Homework on Nonlinear Dynamics and Chaos

PDF submission by email (No late submission is allowed).

Total Marks = 120 marks (15% of your final grade)

First Session (30 marks)

1) For equation $\dot{x} = e^{x} - \cos{(x)}$, in the range of all *x*. Sketch the equation graphically and determine the first 3 fixed points nearest to the x = 0. On each of the fixed point, determine if it is stable or unstable. You may use a simple excel file or any numerical methods to determine the values of the fixed points, as it can not be solved analytically.

6 marks

2) The velocity *v(t)* of an object (mass = *m*) falling to the ground under the gravity force (*g*) and air friction (*k > 0*) is $\dot{v} = g - kv^2/m$. The initial condition is v(0) = 0. Consider the object is far from the ground and it will reach a terminal velocity.

   a) Using the phase diagram, determine the stability of *v* and if this fixed point is stable or unstable.
   b) Obtain the analytical solution of *v(t)* and determine the terminal velocity at *t* → ∞.
   c) Comment on the answers obtained from (a) and (b).

6 marks

3) Determine the type of bifurcation of $\dot{x} = r + x - \ln{(1 + x)}$ and draw its bifurcation diagram when the parameter *r* is varied. For different value of r, sketch the phase diagram to determine its stability.

8 marks

4) Logistic equation known as $\dot{x} = ax - bx^2$ has been used to describe the population growth. Determine type of bifurcation and draw the bifurcation diagram *(x in y-axis and a in x-axis)* for both non-zero *b cases* (b > 0 and b < 0). For different values of *a* and *b,* sketch the phase diagram to determine its stability.

10 marks

Second Session (40 marks)

1. Consider the following nonlinear equations:

$$dx/dt = x - y - x(x^2 + 5y^2) \qquad \text{and} \qquad dy/dt = x + y - y(x^2 + y^2)$$

   a) Classify the type of fixed point at origin (x = 0, y = 0)

(5 marks)

   b) Using the polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) to show that the above question can be rewritten as

$$dr/dt = r[1 - r^2 - r^2\sin^2(2\theta)] \quad \text{and} \quad d\theta/dt = 1 + 4r^2\cos\theta\sin^3\theta$$

(6 marks)

   c) Using equations in 1(b) to show the following characteristics:
      i. dr/dt > 0 (going outward) for all $r \leq r_1 = 1/\sqrt{2}$
      ii. dr/dt < 0 (going inward) for all $r \geq r_2 = 1$
      iii. For $r_1 \leq r \leq r_2$, it has a limiting cycle and no fixed point

(9 marks)

2. Consider $dx/dt = x(1 - x) - h$ has a solution of time-dependent function *x(t)*, where $h \geq 0$ is a constant. Show that a bifurcation occurs at h = H and x = X and classify this bifurcation. Determine the values of H and X.

(5 marks)

3. Classify the following equation that bifurcations occur at μ varies. Sketch the phase diagrams (dθ/dt vs θ) at 5 values of μ = –2, –1, 0, 1 and 2.

$$\frac{d\theta}{dt} = \mu \times \sin\theta - \sin(2\theta)$$

(7 marks)

4. Consider Romeo (R) and Juliet (J) are romantic clones described by a coupled equation: $dR/dt = aR + bJ$ and $dJ/dt = bR + aJ$. Determine the eigenvalue and classify its stability at each of the following 3 conditions:

$$a < b \;; \qquad a < b < 0 \qquad \text{and} \qquad 0 < b < a$$

(8 marks)

Third Session (30 marks) – Coding and Numerical solutions

(The figures below are some output from the textbook, you may not get the same plotting since it is a chaotic system.)

You need to submit your simulated figures and codes you used.

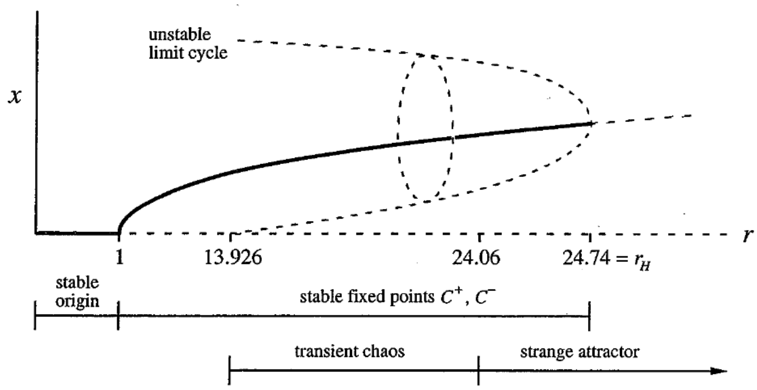
Consider the following Lorentz equations: x(t), y(t) and z(t), where *σ, r, b* are positive constants.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

It is found that the stability is depending on the values of r as shown in Figure below. From r = 13.926 to 24.06, it is known as transient chaos. From r > 24.06 and above, strange attractor is formed. From r > rH = 24.74, there are no more fixed points.

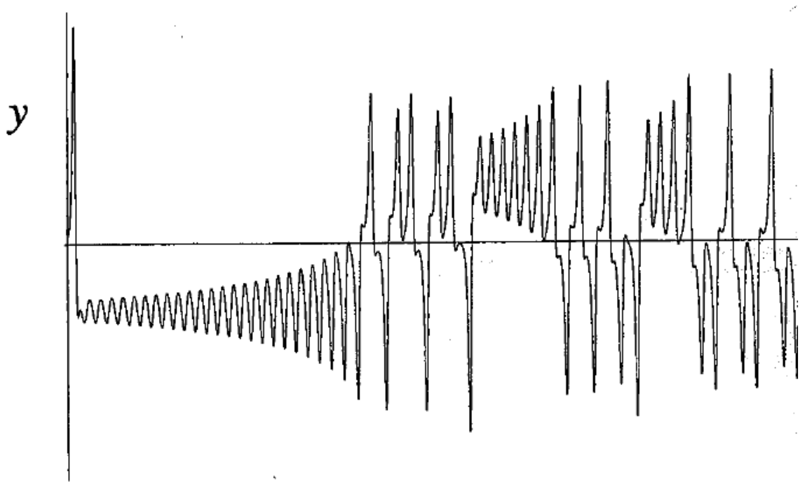


Using numerical method such as Runge Kutta methods, solve the Lorentz equations in the following 3 cases:

https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods#Explicit_Runge%E2%80%93Kutta_methods

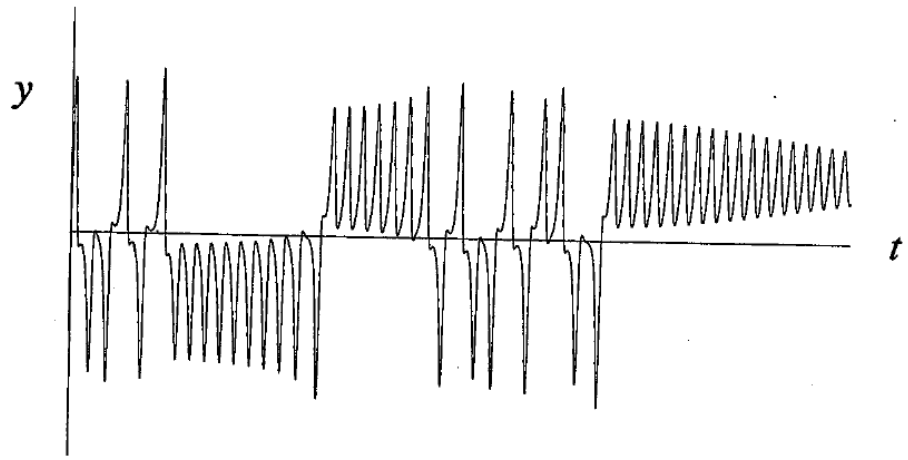
Case1 [Chaos]:

σ = 10, <u>r = 28</u>, b = 8/3, using [x(0), y(0), z(0)] = [0,1,0], show the y(t) as a function of time as below which shows aperiodic behavior at large t.



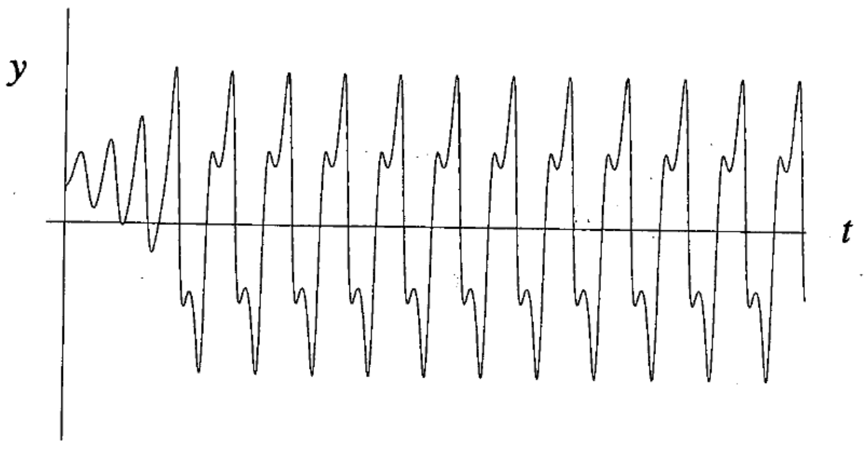
Case 2 [Transient Chaos]:

$\sigma = 10$, r = 21, b = 8/3, using [x(0), y(0), z(0)] = [0,1,0], show the y(t) as a function of time as below with position y(t) at C+ fixed points. You may want to try on using different initial conditions, where the long-time behavior with negative y values (C- fixed points].



Case 3 [Very large r, No chaos]:

$\sigma = 10$, r = 350, b = 8/3, using [x(0), y(0), z(0)] = [0,1,0], show the y(t) as a function of time as below with periodic pattern that no more chaos.



Four Session (20 marks) Self-directed topics

Through online source, please select any non-linear dynamics chaotic system of your interests to share with all.

KPI: short report and oral presentation.

- Presentation (10 marks) – max 3 slides (excluding cover page) – 5 mins (final session) – the ppt slide is to be submitted same as HW due date
- Final Report (10 marks) – max 2 A4 pages (excluding references) – same as HW due date