Linear Regression and Stochastic Gradient Descent

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Download the dataset "housing_scale":

```
In [1]: import requests
        r = requests.get('''https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/regression/housing_scale''')
```

Load the dataset to X,y:

```
In [2]: from sklearn.datasets import load_svmlight_file
        from io import BytesIO
        X, y = load_svmlight_file(f=BytesIO(r.content), n_features=13)
        X = X.toarray()
```

Preprocess, change the shape of X and y:

```
In [3]: import numpy
        n_samples, n_features = X.shape
        X = numpy.column_stack((X, numpy.ones((n_samples, 1))))
        y = y.reshape((-1, 1))
```

Devide the dataset into traning set and validation set(traning size is 3/4,and validation size is rest here):

```
In [4]: from sklearn.model_selection import train_test_split
        X_train, X_val, y_train, y_val = train_test_split(X, y, test_size=0.25)
```

Define regular term coefficients learning rate and max epoches:

```
In [5]: penalty_factor = 0.01
        learning_rate = 0.001
        max_epoch = 3500
        losses_train = []
        losses_val = []
```

Initialize w by different ways(using all zeros here):

```
In [6]: # select different initializing method
        w = numpy.zeros((n_features + 1, 1))  # initialize with zeros
        # w = numpy.random.random((n_features + 1, 1))  # initialize with random numbers
        # w = numpy.random.normal(1, 1, size=(n_features + 1, 1))  # initialize with zero normal distribution
```

Closed solution to linear regression:

$$\hat{w} = (X^T X)^{-1} X^T y$$

```
In [7]: X_train_t=X_train.transpose()
        temp1=numpy.dot(X_train_t,X_train)
        temp2=numpy.dot(numpy.linalg.inv(temp1),X_train_t)
        w=numpy.dot(temp2,y_train)
```

Loss function:

$$L = \frac{1}{2m} \sum_{i=1}^{m} (y_i - X_i \omega)^2$$

or

$$L = \frac{1}{2m} ||y_i - X_i \omega||_2^2$$

```
In [8]: Y_predict_closed = numpy.dot(X_train, w)  # predict under the train set
        loss_train_closed = 1/2*numpy.average(numpy.square(Y_predict_closed-y_train))

        Y_predict_closed = numpy.dot(X_val, w)  # predict under the validation set
        loss_val_closed = 1/2*numpy.average(numpy.square(Y_predict_closed-y_val))

        Y_predict_closed = numpy.dot(X, w)  # under all dataset
        loss_closed=1/2*numpy.average(numpy.square(Y_predict_closed-y))
        print("The loss under the train set is ",loss_train_closed)
        print("The loss under the validation set is ",loss_val_closed)
        print("The loss under all dataset is ",loss_closed)
```

```
The loss under the train set is  11.366724927728239
The loss under the validation set is  10.310000043399386
The loss under all dataset is  11.101499512096295
```

Train and predict by Stochastic Gradient Descent(SGD):

Here are fomulas used below:

we have loss function with L1 regularization（Lasso regression）:

$$L = \frac{1}{2m}\sum_{i=1}^{m}(y_i - X_i\omega)^2 + \frac{\lambda}{m}||\omega||_1$$

or

$$L = \frac{1}{2m}||y_i - X_i\omega||_2^2 + \frac{\lambda}{m}||\omega||_1$$

the gradient is:

$$G = \frac{\partial L}{\partial \omega} = -\frac{1}{m}(\sum_{i=1}^{m} y_i X_i^T - (\sum_{i=1}^{m} X_i^T X_i)\omega - \lambda sgn(\omega))$$

Details about the above derivation:

for above formula, X is a $m \times 13$ matric，and X_i is a row vector which represents a sample with 13 properties, then ω is a 13×1 matric which from ω_1 to ω_{13},so we can seek partial guidance on ω_1（Without loss of generality, We regard that X:$m \times n$ ω:$n \times 1$y:$m \times 1$):

$$\frac{\partial L}{\partial \omega_1} = \frac{\partial \frac{1}{2m}\sum_{i=1}^{m}(y_i - X_i\omega)^2 + \frac{\lambda}{m}||\omega||_1}{\partial \omega_1}$$

$$= \frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)\frac{\partial \sum_{i=1}^{m}(y_i - X_i\omega)}{\partial \omega_1} + \frac{\lambda}{m}sgn(\omega_1)$$

$$= \frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)\frac{\partial(y_1 - X_{11}\omega_1 - X_{12}\omega_2 - \cdots - X_{1n}\omega_n + y_2 - X_{21}\omega_1 - X_{22}\omega_2 - \cdots - X_{2n}\omega_n + \cdots + y_m - X_{m1}\omega_1 - X_{m2}\omega_2 - \cdots - X_{mn}\omega_n)}{\partial \omega_1} + \frac{\lambda}{m}sgn(\omega_1)$$

$$= \frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)(-X_{11} - X_{21} - \cdots - X_{m1}) + \frac{\lambda}{m}sgn(\omega_1)$$

$$= -\frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)\sum_{j=1}^{m}X_{j1} + \frac{\lambda}{m}sgn(\omega_1)$$

We can also get follow formula and so on:

$$\frac{\partial L}{\partial \omega_2} = -\frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)\sum_{j=1}^{m}X_{j2} + \frac{\lambda}{m}sgn(\omega_2)$$

thus:

$$\frac{\partial L}{\partial \omega} = \begin{bmatrix} \frac{\partial L}{\partial \omega_1} \\ \frac{\partial L}{\partial \omega_2} \\ \cdots \\ \frac{\partial L}{\partial \omega_n} \end{bmatrix}$$

and

$$\begin{bmatrix} \sum_{j=1}^{m} X_{j1} \\ \sum_{j=1}^{m} X_{j2} \\ \cdots \\ \sum_{j=1}^{m} X_{jn} \end{bmatrix} = \sum_{i=1}^{m} X_i^T$$

and finally we get:

$$\frac{\partial L}{\partial \omega} = -\frac{1}{m}\sum_{i=1}^{m}(y_i - X_i\omega)X_i^T + \frac{\lambda}{m}sgn(\omega)$$

$$= -\frac{1}{m}(\sum_{i=1}^{m} y_i X_i^T - (\sum_{i=1}^{m} X_i^T X_i)\omega - \lambda sgn(\omega))$$

update

$$\omega = \omega - \eta G$$

pre process:

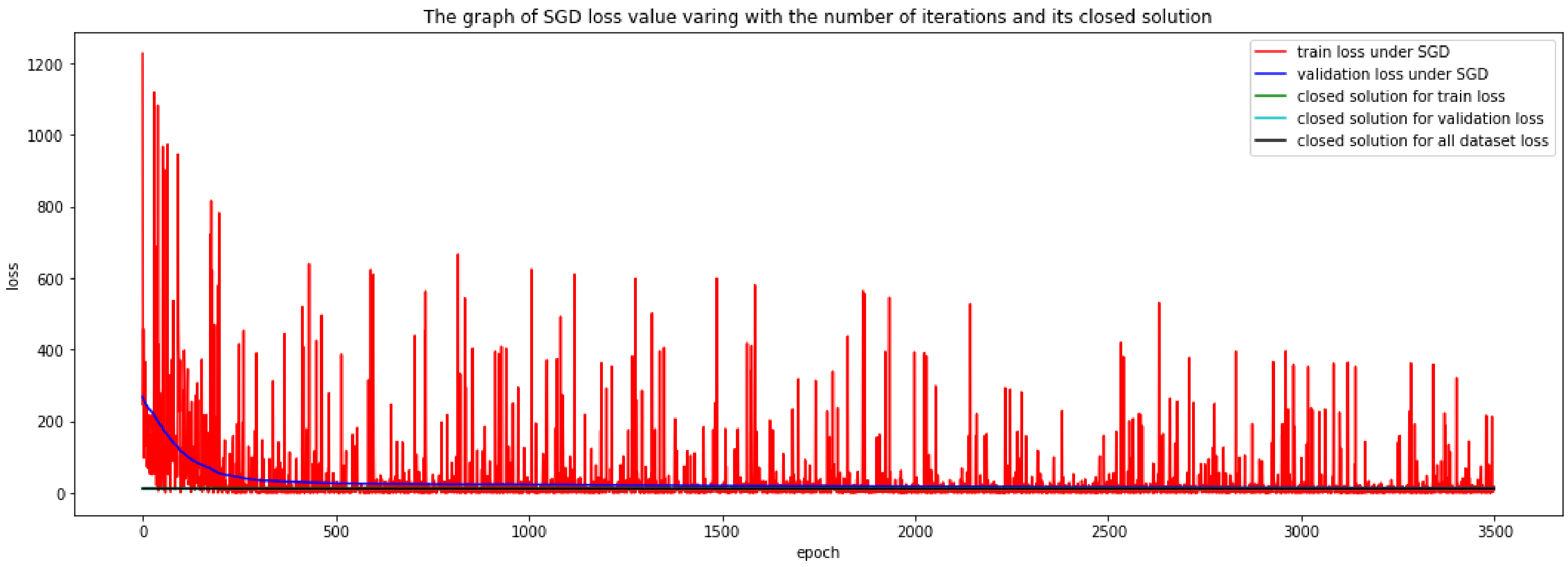
```
In [9]: def sgn(v):
            num=0
            for i in v:
                if i>0:
                    v[num]=1
                if i<0:
                    v[num]=-1
                if i==0:
                    v[num]=0
                num+=1
```

training and iteration:

```
In [10]: import random
         w = numpy.zeros((n_features + 1, 1))  # initialize with zeros
         for epoch in range(max_epoch):
             r=random.randint(0,X_train.shape[0]-1)
             X_train_s=X_train[r]
             y_train_s=y_train[r]
             X_train_s_t=X_train_s.reshape(X_train_s.shape[0],1)#choose one sample

             G = -((y_train_s-numpy.dot(X_train_s,w))*(X_train_s_t)-penalty_factor*w)# calculate the gradient
             G = -G
             w += learning_rate * G  # update the parameters

             Y_predict = numpy.dot(X_train_s, w)  # predict under the train set
             loss_train = 1/2*(numpy.square(Y_predict-y_train_s))# calculate
             losses_train.append(loss_train)

             Y_predict = numpy.dot(X_val, w)  # predict under the validation set
             loss_val = 1/2*numpy.average(numpy.square(Y_predict-y_val)) # calculate
             losses_val.append(loss_val)
```

Plot loss values with the number of iterations:

```
In [11]: %matplotlib inline
         import matplotlib.pyplot as plt
         losses_train_closed=[]
         losses_val_closed=[]
         losses_closed=[]
         plt.figure(figsize=(18, 6))
         plt.plot(losses_train, "-", color="r", label="train loss under SGD")
         plt.plot(losses_val, "-", color="b", label="validation loss under SGD")
         for i in range(3500):
             losses_train_closed.append(loss_train_closed)
             losses_val_closed.append(loss_val_closed)
             losses_closed.append(loss_closed)
         plt.plot(losses_train_closed, "-", color="g", label="closed solution for train loss")
         plt.plot(losses_val_closed, "-", color="c", label="closed solution for validation loss")
         plt.plot(losses_closed, "-", color="k", label="closed solution for all dataset loss")
         plt.xlabel("epoch")
         plt.ylabel("loss")
         plt.legend()
         plt.title("The graph of SGD loss value varing with the number of iterations and its closed solution")

Out[11]: Text(0.5,1,'The graph of SGD loss value varing with the number of iterations and its closed solution')
```



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