Problem 1

(a) There are 8 possible functions f: {a,b,c}⟶{0,1}.

000,100,010,001,110,101,011,111

(b) Consider:

Pow({a,b,c})={$\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}$}

Firstly, both of their number are 8. Secondly, number 1 can be viewed as the position that the elements of {a,b,c} take up, for example, the empty set of Pow({a,b,c}) and the function 000 in (a) have a corresponding relationship.

(c)

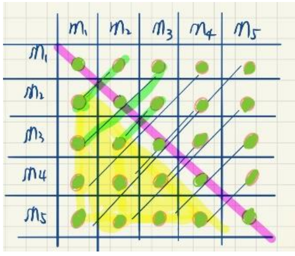
(i) n^m

This is because each of m elements of domain set can map to any of the n elements of codomain set.

(ii) $2^{n\cdot m}$

A relation on $A \times B$ is, by definition, a subset of $A \times B$. Due to $n \cdot m$ elements in $A \times B$, the number of subsets of an n element set is $2^{n\cdot m}$.

(iii) $2^{\frac{m^2+m}{2}}$



As the picture shows,

The number of symmetric relations on A should be :

$$2^{m+\frac{m\cdot(m-1)}{2}} = 2^{m+\frac{m^2-m}{2}} = 2^{\frac{m^2+m}{2}}$$

Problem 2

(a) $S_{2,-3} = \{2m - 3n: m, n \in \mathrm{Z}\}$

When m=0, n=0, S = {0}

When m=1, n=0, S = {2}

When m=0, n=1, S = {-3}

When m=1, n=1, S = {-1}

When m=2, n=1, S = {1}

(b) $S_{12,16} = \{12m + 16n: m, n \in \mathrm{Z}\}$

When m=0, n=0, S = {0}

When m=1, n=0, S = {12}

When m=0, n=1, S = {16}

When m=1, n=1, S = {28}

When m=-1, n=1, S = {4}

(c)

$\because \mathrm{d} = \gcd(\mathrm{x}, \mathrm{y})$

$\therefore \mathrm{x} = k_1 d,\ y = k_2 d$ by definition of greatest common divisor

$\therefore S_{x,y} = \{(mk_1 + nk_2)d: m, n \in \mathrm{Z}\}$

$Suppose\ \ i \in S_{x,y},\ \ it\ means\ i = (m_1 k_1 + n_1 k_2)d$

$\{n: n \in \mathrm{Z}\ and\ d|n\} \Longrightarrow n = k_3 d$ by definition of divisibility

As $(m_1 k_1 + n_1 k_2) \in Z$, from this d|i can be concluded

$\therefore i \in \{n: n \in \mathrm{Z}\ and\ d|n\}$

$\therefore S_{x,y} \subseteq \{n: n \in \mathrm{Z}\ and\ d|n\}$

(d)

$\{n: n \in \mathrm{Z}\ and\ z|n\} \Longrightarrow n = kz$ by definition of divisibility

Z be the smallest positive number in $S_{x,y}$

$\therefore \exists \mathrm{m}, \mathrm{n} \in \mathrm{Z}$, so that $m_1x + n_1y = z$

$Suppose\ \ i \in \{n: n \in \mathrm{Z}\ and\ z|n\}, it\ means\ i = k_1z$

$\therefore i = k_1z = k_1(\ m_1x + n_1y) = k_1\ m_1x + k_1n_1y$

$\because k_1\ m_1\ and\ k_1n_1\ both\ \in \mathrm{Z}$

$\therefore i \in S_{x,y}$

$\therefore \{n: n \in \mathrm{Z}\ and\ z|n\} \subseteq S_{x,y}$

(e)

As mentioned above, $m_1x + n_1y = z$, $m, n \in \mathrm{Z}$ and $\mathrm{x} = k_1d,\ y = k_2d$

$\therefore\ m_1x + n_1y =\ m_1k_1d + n_1k_2d = d(\ m_1k_1 + n_1k_2) = z$

$\therefore d|z$ can be concluded

$\because d > 0\ and\ z > 0$

$\therefore d \leq z$

(f)

$\because \mathrm{d} = \mathrm{gcd(x, y)}$

$\therefore$ This means that $\mathrm{d|x}$ and $d|y$

From this, $\mathrm{d|mx}$ and $d|ny$ can be concluded

$\therefore \mathrm{d}\ \in S_{x,y} = \{mx + ny: m, n \in \mathrm{Z}\}$

Z be the smallest positive number in $S_{x,y}$

$\therefore z \leq d$

Problem 3

(a)

$(A * B) * (A * B)$

$= (A^C \cup B^C) * (A^C \cup B^C)$

$= (A^C \cup B^C)^C \cup (A^C \cup B^C)^C$

$= ((A^C)^C \cap (B^C)^C) \cup ((A^C)^C \cap (B^C)^C)$ *[de Morgan's Laws]*

$= (A \cap B) \cup (A \cap B)$ *[Double complementation]*

$= A \cap B$ *[Idempotence]*

(b) $A * A$

Check:

$A * A$

$= A^C \cup A^C$

$= A^C$ *[Idempotence]*

(c) $(A * A * A) * (A * A * A)$

Check:

$(A * A * A) * (A * A * A)$

$= (A^C * A) * (A^C * A)$

$= ((A^C)^C \cup A^C) * ((A^C)^C \cup A^C)$

$= (A \cup A^C) * (A \cup A^C)$ *[Double complementation]*

$= U * U$ *[Complementation]*

$= U^C \cup U^C$

$= \emptyset \cup \emptyset$

$= \emptyset$

(d) $(A * (B * B)) * (A * (B * B))$

Check:

$(A * (B * B)) * (A * (B * B))$

$= (A * B^C) * (A * B^C)$

$= (A^C \cup B) * (A^C \cup B)$ *[Double complementation]*

$= (A^C \cup B)^C \cup (A^C \cup B)^C$

$= (A \cap B^C) \cup (A \cap B^C)$ *[de Morgan's Laws]*

$= A \cap B^C$ *[Idempotence]*

$= A \setminus B$

Problem 4

(a) When w=ab, v=ba, no z∈ Σ^*can make $v{=}\omega z$

(b) $R^{\leftarrow}(\{abc\})$ =$\{\lambda, a, ab, aba\}$

(c) Proof Reflexive:

When z= λ, $\omega z = \omega$, hence for all $\omega \in \sum^*$: $(\omega, \omega) \in R$ *is true*.

Therefore, it is reflexive.

Proof Antisymmetric:

If $(\omega, v) \in R$ and $(v, \omega) \in R$, then

$v{=}\omega z_1$ and $\omega{=}v z_2$

$\therefore$ $\omega{=}\omega z_1 z_2$, $v{=}v z_2 z_1$

$\therefore z_1 z_2 = \lambda$

$\therefore v{=}\omega$

Therefore, it is antisymmetric.

Proof Transitive:

If there exists υ *such that* $(\omega, \upsilon) \in R$ *and* $(\upsilon, v) \in R$, then

$\upsilon{=}\omega z_1$ and $v{=}\upsilon z_2$

$\therefore v{=}\omega z_1 z_2$

$\because z_1 \in \sum^*, z_2 \in \sum^*$

$\therefore z_1 z_2 \in \sum^*$

$\therefore (\omega, v) \in R$

Therefore, it is transitive.

In summary, R is a partial order.

Problem 5

$d \le z$ and $z \le d$ in Problem 2, from this d=1 can be concluded

$\therefore \exists$m, n ∈ Z, so that $m_1 x + n_1 y = 1$

$\therefore m_1zx + n_1zy = z$

$x|yz \implies yz = kx, k \in \mathrm{Z}$ by definition of divisibility

$\therefore m_1zx + n_1kx = x(m_1z + n_1k) = z$

$\because m_1z + n_1k \in \mathrm{Z}$

Therefore, it follows that $x|z$