# FEA: Element Matrices for the Wave Equation

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# Introduction

- We consider the PDE

$$u_{tt}(x,t) + \lambda u_t(x,t) - u_{xx}(x,t) = p(x,t) \tag{1}$$

defined on the domain $x \in [0,1]$, with homogeneous boundary conditions

$$u(0,t) = 0 \quad \text{or} \quad u_x(0,t) = 0, \tag{2a}$$
$$u(1,t) = 0 \quad \text{or} \quad u_x(1,t) = 0 \tag{2b}$$

and given initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = \dot{u}_0(x). \tag{3}$$

- The damping is assumed to be uniform ($\lambda$ is a constant).

# Introduction

- We will discretize eq. (1) in space using standard techniques.
  - Developing a weak form of the equation.
  - Interpolating the solution from nodal quantities.
  - Deriving mass, damping, and stiffness matrices and a load vector for a representative element.
  - Assembling a single global matrix equation governing the solution.
- The time integration of this equation can then be carried out in a state space with dimension determined by the number of elements used to represent the problem domain.

# Formulation of the weak problem

- Multiplying eq. (1) by an admissible function $v(x,t)$ (i.e., an arbitrary function that satisfies the geometric boundary conditions) and integrating on $x$ gives

$$\int_0^1 vu_{tt}\,\mathrm{d}x + \lambda \int_0^1 vu_t\,\mathrm{d}x - \int_0^1 vu_{xx}\,\mathrm{d}x = \int_0^1 vp\,\mathrm{d}x. \qquad (4)$$

- Integrating by parts,

$$\int_0^1 vu_{xx}\,\mathrm{d}x = \cancelto{0}{vu_x\Big|_0^1} - \int_0^1 v_x u_x\,\mathrm{d}x. \qquad (5)$$

- Hence

$$\int_0^1 vu_{tt}\,\mathrm{d}x + \lambda \int_0^1 vu_t\,\mathrm{d}x + \int_0^1 v_x u_x\,\mathrm{d}x - \int_0^1 vp\,\mathrm{d}x = 0. \qquad (6)$$

# Discretization of the spatial domain

- Substituting the approximate solution $\tilde{u}(x,t)$ for both $u$ and $v$ in the equation above produces

$$\int_0^1 \tilde{u}\tilde{u}_{tt}\ \mathsf{d}x + \lambda \int_0^1 \tilde{u}\tilde{u}_t\ \mathsf{d}x + \int_0^1 \tilde{u}_x^2(x,t)\ \mathsf{d}x - \int_0^1 \tilde{u}p\ \mathsf{d}x \approx 0. \tag{7}$$

- We will seek a $\tilde{u}$ that minimizes the left-hand side of this result. We take this to be of the form

$$\tilde{u}(x,t) = \boldsymbol{\varphi}^{\mathsf{T}}(x)\,\mathbf{u}(t), \tag{8}$$

where $\mathbf{u}$ is a vector of solution values at discrete points $\mathbf{x}$ and $\boldsymbol{\varphi}$ is a vector of functions that interpolate the solution between these nodes.

# Discretization of the spatial domain

- Substituting gives

$$\int_0^1 \boldsymbol{\varphi}^\mathsf{T}\mathbf{u}\boldsymbol{\varphi}^\mathsf{T}\ddot{\mathbf{u}}\,\mathrm{d}x + \lambda\int_0^1 \boldsymbol{\varphi}^\mathsf{T}\mathbf{u}\boldsymbol{\varphi}^\mathsf{T}\dot{\mathbf{u}}\,\mathrm{d}x + \int_0^1 \boldsymbol{\varphi}'^\mathsf{T}\mathbf{u}\boldsymbol{\varphi}'^\mathsf{T}\mathbf{u}\,\mathrm{d}x - \int_0^1 \boldsymbol{\varphi}^\mathsf{T}\mathbf{u}p\,\mathrm{d}x \approx 0 \quad (9)$$

or

$$\mathbf{u}^\mathsf{T}\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^\mathsf{T}\,\mathrm{d}x\,\ddot{\mathbf{u}} + \lambda\mathbf{u}^\mathsf{T}\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^\mathsf{T}\,\mathrm{d}x\,\dot{\mathbf{u}} + \mathbf{u}^\mathsf{T}\int_0^1 \boldsymbol{\varphi}'\boldsymbol{\varphi}'^\mathsf{T}\,\mathrm{d}x\,\mathbf{u} - \mathbf{u}^\mathsf{T}\int_0^1 \boldsymbol{\varphi}\,p\,\mathrm{d}x \approx 0. \quad (10)$$

# Discretization of the spatial domain

- We can write this as

$$\mathbf{u}^{\mathsf{T}}\left(\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\ddot{\mathbf{u}} + \lambda\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\dot{\mathbf{u}} + \int_0^1 \boldsymbol{\varphi}'\boldsymbol{\varphi}'^{\mathsf{T}}\,\mathrm{d}x\,\mathbf{u} - \int_0^1 \boldsymbol{\varphi}\,p\,\mathrm{d}x\right) \approx 0. \quad (11)$$

- For this to be satisfied for $\mathbf{u} \neq \mathbf{0}$, we need the term in parentheses to vanish, resulting in

$$\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\ddot{\mathbf{u}} + \lambda\int_0^1 \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\dot{\mathbf{u}} + \int_0^1 \boldsymbol{\varphi}'\boldsymbol{\varphi}'^{\mathsf{T}}\,\mathrm{d}x\,\mathbf{u} = \int_0^1 \boldsymbol{\varphi}\,p\,\mathrm{d}x. \quad (12)$$

## Interpolation within an element

- Under the assumptions that the problem domain is covered by disjoint elements each defined by two nodes, and that the solution is interpolated within any one element in terms of only the nodes at its ends, we can restrict our attention to a single element with nodes at $x = a$ and $x = b$, and to the corresponding consecutive elements of $\mathbf{u}$.
- Thus, within an element, we consider $u(x,t) = \boldsymbol{\varphi}(x)^{\mathsf{T}}\mathbf{u}_e(t)$ with

$$\boldsymbol{\varphi}(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}, \quad \mathbf{u}_e(t) = \begin{bmatrix} u_a(t) \\ u_b(t) \end{bmatrix}, \tag{13}$$

and the equation

$$\int_a^b \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\ddot{\mathbf{u}}_e + \lambda\int_a^b \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}}\,\mathrm{d}x\,\dot{\mathbf{u}}_e + \int_a^b \boldsymbol{\varphi}'\boldsymbol{\varphi}'^{\mathsf{T}}\,\mathrm{d}x\,\mathbf{u}_e = \int_a^b \boldsymbol{\varphi}\,p\,\mathrm{d}x. \tag{14}$$

# Interpolation within an element

- The integrals depend on only the interpolating functions $\varphi_i$, which we take to be linear and write using a local coordinate $r$, with $r = -1$ corresponding to $x = a$ and $r = 1$ to $x = b$.

$$\varphi_1(r) = \frac{1-r}{2}, \quad \varphi_2(r) = \frac{1+r}{2} \tag{15}$$

- The $x$ and $r$ coordinates are related by

$$r = \frac{2}{b-a}\left(x - \frac{a+b}{2}\right), \quad x = \frac{a+b}{2} + r\frac{b-a}{2}, \tag{16}$$

and so

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{2}{b-a}, \quad \frac{\mathrm{d}x}{\mathrm{d}r} = \frac{b-a}{2}. \tag{17}$$

# Interpolation within an element

- We will evaluate the integrals on $x$ using functions of $r$ and the relations

$$\begin{aligned}\int_a^b f(x)g(x)\ \mathrm{d}x &= \int_{-1}^{1} f(r)g(r)\,\frac{\mathrm{d}x}{\mathrm{d}r}\ \mathrm{d}r \\ &= \frac{b-a}{2}\int_{-1}^{1} f(r)\ \mathrm{d}r,\end{aligned} \tag{18a}$$

$$\begin{aligned}\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}g(x)}{\mathrm{d}x}\ \mathrm{d}x &= \int_{-1}^{1} \frac{\mathrm{d}f(r)}{\mathrm{d}r}\frac{\mathrm{d}g(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2\frac{\mathrm{d}x}{\mathrm{d}r}\ \mathrm{d}r \\ &= \frac{2}{b-a}\int_{-1}^{1}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\frac{\mathrm{d}g(r)}{\mathrm{d}r}\ \mathrm{d}r.\end{aligned} \tag{18b}$$

# Interpolation within an element

- Considering the first integral in eq. (14),

$$\int_a^b \boldsymbol{\varphi}(x)\boldsymbol{\varphi}^\mathsf{T}(x)\ \mathsf{d}x = \frac{b-a}{2}\int_{-1}^{1} \boldsymbol{\varphi}(r)\boldsymbol{\varphi}^\mathsf{T}(r)\ \mathsf{d}r$$
$$= \frac{b-a}{2}\int_{-1}^{1} \begin{bmatrix} \varphi_1^2(r) & \varphi_1(r)\varphi_2(r) \\ \varphi_2(r)\varphi_1(r) & \varphi_2^2(r) \end{bmatrix} \mathsf{d}r. \tag{19}$$

- Similarly,

$$\int_a^b \boldsymbol{\varphi}'(x)\boldsymbol{\varphi}'^\mathsf{T}(x)\ \mathsf{d}x = \frac{2}{b-a}\int_{-1}^{1} \boldsymbol{\varphi}'(r)\boldsymbol{\varphi}'^\mathsf{T}(r)\ \mathsf{d}r$$
$$= \frac{2}{b-a}\int_{-1}^{1} \begin{bmatrix} {\varphi_1'}^2(r) & \varphi_1'(r)\varphi_2'(r) \\ \varphi_2'(r)\varphi_1'(r) & {\varphi_2'}^2(r) \end{bmatrix} \mathsf{d}r. \tag{20}$$

# Interpolation within an element

- We can easily find

$$\int_{-1}^{1} \varphi_1^2(r)\ \mathrm{d}r = \int_{-1}^{1} \tfrac{1}{4}(1-r)^2\ \mathrm{d}r = \tfrac{2}{3}, \tag{21a}$$

$$\int_{-1}^{1} \varphi_1(r)\varphi_2(r)\ \mathrm{d}r = \int_{-1}^{1} \tfrac{1}{4}(1-r)(1+r)\ \mathrm{d}r = \tfrac{1}{3}, \tag{21b}$$

$$\int_{-1}^{1} \varphi_2^2(r)\ \mathrm{d}r = \int_{-1}^{1} \tfrac{1}{4}(1+r)^2\ \mathrm{d}r = \tfrac{2}{3}, \tag{21c}$$

$$\int_{-1}^{1} {\varphi_1'}^2(r)\ \mathrm{d}r = \int_{-1}^{1} \left(-\tfrac{1}{2}\right)^2\ \mathrm{d}r = \tfrac{1}{2}, \tag{21d}$$

$$\int_{-1}^{1} \varphi_1'(r)\varphi_2'(r)\ \mathrm{d}r = \int_{-1}^{1} \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\ \mathrm{d}r = -\tfrac{1}{2}, \tag{21e}$$

$$\int_{-1}^{1} {\varphi_2'}^2(r)\ \mathrm{d}r = \int_{-1}^{1} \left(\tfrac{1}{2}\right)^2\ \mathrm{d}r = \tfrac{1}{2}. \tag{21f}$$

# Interpolation within an element

- Collecting these results, we define the element mass, damping, and stiffness matrices as

$$\mathbf{m} = \int_a^b \boldsymbol{\varphi}\boldsymbol{\varphi}^\mathsf{T} \, \mathrm{d}x = \frac{b-a}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{22a}$$

$$\mathbf{c} = \lambda \int_a^b \boldsymbol{\varphi}\boldsymbol{\varphi}^\mathsf{T} \, \mathrm{d}x = \lambda\mathbf{m} = \lambda\frac{b-a}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{22b}$$

$$\mathbf{k} = \int_a^b \boldsymbol{\varphi}'\boldsymbol{\varphi}'^\mathsf{T} \, \mathrm{d}x = \frac{1}{b-a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{22c}$$

# Interpolation within an element

- We can now write eq. (14) as

$$\mathbf{m}\ddot{\mathbf{u}}_e + \mathbf{c}\dot{\mathbf{u}}_e + \mathbf{k}\mathbf{u}_e = \mathbf{p}_e. \tag{23}$$

- The element load vector

$$\mathbf{p}_e(t) = \int_a^b \boldsymbol{\varphi}\, p \, \mathsf{d}x = \frac{b-a}{2} \int_{-1}^{1} \begin{bmatrix} \varphi_1(r) \\ \varphi_2(r) \end{bmatrix} p(r,t) \, \mathsf{d}r \tag{24}$$

will generally be evaluated numerically.

# Global matrix equation

- From the element matrices, we can assemble the global mass, damping, and stiffness matrices, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$, and the load vector $\mathbf{p}(t)$.
- The global finite element equation to be solved for $\mathbf{u}(t)$ is then

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{p}(t). \tag{25}$$