Maths Review

I. Laws of Logarithms (Section 1)

1. $y = \log_b x \Rightarrow x = b^y$
2. $\log x \Rightarrow \log_2 x$
3. $\log^n x \Rightarrow (\log x)^n$.
4. $\ln x \Rightarrow \log_e x \Rightarrow e = 2.71828$
5. $\log_b (xy) = \log_b x + \log_b y$
6. $\log_b (x/y) = \log_b x - \log_b y$
7. $\log x^y = y \log x$
8. $\log_b x = \dfrac{\log_a x}{\log_a b}$
9. if $0 < x < y \overset{\text{then}}{\Rightarrow} \log_b x < \log_b y$
10. $\log_b 1 = 0$
11. $\log_b b = 1$
12. $\log_b x < x$ iff $x > 0$ && $b \geq 2$
13. $\log_2 1024 = 10$
14. $\ln 2 = \log_e 2 \simeq 0.693$ $(0 < \ln 2 < 1)$
15. $\log e = \log_2 e \simeq 1.44$ $(1 < \log e < 2)$

Problems

1. $(\log n)^k = k \log n$, is not true for $k > 1$

take $k = 2$

$(\log n)^2 = 2 \log n$ iff $n = 1$ or $n = 4$

for ~~take~~ $n = 8$

$(\log 8)^2 = 3^2 = 9$, & $2 \log 8 = 2 \times 3 = 6$

$9 \neq 6$

Geometric

$g_n = r^{(n-1)} \times g_1$ $\quad \boxed{-1 < |r| < 1}$

$S_n = \dfrac{g_1(1-r^n)}{1-r}$

Arithemetic

$a_n = a_1 + (n-1)d$

$S_n = \dfrac{n(a_1 + a_n)}{2} = \dfrac{n(2a_1 + (n-1)d)}{2}$

Show

2. $\log_b(x+y) \neq \log_b^x + \log_b^y$ in general!

$\log_b^x + \log_b^y = \log_b^{xy}$

i.e $\log_b^{(x+y)} = \log_b^{(xy)}$

$x+y = xy.$

$x - xy = -y$

$\dfrac{x(1-y)}{1-y} = \dfrac{-y}{1-y}$

$x = \dfrac{y}{y-1}$, for $y \neq 1$

Substitute

$\log_b^{(x+y)} = \log_b^{(xy)}$

holds true for $x = \dfrac{y}{y-1}$, provided $y \neq 1$

if $y = 1$, $x+y = xy \simeq x+y = x$

can't be true for any x

$\therefore \log_b^{(x+y)} \neq \log_b^x + \log_b^y$

3. Show that for all $n > 2$, $n < n\log n < n^2$

It is also true that for all $n > 4$, $n^2 < 2^n$.

Assume that the inequality holds true for $n = k$.

i.e $k < k\log k < k^2$

We want to show that it's true for $n = k+1$

for $b \geq 2$ & $x > 0 \Rightarrow \log_b^x < x.$

$\Rightarrow \log(k+1) < k+1$ - (*)

Suppose $n = k+1 > 2$, since $n > 2$

$\Rightarrow (k+1)\log(k+1) < (k+1)(k+1) = (k+1)^2$ — (**)

Since $k+1 > 2$, then $\log_2(k+1) > \log_2 2$
$\Rightarrow \log(k+1) > 1$ --- ⊛⊛

Combining ⊛ & ⊛⊛, we have
$1 < \log(k+1) < k+1$

Multiplying all sides by $k+1$.
$k+1 < (k+1)\log(k+1) < (k+1)^2$

∴ The inequality holds true for $n = k+1$.

∴ By induction principle, the inequality $n < \log n < n^2$ for $n > 2$ is true.

⊛ b) For $n > 4$, $n^2 < 2^n$.

for $n = 5$, $5^2 < 2^5$ --- true

- Assume it is true for $n = k$, since $n > 4 \Rightarrow k > 4$
$k^2 < 2^k$, $k > 4$.

- We want to show that it's true for $n = k+1$.

multiplying $k^2 < 2^k$ by 2 we have
$2k^2 < 2^k \cdot 2 = 2^{k+1}$. --- ⊛

For $n > 4$, it can be shown that $(n+1)^2 < 2n^2$

Equivalently for $k > 4$,
$\hookrightarrow (k+1)^2 < 2k^2$ --- ⊛⊛

Combining ⊛ and ⊛⊛, we have
$(k+1)^2 < 2k^2 < 2^{k+1}$.
$\Rightarrow (k+1)^2 < 2^{k+1}$.

∴ For $n > 4$, $n^2 < 2^n$ is true for $n = k+1$.

∴ By induction principle, for $n > 4$, $n^2 < 2^n$ holds true

4. $2^{3n-1} = 32$

$32 = 2^5$

$2^{3n-1} = 2^5$

$3n-1 = 5$

$\frac{3n}{3} = \frac{6}{3}$

$\underline{n = 2}$

Sections, Sets

- Disjoint sets $\Rightarrow$ No common elements $A \cap B = \emptyset$
- Cardinality / size $\Rightarrow |\{2, 4, 8\}| = 3$
- The power set of set A, denoted $P(A)$, is the set whose elements are all the subset of A.
  - ↳ If A has n elements, $P(A)$ has 2^n elements
  - ↳ i.e A set w n elements has 2^n subsets
- If a set A having n elements is totally ordered, then a permutation of A is re-arrangement of the elements of A.

  eg $\{1, 2, 3, 4\} \Rightarrow [1, 2, 4, 3], [4, 3, 2, 1]$
  - The permutation of A that doesn't re-arrange any of the elements is called the identity permutation.
- The notation $C(n, m)$ ~~is rea~~ read as "the number of combinations of n things taken at a time"
  - ↳ the no of m-element subsets of an n-element set.

$$C(n,m) = \frac{n!}{m!(n-m)!}$$

- $P_{n,m} \Rightarrow$ the number of permutations of n things taken m at a time.

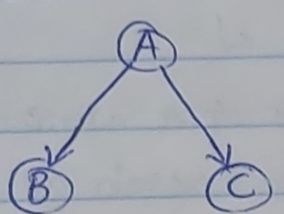
$P_{n,m} = C(n,m)\, m! = \frac{n!}{(n-m)!}$

eg. $P_{10,2} = \frac{10!}{(10-2)!}$

Problem set

1. $\{1,1,2\}$, $\{1,2\}$, $\{2,1\}$ ⇒ are all equal
2. $\{1,\{2,3\}\} \subseteq \{1,2,3,4,5,\dots\}$ FALSE
3. $A = \{1,2\}$, $P(A) = ?$
   $P(A) = \{\emptyset, \{1\}, \{2\}, A\}$
4. $\{1,3,4\}$ all permutations
   ⇒ $\{1,3,4\}, \{1,4,3\}, \{3,1,4\}, \{3,4,1\}, \{4,1,3\}, \{4,3,1\}$
5. $P_{9,5}$? $= \frac{9!}{(9-5)!} = \frac{9!}{4!} = 9\times8\times7\times6\times5$
6. $C(20,3) = \frac{20!}{3!\,(20-3)!} = \frac{20!}{3!\,17!} = \frac{20\times19\times18}{3\times2\times1} = 60\times19$

Section DGF: Directed Graphs and Functions



- A function from a set X to a set Y ⇒ $f: X \to Y$.
  ↳ is a special kind of directed graph w the ff xers
- The objects of the graph f are the elements of X together w the objects of Y.
- Each arrow of f always starts at an element of X and points to an element of Y. If, in f, x points to y we write $x \to y$ or $f(x) = y$.

- In f, no $x \in X$ ever points to more than one element of Y
- In f, every element of X does point to atleast one element of Y.

When $f: X \to Y$ is a fun, X is called its domain, Y its codomain

Suppose $f: X \to Y$ is a fun.

(1) Onto - A f is onto if for each $y \in Y$ there is an element $x \in X$ so that $x \to y$.

(2) Range - The range of f is the set of all $y \in Y$ that are pointed to by one or more x in X; the range is the set of all output values of f.
↳ If the range of f is Y itself, f is onto.

(3) 1-1 function : A function $f: X \to Y$ is 1-1 if, whenever x and x' are distinct elements of X, and $x \to y$ and $x' \to y'$, then y and y' are also distinct elements of Y.

Problem DGE 1.

$f(n) = n^2$, domain $= \mathbb{N}$

Is $f(n)$ 1-1?

Check

Given $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, show $f(n_1) \neq f(n_2)$

~~P~~ Defⁿ

- A function $f: \mathbb{N} \to \mathbb{N}$ is said to be increasing if, whenever $m<n$, we have $f(m) < f(n)$
- A function $g: \mathbb{N} \to \mathbb{N}$ is said to be non-decreasing if, whenever $m<n$, we have $g(m) \leq g(n)$

Problem DGF 2.

Show that the function $f(n) = n^2$, ~~is~~ domain $\mathbb{N}$ is increasing

Solⁿ

For $n, m \in \mathbb{N}$, suppose $n<m$.

We want to show that $f(n) < f(m)$

For $n, m \in \mathbb{N}$, we have $n<m \Rightarrow$ multiply both sides by n

$\hookrightarrow n^2 < mn$ - - - - ⊛

For $n, m \in \mathbb{N}$, we have $n<m \Rightarrow$ multiply both sides by m

$\hookrightarrow nm < m^2$ - - - - - ⊛⊛

$\Rightarrow$ Since $n, m \in \mathbb{N}$ multiplying an inequality by n or m doesn't affect the inequality

Combining ⊛ and ⊛⊛

$n^2 < nm < m^2 \Rightarrow n^2 < m^2$

For $n<m$, we have $f(n) < f(m)$

$\therefore f$ is increasing.

Section SUM:
Summations

1. $\sum_{i=1}^{N} 1 = N$

2. $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

3. $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

4. $\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$

5. $\sum_{i=0}^{N} a^i = \frac{a^{N+1} - 1}{a-1}$

6. $\sum_{i=0}^{N} a^i < \frac{1}{1-a}$ (whenever $0 < a < 1$)

7. $\sum_{i=1}^{N} \frac{1}{i} \simeq \ln 2 \log N$ (the difference b/n these two falls below 0.58 as N tends to infinity)

Problem sum 1.

$$\sum_{i=1}^{N} 2i^2 + 3i - 4$$

$$2 \times \sum_{i=1}^{N} i^2 = \frac{2 \times N(N+1)(2N+1)}{6} = \frac{2N(N+1)(2N+1)}{6}$$

$$3 \times \sum_{i=1}^{N} i = \frac{3 \times N(N+1)}{2} = \frac{3N(N+1)}{2}, \quad 4\sum_{i=1}^{N} 1 = 4$$

$$\frac{2N(N+1)(2N+1)}{6} + \frac{3N(N+1)}{2} - 4N$$

$$\frac{2N(N+1)(2N+1) + 9N(N+1) - 24N}{6}$$

$$\frac{4N^3 + 15N^2 + 13N}{6}$$ //

Section MI : Mathematical Induction

1. Standard Induction

Suppose $\phi(n)$ is a statement depending on n. If

- $\phi(0)$ is true, and
- under the assumption that $n \geq 0$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true, then $\phi(n)$ holds true for all natural numbers n.

→ Basis step

→ Induction step

⇒ As we reason in the induction step, we will typically need to make use of $\phi(n)$ as an assumption; in this context, $\phi(n)$ is called the induction hypothesis.

2. Standard Induction (General Form)

↳ Let $k \geq 0$, suppose $\phi(n)$ is a statement depending on n. If

- ✓ $\phi(k)$ is true, and
- ✓ under the assumption that $n \geq k$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true, then $\phi(n)$ holds true for all natural numbers $n \geq k$

Problem MI.

1, Prove that, for every natural number $n \geq 1$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for $n=1$ $\sum_{i=1} i = 1 = \frac{1(1+1)}{2} = 1$ --- true

Assume it's true for $n=k$

i.e $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

We want to show that it is true for $n=k+1$

$$\sum_{i=1}^{k+1} i = \sum_{i}^{k} i + (k+1) = \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ It holds true for $n=k+1$.

$\therefore$ By mathematical induction principle for $n \geq 1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

2. Show that for every number $n > 4$, $n^2 < 2^n$

~~Total~~ for $n=5$, $5^2 < 2^5$ --- true

Assume it's true for $n=k$

i.e. $k^2 < 2^k$

We want to show that it's true for $n=k+1$

Multiplying both sides of the inequality by 2

$2k^2 < 2^k \cdot 2 \Rightarrow 2k^2 < 2^{k+1}$ --- ⊛

For $n>4$, it can be shown that $(n+1)^2 < 2n^2$

Equivalently for $k>4$, $(k+1)^2 < 2k^2$ --- ⊛⊛

Combining ⊛ and ⊛⊛, we have

$(k+1)^2 < 2k^2 < 2^{k+1}$, for $k>4$

$(k+1)^2 < 2^{k+1}$

∴ For $n>4$, $n^2 < 2^n$ is true for $n=k+1$.

∴ By induction principle

for $n>4$, $n^2 < 2^n$ holds true.

3. Total Induction

Suppose $\phi(n)$ is a statement depending on n and $k \geq 0$

If ✓ $\phi(k)$ is true, and

✓ under the assumption that $n>k$, and that each of $\phi(k)$, $\phi(k+1)$, --- $\phi(n-1)$ are true, you can prove that $\phi(n)$ is also true, then $\phi(n)$ holds true for all $n \geq k$.

Problem MI3

Prove that if $f(n) = 2^n$, then f is increasing

$f(k) = 2^k$, for $k \geq 0$ is increasing

Assume that $n>k$, $f(k) = 2^k$ is increasing

$f(k+1) = 2^{k+1} = 2 \cdot 2^k = 2(f(k))$ is increasing

$f(k+2) = 2^2 f(k)$ --- is increasing

⋮

$f(n-1) = 2^{n-1}$ is increasing

∴ By total induction $f(n)$ holds true for all $n \geq k$

Finite Induction

Suppose $0 \leq k \leq n$, and suppose $\phi(i)$ is a statement depending on i, where $k \leq i \leq n$. If

- $\phi(k)$ is true, and
- under the assumption that $k \leq i < n$ and that $\phi(i)$ is true, you can prove $\phi(i+1)$ is true,

then $\phi(i)$ holds true for all i in $k \leq i \leq n$.

Section BNT: Basic Number Theory

- [divides] $a \mid b \Rightarrow$ a divides b, i.e for some c, $b = ac$.
- [floor and ceiling] $\lfloor a \rfloor$ is the largest integer not greater than a ($\lfloor \cdot \rfloor$ is called the floor function) and $\lceil a \rceil$ is the smallest integer not less than a ($\lceil \cdot \rceil$ is called the ceiling function).

$$-5/4 = -(5/4) = -1 \text{ (Java)}$$

$$\lfloor -5/4 \rfloor = -2 \text{ (Mathematics)}$$

- [greatest common divisor]
- [primes]

Fact ⇒ Every integer > 1 is a product of primes

↳ A prime itself is considered a product of primes)

Fact ⇒ There are infinitely many primes.

- Fibonacci Numbers.

2 $\quad \frac{32 \times 32}{64}$ $\quad \frac{512}{32} = \frac{256}{16} = \frac{\quad}{8}$

$\frac{512}{32} = 16 \qquad \frac{32}{16} = 16$

$1^2 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6$

1. base case $n = 1$.

$1^2 = \frac{1(1+1)(2(1)+1)}{6}$

$1 = \frac{6}{6} = 1$ ✓

2. Assume it holds true for $n = k$

$1^2 + 2^2 + \dots + k^2 = k(k+1)(2k+1)/6$

3. Proof it holds true for $n = k+1$.

LHS $\Rightarrow$ $\underbrace{1^2 + 2^2 + \dots + k^2}_{\text{// iduction hypothesis}} + (k+1)^2$

LHS $= k(k+1)(2k+1)/6 + (k+1)^2$

4. Simplify LHS

$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \cancel{(k+1)}\left[\cancel{\frac{k(2k+1)}{6}}\right] \qquad = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$

$2k^2+3k+4k+6$
$\cancel{(k+3)} + 2\cancel{(k+3)}$
$k(2k+3) + 2(2k+3)$

$= \cancel{(k+1)}\left[\cancel{\frac{2k^2+k}{6}}\right] \qquad = \frac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right]$

$12 \Rightarrow 3 \times 4$
$7 \Rightarrow 3 + 4$

$(2k+3)(k+4) \qquad = \frac{k+1}{6}(2k^2 + 7k + 6)$

$2k+3$
$k+4$
$2k^2 + 3k + 8k + 12$

$2k(2k+1) + 2(k+6) \qquad = \frac{k+1}{6}(2k+3)(k+2)$

$2k^2 +$

$2k^2 + 3k + 4k + 6$
$2k\cancel{(k+1)}(2k+3)$

$= (k+1)(k+2)(2k+3)/6$