

Adaptive Rejection Sampling

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1. Approach

(a) Functions

Modular Functions:

- 1. Deriv_sub: Take the derivative of a function evaluated at x numerically. Basic logic: If x is at the boundary:

$$f'(x) \approx \frac{f(x) - f(x - \epsilon)}{\epsilon}$$

$$f'(x) \approx \frac{f(x + \epsilon) - f(x)}{\epsilon}$$

Otherwise:

$$f'(x) \approx \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon}$$

- 2. check_concave: Check the log-concavity of the density functions. Check if the derivative of function decreases monotonically with increasing x.
- 3. init_mat: We create a matrix called mat to store the information that's collected during the rejection sampling process. The first column is the value of sample x. The second column is function h (which is log of original g function) evaluated at x. The third column is the derivative of function h evaluated at x. This function initializes mat, fills it with starting points provided by the user or starting points we find if they put -Inf or Inf.

```
init_mat <- function(a,b,fun,deriv){  
  #this mat stores x,f(x) and df(x)  
  mat <- matrix(nrow = 2,ncol = 3)  
  mat[1,1] <- a  
  mat[1,2] <- fun(a)  
  if(!is.finite(mat[1,2])){  
    stop('a is not defined on function h', call. = FALSE)  
  }  
  mat[1,3] <- deriv(a,fun,a,b)  
  mat[2,1] <- b  
  mat[2,2] <- fun(b)  
  if(!is.finite(mat[2,2])){  
    stop('b is not defined on function h', call. = FALSE)  
  }  
  mat[2,3] <- deriv(b,fun,a,b)
```

```

return(mat)
}

```

- 4. update_mat: Update mat with x_star, which is the proposed sample points generated by sampling process.

```

update_mat <- function(mat,x_star,fun,deriv,a,b){
  index <- sum(mat[,1] < x_star) #find the location to insert x*
  tmp <- matrix(NA,1,3)
  mat <- rbind(mat,tmp)
  mat[(index+2):nrow(mat),] <- mat[(index+1):(nrow(mat)-1),]
  mat[(index+1),] <- c(x_star,fun(x_star),deriv(x_star,fun,a,b))
  return(mat)
}

```

- 5. init_z: Initialize z, which is a function used to initialize a vector that contains different z values. The input will be the initial mat matrix which contains starting point a and b as well as f(a),f(b),f'(a) and f'(b). We let the first element in the z vector to be the starting point a, and the last element in the z vector to be the starting point b. We calculate the first z given a and b using the formula below.

$$z_j = \frac{h(x_{j+1})h(x_j) - x_{j+1}h'(x_{j+1}) + x_jh'(x_j)}{h'(x_j) - h'(x_{j+1})}$$

Then we put this newly calculated z into the z vector and initialize the z vector.

```

init_z <- function(mat){
  z <- c()
  z[1] <- mat[1,1]
  z[3] <- mat[2,1]
  #if the slopes at a and b are equal, set z as the mean of a and b
  if (abs(mat[2,3]-mat[1,3]) < 1e-9){
    z[2] <- (mat[2,1] + mat[1,1])/2
  } else {
    z[2] <- calc_z(mat)
  }
  return(z)
}

```

- 6. update_z: update_z is a function we use to update the z vector with new z values. Each time we add a x value and recalculate the tangent line based on our current x value, we will get two new z points. we will add these two new z points in our vector. The location in vector z where we add these two new z points is determined by the magnitude of z relative to the existing z points.

```

update_z <- function(z,x_star,mat){
  if (nrow(mat) <= 2){
    stop("something wrong with info matrix", call. = FALSE)
  }
  index <- which(mat[,1] == x_star)
  z <- c(z,NA)
  tmp_mat1 <- mat[(index-1):index,]
  tmp_mat2 <- mat[index:(index+1),]
  if (sum(abs(diff(mat[(index-1):(index+1),3]))) < 1e-9){
    z1 <- mat[(index-1),1] + (mat[(index+1),1] - mat[(index-1),1])/3
    z2 <- mat[(index-1),1] + (mat[(index+1),1] - mat[(index-1),1])/3*2
  } else {
    z1 <- calc_z(tmp_mat1)
    z2 <- calc_z(tmp_mat2)
  }
}

```

```

}
if((round(z1, digits = 6) > round(z2, digits = 6))){
  stop('something wrong with updated z value, check log-convexity!',
       call. = FALSE)
}
z[(index+2):length(z)] <- z[(index+1):(length(z)-1)]
z[index] <- z1
z[index+1] <- z2
return(z)
}

```

- 7. `sample_x`: Generate a new proposed sample point. We use information from `mat` to calculate $u_k(x)$ and $s_k(x)$ and then draw a random number from uniform distribution to determine the range of z we want to sample from. Then we draw another random number from uniform distribution to get the sample x through evaluating the inverse cdf at the value of random draw. The function would return the sample point.

```

sample_x <- function(z, mat, n=1){
  uk1 <- exp(mat[,2]+(z[-1]-mat[,1])*mat[,3])
  uk2 <- exp(mat[,2]+(z[-length(z)]-mat[,1])*mat[,3])
  p <- rep(NA,nrow(mat))
  p <- (uk1-uk2)/mat[,3]

  # normalize p
  p_norm <- p/sum(p)
  w <- runif(n)
  # determine the range we want to sample from
  i <- sum(cumsum(p_norm) < w) + 1

  # sample x using inverse cdf
  samp_x <- (log(p[i]*mat[i,3]*runif(n)+
               exp(mat[i,2]+(z[i]-mat[i,1])*mat[i,3]))-mat[i,2])/mat[i,3]+mat[i,1])
  return(samp_x)
}

```

- 8. `upper_bound`: Calculate the upper bound. We take the `z`, `mat` and the newly generated sample `x` as input. We compare sample `x` with `z` to find which segment of line to calculate from, which is the index of smallest `z` that is larger than sample `x`. Then we use the linear relationship to get the upper bound.

$$u_k(x) = h(x_j) + (x - x_j)h'(x_j)$$

- 9. `lower_bound`: Calculate the lower bound. We take the `mat` and newly generated sample `x` as input. We compare sample `x` with first column of `mat` to find the index of largest `x` that is smaller than sample `x` and use the following formula to get the lower bound.

$$l_k(x) = \frac{(x_{j+1} - x)h(x_j) + (x - x_j)h(x_{j+1})}{x_{j+1} - x_j}$$

Main Part (ars): We try to find the appropriate starting point based on the information user provided and the log-concavity of the density function. We initialize `mat` and `z` vector. While the sample size does not reach our desired number, we repeat the rejection sampling process. First, we check the log-concavity of the density function every time we update the `mat`. Then we generate one proposed sample `x` using `sample_x` function. We draw one random number (`w`) from the uniform distribution and also calculate the upper bound (`u`) and lower bound (`l`) using corresponding functions. Then we determine if:

1. $w \leq \exp(l-u)$, then accept the sample x and store it in the final result vector.
2. $w \leq \exp(h(\text{sample } x)-u)$, then accept the sample x , store it in the final result vector, update the mat and z with this sample x .
3. Otherwise, reject the sample x and update the mat and z with this proposed sample x .

(b) Vectorization

We use matrix to store the updated information and we store z values and the x we sampled as vectors.

2. Testing

(a) Input Checking

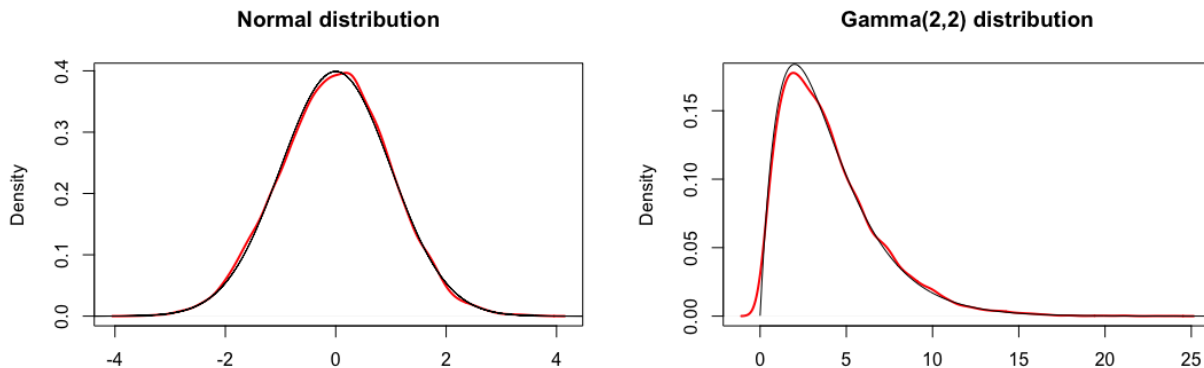
We wrote various tests to check if the inputs are valid, including check if the class of input function is actually a function, check if the starting points and sample size are numbers. We also checked whether the input function is log-concave and whether the function is defined on the given value.

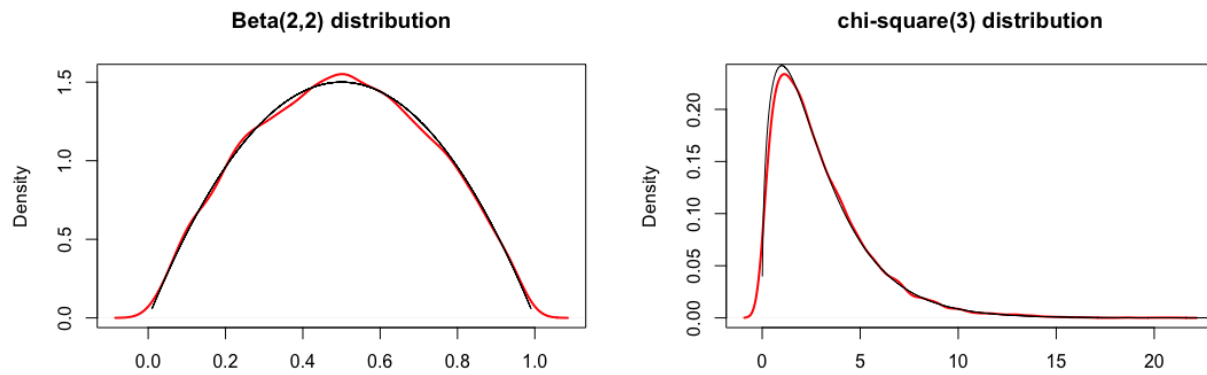
(b) Unit Tests

(c) Formal Testing

We checked several empirical distributions. We compared the samples we got from the adaptive rejection sampling algorithm we wrote with the samples we directly draw from the empirical distributions using the built-in functions in R. We use the `ks.test()` function in R to perform Kolmogorov-Smirnov test on the two samples we had in order to see if the two samples were drawn from the same continuous distribution. The log-concave empirical distributions we tested are: standard normal distribution, $\text{gamma}(2,2)$ distribution, $\text{beta}(2,2)$ distribution, and $\text{chi-square}(3)$ distribution. In the Kolmogorov-Smirnov test, our null hypothesis is that the samples are from the same distribution. In all the test results we have in regards to the above empirical distributions, the p-values are greater than 0.05. This means we fail to reject the null hypothesis. Therefore, we have done a pretty good job in terms of approximating the empirical distributions using adaptive rejection sampling. For the distribution that is not log-concave, we tested the e^{x^2} distribution. And our algorithm successfully gave us an error message.

(d) Graphical Comparison





3. Contributions

Yang, Jue and Winnie worked together to create the first draft of all the modular functions, and the main body of ars function. Then Jue started working on building the test cases for modular functions. And Yang contributed to the test cases included in the ars function. Winnie wrote the codes about how to find the appropriate starting points where $f(x)$ is defined. Then Jue took care of coding about special cases like uniform distribution case. Yang wrote the formal testing. Winnie wrote the overall help page and compiled the first draft of writeup. All of three members contributed to revisions. We worked together to make everything into a package.