ME 303 Winter 2021

Tutorial 02

1. **Fixed point iterations**

   Suppose we have a function $f(x)$ and we want to determine x^* such that $f(x^*) = 0$, then we first rearrange the equation

   $$f(x) = 0 \Rightarrow x = g(x)$$

   There are many ways to do this. Then,

   - Arrange $f(x) = 0$ so we have $x = g(x)$
   - Pick x_0 and set $x_{new} = x_0$
   - Begin iterations - set $n = 1$, then $x_n = g(x_{n-1})$
   - Loop until $|x_n - x_{n-1}| < \text{tol}$

   This algorithm will converge to x^* if your initial guess x_0 is not too far from the optimal point where the following relation is valid

   $$\left| \frac{dg}{dx} \right|_{x=x_0} < 1$$

   (a) It is of interest to find the smallest positive root of $x^2 = 4 - 4x^3$ using MATLAB.

   1. From a sketch, show that the root is near $x = 0.9$.
   2. Try a direct iteration (fixed-point iteration) with the rearrangement $x = (4 - 4x^3)^{1/2}$ and explain what happens.
   3. Try a direct iteration with the rearrangement $x = ((4 - x^2)/4)^{1/3}$

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2. **Bisection method**

The bisection method is one of the simplest algorithms fo finding roots of non-linear equations. It works by creating successively smaller and smaller intervals which contain the root of the equation. Suppose that f is a continuous function on the interval $[a, b]$ with $f(a) > 0$ and $f(b) < 0$, then by the intermediate value theorem, there is some c with $a \leq c \leq b$ with $f(c) = 0$. So to approximate c we define x_0 as the midpoint of the interval, so that $x_0 = (a + b)/2$. Now, if $f(x_0) = 0$, then x_0 is the solution. However, if this is not the case, then we need to modify the interval based on x_0. Specifically, if $f(x_0) > 0$ then we replace a with x_0, i.e., $a = x_0$. Similarly, if $f(x_0) < 0$ then we replace b with x_0, i.e., $b = x_0$. Now that we have defined a new interval, either $[x_0, b]$ or $[a, x_0]$, we still have the root in the interval since we have kept the two ends of the interval having opposite signs. So we repeat this until we have $f(x_0) = 0$ or at least close to 0 with some acceptable level of error. The process can be summarized as follows:

- Choose a and b so that $f(a) > 0$ and $f(b) < 0$
- Set $n = 0$, define $x_0 = a$, and choose an acceptable error level $\epsilon > 0$
- Begin iterations
  1. $n = n + 1$
  2. $x_n = \frac{a+b}{2}$
  3. if $|f(x_n)| < \epsilon$, STOP
  4. if $|f(x_n)| > 0$, then $a = x_n$
  5. if $|f(x_n)| < 0$, then $b = x_n$
- Loop

(a) Use bisection method by hand to obtain the root of $f(x) = e^{-x} + \ln(x)$. Do 4 iterations and round the numbers to 4 decimal places (4DP).

(b) Write a MATLAB code which uses the bisection method to solve $f(x) = \pi$ where $f(x) = x^{2/3}$ for $0 \leq x$. Suppose we have a tolerance of the form $\epsilon = 10^{-n}$, determine the value of n which gives an approximation accurate to 6 decimal places (6DP). The true value to 6DP is 5.568328, when

1. $a = 4.5,\ b = 6$, and the termination condition $|f(x_n)| < \epsilon$ is used
2. $a = 4.5,\ b = 6$, and the termination condition $|x_n - x_{n-1}| < \epsilon$ is used
3. $a = 4.5,\ b = 6$, and the termination condition $|b - a| < \epsilon$ is used

Which termination condition required the smallest value of n? What termination condition was the easiest to ensure that you get the accuracy you want?

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