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Rational framing motions and spatial rational Pythagorean hodograph curves

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ABSTRACT

We propose a new method for constructing rational spatial Pythagorean Hodograph (PH) curves based on determining a suitable rational framing motion. While the spherical component of the framing motion is arbitrary, the translation part is determined be a modestly sized and nicely structured system of linear equations. Rather surprisingly, generic input data will only result in polynomial PH curves. We provide a complete characterization of all cases that admit truly rational (non-polynomial) solutions. Examples illustrate our ideas and relate them to existing literature.

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1. Introduction

The distinctive property of a polynomial Pythagorean-hodograph (PH) curve is that its parametric speed, which specifies the rate of change of arc length with respect to the curve parameter, is a polynomial rather than the square root of a polynomial. This feature endows PH curves with many computational advantages in geometric design, motion control, animation, path planning, and similar applications.

Planar PH curves were first introduced in Farouki and Sakkalis (1990), through direct integration of hodograph components expressed in terms of real preimage polynomials. Subsequently, a complex-variable model for planar PH curves was introduced in Farouki (1994), that greatly facilitates the development of algorithms for their construction and analysis, see e.g. Farouki (2008) for an extensive bibliography.

Spatial PH curves were first considered in Farouki and Sakkalis (1994) using a three polynomial preimage. A characterization of Pythagorean polynomial quadruples in terms of four polynomials was presented, in a different context, in Dietz et al. (1993). Subsequently, two algebraic models for spatial PH curves were based on this characterization: The quaternion and Hopf map representations, as proposed in Choi et al. (2002). These forms are rotation invariant Farouki (2002) and serve as the foundation for many practical constructions of spatial PH curves and associated frames (Farouki, 2008; Farouki et al., 2019).

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The extension from polynomial to rational PH curves is non-trivial, since a rational hodograph does not always yield a rational curve upon integration. To circumvent this difficulty, a different approach was adopted in Fiorot (1994); Pottmann (1995a) based on the dual (line) representation of planar curves. Specifically, plane curves are viewed as the envelopes of one-parameter families of tangent lines, rather than point loci. By assigning rational functions to describe the orientation of these tangents and their distance from the origin, one can construct rational PH curves in a geometrically intuitive manner Pottmann (1995b). A comparison of polynomial and rational planar PH curves can be found in Farouki and Pottmann (1996).

The generalization of rational PH curves from the planar case to the spatial case is also non-trivial, since the dual line representation cannot be simply carried over: In the plane, points and lines are dual elements, but in space the duality is between points and planes, while lines are self-dual elements. This problem was solved in Farouki and Šír (2011) and a construction in two steps was given. In the first step a rational tangent indicatrix of the curve is constructed via stereographic projection. In the second step the osculating planes are obtained with one degree of freedom depending rationally on the parameter. Exploiting the property that every spatial curve has an associated tangent developable surface, the PH curve is determined from its tangent developable as the singular locus, or edge of regression. In Kozak et al. (2014); Farouki and Šír (2020) the first step was modified and the tangent indicatrix was described by a quaternion valued polynomial. In this way a comparatively simpler formula was provided (cf. Equation (5) below). While this formula is quite elegant and provides a full description of all rational PH curves, it is difficult to have an insight in the possible (and frequent) cancellations between the numerator and denominator.

In Farouki and Šír (2011) the theory of rational PH curves was also connected to the theory of polynomial ones, in particular concerning the construction of rotation minimizing frames. Other results on rational spatial PH curves include Bartoň et al. (2010) where spherical rational curves with rational rotation minimizing frame are constructed by applying Möbius transformations in $\mathbb{R}^3$ to piecewise planar PH cubics. In the series of papers Kozak et al. (2014, 2016); Krajnc (2017) the authors exploit the dual representation to interpolate with rational spatial PH curves of low class. In Farouki and Sakkalis (2019) a special form of the rational hodograph is used to construct planar rational PH curves with rational arc-length function. A generalization of this idea to the spatial curves is also suggested and shown on one example.

Our novel approach to rational PH curve starts with the observation that their rational tangent indicatrix gives rise to a rational spherical motion. When composed with the translation along the curve, it is a framing motion of the curve. The spherical component has been called Euler-Rodrigues motion or, equivalently referring to the images of a suitable orthogonal tripod, Euler-Rodrigues frame (Choi and Han, 2002; Kozak et al., 2014).

Writing the framing motion of the yet undetermined rational PH curve in the dual quaternion model of space kinematics, cf. Husty and Schröcker (2012) or Selig (2005, Chapter 11), separating spherical and translational motion component, and selecting appropriate parameters (the spherical motion component, represented by a quaternion polynomial $A \in \mathbb{H}[t]$ and the PH curve's denominator polynomial α) the PH conditions can be expressed by a system of linear equations.

By design, this approach will produce all rational PH curves of a certain maximal degree. In particular, it comprises the usual construction of polynomial PH curves from A via integration. Rather surprisingly, for a generic choice of A and α only polynomial solutions can be obtained. A detailed analysis of the system of linear equations yields a complete characterization of those cases that admit truly rational (non-polynomial) solutions (Theorem 4.6) in terms of the coefficients of the Taylor polynomial of the non-normalized tangent indicatrix at a zero of α and the multiplicity n of this zero.

Our approach not only provides a straightforward and direct method to compute rational PH curves and also polynomial PH curves from the same data. It also sheds new light on rational PH curves and on their relation to polynomial PH curves. Via A and α, essential properties of the resulting family of rational PH curves can be controlled. This and the vector space structure of the solution family is assumed to be beneficial for many applications.

We continue this article by providing some background on the construction of rational and polynomial PH curves and on the dual quaternion model of space kinematics in Section 2. In Section 3 we describe in detail how to compute rational PH curves from the spherical component of a rational framing motion and establish some basic properties of the construction. Existence of non-polynomial PH curves to given input data is discussed in detail in Section 4. In Section 5 we demonstrate how to compute some new and some known examples of rational PH curves and we relate computation methods from literature to our approach.

2. Preliminaries

In this section we recall basic definitions and known results about PH curves, quaternions, dual quaternions and their relation to space kinematics. Let us start with the definition of PH curves. We suggest the analogous wording for both polynomial and rational cases postponing the necessary differences to remarks below.

Definition 2.1. A spatial polynomial (rational) parametric curve $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$ is called *Pythagorean hodograph (PH)* if it has a polynomial (rational) speed. More precisely if there exists a polynomial (rational) function $\sigma(t)$ so that

$$r_1'(t)^2 + r_2'(t)^2 + r_3'(t)^2 = \sigma(t)^2. \tag{1}$$

Note that there is an important difference between the polynomial and the rational case. Unlike the polynomial PH curves, the rational PH curves typically do not possess a rational arc-length function. Indeed, the primitive function of the speed $\int \sigma(t)\mathrm{d}t$ is rational only in special cases (Farouki and Sakkalis, 2019). This might be one of the reasons why in the plane rational PH curves are often defined as the curves having rational offsets (Pottmann, 1995a). We will suggest later that for space curves the analogous geometric property might be the existence of a related framing motion.

Let us also stress that the property to be PH may depend on the parameterization. As a well known example consider the parabola $\mathbf{r}(s) = (s, s^2)$ which is not PH while its rational reparameterization $\mathbf{r}(t) = (\frac{t}{1-t^2}, \frac{t^2}{(1-t^2)^2})$ is PH with speed function $\sigma(t) = \frac{(1+t^2)^2}{(1-t^2)^3}$. Therefore we consider the PH property as an aspect of the particular parameterization rather than of the geometric object. For this reason we will not use non-linear reparameterizations.

Equation (1) implies existence of the unit vector field

$$\mathbf{T}(t) = \left(\frac{r_1'(t)}{\sigma(t)}, \frac{r_2'(t)}{\sigma(t)}, \frac{r_3'(t)}{\sigma(t)} \right)$$

tangent to the curve. $\mathbf{T}(t)$ is defined for all $t \in \mathbb{R}$, possibly with exception of finitely many values. It can be seen as a rational spherical curve called *tangent indicatrix*. In fact, existence of a *rational* tangent indicatrix is equivalent to the PH condition for rational curves.

Next, we introduce the algebra of dual quaternions and the concept of representing motions via certain dual quaternion polynomials, cf. Husty and Schröcker (2012) or Selig (2005, Chapter 11). The skew field of quaternions $\mathbb{H}$ is the associative real algebra generated by the four basis elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ together with the multiplication derived from the generating relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Conjugation of a quaternion $\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ is defined by changing signs of the coefficients at the complex units, i.e. $\mathcal{Q}^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$, and the norm of q is given by $\mathcal{Q}\mathcal{Q}^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{R}$. The algebra $\mathbb{DH}$ of dual quaternions is obtained by adjoining an element ε which commutes with quaternions and has the property $\varepsilon^2 = 0$. A dual quaternion can be written as $\mathcal{H} = \mathcal{P} + \varepsilon\mathcal{D}$ where *primal part* $\mathcal{P}$ and *dual part* $\mathcal{D}$ are quaternions. The dual quaternion conjugate of $\mathcal{H}$ is defined as $\mathcal{H}^* = \mathcal{P}^* + \varepsilon\mathcal{D}^*$ and its norm is $\mathcal{H}\mathcal{H}^* = \mathcal{P}\mathcal{P}^* + \varepsilon(\mathcal{P}\mathcal{D}^* + \mathcal{D}\mathcal{P}^*)$. Note that the norm is a dual number in general and a real number if the *Study condition* $\mathcal{P}\mathcal{D}^* + \mathcal{D}\mathcal{P}^* = 0$ is fulfilled. In this case (and assuming $\mathcal{P} \neq 0$), the multiplicative inverse of $\mathcal{H}$ is given by $\mathcal{H}^{-1} = \mathcal{H}^*/(\mathcal{P}\mathcal{P}^*)$.

Denote by $\mathbb{DH}^\times$ the group of invertible dual quaternions $\mathcal{H} = \mathcal{P} + \varepsilon\mathcal{D}$ that satisfy the *Study condition* $\mathcal{P}\mathcal{D}^* + \mathcal{D}\mathcal{P}^* = 0$. The factor group $\mathbb{DH}^\times/\mathbb{R}^\times$ of $\mathbb{DH}^\times$ modulo the real multiplicative group $\mathbb{R}^\times$ is isomorphic to the group SE(3) of rigid body displacements. The dual quaternion $\mathcal{H} = \mathcal{P} + \varepsilon\mathcal{D} \in \mathbb{DH}^\times$ acts on the point with Cartesian coordinates (x_1, x_2, x_3), embedded into $\mathbb{DH}$ as $\mathbf{x} = 1 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})$, via

$$\mathbf{x} \mapsto \frac{(\mathcal{P} - \varepsilon\mathcal{D})\mathbf{x}(\mathcal{P}^* + \varepsilon\mathcal{D}^*)}{\mathcal{P}\mathcal{P}^*}. \tag{2}$$

Equation (2) describes a single rigid-body displacement in dual quaternions. In order to obtain a motion, primal part $\mathcal{P}$ and dual part $\mathcal{D}$ (and hence also $\mathcal{H} = \mathcal{P} + \varepsilon\mathcal{D}$) should depend on a real motion parameter t. Since we are interested in rational motions, we replace $\mathcal{H}$ by a *dual quaternion polynomial* $\mathcal{C}(t) = \mathcal{A}(t) + \varepsilon\mathcal{B}(t) \in \mathbb{DH}[t]$ in the real variable t that satisfies the polynomial Study condition $\mathcal{A}(t)\mathcal{B}^*(t) + \mathcal{B}(t)\mathcal{A}^*(t) = 0$ and $\mathcal{A}(t) \neq 0$. Polynomials of this type are called *motion polynomials* (Hegedüs et al., 2013). Clearly, the trajectory

$$\frac{(\mathcal{A}(t) - \varepsilon\mathcal{B}(t))x(\mathcal{A}^*(t) + \varepsilon\mathcal{B}^*(t))}{\mathcal{A}(t)\mathcal{A}^*(t)} \tag{3}$$

of a point x is a rational curve and it is well-known that all *rational motions* (that is, motions where all trajectories are rational curves) can be generated in that way (Jüttler, 1993).

The action (2) of dual quaternions (and also the action of motion polynomials) on points can be extended to vectors. Their images depend only on the spherical component $\mathcal{A}(t)$ of the motion $\mathcal{C}(t) = \mathcal{A}(t) + \varepsilon\mathcal{B}(t)$. Embedding the vector $x = (x_1, x_2, x_3)$ into $\mathbb{DH}$ as the *vectorial* quaternion $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, the image of x is given by $\mathcal{A}(t)\mathbf{x}\mathcal{A}^*(t)/(\mathcal{A}(t)\mathcal{A}^*(t))$.

Quaternion polynomials have been proven to be very useful in order to systematically construct polynomial PH curves. All spatial polynomial PH curves can be obtained by taking an arbitrary quaternion valued polynomial $\mathcal{A}(t)$ and a real polynomial $\lambda(t)$ and defining

$$\mathbf{r}(t) = \int \lambda(t)\mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t)\,\mathrm{d}t, \tag{4}$$

(Farouki, 2008). In principle all rational PH curves can be obtained in the same way by allowing $\lambda(t)$ to be rational (Farouki and Šír, 2011, 2020; Farouki and Sakkalis, 2019). The integral (4) however does not need to produce a rational curve and only a (linear) subset of rational functions $\lambda(t)$ may be used. A complete characterization of this subset seems to be a very difficult problem. Note, however, that the linear occurrence of $\lambda(t)$ in (4) indicates that, given two rational PH curves $\tilde{\mathbf{r}}(t)$

and $\hat{\mathbf{r}}(t)$ with the same indicatrix $\mathbf{T}(t)$, their sum $\mathbf{r}(t) = \tilde{\mathbf{r}}(t) + \hat{\mathbf{r}}(t)$ has the same indicatrix and is therefore PH as well. Also for any constant real $c \in \mathbb{R}$ and any constant vector $\mathbf{d} \in \mathbb{R}^3$ the scaled and translated curve $c\mathbf{r}(t) + \mathbf{d}$ has the same indicatrix.

An explicit construction of a general rational PH curve bypassing the integration issue was given in (Farouki and Šír, 2011; Kozak et al., 2014; Farouki and Šír, 2020). Consider any quaternion valued polynomial $\mathcal{A}(t)$ and define the vector fields $\mathbf{v}(t) = \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t)$ and $\mathbf{u}(t) = \mathbf{v}(t) \times \mathbf{v}'(t)$. While $\mathbf{v}(t)$ points in the tangent direction of the curve (it is a multiple of $\mathbf{T}(t)$) the field $\mathbf{u}(t)$ is perpendicular to the osculation plane (it is a multiple of the binormal vector). Then the osculation plane will have the implicit equation $\mathbf{u}(t) \cdot (x_1, x_2, x_3) = f(t)$ where $f(t)$ is an arbitrary rational function. The curve parameterization in terms of $\mathbf{u}(t)$ and $f(t)$ has the form

$$\mathbf{r}(t) = \frac{f(t)\,\mathbf{u}'(t) \times \mathbf{u}''(t) + f'(t)\,\mathbf{u}''(t) \times \mathbf{u}(t) + f''(t)\,\mathbf{u}(t) \times \mathbf{u}'(t)}{\det[\,\mathbf{u}(t), \mathbf{u}'(t), \mathbf{u}''(t)\,]}. \tag{5}$$

Note that for a fixed $A(t)$ the correspondence between $f(t)$ and $\mathbf{r}(t)$ is linear. While this formula is quite elegant and provides a full description of all the rational PH curves, it is difficult to have an insight in the possible cancellations between the numerator and denominator. This is due to the fact that the final denominator of $\mathbf{r}(t)$ comes both from $\det[\,\mathbf{u}(t), \mathbf{u}'(t), \mathbf{u}''(t)\,]$ and from the denominator of $f(t)$ and its derivatives. In particular it is difficult to understand for which $f(t)$ polynomial PH curves occur (as subset of the rational cases).

3. Rational PH curves and framing motions

We are going to study PH curves as trajectories of rational framing motions and we will use the formalism of dual quaternions introduced in the previous section for that purpose. For simplicity of notation we will usually omit the indeterminate t from now on, e.g. we write $\mathbf{r} = \mathbf{r}(t)$ or $\mathcal{A} = \mathcal{A}(t)$.

Lemma 3.1. *Given a rational parametric curve*

$$\mathbf{r} = \frac{1}{\alpha}(r_1, r_2, r_3), \quad \alpha, r_1, r_2, r_3 \in \mathbb{R}[t],$$

any rational motion such that the trajectory of the origin equals $\mathbf{r}$ *is of the form*

$$\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A} \tag{6}$$

where $\mathcal{A} \in \mathbb{H}[t] \setminus \{0\}$ *is arbitrary and* $\mathbf{b} \in \mathbb{H}[t]$ *is of the shape* $\mathbf{b} = -\frac{1}{2}(r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k})$.

Proof. By (3), the translation along $\mathbf{r}$ is given by the motion polynomial $\alpha + \varepsilon\mathbf{b}$ where $\mathbf{b}$ is defined as in the Lemma's statement. For $x = 1$ and $\mathbf{b} = 0$, (3) boils down to 1. Since the origin is embedded into $\mathbb{DH}$ as 1, we see that A fixes the origin and the motion polynomial $\mathcal{C}$ as defined above indeed moves it along $\mathbf{r} = -2\alpha^{-1}\mathbf{b}$.

Conversely, consider a motion polynomial $\mathcal{C} = \mathcal{P} + \varepsilon\mathcal{D} \in \mathbb{DH}[t]$ and assume that $\mathbf{r} = \alpha^{-1}(r_1, r_2, r_3)$ is the trajectory of the origin. The trajectory of the origin under the motion $\mathcal{C}$ is $(\mathcal{P}\mathcal{D}^* - \mathcal{D}\mathcal{P}^*)/(\mathcal{P}\mathcal{P}^*)$ which, by the Study condition $\mathcal{P}\mathcal{D}^* + \mathcal{D}\mathcal{P}^* = 0$, equals $-2\mathcal{D}\mathcal{P}^*/(\mathcal{P}\mathcal{P}^*)$. From this we infer that α divides $\mathcal{P}\mathcal{P}^*$, i.e., $\mathcal{P}\mathcal{P}^* = \lambda\alpha$ with $\lambda \in \mathbb{R}[t]$, whence $\lambda\mathbf{b} = \mathcal{D}\mathcal{P}^*$. Defining $\mathcal{A} := \mathcal{P}$ we have

$$(\alpha + \varepsilon\mathbf{b})\mathcal{A} = \alpha\mathcal{P} + \varepsilon\mathbf{b}\mathcal{P} = \alpha\mathcal{P} + \varepsilon\tfrac{1}{\lambda}D\underbrace{\mathcal{P}^*\mathcal{P}}_{=\lambda\alpha} = \alpha(\mathcal{P} + \varepsilon\mathcal{D}) = \alpha\mathcal{C}$$

so that the two motion polynomials $\mathcal{C}$ and $(\alpha + \varepsilon\mathbf{b})\mathcal{A}$ are indeed equal up to a real polynomial factor and therefore represent the same motion. □

Note that we can assume that $\mathcal{A}$ is reduced, that is, it does not have a real polynomial factor of positive degree. If common real factors of α and $\mathbf{b}\mathcal{A}$ appear, they can be divided off from $\mathcal{C}$ in order to obtain a motion polynomial of lower degree which looks different from (6). Nonetheless, (6) is the general way of writing rational motions to draw the parametric rational curve $\mathbf{r}$ as trajectory of the origin. We will use precisely this form to describe all the rational PH curves.

Lemma 3.2. *The polynomial* $\mathcal{C} = (\alpha + \varepsilon\mathcal{B})\mathcal{A}$ *with* $\alpha \in \mathbb{R}[t]$ *and* $\mathcal{A}, \mathcal{B} \in \mathbb{H}[t]$ *satisfies the Study condition if and only if* $\mathcal{B}$ *is vectorial (and will be denoted* $\mathbf{b}$*).*

Proof. The Study condition boils down to $0 = \alpha\mathcal{A}(\mathcal{B}\mathcal{A})^* + \mathcal{B}\mathcal{A}(\alpha\mathcal{A})^* = \alpha\mathcal{A}\mathcal{A}^*(\mathcal{B} + \mathcal{B}^*)$. This implies $\mathcal{B} + \mathcal{B}^* = 0$, that is $\mathcal{B}$ is vectorial. □

Definition 3.3. A motion polynomial $\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ (or the corresponding rational motion) is called *framing* if the image $\mathcal{A}\mathbf{i}\mathcal{A}^*$ of the vector $\mathbf{i}$ is tangent to the trajectory $\mathbf{r}$ of the origin.

Before we elucidate the relation of the framing motions to the PH curves let us give one more useful definition.

Definition 3.4. We call the polynomial $\mathcal{A} \in \mathbb{H}[t]$ *reduced with respect to* $\mathbf{i} \in \mathbb{H}$ if it is free of real factors of positive degree and of polynomial right factors with coefficients in the sub-algebra of $\mathbb{H}$ which is generated by 1 and $\mathbf{i}$.

Theorem 3.5. *If $\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ is a framing motion then the trajectory $\mathbf{r}$ of the origin is a PH curve. Conversely, if $\mathbf{r}$ is a PH curve, there exists a framing motion $\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ of $\mathbf{r}$ where $\mathcal{A}$ is reduced with respect to $\mathbf{i}$.*

Proof. By Definition 3.3 The motion C is framing if and only if $\mathbf{r}'$ is linearly dependent to $\mathcal{A}\mathbf{i}\mathcal{A}^*$. This implies

$$\frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \pm\frac{\mathcal{A}\mathbf{i}\mathcal{A}^*}{\|\mathcal{A}\mathbf{i}\mathcal{A}^*\|} = \pm\frac{\mathcal{A}\mathbf{i}\mathcal{A}^*}{\mathcal{A}\mathcal{A}^*}.$$

From this we see that the speed function σ of Definition 2.1 is rational and $\mathbf{r}$ is indeed a PH curve.

Assume now conversely that $\mathbf{r} = \alpha^{-1}(r_1, r_2, r_3)$ is a rational PH curve and set $\mathbf{b} = -\frac{1}{2}(r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k})$. The PH-property implies that the normalized hodograph $\mathbf{r}'/\|\mathbf{r}'\|$ is a rational spherical curve. It is well-known (cf. for example the spherical specialization of Li et al. (2016, Theorem 2)) that there exists $\mathcal{A} \in \mathbb{H}[t]$ such that $\mathbf{r}'/\|\mathbf{r}'\| = \mathcal{A}\mathbf{i}\mathcal{A}^*/(\mathcal{A}\mathcal{A}^*)$. Moreover there exist an $\mathcal{A}$ which is reduced with respect to $\mathbf{i}$ and has this property. Indeed if $\mathcal{R} \in \mathbb{H}[t]$ has coefficients in the sub-algebra generated by 1 and $\mathbf{i}$ and is of maximal degree such that $\tilde{\mathcal{A}} = \mathcal{A}\mathcal{R}$ then $\mathcal{A}\mathbf{i}\mathcal{A}^*/(\mathcal{A}\mathcal{A}^*) = \tilde{\mathcal{A}}\mathbf{i}\tilde{\mathcal{A}}^*/(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^*)$. Now $\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ is the sought framing motion. □

The previous theorem allows us to formulate three equivalent systems of linear equations characterizing rational PH curves.

Theorem 3.6. *Rational PH curves are precisely the trajectories of the origin under the rational motions $\mathcal{C} = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ where $\mathcal{A}$ is reduced with respect to $\mathbf{i}$ and one of the following equivalent conditions is satisfied:*

1. $(\alpha\mathbf{b}' - \alpha'\mathbf{b}) \times \mathcal{A}\mathbf{i}\mathcal{A}^* = 0$ (7)
2. $\langle\alpha\mathbf{b}' - \alpha'\mathbf{b}, \mathcal{A}\mathbf{j}\mathcal{A}^*\rangle = \langle\alpha\mathbf{b}' - \alpha'\mathbf{b}, \mathcal{A}\mathbf{k}\mathcal{A}^*\rangle = 0$ (8)
3. *There exists $\mu \in \mathbb{R}[t]$ such that $\alpha\mathbf{b}' - \alpha'\mathbf{b} = \mu\mathcal{A}\mathbf{i}\mathcal{A}^*$.* (9)

Proof. Due to Theorem 3.5 we only need to show that the three conditions are equivalent with $\mathcal{C}$ being framing. We have $\mathbf{r} = -2\alpha^{-1}\mathbf{b}$ implying $\mathbf{r}' = -2\alpha^{-2}(\alpha\mathbf{b}' - \alpha'\mathbf{b})$. Therefore $\mathcal{C}$ is framing if and only if $\alpha\mathbf{b}' - \alpha'\mathbf{b}$ and $\mathcal{A}\mathbf{i}\mathcal{A}^*$ are linearly dependent over the field of rational functions with real coefficients. Now, (7) expresses this linear dependency using the cross product. (8) express the same property via orthogonality to the orthogonal complement $\operatorname{span}(\mathcal{A}\mathbf{i}\mathcal{A}^*)^\perp = \operatorname{span}(\mathcal{A}\mathbf{j}\mathcal{A}^*, \mathcal{A}\mathbf{k}\mathcal{A}^*)$.

In order to obtain (9) we express the linear dependence by existence of prime polynomials μ, $\nu \in \mathbb{R}[t]$ such that

$$\alpha\mathbf{b}' - \alpha'\mathbf{b} = \frac{\mu}{\nu}\mathcal{A}\mathbf{i}\mathcal{A}^*.$$

Because the left-hand side of this equation is polynomial, ν must be a factor of $\mathcal{A}\mathbf{i}\mathcal{A}^\star$. We will show that the assumption that $\mathcal{A}$ is reduced with respect to $\mathbf{i}$ implies that $\mathcal{A}\mathbf{i}\mathcal{A}^\star$ has no non-constant factors and therefore we have $\nu = 1$.

Let us assume that ψ is the unique monic real factor of maximal degree of $\mathcal{A}\mathbf{i}\mathcal{A}^*$. By a simple but useful auxiliary result (Proposition 2.1 of Cheng and Sakkalis (2016) or Lemma 1 of Li et al. (2016)) and because $\mathcal{A}$ is assumed to have no real polynomial factor of positive degree, a positive degree of ψ is only possible if there exists a linear quaternionic right factor $t - \mathcal{H}$ of $\mathcal{A}$ such that $t - \mathcal{H}^*$ is a left factor of $\mathbf{i}\mathcal{A}^*$. In this case, $(t - \mathcal{H})(t - \mathcal{H}^*)$ is a real quadratic factor of ψ.

The linear right factor $t - \mathcal{H}$ must be of a rather special shape. With $\mathcal{H} = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$ we find

$$\mathbf{i}(t - \mathcal{H}^*) = \mathbf{i}(t - (h_0 - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k})) = (t - (h_0 - h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}))\mathbf{i}$$

because t, h_0, and $\mathbf{i}$ commute with $\mathbf{i}$ and $\mathbf{ij} = -\mathbf{ji}$, $\mathbf{ik} = -\mathbf{ki}$. This implies that $t - (h_0 - h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k})$ is a linear left factor of $\mathbf{i}\mathcal{A}^*$. By the factorization theory of quaternionic polynomials (Hegedüs et al., 2013; Li et al., 2019), linear left factors with a given norm polynomial are unique whence

$$h_0 - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k} = h_0 - h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$$

and consequently $h_2 = h_3 = 0$ – a contradiction to the assumption that $\mathcal{A}$ is reduced with respect to $\mathbf{i}$. □

These equations and the relation between framing motions and PH curves stated in Theorems 3.5, 3.6 allow a computational approach to rational PH curves. The basic idea is to prescribe $\alpha \in \mathbb{R}[t]$ and $\mathcal{A} \in \mathbb{H}[t]$ and to compute the unknown coefficients of the polynomial $\mathbf{b} \in \mathbb{H}[t]$ (subject to the constraint $\mathbf{b} + \mathbf{b}^* = 0$) from the linear systems obtained by comparing coefficients of t in (7), (8) or (9). In most cases, the system obtained from (8) is to be preferred because it is the smallest, and the system obtained from (9) is to be avoided because it introduces auxiliary variables (the coefficients of μ) that are not needed. However, for a more detailed analysis in Section 4 we will prefer system (9) which we found to provide the best general insight in solvability issues.

4. Existence of non-polynomial solutions

Given $A \in \mathbb{H}[t]$, reduced with respect to $\mathbf{i}$, and $\alpha \in \mathbb{R}[t]$ we discuss existence of polynomial solutions $\mathbf{b}$ to one of the equivalent equation systems of Theorem 3.6. Special emphasis is put on solutions for $\mathbf{b}$ that lead to non-polynomial (truly rational) PH curves $\mathbf{r} = -2\alpha^{-1}\mathbf{b}$. Our main results are summarized in Theorem 4.6 below.

It will make sense to consider also "trivial" solutions. They are characterized by constant $\mathbf{r}$ whence $\mathbf{r}' = 0$ which, of course, fulfills the PH condition. Trivial solutions have no direct relevance in applications. Nonetheless, the corresponding polynomials $\mathbf{b} \in \mathbb{H}[t]$ are elements of the solution space and can serve as elements of a basis, emphasizing the translation invariance of PH curves.

In case of the well-known polynomial PH curves, solving the systems of Theorem 3.6 can be circumvented by directly integrating $\mathbf{r}' = \lambda \mathcal{A}\mathbf{i}\mathcal{A}^*$ with $\lambda \in \mathbb{R}[t]$, cf. (4). For $\lambda = 0$ we recover trivial solutions (which are polynomial as well), for $\lambda \neq 0$ we will speak of "non-trivial polynomial solutions".

Lemma 4.1. *A solution to the systems of Theorem 3.6 gives rise to a polynomial PH curve if and only if $\alpha \in \mathbb{R}[t]$ is a factor of $\mathbf{b}$. Trivial solutions are obtained precisely for $\mathbf{b} = \alpha\mathbf{b}_0$ where $\mathbf{b}_0 \in \mathbb{H}$ is constant and vectorial.*

Proof. From $\mathbf{b} = \alpha\tilde{\mathbf{b}}$ with $\tilde{\mathbf{b}} \in \mathbb{H}[t]$ we obtain $\mathbf{r} = -2\alpha^{-1}\mathbf{b} = -2\tilde{\mathbf{b}}$ and the curve is indeed polynomial. Conversely, if $\mathbf{r} = -2\alpha^{-1}\mathbf{b}$ is polynomial, then clearly α is a factor of $\mathbf{b}$. The statement on the trivial solutions is trivial. □

Since polynomial PH curves are well-understood, we are mostly interested in truly rational (non-polynomial) solutions. Moreover, it is not our immediate aim to compute solutions of (7), (8), or (9) but to analyze existence and type of solutions. Questions about dimension and basis of the solution space will not be formally treated in this paper but some observations can be found in Section 5.

The system (9) is well-suited for studying solvability and in particular the type of solutions (trivial, polynomial, or non-polynomial). Rather surprisingly, it turns out, that the desired rational solutions only occur in exceptional cases. We proceed by analyzing in detail the system of linear equations (9). In particular we will fully decide for which inputs A and α there exist solutions $\mathbf{b}$ for which the resulting PH curve is truly rational (non-polynomial).

Our main technical tool consists in fixing a monic linear polynomial $\beta \in \mathbb{C}[t]$ and considering (9) in the polynomial basis $(1, \beta, \beta^2, \beta^3, \ldots)$. As we will see, in the interesting cases β will be a factor of α. For this reason we consider the complex extension in order to handle the non-real roots of α as well. If $\beta \in \mathbb{R}[t]$ is real, all the coefficients introduced below are real as well. Otherwise, the coefficients are complex but a simple additional argument in the proof of Theorem 4.6 will guarantee existence of real solutions.

Suppose, that β is a factor of α of multiplicity $n \geq 0$. We express the polynomials $\mathbf{b}$, $\mathcal{A}\mathbf{i}\mathcal{A}^*$, μ and $\gamma := \alpha/\beta^n$ in the basis $(1, \beta, \beta^2, \beta^3, \ldots)$:

$$\mathbf{b} = \sum_i \beta^i \mathbf{b}_i, \quad \mathcal{A}\mathbf{i}\mathcal{A}^* = \sum_i \beta^i \mathbf{f}_i, \quad \mu = \sum_i \beta^i \mu_i, \quad \gamma = \sum_i \beta^i \gamma_i.$$

The vectorial coefficients $\mathbf{f}_i$ and γ_i are determined by the input data. Note that $\mathbf{f}_0 \neq 0$ because $\mathcal{A}$ is reduced with respect to $\mathbf{i}$ and $\gamma_0 \neq 0$ because n is the multiplicity of β as factor of α. The unknowns are b_i which, by Lemma 3.2, are vectorial quaternions and the scalars μ_i. We are looking for solution polynomials, whence only finitely many of the coefficients b_i, μ_i can be different from zero. However, at this point, we do not wish to bound their respective degrees. Thus, we do not restrict the range of the summation variable $i \in \mathbb{N}_0$. With regard to (9), we compute the expressions $\alpha\mathbf{b}' - \alpha'\mathbf{b}$ and $\mu\mathcal{A}\mathbf{i}\mathcal{A}^*$:

$$\begin{aligned}
\alpha\mathbf{b}' - \alpha'\mathbf{b} &= \beta^n\gamma\mathbf{b}' - \beta^n\gamma'\mathbf{b} - n\beta^{n-1}\gamma\mathbf{b} \\
&= \beta^n(\sum_i \beta^i\gamma_i)(\sum_i i\beta^{i-1}\mathbf{b}_i) - \beta^n(\sum_i i\beta^{i-1}\gamma_i)(\sum_i \beta^i\mathbf{b}_i) - n\beta^{n-1}(\sum_i \beta^i\gamma_i)(\sum_i \beta^i\mathbf{b}_i) \\
&= \sum_{i,j} \beta^{i+j+n-1}(j - i - n)\gamma_i\mathbf{b}_j \\
\mu\mathcal{A}\mathbf{i}\mathcal{A}^* &= (\sum_i \beta^i\mu_i)(\sum_i \beta^i\mathbf{f}_i) = \sum_{i,j} \beta^{i+j}\mu_i\mathbf{f}_j.
\end{aligned}$$

Comparing coefficients in this basis, system (9) becomes

$$\sum_{i+j=k-n}(j-i-n)\gamma_i\mathbf{b}_j=\sum_{i=0}^{k-1}\mu_i\mathbf{f}_{k-i-1},\quad k\geq 1. \tag{10}$$

This is a system of homogeneous linear equations with unknowns $\mathbf{b}_i$ and μ_i. In the following lemmas, we will exploit the particular (triangular and symmetric) form of (10) to analyze the solution space depending on n. Let us observe that in all cases $\mathbf{f}_0\neq 0$ by the assumption that $\mathcal{A}$ is reduced with respect to $\mathbf{i}$ and $\gamma_0\neq 0$ as otherwise the multiplicity of β as factor of α would be larger than n. Our main purpose is to discuss existence or non-existence of solutions for $\mathbf{b}$ having β as a factor with certain multiplicity. These can be very simply identified via the annulation of initial coefficients: By Lemma 4.1, solutions with $(\mathbf{b}_0,\mathbf{b}_1,\dots,\mathbf{b}_{n-1})\neq(0,0,\dots,0)$ give rise to non-polynomial PH curves.

Lemma 4.2. *For given $\alpha\in\mathbb{R}[t]$ and $\mathcal{A}\in\mathbb{H}[t]$, reduced with respect to $\mathbf{i}$, consider a monic linear factor $\beta\in\mathbb{C}[t]$ of α of multiplicity $n=1$. There exists a polynomial solution μ, $\mathbf{b}$ of (9) such that β does not divide $\mathbf{b}$ if and only if $\{\mathbf{f}_0,\mathbf{f}_1\}$ are linearly dependent.*

Proof. For $n=1$ the system (10) boils down to

$$\begin{aligned}
-1\gamma_0\mathbf{b}_0&=\mu_0\mathbf{f}_0,\\
0\gamma_0\mathbf{b}_1-2\gamma_1\mathbf{b}_0&=\mu_0\mathbf{f}_1+\mu_1\mathbf{f}_0,\\
1\gamma_0\mathbf{b}_2-1\gamma_1\mathbf{b}_1-3\gamma_2\mathbf{b}_0&=\mu_0\mathbf{f}_2+\mu_1\mathbf{f}_1+\mu_2\mathbf{f}_0,\\
2\gamma_0\mathbf{b}_3+0\gamma_1\mathbf{b}_2-2\gamma_2\mathbf{b}_1-4\gamma_3\mathbf{b}_0&=\mu_0\mathbf{f}_3+\mu_1\mathbf{f}_2+\mu_2\mathbf{f}_1+\mu_3\mathbf{f}_0,\\
&\ \ \vdots
\end{aligned} \tag{11}$$

Contrary to common convention, we do not omit the coefficients 0 and 1 on the left-hand side in order to highlight the system's regular structure.

If $\{\mathbf{f}_0,\mathbf{f}_1\}$ is linearly dependent, we construct a solution with $\mathbf{b}_0\neq 0$. The first vectorial equation is satisfied setting $\mathbf{b}_0=-\mu_0\gamma_0^{-1}\mathbf{f}_0$. Plugging this into the second equation yields

$$\mu_0\mathbf{f}_1+(\mu_1-2\gamma_1\mu_0\gamma_0^{-1})\mathbf{f}_0=0. \tag{12}$$

Since $\mathbf{f}_0$ and $\mathbf{f}_1$ are linearly dependent and $\mathbf{f}_0\neq 0$, there exists $\varphi\in\mathbb{R}$ so that $\mathbf{f}_1+\varphi\mathbf{f}_0=0$. We set $\mu_0=1$ and $\mu_1=\varphi+2\gamma_1\gamma_0^{-1}\varphi_1$ so that the first and second equation are satisfied.

We still need to argue that there exists a solution for the remaining variables $\mathbf{b}_1$, $\mathbf{b}_2$, … and μ_2, μ_3, … with only finitely many of the vectorial quaternions $\mathbf{b}_i$ different from zero.

The further equations can be satisfied, because $\mathbf{b}_2,\mathbf{b}_3,\dots$ appear for the first time one by one and with the nonzero coefficient γ_0. We may therefore arbitrarily assign $\mathbf{b}_1$, μ_2, μ_3, …, and solve the remaining vectorial equations successively for $\mathbf{b}_2$, $\mathbf{b}_3$, … In particular, we are free to select only finitely many μ_i different from zero. Then there exists $M\in\mathbb{N}$ such that $\mu_m=\gamma_m=\mathbf{f}_m=0$ for all $m\geq M$. This implies existence of a solution where only finitely many of the vectorial quaternions $\mathbf{b}_i$ are different from zero as well. By construction, in this solution $\mathbf{b}_0=-\gamma_0^{-1}\mathbf{f}_0\neq 0$ and thus, β does not divide $\mathbf{b}$.

If the set $\{\mathbf{f}_0,\mathbf{f}_1\}$ is linearly independent, (12) implies $\mu_0=\mu_1=0$ whence $\mathbf{b}_0=0$ and β divides all possible solutions for $\mathbf{b}$. The argument for existence of polynomial solutions is the same as above. □

Lemma 4.3. *For given $\alpha\in\mathbb{R}[t]$ and $\mathcal{A}\in\mathbb{H}[t]$, reduced with respect to $\mathbf{i}$, consider a monic linear factor $\beta\in\mathbb{C}[t]$ of α of multiplicity $n=2$. There exists a polynomial solution μ, $\mathbf{b}$ of (9) such that β^2 does not divide $\mathbf{b}$ if and only if $\{\mathbf{f}_0,\mathbf{f}_1,\mathbf{f}_2\}$ are linearly dependent.*

Proof. For $n=2$ the system (10) reads as

$$\begin{aligned}
0&=\mu_0\mathbf{f}_0,\\
-2\gamma_0\mathbf{b}_0&=\mu_0\mathbf{f}_1+\mu_1\mathbf{f}_0,\\
-1\gamma_0\mathbf{b}_1-3\gamma_1\mathbf{b}_0&=\mu_0\mathbf{f}_2+\mu_1\mathbf{f}_1+\mu_2\mathbf{f}_0,\\
0\gamma_0\mathbf{b}_2-2\gamma_1\mathbf{b}_1-4\gamma_2\mathbf{b}_0&=\mu_0\mathbf{f}_3+\mu_1\mathbf{f}_2+\mu_2\mathbf{f}_1+\mu_3\mathbf{f}_0,\\
1\gamma_0\mathbf{b}_3-1\gamma_1\mathbf{b}_2-3\gamma_2\mathbf{b}_1-5\gamma_3\mathbf{b}_0&=\mu_0\mathbf{f}_4+\mu_1\mathbf{f}_3+\mu_2\mathbf{f}_2+\mu_3\mathbf{f}_1+\mu_4\mathbf{f}_0,\\
&\ \ \vdots
\end{aligned} \tag{13}$$

Since $\mathbf{f}_0\neq 0$, the first equation implies $\mu_0=0$ canceling the first terms of all right-hand sides. Solving the second and the third equation for $\mathbf{b}_0$ and $\mathbf{b}_1$ (recall that $\gamma_0\neq 0$) and plugging the result into the fourth equation yields a relation

$$\mu_1\mathbf{f}_2+(\mu_2+\varrho_{11}\mu_1)\mathbf{f}_1+(\mu_3+\varrho_{20}\mu_2+\varrho_{10}\mu_1)\mathbf{f}_0=0 \tag{14}$$

with some coefficients ϱ_{ij} which could be expressed explicitly but there is no need for their particular form.

If $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$ is linearly independent we infer $\mu_1 = \mu_2 = \mu_3 = 0$ whence $\mathbf{b}_0 = \mathbf{b}_1 = 0$ and β^2 necessarily divides all possible solutions $\mathbf{b}$.

If, on the other hand, $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$ is linearly dependent, there exists a triple $(\varphi_0, \varphi_1, \varphi_2) \neq (0,0,0)$ of complex numbers such that $\varphi_0\mathbf{f}_0 + \varphi_1\mathbf{f}_1 + \varphi_2\mathbf{f}_2 = 0$. With $\mu_1 = \varphi_2$, $\mu_2 = \varphi_1 - \varrho_{11}\mu_1$, and $\mu_3 = \varphi_0 - \varrho_{20}\mu_2 - \varrho_{10}\mu_1$, the fourth equation is fulfilled and $(\mu_1, \mu_2, \mu_3) \neq (0,0,0)$. The corresponding solutions for $\mathbf{b}_0$, $\mathbf{b}_1$ can not be simultaneously zero because in such case the left-hand side of the first four equations would be zero, forcing $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$. Solutions for $\mathbf{b}_2$, $\mathbf{b}_3$, … are constructed in the same way as in the proof of Lemma 4.2. More precisely, we may arbitrarily assign $\mathbf{b}_2$, μ_4, μ_5, …, and solve the remaining vectorial equations successively for $\mathbf{b}_3$, $\mathbf{b}_4$, … If only finitely many of μ_4, μ_5, … are different from zero, a solution with only finitely many $\mathbf{b}_2$, $\mathbf{b}_3$, … different from zero exists. In this solution, β^2 does not divide $\mathbf{b}$. □

Lemma 4.4. *For given $\alpha \in \mathbb{R}[t]$ and $\mathcal{A} \in \mathbb{H}[t]$, reduced with respect to $\mathbf{i}$, consider a monic linear factor $\beta \in \mathbb{C}[t]$ of α of multiplicity $n \geq 3$. Then there always exists a polynomial solution μ, $\mathbf{b}$ of (9) such that β^n does not divide $\mathbf{b}$.*

Proof. Let us describe the system (10) for a general $n \geq 3$. The right-hand sides of all equations have a very regular triangular structure. In the i-th equation one new variable μ_{i+1} is introduced with the non-zero coefficient $\mathbf{f}_0$.

The left hand sides are slightly more complicated. The left-hand side of the first $n-1$ equations equals 0. In the n subsequent equations n variables $\mathbf{b}_0$, $\mathbf{b}_1$, …, $\mathbf{b}_{n-1}$ are introduced one by one with non-zero coefficients which is an integer multiple of γ_0. The next equation does not introduce any new variable on the left-hand side. All the remaining equations have a very regular left-hand side introducing always one new variable $\mathbf{b}_{n+i}$ with non-zero coefficient in the $(2n+i)$-th equation.

Based on this general structure we can always construct a non-polynomial solution curve in the following way.

1. Satisfy the equations number $1, \ldots, (n-1)$ by setting $\mu_0 = \mu_1 = \cdots = \mu_{n-2} = 0$.
2. Satisfy the equations number $n, \ldots, (2n-1)$ by expressing the unknowns $\mathbf{b}_0$, $\mathbf{b}_1$, …, $\mathbf{b}_{n-1}$ uniquely in terms of the input coefficients ($\mathbf{f}_i$, γ_i) and the unknowns μ_{n-1}, μ_n, …, μ_{2n-2} (which remain free).
3. Substitute the previous expressions to the equation number $2n$ to obtain one homogeneous vector equation in the $(n+1)$ scalar unknowns μ_{n-1}, μ_n, …, μ_{2n-1}. Set these unknowns to a non-trivial solution which must exist because of $n \geq 3$.
4. Set $\mathbf{b}_n$ and μ_{2n}, μ_{2n+1}, …arbitrarily and solve the remaining equations uniquely for $\mathbf{b}_{n+1}$, $\mathbf{b}_{n+2}$, … If only finitely many of μ_{2n}, μ_{2n+1}, …are different from zero, a solution with only finitely many of $\mathbf{b}_{n+1}$, $\mathbf{b}_{n+2}$, …different from zero does exist.

We claim that in this solution $\mathbf{b}$ does not have β^n as a factor. Indeed, if all $\mathbf{b}_0$, $\mathbf{b}_1$, …, $\mathbf{b}_{n-1}$ are simultaneously zero then $\mu_{n-1} = \mu_n = \cdots = \mu_{2n-1} = 0$ – via equations $n, \ldots, 2n$ – contradicting the non-triviality of the solution in Step 3. □

Before combining Lemmas 4.2–4.4 into the central result of this article, we provide a simple observation that allows us to infer existence of real solutions even if these lemmas allow for $\beta \in \mathbb{C}[t]$.

Lemma 4.5. *If b is a rational function over $\mathbb{C}$ but not polynomial, then either its real part $\frac{1}{2}(b+\overline{b})$ or its imaginary part $\frac{1}{2}(b-\overline{b})$ is non-polynomial as well. (The bar denotes complex conjugation.)*

Proof. If both, real and imaginary part, were polynomial, then so is b. □

Theorem 4.6. *For given $\alpha \in \mathbb{R}[t]$ and $\mathcal{A} \in \mathbb{H}[t]$, reduced with respect to $\mathbf{i}$ the following two conditions are equivalent:*

1. *There exist a polynomial $\mathbf{b} \in \mathbb{H}[t]$ so that $C = (\alpha + \varepsilon\mathbf{b})\mathcal{A}$ is a motion polynomial and the trajectory of the origin is a rational non-polynomial PH curve.*
2. *There exists a (real or complex) root z of α of multiplicity n such that the set $\{\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$ with coefficients $\mathbf{f}_i$ defined by $\mathcal{A}\mathbf{i}\mathcal{A}^* = \sum_i (t-z)^i\mathbf{f}_i$ is linearly dependent. Note that the linear dependence condition is always satisfied if $n \geq 3$.*

Proof. For a real zero z the theorem's statement follows from Theorem 3.6 and Lemmas 4.2–4.4. In case of $z \in \mathbb{C} \setminus \mathbb{R}$, the complex conjugate $\overline{z}$ is also a zero of multiplicity n of α and the solutions $\mathbf{b}$ and $\overline{\mathbf{b}}$ of (9) to z and $\overline{z}$, respectively, are interchanged by complex conjugation. Their linear combinations $\mathbf{b} + \overline{\mathbf{b}}$ and $\mathbf{b} - \overline{\mathbf{b}}$ provide solutions for (7) or (8) as well. They are real and, by Lemma 4.5, at least one of them leads to a non-polynomial PH curve. □

Remark 4.7. Theorem 4.6 talks only about existence of non-polynomial solution curves but does not exclude polynomial solutions. In fact, trivial solutions always exist and non-trivial polynomial solutions exist if the degree of $\mathbf{b}$ is high enough. Lemma 4.1 ensures polynomiality if α is a factor of $\mathbf{b}$. This condition can be encoded by linear homogeneous constraints on the coefficients of $\mathbf{b}$ that can be added, for example, to the system (8). This is an alternative to the usual methods of computing polynomial PH curves.

Theorem 4.6 provides necessary and sufficient conditions for the existence of non-polynomial solutions to data α, $\mathcal{A}$ but does not guarantee that a particular set of conditions can actually be met by suitable α and $\mathcal{A}$. This is not logically needed for the correctness of Theorem 4.6 but, nonetheless, should be clarified. For the construction of non-polynomial solutions it is also of interest to understand the geometry of the exceptional cases of Lemma 4.2 and Lemma 4.3.

The vector valued polynomial function $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^*$ can be interpreted as parametric equation of a curve in the projective plane $\mathbb{P}^2(\mathbb{R})$. We may consider it as the projection of the PH curve's tangent indicatrix into the plane at infinity from the coordinate origin. With this understanding, conditions on linear dependence in Lemmas 4.2 and 4.3, respectively, allow a geometric interpretation. Denote by z the unique (real or complex) zero of the linear factor β of α. Then the vectors $\mathbf{F}(z)$, $\mathbf{F}'(z)$, and $\mathbf{F}''(z)$ are non-zero multiples of $\mathbf{f}_0$, $\mathbf{f}_1$, and $\mathbf{f}_2$, respectively. In other words, linear dependence of $\{\mathbf{f}_0, \mathbf{f}_1\}$ is equivalent to linear dependence of $\{\mathbf{F}(z), \mathbf{F}'(z)\}$ and linear dependence of $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$ is equivalent to linear dependence of $\{\mathbf{F}(z), \mathbf{F}'(z), \mathbf{F}''(z)\}$.

For real z, above linear dependencies generically have the following geometric meaning:

- The set $\{\mathbf{F}(z), \mathbf{F}'(z)\}$ is linearly dependent if the point $\mathbf{F}(z)$ is a cusp of the rational parametric curve $\mathbf{F}$, cf. Bol (1950, p. 2).
- The set $\{\mathbf{F}(z), \mathbf{F}'(z), \mathbf{F}''(z)\}$ is linearly dependent if the point $\mathbf{F}(z)$ is an inflection point of the rational parametric curve $\mathbf{F}$, cf. Bol (1950, p. 4).

The construction of examples with inflection points is straightforward. Starting with a polynomial $\mathcal{A} \in \mathbb{H}[t]$, we define as usual $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^*$ and compute the polynomial $D := \det(\mathbf{F}, \mathbf{F}', \mathbf{F}'')$. The necessary and sufficient condition for linear dependence of $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$ is that β is a linear factor of D. It can easily be fulfilled (cf. Example 5.3).

In order to demonstrate feasibility of the cusp condition, we write $\mathcal{A} = \sum_i \mathcal{A}_i t^i$ with yet undetermined quaternion coefficients $\mathcal{A}_i$ and compute $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^* = \sum_i \mathbf{f}_i t^i$, where $\mathbf{f}_0 = \mathcal{A}_0\mathbf{i}\mathcal{A}_0{}^*$ and $\mathbf{f}_1 = \mathcal{A}_1\mathbf{i}\mathcal{A}_0{}^* + \mathcal{A}_0\mathbf{i}\mathcal{A}_1{}^*$. Solutions to the system of algebraic equations arising from

$$(\mathcal{A}_0\mathbf{i}\mathcal{A}_0{}^*) \times (\mathcal{A}_1\mathbf{i}\mathcal{A}_0{}^* + \mathcal{A}_0\mathbf{i}\mathcal{A}_1{}^*) = 0 \tag{15}$$

will result in a cusp at $t = 0$ (which is as good as any other real parameter value for our purpose). It is no loss of generality to assume $\mathcal{A}_0 = 1$ whence the cusp condition (15) implies that $\mathcal{A}_1$ can be expressed as $\mathcal{A}_1 = a_{10} + a_{11}\mathbf{i}$ with $a_{10}, a_{11} \in \mathbb{R}$. Determining examples with cusps at complex parameter values is more involved. One particular case, taken from Kozak et al. (2014), is provided in Example 5.4 below.

5. Examples and observations

We continue this article by presenting examples, some of them taken from existing literature. The discussion will not only show how the criteria of Theorem 4.6 are met in known examples but also suggest interesting properties of the solution space.

Example 5.1. We consider the polynomials $\mathcal{A} = t^2 - (\mathbf{i} + \mathbf{j})t + \mathbf{k}$ and $\alpha = t^3$ whence necessarily $\beta = t$ and $n = 3$. We are in the case of Lemma 4.4 so that non-polynomial solutions do exist. Solving the system (8) for different degrees of $\mathbf{b}$ we find the following: In case of $\deg \mathbf{b} < 2$, the only solution is $\mathbf{r}_1(t) = (0, 0, 0)^T$. In case of $2 \le \deg \mathbf{b} \le 5$ we obtain the trivial solution $\mathbf{r}_2(t) = (\tau_1, \tau_2, \tau_3)^T = \mathbf{r}_1(t) + (\tau_1, \tau_2, \tau_3)^T$ with constant $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$. For $6 \le \deg \mathbf{b} \le 7$ the solutions space is of dimension four and consists of scalings and translations $\mathbf{r}_6(t) = \mu_6 \mathbf{p}_6(t) + (\tau_1, \tau_2, \tau_3)^T$ of one basis curve

$$\mathbf{p}_6(t) = \frac{1}{t^3}\begin{pmatrix} t^6 + 3t^4 + t^3 + 3t^2 + 1 \\ 12t^4 + t^3 - 12t^2 \\ 3t^5 + t^3 + 3t \end{pmatrix}.$$

Note that non-trivial polynomial PH curves are not among the solutions found thus far as they belong to $\deg \mathbf{b} \ge 8$. In case of $\deg \mathbf{b} = 8$, the solution space is of dimension five, the corresponding rational PH curves are given by $\mathbf{r}_8(t) = \nu_8 \mathbf{p}_8(t) + \mathbf{r}_6(t)$ where $\mathbf{p}_8(t) = \int \mathcal{A}\mathbf{i}\mathcal{A}^* \, \mathrm{d}t$ is the usual quintic polynomial PH curve. The solution space for $\deg \mathbf{b} = 9$ is of dimension six, the corresponding rational PH curves are given by $\mathbf{r}_9(t) = \nu_9 \mathbf{p}_9(t) + \mathbf{r}_8(t)$ where $\mathbf{p}_9(t) = \int t\mathcal{A}\mathbf{i}\mathcal{A}^* \, \mathrm{d}t$.

Remark 5.2. Not unexpectedly, the recursion of Example 5.1 continues as $\mathbf{r}_m(t) = \nu_m \mathbf{p}_m(t) + \mathbf{r}_{m-1}(t)$ where $\mathbf{p}_m(t) = \int t^{m-8}\mathcal{A}\mathbf{i}\mathcal{A}^* \, \mathrm{d}t$. We refrain from formally proving this in this paper. A similar statement for α having several linear factors is a topic of future research. These observations hint at the existence of nested solution spaces with a clear recursive generation.

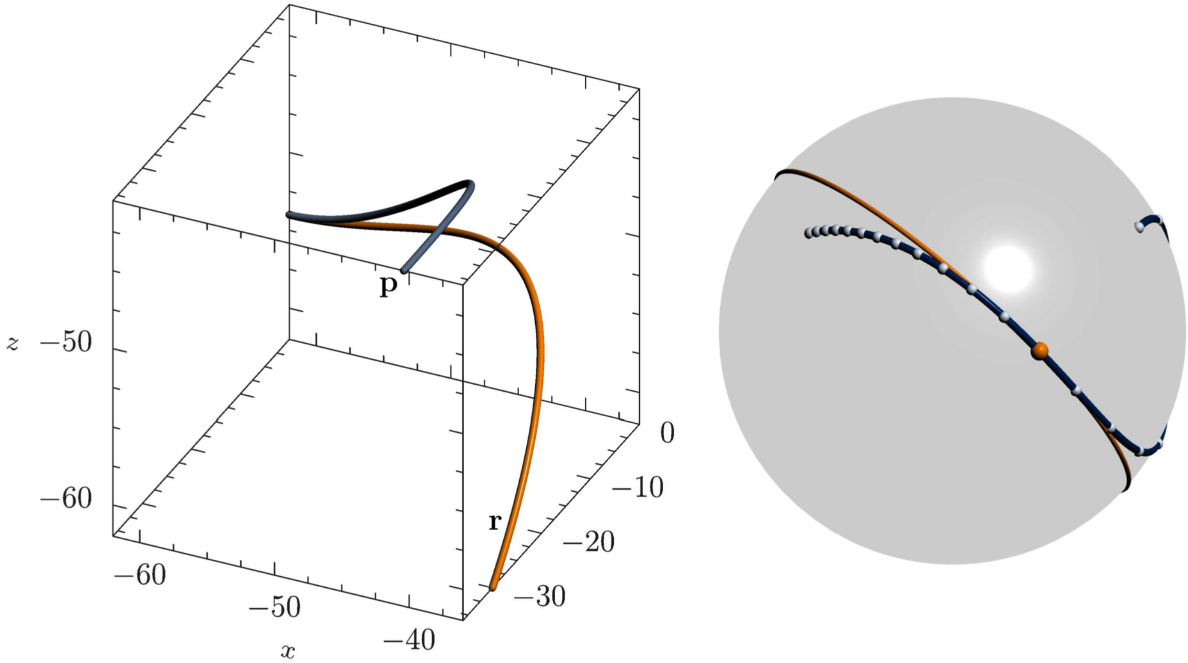

Fig. 1. Rational PH curve $\mathbf{r}$, polynomial PH curve $\mathbf{p}$ (left) and their common spherical tangent indicatrix (right). While the PH curves are parametrized over $[0, 1]$, the parameter interval for the tangent indicatrix is $[-3, 0.5]$ in order to illustrate the inflection point at $t = -1$.

Example 5.3. In this example we pick again $\mathcal{A} = t^2 - (\mathbf{i} + \mathbf{j})t + \mathbf{k}$ but select $\alpha = (t+1)^2$. With $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^*$ and $\beta = t + 1$ we have $n = 2$ and $\mathbf{F} = \mathbf{f}_0 + \mathbf{f}_1\beta + \mathbf{f}_2\beta^2 + \mathbf{f}_3\beta^3 + \mathbf{f}_4\beta^4$ where

$$\mathbf{f}_0 = 4\mathbf{j}, \quad \mathbf{f}_1 = -4\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}, \quad \mathbf{f}_2 = 6\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}, \quad \mathbf{f}_3 = -4\mathbf{i} + 2\mathbf{k}, \quad \mathbf{f}_4 = \mathbf{i}.$$

We observe that $\det(\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2) = 0$ so that we can indeed predict rational solutions by Lemma 4.3. Computing solutions for increasing degree of $\mathbf{b}$ we find $\mathbf{r}_0(t) = \mathbf{r}_1(t) = (0, 0, 0)^T$ for $0 \le \deg \mathbf{b} \le 1$. For $2 \le \deg \mathbf{b} \le 5$ we find the trivial solutions $\mathbf{r}_2(t) = \mathbf{r}_1(t) + (\tau_1, \tau_2, \tau_3) = (\tau_1, \tau_2, \tau_3)$. The first non-trivial solutions occur for $\deg \mathbf{b} = 6$. They are given by $\mathbf{r}_3(t) = \mu_3\mathbf{r}(t) + \mathbf{r}_2(t)$, that is, they are obtained from the prototype solution

$$\mathbf{r}(t) = \frac{1}{(t+1)^2}\begin{pmatrix} -6t^6 - 10t^5 - 2t^4 + 8t^3 - 2t^2 - 70t - 62 \\ -48t^4 - 72t^3 \\ -16t^5 - 26t^4 + 44t^3 - 108t - 54 \end{pmatrix}$$

by scaling and translation. We were not aware of an instance of a rational PH curve satisfying the condition of Lemma 4.3 in prior literature. The rational PH curve $\mathbf{r}(t)$ and a quintic polynomial PH curve $\mathbf{p}(t)$ with identical start point $\mathbf{r}(0) = \mathbf{p}(0)$ are displayed in Fig. 1. The respective end points $\mathbf{r}(1)$ and $\mathbf{p}(1)$ are different and suitable linear combinations of $\mathbf{r}(t)$ and $\mathbf{p}(t)$ would allow us to reach further end points. Adding more rational PH curves with the same tangent indicatrix increases the vector space of solutions and would allow to solve various interpolation problems. The right-hand side in Fig. 1 shows the portion of the tangent indicatrix over the parameter interval $[-3, 0.5]$. As expected, the point to parameter value $t = -1$ is, indeed, a spherical inflection point. The inflection tangent in this geometry, a great circular arc, is displayed as well.

Example 5.4. In Kozak et al. (2014, Theorem 7) the authors present a family of rational PH curves of the particularly low degree three. One example of theirs is

$$\mathbf{r}(t) = \frac{-1}{60(t^2+1)}\begin{pmatrix} t(t^2-4) \\ 2t(3t-1) \\ t(3t+4) \end{pmatrix}. \tag{16}$$

In order to reconstruct it, we use $\alpha = 60(t^2 + 1)$ and $\mathcal{A} = t^2 + 3t\mathbf{i} + 2\mathbf{j} + \mathbf{k} - 1$ which is provided by Kozak et al. (2014, Theorem 7) as well. With $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^*$ and $\beta = t - \mathrm{i}$ we have $n = 1$ and $\mathbf{F} = \mathbf{f}_0 + \mathbf{f}_1\beta + \mathbf{f}_2\beta^2 + \mathbf{f}_3\beta^3 + \mathbf{f}_4\beta^4$ where

$$\mathbf{f}_0 = -10\mathbf{i} - (4 - 12\mathrm{i})\mathbf{j} + (8 + 6\mathrm{i})\mathbf{k}, \quad \mathbf{f}_1 = (10\mathrm{i})\mathbf{i} + (12 + 4\mathrm{i})\mathbf{j} + (6 - 8\mathrm{i})\mathbf{k},$$

$$\mathbf{f}_2 = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{f}_3 = (4\mathrm{i})\mathbf{i}, \quad \mathbf{f}_4 = \mathbf{i}.$$

We observe that $\operatorname{rank}(\mathbf{f}_0, \mathbf{f}_1) = 1$ so that we can indeed predict non-polynomial solutions by Lemma 4.2. Let us proceed by computing solutions for increasing degree of $\mathbf{b}$. For $0 \le \deg \mathbf{b} \le 1$ we obtain $\mathbf{r}_0(t) = \mathbf{r}_1(t) = (0,0,0)^T$. Already for $\deg \mathbf{b} = 2$ we find the translations $\mathbf{r}_2(t) = \mathbf{r}_1(t) + (\tau_1, \tau_2, \tau_3)^T = (\tau_1, \tau_2, \tau_3)$ of $\mathbf{r}_1(t)$. The first non-trivial solution occurs indeed for $\deg \mathbf{b} = 3$. The solution family is obtained from (16) by scaling and translation: $\mathbf{r}_3(t) = \mu_3 \mathbf{r}(t) + \mathbf{r}_2(t)$. Note that our proof of existence of non-polynomial solutions requires complex numbers but actually real solutions are computed from a real system of linear equations.

Example 5.5. In Farouki and Šír (2020, Example 1) a rational PH curve

$$\mathbf{r}(t) = \frac{800}{(t+10)^5} \begin{pmatrix} -13420\,t^5 + 4000\,t^4 + 30000\,t^3 - 50000\,t^2 + 25000\,t \\ 46643\,t^5 - 67850\,t^4 - 7000\,t^3 + 30000\,t^2 \\ 19776\,t^5 - 111200\,t^4 + 126000\,t^3 - 40000\,t^2 \end{pmatrix} \tag{17}$$

is constructed using essentially formula (5). We have $\alpha = (t+10)^5$, $\beta = t + 10$ and $n = 5$ whence we are in the case of Lemma 4.4 again. The spherical motion of the Euler-Rodrigues frame is provided by Farouki and Šír (2020):

$$\mathcal{A} = (7t^2 - 22t + 10) + (-19t^2 + 14t)\mathbf{i} + (-26t^2 + 16t)\mathbf{j} + (-2t^2 + 12t)\mathbf{k}.$$

We have $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^* = \sum_{i=0}^{4} \mathbf{f}_i \beta^i$. Using the system (8) we can now compute solutions for $\mathbf{b}$. Already for $\deg \mathbf{b} = 4$ we find the non-polynomial solution

$$\mathbf{p}_4(t) = \frac{1}{(t+10)^5} \begin{pmatrix} -27t^4 - 538t^3 - 5366t^2 - 26841t - 53680 \\ 96t^4 + 1866t^3 + 18656t^2 + 93286t + 186572 \\ 44t^4 + 786t^3 + 7912t^2 + 39552t + 79104 \end{pmatrix}. \tag{18}$$

Obviously, every uniform scaling of $\mathbf{p}_4(t)$ is a solution as well. For $\deg \mathbf{b} = 5$ we obtain a solution space of dimension four, consisting of the scalings and translations of $\mathbf{p}_4(t)$. The general solution can be written as $\mathbf{r}_5(t) = \mu_4 \mathbf{p}_4(t) + (\tau_1, \tau_2, \tau_3)^T$. The curve $\mathbf{r}(t)$ in Equation (17) is contained in this space with

$$\mu_4 = -20000000, \quad \tau_1 = -10736000, \quad \tau_2 = 37314400, \quad \tau_3 = 15820800.$$

Example 5.6. The paper Farouki and Sakkalis (2019) is devoted principally to the planar rational PH curves with rational arc-length function. Farouki and Sakkalis (2019, Example 9) however deals with spatial curves and the rational PH curve

$$\mathbf{r}(t) = \frac{1}{3t(t-1)} \begin{pmatrix} -4t^5 + 22t^4 - 18t^3 - 10t^2 + 34t \\ -4t^5 + 4t^4 + 12t^3 + 26t^2 - 2t - 24 \\ -2t^5 + 2t^4 + 36t^3 - 32t^2 - 46t + 18 \end{pmatrix} \tag{19}$$

is constructed via the direct integration. It corresponds to the spherical motion given by the polynomial

$$\mathcal{A} = (-t^2 + t - 1) + (t^3 - 2t + 2)\mathbf{i} + (-2t^3 + 3t^2 + t - 1)\mathbf{j} + (-t^3 + 4t^2 - 2t + 2)\mathbf{k}.$$

Moreover, we have $\alpha = 3t(t-1)$, $n = 1$ and $\beta = t$ or $\beta = t - 1$. With $\mathbf{F} := \mathcal{A}\mathbf{i}\mathcal{A}^*$ we have

$$\mathbf{F} = (-8\mathbf{j} + 6\mathbf{k})t^0 + (16\mathbf{j} - 12\mathbf{k})t + \cdots = (-8\mathbf{i} - 4\mathbf{j} + 8\mathbf{k})(t-1)^0 + (-16\mathbf{i} - 8\mathbf{j} + 16\mathbf{k})(t-1)^1 + \cdots$$

and can confirm that both choices for β satisfy the conditions of Lemma 4.2. Once more, we compute solutions via (8). For $2 \le \deg \mathbf{b} \le 4$ we obtain only trivial solutions. For $\deg \mathbf{b} = 5$, the solution space is of dimension four. It consists of all scaled and translated copies of the curve (19).

Let us finish this section with a remarkable additive decomposition of the rational PH curve (19). We have $\mathbf{r}(t) = \mathbf{p}(t) + \mathbf{q}(t)$ where

$$\mathbf{p}(t) = \frac{1}{15t} \begin{pmatrix} -20t^7 + 108t^6 - 135t^5 - 20t^4 - 30t^3 - 120t^2 - 170t \\ -20t^7 + 36t^6 + 30t^5 - 80t^4 - 180t^3 + 360t^2 + 490t + 120 \\ -10t^7 + 18t^6 + 90t^5 - 190t^4 + 210t^3 + 30t^2 - 130t - 90 \end{pmatrix} \quad \text{and}$$

$$\mathbf{q}(t) = \frac{1}{15(t-1)} \begin{pmatrix} 20t^7 - 128t^6 + 243t^5 - 135t^4 + 120t^3 \\ 20t^7 - 56t^6 + 6t^5 + 90t^4 + 120t^3 - 480t^2 + 360 \\ 10t^7 - 28t^6 - 72t^5 + 270t^4 - 390t^3 + 360t^2 - 270 \end{pmatrix},$$

are both rational PH curves with the same rotation $\mathcal{A}$ but with simpler denominator polynomials. We found it computing the solution spaces with the same quaternion polynomial $\mathcal{A}$ but with different choices for the denominator polynomial α. We believe that this additive decomposition of a rational PH curve with denominator $\alpha = 3t(t-1)$ into rational PH curves with respective denominators t and $t-1$ is only one manifestation of a more general pattern. Moreover, we expect that the difference between the minimal degree of a polynomial solution and the minimal degree of a rational PH curve increases with n, and we conjecture that the presence of several linear factors β of α that meet criteria for existence of rational solutions further increase this difference.

6. Conclusion

In this paper we managed to connect the theory of Pythagorean Hodograph curves with the theory of rational framing motions. First benefit of this connection is conceptual. It seems to be appropriate to understand rational spatial PH curves as (trajectories of) framing rational motions (akin curves with rational offsets in the planar case). Second benefit is descriptive and computational. The full set of rational PH curves is described using special motion polynomials in Theorem 3.6. Comparing to the existing results this description contains linear constraints but brings considerable advantages. In particular we were able to understand the cancellation between the numerator and denominator of the PH curves (appearing as a common factor of α and $\mathbf{b}$). In Theorem 4.6 and Remark 4.7 we fully described when this cancellation leads to polynomial curves only.

We are hopeful that our approach will lead to a number of additional results. As a future research we plan to fully describe the basis of the linear system of PH curves with given tangent indicatrix. This opens new possibilities to solve interpolation problems. We have shown how to select infinitely many denominator polynomials α to given $\mathcal{A}$ such that truly rational solutions curves exist. Generically, the only restriction on α is existence of a factor of multiplicity three. We can choose a finitely generated subspace of this vector space of infinite dimension and add (linear) interpolation constraints. Polynomial PH curves are subsumed in this approach but rationality provides additional degrees of freedom. In this context, the better understanding of possible denominators will help us to avoid singular (cuspidal) interpolants. Finally, we plan to connect our results with the theory of the rotation minimizing frames or PH curves with rational arc length.

CRediT authorship contribution statement

The authors contributed in all tasks during the research process and the writing of this paper.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Zbynek Sir reports financial support was provided by Czech Science Foundation. Bahar Kalkan reports financial support was provided by Scientific and Technological Research Council of Turkey.

Data availability

No data was used for the research described in the article.

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