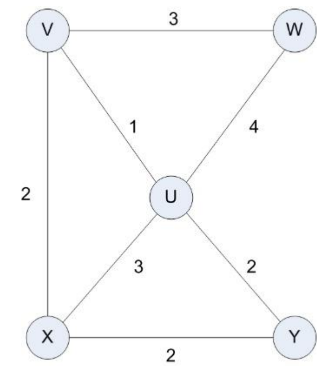
Lab 11

Due Monday 2 pm

1. Carry out the steps of the Slow Dijkstra algorithm to compute the length of the shortest path between vertex V and vertex Y in the graph below. Show all steps and indicate clearly your final answer (which should be a list A[] which shows shortest distances between V and every other vertex).



2. Prove that the Slow Dijkstra algorithm is also correct. The algorithm is reproduced here:

SLOW DIJSKSTRA

Input: A simple connected undirected weighted graph *G* with nonnegative edge weights, determined by a weight function *wt* (*x*,*y*), and a starting vertex *s* of *G*.

Output: Table A of shortest distances d(s,v) from s to v, for each v in V, so A[v] = d(s,v) for each v
Aux Output: Table B with property that B[v] is a shortest path from s to v.

The Algorithm:

$A[s] \leftarrow 0.$
X ← {s} //Basis step

while X ≠ V **do**
{ POOL ← {(v,w) ∈ E | v ∈ X and w ∉ X} }
//here, we have no control over how much of E is searched, leading to slow running time
(v′,w′) ← search POOL for edge (v,w) for which greedy length A[v] + wt(v,w) is minimal
add w′ to X
A[w′] ← A[v′] + wt(v′,w′)

Recall the Lemma shown in the slides. This will be helpful to use in your proof.

Main Lemma. Suppose G is a weighted graph. Suppose $q : s, \ldots, y, z$ is a true shortest path from s to z in G. Then the path $s, \ldots, y$ is a true shortest path from s to y; the path y, z is a true shortest path from y to z; and $d(s, z) = d(s, y) + \mathrm{wt}(y, z)$.

Proof. Let r and t be the following subpaths of q:

$$r : s, \ldots, y \qquad t : y, z.$$

We claim that $d(s, y)$ is equal to the length of r: If not, then there must be a shorter path r' than r that goes from s to y. But then $r' \cup t$ is a shorter path from s to z than q, which contradicts the fact that q is a true shortest path from s to z. For the same reason, $d(y, z)$ is equal to length of t. In particular, $d(y, z) = \mathrm{wt}(y, z)$.

We have:

$$d(s, z) = \mathrm{length}(q) = \mathrm{length}(r) + \mathrm{length}(t) = d(s, y) + \mathrm{wt}(y, z),$$

as required. □

Reason in the following way: Assume that through the first i stages of the iteration in Slow Dijkstra, the values A[u] for each u in X are correct --- that is, for each u in X, A[u] = d(s,u). We consider stage i+1. Let (v',w') be an edge in G with v' in X and w' not in X with the property that for all edges (v,w) in G with v in X and w not in X, we have A[v'] + wt(v',w') ≤ A[v] + wt(v,w). (This is how the algorithm behaves.) We wish to show A[w'] is correct – that is, we wish to show A[w'] = d(s,w'). Assume that this is not the case. Since A[v'] is correct, it follows that A[w'] $\neq$ d(s,w') implies A[w'] > d(s,w').

Let q : s, …, y, z, …, w' be a truly shortest path from s to w'. Let L be the length of q. Assume that z is first vertex in V - X encountered on q, and that y is the predecessor of z on q and must therefore belong to X. The subpath s,…, y must be a shortest path from s to y by the Main Lemma and has length A[y] (we are assuming A[u] is correct for all u in X so far).

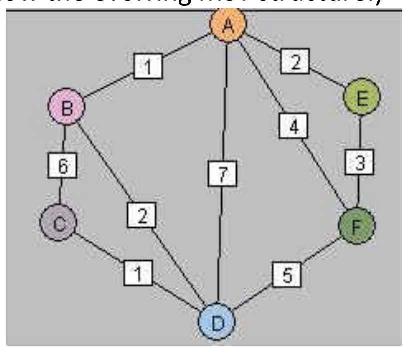
Let q_0 be the path s, …, y, z; we denote its length L_0. Clearly, A[y] ≤ L_0 ≤ L. It suffices to show

$$A[w'] = A[v'] + wt(v',w') \leq L_0$$

Explain the following:

a. Why must it be true that L_0 = A[y] + wt(y,w)?
b. Why must it be true that A[v'] + wt(v',w') ≤ A[y] + wt(y,w)?
c. Why do parts a and b allow us to conclude that we have reached a contradiction, and therefore that A[w'] = d(s,w') after all?

3. Carry out the steps of Kruskal's algorithm for the following weighted graph. Manage the evolution of clusters using a tree-based disjoint sets data structure (as shown at the end of the slides). Also, show how the set T of edges evolves during execution (as shown in the slides). Edges should be sorted initially; to resolve ties, use the alphabetical ordering of vertex names. (You do not need to re-draw the input graph each time to show the evolving MST structure.)



4. Create and implement an algorithm IsPrime(n) that accepts positive integer inputs and outputs true if n is prime, false otherwise. What is the asymptotic running time of your algorithm?

5. Suppose G = (V, E) is an undirected (unweighted) simple graph. A subset U of V is called a *base* for G is every edge in G has at least one endpoint in U. Do the following:
   a. Given G = (V,E) is it true that V itself is a base for G? Explain
   b. Is there a graph G having a base that is the empty set? If so, give an example.
   c. Give an example of a graph having 8 vertices and having a base of size 1.
   d. Give an example of a graph G having 2n vertices with the property that every base for G has size at least n.

   Devise an algorithm to solve the Smallest Base Decision Problem: Given G = (V,E) and a nonnegative integer k, is there a base U for G having size ≤ k? What is the running time of your algorithm?