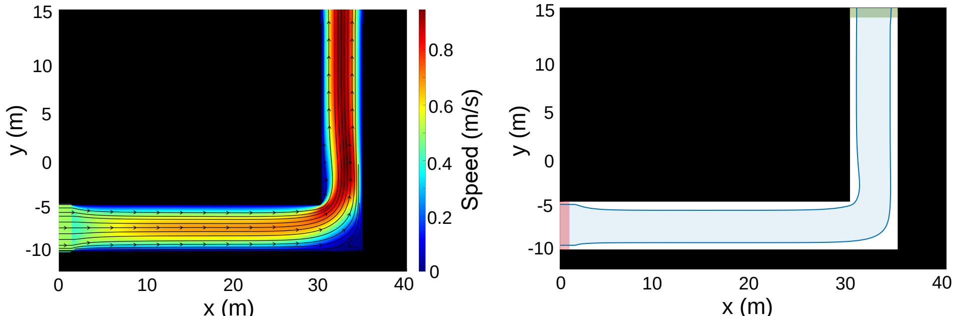
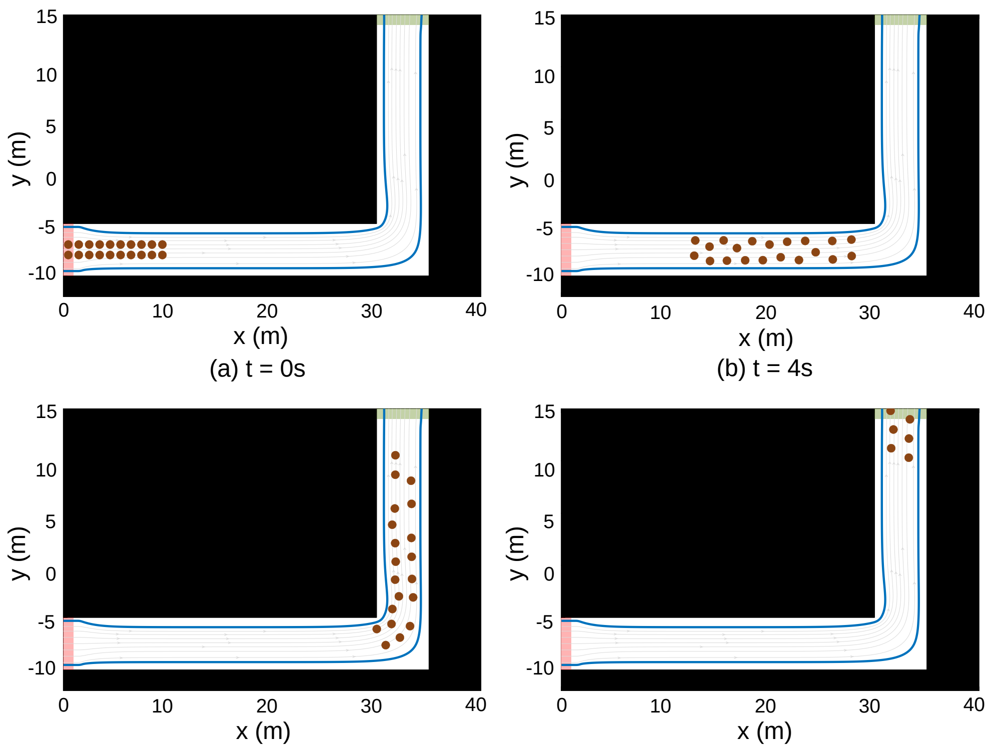
# Supplementary Materials




(a) The solution of NS equations (b) Virtual tube planning based on NS equations

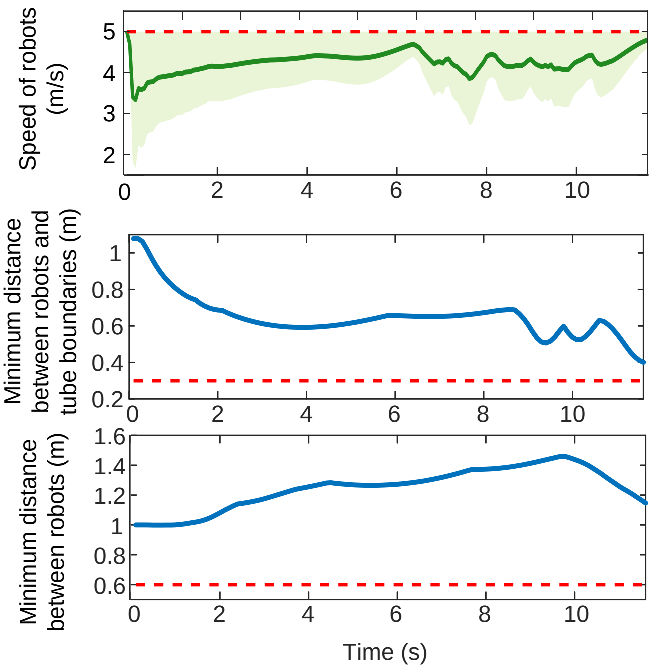
**Supplementary Fig.1. The solution of NS equations and the planned virtual tube based on NS equations of Scene 1.** Scene 1 is the first manually set scene in the paper. The red rectangle represents the start area $\mathcal{C}_{\mathrm{s}}$. The green rectangle represents the goal area $\mathcal{C}_{\mathrm{g}}$. The black areas represent obstacles. The blue curves represent the boundaries of the virtual tube.



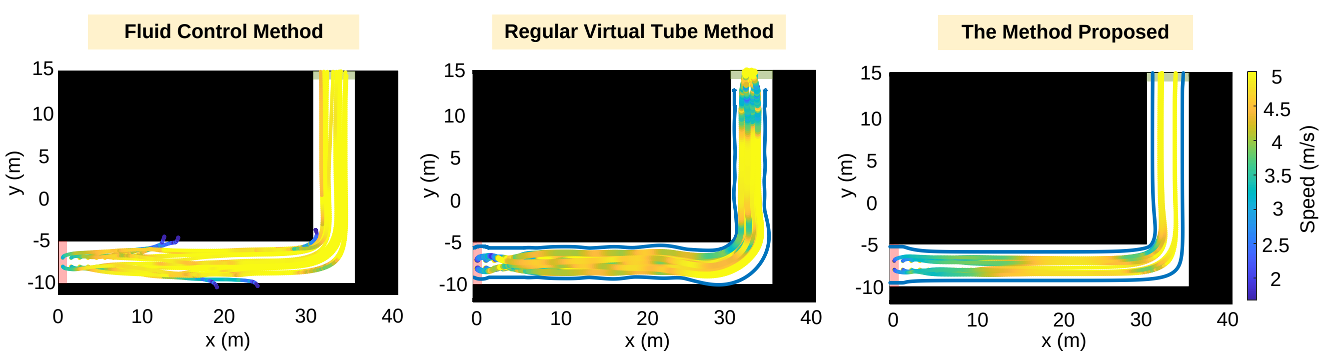

(a) t = 0s (b) t = 4s

(c) t = 8s (d) t = 12s

**Supplementary Fig.2. The passing-through process of 20 robots based on the method proposed under Scene 1.** The light grey curves represent the streamlines. The brown dots represent the robots.




**Supplementary Fig.3. Metrics of the proposed method under Scene 1.** The light green area represents the maximum and minimum speeds. The dark green curve represents the average speed. The blue curve represents the minimum distance with respect to time. The red dotted line represents the maximum speed or the safety distance.




**Supplementary Fig.4. Comparison of passing-through processes based on three methods under Scene 1.** The colors of trajectories correspond to varied speeds where the yellow represents the high speed and the blue represents the low speed.

**Supplementary Appendix. Specific Steps of Virtual Tubes Planning.** This appendix provides a detailed mathematical description of the virtual tube planning method based on the Navier-Stokes (NS) equations, with the symbols defined in Section III-A of the paper.

Consider the fluid flowing from left to right in a two-dimensional Cartesian coordinate system as an example, the specific steps for generating virtual tube boundaries are illustrated as follows.

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- **STEP 1:** The complete streamlines from $\mathcal{C}_0$ to $\mathcal{C}_\mathrm{t}$ are screened out, that is, short streamlines caused by the grid accuracy of the map and boundary conditions are removed. Define the set of selected streamlines as

$$\mathcal{HD}_\mathrm{c} = \{C_{\mathrm{h},i} | \mathbf{p}_{\mathrm{c}_0,i} \in \mathcal{C}_\mathrm{s}, \mathbf{p}_{\mathrm{c}_1,i} \in \mathcal{C}_\mathrm{t}, i = 1, 2, \cdots, M\}.$$

The streamlines in the set $\mathcal{HD}_\mathrm{c}$ are expressed as $C_{\mathrm{hc},i}(t_k), i = 1, 2, \cdots, M_{hd}$.

- **STEP 2:** Arrange all streamlines in the spatial order, then obtain the set of candidate streamlines

$$\begin{aligned}&\mathcal{HD}_{\mathrm{s}}=\{C_{\mathrm{hc},i}|C_{\mathrm{hc},i}\in\mathcal{HD}_{\mathrm{c}},x_{\mathbf{p}_{\mathrm{c}_0,i}}\geq x_{\mathbf{p}_{\mathrm{c}_0,i-1}},\\&i=2,\cdots,M_{hd}\}.\end{aligned}$$

  The streamlines in the set $\mathcal{HD}_{\mathrm{s}}$ are expressed as $C_{\mathrm{hs},i}\left(t_k\right), i=1,2,\cdots,M_{hd}$. Particularly, the sorting condition $x_{\mathbf{p}_{\mathrm{c}_0,i}}\geq x_{\mathbf{p}_{\mathrm{c}_0,i-1}}$ can be replaced by $y_{\mathbf{p}_{\mathrm{c}_0,i}}\geq y_{\mathbf{p}_{\mathrm{c}_0,i-1}}$, which is determined by the position of $\mathcal{C}_{\mathrm{s}}$. The reason for utilizing the coordinates of the starting points of streamlines to spatially order them is that there are no intersections between any pair of streamlines, which is illustrated in *Proposition 1* of the paper.
- **STEP 3:** This step is the key step. The average distance between every two adjacent streamlines is compared to distinguish the boundaries of virtual tubes. When the distance is relatively large, the two streamlines are considered to belong to the boundaries of two different virtual tubes. Assuming the minimum and maximum $x$-coordinate of $C_{\mathrm{hs},i}$ in a two-dimensional coordinate system are $x_{\min,i}$ and $x_{\max,i}$, respectively. Then randomly select $m$ points $\{\mathbf{q}_{i,j}|\mathbf{q}_{i,j}\in C_{\mathrm{hs},i},x_{i,j}\in\left[x_{\min,i},x_{\max,i}\right],i=1,2,\cdots,M_{hd},j=1,2,\cdots,m\}$, where $x_{i,j}$ denotes the $x$-coordinate of $\mathbf{q}_{i,j}$. Correspondingly, the $y$-coordinate of $\mathbf{q}_{i,j}$ is $y_{i,j}$. Let $\mathcal{TB}_{cr}$ and $\mathcal{TB}_{cl}$ represent the set of right and left boundaries of virtual tubes generated by this step respectively, then sets of candidate virtual tube boundaries are

$$\begin{aligned}&\mathcal{TB}_{\mathrm{cl}}=\{C_{\mathrm{hs},i}|C_{\mathrm{hs},i}\in\mathcal{HD}_{\mathrm{s}},\\&\sum_{j=1}^{m}|y_{i,j}-y_{i-1,j}|/m>\Delta_{hd},i=2,3,\cdots,M_{hd}\},\\&\mathcal{TB}_{\mathrm{cr}}=\{C_{\mathrm{hs},i-1}|C_{\mathrm{hs},i-1}\in\mathcal{HD}_{\mathrm{s}},\\&\sum_{j=1}^{m}|y_{i,j}-y_{i-1,j}|/m>\Delta_{hd},i=2,3,\cdots,M_{hd}\},\end{aligned}$$

  where $\Delta_{hd}>0$ is set based on experience. The boundaries are selected based on the principle of *fluid diversion*, where the fluid encountering an obstacle splits into two or more directions from an initially parallel or concentrated streamline, as illustrated in Fig. 2 (a) of the paper.
- **STEP 4:** Filter out virtual tubes that are not excessively narrow to get the set of selected left boundaries

$$\begin{aligned}&\mathcal{TB}_{\mathrm{sl}}=\{C_{\mathrm{tcl},j}|\min\|\mathbf{b}_1-\mathbf{b}_2\|>r_{\mathrm{s}},\mathbf{b}_1\in C_{\mathrm{tcl},j},\\&\mathbf{b}_2\in C_{\mathrm{tcr},j},,j=1,2,\cdots,n_{\mathrm{ct}}\}.\end{aligned}$$

  Similarly, the set of selected right boundaries $\mathcal{TB}_{\mathrm{sr}}$ can be obtained. Heretofore, multiple virtual tubes at the specific time step $t_k,k\in\{1,2,\cdots,n\}$ can be obtained. Then, let $C_{\mathrm{tsl},j}\in\mathcal{TB}_{\mathrm{sl}},C_{\mathrm{tsr},j}\in\mathcal{TB}_{\mathrm{sr}},j=1,2,\cdots,n_{\mathrm{st}}$ represent the left and right boundaries of virtual tubes picked by this step respectively, where $n_{\mathrm{st}}$ denotes the number of virtual tubes filtered out. Subsequently, the width of every virtual tube is defined. To begin with, select $k_{\mathrm{s}}$ points $\{\mathbf{x}_{\mathrm{l},i},i=1,2,\cdots,k_{\mathrm{s}}\}$ on $C_{\mathrm{tsl},j},j=1,2,\cdots,n_{\mathrm{st}}$, which are evenly distributed along the curve. Correspondingly, find $k_{\mathrm{s}}$ points on $C_{\mathrm{tsr},j},j=1,2,\cdots,n_{\mathrm{st}}$, which can be indicated by $\{\mathbf{x}_{\mathrm{r},i}|\left\|\mathbf{x}_{\mathrm{r},i}-\mathbf{x}_{\mathrm{l},i}\right\|\leq\left\|\mathbf{x}-\mathbf{x}_{\mathrm{l},i}\right\|,\forall\mathbf{x}\in C_{\mathrm{tsr},j},\mathbf{x}_{\mathrm{r},i}\in C_{\mathrm{tsr},j},i=1,2,\cdots,k_{\mathrm{s}},j=1,2,\cdots,n_{\mathrm{st}}\}$. Then the *width* of the $j$th virtual tube at $t_k$ is

$$w_{\mathrm{tb},j}\left(t_k\right)=\left(\sum_{i=1}^{k_{\mathrm{s}}}\left\|\mathbf{x}_{\mathrm{r},i}-\mathbf{x}_{\mathrm{l},i}\right\|\right)/k_{\mathrm{s}},j=1,2,\cdots,n_{\mathrm{st}}.$$

- **STEP 5:** Select the virtual tubes with the widest average width from $t\in\{t_1,t_2,\cdots,t_n\}$. The *average width* of virtual tubes at $t_k,k\in\{1,2,\cdots,n\}$ is

$$\bar{w}_{\mathrm{tb}}\left(t_k\right)=\left(\sum_{j=1}^{n_{\mathrm{st}}}w_{\mathrm{tb},j}\left(t_k\right)\right)/n_{\mathrm{st}}.$$

  The set of tube boundaries that are ultimately determined is denoted by $\mathcal{TB}_{\mathrm{fl}}$ and $\mathcal{TB}_{\mathrm{fr}}$.