Simulation of Dynamical Systems
Homework 4: FEA
Due: Friday, 2020.12.25

Recall the nondimensionalized wave equation discussed in lecture,

$$u_{tt}(x,t) + \lambda u_t(x,t) - u_{xx}(x,t) = p(x,t),$$

with x in the domain $[0,1]$, boundary conditions corresponding to fixed ends,

$$u(0,t) = u(1,t) = 0,$$

and initial conditions in the form of a prescribed displacement,

$$u(x,0) = u_0(x), \quad u_t(x,0) = 0.$$

Answer the following questions by carrying out a finite element analysis of this equation. It is not necessary to rederive formulas presented in class, but numerical results you submit should be obtained with a code you write yourself in Python.

1. Solve our usual generalized eigenproblem in $\mathbf{M}$ and $\mathbf{K}$ to find the natural frequencies and natural modes of this system.
   a. Using 100 elements to cover the problem domain, compare the first 6 frequencies and modes to the analytical results $\omega_r = r\pi$, $\phi_r(x) = \sin r\pi x$, $r = 1, 2, \ldots$. Plot and compare both sets of modes.
   b. Starting with a very coarse mesh, study the effect of refining the mesh (i.e., reducing the element size) on the first 3 natural frequencies. Plot these frequencies against the number of elements used.
   c. How many elements are needed to produce a result for the 10th natural frequency that is within 1% of the exact result?
2. Use the results from Problem 1 to examine the system damping.
   a. Find the value of λ needed to produce 1% damping of the first mode.
   b. If λ is set to this value, what will be the damping ratios of the first 5 natural modes?
3. Let the initial conditions have the piecewise-linear form

$$u_0(x) = \begin{cases} A_0 x, & 0 \le x \le 1/2\,, \\ 2A_0(1-x), & 1/2 < x \le 1. \end{cases}$$

   and use λ from Problem 2.
   a. Plot these ICs when $A_0 = 0.05$.
   b. Compute the response to the given ICs using the Runge-Kutta code you wrote for a previous assignment. Plot the response at the three points $x = 0.25, 0.5, 0.75$.
4. Assuming quiescent initial conditions (i.e., $u_0(x) = \dot{u}_0(x) = 0$), find the response to the uniformly distributed, harmonic external excitation

$$p(x,t) = A\cos(\Omega t),$$

with $A = 2$ and $\Omega = 5$. Again use the value of λ you found in Problem 2.

a. Evaluate the element force vector $\mathbf{p}_e(t)$. You may do this either analytically or numerically, but fully explain your work.

b. Compute the response at $x = 0.25, 0.5, 0.75$, showing both the transient and steady-state regimes.

c. Find the magnitude and phase of the steady-state response from your computed time series, and compare these values to those obtained using the frequency response function matrix.