Daniil Manakovskiy
BS18-02

Homework 4

Linear Control Theory
April 4, 2020

Variant: A
Email: d.manakovskiy@innopolis.university
The given dynamics:

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos(\theta) + ml\dot{\theta}^2\sin(\theta) = F \tag{1}$$

$$-\ddot{x}\cos(\theta) + l\ddot{\theta} - g\sin(\theta) = 0 \tag{2}$$

where $M = 5.3, m = 3.2, l = 1.15$

Task A

Write the equation in manipulator form:

$$M(q)\ddot{q} + n(q,\dot{q}) = Bu$$

where $u = F, q = \begin{bmatrix} x & \theta \end{bmatrix}^{\mathbf{T}}$.

Rewriting the equation will lead to:

$$\begin{bmatrix} M+m & -ml\cos(\theta) \\ -\cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} ml\dot{\theta}^2\sin(\theta) \\ -g\sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Substituting values:

$$\begin{bmatrix} 8.5 & -3.68\cos(\theta) \\ -\cos(\theta) & 1.15 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 3.68\dot{\theta}^2\sin(\theta) \\ -9.81\sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Task B

Write the dynamics in the following form:

$$\dot{z} = f(z) + g(z)u$$

where $z = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^{\mathbf{T}}, u = F$

From Task A we can find $\begin{bmatrix} \ddot{x} & \ddot{\theta} \end{bmatrix}^{\mathbf{T}}$ as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} M+m & -ml\cos(\theta) \\ -\cos(\theta) & l \end{bmatrix}^{-\mathbf{1}} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} F - \begin{bmatrix} ml\dot{\theta}^2\sin(\theta) \\ -g\sin(\theta) \end{bmatrix} \right)$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{(M+m)l - ml(\cos(\theta))^2} \begin{bmatrix} l & ml\cos(\theta) \\ \cos(\theta) & M+m \end{bmatrix} \left(\begin{bmatrix} -ml\dot{\theta}^2\sin(\theta) \\ g\sin(\theta) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \right)$$

So we get:

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{-ml^2\dot{\theta}^2\sin(\theta)+mlg\sin(\theta)\cos(\theta)}{(M+m)l-ml\cos(\theta)^2}\\ \frac{-ml\dot{\theta}^2\sin(\theta)\cos(\theta)+(M+m)g\sin(\theta)}{(M+m)l-ml\cos(\theta)^2}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{l}{(M+m)l-ml\cos(\theta)^2}\\ \frac{\cos(\theta)}{(M+m)l-ml\cos(\theta)^2}\end{bmatrix} F$$

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{-ml\dot{\theta}^2\sin(\theta)+mg\sin(\theta)\cos(\theta)}{(M+m)-m\cos(\theta)^2}\\ \frac{-ml\dot{\theta}^2\sin(\theta)\cos(\theta)+(M+m)g\sin(\theta)}{(M+m)l-ml\cos(\theta)^2}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{(M+m)-m\cos(\theta)^2}\\ \frac{\cos(\theta)}{(M+m)l-ml\cos(\theta)^2}\end{bmatrix} F$$

Substituting values we get:

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{-3.68\dot{\theta}^2\sin(\theta)+31.392\sin(\theta)\cos(\theta)}{8.5-3.2\cos(\theta)^2}\\ \frac{-3.68\dot{\theta}^2\sin(\theta)\cos(\theta)+83.385\sin(\theta)}{9.775-3.68\cos(\theta)^2}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{8.5-3.2\cos(\theta)^2}\\ \frac{\cos(\theta)}{9.775-3.68\cos(\theta)^2}\end{bmatrix} F$$

Task C

Linearize the dynamics around equilibrium point $\bar{z} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^{\mathbf{T}}$

$$\delta\dot{z} = A\delta z + B\delta u$$

To perform linearization we need to have equilibrium points $\bar{z}$ and $\bar{u}$.

1. The system has a form:
$$p(z,u) = \dot{z} = f(z) + g(z)u \tag{3}$$

2. The given $\bar{z}$ stands for a stable state, when the mass is at the most upper position, therefore $\bar{u} = 0$ and $p(\bar{z},\bar{u}) = 0$.

3. Knowing
$$\begin{cases}\delta z = z - \bar{z}\\ \delta u = u - \bar{u}\end{cases} \Leftrightarrow \begin{cases} z = \delta z + \bar{z}\\ u = \delta u + \bar{u}\end{cases} \tag{4}$$

   We substitute (4) into (3) getting:

$$(\bar{z} + \delta z)' = p(\delta z + \bar{z}, \delta u + \bar{u})$$

$$\delta\dot{z} = p(\bar{z}, \bar{u}) + \left.\frac{\partial p}{\partial z}\right|_{z=\bar{z}, u=\bar{u}} \delta z + \left.\frac{\partial p}{\partial u}\right|_{z=\bar{z}, u=\bar{u}} \delta u$$

where

$$z = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix};$$

$$p(z,u) = \dot{z} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{-ml\dot{\theta}^2\sin(\theta)+mg\sin(\theta)\cos(\theta)}{(M+m)-m\cos(\theta)^2} + \frac{F}{(M+m)-m\cos(\theta)^2} \\ \frac{-ml\dot{\theta}^2\sin(\theta)\cos(\theta)+(M+m)g\sin(\theta)}{(M+m)l-ml\cos(\theta)^2} + \frac{F\cos(\theta)}{(M+m)l-ml\cos(\theta)^2} \end{bmatrix}$$

4. Now we can calculate Jacobian matrices to get matrices A and B using the following formula:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

All derivatives are taken using derivative calculator.
Calculate A matrix:

$$\left.\left.\frac{\partial p_3}{\partial x_2}\right|_{z=\bar{z}}\right|_{u=\bar{u}} =$$

$$= \left.\left.\frac{m\left(\left(lmz_4^2\cos(z_2) - gm - 2gM\right)\sin^2(z_2) - 2u\cos(z_2)\sin(z_2) - lMz_4^2\cos(z_2) + gM\right)}{\left(m\sin^2(z_2)+M\right)^2}\right|_{z=\bar{0}}\right|_{u=\bar{0}} =$$

$$= \frac{gm}{M}$$

$$\left.\left.\frac{\partial p_3}{\partial x_4}\right|_{z=\bar{z}}\right|_{u=\bar{u}} = -\left.\left.\frac{2lm\sin(z_2)\,z_4}{-m\cos^2(z_2)+m+M}\right|_{z=\bar{0}}\right|_{u=\bar{0}} = 0$$

$$\left.\left.\frac{\partial p_4}{\partial x_2}\right|_{z=\bar{z}}\right|_{u=\bar{u}} =$$

The result of the following calculation doesn't fit, check this link

$$= \left.\left./*\text{result didn't fit. Check link above}*/\right|_{z=\bar{0}}\right|_{u=\bar{0}} = \frac{g(m+M)}{Ml}$$

$$\left.\left.\frac{\partial p_4}{\partial x_4}\right|_{z=\bar{z}}\right|_{u=\bar{u}} = -\left.\left.\frac{2m\cos(z_2)\sin(z_2)\,z_4}{m\sin^2(z_2)+M}\right|_{z=\bar{0}}\right|_{u=\bar{0}} = 0$$

$$\left.\left.\frac{\partial p}{\partial z}\right|_{z=\bar{z}}\right|_{u=\bar{u}} = A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{Ml} & 0 & 0 \end{bmatrix}$$

Calculate B matrix:

$$\left.\left.\frac{\partial p}{\partial u}\right|_{z=\bar{z}}\right|_{u=\bar{u}} = B = \left.\left.\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m\sin^2(z_2) + M} \\ \dfrac{\cos^2(z_2)}{l\left(m\sin^2(z_2) + M\right)} \end{bmatrix}\right|_{z=\bar{0}}\right|_{u=\bar{0}} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M} \\ \dfrac{1}{Ml} \end{bmatrix}$$

5. Finally we have:

$$\delta\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{Ml} & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M} \\ \dfrac{1}{Ml} \end{bmatrix} \delta u$$

$$\delta\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{31.392}{5.3} & 0 & 0 \\ 0 & \frac{83.385}{6.095} & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{5.3} \\ \dfrac{1}{51.993} \end{bmatrix} \delta u$$

$$\delta\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5.923 & 0 & 0 \\ 0 & 13.68 & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 0 \\ 0.189 \\ 0.019 \end{bmatrix} \delta u$$

Task D

Check the stability of linearized system.

Since the system is LTI, we can determine stability using the system's eigenvalues:

```
import numpy as np
np.linalg.eig([
    [0,     0,      1,  0],
    [0,     0,      0,  1],
    [0,     5.923,  0,  0],
    [0,     13.68,  0,  0]
])[0]

>>> array([ 0.        ,  0.        ,  3.6986484, -3.6986484])
```

The system has 1 positive eigenvalue, thus it is unstable.

Task E

Note: *I have read the theory on controllability at this source*
Determine if linearized system obtained in Task C is controllable.

1. Calculate controllability matrix C. Since A's dimension is $4x4$, C is calculated as follows:

$$C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5.923 & 0 & 0 \\ 0 & 13.68 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.189 \\ 0.019 \end{pmatrix} = \begin{pmatrix} 0.189 \\ 0.019 \\ 0 \\ 0 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} 0 & 5.923 & 0 & 0 \\ 0 & 13.68 & 0 & 0 \\ 0 & 0 & 0 & 5.923 \\ 0 & 0 & 0 & 13.68 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.189 \\ 0.019 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.113 \\ 0.26 \end{pmatrix}$$

$$A^3B = \begin{pmatrix} 0 & 0 & 0 & 5.923 \\ 0 & 0 & 0 & 13.68 \\ 0 & 81.027 & 0 & 0 \\ 0 & 187.142 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.189 \\ 0.019 \end{pmatrix} = \begin{pmatrix} 0.113 \\ 0.26 \\ 0 \\ 0 \end{pmatrix}$$

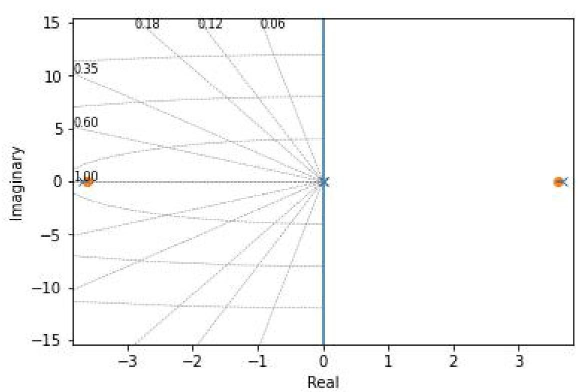
$$C = \begin{bmatrix} 0 & 0.189 & 0 & 0.113 \\ 0 & 0.019 & 0 & 0.26 \\ 0.189 & 0 & 0.113 & 0 \\ 0.019 & 0 & 0.26 & 0 \end{bmatrix}$$

2. Now let find $rank(C)$ using python and numpy:

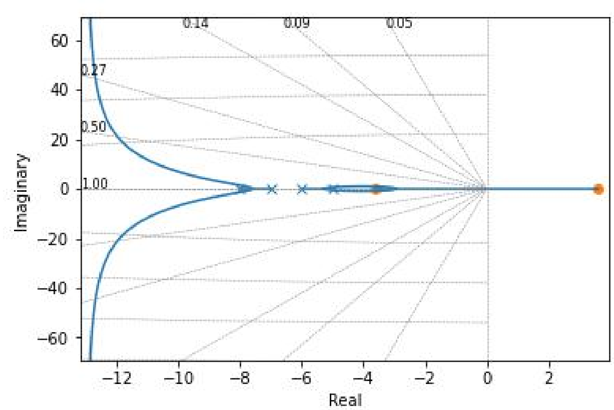
```
import numpy as np
np.linalg.matrix_rank([
    [0,     0.189,  0,      0.113],
    [0,     0.019,  0,      0.26],
    [0.189, 0,      0.113,  0],
    [0.019, 0,      0.26,   0]
])

>>> 4
```

As we can see, controllability matrix has full rank, thus the system is controllable

Task F



(a) Before pole placement



(b) After pole placement

Figure 1: Root-locus

At figure 1a the root-locus for the initial linearized system is shown, there is a pole at positive half-plane. To stabilize the system, we want all poles to have real parts less than 0, and less poles - faster convergence but with rapid changes.

Let use *scipy.signal.place_poles* move poles to $[-5, -6, -7, -8]$ in order to balance between fast convergence and rapid acceleration. Changed root-locus (figure 1b) shows that system is indeed stable, since there are no asymptotes at positive half plane.

Figures 2a, 3a, 4a, 5a, 6a, 7a show the behaviour of controlled system.

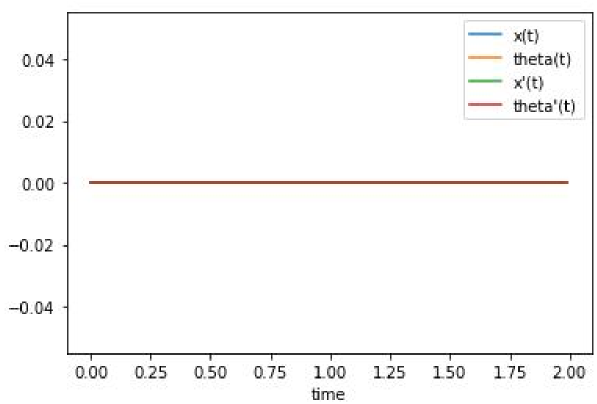
Task G

The LQR controller is shown as python code at Google colab.
Below you can see how the system behave under the LQR controller with different initial conditions $\begin{bmatrix}\dot{x} & \dot{\theta} & \ddot{x} & \ddot{\theta}\end{bmatrix}^{\mathbf{T}}$

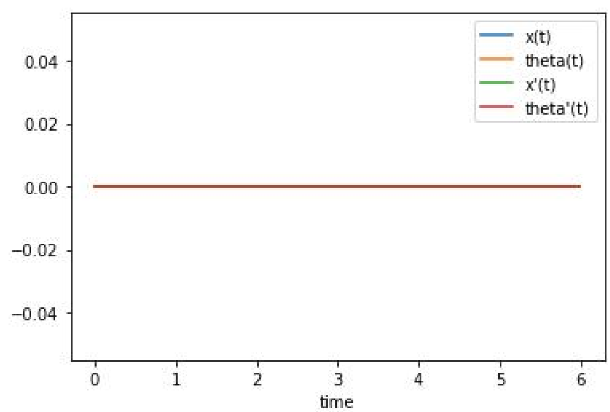
$$Q = 100 * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 \end{pmatrix}$$

For our model it is reasonable to have the following constraint: $-\pi/2 <= \theta <= \pi/2$

From figures 2b, 3b, 4b, 5b, 6b, 7b we can deduce, that state $\bar{0}$ is stable, and for other states the LQR with the given Q and R converges in approx. 6 seconds. Also we see at figure 3b how system behaves if angle in rad is too big which might violate the physics (pendulum shouldn't go through a car) - the graph look pretty reasonable despite very rapid acceleration at the start. Similar rapid acceleration happened at figure 6b, but this can be tuned by changing Q and R according to the actual problem.

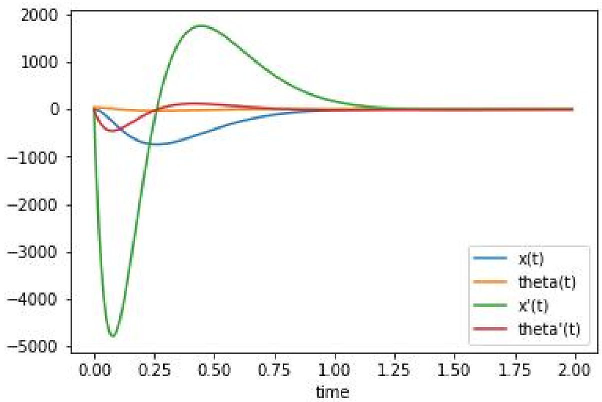


(a) Pole placement

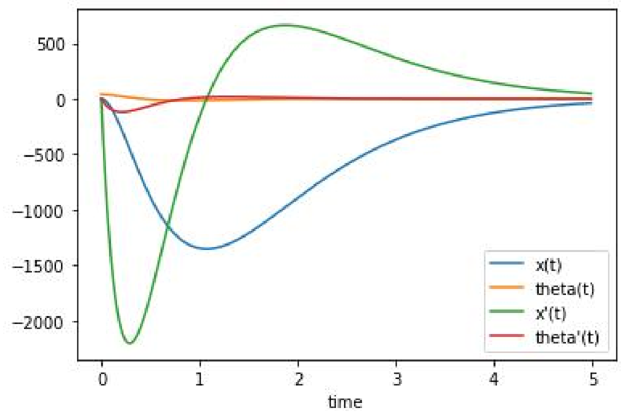


(b) LQR

Figure 2: Stable state [0, 0, 0, 0]

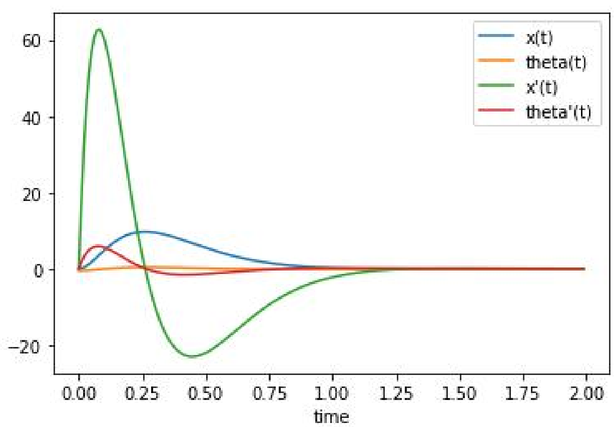


(a) Pole placement

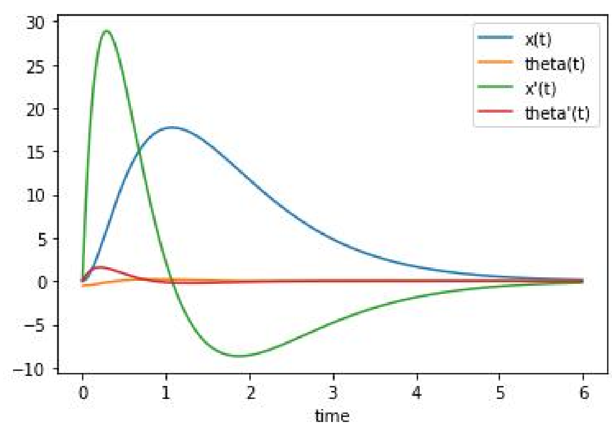


(b) LQR

Figure 3: Big angle: [0, 40, 0, 0]

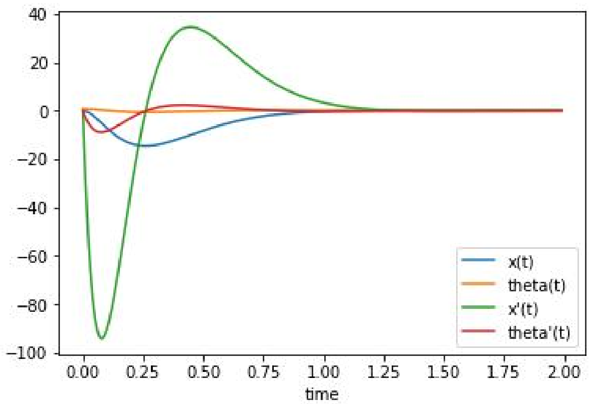


(a) Pole placement

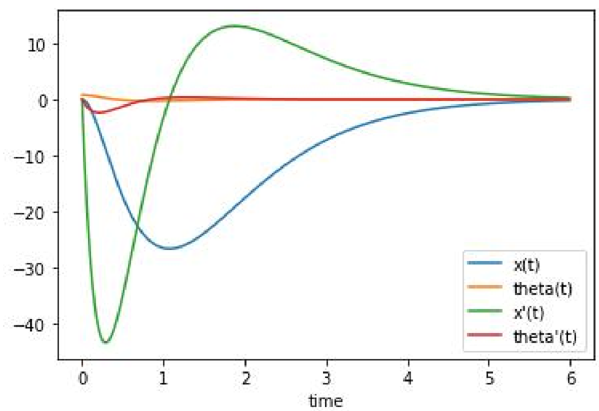


(b) LQR

Figure 4: [0, -π/6, 0, 0]

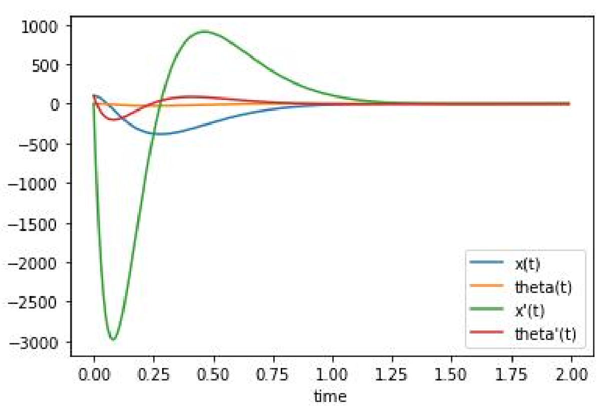


(a) Pole placement

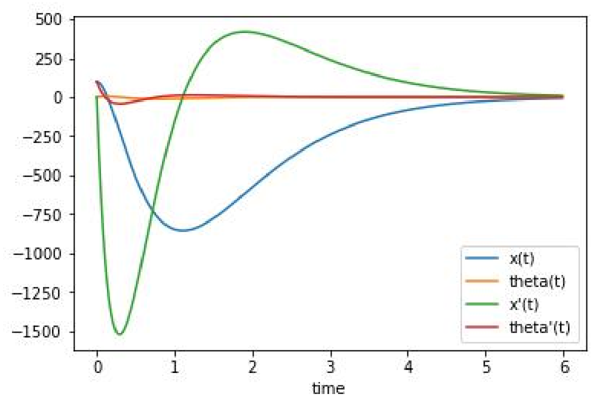


(b) LQR

Figure 5: [0, $\pi/4$, 0, 0]

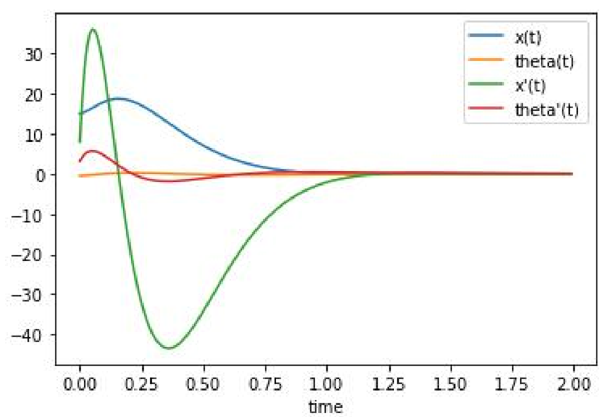


(a) Pole placement

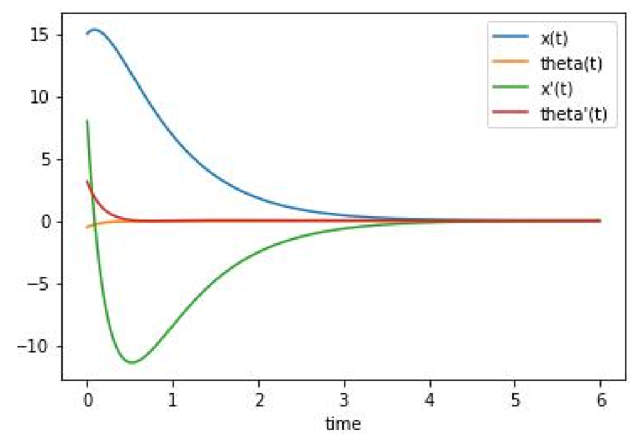


(b) LQR

Figure 6: [100, 0, 0, 100]



(a) Pole placement



(b) LQR

Figure 7: [15, $-\pi/6$, 8, π]