edge (u,v) to T
merge $C(u)$ and $C(v)$ (and update other clusters as needed)

return T

Note: We represent T as a collection of edges; to make it a graph, include as its vertex set all endpoints of the edges in T.

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {…}

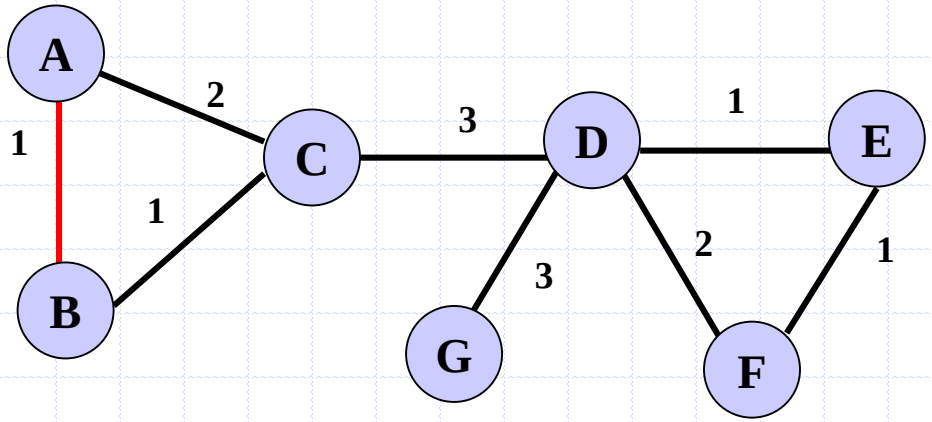


Cluster	*Evolving Values*
C(A)	{A}
C(B)	{B}
C(C)	{C}
C(D)	{D}
C(E)	{E}
C(F)	{F}
C(G)	{G}

Step 1:
Sort the edges and initialize the clusters

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, …}

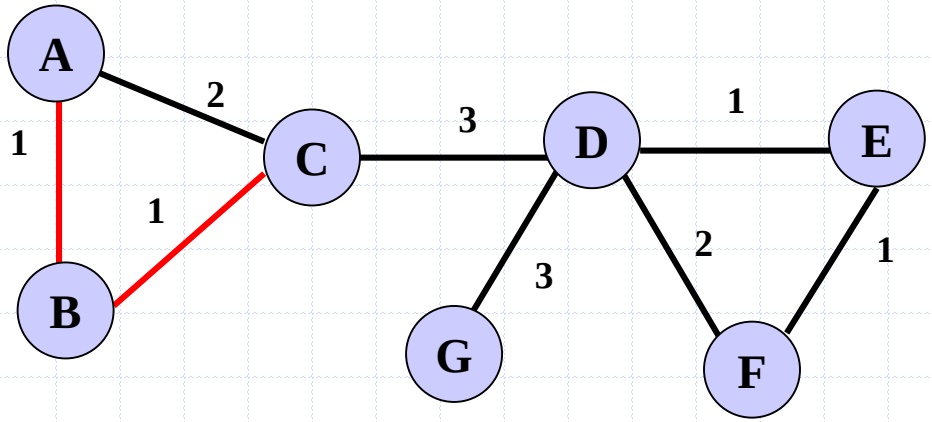


Cluster	*Evolving Values*
C(A)	{A, B}
C(B)	{A, B}
C(C)	{C}
C(D)	{D}
C(E)	{E}
C(F)	{F}
C(G)	{G}

Step 2:
C(A) ≠ C(B)
add AB to T, merge C(A) and C(B)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, …}

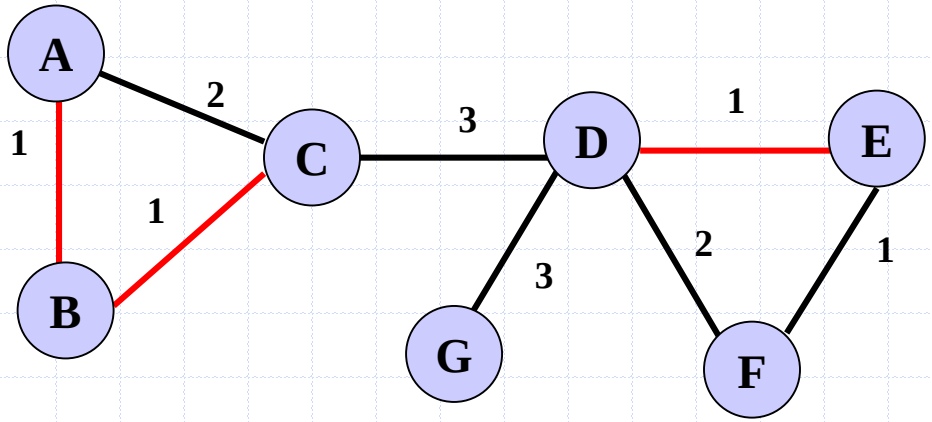


Cluster	*Evolving Values*
C(A)	{A, B, C}
C(B)	{A, B, C}
C(C)	{A, B, C}
C(D)	{D}
C(E)	{E}
C(F)	{F}
C(G)	{G}

Step 3:
C(B) ≠ C(C)
add BC to T, merge C(B) and C(C)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, ...}

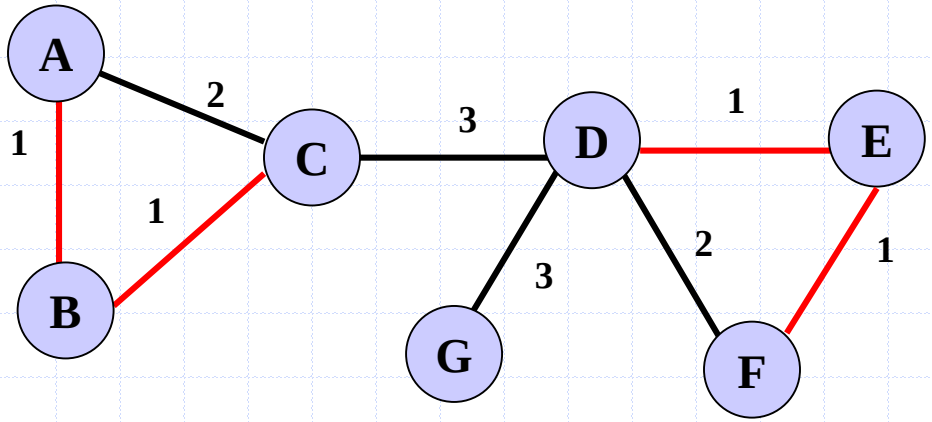


Cluster	*Evolving Values*
C(A)	{A, B, C}
C(B)	{A, B, C}
C(C)	{A, B, C}
C(D)	{D, E}
C(E)	{D, E}
C(F)	{F}
C(G)	{G}

Step 4:
C(D) ≠ C(E)
add DE to T, merge C(D) and C(E)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}

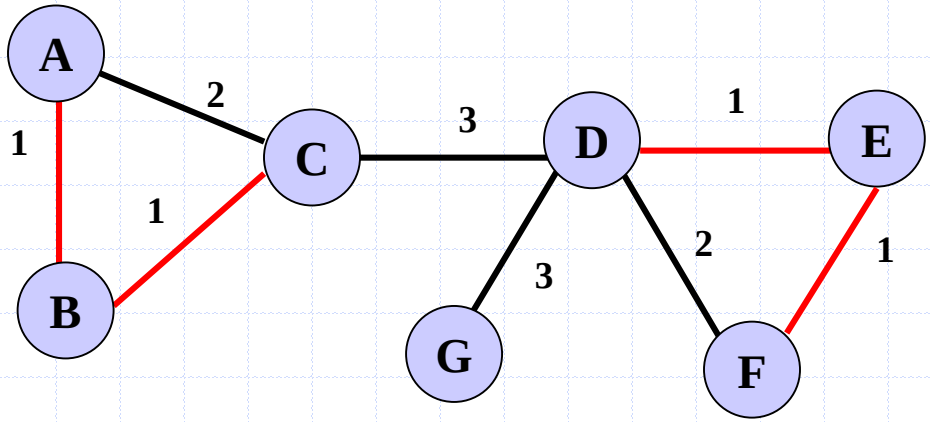


Cluster	*Evolving Values*
C(A)	{A, B, C}
C(B)	{A, B, C}
C(C)	{A, B, C}
C(D)	{D, E, F}
C(E)	{D, E, F}
C(F)	{D, E, F}
C(G)	{G}

Step 5:
C(E) ≠ C(F)
add EF to T, merge C(E) and C(F)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}

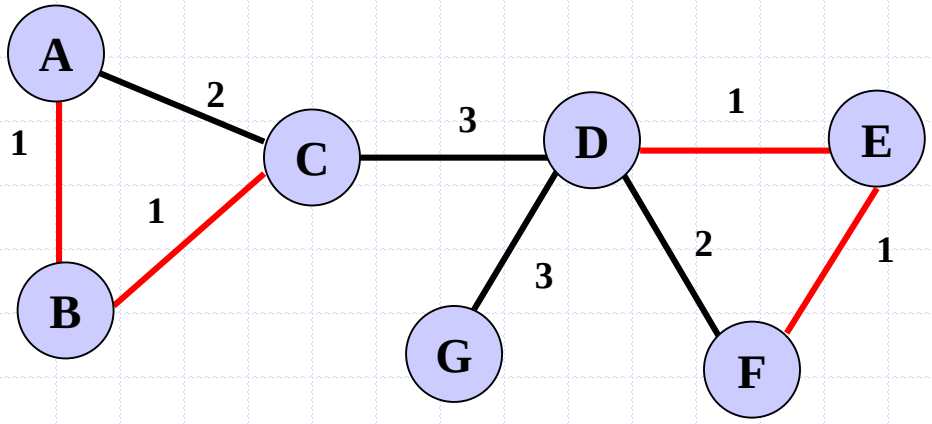


Step 6:
C(A) = C(C) , discard AC

Cluster	*Evolving Values*
C(A)	{A, B, C}
C(B)	{A, B, C}
C(C)	{A, B, C}
C(D)	{D, E, F}
C(E)	{D, E, F}
C(F)	{D, E, F}
C(G)	{G}

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}

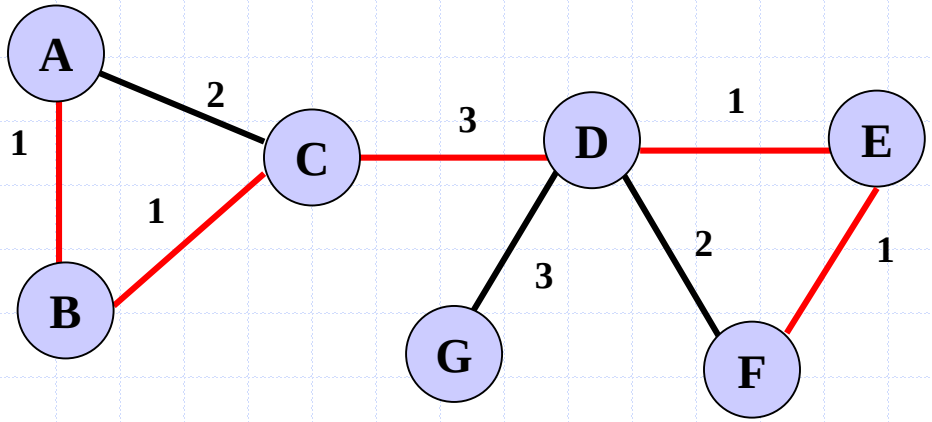


Step 7:
C(D) = C(F) , discard DF

Cluster	*Evolving Values*
C(A)	{A, B, C}
C(B)	{A, B, C}
C(C)	{A, B, C}
C(D)	{D, E, F}
C(E)	{D, E, F}
C(F)	{D, E, F}
C(G)	{G}

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, ...}

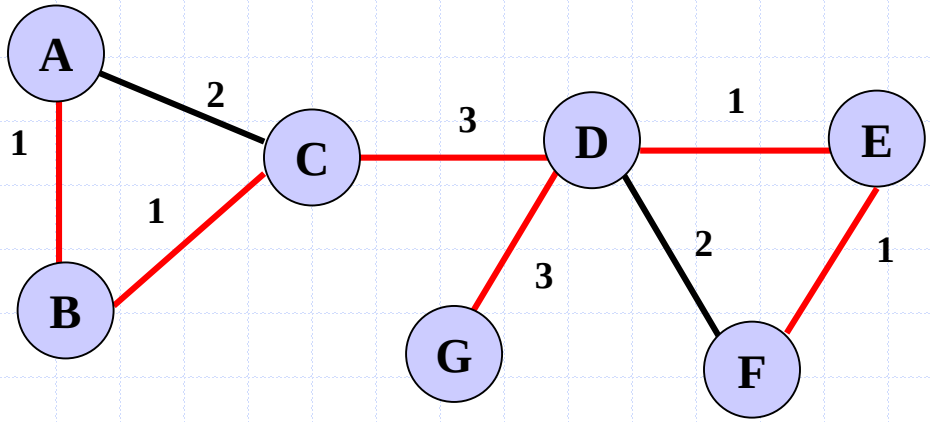


Cluster	*Evolving Values*
C(A)	{A, B, C, D, E, F}
C(B)	{A, B, C, D, E, F}
C(C)	{A, B, C, D, E, F}
C(D)	{A, B, C, D, E, F}
C(E)	{A, B, C, D, E, F}
C(F)	{A, B, C, D, E, F}
C(G)	{G}

Step 7:
C(C) ≠ C(D)
add CD to T, merge C(C) and C(D)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, DG, …}

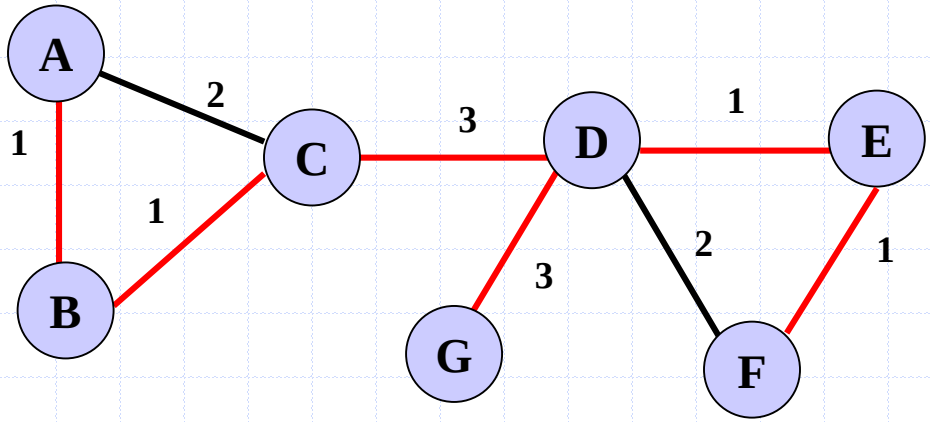


Cluster	*Evolving Values*
C(A)	{A, B, C, D, E, F, G}
C(B)	{A, B, C, D, E, F, G}
C(C)	{A, B, C, D, E, F, G}
C(D)	{A, B, C, D, E, F, G}
C(E)	{A, B, C, D, E, F, G}
C(F)	{A, B, C, D, E, F, G}
C(G)	{A, B, C, D, E, F, G}

Step 8:
C(D) ≠ C(G)
add DG to T, merge C(D) and C(G)

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, DG}



Now we have n-1 = 6 edges in T, the algorithm stops.

Cluster	*Evolving Values*
C(A)	{A, B, C, D, E, F, G}
C(B)	{A, B, C, D, E, F, G}
C(C)	{A, B, C, D, E, F, G}
C(D)	{A, B, C, D, E, F, G}
C(E)	{A, B, C, D, E, F, G}
C(F)	{A, B, C, D, E, F, G}
C(G)	{A, B, C, D, E, F, G}

Correctness: Background Facts

Suppose G = (V, E) is a connected simple graph.

A. Suppose $V_1, V_2, \ldots, V_k$ are disjoint subsets of V with $k > 1$, and that $V = V_1 \cup V_2 \cup \ldots \cup V_k$. Then there is an edge (x,y) in E such that for some $i \neq j$, $x \in V_i$ and $y \in V_j$. (From the labs)

B. Suppose $S = (V_S, E_S)$ and $T = (V_T, E_T)$ are subtrees of G with no vertices in common (in other words, V_S and V_T are disjoint). Then for any edge (x,y) in E for which $x \in V_S$ and $y \in V_T$, the subgraph $U = (V_S \cup V_T, E_S \cup E_T \cup \{(x,y)\})$ is also a tree. (From the labs)

C. Suppose W is a subset of the set E of edges of G and $|W| < n - 1$. Consider the subgraph H of G formed from W by defining the edges of H to be W and defining the vertices of H to be the endpoints of those edges, and assume that H contains no cycle. Then there exists an edge (x,y) in G not in W so that the graph formed by $W \cup \{(x,y)\}$ also contains no cycle.

Proof of Background Fact C

C. Suppose W is a subset of the set E of edges of G and $|W| < n - 1$. Consider the subgraph H of G formed from W by defining the edges of H to be W and defining the vertices of H to be the endpoints of those edges, and assume that H contains no cycle. Then there exists an edge (x,y) in G not in W so that the graph formed by W U {(x,y)} also contains no cycle.

Case 1. *H is connected.* Write $H = (V_H, E_H)$; let $n_H = |V_H|$ and $m_H = |E_H|$. By assumption, $m_H < n-1$. Since H is a tree (since it's connected and acyclic), it follows that $n_H = m_H + 1 \leq n-1 < n$. Therefore, some vertex z of G does not lie in V_H. Since G is connected, given any u in V_H, there is a path from u to z. There is a first vertex y in this path that does not lie in V_H; let x be the vertex adjacent to y, preceding it in the path. Notice that ({y}, {}) is a subtree of G. By the choice of x, y, we can apply Part B to conclude that H U {(x,y)} is also a tree. This establishes the result for Case 1.

Case 2. *H is not connected.* Write $H = (V_H, E_H)$; let $n_H = |V_H|$ and $m_H = |E_H|$. Let $H_1, H_2, \ldots, H_k$ denote the connected components of H where for each j, $H_j = (V_j, E_j)$. $V' = V - V_H$. By Part A there is an edge (x,y) joining two of these vertex sets; assume x belongs to some V_j. If y belongs to V', then we can argue as in Case 1. If y belongs to some V_i, then the result follows from part B.

Correctness

We show that Kruskal produces a spanning tree (proof that it is a *minimal* spanning tree is optional).

1. During execution, for each cluster C, the edges in T that have endpoints in C form a spanning tree in G[C]. Moreover, during execution, if two endpoints in C were joined by a new edge, it would create a cycle (therefore, this step never allowed by the algorithm)

   Proof: This is true when the first edge is examined (one edge, two vertices). Assuming true after the first i stages of the algorithm, at the next stage if edge uv is being examined and C(u) not equal to C(v), then the an edge is added with one vertex in the tree in C(u) and one in the tree in C(v). By Fact B, a new tree is formed that is a spanning tree for C(u) U C(v). For the moreover clause, if two endpoints in a cluster were joined by a new edge, two endpoints of its spanning tree would be joined and the resulting subgraph would contain a cycle.

Correctness

2. The main loop terminates (it is conceivable that after all edges have been examined, T still contains $< n - 1$ edges – this is shown to be impossible).

   Proof: Suppose T still contains $< n - 1$ edges after processing every edge. By Fact C, there is an edge e that can be added to T without creating a cycle. But the algorithm visited every edge (including e) and it rejected e. The algorithm rejects an edge only because it would create a cycle, but e does not create a cycle -- contradiction.

Correctness

3. At the end of the algorithm, T is a spanning tree.

Proof. We have shown that T consists of only disjoint trees, so T contains no cycle, and T has n-1 edges. We first show that T has n vertices.

Let

$$m_T = \#\text{ edges in T and } n_T = \#\text{ vertices in T.}$$

Then $m_T = n-1$. If $n > n_T$, it follows that

$$m_T = n - 1 \geq n_T$$

and so T must contain a cycle (which is impossible). Therefore, T has n vertices and so is a spanning subgraph. Since $m_T = n_T - 1$ and T has no cycles, T must be a tree and is therefore a spanning tree.

OPTIONAL: Proof that Output is a *Minimal* Spanning Tree

We establish the following loop invariant I(i):

At each stage i of the algorithm, if T is the collection of edges obtained so far, there is a minimum spanning tree for G that contains T

Assuming we can establish this loop invariant, then, when the algorithm finishes, it will follow from the loop invariant that there is an MST that contains T. But, as shown in previous slides, when the while loop of the algorithm ends, T has become a spanning tree. It follows that the MST that contains T must be T itself.

Proof of Loop Invariant

The invariant holds at the start – G must have an MST since, as we have shown, it does have a spanning tree, so one such spanning tree must have minimal weight.

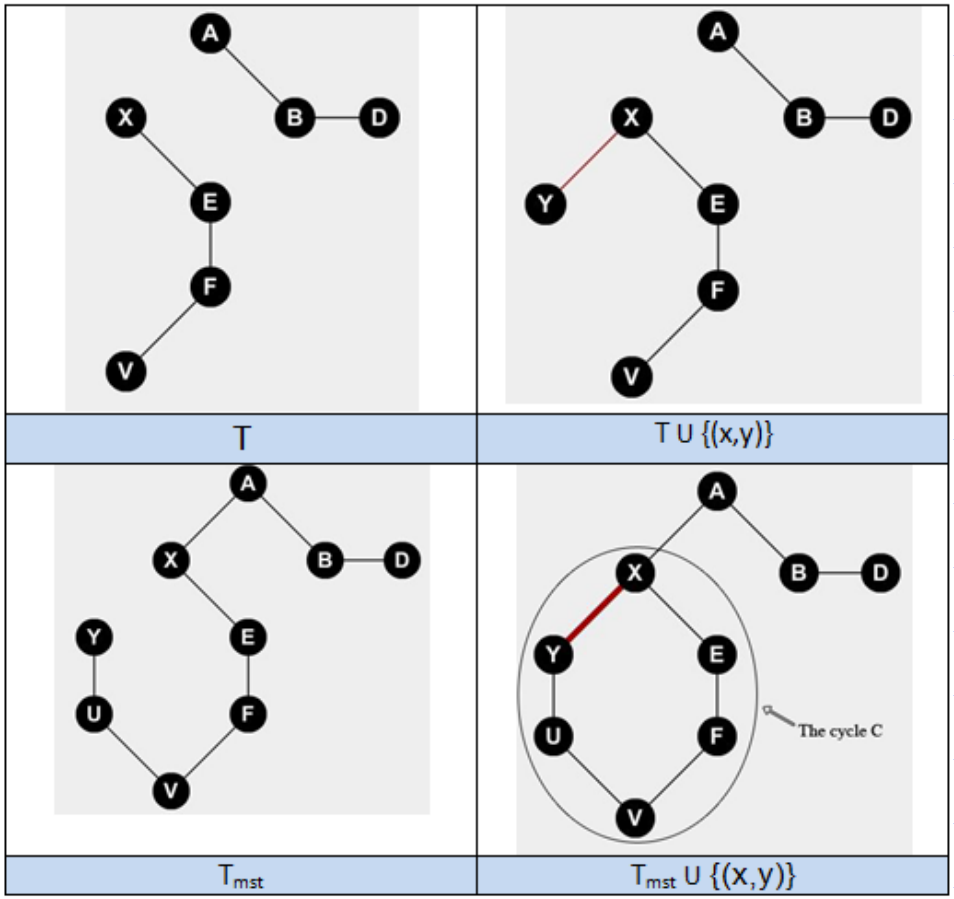
Assuming the invariant holds so far, we consider the next step of the algorithm in which the edge (x,y) is considered; assume that $C(x) \neq C(y)$, so that the algorithm will add (x,y) to T, the set of edges obtained so far. By induction hypothesis, there is an MST for G that contains T; let us denote this MST T_{mst}. We show there is an MST for G that contains $T \cup \{(x,y)\}$.

Proof (continued)

Case I. The new edge (x,y) happens to belong to T_{mst}. In that case, T_{mst} also contains T U {(x,y)}, and the induction step is complete.

Case II. The new edge (x,y) does not belong to T_{mst}. Since T_{mst} is a spanning tree, both x and y are vertices in T_{mst}. Therefore T_{mst} U {(x,y)} contains n edges, and so contains a cycle C, with (x,y) as one of its edges (if (x,y) were not one of the edges of C, then C would be a subgraph of T_{mst}). Recall T U {(x,y)} contains no cycle (by construction). So there must be some (u,v) in C (and also in T_{mst}) that does not belong to T (since edges of C are not a subset of the edges of T). Define T' by

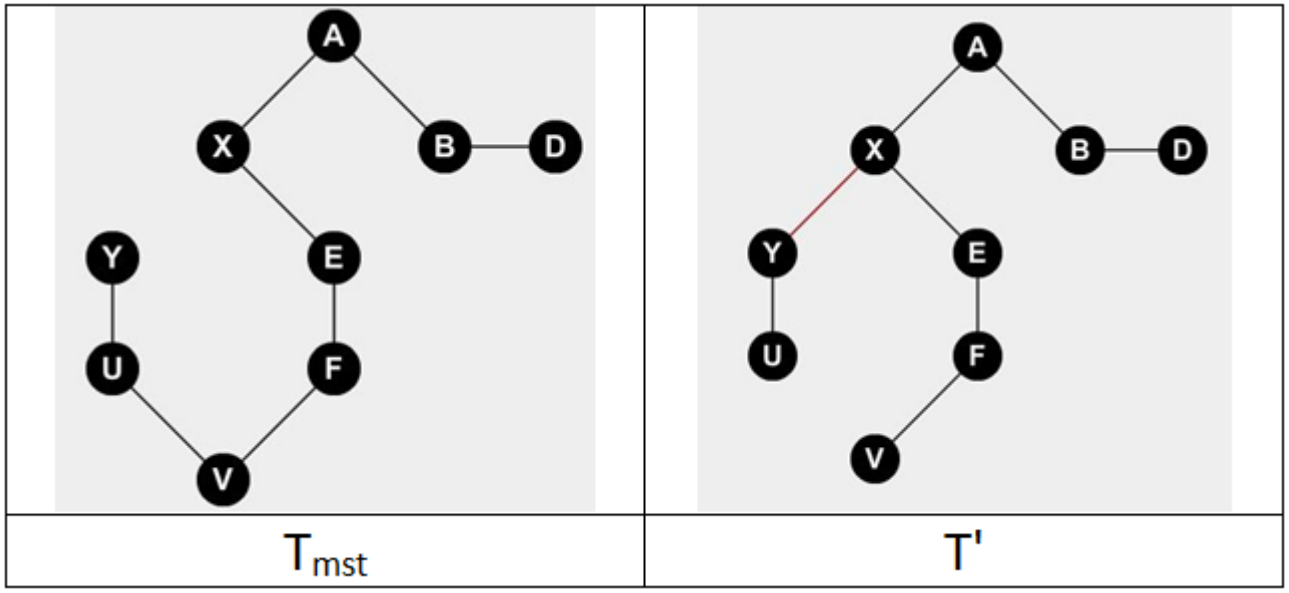
$$T' = T_{mst} \cup \{(x,y)\} - \{(u,v)\}$$



Proof (continued)

Claim 1. T' is a spanning tree.

Proof. Suppose r, s are vertices in G. Since T is a spanning tree, there is a path p from r to s. If this path begins like this: r, . . ., v, u, . . ., then we can replace the occurrence of the edge v,u in p with a path in C from v back to u, and then continue to follow p. Likewise if it begins r, . . ., u, v, … Therefore, T' is connected and clearly has n – 1 edges (since T has that many); it follows that T' is a tree that visits every vertex.



Proof (continued)

Claim 2. T' = T_{mst} U {(x,y)} – {(u,v)} has the same weight as T_{mst}.

Proof. Note that T U {(u,v)} contains no cycle, since T_{mst} includes T U{(u,v)}. So the algorithm could not have already rejected (u,v) [it would reject (u,v) only if (u,v) would introduce a cycle into T]. Since edges are ordered by weight, this means that (u,v) must have weight greater than or equal to wt(x,y). It follows that wt(T') ≤ wt(T_{mst}). But T_{mst} is an MST, so we also have wt(T_{mst}) ≤ wt(T'). This proves Claim 2.

Continuation of Proof of Fact 5. Claims 1 and 2 show that T' is also an MST, but now T' includes both T and (x,y). This establishes the induction step of the proof of the loop invariant I. As observed earlier, we may now conclude that Fact 5 holds and that the algorithm produces an MST.

Running Time of Kruskal

- Computation
  - time to sort edges: O(mlog n)
  - time for while loop O(mn)
    - for each edge (x,y):
      - comparison C(x) = C(y), with a hashtable implementation of sets, is O(n)
      - merging C(x), C(y) costs min{|C(x)|, |C(y)|}, which is O(n).
- Cost of while loop can be improved with a good choice of data structure

DisjointSets Data Structure

- Data structure (U, $\mathscr{C}$) for maintaining a partition of a set into disjoint subsets (this data structure is sometimes called *Partition* rather than DisjointSets)
- General features
  - *Data:*
    - Universe U – the base set that is being partitioned (this set is never altered)
    - Collection $\mathscr{C} = \{X_1, X_2, \ldots, X_n\}$ of subsets of the universe – the subsets are disjoint and their union is U (these subsets are modified when the data structure is used – size of $\mathscr{C}$ shrinks because of repeated union operations)
  - *Operations:*
    - find(x) – returns the subset X_i to which x belongs
    - union(A,B) – replaces the subsets A, B in $\mathscr{C}$ with A U B.

Example

- Initial Structure:
  - U = {1, 2, 3, 4, 5}
  - $X_1 = \{1, 2\}$, $X_2 = \{3\}$, $X_3 = \{4, 5\}$
  - $\mathscr{C} = \{X_1, X_2, X_3\}$
- find Operation:
  find(2) = X_1 find(5) = X_3
- union Operation:
  union(X_1, X_2) = $X_1 \cup X_2 = \{1,2,3\}$
  new value for $\mathscr{C}$ is {{1,2,3}, {4,5}}

Tree-Based Implementation of DisjointSets

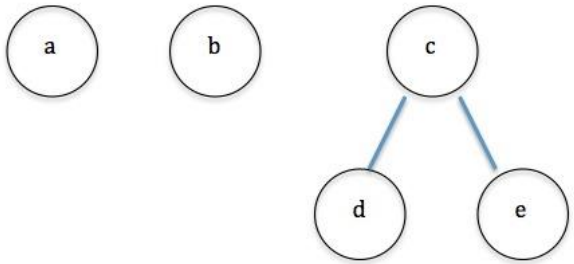
- The elements of each set X in the collection $\mathscr{C}$ are represented by nodes in a tree T_X; the set X itself is referenced by its root r_X.
- find(x) returns the root of the tree to which x belongs
- union(x,y) joins the tree that x belongs to to the tree that y belongs to by pointing root of one to the root of the other.

Example

U = {'a', 'b', 'c', 'd', 'e'}

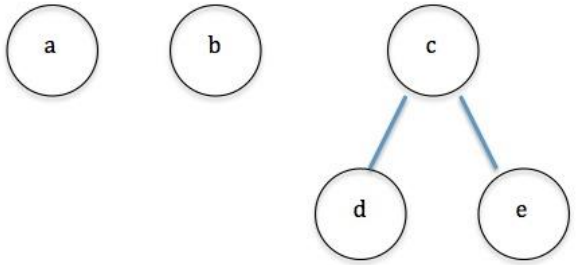
$\mathscr{C}$ = {{'a'}, {'b'}, {'c', 'd', 'e'}}

Tree representations:

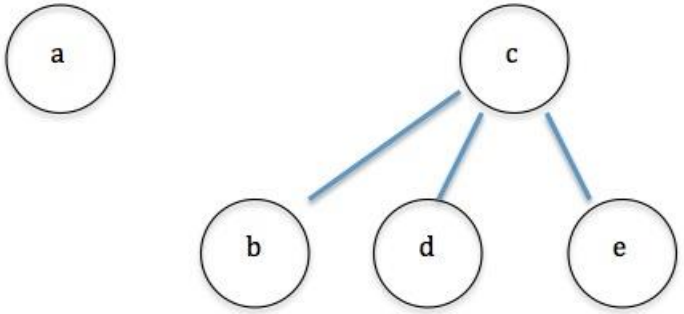


- find('d') returns 'c'

Example (cont)



- union('b', 'c') points root 'b' to 'c'

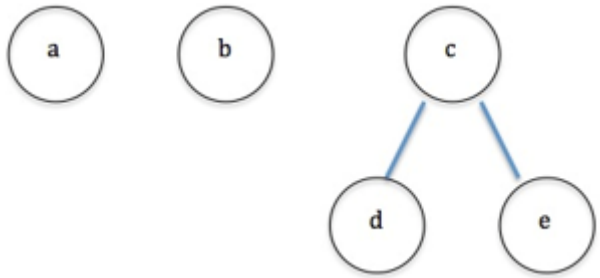


Now, find('b') returns 'c'

Code

```
//handle trees by keeping track of parents only
//whenever a character c is a root, its parent is set to be c itself
HashMap<Character, Character> parents = new HashMap<Character, Character>();
char[] universe;

//find returns the root of tree representing a subset, to which element belongs
//worst case: find requires full depth of tree to locate root of representing tree
public char find(char element) {
    char nextParent = parents.get(element);
    if(nextParent == element) {
        return element;
    } else {
        return find(nextParent);
    }
}
```

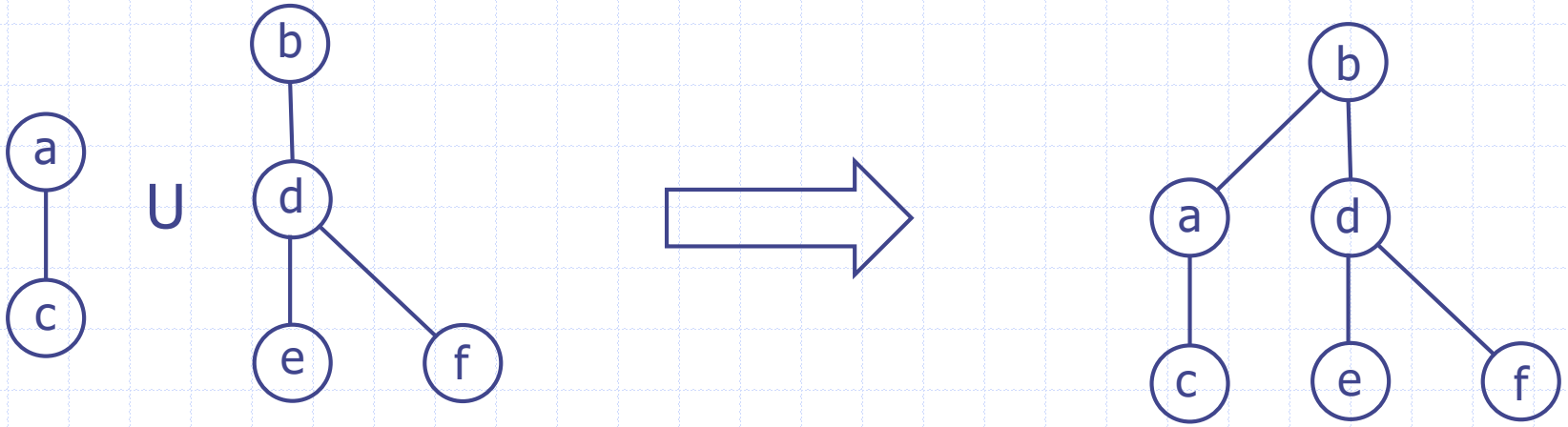


Parents Table	
'a'	'a'
'b'	'b'
'd'	'c'
'e'	'c'
'c'	'c'

Code

```
//union() accepts only tree roots (representing subsets) as arguments
//The method simply points the first root to the second
//In the worst case, resulting tree is taller than original two
public void union(char a_tree, char b_tree) {
    parents.put(a_tree, b_tree);
}
```

To avoid building up trees that are too tall (and therefore imbalanced), an optimization can be used: Always point the shorter tree's root to that of the taller.



Optimized Code

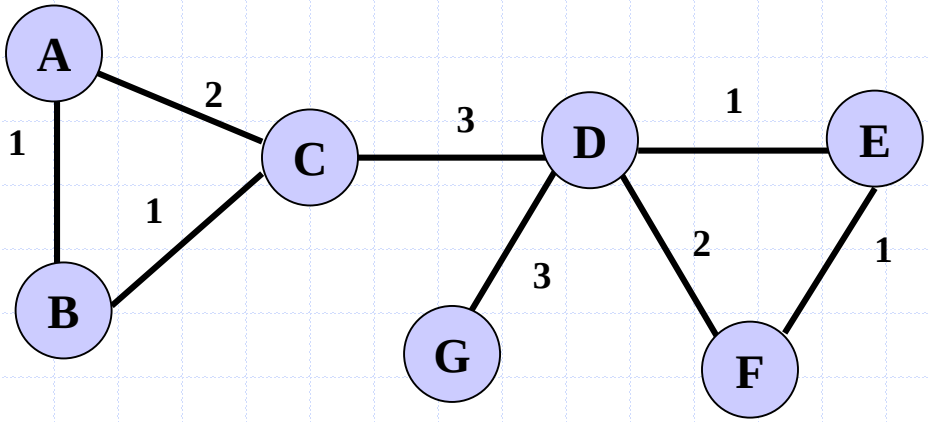
```
HashMap<Character, Character> parents = new HashMap<Character, Character>();
char[] universe;
//keep track of heights of trees
HashMap<Character, Integer> heights = new HashMap<Character, Integer>();

public void union(char a_tree, char b_tree) {
    int height_a = heights.get(a_tree);
    int height_b = heights.get(b_tree);
    if(height_a < height_b) {
        parents.put(a_tree, b_tree);
    } else if(height_b < height_a) {
        parents.put(b_tree, a_tree);
    } else { //height_a == height_b
        parents.put(a_tree, b_tree);
        heights.put(b_tree, height_b + 1); //this is case in which height is increased
    }
}
```

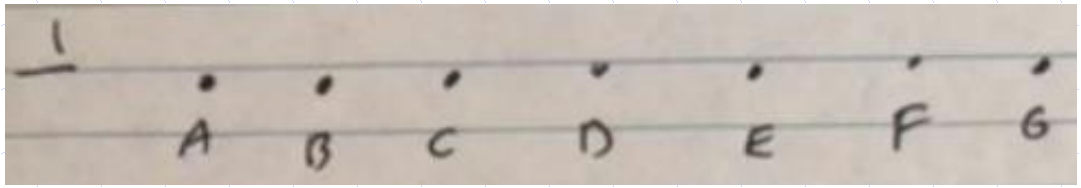
NOTE: With this optimization of union(), find() can be shown to run in O(log n) in the worst case. See https://en.wikipedia.org/wiki/Disjoint-set_data_structure

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {…}

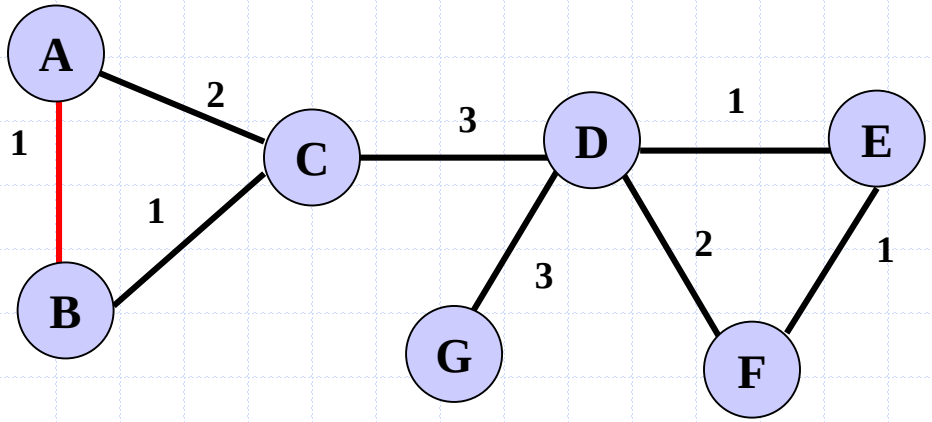


Step 1:
Sort the edges and initialize the clusters

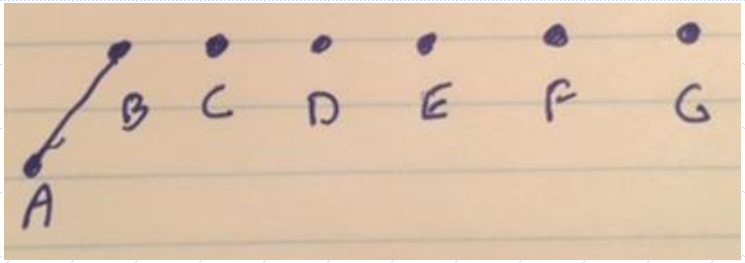


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, …}

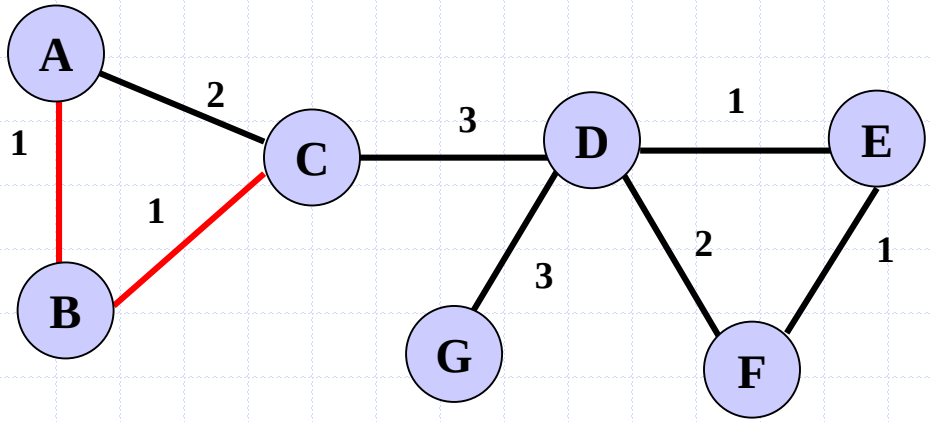


Step 2:
C(A) ≠ C(B)
add AB to T, merge C(A) and C(B)

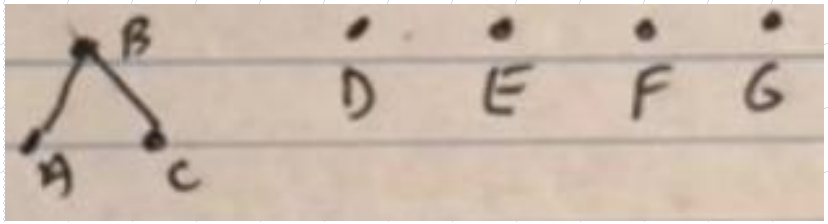


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, …}

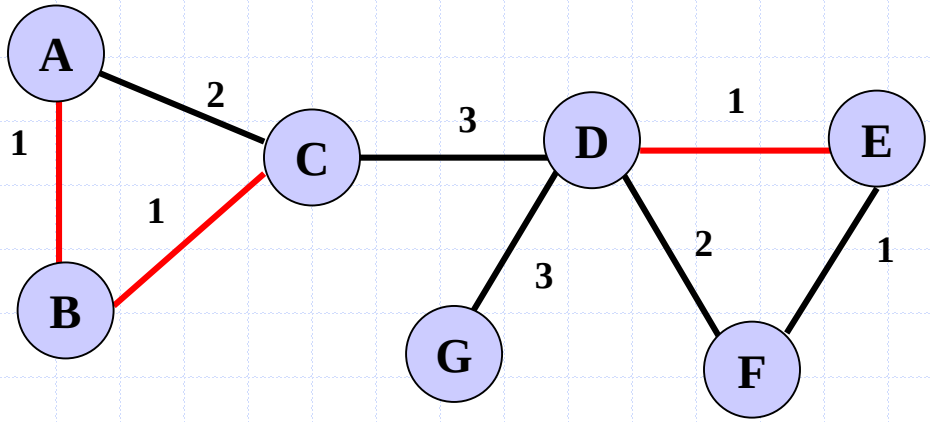


Step 3:
C(B) ≠ C(C)
add BC to T, merge C(B) and C(C)

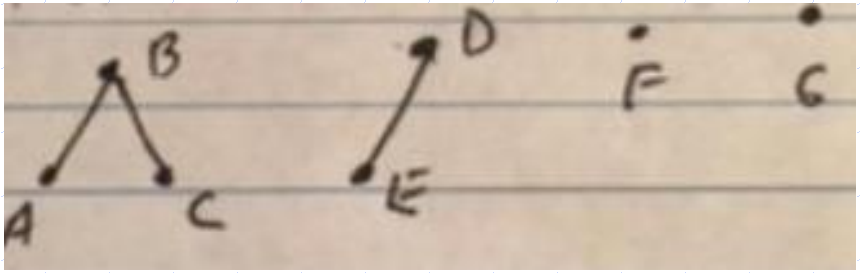


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, ...}

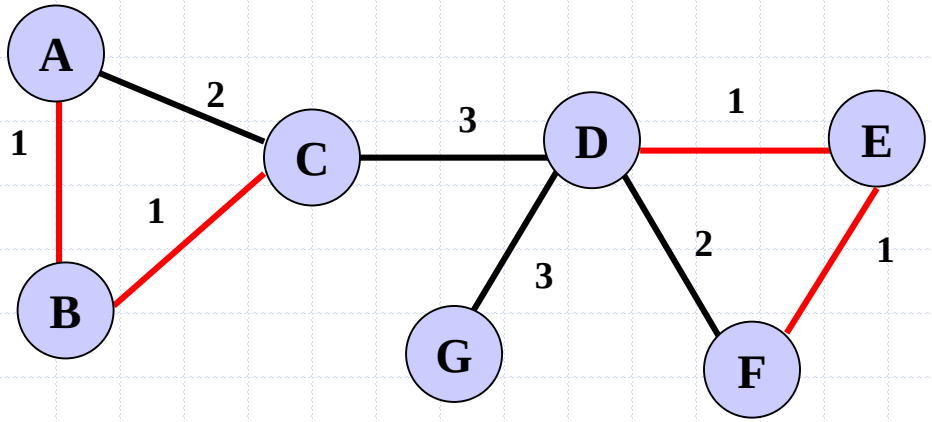


Step 4:
C(D) ≠ C(E)
add DE to T, merge C(D) and C(E)

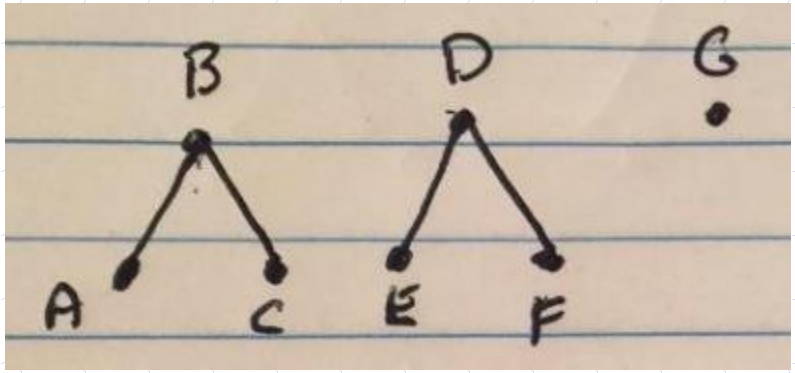


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}



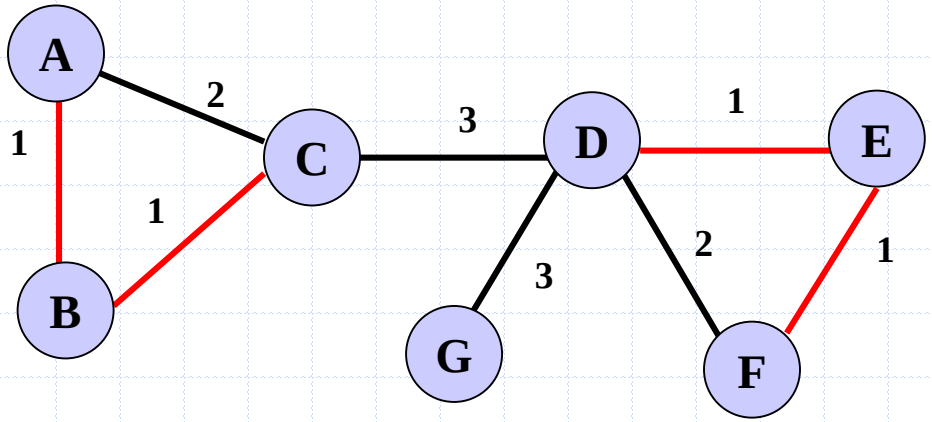
Step 5:
C(E) ≠ C(F)
add EF to T, merge C(E) and C(F)



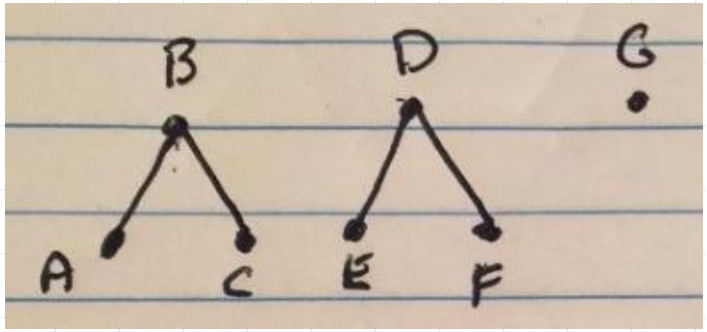
[point root of F to root of E]

Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}

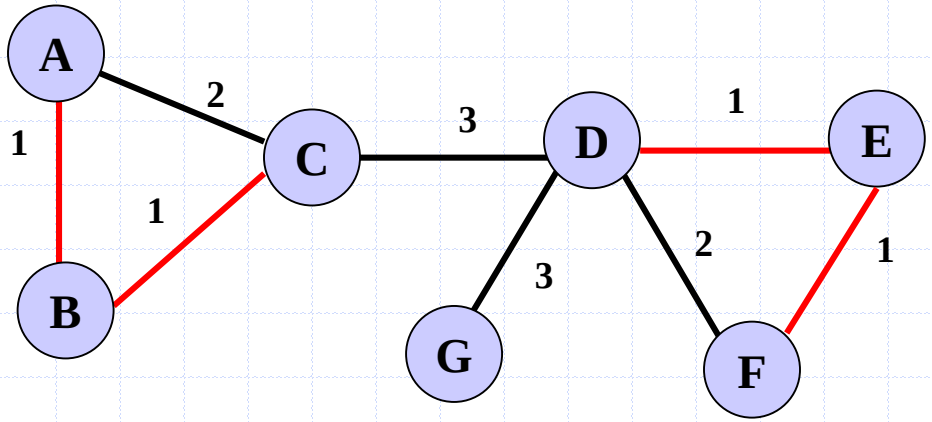


Step 6:
C(A) = C(C) , discard AC

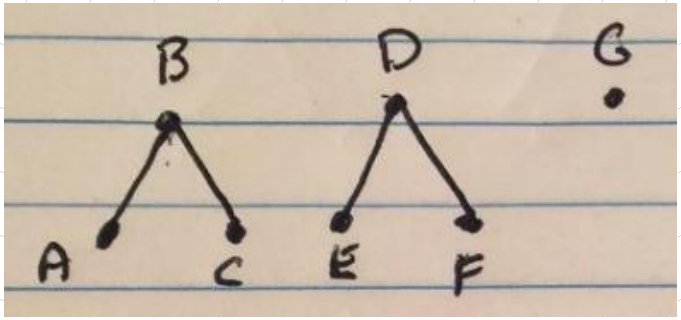


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, …}

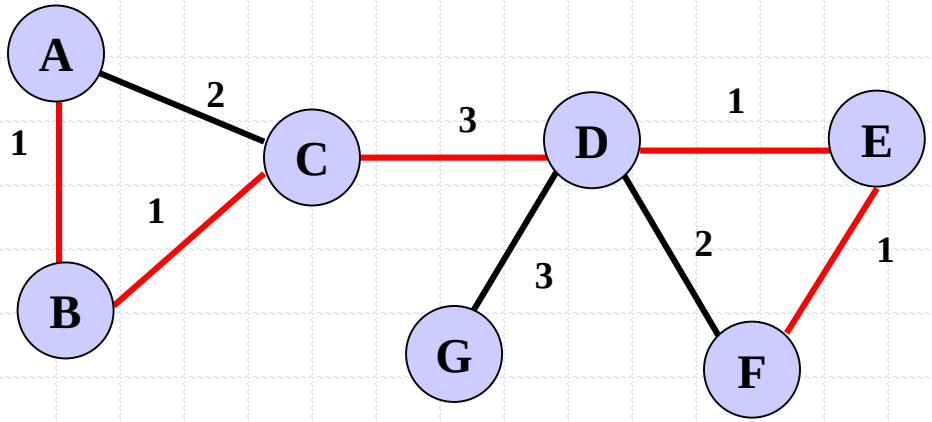


Step 7:
C(D) = C(F) , discard DF

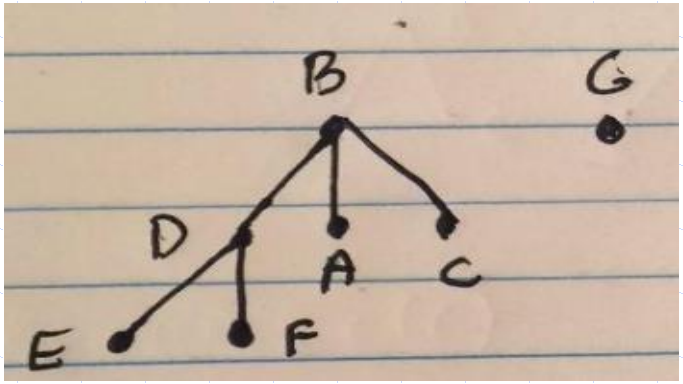


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, …}

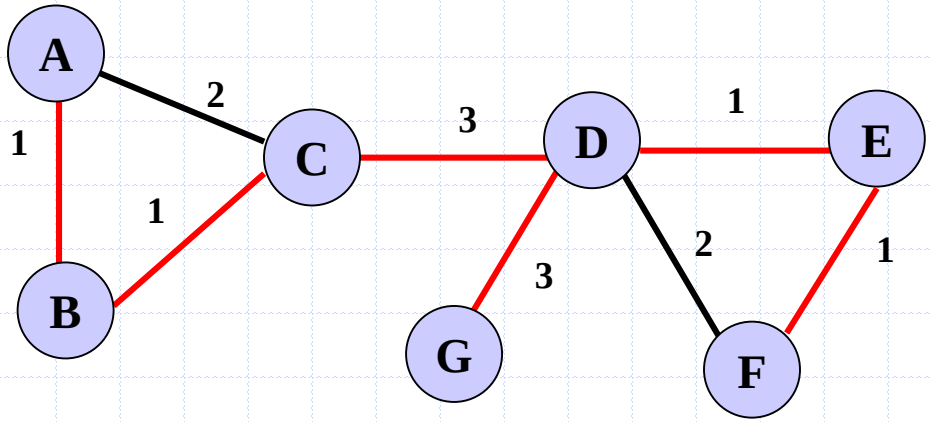


Step 7:
C(C) ≠ C(D)
add CD to T, merge C(C) and C(D)

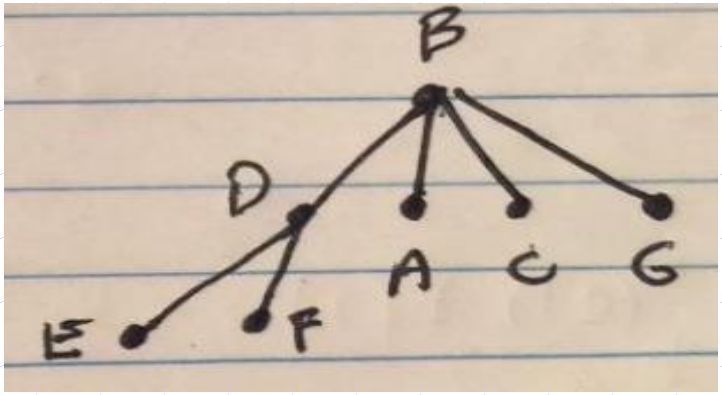


Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, DG, …}

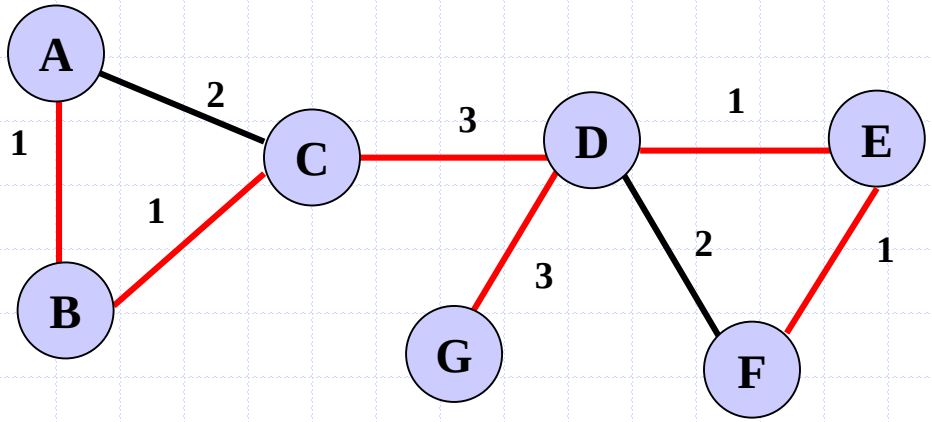


Step 8:
C(D) ≠ C(G)
add DG to T, merge C(D) and C(G)



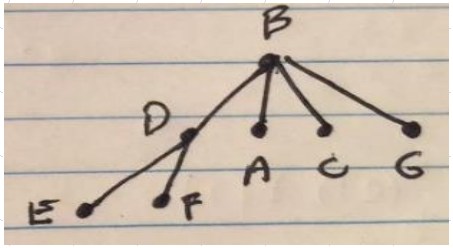
Worked Example

Sorted edges: AB, BC, DE, EF, AC, DF, CD, DG
T = {AB, BC, DE, EF, CD, DG}



Now we have n-1 = 6 edges in T, the algorithm stops.

Important! This disjoint sets data structure that has been built is just for keeping track of clusters – it is not part of final output



Optimized Running Time of Kruskal

- Time to sort edges: O(mlog n)
- Cost of while loop = O(mlog n)
  - loop potentially accesses every edge //O(m)
  - comparison C(x) = C(y) follows these steps:
    //locates roots of representing trees
    - $r_x \leftarrow$ find(x) and $r_y \leftarrow$ find(y) //O(log n)
    - check whether $r_x = r_y$ //O(1)
  - merging C(x), C(y) is done by union() operation //O(1)
    //note: union occurs immediately after checking for equality
    //so the roots $r_x = r_y$ do not need to be found a second time
- Total (optimized) running time for Kruskal: O(mlog n).

The Algorithm:

sort E in increasing order of edge weight

for each vertex v in G, define an elementary cluster $C(v)$ (which will grow) by $C(v) = \{v\}$

$T \leftarrow$ an empty tree // T will eventually become the minimum spanning tree

while T has fewer than $n - 1$ edges **do**

  $(u,v) \leftarrow$ next edge

  $C(v) \leftarrow$ cluster containing v

  $C(u) \leftarrow$ cluster containing u

  if $C(v) \neq C(u)$ **then**
    add edge (u,v) to T
    merge $C(u)$ and $C(v)$ (and update other clusters as needed)

return T

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. A Minimum Spanning Tree can be obtained from a weighted graph G = (V,E) by examining all possible subgraphs of G, and extracting from those that are trees having the smallest sum of edge weights. This procedure runs in $\Omega(2^m)$, where n = |E|.
2. Kruskal's Algorithm is a highly efficient procedure (O(mlog n)) for finding an MST in a graph G. It proceeds by choosing edges with minimum possible weight subject to the constraint that selected edges do not introduce a cycle in the set T of edges obtained so far.

3. *Transcendental Consciousness*, the simplest form of awareness, is the source of effortless right action.
4. *Impulses Within the Transcendental Field.* Effortless, economical, mistake-free creation arises from the self-referral dynamics of the field of pure consciousness.
5. *Wholeness Moving Within Itself.* In Unity Consciousness, optimal solutions arise as an effortless unfoldment within one's unbounded nature.