Model Predictive Control - EL2700

Assignment 3 : Linear Quadratic Regulator

2020

Automatic Control
School of Electrical Engineering and Computer Science
Kungliga Tekniska Högskolan

Introduction

In this task, we will implement LQR controller for the reference tracking, then add integral action to this controller, and finally design an output feedback controller based on Kalman filter. To simplify the task, we use the linear model of the inverted cart pendulum to test the controllers. As before, we will use Python and CasADi to complete task.

PART I: LQR Implementation

We design LQR controller based on the discrete-time linearized dynamics

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + Bu_t + B_w w_t \tag{1}$$
$$y_t = C\mathbf{x}_t \tag{2}$$

where $x_t \in \mathbb{R}^4$, $u_t \in \mathbb{R}$, $w_t \in \mathbb{R}$, and $y_t \in \mathbb{R}$ being system state, control input, disturbance input, and output respectively. The matrices A, B, and C are provided, and the objective is to design an LQR controller with feed forward term that achieves the following three criteria:

1. It moves the cart to the reference point $r = 10$;
2. The cart should move 90% of the reference position within 10 seconds;
3. The pendulum has to be kept within $\pm 10^o$ around the vertical position.

Tracking constant reference r without error is possible if there exist an equilibrium state $\mathbf{x}^{eq}$ and corresponding constant input u^{eq} such that

$$\mathbf{x}^{eq} = A\mathbf{x}^{eq} + Bu^{eq}, \tag{3}$$
$$r = C\mathbf{x}^{eq}. \tag{4}$$

With the incremented state and control variables ($\Delta\mathbf{x}_t = \mathbf{x}_t - \mathbf{x}^{eq}, \Delta u_t = u_t - u^{eq}$), the linear state space model of the system in (1) becomes

$$\Delta\mathbf{x}_{t+1} = A_d\Delta\mathbf{x}_t + B_d\Delta u_t. \tag{5}$$

With the state space model in (5), we can now calculate the optimal gain matrix L, provided by an LQR controller, such that the state feedback controller $u_t = -L\Delta\mathbf{x}_t$ minimizes the infinite horizon quadratic cost function given by

$$J = \sum_{t=0}^{\infty} \Delta\mathbf{x}_t^T Q\Delta\mathbf{x}_t + \Delta u_t^T R\Delta u_t$$

where $Q \succeq 0$ and $R \succ 0$ are symmetric and positive (semi-)definite matrices of appropriate dimensions. These matrices will be used to calculate your LQR feedback gain L. With stabilizing state feedback gain L, we can now find a u^{eq} in (3) of the form

$$u^{eq} = -L\mathbf{x}^{eq} + l_r r$$

where the feed-forward gain l_r satisfies

$$C_d(I - (A_d - B_d L))^{-1} B_d l_r = I.$$

Now combining feedback from the states and feed-forward from the reference, the optimal control to be applied to the original system can be calculated as

$$\begin{aligned} u_t &= -L\Delta\mathbf{x}_t + u^{eq} \\ &= -L\mathbf{x}_t + l_r r \end{aligned}$$

Q1: A key question in LQR design is how to choose the weight matrices Q, and R to ensure desired performance specification. You are encouraged to try different weight matrices and comment on the resulting performance without disturbance.

Q2: Add a small constant disturbance to mimic the presence of horizontal wind force and comment on the tracking ability of the LQR controller.

PART II: Adding integral action

To cope with the disturbance, we will add an integral action to our LQR controller. The integral state dynamics are defined by

$$i_{t+1} = i_t + h(r_t - y_t) = i_t + h(r_t - C_d x_t) \tag{6}$$

By integral action to the state vector, we obtain the system

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ i_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -hC & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ i_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ h \end{bmatrix} r_t + \begin{bmatrix} B_w \\ 0 \end{bmatrix} w_t \tag{7}$$

To incorporate the integral state, we can modify the LQR controller as follows

$$u_t = -L_e \begin{bmatrix} \mathbf{x}_t \\ i_t \end{bmatrix} + l_r r = -L\mathbf{x}_t - l_i i_t + l_r r \tag{8}$$

Since the extended system state consists of both the system state and the integral state, the gains L and l i should be selected to minimize the cost

$$J = \sum_{t=0}^{\infty} \begin{bmatrix} \Delta\mathbf{x}_t \\ \Delta i_t \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & q_i \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_t \\ \Delta i_t \end{bmatrix} + \Delta u_t^T R \Delta u_t \tag{9}$$

where Q_1 , R and q_i are positive definite weight matrices.

Q3: Form the augmented system matrices from (7) by creating the system matrices in `set_augmented_discrete_system` and creating its dynamics in `pendulum_augmented_dynamics`. Adjust the weight matrices Q (with q_i) and R, and design the controller (8) in `lqr_ff_fb_integrator` that meets the desired performance specification. Motivate your choice of the weight matrices. Comment if the controller is eliminating the step disturbance.

Part III: Output feedback controller

We will now implement LQG controller by combining LQR controller with Kalman filter. The only formal requirement for state estimation is that the system is observable. For our cart-pendulum system, probable locations for output matrices are

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}; \tag{10}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{11}$$

We will use a Kalman filter provided by the *filterpy* library. The inputs to this block are:(i) State space model of the system with process noise (w_t) and measurement noise (v_t)

$$\begin{aligned} \mathbf{x}_{t+1} &= A_d \mathbf{x}_t + \begin{bmatrix} B_u & B_w \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix} \\ y_t &= C_d \mathbf{x}_t + \begin{bmatrix} D_u & D_w \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix} + v_t \end{aligned} \tag{12}$$

(ii) Initial guess of the state ($\mathbf{x}_0$); (iii) Process and measurement noise covariance matrices (Q_p , R_n). With the estimated state ($\hat{\mathbf{x}}_t$), we can formulate output feedback controller as:

$$u_t = -L\hat{\mathbf{x}}_t - l_i i_t + l_r r \tag{13}$$

Q4: Compare the performance of the output feedback controller to the state feedback one (from Part II). Try different feasible choices of the matrix C_p and values for Q_p and R_n.

Q5: Comment on the robustness of the output feedback controller.

As always, refer to the Slack workspace `el2700workspace.slack.com` for questions.

Good Luck!