Problem 1

(a) Assume there are $\varphi, \psi \in F$ such that $\varphi \equiv \psi$, it means that $(\varphi \leftrightarrow \psi)$ is a tautology.

Reflexivity:

$\because (\varphi \leftrightarrow \varphi)$ is always a tautology

φ	φ	$(\varphi \leftrightarrow \varphi)$
T	T	T
F	F	T

$\therefore \varphi \equiv \varphi$, $\equiv$ satisfies the property of Reflexivity on F.

Symmetry:

$\because (\varphi \leftrightarrow \psi) = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ for all truth assignments

φ	ψ	$(\varphi \leftrightarrow \psi)$	$(\varphi \rightarrow \psi)$	$(\psi \rightarrow \varphi)$	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$\therefore (\varphi \leftrightarrow \psi) \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, therefore $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$ are always Ture.

$\therefore \varphi \vDash \psi$ and $\psi \vDash \varphi$ can be concluded.

Because $\psi \vDash \varphi$ and $\varphi \vDash \psi$ are True, $(\psi \rightarrow \varphi)$ is a tautology and $(\varphi \rightarrow \psi)$ is a tautology, therefore $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi)$ is a tautology such that $(\psi \leftrightarrow \varphi)$ is a tautology.

$\therefore \psi \equiv \varphi$ can be concluded.

$\therefore \varphi \equiv \psi$, $\equiv$ satisfies the property of Symmetry on F.

Transitivity:

If there are $\varphi, \psi, \phi \in F$ such that $\varphi \equiv \psi$ and $\psi \equiv \phi$, then $(\varphi \leftrightarrow \psi)$ is a tautology and $(\psi \leftrightarrow \phi)$ is a tautology.

According to the truth table:

φ	ψ	ϕ	$(\varphi \leftrightarrow \psi)$	$(\psi \leftrightarrow \phi)$	$(\varphi \leftrightarrow \phi)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	F	F	T	F
T	F	T	F	T	T
F	T	T	T	T	T
F	F	T	T	T	T
F	T	F	T	F	T
F	F	F	T	T	T

This shows $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \vDash (\varphi \leftrightarrow \phi)$

$\therefore (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \rightarrow (\varphi \leftrightarrow \phi)$ is a tautology.

$\therefore (\varphi \leftrightarrow \phi)$ is a tautology, which means $\varphi \equiv \phi$

$\therefore \equiv$ satisfies the property of Transitivity on F.

$\therefore \equiv$ is an equivalence relation on F.

(b) $[\perp] = \{v(\perp): v(\perp) \in P_{ROP}\ and\ v(\perp)$ is a well-formed formulas$\}$

$$[\perp] = \{\cdots, \perp, \neg\perp, (\perp\wedge\perp), (\perp\vee\perp)\cdots\}$$

(c) (i)

$\varphi \equiv \varphi'$ means $(\varphi \leftrightarrow \varphi')$ is a tautology

$\therefore (\varphi \rightarrow \varphi') \wedge (\varphi' \rightarrow \varphi)$ is a tautology such that $(\varphi \rightarrow \varphi')$ and $(\varphi' \rightarrow \varphi)$ are always True (by part(a)).

Proof: $(\neg\varphi \rightarrow \neg\varphi') \wedge (\neg\varphi' \rightarrow \neg\varphi)$ is a tautology.

$(\varphi \rightarrow \varphi')$ is True and $(\varphi \rightarrow \varphi') \equiv (\neg\varphi' \rightarrow \neg\varphi)$ by Contrapositive, hence $(\neg\varphi' \rightarrow \neg\varphi)$ is True.

Similarly, $(\varphi' \rightarrow \varphi) \equiv (\neg\varphi \rightarrow \neg\varphi')$ and $(\varphi \rightarrow \varphi')$ is a tautology, so $(\neg\varphi \rightarrow \neg\varphi')$ is also a tautology.

$\therefore (\neg\varphi \rightarrow \neg\varphi') \wedge (\neg\varphi' \rightarrow \neg\varphi)$ is a tautology.

Therefore, $(\neg\varphi \leftrightarrow \neg\varphi')$ is a tautology, which means $\neg\varphi \equiv \neg\varphi'$.

(ii)

$\varphi \equiv \varphi'$ and $\psi \equiv \psi'$ means $\varphi \vDash \varphi'$, $\varphi' \vDash \varphi$ and $\psi \vDash \psi'$, $\psi' \vDash \psi$ by part (a).

$\therefore$ When $(\varphi \wedge \psi)$ is True, $(\varphi' \wedge \psi')$ is also True and when $(\varphi' \wedge \psi')$ is True, $(\varphi \wedge \psi)$ is True as well.

$\therefore ((\varphi \wedge \psi) \rightarrow (\varphi' \wedge \psi'))$ is a tautology and $((\varphi' \wedge \psi') \rightarrow (\varphi \wedge \psi))$ is also a tautology.

$\therefore ((\varphi \wedge \psi) \rightarrow (\varphi' \wedge \psi')) \wedge ((\varphi' \wedge \psi') \rightarrow (\varphi \wedge \psi))$ is a tautology.

$\therefore ((\varphi \wedge \psi) \leftrightarrow (\varphi' \wedge \psi'))$ is a tautology which means $\varphi \wedge \psi \equiv \varphi' \wedge \psi'$.

(iii)

According to part(c)(ii) above,

$\varphi \equiv \varphi'$ and $\psi \equiv \psi'$ means $\varphi \vDash \varphi'$, $\varphi' \vDash \varphi$ and $\psi \vDash \psi'$, $\psi' \vDash \psi$.

$\therefore$ When $(\varphi \vee \psi)$ is True, $(\varphi' \vee \psi')$ is also True and when $(\varphi' \vee \psi')$ is True, $(\varphi \vee \psi)$ is True as well.

$\therefore ((\varphi \vee \psi) \rightarrow (\varphi' \vee \psi'))$ is a tautology and $((\varphi' \vee \psi') \rightarrow (\varphi \vee \psi))$ is also a tautology.

$\therefore ((\varphi \vee \psi) \rightarrow (\varphi' \vee \psi')) \wedge ((\varphi' \vee \psi') \rightarrow (\varphi \vee \psi))$ is a tautology.

$\therefore ((\varphi \vee \psi) \leftrightarrow (\varphi' \vee \psi'))$ is a tautology which means $\varphi \vee \psi \equiv \varphi' \vee \psi'$.

(d) $F_{\equiv} \coloneqq \{[\varphi]: \varphi \in F\}$

$x \wedge y := [\varphi \wedge \psi]$

$x \vee y := [\varphi \vee \psi]$

$x' = [\neg\varphi]$

$Zero := [\perp]$

$One \coloneqq [\top]$

If $F_{\equiv}$ is a Boolean Algebra, it should follow the rule of commutative, associative, distributive, identity and complementation.

Commutative:

According to the definition, $[\varphi \wedge \psi]$=$[\varphi] \wedge [\psi]$

$[\varphi] \wedge [\psi] \equiv [\psi] \wedge [\varphi]$ by Commutativity

$\therefore [\psi] \wedge [\varphi] = [\psi \wedge \varphi] = [\varphi \wedge \psi]$

$\therefore x \wedge y = y \wedge x$

similarly, $[\varphi \vee \psi]$=$[\varphi] \vee [\psi]$

$[\varphi] \vee [\psi] \equiv [\psi] \vee [\varphi]$ by Commutativity

$\therefore [\psi] \vee [\varphi] = [\psi \vee \varphi] = [\varphi \vee \psi]$

$\therefore x \vee y = y \vee x$

associative:

$[\varphi \wedge \psi] \wedge [\phi] = [(\varphi \wedge \psi) \wedge \phi]$ by the definition

$\equiv [\varphi \wedge (\psi \wedge \phi)]$ by Associativity

$= [\varphi] \wedge [\psi \wedge \phi]$ by the definition

$\therefore (x \wedge y) \wedge z = x \wedge (y \wedge z)$

$[\varphi \vee \psi] \vee [\phi] = [(\varphi \vee \psi) \vee \phi]$ by the definition

$\equiv [\varphi \vee (\psi \vee \phi)]$ by Associativity

$= [\varphi] \vee [\psi \vee \phi]$ by the definition

$\therefore (x \vee y) \vee z = x \vee (y \vee z)$

distributive:

$[\varphi] \wedge [\psi \vee \phi] = [\varphi \wedge (\psi \vee \phi)]$ by the definition

$\equiv [(\varphi \wedge \psi) \vee (\varphi \wedge \phi)]$ by Distributivity

$= [\varphi \wedge \psi] \vee [\varphi \wedge \phi]$ by the definition

$\therefore x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$[\varphi] \vee [\psi \wedge \phi] = [\varphi \vee (\psi \wedge \phi)]$ by the definition

$\equiv [(\varphi \vee \psi) \wedge (\varphi \vee \phi)]$ by Distributivity

$= [\varphi \vee \psi] \wedge [\varphi \vee \phi]$ by the definition

$\therefore x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

identity:

$[\varphi] \vee [\perp] = [\varphi \vee \perp]$ by the definition

$\equiv [\varphi]$ by Identity

$\therefore x \vee 0 = x$

$[\varphi] \wedge [\top] = [\varphi \wedge \top]$ by the definition

$\equiv [\varphi]$ by Identity

$\therefore x \wedge 1 = x$

complementation:

$[\varphi] \vee [\varphi]' = [\varphi \vee \neg\varphi]$ by the definition

$\equiv [\top]$ by Complement

$\therefore x \vee x' = 1$

$[\varphi] \wedge [\varphi]' = [\varphi \wedge \neg\varphi]$ by the definition

$\equiv [\perp]$ by Complement

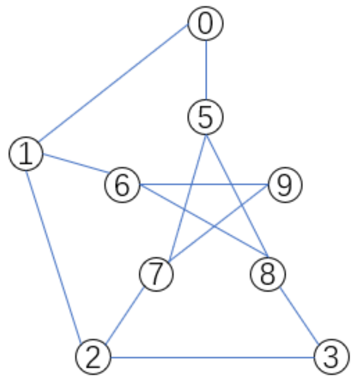
$\therefore x \wedge x' = 0$

Therefore, $F_{\equiv}$ together with the operations defined above forms a Boolean Algebra.

Problem 2

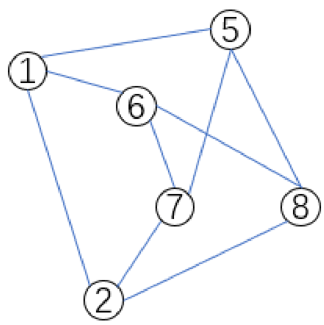
(a) The max degree of K_5 is 4, however the max degree of the Petersen graph is 3. For a graph, edge subdivision means to introduce some new vertices, all of degree 2, by placing them on existing edges, so we can't change the max degree of a graph by subdivision. Therefore, the Petersen graph does not contain a subdivision of K_5.

(b) Step 1: delete the vertex 4 and the edges with the vertex 4: {4,0},{4,9}and{4,3}, then get a subgraph H.



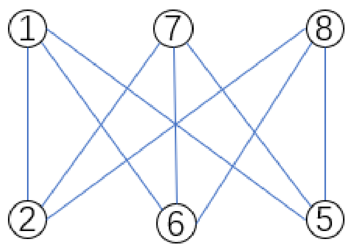
Subgraph H

Step 2: delete the vertex 3 and the edges {3,2} and {3,8}, then replace it by {2,8}.
Step 3: delete the vertex 0 and the edges {0,1} and {0,5}, then replace it by {1,5}.
Step 4: delete the vertex 9 and the edges {9,6} and {9,7}, then replace it by {6,7}.
We can get a subdivision:



Subdivision

If we fix it, we can get $K_{3,3}$:



$K_{3,3}$

Problem 3

(a) (i)

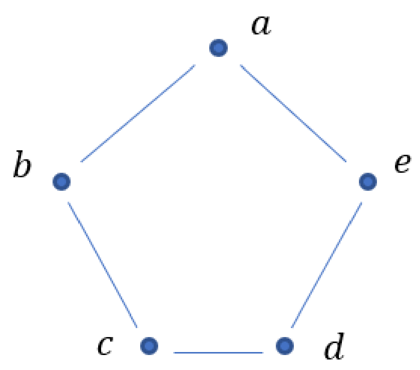
Defence against the Dark Arts: defined as the vertices a.
Potions: defined as the vertices b.
Herbology: defined as the vertices c.
Transfiguration: defined as the vertices d.
Charms: defined as the vertices e.
Every edge $e \in E$ be defined as the classes clash between two vertices connected by an edge.

Graph:

$V = \{a, b, c, d, e\}$

$E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}\}$

Pictorially:



(ii)

1. What is the maximum number of degrees in this graph?
2. Does this graph have the property of Eulerian path?
3. Does this graph have the property of Hamiltonian path?
4. Does this graph have the property of bipartite?
5. Does this graph have the property of connected?
6. What is the minimal number of the chromatic in this graph?
7. What is the cliques number of this graph?

(b) The solution of associated graph problems:

1. $\deg(a) = \deg(b) = \deg(c) = \deg(d) = \deg(e) = 2$
   Therefore, the maximum number of degrees in this graph is 2 and it is a regular graph.
2. Yes, because $\deg(v)$ is even for all $v \in V$.
3. Yes, because we can visit every vertex of graph exactly once.
4. No, because vertices cannot be partitioned into two disjoint set.
5. Yes, because each pair of vertices joined by a path.
6. $\chi(G) = 3$, because it is not bipartite.
7. $\kappa(G) = 2$, because this graph is a cycle C_3.
   The solution of Harry's problem:
   The maximum number of classes he can take is 2, it can be viewed as the problem of cliques number.

Problem 4

(a) $T(n)$

$T(0) = 1$ because empty tree is a only way without node.

$T(1) = 1$

$T(2) = 2$

$T(3) = T(0) \times T(2) + T(1) \times T(1) + T(2) \times T(0) = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5$

$T(4) = T(0) \times T(3) + T(1) \times T(2) + T(2) \times T(1) + T(3) \times T(1)$

$\quad = 1 \times 5 + 1 \times 2 + 2 \times 1 + 5 \times 1 = 14$

Therefore,

$(B)\ T(0) = 1$

$(R)\ \mathrm{T(n)} = \sum_{i=1}^{n} T_{i-1}T_{n-i}$

T_{i-1} represents number of nodes on the left subtree and T_{n-i} represents number of nodes on the right subtree.

(b) According to the recursive definition of the number of nodes count(T):

(B)$count(empty\ \ tree) = 0$

(R)$count(tree) = 1 + count(left\ subtree) + count(right\ subtree)$

Every node of a full binary tree is a fully-internal node or a leaf. If it is a fully-internal node, it should add two extra nodes. If it is a leaf, there is no new node created. Hence, the result of $count(left\ subtree)$ and $count(right\ subtree)$ should be $2n$ or 0 $(n \in N)$. Therefore, 1 plus an even number or 0 should be an odd number, which is the result of $count(T)$.

(c) $B(n)$

$B(0) = 0$

$B(1) = 1$

$B(2) = 0$

$B(3) = 1$

$B(4) = 0$

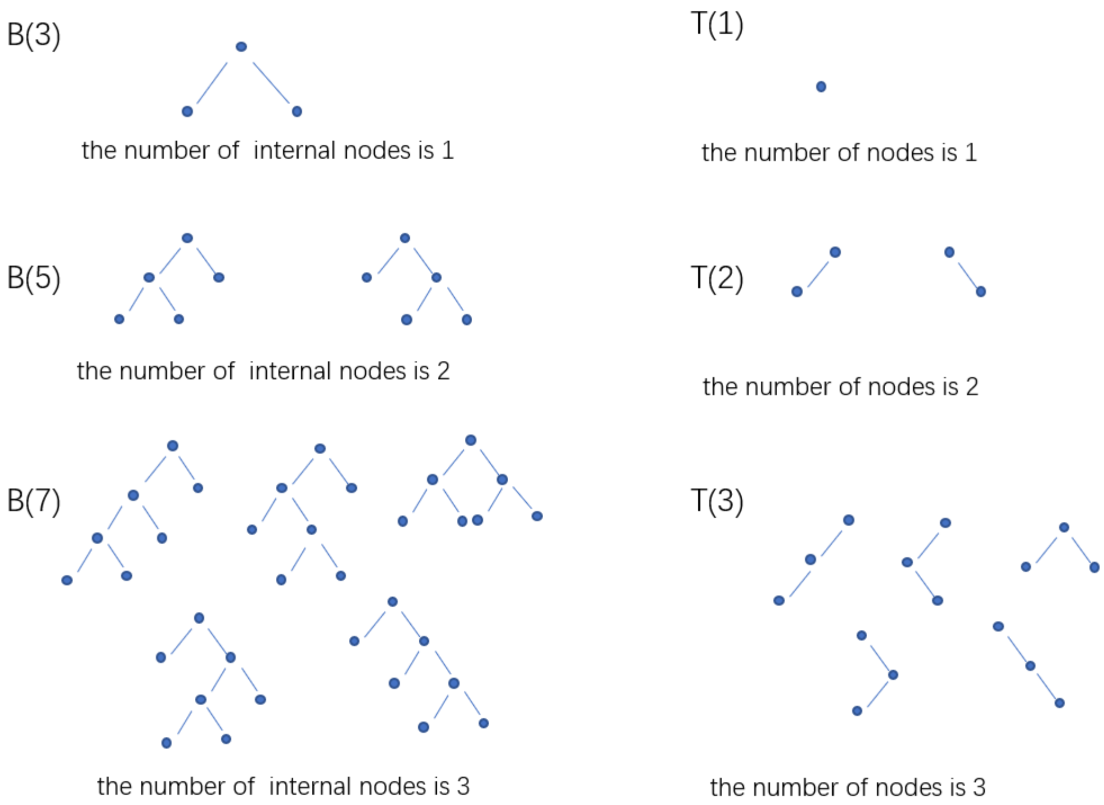
$B(5) = 2$

$B(6) = 0$

$B(7) = 5$

Compared with $T(n)$, we can find when the number of node is an even, B(n) is 0. When the number of node is an odd number, if the number of nodes of T(n) is the same as the number of internal nodes of B(n), their values are equal.

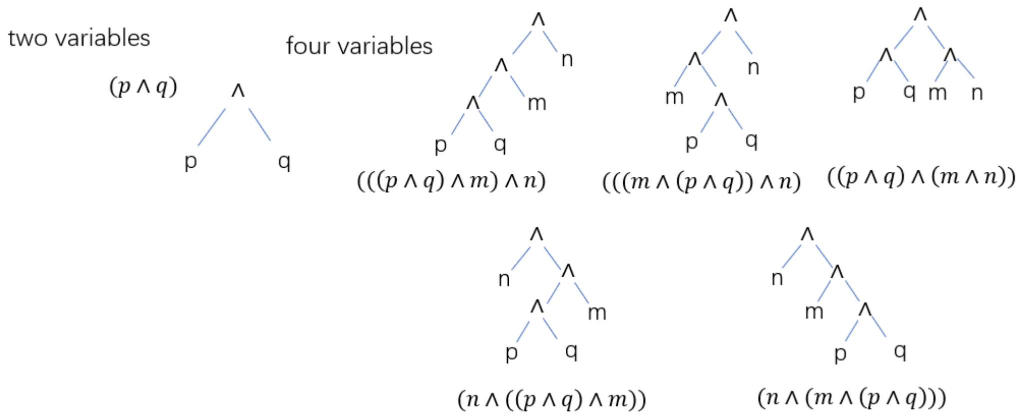
For example,



Hence, we can simplify this problem to find the relationship between the number of nodes of a binary tree and the number of the internal nodes of a full binary tree.
Therefore, we can get:

$$B(n)\begin{cases}0 & ,n\ is\ an\ even\\ T\left(\frac{n-1}{2}\right) & ,n\ is\ an\ odd\end{cases}$$

(d) According to the definition of $F(n)$, we can draw the well-formed-formula binary trees:



It means when the number of propositional variables n is 2, there is only one way to connect them by $\wedge, \vee$ and no need to use brackets. However, when the number of propositional variables n is 4, there are five ways to connect them by $\wedge, \vee$ and brackets. Hence, we can find the realtionshipb between $F(n)$ and $T(n)$,so it also can be seen as the $full\ binary\ tree\ wi$th $(2n-1)$ $nodes$. In order to consider the different condition of $\wedge, \vee$, we use 2^{n-1} to caculate , because there are n-1 blanks between n propositional variables.
Moreover, we also need to consider the Negated normal form. If one propositional variable has two Negated normal forms, then n propositional variables implys 2^n Negated normal forms.
Finally, the sequence of propositional variables could casue different results. For n propositional variables, it should lead to $\sqcap(n,1) = n!$ ways to order them due to no replacement.
Therefore, according to the content above all ,we can get:
$F(n)$= $B(2n-1) \times 2^{n-1} \times 2^n \times n!$

Problem 5

(a) According to the statement of the problem, we can get:

$v_1{}^{-v_2}_{--v_4}$, $v_3{}^{--v_2}_{--v_4}$, $v_2 \begin{smallmatrix}--v_1\\--v_3\\--v_4\end{smallmatrix}$, $v_4 \begin{smallmatrix}--v_1\\--v_2\\--v_3\end{smallmatrix}$

Therefore,
[B]

$p_1(0) = 1\ \ and\ \ p_2(0) = p_3(0) = p_4(0) = 0\ \ when\ \ n = 0$

[R]

$p_1(n+1) = \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n)$

$p_2(n+1) = \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_4(n)$

$p_3(n+1) = \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n)$

$p_4(n+1) = \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_2(n)$

(b) $p_2(n+1) + p_4(n+1) = \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_2(n) =$

$p_1(n) + p_3(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n)$

$$p_1(n) = p_2(n+1) + p_4(n+1) - p_3(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n)$$

$$p_3(n) = p_2(n+1) + p_4(n+1) - p_1(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n)$$

$$\begin{aligned} &p_1(n+1) + p_3(n+1) + p_4(n+1) \\ &= \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_2(n) \\ &= p_2(n) + \frac{2}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) \end{aligned}$$

$$\begin{aligned} &p_1(n+1) + p_3(n+1) + p_2(n+1) \\ &= \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + \frac{1}{3}p_4(n) \\ &= p_4(n) + \frac{2}{3}p_2(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) \end{aligned}$$

$$p_2(n) = p_1(n+1) + p_3(n+1) + p_4(n+1) - \frac{2}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n)$$

$$p_4(n) = p_1(n+1) + p_3(n+1) + p_2(n+1) - \frac{2}{3}p_2(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n)$$

$$\begin{aligned} &p_1(n) + p_2(n) + p_3(n) + p_4(n) \\ &= p_2(n+1) + p_4(n+1) - p_3(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + p_1(n+1) \\ &+ p_3(n+1) + p_4(n+1) - \frac{2}{3}p_4(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) + p_2(n+1) \\ &+ p_4(n+1) - p_1(n) + \frac{1}{3}p_2(n) + \frac{1}{3}p_4(n) + p_1(n+1) + p_3(n+1) \\ &+ p_2(n+1) - \frac{2}{3}p_2(n) + \frac{1}{2}p_1(n) + \frac{1}{2}p_3(n) \\ &= 2p_1(n+1) + 3p_2(n+1) + 2p_3(n+1) + 3p_4(n+1) \end{aligned}$$

$\because p_1(n) + p_2(n) + p_3(n) + p_4(n) = 1$ and $p_1(n+1) = p_3(n+1)$, $p_2(n+1) = p_4(n+1)$

$\therefore p_1(n+1) = p_3(n+1) = \frac{1}{10}$ and $p_2(n+1) = p_4(n+1) = \frac{1}{10}$ when n gets larger.

$\therefore p_1(n) = p_3(n) = \frac{1}{5}$ and $p_2(n) = p_4(n) = \frac{3}{10}$ when n gets larger.

(c) $E(x) = \sum p_{\mathrm{i}} \times x_{\mathrm{i}}$, x_{i} shows the shortest distance between v_1 to v_{i}

	$v_1 \rightarrow v_1$	$v_1 \rightarrow v_2$	$v_1 \rightarrow v_3$	$v_1 \rightarrow v_4$
x_{i}	0	1	2	1
p_{i}	0.2	0.3	0.2	0.3

$$\therefore E(x) = 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.2 + 1 \times 0.3 = 1$$