Problem 1

(a) $\because R_1; R_2 = \{(a,c)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ and $(\mathrm{b,c}) \in R_2\}$
$\therefore (R_1; R_2); R_3 = \{(a,c,d)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ and $(\mathrm{b,c}) \in R_2$ and a *c* with $(\mathrm{b,c}) \in R_2$ and $(\mathrm{c,d}) \in R_3\}$ (by the definition)
Similarly, $R_2; R_3 = \{(c,d)$: there is an *b* with $(\mathrm{c,b}) \in R_2$ and $(\mathrm{b,d}) \in R_3\}$
$\therefore R_1; (R_2; R_3) = \{(a,c,d)$: there is a *b* with $(\mathrm{c,b}) \in R_2$ and $(\mathrm{b,d}) \in R_3$ and a c with $(\mathrm{a,c}) \in R_1$ and $(\mathrm{c,b}) \in R_2\}$ (by the definition)

$$\because R_1, R_2, R_3 \subseteq S \times S$$
$$a, b, c, d \in S$$
$$\therefore (R_1; R_2); R_3 = R_1; (R_2; R_3)$$

(b) According to definition, $I; R_1 = \{(x,a)$: there is a *b* with $(\mathrm{x,b}) \in I$ and $(\mathrm{b,a}) \in R_1\}$ $x \in S$
and $R_1; I = \{(a,x)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ and $(\mathrm{b,x}) \in I\}$ $x \in S$
In order to make $I; R_1 = R_1; I$, $(\mathrm{x,b}) \in I$ and $(\mathrm{b,x}) \in I$ should be both true
$\therefore I = \{(x,x) : x \in S\}$ can be concluded.
$I; R_1 = \{(x,a)$: there is a *x* with $(\mathrm{x,x}) \in I$ and $(\mathrm{x,a}) \in R_1\}$ for $\forall x \in S$, $(\mathrm{x,x}) \in I$

$$\therefore I; R_1 = R_1$$
$$\therefore I; R_1 = R_1; I = R_1 \; where \; I = \{(x,x) : x \in S\}$$

(c) $(R_1; R_2)^{\leftarrow} = \{(c,a)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ and $(\mathrm{b,c}) \in R_2\}$
$R = \{(a,b): a, b \in S\}$ then $R^{\leftarrow} = \{(b,a): a, b \in S\}$
According to definition, $R_1{}^{\leftarrow}; R_2{}^{\leftarrow} = \{(a,c)$: there is a *b* with $(\mathrm{b,a}) \in R_1$ and $(\mathrm{c,b}) \in R_2\}$

$$\therefore (R_1; R_2)^{\leftarrow} \neq R_1{}^{\leftarrow}; R_2{}^{\leftarrow}$$

$R_2{}^{\leftarrow}; R_1{}^{\leftarrow} = \{(c,a)$: there is a *b* with $(\mathrm{b,c}) \in R_2$ and $(\mathrm{a,b}) \in R_1\}$

$$\therefore (R_1; R_2)^{\leftarrow} = R_2{}^{\leftarrow}; R_1{}^{\leftarrow}$$

(d) According to definition,
$(R_1 \cup R_2); R_3 = \{(a,c)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ or $(\mathrm{a,b}) \in R_2$ and $(\mathrm{b,c}) \in R_3\}$
$(R_1; R_3) \cup (R_2; R_3) = \{(a,c)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ and $(\mathrm{b,c}) \in R_3\} \cup \{(a,c)$: there is a *b* with $(\mathrm{a,b}) \in R_2$ and $(\mathrm{b,c}) \in R_3\}$=$\{(a,c)$: there is a *b* with $(\mathrm{a,b}) \in R_1$ or $(\mathrm{a,b}) \in R_2$ and $(\mathrm{b,c}) \in R_3\}$

$$\therefore (R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$$

(e) $R_1; (R_2 \cap R_3) \neq (R_1; R_2) \cap (R_1; R_3)$
Counterexample:
When $R_2 \cap R_3 = \emptyset$, $R_1; (R_2 \cap R_3) = \emptyset$ for example, $R_2 = (2,3), R_3 = (1,3)$
But $R_1 = \{(1,2), (1,1)\}$.According to definition,
$(R_1; R_2) \cap (R_1; R_3)$= (1,3) $\cap$ (1,3)=(1,3) $\neq \emptyset$

$$\therefore R_1; (R_2 \cap R_3) \neq (R_1; R_2) \cap (R_1; R_3)$$

Problem 2

(a) Proof: For all $n \geq 0$, $P(n): R^n = R^{n+1}$

[Base case] $\because R^i = R^{i+1}$

$\therefore R^0 = R^1$

[Inductive case] proof: If $R^k = R^{k+1}$, then $R^{k+1} = R^{k+2}$ $\forall k \geq i$

$R^{k+2} = R^{k+1} \cup (R; R^{k+1})$ (by the definition)

$= R^k \cup (R; R^k)$ (by the inductive hypothesis)

$= R^{k+1}$ (by the definition)

according to part(a), $\exists$ an i such that $R^i = R^{i+1}$

$\therefore R^{k+1} = R^{k+2}$ $\forall k \geq i$

$\therefore R^j = R^i$ $\forall j \geq i$ can be concluded.

(b) When $k > i$, according to part(a), $R^k = R^i$

When $0 \leq k < i$

Proof: $P(n): R^n \subseteq R^{n+1}$, $0 \leq n < i$

[Base case] $R^1 = R^0 \cup (R; R^0)$

$\therefore R^0 \subseteq R^i$

[Inductive case] proof: If $R^{n-1} \subseteq R^n$, then $R^n \subseteq R^{n+1}$

$\because R^{n+1} = R^n \cup (R; R^n)$ (by the definition)

$\therefore R^n \subseteq R^{n+1}$, $P(n)$ *holds*.

Therefore, $R^k \subseteq R^i$ $\forall k \geq 0$

(c) Proof: For all $n \in \mathbb{N}$, $P(n): R^n; R^m = R^{n+m}$, $m \in \mathbb{N}$

[Base case]

$R^0; R^m = I; R^m = \{(x, a)$: there is a x with $(x, x) \in I$ and $(x, a) \in R^m\} = R^m$, $m \in \mathbb{N}$

$\therefore R^0; R^m = R^{0+m}$

[Inductive case]

Proof: If $R^k; R^m = R^{k+m}$, then $R^{k+1}; R^m = R^{k+1+m}$ $\forall k \geq 0$

$R^{k+1+m} = R^{k+m+1}$

$= R^{k+m} \cup (R; R^{k+m})$ (by the definition)

$= (R^k; R^m) \cup [R; (R^k; R^m)]$ (by the inductive hypothesis)

$= (R^k; R^m) \cup [(R; R^k); R^m]$ (by the result from part(a) in Problem 1)

$= [R^k \cup (R; R^k)]; R^m$ (by the result from part(d) in Problem 1)

$= R^{k+1}; R^m$ (by the definition)

$\therefore R^{k+1}; R^m = R^{k+m+1}$ $\forall k \geq 0$

According to Induction, $\forall n \in \mathbb{N}$, $P(n): R^n; R^m = R^{n+m}$, $m \in \mathbb{N}$

(d) $R^i \subseteq S \times S$ be an arbitrary binary relation on a set S, $S \times S$ is a set of all ordered pairs. $|S| = k$ means the number of elements in the set S is k, which is cardinality, so we can view $S \times S$ as a $k \times k$ matrix M_R.

$R^0 = I, R^1$ is M_R, R^2 *is* $M_R \times M_R = M_{R^2}$, R^3 *is* $M_{R^2} \times M_R = M_{R^3}$, R^4 *is* $M_{R^3} \times M_R = M_{R^4}$

Assume k=3, $S = \{a, b, c\}, R = \{(a, b), (b, c), (c, a)\}$, then

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which is $R^2 = \{(a,c),(b,a),(c,b)\}$

$$M_{R^3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is $R^3 = \{(a,a),(b,b),(c,c)\}$

$$M_{R^4 =} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = M_R$$

Therefore, R^i is repetitive when $i > k$.

$\therefore$ If there is $(a,b) \in R^{k+1}$ when $|S| = k$, then $(a,b) \in R^i$ when $0 \le i \le k$

$\therefore if\ |S| = k,\ then\ R^k{=}R^{k+1}$

(e) Assume there are $(a,b) \in R^k$ and $(b,c) \in R^k$

$\therefore (a,c) \in R^k; R^k$ (by the definition from Problem 1)

$\therefore (a,c) \in R^{2k}$ (by the result from part(c))

Due to $|S| = k, k \ge 0$ and $R^k{=}R^{k+1}$ (by the result from part(d)

$\therefore R^{2k} = R^k$ (by the result from part(a))

$\therefore (a,c) \in R^k$

$\therefore R^k$ is transitive when $|S| = k$

(f) $R \subseteq R \cup R^{\leftarrow}$

Assume there is $(a,b) \in R \cup R^{\leftarrow}$, then $(a,b) \in R\ or\ (a,b) \in R^{\leftarrow}$

hence $(b,a) \in R^{\leftarrow}$ or $(b,a) \in R$, therefore $(b,a) \in R \cup R^{\leftarrow}$

$\therefore R \cup R^{\leftarrow}$is symmetric.

$\therefore (R \cup R^{\leftarrow})^k$ is symmetric.

$\because R^k \subseteq (R \cup R^{\leftarrow})^k$

According to part(d), R^k is transitive, so $(R^{\leftarrow})^k$ is transitive as well.

$\therefore (R \cup R^{\leftarrow})^k$ is transitive when $|S| = k$

Assume there are $(a,b) \in R^k$ and $(b,a) \in R^k$ due to symmetric,

$\therefore (a,a) \in R^k; R^k$ (by the definition from Problem 1)

$\therefore (a,a) \in R^{2k}$ (by the result from part(c))

Due to $|S| = k, k \ge 0$ and $R^k{=}R^{k+1}$ (by the result from part(d)

$\therefore R^{2k} = R^k$ (by the result from part(a))

$\therefore (a,a) \in R^k$

$\therefore R^k$ is reflective, so $(R^{\leftarrow})^k$ is reflective as well

$\therefore (R \cup R^{\leftarrow})^k$ is reflective when $|S| = k$

Therefore, $(R \cup R^{\leftarrow})^k$ is an equivalence relation if $|S| = k$.

Problem 3

(a) Recursive definition:

A binary tree is

(B) an empty binary tree

(R) an order tuple (data, left binary tree, right binary tree)

(b) $count(T)$

(B)$count(empty\ \ T) = 0$

(R)$count(data, left\ T, right\ T) = 1 + count(left\ T) + count(right\ T)$

(c) $leaves(T)$

(B) $leaves(empty\ T) = 0$

(B) $leaves(data,\ \ empty\ left\ T, empty\ right\ T) = 1$

(R)$leaves(data,\ \ left\ T, right\ T) = leaves(left\ T) + leaves(right\ T)$

(d) $internal(T)$

(B) $internal(empty\ T) = -1$

(B) $internal(data, empty\ left\ T, empty\ right\ T) = 0$

(R) $internal(data, left\ T, right\ T) = 1 + internal(left\ T) + internal(right\ T)$

(e) Proof $P(T)$ holds for all binary trees T:

[Base case]

$leaves(empty\ T) = 0$ and $internal(empty\ T) = -1$

$\therefore leaves(empty\ T) = 1 + internal(empty\ T)$

$leaves(data, empty\ left\ T, empty\ right\ T) = 1$ and

$internal(data, empty\ left\ T, empty\ right\ T) = 0$

$\therefore leaves(data, empty\ left\ T\ empty\ right\ T)$

$= 1 + internal(data, empty\ left\ T, empty\ right\ T)$

[Inductive case]

Proof: if $leleaves(\ subtreeT) = 1 + internal(subtreeT)$then

$leaves(data\ \ left\ T, right\ T) = 1 + internal(data, left\ T, right\ T)$

$leaves(data, left\ T, right\ T)$

$= leaves(left\ T) + leaves(right\ T)$ (by the definition of $leaves(T)$)

$= 1 + internal(left\ T) + 1 + internal(right\ T)$ (IH)

$= 1 + internal(data, left\ T, right\ T)$ (by the definition of $internal(T)$)

$\therefore P(T)$: $leaves(T) = 1 + internal(T)$ holds for all binary trees T

Problem 4

(a) Propositions:

Ahi: Alpha uses channel hi.

Alo: Alpha uses channel lo.

Bhi: Bravo uses channel hi.

Blo: Bravo uses channel lo.

Chi: Charlie uses channel hi.

Clo: Charlie uses channel lo.

Dhi: Delta uses channel hi.

Dhi: Delta uses channel lo.

True: no interference

False: interfere others

(i) $\varphi_1 = (Ahi \vee Alo) \wedge (Bhi \vee Blo) \wedge (Chi \vee Clo) \wedge (Dhi \vee Dlo)$

(ii) $\varphi_2 = \neg(Ahi \wedge Alo) \wedge \neg(Bhi \wedge Blo) \wedge \neg(Chi \wedge Clo) \wedge \neg(Dhi \wedge Dlo)$

(iii) $\varphi_3 = \neg(Ahi \wedge Bhi) \wedge \neg(Alo \wedge Blo) \wedge \neg(Bhi \wedge Chi) \wedge \neg(Blo \wedge Clo) \wedge \neg(Chi \wedge Dhi) \wedge \neg(Clo \wedge Dlo)$

(b) (i) $\phi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3$

The following row shows a satisfying truth assignment: (T: no interference)

(blank is Fasle)

Ahi	Alo	Bhi	Blo	Chi	Clo	Dhi	Dlo	φ_1	φ_2	φ_3	ϕ
T			T	T			T	T	T	T	T

$\because$ ϕ is not always false.

$\therefore$ ϕ=$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is satisfiable.

(ii) The following rows show all satisfying truth assignments:

Ahi	Alo	Bhi	Blo	Chi	Clo	Dhi	Dlo	φ_1	φ_2	φ_3	ϕ
T			T	T			T	T	T	T	T
	T	T			T	T		T	T	T	T