Strong Induction

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Example 1

Theorem. For all $n \geq 0$, $n^5 - n$ is divisible by 10.

Proof.

Base case $n = 0$.

Clearly, the result holds.

(We may need to consider more cases.)

Induction hypothesis.

Assume the result is true for all values of n in the interval $[0, m]$.

(That is, m is the largest value for which the result is true)

$n^5 - n$ is divisible by 10 for $0 \leq n \leq m$.

Induction step.

We need to prove the result for the next value. That is, we need to prove that $(m + 1)^5 - (m + 1)$ is divisible by 10.

$(m + 1)^5 – (m + 1) = [(m – 1) + 2]^5 – [(m – 1) + 2]$

$= (m – 1)^5 + 10 (m – 1)^4 + 40 (m – 1)^3 + 80 (m – 1)^2 + 80 (m – 1) + 32 – (m + 1) – 2$
$= (m – 1)^5 – (m – 1) + 10$ (polynomial in m).

By induction hypothesis, $(m – 1)^5 – (m – 1)$ is dividable by 10. Hence the proof.

Check where we used the induction hypothesis. For $n = m – 1$.

Note that n MUST satisfy the condition $0 \leq n \leq m$.
That is, $0 \leq m – 1 \leq m$. In other words, $0 \leq m – 1$ or $1 \leq m$.
Therefore base case must include $n = 0$ and $n = 1$. Hence we modify base cases step as follows.

Base cases $n = 0, 1$.

Clearly, the result holds in both cases.

Example 2

Let $G(1) = 1$, $G(2) = 2$, $G(3) = 6$ and $G(n) = (n^3 - 3n^2 + 2n)G(n - 3)$ for $n > 3$.

Theorem. For all $n \geq 1$, $G(n) = n!$.

Proof.

Base cases $n = 1, 2$ and 3.

Clearly, the result holds in all three cases.

Induction hypothesis.

Assume result is true for all values of n in the interval $[0, m]$.

(That is, m is the largest value for which the result is true)

$$G(n) = n! \text{ for } 1 \leq n \leq m.$$

Induction step.

We need to prove the result for the next value. That is, we need to prove that $G(m + 1) = (m + 1)!$.

$G(m + 1) = ((m + 1)^3 - 3(m + 1)^2 + 2(m + 1))G(m - 2) = (m + 1)m(m - 1)G(m - 2)$

By induction hypothesis, $G(m - 2) = (m - 2)!$.

$G(m + 1) = (m+1)m(m-1)G(m - 2)$. Hence the proof.

Check where we used the induction hypothesis. For $n = m - 2$.

Note that n MUST satisfy the condition $1 \leq n \leq m$.

That is, $1 \leq m - 2 \leq m$. In other words, $1 \leq m - 2$ or $3 \leq m$.

Therefore base case must include $n = 1$, $n = 2$ and $n = 3$. Since we have already covered these cases, there is no need to modify.

Example 3

Theorem. Any integer $n \geq 8$ can be expressed as $3x + 5y$ where $x, y \geq 0$.

Proof.

Base case $n = 8$.

$8 = 3x + 5y$ where $x = 1$ and $y = 1$.

Induction hypothesis.

Assume result is true for all values of n in the interval $[8, m]$.

(That is, m is the largest value for which the result is true)

n can be expressed as $3x + 5y$ where $x, y \geq 0$ for $8 \leq n \leq m$

Induction step.

We need to prove the result for the next value. That is, we need to prove that $m + 1$ can be expressed as $3x + 5y$ where $x, y \geq 0$

By induction hypothesis,

$(m - 2)$ can be expressed as $3x + 5y$ where $x, y \geq 0$.

Then $m + 1 = 3(x+1) + 5y$ where $x, y \geq 0$.

Hence the proof.

Check where we used the induction hypothesis. For $n = m - 2$.

Note that n MUST satisfy the condition $8 \leq n \leq m$.

That is, $8 \leq m - 2 \leq m$. In other words, $8 \leq m - 2$ or $10 \leq m$.

Therefore base case must include $n = 8$, $n = 9$ and $n = 10$. Hence we modify base cases step as follows.

Base cases

$8 = 3x + 5y$ where $x = 1$ and $y = 1$.

$9 = 3x + 5y$ where $x = 3$ and $y = 0$.

$10 = 3x + 5y$ where $x = 0$ and $y = 2$.

Importance of checking the base cases

Let $P(1) = 1$, $P(2) = -1$ and $P(n) = P(n - 2)$ for $n > 2$.

Theorem. For all $n \geq 1$, $P(n) > 0$.

Proof.

Base cases $n = 1$.

Clearly, the result holds.

Induction hypothesis.

Assume result is true for all values of n in the interval $[1, m]$.

(That is, m is the largest value for which the result is true)

$P(n) > 0$ for $1 \leq n \leq m$.

Induction step.

$P(m + 1) = P(m - 1) > 0$. Hence the proof.

Check where we used the induction hypothesis. For $n = m - 1$.

Note that n MUST satisfy the condition $1 \leq n \leq m$.

That is, $1 \leq m - 1 \leq m$. In other words, $1 \leq m - 1$ or $2 \leq m$.

Therefore base case must include $n = 1$ and $n = 2$.

Since we cannot prove $P(2) > 0$, one of the base cases, we cannot prove the result through strong induction.

The number of base cases required (Summary)

The smallest value used during Induction to prove the result for m + 1	**Number of cases**
m	1 (Strong induction is not required)
m-1	2
m - 2	3
m - 3	4
m - k	k + 1