CS 435

Design and Analysis of Algorithms*:*
“Discovering the Hidden Dynamics of the Laws of Nature”

Emdad Khan
ekhan@mum.edu
Dept. of Computer Science
Maharishi University of Management
May, 2019

Hilbert’s Agenda

Three questions at the heart of his agenda were:

- Is mathematics *consistent?*
  - Can no statement ever be proven both true and false with the rules of math?
- Is mathematics *complete*?
  - Can every assertion either be proven or disproven with the rules of math?
- Is mathematics *decidable?*
  - Are there definite steps that would prove or disprove any assertion?

Incompleteness Theorem

- Kurt Gödel wrote a paper in 1931 that shook the math world.
- Proved that consistency and completeness in math could not be attained.
- There is no consistent and complete system of formal rules that is comprehensive enough to include arithmetic.
- SCI point: Total Knowledge cannot be fully described in finite relative terms.
- Another important math question:

  Is mathematics *sound?*

  - Is every theorem (proven statement) also "True"? E.g. P ^ not (p) can be proved but the result is "False". Hence, it is not sound.

    (Converse of Completeness – Can we prove Every Statements that are "true"?)

Arithmetic is Incomplete

- Can we come up with enough number of Axioms that can prove all True properties of Model N, the non-negative Integers?
- Consider the following Axioms of NT
  - $\forall x\ (f(x) \mathrel{!=} 0)$
  - $\forall x\ (x + 0 = x)$
  - $\forall x\ (x.o = 0)$
  - $\forall x\ (x < f(x)$ where f is Unary function i.e. $f(x) = x + 1$.
  - …………..
- Unfortunately a complete list of axioms like above does NOT exist to make Number System / Arithmetic Complete.

Decidability Problem

- Alan Turing, sometimes called the “father of computer science”.
- Studied Godel’s work in 1935
- Studied the Problem:

Can one find an algorithm to determine whether a mathematical proposition is true or false or are some propositions undecidable?

Halting Problem

- *Proposition: Given a description of a Turing machine and its initial input, determine whether the program, when executed on this input, ever halts (completes). The alternative is that it runs forever without halting.*
- Recursive – one Turing machine analyzes another Turing machine
- More on this – a few slides later

[These are all part of "Complexity Theory" class – so will not go into very details]

Known Algorithms – P Problems

1. (GCD Problem) The Euclidean Algorithm computes the gcd of two positive integers m <= n. It requires fewer than n steps to output a result. We say it *runs in O(n) time*.
2. (SORTING Problem) MergeSort is an algorithm that accepts as input an array of n integers and outputs an array consisting of the same array elements, but in sorted order. It requires fewer than n2 steps to output a result. We say it runs in O(n2) time.

Both Problems 1 and 2 have algorithms that run in *polynomial time;* this means that the number of steps of processing required is a polynomial function of the input size.

If a problem can be solved using an algorithm that runs in polynomial time, the problem is said to be a *P Problem.*

P Problems are said to have *feasible solutions* because an algorithm that runs in polynomial time can output its results in an acceptable length of time, typically.

Known Algorithms – NP and EXP Problems

3. (SUBSETSUM Problem) Every known algorithm to solve SubsetSum (with input array containing n elements) requires approximately 2^n steps of processing (“exponential time”) to output a result (in the worst case). However, it is still possible that someone will one day find an algorithm that solves SubsetSum in polynomial time.
4. (N X N CHESS Problem) Like SubsetSum, every known algorithmic solution requires exponential time in the worst case. However, it has also been shown that no algorithmic solution could ever be found that runs in polynomial time.

- ❖ Both SubsetSum and n x n Chess are problems that belong to **EXP**. This means that each has an exponential time algorithm that solves it. The possibility remains that SubsetSum may also belong to the smaller class P, but this question remains unsolved. Read about the class **EXP** at http://en.wikipedia.org/wiki/EXPTIME
- ❖ n x n Chess is a special type of problem in **EXP**, known as **EXP-complete**. For this lecture, it suffices to say that **EXP** -complete means that there is no way to devise an algorithm that solves it in polynomial time. See

    http://www.ms.mff.cuni.cz/~truno7am/slozitostHer/chessExptime.pdf

When the only known solutions to a problem are exponential, the problem is said to *have **no known feasible solution.***

Known Algorithms – NP and EXP Problems

- The SubsetSum Problem is also an NP Problem: This means that there is an algorithm A and an integer k so that, given a solution T to a SubsetSum problem instance with input size n, A can verify that T is indeed a correct solution and does so in fewer than n^k steps (i.e. in polynomial time)
- Sample Verification:

  Start with a SubsetSum problem instance $S = \{s_1, s_2, . . ., s_n\}$ and k, where $s_1, ..., s_n$ and k are all positive integers. Given also a solution T: A solution is a subset of S whose elements

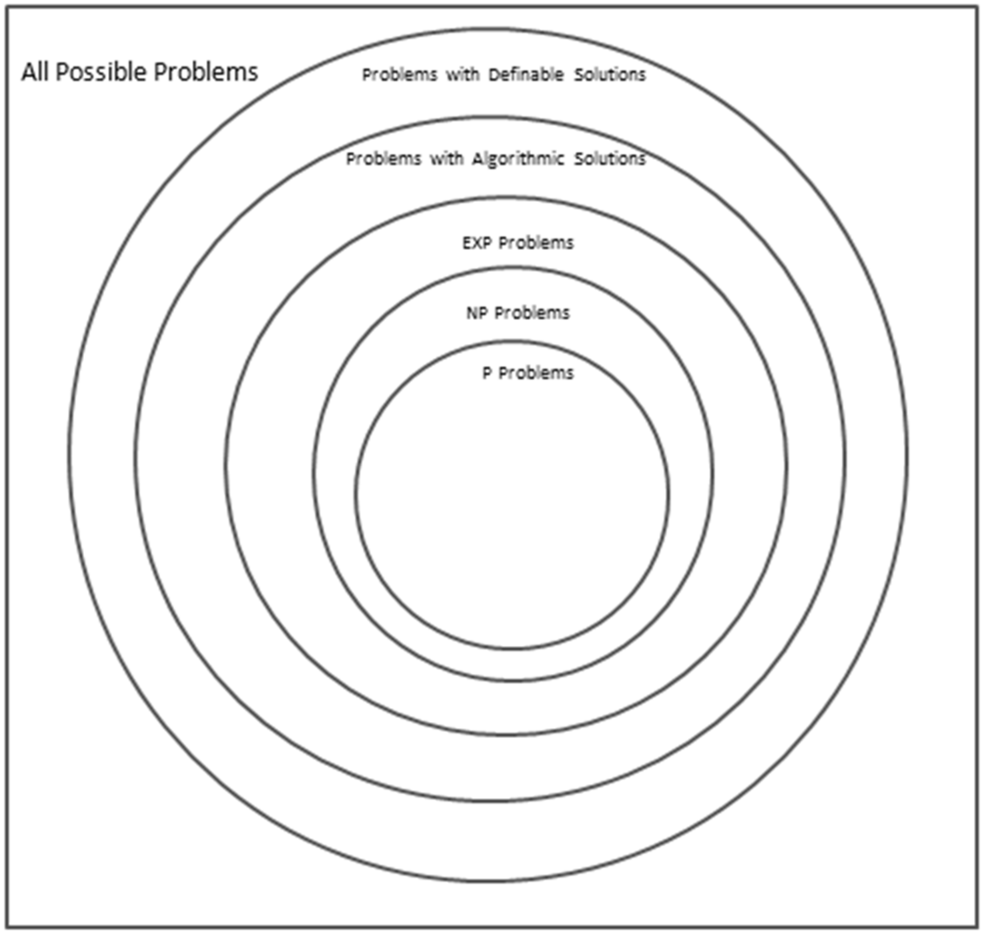
  add up to k.

  Verification:

```
sum = 0;
for each j in T
    sum += j
if(sum == k) return true;
else, return false;
```

- n x n Chess is not known to belong to NP (but one day someone may prove that it does belong to NP). In general, every NP problem belongs to EXP, but whether NP = EXP is not known. See `http://en.wikipedia.org/wiki/EXPTIME`

Classes of Problems


All Possible Problems
Problems with Definable Solutions
Problems with Algorithmic Solutions
EXP Problems
NP Problems
P Problems

The HALTING Problem (already shown)

<u>The Problem</u>: Given a Java program R (having a main method), when R is executed, does R terminate normally?

It is known that *there is no algorithm that could possibly solve the Halting Problem.* Although it is possible to *define* a solution, there is no procedure that could actually *compute* a solution.

Unsolvability of the Halting Problem

Begin with an observation: Every Java program may be encoded as a positive integer in such a way that, from the integer code, the program can be reconstituted.

Unsolvability 2

Encoding Java programs

- Assume < 512 distinct characters are ever used in a Java program, so each character can be represented by a bit string of length 9
- Encode a program by eliminating white space, start with ‘1’, and concatenate the encoded characters one-by-one

```
BigInteger f(BigInteger[] n){
    return n[0].add(n[0]);
}
```

Char	Code	Char	Code	Char	Code
a	000000001	d	000000010	e	000000011
f	000000100	g	000000101	i	000000110
n	000000111	r	000001000	t	000001001
u	000001010	B	000001011	I	000001100
.	000001101	{	000001110	}	000001111
(	000010000	)	000010001	;	000010010
⟨space⟩	000010011	[	000010100	]	000010101
0	000010110				

```
1000001011000000110000000101000001100000000111000001001000000011000000101
000000011000001000000010011000000100000010000000001011000000110000000101
000001100000000111000001001000000011000000101000000011000001000000010100
000010101000010011000000111000010001000001110000001000000000011000001001
000001010000001000000000111000010011000000111000010100000010110000010101
000001101000000001000000010000000010000010000000000111000010100000010110
000010101000010001000010010000001111
```

Unsolvability 3

- With this encoding, we can state the Halting Problem in terms of a function H:

  H(e,n) = 1 if the program

  encoded by e halts when run

  on input n; else H(e,n) = 0.
- The Halting Problem is: What are the output values of H for all inputs e, n?
- A solution for the Halting Problem is *definable* since we can specify a function H that solves it. (More precisely: It can be shown that H is definable in the standard model of arithmetic; if we could know all true statements of arithmetic, we would have all information about H)
- The important question is: Is there an algorithm A that computes the values of H (so that, on input e, n, A outpus H(e,n))? The Halting Problem is said to be *unsolvable* because no such algorithm exists.

Unsolvability 4

- Suppose H is computable. Define another function G as follows:

  G(e) = 1 if H(e,e) = 0

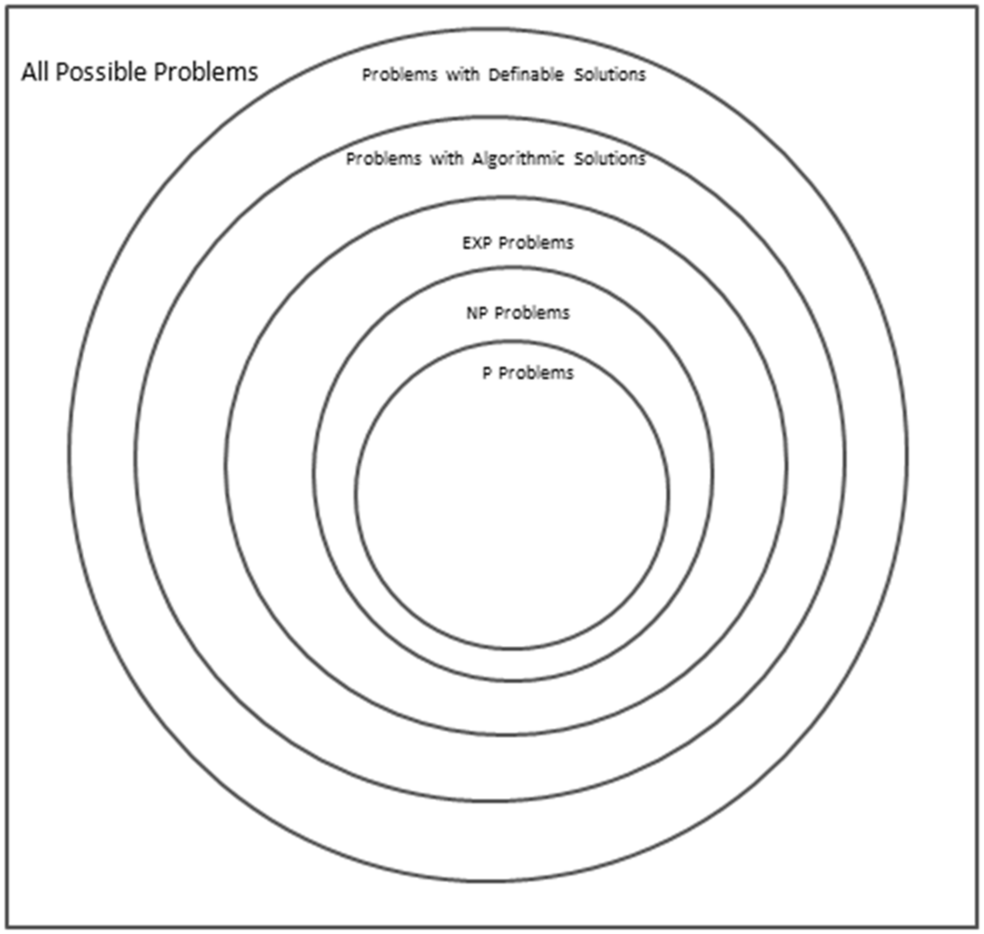
  G(e) is undefined if H(e,e) = 1

- *If H is computable, so is G*: Let P be a program that computes values of H. Create a program Q that computes G like this: On input e, Q runs P on e. If P outputs 0, Q outputs 1 and halts. If P outputs 1, Q goes into an infinite loop. This shows G is computable.

Unsolvability 5

- Use computability of G to obtain a contradiction. Suppose u is the integer that encodes the program Q. We run Q on input u and ask: Does Q halt on input u?
- Suppose Q does eventually halt on input u; then it outputs 1 on this input. By definition H(u,u) = 0. But this means that Q when run on input u, does *not* halt. Impossible.
- Suppose Q does *not* halt on input u. Then H(u,u) =1, which means that when Q is run on input u, it *does* halt. Impossible
- This shows that the assumption that H is computable leads to absurd conclusions. Therefore, H is *not* computable.

Classes of Problems


All Possible Problems
Problems with Definable Solutions
Problems with Algorithmic Solutions
EXP Problems
NP Problems
P Problems

The Full Range of “Problems”

- The Halting Problem has a *definable* solution because there is a definable function H (definable in the standard model of arithmetic) that answers all questions about whether a given Java program terminates on a given input
- Likewise, the abstract notion of “problem” is related to the concept of a function. Speaking generally, every definable function corresponds to (and is a definable solution for) a “problem”.
- More generally, *every* function from f: N -> N (whether or not definable) specifies a problem. But *most* such functions are *not* definable – therefore, most “problems” cannot even be formulated precisely in the language of mathematics.

Summary

- ➢ Algorithm Connects all Course of CS
- ➢ Very Key for your Success in the Industry
- ➢ You will learn how to design and analyze algorithms
- ➢ SCI Can be the Key to Get Full Development of Knowledge and Intelligence In Algorithms and CS.
- ➢ Most Algorithms Used in Industry are P-Problems
- ➢ Many More Very Complex Algorithms Exist (e.g. EXP) many of which Cannot be Solved Today. Using the Knowledge of SCI and Cosmic Computing We would be able to solve such problems in Future

Main Point

It has turned out that the problems that are most needed to be solved for the purpose of developing modern-day software projects happen to lie in the very specialized class of P Problems – problems that have feasible solutions. For these problems, the creativity and intelligence of the industry has been concentrated to a point; the algorithms that have been developed for these are extremely fast and highly optimized. At the same time, this tiny point value reveals how vast is the range of problems that cannot be solved in a feasible way. **These points illustrate that creative expression arises in the collapse of unboundedness to a point, and also that unboundedness itself is beyond the grasp of the intellect.**