Assignment 1 Report
State Feedback Control Design

EL2700 – Model Predictive Control

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PART I: Analytical Task

Introduction

The state space equation for the cart and the inverted pendulum consists of the state variables

$$x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}.$$

But since we are simplifying the system and ignoring the cart states, we are only left with θ and $\dot{\theta}$.

So rewriting the state space with only $x_3 = \theta$ and $x_4 = \dot{\theta}$, and substituting $x_1 = x_2 = 0$ & $w = 0$ we have :

$$\begin{aligned} \dot{x_3} &= x_4 \\ \dot{x_4} &= f_2(x_3, x_4, F) \end{aligned} \tag{1}$$

where, $f_2 = \frac{1}{I+ml^2 - \frac{m^2l^2\cos^2 x_3}{M+m}} \left(\frac{ml\cos x_3}{M+m} F - b_p x_4 + mgl\sin x_3 - \frac{m^2l^2x_4^2 \sin x_3 \cos x_3}{M+m} \right)$

f_2 can be re-written as:

$$f_2 = \frac{1}{I + ml^2 - \frac{m^2l^2\cos^2 x_3}{M(1+\frac{m}{M})}} \left(\frac{ml\cos x_3}{1+\frac{m}{M}} \frac{F}{M} - b_p x_4 + mgl\sin x_3 - \frac{m^2l^2x_4^2 \sin x_3 \cos x_3}{M(1+\frac{m}{M})} \right)$$

Since $M \gg m$, so denominator containing $1 + \frac{m}{M}$ can be approximated as 1. Considering acceleration $u = \frac{F}{m}$ and ignoring the terms with large denominator, f_2 can be further simplified as:

$$f_2 = \frac{1}{I + ml^2} \left(\frac{ml\cos x_3}{1} u - b_p x_4 + mgl\sin x_3 \right)$$

Rewriting (1) into state space with $x_1 = \theta$ and $x_2 = \dot{\theta}$ and substituting simplified f_2, we get:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \underbrace{\begin{bmatrix} x_2 \\ -a_0 \sin x_1 - a_1 x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 u \cos x_1 \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{x},u)} \\ y &= x_1 \end{aligned} \tag{2}$$

where $a_0 = \frac{mgl}{I+ml^2}$, $a_1 = \frac{b}{I+ml^2}$ and $b_0 = \frac{ml}{I+ml^2}$

Linear Model

Substituting $\boldsymbol{x_e} = [x_1 \quad x_2] = [0 \quad 0]$ and $u_e = 0$ in (2), we get $\dot{x_1} = \dot{x_2} = 0$, proving $\boldsymbol{x_e} = [0 \quad 0]^T$ is an equilibrium point.

Using Taylor's series to linearize (2) around $\boldsymbol{x_e}$, where we approximate $x(t) \approx x_e + \Delta x(t)$ and $u(t) \approx u_e + \Delta u(t)$:

$$\frac{d\boldsymbol{f}}{dt}(\boldsymbol{x_e} + \Delta\boldsymbol{x}(t), u_e + \Delta u(t)) \approx \boldsymbol{f}(\boldsymbol{x_e}, u_e) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x_e}, u_e} \Delta\boldsymbol{x}(t) + \left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\boldsymbol{x_e}, u_e} \Delta u(t)$$

Evaluating the above expression at $\boldsymbol{x}_e = [x_1 \quad x_2] = [0 \quad 0]$ and $u_e = 0$ yields:

$$
\begin{aligned}
\frac{d}{dt}\Delta\boldsymbol{x}(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{\boldsymbol{A}_c} \Delta\boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 0 \\ b_0 \end{bmatrix}}_{\boldsymbol{B}_c} \Delta u(t) \\
\Delta y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}_c} \Delta\boldsymbol{x}(t)
\end{aligned}
\tag{3}
$$

Discrete Linear Model

Solving the linear ODE for a forced response $\dot{x}(t) = \boldsymbol{A}_c x(t) + \boldsymbol{B}_c u(t)$ gives us:

$$
x(t) = \mathrm{e}^{\boldsymbol{A}_c(t-t_0)} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{\boldsymbol{A}_c(t-\tau)} \boldsymbol{B}_c u(\tau) d\tau \tag{4}
$$

Substituting $t_0 = kh$ and $t = (k+1)h$, and since u is considered constant within a sampling period, it can be taken out of the integration. As per the zero order hold, $u = u(kh)$

$$
x(kh+h) = \mathrm{e}^{\boldsymbol{A}_c h} x(kh) + \left(\int_{kh}^{(k+1)h} \mathrm{e}^{\boldsymbol{A}_c(kh+h-\tau)} \boldsymbol{B}_c d\tau \right) u(kh)
$$

Rewriting the variable $kh + h - \tau$ as s simplifies the above equation to

$$
x(k+1) = \mathrm{e}^{\boldsymbol{A}_c h} x(k) + \left(\int_{0}^{h} \mathrm{e}^{\boldsymbol{A}_c s} \boldsymbol{B}_c ds \right) u(k) \tag{5}
$$

We can compare (5) with $\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}u_k$, where $\boldsymbol{A} = \mathrm{e}^{\boldsymbol{A}_c h}$ and $\boldsymbol{B} = \int_0^h \mathrm{e}^{\boldsymbol{A}_c s} \boldsymbol{B}_c ds$.

In the equation $\Delta y(t) = \boldsymbol{C}_c \Delta\boldsymbol{x}(t)$, substituting $t = kh$ gives us $y(kh) = \boldsymbol{C}_c \boldsymbol{x}(kh)$. Thus, $\boldsymbol{C} = \boldsymbol{C}_c$

After plugging in all the values for m, g, l, I, b we calculated the value of $\boldsymbol{A}$ and $\boldsymbol{B}$ using MATLAB.

$$
\boldsymbol{A} = \begin{bmatrix}
\mathrm{e}^{-h/4}\left(\cosh\frac{9835^{1/2}h}{20} + \frac{9835^{1/2}\sinh\frac{9835^{1/2}h}{20}}{1967}\right) & \frac{4\times 9835^{1/2}\mathrm{e}^{-h/4}\sinh\frac{9835^{1/2}h}{20}}{1967} \\
\frac{981\times 9835^{1/2}\mathrm{e}^{-h/4}\sinh\frac{9835^{1/2}h}{20}}{19670} & \mathrm{e}^{-h/4}\left(\cosh\frac{9835^{1/2}h}{20} - \frac{9835^{1/2}\sinh\frac{9835^{1/2}h}{20}}{1967}\right)
\end{bmatrix}
$$

$$
\boldsymbol{B} = \begin{bmatrix}
\frac{10\mathrm{e}^{-h/4-\frac{9835^{1/2}h}{20}}\left(9835\mathrm{e}^{\frac{9835^{1/2}h}{10}} + 9835^{1/2}(5\mathrm{e}^{\frac{9835^{1/2}t}{20}} - 5) + 9835\right)}{1929627} - \frac{100}{981} \\
\frac{5\times 9835^{1/2}\mathrm{e}^{-h(\frac{9835^{1/2}+5}{20})}(\mathrm{e}^{\frac{9835^{1/2}h}{10}} - 1)}{1967}
\end{bmatrix}
$$

$\boldsymbol{C}$ is simply $[1 \quad 0]$ and $\boldsymbol{D} = 0$.

Impact of sampling to the pole locations of the discrete-time system

For different values of sampling interval h, pole calculations were calculated. To be precise, all the sampling intervals from 0.001s to 0.1s were tried in steps of 0.001s. The resulting poles of the discrete system is presented in fig. 1.

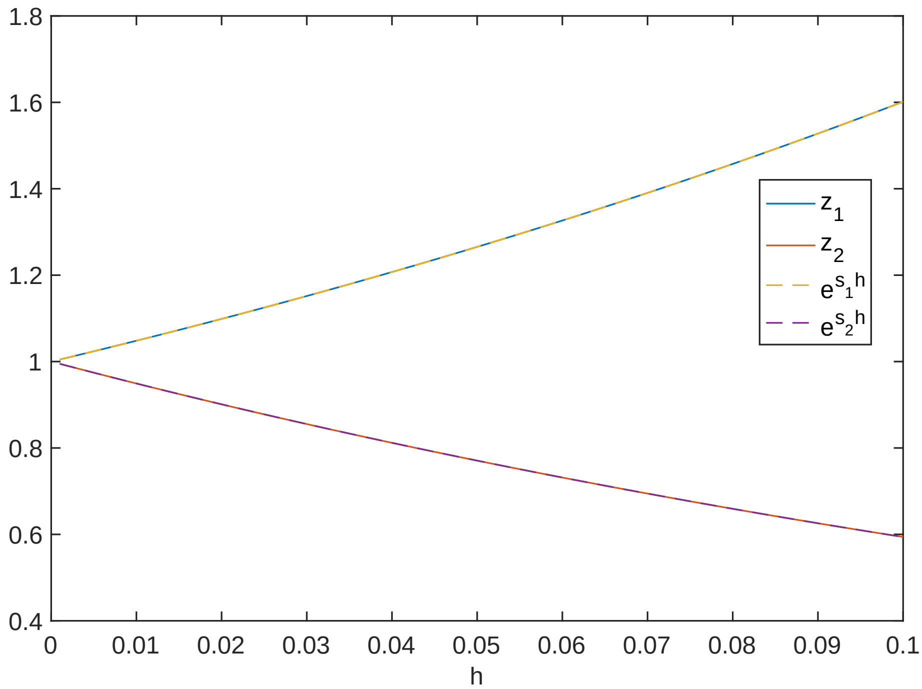

Figure 1: Pole locations for discretized system for different sampling intervals,h

The poles of the continuous system, i.e $\boldsymbol{A_c},\boldsymbol{B_c},\boldsymbol{C_c},\boldsymbol{D_c}$ were found to be lying at 4.708 and -5.208. It has been shown in fig 1 that the discrete-time poles, z_i agree with the relation $z_i = e^{s_i h}$, where, s_i are the continuous-time poles.

PART II: Design Task

Verification of linear continuous-time model

Following the approach explained in section *Linear Model* of Part I, the full model can be linearized at the equilibrium point $\boldsymbol{x}_e = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$, $u_e = 0$ and $w = 0$:

$$\frac{d\boldsymbol{f}}{dt}(\boldsymbol{x_e} + \Delta\boldsymbol{x}(t), u_e + \Delta u(t)) \approx \boldsymbol{f}(\boldsymbol{x_e}, u_e) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x_e},u_e} \Delta\boldsymbol{x}(t) + \left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\boldsymbol{x}_e,u_e} \Delta u(t) \tag{6}$$

$$\boldsymbol{f}(\boldsymbol{x_e}, u_e) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{7}$$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -f_{1a}(x_3)b_c & f_{1b}(\cdot) & f_{1a}(x_3)(-2mlx_4\sin x_3 - \frac{mlb_p\cos x_3}{I+ml^2}) \\ 0 & 0 & 0 & 1 \\ 0 & -f_{2a}(x_3)\frac{mlb_c\cos x_3}{M+m} & f_{2b}(\cdot) & f_{2a}(x_3)(-b_p\frac{-2m^2l^2x_4\cos x_3}{M+m}) \end{bmatrix} \tag{8}$$

where $f_{1a}(x_3) = \frac{1}{M+m-\frac{m^2l^2\cos x_3{}^2}{I+ml^2}}$, $f_{2a}(x_3) = \frac{1}{I+ml^2-\frac{m^2l^2\cos x_3{}^2}{M+m}}$ and $f_{1b}(\cdot)$ and $f_{2b}(\cdot)$ are defined as $f_{1b}(\cdot) = \frac{\partial f1}{\partial x_3}, f_{2b}(\cdot) = \frac{\partial f2}{\partial x_3}$.

By inserting the values for $\boldsymbol{x}_e, u_e$ and w, the above equations lead to

$$\begin{aligned} f_{1a} &= \frac{1}{M+m-\frac{m^2l^2}{I+ml^2}} = v_2 \\ f_{1b} &= \frac{gl^2m^2}{Mml^2+Im+IM} \\ f_{2a} &= \frac{1}{I+ml^2-\frac{m^2l^2}{M+m}} = v_1 \\ f_{2b} &= \frac{glm(M+m)}{Mml^2+Im+IM} \end{aligned} \tag{9}$$

Hence it follows that

$$\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e,u_e} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -b_c v_2 & f_{1b} & v_2\frac{mlb_p}{I+ml^2} \\ 0 & 0 & 0 & 1 \\ 0 & -v_1\frac{mlb_c}{M+m} & f_{2b} & -v_1 b_p \end{bmatrix} = \boldsymbol{A}. \tag{10}$$

$$\left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\boldsymbol{x}_e,u_e} = \left.\left[\begin{bmatrix} 0 \\ \frac{M}{M+m-\frac{l^2m^2\cos(x_3)^2}{ml^2+I}} \\ 0 \\ \frac{Mlm\cos(x_3)}{(M+m)\left(I+l^2m-\frac{l^2m^2\cos(x_3)^2}{M+m}\right)} \end{bmatrix}\right]\right|_{\boldsymbol{x}_e,u_e} = \begin{bmatrix} 0 \\ \frac{M}{M+m-\frac{l^2m^2}{ml^2+I}} \\ 0 \\ \frac{Mlm}{(M+m)\left(I+l^2m-\frac{l^2m^2}{M+m}\right)} \end{bmatrix} = \boldsymbol{B}. \tag{11}$$

The matrices given in (10) and (11) correspond to the given matrices.

Design of State Feedback Controller

The poles of the system were placed at the locations $p_{1,2} = 0.3 \pm 0.0005j$ and $p_{3,4} = 0.72 \pm 0.03j$ (determined by trial-and-error). The response of the system without disturbance is shown in figure 2. The output of the system at $t = 10s$ is $y(t = 10s) = 9.828m$. The maximum angle deviation is $\texttt{max}(|x_3(t)|) = 0.162rad$.

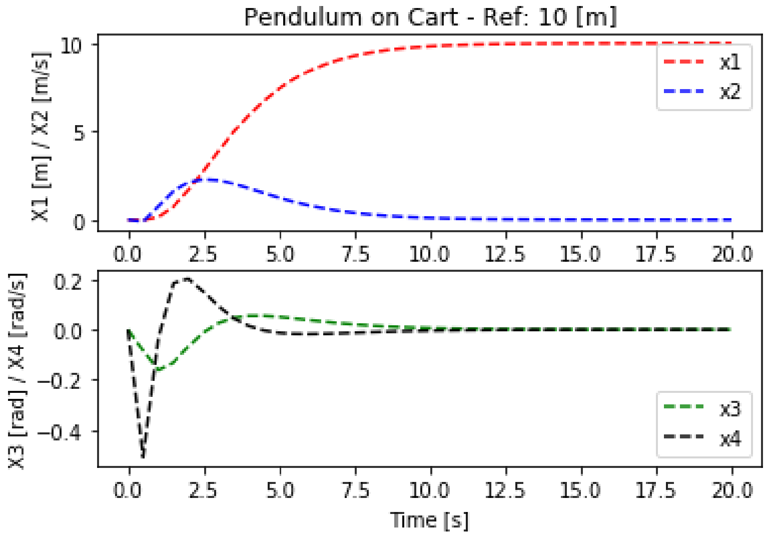

Figure 2: Response of the system without disturbance.

Validation of Feed Forward Gain

The discrete time system in state space form is given by :

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + B\boldsymbol{u}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k \end{aligned} \tag{12}$$

With $D = 0$. Since the control law is found to be $\boldsymbol{u}_k = -L\boldsymbol{x}_k + l_r r_k$, we can substitute that too in eq 14. Taking the z-transform, we can find the transfer function for $\boldsymbol{y}$ to $\boldsymbol{x}$ to be :

$$Y(z) = C(zI - (A - BL))^{-1} B l_r R(z)$$

where $R(z)$ is the z-tranform of $r(k)$. The steady state error is the error between reference and output of the system at the steady state, and using the final-value theorem of z-transform, it can be re-written as :

$$\begin{aligned} y_{ss} = \lim_{k\to\infty} y_k - r_k &= \lim_{z\to 1}(z-1)Y(z) - R(Z) \\ &\Rightarrow \lim_{z\to 1}(z-1)\left(R(z) - C(zI - (A-BL))^{-1} B l_r R(z)\right) \end{aligned}$$

Since we have to find feedforward gain l_r for error-free tracking, $y_{ss} = 0$. For $\lim(z \to 1)$, we get :

$$l_r = \frac{1}{C(I - (A - BL))^{-1} B} \tag{13}$$

Performance in the Presence of Disturbance Input

The response of the system in the presence of disturbance, $w = 0.01$ is given in figure 3. It is visible, that there is a steady state error of $y_{ss} = 3.111m$. This error can be calculated by considering the z-transform of the new system

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + B\boldsymbol{u}_k + B_w w_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k \end{aligned} \tag{14}$$

which results in

$$Y(z) = C(zI - (A - BL))^{-1}(Bl_r R(z) + B_w W(z)),$$

where $W(z)$ is the z-transform of the disturbance. Using the same manipulations as above and the identity given in (13) results in

$$\begin{aligned} y_{ss} &= \lim_{k\to\infty} y_k - r_k = \lim_{z\to 1}(z-1)Y(z) - R(Z) \\ &\Rightarrow \lim_{z\to 1}(z-1)\left(-C(zI - (A - BL))^{-1}B_w W(z)\right) = -C(I - (A - BL)^{-1})B_w w_k \end{aligned}$$

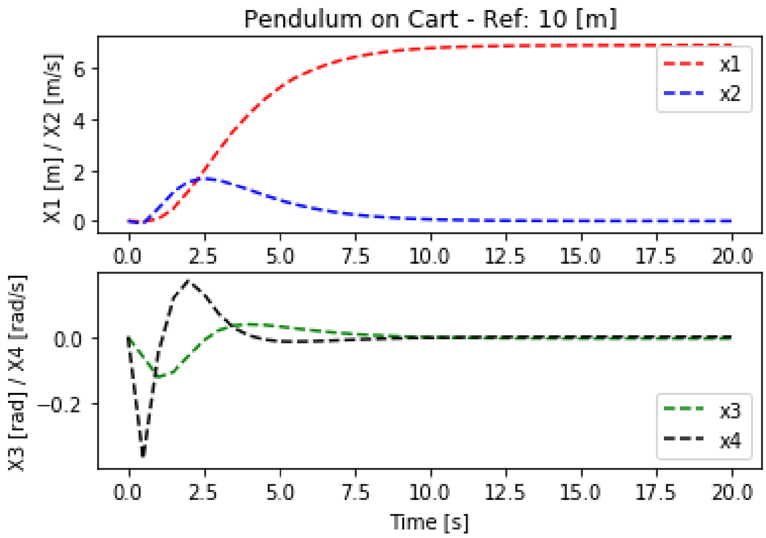

Figure 3: Response of the system with disturbance.