when closer to date, check answers for first session 1b, c, second session 1c.

I get numbers, but what do they all mean?

can just scan this paper as pdf during submission?

First, 1, 2a, b second, 1a, ci, ii,

24/9/23

Goh Jet Wei

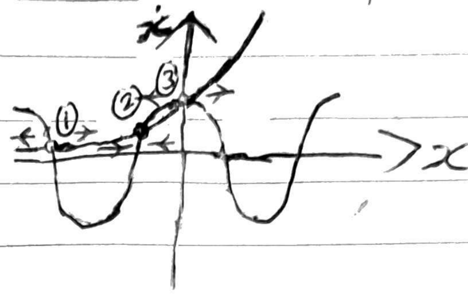
SHARP Honours session Term 1 2023: Homework on Non-Linear Dynamics and Chaos

First session

1. $\dot{x} = e^{x} - \cos x$

$0 = e^{x} - \cos x$, when $\dot{x} = 0$

$e^{x} = \cos x$



Can use desmo graphic calculator online? Cuz Idk how use excel for this actually...

Using Graphing Calculator,

For fixed point 1

Coordinates: $(-4.721, 0.009)$

The left of fixed point 1, $\cos x > e^{x}$, hence $\dot{x} = e^{x} - \cos x < 0$

The right of fixed point 2, $\cos x < e^{x}$, hence $\dot{x} = e^{x} - \cos x > 0$

Hence fixed point 1 is unstable//

For fixed point 2:

Coordinates: $(-1.293, 0.275)$

The left of fixed point 2, $\cos x < e^{x}$, hence $\dot{x} = e^{x} - \cos x > 0$

The right of fixed point 2, $\cos x > e^{x}$, hence $\dot{x} = e^{x} - \cos x < 0$

Hence fixed point 2 is stable//

For fixed point 3:

Coordinates: $(0, 1)$

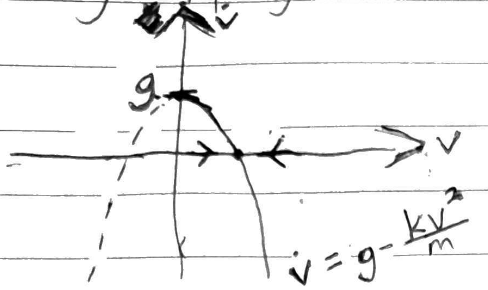
The left of fixed point 3, $\cos x > e^{x}$, hence $\dot{x} = e^{x} - \cos x > 0$

The right of fixed point 3, $\cos x < e^{x}$, hence $\dot{x} = e^{x} - \cos x < 0$

Hence fixed point 3 is unstable//

whats $v(0)$? Is it v or $\dot{v}$? What does it mean by ~~find the~~ determine the stability of v?

2\. a) Using Graphing Calculator,



Fixed point coordinates:

When $\dot{v} = 0$

$$0 = g - \frac{kv^2}{m}$$
$$g = \frac{kv^2}{m}$$
$$v = \sqrt{\frac{gm}{k}} \quad \text{OR} \quad v = -\sqrt{\frac{gm}{k}} \text{ (reject)}$$

Coordinates: $\left(\sqrt{\frac{gm}{k}}, 0\right)$

The left of fixed point, $\dot{v} > 0$

The right of fixed point, $\dot{v} < 0$

Hence, the fixed point is stable//

b)
$$\frac{dv}{dt} = g - \frac{kv^2}{m}$$
$$dt = \frac{dv}{g - \frac{kv^2}{m}}$$

$$\cancel{\int dt = \int \frac{1}{g-\frac{kv^2}{m}} dt}$$
$$\cancel{t = \int \frac{1}{g-\frac{kv^2}{m}} \frac{dv}{dt} dt}$$
$$t = \int \frac{1}{\frac{gm-kv^2}{m}} dv$$
$$= m\int \frac{1}{gm-kv^2} dv$$
$$= m\int \frac{1}{k\left(\frac{gm}{k}-v^2\right)} dv$$
$$= \frac{m}{k}\int \frac{1}{\frac{gm}{k}-v^2} dv$$
$$= \frac{m}{k}\left(\frac{1}{\sqrt{\frac{gm}{k}}}\right)\tan^{-1}\left(\frac{v}{\sqrt{\frac{gm}{k}}}\right) + c$$

$$\int \frac{1}{a^2-u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$
$$= \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$$

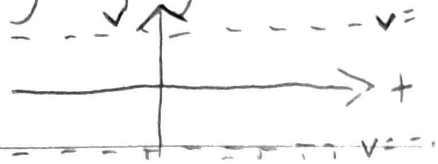
Since $v(0) = 0$, when $t = 0$, $v = 0$;

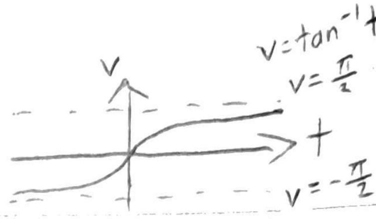
$$0 = 0 + c$$
$$c = 0$$
$$t = \frac{\sqrt{km}}{k\sqrt{g}}\tan^{-1}\left(\frac{v\sqrt{k}}{\sqrt{gm}}\right)$$

$$\tan^{-1}\left(\frac{v\sqrt{k}}{\sqrt{gm}}\right) = \sqrt{\frac{gk}{m}}\, t$$
$$v = \sqrt{\frac{gm}{k}}\tan\sqrt{\frac{gk}{m}}\, t$$

??? shouldn't it be $\tan^{-1}$ $t \to \infty$, $v \to \frac{\pi}{2}$, so terminal velocity. so terminal velocity so Then can say its stable since all falling object will always go to terminal velocity sth like that? Terminal velocity is $\sqrt{\frac{gm}{k}}$?

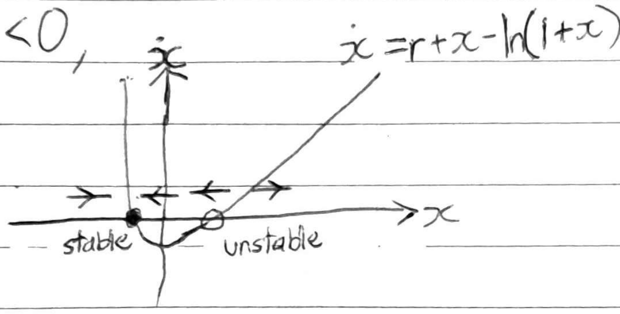
Using Graphing Calculator,





c) The coordinates of the fixed point in (a) is the same as the terminal velocity as $t \to \infty$ in (b), this indicates terminal velocity is a fixed point and is stable. Terminal velocity being a stable fixed point means when an object is in terminal velocity it tend to stay in terminal velocity or go back to it when there is any small disturbances to the object's velocity.

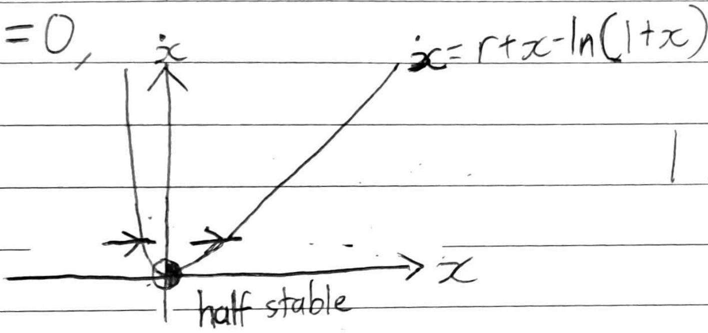
3. $r<0$,



Do I need say each left side of fixed point $\dot{x}>0$ so '→' for each??

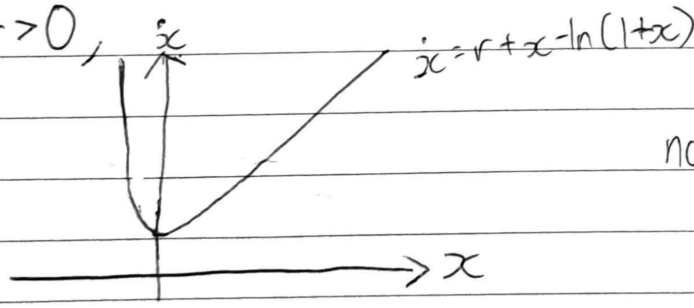
1 stable & 1 unstable fixed point

$r=0$,



1 half stable fixed point

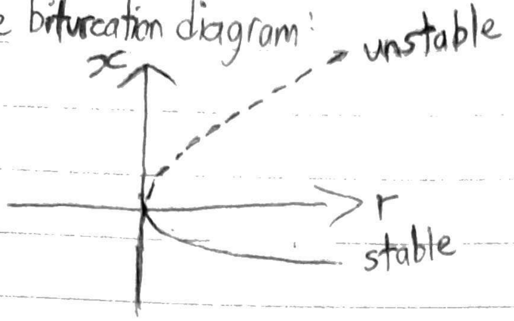
$r>0$,



no fixed points

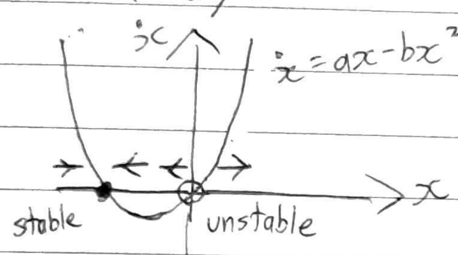
$\dot{x}=r+x-\ln(1+x)$ is saddle-node bifurcation as fixed points are created and destroyed by colliding and mutually annihilating as they move towards each other.

Saddle-node bifurcation diagram:

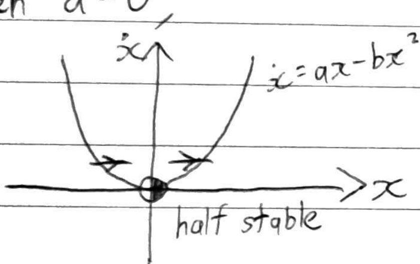


4. For $b<0$,

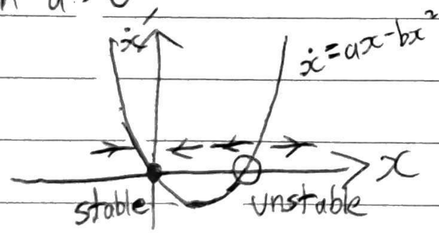
When $a<0$,



When $a=0$,

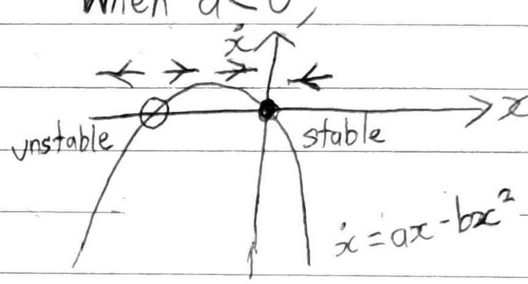


When $a>0$,

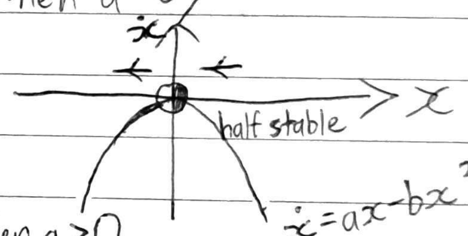


For $b>0$,

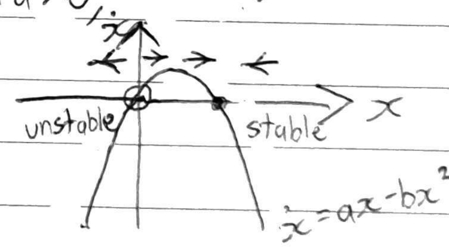
When $a<0$,



When $a=0$,



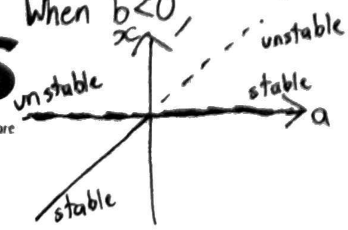
When $a>0$,



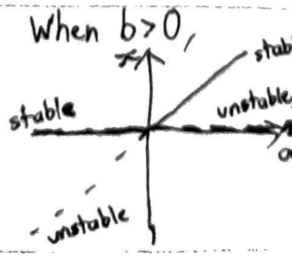
$\dot{x} = ax - bx^2$ is transcritical bifurcation as fixed points changed their stability, and that the 2 fixed points did not disappear bifurcation, they only switched their stability

Transcritical bifurcation diagram:

When $b<0$,



When $b>0$,



qn here (bottom right of page)

Second session

1. a) $\frac{dx}{dt} = x - y - x(x^2+5y^2) = \dot{x}$

$\frac{dy}{dt} = x + y - y(x^2+y^2) = \dot{y}$

Jacobian Matrix: $A = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\dot{x} = x - y - x^3 - 5xy^2$

$\dot{y} = x + y - x^2y - y^3$

$\frac{d\dot{x}}{dx} = 1 - 3x^2 - 5y^2$

$\frac{d\dot{x}}{dy} = -1 - 10xy$

$\frac{d\dot{y}}{dx} = 1 - 2xy$

$\frac{d\dot{y}}{dy} = 1 - x^2 - 3y^2$

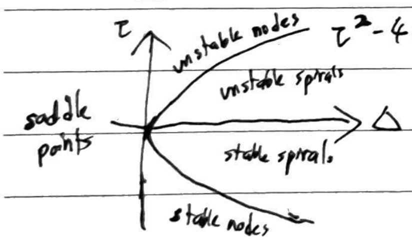
$A = \begin{pmatrix} (1-3x^2-5y^2) & (-1-10xy) \\ (1-2xy) & (1-x^2-3y^2) \end{pmatrix}$

To check fixed point at origin, $(0,0)$, $x=0$ & $y=0$,

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\tau = a + d = 1 + 1 = 2$

$\Delta = ad - bc = 1 - (-1) = 2$



$\tau^2 - 4\Delta = (2^2) - 4(2) = -4 < 0$

Hence, the type of fixed point is a unstable spiral

Formulas:
① $x(t) = x_0 e^{at}$, $y(t) = y_0 e^{at}$
② $x(t) = e^{\lambda t} v$, $y(t) = e^{\lambda t} v$
③ $Av = \lambda v$
④ $\lambda^2 - \tau\lambda + \Delta = 0$
⑤ $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Formulas & deriving them:

$\dot{x} = ax$

$\frac{dx}{dt} = ax$

$\frac{1}{x}\frac{dx}{dt} = a$

$\int \frac{1}{x}\frac{dx}{dt} dt = \int a\, dt$

$\int \frac{1}{x} dx = \int a\, dt$

$\ln|x| = at + c$

$|x| = e^{at+c}$

$|x| = e^{at} \cdot e^c$

Since e^c is just another constant, we let it be x_0,

$\boxed{x(t) = x_0 e^{at}}$

$\dot{x} = ax + by$

$\dot{y} = cx + dy$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\boxed{\dot{\mathbf{x}} = A\mathbf{x}}$

$\mathbf{x}(t) = e^{\lambda t} v$ — ①

Sub ① into $\dot{\mathbf{x}} = A\mathbf{x}$,

$\frac{d}{dt} e^{\lambda t} v = A e^{\lambda t} v$

$\lambda \cancel{e^{\lambda t}} v = A \cancel{e^{\lambda t}} v$

$Av = \lambda v$

v is eigenvector, λ is eigenvalue

Characteristic eqn, $\det(A - \lambda I) = 0$

det: some scalar value λ: eigenvalue
A: square matrix   I: identity matrix of size of A

$\det(A - \lambda I) = 0$

$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$

$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$

How this work?

$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

$(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

Let $\tau = \text{trace}(A) = a + d$,

$\Delta = \det(A) = ad - bc$,

$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$

what formula is this?
Why can just use?

b) $x^2 + y^2 = r^2$

$x\dot{x} + y\dot{y} = r\dot{r}$ — (1)

~~$x = r\cos\theta$, $y = r\sin\theta$~~

$\dot{x} = x - y - x(x^2+5y^2)$

$\dot{y} = x + y - y(x^2+y^2)$ — (2)

Subbing (2) into (1),

$$r\dot{r} = x(x-y-x(x^2+5y^2)) + y(x+y-y(x^2+y^2))$$
$$= x^2 \cancel{-xy} - x^2(x^2+5y^2) \cancel{+xy} + y^2 - y^2(x^2+y^2)$$
$$= x^2 - x^4 - 5x^2y^2 + y^2 - x^2y^2 - y^4$$
$$= x^2 - x^4 + y^2 - y^4 - 6x^2y^2$$
$$= (x^2+y^2) - (x^4+2x^2y^2+y^4) - 4x^2y^2$$
$$= r^2 - (x^2+y^2)(x^2+y^2) - 4x^2y^2, \text{ since } r^2 = x^2+y^2$$
$$= r^2 - r^4 - 4(r^2\cos^2\theta)(r^2\sin^2\theta), \text{ since } x = r\cos\theta \ \& \ y = r\sin\theta$$
$$= r^2 - r^4 - 4r^4(\cos^2\theta\sin^2\theta)$$
$$= r^2 - r^4 - 4r^4(\tfrac{1}{4}\sin^2 2\theta), \text{ since } \sin 2\theta = 2\sin\theta\cos\theta$$

$\sin^2 2\theta = 4\sin^2\theta\cos^2\theta$

$\frac{1}{4}\sin^2 2\theta = \sin^2\theta\cos^2\theta$

$$= r^2 - r^4 - r^4\sin^2 2\theta$$
$$= r^2(1 - r^2 - r^2\sin^2 2\theta)$$
$$\dot{r} = r(1 - r^2 - r^2\sin^2 2\theta) \ // \ \text{(shown)}$$

what formula is this?

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$
$$= \frac{x(x+y-y(x^2+y^2)) - y(x-y-x(x^2+5y^2))}{r^2}$$
$$= \frac{x^2 + xy - xy(x^2+y^2) - (xy - y^2 - xy(x^2+5y^2))}{r^2}$$
$$= \frac{x^2 \cancel{+xy} \cancel{-x^3y} \cancel{-xy^3} \cancel{-xy} + y^2 \cancel{+x^3y} + \overset{4}{\cancel{5}}xy^3}{r^2}$$
$$= \frac{x^2+y^2+4xy^3}{r^2}$$
$$= \frac{r^2 + 4(r\cos\theta)(r^3\sin^3\theta)}{r^2}, \text{ since } r^2 = x^2+y^2$$
$$= \frac{r^2 + 4r^4(\cos\theta)(\sin^3\theta)}{r^2}$$
$$= 1 + 4r^2\cos\theta\sin^3\theta \ // \ \text{(shown)}$$

limit cycle vs closed orbit?
is qn asking about stable or unstable limit cycle?
How to differentiate between stable & unstable limit cycle?
I know Q1 is about chapter 7, about the 2 conditions, but don't quite understand chapter 7... can re explain? haha

c)(i) $\dot{r} = r\left[1 - r^2 - r^2 \sin^2 2\theta\right]$

Since the range of $\sin^2 2\theta$ is $0 \le \sin^2 2\theta \le 1$, assuming largest value of $\sin^2 2\theta = 1$,

why did I assume largest...? but looks like it works tho...

$\dot{r} = r\left[1 - r^2 - r^2(1)\right]$
$= r\left[1 - 2r^2\right]$

Let $\dot{r} = 0$,
$0 = r(1 - 2r^2)$
$r = 0$ OR $2r^2 = 1$
$r^2 = \frac{1}{2}$
$r = \pm \frac{1}{\sqrt{2}}$

how prove again I forgot to show that $\dot{r} > 0$ for all $r \le \frac{1}{\sqrt{2}}$...

(ii) $\dot{r} = r\left[1 - r^2 - r^2 \sin^2 2\theta\right]$

why take minimum??

Since the range of $\sin^2 2\theta$ is $0 \le \sin^2 2\theta \le 1$, assuming smallest value of $\sin^2 2\theta = 0$,
$\dot{r} = r\left[1 - r^2 - r^2(0)\right]$
$= r\left[1 - r^2\right]$

Let $\dot{r} = 0$,
$0 = r\left[1 - r^2\right]$
$r = 0$ OR $r^2 = 1$
$r = \pm 1$

(iii) $r_1 \leq r \leq r_2$

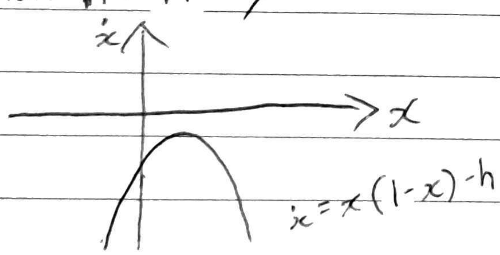
$\frac{1}{\sqrt{2}} \leq r \leq 1$

$\dot{r} = r\left[1 - r^2 - r^2 \sin 2\theta\right]$ continue...

2. $\dot{x} = x(1-x) - h$

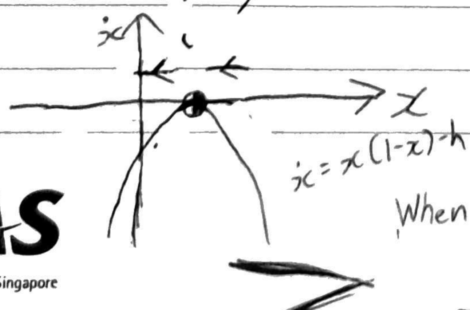
Using Graphing Calculator,

When $h > H$,



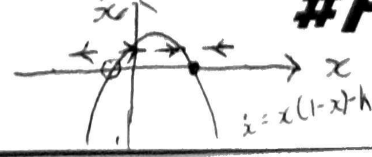
No fixed points

When $h = H$,



1 half stable fixed point

When $h < H$,



1 stable & 1 unstable fixed point

$\dot{x} = x(1-x) - h$ is saddle-node bifurcation as fixed points are created and destroyed by colliding and mutually annihilating as they move towards each other

Values of H and X where the saddle-node bifurcation occured

When $\dot{x} = 0$

$x(1-x) - h = 0$

$x - x^2 - h = 0$

$-x^2 + x - h = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1-4(1)(-h)}}{2}$

$= \frac{-1 \pm \sqrt{1-4h}}{2}$

Discriminant, $b^2 - 4ac = 0$ (only one unique solution)

$1 - 4h = 0$

$h = 0.25$ //

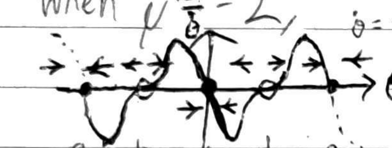
When $h = 0.25$, $x - x^2 - 0.25 = 0$

$x = 0.5$ // (since only one unique solution)

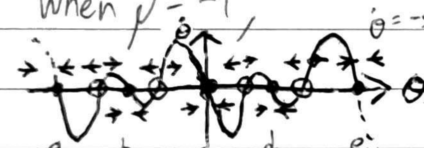
$\dot{\theta} = \mu \sin\theta - \sin 2\theta$

3. Using Graphic Calculator, drawing up to 2 cycles of the graph,

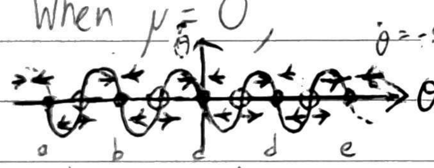
When $\mu = -2$, $\dot{\theta} = -2\sin\theta - \sin 2\theta$



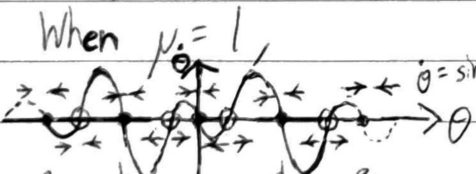
When $\mu = -1$, $\dot{\theta} = -\sin\theta - \sin 2\theta$



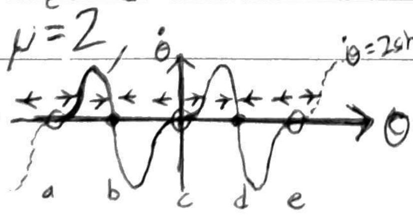
When $\mu = 0$, $\dot{\theta} = -\sin 2\theta$



When $\mu = 1$, $\dot{\theta} = \sin\theta - \sin 2\theta$



When $\mu = 2$, $\dot{\theta} = 2\sin\theta - \sin 2\theta$



Labelling the fixed points a, b, c, d, e from the graph $\dot{\theta} = \mu \sin\theta - \sin 2\theta$ as shown,

1. When $\mu = -2$, fixed points a, c, e are stable while fixed points b and d are unstable.
2. When $\mu = -1$, fixed points a, c, e remain stable, but a pair of unstable fixed points appeared on the left and right side while the fixed points b and d turned stable.
3. When $\mu = 0$, the pair of unstable fixed points that appeared on the left and right side of the now stable fixed points b and d, moves towards the now stable fixed points b and d's neighbouring fixed points, a, c, e.
4. When $\mu = 1$, the pair of unstable fixed points that appeared on the left and right side of the now stable fixed points b and d, moves further towards the now stable fixed points b and d's neighbouring fixed points, a, c, e.
5. When $\mu = 2$, the pair of unstable fixed points beside the stable fixed points d, c, e disappeared, and the fixed points a, c, e became unstable.

- Hence, at $\mu = -1$, 0 and 1, no bifurcation occurs.
- At $\mu = -2$, supercritical pitchfork bifurcation occured as at unstable fixed points b and d, a pair of unstable fixed points appeared, while fixed points b and d themselve became stable.
- At $\mu = 2$, subcritical pitchfork bifurcation occured as at stable fixed points a, c, e the pair of unstable fixed points beside them disappeared, while fixed points a, c, e themselves became unstable.

4. $\frac{dR}{dt} = aR + bJ = \dot{R}$

$\frac{dJ}{dt} = bR + aJ = \dot{J}$

Jarobian matrix: $A = \begin{pmatrix} \frac{d\dot{R}}{dR} & \frac{d\dot{R}}{dJ} \\ \frac{d\dot{J}}{dR} & \frac{d\dot{J}}{dJ} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\frac{d\dot{R}}{dR} = a$

$\frac{d\dot{R}}{dJ} = b$

$\frac{d\dot{J}}{dR} = b$

$\frac{d\dot{J}}{dJ} = a$

$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$\tau = a + a = 2a$

$\Delta = a^2 - b^2$

$\lambda^2 - \tau\lambda + \Delta = 0$

$\lambda^2 - 2a\lambda + a^2 - b^2 = 0$

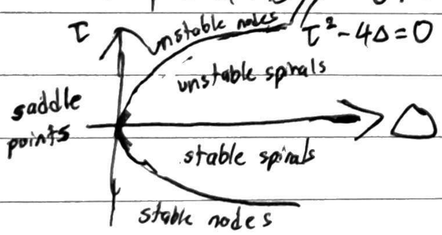
$\lambda = \frac{2a \pm \sqrt{(4a^2) - 4(a^2 - b^2)}}{2}$

$= \frac{2a \pm \sqrt{4b^2}}{2}$

$= \frac{2a \pm 2b}{2}$

$= a \pm b$

$\lambda_1 = a + b$ // OR $\lambda_2 = a - b$ //



$\tau^2 - 4\Delta = (2a)^2 - 4(a^2 - b^2)$

$= 4a^2 - 4a^2 + 4b^2$

$= 4b^2$

$\tau = 2a$, $\Delta = a^2 - b^2$, $\tau^2 - 4\Delta = 4b^2$,

When $a < b$,

$\Delta = a^2 - b^2 < 0$, hence the stability when $a < b$ is saddle point //

When $a < b < 0$,

$\tau = 2a < 0$, $\Delta = a^2 - b^2 > 0$, $\tau^2 - 4\Delta = 4b^2 > 0$,

hence the stability when $a < b < 0$ is stable spiral //

When $0 < b < a$,

$\tau = 2a > 0$, $\Delta = a^2 - b^2 > 0$, $\tau^2 - 4\Delta = 4b^2 > 0$,

hence the stability when $0 < b < a$ is unstable node //